Equivariant Constrained Willmore Tori in S^3

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Dekan:	Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter:	Prof. Dr. Franz Pedit
2. Berichterstatter:	Prof. Dr. Martin U. Schmidt

Contents

Introduction

We investigate conformally immersed tori in S^3 which are critical points of the Willmore functional under variations preserving the conformal type of the torus. These tori are called constrained Willmore. Examples of such tori are constant mean curvature (CMC) tori in a space form and Willmore tori, the later are critical for the Willmore functional with respect to all variations by immersions. Until now no examples of constrained Willmore tori are known which are not CMC tori in a space form or Willmore tori. In order to construct examples we restrict ourselves to immersions with a $1-$parameter group of Möbius symmetries. Such tori are called equivariant. The Euler-Lagrange equation of constrained Willmore tori is an elliptic partial differential equation. By restricting to equivariant tori we reduce this partial differential equation to an ordinary differential equation.

First examples of equivariant Willmore tori were constructed in [**P**]. The Willmore Hopf tori are the preimages of closed elastic curves on S^2 under the Hopf fibration. The conformal type of the torus is given by the length and enclosed area of the curve on S^2. Thus variations preserving the conformal type of the torus must preserve the length and enclosed area of the corresponding curve. Further equivariant Willmore tori were classified in [**FP**]. Another class of examples are the Delaunay tori, which are obtained by rotating elastic curves in the upper half plane, viewed as the hyperbolic plane H^2, around the $x-$axis. These tori are CMC in a space form as shown in [**B**]. Elastic curves in S^2 and H^2 are constructed in [**LS**].

Bohle [**B**] showed that all constrained Willmore tori are of finite type. This means that a certain Riemann surface associated to the conformally immersed torus, its spectral curve, has finite genus. This fact makes it possible to solve the Euler-Lagrange equation of constrained Willmore tori by Riemann theta functions.

In the first chapter we introduce the basic elements of the quaternionic theory for surfaces into S^4 following [**BuFLPP**]. Further, we discuss the lightcone model of the $n-$sphere and the invariants of surface theory in this setup. Moreover, the Weierstrass elliptic functions are defined and some of their properties that are needed in the last

chapter are discussed.

The second chapter deals with general properties of equivariant constrained Willmore tori. We show that every equivariant torus can be interpreted as the preimage of a curve under a certain Riemannian submersion from the round S^3 onto S^2. The parametrization of the surface given by this construction can be chosen to be conformal and the invariants of the surface can be expressed in terms of the invariants of the curve and of the submersion. We compute the Euler-Lagrange equation for the constrained Willmore problem and define an associated family of constrained Willmore surfaces.

We introduce the spectral curve Σ of a general conformally immersed torus $f : T^2 \to S^3$ in the third chapter. We call f a finite gap immersion, if Σ has finite genus. To the spectral curve one can associate a kernel bundle $\mathcal{L}_x \to \Sigma$ for every $x \in T^2$. It is shown in [**BoPP**] that for a fixed $x_0 \in T^2$ the map

$$\Psi : T^2 \to \mathrm{Jac}(\Sigma) \quad x \mapsto \mathcal{L}_x \mathcal{L}_{x_0}^{-1}$$

is a group homomorphism. The immersion can be reconstructed from the spectral curve and $\Psi(T^2)$. We call an immersion simple if it is uniquely determined by these data. An example where this is not the case is given in chapter 5.

As examples of for finite gap immersions we discuss the spectral curves of CMC and constrained Willmore tori in S^3. Following [**H**], [**PS**] and [**BoB**] we show that the spectral curves of CMC tori are hyperelliptic and by [**B**] the spectral curves of constrained Willmore tori are either hyperelliptic or given by a $4-$fold covering of $\mathbb{C}P^1$. We show that if the spectral curve of a simple constrained Willmore torus is hyperelliptic then the torus is CMC in a space form under some further restrictions. Further, we show that simple tori of spectral genus 1 are equivariant. If the spectral genus is 2 we show that simple constrained Willmore tori are either equivariant CMC in a space form.

In the fourth chapter we compute the spectral curve for an equivariant torus which are not necessarily constrained Willmore. It turns out that a finite gap solution have a hyperelliptic spectral curve. This spectral curve has two symmetries which yield reality conditions for the conformal Hopf differential of the torus. In analogy to equivariant harmonic tori into the $2-$sphere it has been conjectured that all equivariant constrained Willmore tori have spectral genus 1, since the known examples are given in terms of elliptic functions. We show that this is not true. Rather, all equivariant and conformally immersed tori of spectral genus 1 are CMC in some $3-$dimensional space form. Moreover they are associated to a Delaunay cylinder which is a surface of

revolution having constant mean curvature in a space form. Furthermore we show that constrained Willmore Hopf tori, which have never constant mean curvature unless they are homogenous, have spectral genus 2. In fact all equivariant and conformally immersed tori with spectral genus 2 are constrained Willmore, if they are non-isothermic. These tori lie all in the associated family of constrained Willmore Hopf cylinders. All other equivariant and constrained Willmore tori have spectral genus 3.

The last chapter deals with the construction of constrained elastic curves in S^2 and H^2 with periodic curvatures. Closed curves in H^2 and S^2 yield tori of revolution and Hopf tori, respectively. The Euler-Lagrange equation for constrained Willmore tori reduces to the Euler-Lagrange equation for constrained elastic curves. As shown in chapter 4 all equivariant constrained Willmore tori of spectral genus ≤ 2 are associated to cylinders build out of these curves. If the curve closes we obtain a torus. Thus we compute the closing condition for the curves and the Willmore energy of the corresponding tori. We end by showing that there exists generically a $1-$ dimensional space of Whitham deformations, i.e., deformations changing the spectral curve, preserving these closing conditions.

CHAPTER I

Basics

1. Quaternionic Theory

Quaternionic holomorphic theory was developed to investigate surfaces in S^3 and S^4. Here we collect some of the basic constructions which are used in the thesis. We refer to [**BuFLPP**] for further reading.

1.1. Quaternions. The Hamiltonian quaternions $\mathbb{H}$ are defined as the unitary associative $\mathbb{R}$-algebra generated by $\mathbb{i}, \mathbb{j}$ and $\mathbb{k}$ with

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -1$$
$$\text{and} \quad \mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i}.$$

$\mathbb{H}$ can be canonically identified with $\mathbb{R}^4$ as a real vector space. The product on $\mathbb{H}$ is not commutative and every non-zero element has a multiplicative inverse. Thus $\mathbb{H}$ is a skew-field and a $4-$ dimensional Division algebra over $\mathbb{R}$. An element $a \in \mathbb{H}$ is given by

$$a = a_0 + a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k}, \quad a_l \in \mathbb{R}.$$

The real part of a is $\mathrm{Re}(a) = a_0$ and the imaginary part is $\mathrm{Im}(a) = a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k}$. We define $\bar{a} := \mathrm{Re}(a) - \mathrm{Im}(a)$ to be the conjugate. Then $a = -\bar{a}$ if and only if $a \in \mathrm{Im}\mathbb{H}$. The space of purely imaginary quaternions $\mathrm{Im}\mathbb{H}$ can be identified with $\mathbb{R}^3$. The quaternions inherit the standard metric from $\mathbb{R}^4$:

$$< a, b >_{\mathbb{R}^4} = Re(a\bar{b}).$$

Further, the quaternionic product restricted to $\mathrm{Im}\mathbb{H}$ is given by

$$a \cdot b = a \times b - < a, b >_{\mathbb{R}^3},$$

where "$\times$" is the vector product of $\mathbb{R}^3$. Thus the multiplication of $a, b \in \mathrm{Im}\mathbb{H}$ is anti-commutative if and only if $a \perp b$ as vectors in $\mathbb{R}^3$.

After fixing an imaginary unit i, with $i^2 = -1$, the quaternions can be identified with $\mathbb{C}^2$. Then $\mathbb{H}$ splits into the direct sum of two complex vector spaces $\mathbb{H} = \mathrm{span}\{1, i\} \oplus \mathrm{span}\{1, i\}^{\perp}$. There is no canonical choice of such a unit but a whole S^2 of such choices.

Lemma. *Let $a, b \in \mathbb{H}$. Then*
(i) $a \cdot b = b \cdot a$ if and only if the imaginary parts of both are linearly

dependent, i.e. if a commutes with b then $a \in span_{\mathbb{R}}\{1, b\}$;
(ii) $a^2 = -1$ *if and only if* $|a|^2 = 1$ *and* $a \in Im\mathbb{H}$.

Nevertheless, the vector spaces $\mathbb{H}$ and $\mathbb{C}^2$ are usually identified by choosing i to be the left multiplication with the quaternionic $\mathbb{i}$. We obtain $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbb{j}$. If not explicitely stated we will fix this identification of $\mathbb{H}$ and $\mathbb{C}^2$ for the rest of the thesis. Another common way to identify $\mathbb{H}$ with $\mathbb{C}^2$ is to let i be the right multiplication by $\mathbb{i}$. Then we have $\mathbb{H} = \mathbb{C} \oplus \mathbb{j}\mathbb{C}$.

1.2. The Lie Group S^3. Since $\mathbb{H} \cong \mathbb{R}^4$ the set of unit length quaternions defines the 3−sphere. Let $a \in S^3$ and consider the right multiplication by a

$$R_a : \mathbb{H} \to \mathbb{H}, \quad R_a(x) = xa.$$

Identifying $\mathbb{H}$ with $\mathbb{C}^2$ there exist $a_1, a_2 \in \mathbb{C}$ with $a = a_1 + a_2\mathbb{j}$. Then the map $\tilde{R}_a : \mathbb{C}^2 \to \mathbb{C}^2$ induced by R_a is given by the $SU(2)$−matrix $A = \begin{pmatrix} a_1 & -\bar{a}_2 \\ a_2 & \bar{a}_1 \end{pmatrix}$ with:

$$\tilde{R}_a \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The map $S^3 \to SU(2) : a \mapsto A$ is a bijective group homomorphism. The Lie algebra of $SU(2)$ can be identified with $\mathrm{Im}\mathbb{H}$ which is the tangent space of S^3 at 1. The Lie bracket for a, $b \in \mathrm{Im}\mathbb{H}$ is given by

$$[a, b] = 2a \times b,$$

which is the anti-commuting part of the quaternionic product.

The well known fact that $SU(2)$ is a double covering of $SO(3)$ translates into out setting as follows: for $a \in S^3$ consider the map

$$g_a : \mathbb{H} \to \mathbb{H}, x \mapsto axa^{-1}.$$

This $\mathbb{R}$−linear map leaves $\mathrm{Im}\mathbb{H}$ invariant and is length preserving. Thus $g_a|_{\mathrm{Im}\mathbb{H}}$ is an element of $SO(3)$. Conversely, every element of $SO(3)$ can be obtained this way. Therefore the map

$$\pi : S^3 \to SO(3), a \mapsto g_a$$

yields the $2:1$ covering map from $S^3 \cong SU(2)$ to $SO(3)$.

1.3. Quaternionic Vector Spaces. A quaternionic vector space V is a $4n$ dimensional real vector space on which $\mathbb{H}$ acts from the right-tas scalar multiplication. Due to the lack of a canonical isomorphism between $\mathbb{H}$ and $\mathbb{C}^2$ it is advantageous to consider an additional complex structure on V compatible with the quaternionic structure. This

enables us to treat V as a complex vector space as well. In other words, we fix $J \in \mathrm{End}_{\mathbb{H}}(V)$ such that $J^2 = -\mathrm{Id}$ and define

$$(x + iy)v := vx + (Jv)y, \quad x, y \in \mathbb{R}.$$

The pair (V, J) is called a complex quaternionic vector space. Such a vector space splits into the $\pm\mathbb{i}-$ eigenspaces of J as every element $v \in V$ can be written as $v = \frac{1}{2}(v - Jv\mathbb{i}) + \frac{1}{2}(v + Jv\mathbb{i})$. Obviously the first term is an element of the $\mathbb{i}-$ eigenspace and the second term an element of the $(-\mathbb{i})-$ eigenspace of J. Thus we have $V = V_+ \oplus V_-$ with

$$V_+ = \{v \in V \mid Jv = v\mathbb{i}\} \quad \text{and} \quad V_- = \{v \in V | Jv = -v\mathbb{i}\}.$$

These eigenspaces are complex vector spaces with respect to J and right multiplication of $\mathbb{j}$ defines an complex isomorphism between V_+ and V_-.

Remark. This splitting of complex quaternionic vector spaces generalizes the splitting of $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbb{j}$ by choosing J to be the left multiplication with the quaternionic $\mathbb{i}$.

1.4. The Quaternionic Projective Space. The $n-$dimensional quaternionic projective space $\mathbb{H}P^n$ is defined, analogously to the real or complex case, to be the space of quaternionic lines in the quaternionic vector space $\mathbb{H}^{n+1}$. For surface theory in S^4 the only interesting case is $n = 1$. Therefore we restrict ourselves to $\mathbb{H}P^1$ here. The general case works analogously.

The space $\mathbb{H}P^1$ is defined to be the space of quaternionic lines in $\mathbb{H}^2$. We have a canonical projection

$$\pi : \mathbb{H}^2 \setminus \{0\} \to \mathbb{H}P^1, x \mapsto [x] = x\mathbb{H}.$$

The manifold structure is given by affine coordinates for $\mathbb{H}P^1$ defined on the open sets $U_1 = \mathbb{H}P^1 \setminus \{[1, 0]\}$ and $U_2 = \mathbb{H}P^1 \setminus \{[0, 1]\}$. On these open sets we define diffeomorphisms

$$g_1 : U_1 \to \mathbb{H} : [x, y] \mapsto xy^{-1}$$

and

$$g_2 : U_2 \to \mathbb{H} : [x, y] \mapsto yx^{-1}.$$

The transition function $g_2 \circ g_1^{-1}$ is given by $x \mapsto x^{-1}$.

It is easy to show that $\mathbb{H}P^1$ is diffeomorphic to S^4. Let $< ., . >$ denote the standard hermitian metric on $\mathbb{H}^2 \cong \mathbb{C}^4$. At any point $x \in \mathbb{H}^2 \setminus \{0\}$ we denote by $(v)^N$ the projection of $v \in \mathbb{H}^2$ to $(x\mathbb{H})^\perp$. Let $v, w \in \mathbb{H}^2$. Then the Fubini-Study metric of $\mathbb{H}P^1$ at a point $[x] \in \mathbb{H}P^1$ is given by

$$< d_x\pi(v), d_x\pi(w) >_{\mathbb{H}P^1} = \frac{1}{<x,x>} Re(< (v)^N, (w)^N >).$$

With respect to this metric $\mathbb{H}P^1$ has constant curvature 4 and is isometric to the round S^4.

We need to have a nice description of the tangent space $T_l\mathbb{H}P^1$. It is easy to show that $\ker(d_x\pi) = l\mathbb{H}$, $x \in l$ and $d_{x\lambda}\pi(v\lambda) = d_x\pi(v)$. Therefore we have a isomorphism

$$d_x\pi : \mathbb{H}^2/l \to T_l\mathbb{H}P^1, \quad l = \pi(x).$$

This isomorphism depends on the choice of x. To eliminate this dependence consider the space $\mathrm{Hom}(l, \mathbb{H}^2/l)$. An element of this space $F \in \mathrm{Hom}(l, \mathbb{H}^2/l)$ is fully determined by its valued at an arbitrary $x \in l$. Thus $\mathrm{Hom}(l, \mathbb{H}^2/l)$ is isomorphic to $\mathbb{H}^2/l$ and the isomorphism

$$\mathrm{Hom}(l, \mathbb{H}^2/l) \to T_l\mathbb{H}P^1, F \mapsto d_x\pi(F(x))$$

is well defined and independent of the choice of $x \in l$.

1.5. Quaternionic Line Bundles.

Definition. A quaternionic vector bundle V of rank n is a real vector bundle of rank $4n$ together with a fiber- preserving action of $\mathbb{H}$ from the right such that the fibers become quaternionic vector spaces.

A complex quaternionic vector bundle is a pair (V, J) consisting of a quaternionic vector bundle and a complex structure J on V, compatible with the quaternionic structure, i.e., $J \in \Gamma(\mathrm{End}_{\mathbb{H}}(V))$, with $J^2 = -\mathrm{Id}$.

Remark. The fibers of any complex quaternionic vector bundle V are complex quaternionic vector spaces. Thus V decomposes into the direct sum of two complex vector bundles

$$V_+ = \{v \in V | Jv = v\mathbb{i}\} \text{ and } V_- = \{v \in V | Jv = -v\mathbb{i}\}.$$

Though $\mathrm{End}_{\mathbb{H}}(V)$ is not a quaternionic vector bundle itself, it is still a real vector bundle and can be decomposed into two complex vector bundles given by the $J-$linear and the $J-$anti-linear endomorphisms. We denote these by $\mathrm{End}_+(V)$ and $\mathrm{End}_-(V)$, respectively.

We restrict ourselves now to quaternionic line bundles. The first observation is:

Lemma. *Let L be a quaternionic line bundle over a Riemann Surface. Then it is isomorphic to the trivial $\mathbb{H}-$bundle.*

Proof. By the transversality theorem there exist a section which intersects the zero-section transversally. As the dimension of the total space is 6, the codimension of any section is 4. Thus a transverse section cannot intersect the zero section at all. But then it is a trivializing section of the line bundle. □

1.6. Conformal Immersions into S^3.

Lemma. *Let M be a Riemann surface. Then there exist a one-to-one correspondence between quaternionic line subbundles $L \subset V := M \times \mathbb{H}^2$ and maps $f : M \to \mathbb{H}P^1 \cong S^4$.*

Proof. Let $\mathcal{T} := \{(x, l) | l \in \mathbb{H}P^1 \text{ and } x \in l\}$ be the tautological bundle of $\mathbb{H}P^1$. To any $f : M \to \mathbb{H}P^1$ one can assign the line bundle $f^*\mathcal{T} \subset V$. Conversely, to a line bundle $L \subset M \times \mathbb{H}^2$ define the map f by $f(x) = L_x \in \mathbb{H}P^1$. □

Instead of dealing with maps $f : M \to S^3$, we consider quaternionic line bundles. Since the target space of the maps we are interested in is $S^3 \subset \mathbb{H}$, $f(x) \in U_1 = \mathbb{H}P^1 \setminus [1, 0]$, for all $x \in M$. This yields that a trivializing section of L is given by $x \in M \mapsto (f(x), 1) \in (f^*\mathcal{T})_x$.

The more convenient line bundle for our proposes is the quotient bundle V/L. The fiber of this bundle is pointwise defined to be the quotient of $\mathbb{H}^2$ by L_p. Let π_L denote the projection from V to V/L. For maps into S^3 the sections given by $\pi_L(1, 0)$ and $\pi_L(0, 1)$ are trivializing sections of V/L. Moreover the map f can be reconstructed as the quotient of these sections since

$$\pi_L(1, 0)f + \pi_L(0, 1) = \pi_L(f, 1) = 0.$$

A related bundle to V/L is the dual bundle L^{-1}. Let α and β be the dual basis of the constant sections $(1, 0)$ and $(0, 1)$ of V. Then the restriction of α and β to L are sections in L^{-1}. If f maps to S^3, then both sections α and β are non-vanishing. Further we have in this case that L^{-1} is isomorphic to V/L. The isomorphism is given by $\alpha \mapsto \pi_L(1, 0)$.

For $\psi \in \Gamma(L)$ we have $d\psi : T(M) \to \mathbb{H}^2$. Let π_L denote the projection from V to V/L. Then $\pi_L(d\psi)$ is a map from $T(M)$ to $T\mathbb{H}P^1$. Further for $\lambda : M \to \mathbb{H}$ we get

$$\pi_L(d(\psi\lambda)) = \pi_L((d\psi)\lambda + \psi(d\lambda)) = \pi_L(d\psi)\lambda.$$

Thus the map $\psi \to \pi_L(d\psi)$ is tensorial and we can define the following:

Definition. Let $f : M \to S^3$, L the associated line bundle and $\psi \in \Gamma(L)$ a trivializing section. The differential δ of L is the element of $\Omega^1\mathrm{Hom}(L, V/L)$ given by $\psi \mapsto \delta(\psi) = \pi_L(d\psi)$ for $\psi \in \Gamma(L)$.

By definition the map $f : M \to \mathbb{H}P^1$ corresponding to the line bundle L is immersed, i.e., df is injective, if and only if δ is injective.

Since we want to study conformal immersions from T^2 to S^3, we give a quaternionic definition of conformality, see [**BuFLPP**]. This definition is shown to coincide with the usual one in case of immersions.

1.7. Definition. Let (M, J) be a Riemann surface. A map $f : M \to \mathbb{H}$ is conformal, if there exists $N, R : M \to \mathrm{Im}\mathbb{H}$ with $N^2 = R^2 = -1$ such that

$$df \circ J = *df = Ndf = -dfR.$$

Remark. If f is an immersion, i.e., df is injective at every point, it is sufficient to claim the existence of either N or R. If furthermore the target space of f is $SU(2) = S^3 = \{v \in \mathbb{H} | |v|^2 = 1\}$ then N and R are the right and left translation of the normal vector of f to the Lie algebra of S^3, which is $\mathrm{Im}\mathbb{H}$.

The following lemma shows that conformal maps in the quaternionic setup are well-defined.

1.8. Lemma. *(Fundamental Lemma)*
Let $U \subset \mathbb{H}$ be a real subspace of dimension 2. Then there exist $N, R \in \mathbb{H}$ such that

1. $N^2 = R^2 = -1$,
2. $NU = U = UR$,
3. $U = \{x \in \mathbb{H} | NxR = x\}$.

The pair (N, R) is unique up to sign and there is only one such pair compatible to a fixed orientation of U. For any pair N and R the sets

$$U := \{x \in \mathbb{H} | NxR = x\}, \quad U^\perp = \{x \in \mathbb{H} | NxR = -x\}$$

are orthogonal real subspaces of dimension 2.

The proof of the lemma can be found in [**BuFLPP**].

Remark. Let (M, J) be a Riemann surface and $f : M \to \mathbb{H}$ be a conformal immersion with normal vectors N and R. Further let $z = x + iy$ be a conformal coordinate on M. Then we have locally

$$df = \lambda dx + N\lambda dy = \lambda dx - \lambda R dy,$$

for a quaternionic valued function λ. This yields $N = df(JX)df(X)^{-1}$, for all $X \in \Gamma(TM)$

A more common way to define a conformal immersion is to require f to be angle preserving, i.e.,

$$|df(X)|^2 = |df(JX)|^2 \text{ and } < df(X), df(JX) >_{\mathbb{R}^4} = 0, \quad \text{for } X \in \Gamma(TM).$$

Since the tangent space of f is real $2-$dimensional at every point, there is a $N \in \mathrm{Im}\mathbb{H}$ with $N^2 = -1$ such that the left multiplication with N preserves the tangent space. We have

$$< df(X), Ndf(X) >_{\mathbb{R}^4} = Re(df(X)\overline{Ndf(X)}) = |df(X)|^2 Re(N) = 0$$
$$\text{and} \quad |df(X)| = |N||df(X)| = |Ndf(X)|.$$

Thus $Ndf(X) = \pm df(JX) = \pm * df(X)$ and both definitions for conformal immersion coincides.

1.9. Quaternionic Holomorphic Structures. Let M be a Riemann surface and let K and $\bar{K}$ be its canonical and anti-canonical bundle. Further let V be a complex quaternionic vector bundle over M with complex structure J. Define

$$KV := K \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(E) | * \omega = J\omega\},$$
$$\bar{K}V := \bar{K} \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(V) | * \omega = -J\omega\}.$$

1.10. Definition. A quaternionic linear map $D : \Gamma(V) \to \Gamma(\bar{K}V)$ is called a holomorphic structure, if for all $\psi \in \Gamma(V)$ and $\lambda : M \to \mathbb{H}$ we have

$$D(\psi\lambda) = (D\psi) \cdot \lambda + \frac{1}{2}(\psi d\lambda + J\psi * d\lambda).$$

Example. Let (L, J) be a complex quaternionic line bundle, and let $\psi \in \Gamma(L)$ be a nowhere vanishing section. Then every other section φ of L is given by $\varphi = \psi\lambda$ for a quaternionic valued function λ. Via Leibniz rule the holomorphic structure is uniquely determined by its value for ψ. Thus there exist a unique holomorphic structure on L for which ψ is holomorphic. In particular we obtain a unique holomorphic structure on V/L for which the sections $\pi_L(1, 0)$ and $\pi_L([0, 1)$ are holomorphic, if f is conformal. Analogously for the bundle L^{-1}.

Unlike the complex case a holomorphic structure on E does not commute with the complex structure J on E in general. We get rather a decomposition into a $J-$linear and a $J-$anti linear part which we denote by $\bar{\partial} = \frac{1}{2}(D - JDJ)$ and $Q = \frac{1}{2}(D + JDJ)$. In contrast to $\bar{\partial}$, the Hopf field Q is not a first order differential operator but a section in $\bar{K}\mathrm{End}_-(L)$ and is called the Hopf field of D.

Let (L,J) be a complex quaternionic line bundle. Because quaternionic line bundles are always trivial, there exist a non vanishing section $\psi \in \Gamma(L)$. Then there exist a unique quaternionic holomorphic structure D such that ψ is holomorphic. As L has rank 1, there exists a quaternionic valued function N with $J\psi = \psi N$ with $N^2 = -1$. The Hopf field Q is tensorial thus it is fully determined by the value of $Q(\psi)$. And we obtain

$$\begin{aligned} 4Q(\psi) &= 2(D + JDJ)(\psi) = 2JDJ(\psi) \\ &= 2JD(\psi N) = 2J(\psi dN)'' \\ &= 2(\psi N dN)'' = \psi(NdN + *dN). \end{aligned} \tag{1.10.1}$$

Analogously to holomorphic structures we can define an anti-holomorphic structure for complex quaternionic vector bundles.

1.11. Definition. A quaternionic linear map $\bar{D} : \Gamma(V) \to \Gamma(KV)$ is called an anti-holomorphic structure, if for all $\psi \in \Gamma(V)$ and $\lambda : M \to \mathbb{H}$ we have

$$\bar{D}(\psi\lambda) = (\bar{D}\psi) \cdot \lambda + \frac{1}{2}(\psi d\lambda - J\psi * d\lambda).$$

Like holomorphic structures an anti-holomorphic structure splits into the $J-$commuting part ∂ and the $J-$anti-commuting part A. Again A is tensorial and is called Hopf field of $\bar{D}$.

Example. Let (V, J) be a complex quaternionic vector bundle, and ∇ be a connection on V. Then ∇ splits into $\nabla = \nabla' + \nabla''$, where $\nabla'' = \frac{1}{2}(\nabla + J * \nabla)$ is a holomorphic structure and $\nabla' = \frac{1}{2}(\nabla - J * \nabla)$ is an anti-holomorphic structure on V.

1.12. The Mean Curvature Sphere Congruence. For a surface $f : M \to S^4$ the conformal Gauß map or mean curvature sphere congruence assigns to every point $x \in M$ the unique $2-$sphere through $f(x)$ with the same tangent space at $f(x)$ and the same mean curvature vector there. We want to define the conformal Gauß map in the quaternionic setting. The first step is to identify the space of $2-$spheres in $S^4 = \mathbb{H}P^1$. Let

$$\mathcal{Z} := \{S \in \text{End } \mathbb{H}^2 | S^2 = -1\},$$

and for a fixed $S \in \mathcal{Z}$ let

$$S' := \{l \in \mathbb{H}P^1 | Sl = l\}.$$

Lemma. *(1) For a given $S \in \mathcal{Z}$ the set S' is a 2-sphere in S^4, i.e., it corresponds to a real 2-plane in $\mathbb{H} = \mathbb{R}^4$ in a suitable affine coordinate.*
(2) Given a 2-sphere A in S^4 there exist a $S \in \mathcal{Z}$, unique up to sign, such that the corresponding S' equals A.
(3) $\mathcal{Z}$ is the set of oriented 2-spheres in S^4.

Next we want to define the conformal Gauß map for a conformal immersion $f : M \to S^4$. Let $L \subset M \times \mathbb{H}^2$ be the corresponding quaternionic line bundle. By the previous lemma the conformal Gauß map of f is equivalent to the choice of a special complex structure S on $V = M \times \mathbb{H}^2$. Further let ∇ be the trivial connection on V. Then we can define a holomorphic structure on V with respect to S by $D = \nabla''$, and like in the last section we denote by Q the corresponding Hopf field, i.e., the $S-$anti-commuting part of D.

Theorem. *Let f be a conformal immersion into S^4 and let $L \subset V$ be the quaternionic line bundle associated to f. Then there exists a unique complex structure $S : M \to End(\mathbb{H}^2)$ on the bundle V such that*

$$SL \subset L, \quad dSL \subset L, \quad Q|_L = 0.$$

Remark. It is easy to show that the first property says that at each point $x \in M$ the sphere S_x goes through L_x, the second means that the tangent space of the sphere S_x and the surface coincides at $f(x)$, and the third property ensures that both sphere and surface have the same mean curvature there.

Since S is a complex structure on V, the trivial connection ∇ of (V, S) splits into $\nabla = \nabla' + \nabla''$, where ∇' is an anti-holomorphic structure and ∇'' is a holomorphic structure on V. The S anti-commuting parts of ∇' and ∇'', i.e., the Hopf fields of ∇, are denoted by A and Q, respectively. We can express A and Q in terms of S.

1.13. Lemma. *The Hopf fields A and Q are given by*

$$4A = SdS + *dS \quad \text{and} \quad 4Q = SdS - *dS.$$

Proof. We have $2\nabla' = \nabla - J * \nabla$ and $2\nabla'' = \nabla + J * \nabla$. Let $\psi : M \to \mathbb{H}^2$. Then $(\nabla S)\psi$ is given by

$$\begin{aligned}(\nabla S)\psi &= \nabla(S\psi) - S\nabla(\psi)\\ &= (\partial + A)S\psi + (\bar\partial + Q)S\psi + S(\partial + A)\psi + S(\bar\partial + Q)\psi\\ &= AS\psi + QS\psi - SA\psi - SQ\psi\\ &= -2S(A + Q)\psi\\ &= 2(*Q - *A)\psi\end{aligned}$$

Thus we obtain

$$SdS = 2(A + Q) \quad \text{and} \quad * dS = 2(A - Q).$$

This proves the statement.

□

To a given conformal immersion in $S^3 \subset \mathbb{H}$ we can associate two quaternionic line bundles L and V/L. As already discussed in section (1.5) we have that $(f, 1)$ is a trivializing section of L and $\pi_L(1, 0)$ and $\pi_L(0, 1)$ are both trivializing sections for V/L. In V/L the original map can be reconstructed as the ratio of these trivializing sections. Furthermore the constant sections $(1, 0)$ and $(0, 1)$ of V are parallel with respect to the trivial connection ∇ on V. To L there exist a canonical complex structure S on V given by its mean curvature sphere congruence. Because $SL \subset L$ and $dSL \subset L$ the projection of S to V/L, denoted by $J = \pi_L S$, is well defined and is a complex structure on V/L. Thus the holomorphic structure on V/L given by $D_{V/L} = \pi_L \nabla''$ is also well defined. With respect to $D_{V/L}$ the quaternionic linear independent sections $\pi_L(1, 0)$ and $\pi_L(0, 1)$ are holomorphic. It turns out that the dimension of the space of holomorphic sections of $D_{V/L}$, which is at least 2, is an important invariant of f.

In the coordinates $V = \operatorname{span}_{\mathbb{H}}\{(f,1),(1,0)\}$ the mean curvature sphere congruence S is computed to be

$$S = \begin{pmatrix} -R & Hf \\ 0 & N \end{pmatrix}, \tag{1.13.1}$$

where H is the mean curvature function and N and R are the left and the right normal vectors of the surface. Thus we obtain on V/L that $S\pi_L(1,0) = \pi_L(1,0)N$ and the formula in (1.10.1) gives:

$$Q\pi_L(1,0) = \pi_L(1,0)(NdN + *dN). \tag{1.13.2}$$

1.14. Surfaces in S^3. Let $(.,.)$ be the indefinite inner product on $V = \mathbb{H}^2$ given by

$$(v,w) := \bar{v}_1 w_2 + \bar{v}_2 w_1.$$

Then the set of isotropic lines $(l,l) = 0$ defines a $S^3 \subset \mathbb{H}P^1$. The $\mathbb{i}-$anti-commuting part of this inner product defines a symplectic structure on $\mathbb{C}^4 \cong \mathbb{H}^2$. Given the basis $(1,0)$ and $(0,1)$ of $\mathbb{H}^2$ and an endomorphism $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its adjoint map with respect to $(.,.)$ is given by $B^* = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$.

Since $(.,.)$ is non-degenerated it defines a isomorphism between V and V^*. For an immersion $f : M \to S^3$ let $L^\perp \subset V^*$ denote the annihilator bundle of L, i.e., $(L, L^\perp) = 0$. Obviously $L^\perp = L$ for surfaces in S^3. Further the following lemma holds.

Lemma. *Let $S^\perp$ denote the mean curvature sphere congruence of $L^\perp$. Then $S^\perp = S^*$.*

Proposition. *Let $L \subset V = M \times \mathbb{H}^2$ be a quaternionic line bundle. The corresponding map is a surface in S^3 if and only if $S^* = S$.*

The proof can be found in [**BuFLPP**]

1.15. Proposition. *For surfaces in S^3 we have the following formulas for the Hopf fields*

$$A^* = -Q \quad \text{and} \quad Q^* = -A.$$

Proof. By Lemma (1.13) we have

$$4A^* = (SdS + *dS)^*.$$

Since $S = S^*$ for maps into S^3, we obtain

$$4A^* = dSS + *dS = -SdS + *dS = -Q.$$

Then the equation $(A^*)^* = A$ yields $Q^* = -A$. □

1.16. The Weierstrass Representation.

1.17. Definition. Let (L, J) and $(\tilde{L}, \tilde{J})$ be complex quaternionic line bundles over a Riemann surface M. A pairing between these bundles is a $\mathbb{R}-$linear and point wise non-degenerate map

$$(.,.) : \tilde{L} \times L \to T^*M \otimes_{\mathbb{R}} \mathbb{H}$$

satisfying

$$(\psi\lambda, \varphi\mu) = \bar{\lambda}(\psi, \varphi)\mu$$

and

$$*(\psi, \varphi) = (J\psi, \varphi) = (\psi, J\varphi),$$

for sections $\psi \in \Gamma(\tilde{L})$, $\varphi \in \Gamma(L)$ and quaternionic valued functions λ and μ.

If such a map exist, then the bundles are called paired. Obviously, the pairing does depend on the order of the bundles appearing in the definition.

Example. Let (L, J) be a complex quaternionic line bundle. Then L and the bundle KL^{-1} with the induced complex structure are paired by evaluation, i.e.,

$$(.,.) : KL^{-1} \times L \to T^*M \otimes_{\mathbb{R}} \mathbb{H}, \quad (\omega, \psi) = \omega(\psi).$$

1.18. Proposition. *Let $\tilde{L}$ and L be paired complex quaternionic line bundles. Then $\tilde{L} \cong KL^{-1}$ as complex quaternionic line bundles.*

Proof. Let $(.,.)$ denote the pairing between L and $\tilde{L}$. The map $i : \tilde{L} \to KL^{-1}$ $\varphi \to (\varphi, .)$ is a well defined isomorphism. □

1.19. Definition. Let $\tilde{L}$ and L be two paired complex quaternionic bundles with holomorphic structures $\tilde{D}$ and D. The holomorphic structures are called compatible with respect to the pairing $(.,.)$, if the following equation holds for all sections $\varphi \in \Gamma(\tilde{L})$ and $\psi \in \Gamma(L)$.

$$d(\varphi, \psi) = (\tilde{D}\varphi \wedge \psi) + (\varphi \wedge D\psi).$$

Here d denotes the exterior derivative, and the wedge products are defined as follows:

$$(\tilde{D}\varphi \wedge \psi)(X, Y) = (\tilde{D}\varphi(X), \psi)(Y) - (\tilde{D}\varphi(Y), \psi)(X)$$
$$(\varphi \wedge D\psi)(X, Y) = (\varphi, D\psi(X))(Y) - (\varphi, D\psi(Y))(X).$$

1.20. Lemma. *Let KL^{-1} and L be complex quaternionic vector bundles and Let D be a holomorphic structure on L. Then there is a unique holomorphic structure $\tilde{D}$ on KL^{-1} such that the holomorphic structures are compatible with respect to the pairing.*

PROOF. Let φ and ψ be trivializing sections in KL^{-1} and L, respectively. The holomorphic structures D and $\tilde{D}$ are compatible if and only if

$$(\tilde{D}\varphi \wedge \psi) = d(\varphi, \psi) - \varphi \wedge D\psi. \tag{1.20.1}$$

Since $\bar{K}K \cong \Lambda^2(M, \mathbb{C})$ by $d\bar{z} \otimes dz \to d\bar{z} \wedge dz$ we have $\tilde{D}\varphi \wedge \psi = (\tilde{D}\varphi, \psi)$, where we evaluate the L^{-1} part of $\tilde{D}\varphi$ at ψ. By formula (1.20.1) and by linearity $\tilde{D}\varphi$ is fully determined by the value $(\tilde{D}\varphi, \psi)$. Thus $\tilde{D}$ is unique. □

1.21. Theorem. *Weierstrass Representation*
Let M be a Riemann surface and $f : M \to \mathbb{H}$ be a conformal immersion. Then there are paired holomorphic bundles KL^{-1} and L with pairing $(.,.)$ and compatible holomorphic structures D an $\tilde{D}$, such that there are holomorphic sections $\varphi \in \Gamma(KL^{-1})$ and $\psi \in \Gamma(L)$ with

$$(\varphi, \psi) = df.$$

This equation is called the Weierstrass representation of f. The bundles and sections are unique up to isomophisms of the bundles as holomorphic quaternionic line bundles.

PROOF. We proof the existence of such a representation first. Let $f : M \to \mathbb{H}$ be a conformal immersion. Consider the associated line bundle $L = f^*\mathcal{T}$, where $\mathcal{T}$ is the tautological bundle of $\mathbb{H}P^1$. A trivializing section of L is given by $\psi = (f, 1)$. The vectors $(1, 0)$ and $(0, 1)$ are a basis of $\mathbb{H}^2$. Let α, β be the corresponding dual basis. The restriction of α and β on L defines sections on the dual bundle L^{-1}. We denotes these sections by α_L and $\beta_L \in \Gamma(L^{-1})$. We want to show that the sections $\varphi := \beta_L d\bar{f} \in \Gamma(KL^{-1})$ and $\psi := (f, 1) \in \Gamma(L)$ are holomorphic sections. Obviously we have

$$(\varphi, \psi) = df.$$

Since ψ is nowhere vanishing, it defines a holomorphic structure on L by

$$D\psi = 0.$$

We have to prove that $\varphi = \beta_L d\bar{f}$ is a holomorphic section of KL^{-1} with respect to the induced holomorphic structure $\tilde{D}$ given by Lemma (1.20). Since $\beta_L(\psi) = 1$, and $D\psi = 0$ we get

$$0 = d(df) = d(\beta_L(d\bar{f}), \psi) = (\tilde{D}\varphi \wedge \psi)$$
$$\text{thus} \quad \tilde{D}\varphi = 0.$$

Let S denote the conformal Gauß map of f. Then S induces a complex structure $\tilde{S}$ on KL^{-1}. We have

$$(\tilde{S}(\beta_L d\bar{f}), \psi) = *df = Ndf = (-(\beta_L d\bar{f})N, \psi) = (\beta_L * d\bar{f}, \psi).$$

This shows that $\beta_L d\bar{f} \in \Gamma(KL^{-1})$.

To proof the uniqueness up to isomorphisms let $\hat{L}$ and $K\hat{L}^{-1}$ be other bundles satisfying the conditions above, i.e., there are holomorphic sections $\hat{\varphi} \in K\hat{L}^{-1}$ and $\hat{\psi} \in \hat{L}$ with

$$(\hat{\varphi}, \hat{\psi}) = df.$$

Let $\hat{D}$ be the holomorphic structure on $\hat{L}$. Since f is an immersion and ψ is a trivializing section of L, we express any section h in $\Gamma(L)$ as $h = \psi\lambda$, where λ is a quaternionic function. Define a linear map

$$I : \Gamma(L) \to \Gamma(\hat{L}), h = \psi\lambda \mapsto \tilde{\psi}\lambda.$$

Obviously I is a quaternionic linear isomorphism. The complex structures on L and $\hat{L}$ are given by the conformal Gauß map S. Thus we have

$$S\psi = -\psi R \quad \text{and} \quad \hat{S}\hat{\psi} = -\hat{\psi}R,$$

and I is compatible with the complex structures. It remains to show that I commutes with the holomorphic structures. This follows by

$$I(Dh) = I((\psi d\lambda)'') = (\tilde{\psi} d\lambda)'' = \hat{D}(I(h)).$$

□

1.22. Corollary. *Let f be a conformal immersion of a Riemann surface into S^3 then the paired bundles KL^{-1} and L are isomorphic as holomorphic bundles.*

Proof. According to the theorem above, there are bundles KL^{-1}, L and sections $\varphi \in \Gamma(KL^{-1})$ and $\psi \in \Gamma(L)$ with $(\varphi, \psi) = df$, such that the bundles and sections are unique up to isomorphism. For $\tilde{\varphi} := \varphi f$ we obtain $(\tilde{\varphi}, \psi) = \bar{f}df$. Further let $\tilde{\psi} = \psi\bar{f}$. The map $-\overline{(.,.)}$ defines also a pairing between KL^{-1} and L. Since f maps into S^3 we get $\overline{\bar{f}df} = -\bar{f}df$. Thus

$$-\overline{(\tilde{\varphi}, \tilde{\psi})} = df.$$

By uniqueness the bundles L and KL^{-1} are isomorphic. Under this isomorphism the holomorphic section ψ of L is mapped to the holomorphic section $\tilde{\varphi}$ of KL^{-1}. Thus the holomorphic structure of KL^{-1} induced by the isomorphism is not the one compatible with the pairing $(.,.)$. □

1.23. Corollary. *Let $f : M \to S^3$ and let L be the corresponding quaternionic line bundle. We can split L into the $\pm\mathbb{i}$ eigenspaces of the mean curvature sphere congruence S, i.e.,*

$$L = E \oplus E\mathbb{j}$$

for a complex line bundle E over M. Then E is a spin bundle of M.

PROOF. For surfaces in S^3 the bundles L and KL^{-1} are holomorphic isomorphic. The section ψ is mapped to $\tilde{\varphi} = \varphi f$ by this isomorphism. By construction

$$S\psi = -\psi R \quad \text{and} \quad S\tilde{\varphi} = -\tilde{\varphi} R$$

Therefore the splittings of L and KL^{-1} into complex bundles are compatible. Since $KL^{-1} = KE^{-1} \oplus KE^{-1}$ we get $KE^{-1} \cong E$ as complex holomorphic bundles. And because $W^{-1} \otimes W = \underline{\mathbb{C}}$, this yields $K = E^2$ as complex holomorphic line bundles. □

1.24. Corollary. *Let $f : M \to S^3$, V/L the corresponding quotient bundle and let π_L denote the projection from V to V/L. Again we can split $V/L = E \oplus E\mathbb{j}$, where E is the $\mathbb{i}-$eigenspace of the the complex structure $J = \pi_L S$, induced by the conformal Gauß map. Then E^{-1} is a spin bundle of M.*

PROOF. For surfaces in S^3 the bundles V/L and L^{-1} are isomorphic as holomorphic line bundles. The isomorphism is determined by mapping the nowhere vanishing holomorphic section $\pi_L(1,0)$ of V/L to the nowhere vanishing holomorphic section α of L^{-1}. The statement follows then from Corollary (1.23). □

2. The Lightcone Model

Another tool to study conformal maps into S^n uses the lightcone model of the conformal S^n. The following is taken from [**BuPP**] and [**B2**].

Consider $\mathbb{R}^{n+1,1}$ together with the Lorenz inner product

$$< w, w >= -w_0^2 + \sum_{i=1}^{n+1} w_i^2.$$

The lightcone in $\mathbb{R}^{n+1,1}$ is defined to be $\mathcal{L} := \{w \mid < w, w >= 0\}$. We obtain a conformal diffeomorphism between the conformal S^n and the projectivized lightcone $\mathbb{P}(\mathcal{L})$ by

$$x \in S^n \mapsto [(1, x)] \in \mathbb{P}(\mathcal{L}).$$

The group of Möbius transformations of S^n is then given by the identity component of $O(n+1,1)_+$ acting isometrically on $\mathbb{R}^{n+1,1}$, where $+$ denotes the component of $O(n+1,1)$ preserving the future lightcone.

All $n-$dimensional space forms can be obtained by a similar construction, namely as intersections of $\mathcal{L}$ with some hyperplane in $\mathbb{R}^{n+1,1}$. Take a nonzero $w_0 \in \mathbb{R}^{n+1,1}$ and define

$$S_{w_0} = \{w \in \mathcal{L} | < w, w_0 > +1 = 0\}.$$

The metric which S_{w_0} inherits from $\mathbb{R}^{n+1,1}$ can be computed to be positive definite and of constant curvature $- < w_0, w_0 >$. For the round

$n-$sphere choose w_0 with $-<w_0,w_0>=1$. In the following we denote this vector by e_0.

We want to derive the conformal invariants of a surface in S^3 which determine it up to conformal transformations. For this we want to define the mean curvature sphere congruence in this set up. As in the quaternionic case this is a map which assigns to every point of the surface the unique oriented $2-$sphere through that point with common tangent space and mean curvature vector. The space of round $m-$spheres in the lightcone model of S^n is given by all possible decompositions of the form

$$\mathbb{R}^{n+1,1} = W \oplus W^{\perp},$$

where W is a $m-$dimensional subspace of $\mathbb{R}^{n+1,1}$ such that $W^{\perp}$ is spacelike. The $m-$dimensional sphere is then obtained by $S_{e_0} \cap W$.

2.1. Lemma. *The mean curvature vector of the $m-$sphere $S_{e_0} \cap W$ at a point $w \in S_{e_0} \cap W$ is*

$$\mathcal{H}_w = -e_0^{\perp} - <e_0^{\perp}, e_0^{\perp}> w,$$

where $w = w^T + w^{\perp}$ denotes the decomposition of a vector with respect to $W \oplus W^{\perp}$.

Proof. The mean curvature vector of a submanifold M is defined to be the trace of second fundamental form:

$$\mathrm{II}(X,Y) = (\nabla_X Y)^{\perp},$$

where X, Y are tangential vector fields of M and ∇ is the Levi-Civita connection of the ambient space. The tangent space of S_{e_0} at x_0 is $T_{x_0}S_{e_0} = \mathrm{span}\{x_0, e_0\}^{\perp}$ and the Levi- Civita connection on S_{e_0} is given by

$$\nabla_X Y = d_X Y + <d_X Y, x_0> \tilde{e}_0 + <d_X Y, \tilde{e}_0> x_0,$$

where $\tilde{e}_0 = e_0 + \frac{1}{2} <e_0, e_0> x_0$. The tangent and normal bundle of the manifold $M = S_{e_0} \cap W$ are

$$T_{x_0}M = T_{x_0}S_{e_0} \cap W, \quad N_{x_0}M = T_{x_0}S_{e_0} \cap W^{\perp}.$$

Now we take a orthonormal basis $X_1, ..., X_m$ of $T_{x_0}M$ and extend the basis to tangential vector fields on M by

$$\tilde{X}_i = X_i + <X_i, x> w(x) + <X_i, w(x)> x,$$

with $w(x) = e_0^T + \frac{1}{2} <e_0^T, e_0^T> x$. Then

$$d_{X_i} X_i = e_0^T + <e_0^T, e_0^T> x_0,$$

and thus using the formula for the Levi-Civita connection we get

$$\nabla_{X_i} X_i|_w = -e_0^{\perp} - <e_0^{\perp}, e_0^{\perp}> w,$$

which is independent of i. □

Now consider a conformal immersion $f : M \to S^3$ from a Riemann surface M into the round S^3, i.e., we choose $w_0 = e_0$ and $n = 3$. Like in the quaternionic case we can associate a line subbundle

$$L \subset M \times \mathbb{R}^{4,1} =: V, \quad L_x = \mathbb{R} f(x)$$

to f. Again we have $\varphi = (1, f)$ as trivializing section of L, which is called the Euclidean lift of f. Any other trivializing section is given by $\varphi\lambda$, with $\lambda : M \to \mathbb{R} \setminus \{0\}$. The change of the induced metric caused by the multiplication with λ can be computed to be $|d\varphi\lambda)|^2 = |d\varphi|^2\lambda^2$. Thus one can always choose a scaling of φ with

$$|d\psi|^2 = |dz|^2,$$

where z is a holomorphic coordinate on M. This section is given by $\psi = e^{-u}(1, f)$ and is called the normalized lift of f, where e^{2u} is the conformal factor of f.

Lemma. *Let $f : M \to S^3$ be a conformal immersion and let $\varphi = (1, f)$ be its Euclidean lift. The mean curvature sphere congruence at $x \in M$ is the $2-$sphere associated to the $4-$dimensional Minkowski space $W_x \subset \mathbb{R}^{4,1}$ given by*

$$W_x = \operatorname{span}\{\varphi(x), d\varphi_x, \varphi_{\bar{z}z}(x)\}.$$

Proof. First, at every point $x \in M$ the vector space W_x is a $4-$ dimensional Minkowski space since

$$< \varphi(x), \varphi_{\bar{z}z}(x) >= - < \varphi_z(x), \varphi_{\bar{z}}(x) > < 0.$$

Therefore $S_{e_0} \cap W_x$ defines a $2-$ sphere. Secondly, the tangent spaces of the immersion and the sphere obviously coincides. It remains to show that also the mean curvature vector at p coincides. The mean curvature of W is given by the formula (2.1). A short computation shows $e_0^{\perp} = e_0 - a\varphi - b\varphi_{\bar{z}z}$, with $a = H^2 - 3$ and $b = 2e^{-2u}$. Using the formula for the Levi-Civita connection, the mean curvature vector of the immersion is

$$\mathcal{H} = 2e^{-2u}\varphi_{\bar{z}z} + 2e^{-2u} < \varphi_{\bar{z}z}, \varphi > \tilde{e}_0 + 2e^{-2u} < \varphi_{\bar{z}z}, \tilde{e}_0 > \varphi.$$

Then the equalities

$$\begin{aligned} < \varphi_{\bar{z}z}, \varphi >= - < \varphi_{\bar{z}}, \varphi_z >= -\tfrac{1}{2}e^{2u} \\ < \varphi_{\bar{z}z}, \tilde{e}_0 >=< \varphi_{\bar{z}z}, e_0 - \tfrac{1}{2}\varphi >= \tfrac{1}{4}e^{2u} \end{aligned}$$

yield

$$\mathcal{H} = 2e^{-2u}\varphi_{\bar{z}z} - e_0 + \varphi,$$

which is exactly the mean curvature vector of W. □

It is obvious that W is independent of all choices (lift and holomorphic coordinate) and is thus a conformal invariant.

We denote by $W^\perp$ the orthogonal complement of W in V. The bundle $W^\perp$ is called the Möbius normal bundle. By definition the metric restricted to it is positive definite. The connection D on $W^\perp$ given by the orthogonal projection of the trivial connection on V is called the normal connection. For surfaces in $S^3 \subset \mathbb{R}^4$ the metrical normal bundle $\perp_f$ is trivial. The isomorphism between this normal bundle and the Möbius normal bundle $W^\perp$ is given by

$$n \in \perp_f \mapsto <\mathcal{H}, n>\varphi + (0, n) =: \hat{n} \in W^\perp. \tag{2.1.1}$$

The section $\hat{n}$ has constant length 1 and gives trivialization of the bundle $W^\perp$.

In order to facilitate calculations we complexify V and choose another basis for $W_\mathbb{C} := W \otimes \mathbb{C}$. Let ψ be the normalized lift of f. Then there is a unique section $\hat{\psi} \in \Gamma(W_\mathbb{C})$ with

$$<\hat{\psi}, \hat{\psi}>= 0, \quad <\hat{\psi}, \psi>= -1, \quad <\hat{\psi}, d\psi>= 0.$$

Further $(\psi, \psi_z, \psi_{\bar{z}}, \hat{\psi}, \hat{n})$ is a frame for $V \otimes \mathbb{C}$, to which we always refer to in the following. Since ψ_{zz} is orthogonal to ψ, ψ_z and $\psi_{\bar{z}}$, there exist complex functions c and q with

$$\psi_{zz} + \frac{c}{2}\psi = q\hat{n}. \tag{2.1.2}$$

This is an inhomogenous Hill's equation and the quantities c and q are Möbius invariants of the immersion f, which are called the Schwarzian derivative and the conformal Hopf differential, respectively. In fact they build a full set of invariants, i.e., two immersions having the same c and q are Möbius equivalent, see [**BuPP**].

In the case at hand the Schwarzian derivative and the conformal Hopf differential are complex valued functions. Given two such functions one can ask whether there exist a surface in S^3 with these invariants. In order to get a surface we need integrability conditions for c and q, the so called Gauß−Codazzi equations. The frame equations are

$$\begin{aligned}
\psi_{zz} &= -\frac{c}{2}\psi + q\hat{n} \\
\psi_{\bar{z}z} &= -|q|^2\psi + \frac{1}{2}\hat{\psi} \\
\hat{\psi}_z &= -2|q|^2 - c\psi_{\bar{z}} + 2q_z \\
\hat{n}_z &= 2q_{\bar{z}}\psi - 2q\psi_{\bar{z}}.
\end{aligned}$$

The Gauß-Codazzi equations are then obtained as the commutativity of the second derivatives of the frame. We obtain

$$c_{\bar{z}} = 4|q|_z^2 + 2\bar{q}_z q - 2\bar{q}q_z$$
$$\operatorname{Im}\left(q_{\bar{z}\bar{z}} + \frac{\bar{c}}{2}q\right) = 0.$$

In particular, if q only depends on 1 parameter, then c also only depends one one parameter. We can express these Möbius geometric invariants also in terms of the metrical ones. The isomorphism in (2.1.1) gives

$$q\hat{n} = Hq\varphi + (0, q\hat{n}),$$

where H is the mean curvature function of the surface. The induced metric of the immersion is:

$$|df|^2 = e^{2u}|dz|^2.$$

Further, the normalized lift is defined to be $\psi = (1, f)e^{-u}$. Inserting all this into (2.1.2), we get

$$q = \mathrm{II}\left(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}\right) e^{-u} \tag{2.1.3}$$

$$c = < H, \mathrm{II}\left(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}\right) > + u_{zz} - (u_z)^2. \tag{2.1.4}$$

Theorem. *Let $U \subset \mathbb{C}$ be a simply connected open set, and let c and q be two functions on U satisfying the Gauß-Codazzi equations for surfaces in S^3. Then there exist a conformal immersion $f : U \to S^3$, unique up to Möbius transformations of S^3, with Schwarzian derivative c and conformal Hopf differential q.*

3. Weierstrass Elliptic Functions

We state in the following the properties of the Weierstrass elliptic functions needed in chapter V. For further reading we refer to [**AS**], where all the proofs can be found.

The Weierstrass elliptic functions are made to investigate all holomorphic maps from a torus into $\mathbb{C}P^1$. It is well known that every compact Riemann surface of genus 1 is biholomorphic to a flat torus, i.e., to $\mathbb{C}/\Gamma$, where $\Gamma := \{2n\omega_1 + 2m\omega_2 |\ 2\omega_1, 2\omega_2 \in \mathbb{C} \text{ real linear independent}\}$ is a lattice in $\mathbb{C}$. Without loss of generality one can always fix $2\omega_1$ to be real. The Weierstrass $\wp-$function on a torus $\mathbb{C}/\Gamma$ is a special holomorphic map of degree 2 to $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Explicitly it is defined as:

$$\wp(z) = \tfrac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\tfrac{1}{(z+m2\omega_1+n2\omega_2)^2} - \tfrac{1}{(m2\omega_1+n2\omega_2)^2}\right).$$

The $\wp$ function is even, holomorphic and doubly periodic, i.e., well defined on the torus. The only pole is at $z = 0 \bmod \Gamma$ with order 2. It

can be shown that all holomorphic maps from the torus to $\mathbb{C} \cup \{\infty\}$ is rational in $\wp$ and its derivative $\wp'$.

The Weierstrass $\wp$-function satisfies a differential equation that realizes every conformal torus as an elliptic curve. Let $\mathbb{C}/\Gamma$ be a torus and Γ the lattice generated by $2\omega_1$ and $2\omega_2$. Define a polynomial of degree 3 by $P_3 = 4x^3 - g_2 x - g_3$ with

$$g_2 = 60 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (2m\omega_1 + 2n\omega_2)^{-4}$$

and

$$g_3 = 140 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (2m\omega_1 + 2n\omega_2)^{-6}.$$

Then the Weierstrass $\wp-$function on $\mathbb{C}/\Gamma$ satisfies

$$(\wp')^2 = P_3(\wp). \tag{3.0.5}$$

If $2\omega_1$ and $2\omega_2$ are real linear independent, then the polynomial P_3 has only simple roots. The constants g_2 and g_3 are called the lattice invariants of Γ. The map $z \mapsto (\wp'(z), \wp(z))$ is a group homomorphism between $(\mathbb{C}/\Gamma, +)$ and the elliptic curve E given by the equation $y^2 = P_3(x)$. The elliptic involution $z \mapsto -z$ is given in this picture by $(x, y) \mapsto (x, -y)$.

On the other hand, given g_2 and g_3 with $g_3^2 - 27g_2^3 \neq 0$, the polynomial P_3 has only simple roots and there exist a lattice Γ such that the solution to the equation (3.0.5) is a $\wp$-function on the torus $\mathbb{C}/\Gamma$. We are interested in the case where both lattice invariants are real. Then all coefficients of the Taylor series of $\wp$ are real. Thus in addition to the elliptic involution we get another symmetry of the curve

$$\bar{\wp}(z) = \wp(\bar{z}). \tag{3.0.6}$$

Other important invariants of the $\wp$-function are the half periods ω_1 and ω_2 and $\omega_3 = \omega_1 + \omega_2$ of Γ. Because the function $\wp'$ is odd and periodic we get for the half periods

$$\wp'(\omega_i) = \wp'(-\omega_i + 2\omega_i) = -\wp'(\omega_i) = 0.$$

Thus $\wp(\omega_i)$ are roots of P_3. For real lattice invariants g_2 and g_3 the 3 roots of P_3 are either all real or we have a real root and a pair of complex conjugate roots. Together with the symmetry given by (3.0.6) we obtain that the lattice Γ is either rectangular or its double covering is rectangular.

If Γ is rectangular then the polynomial P_3 has three real roots. We denote them by e_1,e_2 and e_3. The roots of P_3 are the branch points

of the double covering defined by $\wp$. Without loss of generality we can assume

$$e_1 > e_2 > e_3,$$

with $\wp(\omega_i) = e_i$. Further, we can show $\omega_1 \in \mathbb{R}$, $\omega_2 \in i\mathbb{R}$ and $\omega_3 = \omega_1 + \omega_2$ for the half periods ω_i.

The case that only the double covering of $\mathbb{C}/\Gamma$ is rectangular corresponds to the case where P_3 has only one real and two complex conjugate roots. In this case we denote by ω_1 the half period corresponding to the real root of P_3. For the other half periods we have: $\bar{\omega}_2 = \omega_3$ and $\omega_1 = \omega_2 + \omega_3$. In particular we obtain $i\omega_1 = \omega_1 \bmod \Gamma$.

The set of $z \in \mathbb{C}$ for which $\wp(z)$ is real valued can be determined with the two symmetries of $\wp$, i.e., the property that $\wp$ is even and with equation (3.0.6): if P_3 has three real root we have that $\wp(z)$ is real valued if and only if z lies on the following lines:

$$\mathbb{R} + \Gamma,\ i\mathbb{R} + \Gamma,\ i\mathbb{R} + \omega_1 + \Gamma,\ \mathbb{R} + \omega_3 + \Gamma.$$

In the second case we have that $\wp(z)$ is real if and only if z lies on the lines $\mathbb{R} + \Gamma$ and $i\mathbb{R} + \Gamma$.

For the proposes of the thesis it is necessary to consider two further functions related to the $\wp-$function and some of their properties.

3.0.1. *The Weierstrass ζ-Function.* The Weierstrass ζ-function is defined to be the function with

$$\zeta' = -\wp \text{ and } \lim_{z\to 0} \zeta(z)z = 1.$$

It is an odd function with a simple pole at $z = 0 \bmod \Gamma$. The ζ function is no longer periodic, rather one has

$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i,$$

where $\eta_i := \zeta(\omega_i)$.

In the case of real lattice invariants g_2 and g_3 all coefficients of the power series representation of ζ are real. Thus $\zeta|_{\mathbb{R}}$ is a real function and in particular η_1 is real. Further, the following formulas are valid:

$$\omega_2\eta_1 - \omega_1\eta_2 = \tfrac{1}{2}\pi i,$$

and for $x + y + z = 0$,

$$(\zeta(x) + \zeta(y) + \zeta(z))^2 = \wp(x) + \wp(y) + \wp(z).$$

3.0.2. *The Weierstrass $\sigma-$Function.* The Weierstrass $\sigma-$function is given by a further logarithmic integration of the ζ-function. We have

$$\frac{\sigma'}{\sigma} = \zeta \text{ and } \sigma(0) = 0.$$

All other zeros of σ lie in Γ. Because ζ has additive monodromy with respect to Γ, the monodromy of σ is multiplicative is given by

$$\sigma(z + 2\omega_i) = -e^{2\eta_i(z+\omega_i)}\sigma(z).$$

CHAPTER II

Equivariant Constrained Willmore Tori

4. The Willmore Energy

4.1. Definition. Let M be a compact surface and (S^3, g) be a $3-$dimensional sphere with some metric g. Further let $f : M \to (S^3, g)$ be an immersion. The *Willmore energy* of f is defined to be

$$\mathcal{W}(f) = \int_M (H^2 + \bar{K}) dA,$$

where H is the mean curvature of f in (S^3, g), $\bar{K}$ is the sectional curvature of the tangent plane with respect to g and dA is volume form for the induced metric f^*g.

A conformal immersion $f : M \to (S^3, g)$ is called *Willmore*, if it is a critical point of the Willmore energy W under all variations by immersions. A conformal immersion is called *constrained Willmore*, if it is a critical point of W under conformal variations, see [**BoPetP**] and [**KS**].

Example. Minimal immersions are Willmore and constant mean curvature (CMC) immerisons into a space form are constrained Willmore.

An important property of the Willmore energy is its invariance under conformal changes of the metric of the ambient space, see [**W**]. Note that not only the value of the functional is conformally invariant, but the integrand itself is. Thus, it is reasonable to consider immersions into (S^3, g), where g lies in the conformal class of the round metric.

The physical interpretation of the Willmore energy is the bending energy of the surface. In case g is the round metric on S^3 the Willmore energy of f reduces to

$$W(f) = \int_M (H^2 + 1) dA.$$

Let κ_1 and κ_2 be the principal curvatures of f in the round S^3 and K be its Gaußian curvature. By Gauß- Bonnet $\int_M K dA$ is a topological invariant. We obtain

$$W(f) = \int_M (H^2 + 1) dA = \int_M \tfrac{1}{4}(\kappa_1 - \kappa_2)^2 dA + \int_M K dA.$$

Thus restricted to a given topological type of M the Willmore functional measures the roundness of the immersion.

For the rest of the thesis we restrict ourselves to the case where $M = T^2$ is a $2-$Torus. Since we want to consider constrained Willmore tori, it is necessary to fix the conformal structure of the torus. Every conformal torus is biholomorphic to the torus $\mathbb{C}/\Gamma$, where $\Gamma \subset \mathbb{C}$ is a lattice. This lattice Γ encodes the conformal structure of the torus. Thus conformal variations are those variations preserving the generators of Γ.

The Willmore conjecture states that the minimum of the Willmore energy restricted to the class of immersed tori is attained at $2\pi^2$. Further, if an immersion f has Willmore energy $2\pi^2$, then it is a Möbius transformation of the Clifford torus given by

$$f_{Cliff} : [0, 2\pi] \times [0, 2\pi] \to S^3 \subset \mathbb{C}^2, (\varphi, \theta) \mapsto \tfrac{1}{\sqrt{2}}(e^{i\varphi}, e^{i\theta}).$$

A related conjecture is the Lawson conjecture. It states that the Clifford torus is the only embedded minimal torus in S^3. Very recently both conjectures were proven. The Willmore conjecture by Marquez [**MN**] and Neves and the Lawson conjecture by Brendle [**Br**].

Here a selection of important results towards the Willmore conjecture: An estimate by Li and Yau [**LY**] shows that the Willmore energy of non embedded tori is $\geq 8\pi$. Thus it is only necessary to confirm the Willmore conjecture for embedded tori. Leon Simon [**S**] proved that the minimum of the Willmore energy in the class of tori is attained and that the minimizer is in fact a Willmore torus, i.e., it is a critical point of the Willmore functional under all variations by immersions. Ros [**R**] and Topping [**T**] have independently proven that tori with antipodal symmetry obeys the Willmore conjecture.

More interesting for the integrable systems approach to the Willmore conjecture is the quaternionic Plücker formula proven in [**BuFLPP**]. It gives a lower bound for the Willmore energy depending on the dimension of the space of holomorphic sections of V/L, which is the quotient bundle associated to the immersion, see chapter I. To a conformally immersed torus of constant mean curvature we can associate a Riemann surface, the spectral curve. As a corollary of the Plücker formula we get for CMC tori a lower bound for the Willmore energy depending on the genus g of its spectral curve. Thus CMC tori with Willmore energy below 8π have spectral genus at most 3. Embedded minimal tori have spectral genus $g \leq 6$, since they have Willmore energy $W \leq 16\pi$ by a theorem of [**CW**].

5. Equivariant Maps into the 3−sphere and Seifert Fiber Spaces

We want to investigate conformal immersions $f : T^2 \to S^3$ into the round sphere, which are simple enough to have an explicit description but still provide interesting examples. In this context simple means to have a 1−parameter family of symmetries.

Definition. A map $f : T^2 \to S^3$ is called $\mathbb{R}-$equivariant if there exist group homomorphisms

$$M : \mathbb{R} \to \text{Möb}(S^3), t \mapsto M_t,$$
$$\tilde{M} : \mathbb{R} \to \{\text{conformal transformations of } T^2\}, t \mapsto \tilde{M}_t,$$

such that

$$f \circ \tilde{M}_t = M_t \circ f, \text{ for all } t.$$

Here $\text{Möb}(S^3)$ is the group of Möbiustransformations of S^3.

5.1. Conformal Transformations of T^2. The conformal transformations of the torus $T^2 = \mathbb{C}/\Gamma$ is the subgroup of the conformal transformations of $\mathbb{C}$, compatible with the lattice. Conformal transformations of $\mathbb{C}$ are given by

$$\tilde{M}(z) = az + b, \text{ with } a, b \in \mathbb{C}.$$

$\tilde{M}$ is compatible with the lattice if and only if $a\Gamma = \Gamma$. We are interested in 1−parameter groups of conformal transformations of T^2. The set of $a \in \mathbb{C}$ with $a\Gamma = \Gamma$ is discrete, thus a must be constant for every 1−parameter group. Further the map $t \mapsto \tilde{M}_t$ is by definition a group homomorphism, i.e.,

$$\tilde{M}_{s+t} = \tilde{M}_s \circ \tilde{M}_t$$

therefore we obtain $a = 1$ and $b(t) = tb_0$. Hence all possible 1−parameter groups of conformal transformations of T^2 are given by translations along some line determined by a non zero $b_0 \in \mathbb{C}$, i.e.,

$$\tilde{M}_t(z) = z + tb_0.$$

There are two cases to distinguish. The first case is that we have $tb_0 \notin \Gamma$ for all $t \in \mathbb{R}$. Then the set $\{\tilde{M}_t(0), t \in \mathbb{R}\}$ is dense in $\mathbb{C}/\Gamma$. Equivariant tori with respect to such $\tilde{M}$ are given by Möbius transformations of a point in S^3. Such tori are called homogenous tori and are the product of two circles. We exclude this case from our further considerations.

In the second case we have that there is a $t_0 \in \mathbb{R}$ with $t_0 b_0 \in \Gamma$. Such 1−parameter groups are periodic. We get the following lemma.

Lemma. *Let* $t_0 := \min\{t \in \mathbb{R}_+ \setminus 0 \mid tz_0 \in \Gamma\}$ *and let* $\gamma_1 := t_0 z_0$. *Then there exist a* $\gamma_2 \in \Gamma$ *such that*

$$\operatorname{span}_{\mathbb{Z}}(\gamma_1, \gamma_2) = \Gamma.$$

PROOF. Let ω_1 and ω_2 be the generators of the lattice Γ. The lattice vector γ_1 is then given by $\gamma_1 = k\omega_1 + l\omega_2$, $k, l \in \mathbb{Z}$. Since t_0 is the minimum t such that $tz_0 \in \Gamma$, we get $\gamma_1 = t_0 z_0$ is the lattice vector with the smallest possible absolute value of this form. Therefore k and l are coprime and there exist $a, b \in \mathbb{Z}$ with with $ak + bl = 1$. Let $\gamma_2 := b\omega_1 - a\omega_2$. Then we obtain $\omega_1 = l\gamma_2 + a\gamma_1$ and $\omega_2 = -k\gamma_2 + b\gamma_1$. Hence $\operatorname{span}_{\mathbb{Z}}\{\gamma_1, \gamma_2\} = \operatorname{span}_{\mathbb{Z}}\{\omega_1, \omega_2\} = \Gamma$. □

We can rotate the whole picture by $e^{i\varphi}$ such that $e^{i\varphi} z_0 \in \mathbb{R}$. By the lemma Γ is generated by $\gamma_1 = t_0 e^{i\varphi} z_0 \in \mathbb{R}$ and γ_2. That means we can fix both $z_0 = 1$ and one generator of the lattice Γ to be real at the same time without loss of generality.

5.2. Periodic $1-$Parameter groups of Möb(S^3). Now we turn to the Möbius transformations of S^3. Let $\tilde{M}$ be a periodic $1-$parameter group of conformal transformations of T^2 and let $f : T^2 \to S^3$ be an equivariant map with respect to $\tilde{M}$, i.e., there exist a $1-$parameter group M of Möbius transformations of S^3 such that

$$f \circ \tilde{M} = M \circ f. \tag{5.2.1}$$

Because we restrict ourselves to the case that $\tilde{M}$ is periodic, every group M satisfying (5.2.1) for some map f must also be periodic. It turns out that for every such periodic $1-$parameter group M of Möb(S^3) there exist a unique round metric on the conformal S^3 on which M acts via isometries. Thus we can restrict ourselves to the case where f maps to the round sphere.

Proposition. *Let* $f : T^2 \to S^3$ *be an equivariant map and* $\tilde{M}$ *be a periodic* $1-$*parameter group of conformal transformations of* T^2. *Then a* $1-$*parameter group of Möbius transformations satisfying (5.2.1) is conjugate to the* $1-$*parameter group* M *acting on a unique round* $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ *via isometries of the form:*

$$M_t = \begin{pmatrix} e^{imt} & 0 \\ 0 & e^{int} \end{pmatrix} \in SO(4),$$

with $m, n \in \mathbb{N}$ *and* $gcd(m, n) = 1$.

Remark. Conjugation in the proposition above is equivalent to a Möbius transformation of the ambient space.

PROOF. A model of the conformal $3-$sphere S^3 is given by the projectivized light cone in $\mathbb{R}^{4,1}$, see Chapter I. The group of Möbius transformations of S^3 is known to be $O(4,1)_+$. A connected subgroup of $O(4,1)_+$ must lie in the identity component of the group. Thus we

have in fact here a 1−parameter subgroup of $SO(4,1)$. Because the 1−parameter family is periodic we have that the image of $t \to M_t$ is a compact subgroup of $SO(4,1)$. It is well known that any compact subgroup of $SO(4,1)$ is conjugate to a subgroup of $SO(4)$ which acts on the round $S^3 = S_{e_0} \subset e_0 + e_0^{\perp} \cong \mathbb{R}^4$ via isometries. Further we have that $SO(4) = SO(3) \times SO(3)/\{\pm Id\}$. Thus in a suitable basis of $\mathbb{R}^4$ we get two rotation blocks:

$$M_t = \begin{pmatrix} R_1(t) & 0 \\ 0 & R_2(t) \end{pmatrix}.$$

By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ we get that these rotations are given by $e^{ia_j t}$, $j = 1, 2$. In order to have periodic orbits we need that $a_j \in \mathbb{Q}$. This proves the statement. □

Definition. A map $f : T^2 \to S^3$ is a (m,n)−torus, if it is equivariant with respect to the 1−parameter subgroup $M_t = \begin{pmatrix} e^{imt} & 0 \\ 0 & e^{int} \end{pmatrix}$ of $\mathrm{Möb}(S^3)$ for $m, n \in \mathbb{N}$ coprime.

Remark. Let $f : T^2 \cong \mathbb{C}/\Gamma \to S^3$ be a (m,n)−torus and $z = x + iy$ be a holomorphic coordinate of T^2. Identifying $\mathbb{C}^2$ and $\mathbb{H}$ we get that $M_x = e^{\mathbb{i} l_1 x}(.)e^{\mathbb{i} l_2 x}$ with $l_1 := \frac{m+n}{2}$ and $l_2 = \frac{m-n}{2}$, where the terms are multiplied as quaternions. Then we obtain

$$f(x,y) = e^{\mathbb{i} l_1 x} f(0,y) e^{\mathbb{i} l_2 x}.$$

Further, since $T^2 \cong \mathbb{C}/\Gamma$, we can also consider f as a map from $\mathbb{C}$ to S^3 which is additionally doubly periodic. For equivariant maps we obtain two conditions. The first is that M has to be periodic and the second is a periodicity condition on $f(0,y)$.

5.3. Seifert Fibrations. In order to deal with equivariant tori, we introduce the Seifert fiber spaces. For $m, n \in \mathbb{N}$ corprime, we define the following equivalence relation on $S^3 \subset \mathbb{C}^2$. Let $z = (z_1, z_2), w = (w_1, w_2) \in S^3 \subset \mathbb{C}^2$, then

$$z \sim_{m,n} w \Leftrightarrow \text{ if there exist a } t \text{ such that } z = (e^{imt} w_1, e^{int} w_2).$$

Definition. The triple $F := (S^3, S^3/_{\sim_{m,n}}, \pi_{m,n})$, where $\pi_{m,n}$ maps every point in S^3 to its equivalence class is a (m,n)-Seifert fiber space.

For every point in $S^3/_{\sim_{m,n}}$ we can always find a representative such that the first coordinate is real, since

$$(|z_1| e^{i\varphi}, z_2) \sim_{m,n} (|z_1|, e^{i\frac{n}{m}\varphi} z_2).$$

$S^3/_{\sim_{m,n}}$ will be referred to as the base space or orbit space. It is a regular manifold away from the points $[(1,0)]_{\sim_{m,n}}$ and $[(0,1)]_{\sim_{m,n}}$. If $mn > 1$ both points are singular. For $m = n = 1$ the projection $\pi_{1,1}$ is the Hopf fibration. Thus the base space is the round sphere and has

no singular points. In the case of $m = 0, n = 1$ we have that only $[(0,1)]_{\sim_{m,n}}$ is singular, see [**BW**] for details.

5.4. Example. The two exceptional cases here provide the easiest examples of equivariant tori.
In the first case $m = n = 1$ the torus is obtained as the preimage of a closed curve on S^2 under the Hopf fibration. These tori are called Hopf tori.

In the second case we have $m = 1, n = 0$ and M_t is a rotation. The torus can be constructed by the the rotation of a closed curve in the open upper half plane, viewed as the hyperbolic plane, around the $x-$ axis.

Remark. F is a principal fiber bundle away from the singular points. Let $F^* := F \setminus \{singularpoints\}$. We want to define a connection on F^* such that the curvature of F^* has a geometric meaning for $(m,n)-$Tori. In order to do so we need a metric. It turns out that the right choice of the metric is given by dividing the round metric on S^3 by the fiber length. We denote this new metric by $g_{m,n}$.

Because we can always find a representative of the base space where the first coordinate is real and non negative, we can parametrize the orbit space by

$$\Phi : [0, 2\pi] \times [0, \pi/2] \to S^2 \subset \mathbb{C}^2$$
$$\Phi(\theta, \varphi) = (\cos(\varphi), e^{i\theta} \sin(\varphi)),$$

Lemma. *The round metric on S^3 induces a unique metric on the base space such that $\pi_{\sim m,n}$ is a Riemannian submersion. In the coordinates given by the parametrization above this metric is given by*

$$\Phi^* g_{S^3/\sim_{m,n}} = \frac{\frac{m^2}{4} \sin^2(2\varphi)}{h^2 \circ \Phi} d\theta^2 + d\varphi^2.$$

Proof. It is shown in [**BW**] that the induced metric on the base space given by

$$\Phi^* g_{S^3/\sim_{m,n}}(X, Y) = g_{S^3}(\Phi_*(X)^N, \Phi_*(Y)^N),$$

where $()^N$ denotes the component of the vector orthogonal to the fibers, makes the projection $\pi_{\sim m,n}$ a Riemannian submersion. With basis vector fields $\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\theta}$ and the fiber direction B given by

$$\frac{\partial}{\partial\varphi} = (-\sin(\varphi), e^{i\theta}\cos(\varphi))$$
$$\frac{\partial}{\partial\theta} = (0, ie^{i\theta}\sin(\varphi))$$
$$B = \frac{1}{\sqrt{h}}(im\cos(\varphi), ine^{i\theta}\sin(\varphi))$$

the formula for the metric can be obtained by a simple calculation. □

To abbreviate the calculations, we define $h((z_1, z_2)) := m^2|z_1|^2 + n^2|z_2|^2$ to be the fiber length in the round metric. We divide the round metric g on S^3 by the fiber length at each point and obtain the new metric $g_{m,n} = \frac{1}{h}g$. This metric lies in the conformal class of the round metric. Since h is constant along the fibers, $g_{m,n}$ induces a well defined metric on $S^3/_{\sim_{m,n}}$. We denote this metric also by $g_{m,n} = \frac{1}{h}g_{S^3/_{\sim_{m,n}}}$. Furthermore, together with the fiber direction B the metric $g_{m,n}$ defines a connection on the principle fiber bundle F^*. We want to compute the curvature of this connection.

Lemma. *For the connection $1-$form $\omega = g_{m,n}(., B)$ of the principal fiber bundle F^*, the curvature form is computed to be*

$$\Omega vol_{m,n} = \frac{2nm}{\sqrt{h \circ \Phi}} vol_{m,n},$$

where Ω is a real valued function and $vol_{m,n}$ is the volume form on the base space of F with respect to the metric $g_{m,n}$.

Proof. The connection is well-defined, since ω only depends on the point in the base space. The curvature of the connection is defined to be $d\omega$, which can be computed using Cartan's formula. With

$$\omega(\tfrac{\partial}{\partial\varphi}) = 0$$

and

$$\omega(\tfrac{\partial}{\partial\theta}) = \frac{n\sin^2(\varphi)}{h \circ \Phi}$$

we get

$$\begin{aligned} d\omega(\tfrac{\partial}{\partial\varphi}, \tfrac{\partial}{\partial\theta}) &= \tfrac{\partial}{\partial\varphi}\omega(\tfrac{\partial}{\partial\theta}) \\ &= \frac{n\sin(2\varphi)}{h \circ \Phi} + \frac{n(m^2 - n^2)\sin^2(\varphi)\sin(2\varphi)}{h^2 \circ \Phi} \\ &= \frac{nm^2\sin(2\varphi)}{h^2 \circ \Phi} \end{aligned}$$

The volume form of the metric $g_{m,n}$ is given by

$$vol_{m,n} = \frac{\frac{m}{2}\sin(2\varphi)}{h^{\frac{3}{2}} \circ \Phi} d\varphi \wedge d\theta.$$

Therefore, we obtain

$$\Omega vol_{m,n} = d\omega = \frac{2nm}{\sqrt{h \circ \Phi}} vol_{m,n}.$$

□

Some other curvature functions will be needed in the following.

5.5. Lemma. *Let X be an arbitrary tangential vector field of the base space. The sectional curvature $\bar{K}_{m,n}$ of the plane given by X and the fiber direction in S^3 is a function on the base space given by:*

$$\bar{K}_{m,n} = \frac{n^2 m^2}{h \circ \Phi}.$$

And the Gaußian curvature of metric $g_{m,n}$ on the base space is given by

$$K_{m,n} = 6\frac{m^2 n^2}{h\circ\Phi} - (m^2 + n^2).$$

The proof of the first statement uses the O'Neil formulas, which relate the curvatures of the total space to the curvatures of the base space of a fibration. The second statement is obtained by a straight forward calculation.

5.6. Curves and Equivariant Tori.

Proposition. *There exist an one-to-one correspondence between closed curves in the base space of the $(m,n)-$Seifert fiber space and $(m,n)-$equivariant tori.*

Proof. The preimage of every closed curve with respect to the fibration $\pi_{m,n}$ is a $(m,n)-$torus. And the image of any $(m,n)-$torus under $\pi_{m,n}$ is a closed curve in the base space. □

In every fiber bundle with a connection there exist to any curve a horizontal lift to the total space (in general the lift is not closed). Here we choose the connection given by ω. By definition this means that the tangent of the lift is orthogonal to the fiber direction B. For a conformal parametrization, we still need that the derivatives in both directions have the same length. The fibers are arclength parametrized with respect to $g_{m,n}$. Thus, by arclength parametrization of the curve with respect to the metric $g_{m,n}$, we get a conformal parametrization for the torus. We thus have shown the following:

5.7. Lemma. *Let $l_1 := (m+n)/2$, $l_2 := (m-n)/2$ and let $\tilde{\gamma}$ be an arclength parametrized closed curve on $S^3/\sim_{m,n}$ with respect to $g_{m,n}$. Then there exist a curve γ in S^3 with $\tilde{\gamma} = [\gamma]$ such that the $(m,n)-$equivariant torus given by*

$$f(x,y) = e^{\mathbb{i}l_1 x}\gamma(y)e^{\mathbb{i}l_2 x}$$

is conformally parametrized. The curve γ is called the profile curve of f.

Remark. The horizontal lift of the curve $\tilde{\gamma}$ defines also a horizontal lift of the frame of $\tilde{\gamma}$. The curvature of the lifted curve γ in the direction given by the horizontal lift of the normal vector of $\tilde{\gamma}$ is by definition the curvature of the curve $\tilde{\gamma}$. This statement is valid for both metrics $g_{S^3/\sim_{m,n}}$ and $g_{m,n}$ in the base space. Further the lifted curve γ

has torsion in the fiber direction B, which becomes the binormal vector of γ.

We now show how the invariants of the $(m,n)-$torus are determined by the invariants of the curve $\tilde{\gamma}$ and the fibration $\pi_{m,n}$. The surface in S^3 has two conformal invariants which determine the surface up to Möbius transformations, see section (2) or [**BuPP**]. The first one is the conformal Hopf differential. For a conformally parametrized equivariant torus it is given by the function

$$q = \frac{\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)}{\sqrt{h}}.$$

The second one is the Schwarzian derivative c. In the equivariant case c is determined by q up to a integration constant by the Gauß-Codazzi equations. Thus we will only consider q in the following.

5.8. Proposition. *The conformal Hopf differential q of a $(m,n)-$equivariant torus is given by*

$$q = \frac{1}{4}(\kappa_{m,n} + i\Omega),$$

where $\kappa_{m,n}$ is the curvature of its profile curve $\tilde{\gamma}$ with respect to $g_{m,n}$ and Ω is the curvature function of the $(m,n)-$Seifert fiber space.

Remark. This means that for equivariant tori the conformal Hopf differential q only depends on y and is periodic. On the other hand, if q only depends on one variable, then the corresponding surface is equivariant. This follows from the fact that the conformal Hopf differential together with the Schwarzian derivative determines the surface up to Möbius transformations of S^3. Since c is determined by q up to a constant by the Gauß$-$Codazzi equations, it depends also only on one variable. Then $f(x,y)$ and $f(x+x_0,y)$, for $x_0 \in \mathbb{R}$, has the same q and c. Thus they differ only by a Möbius transformation.

We first compute the curvature and the torsion of the horizontally lifted profile curve γ with respect to the metric $g_{m,n}$ on S^3. We will not distinguish between the normal N_{orm} of the curve γ given by the horizontal lift of the normal of $\tilde{\gamma}$, and the normal of the torus $f = e^{\mathbb{i}l_1 x}\gamma e^{\mathbb{i}l_2 x}$, since the second is just given by $e^{\mathbb{i}l_1 x}N_{orm}(y)e^{\mathbb{i}l_2 x}$. Further we will need to compare the two conformally equivalent metrics on the conformal S^3. We denote by N_{orm} the normal of the curve γ in the round metric and $N_{m,n} = \sqrt{h}N_{orm}$ is the normal of γ with respect to the metric $g_{m,n}$.

5.9. Lemma. *Using the parametrization of a $(m,n)-$torus given in (5.7), the curvature of the profile curve γ with respect to the metric $g_{m,n}$ is*

$$\kappa_{m,n} = \sqrt{h}\kappa_{S^3} - \tfrac{2l_1 l_2}{\sqrt{h}} < \mathbb{i}\gamma\mathbb{i}, N_{orm} >_{S^3} . \tag{5.9.1}$$

The curvature expressed in terms of the corresponding immersion is

$$\kappa_{m,n} = \tfrac{1}{\sqrt{h}} < (f_{yy} - f_{xx}), N_{orm} >_{S^3} . \tag{5.9.2}$$

PROOF. Let $\gamma = \gamma_1 + \mathbb{j}\gamma_2 : I \to S^3 \subset \mathbb{H}$. Since f is conformal, we have that γ is arclength parametrized in the metric $g_{m,n}$. Thus

$$|\gamma'|^2 = |l_1 \mathbb{i}\gamma + l_2 \gamma \mathbb{i}|^2 = m^2|\gamma_1|^2 + n^2|\gamma_2|^2 = h.$$

The change of the Levi-Civita connection due to a conformal change of the metric by $e^{2u} = \frac{1}{h}$ is

$$\nabla^{m,n}_X Y - \nabla_X Y = (X \cdot u)Y + (Y \cdot u)X - g_{S^3}(X, Y) \operatorname{grad} u.$$

The normal vector of the curve in S^3 is given by

$$N_{orm} = \tfrac{1}{h}\gamma\gamma^{-1}\tfrac{\partial\gamma}{\partial x}\gamma^{-1}\tfrac{\partial\gamma}{\partial y} = \tfrac{1}{h}(l_1 i\gamma' + l_2\gamma i\bar{\gamma}\gamma').$$

The length of the vector with respect to $g_{m,n}$ is $\sqrt{h}$, therefore we get that the curvature of γ with respect to the metric $g_{m,n}$ is

$$\begin{aligned}\kappa_{m,n} &= \tfrac{1}{\sqrt{h}} g_{m,n}(\nabla^{m,n}_{\gamma'}\gamma', N_{orm}) \\ &= \tfrac{1}{\sqrt{h}} < \gamma'', N_{orm} >_{S^3} - g_{m,n}(\operatorname{grad} u, N_{m,n})\end{aligned}$$

where $u = -\frac{1}{2} ln(m^2|z_1|^2 + n^2|z_2|^2), (z_1, z_2) \in \mathbb{H}$. Thus

$$\begin{aligned}-\operatorname{grad}(u)|_\gamma &= (m^2|\gamma_1|)^2 + n^2|\gamma_2|^2)(m^2\gamma_1 + n^2\gamma_2\mathbb{j}) \\ &= \tfrac{1}{|g|^2_{S^3}}((l_1 + l_2)^2\gamma_1 + (l_1 - l_2)^2\gamma_2\mathbb{j}) \\ &= \tfrac{1}{|g|^2_{S^3}}((l_1^2 + l_2^2)\gamma - 2l_1 l_2 \mathbb{i}\gamma\mathbb{i}).\end{aligned}$$

So

$$\begin{aligned}\kappa_{m,n} &= \tfrac{1}{\sqrt{h}}(< \gamma'', N_{orm} >_{S^3} - 2l_1 l_2 < \mathbb{i}\gamma\mathbb{i}, N_{orm} >_{S^3}) \\ &= \sqrt{h}\kappa_{S^3} - \tfrac{2l_1 l_2}{\sqrt{h}} < \mathbb{i}\gamma\mathbb{i}, N_{orm} >_{S^3},\end{aligned}$$

where N_{orm} is the normal of the surface in the round metric. By writing out the second derivatives of f we get

$$\kappa_{m,n} = \tfrac{1}{\sqrt{h}} < (f_{yy} - f_{xx}), N_{orm} >_{S^3} .$$

□

5.10. Lemma. *The torsion of γ is $\tau = \frac{1}{2}\Omega$, where $\Omega vol_{m,n}$ is the curvature form of the principle fiber bundle F. In terms of the corresponding torus we have*

$$\tau = \tfrac{mn}{\sqrt{h}} = - < f_{xy}, N_{orm} >_{S^3} .$$

PROOF. The connection between the curvature forms of the domain and the target space of a Riemannian submersion are given by the O'Neil formulas. In our case these formulas give the torsion of the profile curve which is $\frac{1}{2}\Omega$. Thus

$$\tau = \tfrac{mn}{\sqrt{h}} = - < B', N_{orm} >_H = -\tfrac{1}{\sqrt{h}} < f_{xy}, N_{orm} >_{S^3} . \tag{5.10.1}$$

□

Since the conformal Hopf differential q of the conformally parametrized equivariant torus is given by

$$q = \frac{\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)}{\sqrt{h}} = \frac{< f_{xx} - f_{yy} - 2if_{xy}, N_{orm} >}{4\sqrt{h}}$$

this proves Proposition (5.8).

Let T denote the tangent vector of γ. Then we also have $< B', T >=< l_1 i\gamma' + l_2\gamma' i, \gamma' >= 0$, which shows the following.

5.11. Lemma. *The frame of the profile curve γ given by the tangent vector T, the normal vector N_{orm} and the fiber direction B of the Seifert fiber space is its Frénet frame in S^3.*

Example. In the case of tori of revolution, i.e., $m = 1$ and $n = 0$, we have that $4q = \kappa_{1,0} = \kappa$ is real and κ is the curvature of the profile curve γ in the hyperbolic plane. In the case of Hopf tori, i.e., $m = n = 1$, we have that the base space is the round $2-$sphere of constant curvature 4 and the curvature of the Seifert fiber space is 2. Thus $4q = \kappa + 2i$, where κ is the curvature of $\tilde{\gamma} = \pi_{1,1}(\gamma)$ in this $2-$sphere.

The conformal type of the torus can also be derived from the profile curve and the fibration type.

5.12. Lemma. *The conformal type of a conformally parametrized $(m, n)-$torus given by the preimage of a closed curve $\tilde{\gamma}$ in $S^3/_{\sim_{m,n}}$ under $\pi_{m,n}$, which does not go through the singular points of $S^3/_{\sim_{m,n}}$, is given by the lattice Γ generated by:*

$$\omega_1 = 2\pi \text{ and } \omega_2 = ((\int_C \Omega vol_{m,n}) \bmod 2\pi, l),$$

where l is the length of the curve with respect to the metric $g_{m,n}$, $\Omega vol_{m,n}$ is the curvature of the corresponding fiber space and C is a $2-$chain in $S^3/\sim_{m,n}$ with $\partial C = \tilde{\gamma}$.

Proof. As the image of the torus on $S^3/_{\sim_{m,n}}$ under $\pi_{m,n}$ is a closed curve $\tilde{\gamma}$, there exist a x_0 with

$$\gamma(0) = e^{il_1 x_0}\gamma(l)e^{il_2 x_0} = M_{x_0}\gamma(l),$$

which means that $f(0, 0) = f(x_0, l)$. Clearly x_0 only well defined modulo 2π. We obtain that (x_0, l) and $(2\pi, 0)$ are the generators of the lattice Γ. Since $\tilde{\gamma} = \pi_{m,n}(\gamma)$ is arclength parametrized with respect to $g_{m,n}$, l is the length of $\tilde{\gamma}$. It remains to show $x_0 + k = \int_C \Omega vol_{m,n}$, where k is a constant depending on the winding number of the curve around the singularities of $S^3/_{\sim_{m,n}}$. This follows easily from the following more general lemma. □

Lemma. *Let $\omega \in \Omega^1(M, \mathbb{R})$ and $\nabla = d - i\omega$ be a hermitian connection on the topologically trivial bundle $M \times \mathbb{C}$ over a surface M. Further let $\gamma_i, i = 1, 2$ be oriented closed curves on M which give the boundary of a $2-$chain C in M. Then the holonomies α_1 and α_2 along γ_1 and γ_2 considered as complex numbers of length 1 are related by*

$$\alpha_1 = \alpha_2 e^{i\int_C d\omega}$$

Proof. Let $s = e^{if}$ be a parallel section along a curve γ. Then

$$0 = \nabla_{\gamma'} s = e^{if}(idf - iw)(\gamma'),$$

which is equivalent to

$$\alpha_k = e^{i\int_{\gamma_k}\omega}.$$

The formula is then a corollary of Stoke's theorem. □

6. The Willmore Functional for Equivariant Tori

Lemma. *The Willmore functional of an equivariant torus can be computed as:*

$$2\mathcal{W}(f) = \pi \int_0^l (\kappa_{m,n}^2 + 4\bar{K}_{m,n})ds = 16\pi \int_0^l |q|^2 ds,$$

where $\kappa_{m,n}$ is the curvature of the profile curve, $\bar{K}_{m,n} = \frac{1}{4}\Omega^2$ is the sectional curvature of the tangent plane of $S^3/_{\sim_{m,n}}$ and L the length of the profile curve with respect to $g_{m,n}$ and s the corresponding arclength parameter.

This is a simple application of the conformal invariance of the Willmore energy, since the metric $g_{m,n} = \frac{1}{h}g$, where g is the round metric on S^3.

6.1. Theorem. *Let $f : T^2 \cong \mathbb{C}/\Gamma \to S^3$ be a conformally parametrized equivariant immersion and q its conformal Hopf differential. Then f is constrained Willmore if and only if q satisfies the equation:*

$$\begin{aligned} q'' + 8(|q|^2 + C)q - 8\xi q &= 2Re(\lambda q), \\ 2\xi' &= \bar{q}'q - q'\bar{q}. \end{aligned} \tag{6.1.1}$$

where $\lambda \in \mathbb{C}$ is the Lagrange multiplier and ξ is a purely imaginary function and C a real constant.

The real part of equation (6.1.1) is the actual Euler-Lagrange equation. The imaginary part of the equation is the Codazzi equation and the equation on ξ is the Gauß equation, given in [**BuPP**]. In this paper the Euler-Lagrange equations are stated in terms of q and the Schwarzian derivative c of f. By the Gauß-Codazzi equations it turns out that the Schwarzian derivative of an equivariant immersion is given by

$$c = 4|q|^2 + imnH + const.$$

The function ξ we use is a multiple of the imaginary part of c. We have $\xi = i\frac{mn}{4}H$, see formula (2.1.4).

The Gauß-Codazzi equations are already derived in [**BuPP**]. A more general version of the Euler-Lagrange equation can be found there, too. We want to recompute the Euler-Lagrange equation for equivariant tori for which the profile curve does not go through the singular points of the base space. This does not happen for immersed tori of revolution or Hopf tori. The proof of the theorem needs:

6.2. Lemma. *The following formulas hold:*

(1)

$$\nabla^H_{\dot\gamma}\nabla^H_{\gamma'}\gamma' = \nabla^H_{\gamma'}\nabla^H_{\dot\gamma}\gamma' + K g_{m,n}(\dot\gamma, N_{m,n})N_{m,n},$$

where ∇^H is the Levi-Civita connection of $g_{m,n}$ on $S^3/_{\sim_{m,n}}$ and K is the Gaußian curvature.

(2)

$$\kappa_{m,n}\Omega^2 + g_{m,n}(grad(\Omega^2), N_{m,n}) = 4mn\Omega H,$$

where H is the mean curvature of the $(m,n)-$torus.

Proof. The first formula is well known for the curvature tensor of surfaces. For the second formula we need that the mean curvature H is given by

$$2H = \tfrac{1}{h} < f_{xx} + f_{yy}, N_{orm} > .$$

Further

$$\kappa_{m,n}\Omega^2 = \tfrac{4m^2n^2}{h^{3/2}} < f_{yy} - f_{xx}, N_{orm} >_{S^3}$$

and

$$< \mathrm{grad}(u), N_{orm} >_{S^3} = < f_{xx}, N_{orm} >_{S^3},$$

as computed in (5.9). Since the conformal factor of the metric of the torus is given by $4m^2n^2e^{2u} = \Omega^2$, we obtain

$$\mathrm{grad}(\Omega^2) = \frac{8m^2n^2}{h}\,\mathrm{grad}(u).$$

Adding both terms leads to the formula stated in the lemma. □

Now we can proof the theorem.

Proof. Fist we want to compute the Euler-Lagrange equation for equivariant Willmore tori, then we build in the constraints. By symmetrical criticality we can compute the Euler-Lagrange equation for equivariant Willmore tori by considering variations of curves. Let $\tilde\gamma_0$ be any regular and arclength parametrized (with respect to $g_{m,n}$) curve in the base space and let $\tilde\gamma_t(s)$ be a regular variation, i.e., we want $\tilde\gamma_0(s) = \tilde\gamma_0$ and $\tilde\gamma_t$ to be a regular curve for every $t \in [-\epsilon, \epsilon]$. We denote by $()'$ the derivative with respect to the curve parameter and by $\dot{()}$ the derivative with respect to t at $t = 0$. For the calculations it is important to understand what $\dot\kappa_{m,n}$ and $d\dot s$ are. We define the function $v(s,t)$ to be the velocity of the parametrized curves, i.e., we want $\tilde\gamma_t' = v(s,t)T(s,t)$,

Because $\tilde{\gamma}_0$ is arclength parametrized, we have $v(s,0) = v = 1$. The frame equations for curves gives that $\nabla^{m,n}_{\gamma'} T = v\kappa_{m,n} N_{m,n}$, where we use the notations introduced in (5.9). We obtain

$$\dot{\kappa}_{m,n} = g_{m,n}((\gamma'')^{\cdot}, N_{m,n}) - 2\dot{v}\kappa_{m,n} \quad \text{and} \quad \dot{v} = d\dot{s} = g_{m,n}(\dot{\gamma}', T).$$

Now we frequently use the Stokes theorem and the first statement of (6.2) to obtain the Euler-Lagrange equation for equivariant Willmore tori.

$$\begin{aligned}
\tfrac{1}{\pi}\dot{W}(f) &= \int_\gamma (\kappa_H^2 + \Omega^2)^{\cdot} ds + \int_\gamma (\kappa_H^2 + \Omega^2) d\dot{s} \\
&= \int_\gamma (2\kappa''_{m,n} + \kappa^3_{m,n} + 2\kappa_{m,n} K) g_{m,n}(\dot{\gamma}, N_{m,n}) ds \\
&+ \int_\gamma \dot{\Omega}^2 ds + \int_\gamma \Omega^2 d\dot{s} \\
&= \int_\gamma (2\kappa''_{m,n} + \kappa^3_{m,n} + 2\kappa_{m,n} K) g_{m,n}(\dot{\gamma}, N_{m,n}) ds \\
&+ \int_\gamma g_{m,n}(\mathrm{grad}(\Omega^2), \dot{\gamma}) - g_{m,n}(\mathrm{grad}(\Omega^2), g_{m,n}(\dot{\gamma}, \gamma')\gamma') ds \\
&- \int_\gamma \Omega^2 \kappa_{m,n} g_{m,n}(\dot{\gamma}, N_{m,n}) ds \\
&= \int_\gamma (2\kappa''_{m,n} + \kappa^3_{m,n} + \kappa_{m,n}(2K - \Omega^2) g_{m,n}(\dot{\gamma}, N_{m,n}) ds \\
&+ \int_\gamma g_{m,n}(\mathrm{grad}(\Omega^2), N_H)) g_{m,n}(\dot{\gamma}, N_{m,n}) ds.
\end{aligned}$$

Thus an equivariant torus corresponding to the curve $\tilde{\gamma}$ is Willmore if and only if

$$2\kappa''_{m,n} + \kappa^3_{m,n} + \kappa_{m,n}(2K - \Omega^2) + g_{m,n}(\mathrm{grad}(\Omega^2), N_{m,n}) = 0.$$

With the second statement of (6.2) and the formula for the Gaußian curvature of the metric $g_{m,n}$ in (5.5), we get that this equation is equivalent to:

$$2\kappa''_{m,n} + \kappa^3_{m,n} + \kappa_{m,n}(\Omega^2 - 2(m^2 + n^2)) + 4mn\Omega H = 0.$$

The space of conformal constraints for tori which are not isothermic tori is 2−dimensional. For equivariant tori which are not strongly isothermic we can consider the corresponding curve in $S^3/_{\sim_{m,n}}$. Infinitesimal conformal variations of an equivariant torus f, which preserves the equivariance type, preserves the length and the total curvature $\int_C \Omega vol_{m,n}$ of f, see (5.12). This space is also 2 dimensional. Thus all infinitesimal conformal variations of an equivariant torus which are not strongly isothermic preserves the equivariance type.

The variation of the length and the total curvature are given by

$$\dot{l} = \int_\gamma \kappa_{m,n} g_{m,n}(\dot{\gamma}, N),$$

and the variation of the total curvature is

$$\left(\int_C \Omega vol_{m,n}\right)^{\cdot} = \int_\gamma \Omega vol_{m,n}(\dot{\gamma}, .) = \int_\gamma \Omega g_{m,n}(\dot{\gamma}, N).$$

For isothermic tori the space of conformal constraints is only $1-$dimensional. Further there exist a constant $\mu \in S^1$ such that the rotated conformal Hopf differential $q\mu$ is real valued. Therefore Ω is a constant multiple of $\kappa_{m,n}$ and the space of length and total curvature constraint is also 1-dimensional. Thus the Euler-Lagrange equation for constrained Willmore $(m,n)-$tori with Lagrange multipliers λ_1 and λ_2 is:

$$2\kappa''_{m,n} + \kappa^3_{m,n} + \kappa_{m,n}(\Omega^2 - 2(m^2+n^2) + \lambda_1) + 4mn\Omega H + \lambda_2 \frac{mn}{h} = 0.$$

With $4q = \kappa_{m,n} + i\Omega$ and $\xi = i\frac{mn}{4}H$ we get that the latter equation is exactly the real part of the equation we stated above with $C = -\frac{1}{4}(m^2+n^2)$. □

6.3. Example. In the case of tori of revolution, we have $4q = \kappa$, where κ is the curvature of γ in the hyperbolic plane. These tori are always isothermic. The Euler-Lagrange equation reduces to the equation

$$\kappa'' + \tfrac{1}{2}\kappa^3 - \kappa = \lambda_1 \kappa,$$

which is the Euler-Lagrange equation for elastic curves, i.e., critical points of the energy functional $E(\gamma) = \int_\gamma \kappa^2 ds$ with prescribed length. Free elastic curves corresponds to Willmore tori.

For Hopf tori we have that $4q = \kappa + 2i$, where κ is the curvature of the curve $\pi_{1,1}(f)$ in the round S^2 with curvature 4. The Euler-Lagrange equation for constrained Willmore tori reduces to

$$\kappa'' + \tfrac{1}{2}\kappa^3 + 2\kappa = \lambda_1\kappa + \lambda_2.$$

This is the Euler-Lagrange equation for constrained elastic curves in the round S^2 with curvature 4, where $(\lambda_1 + 2)$ is the length and λ_2 is the enclosed area constraint. Note that free elastic curves do not correspond to Willmore-Hopf tori. For free elastic curves we have $\lambda_1 = -2$ and $\lambda_2 = 0$, but for Willmore-Hopf tori $\lambda_1 = \lambda_2 = 0$.

7. Associated Family and Equivariant Constrained Willmore Immersions

Let $f : T^2 \to S^3$ be a conformally immersed constrained Willmore torus with conformal Hopf differential q. Then there exist a circle worth

of constrained Willmore surfaces f_μ, $\mu \in S^1$, to f, the so called constrained Willmore associated family. The conformal Hopf differential of f_μ is given by

$$q_\mu = q\mu.$$

For equivariant tori q_μ satisfies Equation (6.1.1) with the parameters

$$\begin{aligned} C_\mu &= C + \mathrm{Re}(\mu^2 - 1\bar{\lambda}) \\ \xi_\mu &= \xi + \mathrm{Im}(\mu^2 - 1\bar{\lambda}) \\ \lambda_\mu &= \bar{\mu}^2\lambda. \end{aligned}$$

Since q_μ satisfies the Gauß-Codazzi equations, there exist a surface with conformal Hopf differential q_μ and mean curvature $H = \frac{-i4}{mn}\xi$. Since both invariants depends only on one parameter, the surfaces in the associated family of equivariant constrained Willmore surfaces are also equivariant. In general these surfaces are not compact, i.e., $f_\mu : \mathbb{C} \to$ is not doubly periodic , even if the initial surface was.

A special class of constrained Willmore tori are the CMC tori. The following theorem characterizes all equivariant CMC tori. It is a version of a theorem by Richter [**R**].

7.1. Theorem. *An equivariant constrained Willmore torus f is isothermic if and only if f is an equivariant CMC torus in a space form. In particular if $mn \neq 0$, then f is CMC in S^3.*

Proof. A surface is isothermic, i.e., it has a conformal curvature line parametrization, if and only if there exist $\mu \in S^1$ such that $q\mu$ is real valued. That is the case if and only if the imaginary part of q is a constant multiple of the real part of q. Thus we get that f is isothermic if and only if $\xi' \equiv 0$. Since $4\xi = imnH$, it is obvious that these surfaces are CMC in S^3 for $mn \neq 0$. For $mn = 0$ we have constrained Willmore tori of revolution. These are obviously isothermic. In [**B**] it is shown that these tori are CMC in a space form. □

Constrained Willmore tori of revolution are CMC in a space form, thus we refer to these tori as Delaunay tori in the following.

Corollary. *Except for the preimage of a circle, i.e., $q = const$ and the corresponding torus is homogenous, Hopf tori are never isothermic.*

Proof. Since for $m = n = 1$ the curvature Ω of the Seifert fiber space is constant so the imaginary part of q is constant. Thus $\mathrm{Im}(q)$ is a constant multiple of $\mathrm{Re}(q)$ if and only if $\mathrm{Re}(q)$ is constant. But $\mathrm{Re}(q)$ is the curvature of the profile curve on the round S^2, which is constant if and only if the curve is a circle. □

For CMC tori there exists a further associated family - the CMC associated family. By definition isothermic surfaces have a conformal

curvature line parametrization. In this parametrization the second fundamental form is diagonal and thus the Hopf differential is real valued and $\xi = 0$. The Euler-Lagrange equation reduces to

$$q'' + 8q^3 + Cq = Hq,$$

where H is the mean curvature, see [**BuPP**]. The associated family for CMC tori is simply given by $C_r = C + r$ and $H_r = H + r$ for $r \in \mathbb{R}$. Putting both associated families together we obtain

7.2. Theorem. *Every equivariant isothermic torus is in the associated family of a Delaunay cylinder.*

PROOF. First rotate q by μ such that $q\mu$ is real and then choose an $r \in \mathbb{R}$ such that $C = -\frac{1}{4}$. The corresponding q is the conformal Hopf differential of a Delaunay cylinder. By Theorem (3.3) of [**BuPP**] we have that isothermic surfaces with the same conformal Hopf differential lie in the same associated family as isothermic surfaces. This associated family coincides in our case with the associated family of CMC surfaces. □

CHAPTER III

Spectral Curves for Conformal Immersions into S^3

In this chapter we define the spectral curve for a conformal immersion $f : T^2 \to S^3$. Following [**BLPP**] the spectral curve Σ is the Riemann surface parametrizing all so called Darboux transformations of f. If the genus of the spectral curve is finite, then the immersion f is given in terms of algebraic data on Σ. In the following we use the notations of the quaternionic theory introduced in chapter (1).

8. The General Case

8.1. Darboux Transformations.

Definition. Let L be the line bundle associated to the conformal immersion $f : T^2 \to S^4$ and let δ be its differential. A sphere congruence $\tilde{S}$ is a map from T^2 into the space of oriented round 2−spheres in S^4 such that for $x \in T^2$ $f(x) \in \tilde{S}_x$ and such that the tangent space of $\tilde{S}_x$ at $f(x)$ coincides with the one of f at x. In other words

$$\tilde{S}L = L \quad \text{and} \quad * \delta = \tilde{S}\delta = \delta\tilde{S}.$$

Definition. A map $f^\# : T^2 \to S^4$ or its associated line bundle $L^\#$ is called a Darboux transform of the map $f : T^2 \to S^4$, if $f^\#(x)$ is distinct from $f(x)$ for all $x \in T^2$ and there exists a sphere congruence $\tilde{S}$ of f with

$$\tilde{S}L^\# = L^\#, \quad \text{and} \quad * \delta^\# = \tilde{S}\delta^\#,$$

where $\delta^\#$ is the differential of $L^\#$.

If $f^\#(x) = f(x)$ at isolated points $x \in T^2$ then we call $f^\#$ *singular*.

To a conformal immersion $f : T^2 \to S^3 \subset \mathbb{H}P^1$ consider the quotient bundle V/L.

Definition. A section ψ with monodromy of V/L is a section of the pull-back $\widetilde{V/L}$ of V/L to the universal covering $\mathbb{C}$ of $T^2 = \mathbb{C}/\Gamma$ with

$$\gamma^*\psi(z) = \psi(z + \gamma) = \psi(z)h_\gamma, \text{ for all } z \in T^2,$$

where $h : \Gamma \to \mathbb{H}_*$ is a representation. We call h the monodromy representation of ψ and h_γ the monodromy of ψ along γ.

Since h is a representation it is uniquely determined by its value on the generators of the lattice Γ. We denote the generators by γ_1 and γ_2.

Let ∇ be the trivial connection on V and S the mean curvature sphere congruence of f. A holomorphic structure on V is given by ∇'', the $\bar{K}-$part of ∇ with respect to S. Since $SL \subset L$, the projection of ∇'' to V/L defines a canonical holomorphic structure D on V/L. It turns out that all Darboux transformations of f can be obtained by considering holomorphic sections of $(V/L, D)$ with monodromy. We denote the space such sections by $H^0(\widetilde{V/L})$.

8.2. Theorem. *Let $f : T^2 \to S^4$ be a conformal immersion. Then there is a bijective correspondence between the space of (possibly singular) Darboux transforms of f and the space of nontrivial holomorphic sections with monodromy up to scale.*

The proof can be found in [**BLPP**]. It uses the uniqueness of the *prolongation* $\tilde{\psi}$ of a section ψ in $H^0(\widetilde{V/L})$, which is an appropriate lift of the holomorphic section to a section of $\tilde{V} = \mathbb{C} \times \mathbb{H}^2$. Then the corresponding Darboux transform $L^\#$ is given by $\tilde{\psi}\mathbb{H}$. The monodromy representation is the same for the holomorphic section with monodromy in V/L and for its prolongation. Moreover, prolongation is a $\mathbb{H}-$linear map.

8.3. Definition of the Spectral Curve. Consider a section $\psi \in H^0(\widetilde{V/L})$ and let h denote its monodromy representation. For a constant $\lambda \in \mathbb{H}$ the section $\tilde{\psi} = \psi\lambda$ is also holomorphic and induces the same Darboux transform. But its monodromy representation is given by $\tilde{h} = \lambda^{-1}h\lambda$ since

$$\tilde{\psi}(z+\gamma) = \psi(z)h_\gamma\lambda = \psi(z)\lambda\lambda^{-1}h_\gamma\lambda = \tilde{\psi}(z)\tilde{h}_\gamma.$$

Thus we are interested in the conjugacy class of a monodromy representation h as the parameter space for all Darboux transforms of f.

8.4. Definition. The quaternionic spectrum of V/L is the subspace

$$Spec_{\mathbb{H}} \subset \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

of conjugacy classes of possible monodromy representations h of sections in $H^0(\widetilde{V/L})$. In other words, a representation $h : \Gamma \to \mathbb{H}_*$ represents a point $[h] \in Spec_{\mathbb{H}}$ if and only if there exist a non trivial holomorphic section $\psi \in H^0(\widetilde{V/L})$ with

$$\psi(z+\gamma) = \psi(z)h_\gamma \quad \text{for all } z \in \mathbb{C} \text{ and } \gamma \in \Gamma.$$

Let $\psi \in H^0(\widetilde{V/L})$ and let $h : \Gamma \to \mathbb{H}_*$ be its monodromy representation. Since Γ is a lattice, we have

$$h_{\gamma_1} h_{\gamma_2} = h_{\gamma_2} h_{\gamma_1}.$$

Thus the imaginary parts of the quaternions h_{γ_1} and h_{γ_2} are real linearly dependent. Then it is always possible to choose a λ such that $\lambda^{-1} h \lambda : \Gamma \to \mathbb{C}_*$, i.e., for every point $[h] \in Spec_{\mathbb{H}}$ we can always find a representative $h_{\mathbb{C}} : \Gamma \to \mathbb{C}_*$ with complex valued monodromies. Further conjugating $h_{\mathbb{C}}$ by $\mathbb{j} \in \mathbb{H}$ we get the representation $\bar{h}_{\mathbb{C}}$, which is still complex valued. Thus the map

$$p : \mathrm{Hom}(\Gamma, \mathbb{C}_*) \to \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

is $2:1$ away from real representations and we can lift the quaternionic spectrum.

Definition. The lift of the quaternionic spectrum by p

$$Spec(V/L) := p^* Spec_{\mathbb{H}}(V/L)$$

is called the complex spectrum of V/L. With $\rho(h) = \bar{h}$ we get

$$Spec_{\mathbb{H}}(V/L) = Spec(V/L)/\rho.$$

Theorem. *Let $f : T^2 \to S^3$ be a conformal immersion and V/L its quotient bundle. If $h \in Spec(V/L)$ then we have that also $h^{-1} \in Spec(V/L)$. We denote by σ the map*

$$\sigma : Spec(V/L) \to Spec(V/L), h \mapsto h^{-1}.$$

Proof. Let $h \in Spec(V/L)$ and ψ be a holomorphic section of V/L with monodromy h. Then there exist a holomorphic section φ in $K(V/L)^{-1}$ with monodromy such that the pairing between these sections (ψ, φ) has no monodromy. Therefore φ has monodromy h^{-1}. For $f : T^2 \to S^3$ we have further that the bundles V/L and $K(V/L)^{-1}$ are holomorphic isomorphic and so φ induces a holomorphic section in V/L with monodromy h^{-1} and we obtain $h^{-1} \in Spec(V/L)$. □

8.5. Theorem. *For $h \in Spec(V/L)$ the space of holomorphic sections with monodromy h is finite dimensional and generically it is complex $1-$dimensional.*

Theorem. *$Spec(V/L)$ is a complex 1 dimensional analytic variety.*

The proof of both theorems can be found in [**BLPP**]. An analytic variety has a uniquely determined normalization. Thus we can define the following:

8.6. Definition. The spectral curve $\tilde{\Sigma}$ is the normalization of the complex spectrum of V/L.

Remark. The involutions ρ and σ defined on $Spec(V/L)$ given by $\rho(h) = \bar{h}$ and $\sigma(h) = h^{-1}$ induce an anti-holomorphic and a holomorphic involution on $\tilde{\Sigma}$.

The spectral curve defined here is non compact. A criteria when $\tilde{\Sigma}$ can be compactified is given by the following theorem.

8.7. Theorem. *The spectral curve $\tilde{\Sigma}$ can be compactified to a Riemann surface Σ of finite genus if and only if it has two ends interchanged by ρ, i.e., it can be compactified by adding two points. These points are called 0 and ∞.*

The proof can be found in [**BoPP**]. We also call Σ, if it exists, the spectral curve of f. It has always two marked points corresponding to the ends of $\tilde{\Sigma}$. All connected finite type spectral curves are classified in [**BoPP**].

8.8. Theorem. *If the spectral curve of a conformal immersion is connected and has finite genus, then it has the structure of a $n-$fold covering of $\mathbb{C}P^1$.*

Definition. A conformal immerison $f : T^2 \to S^3$ such that its spectral curve has finite genus g is called a finite type immersion and g is called its spectral genus.

It is shown in [**B**] that the spectral curve of a constrained Willmore torus in S^4 has finite genus and is connected. Since our interest lies in equivariant constrained Willmore tori, we restrict ourselves now to conformally immersed tori whose spectral curves are connected and have finite genus.

9. The Kernel Bundle

The spectral curve $\tilde{\Sigma}$ carries a canonical line bundle $\mathcal{L} \subset \tilde{\Sigma} \times \widetilde{\Gamma(V/L)}$. It is called the kernel bundle. The construction of $\mathcal{L}$ works as follows: To a generic point $h \in \tilde{\Sigma}$ there exist a holomorphic section ψ_h in V/L with monodromy h unique up to complex scale. Thus we can assign to a generic $h \in \tilde{\Sigma}$ the complex line given by $\psi_h\mathbb{C}$. Although ψ_h is not well-defined on the torus, the line given by $\psi_h\mathbb{C}$ is. The line bundle $\mathcal{L}$ extends holomorphically through the exceptional points of $\tilde{\Sigma}$, where we have a higher dimensional space of holomorphic sections. The kernel bundle is compatible with the involutions ρ and σ. Since the holomorphic section with monodromy $\bar{h}$ is given by $\psi_h\mathbb{j}$ we have $\rho^*\mathcal{L} = \mathcal{L}\mathbb{j}$.

$\mathcal{L}$ does not extend to Σ by the asymptotic properties of $\tilde{\Sigma}$ shown in [**BoPP**]. This can be repaired by fixing a point $x \in T^2$ and evaluating the holomorphic section ψ_h at each point of Σ in x. In other words,

to a fixed $x \in T^2$ we associate to a generic point $h \in \tilde{\Sigma}$ the line given by $\psi_h(x)\mathbb{C}$. The holomorphic line bundle corresponding to this construction is denoted by $\mathcal{L}_x$, see [**BLPP**].

9.1. Theorem. *The map*

$$\Psi : T^2 \to Jac(\Sigma), x \mapsto \mathcal{L}_x\mathcal{L}_{x_0}^{-1}$$

for a fixed base point $x_0 \in T^2$ is a group homomorphism.

Again, the proof can be found in [**BLPP**]. This theorem means that the conformal immersion f induces a linear map into the Jacobian of the spectral curve. Thus it is possible to construct f explicitly in terms of algebraic data on Σ.

Remark. The bundle $\mathcal{L}$ over $\tilde{\Sigma}$ can be lifted to a line subbundle $\tilde{\mathcal{L}}$ of $\Sigma \times \Gamma(\tilde{V})$ by assigning to a generic point $h \in \tilde{\Sigma}$ the complex line given by the prolongation $\tilde{\psi}_h$ of ψ_h. The monodromy of $\tilde{\psi}_h$ is also h. Further the bundle $\tilde{\mathcal{L}}_x$, for a fixed $x \in T^2$, also extends to Σ.

10. The Reconstruction

Let Σ be a $n-$fold covering of $\mathbb{C}P^1$ with an anti-holomorphic involution ρ. Further fix a real subtorus $Z = \Psi(T^2)$ of dimension $0, 1$ or 2 of the Jacobian of Σ. The question is now how to construct conformal immersions $f : T^2 \to S^3$ with given spectral curve Σ and Z. For $x \in T^2$ the line bundle $\tilde{\mathcal{L}}_x$ over Σ is by construction a complex holomorphic line subbundle of $\Sigma \times \mathbb{C}^4$. Thus it defines a map from Σ to $\mathbb{C}P^3$. A quaternionic structure on $\mathbb{C}^4$ is a real linear endomorphism $\mathbb{j}$ with $\mathbb{j}^2 = -Id$ anti-commuting with i. By fixing such a quaternionic structure $\mathbb{j}$ on $\Sigma \times \mathbb{C}^4$ we obtain a canonical isomorphism between $\mathbb{C}^4$ and $\mathbb{H}^2$. This isomorphism induces a map $\pi_{\mathbb{H}}$ between $\mathbb{C}\mathrm{P}^3$ and $\mathbb{H}P^1$ which is called twistor projection. The main theorem is the following one proven in [**BLPP**].

Theorem. *Let $f : T^2 \to S^3$ be a conformal immersion whose spectral curve Σ has finite genus. Then there exist a map*

$$F : T^2 \times \Sigma \to \mathbb{C}P^3, \tag{10.0.1}$$

such that

- *$F(x, -) : \Sigma \to \mathbb{C}P^3$ is an algebraic curve, for all $x \in T^2$.*
- *The original conformal immersion $f : T^2 \to S^3$ is obtained by the twistor projection of the evaluation of F at the points at infinity:*

$$f = \pi_{\mathbb{H}}F(-, 0) = \pi_{\mathbb{H}}F(-, \infty).$$

For given Σ with marked points 0 and ∞ and a T^2-family of holomorphic line bundles Z in $\mathrm{Jac}(\Sigma)$ the map F is in general not unique. In other words, the the immersion f is in general not uniquely determined by the spectral curve and Z.

Let L be a complex holomorphic line bundle over a Riemann surface M. By Kodaira embedding there exist a holomorphic map s from L to $\mathbb{C}P^n$ if and only if the space of holomorphic sections of the line bundle L^* is at least complex $(n+1)-$dimensional. The space of holomorphic sections of L^* is $(n+1)-$dimensional if and only if the map is unique up to a $PSL(n+1,\mathbb{C})$ action on $\mathbb{C}\mathrm{P}^n$. Further, the line bundles in Z must be compatible with the quaternionic structure $\mathbb{j}$. Elements of $PSL(4,\mathbb{C})$ compatible with $\mathbb{j}$ acts on $\mathbb{H}P^1$ as Möbius transformations. So F is uniquely determined up to a $PSL(4,\mathbb{C})$ action compatible with $\mathbb{j}$ on $\mathbb{C}P^3$ if and only if $\pi_{\mathbb{H}}(F)$ is uniquely determined up to Möbius transformations of S^4.

Definition. A immersion is called simple if the map F is uniquely determined by Σ and Z up to a $PSL(4,\mathbb{C})$ action on $\mathbb{C}P^3$, compatible with the quaternionic structure $\mathbb{j}$.

10.1. Proposition. *Let $f : T^2 \to S^3$ be a simple conformal immersion and V/L the associated quotient bundle, then the space $H^0(V/L)$ is quaternionic $2-$dimensional.*

Proof. If $H^0(V/L) > 2$ then there exist at least 3 quaternionic linear independent holomorphic sections. The quotient of any two of them yield a map $f : T^2 \to \mathbb{H}P^1$. Thus in this case we get at least two maps f and $\tilde{f}$ which are not Möbius equivalent such that the corresponding quotient bundles V/L and $V/\tilde{L}$ are holomorphic isomorphic. Thus f and $\tilde{f}$ have the same spectral curve and Z and the map F cannot be unique. □

10.2. Corollary. *A simple immersion of spectral genus 1 is equivariant.*

Proof. By assumption the map F is unique. Thus the bundle $L = \pi_{\mathbb{H}}F(-,\infty)$ is unique up to Möbius transformations of S^4. If we only consider immersions into a fixed $S^3 \subset S^4$, we obtain that this reconstruction is unique up to Möbius transfomations of S^3. Since the Jacobian of a torus is the torus itself and the set of line bundles compatible with ρ is only a S^1, the map Ψ has a $1-$dimensional kernel. Now let x be the direction in $T^2 = \mathbb{C}/\Gamma$ parametrizing the kernel of Ψ. Then the conformal maps $f(x,y)$ and $f(x+x_0,y)$ have the same spectral curve Σ and Z. Thus there is a Möbius transformation M_{x_0} of S^3 with

$$f(x+x_0,y) = M_{x_0}f(x,y).$$

The map $M : \mathbb{R} \to \text{Möb}(S^3)$ is a group homomorphism, thus f is equivariant. □

10.3. Example 1: The CMC Case. The spectral curve of a conformal immersion of finite type is a $n-$fold covering of $\mathbb{C}P^1$, if it is connected. Thus the easiest non-trivial case is that Σ is a double cover of $\mathbb{C}P^1$. A surface class for which this holds is the class of CMC immersions in S^3 and $\mathbb{R}^3$. For CMC immersions there exist another way to define the spectral curve, namely via the holonomy representation of an associated family of flat connections. Both approaches to the spectral curve are equivalent by the Theorems (4.5) and (6.8.) of [**B**]. In the following, we just explain the constructions and omit the proofs.

This definition of the spectral curve is due to [**H**] and [**BoB**]. Let $f : T^2 \to S^3$ be a conformally immersed torus into the Lie group S^3 with constant mean curvature H. And let $\alpha = \bar{f}df$ be the corresponding Maurer-Cartan form. We denote by V the pull back of the spinor bundle of S^3. This is a complex rank 2 vector bundle with a quaternionic structure j and a symplectic form $\hat{\omega}$.

Let $\alpha = \alpha' + \alpha''$ be the splitting of α into its complex linear and complex anti-linear part.

To f we can associate a family of connections

$$\nabla^\lambda = \nabla + \tfrac{1}{2}(1+\lambda^{-1})(1+iH)\alpha' + \tfrac{1}{2}(1+\lambda)(1-iH)\alpha'), \quad \lambda \in \mathbb{C}_*, \tag{10.3.1}$$

where ∇ is the trivial connection on V. This family of connections is flat if and only if $H \equiv const$. This family of connection has the symmetry

$$\nabla^{\bar{\lambda}^{-1}} = j^{-1}\nabla^\lambda j.$$

Thus for $\lambda \in S^1$ the connection ∇^λ is unitary. An associated family of constant mean curvature surfaces for a given f is obtained by the following theorem.

Theorem. *Let ∇^λ be the family of flat connections given in (10.3.4) and $\lambda_0, \lambda_1 \in S^1$ with $\lambda_0 \neq \lambda_1$. Further let X_λ be a parallel frame of ∇^λ. Then the map $f : \mathbb{C} \to S^3$ given by*

$$f = X_{\lambda_1} X_{\lambda_0}^{-1} \tag{10.3.2}$$

is well defined and has constant mean curvature $H = i\frac{\lambda_0+\lambda_1}{\lambda_0-\lambda_1}$. Further all constant mean curvature immersions in S^3 with doubly periodic metric comes from such a construction. For $\lambda_0 = \lambda_1$ we have

$$f = X_{\lambda_0}^{-1} \frac{\partial X}{\partial \lambda}|_{\lambda_0}, \tag{10.3.3}$$

which yields all CMC immersions with doubly periodic metric in $\mathbb{R}^3$.

The formulas above are called the Sym-Bobenko formulas and the λ_i are called Sym-points. This is Theorem 5 of [**BoB**]. We refer to this paper for the proof. By choosing $\lambda_0 = -\lambda_1$ we obtain a minimal immersion. Thus we can restrict ourselves without loss of generality to the associated family for minimal tori

$$\nabla^\lambda = \nabla + \tfrac{1}{2}(1+\lambda^{-1})\alpha' + \tfrac{1}{2}(1+\lambda)\alpha'', \quad \lambda \in \mathbb{C}_*. \tag{10.3.4}$$

Let $T^2 = \mathbb{C}/\Gamma$ and let $H^\lambda_x(\gamma)$ denote the holonomy of the connection ∇^λ along $\gamma \in \Gamma$ with respect to the base point $x \in T^2$. We choose γ to be one of the generators of Γ.

Proposition. *The holonomy is diagonalizable and has distinct eigenvalues for generic $\lambda \in \mathbb{C}_*$.*

The proof can be found in [**H**]. The spectral curve is now the normalization and compactification of the analytic variety

$$\{(\eta,\lambda) \in \mathbb{C}_*\times\mathbb{C}_* \mid f(\eta,\lambda) = 0\} \qquad \text{with} \qquad f(\eta,\lambda) = \det(H^\lambda_x(\gamma) - \eta Id).$$

Since $H^\lambda_p(\gamma)$ is generically diagonalizable and 2×2, we have that the spectral curve of a constant mean curvature immersion is hyperelliptic. By [**H**] the spectral curve of a minimal immersion is branched over $\lambda = 0$ and $\lambda = \infty$. Thus it is given by

$$\Sigma : \eta^2 = \lambda \Pi_{i=1}^g \bar{q}_i^{-1}(\lambda - q_i)(\lambda - \bar{q}_1^{-1}),$$

where $q_i \neq 0, \infty$, and $\bar{q}_i^{-1}$ are the odd order roots of the function $f(\eta,\lambda)$ without multiplicity, i.e., Σ is a smooth curve.

The change of the base point corresponds to a conjugation of the holonomy matrix. Thus the eigenvalues do not change and the spectral curve is independent of $x \in T^2$. Further since the first fundamental group of T^2 is abelian we obtain that the odd order roots of the function $f(\eta,\lambda)$, and therefore the spectral curve, do not depend on the choice of a generator $\gamma \in \Gamma$.

Next we want to consider the eigenspaces of the holonomy. For a fixed $x \in T^2$ let $\mathcal{L}_x$ denote the line bundle over Σ given by the eigenspace of the holonomy. To be more explicit, at a generic point $(\eta,\lambda) \in \Sigma$ we define the fiber $\mathcal{L}_x|_{(\eta,\lambda)}$ to be the $1-$dimensional eigenspace of $H^\lambda_x(\gamma)$ w.r.t. to the eigenvalue η. The bundle is well-defined and extends holomorphically to a line bundle over all Σ. Let σ denote the hyperelliptic involution on Σ. Then the bundles $\mathcal{L}_x$ and $\sigma^*\mathcal{L}_x$ are subbundles of the trivial bundle $\Sigma \times V_x$. Since V is the pull-back of the spin bundle of S^3, there exists a symplectic form $\hat{\omega}$ on V. Therefore the evaluation of the symplectic form $\hat{\omega}_x$ on $\mathcal{L}_x \otimes \sigma^*\mathcal{L}_x$ defines a holomorphic map to $\mathbb{C}$. Thus $\hat{\omega}_x$ is a section in $\mathcal{L}_x^* \otimes \sigma^*\mathcal{L}_x^*$. It vanishes exactly at those points, where $\mathcal{L}_x$ and $\sigma^*\mathcal{L}_x$ coincides. Therefore the zeros of

$\hat{\omega}_x$ does not depend on x. Obviously $\hat{\omega}_x$ vanishes at branch points of Σ, thus it has at least $2g+2$ zeros.

Definition. Let $f : T^2 \to S^3$ be a conformal immersion of constant mean curvature and Σ its spectral curve. The genus g of Σ is called the geometric spectral genus of f. The arithmetic spectral genus p of f is $\frac{n-1}{2}$, where n is the number of zeros of the symplectic form ω counted with multiplicity.

Obviously we have always $p \geq g$. Let q_i and $\bar{q}_i$ denotes the zeros of ω with multiplicity. Then the equation

$$\eta^2 = \lambda \Pi_{i=1}^{p} \bar{q}_i^{-1}(\lambda - q_i)(\lambda - \bar{q}_i^{-1})$$

defines a (possibly singular) hyperelliptic curve of genus p. We call this curve the arithmetic spectral curve of a conformally immersed CMC torus f. This spectral curve is smooth if and only if $g = p$.

Proposition. *Let ω be the symplectic form defined above. The line bundle $\mathcal{L}_x^*$ has degree $\frac{n}{2}$, where n is the number of zeros of ω counted with multiplicity. Further, the bundle $\mathcal{L}_x$ is nonspecial for all $x \in T^2$, i.e., the line bundle $K\mathcal{L}_x^*$ has no holomorphic sections.*

Obviously the degree of $\mathcal{L}_x^*$ is $\frac{n}{2}$. For the rest of the proof we refer to [**H**].

Proposition. *The map*

$$\Psi : T^2 \to Pic_{p+1}(\Sigma), x \mapsto \mathcal{L}_x$$

is a group homomorphism.

For a given spectral curve and a family of line bundles $\Psi(T^2)$ compatible with all involutions, we can reconstruct by [**H**] an associated family of flat connections of the form (10.3.4) and thus also an associated family of constant mean curvature tori.

Theorem. *Let Σ be a (possibly singular) hyperelliptic curve over $\mathbb{C}P^1$ defined by the equation*

$$\eta^2 = \lambda \Pi_{i=1}^{p} \bar{q}_i^{-1}(\lambda - q_i)(\lambda - \bar{q}_i^{-1}).$$

And let

$$\Psi : T^2 \to Pic_{p+1}(\Sigma), x \mapsto \mathcal{L}_x$$

be a group homomorphism. Suppose all $\mathcal{L}_x \in \Psi(T^2)$ are nonspecial and $\rho^* \mathcal{L}_x = \mathcal{L}_x \mathbb{j}$. Further $\mathcal{L}_x^* \otimes \sigma^* \mathcal{L}_x^* = L(2p+2)$, where $L(2p+2)$ is the bundle of degree $2p+2$ on $\mathbb{C}\mathrm{P}^1$.

Then we can construct a family of flat connections ∇^λ on V for which the lines $\mathcal{L}|_{(\eta,\lambda)}$ and $\mathcal{L}|_{(-\eta,\lambda)}$ define a parallel frame of V w.r.t. ∇^λ. This family has the form

$$\nabla^\lambda = \nabla + \tfrac{1}{2}(1+\lambda^{-1})\alpha' + \tfrac{1}{2}(1+\lambda)\alpha''.$$

10.4. Example 2: The constrained Willmore case. The spectral curve of a constrained Willmore torus is either a double covering or a $4-$fold covering of $\mathbb{C}P^1$. As in the CMC case, we can define an associated family of flat connections. For this we need to state the Euler-Lagrange equations for constrained Willmore surfaces in the quaternionic set up. Recall that to a conformal immersion in S^3 we can associate a line bundle $L \subset V = T^2 \times \mathbb{H}^2$. The trivial connection on V is denoted by ∇ and A denotes the Hopf field of ∇' with respect to the mean curvature sphere congruence S.

Proposition. *Let $f : T^2 \to S^3$ be a conformal immersion. It is constrained Willmore if there exist a $1-$form $\nu \in \Omega^1(\mathcal{R})$ such that*

$$d^{\nabla}(2 * A + \nu) = 0,$$

where $\mathcal{R} = \{B \in End(V) \mid ImB \subset L \subset KerB\}$.

Let A_0 be the $1-$form defined by $2 * A_0 = 2 * A + \nu$. Consider V as a complex rank 4 bundle with complex structure i given by the left multiplication of the quaternionic $\mathbb{i}$ together with an anti-linear endomorphism j which is the right multiplication by the quaternion $\mathbb{j}$. The Euler-Lagrange equation for constrained Willmore tori is equivalent to the flatness of the associated family of $SL(4, \mathbb{C})-$connections given by

$$\nabla^{\mu} = \nabla + (\mu - 1)\frac{1 - iS}{2}A_0 + (\mu^{-1} - 1)\frac{1 + iS}{2}A_0,$$

for $\mu \in \mathbb{C}_*$. This associated family ∇^{μ} has the symmetry

$$\nabla^{\bar{\mu}^{-1}} = j^{-1}\nabla^{\mu} j.$$

Thus for all $\mu \in S^1 \subset \mathbb{C}_*$ the connection ∇^{μ} is quaternionic.

For a fixed point $x \in T^2 = \mathbb{C}/\Gamma$ consider the holonomy representations H^{μ}_x of the associated family ∇^{μ}. The representations are holomorphic in μ and the holonomy H^{μ}_x to different basis points are conjugated. Thus the eigenvalues are independent of $x \in T^2$. Since the first fundamental group of T^2 is abelian, we get that every simple eigenspace of $H^{\mu}_x(\gamma_0)$ for $\gamma_0 \in \Gamma$ is a simultaneous eigenspace for all $\gamma \in \Gamma$. Bohle [**B**] shows

Theorem. *Let $f : T^2 \to S^3$ be a constrained Willmore torus. Then the holonomy representation of the associated family ∇^{μ} belongs to the following 2 cases:*

1. *there is a $\gamma \in \Gamma$ such that the holonomy $H^{\mu}_x(\gamma)$ has 4 distinct eigenvalues for generic $\mu \in \mathbb{C}_*$. These eigenvalues are non-constant in μ.*
2. *all holonomies $H^{\mu}_x(\gamma)$ have a $2-$dimensional common eigenspace with eigenvalue 1 and there is a $\gamma \in \Gamma$ such that $H^{\mu}_p(\gamma)$ has 2 distinct and non constant eigenvalues for generic $\mu \in \mathbb{C}_*$.*

The spectral curve of a constrained Willmore torus is the normalization and compactification of the $1-$dimensional analytic variety

$$\{(\eta,\mu)\in\mathbb{C}_*\times\mathbb{C}_*|f(\eta,\mu)=0\}\qquad\text{with}\qquad f(\eta,\mu)=\det(H_p^\mu(\gamma)-\eta Id).$$

Theorem. *The spectral curve defined by the associated family of flat connections coincides with the spectral curve defined in (8.6).*

This is Theorem (4.5) of [**B**]. Since every constrained Willmore torus is of finite type, the compactification of its spectral curve is well defined and has two marked points 0 and ∞ corresponding to the ends of the spectral curve $\tilde{\Sigma}$. It is a $4-$ fold covering of $\mathbb{C}P^1$, if the holonomy representation of ∇^μ belongs to case (1) and it is hyperelliptic if the holonomy representation of ∇^μ belongs to case (2).

We have two involutions ρ and σ on the spectral curve. Because of the symmetry of ∇^μ we have that ρ covers the involution $\mu\mapsto\bar{\mu}^{-1}$ on $\mathbb{C}P^1$. By the following proposition we get that the involution σ fixes the spectral parameter μ.

Lemma. *Let $f:T^2\to S^3$ be a constrained Willmore torus and ∇^μ the corresponding associated family of flat connections. Further let $H_x^\mu(\gamma)$ denote the holonomy of ∇^μ at a base point $x\in T^2$ along $\gamma\in\Gamma$. If η is an eigenvalue of $H_x^\mu(\gamma)$ then η^{-1} is also an eigenvalue of $H_x^\mu(\gamma)$.*

Proof. Consider the bundle $L^\perp\subset V^*$, where V^* is a complex quaternionic bundle with respect to the complex structure $-i$, see section (1.14). We can define a family of connections $(\nabla^\perp)^\mu$, for $\mu\in\mathbb{C}_*$ on $L^\perp$ by

$$(\nabla^\perp)^\mu=\nabla^\perp+(\mu-1)\frac{1+iS^\perp}{2}A_0^\perp+(\mu^{-1}-1)\frac{1-iS^\perp}{2}A_0^\perp.$$

This family is dual to the family of connections

$$\tilde{\nabla}^\mu=\nabla+(\mu-1)Q_0\frac{1-iS}{2}+(\mu^{-1}-1)Q_0\frac{1+iS}{2},$$

with respect to the indefinite inner product $(.,.)$ defining S^3, since $A^\perp=-Q^*$, $Q^\perp=-A^*$ and $\nu^\perp=-\nu^*$.

Both families of connections are flat if and only if the immersion is constrained Willmore. By duality of these families we have if η is an eigenvalue of the holonomy of $(\nabla^\perp)^\mu$ then η^{-1} is an eigenvalue of the holonomy of $\tilde{\nabla}^\mu$. On other hand, we have $L=L^\perp$ and $S=S^*=S^\perp$ for conformal immersions into S^3. Thus we obtain $(\nabla^\perp)^\mu=\nabla^\mu$. Further we have that ∇^μ is gauge equivalent to $\tilde{\nabla}^\mu$. The gauge is given by

$$2g=(\mu+1)-i(\mu-1)S.$$

This proves the lemma. □

At a generic point $(\eta, \mu) \in \Sigma$ we have that $H^\mu_x(\gamma)$ is diagonalizable and has 2 or 4 distinct eigenvalues. We restrict ourselves now to the case with 4 distinct eigenvalues. The other case corresponds to CMC immersions in a space form. Then there exist basis of $\mathbb{C}^4$ such that $H^\mu_x(\gamma)$ is given by

$$H^\mu_x(\gamma) = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta^{-1} & 0 & 0 \\ 0 & 0 & \tilde{\eta} & 0 \\ 0 & 0 & 0 & \tilde{\eta}^{-1} \end{pmatrix}.$$

For a fixed $x \in T^2$ consider the symplectic form ω on V_x defined in (1.14). Let $\tilde{\mathcal{L}}_x \subset \Sigma \times \mathbb{C}^4$ be the line bundle which at generic points $(\eta, \mu) \in \Sigma$ coincides with the eigenspace of $H^\mu_x(\gamma)$ to the eigenvalue η. Further let $g(\sigma^*\tilde{\mathcal{L}}_x)$ denote the line bundle corresponding to the eigenspace of the holonomy of $\tilde{\nabla}^\mu$ to the eigenvalue η^{-1}.

The evaluation of ω on $\tilde{\mathcal{L}}_x \otimes g(\sigma^*\tilde{\mathcal{L}}_x)|_{(\eta,\mu)}$ is non zero at generic points, where we have 4 distinct eigenvalues, and thus it defines a holomorphic map

$$\tilde{\mathcal{L}}_x \otimes g(\sigma^*\tilde{\mathcal{L}}_x) \to \mathbb{C},$$

which we also denote by ω. This map vanishes, if and only if the lines $\tilde{\mathcal{L}}_x$ and $g(\sigma^*\tilde{\mathcal{L}}_x)$ coalesce at (η, μ). This is indepent on x. The bundles $g(\sigma^*\tilde{\mathcal{L}}_x)$ and $\sigma^*\tilde{\mathcal{L}}_x$ are holomorphic isomorphic. Therefore we obtain

10.5. Lemma. *The bundle $\tilde{\mathcal{L}}_x \otimes \sigma^*\tilde{\mathcal{L}}_x$ is independent of $x \in T^2$ as a complex holomorphic line bundle.*

10.6. Theorem. *Let $f : T^2 \to S^3$ be a simple constrained Willmore immersion with spectral curve Σ and a fix point free anti-holomorphic involution $\rho : (\eta, \mu) \mapsto (\bar{\eta}, \bar{\mu}^{-1})$ on Σ. Further let σ be the involution $(\eta, \mu) \mapsto (\eta^{-1}, \mu)$. If the quotient Σ/σ is $\mathbb{C}P^1$ and $\rho \circ \sigma$ has fixpoints, then f is a CMC immersion in a space form.*

Proof. Let $f : T^2 \to S^3$ be a simple, conformal and constrained Willmore immersion. Let ∇^μ be its associated family of complex flat connections of constrained Willmore surfaces on the bundle $V = T^2 \times \mathbb{H}^2 \cong T^2 \times \mathbb{C}^4$. There are 2 cases to consider.

In the first case the holonomy $H^\mu_x(\gamma)$ of ∇^μ has generically 2 distinct eigenvalues. We obtain that the spectral curve Σ is always hyperelliptic and σ is the hyperelliptic involution. By the Theorems (6.8), (6.9) and (6.10) of [**B**] f is a conformally immersed CMC torus in a space form.

In the second case the holonomy of ∇^μ has generically 4 distinct eigenvalues. So Σ is a $4-$fold covering of $\mathbb{C}P^1$ and has two marked points 0 and ∞ corresponding to the ends of the spectral curve $\tilde{\Sigma}$. These points are fixed under the involution σ and interchanged by

the involution ρ. By assumption the quotient Σ/σ is also $\mathbb{C}P^1$. Thus the spectral curve Σ is a hyperelliptic curve and σ is the hyperelliptic involution. Let λ be a holomorphic coordinate of Σ/σ such that $0 \in \Sigma$ is the point over $\lambda = 0$ and $\infty \in \Sigma$ is the point over $\lambda = \infty$. Since the involutions ρ and σ commute and ρ interchanges the points 0 and ∞ on Σ, ρ also interchanges the points $\lambda = 0$ and $\lambda = \infty$ on Σ/σ and thus it induces an anti-holomorphic involution on Σ/σ. An anti-holomorphic involution on $\mathbb{C}P^1$ interchanging $\lambda = 0$ and $\lambda = \infty$ is either the map $\lambda \mapsto \bar{\lambda}^{-1}$ or the map $\lambda \mapsto -\bar{\lambda}^{-1}$. Since $\rho \circ \sigma$ has fixed points, ρ induces the involution $\lambda \mapsto \bar{\lambda}^{-1}$ on Σ/σ, which fixes the points over $\lambda \in S^1$.

Moreover there exist by assumption a T^2-family of complex line bundles $\tilde{\mathcal{L}}_x$ over Σ. For every $x \in T^2$ the line bundle $\tilde{\mathcal{L}}_x$ is a subbundle of V. By Theorem (10) there is a map

$$F : T^2 \times \Sigma \to \mathbb{C}P^3,$$

such that the line bundle L corresponding to the immersion f can be reconstructed by $L = \pi_{\mathbb{H}} F(-, \infty)$. Thus we can define the quotient bundle V/L and the projection of the $\tilde{\mathcal{L}}_x$ to V/L defines a T^2-family of complex line subbundles $\mathcal{L}_x$ of the topologically trivial complex rank 2 bundle V/L. We want to show that for fixed $x \in T^2$ the bundle $\mathcal{L}_x^*$ has degree $g+1$. Theorem (9.1) states that the degree of $\mathcal{L}_x$ is constant in x. Moreover, since f is simple the map F is unique up to Möbius transformations of S^4. Thus also the complex holomorphic bundle $\pi_L(F(x,-)) = \mathcal{L}_x \subset \Sigma \times (V/L)_x \cong \Sigma \times \mathbb{C}^2$ is unique up to Möbius transformations of $\mathbb{C}P^1$ and therefore $h^0(\mathcal{L}_x^*) = 2$, by the Kodaira embedding theorem. The degree d of $\mathcal{L}_x^*$ satisfies $(g+1) \leq d$, since the symplectic form ω defined in (10.5) is zero at the branch points σ. Thus by the Riemann-Roch theorem the line bundle $\mathcal{L}_x^*$ is non-special and $\deg(\mathcal{L}_x^*) = g+1$. Then, using Theorem (10.3), we can define a family of flat connections ∇^λ, $\lambda \in \mathbb{C}_*$ on V/L of the form

$$\nabla^\lambda = \nabla + \tfrac{1}{2}(1+\lambda^{-1})\omega' + \tfrac{1}{2}(1+\lambda)\omega'',$$

under which $(\pi_{V/L}(F(-,h)), \pi_{V/L}(F(-,\sigma(h)))$ is a parallel frame of the rank two bundle V/L.

The section $\psi_h = \pi_{V/L}(F(-,h))$ is by definition a holomorphic section with monodromy of V/L. Since f is simple, all holomorphic sections of V/L with trivial monodromy are given by the projection of constant sections of V by Proposition (10.1). Constant sections of V are parallel sections of $\nabla^{\mu=1}$, where ∇^μ is the constrained Willmore associated family of flat connections. This is a complex 4 dimensional space. Thus there exist only 2 trivial connections in the ∇^λ family of flat connections on V/L. Since $\rho \circ \sigma$ has fixed points on Σ and ρ induces a involution on the $\mu-$plane which fixes the point $\mu = 1$, the corresponding λ_0 and λ_1 is also fixed under the involution. Hence λ_0

and λ_1 has length 1. The immersion f is given by the quotient of the two holomorphic sections without monodromy. This equals the reconstruction by the Sym-Bobenko formula. Thus f is a CMC immersion in S^3 or $\mathbb{R}^3$ by Theorem (10.3). □

10.7. Corollary. *A conformally immersed CMC torus is simple if and only if its arithmetic spectral genus p equals its geometric spectral genus g.*

Proof. The proof of the theorem shows that for a simple CMC immersion we have $p = g$. On the other hand, the degree of the line bundle $\mathcal{L}_x$ is always $p + 1$. Thus for $p > g$ we obtain by Riemann-Roch that $h^0(\mathcal{L}_x^*) > 2$. Therefore the map F cannot be unique by the Kodaira embedding theorem. □

Corollary. *A simple constrained Willmore immersion in S^3 of spectral genus 1 is CMC in a space form, if $\rho \circ \sigma$ has fixed points.*

Proof. The involution σ is branched over the two ends of the spectral curve. Thus by the Riemann-Hurwitz formula we have that there must be two other branch points and $\Sigma/\sigma \cong \mathbb{C}P^1$. □

Lemma. *Let f be a simple and constrained Willmore conformal immersion in S^3 with even spectral genus. If $\Sigma/\sigma \cong \mathbb{C}P^1$, then the involution $\rho \circ \sigma$ has fixed points.*

Proof. Let Σ be a hyperelliptic spectral curve with hyperelliptic involution σ and a anti-holomorphic involution ρ such that $\rho \circ \sigma$ has no fixed points. Further, Σ has two marked points 0 and ∞ corresponding to the ends of the spectral curve. The involution ρ interchanges these points. Since $\rho \circ \sigma$ has no fixed points, the involution ρ induces an involution on $\mathbb{C}P^1$, which has no fixed points. Let λ be the holomorphic coordinate on $\mathbb{C}P^1$ such that the point $0 \in \Sigma$ lies over $\lambda = 0$ and the point $\infty \in \Sigma$ lies over $\lambda = \infty$. There exist a unique involution on $\mathbb{C}P^1$ interchanging the points $\lambda = 0$ and $\lambda = \infty$ without fixed points, which is

$$\lambda \mapsto -\bar{\lambda}^{-1}$$

In this coordinate Σ is given by

$$\eta^2 = \Pi_{i=1}^{g+1} \bar{q}_i^{-1}(\lambda - q_i)(\lambda + \bar{q}_i^{-1}) =: P(\lambda),$$

where $q_i, -\bar{q}_i^{-1} \in \mathbb{C}$ and $0, \infty$ are the branchpoints of Σ. It is easy to compute that

$$P(-\bar{\lambda}^{-1}) = (-1)^{g+1} \bar{\lambda}^{-(2g+2)} \overline{P(\lambda)}.$$

Therefore a map ρ inducing the involution $\lambda \mapsto -\lambda^{-1}$ on $\mathbb{C}P^1$ is given by

$$(\eta, \lambda) \mapsto (\pm i^{g+1} \bar{\eta} \bar{\lambda}^{-(g+1)}, -\bar{\lambda}^{-1}).$$

For even g this cannot define an involution on Σ. □

Corollary. *A simple constrained Willmore immersion of even spectral genus such that* $\Sigma/\sigma \cong \mathbb{C}P^1$ *is CMC in* S^3 *or* $\mathbb{R}^3$.

Corollary. *A simple constrained Willmore immersion of spectral genus* 2 *is either equivariant or CMC in a space form.*

Proof. The spectral curve of a constrained Willmore immersion is either a double covering or a four-fold covering of $\mathbb{C}P^1$. In the first case we obtain, as before, a CMC immersion in a space form. In the second case we obtain by Riemann-Hurwitz formula that the involution σ has 2 or 6 branch points. If σ has 6 branch points then Σ/σ is $\mathbb{C}P^1$. Because the genus is even, the involution $\rho \circ \sigma$ has fixed points. Therefore we get a CMC torus in a space form.

If σ is 2 branch points then Σ/σ is a torus. Let $x_0, x \in T^2$. Then we have

$$\mathcal{L}_x = E_x \otimes \mathcal{L}_{x_0}.$$

Since the bundle $\mathcal{L}_x^* \otimes \sigma^* \mathcal{L}_x^*$ is independent of x by corollary (10.5), we obtain

$$E_x \otimes \sigma^* E_x = \underline{\mathbb{C}}.$$

Thus $\mathcal{L}_x$ lies in a affine translate of the Prym variety of Σ with respect to σ for all $x \in T^2$. Since the Prym variety is complex 1 dimensional and $\rho^* \mathcal{L}_x = \mathcal{L}_x \mathbb{j}$, the image of the map

$$x \in T^2 \mapsto \mathcal{L}_x \in Jac(\Sigma)$$

is real 1−dimensional. And we obtain by the same arguments as in Corollary (10.2) that f is equivariant.

□

CHAPTER IV

The Equivariant Case

In this chapter we want to restrict ourselves to the case of equivariant conformal immersions $f : T^2 \to S^3$, which are not necessarily constrained Willmore. In this case the induced Dirac operator on the quotient bundle V/L is translational invariant. Thus the partial differential equation $D\psi = 0$, where ψ is a section of V/L with monodromy can be reduced to an ordinary differential equation. We show that the spectral curve of an equivariant torus with finite spectral genus is hyperelliptic. Further we show that equivariant constrained Willmore tori have spectral genus $g \leq 3$.

Recall that since the target space of f is S^3 the section $\varphi := \pi_L(1, 0)$ of V/L is non vanishing and thus it is a trivializing section of the bundle. The holomorphic structure D is determined by $D\varphi = 0$. The projection of the mean curvature sphere congruence S to V/L defines a complex structure J under which D splits into a $J-$commuting part $\bar{\partial}$ and a $J-$anti-commuting part Q. Q is the Hopf field of f.

11. The Hopf Field of a Equivariant Immersion

We want to show how the Hopf field Q is related to the conformal Hopf differential used in the lightcone model. Let $f : T^2 \to S^3$ be a $(m, n)-$equivariant torus. We have shown in proposition (5.8) that the conformal Hopf differential of f is given by $4q = \kappa_{m,n} + i\Omega$, where $\kappa_{m,n}$ is the curvature of the profile curve in the base space of the $(m, n)-$Seifert fiber space with respect to the metric $g_{m,n}$ and Ω is the curvature function of the corresponding principal fiber bundle.

The Hopf field Q is an endomorphism valued $1-$form and the section $\varphi = \pi_L(1, 0)$ is a trivializing section of V/L. Thus Q is uniquely determined by the value of $Q(\varphi)$. We want to switch to another trivializing section which behaves nicer with respect to J in order to compute Q.

11.1. Lemma. *Let f be a conformally immersed torus in S^3 and V/L the corresponding quotient bundle. Then we can choose a trivializing section with monodromy $\psi \in \Gamma(\widetilde{V/L})$ with*

$$J\psi = \psi\mathbb{i}$$
$$\bar\partial\psi = 0,$$

where $\bar\partial$ is the $J-$commuting part of the holomorphic structure D on V/L. This section ψ is uniquely determined up to multiplication by a complex constant and has monodromy ± 1.

Proof. By (1.24) the $\mathbb{i}-$eigenspace E^{-1} of $\pi_L S$ on V/L is a spin bundle. Since $\bar\partial$ defines a complex holomorphic structure on E^{-1}, it is dual to a spin structure and thus there exist a solution to the equation $\bar\partial\psi = 0$ on the double covering of the torus. Thus we obtain a section ψ with monodromy ± 1 satisfying the conditions above. This condition fixes ψ up to multiplication with a complex constant. □

Remark. Since φ is a holomorphic trivializing section of V/L every other section can be written as $\psi = \varphi\lambda$, where λ is quaternionic valued function. Further we have by formula (1.13.1) that $J\varphi = \varphi N$. Here N is the left normal vector of f. Then we have

$$J\psi = J(\varphi\lambda) = (J\varphi)\lambda = \varphi N\lambda = \varphi\lambda\lambda^{-1}N\lambda = \psi\lambda^{-1}N\lambda.$$

Thus $J\psi = \psi\mathbb{i}$ reduces to

$$N = \lambda\mathbb{i}\lambda^{-1}.$$

Since the group S^3 acts transitively on S^2 via

$$x \mapsto \mu x\bar\mu,$$

and $N, i \in S^2$ there exist a function λ, for which the equation above holds. The function λ, as a $\mathbb{H}-$valued function, is not unique. The multiplication by a complex function μ, not necessarily of length 1, from the right does not change the equation.

Lemma. *Let f be a conformally immersed equivariant torus in S^3 and let ψ be the trivializing section of V/L given in (11.1). Then there exist a complex valued function q satisfying*

$$Q\psi = \psi d\bar z q\mathbb{j},$$

where $z = x + iy$ is a holomorphic coordinate of $T^2 = \mathbb{C}/\Gamma$.

Proof. The Hopf differential Q is tensorial, satisfies $*Q = -JQ$ and takes values in $\bar K(V/L)$. Thus there is a quaternionic valued function q with

$$Q(\psi) = \psi d\bar z\tilde q.$$

Further Q is $J-$ anti-commuting and

$$JQ(\psi) = -Q(J\psi) = -Q(\psi\mathbb{i}) = -Q(\psi)\mathbb{i}.$$

This is equivalent to

$$J(\psi d\bar{z}\tilde{q}) = \psi(\mathbb{i}d\bar{z}\tilde{q}) = -\psi(d\bar{z}\tilde{q}\mathbb{i}),$$

thus

$$\mathbb{i}\tilde{q} = -\tilde{q}\mathbb{i}.$$

Since the quaternionic function $\tilde{q}$ anti-commutes with $\mathbb{i}$, it takes values in $\operatorname{span}\{\mathbb{j}, \mathbb{k}\}$. Therefore there is a complex function q with $\tilde{q} = q\mathbb{j}$. □

11.2. Proposition. *For an equivariant conformal immersion $f : T^2 \to S^3$ the function q defined in the previous lemma is the conformal Hopf differential of the torus. In particular q depends only on y and is periodic.*

Proof. The previous Lemma states

$$Q(\psi) = \psi d\bar{z} q\mathbb{j},$$

for a complex valued function q.

We want to compute the function q explicitly. For the section $\varphi = \pi_L(1, 0)$ of V/L we have $J\varphi = \varphi N$. Thus by formula (1.13.2) we obtain

$$4Q(\varphi) = \varphi(NdN - *dN).$$

Since Q is tensorial and there exist a $\mathbb{H}-$valued function λ with $\psi = \varphi\lambda$. We get

$$4Q(\psi) = 4Q(\varphi)\lambda = \varphi(NdN - *dN)\lambda = \psi\lambda^{-1}(NdN - *dN)\lambda. \tag{11.2.1}$$

We want to make a special choice of λ in order to relate q to the conformal Hopf differential of the torus. Then we show that $\psi = \varphi\lambda$ satisfies the conditions of Lemma (11.1).

Let f be a conformally parametrized $(m, n)-$torus in $S^3 \cong SU(2)$ given by

$$f(x, y) = e^{il_1 x}\gamma(y)e^{il_2 x},$$

where $l_1 = \frac{1}{2}(m + n)$, $l_1 = \frac{1}{2}(m - n)$ and γ is the profile curve with $\gamma' \perp f_y$ and $|\gamma'|^2 = |l_1 i\gamma + l_2\gamma i|^2 =: h$, see (5.3). With $su(2) \cong \operatorname{Im}\mathbb{H}$ the normal vector of the surface is given by

$$hN_{orm} = \frac{\partial f}{\partial x}\bar{f}\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial x}\frac{\partial \bar{f}}{\partial y}f$$

and the partial derivatives are

$$\frac{\partial f}{\partial x} = e^{il_1 x}(l_1 i\gamma + l_2\gamma i)e^{il_2 x} = \sqrt{h}\tilde{B}, \qquad \frac{\partial f}{\partial y} = e^{il_1 x}\gamma' e^{il_2 x} = \sqrt{h}\tilde{T}.$$

Recall that by Lemma (5.11) $(\tilde{T}(0, y), N_{orm}(0, y), \tilde{B}(0, y))$ is the Frénet frame of the profile curve γ. Further a frame of the equivariant torus is simply obtained by multiplying every argument of the Frénet frame of γ by $e^{il_1 x}$ from the left and $e^{il_2 x}$ from the right. We denote this frame

of the torus also by $(\tilde{T}, N_{orm}, \tilde{B})$.

Since f is conformal, its left normal vector N is given by

$$*df = Ndf.$$

This function is the right translation of the normal vector of the surface to $su(2) \cong \mathrm{Im}\mathbb{H}$:

$$hN = \frac{\partial f}{\partial y}\frac{\partial \bar{f}}{\partial x} = -\frac{\partial f}{\partial x}\frac{\partial \bar{f}}{\partial y} = N_{orm}\bar{f}.$$

Right translating the frame of the surface to $\mathrm{Im}\mathbb{H}$ we obtain point wise a positive oriented orthonormal basis of $\mathrm{Im}\mathbb{H}$

$$N, \quad T := \frac{\partial f}{\partial y}\bar{f}, \quad B := -\frac{\partial f}{\partial x}\bar{f}.$$

We choose λ to be the function which point wise rotate this basis to the canonical basis $(\mathbb{i}, \mathbb{j}, \mathbb{k})$ of $\mathrm{Im}\mathbb{H}$, i.e., we have

$$\mathbb{i} = \lambda^{-1}N\lambda, \quad \mathbb{j} = \lambda^{-1}T\lambda \quad \mathbb{k} = \lambda^{-1}B\lambda.$$

Obviously, this choice of λ satisfies $J\psi = J\varphi\lambda = \varphi N\lambda = \psi\mathbb{i}$. Thus $Q(\psi) = \psi d\bar{z}q\mathbb{j}$, for a complex function q.

Now we compute $\lambda^{-1}(NdN - *dN)\lambda$. The Frénet equations for the profile curve in S^3 gives:

$$\begin{aligned}
N'_{orm} &= -\sqrt{h}\kappa_{S^3}\tilde{T} + \sqrt{h}\tau_{S^3}\tilde{B} \\
\Rightarrow \frac{\partial N}{\partial y} &= N'_{orm}\bar{f} - N_{orm}\bar{f}\frac{\partial f}{\partial y}\bar{f} \\
&= -\sqrt{h}\kappa T - \sqrt{h}\tau B - \sqrt{h}NT \\
&= -\sqrt{h}\kappa T - \sqrt{h}(\tau+1)B \\
\frac{\partial N}{\partial x} &= l_1(\mathbb{i}N - N\mathbb{i}).
\end{aligned}$$

Inserting into (11.2.1) we get

$$Q\psi = l_1\lambda^{-1}(N\mathbb{i}N + \mathbb{i})\lambda + (\sqrt{h}\kappa + \sqrt{h}(\tau+1)\mathbb{i})\mathbb{j}dz. \tag{11.2.2}$$

The term $N\mathbb{i}N + \mathbb{i}$ is perpendicular to N and purely imaginary. And because (N, T, B) is a orthonormal basis of $\mathrm{Im}\mathbb{H}$ we have

$$N\mathbb{i}N + \mathbb{i} = < N\mathbb{i}N + \mathbb{i}, T > T + < N\mathbb{i}N + \mathbb{i}, B > B.$$

We compute both summands separately. The multiplication by a unit length quaternion does not change the metric on $\mathbb{H} \cong \mathbb{R}^4$. Thus

$$\begin{aligned}
< N\mathbb{i}N + \mathbb{i}, T > &= < N\mathbb{i}N, NTN > + < N\mathbb{i}N, T > \\
&= 2 < N, \mathbb{i}B > = -\frac{2l_2}{\sqrt{h}} < N, if\bar{f} i\bar{f} > .
\end{aligned}$$

And with $N = N_{orm}\bar{f}$ we obtain

$$l_1 < N\mathbb{i}N + \mathbb{i}, T >= -\frac{2l_1l_2}{\sqrt{h}} < N_{orm}, ifi > .$$

Now we turn to the second term:

$$< N\mathbb{i}N + \mathbb{i}, B > = 2 < \mathbb{i}, B >= -2l_1\frac{1}{\sqrt{h}} - 2l_2 < \mathbb{i}, \frac{fi\bar{f}}{\sqrt{h}} >$$
$$= -2l_1\frac{1}{\sqrt{h}} + \frac{2l_2}{\sqrt{h}}Re(ifi\bar{f}).$$

Note that for equivariant tori $Re(ifi\bar{f}) = Re(i\gamma i\bar{\gamma})$. Moreover let $\gamma = \gamma_1 + \mathbb{j}\gamma_2$ with complex functions γ_1 and γ_2. Then we have $h = m^2|\gamma_1|^2 + n^2|\gamma_2|^2$ and of course $|\gamma_1|^2 + |\gamma_2|^2 = 1$. Thus we get

$$< N\mathbb{i}N + \mathbb{i}, B > = -\frac{2}{\sqrt{h}}(m|\gamma_1|^2 - n|\gamma_2|^2)$$
$$\Rightarrow l_1 < N\mathbb{i}N + \mathbb{i}, B > = -\frac{1}{h}((m-n)m|\gamma_1|^2 - (m-n)n|\gamma_2|^2)$$
$$= \tfrac{mn}{\sqrt{h}} - \sqrt{h}.$$

Put these results into (11.2.2) and use the formulas computed in (5.9) and (5.10) we get

$$4q = \sqrt{h}\kappa_{S^3} - \frac{2l_1l_2}{\sqrt{h}} < N_{orm}, i\gamma i > +\mathbb{i}\tfrac{2mn}{\sqrt{h}} = \kappa_{m,n} + \mathbb{i}\tfrac{2mn}{\sqrt{h}}.$$

Because the curvature of the corresponding (m,n)-Seifert bundle is given by $\Omega = \frac{2mn}{\sqrt{h}}$ we obtain

$$4q = \kappa_{m,n} + \mathbb{i}\Omega.$$

It remains to show that we can adjust the section ψ to get $\bar{\partial}\tilde{\psi} = 0$ preserving the property $J\tilde{\psi} = \tilde{\psi}\mathbb{i}$ and the function q. In order to do so, we can still multiply by a real valued function. The condition $\bar{\partial}\tilde{\psi} = 0$ holds if and only if $D(\tilde{\psi})$ anti-commutes with $\mathbb{i}$. We have

$$D(\psi) = D(\varphi\lambda) = (\varphi d\lambda)'' = \psi\lambda^{-1}d\lambda + J\psi\lambda^{-1} * d\lambda = \psi(\lambda^{-1}d\lambda + i\lambda^{-1} * d\lambda).$$

Because of the frame equations of the profile curve we have

$$B'(0,y) = (\sqrt{h}\tau + \sqrt{h})N(0,y).$$

The derivative of the equation $\lambda^{-1}B\lambda = \mathbb{k}$ then gives that

$$\lambda^{-1}(y)\lambda'(y) = \mathbb{j}v(y),$$

with a complex valued function v. Since $\lambda(x,y) = e^{il_1x}\lambda(y)$, we get

$$\lambda^{-1}d\lambda + i\lambda^{-1} * d\lambda = d\bar{z}(l_1\lambda^{-1}i\lambda + \mathbb{j}v(y))$$

Thus for $r = e^{\int_0^y l_1 < \lambda^{-1}i\lambda, \mathbb{i} > ds}$ and $\tilde{\lambda} = \lambda r$ the section $\tilde{\psi} = \varphi\tilde{\lambda}$ is holomorphic with respect to $\bar{\partial}$. □

12. The Spectral Curve of Equivariant Conformal Immersions

12.1. Introducing a Spectral Parameter. The next step towards the spectral curve of an equivariant torus is to investigate holomorphic sections of V/L with monodromy. Let $f : T^2 = \mathbb{C}/\Gamma \to S^3$ be a conformal immersion. Let $\gamma_1 \in \mathbb{R}$ and γ_2 denote the generators of the lattice Γ and let V/L be the quotient bundle associated to f. Recall that the spectral curve is defined to be the normalization of the analytic variety given by

$$\{(h_1, h_2) \subset \mathbb{C}_* \times \mathbb{C}_* | \exists \varphi \in H^0(\widetilde{V/L}) \text{ with monodromy } h_{\gamma_1} = h_1, h_{\gamma_2} = h_2\}.$$

In order to compute the spectral curve, it is thus sufficient to know the generic points.

Let ψ be the trivializing section of V/L with monodromy as in Lemma (11.1) and Proposition (11.2). All sections of V/L are given by ψu, for a quaternionic valued function u, which splits into two complex functions $u = u_1 + \mathbb{j}u_2$, $u_i : \mathbb{C} \to \mathbb{C}$. Then the equation $D(\psi u) = 0$ reduces to

$$\bar{\partial} u + (d\bar{z} q \mathbb{j}) u = 0, \tag{12.1.1}$$

where $\bar{\partial}$ is the ordinary holomorphic structure for functions. By identifying $\mathbb{H}$ with $\mathbb{C} \oplus \mathbb{j}\mathbb{C}$ we get the matrix notation of equation (12.1.1).

$$Du = \begin{pmatrix} \bar{\partial} & \bar{q} \\ -q & \partial \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

Since the potential q depends only on y, the differential operator D is translational invariant, i.e., if $u(x, y) = u_1(x, y) + \mathbb{j}u_2(x, y)$ satisfies $Du = 0$. Then we also have $Du(x + x_0, y) = 0$, for an arbitrary constant $x_0 \in \mathbb{R}$. Obviously both solutions have the same monodromy. By Theorem (8.5) the space of holomorphic sections with monodromies $(h_{\gamma_1}, h_{\gamma_2})$ is generically complex 1 dimensional. Thus, at a generic point of the spectral curve, we have that for all $x_0 \in \mathbb{R}$

$$u(x + x_0, y) = u(x, y) A_{x_0},$$

where A_{x_0} is a complex constant. In particular, we have $u(x, y) = u(0, y) A_x$. Now consider the smooth function $A : \mathbb{R} \to \mathbb{C}$. Since $A_0 = 1$, $A_{\gamma_1} = h_1$ and $A_{x_0 + x_1} = A_{x_0} A_{x_1}$, we have that $A_x = e^{ax}$ with $a \in \mathbb{C}$ and $e^{a\gamma_1} = h_1$.

Thus we can use the following ansatz: $u_i = e^{ax}\tilde{u}_i(y)$, where $a \in \mathbb{C}$ is a constant representing the monodromy in $x-$direction. This yields

$$\begin{aligned}
0 =& \begin{pmatrix} \bar{\partial} & -q \\ \bar{q} & \partial \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
=& \begin{pmatrix} \bar{\partial}(e^{ax}\tilde{u}_1(y)) - q(e^{ax}\tilde{u}_2(y)) \\ \partial(e^{ax}\tilde{u}_2(y)) + \bar{q}(e^{ax}\tilde{u}_1(y)) \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{2}(ae^{ax}\tilde{u}_1(y) + ie^{ax}\frac{\partial \tilde{u}_1(y)}{\partial y}) - q(e^{ax}\tilde{u}_2(y)) \\ \frac{1}{2}(ae^{ax}\tilde{u}_2(y) - ie^{ax}\frac{\partial \tilde{u}_2(y)}{\partial y}) + \bar{q}(e^{ax}\tilde{u}_1(y)) \end{pmatrix}.
\end{aligned}$$

Dividing by e^{ax} we obtain

$$\begin{aligned}
& \begin{pmatrix} \frac{1}{2}(a\tilde{u}_1(y) + i\frac{\partial \tilde{u}_1(y)}{\partial y}) - q(\tilde{u}_2(y)) \\ \frac{1}{2}(a\tilde{u}_2(y) - i\frac{\partial \tilde{u}_2(y)}{\partial y}) + \bar{q}(\tilde{u}_1(y)) \end{pmatrix} = 0 \\
\Leftrightarrow \quad & \begin{pmatrix} -ia\tilde{u}_1(y) + \frac{\partial \tilde{u}_1(y)}{\partial y}) + 2iq(\tilde{u}_2(y)) \\ ia\tilde{u}_2(y) + \frac{\partial \tilde{u}_2(y)}{\partial y}) + 2i\bar{q}(\tilde{u}_1(y)) \end{pmatrix} = 0.
\end{aligned}$$

This is equivalent to the equation

$$\left(\frac{\partial}{\partial y} + \begin{pmatrix} -ia & 2i\bar{q} \\ 2iq & ia \end{pmatrix}\right) \begin{pmatrix} \tilde{u}_2 \\ \tilde{u}_1 \end{pmatrix} = 0. \tag{12.1.2}$$

We have shown the following theorem.

Theorem. *Let $f : T^2 \to S^3$ be an equivariant and conformal immersion and V/L the associated quotient bundle. The spectral curve of f is determined by the kernel of the family of ordinary linear differential operators*

$$D_a := \frac{\partial}{\partial y} + \begin{pmatrix} -ia & 2i\bar{q} \\ 2iq & ia \end{pmatrix}, \quad a \in \mathbb{C}.$$

12.2. Definition and Properties of the Spectral Curve. The spectral curve is given by all possible monodromies of solutions to equation (12.1.2). Since (12.1.2) is an ordinary linear differential equation, we have for arbitrary $a \in \mathbb{C}$ two linear independent solutions. Let $\Phi(a)$ be the fundamental solution matrix to (12.1.2). Then we have

$$\Phi(y + \gamma) = H(a)\Phi(y),$$

where $\gamma \in \mathbb{R}$ is a period of the potential q, and $H(a)$ is a $SL(2, \mathbb{C})$ matrix independent of y. The solutions with monodromy of (12.1.2) are exactly the eigensolutions of $H(a)$. Therefore the spectral curve is the normalization of the variety

$$Spec(V/L) := \{(b, a) | a \in \mathbb{C} \text{ and } b \text{ eigenvalue of } H(a)\}.$$

Lemma. *For generic $a \in \mathbb{C}$ the matrix $H(a)$ is diagonalizable and has distinct eigenvalues.*

PROOF. If $H(a)$ is not diagonalizable, then $H(a)$ still is trigonalizable and has only one eigenvalue 1 or -1 with algebraic multiplicity 2. Since $Spec(V/L)$ is an analytic variety the map

$$p_1 : Spec(V/L) \to \mathbb{C}_*, (b, a) \mapsto b$$

is holomorphic. Therefore, if there exist a open set $U \subset \mathbb{C}$ such that $H(a)$ is not diagonalizable for $a \in U$ then the map p_1 must be constant. Thus all solutions to D_a have either monodromy 1 or -1. Now let $a \in i\mathbb{R}$ and $u_a(y)$ be an eigensolution of D_a. Further let ψ be the trivializing section with monodromy introduced in (11.2). Then $\psi e^{ax} u_a(y)$ is a holomorphic section with monodromy $(e^{a\gamma_1}, 1)$ or $(e^{a\gamma_1}, -1)$ in V/L. In either case for $a \in \frac{\gamma_1 i}{2\pi}\mathbb{Z}$ we obtain infinitely many holomorphic sections with monodromy $(1, -1)$ or $(1, 1)$ which contradicts Theorem (8.5). □

The normalization of $Spec(V/L)$ is a double covering of $\mathbb{C}$, because to every spectral parameter $a \in \mathbb{C}$ we have generically two different eigenvalues $H(a)$ and thus two different points in $Spec(V/L)$. If the spectral curve $\tilde{\Sigma}$ is of finite genus then it can be compactified to Σ by adding two points at infinity, see (8.7). In this case the ends of the spectral curve corresponds to the points over $a = \infty$. Thus we obtain that Σ is a hyperelliptic curve, which is not branched over $a = \infty$.

To be more explicit: The spectral curve of an equivariant immersion of finite genus is defined to be the solution set of the equation

$$\eta^2 = Tr(H(a))^2 - 4 =: P(a),$$

since the eigenvalues of $H(a)$ are given by the roots of the characteristic polynomial and $\det H(a) = 1$. This is a double covering of $\mathbb{C}P^1$ with a as a local coordinate. We denote the hyperelliptic involution by τ. The projection of involutions on Σ defines involutions on $\mathbb{C}P^1$. The spectral curve has the following properties:

- There exist a quaternionic structure

  $$\rho : a \to \bar{a}, (u_1, u_2) \to (-\bar{u}_2, \bar{u}_1),$$

  which comes from the right multiplication by $\mathbb{j}$.

- Because f maps to S^3 we get that $\bar{\partial}$ is a spin structure and thus there exist an involution $\sigma : \Sigma \to \Sigma$ which maps the monodromy h to h^{-1} . With respect to the spectral parameter a we have $\sigma(a) = -a$. On $\mathbb{C}P^1$ the only fixed points of σ are $a = 0$ and $a = \infty$. Thus for equivariant conformal immersions into S^3 the involution σ has 2 or 4 branch points.

- Moreover, we get a third involution $\rho \circ \sigma$ and the induced involution on $\mathbb{C}P^1$ is given by $a \mapsto -\bar{a}$. The imaginary axis of $\mathbb{C}P^1$ is fixed under this involution. On the spectral curve the involution $\rho \circ \sigma$ has fixed points over $a \in i\mathbb{R}$ if and only if

there are branch points of τ over the imaginary axis. This is because $\rho \circ \sigma$ interchanges the points over $a = \infty$. If there are no branch points of τ on the imaginary axis, then by smoothness of $\rho \circ \sigma$, it interchanges the points over all $a \in i\mathbb{R}$.

- The branch points of τ are interchanged by the involutions above. Thus if a is a branch point then $-a$ and $\bar{a}$ and thus also $-\bar{a}$ are branch points of the spectral curve. Therefore the roots of the polynomial $P(a)$ defining the spectral curve has also these symmetries and the polynomial $P(a)$ is even and has real coefficients. If the genus of Σ is even, we have by Riemann-Hurwitz formula that the number of branch points of τ is not divisible by 4 and we always have branch points of τ on the imaginary axis. Thus $\rho \circ \sigma$ has always fix points in this case.

- The two points over $a = \infty$ corresponds to the ends of the spectral curve $\tilde{\Sigma}$.

- Because for $a \in \mathbb{R}$ the connection D_a is a $SU(2)$ connection, $H(a)$ lies also in $SU(2)$. Hence there are no branch points of Σ lying over the real axis.

12.3. Reconstruction of the Equivariant Immersion. The surface can be reconstructed from the spectral curve and the solutions of D_a for suitable $a \in \mathbb{C}$ by the following theorem.

Theorem. *Let $a \in i\mathbb{R}$ and let u be an eigensolution to D_a. Then the map $f : T^2 \to S^3$ determined by*

$$df\bar{f} = -e^{-ax}u(y)^{-1}(dz\mathbb{k})u(y)e^{ax},$$

is conformally immersed and the potential q is the Hopf field of f in a suitable trivialization. Here $z = x + iy$ is a holomorphic coordinate of $T^2 = \mathbb{C}/\Gamma$.

Proof. Since u is the solution of an ordinary differential equation it is non vanishing. Thus the constructed f is an immersion. Further f is conformal because

$$\begin{aligned} *df\bar{f} &= -e^{-ax}u(y)^{-1}(*dz\mathbb{k})u(y)e^{ax} = -e^{-ax}u(y)^{-1}\mathbb{i}(dz\mathbb{k})u(y)e^{ax} \\ &= -(e^{-ax}u(y)^{-1}\mathbb{i}u(y)e^{iax})e^{-ax}u(y)^{-1}(dz\mathbb{k})u(y)e^{ax} \\ &= Ndf\bar{f}. \end{aligned}$$

For equivariant tori in S^3 we have that $\varphi = \pi_L(1, 0)$ is a non vanishing holomorphic section of V/L. Let ψ be the trivializing section of V/L given in Proposition (11.2). In the proof of Proposition (11.2)

we showed that $\psi = \varphi\lambda r$, where r is some real valued function and $\lambda : T^2 \to S^3$ satisfies

$$N = \lambda \mathbb{i} \lambda^{-1} \quad T = \lambda \mathbb{j} \lambda^{-1} \quad B = \lambda \mathbb{k} \lambda^{-1}.$$

Here $T = \frac{\partial f}{\partial y}\bar{f}$ and $B = -\frac{\partial f}{\partial x}\bar{f}$. Further we have $\lambda(x, y) = e^{il_1 x}\lambda(y)$, for a $l_1 \in \mathbb{R}$ and $r = r(y)$. Since φ is holomorphic we have $u := (\lambda(x, y)r(y))^{-1}$ is a eigensolution to the equation $D_{-il_1}u = 0$ and

$$df\bar{f} = -e^{il_1 x}u(y)^{-1}(dz\mathbb{k})u(y)e^{-il_1 x}.$$

Thus by choosing $a = -il_1$ we obtain the original immersion.

Now we want to show that also for arbitrary $a \in i\mathbb{R}$ the Hopf field of f is q in the right trivialization of V/L. Again the section $\varphi = \pi_L(1, 0)$ is a trivializing section of V/L. Thus $\psi := \varphi e^{-ax}u^{-1}$ has by construction the property that $J\psi = \psi\mathbb{i}$. Further ψ satisfies

$$D\psi = (\psi u e^{iax} d(e^{-ax}u^{-1}))'' = -\psi\bar{\partial}(ue^{ax})e^{-ax}u^{-1} = \psi d\bar{z} q \mathbb{j}.$$

This shows $\bar{\partial}\psi = 0$ and q is the Hopf field of f. □

12.4. Corollary. *Let $f : T^2 = \mathbb{C}/\Gamma \to S^3$ be a constrained Willmore immersion with conformal Hopf differential q and left normal vector $N = e^{l_1 ix}N(y)e^{-l_1 ix}$. Let z be a holomorphic coordinate of $\mathbb{C}/\Gamma$. Then its constrained Willmore associated family defined in section (7) can be obtained by rotating the $z-$plane.*

Proof. Let f be a constrained Willmore torus in S^3 with conformal Hopf differential q and let $\tilde{f}$ be a surface in its associated family. Then $\tilde{f}$ is equivariant and lies also in S^3 and by definition its conformal Hopf differential is given by $q\mu^2$, with $\mu^2 \in S^1$. Thus the family of Dirac operators corresponding to $\tilde{f}$ is given by

$$\tilde{D}_a = \frac{\partial}{\partial y} + \begin{pmatrix} -ia & 2\mathbb{i}\bar{q}\bar{\mu}^2 \\ 2\mathbb{i}q\mu^2 & ia \end{pmatrix}.$$

This is gauge equivalent to the Dirac operator of f. The gauge is given by

$$g = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}.$$

By the previous theorem there is a spectral parameter $a \in i\mathbb{R}$ and an eigensolution u to D_a such that

$$df\bar{f} = -e^{-ax}u(y)^{-1}(dz\mathbb{k})u(y)e^{ax}.$$

Since D_a and $\tilde{D}_a$ are gauge equivalent we get $\tilde{u} = \bar{\mu}u(y)e^{ax}$ is a solution to $\tilde{D}_a$. Thus there is a immersion $\hat{f}$ given by

$$d\hat{f}\bar{\hat{f}} = -e^{-ax}u(y)^{-1}(\mu dz\mathbb{k}\bar{\mu})u(y)e^{ax}.$$

Thus by defining $\tilde{z} = z\mu^2$ we obtain

$$d\hat{f}\bar{\hat{f}} = -e^{-ax}u(y)^{-1}(d\tilde{z}\mathbb{k})u(y)e^{ax}.$$

The immersion $\hat{f}$ is equivariant and has conformal Hopf differential $q\mu^2$, since the frame given by $d\hat{f}\bar{\hat{f}}$ and left normal vector N is the Frénet frame of its profile curve. Thus $\hat{f}$ and $\tilde{f}$ are non congruent if and only if they are both isothermic but then they lie in the same associated family as isothermic surfaces by Theorem (3.3) of [**BuPP**]. □

13. The Nonlinear Schrödinger Hierarchy

The definition of the spectral curve for the family of Dirac operators D_a is the same as for the focussing nonlinear Schrödinger equation (NLS). We only consider spectral curves of finite genus, i.e., the spectral curve Σ is a hyperelliptic curve. Then there exist a so-called polynomial Killing field, whose eigenlines for generic $a \in \mathbb{C}P^1$ coincides with the space of eigensolutions of D_a. The equations on the polynomial Killing field gives rise to differential equations on the potential q. Comparing these to the Euler-Lagrange equations of constrained Willmore tori we show that the (arithmetic) spectral genus of equivariant constrained Willmore tori in S^3 is at most 3. In particular, equivariant tori are of spectral genus 1 if and only if they are CMC in a space form and associated to a Delaunay cylinder, in the sense of Theorem (3.3) of [**BuPP**]. They have (arithmetic) spectral genus 2, if and only if they are associated to a constrained Willmore Hopf cylinder as constrained Willmore surfaces.
Hyperelliptic solutions of the NLS equation are constructed in [**Pr**], under some further restrictions. We use a different approach here.

In order to proof the existence of a polynomial Killing field we need the following theorem which can be found in [**BoPP**] (Proposition 3.1).

Theorem. *For a family of elliptic operators, which depends holomorphically on a parameter in a connected* $1-$*dimensional complex manifold* M*, the minimal kernel dimension is generic and attained away from isolated points* $p_i \in N \subset M$*. Further the vector bundle over* $M \setminus N$ *defined by the kernels of the elliptic operators extends through the isolated points with higher dimensional kernel and is holomorphic.*

Corollary. *Let* D_a *be the family of Dirac type operators on* $V := S^1 \times \mathbb{C}^2$*. For a fixed* $a \in \mathbb{C}P^1$ *we define*

$$\mathcal{E}_a := \{\omega \in \Omega^1(End_0(V)) \mid d^{D_a}\omega = 0\},$$

where $End_0(V)$ *denote the trace free endomorphisms of* V *and* d^{D_a} *is the induced differential on* $\Omega^1(End_0(V))$*. Then there is a holomorphic line bundle* $\mathcal{E}$ *over* $\mathbb{C}P^1$ *whose fiber over a generic spectral parameter* $a \in \mathbb{C}P^1$ *coincides with* $\mathcal{E}_a$*.*

Proof. Let $B = \begin{pmatrix} -ia & 2i\bar{q} \\ 2iq & ia \end{pmatrix}$ and y be a coordinate on S^1. Then locally $\tilde{X} \in \Omega^1(\mathrm{End}_0(V))$ can be written as $\tilde{X} = Xdy$, where X is a

section of $\mathrm{End}_0(V)$. The equation $d^{D_a}X = 0$ is equivalent to

$$X' = [X, B].$$

This is a first order ordinary differential equation and therefore X is fully determined by its initial value at $y = 0$. We denote the initial value of X by X_0.

At a generic point $a \in \mathbb{C}P^1$ the holonomy of D_a is diagonalizable and has distinct eigenvalues $\mu^{\pm 1}$. Let L_a^1 and L_a^2 define the corresponding eigenlines. With respect to the splitting $V = L_a^1 \oplus L_a^2$ we get that $X(a) = \begin{pmatrix} -s & v \\ w & s \end{pmatrix}$, where $s \in H^0(L_a^1)$, $v \in H^0(L_a^2 \otimes (L_a^1)^*)$ and $w \in H^0(L_a^1 \otimes (L_a^2)^*)$.

The bundle V is the pull-back of the spin bundle of S^3 and inherits the symplectic structure. Thus by $V = L_a^1 \oplus L_a^2$ we get that $L_a^2 = (L_a^1)^*$. The bundle $L_a^1 \otimes (L_a^2)^* = L_a^1 \otimes L_a^1$ is not trivial and has no holomorphic sections because the monodromy of L_a^1 is not ± 1 and thus L_a^1 is not a spin bundle. We obtain $v = w = 0$. Therefore X is fixed by the initial value of s and $\mathcal{E}$ is a complex line bundle over Σ. □

Remark. Note that at a generic point $a \in \mathbb{C}P^1$ the eigenlines of $X(a)$ are exactly the eigenlines of the holonomy of D_a and $\det X(a) \neq 0$.

For any meromorphic section Y in $\mathcal{E}$ whose only pole over $a = \infty$ we have

$$\deg \mathcal{E} = \sum_{p \in \Sigma} \mathrm{ord}_p Y.$$

Thus if Y is zero at some $a \in \mathbb{C}$ then its pole order at infinity increases and the degree of the the polynomial Y in a increases.

Definition. A non vanishing meromorphic section X of $\mathcal{E}$ with a single pole at $a = \infty$ is called a polynomial Killing field of Σ.

Since $\mathcal{E}$ is a holomorphic vector bundle over $\mathbb{C}P^1$ we have that X is polynomial in $a \in \mathbb{C}P^1$ and thus it is given by

$$X = \sum_{i=0}^{p+1} X_i a^i dy,$$

where X_i is a section in $\Gamma(\mathrm{End}_0(V))$. For $a \in \mathbb{R}$ we have that the holonomy of D_a is $SU(2)-$valued and the corresponding eigenlines are perpendicular. Thus we can choose the section $s \in L_a^1$ to be purely imaginary-valued for $a \in \mathbb{R}$. Then we obtain that the coefficients X_i are also $su(V)$ valued.

Lemma. *Let X be a polynomial Killing field of Σ. The equation $X' = [X, B]$ preserves the polynomial $det(X)$.*

PROOF. Since $\mathrm{tr}(X) = 0$ we get $2\det(X) = \mathrm{tr}X^2$. Differentiating both sides yields

$$(\det X)' = \mathrm{tr}(X')X = \mathrm{tr}([X, B]X) = \mathrm{tr}(XBX - BX^2) = 0.$$

□

The polynomial Killing field X has degree $p+1$. Therefore we get that $\det X$ is a polynomial of degree $2p+2$. The equation

$$\eta^2 = \det X$$

thus defines a possibly singular algebraic curve of genus p.

Theorem. *The normalization of the algebraic curve given by*

$$\eta^2 = detX$$

is the spectral curve.

PROOF. The eigenlines of the polynomial Killing field for generic $a \in \mathbb{C}$ gives exactly the solutions with monodromy to the equation $D_a u = 0$. The branch points of the spectral curve are given by those points where these eigenlines coalesce to an odd order. At these points $a \in \mathbb{C}$ the polynomial Killing field is not diagonalizable and therefore it has only one eigenvalue, which must be 0. Since $\mathrm{tr}X(a) = 0$ and X is non vanishing , we get that $X(a)$ is conjugate to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus $\det X(a)$ has an odd order zero at this point. □

Definition. An equivariant immersion has arithmetic spectral genus p if and only if the corresponding polynomial Killing field has degree $p+1$.

13.1. Lemma. *The coefficient X_{p+1} and X_p of a polynomial Killing field X can be choosen to be*

$$X_{p+1} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad X_p = \begin{pmatrix} -ib & 2i\bar{q} \\ 2iq & ib \end{pmatrix}.$$

PROOF. The degree of the polynomial Killing field is constant in y. The Lax type equation

$$X' = [X, B]$$

yields differential equation on each coefficient X_i of X. Since $B = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} a + \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix}$ we obtain:

$$X_i' = [X_i, \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix}] + [X_{i-1}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}], \quad \text{for } i = 1, ..., p+1,$$

Further since $X_{p+2} = 0$ we get that

$$[X_{p+1}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}] = 0.$$

Thus $X_{p+1} = \begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}$. The term $-s^2$ is the top coefficient of the polynomial $\det X$, which is constant along y. Therefore s is a constant and we can normalize it to be i. Then we obtain

$$X'_{p+1} = [X_{p+1}, \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix}] + [X_p, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}] = 0,$$

and this yields $X_p = \begin{pmatrix} -ib & 2i\bar{q} \\ 2iq & ib \end{pmatrix}$. □

Now we build in the symmetry of the spectral curve coming from an immersion $f : T^2 \to S^3$. It is given by the involution σ, see (12.2). We obtain

$$\det(X(a)) = \det(X(-a)).$$

The branch points are then symmetric with respect to the real axis and the polynomial $\det X$ is even. Since

$$X = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} a^{p+1} + \begin{pmatrix} -ib & 2i\bar{q} \\ 2iq & ib \end{pmatrix} a^p + \text{ lower oder terms},$$

the determinant is given by

$$\det X = a^{2p+2} + ba^{2p+1} + \text{ lower order terms}.$$

Therefore $b = 0$. Further the involution ρ gives that the branch points are symmetric with respect to the imaginary axis, which is satisfied for $X_i \in \Gamma(su(V))$. In particular we have for immersions into S^3 that the sum of all branch points is zero.

Lemma. *An equivariant torus f in S^3 has spectral genus 0 if and only if $q \equiv const \neq 0$, i.e., f is homogenous.*

Proof. The polynomial Killing field of an equivariant torus in S^3 with spectral genus 0 is given by $X = B$. Then we get $X' = [X, B] = 0$ and X is constant. Therefore

$$q = const. \tag{13.1.1}$$

This Killing field has no zeros, if and only if $q \neq 0$. □

13.2. Theorem. *An equivariant torus in S^3 has spectral genus 1 if and only if it is CMC in a space form and not homogenous.*

Proof. Let f be an equivariant conformal immersion. The genus of the spectral curve is 1 if and only if there is a polynomial Killing field of degree 2 satisfying $X' = [X, B]$. By (13.1) we have that such a polynomial Killing field is given by

$$X = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} a^2 + \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix} a + \begin{pmatrix} -ib & 2i\bar{p} \\ 2ip & ib \end{pmatrix}.$$

The equation $X' = [X, B]$ gives:

$$X_0' = \begin{pmatrix} ib' & -2i\bar{p}' \\ -2ip' & -ib' \end{pmatrix} = [X_0, X_1] = \begin{pmatrix} 4p\bar{q} - 4\bar{p}q & 4b\bar{q} \\ -4bq & -4p\bar{q} + 4\bar{p}q \end{pmatrix}$$
$$X_1' = \begin{pmatrix} 0 & -2i\bar{q}' \\ -2iq' & 0 \end{pmatrix} = [X_0, X_2] = \begin{pmatrix} 0 & -4\bar{p} \\ 4p & 0 \end{pmatrix}$$
$$X_2' = 0.$$

Thus we obtain

$$\begin{aligned} 2p &= -iq', \\ p' &= -2ibq, \\ ib' &= 4p\bar{q} - 4\bar{p}q, \end{aligned} \tag{13.2.1}$$

Therefore

$$b = -2|q|^2 - 2c, \text{ for some real constant } c. \tag{13.2.2}$$

This gives

$$q'' + 8(|q|^2 + c)q = 0, \tag{13.2.3}$$

The constrained that $\det X$ is an even polynomial yields

$$q'\bar{q} - \bar{q}'q = 0.$$

This condition is satisfied if and only if there exist a $\mu \in S^1$ such that $q\mu$ is real. Thus we obtain that the function ξ defined in the Euler-Lagrange equation (6.1.1) for constrained Willmore surfaces is constant. Further, if q satisfies equation (13.2.3) then q satisfies also the Euler-Lagrange equation. By the proof of theorem (7.1) the corresponding surface is CMC in a space form.

On the other hand, to a given non constant solution q of the equation

$$q'' + 8(|q|^2 + c)q = 0,$$

defines a polynomial Killing field

$$X = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} a^2 + \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix} a + \begin{pmatrix} -ib & 2i\bar{p} \\ 2ip & -ib \end{pmatrix}$$

with $p = -\frac{i}{2}q'$ and $b = -2|q|^2 - 2c$. By construction X satisfies the equation $X' = [X, B]$. If the so defined polynomial Killing field X would have a zero for $a \in \mathbb{C}$ then there exist another section of $\mathcal{E}$ with degree 1 without any zeros for all $a \in \mathbb{C}$. Then q is constant by the previous lemma. □

Remark. For Delaunay tori we have that $q = \frac{1}{4}\kappa$, where κ is the curvature of the profile curve in the hyperbolic plane. The torus is constrained Willmore if there is a $\lambda_1 \in \mathbb{R}$ such that κ satisfies

$$\kappa'' + \tfrac{1}{2}\kappa^3 + (\lambda_1 - 1)\kappa = 0.$$

This is equivalent to equation (13.2.3) by defining $c = \frac{1}{8}(\lambda_1 - 1)$. The spectral curve is given by the equation $\eta^2 = \det X$. The polynomial $\det X$ is computed to be

$$\begin{aligned} \det X &= (a^2+b)^2 - (2i\bar{q}a + 2i\bar{p})(2iqa + 2ip) \\ &= a^4 + 2a^2 b + 4|q|^2 a^2 + b^2 + 4|p|^2 \\ &= a^4 - 4ca^2 + b^2 + 4|p|^2. \end{aligned} \tag{13.2.4}$$

Since $\det X$ is independent of y we get $b^2 + 4|p|^2 = d \in \mathbb{R}$ is constant. Thus the spectral curve defined here is the elliptic curve defined by the equation

$$(\kappa')^2 = -\tfrac{1}{4}\kappa^4 - 2(\lambda_1 - 1)\kappa^2 - \nu,$$

by defining $a := -i\kappa$, $\eta := i\kappa'$ and $\nu := d$. This equation is a integrated version of the Euler-Lagrange equation for elastic curves in H^2. The spectral curve is regular if and only if the polynomial $\frac{1}{4}\kappa^4 + 2(\lambda_1 - 1)\kappa^2 + \nu$ has only simple roots.

Since all equivariant CMC tori lies in the associated family of Delaunay cylinders, there exist a $\mu \in S^1$ such that $q\mu$ is real. By replacing $\frac{1}{4}\kappa$ by $q\mu$ we obtain the same statement for all equivariant tori of spectral genus 1.

13.3. Theorem. *An equivariant and non-isothermic torus in S^3 has (arithmetic) spectral genus 2 if and only if it lies in the associated family of a constrained Willmore Hopf cylinder.*

Proof. For spectral genus 2 solutions the polynomial Killing field is given by

$$X = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} a^3 + \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix} a^2 + \begin{pmatrix} -ib_1 & 2i\bar{p}_1 \\ 2ip_1 & ib_1 \end{pmatrix} a + \begin{pmatrix} -ib_0 & 2i\bar{p}_0 \\ 2ip_0 & ib_0 \end{pmatrix}$$

The equation $X' = [X, B]$ gives by a straight forward calculation

$$\begin{aligned} 2p_1 &= -iq', \\ ib_1' &= 4p_1\bar{q} - 4\bar{p}_1 q, \\ 2p_0 &= 2b_1 q - ip_1', \\ ib_0' &= -2ip_1'\bar{q} - 2i\bar{p}_1' q, \end{aligned} \tag{13.3.1}$$

Therefore we get

$$\begin{aligned} & b_1 = -2|q|^2 - 2c, \text{ for some real constant } c. \\ \text{and} \quad & b_0 = \bar{q}'q - q'\bar{q} + d, \text{ for some real constant } d. \end{aligned}$$

Thus

$$q''' + 24|q|^2 q' + 8cq' + d = 0. \tag{13.3.2}$$

This equation is known to be the stationary mKdV equation. Further the $\sigma-$symmetry gives:

$$b_0 + 2p_1\bar{q} + 2\bar{p}_1 q = 0 \text{ and } b_0 b_1 + 2p_1\bar{p}_0 + 2\bar{p}_1 p_0 = 0. \tag{13.3.3}$$

The first equation is equivalent to $d = 0$. And the second equation is equivalent to:

$$p_1'\bar{p}_1 - p_1\bar{p}_1' = 0,$$

which holds if and only if there is a $\mu \in S^1$ with $\mu p_1 \in \mathbb{i}\mathbb{R}$. Since $2p_1 = iq'$, we obtain for the corresponding q that there exist a μ with $q\mu = \kappa + ir$ for a real valued funciton κ and a real constant $r \geq 0$. If $r = 0$, the surface is isothermic. An isothermic and constrained Willmore torus has spectral genus 1, because the function ξ in the Euler-Lagrange equation is constant. For $r \neq 0$ the surface with conformal Hopf differential $q = \kappa + ir$ is a Hopf cylinder. The function κ is the curvature of its profile curve. A Hopf cylinder satisfies the stationary mKdV equation with $d = 0$ if and only if its profile curve is constrained elastic, i.e., if and only if the cylinder is constrained Willmore.

To a given solution q of the equation

$$q''' + 24|q|^2q' + 8cq' = 0$$

which does not satisfy equations (13.1.1) and (13.2.3) we can again define a polynomial Killing field

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}a^3 + \begin{pmatrix} 0 & -2i\bar{q} \\ -2iq & 0 \end{pmatrix}a^2 + \begin{pmatrix} ib_1 & -2i\bar{p}_1 \\ -2ip_1 & -ib_1 \end{pmatrix}a + \begin{pmatrix} ib_0 & -2i\bar{p}_0 \\ 2ip_0 & -ib_0 \end{pmatrix}$$

such that the entries satisfy the equations in (13.3.1). Then $X' = [X, B]$ and X has no zeros by assumption. So q has spectral genus 2. □

Remark. In contrast to the Delaunay case the constrained Willmore Hopf tori can have closed solutions with singular spectral curve, which are obtained by simple factor dressing of a circle, see (15.2). Nevertheless the arithmetic genus p of the constrained Willmore Hopf tori satisfies $p \leq 2$, since the profile curve is always constrained elastic.

13.4. Theorem. *Let $f : T^2 \to S^3$ be an equivariant constrained Willmore torus, then its (arithmetic) spectral genus is at most* 3.

Proof. An equivariant torus has arithmetic spectral genus $p = 3$ if and only if it has a polynomial Killing field of the form:

$$X = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}a^4 + \begin{pmatrix} 0 & 2i\bar{q} \\ 2iq & 0 \end{pmatrix}a^3 + \begin{pmatrix} -ib_2 & 2i\bar{p}_2 \\ 2ip_2 & ib_2 \end{pmatrix}a^2 + \begin{pmatrix} -ib_1 & 2i\bar{p}_1 \\ 2ip_1 & ib_1 \end{pmatrix}a + \begin{pmatrix} -ib_0 & 2i\bar{p}_0 \\ 2ip_0 & ib_0 \end{pmatrix}.$$

Again the equation $X' = [X, B]$ yields equations for each entry of the X_i. We obtain

(13.4.1)

$$\begin{aligned}
2p_2 &= -iq' \\
ib_2' &= 4p_2\bar{q} - 4\bar{p}_2 q, \\
\Rightarrow b_2 &= -2|q|^2 - 2c, \text{ for some real constant } c. \\
2p_1 &= 2b_2 q - ip_2' \\
ib_1' &= -2ip_2'\bar{q} - 2i\bar{p}_2' q, \\
\Rightarrow b_1 &= \bar{q}' q - q'\bar{q} + d, \text{ for some real constant } d. \\
2p_0 &= 2b_1 q - ip_1' \\
ib_0' &= -2ip_1'\bar{q} - 2i\bar{p}_1' q, \\
\Rightarrow b_0 &= 6|q|^4 + 2c|q|^2 + \tfrac{1}{2}(q''\bar{q} + \bar{q}'' q - q'\bar{q}') + e, \text{ for some real constant } e.
\end{aligned}$$

By $\sigma-$symmetry we have

$$\begin{aligned}
& b_1 + 2p_2\bar{q} + 2\bar{p}_2 q = 0 \qquad \Leftrightarrow \qquad d = 0, \\
& b_1 b_2 + 2p_0\bar{q} + 2\bar{p}_0 q + 2p_2\bar{p}_1 + 2\bar{p}_2 p_1 = 0, \\
& b_0 b_1 + 2p_1\bar{p}_0 + 2\bar{p}_1 p_0 = 0.
\end{aligned} \tag{13.4.2}$$

Thus we obtain the following differential equation for the conformal Hopf differential q of a spectral genus 3 immersion.

(13.4.3)

$$q'''' + 96|q|^4 q + 16\bar{q}' q' q + 24(q')^2\bar{q} + 8\bar{q}'' q^2 + 32|q|^2 q'' + 8c(q'' + 8|q|^2 q) + 16eq = 0,$$

with real constants c and e.

The second condition of (13.4.2) is equivalent to

$$\operatorname{Im}\left(q'''\bar{q} + 24|q|^2\bar{q}q' + 8c\bar{q}q' + \bar{q}''q'\right) = 0. \tag{13.4.4}$$

Note that this condition holds if q is real. Further if there is a $\mu \in S^1$ with $q'\mu$ is real then the condition above reduces to equation (13.3.2). Thus there is no spectral genus 3 solution with $q = \kappa + ri$ such that $\det X$ is even.

On the other hand the equations in (13.4.1) defines a polynomial Killing field of degree 4 for all solutions of (13.4.3) which do not already satisfy the equations (13.1.1), (13.2.3) and (13.3.2).

We want to show that a solution to the Euler-Lagrange equation of an equivariant constrained Willmore torus satisfies the equation (13.4.3). The Euler-Lagrange equation for equivariant constrained Willmore tori in S^3 stated in (6.1.1) has order 2 thus it is necessary to differentiate twice to obtain:

(13.4.5)

$$q'''' + 24\bar{q}' q' q + 24(q')^2\bar{q} + 4\bar{q}'' q^2 + 20|q|^2 q'' + 8Cq'' - 8\xi q'' + \operatorname{Re}(\lambda q) = 0.$$

Note that if q satisfies the constrained Willmore equation (6.1.1) for a surface in S^3 then it also satisfies (13.4.2). For a constrained Willmore surface we have a whole associated family of solutions to the Euler-Lagrange equation. Inside this associated family we have always a surface such that λ is real. Thus we can fix λ without loss of generality to be real. A computation shows that one can integrate the real and imaginary part of equation (refEL) once and we obtain:

$$\bar{q}'q' = -4|q|^4 + 8\rho^2 - 8C|q|^2 - \lambda q_1^2 - \tilde{d},$$

with $q = q_1 + iq_2$.

Using this equality the equation (13.4.5) is equivalent to the $4-$th flow given in (13.4.3) by defining the constants in (13.4.5) such that

$$\begin{aligned} 16C - 8c + \lambda_1 &= 0, \\ 2d + \tilde{d} + 8C^2 + 8cC &= 0. \end{aligned} \tag{13.4.6}$$

□

The space of spectral genus 3 solutions such that the spectral curve is defined by an even and real polynomial is 8 dimensional but the space of solutions for the Euler-Lagrange equation has only 7 dimensions. Thus not all spectral genus 3 solutions yield constrained Willmore tori. But a solution of a ordinary differential equation is uniquely determined by its initial values. Thus a solution of (13.4.3) yields an equivariant constrained Willmore surface if and only if its initial conditions coincides with the initial values for a solution of the Euler-Lagrange equation.

CHAPTER V

The Construction of Delaunay Tori and Constrained Willmore Hopf Tori

In this chapter we want to derive explicit formulas for the Delaunay and the constrained Willmore Hopf tori. Delaunay tori are surfaces obtained by the rotation of elastic curves in the upper half plane, viewed as the hyperbolic plane, around the $x-$ axis. They have constant mean curvature in a space form. A conformal parametrization corresponds to an arclength parametrization of the curve in the hyperbolic plane. Hopf tori are given by the preimage of a closed curve in the round S^2 under the Hopf fibration. The conformal constraint is equivalent to length and enclosed area constraints for the curve. By Theorems (15.5), (13.2) and (7.1) all equivariant constrained Willmore tori of spectral genus $g \leq 2$ are associated to Delaunay cylinders or Hopf cylinders. The fact that makes these surfaces easy to construct is that the imaginary part of the conformal Hopf differential is either 0 for the Delaunay cylinders or $\frac{1}{2}$ for the Hopf cylinders. Thus we only have to deal with one real function which is the curvature of the curve. In these cases equations (6.1.1) reduce to the Euler-Lagrange equations for (constrained) elastic curves in the hyperbolic plane and the round $2-$sphere with curvature 4, respectively, as shown in (6.3). Free elastic curves in H^2, i.e., critical points of the energy functional for curves without any constraints, corresponds to the minimal Delaunay cylinders, while Willmore Hopf cylinders are given by certain elastic curves.

What we construct in the following are constrained elastic curves in space forms. The Miura transformation of the curvature is the Schwarzian derivative of the curve in the conformal $2-$sphere $\mathbb{C}P^1$. Since the Willmore functional is a conformal invariant functional this setting seems to be more natural than the metric one. We construct the constrained elastic curves by using the Weierstrass elliptic functions. The definitions and properties that are needed were discussed in chapter I, section (3).

In their paper [**LS**] Langer and Singer constructed elastic curves in S^2 and H^2 without the enclosed area constraint. Our result is a generalization of this and uses the Schwarzian derivative instead of the curvature of the curve.

14. Constrained Elastic Curves in Space Forms

14.1. The Scattering Problem. Consider an arclength parametrized closed curve γ into a two dimensional space form of constant curvature G and let κ be its geodesic curvature in the space form. The Euler-Lagrange equation for a constrained elastic curve is then given by:

$$\kappa'' + \frac{1}{2}\kappa^3 + (\mu + G)\kappa + \lambda = 0, \tag{14.1.1}$$

where the real parameters μ and λ can be interpreted as the length constraint and the enclosed area constraint for closed curves, respectively (not to be confused with the spectral parameters in the previous sections). A solution to $\mu = \lambda = 0$ is a free elastic curve in the space form of curvature G. By multiplying the equation with $2\kappa'$ one can integrate the equation once and obtain

$$(\kappa')^2 = -\frac{1}{4}\kappa^4 - (\mu + G)\kappa^2 - 2\lambda\kappa - \nu. \tag{14.1.2}$$

Here ν is a real integration constant. This equation is the well known stationary first order modified Korteweg-de-Vries (mKdV) equation. We denote the polynomial on the right hand side by P_4.

The funciton κ must be real valued in order to be the curvature function of a curve. Such a solution for κ exists if and only if P_4 has real roots. Since we are interested in tori, we need the corresponding curve in S^2 or H^2 to be closed. Thus the solutions of the mKdV equation we are looking for are periodic. A real-valued periodic function always achieves its maximum and minimum and therefore there is a s_0 with $\kappa'(s_0) = 0$. Thus by translation of s we can assume without loss of generality that $s_0 = 0$. So we solve an initial value problem for equation (14.1.1) with initial values

$$\kappa(0) = \kappa_0 \quad \text{and} \quad \kappa'(0) = 0,$$

where κ_0 is a real root of P_4. If κ_0 is a multiple zero of P_4 then it is also a root of $\frac{\partial P_4}{\partial \kappa}$, which is the right hand side of equation (14.1.1). Therefore $\kappa \equiv \kappa_0$ is the unique solution to the given initial value problem by Picard-Lindelöff.

The so called asymptotic solutions are obtained if P_4 has multiple roots and we choose the initial value κ_0 to be a simple root.

Proposition. *Asymtotic solutions with $\lambda = 0$ are never periodic.*

Proof. For $\lambda = 0$ we have the differential equation

$$(\kappa')^2 = -\tfrac{1}{4}\kappa^4 - 2(\mu + G)\kappa^2 - \nu.$$

The polynomial on the right hand side is even. Asymptotic solutions are obtained if this polynomial has multiple roots. In order to obtain

non constant solutions we need at least 1 simple root of P_4. By symmetry the only case to consider is that the multiple root of P_4 is at $\kappa = 0$ with multiplicity 2 and we have 2 simple roots for $\kappa = \pm\kappa_0$.

We solve a initial value problem for the differential equation of second order

$$\kappa'' + \tfrac{1}{2}\kappa^3 + (\mu + G)\kappa = 0,$$

with initial value $\kappa(0) = \kappa_0$ and $\kappa'(0) = 0$. At $\kappa(0)$ we obtain that $\kappa''(0) = \frac{\partial(\kappa')^2}{\partial\kappa} < 0$. Thus there exist a $\epsilon > 0$ with $\kappa'(t) < 0$ for $t \in (0, \epsilon)$. The curvature function κ decreases monotonically for $t \in (0, \epsilon)$. Let $T := \sup\{\epsilon \in \mathbb{R}_+ | \kappa'(t) < 0 \text{ for } t \in (0, \epsilon)\}$. If $T < \infty$, then $\kappa'(T) = 0$ and we obtain $\kappa(T)$ is a root of P_4. Since κ is continuos, we obtain $\kappa(T) = 0$, which is a multiple root. By Picard-Linderlöff we get then that $\kappa(t) \equiv 0$ is the unique solution to the initial value problem $\kappa'(T) = \kappa(T) = 0$. This contradicts $\kappa(0) = \kappa_0 \neq 0$. Therefore $T = \infty$ and $\kappa(t)$ is not periodic. □

Corollary. *Equivariant tori of spectral genus* 1 *are simple.*

Proof. The spectral curve of a Delaunay torus is given by

$$(\kappa')^2 = -\tfrac{1}{4}\kappa^4 - 2(\mu + G)\kappa^2 - \nu.$$

Solutions where the polynomial on the right hand side has multiple roots are either constant, which are of spectral genus 0, or non periodic. The polynomial P_4 has therefore only simple roots. Thus by Corollary (10.7) we get that the torus is simple. For other tori of spectral genus 1 there exist a $\mu \in S^1$ such that $q\mu$ is real valued, where q is the conformal Hopf differential of the immersion. By replacing $\frac{1}{4}\kappa$ by $q\mu$, we obtain that all CMC tori of spectral genus 1 are simple. □

We exclude the case that P_4 can have multiple roots in the following.

For polynomials of degree 4 there exist an explicit algorithm to compute their roots. The procedure is the following: Assign to every polynomial of degree 4 a polynomial of degree 3, which is called the cubic resolvent, and compute the roots of the first polynomial out of the roots of the second.

The cubic resolvent of P_4 is given by:

$$cr_z = z^3 + 8(\mu + G)z^2 + 16((\mu + G)^2 - \nu)z - 64\lambda^2. \tag{14.1.3}$$

By substituting $z = x - \frac{8}{3}(\mu + G)$ we get the normal form of the polynomial, i.e., $cr_x = x^3 + px + q$ with

$$p = -\tfrac{16}{3}(\mu + G)^2 - 16\nu$$

and

$$q = -\tfrac{128}{27}(\mu + G)^3 + \tfrac{128}{3}\nu(\mu + G) - 64\lambda^2.$$

The properties of the roots of the cubic resolvent heavily depend on the sign of its discriminant

$$D = \tfrac{1}{4}q^2 + \tfrac{1}{27}p^3.$$

There are 3 cases to consider:

- $D = 0$: there exist multiple roots of P_4.
- $D < 0$: (*orbitlike*) the cubic resolvent cr_z has 3 real roots. If all of them are non-negative, then P_4 has 4 real roots. Otherwise P_4 has no real roots and there is no real solution for κ.
- $D > 0$: (*wavelike*) the cubic resolvent has 1 real root and P_4 has 2 real roots.

Lemma. *Let P_4 be the real polynomial of degree 4 given in (14.1.2) with only simple roots and let cr_z be its cubic resolvent. Then P_4 has real roots if and only if all real roots of cr_z are non-negative.*

Proof. Let e_1, e_2 and e_3 be the roots of cr_z. Then the cubic resolvent can be written as $cr_z = (z - e_1)(z - e_2)(z - e_3)$. We obtain

$$cr_z(0) = -e_1e_2e_3 = -64\lambda^2 \leq 0.$$

For $D > 0$ there is only 1 real root and a pair of complex conjugate roots of cr_z. Therefore the real root must be non negative. For $D < 0$ all roots of cr_z must be non-negative in order to obtain real roots of P_4. □

Now we fix an arclength parametrized constrained elastic curve γ in a 2−dimensional space form of constant curvature G. This curve has a real valued curvature function κ satisfying the stationary mKdV equation for fixed constants μ, λ and ν. The solutions to the mKdV equation can be transformed via Miura transformation to solutions of the KdV equation. A geometric way to do this is described in [**BuPP**]. It works as follows: Take a curve γ in $\mathbb{R}^2$, H^2 orS^2. By interpreting these space forms as subsets of $\mathbb{C}P^1$, the curve γ can be lifted to a curve $\tilde{\gamma}$ in $\mathbb{C}^2$ (not necessarily closed) with respect to the canonical projection from $\mathbb{C}^2$ to $\mathbb{C}P^1$. Then there exist a complex function a with $\hat{\gamma} := a\tilde{\gamma}$ such that $\det_{\mathbb{C}}(\hat{\gamma}, \hat{\gamma}') = 1$. Thus $\hat{\gamma}''$ and $\hat{\gamma}$ are linear dependent over $\mathbb{C}$ and there exist a complex function q with

$$\hat{\gamma}'' + q\hat{\gamma} = 0. \tag{14.1.4}$$

An equations of this type is called Hill's equation.

Remark. We call q the Schwarzian derivative of the curve γ. It is invariant under Möbius transformations of $\mathbb{C}P^1$. To a given q we can find by solving (14.1.4) a curve with Schwarzian derivative q unique up to Möbius transformations of $\mathbb{C}P^1$.

Lemma. *Let γ be a regular and arclength parametrized curve in a space form of constant curvature G and let κ be its geodesic curvature. Then the Schwarzian derivative q of γ is given by*

$$q = \frac{i\kappa'}{2} + \frac{\kappa^2}{4} + \frac{G}{4}.$$

PROOF. It is only necessary to consider the normalized cases $G = 0, 1, -1$.

G = 0. Let $\gamma : I \to \mathbb{R}^2 = \mathbb{C}$ be a regular and arclength parametrized curve. Using affine coordinates we get that $[\gamma, 1]$ is a curve in $\mathbb{C}P^1$. Thus $\tilde{\gamma} := (\gamma, 1)$ is a lift of the curve γ to $\mathbb{C}^2$. Then $\hat{\gamma} = \frac{1}{\sqrt{i\gamma'}}(\gamma, 1)$ is the lift of γ with the property that $\det(\hat{\gamma}, \hat{\gamma}') = 1$. Therefore $\hat{\gamma}$ and $\hat{\gamma}''$ are linearly dependent and there exist a complex function q with

$$\hat{\gamma}'' + q\hat{\gamma} = 0.$$

The frame equations of a plane curve is given by

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix},$$

where $T = \gamma'$ is the tangent vector, $N = iT$ is the normal vector and κ is the curvature of the curve. Thus we have $\gamma'' = i\kappa\gamma'$ and we can compute

$$\begin{aligned}
\hat{\gamma}' &= \begin{pmatrix} \frac{-\gamma''}{2\gamma'\sqrt{i\gamma'}}\gamma + \frac{1}{\sqrt{i\gamma'}}\gamma' \\ \frac{-\gamma''}{2\gamma'\sqrt{i\gamma'}} \end{pmatrix} = \begin{pmatrix} \frac{-i\kappa}{2\sqrt{i\gamma'}}\gamma + \frac{1}{\sqrt{i\gamma'}}\gamma' \\ \frac{-i\kappa}{2\sqrt{i\gamma'}} \end{pmatrix} \\
\hat{\gamma}'' &= \begin{pmatrix} \left(\frac{-i\kappa'}{2\sqrt{i\gamma'}} - \frac{\kappa^2\gamma'}{4\gamma'\sqrt{i\gamma'}}\right)\gamma \\ \frac{-i\kappa'}{2\sqrt{i\gamma'}} - \frac{\kappa^2\gamma'}{4\gamma'\sqrt{i\gamma'}} \end{pmatrix} = \begin{pmatrix} (\frac{-i\kappa'}{2\sqrt{i\gamma'}} - \frac{\kappa^2}{4\sqrt{i\gamma'}})\gamma \\ \frac{-i\kappa'}{2\sqrt{i\gamma'}} - \frac{\kappa^2}{4\sqrt{i\gamma'}} \end{pmatrix} \\
&= (-i\tfrac{\kappa'}{2} - \tfrac{\kappa^2}{4})\hat{\gamma}.
\end{aligned}$$

This yields $q = i\frac{\kappa'}{2} + \frac{\kappa^2}{4}$.

G = 1. Let $\gamma : I \to S^2$ be a regular and arclength parametrized curve in the 2−sphere. In the quaternionic language the Hopf fibration is given by

$$\pi : S^3 \subset \mathbb{H} \to S^2 \subset \ \mathrm{Im}\mathbb{H}, x \mapsto \bar{x}\mathbb{i}x.$$

Let η the horizontal lift of γ to S^3 with respect to π, i.e., $\eta i\bar{\eta} = \gamma$ and $< i\eta, \eta' >= 0$. Since η' is a quaternionic valued function and η is non vanishing, there exist a quaternionic function $u : I \to \mathbb{H}$ such that $\eta' = \frac{1}{2}u\eta$. Because $< \eta, \eta' >= 0$ and $< i\eta, \eta' >= 0$ we have that $u : I \to \mathrm{span}\{\mathbb{i}, \mathbb{j}\}$. Further

$$\begin{aligned}
&T = \gamma' = \bar{\eta}'i\eta + \bar{\eta}i\eta' = \bar{\eta}iu\eta, \\
&\Rightarrow |u|^2 = 1.
\end{aligned}$$

Thus there exists a real valued function θ with $u = e^{\mathbb{i}\theta}\mathbb{j}$. The normal vector of the curve γ in the 2−sphere is the vector which is simultaneously perpendicular to T and γ. Hence it is given by

$$N = \gamma \times T = \gamma T = -\bar{\eta} u \eta,$$

where $\times$ denotes the vector product in $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$. The geodesic curvature of γ is then given by

$$\kappa = < \gamma'', N > = \theta'.$$

Now consider $\mathbb{H} \cong \mathbb{C}^2$ as a complex vector space. Then $\eta = \eta_1 + \eta_2\mathbb{j} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ with complex funtions η_1 and η_2 and

$$\eta' = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}' = \frac{1}{2} e^{i\theta} \begin{pmatrix} -\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix}.$$

Therefore $\hat{\gamma} = \sqrt{2} e^{-i\theta/2} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ is a lift of γ to $\mathbb{C}^2$ with $\det(\hat{\gamma}, \hat{\gamma}') = 1$. To obtain the Schwarzian derivative we have to compute the second derivative of $\hat{\gamma}$.

$$\begin{aligned}
\hat{\gamma}' &= -i\tfrac{\theta'}{2}\sqrt{2}e^{-i\theta/2}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \sqrt{2}e^{-i\theta/2}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}' \\
&= -i\tfrac{\kappa}{2}\sqrt{2}e^{-i\theta/2}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \frac{\sqrt{2}}{2}e^{i\theta/2}\begin{pmatrix} -\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix} \\
\Rightarrow \hat{\gamma}'' &= (-i\tfrac{\kappa'}{2} - \tfrac{\kappa^2}{4})\sqrt{2}e^{-i\theta/2}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \tfrac{1}{4}\sqrt{2}e^{-i\theta/2}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\
&= -i\tfrac{\kappa}{2} - \tfrac{\kappa^2}{4} - \tfrac{1}{4}\hat{\gamma}.
\end{aligned}$$

And we get $q = i\frac{\kappa}{2} + \frac{\kappa^2}{4} + \frac{1}{4}$.

G = -1. Consider the hyperbolic plane H^2 as the upper half plane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ of $\mathbb{R}^2$ together with the metric $g_H = \frac{1}{y^2}(dx^2 + dy^2)$. Obviously, this metric lies in the conformal class of the euclidiean metric of $\mathbb{R}^2$. Let $\gamma = (\gamma_1, \gamma_2)$ be a regular and arclength parametrized curve in the upper half plane with respect to the euclidean metric. Then $\gamma \circ \varphi$ with $\varphi' = \gamma_2(\varphi)$ is an arclength parametrized curve in the hyperbolic plane. Like in the $G = 0$ case we have that $\tilde{\gamma} = (\gamma \circ \varphi, 1)$ is a lift of $\gamma \circ \varphi$ to $\mathbb{C}^2$. Therefore $\hat{\gamma} = \frac{1}{\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\tilde{\gamma}$ satisfies $\det(\hat{\gamma}(\varphi), (\hat{\gamma}(\varphi))') = 1$. The geodesic curvature κ of $\gamma(\varphi)$ in the hyperbolic plane is given by

$$\kappa = g_H(\nabla^H_{(\gamma(\varphi))'}(\gamma(\varphi))', N_H),$$

where ∇^H is the Levi-Civita connection of H^2 and N_H the normal vector of the curve in H^2. Because the hyperbolic metric lies in the same conformal class as the euclidean metric we have that $N_H = \gamma_2(\varphi)N$, where N is the normal vector of the curve in the $\mathbb{R}^2$ metric. Further

the change of the Levi-Civita connection due to a conformal change of the metric by $e^{2\lambda}$ is known to be

$$\nabla^{\lambda}_X Y = \nabla_X Y - (X\cdot\lambda)Y - (Y\cdot\lambda)X - < X, Y > \operatorname{grad}(\lambda).$$

Here $\lambda = -\ln(y)$. Thus we have that the geodesic curvature of the curve κ in H^2 is given by

$$\kappa(\varphi) = \gamma_2(\varphi)\kappa_E(\varphi) + \gamma_1'(\varphi)$$

and $\kappa_E(\varphi) =< \gamma''(\varphi), i\gamma'(\varphi) >$ is the curvature of the curve in the euclidean metric. We have $\gamma_1''(\varphi) = -\kappa_E(\varphi)\gamma_2'(\varphi)$ and $\gamma_2''(\varphi) = \kappa_E(\varphi)\gamma_1'(\varphi)$ and $\kappa'(\varphi) = \gamma_2(\varphi)^2\kappa_E'(\varphi)$. And now $\hat{\gamma}''$ is computed to be

$$\begin{aligned}
(\hat{\gamma})' &= \left(\frac{1}{\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\right)'\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix} + \frac{1}{\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma_2(\varphi)\gamma'(\varphi)\\0\end{pmatrix}\\
&= -\frac{\gamma_2(\varphi)(\gamma_2'(\varphi)\gamma'(\varphi)+\gamma_2(\varphi)\gamma''(\varphi))}{2\gamma_2(\varphi)\gamma'(\varphi)\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix} + \begin{pmatrix}-i\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}\\0\end{pmatrix}\\
&= -\frac{(\gamma_2'(\varphi)+i\kappa_E\gamma_2(\varphi))}{2\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix} + \begin{pmatrix}-i\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}\\0\end{pmatrix}\\
\Rightarrow\quad (\hat{\gamma})'' &= -\frac{(\gamma_2(\varphi)\gamma_2''(\varphi)+i\kappa_E'(\varphi)\gamma_2^2(\varphi)+i\kappa_E(\varphi)\gamma_2(\varphi)\gamma_2'(\varphi))}{2\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix}\\
&\quad + \frac{(\gamma_2'(\varphi)+i\kappa_E(\varphi)\gamma_2(\varphi))^2}{4\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix}\\
&= -\frac{(\gamma_2(\varphi)\kappa_E(\varphi)\gamma_1'(\varphi)+i\kappa_E'(\varphi)\gamma_2^2(\varphi)+i\kappa_E(\varphi)\gamma_2(\varphi)\gamma_2'(\varphi))}{2\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix}\\
&\quad + \frac{((\gamma_2'(\varphi))^2-\kappa_E^2(\varphi)(\gamma_2(\varphi))^2+2i\kappa_E(\varphi)\gamma_2(\varphi)\gamma_2'(\varphi))}{4\sqrt{i\gamma_2(\varphi)\gamma'(\varphi)}}\begin{pmatrix}\gamma(\varphi)\\1\end{pmatrix}\\
&= \frac{1}{4}\left(-2\gamma_2(\varphi)\kappa_E(\varphi)\gamma_1'(\varphi) - 2i\kappa_E'(\varphi)\gamma_2^2(\varphi) + 1 - (\gamma_1'(\varphi))^2 - \kappa_E^2(\varphi)(\gamma_2(\varphi))^2\right)\hat{\gamma}.
\end{aligned}$$

Since

$$\kappa^2(\varphi) = \gamma_2^2(\varphi)\kappa_E^2(\varphi) + \gamma_1'^2(\varphi) + 2\gamma_2(\varphi)\gamma_1'(\varphi)\kappa_E(\varphi),$$

we get

$$q = i\tfrac{\kappa'}{2} + \tfrac{\kappa^2}{4} - \tfrac{1}{4}.$$

□

Lemma. *Let γ be an arclength parametrized curve in a space form of curvature G. Let κ be its geodesic curvature and q its Schwarzian derivative. If the curve is constrained elastic, i.e., κ is a real solution of the stationary mKdV equation (14.1.2) with real constants λ, μ, G and ν, then q satisfies the stationary KdV equation*

$$(q')^2 + 2q^3 + cq^2 + 2dq + e = 0,$$

with real constants c, d and e given by

$$\begin{aligned} c &= \mu - \tfrac{G}{2} \\ d &= -\tfrac{\nu}{4} - \tfrac{G^2}{16} - \mu\tfrac{G}{4} \\ e &= cd + \tfrac{\lambda^2}{4} + \tfrac{\mu^2 G}{4} - \tfrac{\nu G}{4}. \end{aligned}$$

PROOF. Let γ be an arclength parametrized and constrained elastic curve in a space form of constant curvature G and let κ be its geodesic curvature and $q = i\frac{\kappa'}{2} + \frac{\kappa^2}{4} + \frac{G}{4}$ be its schwarzian derivative. Then κ satisfies the equations:

$$\kappa'' + \tfrac{1}{2}\kappa^3 + (\mu + G)\kappa + \lambda = 0,$$

and

$$(\kappa')^2 + \tfrac{1}{4}\kappa^4 + (\mu + G)\kappa^2 + 2\lambda\kappa + \nu = 0.$$

Therefore we can compute $(q')^2$.

$$\begin{aligned} q' &= \frac{i}{2}\kappa'' + \frac{1}{2}\kappa'\kappa = -\frac{i}{4}\kappa^3 - \frac{(\mu+G)i}{2}\kappa - \frac{i\lambda}{2} + \frac{1}{2}\kappa'\kappa, \\ \Rightarrow (q')^2 &= -\frac{1}{8}\kappa^6 - \frac{(\mu+G)^2}{4}\kappa^2 - \frac{\lambda^2}{4} - \frac{\nu}{4}\kappa^2 - \frac{(\mu+G)}{2}\kappa^4 - \frac{3\lambda}{4}\kappa^3 - \frac{i}{4}\kappa'\kappa^4 \\ &\quad - \frac{(\mu+G)\lambda}{2}\kappa - \frac{(\mu+G)i}{2}\kappa^2\kappa' - \frac{i\lambda}{2}\kappa'\kappa. \end{aligned}$$

Moreover q^2 and q^3 are computed to be

$$\begin{aligned} q^2 &= \left(\frac{i}{2}\kappa' + \frac{1}{4}\kappa^2 + \frac{G}{4}\right)^2 \\ &= -\frac{1}{4}(\kappa')^2 + \frac{1}{16}\kappa^4 + \frac{G^2}{16} + \frac{i}{4}\kappa'\kappa^2 + \frac{Gi}{4}\kappa' + \frac{G}{8}\kappa^2 \\ &= \frac{1}{8}\kappa^4 + \frac{(\mu+G)}{4}\kappa^2 + \frac{1}{2}\lambda\kappa + \frac{\nu}{4} + \frac{G^2}{16} + \frac{i}{4}\kappa'\kappa^2 + \frac{Gi}{4}\kappa' + \frac{G}{8}\kappa^2, \\ q^3 &= \left(\frac{i\kappa'}{2} + \frac{\kappa^2}{4} + \frac{G}{4}\right)\left(\frac{1}{8}\kappa^4 + \frac{(\mu+G)}{4}\kappa^2 + \frac{\lambda}{2}\kappa + \frac{\nu}{4} + \frac{G^2}{16} + \frac{i}{4}\kappa'\kappa^2 + \frac{Gi}{4}\kappa' + \frac{G}{8}\kappa^2\right) \\ &= \frac{1}{16}\kappa^6 + \frac{i}{8}\kappa'\kappa^4 + \frac{3\mu}{16}\kappa^4 + \frac{9G}{32}\kappa^4 + \frac{3\lambda}{8}\kappa^3 \\ &\quad + \frac{i(\mu+2G)}{8}\kappa'\kappa^2 + \frac{3\nu}{16}\kappa^2 + \frac{3G^2}{64}\kappa^2 + \frac{3(\mu+G)G}{16}\kappa^2 + \frac{iG}{16}\kappa'\kappa^2 + \frac{i\lambda}{4}\kappa'\kappa \\ &\quad + \frac{i\nu}{8}\kappa' + \frac{3iG^2}{32}\kappa' + \frac{\lambda G}{8}\kappa + \frac{3\nu G}{16} + \frac{G^3}{64} + \frac{\lambda G}{4}\kappa. \end{aligned}$$

Thus

$$\begin{aligned}(q')^2+2q^3 = &-\frac{(\mu-\frac{G}{2})}{8}\kappa^4-\frac{\mu^2}{4}\kappa^2\\ &-\frac{i\mu}{4}\kappa'\kappa^2+\frac{\nu}{8}\kappa^2+\frac{7G^2}{32}\kappa^2-\frac{\mu G}{8}\kappa^2+\frac{iG}{8}\kappa'\kappa^2-\frac{\mu\lambda}{2}\kappa\\ &+\frac{i\nu}{4}\kappa'+\frac{3iG^2}{16}\kappa'+\frac{3\nu G}{8}+\frac{G^3}{32}+\frac{\lambda G}{4}\kappa-\frac{\lambda^2}{4}.\end{aligned}$$

In order to cancel the highest order terms we need to choose $c=(\mu-\frac{G}{2})$ and obtain

$$\begin{aligned}\Rightarrow (q')^2+2q^3+(\mu-\tfrac{G}{2})q^2 = &\frac{\nu}{8}\kappa^2+\frac{G^2}{32}\kappa^2+\frac{i\nu}{4}\kappa'+\frac{iG^2}{16}\kappa'+\frac{\nu G}{4}-\frac{\lambda^2}{4}\\ &+\frac{\mu\nu}{4}+\frac{\mu G^2}{16}+\frac{\mu Gi}{4}\kappa'+\frac{\mu G}{8}\kappa^2\\ &=\left(\frac{\nu}{2}+\frac{G^2}{8}+\mu\frac{G}{2}\right)q+\left(\mu-\frac{G}{2}\right)\left(\frac{\nu}{4}+\frac{G^2}{16}+\mu\frac{G}{4}\right)\\ &-\frac{\lambda^2}{4}-\frac{\mu^2 G}{4}+\frac{\nu G}{4}.\end{aligned}$$

So we get

$$\begin{aligned}c &= \mu-\tfrac{G}{2}\\ d &= -\tfrac{\nu}{4}-\tfrac{G^2}{16}-\mu\tfrac{G}{4}\\ e &= cd+\tfrac{\lambda^2}{4}+\tfrac{\mu^2 G}{4}-\tfrac{\nu G}{4}.\end{aligned}$$

□

By substituting $\tilde{q}=q+\frac{1}{6}c$ we obtain:

$$(\tilde{q}')^2=-2\tilde{q}^3+2g_2\tilde{q}-4g_3, \tag{14.1.5}$$

with

$$\begin{aligned}g_2 &= \frac{c^2}{12}-d=\frac{(\mu+G)^2}{12}+\frac{\nu}{4}\\ g_3 &= -\frac{cd}{12}+\frac{e}{4}+\frac{1}{6^3}c^3=\frac{1}{216}(\mu+G)^3+\frac{1}{16}\lambda^2-\frac{1}{24}\nu(\mu+G).\end{aligned}$$

In particular, we have that both constants are real. We denote the polynomial on the righthand side by $\tilde{P}_3$. Substituting $\tilde{q}=-\frac{1}{8}x$ in $\tilde{P}_3$ we get the polynomial $\frac{1}{256}x^3-\frac{1}{3}g_2x-4g_3$. Then $\frac{1}{256}\tilde{P}_3(x)$ is the normal form of the cubic resolvent of P_4 given in (14.1.3). So there are the same 3 cases for the number of roots as before.

Equation (14.1.5) is solved by the Weierstrass $\wp$−function

$$\tilde{q}(s)=-2\wp(s+s_0),$$

for some constant $s_0\in\mathbb{C}_*$. The $\wp$−function is defined on a torus $\mathbb{C}/\Gamma$ with a lattice Γ determined by the lattice invariants g_2 and g_3. Because

g_2 and g_3 are real, Γ is rectangular or its double covering is rectangular, see chapter I section (3). As we started with an elastic curve, we already know that

$$\tilde{q}(s) = \frac{i\kappa'}{2} + \frac{\kappa^2}{4} + \frac{1}{6}(\mu + G). \tag{14.1.6}$$

Since we have chosen $\kappa'(0) = 0$ we get: $-2\wp(s_0) = \tilde{q}(0) \in \mathbb{R}$. Because the double covering of Γ is rectangular and $\tilde{q}$ is not a real valued function we obtain $s_0 \in i\mathbb{R}_*$ or $s_0 \in i\mathbb{R}_* + \omega_1$, where ω_1 the real half period of Γ. If $s_0 \in i\mathbb{R}_* + \omega_1$, let $\tilde{q}(s) = -2\wp(s + s_0)$ and $\tilde{q}_1(s) = -2\wp(s - \omega_1 + s_0) = -2\wp(s + \tilde{s}_0)$ with $\tilde{s}_0 \in i\mathbb{R}$. Then we get by Picard-Lindelöff $\tilde{q}(s) = \tilde{q}_1(s + \omega_1)$, i.e., both functions differ only by a translation of s. Thus without loss of generality we can choose $s_0 \in i\mathbb{R}_*$.

Let $\tilde{q} = -2\wp(s + s_0)$ as before, then we can reconstruct a whole family of curves γ_E with Schwarzian derivative $q_E = (\tilde{q} - E)$, for an arbitrary parameter $E \in \mathbb{C}$:
Let π denote the projection from $\mathbb{C}^2$ to $\mathbb{C}P^1$

$$(z, w) \mapsto [z, w].$$

Any curve γ_E in $\mathbb{C}P^1$ with Schwarzian derivative $q_E = (\tilde{q} - E)$ has a lift $\hat{\gamma}_E$ into $\mathbb{C}^2$ satisfying the equation

$$\hat{\gamma}_E'' + q_E\hat{\gamma}_E = 0$$

or equivalently

$$\hat{\gamma}_E'' + \tilde{q}\hat{\gamma}_E = E\hat{\gamma}_E, \tag{14.1.7}$$

where $\hat{\gamma}_E = (\hat{\gamma}_E^1, \hat{\gamma}_E^2)$ with complex valued functions $\hat{\gamma}_E^i$, $i = 1, 2$. Thus $\hat{\gamma}_E^i$, $i = 1, 2$ also satisfies the equation

$$(\hat{\gamma}_E^i)'' + \tilde{q}\hat{\gamma}_E^i = E\hat{\gamma}_E^i.$$

On the other hand, if for a $E \in \mathbb{C}$ there exist two complex linear independent solutions $\hat{\gamma}_E^i$ for the equation[1]

$$(\hat{\gamma}_E^i)'' + \tilde{q}\hat{\gamma}_E^i = E\hat{\gamma}_E^i,$$

then the curve on $\mathbb{C}P^1$ given by

$$\gamma_E = \pi((\hat{\gamma}_E^1, \hat{\gamma}_E^2)) = [\hat{\gamma}_E^1, \hat{\gamma}_E^2] \tag{14.1.8}$$

has Schwarzian derivative q_E. The functions $\hat{\gamma}_E^i$ can be expressed in terms of the Weierstrass ζ and σ functions, since $\tilde{q} = -2\wp(x + x_0)$. We

[1]At points $E \in \mathbb{C}$ where only 1 solution with monodromy exits, which happens exactly at branch points of the Weierstrass $\wp$ function, there exist another reconstruction for the curve. We will show that this case never occurs for curves corresponding to equivariant tori.

have

$$\begin{aligned}\hat{\gamma}^1_E &= \frac{\sigma(x+x_0-\rho)}{\sigma(x+x_0)\sigma(-\rho)}e^{\zeta(\rho)(x+x_0)}\\ \hat{\gamma}^2_E &= \frac{\sigma(x+x_0+\rho)}{\sigma(x+x_0)\sigma(\rho)}e^{\zeta(-\rho)(x+x_0)}, \quad \text{with } \wp(\rho)=E.\end{aligned} \tag{14.1.9}$$

By construction the functions $\hat{\gamma}^i_E$, $i=1,2$ have no poles, since the only zero of the Weierstrass σ function is at 0 but $x_0 \in i\mathbb{R}_*$ and $x \in \mathbb{R}$. If E is chosen such that $\rho \neq -\rho \bmod \Gamma$, then both functions are complex linear independent. Further, $\hat{\gamma}^i_E$, $i=1,2$ have no common zeros, if they are linear independent. Hence the curve is well defined. Recall that the Schwarzian derivative of a constrained elastic curve in the space form of constant curvature G with parameters λ, μ and ν is given by $q = \frac{i\kappa'}{2} + \frac{\kappa^2}{4} + \frac{G}{4}$, and

$$\tilde{q} = q + \tfrac{1}{6}(\mu - \tfrac{G}{2}).$$

Note that the Schwarzian derivative determines the curve up to Möbius transformations of $\mathbb{C}P^1$. Thus, by choosing $E = \frac{1}{6}(\mu - \frac{G}{2}) \in \mathbb{R}$, we obtain that the corresponding curve $\hat{\gamma}_E$ is a Möbius transformation of the constrained elastic curve we started with, if E is not a branch point of the $\wp-$function. The space of Möbius transformations of $\mathbb{C}P^1$ is $3-$dimensional. Thus the Möbius transformation needed to map one curve on the other is fully determined by fixing the initial point, the initial tangent vector of the curve and by the infinity point of $\mathbb{C}P^1$.

14.2. The Inverse Problem. Now we want to construct all constrained elastic curves in $2-$dimensional space forms with regular spectral curves leading to equivariant constrained Willmore tori. As we have seen in the last section all these curves have a Schwarzian derivative q solving the stationary KdV equation

$$q'' = 3q^2 + cq + d,$$

for real constants c and d. The solutions are given by $q = -2\wp - \frac{1}{6}c$, where $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{14.2.1}$$

with real constants g_2 and g_3. Thus, we first write down all such KdV solutions that are putative Schwarzian derivatives. It is already shown that these solutions are given by $\wp-$functions to special tori $\mathbb{C}/\Gamma$. A necessary condition, in order to get a real valued curvature function for the curve, is that the lattice invariants g_2 and g_3 are real. We also need that the discriminant $D = g_3^2 - 27g_2^3$ is nonzero in order to have two linear independent generators for the lattice. Then the polynomial

$$P_3 = 4x^3 - g_2x - g_3$$

has only simple roots.

We fix the space form of constant curvature G in which we want to construct the constrained elastic curve and choose a $E \in \mathbb{R}$ which is not a root of P_3, see (14.1.9). Then we obtain the coefficients of P_4 by:

$$\begin{aligned} \mu &= 6E + 3G \\ \nu &= 4g_2 - \frac{(\mu+G)^2}{3} \\ \lambda^2 &= 16g_3 - \frac{2(\mu+G)^3}{27} + \frac{2}{3}\nu(\mu+G). \end{aligned} \tag{14.2.2}$$

Lemma. *The stationary mKdV equation (14.1.2) with real parameters $(\mu+G)$, λ and ν has real solutions, if and only if $\frac{1}{6}(\mu+G)$ is less or equal to all real roots of the polynomial P_3. Equality holds if and only if $\lambda = 0$.*

Proof. The stationary mKdV equation (14.1.2) has real solutions if and only if the corresponding polynomial P_4 has real roots. For this it is necessary that the real roots of the cubic resolvent cr_z, given in (14.1.3), are non-negative. In order to obtain the normal form of cr_z we substitute z by $x = z - \frac{8}{3}(\mu+G)$. And by substituting $y = 16x$ we obtain the polynomial P_3 defining the Weierstrass $\wp-$function. Thus $z \geq 0$ is equivalent to $y \geq \frac{1}{6}(\mu+G)$ and all roots of P_3 must be greater or equal to $\frac{1}{6}(\mu+G)$. Further

$$P_3(\tfrac{1}{6}(\mu+G)) = -\tfrac{1}{16}\lambda^2 \leq 0, \tag{14.2.3}$$

and equality holds if and only if $\lambda = 0$. □

Since $-cr_z(0) = 64\lambda^2$, the product of all roots is positive. Thus in the case of $D < 0$, where we have 3 real roots of cr_z, it enough to show that two roots of cr_z are non-negative. This is equivalent to the fact that all critical points of cr_z are positive, i.e., all zeros of its derivative are positive. The derivative of cr_z is $3z^2+16(\mu+G)z+16((\mu+G)^2-\nu)$. Thus the condition needed is computed to be:

$$(\mu+G) < 0 \text{ and } \tfrac{1}{3}(\mu+G)^2 \leq \nu \leq (\mu+G)^2.$$

Remark. Applying the conditions above to the case of free elastic curves on S^2, i.e., $G > 0$, and $\lambda = \mu = 0$, we get that there are no orbitlike free elastic curves on S^2. In the case of Willmore Hopf tori we have that $G > 0$, $\lambda = 0$ and $(\mu+G) = \frac{1}{2}G > 0$. Thus there are also no orbitlike curves on S^2 corresponding to Willmore Hopf tori.

In order to reconstruct the curve we use the following compatibility condition which states that the conditions above are enough to get that $q = -2\wp(x+x_0) - \frac{1}{6}c$ is the Miura transformation of the curvature function of a constrained elastic curve.

Lemma. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 \neq 0$. And let $\wp$ be the Weierstrass function with respect to the lattice $\Gamma \subset \mathbb{C}$ given*

by the lattice invariants g_2 and g_3. Then there is a function $\kappa : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{C} \setminus \mathbb{R}$ with

$$\wp(x + x_0) = -i\frac{\kappa'(x)}{4} - \frac{\kappa(x)^2}{8} - b,$$

where b is a real constant. Moreover, κ is a stationary mKdV solution with coefficients determined by g_2, g_3 with the formulas given in (14.2.2).

Remark. Since $\wp(x + x_0)$ is a periodic function, κ is also periodic and achieves its maximum and minimum. Thus we can always choose $\kappa'(0) = 0$, which means that we choose a x_0 such that $\wp(x_0) \in \mathbb{R}$. Assume that there exist an arclength parametrized constrained elastic curve in a space form of constant curvature G with $\wp(x + x_0)$ as its Schwarzian derivative. Then b would have to be real and because

$$\wp(x + x_0) = -\tfrac{1}{2}q + \tfrac{1}{12}c = -i\tfrac{\kappa'(x)}{4} - \tfrac{\kappa(x)^2}{8} - \tfrac{G}{4} - \tfrac{1}{12}(\mu - \tfrac{G}{2}),$$

we obtain $b = \frac{1}{12}(\mu + G)$.

Proof. We will first proof that $\wp(x + x_0)$ has the right form. Details on elliptic functions were discussed in chapter I section (3). By differentiating the differential equation defining the Weierstrass $\wp$ function we get another differential equation for $\wp$, namely

$$\wp''(x + x_0) = 6\wp(x + x_0)^2 - \tfrac{1}{2}g_2. \tag{14.2.4}$$

Consider now only the points $z \in \mathbb{C}/\Gamma$ with $\wp - \bar{\wp} \neq 0$. A reformulation of our statement is:

$$\wp + \bar{\wp} = (\zeta - \bar{\zeta} + const_1)^2 + const_2,$$

with $const_1$ purely imaginary, $const_2$ real and ζ is the Weierstrass $\zeta-$function. Then we can define

$$\kappa := -2i(\zeta - \bar{\zeta} + const_1). \tag{14.2.5}$$

With (14.2.1) and (14.2.4) we obtain

$$2(\bar{\wp} - \wp)^3 = (\wp'' + \bar{\wp}'')(\bar{\wp} - \wp) + (\wp')^2 - (\bar{\wp}')^2.$$

This is equivalent to

$$2(\bar{\wp} - \wp) = \frac{\wp'' + \bar{\wp}''}{\bar{\wp} - \wp} + \frac{(\wp')^2 - (\bar{\wp}')^2}{(\bar{\wp} - \wp)^2}.$$

By integration we get

$$2(\zeta - \bar{\zeta} + const_1) = \frac{\wp' + \bar{\wp}'}{\bar{\wp} - \wp},$$

with a purely imaginary integration constant $const_1$. Thus

$$\wp' + \bar{\wp}' = 2(\bar{\wp} - \wp)(\zeta - \bar{\zeta} + const_1).$$

Integrate again we obtain

$$\wp + \bar{\wp} = ((\zeta - \bar{\zeta}) + const_1)^2 + const_2,$$

with a real integration constant $const_2$. The functions $\wp$ and $\bar{\wp}$ is holomorphic and anti-holomorphic, respectively, therefore we get that the derivative of $\wp$ with respect to $z = x + iy$ and of $\bar{\wp}$ with respect to $\bar{z}$ is the same as the derivative of $\wp$ and $\bar{\wp}$ with respect to x. Thus by replacing $\wp$ by $\wp(x + x_0)$ this proves the statement. Since all the functions we consider are continuous the equation above is still valid at the boundaries in the $x-$ direction. Thus it is necessary to choose a x_0 which does not lie on the real axis or on a parallel translate of it by a half lattice point. These choices of x_0 does not lead to an arclength parametrized curve, since q would be real valued.

Now we show that κ defined by equation (14.2.5) is mKdV stationary with coefficients compatible with (14.1.5) and (14.2.2). We have $\wp = -i\frac{\kappa'(x)}{4} - \frac{\kappa(x)^2}{8} - b$ and therefore

$$\wp(x + x_0)'' = -i\frac{1}{4}\kappa'''(x) - \frac{1}{4}\kappa''(x)\kappa(x) - \frac{1}{4}(\kappa'(x))^2$$
$$6\wp(x + x_0)^2 = \frac{3i}{8}\kappa'\kappa^2 + 3ib\kappa' - \frac{3}{8}\kappa'^2 + \frac{3}{32}\kappa^4 + 6b^2 + \frac{3}{2}b\kappa^2.$$

Now put this into (14.2.4) and take the imaginary part we obtain

$$\kappa''' + \frac{3}{2}\kappa'\kappa^2 + 12b\kappa' = 0, \tag{14.2.6}$$

which shows that κ is mKdV stationary and we get that

$$12b = (\mu + G).$$

The other coefficients can be obtained by a straight forward computation. □

15. Construction of Tori

In this section we use the constructions of constrained elastic curves in order to get examples of equivariant constrained Willmore tori. There exist the following cases:

- Delaunay tori:
  - orbitlike
  - wavelike,
- constrained Willmore Hopf tori:
  - orbitlike
  - wavelike.

The surface corresponding to a constrained elastic curve is a torus if and only if the curve is closed. We first show how the "surface" spectral curve is connected to the "curve" spectral curve. Then we

single out the closed constrained elastic curves and compute the conformal types and the Willmore energies of the corresponding tori. We close by introducing a 1−dimensional deformation of Delaunay and constrained Willmore Hopf tori preserving the closing condition, the so called Whitham deformations.

15.1. The Surface Spectral Curve. We want to understand the connection between the spectral curves of the equivariant tori of genus $g \leq 2$ and the spectral curves of the corresponding constrained elastic curves in space forms. Further, we want to compute their spectral genus. The spectral genus of an equivariant torus is defined to be the genus of the spectral curve corresponding to the Dirac equation (12.1.2). The spectral curve of the curve in the KdV setup is the torus defined by the lattice invariants g_2 and g_3. This curve has always genus 1. It turns out that the surface spectral curve is a double covering of the spectral curve we use in the KdV setup to construct the curves. The branch points of this double cover determines the genus of the "surface" spectral curve. While the spectral curves of the Delaunay tori have also genus 1, the spectral curves of constrained Willmore Hopf tori have genus 2.

Delaunay tori have constant mean curvature in a space form. To obtain the right space form we need to consider the fix point set of the involution $\rho \circ \sigma$, see sections (12.2) and (10.3). Fixed points of this involution have unitary monodromy. For a CMC torus in S^3 the involution $\rho \circ \sigma$ has fixed points and the two Sym-points, see Theorem (10.3), are fixed points by this involution. CMC tori in $\mathbb{R}^3$ are obtained as the limit of the S^3 case if the two Sym-points coincides. For CMC tori in H^3 with mean curvature $H \geq 1$, we need that $\rho \circ \sigma$ has fixed points but the two Sym-points are not fixed points. If the involution $\rho \circ \sigma$ has no fix points, then we have CMC tori in H^3 with mean curvature $|H| < 1$. We show that for Delaunay tori the involution $\rho \circ \sigma$ has fixed points if and only if the corresponding curve is orbitlike and $\rho \circ \sigma$ has no fixed points if and only if the curve is wavelike.

We need first a matrix version of the Miura transformation which gauges the Dirac equation (12.1.2) into the matrix version of the KdV equation. The Hill's equation (14.1.7) can be written in matrix form as:

$$\begin{pmatrix} \psi' \\ \psi \end{pmatrix}' + \begin{pmatrix} 0 & q-\eta \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi' \\ \psi \end{pmatrix} = 0.$$

Proposition. *The spectral curve of a Delaunay torus has genus* 1 *and is a double covering of the corresponding "curve" spectral curve.*

PROOF. The spectral curve of the Delaunay torus is determined by the kernel of the operator

$$D = \partial_y + \begin{pmatrix} -ia & i\frac{\kappa}{2} \\ i\frac{\kappa}{2} & ia \end{pmatrix},$$

where $a \in \mathbb{C}$ is the spectral parameter. The KdV operator with spectral parameter $\eta \in \mathbb{C}$ is given by

$$L = \partial_y + \begin{pmatrix} 0 & q-\eta \\ -1 & 0 \end{pmatrix}$$

We define a transformation of the spectral parameters $\eta = -a^2 - \frac{1}{4}$, which is a double covering of the $\eta-$plane by the $a-$plane. Further we have $q = i\frac{\kappa'}{2} + \frac{\kappa^2}{4} - \frac{1}{4}$, which is just the Miura transformation of the curvature function κ. Then these operators are gauge equivalent and the gauge transformation from L to D is given by

$$g = \begin{pmatrix} -i\frac{\kappa}{2} - ia & -i\frac{\kappa}{2} + ia \\ 1 & 1 \end{pmatrix},$$

for $a \in \mathbb{C}_*$.

The double covering of the parameter planes can be extended to $a = 0$ and $a = \infty$ and is branched at $\eta = -\frac{1}{4}$ and $\eta = \infty$.

The surface spectral curve is a hyperelliptic curve over the $a-$plane. Its branch points are determined by the branch points of the curve spectral curve and the branch points of the double covering of the parameters. The curve spectral curve has 4 branch points which are the branch points of the corresponding Weierstrass $\wp-$function. If any of these branch points coincides with the branch points of the parameter covering, then the branch point becomes a regular point of the spectral curve of D. Otherwise every branch point of the spectral curve of L makes 2 branch points of D. The point $\eta = \infty$ is a common branch point of the spectral curve of L and $\eta = -a^2 - \frac{1}{4}$, thus $a = \infty$ is not a branch point of the spectral curve of D. Therefore all branch points $\eta \in \mathbb{C} \setminus \{-\frac{1}{4}\}$ doubles over a and D has 4 or 6 branch points which corresponds to spectral genus 1 and 2, respectively. The spectral genus of the surface depends on whether $\eta = -1/4$ is a branch point of the curve spectral curve. We have

$$q + \tfrac{1}{6}(\mu + \tfrac{1}{2}) = -2\wp(x + x_0) \quad \text{and} \quad E = \eta + \tfrac{1}{6}(\mu + \tfrac{1}{2}).$$

Hence $\eta = -1/4$ is a branch point if and only if $\frac{1}{6}(\mu - 1)$ is a branch point of the Weierstrass $\wp-$function, i.e., a root of the polynomial P_3. By equation (14.2) we have

$$P_3(\tfrac{1}{6}(\mu - 1)) = -\frac{1}{16}\lambda^2.$$

Thus $\frac{1}{6}(\mu - 1)$ is a root of P_3 if and only if $\lambda = 0$, which always holds for Delaunay tori. We obtain that in this case the spectral curve of D

has always 4 branch points and is therefore of genus 1.

□

Proposition. *The involution $\rho \circ \sigma$ has fixed points if and only the curve is orbitlike and it has no fixed points if and only if the curve is wavelike.*

PROOF. The double covering of the parameters gives that a is purely imaginary if and only if $\eta \in \mathbb{R}$ and $\eta \geq -\frac{1}{4}$. Thus the surface spectral curve has branch points over $a \in i\mathbb{R}$ if and only if the curve spectral curve is branched over $\eta \in \mathbb{R}$, $\eta > -\frac{1}{4}$. In the case of wavelike curves the curve spectral curve has only 1 real branch point over $\eta = -\frac{1}{4}$, which vanishes over the parameter a. There fore there is no branch point of the surface spectral curve over $a \in i\mathbb{R}$. The curve spectral curve of orbitlike curves has 3 real roots and by (14.2). All roots are greater or equal to $\frac{1}{6}(\mu - 1)$. Thus the corresponding η is real and satisfies $\eta > -\frac{1}{4}$. Thus all branch points of the surface spectral curve lie over $a \in i\mathbb{R}$. □

Proposition. *The spectral curve of a constrained Willmore Hopf torus has genus* 2 *and is a double covering of the corresponding curve spectral curve.*

PROOF. For Hopf tori we have a slightly different Dirac operator.

$$\tilde{D} = \partial_y + \begin{pmatrix} -ia & i\frac{\kappa}{2} - 1 \\ i\frac{\kappa}{2} + 1 & ia \end{pmatrix}.$$

Now the potential is not purely imaginary anymore and although the real part is only a constant, we get that the spectral genus increases by the gauge. Like in the $G = -1$ case we want to gauge the KdV operator L to the operator $\tilde{D}$. The gauge is essentially the same

$$\tilde{g} = \begin{pmatrix} -i\frac{\kappa}{2} + 1 - ia & -i\frac{\kappa}{2} - 1 + ia \\ 1 & 1 \end{pmatrix}.$$

But we need to define $q = i\frac{\kappa'}{2} + \frac{\kappa^2}{4} + \frac{1}{4}$ and $\eta = -a^2 - \frac{3}{4}$.

The difference to the Delaunay case is that the parameter covering is now branched over $\eta = -\frac{3}{4}$ and $\eta = \infty$. Again, the branch point over $\eta = \infty$ vanishes and if we have that the spectral curve of L is branched over $\eta = -\frac{3}{4}$, then the spectral genus of the surface would be 1 and otherwise we get spectral genus 2. We need to compute whether $E = -\frac{3}{4} + \frac{1}{6}(\mu - \frac{1}{2}) = \frac{1}{6}(\mu - 5)$ is a root of the polynomial $P_3 = 4x^3 - g_2 x - g_3$. This is never the case, because if e is a root of P_3 then it satisfies the condition

$$e \geq \frac{1}{6}(\mu + 1),$$

by Lemma (14.2), but $\frac{1}{6}(\mu - 5) < \frac{1}{6}(\mu + 1)$. Thus constrained Willmore Hopf tori have always spectral genus 2. □

Remark. If we only consider constrained elastic curves in S^2, its spectral curve as a curve in the round 2−sphere (instead of $\mathbb{C}P^1$) is determined by the eigensolutions of D and the potential q is the curvature of the curve as in the $G = -1$ case. Therefore, we get that the curves have spectral genus 1 if and only if they are elastic, i.e., $\lambda = 0$ and spectral genus 2 if and only if they are constrained elastic, i.e., $\lambda \neq 0$.

Corollary. *Let f be a constrained Willmore Hopf torus and D the corresponding Dirac operator. Moreover let Φ_a be the fundamental solution matrix to $D_a u = 0$. Then its profile curve in S^2 can be obtained by*

$$\gamma = \Phi_{a_0}\Phi_{-a_0}^{-1},$$

for $a_0 \in \mathbb{R}_*$. Further the holonomy matrix of $D_{\pm a_0}$ is $\pm Id$.

Proof. The curve to a given $E = \wp(\rho)$ is given by the quotient of the two linear independent solutions of the equation $L_\eta u = 0$, with $E = \eta + \frac{1}{6}(\mu - \frac{G}{2})$. Because D and L is gauge equivalent, a curve to a given E is also constructed by the quotient of the fundamental solutions to the equations $D_{a_0}u = 0$ and $D_{-a_0}u = 0$, with $E = -a_0^2 - \frac{3}{4} + \frac{1}{6}(\mu - \frac{1}{2})$ smaller than all roots of P_3. Thus $a_0 \in \mathbb{R}_*$. Further the holonomy of L_η is $\pm Id$, thus holonomy of $D_{\pm a_0}$ is also $\pm Id$. □

15.2. Corollary. *There exist asymptotic solutions, i.e., non-constant solutions to equation (14.1.2) where the polynomial P_4 has multiple roots, which yield closed curves in S^2.*

Proof. These curves are obtained by simple factor dressing of a multi-covered circle. To be more concrete, let γ be a circle in the round S^2 of curvature $\kappa \equiv 1$, i.e., $q = \frac{1}{2}$. Then the corresponding operator D is given by:

$$D = \partial_y + \begin{pmatrix} -ia & \frac{1}{2}i - 1 \\ \frac{1}{2}i + 1 & ia \end{pmatrix},$$

and its polynomial Killing field is given by

$$X_0 = X = \begin{pmatrix} -ia & \frac{1}{2}i - 1 \\ \frac{1}{2}i + 1 & ia \end{pmatrix} dy.$$

The eigenvalues of the holonomy matrix are $e^{\pm iL\sqrt{a^2+\frac{5}{4}}}$, where $L = \sqrt{2}\pi$ is the length of γ. The branch points of the spectral curve are given by $a = \pm\sqrt{\frac{5}{4}}i$. And γ can be reconstructed at $a_0 = \sqrt{\frac{3}{4}}$. A curve reconstructed with respect to the spectral parameter $a \in \mathbb{R}$ closes after n periods of the original curve γ if and only if the eigenvalue of the holonomy satisfies $\sqrt{2}\sqrt{a^2 + \frac{5}{4}} = \frac{2m}{n}$. Then for the double covered circle γ, i.e., $n = 2$ and $m = 1$ the holonomy matrix is $\pm Id$ at $\alpha = \pm\sqrt{\frac{3}{4}}i$.

We change the initial values of the fundamental solution matrix. Instead of $\varphi_a(0) = Id$ we choose $\tilde{\varphi}_a(0) = h$, with $h = \begin{pmatrix} \sqrt{\frac{a-\alpha}{a+\alpha}} & 0 \\ 0 & \sqrt{\frac{a+\alpha}{a-\alpha}} \end{pmatrix}$. Then $\tilde{\varphi}_a = h\varphi_a$ is a solution of $D_a\tilde{\varphi}_a = 0$ with this new intial value. The monodromy matrix of $\tilde{\varphi}_a$ and then also its eigenlines with respect to the basis given by h are obtained by conjugation with h. Although the gauge has a pole in $\pm\alpha$, the holonomy of D_a with respect to $\tilde{\varphi}_a$ is well defined for all $a \in \mathbb{C}$, since the monodromy of φ_α is $\pm Id$. Thus also the initial value of the polynomial Killing field is conjugated by h. Let $\tilde{X}_0 = hX_0h^{-1}$. We can compute $\tilde{X}_0$ to be

$$\tilde{X}_0 = \begin{pmatrix} -ia & -\frac{a+\alpha}{a-\alpha} \\ \frac{a-\alpha}{a+\alpha} & ia \end{pmatrix},$$

which has a pole at $a = \pm\alpha$. In order to obtain the normal form of a polynomial Killing field we multiply $\tilde{X}_0$ by $(a+\alpha)(a-\alpha)$. This yields the initial value of a non vanishing polynomial Killing field $\hat{X}$ of degree 3. Thus the corresponding solution is mKdV stationary and corresponds to constrained elastic curves in S^2. The polynomial $\det\hat{X}$ has double roots at $a = \pm\alpha$. Thus also the polynomial P_3 has multiple roots. Since the monodromy of the fundamental solutions $\varphi_{\pm a_0}$ is conjugated by h, the reconstructed curve remains closed. □

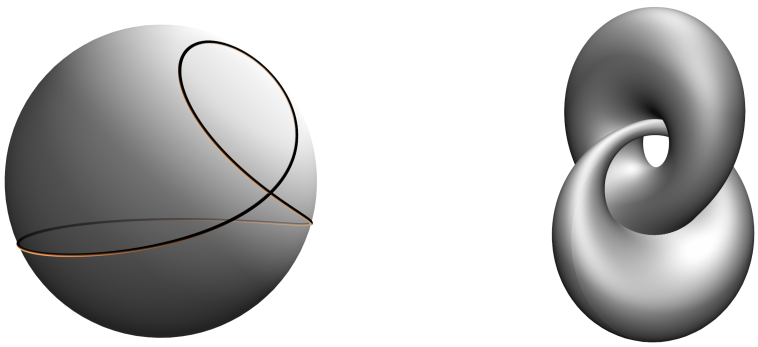

FIGURE 15.2.1. Constrained elastic curve with singular spectral curve in S^2 and corresponding constrained Willmore Hopf torus.

15.3. Closing Conditions. To obtain closing conditions for the curves γ_E defined in (14.1.8) we compute their monodromy. The curve γ_E closes if and only if the monodromy is a rotation by a rational angle. We fix a lattice Γ in $\mathbb{C}$ with real lattice invariants g_2 and g_3 and get a $\wp-$function with respect to this lattice. We denote by ω_i, $i = 1, 2, 3$, the half periods of Γ and fix ω_1 to be the half period lying on the real

axis. Then we always obtain for real g_2 and g_3 always a half lattice point on the imaginary axis, which we denote by ω_3. In the case of $D > 0$ we have $\omega_1 = \omega_3 \bmod \Gamma$. Recall that γ_E is given by

$$\gamma_E = [\hat{\gamma}_E^1, \hat{\gamma}_E^2],$$

where $\hat{\gamma}_E^1$ and $\hat{\gamma}_E^2$ are certain complex functions given in (14.1.9), provided E is not a branch point of the $\wp$-function. Further let ζ be the Weierstrass ζ-function and define $\eta_1 = \zeta(\omega_1)$, see chapter I section (3). This is real because the lattice invariants g_2 and g_3 are real. We compute the monodromy of both components separately with the formulas for the monodromy of the Weierstrass σ function and obtain:

$$\hat{\gamma}_E^1(x + 2\omega_1) = e^{-2\eta_1\rho + 2\zeta(\rho)\omega_1}\hat{\gamma}_E^1(x)$$
$$\hat{\gamma}_E^2(x + 2\omega_1) = e^{2\eta_1\rho - 2\zeta(\rho)\omega_1}\hat{\gamma}_E^2(x).$$

The monodromy of the γ_E is the quotient of the both monodromies computed here. Therefore we get that the curve closes after n periods if and only if there exist a $m \in \mathbb{Z}$, with (m, n) coprime, such that

$$e^{4\eta_1\rho - 4\zeta(\rho)\omega_1} = e^{\frac{2m}{n}\pi i}.$$

The integer m is the winding number of the curve. The equation above is equivalent to

$$\eta_1\rho - \zeta(\rho)\omega_1 = \frac{m}{2n}\pi i.$$

We distinguish in the following between the 4 cases mentioned at the beginning of the section. We want to show that we have always infinite many Delaunay and constrained Willmore Hopf tori.

We start with the constrained Willmore Hopf tori.

15.4. Proposition. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 > 0$. And let γ_E be the family of curves in the round $2-$sphere of curvature $G > 0$ given by (14.1.8) with respect to these constants. Then there exist infinitely many closed curves in that family for E smaller than all roots of P_3. In particular, E is never a branch point of the $\wp-$function.*

Proof. This is the case of wavelike constrained elastic curves in the round $2-$sphere. The polynomial P_3 has 1 real root and a pair of complex conjugate roots. We denote the real root by e. Let ω_1 be the real half lattice point of Γ. Then $\wp(\omega_1) = e$. In this case we have $\omega_3 = \omega_1 \bmod \Gamma$. We vary E to close the curves. Since $G > 0$ we have that $E = \frac{1}{6}(\mu - \frac{G}{2}) < \frac{1}{6}(\mu + G)$. Lemma (14.2) states that $\frac{1}{6}(\mu + G)$ is less or equal to all real roots of P_3. Thus here we have that $E < e$. The Weierstrass $\wp-$ function is even and $\lim_{x\to\infty} \wp(ix) = -\infty$, therefore we obtain $\rho \in i\mathbb{R}$. For fixed real invariants g_2 and g_3 we get that η_1

and ω_1 are real constants. Further for $\rho \in i\mathbb{R}$ then we get $\zeta(\rho) \in i\mathbb{R}$, too. Thus it is possible to define a map

$$g : i\mathbb{R} \to i\mathbb{R}, g(\rho) = \eta_1\rho - \zeta(\rho)\omega_1,$$

and $g(i\mathbb{R})$ is a nontrivial interval since

$$\lim_{\rho\to\pm 0} g(\rho) = \pm\infty \text{ and } g(\omega_3) = 0.$$

The rational numbers is a dense subset of the real numbers, thus there exist always infinite many closed solutions in this class.

□

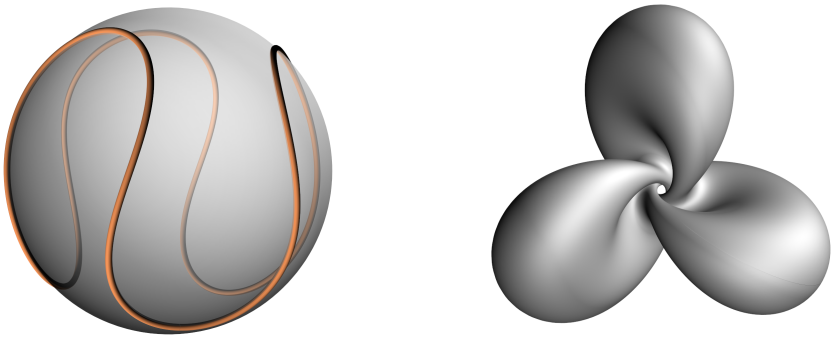

FIGURE 15.4.1. Wavelike elastic curve in S^2 to parameters $\mu = -\frac{1}{2}$ and $\lambda = 0$ in S^2 and corresponding Willmore Hopf torus.

15.5. Proposition. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 < 0$. And let γ_E be a family of curves in the round S^2 as before. Then there exist infinitely many closed curves in that family for E smaller than all roots of P_3. In particular, E is never a branch point of the $\wp$-function.*

PROOF. Now we consider orbitlike constrained elastic curves in the 2−sphere. The polynomial P_3 has 3 distinct real roots. Again we have that E must be smaller than the roots of P_3 by Lemma (14.2) and therefore $\rho \in i\mathbb{R}$. Further the map

$$g : i\mathbb{R} \to i\mathbb{R}, g(\rho) = \eta_1\rho - \zeta(\rho)\omega_1$$

is still well defined and $g(i\mathbb{R})$ is a non trivial interval, but $g(\omega_3) = \frac{1}{2}\pi i$. Therefore we obtain infinite many closed solutions.

□

Now we turn to the case of Delaunay tori, i.e., we have $G = -1$ and $\lambda = 0$.

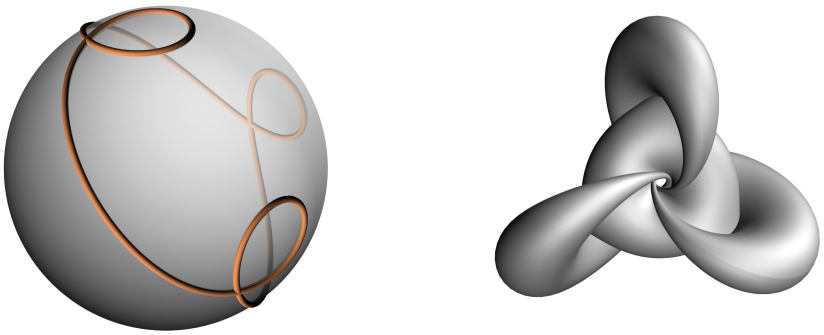

FIGURE 15.5.1. Orbitlike constrained elastic curve in S^2 and corresponding constrained Willmore Hopf torus.

15.6. Proposition. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 > 0$ and let γ_E be the family of curves as in (15.4) in the space form of curvature $G < 0$. Then there exist at most one closed curve in that family. Also in this case E is never a branch point of the $\wp$-function.*

PROOF. In the case of Delaunay tori we have $\lambda = 0$. This yields by Lemma (14.2)

$$P_3(\tfrac{1}{6}(\mu + G)) = -\tfrac{1}{16}\lambda^2 = 0.$$

Thus $\frac{1}{6}(\mu + G)$ is the smallest root of P_3. Since $G < 0$ we have $E > E + \frac{1}{4}G = \frac{1}{6}(\mu + G)$ thus we get that ρ with $\wp(\rho) = E$ does not lie on the imaginary axis. In the case of wavelike solutions we have that

$$e_0 = \tfrac{1}{6}(\mu + G)$$

is in fact the only real root of P_3 and because of $E > e_0$ we get $\rho \in \mathbb{R}$ and thus $\zeta(\rho) \in \mathbb{R}$: Therefore the only chance to get a closed solution is that

$$\rho\eta_1 - \zeta(\rho)\omega_1 = 0.$$

The solution holds obviously for $\rho = \omega_1$ but this choice contradicts the fact that $E > E_0$. The closing condition can be interpreted as the intersection of the line given by $\rho \mapsto \rho\frac{\eta_1}{\omega_1}$ with the graph of the function $\zeta|_{\mathbb{R}}$. The function $\zeta|_{\mathbb{R}}$ is anti-symmetric with respect to ω_1 and has a simple pole in 0 and is convex for $\rho < \omega_1$ and concav for $\rho > \omega_1$. Thus there exist two other intersection points if and only if $-\wp(\omega_1) = -(E + \frac{1}{4}G) > \frac{\eta_1}{\omega_1}$. Otherwise there are no other intersection points and no closed curves. This condition is never valid for $(\mu+G) > 0$ by (15.8.2) in the next section. Because in this case by (14.2) there are no orbitlike curves there are never closed elastic curves in H^2 with $(\mu + G) > 0$. □

Example. A closed curve in this class corresponds to a constrained Willmore torus obtained by glueing two CMC cylinder in H^3 with mean curvature $|H| < 1$ at the infinity border of the hyperbolic 3−space. An example of such a torus is shown in figure (15.6.1). The curve is closed since the closing condition is a real valued and continuos function in all parameters. In the left picture this function is negative and in the right picture it is positive. Thus there exist a zero for the closing condition.

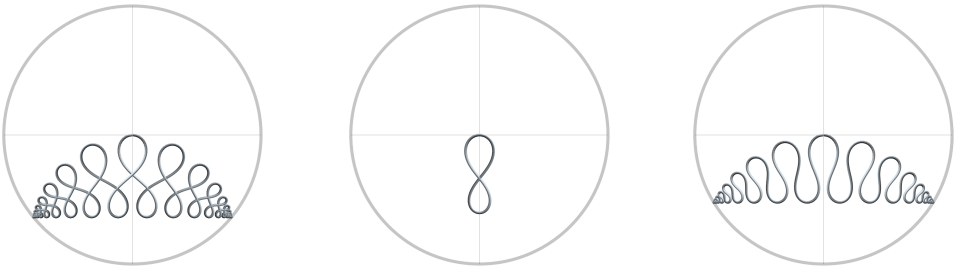

FIGURE 15.6.1. Wavelike elastic curves in H^2.

15.7. Proposition. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 < 0$ and let γ_E be the family of curves as in (15.4) in the space form of curvature $G < 0$. Then there exist infinitely many closed curves in this family.*

PROOF. In the case of orbitlike constrained elastic curves we get infinite many closed curves. To show this note that in the case of orbitslike curves the polynomial P_3 has three real roots and thus we can choose a $E > \frac{1}{6}(\mu + G)$ such that $P_3(E) < 0$ by varying $G < 0$. The corresponding ρ satisfies $\rho = \tilde{\rho} + \omega_1$ with $\tilde{\rho} \in i\mathbb{R}$ and

$$\overline{\zeta(\tilde{\rho} + \omega_1)} = -\zeta(\tilde{\rho} - \omega_1) = -\zeta(\tilde{\rho} + \omega_1) + 2\eta_1.$$

Thus the function

$$g(\rho) = \rho\eta_1 - \zeta(\rho)\omega_1$$

is purely imaginary. Further $g(\omega_2) = \frac{1}{2}\pi i$ and $g(\omega_1) = 0$. By the same argument as in (15.4) we get infinite many closed solutions. □

Proposition. *The choice of $\rho \in i\mathbb{R} + \omega_1$ such that $E = \wp(\rho)$ is not a branch point of $\wp$ yield CMC cylinders in S^3. If E is a branch point, we get CMC cylinders in $\mathbb{R}^3$. Further the choice of $\rho \in \mathbb{R}$ or $\rho \in \mathbb{R} + \omega_2$, which is not branched, yield CMC cylinders in H^3 with mean curvature $H > 1$.*

Remark. It is well known that there are no CMC tori in $\mathbb{R}^3$ with spectral genus 1. Thus, in order to get all Delaunay tori it is not necessary to consider the case where E is a branch point of $\wp$.

PROOF. Since the corresponding curves are orbitlike, the involution $\rho \circ \sigma$ has fixpoints. This exclude the case of CMC tori in H^3 with mean curvature $|H| < 1$. The surface spectral curve is obtained by a double covering of the curve spectral curve. The parameter covering is given by $\eta = a^2 - \frac{1}{4}$. Since the surfaces have constant mean curvature in a space form, we can also consider the involution σ as the elliptic involution. Then we can use the Sym-Bobenko formula to reconstruct the immersion. For $\rho \in i\mathbb{R} + \omega_1$ such that E is not a branch point of $\wp$, we have that the corresponding parameter $\pm a$ are not branch points of the surface spectral curve. Thus we have 4 complex independent solutions with monodromy to the equation $D_{\pm a}u = 0$. This corresponds to the case of 2 Sym-points. Further, these solutions have unitary monodromy, thus the Sym-points are fixed under the involution $\rho \circ \sigma$. And we obtain that the corresponding surface is CMC in S^3.

If we choose E to be a branch point of $\wp$, then we have only 2 complex independent solutions with monodromy to the equation $D_{\pm a}u = 0$ which are interchanged by the involution σ. Thus in this case we have only 1 Sym-point and the corresponding surface is CMC in $\mathbb{R}^3$.

For $\rho \in \mathbb{R}$ or $\rho \in \mathbb{R} + \omega_2$ and $\wp$ not branched in E, the monodromies of the 4 complex linear independent solutions to $D_{\pm a}u = 0$ are real. Thus these solutions are not fixed by the involution $\rho \circ \sigma$ and we obtain CMC cylinders in H^3 with mean curvature $H > 1$. By the same arguments as in the wavelike case there is at most 1 closed solution in this case. □

15.8. Conformal Type and Willmore Energy. The Willmore energy of the torus is determined by the bending energy of the curve, the parameter μ and the curvature G of the space form. The curves γ_E are Möbius transformations of arclength parametrized curves $\tilde{\gamma}_E$, for which $q = \frac{1}{2}i\kappa' + \frac{1}{4}\kappa^2 + \frac{1}{4}G$, where κ is the geodesic curvature of $\tilde{\gamma}_E$. Thus the conformal type and the Willmore energy of the corresponding tori can be computed. Recall that the Schwarzian derivative q of such a torus is a stationary KdV solution and it is given by

$$q = -2\wp(x + x_0) - \tfrac{1}{6}(\mu - \tfrac{1}{2}G).$$

Thus the integral of the real part of the Weierstrass $\wp$ function, i.e., the real part of Weierstrass $\zeta-$ function determines the bending energy of the curve. We have

$$\int_\gamma (\kappa^2 + \tfrac{2}{3}(\mu + G))ds = 8n(Re(\zeta(x - x_0 + 2\omega_1) - \zeta(x - x_0))) = 16n\eta_1,$$

if the curve closes after n periods of $\wp$. In particular we have

$$\int_\gamma \kappa^2 ds = 16n\eta_1 - \tfrac{4}{3}(\mu + G)n\omega_1 \geq 0 \tag{15.8.1}$$

thus

$$\frac{\eta_1}{\omega_1} > \tfrac{1}{12}(\mu + G). \tag{15.8.2}$$

We need to deal with both cases separately. We start with Delaunay tori, i.e., $G = -1$ and $\lambda = 0$. The conformal type of a torus of revolution is rectangular by construction and the vectors generating the lattice $\Gamma \in \mathbb{C}$ are given by

$$z_1 = 2\pi \quad \text{and} \quad z_2 = il$$

where l is the length of the curve. As the curve is arclength parametrized, we get that the length of the curve is $2n\omega_1$. The Willmore energy is

$$\mathcal{W}(f) = \tfrac{1}{2}\pi \int_\gamma \kappa^2 ds,$$

where κ is the geodesic curvature of the curve in the hyperbolic plane. Thus

$$\mathcal{W}(f) = 8n\pi\eta_1 - \tfrac{2}{3}n\pi(\mu - 1)\omega_1.$$

For constrained Willmore Hopf tori we choose the space form $G = 4$ due to the formula in Example (6.3). The Willmore energy is given by

$$\mathcal{W}(f) = \tfrac{1}{2}\pi \int_\gamma (\kappa^2 + 4) ds,$$

and κ is here the geodesic curvature of the curve in the round S^2 metric of curvature 4. The Willmore energy can be computed to be

$$\mathcal{W}(f) = 8n\pi\eta_1 - \tfrac{2}{3}n\pi(\mu + 1)\omega_1.$$

The conformal type of a Hopf torus is a bit more complicated to compute than for Delaunay tori because apart from the length we need the enclosed area A of the curve, see Lemma (5.12). The total area of the 2−sphere of curvature 4 is π. Since there are no singularities in the case of Hopf tori we get that the lattice is generated by

$$z_1 = 2\pi \quad \text{and} \quad z_2 = 2A \bmod 2\pi + il.$$

The enclosed area of a curve is given by $A = \frac{1}{2}\pi m - \frac{1}{4}\int_\gamma \kappa ds$ by the Gauß-Bonnet theorem, where m is the winding number of the curve. On the other hand we have: $\mathrm{Im}\zeta(x + x_0) = \frac{\kappa}{4} - \frac{\kappa(0)}{4} - i\zeta(x_0)$. Thus:

$$\begin{aligned}
\tfrac{1}{2}\int_\gamma \kappa ds - 2n\omega_1(\tfrac{1}{2}\kappa(0) - 2i\zeta(x_0)) &= 2\mathrm{Im}(ln\sigma(x + x_0 + 2n\omega_1) - ln\sigma(x + x_0)) \\
&= 2\mathrm{Im}\left(ln\left(\frac{-e^{2n\eta_1(x+x_0+\omega_1)}\sigma(x + x_0)}{\sigma(x + x_0)}\right)\right) \\
&= 2\mathrm{Im}\left(ln\left(e^{i\pi}e^{2n\eta_1(x+x_0+\omega_1)}\right)\right).
\end{aligned}$$

The logarithm is only well defined modulus $2\pi i$. We obtain

$$\tfrac{1}{2}\int_\gamma \kappa ds - 2n\omega_1(\tfrac{1}{2}\kappa(0) - 2i\zeta(x_0)) = (2\pi - 4ni\eta_1 x_0) \bmod 4\pi.$$

Therefore $(2A \bmod 2\pi)$ is given by:

$$\big(\pi m - 4in\eta_1 x_0 - 2n\omega_1(\tfrac{1}{2}\kappa(0) - 2i\zeta(x_0)\big) \bmod 2\pi.$$

We have shown the following theorem.

Theorem. *Let $f : T^2 \to S^3$ be either a Delaunay torus or a constrained Willmore Hopf torus determined by the formulas (14.1.8). Then we have the following.*

- *If f is a Delaunay torus, then its conformal class is given by the lattice generated by $z_1 = 2\pi$ and $z_2 = il$. Further the Willmore energy of f is given by*
$$\mathcal{W}(f) = 8n\pi\eta_1 - \tfrac{2}{3}n\pi(\mu - 1)\omega_1.$$
- *If f is a constrained Willmore Hopf torus, then its conformal class is given by the lattice generated by $z_1 = 2\pi$ and $z_2 = 2A \bmod 2\pi + il$. Further the Willmore energy of f is given by*
$$\mathcal{W}(f) = 8n\pi\eta_1 - \tfrac{2}{3}n\pi(\mu + 1)\omega_1.$$

Here $l = 2n\omega_1$ denotes the length of the curve in the respective space form and n is the lobe number. The enclosed area of the curve in S^2, $2A \bmod 2\pi$, is given by

$$2A \bmod 2\pi = \big(\pi m - 4in\eta_1 x_0 - 2n\omega_1(\tfrac{1}{2}\kappa(0) - 2i\zeta(x_0)\big) \bmod 2\pi.$$

15.9. Deformation of Constrained Elastic Curves. In order to get a better understanding of the moduli space of closed elastic curves, we want to investigate deformations of these. Deformations preserving the spectral curve are excluded as these also preserves the Willmore energy. It turns out that there exist generically a $1-$dimensional family of deformations of the spectral curves of constrained elastic curves, the so called Whitham deformations. These deformations preserves the closing of the curves.

We can assume with out loss of generality that the spectral torus of a constrained elastic curve is generated by $2\omega_1 = 1$ and $2\omega_2 = \tau$. The scaling of the lattice corresponds to the scaling of the curve and of the space form. As the invariants g_2 and g_3 are real we have $\tau \in i\mathbb{R}$ or $\tau \in 1/2 + i\mathbb{R}$, depending on whether the curve is orbitlike or wavelike. There exist Fourier series expansion for the functions $\wp$, ζ and η_1 with respect to τ. For $\rho \in i\mathbb{R}$ consider the function $g : i\mathbb{R}\times i\mathbb{R} \to i\mathbb{R}$, $g(\rho,\tau) = \rho\eta_1(\tau) - \zeta(\rho,\tau)$ which is smooth in both arguments. The closing conditions, (15.3), imply that the value of the function is rational. Thus, it has to remain constant if we vary τ. A geometric interpretation of this fact is that the quotient of the lobe number and

the winding number of the curve is constant along the deformation. The implicit function theorem states that if we have $\wp(\rho, \tau) \neq \eta_1$, we can write the level lines of the the function g as a graph over τ, i.e., there exist a differentiable function $\rho(\tau)$ such that $g(\rho(\tau), \tau) = const.$ This results in a 1-dimensional deformation of the spectral curve. For $\rho \in i\mathbb{R} + \omega_1$ the deformation can be defined analogously.

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