Three–Dimensional Homogeneous Spaces and their Application in General Relativity

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Introduction

> To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part.
>
> *David Hilbert's* 6^{th} *problem*

THE TEXT IN HAND is build up by three main parts and is concerned with a special class of homogeneous spaces and their application. The first part contains some old and new facts about Riemannian homogeneous spaces. The second part contains a classification result for Riemmanian homogeneous spaces in dimension three. And in the third part we will use those 3–dimensional manifolds and the theory of general relativity to construct homogeneous but non-isotropic cosmological models.

A Riemannian homogeneous space is a pair (M, G) where M is a manifold and G a smooth Lie group acting on M effectively, transitively and with compact isotropy groups. The last property allows M to admit Riemannian metrics which are invariant under the action of G, thus all elements of G may be regarded as isometries. As a consequence many geometric quantities are already determined by their values in a certain point, e.g. the curvature of a homogeneous metric. We will discuss some of those properties at the beginning of the first chapter. After that we study a special class of vector fields on a Riemannian homogeneous space which we will call *homogeneous vector fields*. Those vector fields arise if the isotropy representation has a non trivial eigenspace to the eigenvalue 1. They can be also described as vector fields which are invariant under the induced action of G on the tangent bundle of M. The set of homogeneous vector fields is in natural way a finite dimensional vector space and therefore it induces an involutive distribution $\mathcal{D}$ on M. This gives a foliation of M which turns out to have the structure of a principal bundle. The main theorem can be stated as follows

Theorem 1. *Let (M, G) be a Riemannian homogeneous space and $\mathcal{D}$ defined as above. Then there is a group H acting freely and properly on M such that $\pi\colon M \to B := M/H$ is an H–principal bundle over a homogeneous space B. Moreover the fibers of π are the maximal connected integral manifolds of $\mathcal{D}$ and G acts on B such that π is an equivariant map.*

This principal bundle inherits a natural connection $\mathcal{H}$ which can be constructed as the orthogonal complement to $\mathcal{D}$ with respect to a G–invariant metric. This horizontal distribution is unique in the sense, that for every G–invariant metric $\mathcal{H}$ is the orthogonal complement to $\mathcal{D}$.

Finally in the end of the first chapter we study extensions and semidirect products of Lie groups which we will use in the classification of Riemannian homogeneous spaces in dimension three. We are going to develop techniques to determine all possible extensions $1 \to K \to G \to H \to 1$ where K and H are known. One way to face this extension problem is to linearize it, i.e. to determine first the extension $1 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{h} \to 1$ of their Lie algebras. In certain circumstances those extensions are encoded in a second cohomology $H^2(\mathfrak{h}; \mathfrak{k})$ of $\mathfrak{h}$ with coefficients in $\mathfrak{k}$.

In addition we would like to point out that we used examples sparingly in the first chapter since the following part will contain of a lot them as well as application for the results of Chapter 1.

In the second chapter we classify the Riemannian homogeneous spaces (M, G) such that M is simply connected and $\dim M = 3$. Some of those spaces gained much popularity in the last decades. In [Thu97] William Thurston classified the Riemannian homogeneous spaces which admit compact quotients, i.e. there is a discrete cocompact group $\Gamma \subset G$. The maximal ones are known as the *eight Thurston geometries* which became famous since Thurston proposed in 1982 his geometrization conjecture and which was proven by Grigori Perelmann in 2002. At this point we have to mention also Peter Scott's beautiful article [Sco83] where he explains in great detail all eight Thurston geometries.

Now it seems natural to ask what happens if we remove the condition about the existence of cocompact discrete subgroups Γ. In [Sco83] there is a remark that Kulkarni carried out the classification of Riemannian homogeneous spaces but which has not been published. In general it seems hard to find any proof of the classification and even the proofs for the eight Thurston geometries in [Thu97] and [Sco83] are a little sketchy and therefore we decided to find a rigorous proof.

Let us now give an overview how the classification works. For this let K be a compact isotropy group in G. Since K is compact we have that K is a subgroup of $\mathbf{SO}(3)$ and we obtain $\dim K \in \{0, 1, 3\}$ and thus $\dim G \in \{3, 4, 6\}$. If $\dim K = 3$ any G–invariant metric μ has constant sectional, hence (M, μ) is isometric to either the sphere with its round metric, the euclidean three dimensional space or to the three dimensional hyperbolic space.

If $\dim K = 1$ things are more complicated. Because K is compact and connected it has to be isomorphic to $\mathbf{SO}(2)$ and the isotropy action of K in a tangent space has a 1–dimensional Eigenspace to the Eigenvalue 1. Applying Theorem 1, we obtain a principal bundle $\pi\colon M \to B$ over a surface B where the fiber group H is either $\mathbb{R}$ or $\mathbf{SO}(2)$. Moreover B is a Riemmanian homogeneous space for itself, say (B, G_B), and those are determined very quickly as one can see by Lemma B.4 of Appendix B:

$$(S^2, \mathbf{SO}(3)),\ (\mathbb{R}^2, \mathbf{E}_0(2)),\ (D^2, \mathbf{SO}^+(2, 1))$$

where $\mathbf{E}_0(2) = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ and $\mathbf{SO}^+(2, 1)$ is the isometry group of the 3–dimensional Minkowski space which preserves the future cone. Now the classification splits into two cases: the one where the natural connection is flat and the other where the horizontal distribution is not flat. In the flat case we obtain the extension

$$1 \longrightarrow G_B \longrightarrow G \longrightarrow H \longrightarrow 1.$$

where $G_B \in \{\mathbf{SO}(2), \mathbf{E}_0(2), \mathbf{SO}^+(2, 1)\}$ and $H = \mathbb{R}$. After some work we will see (M, G) is isomorphic to

$$(S^2 \times \mathbb{R}, \mathbf{SO}(3) \times \mathbb{R}),\ (D^2 \times \mathbb{R}, \mathbf{SO}^+(2, 1) \times \mathbb{R}),\ (\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \times \mathbb{R}),\ (\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes \mathbb{R})$$

where in the first three cases the action is obvious and the last case is explained in detail in Example 2.16. The space which does not admit a compact quotient is this last space. On the other side if the bundle is not flat one gets the central extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow G_B \longrightarrow 1.$$

Using results about the extension problem obtained in the first chapter we deduce (M, G) is isomorphic to

$$(S^3, \mathbf{U}(2)),\ (D^2 \times \mathbb{R}, \Gamma),\ (\mathbb{R}^2 \times \mathbb{R}, \mathbf{Nil} \rtimes \mathbf{SO}(2)).$$

We refer the reader to Propositions 2.24, 2.26 and 2.27 respectively for more informations about those groups. All of the latter spaces admit compact quotients and therefore they can be found in the classification in [Thu97] and in the discussion in [Sco83]. This finishes the case $\dim K = 1$.

For the remaining case $\dim K = 0$ we see that M has to be a simply connected Lie group which are in one–to–one correspondence to 3–dimensional Lie algebras. This classification has be done already by Luigi Bianchi in the 19^{th} century, see [Bia02]. We would like to mention also the Diploma thesis of Manuel Glas [Gla08] where he revisited Bianchi's proof in a more modern, coordinate–free way. Also John Milnor did classify in [Mil76] very elegantly the 3–dimensional unimodular Lie algberas. In this work here we present an alternative way to determine those Lie algebras which is more in the spirit of preceeding sections.

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}'$ its derived Lie algebra which is an ideal in $\mathfrak{g}$ we obtain an extension of Lie algebras

$$1 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 1$$

for $\mathfrak{h} = \mathfrak{g}/\mathfrak{g}'$. Since the dimension of $\mathfrak{g}'$ is invariant under isomorphisms we may sort the Lie algebras by the dimension of their derived Lie algebras $\mathfrak{g}'$. If $\dim \mathfrak{g}' = 0$ then $\mathfrak{g}$ is the abelian Lie algebra. If $\dim \mathfrak{g}' = 1$ then $\mathfrak{g}'$ is the abelian Lie algebra $\mathbb{R}$ and it turns out that $\mathfrak{h}$ has to be the abelian two–dimensional Lie algebra $\mathbb{R}^2$. The extensions $0 \to \mathbb{R} \to \mathfrak{g} \to \mathbb{R}^2 \to 0$ may be computed through the second cohomology $H^2(\mathbb{R}^2; \mathbb{R})$ where $\mathbb{R}^2$ acts on $\mathbb{R}$ through a certain representations. The next case is $\dim \mathfrak{g}' = 2$ and again a short computations shows that $\mathfrak{g}'$ has to be abelian and since $\dim \mathfrak{h} = 1$ we obtain the extension $0 \to \mathbb{R}^2 \to \mathfrak{g} \to \mathbb{R} \to 0$ which does always split since $\mathbb{R}$ is abelian, thus $\mathfrak{g} = \mathbb{R}^2 \rtimes_\pi \mathbb{R}$ for a certain representation $\pi \colon \mathbb{R} \to \mathbf{GL}(\mathbb{R}^2)$ which has to be determined. Finally $\dim \mathfrak{g}' = 3$ means that $\mathfrak{g}$ is unimodular and there we use Milnor's idea in [Mil76]. The full list of 3– dimensional Lie algebras and their groups together with a comparison to Bianchi's list can be found after Proposition 2.32. At this point we forgo to list all 3–dimensional homogeneous Riemannian spaces and refer the reader to the section The list of 3–dimensional geometries.

Finally the last chapter is about an application of Riemannian homogeneous spaces in the field of cosmology. Since Einstein's publication of his groundbreaking theory of gravitation in 1916, the field of modern cosmology in physics began to grow rapidly and one aim of this branch is to study the dynamics of the universe based on the laws of general relativity. The mathematical ingredients of this theory consist of a Lorentzian manifold (L, h) and a symmetric bilinear form T on L called the stress–energy tensor and it represents the energy and the mass of the physical system which we would like to describe. For physical reasons T has to be divergence free on (L, h). Einstein's theory now demands to solve the equation

$$\mathrm{Ein} := \mathrm{Ric} - \tfrac{1}{2} R h = T$$

on L where Ric is the Ricci curvature of h and R its scalar curvature. The physical system in our situation is the whole universe and usually this is modeled as a perfect fluid or as a dust cloud like in the Robertson–Walker cosmology, see [Bar83] for a nice and comprehensive introduction. Moreover nowadays one assumes that besides the visible matter like the stellar objects the universe if filled up with so called *dark matter* and *dark energy*. The latter objects make up to 95% on the content of the universe and in particular the dark energy about 72% (see [Hin10]). Furthermore dark energy is modeled mathematically by a *cosmological constant* Λ and therefore we would like to concentrate our studies to a universe where there is no mass but a lot of dark energy. Thus the stress–energy tensor T simplifies to $T = \Lambda h$ which is clearly divergence free. Note that $\Lambda < 0$ corresponds physically speaking to a positive cosmological constant since usually the physicist do not see the cosmological constant as part of T but rather as a separate term in the Einstein equation which stands on the side of the geometric quantities.

Now solving the Einstein equation turns out to be a pretty hard task and therefore one assumes symmetry conditions as it will be in our case. So we will consider $L = I \times M$ where I is an interval in $\mathbb{R}$ and M is a simply connected 3–dimensional Riemannian homogeneous space where G is the group acting on M. Moreover let $\mu \colon I \to \mathcal{S}^+$ be a curve of Riemannian homogeneous metrics on (M, G) and we are searching for those curves such that $h := -\mathrm{d}t^2 + \mu(t)$ fulfills the Einstein equation $\mathrm{Ein} = \Lambda h$. Since M is a homogeneous space and $\mu(t)$ is a homogeneous metric the complicated partial differential equation for h reduces to a second–order ordinary differential equation for μ on the phase space $\mathcal{P} = \mathcal{S}^+ \times \mathbf{Sym}$ where $\mathbf{Sym}$ are the G–invariant symmetric bilinear forms on M. It is easy to see that $\mathcal{P}$ is a symmetric space, hence a finite–dimensional manifold. Let $V \subset \mathcal{P}$ be set of $(\mu, \kappa) \in \mathcal{P}$ such that $R + H^2 - |\kappa|^2 + 2\Lambda = 0$ where R is the scalar curvature of (M, μ), $H = \mathrm{tr}_{\mu}(\kappa)$ and $|\kappa|$ the norm of κ with respect to the induced metric on $\mathbf{Sym}$ by μ. It depends on (M, G) whether V is a smooth submanifold of $\mathcal{P}$, however V is always a semialgebraic set. We have the following result concerning solutions to the Einstein equation

Theorem 2. *Suppose $(\mu_0, \kappa_0) \in V$ and $\mathrm{div}_{\mu_0}\kappa_0 = 0$ then the maximal integral curve $(\mu(t), \kappa(t))$ through (μ_0, κ_0) to the vector field*

$$\mathcal{X}_{(\mu,\kappa)} = \begin{pmatrix} -2\kappa \\ \mathrm{Ric} + \Lambda\mu + H\kappa - 2\kappa^2 \end{pmatrix}$$

on $\mathcal{P}$, where Ric is the Ricci curvature of (M, μ), is a solution to $\mathrm{Ein} = \Lambda h$ on $L = I \times M$ where $h = -\mathrm{d}t^2 + \mu(t)$.

The quantity $H(t)$ has a nice geometric interpretation: it is the mean curvature of $(t \times M, \mu(t)) \subset (L, h)$ and its physically interpretation is that it is the Hubble constant. Now the maximal interval I where the solution $\mu(t)$ lives on, depends on the behavior of H, namely we have that solutions exists as long as H is bounded, see Proposition 3.10.

Finally we show the following result concerning the asymptotic behavior of the solution $\mu(t)$ on a special Riemannian homogeneous space. If $(M, G) = (S^3, \mathbf{U}(2))$ and $\Lambda = -3$ (note again that this means a positive cosmological constant) then the round metric β with scalar curvature $R = 6$ lies in V as $(\beta, 0)$. Clearly β is $\mathbf{U}(2)$–invariant and $(\cosh^2(t)\beta, -\cosh(t)\sinh(t)\beta)$ for $t \in \mathbb{R}$ is a solution to $\mathcal{X}$ through $(\beta, 0)$. The metric $h = -\mathrm{d}t^2 + \cosh^2(t)\beta$ is known as the *De–Sitter solution*. In [Wal83] Robert Wald showed that a positive cosmological constant isotropize an initially expanding universe meaning that the solution looks on late times locally like the De–Sitter solution. We will try to make this statement more precise in terms of dynamical systems and show that for all initial expanding solutions with initial volume big enough, the De–Sitter solution is a good candidate for an attractor. More precisely: let $\mathcal{A} = \{(\cosh^2(t)\beta, -\cosh(t)\sinh(t)\beta) : t \in \mathbb{R}\} \subset V$ be the set of the De–Sitter solution. Then we obtain

Theorem 3. *The function $L\colon V \to \mathbb{R}$, $L(\mu, \kappa) := v^{\frac{2}{3}}(\frac{2}{3}H^2 - 6)$ is a Lyapunov function for the dynamical system induced by $\mathcal{X}$ and the global minima of L are given by the set $\mathcal{A}$.*

Now if $(\mu(t), \kappa(t))$ is a solution of $\mathcal{X}$ on V starting at $(\mu_0, \kappa_0) \in V$ such that the volume of μ is bigger than 1 and $\mathrm{tr}_{\mu_0}\kappa_0 < 0$, then $L(\mu(t), \kappa(t))$ converges to $\inf_t L(\mu(t), \kappa(t))$ and therefore it is reasonable to conjecture that $\inf_t L(\mu(t), \kappa(t)) = \min L$.

Finally there are of course other approaches to homogeneous cosmological solutions which are popular among physicist. Usually they use as homogeneous spaces the Bianchi Lie groups which may be regarded as the minimal geometries in the spirit of William Thurston (in comparison to the maximal geometries in [Thu97]). It turns out that every Riemannian homogeneous space except $(S^2 \times \mathbb{R}, \mathbf{SO}(3) \times \mathbb{R})$ contains a 3–dimensional subgroup acting transitively on M, such that it is enough to know the dynamics of this minimal geometry. For an introduction to this approach we recommend [WJ97].

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1 Homogeneous Riemannian Spaces

This first chapter is intended to develop the necessary tools and techniques for homogeneous Riemannian spaces which are the central objects of this work. Although some of the following statements are true for general homogeneous spaces, we will not mention this explicitly. We will start with the basic theory of Riemannian homogeneous spaces and their geometric properties. We will proceed to discuss some special vector fields and the consequences of their existence. The next step will be to study (central) extensions of Lie groups and their relation to semidirect products and we close the chapter with a discussion about the set of homogeneous metrics.

Throughout this chapter the letters G, K and H will denote smooth Lie groups, where smooth means C^∞ and with $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ we denote their Lie algebras respectively. In general all differentiable objects in this work are smooth if not otherwise stated. Furthermore M will always be a smooth, connected manifold of dimension d_M and we fix a point $m_0 \in M$. If G acts from the left on M in a smooth way, then the action will be denoted by $\theta\colon G \times M \to M$ and we abbreviate $\theta(g, m) =: g.m$. Finally we denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M equipped with the compact–open topology.

Introduction to homogeneous Riemannian spaces

A homogeneous Riemannian space is a special homogeneous space endowed with Riemannian metrics which are invariant under some transitive action of a Lie group. Hence those spaces possess many symmetries.

Definition 1.1. A pair (M, G) is a *homogeneous Riemannian space*, if G acts on M smoothly, transitively and effectively with compact isotropy groups. Henceforth (M, G) will always denote a Riemannian homogeneous space if not otherwise stated.

Remark 1.2.

(a) Since G acts effectively on M we have an injective homomorphism $\delta\colon G \to \mathrm{Diff}(M)$, $g \mapsto (m \mapsto g.m)$. And because the action is smooth (in particular continuous), we have that δ is continuous. Often, we will not distinguish between G and its image $\delta(G)$ and we will show later, that $\delta(G)$ is closed in $\mathrm{Diff}(M)$. Therefore we can see G as a transformation group of M.

(b) The action θ induces a fiber wise linear action on the tangent bundle $\pi\colon TM \to M$. We define $\Theta\colon G \times TM \to TM$, by $\Theta(g,\xi) := D\theta_{(g,\pi(\xi))}(\iota(\xi))$, where D denotes the differential and ι is the natural inclusion of $TM_{\pi(\xi)}$ in $T(G\times M)_{(g,\pi(\xi))} \cong TG_g \oplus TM_{\pi(\xi)}$. It is easy to see that $\Theta(g,\xi) = (Dg)_{\pi(\xi)}(\xi)$. We will write $g.\xi$ for $(Dg)_{\pi(\xi)}(\xi)$ and although we used this abbreviation for the action θ there will be no confusion which action is meant, since it will be clear from the context.

(c) We will denote the (compact) isotropy group of G acting on M in m_0 by K. By a well–known theorem M is diffeomorphic to G/K and G is a K–principal fiber bundle over M with projection map $p\colon G \to M$, $g \mapsto g.m_0$ and therefore we set $P := (\mathrm{D}p)_e\colon \mathfrak{g} \to TM_{m_0}$. Of course P induces an isomorphism from $\mathfrak{g}/\mathfrak{k}$ to TM_{m_0}.

(d) One could define a Riemannian homogeneous space in purely 'algebraic' terms, i.e. we say a pair (G,K) is an *algebraic Riemannian homogeneous space* if K is a compact subgroup of G. Of course $(G/K, G)$ is a Riemannian homogeneous space and vice versa a Riemannian homogeneous space by our first definition is an algebraic Riemannian homogeneous space just by taking K as in (c). Hence both definitions are equivalent.

Examples 1.3.

(a) Consider the pairs $(\mathbb{R}^n, \mathbf{E}(n))$, $(S^n, \mathbf{O}(n))$ and $(D^n, \mathbf{H}(n))$, where $\mathrm{E}(n)$, $\mathbf{O}(n)$ and $\mathbf{H}(n)$ are the (maximal) isometry groups of the standard metrics on $\mathbb{R}^n$, on the n–dimensional sphere and on the n–dimensional hyberbolic disk respectively.

(b) Every Lie group G is a Riemannian homogeneous space given by (G,G) where G acts on itself by left-multiplication.

(c) Let (M_i, G_i), $i \in \{1,\dots,n\}$ be Riemannian homogeneous spaces. Then the product $(M_1 \times \cdots \times M_n, G_1 \times \cdots \times G_n)$ is a Riemannian homogeneous space as well (equipped with the obvious action).

Now we would like to prove, that the space of Riemannian metrics, which are invariant under $G \subseteq \mathrm{Diff}(M)$ is not empty. Indeed, we will see later, that the space of such metrics is a symmetric space, hence finite–dimensional. Moreover these metrics will be a useful tool to prove different statements about geometric topics for Riemannian homogeneous spaces. Although the following proposition is well–known, we would like to recall the proof anyways.

Proposition 1.4. *There exists a G–invariant Riemannian metric μ on (M, G), i.e. for all $g \in G$ we have $g^*(\mu) = \mu$, where $g^*(\mu)$ is the pullback of the metric μ by g seen as a diffeomorphism of M.*

Proof. Note that K acts linear on TM_{m_0} by Remark 1.2. First let s_K be a K–invariant euclidean inner product on TM_{m_0}. We will prove at the end the existence of such inner products. Of course set now $\mu_{m_0}(\xi_1, \xi_2) := s_K(\xi_1, \xi_2)$ for $\xi_1, \xi_2 \in TM_{m_0}$ and for $m \in M$ we define

$$\mu_m(\eta_1, \eta_2) := \mu_{m_0}(g.\eta_1, g.\eta_2)$$

for $g \in G$ such that $g.m = m_0$. This is well–defined, since for any other $g' \in G$ with $g'.m = m_0$ we have $g'g^{-1} \in K$ and because s_K is K–invariant. This makes μ to an inner product at every point $m \in M$. So it remains to show, that μ is smooth. Around every point in M there is an open neighborhood U, such that there is a section $\sigma\colon U \to G$ of $p\colon G \to M$. Suppose X, Y are vector fields on U. Then the map $m \mapsto \mu_{m_0}(\sigma(m)^{-1}.X_m, \sigma(m)^{-1}.Y_m)$ is smooth on U. Obviously μ is G–invariant, hence it remains to show that there exists such an s_K.

Let s be an arbitrary euclidean inner product on TM_{m_0} and let $\mathrm{d}k$ be a right–invariant volume form on K. For $\xi_1, \xi_2 \in TM_{m_0}$ define

$$s_K(\xi_1, \xi_2) := \int_K s(k.\xi_1, k.\xi_2)\mathrm{d}k,$$

which is well–defined, since K is compact. Obviously s_K is K–invariant (since $\mathrm{d}k$ is K–invariant) and defines an euclidean inner product on TM_{m_0}. ∎

Corollary 1.5. *There is an embedding of K into $\mathbf{O}(n)$, hence K is a Lie subgroup of $\mathbf{O}(n)$.*

Proof. Let μ be a G–invariant metric. Then the map $\Phi\colon K \to \mathrm{Aut}(TM_{m_0})$, $k \mapsto (Dk)_{m_0}$ is a smooth group homomorphism. But by Proposition 1.4 $\Phi(k)$ is a linear isometry of (TM_{m_0}, μ_{m_0}), hence $\Phi(k) \in \mathbf{O}(TM_{m_0}, \mu) \cong \mathbf{O}(n)$. Note that every element of K is uniquely determined by the linear map $(Dk)_{m_0}$, therefore Φ is injective, and since K is compact, Φ is an embedding. ∎

Remark 1.6. (a) In the proof of Proposition 1.4 we saw, that every K–invariant scalar product on TM_{m_0} induces a G–invariant metric on M. Of course vice versa every G–invariant metric gives a K–invariant inner product on TM_{m_0}. This will be the key to see that the space of G–invariant metrics on M is a symmetric space.

(b) Let μ be a G–invariant metric for a Riemannian homogeneous space (M, G). Then by [MS39] the group of isometries $I(M, \mu) \subseteq \mathrm{Diff}(M)$ equipped with the CO–topology carries a finite–dimensional Lie group structure. Furthermore it follows from [KN63, p.48] that $I(M, \mu)$ is closed in $\mathrm{Diff}(M)$. We will show that G is closed in $I(M, \mu)$, so we can see G as a closed Lie subgroup of the group of isometries of the Riemannian manifold (M, μ). Of course it follows then that G is closed in $\mathrm{Diff}(M)$ as well.

Proposition 1.7. *Let μ be any G–invariant metric on (M, G). Then G is a closed subgroup of $I(M, \mu)$.*

Proof. Here we want to distinguish between G and its image under δ (see Remark 1.2), because there are two different topologies involved in the proof, namely that of G and that of $I(M, \mu) \subset \mathrm{Diff}(M)$. Let $(g_n)_n$ be a sequence in G such that $(\delta(g_n))_n$ converges in $I(M, \mu)$, i.e. in the CO–topology. Let $g \in G$ be such that $f\delta(g)(m_0) = m_0$ and let C be a compact neighborhood around m_0. By Remark 1.2 G is a principal fiber bundle over M with compact fiber, hence $p^{-1}(C)$ is compact. Now furthermore we have that (see again e.g. in [Sch08, p.15]) $(g_n g).m_0$ converges to m_0 and hence, we can assume $(g_n g).m_0 \in C$ and therefore $g_n g \in p^{-1}(C)$ for all $n \in \mathbb{N}$. Because $p^{-1}(C)$ is compact we can assume that $g_n g$ converges to some $k_0 \in K$ in G. By Remark 1.2 δ is continuous and we may compute

$$f\delta(g) = \lim_{n\to\infty} \delta(g_n g) = \delta\left(\lim_{n\to\infty} g_n g\right) = \delta(k_0),$$

hence f lies in the image of δ. ∎

Corollary 1.8. *The action θ of G on M is a proper action.*

Proof. For a definition of proper action see Definition B.1 in Appendix B. From [Sch08] we have that $I(M, \mu)$ is acting proper on M for a G–invariant metric μ. Since the action of $G \subset I(M, \mu)$ is the restricted action of the full isometry group of μ and $G \times M$ is a closed subspace of $I(M, \mu) \times M$, we have that G acts properly on M (because restriction of proper maps on closed subsets are again proper maps). ∎

We mentioned before that G is closed in Diff(M), which follows now from Proposition 1.7 and the fact that isometry groups are closed in Diff(M). Moreover the proposition above shows, that δ is an embedding of G in an isometry group of a G–invariant metric.

One aim of this work will be to classify the homogeneous Riemannian spaces in dimension 3. This means we must determine the possible pairs (M, G). Note that G must not be connected, but it will be easier to look first at the connected component of the identity of G which is a closed subgroup and acts still transitive on M. After this is done, one has to deduce the whole group G which can act on M, such that (M, G) is a Riemannian homogeneous spaces. But first we would like to give a definition, when two Riemannian homogeneous spaces are isomorphic in order to speak about a classification. This leads to the next

Definition 1.9. Let (M_i, G_i) $i = 1, 2$ be two Riemannian homogeneous spaces . Then we say that (M_1, G_1) is *isomorphic* or *equivariant diffeomorphic* to (M_2, G_2), if there exists a diffeomorphism $F\colon M_1 \to M_2$ and a group isomorphism $f\colon G_1 \to G_2$, such that for all $m_1 \in M_1$ and $g_1 \in G_1$ we have $F(g_1.m_1) = f(g_1).F(m_1)$.

And as mentioned above we have the

Proposition 1.10. *If G_0 is the connected component of G which contains the neutral element e, then G_0 is a closed Lie subgroup of G and (M, G_0) is a Riemannian homogeneous space with the restricted action of G on M to G_0.*

Proof. It is clear that G_0 is closed in G. Hence, to show that G_0 is a Lie subgroup of G, it is sufficient to show that G_0 is a subgroup. Let $g_1, g_2 \in G_0$ and let α be a smooth curve connecting e with g_2. Then the curve α^{-1} lies entirely in G_0 again and connects e with g_2^{-1}, hence $g_2^{-1} \in G_0$. Furthermore the curve $g_1\alpha$ connects g_1 with g_1g_2 and therefore $g_1g_2 \in G_0$.

Now we restrict the action of G on M to G_0, which is of course still effective. Since isotropy groups are closed and since K is compact, we have that the isotropy group of G_0 in m_0, which is given by $G_0 \cap K$, is compact as well. So it remains to show that G_0 acts transitively on M. Because $p\colon G \to M$ is an open map, we have that $G_0.m_0$ is open in M. Let $R := M/G_0$ be the set of all disjoint orbits by the action of G_0 on M, which are all open sets (to see that, note that if $m = g.m_0$ for some $g \in G$, then $G_0.m = p(G_0g)$, which is clearly an open set). Since M is connected and $M = \dot{\bigcup}_{O \in R} O$ we deduce that $M = G_0.m_0$. ■

Remark 1.11. We showed that G_0 is a subgroup, but indeed G_0 is actually a normal subgroup of G: take $g \in G$, then $gG_0g^{-1} \subseteq G_0$, since in gG_0g^{-1} lies the element e, and the conjugation with g is a diffeomorphism of G meaning that gG_0g^{-1} must be a connected component again. Hence G/G_0 possesses a group structure which is as a set equal to $\pi_0(G)$, where $\pi_i(M)$ is the i–th ($i \in \mathbb{N}_0$) homotopy group of the manifold M. Therefore $\pi_0(G)$ has a group structure. Then it is easy to see, that the long exact homotopy sequence of fiber bundles (see in [Hat02, p. 376] for a very general version) can be continued in terms of exactness of groups to the 0–th homotopy. We obtain with Remark 1.2 the

Lemma 1.12. *For the fiber bundle $p\colon G \to M$ induced by (M, G) we have a long exact sequence of groups*

$$\begin{aligned} \dots &\to \pi_n(K, e) \to \pi_n(G, e) \to \pi_n(M, m_0) \to \pi_{n-1}(K, e) \to \dots \\ \dots &\to \pi_1(M, m_0) \to \pi_0(K, e) \to \pi_0(G, e) \to 1 \end{aligned}$$

This implies a useful

Corollary 1.13. *For (M, G) with M simply connected we conclude by Lemma 1.12 $\pi_0(K) \cong \pi_0(G)$ and since K is compact there are only finitely many connected components of G. Furthermore if G is connected, so is K.*

Example 1.14. Suppose now $(M, G_0) = (S^n, \mathbf{SO}(n+1))$. Hence the isotropy group K is a closed subgroup of $\mathbf{O}(n)$ and has the same dimension. Then of course $K_0 = \mathbf{SO}(n)$ and the only other possibility for K is $\mathbf{O}(n)$ and therefore we have that (M, G) is isomorphic to either $(S^n, \mathbf{SO}(n+1))$ if $G_0 = G$ or $(S^n, \mathbf{O}(n+1))$ otherwise.

Finally we would like to phrase and prove a well–known proposition which concerns the orientability of Riemannian homogeneous spaces .

Proposition 1.15. *(M, G) is orientable if K is connected. Furthermore if there is an orientation Ω on M and μ a G–invariant metric, then the Riemannian volume form* vol *is G–invariant.*

Proof. Choose a non–zero n–form $\Omega_{m_0} \in \bigwedge^d(TM^*_{m_0})$ ($d := d_M = \dim M$). Then since K is a connected subgroup of $\mathbf{SO}(d)$ the action of K on $\bigwedge^d(TM^*_{m_0})$ leaves Ω_{m_0} invariant. For a $g \in G$ with $g.m = m_0$ where $m \in M$ we define $\Omega_m := g^*(\Omega_{m_0})$. This definition is well–defined, since for any $g' \in G$ with $g'm = m_0$ there is a $k \in K$ with $g' = kg$. Hence $(g')^*(\Omega_{m_0}) = g^*(k^*(\Omega_{m_0})) = g^*(\Omega_{m_0})$. As in Proposition 1.4 a local section of the K–principal bundle $p\colon G \to M$ shows that Ω is a smooth d–form on M and obviously Ω has no zeros. Hence the bundle $\bigwedge^d TM^*$ is trivial and Ω represents an orientation on M. If Ω is an orientation on (M, G) and μ a G–invariant metric, then the Riemannian volume form is defined by $\mathrm{vol} = *(1)$, where $*$ is the Hodge star operator on the oriented Riemannian manifold (M, μ) with orientation Ω and 1 is the constant function with value 1 on M. Since $g \in G$ is an isometry on (M, μ) the star operator commutes with the action of g, i.e. $g^*(*\eta) = *(g^*(\eta))$ for $\eta \in \bigwedge^j TM^*$, $j = 0, \ldots, n$. Clearly the constant function 1 on M is G–invariant and therefore vol is G–invariant. ■

Geometry of Riemannian homogeneous spaces

Now let us proceed with Riemannian homogeneous spaces equipped with G–invariant metrics. The geometric intuition for Riemannian homogeneous spaces should be the following: geometric objects (e.g. tensors which depend on a G–invariant metric) do depend only by their value in a single point. To support this geometric intuition we prove first the

Proposition 1.16. *Let μ a G–invariant metric for (M, G). Then (M, μ) is a complete Riemannian manifold.*

Proof. It is sufficient to prove that there is an $\varepsilon > 0$ such that for all $m \in M$ and all $\eta \in TM_m$ with $|\eta| = 1$ the geodesic through m with tangent vector η, denoted by γ_η lives at least on the interval $(-\varepsilon, \varepsilon)$.

Suppose first, that this statement is true. Then all geodesics γ_η with $|\eta| = 1$ are defined on the reals $\mathbb{R}$. Let γ_η be a geodesic with unit vector η and let (t_-, t_+) be the maximal interval on which γ_η is defined. If $t_+ < \infty$ then choose $t_0 < t_+$ such that $t_+ - t_0 < \varepsilon/2$. The geodesic γ with $\gamma(0) = \gamma_\eta(t_0)$ and $\dot{\gamma}(0) = \dot{\gamma}_\eta(t_0)$ (which is a vector of unit length) lives at least on the interval $(-\varepsilon, \varepsilon)$. By the uniqueness of geodesics γ_η lives at least on $(t_-, t_+ + \varepsilon/2)$, which is a contradiction to the maximality of t_+. Same argument holds for t_-. If γ_η is a geodesic such that $\eta \neq 0$ is not of unit length and if (t_-, t_+) is the maximal interval of existence, then the geodesic $\gamma_{\eta/|\eta|}$ is defined on $(t_-|\eta|, t_+|\eta|) = \mathbb{R}$, hence γ_η is defined on the entire real line.

Now it remains to show the claim at the beginning of the proposition. By [Car92, p.64] the claim is true for m_0 and all $\xi \in TM_{m_0}$ of unit length. Let γ_η be a geodesic starting at $m \in M$ with unit vector $\eta \in TM_m$ defined on the interval I. Then there is a $g \in G$ and a unit vector $\xi \in TM_{m_0}$ such that $g.m_0 = m$ and $g.\xi = \eta$. Since g maps geodesics into geodesics it follows that $g.\gamma_\xi$ is a geodesic which is defined at least on the interval $(-\varepsilon, \varepsilon)$. So we have now $g.\gamma_\xi(0) = m$ and the tangent vector of $g.\gamma_\xi$ in 0 is $g.\xi = \eta$, hence by uniqueness of geodesics $g.\gamma_\xi = \gamma_\eta$ and $(-\varepsilon, \varepsilon) \subseteq I$. ■

Definition 1.17. For $p \in \mathbb{N}_0$ we define $\mathcal{M}_p$ as the set of $(0, p)$–tensor fields on M. We call a tensor $T \in \mathcal{M}_p$ on (M, G) *G–invariant* if for all $g \in G$ we have $g^*(T) = T$. The set of all those G–invariant p–linear forms on (M, G) will be denoted by $\mathcal{M}_p^G$. Furthermore if a group K is acting linear on a vector space V, we denote by $\mathcal{M}_p^K(V)$ the set of all K–invariant p–linear forms on V.

Proposition 1.18. *On* (M, G) *we have* $\mathcal{M}_p^G \cong \mathcal{M}_p^K(TM_{m_0})$ *as vector spaces.*

Proof. Define the $\mathbb{R}$–linear evaluation map $\Phi\colon \mathcal{M}_p^G \to \mathcal{M}_p^K(TM_{m_0})$ by $T \mapsto T_{m_0}$. First Φ is well–defined since T is G-invariant and therefore $\Phi(T)$ is invariant by the linear action of K on TM_{m_0}. Clearly Φ is injective, since if $\Phi(T) = 0$ then by the G–invariance we conclude that $T = 0$. Copying the first part of the proof of Proposition 1.4 shows that Φ is onto. ∎

Next we show that taking traces of G–invariant tensors with respect to G–invariant metrics gives us G–invariant tensors back.

Proposition 1.19. *Let* $T \in \mathcal{M}_p^G$ $(p \geq 2)$ *and let* $\operatorname{tr}_\mu^{k,l} T$ $(k, l \leq p)$ *be the trace of* T *in the k-th and l-th entry with respect to* μ*. Then* $\operatorname{tr}_\mu^{k,l} T \in \mathcal{M}_{p-2}^G$.

Proof. Let $m \in M$ and $(e_1, \ldots, e_d)$ an orthonormal basis for (TM_m, μ_m) (recall $d = d_M = \dim M$). Let $\xi_1, \ldots, \xi_{p-2} \in TM_m$ and set $\operatorname{tr} T := \operatorname{tr}_\mu^{k,l} T$, where we assume for simplicity $k = 1, l = 2$. Observe that $(g.e_1, \ldots g.e_d)$ is an orthonormal basis for $(TM_{g.m}, \mu_{g.m})$ and therefore

$$
\begin{aligned}
g^*(\operatorname{tr} T)(\xi_1, \ldots, \xi_{p-2}) &= \sum_{i=1}^d T(g.e_i, g.e_i, g.\xi_1, \ldots, g.\xi_{p-2}) \\
&= \sum_{i=1}^d g^*(T)(e_i, e_i, \xi_1 \ldots, \xi_{p-2}) \\
&= \sum_{i=1}^d T(e_i, e_i, \xi_1 \ldots, \xi_{p-2}) = \operatorname{tr} T(\xi_1, \ldots, \xi_{p-2}),
\end{aligned}
$$

for $g \in G$. ∎

Corollary 1.20. *Let* μ *be a G–invariant metric for* (M, G)*. Then the Ricci curvature* Ric *of* μ *is G–invariant and the scalar curvature R is constant on M.*

Proof. Let $\mathrm{Rm} \in \mathcal{M}_4$ be the Riemann tensor of μ. We show $\mathrm{Rm} \in \mathcal{M}_4^G$ and the corollary follows with Proposition 1.19. Let ∇ be the Levi–Civita connection of μ. Then, since g is an isometry, we have for X, Y vector fields on M, $\nabla_{g_*X} g_*Y = g_*(\nabla_X Y)$ where g_*X is the pushforward of X by the diffeomorphism $g \in G$ and of course we have $[g_*X, g_*Y] = g_*[X, Y]$.

Then clearly we have $\mathrm{Rm}(g_*X, g_*Y, g_*W, g_*Z) = \mathrm{Rm}(X, Y, W, Z) \circ g^{-1}$ for vector fields X, Y, W and Z on M. In addition we can write $g^*(\mathrm{Rm})(X, Y, W, Z) = \mathrm{Rm}(g_*X, g_*Y, g_*W, g_*Z) \circ g$ and the corollary follows. ■

Proposition 1.21. *Let $T \in \mathcal{M}_p^G$ and ∇ the Levi–Civita connection to μ. Then we have $\nabla T \in \mathcal{M}_p^G$.*

Proof. Let $\xi, \eta_1, \dots, \eta_p \in TM_m$ and let $X, Y_1, \dots, Y_p$ be continuations on M respectively. If $g \in G$ and $g_*(Y)$ denotes the pushforward of a vector field Y we obtain $T(g_*(Y_1), \dots, g_*(Y_p)) = T(Y_1, \dots, Y_p) \circ g^{-1}$ as functions on M since T is G–invariant. It follows that

$$g^*(\nabla T)(\xi, \eta_1, \dots \eta_p) = (\nabla_{g_*(X)}T)(g_*(Y_1), \dots, g_*(Y_p))(g.m)$$

and

$$\begin{aligned}
(\nabla_{g_*(X)}T)(g_*(Y_1), \dots, g_*(Y_p)) &= g_*(X)(T(g_*(Y_1), \dots, g_*(Y_p)) - \sum_{i=1}^{p} T(g_*(Y_1), \dots, \nabla_{g_*(X)}g_*(Y_i), \dots, Y_p) \\
&= g_*(X)(T(g_*(Y_1), \dots, g_*(Y_p)) - \sum_{i=1}^{p} T(Y_1, \dots, \nabla_X Y_i, \dots, Y_p) \circ g^{-1} \\
&= X(T(Y_1, \dots, Y_p)) \circ g^{-1} - \sum_{i=1}^{p} T(Y_1, \dots, \nabla_X Y_i, \dots, Y_p) \circ g^{-1}
\end{aligned}$$

hence

$$\begin{aligned}
g^*(\nabla T)(\xi, \eta_1, \dots \eta_p) &= (\nabla_{g_*(X)}T)(g_*(Y_1), \dots, g_*(Y_p))(g.m) = (\nabla_X T(Y_1, \dots, Y_p))(m) \\
&= (\nabla_\xi T)(\eta_1, \dots, \eta_p)
\end{aligned}$$

■

A crucial fact for Riemannian homogeneous space is that geometric quantities induced by a G–invariant metric depend not alone on a single point but even more on the algebraic structure of G, i.e. on Lie algebra $\mathfrak{g}$. But to understand this we have to make some observations first.

Proposition 1.22. *For the adjoint representation of G, $\mathrm{Ad}\colon G \to \mathbf{Aut}(\mathfrak{g})$ we have $P \circ \mathrm{Ad}_k = k \circ P$ for all $k \in K$, where P is defined in Remark 1.2*

Proof. Let $g\colon (-\varepsilon, \varepsilon) \to G$ be a curve with $g(0) = e$. Then for $k \in K$ we have for all t

$$p(kg(t)k^{-1}) = kg(t).m_0 = k.(p(g(t))).$$

Taking the derivative at $t = 0$ on both sides proves the proposition. ■

Proposition 1.23. *There is a linear,* Ad_K*–invariant subspace* $\mathfrak{p}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{p}$ *is a complement for* $\mathfrak{k}$*.*

Proof. Since K is compact and acts linear on $\mathfrak{g}$ via Ad, we can - as in Proposition 1.4 - construct an Ad_K–invariant scalar product s_K on $\mathfrak{g}$. Taking $\mathfrak{p} := \mathfrak{k}^\perp$ completes the proof. ∎

Remark 1.24. For this section we want to fix once and for all such a $\mathfrak{p}$, which is via P isomorphic to TM_{m_0}. Now every G–invariant tensor T corresponds to an Ad_K–invariant tensor t on $\mathfrak{p}$: If $T \in \mathcal{M}_p^G$ is given, then set $t := P^*(\Phi(T))$ and with Proposition 1.22 we have for all $k \in K$

$$\mathrm{Ad}_k^* t = (P \circ \mathrm{Ad}_k)^* \hat{T} = (k \circ P)^* \hat{T} = P^* \hat{T} = t.$$

where $\hat{T} = \Phi(T)$. Vice versa given an Ad_K–invariant tensor t on $\mathfrak{p}$, the tensor $T_{m_0} := ((P|\mathfrak{p})^{-1})^* t$ is K–invariant, since

$$k^* T_{m_0} = ((P|\mathfrak{p})^{-1} \circ k)^* t = (\mathrm{Ad}_k \circ (P|\mathfrak{p})^{-1})^* t = T_{m_0}.$$

by Propostition 1.22 again and with Proposition 1.18 this tensor defines a G–invariant tensor T on M. Consequently a G–invariant metric μ on M defines a Ad_K–invariant scalar product $P^*\mu$ on $\mathfrak{p}$ which we will denote henceforth by s.

And finally for the formulas for the curvatures (which we will not prove, since there are complete proofs given in [Bes08]) we have to introduce some basic notations for Lie groups.

Definition 1.25. We define for $v, w \in \mathfrak{g}$ the *Killing–Cartan form* by $B(v, w) := \mathrm{tr}\,(\mathrm{ad}_v \circ \mathrm{ad}_w)$, where $\mathrm{ad}\colon \mathfrak{g} \to \mathbf{Aut}(\mathfrak{g})$ is the adjoint representation of the Lie algebra of $\mathfrak{g}$, $ad_v(w) = [v, w]$. For a Riemannian homogeneous space (M, G) we let $z \in \mathfrak{p}$ such that $s(z, v) = \mathrm{tr}\,(\mathrm{ad}_v)$ for all $v \in \mathfrak{p}$.

And this leads to

Lemma 1.26 (see Ch. 7C in [Bes08]). *Let (M, G) be a Riemannian homogeneous space and μ a G–invariant metric on M. Let* Ric *be the Ricci–tensor of μ, which is a G–invariant symmetric tensor. By Remark 1.24 let* ric *the corresponding Ad_K–invariant symmetric bilinear form on $\mathfrak{p}$. Then we have for all $v \in \mathfrak{p}$ and $z \in \mathfrak{p}$ like in the definition above*

$$\mathrm{ric}(v,v) = -\frac{1}{2}\sum_{i=1}^{p} \left|[v,e_i]_{\mathfrak{p}}\right|^2 - \frac{1}{2}B(v,v) + \frac{1}{4}\sum_{i,j=1}^{p} s\left([e_i,e_j]_{\mathfrak{p}}, v\right)^2 - s\left([z,v]_{\mathfrak{p}}, v\right)$$

where $[v,w]_{\mathfrak{p}}$ means the $\mathfrak{p}$–component of $[v,w]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and where $(e_1, \ldots, e_p)$ is an orthonormal basis of $(\mathfrak{p}, s)$ with $p := \dim \mathfrak{p}$.

As noticed in Remark 1.24 the scalar curvature R of μ is constant on M and is given by

$$R = -\frac{1}{4}\sum_{i,j=1}^{p} \left|[e_i,e_j]_{\mathfrak{p}}\right|^2 - \frac{1}{2}\mathrm{tr}_s B - |z|^2$$

where tr_s is the trace taken with respect to the scalar product s.

Homogeneous vector fields

There are two special kinds of vector fields on a Riemannian homogeneous space . One kind which is called the *fundamental vector field* and can always be defined for a smooth transformation group. The other type we will call *homogeneous vector field* which exists in general not on every Riemannian homogeneous space unless it is the zero vector field. Let us start with the definition of the fundamental vector fields.

Definition 1.27. For $v \in \mathfrak{g}$ we define the corresponding *fundamental vector field X_v* of v on M by

$$(X_v)_m := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(e^{-tv}.m\right)$$

for $m \in M$ where $v \mapsto e^v$ is the exponential map of G.

Proposition 1.28. *Let $v \in \mathfrak{g}$ and X_v its fundamental vector field on M.*

(a) The map $\Phi\colon \mathfrak{g} \to \mathcal{V}(M)$, $\Phi(v) := X_v$ is an injective Lie algebra homomorphism, where $\mathcal{V}(M)$ is the space of vector fields on M.

(b) For any G–invariant metric μ the fundamental vector fields are Killing fields on (M, μ).

Proof. For $m \in M$ define the map $\theta^m : G \to M$ as $\theta^m(g) := g^{-1}.m$. First let us note, that X_v is smooth, since it is the vector field to the flow $(t, m) \mapsto e^{-tv}.m$. Furthermore by definition we have $(X_v)_m = D(\theta^m)_e(v_e)$ and therefore we see that Φ is a linear map, where we consider v as a left invariant vector field on G. But Φ is injective since $X_v = 0$ implies that every point of the flow of X_v is a stationary point, hence $e^{-tv}.m = m$ for all $t \in \mathbb{R}$ and all $m \in M$. And since G acts effectively on M we have e^{-tv} is equal to the neutral element of G for all $t \in \mathbb{R}$ which forces v to be zero. This shows that Φ is an injective homomorphism of vector spaces. For $g \in G$ we compute ($L_g : G \to G$ shall denote the left translation on G by g)

$$D(\theta^m)_g(v_g) = D(\theta^m \circ L_g)_e(v_e) = D(\theta^{g^{-1}.m})_e(v_e) = (X_v)_{g^{-1}.m}$$

and this shows that v and X_v are θ^m–related which for $w \in \mathfrak{g}$ implies $(X_{[v,w]})_m = [X_v, X_w]_m$ for all $m \in M$. Hence Φ is a Lie homomorphism and $\mathfrak{g}$ is embedded in $\mathcal{V}(M)$ via Φ as a Lie subalgebra. This completes the proof of (a).

For (b) just choose a G–invariant metric. Since G is a closed subgroup of the isometry group of this G–invariant metric we have, that the flows of the fundamental vector fields are isometries. Hence the corresponding vector fields are Killing fields. ■

Let us now discuss the other class of special vector fields on a Riemannian homogeneous space .

Definition 1.29. We call a vector field $X \in \mathcal{V}(M)$ a *G–invariant vector field* or a *homogeneous vector field* on (M, G) if for every $g \in G$ we have $g_*(X) = X$, i.e. $g_*(X)_m = (Dg)_{g^{-1}.m}(X_{g^{-1}.m}) = X_m$ for all $m \in M$ (or equivalently $g.X_m = X_{g.m}$). Furthermore we denote the vector space of homogeneous vector fields by $\mathcal{V}^G(M)$.

Remark 1.30. Of course a non–trivial homogeneous vector field has no zeros and if a G–invariant metric is chosen, it has constant length. Hence the tangent bundle gives a restriction to the existence of homogeneous vector fields. For example every such tangent bundle has therefore vanishing Euler class if we choose an orientation with Proposition 1.15 and consequently some Riemannian homogeneous space like the even–dimensional spheres do not possess such non–trivial vector fields. As always for Riemannian homogeneous spaces the existence of such vector fields can be expressed in algebraic terms.

Proposition 1.31. *We have $\mathcal{V}^G(M) \cong \cap_{k \in K} \mathrm{Eig}(k, 1)$, where $\mathrm{Eig}(k, 1)$ is the eigenspace of the linear map $k : TM_{m_0} \to TM_{m_0}$ with eigenvalue 1. With Proposition 1.22 we conclude that $\mathcal{V}^G(M) \cong \cap_{k \in K} \mathrm{Eig}(\mathrm{Ad}_k|\mathfrak{p}, 1) \subseteq \mathfrak{p}$.*

Proof. Consider the $\mathbb{R}$–linear evaluation map $\Phi\colon \mathcal{V}^G(M) \to TM_{m_0}$, $X \mapsto X_{m_0}$. Then $\Phi(X)$ is K–invariant which means $k\Phi(X) = \Phi(X)$ for all $k \in K$ hence $\Phi(X) \in \mathrm{Eig}(K,1) := \cap_{k\in K}\mathrm{Eig}(k,1)$. If $\Phi(X) = 0$ then $X = 0$ by the G–invariance. On the other hand if $\xi \in \mathrm{Eig}(K,1)$ the map $m \mapsto X_m := g.\xi$ for $g \in G$ with $g.m_0 = m$ defines a vector field on M. It is well–defined since ξ is K–invariant and X is smooth if we pick a local section like in the proof of Proposition 1.4. Hence Φ is an isomorphism onto $\mathrm{Eig}(K,1)$. ∎

Proposition 1.32. *Let (M_i, G_i) for $i = 1, 2$ be two Riemannian homogeneous spaces which are isomorphic. If $f\colon M_1 \to M_2$ is the diffeomorphism corresponding to the isomorphism and X a homogeneous vector field on M_1 then the pushforward of X under f, $f_*(X)$ is a homogeneous vector field on M_2.*

Proof. Let $F\colon G_1 \to G_2$ denote a Lie group isomorphism such that $f(g_1.m_1) = F(g_1).f(m_1)$ for $m_1 \in M_1$ and $g_1 \in G_1$. For $g_2 \in G_2$ there is a $g_1 \in G_1$ let $F(g_1) = g_2$ thus $g_2 \circ f = f \circ g_1$. Therefore we obtain

$$(g_2)_*(f_*(X)) = (g_2 \circ f)_*(X) = (f \circ g_1)_*(X) = f_*((g_1)_*(X)) = f_*(X).$$

for each $g_2 \in G_2$.

∎

Proposition 1.33. *Homogeneous vector fields are complete and their flow commute with elements of G.*

Proof. Let X be a homogeneous vector field on (M, G). Like in the proof of Proposition 1.16 it is sufficient to find an $\varepsilon > 0$ such that every integral curve of a homogeneous vector field exists at least on the interval $(-\varepsilon, \varepsilon)$. Clearly there is an $\varepsilon > 0$ such that $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = m_0$ fulfills $\dot\alpha = X \circ \alpha$. Let $m \in M$ be a arbitrary point and $g \in G$ such that $g.m_0 = m$. Then for $\beta := g \circ \alpha$ we have

$$\dot\beta(t) = g.X_{\alpha(t)} = X_{g.\alpha(t)} = (X \circ \beta)(t)$$

for all $t \in (-\varepsilon, \varepsilon)$. Hence β is a solution to X with $\beta(0) = m$ which exists at least on $(-\varepsilon, \varepsilon)$. This also shows that the flow φ^t of X commutes with a $g \in G$, i.e. $g \circ \varphi^t = \varphi^t \circ g$ for all $t \in \mathbb{R}$. ∎

Note that every homogeneous vector field defines by Proposition 1.31 a vector $v \in \mathfrak{p} \subseteq \mathfrak{g}$ but the corresponding fundamental vector field V must not coincide with the homogeneous vector field. The obstruction is that the corresponding flow has to be a family of isometries for some G–invariant metric.

Proposition 1.34. *Let $v \in \mathfrak{g}$. If X_v is a homogeneous vector field then v lies in the center of $\mathfrak{g}$. If G is connected the converse is also true. Moreover if $Z(G)$ denotes the center of G and $\mathfrak{z}$ its Lie algebra, then there is an embedding of $\mathfrak{z}$ into $\mathcal{V}^G(M)$, namely sending $z \in \mathfrak{z}$ to its fundamental vector field X_z on M.*

Proof. Suppose that X_v is a homogeneous vector field and let X_w be the fundamental vector field for $w \in \mathfrak{g}$. Since X_v is homogeneous the flow of X_v commutes with all elements of G, in particular with the flow of X_w. Hence $[X_v, X_w] = 0$ which implies $[v, w] = 0$ for all $w \in \mathfrak{g}$.

Now suppose G is connected and $v \in \mathfrak{z}$. There are open neighborhoods U' and U in $\mathfrak{g}$ and G respectively such that the exponential map is a diffeomorphism from U' to U with $0 \in U'$. Then for every $g \in U$ there is a $w \in U'$ such that $e^w = g$ and it follows, since $[v, w] = 0$, that $ge^{tv}g^{-1} = e^w e^{tv} e^{-w} = e^{tv}$ for all $t \in \mathbb{R}$. Hence e^{tv} commutes with all elements of U and since G is connected, U generates G and therefore e^{tv} commutes with all elements of G, i.e. $e^{tv} \in Z(G)$ for all $t \in \mathbb{R}$. Clearly the flow of X_v commutes with the action of G on M and this implies, that X_v is a homogeneous vector field. This shows also that $\mathfrak{z}$ embeds into $\mathcal{V}^G(M)$. ■

Corollary 1.35. *If $\dim Z(G) > 0$, then $\mathcal{V}^G(M)$ is non–trivial.*

Remark 1.36.

(a) Suppose $\mathcal{V}^G(M)$ is not empty. Then, since $\mathcal{V}^G(M)$ has finite dimension, say d, we choose a basis $X_1, \dots, X_d$. The distribution $\mathcal{D} := \operatorname{span}_{\mathbb{R}}(X_1, \dots, X_d)$ of TM do not depend of the chosen basis, hence TM possess a d–dimensional trivial subbundle associated to the $\mathbb{R}$–vector space of homogeneous vector fields $\mathcal{V}^G(M)$. Thereby note that $\mathcal{D}$ is an involutive distribution: first, if $g \in G$ then $g.[X_i, X_j] = [g_* X_i, g_* X_j] \circ g = [X_i, X_j] \circ g$, hence the Lie bracket of homogeneous vector fields is again a homogeneous vector field and therefore $[X_i, X_j]$ takes values in $\mathcal{D}$. If X is now an arbitrary vector field with values in $\mathcal{D}$ then $X = \sum_{i=1}^d f_i X_i$ for smooth functions f_i on M. And of course one has that $[f_i X_i, f_j X_j] = f_i f_j [X_i, X_j] + f_i X_i(f_j) X_j + f_j X_j(f_i) X_i$ takes values in $\mathcal{D}$ as well. If we sum up the previous arguments we have that the Lie bracket $[X, Y]$ of vector fields X and Y with values in $\mathcal{D}$ takes values in $\mathcal{D}$ hence $\mathcal{D}$ is involutive. By [War83, p. 48] there exists through every point $m \in M$ a maximal connected integral manifold of $\mathcal{D}$.

At this point we want to emphasize the different notions of submanifolds. We call a pair (S, σ) where S is a manifold and $\sigma \colon S \to M$ is a smooth injective immersion simply a *submanifold of* M. If σ is an embedding, i.e. a one–to–one immersion, which is also a homeomorphism, then we call (S, σ) an *embedded submanifold of M*. If we say that a subset $S \subset M$ is a (embedded) submanifold we will always mean that the pair (S, i), where $i \colon S \to M$ is the inclusion is a (embedded) submanifold.

(b) For $m \in M$ we have $\mathcal{D}_m = \mathrm{Eig}(K_m, 1)$ where K_m is the isotropy group in m by the action of G on M and $\mathrm{Eig}(K_m, 1) = \cap_{k \in K_m} \mathrm{Eig}(k, 1)$. To see this choose a $\xi \in \mathcal{D}_m$ and a homogeneous vector field X such that $X_m = \xi$. Surely there is a $g \in G$ with $g.m = m_0$ and therefore we obtain $K_m = gKg^{-1}$. But this implies for all $k \in K_m$

$$k.X_m = (kg).X_{m_0} = (gk').X_{m_0} = X_m$$

for a $k' \in K$ and we see that $\mathcal{D}_m \subset \mathrm{Eig}(K_m, 1)$. On the other hand if $\xi \in \mathrm{Eig}(K_m, 1)$ then the vector $g.\xi \in TM_{m_0}$ is invariant under K as the computation

$$k.(g.\xi) = (kg).\xi = (gk').\xi = g.\xi$$

shows. For the homogeneous vector field X with $X_{m_0} = g.\xi$ we have $X_m = \xi$ and this implies $\mathrm{Eig}(K_m, 1) \subset \mathcal{D}_m$.

(c) Fix $m \in M$ and let $\mathrm{Fix}(K_m) \subset M$ be the fixed point set of the elements of the isotropy group in m, namely K_m, i.e. $\mathrm{Fix}(K_m) = \{m' \in M : k.m' = m' \text{ for all } k \in K_m\}$. Then F_m is a d–dimensional closed embedded submanifold of M. To see this choose a G–invariant metric μ on (M, G) then by [Kob95, p. 59] all connected components of F_m are closed, totally geodesic, embedded submanifolds of (M, μ).

Theorem 1.37. *We apply the notations of Remark 1.36 and we suppose $\mathcal{D}$ is not empty. Then the connected component F of $\mathrm{Fix}(K_m)$ with $m \in F$ is the maximal connected integral manifold for $\mathcal{D}$ through m. Moreover the following holds:*

(a) The normalizer of K_m in G, namely $N_G(K_m)$ acts transitively on $\mathrm{Fix}(K_m)$ with isotropy group K_m in m.

(b) There is a closed subgroup N_F of $N_G(K_m)$ such that N_F acts transitively on F with isotropy group K_m in m. Hence N_F has the same dimension as $N_G(K_m)$ but maybe different connected components. If K_m is connected then N_F is the identity component of $N_G(K_m)$.

(c) Since K_m is normal in N_F the action of N_F on F descends to a free and transitive action of $H := N_F/K_m$ on F.

Proof. By Remark 1.36 F is a closed embedded submanifold of M. First, let us show that F is indeed an integral manifold for $\mathcal{D}$ through m. Choose a $m' \in F$, $\xi' \in TF_{m'}$ and a curve

$\alpha\colon (-\varepsilon, \varepsilon) \to F$ ($\varepsilon > 0$) such that $\dot\alpha(0) = \xi'$. Hence α is invariant under K_m, i.e. $k \circ \alpha = \alpha$ for all $k \in K_m$ and this implies $k.\xi' = \xi'$ for all $k \in K_m$. Set $\xi := g^{-1}.\xi'$ for a $g \in G$ with $g.m = m'$. Now observe that K_m fixes the point m' and since both are conjugated to each other as well as $K_m \subset K_{m'}$ we get $K_m = K_{m'}$. But on the other hand K_m is conjugated to $K_{m'}$, say $K_m = g^{-1}K_{m'}g$, which implies $K_m = g^{-1}K_m g$ for every $g \in G$ with $g.m = m'$ and hence g lies in the normalizer of K_m in G. Now for all $k \in K_m$ we obtain

$$k.\xi = kg^{-1}.\xi' = g^{-1}k'.\xi' = \xi$$

and therefore there is a homogeneous vector field X with $X_m = \xi$. But this means that $\xi' = g.\xi = g.X_m = X_{g.m} \in \mathcal{D}_{m'}$ and consequently $TF_{m'} \subset \mathcal{D}_{m'}$. It remains to show $\mathcal{D}_{m'} \subset TF_{m'}$. So take an arbitrary $\xi' \in \mathcal{D}_{m'}$ and a corresponding homogeneous vector field X with $X_{m'} = \xi'$. Let α be the integral curve of X through m. Choose again a $g \in G$ with $g.m = m'$ and as seen already before $g \circ \alpha$ is the integral curve of X through m'. Now we conclude with Proposition 1.33 that $k \circ \alpha = \alpha$ for all $k \in K_m$, hence

$$k \circ g \circ \alpha = (kg) \circ \alpha = (gk') \circ \alpha = g \circ \alpha$$

for a $k' \in K_m$ and this implies that the image of $g \circ \alpha$ lies in F, since F is a connected component and $m' \in F$. Therefore $\xi' \in TF_{m'}$ and this verifies the claim that F in an integral manifold for $\mathcal{D}$ through m.

As mentioned in Remark 1.36 there is a maximal integral manifold (S, σ) through m for $\mathcal{D}$. Of course this must not be an embedded submanifold, which means, that $\sigma(S)$ must not have the subspace topology of M. Following [War83, p.48] we have $F \subset \sigma(S)$ by the maximality of S and since we showed that F is an integral manifold for $\mathcal{D}$ through m. We would like to show $\sigma(S) \subset F$. Note we proved already that an integral curve of a homogeneous vector field starting at a point in F stays in F. Therefore we will show that every point in $\sigma(S)$ can be connected by a broken integral path to m (see Appendix A for a definition of broken integral paths). Choose homogeneous vector fields $X_1, \ldots, X_d$ which span the distribution $\mathcal{D}$ and $Y_1, \ldots, Y_d$ vector fields on S such that $D\sigma(Y_j) = X_j \circ \sigma$ for all $j = 1, \ldots, d$. Since σ is an immersion the vector fields $Y_1, \ldots, Y_d$ trivialize the tangent bundle of S. By Lemma A.3 of the appendix we can connect every point $s \in S$ to $s_0 \in S$ with a broken integral path for $Y_1, \ldots, Y_d$ where s_0 is the unique point with $\sigma(s_0) = m$. Therefore every point in $\sigma(S)$ can be connected by a broken integral path for $X_1, \ldots, X_d$ with m and this implies that $\sigma(S) \subset F$ since the integral curves of homogeneous vector fields lie in F if we start those at a point in F. Finally this proves that $\sigma(S) = F$ and F is the unique (up to equivalence of submanifolds) maximal integral manifold for $\mathcal{D}$ through m.

We proceed proving part (a) of the theorem. First, $N_G(K_m)$ is indeed a closed Lie subgroup of G since K_m is closed. Now we point out that $N_G(K_m)$ acts on $\mathrm{Fix}(K_m)$. If $g \in N_G(K_m)$

then we have for $m' \in \mathrm{Fix}(K_m)$ that $k.g.m' = g.k'.m' = g.m'$ for a $k' \in K_m$ and this implies $g.m' \in \mathrm{Fix}(K_m)$. As we have seen before if $g \in G$ maps m to a point in $\mathrm{Fix}(K_m)$ we have that $gK_mg^{-1} = K_m$, i.e. $g \in N_G(K_m)$ and since G acts transitively on M it follows that $N_G(K_m)$ acts transitively on $\mathrm{Fix}(K_m)$. Clearly, by definition we have that $K_m \subset N_G(K_m)$, thus the isotropy group of $N_G(K_m)$ in m is K_m. This proves part (a).

Define now the subset $N := N_F := \{g \in G : g.m \in F\}$. We will show that N is a closed subgroup of $N_G(K_m)$ hence a closed subgroup of G as well. Clearly by the above discussion N is a subset of $N_G(K_m)$ since every element of N maps m into $F \subset \mathrm{Fix}(K_m)$. To prove that N is a indeed a group, we choose a G–invariant metric μ and with Remark 1.36 we have that F is a totally geodesic closed submanifold of M. Obviously F equipped with the metric induced by μ is a complete Riemannian manifold, since (M, μ) is complete. Now take $g \in N$ and a geodesic $\gamma\colon [0, 1] \to F$ connecting m and $g.m \in F$. Define $\gamma^-(t) := \gamma(1 - t)$, which is again a geodesic and therefore $g^{-1} \circ \gamma^-$ is a geodesic as well connecting m with $g^{-1}.m$. We have to show $g^{-1}.m \in F$. There is a homogeneous vector field X with $X_{g.m} = \dot\gamma(1)$ and therefore the derivative of $g^{-1} \circ \gamma^-$ at $t = 0$ is given by $-g^{-1}.X_{g.m} = -X_m$ and since F is totally geodesic we conclude that $g^{-1} \circ \gamma^-$ lies completely in F thus in particular $g^{-1}.m$. If we take another element $\hat g \in N$, then the geodesic $\hat g \circ \gamma$ connects $\hat g.m$ with $\hat g g.m$ and since γ lies in F and $\hat g \in N$ the curve $\hat g \circ \gamma$ lies in connected component of $\mathrm{Fix}(K_m)$ which contains m, namely F. This shows that $\hat g g.m \in F$ and finally N is indeed a subgroup of $N_G(K_m)$ or G either. Let (g_n) be a sequence in N which converges in $N_G(K_m)$ say to g_0. Then we conclude that $g_n.m$ converges to $g_0.m$ which lies in F because F is closed. Hence $g_0.m \in F$ and $g_0 \in N$. Indeed N acts on F since every point in F is given by $g.m$ for a suitable $g \in N$ and since N is a group we have an action of N on F. This argument show also that N acts transitively on F. Clearly K_m is a subgroup of N_F and as above the isotropy group of N in m is exactly K_m. This shows that $N_G(K_m)$ and N have the same dimension but possibly different connected component where $N_G(K_m)$ is the bigger group. If K_m is connected it follows from Lemma 1.12 that N is connected, since F is connected too. Then N must be the identity component of $N_G(K_m)$. This completes the proof of part (b). In addition we would like to note that the definition of N does not depend on m, since it easily seen that $N_F = \{g \in G : \mathrm{im}\,(g|F) \subset F\}$.

Clearly K_m is a normal subgroup of N and therefore $H := N/K_m$ is a group again. Since K_m is the isotropy group of N acting on F it follows that this action descends to a free and transitive action of the quotient H on F. This ends the proof of the theorem above. ■

Because the action of $H = N_F/K_m$ on F is free we would like to have, that this groups acts on the other integral manifolds of $\mathcal{D}$ free and transitive and properly as well, hence we can pass to the induced H–principal fiber bundle, where the base space is given by the orbit space of the H–action. But even if this is true, it is not clear in the case where H is not compact that the orbit space is a nice manifold, e.g. it can loose the Hausdorff property. We would like to discuss these questions in the following propositions and corollaries.

Proposition 1.38. *Let $\mathcal{D}$ be non–trivial. Then there is a free and proper action of the group $H := N_F/K$ (notations as in Theorem 1.37) on M, such that the orbits are the maximal connected integral manifolds of $\mathcal{D}$, where F is the maximal connected integral manifold through m_0 and K its isotropy group.*

Proof. We shall use the notation of Theorem 1.37. Let F_1 and F_2 be two maximal connected integral manifolds for $\mathcal{D}$. Let K_i denote the isotropy group of F_i $(i = 1, 2)$ and denote by N_i the group N_{F_i} of Theorem 1.37, hence $F_i \cong N_i/K_i$ induced by the action of N_i on F_i. First we show that if a $g \in G$ maps a point of F_1 into F_2, then g maps all points of F_1 into F_2. For this choose a G–invariant metric μ on (M, G) and remember that F_i is a totally geodesic and complete submanifold of (M, μ). Suppose that $g.m_1 = m_2$ for $m_i \in F_i$ (as always for $i = 1, 2$) and take an arbitrary $m \in F_1$, hence we would like to show $g.m \in F_2$. Since F_1 is complete there is a geodesic $\gamma\colon [0, 1] \to F_1$ connecting m_1 with m and furthermore there is a homogeneous vector field X such that $X_{m_1} = \dot\gamma(0)$. Now g is an isometry of (M, μ) and therefore $g\circ\gamma$ is a geodesic starting at m_2 with tangent vector $g.\dot\gamma(0) = g.X_{m_1} = X_{m_2} \in T(F_2)_{m_2}$. Hence $g \circ \gamma$ lies completely in F_2 and the endpoint is given by $g.m$, which proves our claim.

This helps us to show that N_1 is conjugated to N_2 in G. Let $c_g\colon N_1 \to G$ be the conjugation map by g restricted to N_1. Since N_1 and N_2 have the same dimension and are closed subgroups it is sufficient to show $c_g(N_1) \subset N_2$, i.e. $c_g(g_1).m_2 \in F_2$ for $g_1 \in N_1$. Let $g \in G$ be an element which maps F_1 into F_2 then we see indeed $c_g(g_1).m_2 \in F_2$. Since all isotropy groups of G acting on M are conjugated to each other we have with the same g that $c_g(K_1) = K_2$. But this means that H_1 is isomorphic to H_2, where $H_i = N_i/K_i$, thus it seems now possible that there a single group acting on M such that the orbits are the maximal connected integral manifolds for $\mathcal{D}$.

Let F be the maximal integral manifold through m_0 and K its isotropy group. Now define the map

$$\tilde\eta\colon N_F \times M \to M, \quad (g_F, m) \mapsto \theta(c_g(g_F), m) = c_g(g_F).m$$

for a $g \in G$ with $g.m_0 = m$. First of all this map is well–defined, since any other g' with $g'.m_0 = m$ differs from g by an element of K, i.e, $g' = gk$ for $k \in K$. But then we have

$$gkg_Fk^{-1}g^{-1}.m = gkg_Fg^{-1}.m = gg_Fg^{-1}.m$$

since $g^{-1}.m \in F$ and $g_Fg^{-1}.m \in F$. The map $\tilde\eta$ is smooth, which can been seen easily by taking a local section of the bundle $p\colon G \to M$. This map is indeed an action of H on M since θ is an action and the conjugation is a homomorphism. Now let us pass to the induced map $\eta\colon H \times M \to M$ given by $\eta(h, m) = \tilde\eta(g_F, m)$ for a $g_F \in N_F$ representing h in H. This is again well–defined, since any other representative differs from g_F by an element of K and the same computation as above shows that the definition does not depend on the choice of g_F. Clearly η is smooth and is an action for H on M.

Finally it remains to show, that the orbits of H on M are the maximal connected integral manifolds for $\mathcal{D}$. Since the action is free and proper we have that $H.m$ is an embedded submanifold of M diffeomorphic to H and therefore it is sufficient to show that $H.m = F_m$ where F_m is the maximal connected integral manifold through $m \in M$. But we have $H.m = gN_F g^{-1}.m = N_{F_m}.m = F_m$ by Theorem 1.37 and the statements proven above in this proposition (note that the dots represent different actions on M). ∎

Corollary 1.39. *The action of H on M from Proposition 1.38 induces an H–principal fiber bundle $\pi\colon M \to B = M/H$ such that B is diffeomorphic to the homogeneous space G/N_F. More precisely, G commutes with the action of H on M and descends to an action of G on B with isotropy group isomorphic to N_F, such that π is G–equivariant.*

Proof. By Theorem B.3 of Appendix B and Proposition 1.38 we have that the orbit space $B = M/H$ is indeed a smooth manifold. Now we show that the actions of G and H on M commute. Let $m \in M$, $h \in H$, $g \in G$ and choose a $g_F \in N_F$ which represents h. Then we have $g.{}_\theta(h.{}_\eta m) = g.(c_{g_0}(g_F).m)$ for $g_0.m_0 = m$ or equivalently $(gg_0g_Fg_0^{-1}).m$. Set $m' := gg_0.m_0$, then $N_{F_{m'}} = (gg_0)G_F(gg_0)^{-1}$ by the proof of Proposition 1.38, hence $(gg_0)g_F = g'(gg_0)$ for a $g' \in G_{F_{m'}}$. This leads to $(gg_0g_Fg_0^{-1}).m = g'g.m'$ which can be written as $(gg_0)g_F(gg_0)^{-1}.(g.m)$ and since $gg_0.m_0 = m' = g.m$ we have that $(gg_0)g_F(gg_0)^{-1}.(g.m) = h.{}_\eta(g.{}_\theta m)$. It follows that the action of G on M descends to an action on B: Define $\theta'\colon G \times B \to B$ as $\theta'(g, b) := \pi(g.m)$ for a $m \in M$ with $\pi(m) = b$. This map is well–defined since any other m' with $\pi(m') = b$ is related to m by $m' = h.m$ for $h \in H$. Then we have $\pi(g.m') = \pi(g.h.m) = \pi(h.g.m) = \pi(g.m)$ (note again the different action denoted by the dots), hence θ' is well–defined. The action is smooth because $\pi\colon M \to B$ is a principal fiber bundle and the involved actions are smooth too. Moreover θ' acts on B transitive, since θ acts on M transitive and therefore B is diffeomorphic to G modulo the isotropy group of θ' in a point of B, say $b_0 := \pi(m_0)$. Thus suppose $g.b_0 = b_0$ for a $g \in G$ then $g.(h.m_0) = m_0$ or equivalently $gg_F.m_0 = m_0$ for a $g_F \in N_F$, thus $g = g_F^{-1}k \in N_F$ for a $k \in K \subset N_F$ and it follows that G/N_F is diffeomorphic to $B = M/H$. ∎

Remark 1.40. Note that G/N_F is merely a homogeneous space, since it is not clear whether N_F is compact. Note moreover, that the action of G on B has not to be effective. But if C denotes the maximal normal subgroup of G contained in N_F, we have that G/C acts on B effectively with isotropy group N_F/C.

Corollary 1.41. *The Lie algebra of fundamental vector fields of H acting on M is isomorphic to the Lie algebra $\mathcal{V}^G(M)$. Hence the Lie algebra of H is isomorphic to $\mathcal{V}^G(M)$.*

Proof. Clearly the Lie algebra $\mathfrak{h}$ of H has the same dimension as $\mathcal{V}^G(M)$ and since the actions of G and H commute, we have that the fundamental vector fields of $\mathfrak{h}$ are G–invariant. Since the map sending a Lie algebra element to its fundamental vector field is an injective Lie homomorphism we conclude that $\mathfrak{h}$ is isomorphic to $\mathcal{V}^G(M)$. ∎

Equivariant principal bundles

As we saw in the previous section a Riemannian homogeneous space which allows homogeneous vector fields comes with a principal bundle between homogeneous spaces such that the projection map is equivariant. Here we would like to study the properties of those objects more precisely.

Definition 1.42. Let $\pi\colon M \to B$ be an H–principal bundle where H is a Lie group and B a smooth manifold. We say that $\pi\colon M \to B$ is an *G–equivariant H–principal bundle* (or simply *equivariant principal bundle*) if the actions of G and H commute. Therefore we regard the action of H on M as a right action defined by $m.h := h^{-1}.m$.

Although many of the statements below would work for equivariant principal bundles in general, we would like to restrict our attention to those which arise from Corollary 1.39. Henceforth we will discuss only such equivariant principal bundles.

In the following lines we would like to deal with connections on equivariant principal bundles. As one could expect there is a G–invariant connection on the H–principal bundle $\pi\colon M \to B$. But before proving the existence of such a connection we have to phrase an elementary proposition in linear algebra.

Proposition 1.43. *Let (V, σ) be a euclidean vector space. We denote the linear isometries of (V, σ) by $\mathbf{O}(V)$. Suppose $K \subset \mathbf{O}(V)$ is a closed subgroup and that the subspace $\mathrm{Eig}(K, 1) := \cap_{k\in K}\mathrm{Eig}(k, 1)$ has positive dimension. Suppose furthermore that $f \in \mathbf{Aut}(V)$ normalizes K, i.e. $fKf^{-1} \subset K$. Set $U := \mathrm{Eig}(K, 1)$ and $W := U^{\perp}$ then f respects the orthogonal decomposition of V.*

Proof. First we show that f maps U into itself. For a $u \in U$ we see that $kf(u) = f\hat{k}(u) = f(u)$ for all $k \in K$ as well as for some $\hat{k} \in K$ and therefore $f(u) \in \mathrm{Eig}(K, 1)$. We denote by f^* the adjoint map of f with respect to σ. Note that $k^* = k^{-1} \in K$ for every $k \in K$. Then f^* normalizes K as well as can be seen by the computation

$$f^*k = (k^*f)^* = (f\hat{k}^*)^* = \hat{k}f^*$$

for all $k \in K$ and some $\hat{k} \in K$; hence f^* maps U into itself too. From this we conclude that

$$\sigma(u, f(w)) = \sigma(f^*(u), w) = 0$$

for $w \in W$ and for all $u \in U$ since $f^*(u) \in U$. Therefore $f(w) \in U^\perp = W$. ∎

> **Proposition 1.44.** *There is a G–invariant connection on the equivariant principal bundle $\pi\colon M \to B$, i.e. there is a distribution $\mathcal{H}$ on M such that $\mathcal{H}$ is in every point a complement to the tangent space of orbits of H and is invariant under the actions of G and H. The connection is unique in the sense, that $\mathcal{H}$ is the orthogonal complement of $\mathcal{D}$ for every G–invariant metric.*

Proof. Let $\mathcal{D}$ be the distribution defined in Remark 1.36. Then $\mathcal{D}$ along an orbit of H is exactly the tangent space of this orbit as seen in Proposition 1.38. Choose a G–invariant metric μ of (M, G) and define $\mathcal{H} := \mathcal{D}^\perp$ which is clearly a smooth distribution on M. Since $\mathcal{D}$ is defined through G–invariant vector fields we have that $\mathcal{D}$ is invariant under the action of G on TM. And because the elements of G are isometries it follows that every $g \in G$ preserves the orthogonal splitting of $TM = \mathcal{D} \oplus \mathcal{H}$; in particular $\mathcal{H}$ is G–invariant. The definition of a connection requires now that $\mathcal{H}$ is invariant under H. Let $m_0 \in M$, $h \in H$ and $g \in G$ such that $g.m_0.h = m_0$. Let $F\colon M \to M$ be the diffeomorphism $F(m) := g.m.h$ and f the derivative of F in m_0 which is an automorphism of TM_{m_0}. Set $K := K_{m_0}$, $F_{m_0} := H.m_0$ and we see that g maps m_0 into F_{m_0}, which means that g normalizes K. It follows that f normalizes K since $k(g.m_0.h) = m_0$ and by the chain rule $kf = f\hat{k}$ for $k, \hat{k} \in K$. Since $\mathcal{H}_{m_0}$ is the orthogonal complement of $\mathrm{Eig}(K, 1) = \mathcal{D}_{m_0}$ in (TM_{m_0}, μ_{m_0}) we can apply Proposition 1.43 to see that f respects the orthogonal decomposition $TM_{m_0} = \mathcal{D}_{m_0} \oplus \mathcal{H}_{m_0}$. Now since G respects the orthogonal splitting of $TM = \mathcal{D} \oplus \mathcal{H}$ and because $\xi.h = g^{-1} \circ f(\xi)$ for $\xi \in \mathcal{H}_{m_0}$ we see that $\xi.h \in \mathcal{H}_{m_0.h}$. Hence $\mathcal{H}_{m_0}.h \subset \mathcal{H}_{m_0.h}$ and because the derivative of h seen as a diffeomorphism on M is injective we conclude $\mathcal{H}_{m_0}.h = \mathcal{H}_{m_0.h}$. Therefore $\mathcal{H}$ is invariant under H, hence $\mathcal{H}$ is indeed a connection on $\pi\colon M \to B$.

Now let μ_0 be a G–invariant metric on M beside μ. We have to show $\mathcal{H} \subset \mathcal{D}^{\perp_0}$ where $\perp_0$ indicates the orthogonal complement with respect to the metric μ_0. Since $\mathcal{H}$ is G–invariant and G acts by isometries on M for the metrics μ and μ_0 we have merely to show $\mathcal{H}_{m_0} \subset \mathcal{D}_{m_0}^{\perp_0}$. Let f_0 be the self–adjoint automorphism of TM_0 such that $\mu_0(\xi_1, \xi_2) = \mu(f_0(\xi_1), \xi_2)$ for $\xi_1, \xi_2 \in TM_{m_0}$. For $\eta \in \mathcal{D}_{m_0}$ and all $\xi \in TM_{m_0}$ as well as all $k \in K$ we have

$$\mu(kf_0(\eta), \xi) = \mu(f_0(\eta), k^{-1}\xi) = \mu_0(\eta, k^{-1}\xi) = \mu_0(\eta, \xi) = \mu(f_0(\eta), \xi),$$

since μ, μ_0 are invariant under K and $\mathcal{D}_{m_0}$ is fixed by K. This implies $k.f_0(\eta) = f_0(\eta)$ and hence $f_0(\eta) \in \mathrm{Eig}(K, 1) = \mathcal{D}_{m_0}$ for all $\eta \in \mathcal{D}_{m_0}$. Now if $\eta \in \mathcal{D}_{m_0}$ and $\xi \in \mathcal{H}_{m_0}$ we see that

$$\mu_0(\eta, \xi) = \mu(f_0(\eta), \xi) = 0$$

since $f_0(\eta) \in \mathcal{D}_{m_0} = \mathcal{H}_{m_0}^{\perp}$ and this implies $\mathcal{H} \subset \mathcal{D}^{\perp_0}$.

∎

Remark 1.45.

(a) We denote by $\Omega^k(M, V) := \Omega^k(M, \mathbb{R}) \otimes V$ the k–forms on M with values in a vector space V. If H is acting on M from the right we denote by $(\alpha)h_*$ the pullback of α by h, i.e. for tangent vectors $\xi_1, \dots, \xi_k$ on TM_m we have

$$((\alpha)h_*)_m(\xi_1, \dots, \xi_k) = \alpha_{m.h}(\xi_1.h, \dots, \xi_k.h).$$

(b) There is a unique connection form $\omega \in \Omega^1(M, \mathfrak{h})$ for the connection $\mathcal{H}$ on $\pi\colon M \to B$ of Proposition 1.44 (see e.g. [Bau09, p.77]) such that $\ker \omega = \mathcal{H}$ and for every $h \in H$ we have $(\omega)h_* = \mathrm{Ad}_H(h^{-1}) \circ \omega$ as well as $\omega(X_w) = w$ for $w \in \mathfrak{h}$.

Proposition 1.46. *The connection form ω of Remark 1.45 is G–invariant. Conversely every G–invariant connection form on $\pi\colon M \to B$ defines a G–invariant connection $\mathcal{H}$.*

Proof. First we have to show that $g_*(\omega) = \omega$ for every $g \in G$. Let $\xi \in TM_m$ for $m \in M$. Then there is a uniquely determined decomposition of $\xi = \xi_{\mathcal{D}} + \xi_{\mathcal{H}}$ with $\xi_{\mathcal{D}} \in \mathcal{D}_m$ and $\xi_{\mathcal{H}} \in \mathcal{H}_m$. Surely there is a fundamental vector field X_w on M for $w \in \mathfrak{h}$ such that $(X_w)_m = \xi_{\mathcal{D}}$ and such that X_w is G–invariant (since G and H commute). Hence

$$g_*(\omega)_m(\xi) = \omega_{g.m}(g.\xi) = \omega_{g.m}(g.\xi_{\mathcal{D}}) = \omega_{g.m}((X_w)_{g.m}) = w = \omega_m(X_w) = \omega_m(\xi).$$

On the other side, if ω is a G–invariant connection form on $\pi\colon M \to B$ then set $\mathcal{H} := \ker \omega$. Since ω is a connection form, $\mathcal{H}$ is connection. If $\xi \in \mathcal{H}_m$ then $\omega_{g.m}(g.\xi) = 0$ since ω is G–invariant. Hence $g.\xi \in \mathcal{H}_{g.m}$ and it follows that $g.\mathcal{H}_m = \mathcal{H}_{g.m}$. ∎

Remark 1.47.

(a) Let $\alpha \in \Omega^k(M, V)$ and V a vector space. If $V = \mathbb{R}$ then we have the usual exterior derivative $\mathrm{d}\alpha\colon \Omega^k(M, \mathbb{R}) \to \Omega^{k+1}(M, \mathbb{R})$. We can extend d to map $\mathrm{d}\colon \Omega^k(M, V) \to \Omega^{k+1}(M, V)$ by setting $\mathrm{d}(\alpha \otimes v) := \mathrm{d}\alpha \otimes v$ for $\alpha \in \Omega^k(M, \mathbb{R})$, $v \in V$ and using the universal property of tensor product we extend this to $\Omega^k(M, \mathbb{R}) \otimes V$.

(b) Suppose $\mathfrak{l}$ is a Lie algebra with Lie bracket denoted by $[\cdot, \cdot]$. Then $[\cdot, \cdot]$ is a bilinear map $\mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ which fulfills the Jacobi identity. We use the Lie bracket to define the *commutator* of forms on M with values in $\mathfrak{l}$. Consider $\alpha \in \Omega^k(M, \mathbb{R})$ and $\beta \in \Omega^l(M, \mathbb{R})$ as well as $v, w \in \mathfrak{l}$. Then define

$$[(\alpha \otimes v) \wedge (\beta \otimes w)] := \alpha \wedge \beta \otimes [v, w]$$

and extend this to a map $[\cdot \wedge \cdot]\colon \Omega^k(M, \mathfrak{l}) \times \Omega^l(M, \mathfrak{l}) \to \Omega^{k+l}(M, \mathfrak{l})$ using again the universal property of tensor products.. Directly from the definition we have the following properties for $\alpha \in \Omega^k(M, \mathfrak{l})$ and $\beta \in \Omega^l(M, \mathfrak{l})$

(i) $[\alpha \wedge \beta] = (-1)^{kl+1}[\beta \wedge \alpha]$,

(ii) $\mathrm{d}[\alpha \wedge \beta] = [\mathrm{d}\alpha \wedge \beta] + (-1)^k[\alpha \wedge \mathrm{d}\beta]$.

(c) Let ω be the connection form of Remark 1.45. There is an $\mathfrak{h}$–valued 2–form $F^\omega \in \Omega^2(M, \mathfrak{h})$ called the *curvature of* ω which is defined by

$$F^\omega := \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega].$$

We shall call a connection ω on a principal bundle with $F^\omega = 0$ a *flat*.

The integrability of the horizontal distribution is determined by the curvature F^ω of the connection form ω. Let $\pi_{\mathcal{D}}\colon TM \to \mathcal{D}$ be the vertical projection, then for Y_1 and Y_2 horizontal vector fields we have

$$\pi_{\mathcal{D}}\Big([(Y_1, Y_2)]_m\Big) = X_v$$

where X_v is the fundamental vector field to $v = -F^\omega((Y_1)_m, (Y_2)_m) \in \mathfrak{h}$ which follows from

$$F^\omega(Y_1, Y_2) = \mathrm{d}\omega(Y_1, Y_2) = -\omega([Y_1, Y_2])$$

since Y_1 and Y_2 are horizontal. Therefore $\mathcal{H}$ is integrable if ω is flat.

Extensions of Lie Groups and Semidirect Products

Here we would like to discuss some properties of short exact sequences of connected Lie groups and their relation to semidirect products of Lie groups. Therefore let us start with a

Definition 1.48. Let K, G and H be Lie groups as well as $i\colon K \to G$ a monomorphism and $\pi\colon G \to H$ an epimorphism. We say G *is an extension of* K *by* H (or simply an *extension*) if

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1$$

is a short exact sequence of Lie groups and we will indicate an extension like this by (i, π). We shall say an extension π *splits* if there is a section $\sigma\colon H \to G$ of $\pi\colon G \to H$, i.e. $\pi \circ \sigma = \mathrm{id}_H$.

Clearly every epimorphism $\pi\colon G \to H$ induces an extension of Lie groups

$$1 \longrightarrow \ker\pi \longrightarrow G \longrightarrow H \longrightarrow 1,$$

where we take the inclusion of $\ker\pi$ into G as the monomorphism. And every extension is determined by an epimorphism modulo the isomorphism class of the kernel. Two extensions of K by H say (i_1, π_1) and (i_2, π_2) are *isomorphic* or *equivalent* if there are $\psi\colon G_1 \to G_2$, $\tau \in \mathbf{Aut}(K)$ and $\varphi \in \mathbf{Aut}(H)$ such that the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & K \arrow[r, "i_1"] \arrow[d, "\tau"'] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\psi"'] & H \arrow[r] \arrow[d, "\varphi"'] & 1 \arrow[d] \\
1 \arrow[r] & K \arrow[r, "i_2"] & G_1 \arrow[r, "\pi_2"] & H \arrow[r] & 1
\end{tikzcd}$$

and by the five–lemma ψ is an isomorphism.

Every extension of Lie groups like in the definition above may be seen as a K–principal bundle $\pi\colon G \to H$. Thus the splitting map corresponds to a global section of this bundle, hence G is diffeomorphic to the product manifold $K \times H$ (even more we have that the principal bundle is the trivial one). But the crucial fact is that G has not to be the product group in general. However it is nearly a product of groups which we will call a semidirect product. In the purely algebraic setting there no subtlety to define a semidirect product, but in the case of Lie groups we have to be more careful since there is a Lie group structure involved there. Now let us first phrase the definition and discuss afterwards what we have to do to understand the definition more precisely.

Definition 1.49. Let G and H be Lie groups such that G is connected and $\rho\colon H \to \mathbf{Aut}(G)$ a homomorphism of groups, such that the map $H \times G \to G$, $(h, g) \mapsto \rho_h(g)$ is smooth. Then the *semidirect product of Lie groups* $G \rtimes_\rho H$ is given as follows: take the manifold $G \times H$ and define a group structure by

$$(g_1, h_1)(g_2, h_2) = (g_1\rho_{h_1}(g_2), h_1h_2).$$

A straightforward computation shows that $G \rtimes_\rho H$ is indeed a Lie group.

Remark 1.50. The crucial part of this definition is the group $\mathbf{Aut}(G)$. Clearly in algebraic terms it is a group, but if G is a Lie group the question arises if there is a natural Lie group structure on $\mathbf{Aut}(G)$. One of the reasons is that it would be more comfortable to define ρ as a Lie group homomorphism.

In [War83, p. 119] $\mathbf{Aut}(G)$ can be given a Lie group structure if G is simply connected. In that case we have an isomorphism of groups between $\mathbf{Aut}(G)$ and $\mathbf{Aut}(\mathfrak{g})$ which is a closed subgroup of the group of invertible endomorphism of $\mathfrak{g}$ and therefore we may transfer the Lie group structure of $\mathbf{Aut}(\mathfrak{g})$ onto $\mathbf{Aut}(G)$ via this isomorphism. Of course the question now is how to install a Lie group structure in the case where G is not simply connected.

Let G be a connected group and $\tilde{G}$ its universal cover group with $\pi\colon \tilde{G} \to G$ the covering homomorphism. As is generally known the kernel of π is isomorphic to the fundamental group $\pi_1(G)$ of G which in turn is isomorphic to the group of decktransformations of the universal covering $\pi\colon \tilde{G} \to G$. Define

$$\mathbf{Aut}(\tilde{G}, G) := \{\tilde{\varphi} \in \mathbf{Aut}(\tilde{G}) : \tilde{\varphi}(\pi_1(G)) = \pi_1(G)\}$$

which is a closed subgroup of $\mathbf{Aut}(\tilde{G})$ hence itself a Lie group. We claim that $\mathbf{Aut}(\tilde{G}, G)$ is isomorphic to $\mathbf{Aut}(G)$ as a group. First define the map $F\colon \mathbf{Aut}(\tilde{G}, G) \to \mathbf{Aut}(G)$ as follows: for $\tilde{\varphi} \in \mathbf{Aut}(\tilde{G}, G)$ and $g \in G$ let $F(\tilde{\varphi})(g) := \pi \circ \tilde{\varphi}(\tilde{g})$ for a $\tilde{g} \in \pi^{-1}(g)$. This map is well–defined since $\tilde{\varphi}$ is a homomorphism and sends $\pi_1(G)$ into itself. We would like to abbreviate $F(\tilde{\varphi})$ as φ. Then the diagram below is commutative and clearly φ is a Lie group homomorphism.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\varphi}} & \tilde{G} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ G & \xrightarrow{\varphi} & G \end{array}$$

Hence F is well–defined. Moreover F is a group homomorphism because

$$F(\tilde{\varphi}_1 \circ \tilde{\varphi}_2)(g) \quad = \quad \pi \circ \tilde{\varphi}_1 \circ \tilde{\varphi}_2(\tilde{g}) = \varphi_1 \circ \varphi_2(g)$$

since the above diagram commutes. We will show that F is bijective. If $\varphi \in \mathbf{Aut}(G)$ then since $\tilde{G}$ is simply connected we may lift $\varphi \circ \pi\colon \tilde{G} \to G$ to a unique map $\tilde{\varphi}\colon \tilde{G} \to \tilde{G}$ such that $\tilde{\varphi}(\tilde{e}) = \tilde{e}$ where $\tilde{e}$ is the neutral element of $\tilde{G}$. Therefore the above diagram commutes again and $\tilde{\varphi}$ is indeed a smooth automorphism. For an element $\tilde{g} \in \pi_1(G) = \ker \pi$ we compute that $\pi \circ \tilde{\varphi}(\tilde{g}) = \varphi \circ \pi(\tilde{g}) = e$, i.e. $\tilde{\varphi}(\pi_1(G)) \subset \pi_1(G)$ and since φ is bijective and F a group homomorphism we conclude $\tilde{\varphi}(\pi_1(G)) = \pi_1(G)$ and finally we obtain $\tilde{\varphi} \in \mathbf{Aut}(\tilde{G}, G)$. Moreover since the diagram commutes $F(\tilde{\varphi})(g) = \pi \circ \tilde{\varphi}(\tilde{g}) = \varphi \circ \pi(\tilde{g}) = \varphi(g)$ hence F is onto. If $F(\tilde{\varphi}) = \mathrm{id}_G$ then $\tilde{\varphi}$ has to be a decktransformation such that $\tilde{\varphi}(\tilde{e}) = \tilde{e}$ which implies that $\tilde{\varphi} = \mathrm{id}_{\tilde{G}}$. Thus F is bijective and we carry the Lie group structure of $\mathbf{Aut}(\tilde{G}, G)$ over to $\mathbf{Aut}(G)$.

Hochschild showed in [Hoc52] that the topology of $\mathbf{Aut}(G)$ defined above coincide with the natural compact–open topology on it. Furthermore he showed in the same article, that if $\pi_0(G)$ is finitely generated then $\mathbf{Aut}(G)$ has still a Lie group structure which underlying topology coincides with the compact–open topology. Therefore if $H \times G \to G$ is smooth as in the definition of the semidirect product then, in the case where $\pi_0(G)$ is finitely generated, the map $\rho\colon H \to \mathbf{Aut}(G)$ is a Lie group homomorphism. As we have to deal mostly with Lie groups where $\pi_0(G)$ is finitely generated, we assume henceforth without further mentioning that π_0 is finitely generated for any group.

Proposition 1.51. *Let $1 \to K \to G \xrightarrow{\pi} H \to 1$ be an extension which splits via a map $\sigma\colon H \to G$. We see K as a closed subgroup of G. Then G is isomorphic to $K \rtimes_\rho H$, where the $\rho\colon H \to \mathbf{Aut}(K)$ is given by $\rho_h(k) := \sigma(h)k\sigma(h)^{-1}$. Therefore we write also $K \rtimes_\sigma H$ since σ determines the representation ρ.*

Proof. Define $\rho(h)(k) := \sigma(h)k\sigma(h)^{-1}$ which lies indeed in K since $\pi(\rho(h)(k)) = hh^{-1} = e$. Clearly the map $H \times K \to K$, $(h,k) \mapsto \rho_h(k)$ is a smooth homomorphism and we consider the map $\Phi\colon K \rtimes_\rho H \to G$, $\Phi(k,h) := k\sigma(h)$ which is a smooth map. And we compute

$$\begin{aligned}\Phi((k_1,h_1)(k_2,h_2)) &= \Phi(k_1\rho_{h_1}(k_2), h_1h_2) = k_1\rho_{h_1}(k_2)\sigma(h_1)\sigma(h_2)\\ &= k_1\sigma(h_1)k_2\sigma(h_2) = \Phi(k_1,h_1)\Phi(k_2,h_2).\end{aligned}$$

which shows that F is a homomorphism. On the other side define $\Psi\colon G \to K \rtimes_\rho H$ to be $\Psi(g) := (g\sigma(\pi(g^{-1})), \pi(g))$. We have $\pi(g\sigma(\pi(g^{-1})) = e$ hence $g\sigma(\pi(g^{-1}) \in K$ and Ψ is a well–defined smooth map. The computation

$$\begin{aligned}\Psi(g_1g_2) &= (g_1g_2\sigma\pi(g_2)^{-1}\sigma\pi(g_1)^{-1}, \pi(g_1)\pi(g_2))\\ &= (g_1\sigma\pi(g_1)^{-1}\sigma\pi(g_1)g_2\sigma\pi(g_2)^{-1}\sigma\pi(g_1)^{-1}, \pi(g_1)\pi(g_2))\\ &= (g_1\sigma\pi(g_1)^{-1}, \pi(g_1))(g_2\sigma\pi(g_2)^{-1}, \pi(g_2)) = \Psi(g_1)\Psi(g_2).\end{aligned}$$

shows that Ψ is a homomorphism as well. We claim that Ψ is the inverse map for Φ. We show this by conducting the following two boring computations.

$$\Psi \circ \Phi(k,h) = \Psi(k\sigma(h)) = (k\sigma(h)\sigma(h)^{-1}, h) = (k,h)$$

and

$$\Phi \circ \Psi(g) = \Phi(g\sigma\pi(g^{-1}), \pi(g)) = g\sigma\pi(g^{-1})\sigma\pi(g) = g.$$

■

Suppose we have that G acts on M and H acts on a manifold N and suppose furthermore that there is a representation of H on G, i.e. there is a Lie group homomorphism $\rho\colon H \to \mathbf{Aut}(G)$. Now we are interested how the semidirect product $G \rtimes_\rho H$ may act on $M \times N$. We say $G \rtimes_\rho H$ acts *semidirect on* $M \times N$ if there is an action of H on M by diffeomorphisms such that $c_h(g) := h \circ g \circ h^{-1} = \rho_h(g)$. Then the action of $G \rtimes_\rho H$ on $M \times N$ may be defined as

$(g, h).(m, n) := (gh.m, h.n)$, where gh is the composition of the diffeomorphism g and h on M. Indeed this is an action since $(e, e).(m, n) = (m, n)$ and

$$\begin{aligned}(g_1, h_1).\,[(g_2, h_2).(m, n)] &= (g_1, h_1)(g_2h_2.m, h_2.n) = (g_1h_1g_2h_2.m, h_1h_2.m)\\ &= (g_1h_1g_2h_1^{-1}h_1h_2.m, h_1h_2.n) = (g_1\rho_{h_1}(g_2)h_1h_2.m, h_1h_2.n)\\ &= (g_1\rho_{h_1}(g_2), h_1h_2).(m, n) = (g_1, h_1)(g_2, h_2).(m, n).\end{aligned}$$

As a first consequence we would phrase some statements about equivariant principal bundles with flat connection.

Remark 1.52. Let $\pi\colon M \to B$ be the G–equivariant H–principal bundle from Corollary 1.39 and suppose that the connection of Proposition 1.44 is flat. Suppose moreover that M is simply connected and G is connected. Then by Lemma 1.12 the isotropy group K in $m_0 \in M$ is connected and with Theorem 1.37 we conclude that H is connected. Using this and the long exact homotopy sequence of Lemma 1.12 again we see that B has to be simply connected. Therefore the H–principal bundle has to be H–equivariant isomorphic to $B \times H$ with H acting by right–multiplication on the second factor and the connection to be the trivial connection. Observe that H has to be simply connected since M and B are. If $\Psi\colon M \to B \times H$ is this H–equivariant isomorphism we are to able to define an G–action on $B \times H$ which commutes with H and is a G–equivariant isomorphism between (M, G) and $(B \times H, G)$. We define this action $\tilde{\theta}$ of G on $B \times H$ such that the diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\theta} & M \\ {\scriptstyle \mathrm{id}\times\Psi}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ G \times (B \times H) & \xrightarrow{\tilde{\theta}} & B \times H \end{array}$$

becomes commutative.

Proposition 1.53. *We assume the conditions of Remark 1.52. In that case there is an extension*

$$1 \longrightarrow G_B \longrightarrow G \xrightarrow{\Phi} H \longrightarrow 1,$$

where G_B is a connected and closed Lie subgroup of G acting on $B \times e = B$ effectively, transitively with isotropy group K in (b_0, e).

Proof. We may assume that $\Psi(m_0) = (b_0, e)$ where $b_0 \in B$ and e the neutral element of H. Denote by π_H the projection $B \times H \to H$ and if $g \in G$ let g_H denote the composition $\pi_H \circ g\colon B \times H \to H$. But the map g_H can be seen as a map from H to itself since we will show that it does not depend on the B–factor. We choose a fixed $h \in H$ and a curve $\alpha\colon (-\varepsilon, \varepsilon) \to B \times h$ where $\dot{\alpha}(0)$ is a horizontal vector in $T(B \times H)_{(b,h)}$. Since g maps horizontal vectors into horizontal vectors and the differential of π_H maps horizontal vectors to zero, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_H \circ \alpha = 0$$

(note that we used that the principal bundle $B \times H \to B$ is equipped with the trivial flat connection). Hence the map $b \mapsto g_H(b, h)$ is constant on B which means that g_H does not depend on B. Therefore define the map $\Phi\colon G \to H$ by $\Phi(g) := g_H(b, e)$ for a $b \in B$. We claim that Φ is a group homomorphism. Let $g_1, g_2 \in G$ and let $h_i := \Phi(g_i)$ for $i = 1, 2$. Now $g_2(b, e) = (b_2, h_2)$ for some $b_2 \in B$, i.e. that $h_2 = \pi_H \circ g_2(b, e)$. Then we compute that

$$\Phi(g_1 g_2) = \pi_H(g_1(b_2, h_2)) = \pi_H(g_1(b_2, e).h_2) = h_1 h_2 = \Phi(g_1)\Phi(g_2),$$

since the actions of G and H commute and π_H is H-equivariant. Moreover Φ is smooth since Φ is composed through smooth maps, more precisely Φ is the composition of the evaluation map and π_H. If $h \in H$ then there is a $g \in G$ such that $g(b_0, e) = (b_0, e).h = (b_0, h)$ and consequently $\Phi(g) = h$ therefore Φ is an epimorphism.

Every element of $G_B = \ker \Phi$ acts clearly on $B \times e$. The group acts effectively since if $g(b, e) = (b, e)$ for all $b \in B$ then $g(b, h) = (b, h)$ for all $(b, h) \in B \times H$ since $g(b, h) = g(b, e).h = (b, e).h = (b, h)$. The kernel acts transitively since for $(b, e) \in B \times e$ we have a $g \in G$ such that $g(b_0, e) = (b, e)$ which means by definition $g \in G_B$. Since H is simply connected and G is connected we use again Lemma 1.12 to conclude that G_B is connected. Same arguments apply to see that the isotropy group K' in (b_0, e) of G_B is connected. And since $\dim K = \dim K'$, $K \subset K'$ and the fact that K as well as K' is connected we have that $K = K'$. ■

Theorem 1.54. *Suppose $\sigma\colon H \to G$ splits the extension of Proposition 1.53. Then $(B \times H, G)$ is equivariant isomorphic to $(B \times H, G_B \rtimes_\sigma H)$ with a semidirect action of $G_B \rtimes_\sigma H$ on $B \times H$.*

Proof. From Proposition 1.51 G is the semidirect product $G_B \rtimes_\sigma H$ and the action of H on G_B is given by the map $\rho\colon H \to \mathbf{Aut}(G_B)$, $\rho(h)(g) = \sigma(h) g \sigma(h)^{-1}$. By definition the group G_B acts on $B \times e$. The action of $\sigma(h)$ on $B \times H$ in the H–factor is given as left–multiplication by h. This can be easily seen as one computes $\pi_H \circ \sigma(h)(b', h') = hh'$. Hence the element $\sigma(h)(b, e)h^{-1}$ lies in $B \times e$ and therefore there is a map $f\colon H \to \mathrm{Diff}(B)$ such that $(f(h)(b), e) = \sigma(h)(b, e)h^{-1}$. Since G_B is a subgroup of $\mathrm{Diff}(B)$ we have $f(h) \circ g \circ f(h)^{-1} = \rho(h)(g)$ and this defines a semidirect action of $G_B \rtimes_\sigma H$ on $B \times H$: for $(g,h) \in G_B \rtimes H$ this action is given by

$$\begin{aligned}(g,h)(b,h') &= g(f(h)(b),e)hh' \\ &= g\sigma(h)(b,e)h^{-1}hh' \\ &= g\sigma(h)(b,e)h' \\ &= g\sigma(h)(b,h') \\ &= \Phi(g,h)(b,h')\end{aligned}$$

where $\Phi\colon G_B \rtimes H \to G_B \rtimes H$ is the map from Proposition 1.51, hence this defines the same action. Finally we choose the identity map $\mathrm{id}\colon B \times H \to B \times H$ as a diffeomorphism to see that the identity is Φ–equivariant. ■

A direct product of groups may be considered as a trivial semidirect product if we choose $\rho\colon H \to \mathbf{Aut}(K)$ to be the trivial representation, i.e. $\rho(h) = \mathrm{id}_K$ for all $h \in H$. However, sometimes a direct product can appear as a non–trivial semidirect product as the following simple example shows. As we know $\mathbf{SO}(2n+1)$ is a normal subgroup of $\mathbf{O}(2n+1)$ and thus we obtain an extension

$$1 \longrightarrow \mathbf{SO}(2n+1) \longrightarrow \mathbf{O}(2n+1) \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

where we have the determinant to be the epimorphism from $\mathbf{O}(2n+1)$ to $\mathbb{Z}_2$. Then we may take two splitting maps into account. First $\sigma_1\colon \mathbb{Z}_2 \to \mathbf{O}(2n+1)$, $\sigma_1(-1) = -E_{2n+1}$, where E_{2n+1} is the identity matrix in $\mathbb{R}^{2n+1}$ which is clearly a splitting map. But the corresponding representation $\rho_1\colon \mathbb{Z}_2 \to \mathbf{Aut}(\mathbf{SO}(2n+1))$ is the trivial one since $-E_{2n+1}$ commutes with all elements of $\mathbf{SO}(2n+1)$ and therefore $\mathbf{O}(2n+1) = \mathbf{SO}(2n+1) \times \mathbb{Z}_2$. On the other hand $\sigma_2\colon \mathbb{Z}_2 \to \mathbf{O}(2n+1)$ defined by $\sigma_2(-1) = \mathrm{diag}(-1,1,\ldots,1) =: D$ splits the exact sequence as well but the representation does not act by the identity on $\mathbf{SO}(2n+1)$. Hence we have also $\mathbf{O}(2n+1) = \mathbf{SO}(2n+1) \rtimes_{\sigma_2} \mathbb{Z}_2$. The reason for this is that the representation ρ_2 is equal to a representation by inner automorphisms, more precisely for $S \in \mathbf{SO}(2n+1)$ we have $DSD^{-1} = -DS(-D^{-1})$ and $-D \in \mathbf{SO}(2n+1)$. We will show below that a semidirect product with a representation by inner automorphisms is isomorphic to the direct product of groups.

But first let us say a few words about the groups of inner and outer automorphisms of a connected Lie group K. The inner automorphism of K are defined by

$$\mathbf{Inn}(K) := \{\varphi \in \mathbf{Aut}(K) : \exists k \in K,\ \varphi = c_k\}$$

where $c_k\colon K \to K$ is the conjugation map. If $\varphi \in \mathbf{Aut}(K)$ then $\varphi c_k \varphi^{-1} = c_{\varphi(k)}$ hence $\mathbf{Inn}(K)$ is a normal subgroup of $\mathbf{Aut}(K)$ and the quotient $\mathbf{Out}(K) = \mathbf{Aut}(K)/\mathbf{Inn}(K)$ is called *outer* *automorphism* group. Thus this induces an extension

$$1 \longrightarrow \mathbf{Inn}(K) \longrightarrow \mathbf{Aut}(K) \longrightarrow \mathbf{Out}(K) \longrightarrow 1.$$

Moreover the conjugation map $c\colon K \to \mathbf{Inn}(K)$, $k \mapsto c_k$ is clearly onto and the kernel is $Z(K)$ the center of K, and again we obtain an extension

$$1 \longrightarrow Z(K) \longrightarrow K \longrightarrow \mathbf{Inn}(K) \longrightarrow 1.$$

Note that $\mathbf{Inn}(K)$ is connected if K is. The next proposition will show us that representations which are given by inner automorphisms create semidirect products which are trivial.

Proposition 1.55. *Suppose $\rho\colon H \to \mathbf{Inn}(K)$ is a representation of H on K. Assume that there is a homomorphism $\lambda\colon H \to K$ which is a lift of ρ for $c\colon K \to \mathbf{Inn}(K)$, i.e. $\rho_h = c_{\lambda(h)}$. Then $K \rtimes_\rho H$ is isomorphic to $K \times H$.*

Proof. Define the smooth map $\Phi\colon K \times H \to K \rtimes_\rho H$ by $\Phi(k,h) := (k\lambda(h)^{-1}, h)$. We claim that Φ is a homomorphism.

$$\begin{aligned}
\Phi\left((k_1,h_1)(k_2,h_2)\right) &= \Phi(k_1k_2,h_1h_2) = (k_1k_2\lambda(h_1h_2)^{-1},h_1h_2) = (k_1k_2\lambda(h_2)^{-1}\lambda(h_1)^{-1},h_1h_2)\\
&= (k_1\lambda(h_1)^{-1}\lambda(h_1)k_2\lambda(h_2)^{-1}\lambda(h_1)^{-1},h_1h_2)\\
&= (k_1\lambda(h_1)^{-1}\rho_{h_1}(k_2\lambda(h_2)^{-1}),h_1h_2)\\
&= (k_1\lambda(h_1)^{-1},h_1)(k_2\lambda(h_2)^{-1},h_2)\\
&= \Phi(k_1,h_1)\Phi(k_2,h_2).
\end{aligned}$$

The inverse map is given by $\Psi\colon K \rtimes_\rho H \to K \times H$, $\Psi(k,h) = (k\lambda(h),h)$. First Ψ is clearly smooth and Ψ is the inverse map of Φ:

$$\Psi \circ \Phi(k,h) = \Psi(k\lambda(h)^{-1},h) = (k,h)$$

and

$$\Phi \circ \Psi(k,h) = \Phi(k\lambda(h),h) = (k,h).$$

∎

We say a Lie group is *complete* if the center of K is trivial and $\mathbf{Aut}(K) = \mathbf{Inn}(K)$, thus $\mathbf{Inn}(K) \cong K$ via the conjugation map. In addition we say that K is *almost complete* if the center is trivial and the connected component of the identity $\mathbf{Aut}_0(K)$ of $\mathbf{Aut}(K)$ consists only of inner automorphisms.

Proposition 1.56. *Assume that we have an extension*

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

such that K is a complete group. Then the sequence splits and we have $G \cong K \times H$ with H being isomorphic to the centralizer of K in G. If K is almost complete and G connected the we obtain the same statement.

Proof. Consider the conjugation map $c\colon G \to \mathbf{Aut}(K)$, $g \mapsto c_g$ which can be considered as a map from $G \to K$ since K is complete. The kernel of this map is the centralizer $C_G(K)$ of K in G. Therefore we obtain another extension

$$1 \longrightarrow C_G(K) \longrightarrow G \longrightarrow K \longrightarrow 1.$$

Now the inclusion map from K into G is a splitting map. But conjugation on $C_G(K)$ by elements of K yields a trivial representation of K on $C_G(K)$ and we conclude $G \cong C_G(K) \times K$. Finally we have $C_G(K) \cong G/K \cong H$.

If K is almost complete and G connected then the image of G under the conjugation map lies in $\mathbf{Aut}_0(K)$. Repeating the arguments from above we obtain the proposition. ∎

Central Extensions

A central extension is a special case of an extension, but nevertheless we would like to dedicate an extra section for those short exact sequences. One reason is that central extensions are classified through a second cohomology group with abelian groups as coefficients.

Definition 1.57. An extension (i, π) with $i\colon K \to G$ and $\pi\colon G \to H$ is called a *central extension* if $i(K)$ lies in the center of G (in particular K has to be abelian).

Example 1.58. If G is a connected Lie group and $\tilde{G}$ is its universal covering group we obtain the natural central extension

$$1 \longrightarrow \pi_1(G) \longrightarrow \tilde{G} \to G \to 1$$

since fundamental groups of Lie groups are discrete subgroups of the center of its universal cover group. Clearly every group with non–trivial center forms a (non–trivial) central extension. A concrete example is given as follows: let $i\colon \mathbf{U}(1) \to \mathbf{U}(2)$ be given by mapping a $z \in \mathbf{U}(1)$ to the diagonal matrix $z \cdot E_2$ and $\pi\colon \mathbf{U}(2) \to (\mathbf{SU}(2)/\pm E_2)$ by $A \mapsto [\det(A)^{-\frac{1}{2}}A]$ which is a well–defined Lie homomorphism since the root of a complex number is unique modulo ± 1. But $\mathbf{SU}(2)/\pm E_2$ is isomorphic to $\mathbf{SO}(3)$ which can been seen by the orthogonal representation of $\mathbf{SU}(2)$ (or just note that the both groups have the same Lie algebras, hence – since $\mathbf{SU}(2)$ is simply connected – $\mathbf{SU}(2)$ is the universal cover group of $\mathbf{SO}(3)$ and since $\pi_1(\mathbf{SO}(3)) = \mathbb{Z}_2$ we have clearly $\mathbf{SU}(2)/\pm E_2 = \mathbf{SO}(3)$). Apparently $\mathbf{U}(1)$ is abelian and $\operatorname{im} i$ lies in the center of $\mathbf{U}(2)$ since every matrix commutes with multiples of the identity. Moreover $\pi(z \cdot E_2) = [\pm E_2]$ thus $\operatorname{im} i \subset \ker \pi$. Otherwise if $\pi(A) = [\pm E_2]$ then $A = \pm\sqrt{\det A} E_2 \in \operatorname{im} i$ which finally proves that (i, π) is a central extension.

Proposition 1.59. *If a central extension* $1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$ *splits, then* $G \cong K \times H$.

Proof. If $\sigma\colon H \to G$ is a splitting map, the associated representation $\rho\colon H \to \mathbf{Aut}(K)$ given by $\rho_h(k) = \sigma(h)i(k)\sigma(h)^{-1}$ is the trivial one, since $i(k)$ lies in the center of G. ■

We assume henceforth in this section that all Lie groups are *connected*. Suppose K and H are given, we would like to determine the group G which fits to such a central extension as in the definition above. We cannot expect that such an extension will split, like the central extension of Example 1.58

$$1 \longrightarrow \mathbf{U}(1) \longrightarrow \mathbf{U}(2) \longrightarrow \mathbf{SO}(3) \longrightarrow 1$$

shows. Say if the extension would split the group $\mathbf{U}(2)$ would be isomorphic to $\mathbf{U}(1)\times\mathbf{SO}(3)$ which cannot be since the fundamental group of $\mathbf{U}(2)$ is isomorphic to $\mathbb{Z}$ and that of $\mathbf{U}(1) \times \mathbf{SO}(3)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. One method to crack this extension problem is to go though the universal cover of G. The (abelian) fundamental groups of K, H and G build a(n) (central) extension for their own and thus knowing the fundamental group of G together with the knowledge of the universal cover group, as well as how $\pi_1(G)$ lies in $\tilde{G}$ determines the group G uniquely. All these facts are already encoded in a central extension as the next proposition shows us. Although this proposition works for extensions in general, we would like to put it nevertheless in this section about central extensions since we will use it solely for central extensions.

Proposition 1.60. *Let $1 \longrightarrow K \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1$ be an extension. Denote by $\pi_K\colon \tilde{K} \to K$, $\pi_G\colon \tilde{G} \to G$ and $\pi_H\colon \tilde{H} \to H$ the universal coverings of K, G and H respectively. Let $\tilde{i}$ and $\tilde{\pi}$ be the lifts of the maps i and π respectively. Then the following diagram commutes*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \pi_1(K) \arrow[r, "i_*"] \arrow[d] & \pi_1(G) \arrow[r, "\pi_*"] \arrow[d] & \pi_1(H) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \tilde{K} \arrow[r, "\tilde{i}"] \arrow[d, "\pi_K"'] & \tilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d, "\pi_G"'] & \tilde{H} \arrow[r] \arrow[d, "\pi_H"'] & 1 \\
1 \arrow[r] & K \arrow[r, "i"] \arrow[d] & G \arrow[r, "\pi"] \arrow[d] & H \arrow[r] \arrow[d] & 1. \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

and is exact in every horizontal as well as in every vertical direction where the fundamental groups are identified naturally with the kernels of the covering homomorphisms and i_* as well as π_* are the restricted maps of $\tilde{i}$ and $\tilde{\pi}$ respectively.

Proof. First, the vertical directions are clearly (central) extensions which are induced by the covering homomorphisms. By Lemma B.7 of Appendix B the corresponding subdiagrams commute which implies that the whole diagram commutes. Thus it remains to check the exactness of the horizontal sequences. Since the groups are connected $\pi_0(K) = 1$ and by Lemma B.11 of Appendix B we have $\pi_2(H) = 1$, thus by Lemma 1.12 the upper horizontal sequence is exact.

If we extend the diagram at the top and at the bottom commutatively by adding the trivial map $1 \mapsto 1$ we obtain with the two four–lemmas that $\tilde{i}$ is injective and $\tilde{\pi}$ is surjective. Hence the last thing to check is $\operatorname{im} \tilde{i} = \ker \tilde{\pi}$. The map $\tilde{\pi} \circ \tilde{i}\colon \tilde{K} \to \tilde{H}$ is uniquely determined by its derivative in the neutral element since $\tilde{K}$ is connected (see [War83, p. 101]). But the derivative in the neutral element of $\tilde{\pi} \circ \tilde{i}$ is the same as the derivative of $\pi \circ i$. And the map $\pi \circ i$ is the constant homomorphism thus $\tilde{\pi} \circ \tilde{i}$ is the constant homomorphism as well. We conclude $\operatorname{im} \tilde{i} \subset \ker \tilde{\pi}$. But the homomorphism $\tilde{i}\colon \tilde{K} \to \ker \tilde{\pi}$ has an injective derivative in the neutral element and since K has the same dimension as $\ker \tilde{\pi}$ the derivative is actually an isomorphism. Thus $\tilde{i}$ is a covering map from $\tilde{K}$ to ker˜*pi*, but $\tilde{K}$ is simply connected and we conclude $\operatorname{im} \tilde{i} = \ker \tilde{\pi}$. ■

We say a central extension $1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$ acts on a K–principal bundle

$p\colon M \to B$, if G acts on M such that restricted to $K \subset G$ we get the action of K on the principal bundle and H acts on B with the property that $p(g.m) = \pi(g).p(m)$ for all $m \in M$ and $g \in G$. Note moreover that if G is acting on M then any cover group of G acts on M through the covering homomorphism.

Proposition 1.61. *Suppose* $1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$ *acts on the trivial K–principal bundle $p\colon B \times K \to B$ and suppose futhermore that the central extension has a trivial universal cover extension (described in Proposition 1.60). Then G is isomorphic to $(\tilde{K} \times \tilde{H})/\pi_1(G)$ and the action of G on $B \times K$ is induced by the component wise action of $\tilde{H} \times \tilde{K}$ on $B \times K$.*

Proof. If the universal cover extension of $1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$ splits then clearly G is isomorphic to $(\tilde{H} \times \tilde{K})/\Gamma$ where $\Gamma := \ker \pi_G \cong \pi_1(G)$ (we adopt the notation of Proposition 1.60). The action of $\tilde{H} \times \tilde{K}$ is given through the covering homomorphisms $\pi_G\colon \tilde{H} \times \tilde{K} \to G$ as described above. And since $\tilde{H} \times \tilde{K}$ is a direct product, we have to check the actions of $(\tilde{h}, e)$ and $(e, \tilde{k})$ on $M = B \times K$ for $\tilde{h} \in \tilde{H}$ and $\tilde{k} \in \tilde{K}$. First note that $(e, \tilde{k}).(b, k) = \pi_G((e, \tilde{k})).(b, k)$ for $(b, k) \in B \times K$ by definition. Chasing through the diagram of 1.60 we see

$$\pi_G((e, \tilde{k})) = \pi_G \circ \tilde{i}(\tilde{k}) = i \circ \pi_K(\tilde{k})$$

which implies $(e, \tilde{k}).(b, k) = i \circ \pi_K(\tilde{k}).(b, k) = (b, i \circ \pi_K(\tilde{k})k)$ since $i \circ \pi_K(\tilde{k}) \in K$ and the action of G restricted to K is exactly the action of K on the trivial K–principal bundle $p\colon B \times K \to B$. Now let $(\tilde{h}, e) \in \tilde{K} \times \tilde{H}$ then again $(\tilde{h}, e).(b, k) = \pi_G(\tilde{h}, e)).(b, k)$. Using the commutativity of the diagram im Proposition 1.60 we get

$$p(\pi_G(\tilde{h}, e).(b, k)) = \pi(\pi_G(\tilde{h}, e)).b = \pi_H(\tilde{\pi}(\tilde{h}, e)).b = \pi_H(\tilde{h}).b$$

hence, since p is the projection to the B–factor, we get finally

$$(\tilde{h}, e)(b, k) = (\pi_H(\tilde{h}).b, k).$$

Thus to sum up, the action of $\tilde{H} \times \tilde{K}$ is the component wise action on $B \times K$ induced by the covering maps. Therefore the action of $[\tilde{h}, \tilde{k}] \in \tilde{H} \times \tilde{K}/\Gamma$ on (b, k) is given by $(\tilde{h}.b, \tilde{k}.k)$ for any representative $(\tilde{h}, \tilde{k})$ of $[\tilde{h}, \tilde{k}]$. ∎

Central Extensions of Lie algebras

Given a central extension of Lie groups Proposition 1.60 yields another central extension of simply connected groups. But those can be fully understood if one understands the central extension of the corresponding Lie algebras. We will prove that their isomorphism classes are given by the elements of a finite dimensional vector space.

Definition 1.62. Let $i\colon \mathfrak{k} \to \mathfrak{g}$ and $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be Lie algebra homomorphisms. We shall say the pair (i, π) or the sequence

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$$

is a *central extension of* $\mathfrak{k}$ *by* $\mathfrak{h}$ (or simply a *central extension*) if the sequence is exact, i is injective, π surjective and $i(K)$ lies in the center of $\mathfrak{g}$.

As in the case of extensions of Lie groups sometimes we will not distinguish between $\mathfrak{k}$ and its image $\imath(\mathfrak{k})$ in $\mathfrak{g}$. We say two central extensions are isomorphic if there is a Lie algebra homomorphism $\Phi\colon \mathfrak{g} \to \mathfrak{g}'$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{k} \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\Phi"] & \mathfrak{h} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathfrak{k} \arrow[r, "i'"] & \mathfrak{g}' \arrow[r, "\pi'"] & \mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

Note that Φ has to be an isomorphism by the five lemma. Let $\mathcal{E}(\mathfrak{k}, \mathfrak{h})$ be the set of central extensions $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0$ modulo isomorphisms.

Finally let us note that, in this section, we regard $\mathfrak{k}$ as the trivial $\mathfrak{h}$ module (see Appendix B, section Cohomology of Lie Algebras).

Remark 1.63. Let $\sigma\colon \mathfrak{h} \to \mathfrak{g}$ be a linear map for the extension $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$ such that $\pi \circ \sigma = \mathrm{id}_{\mathfrak{h}}$ (and we regard $\mathfrak{k}$ as a subliealgebra of $\mathfrak{g}$). We define a map $\omega_\sigma\colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{k}$ by $\omega_\sigma(x, y) := [\sigma(x), \sigma(y)] - \sigma([x, y])$. Indeed we have $\pi \circ \omega_\sigma(x, y) = [\pi \circ \sigma(x), \pi \circ \sigma(y)] - \pi \circ \sigma([x, y]) = 0$, since π is a Lie algebra homomorphism, thus ω_σ has values in $\mathfrak{k}$. Obviously ω_σ is a 2–form with values in $\mathfrak{k}$ and ω_σ is the trivial map iff σ is a splitting map for the central extension. For $x, y, z \in \mathfrak{h}$ we have

$$\omega_\sigma([x, y], z) + \omega_\sigma([y, z], x) + \omega_\sigma([z, x], y) = 0.$$

To see this we remark first that

$$\omega_\sigma([x, y], z) + \omega_\sigma([y, z], x) + \omega_\sigma([z, x], y) = [\sigma([x, y]), \sigma(z)] + [\sigma([y, z]), \sigma(x)] + [\sigma([y, z]), \sigma(x)]$$

since the other terms vanish due to the linearity of σ and the Jacobi identity in $\mathfrak{h}$. Writing $\sigma([x, y]) = [\sigma(x), \sigma(y)] - \omega_\sigma(x, y)$ as well as noticing that $\omega_\sigma(x, y)$ is a central element of $\mathfrak{g}$ we conclude by using the Jacobi identity in $\mathfrak{k}$ that the right hand side of the equation above vanishes. Thus ω_σ defines an element $c \in H^2$. The class c is independent of σ: let $\sigma'\colon \mathfrak{h} \to \mathfrak{g}$ be another map such that $\pi \circ \sigma' = \mathrm{id}_{\mathfrak{h}}$ and set $\alpha := \sigma - \sigma'$ which is an element of

$\mathbf{Hom}(\mathfrak{h},\mathfrak{k}) = C^1$ (using the notations of Appendix A). Further

$$\begin{aligned}\omega_\sigma(x,y) - \omega_{\sigma'}(x,y) &= [\sigma(x),\sigma(y)] - [\sigma'(x),\sigma'(y)] - \alpha([x,y]) \\ &= -[\alpha(x),\alpha(y)] + [\alpha(x),\sigma(y)] + [\sigma(x),\alpha(y)] - \alpha([x,y]) \\ &= -\alpha([x,y]) = (\delta^1\alpha)(x,y)\end{aligned}$$

since α takes values in the center of $\mathfrak{g}$, which shows c is independent of the linear map σ.

If $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g}' \xrightarrow{\pi'} \mathfrak{h} \longrightarrow 0$ is an isomorphic central extension via a map $\Phi\colon \mathfrak{g} \to \mathfrak{g}'$, then its cohomology class in $H^2(\mathfrak{h};\mathfrak{k})$ is equal to c from above. To see this, let $\omega_\sigma\colon \mathfrak{h}\times\mathfrak{h} \to \mathfrak{k}$ be a representative of c where $\sigma\colon \mathfrak{h} \to \mathfrak{g}$ is as above. Then $\omega_{\Phi\circ\sigma}$ is a cocycle representing the central extension of $\mathfrak{g}'$. But clearly $\omega_{\Phi\circ\sigma} = \Phi \circ \omega_\sigma$ and Φ is the identity on $\mathfrak{k}$ by definition of isomorphism of central extensions, which implies in fact $\omega_{\Phi\circ\sigma} = \omega_\sigma$.

Corollary 1.64. *Define a map $\Psi\colon \mathcal{E}(\mathfrak{k},\mathfrak{h}) \to H^2(\mathfrak{h};\mathfrak{k})$ as follows: If $e \in E(\mathfrak{k},\mathfrak{h})$ and choose a representative, i.e. a central extension representing e and let $\Psi(e)$ be the cohomology class $c \in H^2(\mathfrak{h};\mathfrak{k})$ as in Remark 1.63. There we showed also that this map is well–defined, i.e. it does not depend on the chosen representative of e.*

Proposition 1.65. *Let $\omega \in C^2 = \mathbf{Hom}(\wedge^2\mathfrak{h},\mathfrak{k})$ be a cochain (see Appendix A). Then the vector space $\mathfrak{k}\oplus\mathfrak{h}$ together with the vector–valued 2–form*

$$[(x_1,y_1),(x_2,y_2)] := (\omega(y_1,y_2),[y_1,y_2]).$$

is a Lie algebra denoted by $\mathfrak{k}\times_\omega\mathfrak{h}$. Moreover there is a central extension of $\mathfrak{k}$ by $\mathfrak{h}$

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{k}\times_\omega\mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$$

with the natural inclusion and natural projection as Lie homomorphisms. Additionally if $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$ is a central extension and $\sigma\colon \mathfrak{h} \to \mathfrak{k}$ a linear map such that $\pi\circ\sigma = \mathrm{id}_\mathfrak{h}$, then $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0$ and $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{k}\times_{\omega_\sigma}\mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$ are isomorphic, where ω_σ is defined as in Remark 1.63.

Proof. Surely the sequence $0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{k}\oplus\mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$ is an exact sequence of vector spaces. It remains to show that $\mathfrak{k}\oplus\mathfrak{h}$ with the Lie bracket defined above is indeed a Lie algebra and that the natural maps are Lie algebra homomorphisms. Let $x_1,x_2,x_3 \in \mathfrak{k}$ and $y_1,y_2,y_3 \in \mathfrak{h}$. Then

$$\Big[[(x_1,y_1),(x_2,y_2)],(x_3,y_3)\Big] = \Big[(\omega(y_1,y_2),[y_1,y_2]),(x_3,y_3)\Big] = \Big(\omega([y_1,y_2],y_3),\Big[[y_1,y_2],y_3\Big]\Big).$$

Using that ω is a cochain and the Jacobi identity on $\mathfrak{h}$ we obtain the claim. Since the Lie bracket of vectors $(x_1, 0)$ and $(x_2, 0)$ vanish and since $\mathfrak{k}$ is abelian, the natural inclusion is a Lie algebra homomorphism. If $\pi\colon \mathfrak{k} \oplus \mathfrak{h} \to \mathfrak{h}$ is the natural projection we compute

$$\pi\Big([(x_1, y_1), (x_2, y_2)]\Big) = [y_1, y_2] = [\pi(x_1, y_1), \pi(x_2, y_2)]$$

which shows that π is a Lie homomorphism.

We consider the linear map $\Phi\colon \mathfrak{k} \times_{\omega_\sigma} \mathfrak{h} \to \mathfrak{g}$, $\Phi(x, y) = x + \sigma(y)$ and we compute

$$\begin{aligned} \Phi([(x_1, y_1), (x_2, y_2)]) &= \Phi(\omega_\sigma(y_1, y_2), [y_1, y_2]) = \omega_\sigma(y_1, y_2) + \sigma([y_1, y_2]) = [\sigma(y_1), \sigma(y_2)] \\ &= [x_1 + \sigma(y_1), x_2 + \sigma(y_2)] = [\Phi(x_1, y_1), \Phi(x_2, y_2)]. \end{aligned}$$

Hence since $\Phi(x, 0) = x$ and $\pi \circ \Phi(0, y) = y$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{k} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathfrak{k} \times_{\omega_\sigma} \mathfrak{h} \arrow[r] \arrow[d, "\Phi"] & \mathfrak{h} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathfrak{k} \arrow[r] & \mathfrak{g} \arrow[r, "\pi"] & \mathfrak{h} \arrow[r] & 0 \end{tikzcd}$$

commutes which implies that those two extensions are isomorphic. ∎

Proposition 1.66. *The map $\Psi\colon \mathcal{E}(\mathfrak{k}, \mathfrak{h}) \to H^2(\mathfrak{h}; \mathfrak{k})$ is bijective. The inverse map is given by Proposition 1.65, i.e. given a $c \in H^2(\mathfrak{h}; \mathfrak{k})$ choose a cochain $\omega \in \mathbf{Hom}(\wedge^2\mathfrak{h}, \mathfrak{k})$ representing c and take the isomorphism class of the central extension induced by ω as in Proposition 1.65.*

Proof. Let $\Psi^{-1}\colon H^2(\mathfrak{h}; \mathfrak{k}) \to \mathcal{E}(\mathfrak{k}, \mathfrak{h})$ be the map introduced in this proposition. First we show Ψ^{-1} is well–defined. Thus let ω' be another cochain such that there is a linear map $\alpha \in \mathbf{Hom}(\mathfrak{h}, \mathfrak{k})$ with $\omega' = \omega + \delta^1(\alpha)$. Define $\Phi\colon \mathfrak{k} \times_\omega \mathfrak{h} \to \mathfrak{k} \times_{\omega'} \mathfrak{h}$ as $\Phi(x, y) := (x + \alpha(y), y)$ which is obviously an isomorphism of vector spaces. Moreover Φ respects the Lie bracket

$$\begin{aligned} \Phi([(x_1, y_1), (x_2, y_2)]) &= \Phi(\omega(y_1, y_2), [y_1, y_2]) = (\omega(y_1, y_2) - \alpha([y_1, y_2]), [y_1, y_2]) \\ &= (\omega'(y_1, y_2), [y_1, y_2]) = [(x_1 + \alpha(y_1), y_1), (x_2 + \alpha(y_2), y_2)] \\ &= [\Phi(x_1, y_1), \Phi(x_2, y_2)]. \end{aligned}$$

Finally this map induces an isomorphism between the two extensions determined by ω and ω', thus Ψ^{-1} is well–defined.

It remains to check, that Ψ^{-1} is the inverse map to Ψ. Let $e \in \mathcal{E}(\mathfrak{k}, \mathfrak{h})$ and let $1 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 1$ be a representative of e. Take $\sigma\colon \mathfrak{h} \to \mathfrak{g}$ such that $\pi \circ \sigma = \mathrm{id}_\mathfrak{h}$ and define ω_σ as in Remark 1.63. Then $\Psi(e) = [\omega_\sigma] \in H^2(\mathfrak{h}; \mathfrak{k})$ and by Proposition 1.65 $\Psi^{-1}([\omega_\sigma]) = e$. On the other side for $c \in H^2(\mathfrak{h}; \mathfrak{k})$ write $c = [\omega]$. The isomorphism class of $\Psi^{-1}(c)$ is represented by the central extension $1 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{k} \times_\omega \mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 1$. Choose $\sigma\colon \mathfrak{h} \to \mathfrak{k} \times_\omega \mathfrak{h}$ by $\sigma(y) := (0, y)$ and we obtain $\omega_\sigma(y_1, y_2) = (\omega(y_1, y_2), 0)$, thus $\omega_\sigma = \omega$ since $\mathfrak{k}$ is embedded as $x \mapsto (x, 0)$ hence $\Psi([\omega_\sigma]) = c$. ■

Corollary 1.67. *The zero element in $H^2(\mathfrak{h}, \mathfrak{k})$ corresponds under the bijection of Proposition 1.66 to the trivial central extension.*

Proof. The zero class is represented by the trivial map $\omega = 0$ and $\mathfrak{k} \times_\omega \mathfrak{h}$ is the product Lie algebra. ■

Example 1.68. Let $\mathfrak{k} = \mathbb{R}$ and $\mathfrak{e}(2)$ the Lie algebra of the isometry group of the euclidean plane. The latter Lie algebra is given by a semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ where

$$t \mapsto \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$$

is the action of $\mathbb{R}$ on $\mathbb{R}^2$, see Appendix A in the section Lie Algebras for the definition of semidirect products of Lie algebras. To see that this is the Lie algebra of $\mathbf{E}(2)$ we use Lemma A.6 of Appendix A. We would like to compute $H^2(\mathfrak{e}(2); \mathbb{R})$ where again $\mathbb{R}$ is the trivial $\mathfrak{e}(2)$–module. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^2 \rtimes \mathbb{R}$ seen as $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. The Lie bracket is given by

$$[e_1, e_2] = 0, \; [e_1, e_3] = -e_2, \; [e_2, e_3] = e_1.$$

Every $\omega \in C^2 = \mathbf{Hom}(\wedge^2(\mathfrak{e}(2)), \mathbb{R})$ is given in the standard basis by the matrix (where ω is seen as a bilinear form on $\mathfrak{e}(2)$)

$$(\omega_{ij}) = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}.$$

with $\omega_{ij} = \omega(e_i, e_j)$. Clearly $\delta\omega$ is a 3–form and since $\mathfrak{e}(2)$ is 3–dimensional, the space of 3–forms is one–dimensional thus we have only to check the value $\delta\omega(e_1, e_2, e_3)$. But

$$\delta\omega(e_1, e_2, e_3) = \omega([e_1, e_2], e_3) + \omega([e_2, e_3], e_1) + \omega([e_3, e_1], e_2) = 0.$$

If $\alpha \in C^1 = \mathbf{Hom}(\mathfrak{e}(2), \mathbb{R})$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_i = \alpha(e_i)$ its coordinate representation, the 2–form $\delta\alpha$ is then given by

$$\delta\alpha = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}.$$

This implies that for every element of $H^2(\mathfrak{e}(2);\mathbb{R})$ there is a unique representative of the form

$$\omega_\lambda = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for a $\lambda \in \mathbb{R}$ and therefore $H^2(\mathfrak{e}(2);\mathbb{R}) \cong \mathbb{R}$ via the isomorphism $\lambda \mapsto [\omega_\lambda]$. The respective central extensions are given by the Lie algebras $\mathbb{R} \times_{\omega_\lambda} \mathfrak{e}(2)$ with the natural inclusion of $\mathbb{R}$ and natural projection onto the $\mathfrak{e}(2)$–factor. Let e_0, e_1, e_2, e_3 be the canonical basis of $\mathbb{R} \times_{\omega_\lambda} \mathfrak{e}(2) = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$ where e_1, e_2, e_3 are chose as above. The Lie bracket is given then by

$$\begin{aligned} [e_0, e_i] &= 0,\ i = 1, \ldots, 3 \\ [e_1, e_2] &= \lambda e_0,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1. \end{aligned}$$

Finally from here we may compute $H^1(\mathfrak{e}(2);\mathbb{R})$ easily, since the image $\delta(C^1)$ is two–dimensional (see above), hence the cocycles has to have dimension one. Moreover $\delta\colon C^0 \to C^1$ is the zero map and this implies $\mathfrak{e}(2)/[\mathfrak{e}(2), \mathfrak{e}(2)] = H^1(\mathfrak{e}(2);\mathbb{R}) = \mathbb{R}$.

For our purposes it is sufficient to have a weaker notion of isomorphism classes. We shall say that two central extensions are *weakly isomorphic* if the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{k} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{h} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{k} \arrow[r] & \mathfrak{g}' \arrow[r] & \mathfrak{h} \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are isomorphisms. For two cochains ω_1 and ω_2 we shall say that ω_1 is (weakly) isomorphic to ω_2 if their central extensions are. The set of those isomorphism classes shall be denoted by $\mathcal{W}(\mathfrak{k}, \mathfrak{h})$. Obviously isomorphic central extensions are weakly isomorphic and therefore computing the second cohomology gives us an 'upper bound' for $\mathcal{W}(\mathfrak{k}, \mathfrak{h})$. So as a next step we would like to clarify the relation between $H^2(\mathfrak{h};\mathfrak{k})$ and $\mathcal{W}(\mathfrak{k}, \mathfrak{h})$.

Proposition 1.69. *Let* $(\varphi, \psi) \in \mathbf{A} := \mathbf{Aut}(\mathfrak{k}) \times \mathbf{Aut}(\mathfrak{h})$ *and* $\omega \in \mathbf{Hom}(\wedge^2\mathfrak{h}, \mathfrak{k})$ *be a cochain. The* 2*–form*

$$((\varphi, \psi).\omega)(x, y) := \varphi(\omega(\psi^{-1}(x), \psi^{-1}(y)))$$

is a cochain again. This defines a linear action of $\mathbf{A}$ *on* $H^2(\mathfrak{h};\mathfrak{k})$.

Proof. It is pretty clear that $(\varphi, \psi).\omega$ is a cochain again since φ and ψ are Lie homomorphisms and ω a cochains. Let $\omega' \in [\omega]$ and $\alpha \in \mathbf{Hom}(\mathfrak{h}, \mathfrak{k})$ such that $\omega = \omega' + \delta\alpha$. Then $(\varphi, \psi).\omega =$

$(\varphi,\psi).\omega' + (\varphi,\psi).(\delta\alpha)$ and

$$\begin{aligned}(\varphi,\psi).(\delta\alpha)(x,y) &= \varphi\circ\delta\alpha(\psi^{-1}(x),\psi^{-1}(y)) = -\varphi\circ\alpha([\psi^{-1}(x),\psi^{-1}(y)])\\ &= \delta(\varphi\circ\alpha\circ\psi^{-1})(x,y)\end{aligned}$$

thus $[(\varphi,\psi).\omega] = [(\varphi,\psi).\omega']$. This defines a linear action, since the action on the cochain level is already linear. ∎

Proposition 1.70. *There is a bijection between $\mathcal{W}(\mathfrak{k},\mathfrak{h})$ and $H^2(\mathfrak{h};\mathfrak{k})/A$.*

Proof. Let $\Phi\colon H^2(\mathfrak{h};\mathfrak{k}) \to \mathcal{W}(\mathfrak{k},\mathfrak{h})$ be the map which assigns a representative ω of $c \in H^2(\mathfrak{h};\mathfrak{k})$ its isomorphism class $w = [\omega]$ in $\mathcal{W}(\mathfrak{k},\mathfrak{h})$. This is well–defined, since any other $\omega' \in c$ is weakly isomorphic to ω hence $\omega' \in w$. Clearly Φ is surjective; see Proposition 1.65. Now assume that $\Phi(c_1) = \Phi(c_2)$ and let $\omega_i \in c_i$ be representatives for $i = 1, 2$ thus ω_1 and ω_2 are weakly isomorphic. Hence there is $(\varphi,\psi) \in \mathbf{Aut}(\mathfrak{k}) \times \mathbf{Aut}(\mathfrak{h})$ and $\Phi\colon \mathfrak{k}\times_{\omega_1}\mathfrak{h} \to \mathfrak{k}\times_{\omega_2}\mathfrak{h}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{k} \arrow[r] \arrow[d, "\varphi"] & \mathfrak{k}\times_{\omega_1}\mathfrak{h} \arrow[r] \arrow[d, "\Phi"] & \mathfrak{h} \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathfrak{k} \arrow[r] & \mathfrak{k}\times_{\omega_2}\mathfrak{h} \arrow[r, "\pi"] & \mathfrak{h} \arrow[r] & 0 \end{tikzcd}$$

commutes. By the commutativity of the diagram there is an $\alpha \in \mathbf{Hom}(\mathfrak{h},\mathfrak{k})$ such that $\Phi(x,y) = (\varphi(x) + \alpha(y), \psi(y))$. Moreover

$$\Phi([(x_1,y_1),(x_2,y_2)]) = \Phi(\omega_1(y_1,y_2),[y_1,y_2]) = (\varphi\circ\omega_1(y_1,y_2) + \alpha([y_1,y_2]),\psi([y_1,y_2]))$$

and on the other side

$$\begin{aligned}[\Phi(x_1,y_1),\Phi(x_2,y_2)] &= [(\varphi(x_1)+\alpha(y_1),\psi(y_1)),(\varphi(x_2)+\alpha(y_2),\psi(y_2))]\\ &= (\omega_2(\psi(y_1),\psi(y_2)),[\psi(y_1),\psi(y_2)])\end{aligned}$$

Thus $(\varphi,\psi).\circ\omega_1 = \omega_2 + \delta(\alpha\circ\psi^{-1})$ and this implies of course $(\varphi,\psi).c_1 = c_2$. ∎

Remark 1.71. Back to our Example 1.68. There we have computed that $H^2(\mathfrak{e}(2);\mathbb{R}) = \mathbb{R}$ and now by Proposition 1.70 $\mathbf{A} := \mathbb{R}^* \times \mathbf{Aut}(\mathfrak{e}(2))$ where $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$. Since the action is linear, the trivial central extensions is an orbit. Except for the trivial orbit there is one other orbit given by $\mathbb{R}^*$. If $\lambda \in H^2(\mathfrak{e}(2);\mathbb{R})$ with $\lambda \neq 0$ choose the automorphism $(\lambda^{-1},\mathrm{id})$ and this shows that every λ is equivalent to $1 \in \mathbb{R}$ under the action of $\mathbf{A}$. Hence $|\mathcal{W}(\mathfrak{k},\mathfrak{h})| = 2$ by Proposition 1.70 and the Lie algebras are determined by ω_0 and ω_1 of Example 1.68. The corresponding simply connected Lie group to ω_0 is given by $\mathbb{R}\times\widetilde{\mathbf{E}(2)}$ where $\widetilde{\mathbf{E}(2)}$ is the universal cover group to $\mathbf{E}(2)$. The group for ω_1 is a bit more complicated. Let **Nil** denote the 3–dimensional Heisenberg group which is simply connected. We regard **Nil** as $\mathbb{R}^3$ equipped with a special multiplication (see [Sco83, p. 467] for more details). We define an action of $\mathbb{R}$ on **Nil** as in [Sco83, p. 467]: for $\theta \in \mathbb{R}$ and $(\mathbf{x}, z) \in \mathbf{Nil}$ set

$$\theta.(\mathbf{x}, z) := (\rho_\theta(\mathbf{x}), z + \tfrac{1}{2}s(cy^2 - cx^2 - 2sxy))$$

where $\rho_\theta(\mathbf{x})$ rotates $\mathbf{x}$ through θ, $\mathbf{x} = (x, y)$, $s := \sin\theta$ and $c := \cos\theta$. Set $\tilde{N} := \mathbf{Nil} \rtimes_\theta \mathbb{R}$ which is simply connected and its Lie algebra is $\mathbb{R} \times_{\omega_1} \mathfrak{e}(2)$ (see Appendix B).

The Space of Homogeneous Metrics

In general the space of Riemannian metrics on an arbitrary manifolds is very big and is always infinite dimensional. But if G is acting on M transitively with compact isotropy groups there is a nice finite dimensional submanifold in the space of Riemannian metrics such that every element there is a G–invariant metric. Let $\mathcal{RM}$ denote the space of Riemannian metrics on the manifold M. Then G is acting by pullbacks on $\mathcal{RM}$, i.e. $g.\mu := g^*(\mu)$. Denote by $\mathcal{S}^+$ the fixed–point set of this action.

Definition 1.72. We shall call $\mathcal{S}^+$ the *space of G–invariant metrics*. This set contains by definition Riemannian metrics which are invariant under the action of G which means that the elements of G are isometries for those metrics.

Remark 1.73. Let $\mathcal{S}^+_K$ be the space of K–invariant scalar products on TM_{m_0}. Then there is a bijection between $\mathcal{S}^+_K$ and $\mathcal{S}^+$ (see Remark 1.6). The space $\mathcal{S}^+_K$ carries a nice geometric structure, indeed it is a symmetric space.

Let V be a real vector space of finite dimension and K a compact group acting faithfully and linear on V. Therefore we may regard K as a proper closed subgroup of $\mathbf{GL}(V)$. Let $S^+(V)$ be the space of euclidean products on V which is an open cone in a real vector space of finite dimension, thus in particular a finite dimensional manifold. This space is a homogeneous space itself: the group $\mathbf{GL}(V)$ acts on $V^* \otimes V^*$ naturally and this action descends to a transitive action of $\mathbf{GL}(V)$ on $S^+(V)$. The isotropy group is in $\mu_0 \in \mathcal{S}^+(V)$ is $\mathbf{O}(V, \mu_0)$ hence $S^+(V) = \mathbf{GL}(V)/\mathbf{O}(V, \mu_0)$ which is a symmetric space in particular a Riemannian homogeneous space.

This space has a tautological Riemannian metric: the tangent bundle of $\mathcal{S}^+(V)$ is given by the manifold $\mathcal{S}^+(V) \times \mathbf{Sym}(V)$ where $\mathbf{Sym}(V)$ is the vector space of symmetric bilinear forms. Let $\mu \in \mathcal{S}^+(V)$ and $\sigma_1, \sigma_2 \in T\mathcal{S}^+(V)_\mu = \mathbf{Sym}(V)$. Denote by $\sigma^\sharp$ the endomorphism of

V given by $\mu(\sigma^\sharp(v), w) = \sigma(v, w)$ where $\sigma \in \mathbf{Sym}(V)$. Then define

$$\langle \sigma_1, \sigma_2 \rangle_\mu := \operatorname{tr}(\sigma_1^\sharp \circ \sigma_2^\sharp)$$

which is a euclidean scalar product on $\mathbf{Sym}(V)$ for every μ. If $(e_1, \dots, e_n)$ is an orthonormal basis of (V, μ), then we obtain by definition

$$\langle \sigma_1, \sigma_2 \rangle_\mu = \sum_{i,j=1}^{n} \sigma_1(e_i, e_j)\sigma_2(e_i, e_j)$$

and therefore

$$|\sigma|_\mu^2 := \langle \sigma, \sigma \rangle_\mu = \sum_{i=1}^{n} \lambda_i^2$$

where the λ_i are the eigenvalues of $\sigma^\sharp$. Finally the euclidean scalar product in the tangent space $T\mathcal{S}^+(V)_\mu = \mathbf{Sym}(V)$ is the induced product from μ on $V^* \otimes V^*$ restricted to the symmetric bilinear forms.

Proposition 1.74. *The metric $\langle \cdot, \cdot \rangle$ is a $\mathbf{GL}(V)$–invariant metric on $\mathcal{S}^+(V)$.*

Proof. Let $g \in \mathbf{GL}(V)$ and $\mu \in \mathcal{S}^+(V)$ and let $g^*(\langle \cdot, \cdot \rangle)$ be the pullback of the metric by the diffeomorphism $g \colon \mathcal{S}^+(V) \to \mathcal{S}^+(V)$. For $\sigma_1, \sigma_2 \in T\mathcal{S}^+(V)_\mu$ we have

$$g^*(\langle \cdot, \cdot \rangle)_\mu(\sigma_1, \sigma_2) = \langle \mathrm{D}g(\sigma_1), \mathrm{D}g(\sigma_2) \rangle_{g.\mu}$$

and note that $\mathrm{D}g(\sigma) = g.\sigma$ is the action of $\mathbf{GL}(V)$ on $V^* \otimes V^*$. Let $(g.\sigma_i)^\natural$ be the endomorphism of $g.\sigma_i$ with respect to $g.\mu$ and $\sigma_i^\sharp$ the endomorphism of σ_i with respect to μ. Then

$$\begin{aligned}(g.\mu)((g.\sigma_i)^\natural(v), w) &= g.\sigma_i(v, w) = \sigma_i(g^{-1}(v), g^{-1}(w)) = \mu(\sigma_i^\sharp(g^{-1}(v)), g^{-1}(w)) \\ &= (g.\mu)(g(\sigma_i^\sharp(g^{-1}(v))), w)\end{aligned}$$

for all $v, w \in V$, hence $(g.\sigma_i)^\natural = g(\sigma_i^\sharp)g^{-1}$ and consequently

$$\langle g.\sigma_1, g.\sigma_2 \rangle_{g.\mu} = \operatorname{tr}((g.\sigma_1)^\natural (g.\sigma_2)^\natural) = \operatorname{tr}(g\sigma_1^\sharp g^{-1} g\sigma_2^\sharp g^{-1}) = \operatorname{tr}(\sigma_1^\sharp \sigma_2^\sharp) = \langle \sigma_1, \sigma_2 \rangle_\mu.$$

∎

Let $\mathcal{S}_K^+(V)$ be fixed–point set of the induced K–action on $\mathcal{S}^+(V)$. Then we obtain the

Corollary 1.75. *The space $S_K^+(V)$ is an embedded, totally geodesic and closed submanifold of $S^+(V)$ and therefore a symmetric space as well.*

Proof. The group K acts on $(\mathcal{S}^+(V), \langle\cdot,\cdot\rangle)$ by isometries. Using the theorem in [Kob95, p.59] like Remark 1.36 we obtain the claim (since the submanifold is totally geodesic, it inherits the geodesics symmetries from the big space). ∎

So our space $\mathcal{S}_K^+$ from the beginning of this section is a symmetric space. Since $\mathcal{S}_K^+(V)$ is convex it is contractible and therefore the tangent bundle is trivial, say $\mathcal{P} := T\mathcal{S}_K^+(V) = \mathcal{S}_K^+(V) \times \mathbf{Sym}_K(V)$ where $\mathbf{Sym}_K(V)$ are the symmetric bilinear forms on V invariant under the induced action of K on $V^* \otimes V^*$. Thus the topology of $\mathcal{P}$ is the relative topology of $\mathbf{Sym}(V) \times \mathbf{Sym}(V)$. Further we are interested to determine compact sets in $\mathcal{P}$ and therefore we will use a certain norm on $\mathbf{Sym}_K(V)$.

Remark 1.76. Let β be a background metric on V and $\kappa \in \mathbf{Sym}(V)$. Then we define the *maximum norm* of κ by

$$|\kappa|_\infty := \sup_{\|v\|_\beta = 1} \|\kappa^\natural(v)\|$$

where $\|v\|^2 = \beta(v,v)$ for $v \in V$ and $\kappa^\natural$ is the self–adjoint endomorphism to κ with respect to β. Obviously $|\kappa|_\infty = |\lambda_{\max}|$ where $|\lambda_{\max}| := \max_i\{|\lambda_i| : \lambda_i \text{ is Eigenvalue of } \kappa^\natural\}$. If we denote by $|\kappa|_\beta := \langle\kappa,\kappa\rangle_\beta$ we obtain

$$|\kappa|_\infty \leq |\kappa|_\beta.$$

Let $\mu \in \mathcal{S}^+(V) \subset \mathbf{Sym}(V)$ and $(e_1,\ldots,e_n)$ be an orthonormal basis for (V,β) such that $\mu^\natural(e_i) = \lambda_i e_i$ for all i, hence λ_i are positive real numbers. The vectors $(\sqrt{\lambda_1}^{-1}e_1,\ldots,\sqrt{\lambda_n}^{-1}e_n)$ are an orthonormal basis of (V,μ) and thus we compute

$$\begin{aligned} |\kappa|_\beta^2 &= \sum_{i,j=1}^n \kappa(e_i,e_j)^2 = \sum_{i,j=1}^n \lambda_i\lambda_j\kappa\left(\sqrt{\lambda_i}^{-1}e_i, \sqrt{\lambda_j}^{-1}e_j\right)^2 \\ &\leq |\mu|_\infty^2 |\kappa|_\mu^2 \end{aligned}$$

which sums up to

$$|\kappa|_\infty \leq |\kappa|_\mu |\mu|_\infty.$$

Example 1.77. Let $K = \mathbf{SO}(2)$ acting linear on a three dimensional vector space V. Using Proposition 1.4 let β be a K–invariant scalar product on V. Then Proposition 2.8 tells us that there are subspaces L and U of V such that $\dim L = 1$, $L \oplus U = V$, $kl = l$ for all $l \in L$ and L is the orthogonal complement to U with respect to β. If $u \in U$ we see that $\beta(ku,l) = 0$ for all l and therefore $k \in K$ maps U to itself. Thus $V = L \oplus U$ is a decomposition into irreducible subspaces. Let μ be another K–invariant scalar product on V and $\mu^\natural$ as above. Then $\mu^\sharp$ is K–equivariant since

$$\beta(\mu^\natural(kv), w) = \mu(kv, w) = \mu(v, k^{-1}w) = \beta(\mu^\natural(v), k^{-1}w) = \beta(k\mu^\natural(v), w)$$

and therefore $\mu^\sharp$ respects the decomposition $L \oplus U$. By Schur's Lemma $\mu^\sharp$ is given as

$$\begin{pmatrix} \lambda_1 \mathrm{id}_L & 0 \\ 0 & \lambda_2 \mathrm{id}_U \end{pmatrix}$$

for $\lambda_1, \lambda_2 > 0$ since μ is positive–definite. If $l \in L$ with $\beta(l, l) = 1$ then there is a diffeomorphism between $\mathcal{S}^+_k(V)$ and $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ given by

$$\mu \mapsto (\mu(l, l), \mathrm{tr}_\beta(\mu) - \mu(l, l)) = (\lambda_1, 2\lambda_2).$$

As a symmetric space $\mathcal{S}^+_K(V)$ is given by the subgroup $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times K$ of $\mathbf{GL}^+(V)$ acting on V by $(r_1, r_2, k)(l + u) = r_1 l + r_2 ku$. In particular this group is abelian and by Lemma 1.26 $\mathcal{S}^+_K(V)$ is flat if we restrict the metric from $\mathcal{S}^+(V)$ to $\mathcal{S}^+_K(V)$. Analogous we may identify $\mathbf{Sym}_K(V)$ with $\mathbb{R} \times \mathbb{R}$.

2

Homogeneous Three–Manifolds

Like in the last chapter we would like to fix some notations. As usual we denote by (M, G) a Riemannian homogeneous space and we denote the isotropy group in $m_0 \in M$ by K. Furthermore we demand that M is simply connected and of dimension three. We already noticed in the introduction that we are going to classify the simply-connected 3–dimensional Riemannian homogeneous spaces.

Preliminary Considerations

First, the maximal dimension d_G of G cannot exceed 6 since the maximal dimension of the isotropy group is 3 which is the dimension of $\mathbf{O}(3)$ (see Corollary 1.5). Accordingly we have $3 \leq d_G \leq 6$ but we can exclude one case due to the Lie group structure of $\mathbf{O}(3)$.

Proposition 2.1. *There is no two dimensional subgroup of* $\mathbf{O}(3)$.

Proof. Suppose K is a Lie subgroup of $\mathbf{O}(3)$ of dimension two. Hence there is a two dimensional Lie subalgebra $\mathfrak{k}$ of the Lie algebra of $\mathbf{O}(3)$ which is isomorphic to $\mathbb{R}^3$ with the cross product denoted by $\times$. Let (b_1, b_2) be a basis of $\mathfrak{k}$, then $b_1 \times b_2$ lies in the orthogonal complement of $\mathfrak{k}$ with respect to the euclidean product on $\mathbb{R}^3$. There we have a contradiction to the fact that $\mathfrak{k}$ has to be a Lie subalgebra. ∎

And this leads us to the following definition.

Definition 2.2. Let (M, G) be a Riemannian homogeneous space with $\dim M = 3$ and M simply connected which we will call henceforth a *geometry*. We shall say a geometry (M, G)

is *isotropic* if $\dim G = 6$, *rotationally symmetric* if $\dim G = 4$ and otherwise we call (M, G) a *Bianchi group*.

In order to classify the geometries it is very useful to work with a connected Lie group G. As we saw in Proposition 1.10 the connected component of the identity G_0 of G acts still transitive on M. After knowing this component one has to deduce the possible other components. For that one could ask what is the biggest group of the same dimension where G_0 lies as the identity component.

Definition 2.3. Let (M_1, G_1) and (M_2, G_2) be two Riemannian homogeneous spaces . We say (M_1, G_1) is contained in (M_2, G_2), indicated by $(M_1, G_1) \leq (M_2, G_2)$, if there is a diffeomorphism $F\colon M_1 \to M_2$ and a monomorphism of Lie groups $f\colon G_1 \to G_2$ such that F is f–invariant, i.e. $F(g_1.m_1) = f(g_1).F(m_1)$ for all $g_1 \in G_1$ and $m_1 \in M_1$. We say (M^*, G^*) is *maximal* if for every (M, G) with $(M^*, G^*) \leq (M, G)$ we have that (M, G) is isomorphic to (M^*, G^*). Furthermore we shall say that (M^*, G^*) is *maximal in a fixed dimension* if for every (M, G) with $(M^*, G^*) \leq (M, G)$ with $\dim G = \dim G^*$ we have that (M, G) is isomorphic to (M^*, G^*).

Proposition 2.4. *If (M, G) is an isotropic geometry, then G can only have one or two components. If (M, G) is a rotationally symmetric geometry the number of possible components are* 1, 2 *or* 4.

Proof. Since M is simply connected we have by Lemma 1.12 that $\pi_0(K)$ is isomorphic to $\pi_0(G)$. Hence if (M, G) is isotropic then K is a subgroup of $\mathbf{O}(3)$ with the same dimension. It follows that K can have only one or two components. Now let us turn to the case where (M, G) is rotationally symmetric. If $K \cap (\mathbf{O}(3) \setminus \mathbf{SO}(3)) \neq \emptyset$ then $K_1 := K \cap \mathbf{SO}(3)$ lies in K with index two. This follows from the short exact sequence

$$1 \longrightarrow \mathbf{SO}(3) \longrightarrow \mathbf{O}(3) \longrightarrow \pi_0(\mathbf{O}(3)) \longrightarrow 1$$

where $\pi_0(\mathbf{O}(3)) \cong \mathbb{Z}_2$. Hence we have an exact sequence for K

$$1 \longrightarrow K_1 \longrightarrow K \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

since $K \cap (\mathbf{O}(3) \setminus \mathbf{SO}(3)) \neq \emptyset$. Therefore it is sufficient to understand the number of connected components of K_1. Now suppose $K \subset \mathbf{SO}(3)$ and $\dim K = 1$. Clearly we have $K_0 \cong S^1$ and $\pi_0(K)$ is finite, since K is compact. And obviously $\mathbf{SO}(3)/K_0$ is a two–sphere which induces a covering $\pi\colon S^2 \to \mathbf{SO}(3)/K$ with fiber $\pi_0(K)$, hence $\mathbf{SO}(3)/K \cong S^2/\pi_0(K)$. Using Lemma A.8 we obtain that $S^2/\pi_0(K)$ is either diffeomorphic to S^2 or to the projective plane (observe that $\pi_0(K)$ is compact and therefore it acts properly on S^2). Hence $\pi_0(K)$ is either trivial or of order two. ∎

Isotropic geometries

This section is devoted to classify the isotropic geometries. Before stating the main theorem for this section we would like to introduce the isometry groups of the Riemannian spaces of constant sectional curvature.

Example 2.5. (a) We start with the Riemannian manifold $(M, \mu) = (\mathbb{R}^n, \delta)$ where δ is the standard euclidean metric on $\mathbb{R}^n$. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $(\mathbb{R}^n, \delta)$ then the map $x \mapsto f(x) - (0)$ is still an isometry and fixes the origin. Thus we may assume $f(0) = 0$. Set $A := (\mathrm{D}f)_0 \in \mathbf{O}(n)$ (see Corollary 2.21) where $(\mathrm{D}f)_0\colon T\mathbb{R}^n_0 \to T\mathbb{R}^n_0$ is the derivative of f in 0. But the map $f_A\colon \mathbb{R}^n \to \mathbb{R}^n$, $f_A(x) = A \cdot x$ is an isometry which fixes the origin and has same derivate in 0 as f. Thus $f = f_A$ and if $f(0) =: a$ we have $f(x) = Ax + a$.

Let G be the isometry group of $(\mathbb{R}^n, \delta)$. The argument above shows that there is a surjective map $\pi\colon G \to \mathbf{O}(n)$ which assigns every isometry its linear part, i.e. if $f(x) = Ax + a$ for $A \in \mathbf{O}(n)$ and $a \in \mathbb{R}^n$ we have $\pi(f) = A$. Since we regard G as a subgroup of $\mathrm{Diff}(\mathbb{R}^n)$ we obtain that π is a homomorphism. Clearly the kernel of π is isomorphic to $\mathbb{R}^n$ where we identify $a \in \mathbb{R}^n$ in $\ker \pi$ with a translation on $\mathbb{R}^n$ by the vector a. This shows that we get the extension

$$1 \longrightarrow \mathbb{R}^n \longrightarrow G \longrightarrow \mathbf{O}(n) \longrightarrow 1,$$

thereby the group $\mathbf{O}(n)$ is the isotropy group of G in $0 \in \mathbb{R}^n$. This extension splits via the map $\sigma\colon \mathbf{O}(n) \to G$, $\sigma(A) = f_A$ where $f_A(x) = Ax$ for $A \in \mathbf{O}(n)$ and $x \in \mathbb{R}^n$. Thus with Proposition 1.51 we obtain $G \cong \mathbb{R}^n \rtimes_\rho \mathbf{O}(n)$ where $\rho\colon \mathbf{O}(n) \to \mathbf{GL}(\mathbb{R}^n)$ is given by

$$\rho_A(a) = \sigma(A) a \sigma(A)^{-1}$$

and as a diffeomorphism of $\mathbb{R}^n$ we get

$$\rho_A(a)(x) = \sigma(A) a \sigma(A)^{-1}(x) = \sigma(A)(A^{-1}x + a) = x + Aa$$

thus the translation is given by $A \cdot a$ and $\rho\colon \mathbf{O}(n) \to \mathbf{GL}(\mathbb{R}^n)$ is the standard action of $\mathbf{O}(n)$ on $\mathbb{R}^n$. We define $\mathbf{E}(n) := \mathbb{R}^n \rtimes_\rho \mathbf{O}(n)$.

(b) Let $(M, \mu) = (S^n, \mu_S)$ where μ_S is the round metric on S^n and G its full isometry group. We could have used the following method also in part (a), however we believe that the method above fits better in the context of this thesis.

Obviously we have $\mathbf{O}(n+1) \subset G$, where the orthogonal group acts by the standard action of S^n. If $g \in G$ and N is the north pole of S^n then there is a $A \in \mathbf{O}(n+1)$ such that

$g \circ A(N) = N$. The derivative $(\mathrm{D}g \circ A)_N \colon TS^n_N \to TS^n_N$ lies in $\mathbf{O}(n)$, therefore there is a $K \in \mathbf{O}(n)$ such that $g \circ A = K$ (observe that we used here that $\mathbf{O}(n)$ may be identified with the isotropy group in N from the action of $\mathbf{O}(n+1)$ on S^n). Hence $G \subset \mathbf{O}(n+1)$ and therefore $G = \mathbf{O}(n+1)$.

(c) In the last example we would like to introduce the hyperbolic space with its isometry group. Let first $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^{n+1}$ with Lorentzian signature such that we have $\langle e_{n+1}, e_{n+1} \rangle = -1$ for $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Let $\mathbf{O}(n,1)$ be the group of linear automorphisms of $\mathbb{R}^n$ which preserve $\langle \cdot, \cdot \rangle$ and let D^n be the set of all $x \in \mathbb{R}^{n+1}$ such that $\langle x, x \rangle = -1$ and $x_{n+1} > 0$ where $x = (x_1, \ldots, x_{n+1})$. Using the stereographic projection of D^n through the point $(0, \ldots, -1)$ we obtain that D^n is diffeomorphic to the unit ball in $\mathbb{R}^n$. The subgroup $\mathbf{O}^+(n,1)$ of $\mathbf{O}(n,1)$ which preserves the sign of the last coordinate acts transitively on D^n and the prove goes analogous to that one of $\mathbf{O}(n)$ acting on S^n transitively. The connected component of $\mathbf{O}^+(n,1)$ are given by the linear maps of determinant equal to 1 and we denote this group by $\mathbf{SO}^+(n,1)$. The isotropy group of the point $e_{n+1} \in D^n$ is isomorphic to $\mathbf{O}(n)$. The restriction of $\langle \cdot, \cdot \rangle$ to D^n induces a Riemannian metric μ_D on D^n of constant sectional curvature equal to -1 and is invariant under $\mathbf{O}^+(n,1)$. Same arguments as in part (b) show that $\mathbf{O}(n,1)$ is the full isometry group of (D^n, μ_D).

Theorem 2.6. *Let (M,G) be an isotropic geometry with G connected. Then (M,G) is isomorphic to one of the following geometries:*

$$(\mathbb{R}^3, \mathbf{E}_0(3)), \quad (S^3, \mathbf{SO}(3)), \quad (D^3, \mathbf{SO}^+(3,1)).$$

Proof. First we define $(M_\bullet, G_\bullet)$ to be one of the Riemannian homogeneous spaces mentioned in the theorem. Let μ be a G–invariant metric. Since K is isomorphic to $\mathbf{SO}(3)$, we have that μ has constant sectional curvature. Because M is simply connected there is an isometry $F \colon (M, \mu) \to (M_\bullet, \mu_\bullet)$ where $\mu_\bullet$ is the standard metric of $M_\bullet$. Observe that $G_\bullet$ is equal to the connected isometry group of $(M_\bullet, \mu_\bullet)$. Now define a group homomorphism $f \colon G \to G_\bullet$ by $f(g) := F \circ g \circ F^{-1}$ which is well–defined, since g and F are isometries. First note that f is continuous. If (g_n) is a convergent sequence of isometries in G with limit $g \in G$, then $f(g_n)$ converges pointwise on $M_\bullet$ to $f(g)$ since g_n converges pointwise on M. Hence $f(g_n)$ converges in the topology of $G_\bullet$ to $f(g)$. Since f is a continuous homomorphism between Lie groups we have that f is smooth, and since f^{-1} is smooth as well (the inverse map is conjugating every element in $G_\bullet$ with F^{-1}) it follows that f is an isomorphism. Obviously F is f–equivariant and the theorem is proven. ■

Corollary 2.7. *The maximal isotropic geometries are given by*

$$(\mathbb{R}^3, \mathbf{E}(3)), \quad (S^3, \mathbf{O}(4)) \quad \textit{and} \quad (D^3, \mathbf{O}^+(3,1))$$

Proof. Every isotropic geometry is contained in one of the above geometries. This follows because those groups are exactly the isometry groups of the standard metrics on the respective manifolds. And as seen in Theorem 2.6 every 6–dimensional geometry induces a metric of constant sectional curvature and the group in question must be a closed subgroup of those which are specified in this corollary. ■

Thus we have completed the classification of the isotropic geometries in dimension three.

Rotationally symmetric geometries

The rotationally symmetric geometries have a one–dimensional isotropy group. We suppose first that G is connected. Since M is simply connected it follows by Lemma 1.12 that K is connected too. Hence K is a connected, one–dimensional, compact group and therefore K is isomorphic to $\mathbf{SO}(2)$. Moreover K fixes the point m_0 and so K acts linear on the tangent space TM_{m_0}. We choose for this section once and for all a G–invariant metric μ on M and therefore K acts by linear isometries on the euclidean vector space (TM_{m_0}, μ_{m_0}). Before we collect some useful properties in order to classify the geometries in this section, we would like to mention that we combine ideas from the first chapter and from [Thu97].

Proposition 2.8. *A linear, effective and isometric action of* $\mathbf{SO}(2)$ *on a three–dimensional euclidean vector* $(V, \langle , \rangle)$ *space has a unique rotation axis, i.e. there is a unique one–dimensional subspace L of V such that every element of K fixes every point on L.*

Proof. Surely there is an elementary proof of this fact just by using some linear algebra, but we would like to show a different way to prove this proposition. Since the action of $K := \mathbf{SO}(2)$ is isometric it acts (effectively) on the two–sphere S^2 of the euclidean vector space V. Let X denote a fundamental vector field of this action on S^2. Then X has to have a zero and it follows that the corresponding flow has a fixed point say $s_0 \in S^2$. But since K is the image of $\mathbb{R}$ under the exponential map, it follows that every element in K fixes this point s_0. It remains to show that the linear subspace $L := \mathbb{R}s_0$ is the unique rotation axis mentioned in the proposition. But a non–trivial element of $K \subset \mathbf{SO}(3)$ can have at most a one–dimensional eigenspace to the eigenvalue 1. ■

Corollary 2.9. *A rotationally symmetric geometry (M, G) posses a one–dimensional space of homogeneous vector fields. This follows by Proposition 1.31 and by Remark 1.36 we have a one–dimensional involutive distribution $\mathcal{D}$ on M.*

Corollary 2.10. *A rotationally symmetric geometry (M, G) is an $\mathbb{R}$ or an $\mathbf{SO}(2)$ principal–bundle $\pi\colon M \to B$ over a smooth surface B.*

Proof. By Corollary 1.39 we have that M is a H–principal bundle over a smooth surface since the $\mathcal{D}$ is one–dimensional. Since K is connected we have again by Theorem 1.37 that N_F must be a two–dimensional, connected Lie group. Hence $H = N_F/K$ is connected and therefore the fiber group is either $\mathbb{R}$ or $\mathbf{SO}(2)$. ∎

Proposition 2.11. *The extension of Theorem 1.37 for rotationally symmetric geometries*

$$1 \longrightarrow \mathbf{SO}(2) \longrightarrow N_F \longrightarrow H \longrightarrow 1,$$

where F is the maximal connected integral manifold for $\mathcal{D}$ through m_0 is a central extension.

Proof. This is actually a general fact of an extension of $\mathbf{SO}(2)$ and there is no special property of the rotationally symmetric geometries involved there as long as N_F is connected. As usual we regard $\mathbf{SO}(2)$ as a closed Lie subgroup of N_F. Then since $\mathbf{SO}(2)$ lies normal in N_F the conjugation $c\colon N_F \to \mathbf{Aut}(\mathbf{SO}(2))$ by elements of N_F yields an automorphism of $\mathbf{SO}(2)$. But the automorphism group of $\mathbf{SO}(2)$ is $\mathbb{Z}_2$ and since c is continuous and N_F connected we obtain $c_g = \mathrm{id}_{\mathbf{SO}(2)}$ for all $g \in N_F$. This implies of course that $\mathbf{SO}(2)$ lies in the center of N_F. ∎

The comfortable situation here is that the integral manifolds of $\mathcal{D}$ are given by the flow of a G–invariant vector field. Therefore it makes sense to study first some properties about G–invariant vector fields on rotationally symmetric geometries.

Proposition 2.12. *Let X be a homogeneous vector field of a rotationally symmetric geometry. If μ is a G–invariant metric on (M, G) the flow lines of X are geodesics.*

Proof. Let ∇ denote the Levi-Civita connection for the metric μ. We would like to show $\nabla_X X = 0$. Fix a point $m \in M$ and denote by K_m the isotropy group m of G acting on M. By Remark 1.36 X_m is invariant under the action of K_m and since those elements are isometries for μ we obtain for all $k \in K_m$

$$k_* \left(\nabla_{X_m} X\right) = \nabla_{X_m} X$$

where $k_*(X)$ is the pushforward of the vector field X by k. The last computation shows that $\nabla_{X_m}X \in \mathcal{D}_m$, hence there is a $\lambda \in \mathbb{R}$ with $\nabla_{X_m}X = \lambda X_m$, since $\mathcal{D}$ is a one–dimensional distribution and X is a non–zero vector field. But on the other side $\nabla_{X_m}X$ lies in the orthogonal complement of $\mathcal{D}_m$ as the following lines show. Clearly the norm of X with respect to μ is a constant function on M and therefore we obtain

$$\mu_m(\nabla_{X_m}X, X_m) = X_m\left(|X|^2\right) - \mu_m(X_m, \nabla_{X_m}X) = -\mu_m(\nabla_{X_m}X, X_m).$$

which again implies $\mu_m(\nabla_{X_m}X, X_m) = 0$. We conclude $\lambda = 0$ and since m was arbitrary the proposition follows. ■

Proposition 2.13. *Let X be a G–invariant vector field on (M, G). Let* $\operatorname{div}X$ *be the divergence of X with respect to the G–invariant metric μ. Then* $\operatorname{div}X$ *is a constant function on M and does not depend on the G–invariant metric. Finally* $\operatorname{div}X = 0$ *iff X is a Killing field such that the flow is a one–parameter subgroup of G.*

Proof. By Proposition 1.15 M is orientable. Therefore $\operatorname{div}X$ can be expressed by

$$\mathcal{L}_X\mathrm{vol} = \operatorname{div}(X)\cdot\mathrm{vol}$$

for a Riemannian volume form vol (different Riemannian volume forms on orientable manifolds differ only by a sign, which does not affect the divergence). Since vol is a 3–form we deduce $\mathcal{L}_X\mathrm{vol} = \mathrm{d}(i_X\mathrm{vol})$ and therefore if $g \in G$ we obtain

$$g^*\left(\mathcal{L}_X\mathrm{vol}\right) = g^*(\mathrm{d}(i_X\mathrm{vol})) = \mathrm{d}(g^*(i_X\mathrm{vol}))$$

but $i_X\mathrm{vol}$ is G–invariant since G acts on (M, μ) by isometries and X is a homogeneous vector field, hence the above Lie derivative is G–invariant. We conclude that $\operatorname{div}(X)\cdot\mathrm{vol}$ has to be G–invariant as well and this implies that $\operatorname{div}(X)$ is a constant function on M. If vol' is a Riemannian volume form with respect to another G–invariant metric μ', then by the G–invariance there is a $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $\mathrm{vol}' = \lambda\cdot\mathrm{vol}$ which does not affect the value of $\operatorname{div}X$.

By Proposition 1.18 the tensor $\mathcal{L}_X\mu$ is fully determined by its bilinear form $\beta = (\mathcal{L}_X\mu)_{m_0}$ on TM_{m_0}. Let $X \neq 0$ and e_1, e_2, e_3 be an orthonormal basis such that $e_1 = X_m/|X_m|$ and $e_2, e_3 \in \mathcal{H}_{m_0} = \mathcal{D}^{\perp}_{m_0}$. Note that K acts on the unit circle in $\mathcal{H}_{m_0}$ transitively. First we observe that $i_{e_1}\beta = 0$ since

$$i_{e_1}\beta(e_i) = (\mathcal{L}_X\mu)(e_1, e_i) = \mu_{m_0}\left(\nabla_{e_1}X, e_i\right) + \mu_{m_0}\left(e_1, \nabla_{e_i}X\right) = 0$$

because $\nabla_{e_1} X = 0$ by Proposition 2.12 and

$$\mu_{m_0}(e_1, \nabla_{e_i} X) = \frac{1}{2|X_m|} e_i (\mu(X, X)) = 0.$$

The form β is clearly symmetric and we saw above that $\beta(e_1, e_i) = 0$. Therefore we may assume that β is diagonal in the basis e_1, e_2, e_3 with eigenvalues $\lambda_1 = 0, \lambda_2$ and λ_3. But since β is K–invariant and K acts on the unit circle of $\mathcal{H}_{m_0}$ transitively we have $\lambda_2 = \lambda_3 =: \lambda$. Finally we get

$$\operatorname{div} X = \frac{1}{2} \operatorname{tr}(\mathcal{L}_X \mu) = \lambda,$$

hence $\operatorname{div} X = 0$ iff $\mathcal{L}_X \mu = 0$.

Suppose X is a Killing field and φ^t its flow. Let $g \colon \mathbb{R} \to G$ be a smooth curve such that $g_t \circ \varphi^t(m_0) = m_0$ for all $t \in \mathbb{R}$. Then $g_t \circ \varphi^t$ is an isometry and let $f_t \colon TM_{m_0} \to TM_{m_0}$ denote its derivative in m_0 which is a linear isometry of (TM_{m_0}, μ_{m_0}). By the G–invariance of X we conclude $f_t(X_{m_0}) = X_{m_0}$ and $\det f_t = 1$ for all $t \in \mathbb{R}$, since f_t is an isometry, f_0 is the identity map and f_t depends smoothly on t. Therefore f_t acts on $X_{m_0}^{\perp}$ orientation preserving and isometrically, hence it must be an element k_t of K and we finally conlcude $\varphi^t = (g_t)^{-1} k_t \in G$ for $k_t \in K$. ∎

Remark 2.14. Let ω be the induced G–invariant connection form on $\pi \colon M \to B$ from Proposition 1.46. Since the Lie algebra of $\mathbb{R}$ and $\mathbf{SO}(2)$ is $\mathbb{R}$, we consider ω as a scalar 1–form on M. Observe that by Proposition 1.18 the space of G–invariant one–forms is one dimensional, since they are fixed points under the induced action of K on $TM^*_{m_0}$ via the G–invariant metric μ.

As a next step we would like to deduce how the G–invariant metrics on rotational geometries look like, since this will help us to understand the group G better.

Proposition 2.15. *Let X_l be a fundamental vector field of H where $l \neq 0$ is an element of the Lie algebra $\mathfrak{h} = \mathbb{R}$ (observe that X_l is a homogeneous vector field by Proposition 1.41). If $\varphi \colon \mathbb{R} \times M \to M$ denotes the flow of X_l on M and $\Pi \colon TM \to TM$ the fiberwise linear map which maps a tangent vector into its horizontal component, then we have*

$$(\varphi^t)^*(\mu) = \frac{\lambda^2}{l^2} \omega \otimes \omega + e^{\kappa t} (\Pi^*(\mu))$$

where $\lambda := |X_l|$ and $\kappa = \operatorname{div} X_l$.

Proof. Both sides of the equation in the proposition are G–invariant and therefore it is sufficient to prove the equality only in one point, say in m_0. Set $X := X_l$ and let $g \colon \mathbb{R} \to G$

be a smooth curve such that $g_t \circ \varphi^t(m_0) = m_0$ for all $t \in \mathbb{R}$. It follows, since g_t is an isometry for every t, that

$$(g_t \circ \varphi^t)^*(\mu) = (\varphi^t)^*(\mu),$$

and if $f_t : TM_{m_0} \to TM_{m_0}$ denotes the derivative of $g_t \circ \varphi^t$ in m_0 we may write

$$(\varphi^t)^*(\mu)_{m_0} = (f_t)^*(\mu_{m_0}).$$

Let us set $\sigma := \mu_{m_0}$ and $V := TM_{m_0}$ as well as U to be the span of X_{m_0} in V. Note that $g_t \in N_F$ where F is the orbit of m_0 under the action of the fiber group. By Proposition 2.11 g_t commutes with every element of K and therefore f_t commutes as well with K. Using Proposition 1.43 f_t respects the orthogonal decomposition $V = U \oplus W$ where $W = U^\perp$. With respect to this orthogonal decomposition f_t is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & F_t \end{pmatrix}$$

and F_t is a an invertible endomorphism of W. Note that $\det f_t > 0$ since $f_0 = \mathrm{id}_V$ and this implies $\det F_t > 0$ for all $t \in \mathbb{R}$. Moreover in the orthogonal decomposition an element of K is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$$

for $S \in \mathbf{SO}(W, \sigma_W)$, where σ_W is the induced scalar product on W from (V, σ). Thus F_t commutes with all $S \in \mathbf{SO}(V, \sigma_W)$ and preserves every orientation of W. By the polar decomposition of endomorphisms there is an $S_t \in \mathbf{SO}(V, \sigma_W)$ such that $F_t S_t$ is a self–adjoint endomorphism which commutes with $\mathbf{SO}(V, \sigma_W)$. But this implies that $F_t S_t$ is a multiple of the identity and therefore there is an $\varepsilon_t > 0$ and a $S_t \in \mathbf{SO}(V, \sigma_W)$ such that $F_t = \varepsilon_t S_t$ with $\varepsilon_t^2 = \det F_t$. Note that all t–dependences are smooth. Let $D_t : TM_{m_0} \to TM_{\varphi^t(m_0)}$ be the differential of φ^t in m_0. If (e_1, e_2, e_3) is an orthonormal basis of (V, σ) then $(D_t(e_1), D_t(\varepsilon_t^{-1} e_2), D_t(\varepsilon_t^{-1} e_3))$ is an orthonormal basis of $(TM_{\varphi^t(m_0)}, \mu_{\varphi^t(m_0)})$ since $D_t = (g_t)^{-1} f_t$. Furthermore differentating both sides of $(\varphi^t)^*(\mu)_{m_0} = (f_t)^*(\mu_{m_0})$ with respect to t yields

$$(\varphi^t)^* \left(\mathcal{L}_X \mu\right)_{m_0} (w_1, w_2) = 2\varepsilon_t \dot{\varepsilon}_t \sigma(w_1, w_2)$$

for $w_1, w_2 \in W$. If we take the trace of the left side in (W, σ_W) we obtain $2\varepsilon_t^2 \mathrm{div} X$ since we have seen in Proposition 2.13 that $\mathcal{L}_X \mu(X, X) = 0$. Taking the same trace of the right side yields $4\varepsilon_t \dot{\varepsilon}_t$ so we end up to solve the differential equation

$$\dot{\varepsilon}_t = \frac{\mathrm{div} X}{2} \varepsilon_t$$

with $\varepsilon_0 = 1$. We conclude $\varepsilon_t = e^{\frac{1}{2} \mathrm{div} X \cdot t}$.

Finally we have for $w_1, w_2 \in W$

$$(\varphi^t)^*(\mu)_{m_0}(X_{m_0}, X_{m_0}) = \lambda^2 = \frac{\lambda^2}{l^2}\omega_{m_0} \otimes \omega_{m_0}(X_{m_0}, X_{m_0})$$

as well as

$$(\varphi^t)^*(\mu)_{m_0}(X_{m_0}, w_1) = 0$$

and

$$(\varphi^t)^*(\mu)_{m_0}(w_1, w_2) = e^{\kappa t}\sigma(w_1, w_2) = e^{\kappa t}\left(\Pi^*(\mu)\right)_{m_0}(w_1, w_2)$$

■

The curvature F^ω from Remark 1.47 is equal to $\mathrm{d}\omega$ (where d is the usual exterior derivative of forms) since the commutator $[\omega \wedge \omega]$ vanishes because the Lie algebra is abelian. Consequently we have to study the case where ω is closed, i.e. the bundle is flat and where $\mathrm{d}\omega$ is not closed. We will handle the flat case first.

But before we hop into the classification of rotational geometries with flat bundles we have to study a certain class of geometries in order to complete the proof Theorem 2.17.

Example 2.16. Let $\kappa \in \mathbb{R}$ and set $G_\kappa := \mathbf{E}_0(2) \rtimes_{\rho_\kappa} \mathbb{R}$ where $\rho_\kappa \colon \mathbb{R} \to \mathbf{Aut}_0(\mathbf{E}_0(2))$ is given by $(\rho_\kappa)_s(a, A) := (e^{-\frac{1}{2}\kappa s}a, A)$ for $s \in \mathbb{R}$ and $(a, A) \in \mathbf{E}_0(2) = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$. For the sake of convenience we omit the subscript κ for ρ_κ and G_κ but keep in mind that this map is determined by a fixed κ. The map ρ_s is indeed an automorphism of $\mathbf{E}_0(2)$ and it remains to check that ρ is a homomorphism. For $s, r \in \mathbb{R}$ we compute

$$\rho_{s+r}(a, A) = (e^{-\frac{1}{2}\kappa(s+r)}a, A) = \rho_s(e^{-\frac{1}{2}\kappa r}a, A) = \rho_s \circ \rho_r(a, A).$$

Since ρ is continuous with respect to the compact–open topology we obtain that ρ is a Lie group homomorphism.

Consider the manifold $M = \mathbb{R}^2 \times \mathbb{R}$ and denote by (x, t) a point in M where $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Then $\mathbf{E}_0(2)$ acts on $\mathbb{R}^2$ in the natural way and $\mathbb{R}$ on $\mathbb{R}$ by translations. Define an action of $\mathbb{R}$ on $\mathbb{R}^2$ by $s.x := e^{-\frac{1}{2}\kappa s}x$ and thus we regard $s \in \mathbb{R}$ as a diffeomorphism of $\mathbb{R}^2$ by this action. If $c \colon \mathbb{R} \to \mathrm{Diff}\left(\mathbb{R}^2\right)$ is defined by c_s to be the conjugation by the diffeomorphism s in the diffeomorphism group $\mathrm{Diff}\left(\mathbb{R}^2\right)$ then we obtain

$$c_s(a, A)(x) = s.(a + e^{\frac{1}{2}\kappa s}Ax) = e^{-\frac{1}{2}\kappa s}a + Ax = \rho_s(a, A)(x).$$

Thus according to the paragraph before, G can act semidirectly on M which is induced by the action of $\mathbb{R}$ on M. Consequently the action of G on M is given by

$$(a, A, s).(x, t) := \left(e^{-\frac{1}{2}\kappa s}Ax + a, t + s\right).$$

First we show that (M, G) is a rotationally symmetric geometry. The group G acts on M transitively since every point (x, t) is the image of $(0, 0)$ by the group element (x, E_2, t) and it acts effectively since $(a, A, s).(x, t) = (x, t)$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ implies $s = 0$ and $Ax + a = x$ hence $a = 0$ as well as $A = E_2$. Surely M is simply connected and of dimension 3 and G is a 4–dimensional Lie group.

We fix the point $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$ and we would like to compute the isotropy group K in that point. We are done if we show that K is isomorphic to $\mathbf{SO}(2)$. Observe that K has to be connected and of dimension one and so the only possibilities are $\mathbf{SO}(2)$ or $\mathbb{R}$ and therefore it is sufficient to show that K has a subgroup isomorphic to $\mathbf{SO}(2)$. But the elements $(0, A, 0)$ for all $A \in \mathbf{SO}(2)$ fix the point $(0, 0)$. Altogether this shows that (M, G) is a rotationally symmetric geometry.

But now we do not know if some of the (M, G_κ) are equivariant isomorphic to each other. Indeed we have (M, G_κ) is isomorphic to (M, G_1) if $\kappa \neq 0$ as Lemma B.6 of Appendix B shows.

If we choose the global chart induced by the coordinates $(x, t) \in M$ we define a metric in this chart by

$$\mu(x, t) = \begin{pmatrix} e^{\kappa \cdot t} E_2 & 0 \\ 0 & 1 \end{pmatrix}.$$

This Riemannian metric is invariant under G since the Jacobian of a diffeomorphism $(a, A, s) \in G$ in a point (x, t) is given by

$$\begin{pmatrix} e^{-\frac{1}{2}\kappa \cdot s} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The Riemannian volume form vol to μ with respect to the standard orientation $\mathrm{d}x \wedge \mathrm{d}t$ is clearly $\mathrm{vol}(x, t) = \sqrt{\det \mu(x, t)}\, \mathrm{d}x \wedge \mathrm{d}t = e^{\kappa \cdot t} \mathrm{d}x \wedge \mathrm{d}t$. With that in mind we would like to compute the divergence of the global, G–invariant vector field ∂_t. The flow is given by $\varphi^\varepsilon(x, t) = (x, t + \varepsilon)$ and therefore $(\varphi^\varepsilon)^*(\mathrm{vol})(x, t) = e^{\kappa(t+\varepsilon)} \mathrm{d}x \wedge \mathrm{d}t$. Differentating this equation in $\varepsilon = 0$ yields

$$\mathcal{L}_{\partial_t} \mathrm{vol} = \kappa \cdot \mathrm{vol}$$

hence $\mathrm{div} \partial_t = \kappa$.

Lemma B.6 of Appendix B shows that (M, G_κ) is equivariant diffeomorphic to (M, G_1) for $\kappa \neq 0$. Moreover (M, G_0) and (M, G_1) cannot be isomorphic since G_0 has a non–trivial center and G_1 is centerless, see Lemma B.18 of Appendix B. The main difference between these two geometries is that on the one hand G_0 has a subgroup isomorphic to $\mathbb{R}^3$ which still acts transitively; thus all homogeneous metrics are flat and on the other hand the geometry (M, G_1) admits an hyperbolic metric, hence G_1 is a subgroup of $\mathbf{H}(3)$. To see the last assertion we consider the metric

$$\mu_{(x,t)} = \begin{pmatrix} e^t E_2 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\mathbb{R}^2 \times \mathbb{R}$ which is invariant under G_1. Define $H := \mathbb{R}^2 \times \mathbb{R}_{>0}$ and $M = \mathbb{R}^2 \times \mathbb{R}$ as well as the diffeomorphism $F\colon H \to M$, $(x,s) \mapsto (2x, -2\ln(s))$. The Jacobian $\mathrm{D}F$ of F in (x,s) is given as

$$\begin{pmatrix} 2E_2 & 0 \\ 0 & -\frac{2}{s} \end{pmatrix}$$

and therefore the pulled back metric $F^*(\mu)$ in (x,s) has the form

$$\mathrm{D}F\mu\mathrm{D}F^T = \begin{pmatrix} 2E_2 & 0 \\ 0 & -\frac{2}{s} \end{pmatrix}\begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2E_2 & 0 \\ 0 & -\frac{2}{s} \end{pmatrix} = \frac{4}{s^2}\begin{pmatrix} E_2 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the standard hyperbolic metric on H.

Theorem 2.17. *Let $\pi\colon M \to B$ be the bundle obtained by Corollary 2.10 and ω the G–invariant connection form mentioned above. Suppose that $F^\omega = \mathrm{d}\omega = 0$. Then (M,G) is equivariant isomorphic to one of the following Riemannian homogeneous spaces :*

$(S^2 \times \mathbb{R},\ \mathbf{SO}(3) \times \mathbb{R})$,

$(D^2 \times \mathbb{R},\ \mathbf{SO}^+(2,1) \times \mathbb{R})$,

$(\mathbb{R}^2 \times \mathbb{R},\ \mathbf{E}_0(2) \times \mathbb{R})$,

$(\mathbb{R}^2 \times \mathbb{R},\ \mathbf{E}_0(2) \rtimes_1 \mathbb{R})$,

where the last Riemannian homogeneous space is the one of Example 2.16 for $\kappa = 1$.

Proof. In Remark 1.52 we saw that the fiber group has to be simply connected which exclude the case of an $\mathbf{SO}(2)$–bundle, hence the fiber group is $\mathbb{R}$. The extension of Proposition 1.53 splits, since $\mathbb{R}$ is abelian and simply connected and therefore we may apply Theorem 1.54. It follows that (M,G) is equivariant isomorphic to $(B\times\mathbb{R}, G_B \rtimes \mathbb{R})$. Furthermore we conclude with Proposition 1.53 that (B, G_B) is a Riemannian homogeneous space such that B is simply connected, G_B connected and $\dim G_B = 3$. But those spaces can be easily classified as one may see with Lemma B.4. Hence (B, G_B) is equivariant isomorphic to $(S^2, \mathbf{SO}(3))$, $(\mathbb{R}^2, \mathbf{E}_0(2))$ or $(D^2, \mathbf{SO}^+(2,1))$ and this implies that (M,G) is isomorphic to either $(S^2 \times \mathbb{R}, \mathbf{SO}(3) \rtimes \mathbb{R})$, or $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes \mathbb{R})$ or $(D^2 \times \mathbb{R}, \mathbf{SO}^+(2,1) \rtimes \mathbb{R})$ for some representations for the semidirect product. The final step of this proof is to study all possible representations such that (M,G) is a Riemannian homogeneous space . By Lemma B.5 of Appendix B we have that $\mathbf{SO}(3)$

is a complete group and $\mathbf{SO}^+(2,1)$ is almost complete. Using Proposition 1.56 and the fact that G has to be connected, we deduce that there are no semidirect products which are not isomorphic to direct products.

It remains to check the case $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes \mathbb{R})$ where we have to answer the question what semidirect structure the group possess. For that we go a step back to the extension $1 \longrightarrow G_{\mathbb{R}^2} \longrightarrow G \xrightarrow{\pi} \mathbb{R} \longrightarrow 1$ where we recall from Theorem 1.54 that the value of π indicates the translation in the fiber. Hence we have to find a splitting map $\sigma\colon \mathbb{R} \to G$ which corresponds to a one–parameter subgroup of G and is a translation in the second argument. Note that two different splitting maps gives us two semidirect products which are isomorphic and therefore it is sufficient to find any splitting map. Furthermore observe that the group structure on G have to coincide with the composition of diffeomorphism of $\mathbb{R}^2 \times \mathbb{R}$. Finding such a splitting map $\sigma\colon \mathbb{R} \to G$ means to know that σ_s for $s \in \mathbb{R}$ is an isometry and contained in G. Finally this is what we aiming for in the next section.

We choose the canonical global chart for $\mathbb{R}^2 \times \mathbb{R}$ such that a point there is given by the pair (x,t) where $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Since the principal bundle is trivial we have $\omega = \mathrm{d}t$ and a fundamental vector field of the action of the fiber group is given by the coordinate vector field ∂_t where $\varphi^\varepsilon(x,t) = (x, t+\varepsilon)$ is its flow. We may assume $|\partial_t| = 1$ by rescaling μ and clearly we have $\omega(\partial_t) = 1$. If we denote by ∂_i $(i = 1,2)$ the coordinate vector fields of the x–coordinate and if we use Proposition 2.15 we deduce for $i, j = 1,2$

$$\mu_{(x,t)}(\partial_t, \partial_t) = 1, \;\; \mu_{(x,t)}(\partial_t, \partial_i) = 0, \;\; \mu_{(x,t)}(\partial_i, \partial_j) = e^{\kappa t}(\mu_{(x,0)})_{ij}.$$

for some $\kappa \in \mathbb{R}$. But the restriction of μ on $\mathbb{R}^2 \times 0$ has to be invariant under $\mathbf{E}_0(2)$ which means that μ is a multiple of the euclidean flat metric on $\mathbb{R}^2$ and we assume by rescaling μ in the $\mathbb{R}^2$–direction once more, that ∂_i has length equal to one. To sum up the metric μ has the coordinate representation

$$\mu_{ij}(x,t) = \begin{pmatrix} e^{\kappa t} & 0 & 0 \\ 0 & e^{\kappa t} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\sigma_s\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ be given by $\sigma_s(x,t) = (e^{-\frac{\kappa}{2}s}x, t+s)$. The Jacobian of σ_s is given by

$$(\mathrm{D}\sigma_s)_{(x,t)} = \begin{pmatrix} e^{-\frac{\kappa}{2}s} & 0 & 0 \\ 0 & e^{-\frac{\kappa}{2}s} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and σ_s is an isometry since

$$\begin{aligned}
((\sigma_s)^*(\mu))_{ij}(x,t) &= \begin{pmatrix} e^{-\frac{\kappa}{2}s} & 0 & 0 \\ 0 & e^{-\frac{\kappa}{2}s} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\kappa(s+t)} & 0 & 0 \\ 0 & e^{\kappa(s+t)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\kappa}{2}s} & 0 & 0 \\ 0 & e^{-\frac{\kappa}{2}s} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{\kappa t} & 0 & 0 \\ 0 & e^{\kappa t} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mu_{ij}(x,t).
\end{aligned}$$

Moreover σ_s is contained in G since it commutes with the flow of ∂_t which means that the derivative of σ_s maps ∂_t into itself and this implies that $\sigma_s \in G$ (every isometry which maps ∂_t to ∂_t is an element of G). For $s_1, s_2 \in \mathbb{R}$ we compute

$$\sigma_{s_1} \circ \sigma_{s_2}(x,t) = (e^{-\frac{\kappa}{2}(s_1+s_2)}x, t + s_1 + s_2) = \sigma_{s_1+s_2}(x,t)$$

and this shows $\sigma \colon \mathbb{R} \to G$, $s \mapsto \sigma_s$ is a Lie group homomorphism. (Note that σ is at least continuous in the CO–topology of $\mathrm{Diff}(\mathbb{R}^2 \times \mathbb{R})$ and has image in G. Thus σ is a continuous map between Lie groups hence smooth.) Finally and obviously σ is a splitting map for π. Thus $G \cong \mathbf{E}_0(2) \rtimes_\sigma \mathbb{R}$ where the representation of $\mathbb{R}$ on $\mathbf{E}_0(2)$ is given by

$$\rho_s(a,A) = \sigma_s \circ (a,A) \circ \sigma_{-s} = (e^{-\frac{1}{2}\kappa s}a, A)$$

and by Theorem 1.54 the action of G on $\mathbb{R}^2 \times \mathbb{R}$ is given by

$$(a,A,s).(x,t) = (e^{-\frac{1}{2}\kappa s}Ax + a, t + s).$$

And we conclude that this geometry $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\sigma \mathbb{R})$ is precisely the geometry of Example 2.16. ■

Thus the remaining rotational geometries are determined by $\mathrm{d}\omega \neq 0$ which we will assume henceforth. Since the pullback of a diffeomorphism $g \in G$ commutes with exterior derivative of forms we conclude $\mathrm{d}\omega$ is G–invariant since ω is. This means as usual that the form is determined by a 2–form δ on the vector space TM_{m_0}

Proposition 2.18. *With respect to the decomposition $TM_{m_0} = \mathcal{D}_{m_0} \oplus \mathcal{H}_{m_0}$ the 2–form δ from above is given by*

$$\begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}$$

where Δ is a non–degenerated 2-form on $\mathcal{H}_{m_0}$.

Proof. Since $\mathcal{D}_{m_0}$ is one–dimensional we obtain $\delta(\xi_1, \xi_2) = 0$ for all $\xi_1, \xi_2 \in \mathcal{D}_{m_0}$. Thus fix a $\xi \in \mathcal{D}_{m_0}$ and $\eta \in \mathcal{H}_{m_0}$. Suppose furthermore that Y is a local horizontal extension of η and X the homogeneous vector field with $X(m_0) = \xi$. Then we obtain

$$\mathrm{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]).$$

Since ω measure the $\mathcal{D}$–component of tangent vectors we have to compute that component for $[X, Y]$. With ∇ being the Levi–Civita connection to μ we may rewrite the Lie bracket to $[X, Y] = \nabla_X Y - \nabla_Y X$. Since the decomposition $TM_{m_0} = \mathcal{D}_{m_0} \oplus \mathcal{H}_{m_0}$ is an orthogonal one for μ we will check the term $\mu(X, \nabla_X Y - \nabla_Y X)$ to determine the $\mathcal{D}$–component. First we see

$$\mu(X, \nabla_X Y) = X(\mu(X, Y)) - \mu(\nabla_X X, Y) = 0,$$

because $\nabla_X X = 0$ by Proposition 2.12 and $\mu(X, Y) = 0$ around m_0. And clearly we have

$$\mu(X, \nabla_Y X) = \tfrac{1}{2} Y|X|^2 = 0$$

since X is homogeneous and has therefore constant length with respect to μ. This implies $\mathrm{d}\omega(X, Y) = 0$ around m_0 and then of course $\delta(\xi, \eta) = 0$. We assumed $\mathrm{d}\omega \neq 0$, thus δ restricted to the 2–dimensional plane $\mathcal{H}_{m_0}$ has to be non–degenerated otherwise we would have $\mathrm{d}\omega = 0$. ∎

Let X be a homogeneous vector field on M such that $\omega(X) = 1$. By rescaling μ the metric remains G–invariant and we may achieve $\omega_m(\xi) = \mu_m(X_m, \xi)$ for all $\xi \in TM_m$ and all $m \in M$. The very first step to handle the case $\mathrm{d}\omega \neq 0$ is to show that X is a Killing field.

Proposition 2.19. *If* $\mathrm{d}\omega \neq 0$ *then* $\mathrm{div} X = 0$ *and by Proposition 2.13 X is a Killing field.*

Proof. Note that the flow of X on M is given by $\varphi^t(m) = m.e^{t \cdot 1}$ for $t \in \mathbb{R}$ and $m \in M$ since $\omega(X) = 1$ (Note: in that context here e denotes the exponential map of $H \in \{\mathbb{R}, \mathbf{SO}(2)\}$). Since H is abelian the adjoint representation $\mathrm{Ad}\colon H \to \mathbf{Aut}(\mathfrak{h})$ is the trivial one. Thus $(\varphi^t)^*\omega = (\omega)^*(e^{t \cdot 1}) = \omega$ for all $t \in \mathbb{R}$ and this implies $\mathcal{L}_X \omega = 0$. Since pullbacks commutes with the exterior derivatives of forms we deduce $\mathcal{L}_X \mathrm{d}\omega = 0$. On the other side we saw in the proof of Proposition 2.15 that the derivative $D_t\colon \mathcal{H}_m \to \mathcal{H}_{\varphi^t(m)}$ of φ^t restricted to the horizontal plane field is given by $D_t = \varepsilon_t g_t k_t$ for some $g_t \in G$, $k_t \in K$ and $\varepsilon_t = e^{\frac{1}{2}\kappa \cdot t}$ where $\kappa = \mathrm{div} X$ with respect to μ. Putting all this together we obtain for $\eta_1, \eta_2 \in \mathcal{H}_{m_0}$

$$(\varphi^t)^*(\mathrm{d}\omega)(\eta_1, \eta_2) = \varepsilon_t^2 (g_t k_t)^*(\mathrm{d}\omega)(\eta_1, \eta_2) = \varepsilon_t^2 \mathrm{d}\omega(\eta_1, \eta_2) = \varepsilon_t^2 \Delta(\eta_1, \eta_2)$$

for Δ from Proposition 2.18. Due to that

$$0 = (\mathcal{L}_X \mathrm{d}\omega)(\eta_1, \eta_2) = -\kappa\, \Delta(\eta_1, \eta_2)$$

and since $\Delta(\eta_1, \eta_2) \neq 0$ for η_1 and η_2 linear independent we conclude $\kappa = 0$. ∎

Corollary 2.20. *Let ∇ be the Levi–Civita connection of a G-invariant metric μ of (M, G) and X a homogeneous vector field such that $\omega(X) = 1$. If* vol *is a volume form of (M, μ) and $\times$ the vector product of TM induced by* vol *and μ then there exists a $\delta \neq 0$ such that*

$$\nabla_\xi X = \frac{\delta}{2|X|}(X_{m_0} \times \xi)$$

for all $\xi \in TM_{m_0}$. In particular $\nabla_\xi X$ is horizontal.

Proof. Let $e_0 := |X|^{-1} X_{m_0}$ and choose $e_1, e_2 \in \mathcal{H}_{m_0}$ such that (e_0, e_1, e_2) is a positively oriented orthonormal base of (TM_{m_0}, μ_{m_0}) (which is possible since e_0 is orthogonal to $\mathcal{H}_{m_0}$). Moreover let E_1, E_2 be horizontal extensions of e_1, e_2 respectively. Since X is a Killing field ∇X is a self–adjoint endomorphism on TM_{m_0} with respect to μ. First $\nabla_\xi X$ is horizontal since $\xi|X|^2 = 0$ and therefore $\nabla_{e_1} X$ has to be a multiple of e_2 since ∇X is a skew self–adjoint. We compute

$$\begin{aligned}
2\mu(\nabla_{e_1} X, e_2) &= \mu(\nabla_{e_1} X, e_2) - \mu(\nabla_{e_2} X, e_1) \\
&= -\mu(X_{m_0}, \nabla_{e_1} E_2) + \mu(X_{m_0}, \nabla_{e_2} E_1) \\
&= \mu(X_{m_0}, \nabla_{e_2} E_1 - \nabla_{e_1} E_2) \\
&= -\mu(X_{m_0}, [E_1, E_2]_{m_0}) \\
&= -|X|^2 \omega([E_1, E_2]_{m_0}) = |X|^2 \mathrm{d}\omega(e_1, e_2) =: \delta
\end{aligned}$$

and $\delta \neq 0$ since ω is not flat. Hence $\nabla_{e_1} X = \frac{\delta}{2} e_2 = \frac{\delta}{2|X|}(X_{m_0} \times e_1)$. And since ∇X is skew self–adjoint we get further $\nabla_{e_2} X = -\frac{\delta}{2} e_1 = \frac{\delta}{2|X|}(X_{m_0} \times e_2)$ ∎

Corollary 2.21. *The abelian group H is a closed subgroup of the center $Z(G)$ of G. The extension of Proposition 2.11 splits, i.e. N_F is isomorphic to $K \times H$, where F is the orbit of H through m_0.*

Proof. By Proposition 2.19 the group H acts on (M, μ) by isometries. Note that by Corollary 1.41 the homogeneous vector fields are precisely the fundamental vector fields of H acting on M. Thus if X is a homogeneous vector field we have $h_*(X) = X$ for all $h \in H$. Choose a $g \in G$ such that $g.m_0.h = m_0$ and let f denote the derivative of the map $m \mapsto g.m.h$ in m_0. Then f is an isometry of (TM_{m_0}, μ_{m_0}) and $f(X_{m_0}) = X_{m_0}$. Therefore there is a $k \in K$ such that $f = k$ and this implies that H is contained in G (observe that we used that G and H are subgroups of the diffeomorphism group of M). H is closed since it acts on (M, μ) by isometries and since the orbits are closed (see Theorem 1.37 and Proposition 1.38). By Proposition 1.34 H is a subgroup of the center of G (note that G is connected). Hence H is a subgroup of N_F (see Theorem **??** for the definition of N_F) and therefore the central extension of Proposition 2.11 splits and with Proposition 1.59 the claim follows. ∎

Recall from Corollary 1.39 that the group G acts on B transitive with isotropy group N_F in the point $b_0 := \pi(m_0)$ thus B is diffeomorphic to G/N_F. Then there is an action of G/H on B induced by the action of G. More precisely if $gH \in G/H$ then define $(gH).b := \pi(g.m)$ for $m \in M$ with $\pi(m) = b$. Since H lies in the center of G this action is well–defined.

Proposition 2.22. *The group $G' := G/H$ acts on B effectively, transitively with isotropy group K' isomorphic to K. Thus (B, G') is a Riemannian homogeneous space and $\pi\colon M \to B$ is an equivariant map. Moreover the projection map $\pi\colon G \to G'$ restricted to K is an isomorphism from K to K'.*

Proof. Clearly G' acts on B transitively since G does. For $g' \in G'$ suppose $g'.b = b$ for all $b \in B$. If $g \in G$ represents g' and $g'.b_0 = b_0$ then there is an $h \in H$ such that $g.m_0 = h.m_0$ where $\pi(m_0) = b_0$ (note since H lies in the center of G it does not matter if H acts from the right or from the left). Thus $g = hk$ for a $k \in K$ which fixes the fiber over b_0. But since g and h map points of a fiber into the same fiber, then the same holds for k. And gh acts on B as the identity and thus k has to act on B as the identity. Let $k\colon TM_{m_0} \to TM_{m_0}$ be the derivative of k in m_0 acting on M. Therefore $\pi \circ k = \pi$ and if $\xi \in \mathcal{H}_{m_0}$ we conclude $D\pi_{m_0}(k.\xi) = D\pi_{m_0}(\xi)$. But $D\pi_{m_0}$ restricted to $\mathcal{H}_{m_0}$ is an isomorphism to TB_{b_0} and this implies $k.\xi = \xi$. Since k fixes the vectors of $\mathcal{D}_{m_0}$ we obtain that k is the identity map of TM_{m_0} and by Corollary 1.5 k has to be the identity map on M. Finally this shows that $g = h$ and thus $G' = G/H$ acts on B effectively with isotropy group N_F/H. By Corollary 2.21 N_F/H is isomorphic to K and π restricted to K maps K into $N_F/H = K'$. ∎

Corollary 2.23. *There is a G'–invariant volume form vol' on B such that $\pi^*(\mathrm{vol}') = \lambda \mathrm{d}\omega$ for a $\lambda \neq 0$.*

Proof. If μ is a G–invariant metric on (M, G) then, since H is a subgroup of G there is a G'–invariant metric μ' on (B, G') such that π is a Riemannian submersion. Let vol' be a G'–invariant Riemannian volume form on (B, G'). Then $\pi^*(\mathrm{vol}')$ is a G–invariant 2–form on M. Let us compute this 2–form with respect to the orthogonal decomposition $\mathcal{D}_{m_0} \oplus \mathcal{H}_{m_0}$. Clearly we have

$$\pi^*(\mathrm{vol}) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta' \end{pmatrix}$$

where $\Delta' = \lambda\Delta$ for a $\lambda \neq 0$ and Δ from Proposition 2.18 (since the space of 2–forms on $\mathcal{H}_{m_0}$ is one–dimensional). And because $\pi^*(\mathrm{vol}')$ and $\mathrm{d}\omega$ are G–invariant we conclude $\pi^*(\mathrm{vol}') = \lambda \mathrm{d}\omega$. ∎

Now since G' is a 3–dimensional connected group and B is simply connected, the space (B, G') is equivariant diffeomorphic to either of one the geometries determined in Lemma

B.4 of Appendix B. Moreover some of the base spaces are contractible which is a strong restriction to the topology of the fiber bundle $\pi\colon M \to B$ as well as for the fiber group H. Therefore if G' and H is known, the group G fits into an extension of H by G'. This means, that we have to determine the extensions $1 \longrightarrow H \longrightarrow G \longrightarrow G' \longrightarrow 1$ where G' and H is known. We will treat each case of (B, G') separately and we will begin with the spherical geometry.

Proposition 2.24. *Let (B, G') be equivariant diffeomorphic to $(S^2, \mathbf{SO}(3))$. Then (M, G) is isomorphic to $(S^3, \mathbf{U}(2))$.*

Proof. First we would like to determine the fiber group H. Suppose $H = \mathbb{R}$. There is a global section $s\colon S^2 \to M$ (see [Bau09, p. 47]) of the H–principal bundle $\pi\colon M \to S^2$. Thus $s^*(\omega)$ is a 1–form on S^2 such that $d(s^*(\omega)) = s^*(\mathrm{d}\omega)$. Pick a G'–invariant volume form vol' on S^2 like in Corollary 2.23, thus $\pi^*(\mathrm{vol}') = \lambda \mathrm{d}\omega$. Hence

$$d(s^*(\omega)) = s^*(\mathrm{d}\omega) = \tfrac{1}{\lambda}(\pi \circ s)^*(\mathrm{vol}') = \tfrac{1}{\lambda}\mathrm{vol}'$$

and we see that vol' has to be exact. But using Stokes' theorem we conclude that no volume form on a closed manifold is exact and we obtain a contradiction to the assumption $H = \mathbb{R}$ thus $H = \mathbf{SO}(2)$.

Next we compute the fundamental group of G. We have two fibrations here to use; the one is given by $H \longrightarrow M \longrightarrow B$ and the other by $K \longrightarrow G \longrightarrow M$. Using Lemma 1.12 and put in the given data $B = S^2$ and $H = \mathbf{SO}(2) = K$ we obtain the exact sequences

$$\ldots \longrightarrow \pi_2(\mathbf{SO}(2)) \longrightarrow \pi_2(M) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(\mathbf{SO}(2)) \longrightarrow \pi_1(M) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow \pi_2(M) \longrightarrow \pi_1(\mathbf{SO}(2)) \longrightarrow \pi_1(G) \longrightarrow \pi_1(M) \longrightarrow \ldots$$

which imply first the extension

$$1 \longrightarrow \pi_2(M) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1.$$

Hence $\pi_2(M)$ has to be a subgroup of $\mathbb{Z}$ thus of the form $n\mathbb{Z}$ but since $\mathbb{Z}/\pi_2(M)$ is isomorphic to $\mathbb{Z}$ we decduce $n = 0$ and $\pi_2(M)$ is trivial. Plugging this into the second sequence we obtain the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(G) \longrightarrow 1$$

thus $\pi_1(G) = \mathbb{Z}$. In addition following Proposition 2.22 we obtain the central extension

$$1 \longrightarrow \mathbf{SO}(2) \longrightarrow G \longrightarrow \mathbf{SO}(3) \longrightarrow 1$$

and if $\tilde{G}$ denotes the universal cover group of G there is an another central extension given by

$$1 \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \longrightarrow \mathbf{SU}(2) \longrightarrow 1$$

from Proposition 1.60. We claim that this extension splits. So as to find such a splitting map we may linearize this problem since $\mathbf{SU}(2)$ is simply connected. Thus we have to check if the induced central extension $1 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{so}(3) \longrightarrow 1$ splits. But $\mathfrak{so}(3)$ is semi–simple and by Whitehead's second lemma (cf. [Wei97, p. 246]) $H^2(\mathfrak{so}(3);\mathbb{R}) = 0$ which means that the central extension of Lie algebras splits, hence $\tilde{G}$ is isomorphic to $\mathbb{R} \times \mathbf{SU}(2)$ and the related central extension is the trivial one. The group G is isomorphic to $\mathbb{R} \times \mathbf{SU}(2)/\pi_1(G)$ and therefore we have to determine how $\pi_1(G)$ lies in the center $Z(\mathbb{R} \times \mathbf{SU}(2)) = \mathbb{R} \times \mathbb{Z}_2$. Let $(c, \pm 1) \in \pi_1(G)$ denotes a generator. By Proposition 1.60 the maps of the extension

$$1 \longrightarrow \pi_1(\mathbf{SO}(2)) \longrightarrow \pi_1(G) \longrightarrow \pi_1(\mathbf{SO}(3)) \longrightarrow 1$$

are given as the restriction of the natural inclusion $i\colon \mathbb{R} \to \mathbb{R} \times \mathbf{SU}(2)$ and as the natural projection $\pi\colon \mathbb{R} \times \mathbf{SU}(2) \to \mathbf{SU}(2)$, since the central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \to \mathbf{SU}(2) \longrightarrow 1$$

splits. Therefore a generator of $\pi_1(G)$ must be of the form $(c, -1) \in \mathbb{R} \times \mathbb{Z}_2 = Z(\mathbb{R} \times \mathbf{SU}(2))$. Finally $c \neq 0$ since otherwise $\pi_1(G)$ would be isomorphic to $\mathbb{Z}_2$. With Lemma B.12 we deduce that G is isomorphic to $\mathbf{U}(2)$ and with Lemma B.13 of Appendix B the proposition follows. ∎

Remark 2.25. For the next Proposition it is usefull to introduce briefly the group $\widetilde{\mathbf{SL}(2,\mathbb{R})} =: \widetilde{\mathbf{SL}}$. For a more geometric and detailed discussion about this group see [Sco83]. We start with the group $\mathbf{SL}(2,\mathbb{R})$ which is defined as a matrix group and we define $\widetilde{\mathbf{SL}}$ as the universal cover group of $\mathbf{SL}(2,\mathbb{R})$. The center of $\mathbf{SL}(2,\mathbb{R})$ is isomorphic to $\mathbb{Z}_2$ and the quotient $\mathbf{PSL}(2,\mathbb{R}) := \mathbf{SL}(2,\mathbb{R})/\mathbb{Z}_2$ is isomorphic to $\mathbf{SO}^+(2,1)$. Since $(D^2, \mathbf{SO}^+(2,1))$ is a Riemannian homogeneous space with isotropy groups isomorphic to $\mathbf{SO}(2)$ the group $\mathbf{PSL}(2,\mathbb{R}) \cong \mathbf{SO}^+(2,1)$ may be identified as a manifold with the circle bundle of the standard hyperbolic space which is diffeomorphic to $D^2 \times S^1$. Therefore $\pi_1(\mathbf{PSL}(2,\mathbb{R})) \cong \mathbb{Z}$.

The group $\widetilde{\mathbf{SL}}$ is also the universal covering group of $\mathbf{PSL}(2,\mathbb{R})$ since $\mathbf{SL}(2,\mathbb{R})$ is a covering group for $\mathbf{PSL}(2,\mathbb{R})$ and with Lemma B.9 of Appendix B we see that $Z(\widetilde{\mathbf{SL}}) \cong \pi_1(\mathbf{PSL}(2,\mathbb{R})) \cong \mathbb{Z}$ because $\mathbf{PSL}(2,\mathbb{R})$ is centerless.

Proposition 2.26. *Let (B, G') be equivariant diffeomorphic to $(D^2, \mathbf{SO}^+(2,1))$. Then (M, G) is isomorphic to $(D^2 \times \mathbb{R}, \Gamma)$ where $\Gamma := (\mathbb{R} \times \widetilde{\mathbf{SL}})/\mathbb{Z})$, $\widetilde{\mathbf{SL}}$ is the universal cover group of $\mathbf{SL}(2, \mathbb{R})$ and $\mathbb{Z}$ is lying in $Z(\mathbb{R} \times \widetilde{\mathbf{SL}}) = \mathbb{R} \times \mathbb{Z}$ as $\mathbb{Z}(1,1)$. The action of Γ on $D^2 \times \mathbb{R}$ is induced by the componentwise action of $\mathbb{R} \times \widetilde{\mathbf{SL}}$ on $D^2 \times \mathbb{R}$ (note that since $\widetilde{\mathbf{SL}}$ is the universal covering group of $\mathbf{PSL}(2, \mathbb{R}) \cong \mathbf{SO}^+(2,1)$ it acts on D^2 through the action of $\mathbf{SO}^+(2,1)$ on D^2 and compare Proposition 1.61).*

Proof. Since the base space of the principal bundle $\pi\colon M \to D^2$ is contractible, it has to be trivial. Therefore we exclude the case of the $\mathbf{SO}(2)$ as fiber and conclude that M is isomorphic to the trivial $\mathbb{R}$–bundle $D^2 \times \mathbb{R}$. Hence $\pi_2(M) = 0$ and applying Lemma 1.12 on the $\mathbf{SO}(2)$–principal bundle $\mathbf{SO}(2) \longrightarrow G \longrightarrow M$ we deduce $\pi_1(G) \cong \mathbb{Z}$. Proposition 1.60 yields the central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \longrightarrow \widetilde{\mathbf{SL}} \longrightarrow 1$$

where $\tilde{G}$ is the universal cover group of G. Again like in Proposition 2.24 the Lie algebra $\mathfrak{sl}(2)$ of $\mathbf{SO}^+(2,1)$ is semi–simple and hence by Whitehead's Lemma the respective extension of Lie algebras splits which means that the central extension above is the trivial one. We conclude $\tilde{G}$ is isomorphic to $\mathbb{R} \times \widetilde{\mathbf{SL}}$ and thus $G \cong (\mathbb{R} \times \widetilde{\mathbf{SL}})/\pi_1(G)$. The fundamental group $\pi_1(G)$ is a subgroup of the center $\mathbb{R} \times \mathbb{Z}$. Since the fundamental group of $\mathbf{SO}^+(2,1) = \mathbf{PSL}(2, \mathbb{R})$ is exactly the center of $\widetilde{\mathbf{SL}}$ (see Remark 2.25) a generator of $\pi_1(G)$ has to have the form $(c, 1) \in \mathbb{R} \times \mathbb{Z}$ where 1 is a generator of $\mathbb{Z}$. If $c = 0$ G would be isomorphic to $\mathbf{SO}^+(2,1) \times \mathbb{R}$ with component wise action on $D^2 \times \mathbb{R}$ (see Proposition 1.61). By Proposition 1.44 the horizontal distribution is given as the orthogonal complement to the fibers $\{x\} \times \mathbb{R}$ for $x \in D^2$ by a $\mathbb{R} \times \mathbf{SO}^+(2,1)$–invariant metric. But then clearly the horizontal distribution is integrable and therefore flat which contradicts the assumption of non–flatness. Hence $c \neq 0$. Consider the automorphism $\varphi \in \mathbf{Aut}(\mathbb{R} \times \widetilde{\mathbf{SL}})$ given by $\varphi(r, g) := (\frac{r}{c}, g)$. This maps $\pi_1(G)$ to the subgroup $\mathbb{Z}(1,1)$ of $\mathbb{R} \times \mathbb{Z}$ and the quotients by those subgroups of $\tilde{G} = \mathbb{R} \times \widetilde{\mathbf{SL}}$ are isomorphic, cf. Lemma B.8. ■

Proposition 2.27. *Let (B', G) be equivariant diffeomorphic to $(\mathbb{R}^2, \mathbf{E}_0(2))$. Then (M, G) is isomorphic to $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{Nil} \rtimes \mathbf{SO}(2))$.*

Proof. As in the previous Proposition the bundle $\pi\colon M \to \mathbb{R}^2$ is trivial and therefore an $\mathbb{R}$–principal bundle. Repeating the arguments of Proposition 2.26 we obtain $\pi_1(G) \cong \mathbb{Z}$ and the central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \longrightarrow \widetilde{\mathbf{E}_0(2)} \longrightarrow 1$$

with associated central extension of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{e}(2) \longrightarrow 0.$$

In Example 1.71 we computed the weak isomorphism classes of this extension which are given by the Lie algebras $\mathbb{R} \times \mathfrak{e}(2)$ and $\mathbb{R} \times_{\omega_1} \mathfrak{e}(2)$ with their canonical central extensions. Suppose first the central extension is weakly isomorphic to

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \times \mathfrak{e}(2) \longrightarrow \mathfrak{e}(2) \longrightarrow 0.$$

Therefore the central extension of simply connected groups is induced by $\mathbb{R} \times \widetilde{\mathbf{E}_0(2)}$ with its canonical maps. Then the center is isomorphic to $\mathbb{R} \times \mathbb{Z}$ (see Lemma B.16 of Appendix B for the center of $\widetilde{\mathbf{E}_0(2)}$) embedded as $(t, l) \mapsto (t, 0, 2\pi l)$ and a generator γ_c of $\pi_1(G)$ has to be of the form $(c, 0, 2\pi)$ for $c \in \mathbb{R}$. The groups $G_c := (\mathbb{R} \times \widetilde{\mathbf{E}_0(2)})/(\mathbb{Z}\gamma_c)$ are all isomorphic to each other. According to Lemma B.8 we have to find an automorphism $\varphi_c \colon \mathbb{R} \times \widetilde{\mathbf{E}_0(2)} \to \mathbb{R} \times \widetilde{\mathbf{E}_0(2)}$ such that $\varphi_c(\gamma_c) = \gamma_0$. Choose φ_c as the linear map defined by the matrix

$$\begin{pmatrix} 1 & 0 & -c/2\pi \\ 0 & E_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

in the standard basis of $\mathbb{R}^4 = \mathbb{R} \times (\mathbb{R}^2 \rtimes \mathbb{R})$. A simple calculation shows that φ_c is a Lie group isomorphism. Thus here we have to deal with the group $G_0 = \mathbb{R} \times \mathbf{E}_0(2)$ with the product action on $\mathbb{R} \times \mathbb{R}^2$ (see Proposition 1.61). But then the horizontal distribution is integrable by Proposition 1.44, which contradics the assumption of non–flatness.

Therefore we are concerned with the central extension of $\mathbb{R} \times_{\omega_1} \mathfrak{e}(2)$. The group $\mathbf{Nil} \rtimes \mathbb{R}$ from Remark 1.71 is up to isomorphism the simply connected group to the Lie algebra $\mathbb{R} \times_{\omega_1} \mathfrak{e}(2)$. The projection $\mathbf{Nil} \rtimes \mathbb{R} \to \widetilde{\mathbf{E}_0(2)}$ is given by the map $(\mathbf{x}, z, \theta) \mapsto (\mathbf{x}, \theta)$ as can be deduced by the central extension of the Lie algebras. The center of $\mathbf{Nil} \rtimes \mathbb{R}$ is isomorphic to $\mathbb{R} \times \mathbb{Z}$ (see B.17) embedded as $(z, l) \mapsto (0, z, 2\pi l)$. The generator of $\pi_1(G)$ in $\mathbf{Nil} \rtimes \mathbb{R}$ has to have the form $\gamma_c := (0, c, 2\pi)$ for a $c \in \mathbb{R}$. But as above the groups $G_c := (\mathbf{Nil} \rtimes \mathbb{R})/(\mathbb{Z}\gamma_c)$ are all isomorphic to each other. Let $\varphi_c \colon \mathbf{Nil} \rtimes \mathbb{R} \to \mathbf{Nil} \rtimes \mathbb{R}$ be the linear map from $\mathbb{R}^4$ to itself given by

$$\begin{pmatrix} E_2 & 0 & 0 \\ 0 & 1 & -c/2\pi \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously $\varphi_c(\gamma_c) = \gamma_0$ and φ_c is a Lie group isomorphism. So the group here is given by $G \cong \mathbf{Nil} \rtimes \mathbf{SO}(2)$ and by Proposition 2.22 the group K is given by the $\mathbf{SO}(2)$ part. ∎

Now that we found all possible rotationally symmetric geometries we have to check which geometries are isomorphic.

Theorem 2.28. *Let (M, G) be a rotationally symmetric geometry. Then (M, G) is equivariant diffeomorphic to exactly one of the following seven geometries*

Flat Connection	*Non-Flat Connection*
$(S^2 \times \mathbb{R},\ \mathbf{SO}(2) \times \mathbb{R})$	$(S^3,\ \mathbf{U}(2))$
$(D^2 \times \mathbb{R},\ \mathbf{SO}^+(2,1) \times \mathbb{R})$	$(D^2 \times \mathbb{R},\ \Gamma)$
$(\mathbb{R}^2 \times \mathbb{R},\ \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R}),\ \kappa = 0, 1$	$(\mathbb{R}^2 \times \mathbb{R},\ \mathbf{Nil} \rtimes \mathbf{SO}(2))$

Proof. We will begin to show that a rotationally symmetric geometry with flat connection cannot be isomorphic to one with a non–flat connection. Suppose (M, G) and (M', G') are isomorphic geometries, i.e. there are isomorphism $F : G \to G'$ and $f : M \to M'$ such that f is equivariant with respect to F. Then M and M' are principal bundles over surfaces B and B' respectively. Let X be a fundamental vector field of the bundle $\pi : M \to B$, hence X is G–invariant. The pushforward $f_*(X)$ is G'–invariant as can be seen by the following computation: it is sufficient to choose a point $m'_0 \in M'$ and to show that $f_*(X)$ is invariant under the isotropy group K' in that point since f is F–equivariant. Let K be the isotropy group of (M, G) in the point m_0. Then it follows that $F(K) =: K'$ is the isotropy of $f(m_0) =: m'_0$. For all $k' \in K'$ we obtain

$$k'.(f_*(X)_{m'_0}) = k'.(Df_{m_0}(X_{m_0})) = D(k' \circ f)_{m_0}(X_{m_0}) = D(f \circ k)_{m_0}(X_{m_0}) = Df_{m_0}(X_{m_0}) = f_*(X)_{m'_0}$$

for $k \in K$ with $k' = F(k)$. Thus $f_*(X)$ is a fundamental vector field of the principal bundle $\pi' : M' \to B'$. On the other hand if μ' is a G'–invariant metric then $\mu := f^*(\mu')$ is G–invariant. From Proposition 1.44 we know that the connection $\mathcal{H}$ of $\pi : M \to B$ and the connection $\mathcal{H}'$ of $\pi' : M' \to B'$ are the orthogonal complements of $\mathbb{R}X$ and $\mathbb{R}f_*(X)$ respectively for any invariant metric. But then f maps $\mathcal{H}$ into $\mathcal{H}'$ isomorphically, since for all $\xi \in \mathcal{H}_m$ and $m' := f(m)$

$$\mu'\left(Df_m(\xi), f_*(X)_{m'}\right) = \mu'(Df_m(\xi), Df_m(X_m)) = \mu(\xi, X_m) = 0.$$

This implies that the pushforward $f_*(X_H)$ of a horizontal vector field X_H on M is a horizontal vector field on M'. Suppose $\mathcal{H}'$ is flat and X_1, X_2 are horizontal vector fields of $\mathcal{H}$. Since the derivative of f is a Lie algebra isomorphism from the vector fields on M to those on M' we see that $f_*([X_1, X_2]) = [f_*(X_1), f_*(X_2)]$ is horizontal hence $\mathcal{H}$ is flat as well.

We start to check the flat case first. The geometry of $S^2 \times \mathbb{R}$ is surely not isomorphic to the other flat ones because of the topological type of the manifold. To compare the other geometries, we will show that the Lie groups cannot be isomorphic. This in turn can be determined through their Lie algebras. We will use two invariants of isomorphism classes of Lie algebras, namely its derived algebras and its center.

If $\mathfrak{g}$ is a Lie algebra, denote by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ its derived Lie algebra. Let $\mathfrak{g}_1$ be the Lie algebra of $\mathbf{SO}^+(2,1) \times \mathbb{R}$ and $\mathfrak{g}_2^\kappa$ the Lie algebra of $\mathbf{E}_0(2) \rtimes_\kappa \mathbb{R}$. We already showed that the geometries $\mathbf{E}_0(2) \rtimes_\kappa \mathbb{R}$ for $\kappa \in \{0,1\}$ are non-isomorphic to each other. Hence we have to compare the Lie algebra $\mathfrak{g}_1$ with $\mathfrak{g}_2^\kappa$. We have $\mathfrak{g}_1 = \mathbb{R} \times \mathfrak{sl}(2)$, where the $\mathbb{R}$–factor is the center of the Lie algebra. Thus $\mathfrak{g}_1$ cannot be isomorphic to $\mathfrak{g}_2^\kappa$ for $\kappa = 1$ since by Lemma B.18 of Appendix B the corresponding simply–connected group of the latter Lie algebra has trivial center. Write $\mathfrak{g}_2 := \mathfrak{g}_2^0 = \mathbb{R} \times \mathfrak{e}(2)$ and we see that $\mathfrak{g}_2' = [\mathfrak{e}(2), \mathfrak{e}(2)]$ since $\mathbb{R}$ is the center of $\mathfrak{g}_2$. Same holds for $\mathfrak{g}_1' = [\mathfrak{sl}(2), \mathfrak{sl}(2)]$. But since $\mathfrak{sl}(2)$ is unimodular we have $\mathfrak{sl}(2)' = \mathfrak{sl}(2)$ and $\mathfrak{e}(2)'$ is only two–dimensional since $H^1(\mathfrak{e}(2); \mathbb{R}) = \mathfrak{e}(2)/\mathfrak{e}(2)' = \mathbb{R}$ (see Example 1.68). Thus $\mathfrak{g}_1'$ cannot be isomorphic to $\mathfrak{g}_2'$ and therefore the geometries are distinct.

Let us turn our attention to the non–flat cases. The geometry of $\mathbf{U}(2)$ cannot be isomorphic to the other two since the group is compact. The group Γ has Lie algebra $\mathfrak{g}_1$. Denote by $\mathfrak{g}_3 = \mathbb{R} \times_{\omega_1} \mathfrak{e}(2)$ the Lie algebra of $\mathbf{Nil} \rtimes \mathbf{SO}(2)$. Let $e_0, \ldots, e_3$ be the canonical basis for $\mathfrak{g}_3$ if we regard it as $\mathbb{R}^4$, where e_0 spans the center of $\mathfrak{g}_3$. The first derivative $\mathfrak{g}_3'$ is spanned by the elements of the form $[e_i, e_j]$ for $i, j = 0, \ldots, 3$. Thus $\mathfrak{g}_3' = \langle e_0, e_1, e_2 \rangle$ (see Example 1.68) and this implies $\mathfrak{g}_3'' = 0$ whereas $\mathfrak{g}_1'' = \mathfrak{sl}(2)$. ∎

Bianchi groups

Here the isotropy group is trivial which means that G is 3–dimensional and is acting freely on M and thus we may assume that M is a 3–dimensional simply connected Lie group. To understand those groups it is sufficient to know the zoo of 3–dimensional Lie algebras, since they are in one-to-one correspondence to simply connected groups modulo isomorphisms.

There were many successful attempts to determine the 3–dimensional Lie algebras. First, we would like to mention Bianchi's classification [Bia02] and [Gla08] for a revisited, more modern version of Bianchi's proof. Second, Milnor classified in [Mil76] all 3–dimensional unimodular Lie groups with a very nice proof. Of course there are many other articles on 3–dimensional Lie algebras which can by found though the literature and the internet very easily.

We would like to give here an alternative proof which is a bit more in the flavor of this work. Our approach will be through extensions of Lie algebras by their derived Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic Lie algebras then its first derivatives $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as well which makes the dimension of $\mathfrak{g}'$ an invariant. Hence we may classify the 3–dimensional Lie algebras based on the dimension of its first derivatives. In the following $\mathfrak{g}$ will always denote a 3–dimensional Lie algebra.

But before plunging into the classification we develop a notation for 3–dimensional Lie algebras. If (e_1, e_2, e_3) is a Basis of $\mathfrak{g}$ then the Lie algebra is fully determined by the vectors $[e_i, e_j]$. Write $[e_i, e_j] = C_{ij}^k e_k$ and furthermore write $\mathbf{i}$ for e_i. Then $[\mathbf{i}, \mathbf{j}] = C_{ij}^k \mathbf{k}$ and the only interesting combinations are $C_{12}^k \mathbf{k}$, $C_{13}^k \mathbf{k}$ and $C_{23}^k \mathbf{k}$. We denote this Lie algebra by $\mathfrak{b}(C_{12}^k \mathbf{k}, C_{13}^k \mathbf{k}, C_{23}^k \mathbf{k})$. For example, suppose the Lie brackets are given by

$$[e_1, e_2] = he_2 - e_3, \quad [e_1, e_3] = e_2 + he_3, \quad [e_2, e_3] = 0$$

for $h \geq 0$. We would denote this Lie algebra by $\mathfrak{b}(h\mathbf{2} - \mathbf{3}, \mathbf{2} + h\mathbf{3}, 0)$.

Proposition 2.29. *Suppose* $\mathfrak{g}' = 0$, *then* $\mathfrak{g} = \mathbb{R}^3$ *is the abelian* 3*–dimensional Lie algebra* $\mathfrak{b}(0, 0, 0)$.

Proof. If $\mathfrak{g}' = 0$ then $[\mathfrak{g}, \mathfrak{g}] = 0$, hence $\mathfrak{g}$ is abelian. ■

Proposition 2.30. *Suppose* $\mathfrak{g}'$ *is* 1*-dimensional, hence isomorphic to* $\mathbb{R}$. *Then* $\mathfrak{g}$ *is isomorphic either to* $\mathfrak{b}(\mathbf{1}, 0, 0)$ *or* $\mathfrak{b}(0, 0, \mathbf{1})$.

Proof. The condition $\dim \mathfrak{g}' = 1$ implies the extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 1$$

where $\mathfrak{h} = \mathfrak{g}/\mathfrak{g}'$. Then, since $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, $\mathfrak{h}$ is a 2–dimensional Lie algebra. There are only two types of them: the abelian one $\mathbb{R}^2$ and then one, which is not abelian but solvable. We name the latter one by $\mathfrak{h}^2$ where $\mathbb{R}^2$ is the underlying vector space and if e_1 and e_2 are the canonical basis of $\mathbb{R}^2$ the Lie bracket is given by $[e_1, e_2] = e_1$. We saw in Proposition 1.66 that isomorphism classes of central extensions are encoded by the second cohomology of that extension. The same proposition is true for extensions $0 \to \mathfrak{i} \to \mathfrak{g} \xrightarrow{p} \mathfrak{h} \to 0$ where $\mathfrak{i}$ is abelian (see [Kna88, pp. 161]). As to compute that cohomology we need to regard $\mathfrak{i}$ as an $\mathfrak{h}$–module in the following way: Let $\pi \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{i})$ given by $\pi(X)(Y) := [X', Y]$ where $p(X') = X$. This is well–defined since $\mathfrak{i}$ is abelian and the Jacobi identity shows that π is indeed an action of $\mathfrak{h}$ on $\mathfrak{i}$. We will denote the corresponding k–th cohomology group by $H^2(\mathfrak{h}; \mathfrak{i}_\pi)$ (cf. Appendix A, in Cohomology of Lie Algebras). Thus every extension comes with such a π and so does the extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 1.$$

Suppose first $\mathfrak{h} = \mathbb{R}^2$. Since $\mathfrak{gl}(\mathbb{R})$ is abelian, π has to be simply a 1–form on $\mathbb{R}^2$. Moreover $C^2 = \mathbf{Hom}(\wedge^2 \mathbb{R}^2, \mathbb{R})$ which is isomorphic to $\mathbb{R}$ via

$$\lambda \mapsto \omega_\lambda := \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

so we conclude that all elements of C^2 are cocycles since there are no non–trivial 3–forms on $\mathbb{R}^2$. The coboundary map $\delta\colon C^1 \to C^2$ (for the definition see Appendix A) is given by $\delta(\alpha) = \pi \wedge \alpha$. We distinguish two cases whether π is trivial or not.

(a) If π is not trivial, then δ is surjective and consequently $H^2(\mathbb{R}^2;\mathbb{R}_\pi) = 0$. The zero element in the second cohomology represents the extension induced by the semidirect product of Lie algebras $\mathbb{R} \rtimes_\pi \mathbb{R}^2$ (note that in this case π is a map from $\mathbb{R}^2$ into the derivations of $\mathbb{R}$ which is $\mathfrak{gl}(1)$ and therefore π induces a semidirect product, see Appendix A in Lie Algebras). Let $A \in \mathbf{GL}(\mathbb{R}^2)$ such that $\pi \circ A(e_1) = 1$ and $\pi \circ A(e_2) = 0$ and denote this map by π_0. Then every $\mathbb{R} \rtimes_\pi \mathbb{R}^2$ is isomorphic to $\mathbb{R} \rtimes_{\pi_0} \mathbb{R}^2$ as Lie algebras by the linear map $\mathrm{id} \times A\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$. Regarding $\mathbb{R} \rtimes_{\pi_0} \mathbb{R}^2$ as $\mathbb{R}^3$ and if (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ such that $\mathfrak{g}'$ is spanned by e_1 we obtain the relations
$$[e_1, e_2] = e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0$$
and we see $\mathbb{R} \rtimes_{\pi_0} \mathbb{R}^2$ is isomorphic to $\mathfrak{h}^2 \times \mathbb{R}$. This Lie algebra is not unimodular since $\operatorname{tr} \operatorname{ad}_{e_2} \neq 0$.

(b) Suppose π is trivial. Then the extension is central and $\delta\colon C^1 \to C^2$ is the trivial map. Thus $H^2(\mathbb{R}^2;\mathbb{R}) = \mathbb{R}$ via the isomorphism given above. From Proposition 1.65 we deduce the extensions which come in question are those induced by the Lie algebras $\mathbb{R} \times_{\omega_\lambda} \mathbb{R}^2$. We may assume $\lambda \neq 0$ since otherwise we would obtain the abelian Lie algebra $\mathbb{R}^3$. Let $f_\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map determined by $f_\lambda(e_2) = \lambda^{-1} e_2$ and $f_\lambda(e_3) = e_3$. Then $\mathrm{id} \times f_\lambda\colon \mathbb{R} \times_{\omega_1} \mathbb{R}^2 \to \mathbb{R} \times_{\omega_\lambda} \mathbb{R}^2$ is a Lie algebra isomorphism. Choosing (e_1, e_2, e_3) like above we obtain the relations
$$[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = e_1.$$
Since all ad_{e_i} are trace free we conclude that this Lie algebra is unimodular and therefore it is not isomorphic to the previous one.

Assume now $\mathfrak{h} = \mathfrak{h}^2$ hence there is a basis (e_2, e_3) of $\mathfrak{h}^2$ such that $[e_2, e_3] = e_2$. We claim that there is no Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}'$ is 1–dimensional and $\mathfrak{g}/\mathfrak{g}' = \mathfrak{h}^2$. Let therefore $p\colon \mathfrak{g} \to \mathfrak{h}^2$ be the Lie epimorphism from the extension. Choose $\varepsilon_i \in p^{-1}(e_i)$ for $i = 2, 3$. Then by assumption $[\varepsilon_1, \varepsilon_2] \in \mathfrak{g}' = \ker p$ but $p([\varepsilon_2, \varepsilon_3]) = e_2 \neq 0$. ∎

Proposition 2.31. *Let $\mathfrak{g}'$ be 2–dimensional. Then $\mathfrak{g}$ is isomorphic to either $\mathfrak{b}(0, \mathbf{1}, \lambda\mathbf{2})$ for $\lambda \in [-1, 1] \setminus \{0\}$ or $\mathfrak{b}(0, \mathbf{1}, \mathbf{1} + \mathbf{2})$ or $\mathfrak{b}(0, h\mathbf{1} - \mathbf{2}, \mathbf{1} + h\mathbf{2})$, $h \geq 0$.*

Proof. The assumption implies the extension

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R} \longrightarrow 0$$

which is a splitting extensions, since $\mathbb{R}$ is 1–dimensional. Thus using Lemma A.7 we have $\mathfrak{g} = \mathfrak{g}' \rtimes_\pi \mathbb{R}$ for a $\pi\colon \mathbb{R} \to \mathbf{Der}(\mathfrak{g}')$ and π determines the Lie bracket between elements of $\mathbb{R}$ and $\mathfrak{g}'$. Choose a non–zero element $e_3 \in \mathbb{R}$, then $\pi(e_3)(X) = [e_3, X]$ where we regard e_3 as $e_3 \in \mathfrak{g}' \rtimes \mathbb{R}$. This means that $\pi(e_3)$ is the adjoint map $\mathrm{ad}_{e_3} =: f$ (all the other brackets are known in the semidirect product). Therefore $f\colon \mathfrak{g}' \to \mathfrak{g}'$ is well–defined linear map. Note that $\mathfrak{g}'$ has to be abelian. Suppose $\mathfrak{g}' \cong \mathfrak{h}^2$ from above. Then there is a basis (Y_1, Y_2) of $\mathfrak{g}'$ such that $[Y_1, Y_2] = Y_1$ and ad_{Y_2} is not tracefree since $\mathrm{ad}_{Y_2}(Y_1) = -Y_1$. But this is a contradiction as Lemma A.5 shows.

Hence we may assume that $\mathfrak{g}'$ is the abelian Lie algebra $\mathbb{R}^2$. Then f has to be an isomorphism. If there would be a $Y_1 \in \mathfrak{g}'$ such that $f(Y_1) = 0$ then choose another non–zero vector Y_2 such that (Y_1, Y_2) is a basis for $\mathfrak{g}'$. Hence $\mathfrak{g}'$ is spanned by $[Y_i, e_3] = f(Y_i)$ and this implies that $\mathfrak{g}'$ would have dimension smaller then 2, which contradicts the assumption. Furthermore if $A \in \mathbf{GL}(\mathbb{R}^2)$ and $0 \neq \lambda \in \mathbb{R}$ then the map $A \times \lambda\colon \mathfrak{g}' \rtimes \mathbb{R} \to \mathfrak{g}' \rtimes \mathbb{R}$, $(Y, x) \mapsto (A(Y), \lambda x)$ is an isomorphism between the Lie algebras induced by the maps f and $\lambda^{-1} c_A(f)$, where c_A is the conjugation of f by A in $\mathbf{GL}(\mathbb{R}^2)$. Thus we have to examine the conjugacy class of $f \in \mathbf{GL}(\mathbb{R}^2)$. We will distinguish between the eigenvalues of $f \in \mathbf{GL}(\mathbb{R}^2)$:

(a) f is semi–simple, say with real eigenvalues λ_1, λ_2, which are not zero since f is an isomorphism. Hence there is an eigenvector basis of f, say (e_1, e_2), of $\mathbb{R}^2$ where e_i is an eigenvector of λ_i. Since we may multiply f with a $\lambda \neq 0$ while staying in the same isomorphism class of Lie algebras we may go over to $\lambda_1^{-1} f$. Thus the relations for the Lie bracket are given by

$$[e_3, e_1] = e_1, \quad [e_3, e_2] = \lambda e_2, \quad [e_1, e_2] = 0.$$

for $\lambda = \lambda_2/\lambda_1$ and we call that map f_λ. Moreover let $\mathfrak{b}(\lambda)$ be the Lie algebra defined by f_λ for $\lambda \neq 0$. The next step will be to check what $\mathfrak{b}(\lambda)$ are isomorphic. Set $\mathfrak{b}_i = \mathfrak{b}(\lambda_i)$ for $i = 1, 2$ and $\lambda_1 \neq \lambda_2$. Suppose $\Phi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ is an Lie algebra isomorphism, then Φ restricted to $\mathfrak{b}_1'$ is an isomorphism between $\mathfrak{b}_1'$ and $\mathfrak{b}_2'$. Hence $\Phi|\mathfrak{b}_1' =: A \in \mathbf{GL}(\mathbb{R}^2)$ and for $v \in \mathbb{R}^2$ we obtain on the one hand

$$[\Phi(e_3), \Phi(v)]_{\mathfrak{b}_2} = \Phi([e_3, v]_{\mathfrak{b}_1}) = A f_{\lambda_1}(v)$$

and on the other hand

$$[\Phi(e_3), \Phi(v)]_{\mathfrak{b}_2} = k \cdot [e_3, A(v)]_{\mathfrak{b}_2} = k \cdot f_{\lambda_2} A(v)$$

where $0 \neq k \in \mathbb{R}$ and $\Phi(e_3) = ke_3 + w$, $w \in \mathfrak{b}_2'$ (observe that k cannot be zero since otherwise Φ would be singular). Thus f_{λ_2} is conjugated to $k^{-1} \cdot f_{\lambda_1}$ in $\mathbf{GL}(\mathbb{R}^2)$. Therefore those maps must have the same eigenvalues and that is why we deduce $k = \lambda_2$ and $\lambda_2\lambda_1 = 1$. On the other side if $\lambda_2 = \lambda_1^{-1}$ then the following change of basis

$$e_3 \mapsto -\lambda_1 e_3, \quad e_1 \mapsto -e_2, \quad e_2 \mapsto -e_1$$

shows that $\mathfrak{b}(\lambda_2)$ is isomorphic to $\mathfrak{b}(\lambda_1)$. To sum up, $\mathfrak{b}_1$ is isomorphic to $\mathfrak{b}_2$ iff $\lambda_1\lambda_2 = 1$. We remark that $\mathfrak{b}(0, \mathbf{1}, \lambda\mathbf{2})$ becomes $\mathfrak{b}(\mathbf{1}, 0, 0)$ for $\lambda = 0$.

(b) f has an eigenvalue $\lambda \neq 0$ of multiplicity two, but f is not a multiple of the identity. Hence there is a basis (e_1, e_2) such that f is given by the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We conclude that the relations here are given by

$$[e_3, e_1] = \lambda e_1, \quad [e_3, e_2] = e_1 + \lambda e_2, \quad [e_1, e_2] = 0.$$

These Lie algebras are isomorphic to the one given by the relations

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2, \quad [e_1, e_2] = 0.$$

We see this, if we consider the linear map induced by the following change of basis

$$e_1 \mapsto -e_1, \quad e_2 \mapsto -\lambda e_2, \quad e_3 \mapsto -\lambda^{-1} e_3$$

(c) In the latter cases, the characteristic polynom of f had a real zero. Thus the remaining case is where f has a complex non–real eigenvalue λ, but since f is a real automorphism the complex conjugate $\bar{\lambda}$ is as well an eigenvalue. Therefore there is a complex eigenvector basis (z, w) of $\mathbb{C}^2$ such that

$$f^c(z) = \lambda z, \quad f^c(w) = \bar{\lambda} w,$$

where $f^c \in \mathbf{GL}(\mathbb{C}^2)$ is the complexification of $f \in \mathbf{GL}(\mathbb{R}^2)$, since λ is not real and therefore $\lambda \neq \bar{\lambda}$. Observe that $f^c(\bar{z}) = \bar{\lambda}\bar{z}$, hence we may choose $(z, \bar{z})$ as an eigenvector basis, where z and $\bar{z}$ is linear independent because $\lambda \neq \bar{\lambda}$. Write $z = e_1 + ie_2$ for $e_1, e_2 \in \mathbb{R}^2$, then (e_1, e_2) is a basis for $\mathbb{R}^2$ since otherwise $(z, \bar{z})$ is not a basis for $\mathbb{C}^2$. Moreover write $\lambda = \alpha + i\beta$ ($\beta \neq 0$) then $\lambda z = (\alpha e_1 - \beta e_2) + i(\beta e_1 + \alpha e_2)$ and

$$\begin{aligned} f(e_1) &= \mathbf{Re}(f^c(z)) = \mathbf{Re}(\lambda z) = \alpha e_1 - \beta e_2 \\ f(e_2) &= \mathbf{Im}(f^c(z)) = \mathbf{Im}(\lambda z) = \beta e_1 + \alpha e_2. \end{aligned}$$

As mentioned above $\beta^{-1}f$ induces the same Lie algebra as f. Thus the Lie bracket relations are given by

$$[e_3, e_1] = he_1 - e_2, \quad [e_3, e_2] = e_1 + he_2, \quad [e_1, e_2] = 0.$$

for $h := \alpha/\beta \in \mathbb{R}$ and we write $\mathfrak{b}(h)$ for this Lie algebra. This Lie algebra is isomorphic to

$$[e_1, e_2] = 0, \quad [e_1, e_3] = he_1 - e_2, \quad [e_2, e_3] = e_1 + he_2$$

by the linear transformation $e_i \mapsto -e_i$ for $i = 1, 2, 3$. Moreover the linear map of $\mathbb{R}^3$ defined by

$$e_1 \mapsto e_2, \quad e_2 \mapsto e_1, \quad e_3 \mapsto -e_3$$

shows that $\mathfrak{b}(h)$ isomorphic to $\mathfrak{b}(-h)$. As a last step, let us show that $\mathfrak{b}(h_1)$ is not isomorphic to $\mathfrak{b}(h_2)$ for $h_i \geq 0$ and $h_1 \neq h_2$. Suppose they are isomorphic. Then as in (b) the matrices

$$\begin{pmatrix} h_1 & 1 \\ -1 & h_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} kh_2 & k \\ -k & kh_2 \end{pmatrix}$$

have to be similar for a $k \neq 0$, thus they must have the same eigenvalues. The eigenvalues of the first matrix are $h_1 \pm i$ and of the second $kh_2 \pm ki$ which is a contradiction to $h_i \geq 0$ and $h_1 \neq h_2$.

We note that the cases (a), (b) and (c) cannot be isomorphic to each other since the corresponding representations lie in different conjugacy classes as the following lines show (see also part (a)). If $\Phi\colon \mathfrak{b}_1 \to \mathfrak{b}_2$ where $\mathfrak{b}_i := \mathbb{R}^2 \rtimes_{f_i} \mathbb{R}$ is an isomorphism then $\Phi|\mathfrak{b}_1'\colon \mathfrak{b}_1' \to \mathfrak{b}_2'$ is an Lie algebra isomorphism but since $\mathfrak{b}_i'$ is abelian we have $A := \Phi|\mathfrak{b}_1 \in \mathbf{GL}(\mathbb{R}^2)$. Furthermore observe that for $e_3 \in \mathfrak{b}_1$ as above we have $\Phi(e_3) = \lambda e_3 + y$ for a non–zero $\lambda \in \mathbb{R}$ (since otherwise Φ would map $\mathfrak{b}_1$ onto $\mathfrak{b}_2'$) and $y \in \mathfrak{b}_1'$. Now we may compute for $x \in \mathfrak{b}_1'$

$$\Phi([e_3, x]) = \Phi \circ f_1(x) = A \circ f_1(x).$$

On the other since Φ is a Lie algebra homomorphism we obtain

$$\Phi([e_3, x]) = [\Phi(e_3), \Phi(x)] = \lambda[e_3, \Phi(x)] = \lambda f_2 \circ A(x)$$

and this shows that $f_2 = \lambda^{-1}c_A(f_1)$. ∎

Proposition 2.32. *If $\mathfrak{g}' = \mathfrak{g}$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2)$ or $\mathfrak{so}(3)$.*

Proof. $\mathfrak{g}$ is unimodular by Lemma A.5. We know for $\mathfrak{so}(3) = (\mathbb{R}^3, \times)$ we have $\mathfrak{so}(3)' = \mathfrak{so}(3)$. Following [Mil76] $\mathfrak{sl}(2)$ is the only other 3–dimensional unimodular Lie algebra such that $\mathfrak{sl}(2)' = \mathfrak{sl}(2)$. To see this, choose a scalar product on $\mathfrak{g}$ and define the vector product $\times$ on $\mathfrak{g}$ such that $(\mathfrak{g}, \times) \cong \mathfrak{so}(3)$. The self–adjoint map $L\colon \mathfrak{g} \to \mathfrak{g}$, $L(x \times y) = [x, y]$ has to be invertible and this implies that $\mathfrak{g} \cong \mathfrak{so}(3)$ or $\mathfrak{g} \cong \mathfrak{sl}(2)$. ■

We would like to compare those Lie algebras with the unimodular ones which was found and named in [Mil76] as well as with those from the Bianchi list. In the first row we indicate the simply connected Lie groups instead of the Lie algebras like in [Mil76].

Milnor	Bianchi	$\mathfrak{b}(*,*,*)$
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	I	$\mathfrak{b}(0,0,0)$
Heisenberg group	II	$\mathfrak{b}(0,0,\mathbf{1})$
–	III=VI$_{-1}$	$\mathfrak{b}(\mathbf{1},0,0)$
–	IV	$\mathfrak{b}(0,\mathbf{1},\mathbf{1}+\mathbf{2})$
–	V	$\mathfrak{b}(0,\mathbf{1},\lambda\mathbf{2})$, $\lambda = 1$
$E(1,1)$	VI$_0$	$\mathfrak{b}\,(0,\mathbf{1},\lambda\mathbf{2})$, $\lambda = -1$
–	VI$_h$, $-1 \neq h \neq 0$	$\mathfrak{b}\,(0,\mathbf{1},\lambda\mathbf{2})$, $\lambda = \frac{1+h}{1-h}$
$\widetilde{E(2)}$	VII$_0$	$\mathfrak{b}(0, h\mathbf{1}-\mathbf{2}, \mathbf{1}+h\mathbf{2})$, $h = 0$
–	VII$_h$, $h > 0$	$\mathfrak{b}(0, h\mathbf{1}-\mathbf{2}, \mathbf{1}+h\mathbf{2})$, $h > 0$
$\widetilde{\mathbf{SL}(2)}$	VIII	$\mathfrak{sl}(2)$
$\mathbf{SU}(2)$	IX	$\mathfrak{so}(3)$

Finally as the last step in this section we would like to determine the simply connected groups $B(*,*,*)$ to the Lie algebras $\mathfrak{b}(*,*,*)$. We will give the list of groups without calculating their Lie algebras, since this is an easy task to accomplish. Some of them are pointed out in the table above.

(1) $\mathfrak{b}(0,0,0)$: this is clearly the abelian Lie group $\mathbb{R}^3$.

(2) $\mathfrak{b}(0,0,\mathbf{1})$: the Heisenberg group, see e.g. in [Sco83].

(3) $\mathfrak{b}(\mathbf{1},0,0)$: Let $\rho^\lambda\colon \mathbb{R} \to \mathbf{Aut}(\mathbb{R}^2)$ be the group homomorphism

$$\rho^\lambda_z := \begin{pmatrix} e^z & 0 \\ 0 & e^{\lambda z} \end{pmatrix}$$

for $\lambda \in \mathbb{R}$. Define the group $B(0,\mathbf{1},\lambda\mathbf{2}) =: B(\lambda) := \mathbb{R}^2 \rtimes_{\rho^\lambda} \mathbb{R}$. Then the Lie algebra of $B(0)$ is isomorphic to $\mathfrak{b}(\mathbf{1},0,0)$.

(4) $\mathfrak{b}(0, \mathbf{1}, \mathbf{1}+\mathbf{2})$: Here define the group homomorphism $\rho\colon \mathbb{R} \to \mathbf{Aut}(\mathbb{R}^2)$ by

$$\rho_z := \begin{pmatrix} e^z & ze^z \\ 0 & e^z \end{pmatrix}$$

and the group $B(0, \mathbf{1}, \mathbf{1}+\mathbf{2}) := \mathbb{R}^2 \rtimes_\rho \mathbb{R}$.

(5) $\mathfrak{b}(0, \mathbf{1}, \lambda\mathbf{2})$: the associated group is given by $B(\lambda)$, see (3).

(6) $\mathfrak{b}(0, h\mathbf{1}-\mathbf{2}, \mathbf{1}+h\mathbf{2})$: here we have perhaps the most complicated semidirect product. Define $\rho^h\colon \mathbb{R} \to \mathbf{Aut}(\mathbb{R}^2)$ by

$$\rho^h_t := e^{ht} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

for $h \geq 0$ and finally $B(h\mathbf{2}-\mathbf{3}, \mathbf{2}+h\mathbf{3}, 0) := \mathbb{R}^2 \rtimes_\rho \mathbb{R}$.

(7) $\mathfrak{sl}(2)$, $\mathfrak{so}(3)$: of course the groups here are $\widetilde{\mathbf{SL}(2)}$ and $\mathbf{SU}(2)$ respectively.

The list of 3–dimensional geometries

We would like to sum up the 3–dimensional geometries and discuss some connections among them. First let us start with an overview.

(i) *isotropic geometries* ($\dim K = 3$).

$$(\mathbb{R}^3, \mathbf{E}_0(3)) \;\; (S^3, \mathbf{SO}(4) \;\; (D^3, \mathbf{H}_0(3))$$

(ii) *rotationally symmetric geometries* ($\dim K = 1$).

flat	non–flat
$(S^2 \times \mathbb{R}, \mathbf{SO}(3) \times \mathbb{R})$	$(S^3, \mathbf{U}(2))$
$(D^2 \times \mathbb{R}, \mathbf{SO}^+(2,1) \times \mathbb{R})$	$(D^2 \times \mathbb{R}, \Gamma)$
$(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R})$, $\kappa = 0, 1$	$(\mathbb{R}^2 \times \mathbb{R}, \mathbf{Nil} \rtimes \mathbf{SO}(2))$

(iii) *Bianchi Groups* ($\dim K = 0$). We sort the groups by the dimension of its derived Lie algebra.

$\dim \mathfrak{b}'$	0	1	2	3
	$\mathbb{R}^3$	$\mathbf{Nil}$	$B(\lambda)$, $\lambda \neq 0$, $\lvert\lambda\rvert \leq 1$	$\mathfrak{so}(3)$
		$B(0)$	$B(0, h\mathbf{1}-\mathbf{2}, \mathbf{1}+h\mathbf{2})$, $h \geq 0$	$\mathfrak{sl}(2)$
			$B(0, \mathbf{1}, \mathbf{1}+\mathbf{2})$	

First note that $S^2 \times \mathbb{R}$ cannot carry *any* Lie group structure. It is easy to see that there is no 3–dimensional Lie subgroup of $\mathbf{SO}(3) \times \mathbb{R}$ which acts transitive on $S^2 \times \mathbb{R}$, but to exclude that there is no Lie group structure on $S^2 \times \mathbb{R}$ we utilize the topological fact that for a Lie group G we have $\pi_2(G) = 0$, but $\pi_2(S^2 \times \mathbb{R}) = \mathbb{Z}$.

Any other geometry admits a 3–dimensional subgroup which acts on M transitive. However every geometry except $(S^2 \times \mathbb{R}, \mathbf{SO}(3) \times \mathbb{R})$ owns a 3–dimensional subgroup which acts transitively on M. The universal cover of this group has to be a Bianchi group. Thus we may assign to every geometry a 3–dimensional Lie algebra, but not in a unique way.

Let us recall the *Thurston–Geometries*. These are geometries (M, G) which admit a compact quotients, i.e. there is discrete subgroup $\Gamma \subset G$ which acts free and properly discontinuously on M such that M/Γ is compact. Such groups Γ are called *cocompact groups*. Moreover the geometry (M, G) should be *maximal* in the sense that there is no bigger group G' such that G is a subgroup of G' and (M, G') a geometry with the restricted action. But the maximality condition is not important for us and therefore we would like to focus on the geometries with compact quotients. As one may take from [Thu97], [Sco83] and [Mil76] the list of geometries with compact quotients is given by

$\dim K$	
3	$(\mathbb{R}^3, \mathbf{E}_0(3))$
	$(S^3, \mathbf{SO}(4))$
	$(D^3, \mathbf{SO}^+(3,1))$
1	$(S^2 \times \mathbb{R}, \mathbf{SO}(3) \times \mathbb{R}))$
	$(D^2 \times \mathbb{R}, \mathbf{SO}^+(2,1) \times \mathbb{R})$
	$(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \times \mathbb{R})$
	$(S^3, \mathbf{U}(2))$
	$(D^2 \times \mathbb{R}, \Gamma)$
	$(\mathbb{R}^2 \times \mathbb{R}, \mathbf{Nil} \rtimes \mathbf{SO}(2))$
0	$\mathbb{R}^3$
	$\mathbf{Nil}$
	$B(-1) = \mathbf{E}_0(1,1)$
	$\widetilde{\mathbf{E}_0(2)}$
	$\widetilde{\mathbf{SL}(2)}$
	$\mathbf{SU}(2)$

where $B(-1)$ is also known as the *Sol–geometry* (note that a Lie group is unimodular if it admits a compact quotient). The group listed in the case of $\dim K = 0$ are also called of *class A*. This comes from the Bianchi classification where one distinguish between unimodular (class A) and non–unimodular (class B) groups. From [Mil76, Lemma 6.2] we see that the class B Lie groups cannot admit compact quotients since otherwise they would be unimodular. The only other geometries to question are the spaces $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R})$ for $\kappa > 0$ (clearly the isotropic geometries possess compact quotients). The next Proposition clarifies the situation

Proposition 2.33. *Let X be a homogeneous vector field on a geometry (M, G). If (M, G) admits compact quotients then* $\mathrm{div} X$ *with respect to a G–invariant metric is zero.*

Proof. Suppose $\Gamma \subset G$ is a cocompact group and $\check{M} := M/\Gamma$ is the induced compact manifold. Since X is G–invariant thus in particular Γ–invariant we may define a vector field

$$\check{X}_{\check{m}} := (\mathrm{D}\pi)_m(X_m)$$

where $\pi \colon M \to \check{M}$ is the covering map and $\pi(m) = \check{m}$. This is well–defined by the Γ–invariance of the vector field X. Let μ be a G–invariant metric on (M, G) and $\check{\mu}$ the induced metric on $\check{M}$. Then $\mathrm{div}_\mu X = \pi^*(\mathrm{div}_{\check{\mu}} \check{X})$. To see this let U be an open subset of M such that $p := \pi|U$ is a local isometry. We choose local orthonormal fields (E_1, E_2, E_3) on U and then clearly $p_*(E_i)$ for $i = 1, 2, 3$ are orthonormal vector fields on $\check{U} := p(U)$. Furthermore note that $p_*(X) = \check{X}$ and hence we obtain

$$\mathrm{div}_{\check{\mu}} \check{X} = \sum_i \check{\mu}(\check{\nabla}_{p_*(E_i)} \check{X}, p_*(E_i)) = \sum_i \check{\mu}(p_*(\nabla_{E_i} X), p_*(E_i)) = \sum_i \mu(\nabla_{E_i} X, E_i) \circ p^{-1}$$

on $\check{U}$. Since π is a local isometry we deduce $\mathrm{div}_\mu X = \pi^*(\mathrm{div}_{\check{\mu}} \check{X})$ and so $\mathrm{div} \check{X}$ is constant on $\check{M}$. Using Stokes' theorem we obtain $\mathrm{div} \check{X} = 0$, hence $\mathrm{div} X = 0$ ∎

In Example 2.16 we saw that ∂_t is a homogeneous vector field for $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R})$ and there we computed that $\mathrm{div} \partial_t = \kappa$, hence for $\kappa > 0$ those geometries can not admit compact quotients.

Definition 2.34. We shall call a geometry (M, G) of *class A* if it admits compact quotients. Hence it has to be one of the list above.

We remark that all groups G of a class A geometry are unimodular which means that G admit a biinvariant volume form. Moreover we see that most Lie algebras are given as $\mathfrak{g} = \mathfrak{h} \times \mathbb{R}$ where $\mathfrak{h}$ is unimodular and $\mathbb{R}$ is the center of $\mathfrak{g}$ (then clearly ad_x is trace–free for all $x \in \mathfrak{g}$).

3
Homogeneous Cosmological Models

MANY PHENOMENONS in physics on big scales involve Einstein's general theory of relativity as for example to describe the dynamics of the universe. Using general relativity, we have to solve a very complicated partial differential equation on a manifold. For that reason we may impose symmetry conditions in order to simplify the problem.

We do not want to explain at this point how the theory of general relativity in detail works but rather discuss some cornerstones. From the mathematical viewpoint general relativity is very quickly introduced. The basis is always a 4–dimensional Lorentzian manifold, say (L, h). Suppose we would like to model a physical system such as a star or a whole universe. All physical objects in that system which may interact with the gravitational field will be packed into a symmetric, divergence free tensor T (the divergence with respect to h). Define the *Einstein–Tensor* by $\mathrm{Ein}(h) := \mathrm{Ric}\,(h) - \frac{1}{2}R(h)h$ where Ric is the Ricci–curvature of h and $R(h)$ is the scalar curvature of h. The objects which we would like to determine depend on the problem: sometimes the manifold L as well as the metric h is wanted, sometimes L is given and h is wanted. Suppose L and T are given (like it will be in our case), then we try to find a Lorentzian metric h which solves the equation

$$\mathrm{Ein}(h) = T.$$

Note that T can depend on h as well. Moreover, unlike in physics, we would like to put the possibly cosmological constant into the tensor T. If $T = \Lambda h$ where $0 \neq \Lambda \in \mathbb{R}$ is called *cosmological constant* and the equation $\mathrm{Ein}(h) = \Lambda h$ is equivalent to the equation $\mathrm{Ric}\,(h) = -\Lambda h$.

Our aim will be to model the universe where physics are based on the laws of general relativity. The classical models are known among others under the name *Robertson–Walker* models where the universe (stars, galaxies, clusters of galaxies) are assumed to behave like a perfect fluid. Moreover there is made the assumption that the universe looks isotropic which means geometrically that there are so many symmetries such that the sectional curvature of the spatial parts of the spacetimes are constant (this implies in particular that that the spatial parts are homogeneous). We will try to extend these models. But in contrast to the Robertson–Walker model we will deal solely with the case where there is no matter but a lot of dark energy. Therefore the word 'extend' is maybe not a precise description, but here we mean to extend the models in the sense that we drop the isotropy condition, however, we retain the homogeneity of the spatial parts.

The first attempts in that field were made by Gödel and Taub in the late 50's where they used the Bianchi groups and the so called Kantowski–Sachs geometry to model a homogeneous but anisotropic universe. They originally used that models to reveal some interesting causality effects of general relativity. Surely there are more physical reasons to examine those models but from a mathematical viewpoint they are interesting too, since they provide a class of solutions to the Einstein equations which are in general really hard to solve. The key fact is that by the symmetries the complicated partial differential equation system reduce to a (still hard but treatable) ODE system.

Homogeneous cosmologies

The name homogeneous spacetime is maybe misleading since we mean that a spacetime should be merely spatially homogeneous. We make this precise in the following definition. On a first glimpse this definition looks a little weird but we will explain afterwards why the naming is reasonable.

Definition 3.1. Let (M, G) be a geometry[1] and $\mu\colon I \to \mathcal{S}^+$ a curve of G–invariant metrics where I is an interval in $\mathbb{R}$ (for the definition of $\mathcal{S}^+$ consider the section THE SPACE OF HOMOGENEOUS METRICS of Chapter 1). We call (M, G, μ) a *homogeneous spacetime*.

Remark 3.2. We are able to craft a Lorentzian manifold from the data (M, G, μ). Set $L := I \times M$ and $h := -\mathrm{d}t^2 + \mu(t)$ as the Lorentzian metric where t is the standard time–function on $I \subset \mathbb{R}$, i.e. t is the projection from $I \times M$ onto the first factor. The group G is acting on L by $\mathrm{id} \times g\colon I \times M \to I \times M$ for $g \in G$. Then the submanifolds $t \times M$ are Riemannian homogeneous spaces by the action of G and $t \times M$ is equipped with the metric $\mu(t)$ which turns it into a Riemannian manifold. Thus $t \times M$ are spatial submanifolds of (L, h). We denote with T the vector field to the flow $\varphi^s(t, m) = (t + s, m)$ on $I \times M$ and it follows that $h(T, T) = -1$. Let $\kappa(t)$

[1] Recall that a *geometry* is a Riemannian homogeneous space, such that the manifold is simply connected and of dimension 3

be the second fundamental form of $(t \times M, \mu(t)) \subset (L, h)$ with respect to T, $H(t) := \operatorname{tr}_{\mu(t)}\kappa(t)$ its mean curvature. By definition we have $\kappa(t)(\xi, \eta) = -h(\xi, \nabla_\eta T)$ where ∇ is the Levi–Civita connection of h. Then T is invariant under the action of G on L and T lies always orthogonal with respect to h on the slices $t \times M$. Finally this implies that $\kappa(t)$ is a G–invariant tensor M and therefore $\kappa\colon I \to \mathbf{Sym}_G(M) \cong \mathbf{Sym}_K(TM_{m_0})$ where $\mathbf{Sym}_G(M)$ are the G–invariant symmetric 2–tensors on M and $\mathbf{Sym}_K(TM_{m_0})$ are the K–invariant symmetric bilinear–forms on TM_{m_0} where $m_0 \in M$ and K is the isotropy group of m_0. The *shape operator* $S(t)$ of $\kappa(t)$ is defined as $\kappa(t)(\xi, \eta) = h(S(t)(\xi), \eta)$ and this implies that $\mu(t)(S(t)(\xi), \eta) = \kappa(t)(\xi, \eta)$ as well as $S(t)(\xi) = -\nabla_\xi T$. We will write more intuitively $\kappa(t)^\sharp$ for $S(t)$.

Clearly as a consequence of Proposition 1.19 the mean curvature $H(t)$ is constant on $t \times M$ and therefore solely a time–dependent geometric quantity. Let us fix some more notation at this point: we shall describe with $\operatorname{Ric}(t)$ the Ricci–curvature and $R(t)$ the scalar curvature of $(t \times M, \mu(t)) =: M(t)$. We regard those objects always as quantities on M rather on $t \times M$. More precisely we have a curve $\operatorname{Ric} : I \to \mathbf{Sym}_G(M)$ since $\operatorname{Ric}(t)$ is G–invariant and $R(t)$ is a constant function on $M(t)$. And if an object, like $\operatorname{Ric}(t)$, is G–invariant we would like to identify it with the object which is given in a standard point $m_0 \in M$. Thus in the case of $\operatorname{Ric}(t) \in \mathbf{Sym}_G(M)$ we may pass over to the symmetric bilinear form $\operatorname{Ric}(t)_{m_0} \in \mathbf{Sym}(TM_{m_0})$ on TM_{m_0} but mark it again with $\operatorname{Ric}(t)$.

Definition 3.3. A homogeneous spacetime (M, G, μ) is a *(homogeneous) cosmological model with (cosmological) constant* Λ if $\operatorname{Ein}(h) = \Lambda h$ on (L, h) for a $\Lambda \in \mathbb{R}$ (see Remark 3.2 for (L, h)).

The manifold (L, h) is merely build by objects from the triple (M, G, μ) and therefore we expect that the condition $\operatorname{Ric}(h)$ can be reformulated to conditions for the triple (M, G, μ).

Proposition 3.4. *A homogeneous spacetime (M, G, μ) is a homogeneous cosmological model with constant Λ iff the following conditions are fullfilled for all $t \in I$*

(a) $R(t) + H(t)^2 - |\kappa(t)|^2 + 2\Lambda = 0$

(b) $\mathrm{div}_{\mu(t)} \kappa(t) = 0$

(c) $\dot{\mu}(t) = -2\kappa(t)$

(d) $\dot{\kappa}(t) = \mathrm{Ric}\,(t) + \Lambda\mu(t) + H(t)\kappa(t) - 2(\kappa(t)^\sharp)^2$

where $|\kappa(t)|^2 = \langle \kappa(t), \kappa(t)\rangle_{\mu(t)}$ and $\kappa(t)^\sharp$ is the self–adjoint shape operator given by $\mu(t)(\kappa(t)^\sharp(v), w) = \kappa(t)(v, w)$ (see after Remark 1.73). With $(\kappa(t)^\sharp)^2$ we would like to denote the symmetric bilinear form of the endomorphism $\kappa(t)^\sharp \circ \kappa(t)^\sharp$ with respect to $\mu(t)$, thus $(\kappa(t)^\sharp)^2(v, w) = \mu(t)((\kappa^\sharp)^2(v), w) = \mu(t)(\kappa(t)^\sharp(v), \kappa^\sharp(w))$. Moreover observe with our convention of Remark 3.2 these equations are regarded in an arbitrary but fixed point $m_0 \in M$ since of their G–invariance.

Proof. See [Zeg11] for a proof. It is a straightforward computation using the Gauss and Codazzi equations for the submanifolds $M(t) \subset (L, h)$. Part (a) is $(\mathrm{Ein}(h) - \Lambda h)(T, T) = 0$, part (b) are the mixed directions $(\mathrm{Ein}(h) - \Lambda h)(T, X) =$ for a spatial vector field, i.e. X has no T–component and finally the last two equations are given by $(\mathrm{Ein}(h) - \Lambda h)(X, Y) = 0$ for X, Y spatial vector fields on L. ∎

These four equations may be interpreted as the flow lines of a vector field on a subset of the tangent bundle of $\mathcal{S}^+(TM_{m_0})$ which is isomorphic to $\mathcal{P} := \mathcal{S}^+(TM_{m_0}) \times \mathbf{Sym}(TM_{m_0})$. The tangent space in a point $(\mu, \kappa) \in \mathcal{P}$ is given by $\mathbf{Sym}(TM_{m_0}) \times \mathbf{Sym}(TM_{m_0})$. Thus a vector field $\mathcal{X}$ on $\mathcal{P}$ in (μ, κ) is given by a pair of symmetric bilinear forms on TM_{m_0}. Moreover since $\mathcal{P}$ is for itself a tangent bundle we may equip $T\mathcal{P}$ with a natural Riemannian metric: for $(\sigma_1, \sigma_2), (\tau_1, \tau_2) \in T\mathcal{P}_{(\mu,\kappa)}$ define

$$\langle\langle(\sigma_1, \sigma_2), (\tau_1, \tau_2)\rangle\rangle_{(\mu,\kappa)} := \langle\sigma_1, \tau_1\rangle_\mu + \langle\sigma_2, \tau_2\rangle_\mu.$$

Definition 3.5. Let (M, G) be a Riemannian homogeneous space. For a $\Lambda \in \mathbb{R}$ we call the vector field $\mathcal{X}^\Lambda$ on $\mathcal{P}$ given by

$$\mathcal{X}^\Lambda_{(\mu,\kappa)} := (-2\kappa, \mathrm{Ric}\,(\mu) + \Lambda\mu + H\kappa - 2(\kappa^\sharp)^2)$$

the *Einstein vector field* (to the constant Λ) where $H = \mathrm{tr}_\mu \kappa$ and $\kappa^\sharp$ are taken with respect to μ and its flow the *Einstein flow* (note that $\mathrm{Ric}\,(\mu)$ is the Ricci curvature of μ as a G–invariant metric).

If $t \mapsto (\mu(t), \kappa(t))$ is a flow line of $\mathcal{X}^\Lambda$ then the curve $t \mapsto \mu(t)$ fullfills condition (c) and (d) of Proposition 3.4. Let us study now condition (a) of Proposition 3.4 more precisely. Define the smooth function $F^\Lambda \colon \mathcal{P} \to \mathbb{R}$, $F^\Lambda(\mu, \kappa) := R(\mu) + H^2 - |\kappa|^2 + 2\Lambda$, where $R(\mu)$ is the scalar curvature of μ is (seen as G–invariant metric on M) and $|\kappa|$ is the norm of κ with respect to μ. Let us stipulate to omit the superscript Λ.

Proposition 3.6. *Suppose (M, G) is a Riemannian homogeneous space and let $F \colon \mathcal{P} \to \mathbb{R}$ be the function defined above for an arbitrary $\Lambda \in \mathbb{R}$. Moreover we assume that the homogeneous vector fields are divergence free with respect to a (and therefore any) G–invariant metric. Then 0 is a regular value of F if $\Lambda \neq 0$ or if $\Lambda = 0$ and (M, G) does not admit a G–invariant Ricci–flat metric.*

Proof. Let $\alpha \colon I \to \mathcal{P}$ be a smooth curve where $0 \in I \subset \mathbb{R}$ is an interval. Write $\alpha(t) = (\mu(t), \kappa(t))$, $\alpha(0) = (\mu, \kappa)$ and $\dot\alpha(0) = (\sigma_1, \sigma_2)$. Then $DF_{(\mu,\kappa)}(\sigma_1, \sigma_2) = \frac{\mathrm{d}}{\mathrm{d}t} F \circ \alpha(0)$. First consider the scalar curvature as a function of $\mathcal{P}$ into the reals, i.e. $(\mu, \kappa) \mapsto R(\mu)$, which does not depend on κ. Consider the curve $\mu(t)$ now as a curve of Riemannian metrics on M globally. Then by [Bes08, p. 63] we obtain in $t = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\alpha(t)) = \frac{\mathrm{d}}{\mathrm{d}t} R(\mu(t)) = \Delta_\mu(\mathrm{tr}_{\,\mu}(\sigma_1)) + \mathrm{div}_\mu(\mathrm{div}_\mu \sigma_1) - \langle \mathrm{Ric}\,(\mu), \sigma_1 \rangle_\mu$$

where all objects are globally defined on M. Proposition 1.19 implies that $\mathrm{tr}_{\,\mu}\sigma_1$ is a constant function since σ_1 is G–invariant, hence $\Delta \mathrm{tr}\,\sigma_1 = 0$.

Using Proposition 1.21 and Proposition 1.19 $\mathrm{div}\sigma_1$ is a G–invariant 1-form and therefore there is a homogeneous vector field X which is the dual to $\mathrm{div}\sigma_1$ with respect to μ. From the definition we have $\mathrm{div}(\mathrm{div}\sigma_1) = \mathrm{div}X$ and by Proposition 2.13 $\mathrm{div}X$ is a constant which is zero because of the assumption (for every G–invariant metric). We sum up and conclude that $\mathrm{div}(\mathrm{div}\sigma_1)$ vanishes. Observe that $\langle \mathrm{Ric}\,(\mu), \sigma_1 \rangle_\mu$ is constant as well, since it is G–invariant and therefore the gradient of R with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ is given by

$$(\nabla R)_{(\mu,\kappa)} = (-\mathrm{Ric}\,(\mu), 0)$$

(note that we passed again from the global object $\mathrm{Ric}\,(\mu)$ to $\mathrm{Ric}\,(\mu)_{m_0}$ and ∇R is a vector field on $\mathcal{P}$).

Next we differentiate the function $(\mu, \kappa) \mapsto H^2 = (\mathrm{tr}_{\,\mu}\kappa)^2$. Obviously $H = \mathrm{tr}_{\,\mu}(\kappa) = \mathrm{tr}\,(\kappa^\sharp)$ where $\kappa^\sharp$ is taken with respect to μ. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{tr}_{\,\mu(t)} \kappa(t) = \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t} \kappa(t)^\sharp \right)$$

where $\mu(t)(\kappa(t)^\sharp(\xi), \eta) = \kappa(t)(\xi, \eta)$ so $\kappa(t)^\sharp$ means always that we take the $\sharp$ of $\kappa(t)$ with respect to $\mu(t)$. This gives us a possibility to compute $\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp$ by differentating both sides in

$t = 0$. We end up with the equation

$$\sigma_1(\kappa^\sharp(\xi), \eta) + \mu\left(\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp_{t=0}(\xi), \eta\right) = \sigma_2(\xi, \eta)$$

and this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp_{t=0} = \sigma_2^\sharp - \sigma_1^\sharp\kappa^\sharp$$

(where all $\sharp$ are taken with respect to μ). And therefore we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,(\kappa^\sharp) = \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp\right) = \mathrm{tr}\,(\sigma_2^\sharp) - \mathrm{tr}\,(\sigma_1^\sharp\kappa^\sharp) = \langle\mu, \sigma_2\rangle - \langle\kappa, \sigma_1\rangle = \langle\langle(-\kappa, \mu), (\sigma_1, \sigma_2)\rangle\rangle,$$

so the gradient of the function $(\mu, \kappa) \mapsto H^2 = \mathrm{tr}_\mu(\kappa)^2$ in (μ, κ) is given by

$$2H\nabla H_{(\mu,\kappa)} = 2H(-\kappa, \mu).$$

The last function is given by $(\mu, \kappa) \mapsto |\kappa|^2 = \mathrm{tr}\,((\kappa^\sharp)^2)$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\kappa(t)^\sharp)^2_{t=0} &= \kappa^\sharp\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp_{t=0}\kappa^\sharp \\ &= \kappa^\sharp\sigma_2^\sharp - \kappa^\sharp\sigma_1^\sharp\kappa^\sharp + \sigma_2^\sharp\kappa^\sharp - \sigma_1^\sharp\kappa^\sharp\kappa^\sharp\end{aligned}$$

and taking the trace yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,((\kappa(t)^\sharp)^2)_{t=0} &= 2\mathrm{tr}\,(\kappa^\sharp\sigma_2^\sharp) - 2\mathrm{tr}\,((\kappa^\sharp)^2\sigma_1^\sharp) \\ &= 2\langle\kappa, \sigma_2\rangle - 2\langle(\kappa^\sharp)^2, \sigma_1\rangle\end{aligned}$$

and we conclude

$$\nabla|\kappa^2|_{(\mu,\kappa)} = 2(-(\kappa^\sharp)^2, \kappa).$$

So finally the gradient of F is given by

$$\nabla F_{(\mu,\kappa)} = (-\mathrm{Ric}\,(\mu) - 2H\kappa + 2(\kappa^\sharp)^2, 2H\mu - 2\kappa)$$

which vanishes exactly at the points $(\mu, 0)$ where μ is a Ricci-flat metric, since the last row implies $\kappa = 0$ which implies $\mathrm{Ric}\,(\mu) = 0$ from the first row. ■

Remark 3.7. Proposition 3.6 shows that under the conditions there, the set $(\mu, \kappa) \in \mathcal{P}$ such that $R(\mu) + H^2 - |\kappa|^2 + 2\Lambda = 0$ is a closed embedded hypersurface. We denote the set $V := F^{-1}(0)$ where the V stands for *variety*. Now let us discuss the assumptions of Proposition 3.6.

(a) If a geometry (M, G) admits a Ricci–flat metric μ then since M is 3–dimensional the sectional curvature of (M, μ) is fully determined by its Ricci–tensor and therefore μ has constant sectional curvature equal to zero. This implies that $M = \mathbb{R}^3$ and μ is the standard euclidean metric on $\mathbb{R}^3$ which forces G to be a subgroup of $\mathbf{E}_0(3)$ (since by Proposition 1.7 G has to be a closed subgroup of $I(M, \mu) = \mathbf{E}(3)$).

(b) For class A geometries (Definition 2.34) the divergence of homogeneous vector fields is always zero, see Proposition 2.33. Thus under the assumption that (M, G) is a class A geometry and $\Lambda \neq 0$ or $\Lambda = 0$, but without a homogeneous Ricci–flat metric the set V is a smooth embedded manifold. In those cases the vector field $\mathcal{X}$ lies tangential, since

$$\begin{aligned} \mathrm{d}F_{(\mu,\kappa)}(\mathcal{X}_{(\mu,\kappa)}) &= 2\langle \mathrm{Ric}\,(\mu), \kappa\rangle + 4H|\kappa|^2 - 4\langle (\kappa^\sharp)^2, \kappa\rangle + 2HR(\mu) + 6H\Lambda + 2H^3 \\ &- 4H|\kappa|^2 - 2\langle \kappa, \mathrm{Ric}\,(\mu)\rangle - 2H\Lambda - 2H|\kappa|^2 + 4\langle (\kappa^\sharp)^2, \kappa\rangle \\ &= 2H(R(\mu) + H^2 - |\kappa|^2 + 2\Lambda) = 2H \cdot F(\mu, \kappa) \end{aligned}$$

hence for $(\mu, \kappa) \in V$ $\mathcal{X}$ is tangential to V. Moreover this means that for any geometry the integral curve of $\mathcal{X}$ with starting point $(\mu, \kappa) \in V$ stays in V, i.e. the flow preserves the condition V . However, we will show in Proposition 3.8 that the conditions $F = 0$ and $\mathrm{div}_\mu \kappa = 0$ are preserved by using the contracted second Bianchi identity.

Proposition 3.8 (Lemma 2.4 in [EW00]). *Let (M, G) be a geometry and $(\mu(t), \kappa(t))_{t\in I}$ an integral curve of $\mathcal{X}^\Lambda$ such that $(\mu_0, \kappa_0) := (\mu(0), \kappa(0)) \in V$ and $\mathrm{div}_{\mu_0}\kappa_0 = 0$, then $(\mu(t), \kappa(t)) \in V$ and $\mathrm{div}_{\mu(t)}\kappa(t) = 0$ for all $t \in I$.*

Proof. Set $L := I \times M$ and $h := -\mathrm{d}t^2 + \mu(t)$ and choose a point $(t_0, m_0) \in L$. Let us fix some notations

∇	the Levi–Civita connection of (L, h)
∇^t	the Levi–Civita connection of $(M, \mu(t))$
S	scalar curvature of h
Ric	the Ricci–curvature of (L, h)
Ric (t)	the Ricci curvature of $(M, \mu(t))$
X	a vector field on M
$\hat{X}$	the spatial lift of X, i.e. $X_{(t,m)} = X_m$
T	the vector field ∂_t on L
e_0	$T_{(t_0,m_0)}$
β	a background metric on M
$v(t)$	defined as $\sqrt{\det \mu(t)}$, where the determinant is with respect ot β
(e_1, e_2, e_3)	an orthonormal base in TM_{m_0} with respect to $\mu(t_0)$
E_i	local parallel continuations of e_i on M with respect to $\mu(t_0)$

and we may not distinguish between vectors $\xi \in TM_{m_0}$ and $\xi \in TL_{(t_0,m_0)}$. Clearly (e_0, e_1, e_2, e_3) is an orthonormal base of (L, h) in the point (t_0, m_0). Moreover note that S is constant along $t_0 \times M$, since G acts isometrically on L and transitive on M. Thus the contracted Bianchi identity divRic $= \frac{1}{2} dS$ yields for $\xi := \hat{X}_{(t_0,m_0)}$ (where X is a local parallel continuation of $\xi \in TM_{m_0}$ with respect to $\mu(t_0)$)

$$0 = \mathrm{divRic}\,(\xi) = -(\nabla_{e_0}\mathrm{Ric}\,)(e_0, \xi) + \sum_{i=1}^{3} (\nabla_{e_i}\mathrm{Ric}\,)(e_i, \xi)$$

(mind the minus sign due to the Lorentzian signature of h). Further we see

$$\begin{aligned} (\nabla_{e_0}\mathrm{Ric}\,)(e_0, \xi) &= e_0(\mathrm{Ric}\,(T, \hat{X})) - \mathrm{Ric}\,(\nabla_{e_0}T, \xi) - \mathrm{Ric}\,(e_0, \nabla_{e_0}\hat{X}) \\ &= e_0(\mathrm{Ric}\,(T, \hat{X})) + \mathrm{Ric}\,(e_0, \kappa(t_0)^\sharp(\xi)) \end{aligned}$$

since $\nabla_{e_0}T = 0$ and $[\hat{X}, T] = 0$ as well as $\nabla_\xi T = -\kappa(t_0)^\sharp(\xi)$. For $i = 1, 2, 3$ we obtain

$$(\nabla_{e_i}\mathrm{Ric}\,)(e_i, \xi) = e_i(\mathrm{Ric}\,(\hat{E}_i, \hat{X})) - \mathrm{Ric}\,(\nabla_{e_i}\hat{E}_i, \xi) - \mathrm{Ric}\,(e_i, \nabla_{e_i}\hat{X})$$

and we recall

$$\mathrm{Ric}\,(e_i, e_j) = \mathrm{Ric}\,(t)(e_i, e_j) - \dot{\kappa}(t)(e_i, e_j) + H(t)\kappa(t)(e_i, e_j) - 2(\kappa(t)^\sharp)^2(e_i, e_j),$$

see e.g. Fact 9.3 of [Zeg11]. Since $(\mu(t), \kappa(t))$ is an integral curve of $\mathcal{X}^\Lambda$ we obtain

$$\mathrm{Ric}\,(e_i, e_j) = -\Lambda\mu(t_0)(e_i, e_j).$$

which implies

$$e_i(\mathrm{Ric}\,(\hat{E}_i, \hat{X})) = -\Lambda e_i(\mu(t_0)(E_i, X)) = 0.$$

Furthermore we have $\nabla_{e_i}\hat{E}_i = \widehat{\nabla^{t_0}_{e_i}E_i} - \kappa(t_0)(e_i, e_i)e_0 = -\kappa(t_0)(e_i, e_i)e_0$. All this implies

$$\sum_{i=1}^{3}(\nabla_{e_i}\mathrm{Ric}\,)(e_i, \xi) = H(t_0)\mathrm{Ric}\,(e_0, \xi) + \mathrm{Ric}\,(\kappa(t_0)^\sharp(\xi), e_0).$$

From the contracted second Bianchi identity we therefore get

$$\partial_t(\mathrm{Ric}\,(T, \hat{X})) = e_0(\mathrm{Ric}\,(T, \hat{X})) = H(t_0)\mathrm{Ric}\,(e_0, \xi).$$

Moreover note

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \frac{1}{2\sqrt{\det\mu(t)}}\frac{\mathrm{d}}{\mathrm{d}t}\det(\mu(t)) = \frac{1}{2\sqrt{\det\mu(t)}}\mathrm{tr}_{\,\mu(t)}(-2\kappa(t))\det(\mu(t)) = -H(t)v(t)$$

and if we sum up we obtain

$$\partial_t(v\cdot\mathrm{Ric}\,(T, \hat{X})) = -Hv\mathrm{Ric}\,(T, \hat{X}) + v\partial_t(\mathrm{Ric}\,(T, \hat{X})) = 0$$

but since $\mathrm{Ric}\,(e_0, \xi) = \mathrm{div}_{\mu(t_0)}\kappa(t_0)(\xi)$ and $\mathrm{Ric}\,(T_{(0,m_0)}, \hat{X}_{(0,m_0)}) = \mathrm{div}_{\mu(0)}\kappa(0)(\hat{X}_{(0,m_0)}) = 0$ we have $\mathrm{Ric}\,(T, \hat{X}) \equiv 0$ which implies $\mathrm{div}_{\mu(t)}\kappa(t) = 0$ for all $t \in I$.

Since $S = \mathrm{tr}_{\,h}\mathrm{Ric}\; = -\mathrm{Ric}\,(T, T) - 3\Lambda$ we obtain $\mathrm{d}S(e_0) = -(\nabla_{e_0}\mathrm{Ric}\,)(e_0, e_0)$ and using the contracted Bianchi identity again as well as $\mathrm{Ric}\,(T, \hat{X}) = \mathrm{div}\kappa(X) = 0$ we end up with

$$\begin{aligned}
\frac{1}{2}(\nabla_{e_0}\mathrm{Ric}\,)(e_0, e_0) &= \sum_{i=1}^{3}(\nabla_{e_i}\mathrm{Ric}\,)(e_i, e_0) = -\sum_{i=1}^{3}\Big(\mathrm{Ric}\,(\nabla_{e_i}\hat{E}_i, e_0) + \mathrm{Ric}\,(e_i, \nabla_{e_i}T)\Big)\\
&= \sum_{i=1}^{3}\Big(\kappa(t_0)(e_i, e_i)\mathrm{Ric}\,(e_0, e_0) + \mathrm{Ric}\,(e_i, \kappa(t_0)^\sharp(e_i))\Big)\\
&= \sum_{i=1}^{3}\Big(\kappa(t_0)(e_i, e_i)\mathrm{Ric}\,(e_0, e_0) - \Lambda\mu(t_0)(\kappa(t_0)^\sharp(e_i), e_i)\Big)\\
&= H(t_0)\mathrm{Ric}\,(e_0, e_0) - \Lambda H(t_0)\\
&= H(t_0)(\mathrm{Ric}\,(e_0, e_0) - \Lambda)
\end{aligned}$$

hence

$$\begin{aligned}
\partial_t((\mathrm{Ric}\,(T, T) - \Lambda)v^2) &= T(\mathrm{Ric}\,(T, T))v^2 - 2(\mathrm{Ric}\,(T, T) - \Lambda)Hv^2\\
&= 2H(\mathrm{Ric}\,(T, T) - \Lambda)v^2 - 2H(\mathrm{Ric}\,(T, T) - \Lambda)v^2\\
&= 0
\end{aligned}$$

and we conclude $(\mathrm{Ric}\,(T,T) - \Lambda)v^2$ is constant in t. On the other hand we have from the Gauss equation

$$S + 2\mathrm{Ric}\,(T,T) = R + H^2 - |\kappa|^2$$

and this combines to

$$\mathrm{Ric}\,(T,T) - \Lambda = R + H^2 - |\kappa|^2 + 2\Lambda$$

hence $R + H^2 - |\kappa|^2 + 2\Lambda = 0$ for all $t \in I$ since it is zero at $t = 0$. ∎

Thus the homogeneous cosmological model (M, G, μ) is determined by the data $(M, G, \mu(0), \dot{\mu}(0))$. Vice versa every (M, G, μ_0, κ_0) with $\mathrm{div}_{\mu_0}\kappa_0 = 0$ and $(\mu_0, \kappa_0) \in V$ induce clearly a homogeneous model in the following way.

Definition 3.9. Let (M, G) be a Riemannian homogeneous space and $\Delta(M, G) := \Delta := \{(\mu, \kappa) \in \mathcal{P} : \mathrm{div}_\mu \kappa = 0\}$. If $(\mu, \kappa) \in \mathcal{V} \cap \Delta$ then we call (M, G, μ, κ) an *initial data set*. To every initial data set we assign a homogeneous cosmological model with cosmological constant Λ by the following construction: let $(\mu(t), \kappa(t))$ be the maximal integral curve of $\mathcal{X}^\Lambda$ and using Proposition 3.4 we see that $(M, G, \mu(t))$ is a homogeneous cosmological model.

The Einstein flow

Let $\Phi \colon \Omega \to V \cap \Delta$ be the Einstein flow of $\mathcal{X}^\Lambda$ where $\Omega :=\subset \mathbb{R} \times (V \cap \Delta)$ and we will often denote $\Phi(t, (\mu, \kappa))$ by $\Phi^t(\mu, \kappa)$. We write $I(\mu, \kappa) = (t_-, t_+)$ for the maximal interval of the integral curve of $\mathcal{X}$ through (μ, κ) and $I^+(\mu, \kappa) := [0, t_+)$ as well as $I^-(\mu, \kappa) := (t_-, 0]$.

Proposition 3.10. *Suppose $(\mu, \kappa) \in \Delta \cap V$ for a $\Lambda \in \mathbb{R}$ and let $(\mu(t), \kappa(t))$ the integral curve of the Einstein vector field through (μ, κ). If $H(t) := \mathrm{tr}\,_{\mu(t)}\kappa(t)$ and*

$$\sup_{t \in I^+(\mu,\kappa)} H(t) < \infty$$

then $t_+ = \infty$.

Proof. We are following the arguments in [Ren94]. Suppose $t_+ < \infty$ and fix a background metric β on TM_{m_0}. First we compute the evolution equation for $H(t) = \mathrm{tr}\,_{\mu(t)}\kappa(t)$ and there we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,_{\mu(t)}\kappa(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,(\kappa(t)^\sharp) = \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp\right)$$

and from the proof of Proposition 3.6 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa(t)^\sharp = \dot{\kappa}(t)^\sharp + 2(\kappa(t)^\sharp)^2$$

hence

$$\dot H(t) = \operatorname{tr}_{\mu(t)}\dot\kappa(t) + 2|\kappa(t)|^2 = R(t) + 3\Lambda + H(t)^2 - 2|\kappa(t)|^2 + 2|\kappa(t)|^2 = R(t) + 3\Lambda + H(t)^2$$

and since $(\mu(t), \kappa(t)) \in V$ we end up with

$$\dot H(t) = |\kappa(t)|^2 + \Lambda. \tag{3.1}$$

Furthermore we conclude

$$H(t) = H(0) + t\Lambda + \int_0^t |\kappa(s)|^2 \mathrm{d}s$$

and since t_+ as well as $\sup H < \infty$ is finite we conclude

$$C_0 := \sup_{t\in I^+(\mu,\kappa)} \int_0^t |\kappa(s)|^2 \mathrm{d}s < \infty.$$

Let d be the Riemannian distance of the symmetric space $\mathcal{S}^+(TM_{m_0})$ endowed with the complete (Proposition 1.16) tautological Riemannian metric from Remark 1.74. Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} d(\mu, \mu(t)) &\leq \int_0^t \sqrt{\langle \dot\mu(s), \dot\mu(s)\rangle_{\mu(s)}}\mathrm{d}s = t\int_0^t |\dot\mu(s)|^2 \mathrm{d}s \\ &= 4t\int_0^t |\kappa(s)|^2 \mathrm{d}s \leq C_1 < \infty \end{aligned}$$

since t_+ is finite. Thus the curve $\mu(t)$ stays in a bounded and closed and therefore compact subset of $\mathcal{S}^+(TM_{m_0})$.

Moreover we have $\mu(t), \kappa(t) \in \mathbf{Sym}(TM_{m_0})$ thus we can integrate the curves *component wise* (in any basis). This yields

$$\mu(t) = \mu(0) - 2\int_0^t \kappa(s)\mathrm{d}s.$$

We use the norm $|\cdot|_\infty$ of Remark 1.76 such that we obtain

$$|\mu(t)|_\infty \leq |\mu|_\infty + 2\int_0^t |\kappa(s)|_\infty \mathrm{d}s$$

(note that $|\mu(t)|_\infty$ is continuous in t) and again with Remark 1.76

$$|\mu(t)|_\infty \leq |\mu|_\infty + 2\int_0^t |\kappa(s)|_{\mu(s)} |\mu(s)|_\infty \mathrm{d}s.$$

Applying Gronwall's Lemma yields

$$|\mu(t)|_\infty \leq |\mu|_\infty \exp\left(2\int_0^t |\kappa(s)|_{\mu(s)} \mathrm{d}s\right) \leq C_2 < \infty$$

for all $t \in [0, t_+)$, which implies that all eigenvalues of $\mu(t)$ are uniformly bounded from above. On the other hand if $v(t)^2 = \det_\beta(\mu(t))$ and $\lambda_1(t) \leq \lambda_2(t) \leq \lambda_3(t)$ are the eigenvalues of $\mu(t)$ then from Proposition 3.8 we have

$$v(t) = v(0)\exp\left(-\int_0^t H(s)\mathrm{d}s\right) \geq C_4 > 0$$

since H is bounded from above. Therefore $\lambda_1\lambda_2\lambda_3 \geq C_4 > 0$ for all $t \in [0, t_+)$ and we combine this to

$$\frac{1}{\lambda_1} \leq \frac{1}{C_4}\lambda_2\lambda_3 \leq \frac{1}{C_4}\lambda_3^2 \leq \frac{C_2^2}{C_4} < \infty$$

for all $t \in [0, t_+)$. We use this to show that $R(t)$ is bounded from above: from Lemma 1.26 we have $R(t) \leq -\frac{1}{2}\operatorname{tr}_{\mu(t)} B$ where B is the Killing–Cartan–form of G restricted to a complement of the isotropy group and pulled over to TM_{m_0}. If (e_1, e_2, e_3) is an β–orthonormal basis such that $\mu(t)^\sharp(e_i) = \lambda_i(t)e_i$ where $\mu(t)^\sharp$ is taken with respect to β then

$$|\operatorname{tr}_{\mu(t)} B| = \left|\sum_{i=1}^3 \beta(B^\sharp(\mu(t)^\sharp)^{-1}(e_i), e_i)\right| \leq \frac{1}{\lambda_1(t)}\sum_{i=1}^3 |B(e_i, e_i)| \leq C|B|_\infty$$

hence $R(t)$ is bounded from above. The equation $R + H^2 - |\kappa|^2 + 2\Lambda = 0$ implies then that $|\kappa(t)|^2$ is bounded uniformly and finally this yields

$$|\kappa(t)|_\infty \leq |\kappa(t)| \cdot |\mu(t)|_\infty < \infty.$$

Hence the solution $(\mu(t), \kappa(t))$ of $\mathcal{X}$ stays in a compact set of $\mathcal{P}$ which contradicts the assumption $t_+ < \infty$. ∎

Rotationally symmetric cosmological models

Now we would like to study the Einstein flow for rotational symmetric geometries of class A. The reason to choose class A models is that the space of G–invariant metrics is two dimensional, as we will show next, and this makes the analysis of the Einstein flow a lot easier. Remember from Proposition 2.33 that a homogeneous vector field X is divergence free with respect to any G–invariant metric and therefore it is a Killing field (for any homogeneous metric), see Proposition 2.13. Moreover Proposition 2.10 tells us that a rotational symmetric geometry (M, G) induces a principal fiber bundle $\pi\colon M \to B$ with fiber $\mathbb{R}$ or $\mathbf{SO}(2)$ over a surface B. The fibers are given by the integral curves of a homogeneous vector field. Let ω be the connection 1-form of Remark 2.14.

Proposition 3.11. *Let (M,G) be a class A geometry and μ a G–invariant metric on (M,G). Then there is a metric μ_B on B of constant curvature and there is a G–invariant 1–form α on M such that*

$$\mu = \alpha \otimes \alpha + \pi^*(\mu_B).$$

In particular $\pi\colon (M,\mu) \to (B,\mu_B)$ is a Riemannian submersion.

Proof. Let X be a homogeneous Killing field with $\mu(X,X) = 1$. Thus the flow is a one–parameter curve of isometries lying in G. This allows us to define μ_B on B as follows: if $b \in B$ then for all $m \in \pi^{-1}(b)$ we have that $\Phi_m := \mathrm{D}\pi_m\colon \mathcal{H}_m \to TB_b$ is an isomorphism. For $\eta_1, \eta_2 \in TB_b$ define then

$$(\mu_B)_b(\eta_1,\eta_2) := \mu_m(\Phi_m^{-1}(\eta_1), \Phi_m^{-1}(\eta_2))$$

for an $m \in \pi^{-1}(b)$. Since any points of the fiber over b can be connected by the flow of X which are isometries we see that the definition above is well–defined. Recalling Proposition 2.22 the group G acts transitive on B as well since elements of G commute with the flow of X, i.e. $g.b := \pi(g.m)$ for an $m \in \pi^{-1}(b)$. Then μ_B is invariant under G since for $g \in G$, $m \in \pi^{-1}(b)$, $\eta_1, \eta_2 \in TB_b$ we compute

$$\begin{aligned} g^*(\mu_B)_b(\eta_1,\eta_2) &= \mu_{g.m}(\Phi_{g.m}^{-1}(g.\eta_1), \Phi_{g.m}^{-1}(g.\eta_2)) = \mu_{g.m}(g.\Phi_m^{-1}(\eta_1), g.\Phi_m^{-1}(\eta_2)) \\ &= \mu_m(\Phi_m^{-1}(\eta_1), \Phi_m^{-1}(\eta_2)) = (\mu_B)_b(\eta_1,\eta_2) \end{aligned}$$

and we conclude that μ_B has constant curvature. Since $TM_m = \mathcal{D}_m \oplus \mathcal{H}_m$ and $\mathcal{H}_m$ is orthogonal to $\mathcal{D}_m$ (see Proposition 1.44) we have $\mu(\xi,\zeta) = 0$ if $\zeta \in \mathcal{H}_m$ and $\xi \in \mathcal{D}_m$. For all $\zeta_1,\zeta_2 \in \mathcal{H}_m$ we have obviously $\mu_m(\zeta_1,\zeta_2) = \pi^*(\mu_B)_m(\zeta_1,\zeta_2)$. Then $\lambda := \omega(X) \neq 0$ and we have

$$\mu(X,X) = 1 = \frac{1}{\lambda^2}\omega \otimes \omega(X,X)$$

and therefore we set $\alpha := \frac{1}{\lambda}\omega$. Finally $\mu(X_m,\zeta) = 0 = \alpha \otimes \alpha(X,\zeta) + \pi^*(\mu_B)(X_m,\zeta)$ for all $\zeta \in \mathcal{H}_m$. ∎

The Riemannian submersion of Proposition 3.11 will help us to compute the Ricci curvature of (M,G) equipped with a G–invariant metric. We will not use Lemma 1.26 since we would need a good knowledge of G. Of course we know all possible groups G and we could compute the curvature case by case but we would like to keep things as general as possible. On the contrary using the form of the metric in Proposition 3.11 and the fact that π is a Riemannian submersion onto a surface of constant curvature, the Ricci curvature is fully determined by the curvature of the surface and the curvature of the principal bundle $\pi\colon M \to B$. Since μ_B has constant curvature there is a $c \in \mathbb{R}$ such that $\mathrm{Ric}\,(\mu_B) = c\mu_B$.

Remark 3.12. Especially the metric $\beta_c := \omega \otimes \omega + \pi^*(\mu_B)$ is G–invariant where μ_B is a metric of constant curvature such that $\mathrm{Ric}\,(\mu_B) = c \cdot \mu_B$ on B. We see that the fundamental vector field X with $\omega(X) = 1$ has length 1 in this metric.

Remark 3.13. The curvature formulae for Riemannian submersions may be found originally in [O'N66] and summarized in [Bes08, Ch. 9]. We will use the notation of the latter book. Let $\mu = \alpha^2 + \pi^*(\mu_B)$ be a G–invariant metric like in Proposition 3.11 and let X be a homogeneous vector field with $\omega(X) = 1$. We recall the definitions of the tensors A and T: For a vector field W on the principal bundle $\pi\colon M \to B$ with connection $\mathcal{H}$ we denote by W^V and by W^H the vertical and the horizontal component of W respectively. Then we have

$$A_{W_1}W_2 := (\nabla_{W_1^H} W_2^V)^H + (\nabla_{W_1^H} W_2^H)^V$$

and

$$T_{W_1}W_2 := (\nabla_{W_1^V} W_2^V)^H + (\nabla_{W_1^V} W_2^H)^V.$$

For all vector fields W and horizontal fields Z we see $T_Z W = 0$. Since the fibers are geodesics we have $T_X X = 0$. And if Z is horizontal we obtain $T_X Z = 0$ as we can see if we use again that the fibers are geodesics:

$$\mu(T_X Z, X) = \mu(\nabla_X Z, X) = -\mu(Z, \nabla_X X) = 0.$$

And since T is a tensor we conclude $T = 0$. From Proposition 9.24 of [Bes08, p. 240] we obtain for horizontal vector fields Z_1 and Z_2 that $2(A_{Z_1}Z_2)$ is the vertical component of $[Z_1, Z_2]$. Furthermore observe $\omega(Y)|X|^2 = \mu(X, Y)$ for all vector fields Y on TM since the decomposition $TM = \mathcal{D} \oplus \mathcal{H}$ is orthogonal. But with this we compute

$$\mu(A_{Z_1}Z_2, X) = \tfrac{1}{2}\mu(X, [Z_1, Z_2]) = \tfrac{1}{2}|X|^2\omega([Z_1, Z_2]) = -\tfrac{1}{2}|X|^2 \mathrm{d}\omega(Z_1, Z_2)$$

hence

$$A_{Z_1}Z_2 = -\tfrac{1}{2}\mathrm{d}\omega(Z_1, Z_2)X.$$

Since the fiber group is abelian, the two–form $\mathrm{d}\omega$ is the curvature of the bundle. Let $e_1, e_2 \in \mathcal{H}_{m_0} \subset TM_{m_0}$ be such that they are orthonormal with respect to μ and $\delta_\mu := \delta := |\mathrm{d}\omega|_\mu = \sqrt{\mathrm{d}\omega(e_1, e_2)^2}$ (observe that δ depends on the metric). Set $e_0 := X_{m_0}/|X|$ and complement it with e_1, e_2 to an orthonormal base (e_0, e_1, e_2) of TM_{m_0}.

Proposition 3.14. *Let $\mu = \alpha^2 + \pi^*(\mu_B)$ be a G–invariant metric on (M, G) with $\mathrm{Ric}(\mu_B) = c \cdot \mu_B$ on B and X a homogeneous vector field such that $\omega(X) = 1$. With the notations of Remark 3.13 the Ricci curvature* Ric *of μ is given by*

$$\begin{aligned} \mathrm{Ric}(X_{m_0}, X_{m_0}) &= \tfrac{1}{2}\delta^2|X|^4 \\ \mathrm{Ric}(X_{m_0}, \zeta) &= 0 \\ \mathrm{Ric}(\zeta_1, \zeta_2) &= \left(c - \frac{\delta^2}{2}|X|^2\right)\mu_B(\zeta_1, \zeta_2) \end{aligned}$$

for $\zeta, \zeta_1, \zeta_2 \in \mathcal{H}_{m_0}$ and we identify $\zeta \in \mathcal{H}_{m_0}$ with $\mathrm{D}\pi(\zeta) \in TB_{\pi(m_0)}$. In particular the scalar curvature of μ is given as

$$R = R(\mu) = 2c - \frac{\delta^2|X|^2}{2}.$$

Proof. Set $\xi := X_{m_0}$. From Proposition 9.36 in [Bes08, p.244] we obtain

$$\mathrm{Ric}(\xi, \xi) = |A_{e_1}\xi|^2 + |A_{e_2}\xi|^2.$$

If E_1 and E_2 are horizontal continuations of e_1 and e_2 respectively then we have for $i, j = 1, 2$

$$\mu(A_{e_i}e_j, \xi) = \mu(\nabla_{e_i}E_j, \xi) = -\mu(e_j, \nabla_{e_i}X) = -\mu(e_j, A_{e_i}\xi),$$

using Remark 3.13 we end up with

$$A_{e_1}\xi = \frac{\delta|\xi|^2}{2}e_2 \quad \text{and} \quad A_{e_2}\xi = -\frac{\delta|\xi|^2}{2}e_1$$

thus $\mathrm{Ric}(\xi, \xi) = \frac{1}{2}\delta^2|\xi|^4$. By the formula of $\mathrm{Ric}(e_0, e_j)$ in [Bes08] we see those terms are zero. For the last term we have

$$\mathrm{Ric}(e_i, e_j) = c \cdot \delta_{ij} - \frac{2}{|\xi|^2}\mu(A_{e_i}\xi, A_{e_j}\xi) = \left(c - \frac{\delta^2}{2}|\xi|^2\right)\delta_{ij}.$$

This shows the statement for the Ricci curvature. Further we have

$$R(\mu) = \mathrm{Ric}(e_0, e_0) + \mathrm{Ric}(e_1, e_1) + \mathrm{Ric}(e_2, e_2) = \frac{\delta^2|\xi|^2}{2} + 2\left(c - \frac{\delta^2|X|^2}{2}\right) = 2c - \frac{\delta^2|\xi|^2}{2}.$$

∎

Remark 3.15. We see from Proposition 3.14 that the only rotational symmetric geometry which admit an Einstein metric is $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \times \mathbb{R})$ or $(S^3, \mathbf{U}(2))$. If $\delta = 0$ then it follows from the horizontal directions $c = 0$ and (M, G) has to be $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \times \mathbb{R})$. If $\delta \neq 0$ then we conclude $c = \delta^2|X|^2 > 0$ which means that the principal bundle is not flat and the base space has to admit a metric with constant positive curvature, i.e. $B = S^2$ and therefore (M, G) has to be $(S^3, \mathbf{U}(2))$.

Fix now a background metric $\beta := \omega^2 + \pi^*(\beta_B)$ for every rotational symmetric geometry. Clearly we cannot use the same β for all geometries since μ_B depends on the base space B. Note that $\beta(X, X) = 1$.

Remark 3.16. The isotropy group K of (M, G) is isomorphic to $\mathbf{SO}(2)$ and acts on $V = TM_{m_0}$ linear. Recalling Example 1.77 we see that $\mathcal{S}_K^+$ is two dimensional. We will now identify $\mathcal{S}_K^+$ with $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ like in Example 1.77 with respect to the background metric β. Thus for every G–invariant metric μ there are $x, y > 0$ such that

$$\mu = x\omega^2 + y\pi^*(\beta_B).$$

Furthermore the tangent space in every point of $\mathcal{S}_K^+(V)$ is canonically given by $\mathbf{Sym}_K(V)$ and we identify it with $\mathbb{R}^2$ as follows: If $\kappa \in \mathbf{Sym}_K(V)$ then there are $w, z \in \mathbb{R}$ such that

$$\kappa = w\omega^2 + z\pi^*(\beta_B).$$

From now on we fix $\delta_0 := |\mathrm{d}\omega|_\beta$ and $c_0 \in \mathbb{R}$ is such that $\mathrm{Ric}\,(\beta_B) = c_0 \cdot \beta_B$.

Proposition 3.17. *If $\mu = x\omega^2 + y\pi^*(\beta_B)$ then with the above identification* $\mathrm{Ric}\,(\mu)$ *is given by $w\omega^2 + z\pi^*(\beta_B)$ where*

$$\begin{aligned} w &= \delta_0^2 \frac{x^2}{2y^2} \\ z &= c_0 - \frac{\delta_0^2 x}{2y} \end{aligned}$$

and the scalar curvature can be expressed by

$$R = \frac{2c}{y} - \frac{\delta_0^2 x}{2y^2}.$$

Proof. First let us transform the constants δ and c to δ_0 and c_0. If (e_1, e_2) is a basis of $\mathcal{H}_{m_0}$ with respect to $\pi^*(\beta_B)$ then $(\sqrt{y}^{-1}e_1, \sqrt{y}^{-1}e_2)$ is an orthonormal basis of $\mathcal{H}_{m_0}$ with respect to μ, thus $\delta^2 = y^{-2}\delta_0^2$. Further $\mathrm{Ric}\,(y\beta_B) = \mathrm{Ric}\,(\beta_B) = c_0 y^{-1}(y\beta_B)$ thus $c = c_o y^{-1}$. Using Proposition 3.14 we see $w = \delta_0^2 \frac{x^2}{2y^2}$ and

$$\mathrm{Ric}\,(e_i, e_j) = \left(\frac{c_0}{y} - \frac{\delta_0^2 x}{2y^2}\right) y\delta_{ij}$$

hence $z = c_0 - \frac{\delta_0^2 x}{2y}$. ∎

Proposition 3.18. *If κ is a G–invariant symmetric bilinear form on a rotational symmetric geometry of class A then for every G-invariant metric μ we have* $\operatorname{div}_\mu \kappa = 0$.

Proof. In m_0 there are $w, z \in \mathbb{R}$ such that $\kappa = w\omega^2 + z\pi^*(\beta_B)$ and since both sides are G–invariant this equation holds globally on M. Let $e_0 := X_{m_0}$ and let e_1, e_2 be on orthonormal basis of $\mathcal{H}_{m_0}$ with respect to $\pi^*(\beta_B)$ such that (e_0, e_1, e_2) is an orthonormal basis of TM_{m_0} for β. Suppose furthermore $\mu = x\omega^2 + y\pi^*(\beta_B)$ for $x, y > 0$. Let E_1 and E_2 be horizontal extensions of e_1 and e_2 respectively. The 1–form $\operatorname{div}\kappa$ is fully determined by $\operatorname{div}\kappa(e_0)$ since it is G–invariant. We obtain

$$\operatorname{div}\kappa(e_0) = \frac{1}{x}(\nabla_{e_0}\kappa)(e_0, e_0) + \frac{1}{y}\sum_{i=1}^{2}(\nabla_{e_i}\kappa)(e_i, e_0).$$

Further we have

$$(\nabla_{e_0}\kappa)(e_0, e_0) = e_0(\kappa(X, X)) - 2\kappa(\nabla_{e_0}X, e_0) = 0$$

and for $i = 1, 2$

$$(\nabla_{e_i}\kappa)(e_i, e_0) = e_i(\kappa(E_i, X)) - \kappa(\nabla_{e_i}E_i, e_0) - \kappa(e_i, \nabla_{e_i}X) = 0$$

since $\kappa(E_i, X) = 0$, $\kappa(\nabla_{e_i}E_i, e_0) = w\beta(A_{e_i}e_i, e_0) = 0$ and $\kappa(e_i, \nabla_{e_i}X) = z\beta(e_i, A_{e_i}e_0) = 0$ ∎

Theorem 3.19. *Let (M, G) be a rotational symmetric geometry of class A. The dynamical system on $\mathcal{P}$ induced from $\mathcal{X}$ is equivalent to the dynamical system on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ defined by the vector field*

$$\mathcal{X}(x, y, w, z) = \begin{pmatrix} -2w \\ -2z \\ \delta_0^2 \frac{x^2}{2y^2} + \Lambda x - \frac{w^2}{x} + 2\frac{zw}{y} \\ c_0 - \delta_0^2 \frac{x}{2y} + \Lambda y + \frac{zw}{x} \end{pmatrix}$$

and $V \cap \Delta$ is given as the zero set

$$2cxy - \tfrac{1}{2}\delta_0^2 x^2 + 2z^2 x + 4ywz + 2\Lambda x y^2 = 0.$$

Proof. Let $\mu^\sharp$ and $\kappa^\sharp$ be the self–adjoint endomorphism of μ with respect to β, then $H = \operatorname{tr}_\mu\kappa = \operatorname{tr}_\beta((\mu^\sharp)^{-1}\kappa^\sharp)$. Suppose $\mu = x\omega^2 + y\pi^*(\beta_B)$ and $\kappa = w\omega^2 + z\pi^*(\beta_B)$, then $H = \frac{w}{x} + 2\frac{z}{y}$ and the bilinear form $(\kappa^\sharp)^2$ is given as $\frac{w^2}{x}\omega^2 + \frac{z^2}{y}\pi^*(\beta_B)$. Further we have $|\kappa|^2 = \frac{w^2}{x^2} + 2\frac{z^2}{y^2}$ and by Proposition 3.18 Δ is the whole space $\mathcal{P} = \mathcal{S}_K^+(TM_{m_0}) \times \mathbf{Sym}_K(TM_{m_0})$. Plugging all this together gives the formulae stated in the theorem. ∎

Remark 3.20. As a next step we would like to study the dynamics of a specific geometry and for that we choose the geometry $(S^3, \mathbf{U}(2))$. We regard of course $S^3 \subset \mathbb{C}^2$ as the set of unit vectors with respect to the hermitian inner product. Consider the real inner product $(z_1, z_2) \cdot (w_1, w_2) = \mathbf{Re}(z_1\overline{w}_1 + z_2\overline{w}_2)$ which is the standard real euclidean product on $\mathbb{C}^2 = \mathbb{R}^4$. Surely $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2) \cdot (z_1, z_2) = 1\}$ and the induced Riemannian metric on the sphere ist the round one with curvature equal to 1 which we will denote by β. Moreover the sectional curvature of β is constant and equal to one, which implies $\mathrm{Ric}\,(\beta) = 2\beta$. Clearly β is invariant under $\mathbf{U}(2)$ since it is invariant under $\mathbf{SO}(4)$ and the S^1–principal bundle $\pi\colon S^3 \to S^2$ is given by the action of center $H = \mathbf{U}(1) = \{z \cdot E_2 : z \in S^1\}$, thus it is the Hopf–fibration. A homogeneous vector field is the fundamental vector field of $1 \in S^1$ by the action H on S^3 is given as by $X_{(z_1,z_2)} = i(z_1, z_2) \in TS^3_{(z_1,z_2)}$. Thus its length with respect to β is $|z_1|^2 + |z_2|^2 = 1$. This data is sufficient to determine the curvature of the connection induced the isotropy group: apparently the 1–form $\omega(\xi) := \beta(X_{(z_1,z_2)}, \xi)$ where $\xi \in TS^3_{(z_1,z_2)}$ is $\mathbf{U}(2)$–invariant such that $\omega(X) = 1$ and with Proposition 3.14 and the fact that $\mathrm{Ric}\,(\beta) = 2\beta$ we see $\delta_0^2 = 4$ and therefore $c_0 = 4$.

Corollary 3.21. *The Einstein vector field of* $(S^3, \mathbf{U}(2))$ *on* $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ *is given by*

$$\mathcal{X}(x, y, w, z) = \begin{pmatrix} -2w \\ -2z \\ 2\frac{x^2}{y^2} + \Lambda x - \frac{w^2}{x} + 2\frac{zw}{y} \\ 4 - 2\frac{x}{y} + \Lambda y + \frac{zw}{x} \end{pmatrix}$$

and V *is the zero set*

$$8xy - 2x^2 + 2z^2x + 4ywz + 2\Lambda xy^2 = 0.$$

Remark 3.22. We choose $\Lambda = -3$ since then we have the feature that $(1, 1, 0, 0) \in V$ which represents the pair $(\beta, 0) \in \mathcal{P}$. If $\mathrm{Ric}\,(\beta) = 2\beta$ then $R = 6$ hence for $(\beta, 0)$ we have $R + H^2 - |\kappa|^2 - 6 = 0$. This gives the possibility to construct a special solution: suppose $\mu(t) = f^2(t)\beta$ for a smooth positive function f on an interval I then $\kappa = -\dot{f}f\beta$. Then we obtain the condition

$$8f^4 - 2f^4 + 2\dot{f}^2f^4 + 4\dot{f}^2f^4 - 6f^6 = 0$$

since $x = y = f^2$ and $w = z = -\dot{f}f$. This simplifies to $(\dot{f})^2 = f^2 - 1$ since $f > 0$. We consider now the ordinary differential equation

$$\dot{f} = \sqrt{f^2 - 1}$$

such that $f(1) = 1$. The constant function $f = 1$ and the function $f(t) = \cosh(t)$ where $I = \mathbb{R}$ are solutions of the ODE with $f(1) = 1$ (observe that $x \mapsto \sqrt{x^2 - 1}$ is not Lipschitz at $x = 1$ and therefore there is no uniqueness).

For $\Lambda = -3$ the constant curve $(\beta, 0)$ is not an integral curve of $\mathcal{X}$. But the curve $(\cosh^2(t)\beta, -\frac{1}{2}\sinh(2t)\beta)$ is an integral curve of $\mathcal{X}$. We denote the image of that curve by $\mathcal{A}$ since the set will be a good candidate for an attractor for the dynamical system of $\mathcal{X}$. We remark finally that the solution $\mu(t) = \cosh^2(t) \cdot \beta$ is known as the *De–Sitter solution*. This is also the solution to the dynamical system of $\mathcal{X}^\Lambda$ on the geometry $(S^3, \mathbf{SO}(4))$ with $\Lambda < 0$. Also note that the value of Λ can be choose freely as long as the scalar is negative. By rescaling the metrics one may obtain solutions for all $\Lambda \in \mathbb{R}$ with $\Lambda < 0$.

Definition 3.23. Let $L\colon V \to \mathbb{R}$ be the function $L(\mu, \kappa) := v^{\frac{2}{3}}\left(\frac{2}{3}H^2 - 6\right)$ where $v := \sqrt{\det_\beta \mu}$ and $H := \operatorname{tr}_\mu \kappa$. We will show that L is a Lyapunov function for $\mathcal{X}$ on a certain subset of V.

Proposition 3.24. *If $(\mu(t), \kappa(t))$ is an integral curve of $\mathcal{X}$ on V then*

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\mu(t), \kappa(t)) = \tfrac{2}{3} v^{\frac{2}{3}} H(t) |\mathring{\kappa}(t)|^2$$

where $\mathring{\kappa}(t)$ is the trace free part of $\kappa(t)$ with respect to $\mu(t)$.

Proof. We know $\dot{v} = -Hv$ and $\dot{H} = |\kappa|^2 - 3$, see the proofs of Proposition 3.8 and 3.10 respectively. So we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} L &= \tfrac{2}{3} v^{-\frac{1}{3}} \dot{v} \left(\tfrac{2}{3}H^2 - 6\right) + v^{\frac{2}{3}}\left(\tfrac{4}{3} H \dot{H}\right) \\
&= \tfrac{2}{3} v^{\frac{2}{3}} H \left(-\frac{2}{3}H^2 + 6\right) + \tfrac{2}{3} v^{\frac{2}{3}} H \left(2|\kappa|^2 - 6\right) \\
&= \tfrac{4}{3} v^{\frac{2}{3}} H \left(|\kappa|^2 - \frac{1}{3}H^2\right) \\
&= \tfrac{4}{3} v^{\frac{2}{3}} H |\mathring{\kappa}|^2
\end{aligned}$$

■

We will show in the next lines that under certain circumstances if $H(0) < 0$ then $H(t) < 0$ for all $t \in I^+$ and in that case Proposition 3.24 shows that L is a Lyapunov function. Furthermore the set $\mathcal{A}$ represents the absolute minima of the function L. To show these facts we start with a proposition about the scalar curvature.

Proposition 3.25. *In the case of* $(S^3, \mathbf{U}(2))$ *the scalar curvature can be estimated by*

$$R(\mu) \leq \frac{6}{\sqrt[3]{v(\mu)^2}}$$

where $v(\mu) = \sqrt{\det_\beta \mu}$.

Proof. We use the identification $\mathcal{S}^+_K(V)$ with $N := \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ thus we have $R(x,y) = \frac{8}{y} - \frac{2x}{y^2}$ and the volume element is given as $v(x,y) = \sqrt{xy^2}$. For $c_0 > 0$ the equation $v(x,y) = c_0$ defines a smooth one–dimensional submanifold S of N and we will prove that R has a maximum along $v(x,y) = c_0$. On S we have

$$R(x,y) = R\left(x, \frac{c_0}{\sqrt{x}}\right) = R(x) = \frac{8}{c_0}\sqrt{x} - \frac{2}{c_0^2}x^2$$

and the first derivative is

$$R'(x) = \frac{4}{c_0^2}\left(\frac{c_0}{\sqrt{x}} - x\right)$$

and therefore $x_0 = c_0^{\frac{2}{3}}$ is the only zero of R'. The second derivative is

$$R''(x) = -4\left(\frac{1}{2c_0 x^{\frac{3}{2}}} + \frac{1}{c_0^2}\right) < 0$$

and this implies x_0 is a maximum. Thus the point $\left(c_0^{\frac{2}{3}}, c_0^{\frac{2}{3}}\right) \in S$ is the maximum of R along S, which represents the round metric on S^3 of volume c_0. Hence for all $(x,y) \in S$ we have

$$R(x,y) \leq R(x_0, y_0) = \frac{6}{c_0^{\frac{2}{3}}}$$

∎

Proposition 3.26. *If* $(\mu_0, \kappa_0) \in V$ *such that* $v_0 = \sqrt{\det_\beta \mu_0} \geq 1$ *and* $H_0 := \operatorname{tr}_{\mu_0} \kappa_0 < 0$, *then if* $(\mu(t), \kappa(t))$ *is the integral curve of* $\mathcal{X}$ *starting at* (μ_0, κ_0) *then* $H(t) = \operatorname{tr}_{\mu(t)} \kappa(t) < 0$ *for all* $t \in I^+(\mu_0, \kappa_0)$. *Therefore* $t_+ = \infty$.

Proof. In this proof we are following the ideas in [Wal83]. Suppose there is a smallest $t_0 \in I^+(\mu_0, \kappa_0)$ such that $H(t_0) = 0$ (otherwise we are done). Then at that point we have $0 = |\kappa|^2 + 6 - R$ since the solution lies on V and with Proposition 3.25 we obtain

$$0 \geq |\kappa|^2 + 6\left(1 - \frac{1}{\sqrt[3]{v^2}}\right) \geq 6\left(1 - \frac{1}{\sqrt[3]{v^2}}\right).$$

Further we saw in the proof of Proposition 3.10 that $v(t) = v_0 \exp\left(-\int_0^t H(s)\mathrm{d}s\right)$ and by assumption we have $H(t) < 0$ for all $0 \le t < t_0$ which implies $v(t_0) > v_0$. Finally this leads to

$$0 \ge 6\left(1 - \frac{1}{\sqrt[3]{v^2}}\right) > 6\left(1 - \frac{1}{\sqrt[3]{v_0^2}}\right) \ge 0$$

and that is a contradiction to $H(t_0) = 0$. Proposition 3.10 tells us that $t_+ = \infty$. ∎

Remark 3.27. With Proposition 3.25 L is bounded from below:

$$L = v^{\frac{2}{3}}\left(\tfrac{2}{3}H^2 - 6\right) = v^{\frac{2}{3}}\left(|\mathring{\kappa}|^2 - R\right) \ge -v^{\frac{2}{3}}\frac{6}{v^{\frac{2}{3}}} \ge -6$$

and the minima are attained by elements of $\mathcal{A}$. If $(\mu, \kappa) \in \mathcal{A}$ then $\mu = \cosh^2(t)\beta$ and $\kappa = -\cosh(t)\sinh(t)\beta$ for $t \in \mathbb{R}$ which leads to $v^{\frac{2}{3}} = \sqrt[3]{\det_\beta \mu} = \sqrt[3]{\cosh^6(t)} = \cosh^2(t)$ and

$$H = -3\frac{\sinh(t)\cosh(t)}{\cosh^2(t)} = -3\tanh(t)$$

and therefore in (μ, κ) we have

$$L = \cosh^2(t)\left(\frac{2}{3}9\tanh^2(t) - 6\right) = -6\cosh^2(t)(1 - \tanh^2(t)) = -6\cosh^2(t)\frac{1}{\cosh^2(t)} = -6.$$

It remains to show that if $L(\mu, \kappa) = -6$ then $(\mu, \kappa) \in \mathcal{A}$. However observe that L is not bounded from above.

Proposition 3.28. *The critical points of L are contained in $\mathcal{A}$.*

Proof. Let Φ be defined as L but on the whole space $\mathcal{P}$. Furthermore let $\nabla\Phi$ be the gradient of Φ with respect to the tautological metric on $\mathcal{P}$, see after Proposition 3.4. We compute

$$\nabla\Phi = \frac{2}{3}v^{-\frac{1}{3}}\nabla v\left(\frac{2}{3}H^2 - 6\right) + \frac{4}{3}v^{\frac{2}{3}}H\nabla H.$$

The derivative of the volume form v in μ in the direction of h is given as $\mathrm{d}v_\mu(h) = \frac{1}{2}v\operatorname{tr}_\mu h$ and hence the function $(\mu, \kappa) \mapsto v(\mu)$ has in $(\mu, \kappa) \in \mathcal{R}$ the gradient

$$\nabla v = \frac{1}{2}v\begin{pmatrix}\mu\\0\end{pmatrix}$$

since $\operatorname{tr}_\mu(h) = \langle \mu, h\rangle_\mu$. In Proposition 3.6 we saw

$$\nabla H = \begin{pmatrix} -\kappa \\ \mu \end{pmatrix}.$$

This yields

$$\nabla \Phi = \frac{2}{3} v^{\frac{2}{3}} \begin{pmatrix} \left(\frac{1}{3}H^2 - 3\right)\mu - 2H\kappa \\ 2H\mu \end{pmatrix}.$$

On the other hand V is the zero set of F from Proposition 3.6 and there we computed

$$\nabla F = \begin{pmatrix} -\mathrm{Ric} - 2H\kappa + 2(\kappa^\sharp)^2 \\ 2H\mu - 2\kappa \end{pmatrix}.$$

The critical points of Φ on V are given by those points $(\mu, \kappa) \in V$ such that there is a $\lambda \in \mathbb{R}$ and

$$\nabla\Phi = \lambda \nabla F.$$

Suppose first that $H \neq 0$ then from the second row we deduce

$$\left(1 - \frac{2}{3\lambda} v^{\frac{2}{3}}\right) H\mu = \kappa$$

and this implies $\lambda = v^{\frac{2}{3}}$. If we put this into the first row we obtain

$$\mathrm{Ric} = 2\left(1 - \frac{1}{9}H^2\right)\mu,$$

thus μ and κ have to be a multiple of β. Thus suppose $\mu = f^2\beta$ and $\kappa = h\beta$ and since $(\mu, \kappa) \in V$ we obtain

$$f^2(1 - f^2) + h^2 = 0.$$

First this shows that $f \geq 1$ since otherwise $1 > f^2 > 0$, $h^2 > 0$ (since $H \neq 0$) as well as $(1 - f^2) > 0$ therefore $h = 0$ which is a contradiction. Now there is a $t \in \mathbb{R}$ such that $f = \cosh(t)$ and this yields $f^2(1 - f^2) = -\cosh^2(t)\sinh^2(t)$ so $h = \pm\sinh(t)\cosh(t)$. If $h = -\cosh(t)\sinh(t)$ we are done otherwise go from t to $-t$ since cosh is a symmetric function and thus $(\mu, \kappa) \in \mathcal{A}$.

If $H = 0$ then we see from above $\kappa = 0$ and

$$\mathrm{Ric} = 2\mu$$

Therefore μ is an Einstein metric with sectional curvature 1 hence $\mu = \beta$. So $(\beta, 0) \in \mathcal{A}$ for $t = 0$. ■

Remark 3.29. If $(\mu, \kappa) \in L^{-1}(-6)$ then (μ, κ) is a minimum of L and therefore a critical point. Hence $(\mu, \kappa) \in \mathcal{A}$.

Moreover with Proposition 3.24 we see that $L(\mu(t), \kappa(t)) \to \inf_{s \in I(\mu,\kappa)} L(\mu(s), \kappa(s)) \geq -6$ for $t \to \infty$ if $v(\mu) \geq 1$ and $\operatorname{tr}_\mu \kappa < 0$. It seems therefore reasonable to conjecture that $\mathcal{A}$ is (at least locally) an attractor.

$\mathcal{A}$

Miscellaneous Topics

In this appendix, we want to provide some lemmas which are helpfull to prove some statements in the sections of the preceeding chapters.

Homogeneous vector fields

Definition A.1. Let $e_1, \dots, e_d$ be the canonical basis for $\mathbb{R}^d$. We say a curve $\gamma\colon [0,1] \to \mathbb{R}^d$ is an *edge path* if $\gamma(0) = 0$ and the following holds

(a) there are closed intervalls $I_j = [t_j^-, t_j^+] \subset [0,1]$ ($j = 1, \dots, l$, $t_j^- < t_j^+$) with $t_1^- = 0$, $t_j^+ = t_{j+1}^-$ ($j = 1, \dots, l-1$) and $t_l^+ = 1$ (hence $[0,1]$ is the union of all I_j).

(b) for every j there is a $k_j \in \{1, \dots, d\}$ such that $\gamma \,|\, I_j$ is given by $t \mapsto \gamma(t_j^-) + (t - t_j^-)e_{k_j}$ for $t \in I_j$.

Obviously every point in an open connected neighboorhood of $0 \in \mathbb{R}^d$ can be connected with 0 by an edge path, since the set of such points is open and closed.

Now let S be a d–dimensional connected smooth manifold and let $X_1, \dots X_k$ be vector fields on S. If φ_j denotes the local flow of X_j and $s_0 \in S$ then there is an open neighboorhood V of $0 \in \mathbb{R}^k$, such that the map

$$\Phi_{s_0}\colon V \to M, \quad \Phi_{s_0}(t_1, \dots, t_k) := \varphi_1^{t_1} \circ \cdots \circ \varphi_k^{t_k}(s_0)$$

is well–defined.

Definition A.2. Let S be a d–dimensional connected smooth manifold and $X_1, \dots, X_k$ vector fields on S. We say a curve $\alpha\colon [0,1] \to S$ is a *basic broken integral path of* $X_1, \dots, X_k$ if α is given as the image of a edge path γ under the map $\Phi_{\alpha(0)}$, i.e. $\alpha = \Phi_{\alpha(0)} \circ \gamma$. Furthermore we call α a *broken integral path of* $X_1, \ldots, X_k$ if α is a concatenation of basic broken integral paths of those vector fields.

Lemma A.3. *Let S be a d–dimensional connected manifold with trivial tangent bundle TS. Suppose $X_1, \ldots, X_d$ are trivializations of TS. Fix a point $s_0 \in S$. Then every point in S can be joined by a broken integral curve of $X_1, \ldots, X_d$ starting at s_0.*

Proof. Let Σ be the set of points of S which can be joined by a broken integral path of $X_1, \ldots, X_d$ with s_0. Of course $\Sigma \neq \emptyset$ and we show that Σ is clopen which concludes the proof, since S is connected. For $s \in \Sigma$ there is the map $\Phi_s \colon V \to M$ described above where $V \subset \mathbb{R}^d$ is a open neighboorhood of the origin. Moreover we have

$$\frac{\partial \Phi_s}{\partial t_j}(0, \ldots, 0) = X_j(s)$$

hence the derivative $D\Phi_s(0, \ldots, 0)$ is invertible, since the vector fields $X_1, \ldots, X_d$ trivialize the tangent bundle and by the inverse function theorem there is a open neighboorhood $U \subset V$ of $0 \in \mathbb{R}^d$ such that $\Phi_s \colon U \to \Phi_s(U)$ is a diffeomorphism. We can assume that U is connected and hence every point in U can be connected to 0 by an edge path, which means that we can connect s with every point in $\Phi_s(U)$ by an basic broken integral path. Hence we have that the open set $\Phi_s(U)$ is contained in Σ and this proves that Σ is open. Otherwise if $s' \in S \setminus \Sigma$ then there is as well an open neighboorhood $\Phi_{s'}(U')$ of s' with the same properties. If $\Phi_{s'}(U') \cap \Sigma \neq \emptyset$ we could connect s' to s with a broken integral path. Therefore we have that Σ is closed. ∎

Cohomology of Lie algebras

Suppose $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{K}$ and V a $\mathbb{K}$–vector space. We say V is a $\mathfrak{g}$–module if there is a homomorphism of Lie algebras $\theta \colon \mathfrak{g} \to \mathfrak{gl}(V)$ with $\mathfrak{gl}(V)$ as the Lie algbra of the invertible linear endomorphisms of V. As for groups we would like to write $x.v$ for the action $\theta(x, v)$ where $x \in \mathfrak{g}$ and $v \in V$. Suppose henceforth that V is a $\mathfrak{g}$–module.

For $k \in \mathbb{Z}$ and $k \geq 0$ define the vector spaces $C^k := \mathbf{Hom}(\wedge^k \mathfrak{g}, M)$ and $C^k = \{0\}$ for $k < 0$. Moreover consider the linear maps $\delta^k \colon C^k \to C^{k+1}$ defined as

$$\delta^k(\alpha)(x_0, \ldots, x_k) := \sum_{i=0}^{k} (-1)^i x_i.\alpha(\ldots, \hat{x}_i, \ldots) + \sum_{i=0}^{k} \sum_{j=0}^{i-1} (-1)^{j+i} \alpha([x_j, x_i], \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots)$$

for $k \geq 0$ and δ^k the zero map if $k < 0$ where $\hat{x}_i$ means we omit the vector x_i in the $i - th$ entry.

Lemma A.4. *The pair $(C^k, \delta^k)_{k\in\mathbb{Z}}$ is a cochain complex.*

The k–th cohomology $H^k(\mathfrak{g}, V)$ of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$–module V is the k-th cohomology induced by the cochain complex (C^k, δ^k) of Lemma A.4.

As an example we would like to determine the meaning of the zeroth and first cohomology group with values in the trivial $\mathfrak{g}$–module $\mathbb{R}$. Clearly $C^0 = \mathbb{R}$ and δ^0 is the zero map as well as δ^{-1}, hence $H^0(\mathfrak{g}; \mathbb{R}) = \mathbb{R}$. For $\alpha \in C^1 = \mathfrak{g}^*$ we have

$$(\delta^1\alpha)(x, y) = -\alpha([x, y])$$

and thus α is a cocycle iff $\alpha([x, y]) = 0$ for all $x, y \in \mathfrak{g}$. Here $H^1(\mathfrak{g}; \mathbb{R})$ is just the set of cocycles in C^1 since δ^0 is the trivial map. The map $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \to H^1(\mathfrak{g}; \mathbb{R})$, $\alpha \mapsto (x \mapsto \alpha([x])$ where $[x]$ is the projection of x into $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is well–defined and an isomorphism. And since $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ is natural isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ we have

$$H^1(\mathfrak{g}; \mathbb{R}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Lie algebras

This section answers the purpose all around Lie algebras. Let $\mathfrak{g}$ be a Lie algebra of arbitary dimension over the reals. We denote by $\mathfrak{g}'$ the derived algebra of $\mathfrak{g}$, hence $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$.

Lemma A.5. *For every $x \in \mathfrak{g}'$ the adjoing map ad_x is tracefree. In particular every algebra with $\mathfrak{g}' = \mathfrak{g}$ is unimodular.*

Proof. If $x \in \mathfrak{g}'$ then they are $y_1, y_2 \in \mathfrak{g}$ such that $x = [y_1, y_2]$. This leads to

$$\mathrm{ad}_x = \mathrm{ad}_{[y_1,y_2]} = \mathrm{ad}_{y_1}\mathrm{ad}_{y_2} - \mathrm{ad}_{y_2}\mathrm{ad}_{y_1}$$

where the right hand side is clearly tracefree. ∎

Let $\mathfrak{k}$ and $\mathfrak{h}$ be two Lie algebras and let $\mathbf{Der}(\mathfrak{k})$ be the Lie algebra of Derivations of $\mathfrak{k}$, i.e. for $f \in \mathbf{Der}(\mathfrak{k})$ and $k_1, k_2 \in \mathfrak{k}$ we have

$$f[k_1, k_2] = [f(k_1), k_2] + [k_1, f(k_2)].$$

If $\rho\colon \mathfrak{h} \to \mathbf{Der}(\mathfrak{k})$ is a Lie algebra homomorphism then we may define the *semidirect product* $\mathfrak{k} \rtimes_\rho \mathfrak{h}$ of $\mathfrak{k}$ and $\mathfrak{h}$ with respect to ρ by the following: as a vector space $\mathfrak{k} \rtimes_\rho \mathfrak{h}$ is given by $\mathfrak{k} \oplus \mathfrak{h}$ where we regard $\mathfrak{k}$ and $\mathfrak{h}$ as vector spaces. The Lie bracket is defined by

$$[(k_1, h_1), (k_2, h_2)] := \big([k_1, k_2] + h_1 \cdot k_2 - h_2 \cdot k_1, [h_1, h_2]\big)$$

where $k_1, k_2 \in \mathfrak{k}$, $h_1, h_2 \in \mathfrak{h}$ and where $h_i \cdot k_j = \rho(h_i)(k_j)$ $(i, j \in \{1,2\})$. It is easy to see, that this gives a Lie bracket on $\mathfrak{k} \oplus \mathfrak{h}$, and we define $\mathfrak{k} \rtimes_\rho \mathfrak{h} := (\mathfrak{k} \oplus \mathfrak{h}, [\cdot,\cdot])$. If K is simply connected then let $\Phi\colon \mathbf{Aut}(K) \to \mathbf{Aut}(\mathfrak{k})$ denotes the natural isomorphism $\varphi \mapsto (\mathrm{D}\varphi)_e$ from Remark 1.50. We obtain the

Lemma A.6. *Suppose K and H are Lie groups, K simply connected and let $r\colon H \to \mathbf{Aut}(K)$ be a smooth homomorphism. Define $G := K \rtimes_r H$ and let $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}$ is isomophic to $\mathfrak{k} \rtimes_\rho \mathfrak{h}$ where $\mathfrak{k}, \mathfrak{h}$ are the Lie algebras of K, H respectively and $\rho\colon \mathfrak{h} \to \mathbf{Der}(\mathfrak{k})$ is given as $\rho := \mathrm{D}(\Phi \circ r)_e$*

Proof. Note that the Lie algebra of $\mathbf{Aut}(K)$ is $\mathbf{Der}(\mathfrak{k})$: We have $\mathbf{Aut}(K) \cong \mathbf{Aut}(\mathfrak{k})$ via Φ since K is simply connected and the Lie algebra of the latter one is $\mathbf{Der}(\mathfrak{k})$. Therefore $\rho = \mathrm{D}(\Phi \circ r)_e\colon \mathfrak{h} \to \mathbf{Der}(\mathfrak{k})$ since $\Phi \circ r(e) = \mathrm{id}_\mathfrak{k}$. Moreover $\mathfrak{g}$ is $\mathfrak{k} \oplus \mathfrak{h}$ as a vector space since $K \times H$ is the underlying manifold of G and if $\eta_1, \eta_2 \in \mathfrak{k}$, $\xi_1, \xi_2 \in \mathfrak{h}$ then $[(\eta_1, 0), (\eta_2, 0)] = [\eta_1, \eta_2]_\mathfrak{k}$ as well as $[(0, \xi_1), (0, \xi_2)] = [\xi_1, \xi_2]_\mathfrak{h}$ since $K \times e$ and $e \times H$ are subgroups of $K \rtimes_r H$. It remains to check the term $[(\eta, 0), (0, \xi)]$. Let $(\alpha(t), e)$ and $(e, \beta(s))$ be curves on G through (e,e) tangent to $(\eta, 0) \in \mathfrak{g}$ and $(0, \xi) \in \mathfrak{g}$ respectively. Then

$$[(\eta,0),(0,\xi)] = -\frac{\partial^2}{\partial s \partial t}(\beta(s), e)(\alpha(t), 0)(\beta^{-1}(s), 0)$$

in $(s,t) = (0,0)$. But

$$(\beta(s), e)(\alpha(t), 0)(\beta^{-1}(s), 0) = (r_{\beta(s)}(\alpha(t)), e)$$

and taking the t–derivative in $t = 0$ yields

$$\mathrm{D}(r_{\beta(s)})_e(\eta) = (\Phi \circ r)(\beta(s))(\eta)$$

and finally taking the s–derivative in $s = 0$ gives us

$$[(\eta,0),(0,\xi)] = -\mathrm{D}(\Phi \circ r)_e(\xi)(\eta) = -\rho(\xi)(\eta) = -\xi \cdot \eta.$$

From the bilinearity of the bracket we obtain the formula above for a semidirect product $\mathfrak{k} \rtimes_\rho \mathfrak{h}$ where the identity map is an Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{k} \rtimes_\rho \mathfrak{h}$. ■

As for groups we obtain a splitting theorem for extensions of Lie algebras.

Lemma A.7. *Suppose $0 \to \mathfrak{k} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0$ is short exact sequence of Lie algebras and let $\sigma\colon \mathfrak{h} \to \mathfrak{g}$ be a splitting map, i.e. it is a Lie homomorphism such that $\pi \circ \sigma = \mathrm{id}_\mathfrak{h}$. Then the map $\rho\colon \mathfrak{h} \to \mathbf{Der}(\mathfrak{k})$, $\rho(X)(Y) := [\sigma(X), Y]$ is well–defined and a Lie algebra homomorphism and $\mathfrak{g}$ is isomorphic to $\mathfrak{k} \rtimes_\rho \mathfrak{h}$.*

Proof. First we prove that ρ is well–defined, i.e. $\rho(X) \in \mathbf{Der}(\mathfrak{k})$ for all $X \in \mathfrak{h}$. For $Y \in \mathfrak{k}$ and $X \in \mathfrak{h}$ we see $\rho(X)(Y) \in \mathfrak{k}$ since $\pi([\sigma(X), Y]) = 0$. We compute by using the Jacobi identity

$$\begin{aligned}\rho(X)[Y_1, Y_2] &= [\sigma(X), [Y_1, Y_2]] = -[Y_1, [Y_2, \sigma(X)]] - [Y_2, [\sigma(X), Y_1]] \\ &= [[\sigma(X), Y_1], Y_2] + [Y_1, [\sigma(X), Y_2]] = [\rho(X)(Y_1), Y_2] + [Y_1, \rho(X)(Y_2)],\end{aligned}$$

for $X \in \mathfrak{h}$ and $Y_1, Y_2 \in \mathfrak{k}$. Moreover ρ is a Lie algebra homomorphisms since for $X_1, X_2 \in \mathfrak{h}$ and $Y \in \mathfrak{k}$

$$\begin{aligned}\rho([X_1, X_2])(Y) &= [\sigma([X_1, X_2]), Y] = [[\sigma(X_1), \sigma(X_2)], Y] \\ &= [\sigma(X_1), [\sigma(X_2), Y] - [\sigma(X_2), [\sigma(X_1), Y]] \\ &= \rho(X_1) \circ \rho(X_2)(Y) - \rho(X_2) \circ \rho(X_1)(Y) \\ &= [\rho(X_1), \rho(X_2)](Y)\end{aligned}$$

where we used that the Jacobi identity again and that σ is a Lie algebra homomorphism.

We define the linear map $\Phi\colon \mathfrak{k} \rtimes_\rho \mathfrak{h} \to \mathfrak{g}$, $\Phi(Y, X) := \sigma(X) + Y$. Further Φ is an isomorphism of vector spaces: if $\Phi(Y, X) = 0$ then $\sigma(X) = -Y$ applying π on both sides yields $X = 0$ which then implies $Y = 0$. Hence Φ is bijective. It remains to check that Φ is a Lie algebra homomorphism. If $(Y_1, X_1), (Y_2, X_2) \in \mathfrak{k} \rtimes_\rho \mathfrak{h}$ then

$$\begin{aligned}\Phi([(Y_1, X_1), (Y_2, X_2)]) &= \Phi(([Y_1, Y_2] + X_1 \cdot Y_2 - X_2 \cdot Y_1, [X_1, X_2]) \\ &= [Y_1, Y_2] + X_1 \cdot Y_2 - X_2 \cdot Y_1 + [\sigma(X_1), \sigma(X_2)] \\ &= [Y_1, Y_2] + [\sigma(X_1), Y_2] + [Y_1, \sigma(X_2)] + [\sigma(X_1), \sigma(X_2)] \\ &= [\sigma(X_1) + Y_1, \sigma(X_2) + Y_2] \\ &= [\Phi(Y_1, X_1), \Phi(Y_2, X_2)].\end{aligned}$$

∎

Smooth quotients of S^2

Suppose $\Gamma \subset \mathrm{Diff}(S^2)$ is a discrete subgroup which acts properly and freely on S^2, i.e. $N := S^2/\Gamma$ is a smooth surface. Then we obtain the following result.

Lemma A.8. *The space N is either diffeomorphic to S^2 or to the projective plane, i.e.* $\Gamma = \{\mathrm{id}_{S^2}\}$ *or* $\Gamma = \{\pm\mathrm{id}_{S^2}\}$.

Proof. Let $\pi\colon S^2 \to N$ be the covering map induced by the action of Γ on S^2. Since N is a smooth surface it admits a Riemannian metric of constant curvature say μ_N and thus $\pi^*(\mu_N)$

has constant curvature on S^2, which shows that $\Gamma \subset \mathbf{O}(3)$. If $\Gamma = \{\mathrm{id}\}$ we are done, so we assume that Γ contains more elements than the identity map. Let $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ then there is a line L in $\mathbb{R}^3$ such that $\gamma(l) = \pm l$ for all $l \in L$. Since γ acts freely it has no fixed point and therefore $\gamma(l) = -l$. But this implies that $\gamma^2 = \mathrm{id}$ since otherwise an $l \in L$ would be a fixed point of γ. Let $E := L^\perp$ then $\gamma|E$ has to be a rotation since otherwise it would have again a fixed point. But then $\gamma|E = -\mathrm{id}_E$ because of γ^2, hence $\gamma = -\mathrm{id}$. Thus N is diffeomorphic to either S^2 if $\Gamma = \{\mathrm{id}\}$ or to the projective plane if $\Gamma = \{\pm\mathrm{id}\}$. ∎

B

Lie Groups

THIS APPENDIX IS DEVOTED TO the properties of Lie groups and their actions on manifolds. Here, the letter G will denote always a smooth Lie group. The letter M shall be a smooth manifold, where G acts (from the left) on M, i.e. we have a smooth map $\theta\colon G \times M \to M$ and we write $g.m$ for $\theta(g, m)$. In general we want to denote any actions of G always with a dot.

Transformation groups

Definition B.1. We call θ a *proper* action of G on M if the map $G \times M \to M \times M$, $(g, m) \mapsto (m, g.m)$ is a proper map, i.e. the preimage of any compact set is compact.

Lemma B.2. *The following statements are equivalent*

(a) θ is a proper action of G on M

(b) If $(m_k)_{k\in\mathbb{N}}$ is a convergent sequence in M and $(g_k)_{k\in\mathbb{N}}$ a sequence in G such that $(g_k.m_k)_{k\in\mathbb{N}}$ converges in M then there is a subsequence $(g_{k_l})_{l\in\mathbb{N}}$ of $(g_k)_{k\in\mathbb{N}}$ which converges in G.

Proof. See [Lee03, p. 216] for a proof. ∎

Theorem B.3. *Suppose that G acts properly and freely on M. Then the orbit space M/G is a topological manifold of dimension* $\dim M - \dim G$, *and has a unique smooth structure with the property that the quotient map $\pi\colon M \to M/G$ is a smooth submersion.*

Proof. See [Lee03] for a proof. ■

Lemma B.4. *Let M be a two–dimensional simply connected surface and G a connected three–dimensional group such that (M, G) is a Riemannian homogeneous space . Then (M, G) is equivariant isomorphic to one of the following Riemannian homogeneous spaces .*

$$(S^2, \mathbf{SO}(3)),\ (\mathbb{R}^2, \mathbf{E}_0(2)),\ (D^2, \mathbf{SO}_0^+(2,1)).$$

Proof. Let μ be a G–invariant metric on M. Since G acts transitively the Gaussian curvature is constant and after rescaling the metric, the curvature is equal to $1, 0$ or -1. It follows that (M, μ) is isometric to either S^2, $\mathbb{R}^2$ or D^2 equipped with their standard metric. Following the arguments of Theorem 2.6 we obtain the claim of this lemma. ■

Complete groups

We recall that a connected group K is called a complete group if K is centerless and $\mathbf{Aut}(K) = \mathbf{Inn}(K) \cong K$ and K is called almost complete if it is centerless and $\mathbf{Aut}_0(K) = \mathbf{Inn}(K)$. We would like to show now two examples of complete groups.

Lemma B.5. *The Lie group $\mathbf{SO}(3)$ is complete and $\mathbf{SO}^+(2,1)$ is almost complete.*

Proof. The Lie algebra $\mathfrak{so}(3)$ of $\mathbf{SO}(3)$ may be identified with $(\mathbb{R}^3, \times)$, where $\times$ is the cross product of $\mathbb{R}^3$. The universal cover group of $\mathbf{SO}(3)$ is $\mathbf{SU}(2)$ and the fundamental group of $\mathbf{SO}(3)$ is isomorphic to $\mathbb{Z}_2$. But the fundamental group has to be a subgroup of the center of $\mathbf{SU}(2)$ which is isomorphic to $\mathbb{Z}_2$ and embedded in $\mathbf{SU}(2)$ by $\pm E_2$ where E_2 the identity matrix on $\mathbb{C}^2$. Hence the fundamental group of $\mathbf{SO}(3)$ can be identified with $\{\pm E_2\} \subset \mathbf{SU}(2)$. We saw in Remark 1.50 that $\mathbf{Aut}(\mathbf{SU}(2))$ is isomorpic to $\mathbf{Aut}(\mathfrak{so}(3))$ and $\mathbf{Aut}(\mathbf{SO}(3))$ to $\mathbf{Aut}(\mathbf{SU}(2), \mathbf{SO}(3))$. Thus consequently the first step is to compute $\mathbf{Aut}(\mathfrak{so}(3))$.

For $v, w \in \mathbb{R}^3$ we have that $v \times w = *(v \wedge w)$, where $*$ is the star operator with respect to the standard euclidean inner product $\langle \cdot, \cdot \rangle$ and the standard orientation given by the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ to be positive oriented. For an endomorphism f of $\mathbb{R}^3$ we denote by f^* the adjoint endomorphism of f with respect to $\langle \cdot, \cdot \rangle$. Let $v, w, z \in \mathbb{R}^3$ and set $\omega = e_1 \wedge e_2 \wedge e_3$ be the standard form. Then the computation

$$\begin{aligned}
\langle f^*(f(v) \times f(w)), z\rangle\omega &= \langle f(v) \times f(w), f(z)\rangle\omega = \langle *(f(v) \wedge f(w)), f(z)\rangle\omega \\
&= f(v) \wedge f(w) \wedge f(z) = (\det f)(v \wedge w \wedge z) \\
&= \langle (\det f) * (v \wedge w), z\rangle\omega = \langle (\det f)(v \times w), z\rangle\omega.
\end{aligned}$$

shows that $f^*(f(v) \times f(w)) = \det f(v \times w)$. If f is non–singular we have $f(v) \times f(w) = \det f(f^*)^{-1}(v \times w)$ and hence if $f \in \mathbf{Aut}(\mathfrak{so}(3))$ then $ff^* = \det f \cdot \mathrm{id}$. The last equation implies $\det f = 1$ and therefore f is an orthogonal map. Thus we figured out that $\mathbf{Aut}(\mathfrak{so}(3)) = \mathbf{SO}(3)$ since obviously every element of $\mathbf{SO}(3)$ respects the cross product. As we have seen in Remark 1.50 since $\mathbf{SO}(3)$ is connected the automorphism group $\mathbf{Aut}(\mathbf{SO}(3))$ is build by those automorphisms of $\mathbf{SU}(2)$ which maps the fundamental group of $\mathbf{SO}(3)$ into itself. But $\pi_1(\mathbf{SO}(3))$ is exactly the center of $\mathbf{SU}(2)$ and since automorphisms map the center into itself bijectively we obtain $\mathbf{Aut}(\mathbf{SO}(3)) \cong \mathbf{SO}(3)$. Now, since $\mathbf{SO}(3)$ is centerless,

$$\mathbf{SO}(3) \cong \mathbf{Inn}(\mathbf{SO}(3)) \subset \mathbf{Aut}(\mathbf{SO}(3)) \cong \mathbf{SO}(3).$$

which shows $\mathbf{Inn}(\mathbf{SO}(3)) = \mathbf{Aut}(\mathbf{SO}(3))$ and we conclude that $\mathbf{SO}(3)$ is indeed complete.

The universal cover group of $\mathbf{SO}^+_0(2,1)$ is $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ and as above $\pi_1(\mathbf{SO}^+(2,1))$ is the center of $\widetilde{\mathbf{SL}(2,\mathbb{R})}$. The Lie algebra $\mathfrak{so}^+(2,1)$ of $\mathbf{SO}^+(2,1)$ is isomorphic to $(\mathbb{R}^3, \overline{\times})$ where $v \overline{\times} w := *(v \wedge w)$ and the star operator is taken with respect to the standard Lorentzian metric on $\mathbb{R}^3$ with standard orientation defined by the canonical basis. Following the arguments above we obtain that an automorphism of $\mathfrak{so}^+(2,1)$ has to satisfy the condition $ff^* = \mathrm{id}$ where f^* is the adjoint map of f with respect to the standard Lorentzian metric on $\mathbb{R}^3$. Therefore we get in the same way as above $\mathbf{Aut}(\mathfrak{so}^+(2,1)) = \mathbf{SO}(2,1)$ which is a non–connected group. Since the center of $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ is isomorphic to the fundamental group of $\mathbf{SO}^+(2,1)$ we conclude $\mathbf{Aut}(\mathbf{SO}^+(2,1)) \cong \mathbf{SO}(2,1)$. Therefore $\mathbf{Aut}_0(\mathbf{SO}^+(2,1)) \cong \mathbf{SO}^+(2,1)$. Since $\mathbf{SO}^+(2,1)$ is centerless we argue as above to obtain $\mathbf{Aut}_0(\mathbf{SO}^+(2,1)) = \mathbf{Inn}(\mathbf{SO}^+(2,1))$ and this shows $\mathbf{SO}^+(2,1)$ is almost complete. ∎

The Spaces $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R})$

Lemma B.6. *The Riemannian homogeneous spaces* $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R})$ *and* $(\mathbb{R}^2 \times \mathbb{R}, \mathbf{E}_0(2) \rtimes_1 \mathbb{R})$ *for* $\kappa \neq 0$ *from Example 2.16 are equivariant isomorphic.*

Proof. Set $M := \mathbb{R}^2 \times \mathbb{R}$ and $G_\kappa := \mathbf{E}_0(2) \rtimes_\kappa \mathbb{R}$ and denote a point of M by (x,t) for $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Define the maps $f\colon M \to M$, $f(x,t) := (x, \kappa t)$ and $F\colon G_\kappa \to G_1$, $F(a,A,s) := (a,A,\kappa s)$. Clearly f is a diffeomorphism of M and F is a homomorphism as the computation

$$\begin{aligned}
F\Big((a,A,s) \cdot_\kappa (b,B,r)\Big) &= F\Big(a + e^{-\frac{1}{2}\kappa s}Ab, AB, s+r\Big)\\
&= \Big(a + e^{-\frac{1}{2}\kappa s}Ab, AB, \kappa s + \kappa r\Big)\\
&= (a,A,\kappa s) \cdot (b,B,\kappa r)\\
&= F(a,A,s) \cdot F(b,B,r)
\end{aligned}$$

shows. By Example 2.16 we know how G_κ acts on M and therefore we obtain

$$\begin{aligned} f\Big((a,A,r).(x,t)\Big) &= f(e^{-\frac{1}{2}\kappa r}Ax+a, t+r) \\ &= (e^{-\frac{1}{2}\kappa r}Ax+a, \kappa t+\kappa r) \\ &= (a,A,\kappa r).(x,\kappa t) \\ &= F(a,A,r).f(x,t). \end{aligned}$$

The map $F^{-1}\colon G_1 \to G_\kappa$, $(a,A,s) \mapsto (a,A,\kappa^{-1}s)$ is the inverse map to F. ■

Universal covering group

Suppose we have a homomorphism of connected Lie groups $\varphi\colon G \to H$ and let $\pi_G\colon \tilde{G} \to G$ as well as $\pi_H\colon \tilde{H} \to H$ the universal coverings of G and H respectively. Then, since $\tilde{G}$ and $\tilde{H}$ are simply connected, there is a unique lift $\tilde{\varphi}\colon \tilde{G} \to \tilde{H}$ of $\varphi\colon G \to H$ such that it is a Lie homomorphism and the diagram

$$\begin{tikzcd} \tilde{G} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi_G"] & \tilde{H} \arrow[d, "\pi_H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes. Furthermore it is well–known that there is a canonical isomorphism of groups $\pi_1(G) \to \ker \pi_G$ which is given by $[\alpha] \mapsto \tilde{\alpha}(1)$ where $\alpha\colon [0,1] \to G$ is a closed path and $\tilde{\alpha}$ its lift such that $\tilde{\alpha}(0) = \tilde{e}_G$.

Lemma B.7. *The following diagramm is commutative*

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \pi_1(G) \arrow[r] \arrow[d, "\varphi_*"] & \tilde{G} \arrow[r, "\pi_G"] \arrow[d, "\tilde{\varphi}"] & G \arrow[r] \arrow[d, "\varphi"] & 1 \arrow[d] \\ 1 \arrow[r] & \pi_1(H) \arrow[r] & \tilde{H} \arrow[r, "\pi_H"] & H \arrow[r] & 1 \end{tikzcd}$$

where the fundamental groups are identified with the kernels of the covering homomorphisms as described above and where $\varphi_*\colon \pi_1(G) \to \pi_1(H)$ is the homomorphism on the fundamental groups induced by φ.

Proof. Clearly by construction we have $\varphi \circ \pi_G = \pi_H \circ \tilde{\varphi}$. Note that this implies that $\tilde{\varphi}$ restricted to the $\ker \pi_G$ maps into $\ker \pi_H$. Denote this restricted map again by $\tilde{\varphi}$. Then the diagramm

$$\begin{array}{ccc} \pi_1(G) & \xrightarrow{\varphi_*} & \pi_1(H) \\ \Big\downarrow{\scriptstyle i_G} & & \Big\downarrow{\scriptstyle i_H} \\ \ker \pi_G & \xrightarrow{\tilde{\varphi}} & \ker \pi_H \end{array}$$

commutes, where the vertical maps are given by the isomophism described at the beginning of this section. To see this take a homotopy class $[\alpha] \in \pi_1(G)$ where $\alpha\colon [0,1] \to G$ is a closed curve representing the chosen class. Then $i_H \circ \varphi_*([\alpha]) = \widetilde{\varphi \circ \alpha}(1)$. If we go the other way we obtain $\tilde{\varphi} \circ i_G([\alpha]) = \tilde{\varphi} \circ \tilde{\alpha}(1)$. But the curves $\widetilde{\varphi \circ \alpha}$ and $\tilde{\varphi} \circ \tilde{\alpha}$ are both lifts of the curve $\varphi \circ \alpha$ for the covering $\pi_H\colon \tilde{H} \to H$ (observe $\tilde{\varphi} \circ \tilde{\alpha}(0) = \widetilde{\varphi \circ \alpha}(0) = \tilde{e}_H$ and $\pi_H \circ \tilde{\varphi} \circ \tilde{\alpha} = \varphi \circ \pi_G \circ \tilde{\alpha} = \varphi \circ \alpha$), thus $\tilde{\varphi} \circ \tilde{\alpha}(1) = \widetilde{\varphi \circ \alpha}(1)$. We conclude that the last diagramm commutes which completes the proof. ∎

The next lemma clarifies when two discrete subgroups of the center of a simply connected group induce isomorphic quotients. It is not sufficient to know the isomorphism class of the discrete subgroup as the example of $G = \mathbb{R} \times \mathbf{SU}(2)$ shows: the center is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$ and the subgroups $\mathbb{Z}(1,1)$ and $\mathbb{Z}(1,-1)$ are both isomorphic to $\mathbb{Z}$ (we consider $\mathbb{Z}_2$ as a multiplicative abelian group) but the quotients are $S^1 \times \mathbf{SU}(2)$ and $\mathbf{U}(2)$ (see Lemma B.12) respectively which are not isomorphic, since e.g. the centers are not isomophic.

Lemma B.8. *Let $\tilde{G}$ be a simply connected Lie group and $\Gamma_1, \Gamma_2 \subset Z(\tilde{G})$ two discrete subgroups. The quotients $G_1 := \tilde{G}/\Gamma_1$ and $G_2 = \tilde{G}/\Gamma_2$ are isomorphic iff there is an automorphism $\tilde{\varphi} \in \mathbf{Aut}(\tilde{G})$ such that $\tilde{\varphi}(\Gamma_1) = \Gamma_2$.*

Proof. Suppose first there is an isomorphism $\varphi\colon G_1 \to G_2$ and let $\pi_i\colon \tilde{G} \to G_i$ be the covering homomorphism for $i = 1, 2$. Since $\tilde{G}$ is simply connected we may lift the homomorphism $\varphi \circ \pi_1\colon \tilde{G} \to G_2$ to a homomorphism $\tilde{\varphi}\colon \tilde{G} \to \tilde{G}$ such that the diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\varphi}} & \tilde{G} \\ {\scriptstyle \pi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_2} \\ G_1 & \xrightarrow{\varphi} & G_2 \end{array}$$

commutes. For φ^{-1} we may proceed as for φ, however we have $\pi_1 \circ \widetilde{\varphi^{-1}} = \varphi^{-1} \circ \pi_2$. We claim that $\tilde{\varphi} \in \mathbf{Aut}(\tilde{G})$ and $\tilde{\varphi}^{-1} = \widetilde{\varphi^{-1}}$. Consider therefore map the $F\colon \tilde{G} \to \tilde{G}$,

$$F(\tilde{g}) := \tilde{g}^{-1} \cdot \left(\widetilde{\varphi^{-1}} \circ \tilde{\varphi}(\tilde{g})\right)$$

which is smooth and has image in $\Gamma_1 \subset \tilde{G}$:

$$\pi_1 \circ F(\tilde{g}) = \pi_1(\tilde{g}^{-1}) \cdot \Big(\pi_1 \circ \widetilde{\varphi^{-1}} \circ \tilde{\varphi}(\tilde{g})\Big) = \pi_1(\tilde{g}^{-1}) \cdot \Big(\varphi^{-1} \circ \varphi \circ \pi_1(\tilde{g})\Big) = e.$$

Since Γ_1 is discrete and $F(e) = e$ we conclude $\widetilde{\varphi^{-1}} \circ \tilde{\varphi} = \mathrm{id}_{\tilde{G}}$. A similar argument provides $\tilde{\varphi} \circ \widetilde{\varphi} = \mathrm{id}_{\tilde{G}}$. If $\gamma_1 \in \Gamma_1$ then $\pi_2 \circ \tilde{\varphi}(\gamma_1) = \varphi(e) = e$ hence $\tilde{\varphi}(\gamma_1) \in \Gamma_2$ and also $\tilde{\varphi}^{-1}(\gamma_2) \in \Gamma_1$ for $\gamma_2 \in \Gamma_2$. This shows $\tilde{\varphi}(\Gamma_1) = \Gamma_2$.

If $\tilde{\varphi} \in \mathbf{Aut}(\tilde{G})$ such that $\tilde{\varphi}(\Gamma_1) = \Gamma_2$ then define a map $\varphi\colon G_1 \to G_2, \varphi([\tilde{g}]_1) := [\tilde{\varphi}(\tilde{g})]_2$, where we regard $G_i = \tilde{G}/\Gamma_i$. This map is well–defined since $\tilde{\varphi}(\Gamma_1) = \Gamma_2$ and is a homomorphism since $\tilde{\varphi}$ is one. Note that $\tilde{\varphi}^{-1}(\Gamma_2) = \Gamma_1$ and therefore the map $\varphi^{-1}([\tilde{g}]_2) := [\tilde{\varphi}^{-1}(\tilde{g})]_1$ is well–defined and the inverse map to φ. Hence φ is an isomorphism. ∎

Lemma B.9. *Let G be a connected Lie group and $\tilde{G}$ its universal covering group. Then $Z(G) \cong Z(\tilde{G})/\pi_1(G)$, where $Z(G)$ denotes the center of G.*

Proof. Let $\pi\colon \tilde{G} \to G$ denote the covering homomorphism. We have $\pi_1(G) = \ker \pi \subset Z(\tilde{G})$ and the restriction of π to $Z(\tilde{G})$ has image in $Z(G)$, therefore $p := \pi|Z(\tilde{G})\colon Z(\tilde{G}) \to Z(G)$ is well–defined. If $z \in Z(G)$ and $\tilde{z} \in \pi^{-1}(z)$ then we claim that $\tilde{z} \in Z(\tilde{G})$. For that consider the smooth map $\Phi\colon \tilde{G} \to \tilde{G}, \Phi(\tilde{g}) := \tilde{g}\tilde{z}\tilde{g}^{-1}\tilde{z}^{-1}$ and we see that $\pi \circ \Phi(\tilde{g}) = e$ since $\pi(\tilde{z}) = z \in Z(G)$, thus $\Phi(\tilde{g}) \in \ker \pi = \pi_1(G)$. The fundamental group is a discrete subgroup of $\tilde{G}$ and since $\Phi(\tilde{e}) = \tilde{e}$ we obtain $\tilde{z} \in Z(\tilde{G})$ and this shows that p is onto. Clearly if $p(\tilde{z}) = e$ then $\tilde{z} \in \ker \pi = \pi_1(G)$, hence $Z(G) \cong Z(\tilde{G})/\pi_1(G)$. ∎

About homotopy groups of Lie groups

The Lie group structure of a connected Lie group G makes it possible to elicit some properties of its homotopy groups, namely that $\pi_2(G)$ vanishes. We would like to give an outline how this can be proven.

First let us start with the observation that we have the fibration $\Omega G \to PG \xrightarrow{\pi} G$, where PG is the space of all paths in G starting at the neutral element e and $\pi\colon PG \to G$ is the map which assigns to each element of PG its endpoint on G. The space ΩG is the fiber of π over e, i.e. it is the loop space of G in the point e. Both ΩG and PG are equipped with the compact–open topology and this makes π a indeed to a fibration. The path space PG is contractible and the long homotopy–sequence of the fibration

$$\dots \to \pi_k(PG) \to \pi_k(G) \to \pi_{k-1}(\Omega G) \to \pi_{k-1}(PG) \to \dots$$

implies

$$\pi_{k-1}(\Omega G) \cong \pi_k(G).$$

Moreover note that if G is simply connected we have $\pi_0(\Omega G) = 1$ since PG is contractible. Now with a theorem of R. Bott using Morse theory, cf. [Mil63, p. 116], we see that we have some knowledge about the CW–strucure of ΩG:

Lemma B.10. *Let G be a compact, simply connected Lie group. Then the loop space ΩG has the homotopy type of CW–complex with no odd dimensional cells.*

This shows that $\pi_1(\Omega G) = 1$ using the cellular approximation theorem and therefore $\pi_2(G) \cong \pi_1(\Omega G) = 1$ for G compact and simply connected. Indeed if we are interested solely on the homotopy groups of G, we may assume without loss of generality that G is compact and simply connected as the following lines will show us.

First, it is easy to see that $\pi_k(\tilde{G}) \cong \pi_k(G)$ for $k \geq 2$ where $\tilde{G}$ is the universal cover of G. So we may assume that G is simply connected. Second, Theorem 6 of [Iwa49] states that every Lie group contains a compact subgroup K which is a deformation retract of G and therefore K is simply connected and compact with same homotopy groups as G. This finally leads to the

Lemma B.11. *For every connected Lie group G the second homotopy group $\pi_2(G)$ vanishes.*

The group $\mathbf{U}(2)$

We show that the universal cover group of $\mathbf{U}(2)$ is $\mathbb{R} \times \mathbf{SU}(2)$ and we will determine some quotients by one–dimensional subgroups.

Lemma B.12. *Let $c \in \mathbb{R}$ such that $c \neq 0$. Furthermore define the subgroup $Z := \mathbb{Z}(c, -E_2) \subset \mathbb{R} \times \mathbf{SU}(2)$ which is discrete and lies in the center of $\mathbb{R} \times \mathbf{SU}(2)$. Then $\mathbf{U}(2)$ is isomophic to $(\mathbb{R} \times \mathbf{SU}(2)/Z)$ and therefore $\mathbb{R} \times \mathbf{SU}(2)$ is the universal cover group of $\mathbf{U}(2)$.*

Proof. Consider the homomorphism $\Phi\colon \mathbb{R} \times \mathbf{SU}(2) \to \mathbf{U}(2)$, $(x, S) \mapsto e^{i\pi\frac{x}{c}}S$. Φ is onto since if $U \in \mathbf{U}(2)$ then $\det U \in \mathbf{U}(1)$ and there is an $x \in \mathbb{R}$ such that $e^{i2\pi x/c} = \det U$. The pair $(x, e^{-i\pi x/c}U)$ lies in $\mathbb{R} \times \mathbf{SU}(2)$ and its image under Φ is U. Next we would like to show that the kernel of Φ is given by $\mathbb{Z}(c, -E_2)$. Therefore if $e^{i\pi x/c}S = E_2$ we deduce by applying the determinant $e^{i2\pi x/c} = 1$ thus $x = kc$ for a $k \in \mathbb{Z}$ and $S = e^{-i\pi k}E_2 = (-1)^k E_2$. We conclude $(kc, (-1)^k E_2) \in \mathbb{Z}(c, -E_2)$ and the kernel lies in Z. On the other hand Z is clearly a subset of the kernel of Φ. ∎

Lemma B.13. *Suppose K is a closed subgroup of $\mathbf{U}(2)$ isomorphic to $\mathbf{U}(1)$. The only homogeneous space on which $\mathbf{U}(2)$ acts transitively and effectively with isotropy group K is equivariantly diffeomorphic to $S^3 \subset \mathbb{C}^2$ (where $\mathbf{U}(2)$ acts through the restricted linear action of $\mathbb{C}^2$).*

Proof. We would like to work with the linear (faithful) representation of $\mathbf{U}(2)$ on $\mathbb{C}^2$. The quotients $\mathbf{U}(2)/K$ and $\mathbf{U}(2)/K'$ are equivariantly diffeomorphic if K' is a conjugated subgroup of K in $\mathbf{U}(2)$. An element of $\mathbf{U}(2)$ is after a conjugation in $\mathbf{U}(2)$ given by the matrix

$$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} =: \operatorname{diag}(z_1, z_2)$$

where $z_1, z_2 \in \mathbf{U}(1)$. Now since K is abelian we may diagonalize every element in K by a conjugation with the same matrix of $\mathbf{U}(2)$. Thus without loss of generality we assume that the elements of K are given by diagonal matrices (which means that every subgroup isomorphic to $\mathbf{U}(1)$ of $\mathbf{U}(2)$ is conjugated to a subgroup of a torus in $\mathbf{U}(2)$). Suppose $\Phi\colon \mathbf{U}(1) \to K$ is an isomorphism. Then there have to be homomorphisms $\Phi_1, \Phi_2\colon \mathbf{U}(1) \to \mathbf{U}(1)$ such that $\Phi(z) = \operatorname{diag}(\Phi_1(z), \Phi_2(z))$. But homomorphisms $\mathbf{U}(1) \to \mathbf{U}(1)$ are given by $z \mapsto z^n$ for $n \in \mathbb{Z}$. Clearly $\Phi\colon K \to \mathbf{U}(2)$ is an isomorphism onto its image if $n = 0$ and $m = 1$ or $n = 1$ and $m = 0$. Therefore suppose $n \neq 0 \neq m$. Then Φ is an isomorphism iff $\gcd(n, m) = 1$. To see this suppose $d := \gcd(m, n) \neq 1$ and let $l_1, l_2 \in \mathbb{N}$ such that $n = l_1 d$ as well as $m = l_2 d$, in particular $l_1 < n$. Then $z_0 := e^{2\pi i \frac{l_1}{n}}$ is an nth root of unity as well as an mth root of unity since $z_0^m = e^{2\pi i \frac{l_1}{n} m} = e^{2\pi i \frac{l_1}{l_1} l_2} = e^{2\pi i l_2} = 1$, hence Φ is not injective. Now on the other side suppose $\gcd(n, m) = 1$ where n, m are non–zero and let $1 \neq z_0 \in \ker\Phi$. Thus $z_0^n = 1$ and we may assume $n > 0$. Hence $z_0 = e^{2\pi i \frac{l}{n}}$ for an integer $0 < l < n$. Moreover we have $z_0^m = 1$ which means $e^{2\pi i \frac{l}{n} m} = 1$ and this implies $n \,|\, lm$. But then $n = \gcd(lm, n) = \gcd(l, n)\gcd(m, n) = \gcd(l, n)$ and since $l < n$ this is a contradiction to $z_0 \neq 1$. We conclude that $\ker\Phi$ is trivial.

Clearly if $K \cap Z(\mathbf{U}(2)) \neq \{E_2\}$ then $\mathbf{U}(2)$ is not acting effectively on $\mathbf{U}(2)/K$ therefore we exclude first the case $n = m = 1$. Now suppose $n \neq 0 \neq m$ and $\gcd(n, m) = 1$. We may write an element of K as $z^n \operatorname{diag}(1, z^{m-n})$ where $m - n \neq 0$. The number $z_0 = e^{2\pi i \frac{1}{m-n}}$ is an $|m - n|$th root but $z_0^n \neq 1$ since $\gcd(m - n, n) = \gcd(m, n) = 1$ and therefore $\frac{n}{m-n}$ is not an integer. We conclude that the element $\operatorname{diag}(z_0^n, z_0^m)$ lies in $K \cap Z(\mathbf{U}(1))$ and this implies that in that case $\mathbf{U}(2)$ would not act effectively on $\mathbf{U}(2)/K$. The remaining cases are $n = 0$ and $m = 1$ or the other way around. Here $K \cap Z(\mathbf{U}(2)) = \{E_2\}$ and the quotient $\mathbf{U}(2)/K$ is diffeomorphic to $S^3 \subset \mathbb{C}^2$ with the canonical action of $\mathbf{U}(2)$ on S^3.

∎

The Lie Algebra of $\mathbf{Nil} \rtimes \mathbb{R}$

We defined $\mathbf{Nil} \rtimes \mathbb{R}$ in Remark 1.71. The diffeomorphism type is $\mathbb{R}^4$ and the neutral element is given by $0 \in \mathbb{R}^4$. We recall the multiplicative structure of $\mathbf{Nil} \rtimes \mathbb{R}$. Write $(\mathbf{x}, z, \theta)$ for an element of $\mathbf{Nil} \rtimes \mathbb{R}$ where $\mathbf{x} = (x_1, x_2)$ and $(\mathbf{x}, z)$ belongs to the $\mathbf{Nil}$ part. Define a curve of quadratic forms $\tilde{\beta}\colon \mathbb{R} \to \mathbf{Bil}(\mathbb{R}^2)$ such that $\tilde{\beta}_\theta$ is given in the standard basis of $\mathbb{R}^2$ by

$$\frac{1}{2} s \begin{pmatrix} -c & -s \\ -s & c \end{pmatrix}$$

where $s = \sin\theta$ and $c = \cos\theta$ and let β_θ be its quadratic form. Finally define for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ another bilinear form $n(\mathbf{x}, \mathbf{y}) := x_1 y_2$. With this notation the multiplication in $\mathbf{Nil} \rtimes \mathbb{R}$ may be described as

$$(\mathbf{x}, z, \theta) \cdot (\mathbf{y}, w, \phi) = (\mathbf{x} + \rho_\theta(\mathbf{y}), z + w + \beta_\theta(\mathbf{y}) + n(\mathbf{x}, \rho_\theta(\mathbf{y})), \theta + \phi).$$

Let $\mathfrak{g}$ be its Lie algebra. We define

$$E_i := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (te_i, 0, 0)$$

for $i = 1, 2$ where e_i is the i–th canonical basis vector of $\mathbb{R}^2$ and

$$E_3 := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (0, t, 0), \quad E_4 := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (0, 0, t).$$

To compute the Lie brackets of the E_i's we have to know the inverse elements of the defining curves. If $G \rtimes H$ is a semidirect product the inverse of (g, e) and (e, h) is given by (g^{-1}, e) and (e, h^{-1}) respectively. Thus the inverse of $(0, 0, t)$ is $(0, 0, -t)$. The inverse of the remaining curves are given by $(-te_i, 0, 0)$ and $(0, t, 0)$ (one way to see this very quickly is to look at the linear representation of $\mathbf{Nil}$ given by the Heisenberg group). We compute

$$\begin{aligned}
(te_1, 0, 0) \cdot (re_2, 0, 0) \cdot (-te_1, 0, 0) &= (re_2, rt, 0) \\
(te_1, 0, 0) \cdot (0, r, 0) \cdot (-te_1, 0, 0) &= (0, r, 0) \\
(te_1, 0, 0) \cdot (0, 0, r) \cdot (-te_1, 0, 0) &= (te_1 - \rho_r(te_1), -t^2\beta_r(e_1) - t^2 n(e_1, \rho_r(e_1)), r) \\
(te_2, 0, 0) \cdot (0, r, 0) \cdot (-te_2, 0, 0) &= (0, r, 0) \\
(te_2, 0, 0) \cdot (0, 0, r) \cdot (-te_2, 0, 0) &= (te_2 - \rho_r(te_2), -t^2\beta_r(e_2) - t^2 n(e_2, \rho_r(e_2)), r) \\
(0, t, 0) \cdot (0, 0, r) \cdot (0, -t, 0) &= (0, 0, r).
\end{aligned}$$

Deriving those equation first with respect to r in $r = 0$ then with respect to t in $t = 0$ we obtain the Lie brackets

$$\begin{aligned}
[E_1, E_2] &= E_3 \\
[E_1, E_3] &= 0 \\
[E_1, E_4] &= -E_2 \\
[E_2, E_3] &= 0 \\
[E_2, E_4] &= E_1 \\
[E_3, E_4] &= 0
\end{aligned}$$

The linear map $\Phi\colon \mathbb{R} \times_{\omega_1} \mathfrak{e}(2) \to \mathfrak{g}$ defined by

$$\begin{aligned}
e_0 &\mapsto E_3 \\
e_1 &\mapsto E_1 \\
e_2 &\mapsto E_2 \\
e_3 &\mapsto E_4
\end{aligned}$$

is obviously a Lie algebra isomorphism.

The center of semidirect products

Let G and H be arbitary groups and $G \rtimes_\varrho H$ the semidirect product of G and H where $\varrho\colon H \to \mathbf{Aut}(G)$ is a representation of H in G. Define $\Gamma := \cap_{h \in H} \mathrm{Fix}(\varrho_h) \subset G$ where $\mathrm{Fix}(\varrho_h)$ is the fixed point set of the automorphism $\varrho_h\colon G \to G$. Clearly Γ is a subgroup of G. Further define $\Delta := \varrho^{-1}(\mathrm{im}\,(c|\Gamma))$ which is a subgroup of H where $c\colon G \to \mathbf{Inn}(G)$ is the conjugation map $g_0 \mapsto (g \mapsto g_0 g g_0^{-1})$. Obviously $\Gamma \times \Delta$ is a subgroup of $G \rtimes_\varrho H$.

Lemma B.14. *The map $\Phi\colon \Gamma \times \Delta \to \mathbf{Inn}(G)$, $\Phi(g, h) := \varrho_h c_g$ is a homomorphism of groups.*

Proof. For $(g_0, h_0) \in \Gamma \times \Delta$ the elements ϱ_{h_0} and c_{g_0} commute in $\mathbf{Inn}(G)$ since have

$$\varrho_{h_0} c_{g_0} = c_{\varrho_{h_0}(g_0)} \varrho_{h_0} = c_{g_0} \varrho_{h_0}$$

since $g_0 \in \Gamma = \cap_{h \in H} \mathrm{Fix}(\varrho_h)$. Thus for $(g_i, h_i) \in \Gamma \times \Delta$ $(i = 1, 2)$ we obtain

$$\varrho_{h_1 h_2} c_{g_1 g_2} = \varrho_{h_1} \varrho_{h_2} c_{g_1} c_{g_2} = \varrho_{h_1} c_{g_1} \varrho_{h_2} c_{g_2}$$

which shows that Φ is a homomorphism. ∎

Lemma B.15. *For* $\Psi := \Phi|(\Gamma \times (\Delta \cap Z(H)))$ *we have* $Z(G \rtimes_{\varrho} H) = \ker \Psi$.

Proof. Let $(g_0, h_0) \in \ker \Psi$ which means $\varrho_{h_0} = c_{g_0}^{-1}$. Then for all $(g, h) \in G \rtimes_{\varrho} H$

$$(g_0, h_0)(g, h) = (g_0 \varrho_{h_0}(g), h_0 h) = (g_0 g_0^{-1} g g_0, h h_0) = (g g_0, h h_0) = (g \varrho_h(g_0), h h_0) = (g, h)(g_0, h_0),$$

thus $(g_0, h_0) \in Z(G \rtimes_{\varrho} H)$. On the other hand if $(g_0, h_0) \in Z(G \rtimes_{\varrho} H)$ then we obtain the equations

$$h h_0 = h_0 h, \quad g_0 \varrho_{h_0}(g) = g \varrho_h(g_0)$$

for all $(g, h) \in G \rtimes_{\varrho} H$. The first equations implies $h_0 \in Z(H)$. Choosing $g = e$ in the second equations yields $g_0 = \varrho_h(g_0)$ for all $h \in H$, hence $g_0 \in \Gamma$. On the other side choosing $h = e$ in the same equation yields $\varrho_{h_0}(g) = g_0^{-1} g g_0 = c_{g_0}^{-1}(g)$. Hence $h_0 \in \Delta \cap Z(H)$ and $\varrho_{h_0} c_{g_0} = \mathrm{id}_G$. ■

We would like to compute the center of some semidirect products.

Lemma B.16. *The center of* $\mathbf{E} := \widetilde{\mathbf{E}_0(2)}$ *is isomorphic to* $\mathbb{Z}$ *and is embededd like* $l \mapsto (0, 2\pi l) \in \mathbf{E}$.

Proof. The group $\mathbf{E}$ is the semidirect product $\mathbb{R}^2 \rtimes_{\varrho} \mathbb{R}$ where the action $\varrho\colon \mathbb{R} \to \mathbf{Aut}(\mathbb{R}^2)$ is given by $\theta.a := \varrho_\theta(a) = \rho_\theta(a)$ (we recall that ρ_θ is the rotation in $\mathbb{R}^2$ around the origin with rotation angle θ). If $\theta.a = a$ for all $\theta \in \mathbb{R}$ then $a = 0$ ($\theta = \pi$) and Γ is trivial. Then by definition Δ is the kernel of ϱ which $2\pi\mathbb{Z} \subset \mathbb{R}$. ■

Lemma B.17. *The center of* $\mathbf{Nil} \rtimes_\theta \mathbb{R}$ *from Example 1.68 is isomorphic to* $\mathbb{R} \times \mathbb{Z}$ *and is embedded as* $(z, l) \mapsto (0, 0, z, 2\pi l)$.

Proof. The representation of $\mathbb{R}$ in $\mathbf{Nil}$ is given by

$$\theta.(\mathbf{x}, z) := \varrho_\theta(\mathbf{x}, z) = (\rho_\theta(\mathbf{x}), z + \beta_\theta(\mathbf{x})).$$

Let $(\mathbf{x}, z) \in \mathbf{Nil}$ such that $\theta.(\mathbf{x}, z) = (\mathbf{x}, z)$ for all $\theta \in \mathbb{R}$. Then $\rho_\theta(\mathbf{x}) = \mathbf{x}$ for all $\theta \in \mathbb{R}$. This forces $\mathbf{x}$ to be zero and therefore z can be an arbitrary real number. Thus $\Gamma = \{(0, 0, z) : z \in \mathbb{R}\}$ which is exactly the center of $\mathbf{Nil}$ and therefore $\mathrm{im}\,(c|\Gamma) \subset \mathbf{Inn}(\mathbf{Nil})$ is trivial. This means that Δ is given as the kernel of ϱ which is $2\pi\mathbb{Z} \subset \mathbb{R}$. ■

‖ **Lemma B.18.** *The center of* $\mathbf{E}_0(2) \rtimes_\kappa \mathbb{R}$ *from Example 2.16 is trivial for* $\kappa > 0$.

Proof. The representation here of $\mathbb{R}$ in $\mathbf{E}_0(2)$ is given by

$$t.(a, A) := (e^{-\frac{1}{2}\kappa t} a, A)$$

for $(a, A) \in \mathbf{E}_0(2) = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ and $t \in \mathbb{R}$. The group Γ contains elements of $\mathbf{E}_0(2)$ which fulfill $(e^{-\frac{1}{2}\kappa t} a, A) = (a, A)$ for all $t \in \mathbb{R}$. Since $\kappa > 0$ we deduce $a = 0$. The inner automorphism of $(0, A)$ on $\mathbf{E}_0(2)$ is given by

$$(0, A)(b, B)(0, A^{-1}) = (Ab, B).$$

We conclude $t.(b, B) = c_{(0,A)}(b, B)$ iff $(e^{-\frac{1}{2}\kappa t} b, B) = (Ab, B)$ which forces $t = 0$. Thus $\Delta = \{0\}$ and the kernel of $\Phi\colon \Gamma \times \Delta \to \mathbf{Inn}(\mathbf{E}_0(2))$ is given by the elements of $(0, A) \in \Gamma$ such $c_{(0,A)}$ is the identity on $\mathbf{E}_0(2)$. This implies $Ab = b$ for all $b \in \mathbb{R}^2$, hence A is the identity which leads to the claim. ■

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