Discrete Time and Continuous Time Dynamic Mean-Variance Analysis

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Abstract

Contrary to static mean-variance analysis, very few papers have dealt with dynamic mean-variance analysis. Here, the mean-variance efficient self-financing portfolio strategy is derived for n risky assets in discrete and continuous time. In the discrete setting, the resulting portfolio is mean-variance efficient in a dynamic sense. It is shown that the optimal strategy for n risky assets may be dominated if the expected terminal wealth is constrained to exactly attain a certain goal instead of exceeding the goal. The optimal strategy for n risky assets can be decomposed into a locally mean-variance efficient strategy and a strategy that ensures optimum diversification across time. In continuous time, a dynamically mean-variance efficient portfolio is infeasible due to the constraint on the expected *level* of terminal wealth. A modified problem where mean and variance are determined at $t = 0$ was solved by Richardson (1989). The solution is discussed and generalized for a market with n risky assets. Moreover, a dynamically optimal strategy is presented for the objective of minimizing the expected quadratic deviation from a certain target level subject to a given mean. This strategy equals that of the first objective. The strategy can be reinterpreted as a two-fund strategy in the growth optimum portfolio and the risk-free asset.

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1 Introduction

The problem of minimizing the variance of wealth at the end of one period when the expectation is constrained to a specified goal has been extensively studied in a static setting since the pioneering papers of Markowitz (1952), (1959). Surprisingly little has been said about multi-period or dynamic mean-variance analysis. Tobin (1965) solved for the optimal mean-variance efficient portfolio in a multi-period setting under the assumption that the portfolio is not adjusted according to new information over time. Mossin (1968) derived the optimal strategy by means of dynamic programming for two assets and two periods to analyze when myopic behavior is optimal. In the seventies, Hakansson (1971, 1979), Samuelson (1971) and other papers dealt with the question whether maximizing the average mean or the growth rate of wealth is consistent with utility theory. Subsequently, there was an extensive literature on the latter goal since maximizing the growth rate is equivalent to maximizing the logarithm of final wealth so that risk is also considered in the optimization problem. However, still many investors regard mean and variance as the relevant risk measures and managers are judged according to these portfolio features. The portfolio performance is usually not assessed after one period or on a day-to-day basis but over a longer time period. Thus, investors face the problem of minimizing the variance of wealth at the end of a predetermined investment horizon given that the expected terminal value meets a certain goal. Until the investment horizon, the portfolio may be rebalanced in a self-financing way.

In this paper, the latter problem is dealt with in a discrete time and in a continuous time setting. The dynamic mean-variance efficient self-financing portfolio strategy is derived for n assets.

In the discrete time discrete space setting, a dynamic programming approach yields the strategy which minimizes the variance of final wealth every period while keeping the expected final wealth at the required level. It is analyzed whether placing a constraint on the lower bound or on the value to be attained exactly by the expected terminal wealth fundamentally changes the solution. It is shown that the mean-variance efficient strategy for n risky assets can be found by decomposing the problem into two parts. At first, the locally mean-variance efficient portfolio is determined, and then the strategy that ensures optimum diversification across time is derived.

Unfortunately, in continuous time a dynamically mean-variance efficient portfolio is infeasible due to the constraint on the expected *level* of terminal wealth. However, a strategy that minimizes the variance of final wealth at the beginning of the investment horizon subject to a predetermined final wealth expected at time zero is feasible. The dynamic programming approach is not suitable to this problem formulation. Richardson (1989), Duffie and Richardson (1991), and Korn (1998) decompose the latter problem into a static optimization problem and into a martingale representation problem in the spirit of Pliska (1990). At first, the terminal portfolio value that minimizes the variance is identified within all possible terminal values which attain the required expectation. Then, the admissible strategy that generates this terminal value is determined using martingale theory. Richardson applies Hilbert space projection theory to obtain a closed-form solution for the optimal trading policy and the variance of terminal wealth for a market where there is one risky asset and a risk-free bond. A slightly different approach is taken by Korn (1998) who considers n stocks and places an additional constraint of non-negativity on final wealth. However, a price is paid for the latter advantage due to the lack of a closed-form solution. The optimal policy of the modified problem is discussed and generalized for a market with n risky assets. As time passes, there exist strategies that lead to a lower terminal variance and higher expected value. However, at time zero, these strategies do not satisfy the constraint on the terminal expectation.

Alternatively, the optimal strategy for an investor who continuously minimizes the expected quadratic deviation from a target level is derived for n assets with an optimal control approach. The target level is set so that the mean terminal wealth achieves a predetermined value. It turns out that this objective leads to the same optimal policy as minimizing the unconditional variance. Interestingly, the optimal strategy can be interpreted as a two-fund strategy, where the two funds are the growth optimum portfolio and the risk-free asset.

The paper is organized as follows. In Section 2.1 the discrete framework is presented. After a problem formulation in Section 2.2, Section 2.3 presents the solution for a market with one risky asset. Section 2.4 generalizes the analysis for n stocks. In Section 3.1 the basic continuous time model is presented. The main problem is stated in Section 3.2, followed in Section 3.2.1. by the description of the minimum variance problem, and in Section 3.2.2 by the formulation of the minimum deviation problem. The results for one risky asset are presented in Section 3.3.1 and 3.3.2, respectively. Section 3.3.3 shows how to decompose the optimal strategy into a two-fund strategy. The solution for the general case of n assets is given in Section 3.4. Section 3.5 concludes.

2 The Discrete Time Model

2.1 The Basic Model

There exist n correlated assets $S_1, .., S_n$ each of which is driven by the same class of discrete time discrete space processes. The (column) vector of prices is given by $\mathbf{S} = (S_1, .., S_n)'$ with constant mean rate of return per period $\boldsymbol{\mu} = (\mu_1, .., \mu_n)'$ and constant $\boldsymbol{\sigma} = (\sigma)_{ij}$. Let $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ be the non-singular covariance matrix, and $\frac{\Delta \mathbf{S}_{t+1}}{\mathbf{S}_t}$ the vector of the rates of return at time t $\frac{\Delta \mathbf{S}_{t+1}}{\mathbf{S}_t} = \left(\frac{S_{it+1} - S_{it}}{S_{it}}, .., \frac{S_{nt+1} - S_{nt}}{S_{nt}} \right)'$. For simplicity, time period Δt is set equal to 1.

Furthermore, there exists a risk-free security with interest rate r per period. The market is complete.

A portfolio strategy is a vector control process $(\mathbf{w}_t, t \geq 0), \mathbf{w} = (w_i, .., w_n)'$ with $w_i(t)$ as the total amount of wealth invested in asset i at time t such that $\mathbf{w}'\mathbf{1}$ with $\mathbf{1} = (1, .., 1)'$ is the total amount of wealth invested in risky assets. It is assumed that $\mathbf{w}$ is admissible and non-anticipating.

$X_t^{\mathbf{w}}$ denotes the investor's wealth at time t under the strategy $\mathbf{w}$. The portfolio is rebalanced in a self-financing way:

$$\begin{aligned} X_{t+1}^{\mathbf{w}} &= \mathbf{w}_t' \left(\mathbf{1} + \frac{\Delta \mathbf{S}_{t+1}}{\mathbf{S}_t} \right) + \left(X_t^{\mathbf{w}} - \mathbf{w}_t' \mathbf{1} \right) (1 + r) \\ &= \mathbf{w}_t' \left(\frac{\Delta \mathbf{S}_{t+1}}{\mathbf{S}_t} - r\mathbf{1} \right) + X_t^{\mathbf{w}} (1 + r) \end{aligned}$$

2.2 Problem Statement

The investor's objective is to find a strategy that minimizes the variance of wealth at time horizon T subject to the constraint that the expectation of terminal wealth is never below the goal M:

$$var(X_T \mid X_t) = \inf_{\mathbf{w}} var\left(X_T^{\mathbf{w}} \mid X_t^{\mathbf{w}} \right) \tag{1}$$

subject to

$$E(X_T \mid X_t) \geq M \tag{2}$$

The optimal strategy is solved for via dynamic programming.

If the initial wealth of the investor X_0 exceeds the present value of the goal M then X_0 is invested entirely in the risk-free asset so that the investor does not face any risk. The portfolio is not rebalanced and the final wealth is $X_0(1+r)^T \geq M$. Since this case is trivial, it is not dealt with in the sequel.

2.3 Case 1: $n = 1$

In a market with one stock and a risk-free asset, there is no optimization problem in a one period model because the constraint determines how many shares of the risky asset are to be bought. However, in a dynamic framework there exists an optimization problem due to diversification across time. In the following, the optimal strategy will be determined for $n = 1$, such that $\boldsymbol{\mu} = \mu$, $\Sigma = \sigma^2$, and $\mathbf{w} = w$. To avoid triviality, it is assumed that $\mu > r$. The asset follows a multiplicative binomial process.
Recursively solving problem (1) yields the optimal amount to be invested in the risky asset at time $T-1$

$$w^*(T-1) = (M - X_{T-1}(1+r)) \frac{\pi}{\pi\sigma^2\pi} \tag{3}$$

where

$$\pi = \frac{\mu - r}{\sigma^2}.$$

At all prior dates $t < T-1$ the optimal amount is given by

$$w^*(t) = \left(M - X_t(1+r)^{T-t}\right)(1+r)^{-(T-(t+1))} \frac{\pi}{1+\pi\sigma^2\pi} \tag{4}$$

The minimum variance conditioned on the information at time $T-1$ and $t < T-1$, respectively, is

$$var(X_T \mid X_{T-1}) = \frac{1}{\pi\sigma^2\pi}(M - X_{T-1}(1+r))^2 \tag{5}$$

$$var(X_T \mid X_t) = \frac{1}{\pi\sigma^2\pi(1+\pi\sigma^2\pi)}\left(M - X_t(1+r)^{T-t}\right)^2. \tag{6}$$

The optimal amount invested in the risky asset is proportional to the difference of the present value of the goal and wealth. The conditional variance and the amount invested in the risky asset increase with decreasing time to maturity because there is less time left to attain the goal. Furthermore, both

values are the smaller, the nearer the wealth at the goal, and the variance decreases when the mean rate of return increases.
The optimal strategy ensures that the portfolio value never exceeds the discounted required mean terminal value ($M > X_t (1+r)^{T-t}$) for all $t < T$:

$$\begin{aligned} X_{t+1} &= w^*(t) \left(\frac{\Delta S_{t+1}}{S_t} - r \right) + X_t (1+r) \\ &= \left(\left(M - X_t (1+r)^{T-t} \right) \frac{\pi \left(\frac{\Delta S_{t+1}}{S_t} - r \right)}{1 + \pi \sigma^2 \pi} + X_t (1+r)^{T-t} \right) (1+r)^{-(T-(t+1))} \end{aligned}$$

and therefore

$$X_t (1+r)^{T-t} < M \Rightarrow X_{t+1} (1+r)^{T-(t+1)} < M$$

due to

$$\frac{\pi \left(\frac{\Delta S}{S} - r \right)}{1 + \pi \sigma^2 \pi} < 1 \tag{7}$$

for all realizations of $\frac{\Delta S}{S}$ in a complete and arbitrage-free market.
This can be easily checked. Let q be the probability of an up-move, and let r^u, r^d be the realizations of $\frac{\Delta S}{S}$ in case of an up-move and down move, respectively. Since $\left(r^d - r \right) < 0$ in an arbitrage-free market ($r^u > r > r^d$), only $\frac{\Delta S}{S} = r^u$ has to be checked. (7) then becomes $(\mu - r)(r^u - r) < \sigma^2 + (\mu - r)^2 \Rightarrow r(\mu - r^u) < E(\frac{\Delta S}{S}^2) - \mu r^u \Rightarrow r(1-q)(r^d - r^u) < (1-q) r^d \left(r^d - r^u \right) \Rightarrow r^d < r.$ $\square$
Since $M > X_t (1+r)^{T-t}$ for all $t < T$, the variance ((5), (6)) is never positive if the investor is able to meet the goal by investing entirely in bonds. Furthermore, short-selling is never optimal. The final portfolio value is given by

$$\begin{aligned} X_T &= w^*(T-1) \left(\frac{\Delta S}{S} - r \right) + X_{T-1} (1+r) \\ &= (M - X_{T-1} (1+r)) \frac{1}{\mu - r} \left(\frac{\Delta S}{S} - r \right) + X_{T-1} (1+r) . \end{aligned}$$

It exceeds the required expected value in case of an up-move and is below M in case of a down move. Hence, the optimal strategy satisfies at time $T-1$

constraint (2). At all periods prior to $T-1$ the minimum variance policy that leads to the portfolio value at time $T-1$ is derived.
The optimal strategy ensures that constraint (2) is binding

$$E(X_T \mid X_t) = M$$

since this leads to minimum variance in a complete market.
Hence, in a multi-period framework, there is diversification across time for $n = 1$. It is optimal to invest less in the risky asset when the portfolio value is close to the required expected final value and when there is much time left.

2.4 Case 2: $n > 1$

In this section, problem (1) is solved for n risky and one risk-free asset.
The optimal strategy achieves an expected value $E(X_T \mid X_t) > M$ for some stock price processes and parameters even if the market is complete. However, if $E(X_T \mid X_t) > M$ no 'nice' formula for the optimal portfolio strategy is obtained. In order to compare the optimal strategy for n assets with the strategy for one asset, only those optimal strategies that result in $E(X_T \mid X_t) = M$ are dealt with.
The optimal control vectors at time $T-1$ and $t < T-1$, respectively are given by

$$\mathbf{w}_{T-1}^* = \left(M - X_{T-1}\left(1+r\right)\right)\frac{\boldsymbol{\pi}}{\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}}$$

$$\mathbf{w}_t^* = \left(M - X_t\left(1+r\right)^{T-t}\right)\left(1+r\right)^{-(T-(t+1))}\frac{\boldsymbol{\pi}}{1+\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}} \tag{8}$$

where

$$\boldsymbol{\pi} = \Sigma^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}\right).$$

The corresponding minimum variances conditioned on the information at time $T-1$ and $t < T-1$, respectively are

$$var(X_T \mid X_{T-1}) = \frac{1}{\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}}\left(M - X_{T-1}\left(1+r\right)\right)^2$$

$$var(X_T \mid X_t) = \frac{1}{\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}\left(1+\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}\right)}\left(M - X_t\left(1+r\right)^{T-t}\right)^2.$$

For $M > X_0 (1+r)^T$ the amount invested in each stock is proportional to the difference of the present value of the goal and the portfolio value. The optimal strategy implies that wealth never exceeds the present value of the required mean: $M > X_t (1+r)^{T-t}$ for all $t < T$.
Interestingly, the above result can also be derived by decomposing the portfolio problem. First, the investor solves for the locally mean-variance efficient portfolio. The variance of the portfolio rate of return per period is minimized subject to the mean rate of return per period being constrained to a predetermined value μ_{PF}. Let $\boldsymbol{\theta} = (\theta_i, .., \theta_n)$ denote the vector of portfolio weights, i. e. the proportion of wealth invested in each asset of the total wealth invested in the portfolio. The problem to be solved is

$$\sigma_{PF}^2 = \min_{\boldsymbol{\theta}} \ \boldsymbol{\theta}' \Sigma \boldsymbol{\theta}$$

subject to

$$\boldsymbol{\theta}' \Sigma \boldsymbol{\pi} + r = \mu_{PF}.$$

This yields the vector of optimal portfolio weights

$$\boldsymbol{\theta}^* = \frac{\mu_{PF} - r}{\boldsymbol{\pi}' \Sigma \boldsymbol{\pi}} \boldsymbol{\pi}$$

with the corresponding variance of the rate of return per period of the portfolio with mean rate of return μ_{PF}

$$\sigma_{PF}^2 = \frac{(\mu_{PF} - r)^2}{\boldsymbol{\pi}' \Sigma \boldsymbol{\pi}}.$$

Secondly, optimization problem (1) is solved for just two funds, the portfolio with parameters μ_{PF}, σ_{PF}^2 and the risk-free asset. The optimal amount invested in the risky portfolio is determined by (cf. (4))

$$\begin{aligned} w^*(t) &= \left(M - X_t (1+r)^{T-t} \right) (1+r)^{-(T-(t+1))} \frac{\pi_{PF}}{1 + \pi_{PF} \sigma_{PF}^2 \pi_{PF}} \\ &= \left(M - X_t (1+r)^{T-t} \right) (1+r)^{-(T-(t+1))} \frac{\pi_{PF}}{1 + \boldsymbol{\pi}' \Sigma \boldsymbol{\pi}} \end{aligned}$$

where

$$\pi_{PF} = \frac{\mu_{PF} - r}{\sigma_{PF}^2} = \frac{\boldsymbol{\pi}' \Sigma \boldsymbol{\pi}}{\mu_{PF} - r}.$$

Consequently, the amount invested in each asset $w^*(t)\boldsymbol{\theta}^*$ is given by (8). Hence, problem (1) for n assets can be reduced to a diversification-across-time and a diversification-across-assets subproblem: First, the locally mean-variance efficient portfolio is solved for, such that the variance of the rate of return is minimized for a given mean rate of return. Thereafter, the optimal strategy for the minimum variance of wealth at the investment horizon subject to a predetermined mean terminal wealth is obtained for an investor who invests in two funds: The locally mean-variance efficient portfolio and the risk-free asset. The required mean rate of return μ_{PF} does not enter the optimal solution and can be chosen arbitrarily. The amount of wealth invested into the locally mean-variance efficient portfolio is, however, determined by the required expected terminal wealth.

As stated above, strategy (8) is only optimal if $E(X_T \mid X_t)$ always exactly equals its lower bound M. A prerequisite for this is that there are no realizations of stock prices such that $\boldsymbol{\pi}' \left(\frac{\Delta \mathbf{S}}{\mathbf{S}} - r\mathbf{1}\right) \geq 1 + \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$. Otherwise, strategy (8) solves the modified problem

$$\inf_{\mathbf{w}} var\left(X_T^{\mathbf{w}} \mid X_t^{\mathbf{w}}\right) \tag{9}$$

subject to

$$E(X_T \mid X_t) = M. \tag{10}$$

Since this modified problem constrains the conditional expected terminal value to exactly match goal M, the portfolio value X_t is above the present value of the goal $M\,(1+r)^{-(T-t)}$ for some stock price paths. Hence, in these cases the variance is positive even though rebalancing the portfolio such that everything is invested at the risk-free rate leads to a deterministic final wealth that is above M. This strategy is clearly counter-intuitive. There exists a strategy which results in a deterministic final value above the required mean final value. However, this strategy is not pursued but a strategy that involves some risk in order to meet the goal exactly. Hence, the optimal strategy throws money away on average.

Consequently, while solving problem (9) subject to the equality constraint (10) for $n = 1$ results in the same optimal portfolio as solving (9) subject to the inequality constraint (2), the optimal strategy may differ for $n > 1$ risky assets. If $\boldsymbol{\pi}' \left(\frac{\Delta \mathbf{S}}{\mathbf{S}} - r\mathbf{1}\right) < 1 + \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$ for all stock price realizations, then the solutions of both problems coincide. Otherwise, the equality constraint (10) is clearly not appropriate for investors who prefer more wealth to less since for some stock price paths a risky strategy is followed in order to give money away on average. Hence, in the general case, it is inevitable to solve problem (1) with constraint (2) to obtain a policy which is not dominated.

3 The Continuous Time Model

3.1 The Basic Model

There exist n correlated stocks generated by n independent Brownian motions. The prices of these stocks are assumed to evolve as

$$dS_i = S_i \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j$$

where μ_i and σ_{ij} are constants for $i, j = 1, .., n$ and W_j is a standard independent Brownian motion. The market is frictionless.
Furthermore, there exists a risk-free bond whose price evolves according to

$$dB = \rho B dt$$

where ρ denotes the constant risk-free interest rate.
The self-financing portfolio strategy is an admissible and non-anticipating vector control process $(\mathbf{w}_t, t \geq 0), \mathbf{w} = (w_i, ..., w_n)'$ with $w_i(t)$ as the total amount of wealth invested in asset i at time t. Under the strategy $\mathbf{w}$, the investor's wealth $X_t^{\mathbf{w}}$ follows the process

$$\begin{aligned} dX^{\mathbf{w}} &= \sum_{i=1}^{n} w_i \frac{dS_i}{S_i} + (X^{\mathbf{w}} - \sum_{i=1}^{n} w_i) \frac{dB}{B} \\ &= \left(\mathbf{w}' \left(\boldsymbol{\mu} - \rho \mathbf{1}\right) + \rho X^{\mathbf{w}}\right) dt + \mathbf{w}' \boldsymbol{\sigma} \mathbf{dW} \end{aligned}$$

where $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)'$, $\boldsymbol{\sigma} = (\sigma)_{ij}$, and $\mathbf{W} = (W_1, .., W_n)'$ denotes a standard n dimensional Brownian motion defined on the complete probability space.
Let $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ be the non-singular covariance matrix, and let the vectors γ and π be defined by

$$\boldsymbol{\gamma} = \boldsymbol{\sigma}^{-1} \left(\boldsymbol{\mu} - \rho \mathbf{1}\right)$$

$$\boldsymbol{\pi} = \Sigma^{-1} \left(\boldsymbol{\mu} - \rho \mathbf{1}\right) = \boldsymbol{\sigma}'^{-1} \boldsymbol{\gamma}.$$

According to Itô's Lemma, a function $\Psi(t, x)$ that is at least twice differentiable in x and once in t evolves as

$$d\Psi = \Psi_t dt + \Psi_x \left(\left(\mathbf{w}'\Sigma\boldsymbol{\pi} + \rho x \right) dt + \mathbf{w}'\boldsymbol{\sigma}\mathbf{dW} \right) + \frac{1}{2}\Psi_{xx}\mathbf{w}'\Sigma\mathbf{w}dt.$$

The generator of the wealth process can therefore be written as

$$\mathcal{D}^{\mathbf{w}}\Psi = \Psi_t + \Psi_x \left(\mathbf{w}'\Sigma\boldsymbol{\pi} + \rho x \right) + \frac{1}{2}\Psi_{xx}\mathbf{w}'\Sigma\mathbf{w}dt. \tag{11}$$

3.2 Problem Statement

In the discrete time discrete space setting, the terminal variance is dynamically minimized subject to a specified conditional expected terminal value. Solving this problem results in unbounded strategies in continuous time. If the portfolio value an instant before T is below the desired conditional mean M then the optimal portfolio must consist of an unbounded number of risky assets. This leads to unbounded variance. Consequently, as long as the investor desires a certain expected level of wealth and not a certain expected rate of return, a policy that ensures a conditional expected value M is infeasible. Due to this infeasibility, the continuous time solution cannot be derived by the limiting case of the discrete time solution and the problem posed is different.
In continuous time, the investor requires a certain unconditional expected terminal value specified at date $t = 0$, i. e. $E(X_T \mid X_0)$ should meet the goal M. At time $t > 0$ there are no constraints on the conditional expected final wealth. Two different criteria are dealt with in the sequel. The first objective is to find the self-financing continuously rebalanced portfolio that minimizes the (unconditional) variance $var(X_T \mid X_0)$ of terminal wealth under the constraint that $E(X_T \mid X_0)$ be equal to M. In the second model, the expected quadratic deviation of the terminal portfolio value from a target level Γ is continuously minimized. The target level is chosen such that M is a lower bound for the expected final wealth $E(X_T \mid X_0)$.

3.2.1 Minimum Variance

The minimum unconditional variance of terminal wealth subject to a predetermined unconditional expected terminal portfolio value is the solution of the optimization problem

$$var(X_T) = \inf_{\mathbf{w}} var(X_T^{\mathbf{w}} \mid X_0) \tag{12}$$

subject to

$$E(X_T \mid X_0) = M. \tag{13}$$

3.2.2 Minimum Quadratic Deviation From a Target Level

The investor's objective is to minimize the conditional expected quadratic deviation of terminal wealth from a target level Γ under the restriction that M is a lower boundary for the mean final wealth $E(X_T \mid X_0)$. The optimal value function of the mean squared deviation and the corresponding strategy is the solution of problem

$$\Psi(t,x) = \inf_{\mathbf{w}} E\left((X_T^{\mathbf{w}} - \Gamma)^2 \mid X_t^{\mathbf{w}}\right) = \inf_{\mathbf{w}} \Psi^{\mathbf{w}}(t,x) \tag{14}$$

subject to

$$\begin{aligned} \Psi(0, M\exp(-\rho T)) &= 0 \\ \Psi(t, \Gamma\exp(-\rho\,(T-t))) &= 0 \\ E(X_T \vert\, X_0) &\geq M. \end{aligned} \tag{15}$$

The first boundary condition ensures that all wealth is placed in the risk-free asset if the initial wealth equals the present value of the goal M. The second condition guarantees that the quadratic deviation of terminal wealth from the target level Γ is zero as soon as the target level can be reached by investing in the risk-free asset. This boundary condition presupposes $\Gamma > M$. The appropriate optimality equation for $\Psi^{\mathbf{w}}(t,x)$ is

$$\inf_{\mathbf{w}} \mathcal{D}^{\mathbf{w}} \Psi(t,x) = 0. \tag{16}$$

Assuming that a solution exists that satisfies $\Psi_x < 0, \Psi_{xx} > 0$, the optimal control vector is given by

$$\mathbf{w}^* = \arg\inf_{\mathbf{w}} \Psi^{\mathbf{w}}(t,x) = -\frac{\Psi_x}{\Psi_{xx}} \boldsymbol{\pi}. \tag{17}$$

Equation (17) is placed in (11) and (16). It remains to solve the nonlinear partial differential equation subject to the boundary conditions.

3.3 Case 1: $n = 1$

In this section, the optimal strategy and value functions of the two objectives are presented for one risky and a risk-free asset with $\mu > \rho$.

3.3.1 Minimum Variance

Richardson (1989) solved the minimum variance problem (12) for $n = 1$. Richardson does not use an optimal control approach but decomposes the problem: The minimum variance of terminal wealth under the requirement is explicitly determined with the help of convex programming and Hilbert space projection theory. Then, the strategy that leads to the terminal wealth which generates the minimum variance is derived via martingale theory. The following solution is presented for $M > X_0 \exp(\rho T)$

$$\begin{aligned} var(X_T) &= \frac{(M - X_0 \exp(\rho T))^2}{\exp(\gamma^2 T) - 1} \\ w^*(t) &= \left(\bar{\Gamma} \exp(-\rho(T-t)) - X_t\right) \pi \end{aligned} \tag{18}$$

where

$$\bar{\Gamma} = \frac{M - X_0 \exp(\rho T) \exp(-\gamma^2 T)}{1 - \exp(-\gamma^2 T)}$$

$$\gamma = \sigma^{-1} (\mu - \rho)$$

and

$$\pi = \Sigma^{-1} (\mu - \rho) = \sigma^{-1} \gamma.$$

The minimum variance and the amount invested in the risky asset increase with decreasing investment horizon because there is less time to attain the goal. Furthermore, the variance and w^* decrease with increasing drift and decreasing instantaneous volatility.
With the above solution as a starting point, the conditional variance of final wealth is obtained

$$var(X_T \mid X_t) = \left(\bar{\Gamma} - X_t \exp(\rho(T-t))\right)^2 \exp(-\gamma^2 (T-t)) \left(1 - \exp(-\gamma^2 (T-t)\right)$$

and

$$var(X_T \mid X_0) = \left(\bar{\Gamma} - X_0 \exp(\rho T)\right)^2 \exp(-\gamma^2 T) \left(1 - \exp(-\gamma^2 T)\right). \tag{19}$$

The wealth process when pursuing strategy (18) evolves according to

$$
\begin{aligned}
dX &= (w\Sigma\pi + \rho X)\, dt + w\sigma dW \\
&= \left(\pi\Sigma\pi \left(\bar{\Gamma}\exp(-\rho(T-t) - X_t\right) + \rho X\right) dt + \gamma(\bar{\Gamma}\exp(-\rho(T-t) - X_t)dW.
\end{aligned}
$$

This stochastic differential equation can be solved by standard methods to yield

$$
X_t = \bar{\Gamma}\exp(-\rho\,(T-t)) - \left(\bar{\Gamma}\exp(-\rho\,(T-t)) - X_0\exp(\rho t)\right)\exp\left(-\gamma W_t - \frac{3}{2}\gamma^2 t\right). \tag{20}
$$

The conditional expected terminal value at time $t > 0$

$$
E(X_T \mid X_t) = \bar{\Gamma} - \left(\bar{\Gamma} - X_t\exp(\rho\,(T-t))\right)\exp(-\gamma^2\,(T-t))
$$

does not match the goal M as long as the value at time t differs from its expected value $E(X_t \mid X_0)$. Hence, the expected final wealth exceeds M if $X_t > E(X_t \mid X_0)$.

As stated above, a strategy that continuously minimizes the variance of terminal wealth subject to a given conditional expected final wealth is infeasible. It is striking that the investor does not stop investing in the risky asset as soon as goal M is attained by investing in the risk-free asset ($X_t = M\exp(-\rho(T-t))$). This strategy has zero conditional variance. Rather, the investor backs out of the risky asset as soon as

$$
X_t = \bar{\Gamma}\exp(-\rho(T-t)) > M\exp(-\rho(T-t)).
$$

This seemingly striking investment policy is caused by constraint (13). If the portfolio value is invested entirely in the risk-free asset as soon as $X_t = M\exp(-\rho(T-t))$, then the constraint is never satisfied since terminal wealth never exceeds M. As soon as X reaches the present value of the goal the investor opts out of the risky investment so that $X_T = M$. Since it is possible that $X_t \leq M\exp(-\rho(T-t))$ for all $t < T$ there exist realizations of X_T such that $X_T < M$. Hence, the expected final wealth $E(X_T)$ is below M when pursuing a strategy such that wealth is invested risk-free as soon as X_t equals the present value of M.

In fact, $X_t = \bar{\Gamma}\exp(-\rho(T-t))$ which causes no more risky investment according to the optimal strategy (18) requires $W_t = \infty$. Therefore, (almost) always some wealth is invested in the risky asset.

3.3.2 Minimum Quadratic Deviation From a Target Level

The minimum conditional expected quadratic deviation of X_T from a target level Γ given that $E(X_T \mid X_0) \geq M$ and the corresponding optimal strategy is derived for $n = 1$ using a two step procedure. First, the unconstrained problem is solved with the dynamic programming approach (16) and then the target level Γ is determined such that constraint (15) with $\Gamma > M$ is satisfied.

1. The solution of the nonlinear partial differential equation is

$$\Psi(t, X_t) = (\Gamma - X_t \exp(\rho\,(T-t)))^2 \exp(-\gamma^2\,(T-t)) \tag{21}$$

and the optimal strategy is given by

$$w^*(t) = (\Gamma \exp(-\rho\,(T-t)) - X_t)\,\pi. \tag{22}$$

When this policy is followed, the portfolio value at time t can be expressed as

$$\begin{aligned} X_t &= \Gamma \exp(-\rho\,(T-t)) \qquad (23) \\ &\quad - (\Gamma \exp(-\rho\,(T-t)) - X_0 \exp(\rho t)) \exp\left(-\gamma W_t - \frac{3}{2}\gamma^2 t\right). \end{aligned}$$

2. The constraint on $E(X_T \mid X_0)$ now determines the target level Γ:

$$\begin{aligned} E(X_T\,|X_0) &= \Gamma - (\Gamma - X_0 \exp(\rho T)) \exp\left(-\gamma^2 T\right) \geq M \\ \Gamma &\geq \frac{M - X_0 \exp(\rho T) \exp\left(-\gamma^2 T\right)}{1 - \exp\left(-\gamma^2 T\right)}. \end{aligned}$$

The optimal value function (21) decreases with decreasing Γ. Therefore, Γ is set equal to its lower bound such that $\Gamma = \bar{\Gamma}$. Hence, a strategy that minimizes the variance of terminal wealth at time $t = 0$ is equivalent to a strategy that continuously minimizes the expected quadratic deviation from a target level Γ, if the two strategies are constrained to ensure the same mean final wealth.

3.3.3 Two Fund Strategy

Note that the optimal strategy for both objectives can be viewed as a strategy in two funds: the risk-free asset and the optimal growth fund. The optimal growth fund maximizes the mean growth rate of wealth if a proportional strategy is pursued.
Let $\tilde{\pi}(t)Y_t$ be the amount invested in the risky asset at time t with Y_0 being the initial wealth. The wealth process for strategy $\tilde{\pi}Y_t$ evolves according to

$$dY^{\tilde{\pi}} = Y^{\tilde{\pi}} \left(\tilde{\pi}\Sigma\pi + \rho\right) dt + Y^{\tilde{\pi}}\tilde{\pi}\sigma dW. \tag{24}$$

This stochastic differential equation yields

$$Y_t^{\tilde{\pi}} = Y_0 \exp\left(\left(\tilde{\pi}\Sigma\pi + \rho - \frac{1}{2}\tilde{\pi}\Sigma\tilde{\pi}\right) t + \tilde{\pi}\sigma W_t\right).$$

The solution of problem

$$\sup_{\tilde{\pi}} E(\ln Y_t^{\tilde{\pi}})$$

determines the strategy that maximizes the mean growth rate of wealth:

$$\tilde{\pi}^* = \pi.$$

Strategy (18) is equivalent to a strategy in the risk-free bond and an optimal growth fund. At time $t = 0$ the amount $\Gamma \exp(-\rho T)$ is invested in bonds, and the amount $-\left(\Gamma \exp(-\rho T) - X_0\right)$ is invested in a risky fund. Since the latter amount is negative, the fund is sold short. The optimal strategy of the fund is to invest $-\pi F$ in the risky asset where $F_0 = (\Gamma \exp(-\rho T) - X_0)$ denotes the initial fund value. The fund value then follows the process

$$dF = F\left(-\pi\Sigma\pi + \rho\right) dt - F\pi\sigma dW.$$

Integration leads to

$$F_t = F_0 \exp(\rho t) \exp\left(-\gamma W_t - \frac{3}{2}\gamma^2 t\right).$$

When pursuing the two-fund-strategy the portfolio value at time t becomes (20):

$$X_t = \Gamma \exp(-\rho(T-t)) - (\Gamma \exp(-\rho T) - X_0) \exp(\rho t) \exp\left(-\gamma W_t - \frac{3}{2}\gamma^2 t\right).$$

The investor determines initially how much wealth is placed into the risk-free fund and how much of the risky fund is sold short. Thereafter, he only rebalances the risky fund. Since the fund is sold short and the fund invests $-\pi F$ in the risky asset, the investor actually pursues the growth optimum strategy πF.
This growth optimum strategy not only maximizes the mean growth rate of wealth but also minimizes the expected time to reach a certain goal as Merton (1990, Chapter 6) shows. Actually, the investor's goal to be reached is a level of wealth that requires no more risky investment and therefore has zero conditional variance. Pursuing the growth optimum strategy whereby investing all wealth leads to mean terminal wealth (see (24)):

$$E(X_T^\pi | X_0) = X_0 \exp\left(\left(\gamma^2 + \rho\right) T\right) \neq M.$$

Hence, a certain amount has to be invested in the risk-free asset such that the constraint is satisfied. The remaining part is invested according to the growth optimum strategy. The amount to be invested turns out to be Γ. Obviously, this mixed strategy leads to a lower variance than the pure growth optimum strategy.

3.4 Case 2: $n > 1$

The problem of continuously minimizing the expected quadratic deviation of X_T from a target level Γ given that $E(X_T \mid X_0) \geq M$ is solved for n risky assets. The two step procedure is the same as for $n = 1$.

1. The nonlinear partial differential equation (16) yields the optimal value function

   $$\Psi(t, X_t) = (\Gamma - X_t \exp(\rho (T-t)))^2 \exp(-\boldsymbol{\pi}' \Sigma \boldsymbol{\pi} (T-t))$$

   where

   $$\boldsymbol{\pi} = \Sigma^{-1} (\boldsymbol{\mu} - \rho \mathbf{1}) = \boldsymbol{\sigma}'^{-1} \boldsymbol{\gamma}.$$

   The optimal control vector is given by

   $$\mathbf{w}_t^* = (\Gamma \exp(-\rho (T-t)) - X_t) \boldsymbol{\pi}. \tag{25}$$

2. In order to determine the minimum target level Γ that satisfies the constraint on $E(X_T \mid X_0)$, the wealth process is derived:

$$
\begin{aligned}
X_t &= \Gamma \exp(-\rho\,(T-t)) \\
&\quad - (\Gamma \exp(-\rho\,(T-t)) - X_0 \exp(\rho t)) \exp\left(-\boldsymbol{\gamma}'\mathbf{W}_t - \frac{3}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}t\right).
\end{aligned}
$$

The optimal target level

$$\Gamma = \frac{M - X_0 \exp(\rho T) \exp\left(-\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}T\right)}{1 - \exp\left(-\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}T\right)}$$

follows from

$$E(X_T \mid X_0) = \Gamma - (\Gamma - X_0 \exp(\rho T)) \exp\left(-\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}T\right) \geq M.$$

Since maximizing $\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$ minimizes the value function, problem (14) for $n > 1$ can be reinterpreted as two subproblems to be solved sequentially as in the discrete time setting. In a first step, the instantaneous mean-variance efficient portfolio is derived. The instantaneous variance of a portfolio is minimized subject to a given instantaneous drift μ_{PF}. Let $\boldsymbol{\theta} = (\theta_i, .., \theta_n)$ be the vector of portfolio weights. In order to find the efficient portfolio the problem

$$\sigma_{PF}^2 = \min_{\boldsymbol{\theta}}\ \boldsymbol{\theta}'\Sigma\boldsymbol{\theta}$$

subject to

$$\mu_{PF} = \boldsymbol{\theta}'\Sigma\boldsymbol{\pi} + \rho$$

is solved. This leads to the vector of optimal portfolio weights

$$\boldsymbol{\theta}^* = \frac{\mu_{PF} - r}{\boldsymbol{\pi}'\Sigma\boldsymbol{\pi}}\boldsymbol{\pi}.$$

In a second step, problem (14) is solved for just two funds, the instantaneous mean-variance efficient portfolio and the risk-free asset. The optimal amount invested in the portfolio is

$$w^*(t) = (\Gamma \exp(-\rho\,(T-t)) - X_t)\,\frac{\mu_{PF} - r}{\sigma^2_{PF}}$$

The amount invested in each asset $w^*(t)\boldsymbol{\theta}^*$ is then given by (25).
The general solution for the minimum variance problem (12) can be derived in an analogous way.

4 Conclusion

In a discrete time setting, the dynamically mean-variance efficient portfolio is derived. In a model with only one risky asset, placing a lower bound on the mean terminal wealth yields the same solution as restricting mean terminal wealth to exactly meet a certain goal. Otherwise, these different problem formulations matter and may result in different optimal strategies. Constraining mean terminal wealth to achieve the goal exactly may lead to dominated policies. It is shown how the mean-variance efficient strategy for n risky assets can be decomposed into a locally mean-variance efficient strategy and a strategy that diversifies risk across time.
In a continuous time setting, a dynamically mean-variance efficient portfolio is infeasible. The solution to the problem of minimizing the unconditional variance when the unconditional expected terminal wealth is supposed to exactly match a certain goal is discussed. Furthermore, the optimal strategy of an investor who continuously minimizes the expected quadratic deviation from a target level is derived. The target level is set so that the unconditional expected terminal wealth attains a specified goal. The resulting strategy turns out to be the same as that of the first objective. Moreover, the strategy can be reinterpreted as a two-fund strategy whereby the two funds are the growth optimum portfolio and the risk-free asset.

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