The adiabatic limit of Schrödinger operators on fibre bundles

DISSERTATION

der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
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Tübingen 2013

Tag der mündlichen Qualifikation:	19.02.2014
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Danksagung

Ich bedanke mich ganz herzlich bei allen die mich dabei begleitet haben für die wunderbare Zeit, die ich während des Schreibens dieser Arbeit hatte.

Mein besonderer Dank gilt Stefan Teufel für die gute Betreuung und Förderung sowie viele interessante Diskussionen auch über das Thema dieser Arbeit hinaus.

Stefan Haag danke ich für das aufmerksame Lesen des Manuskriptes.

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Zusammenfassung

Das Thema dieser Dissertation sind Schrödingeroperatoren im adiabatischen Limes. Der adiabatische Limes eines Faserbündels riemannscher Mannigfaltigkeiten bezeichnet dabei den Skalierungslimes, in dem das Verhältnis der Längen von Vektoren tangential an die Fasern und solchen orthogonal dazu proportional zu $\varepsilon \ll 1$ ist. Genauer gesagt werden Faserbündel $F \to M \overset{\pi}{\to} B$ von Mannigfaltigkeiten mit Rand, wobei die typische Faser F kompakt ist und die Basis B randlos, betrachtet. Ein solches Bündel ist eine riemannsche Submersion wenn M und B riemannsche Metriken tragen, so dass das Differenzial π_* eine Isometrie $TM/\ker \pi_* \to TB$ induziert. Die Metrik auf M hat dann die Form $g = g_F + \pi^* g_B$ und der adiabatische Limes wird durch die Familie von Metriken $g_\varepsilon = g_F + \varepsilon^{-2}\pi^* g_B$ beschrieben.

Auf Faserbündeln dieser Art werden nun Schrödingeroperatoren

$$H = -\Delta_{g_\varepsilon} + V$$

mit Dirichlet-Randbedingungen und glattem Potential $V \in \mathscr{C}_b^\infty(M)$ untersucht. Dabei werden auch Störungen dieser durch Differenzialoperatoren εH_1, die etwa eine kleine Störung der Metrik g_ε modellieren können, zugelassen.

Die Aufspaltung der Richtungen auf M in vertikale, welche tangential an die Fasern sind, und horizontale, orthogonal dazu, führt zu einer Aufspaltung des Laplace-Operators

$$\Delta_{g_\varepsilon} = \varepsilon^2 \Delta_h + \Delta_F \,,$$

wobei Δ_F faserweise als der Laplace-Operator der induzierten Metrik wirkt. Damit ist für jedes $x \in B$ der Operator

$$H_F(x) := -\Delta_F|_{F_x} + V$$

auf der Faser F_x unabhängig von ε. Mit Dirichlet-Randbedingungen ist dies ein selbstadjungierter Operator, dessen Spektrum wegen der Kompaktheit der Faser eine unbeschränkte Folge von reellen Eigenwerten endlicher Entartung bildet. Ein Eigenband ist eine Funktion $\lambda\colon B \to \mathbb{R}$ mit $\lambda(x) \in \sigma(H_F(x))$. Zu einem solchen Band gehört eine Projektion P auf $L^2(M)$, welche faserweise durch die spektrale Projektion von $H_F(x)$ zu $\lambda(x)$ gegeben ist. Ist λ auf geeignete Weise isoliert vom Rest des Spektrums (siehe Kapitel 2, Voraussetzung 3), so ist λ stetig und auch P hängt stetig von x ab (siehe Lemma 2.9). Das Bild von P besteht dann aus L^2-Schnitten in einem Vektorbündel $\mathcal{E} \to B$ von endlichem Rang, dessen Faser über x genau der $\lambda(x)$-Eigenraum von $H_F(x)$ ist.

In Kapitel 2 wird untersucht inwiefern H durch den adiabatischen Operator $H_\mathrm{a} := PHP = -\varepsilon^2 P\Delta_h P + \lambda_0$ auf $L^2(\mathcal{E})$ approximiert werden kann. Eine solche Approximation kann noch verbessert werden durch die Konstruktion einer super-adiabischen Projektion P_ε, deren Differenz zu P von der Ordnung ε ist. Unter Beschränktheitsannahmen an die Geometrie von (M, g) (siehe Voraussetzung 1) und H_1 (siehe Voraussetzung 2) wird gezeigt, dass für geeignete Anfangsbedingungen die von H erzeugte unitäre Gruppe sehr gut durch einen effektiven Operator auf $L^2(\mathcal{E})$ beschrieben wird.

Theorem. *Es gelten die Voraussetzungen 1-3 für (M, g), H und das Eigenband λ. Dann existiert für alle $N \in \mathbb{N}$ und $\Lambda > 0$ eine Projektion P_ε, eine unitäre Abbildung U_ε mit $U_\varepsilon P = P_\varepsilon U_\varepsilon$ und Konstanten C, $\varepsilon_0 > 0$, so dass der effektive Operator $H_\mathrm{eff} := U_\varepsilon^* P_\varepsilon H P_\varepsilon U_\varepsilon$ selbstadjungiert auf $U_\varepsilon^* P_\varepsilon D(H) \subset L^2(\mathcal{E})$ ist und*

$$\left\| \left(\mathrm{e}^{-\mathrm{i}Ht} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_\mathrm{eff}t} U_\varepsilon^*\right) P_\varepsilon 1_{(-\infty,\Lambda]}(H) \right\|_{\mathscr{L}(\mathscr{H})} \leq C\varepsilon^{N+1}\, |t|$$

für alle $\varepsilon \leq \varepsilon_0$ gilt.

Des Weiteren werden bestimmte Teile des Spektrums von H durch den selben effektiven Operator approximiert.

Theorem. *Unter den Voraussetzungen des vorherigen Satzes existieren für jedes $\delta > 0$ Konstanten C und $\varepsilon_0 > 0$, so dass für jedes $\mu \in \sigma(H_\mathrm{eff})$ mit $\mu \leq \Lambda - \delta$ und alle $\varepsilon < \varepsilon_0$ gilt:*

$$\mathrm{dist}(\mu, \sigma(H)) \leq C\varepsilon^{N+1}\,.$$

Für den Spezialfall des Eigenbandes $\lambda_0(x) := \min \sigma(H_F(x))$ wird sogar das gesamte Spektrum von H unterhalb von

$$\Lambda_1 := \inf_{x \in B} \min \big(\sigma(H_F(x)) \setminus \lambda_0(x)\big)$$

durch H_{eff} approximiert, wie aus Theorem 2.20 folgt.

Theorem. *Sei $N \in \mathbb{N}$, $\Lambda = \Lambda_1$ und H_{eff} der entsprechende effektive Operator. Sei $-\varepsilon^2 \Delta_h + \varepsilon H_1 \geq -c\varepsilon$ für ein $c \geq 0$, dann gibt es für jedes $\delta > 0$ Konstanten C und $\varepsilon_0 > 0$, so dass für alle $\varepsilon < \varepsilon_0$ gilt*

$$\operatorname{dist}\big(\sigma(H) \cap (-\infty, \Lambda - \delta], \sigma(H_{\text{eff}}) \cap (-\infty, \Lambda - \delta]\big) \leq C\varepsilon^{N+1}\,.$$

In Kapitel 3 wird die allgemeine Theorie des vorherigen Kapitels anhand von Beispielen konkretisiert und verfeinert. Dadurch zeigen sich Bezüge zu einer reichhaltigen Literatur zu verschiedenen Aspekten des adiabatischen Limes, die in Abschnitt 1.2.1 ausführlicher dargestellt wird. Es wird gezeigt, dass bestimmte Arten von Einbettungen $M \hookrightarrow \mathbb{R}^k$ zu Operatoren führen, die alle in Kapitel 2 verwendeten Voraussetzungen erfüllen. Spezialfälle solcher Einbettungen wurden von einer Vielzahl von Autoren studiert, wie ebenfalls in Abschnitt 1.2.1 erläutert wird. Im Besonderen werden Laplace-Operatoren der von solchen Einbettungen induzierten Metriken, die dem Fall $V = 0$ entsprechen, behandelt. Das Augenmerk liegt dabei auf dem Eigenband λ_0 und der Asymptotik für niedrige Energien. Diese wird in besonderem Maße durch den adiabatischen Operator H_{a} bestimmt, wie aus der Analyse von P_ε folgt (vergleiche Abschnitt 3.2).

Theorem. *Sei $\Lambda_0 := \inf_{x \in B} \lambda_0(x)$ und $0 < \alpha \leq 2$. Für jedes $C > 0$ existiert $\varepsilon_0 > 0$, so dass für alle $\varepsilon \leq \varepsilon_0$*

$$\operatorname{dist}\big(\sigma(H) \cap (-\infty, \Lambda_0 + C\varepsilon^\alpha), \sigma(H_{\text{a}}) \cap (-\infty, \Lambda_0 + C\varepsilon^\alpha)\big) = \mathcal{O}(\varepsilon^{2+\alpha/2})\,.$$

Ist $\varepsilon^\alpha \lambda$ ein einfacher Eigenwert von H_{a} mit

$$\operatorname{dist}(\varepsilon^\alpha \lambda, \sigma(H_{\text{a}}) \setminus \varepsilon^\alpha) \geq C_\lambda > 0$$

dann gibt es einen einzigen einfachen Eigenwert $\varepsilon^\alpha \mu$ von H mit $|\mu - \lambda| = \mathcal{O}(\varepsilon^2)$.

Eine natürliche Skala für die Betrachtung niedriger Energien ist somit durch die typischen Abstände der Eigenwerte von H_a gegeben, falls diese existieren. Ist λ_0 konstant, so ist diese Skala durch $\alpha = 2$ gegeben. Hat hingegen λ_0 ein einziges quadratisches Minimum, so verhalten sich die niedrigen Eigenwerte von H_a wie die eines harmonischen Oszillators und die passende Skala ist $\alpha = 1$.

Der verbleibende Teil des Kapitels widmet sich der Untersuchung der Eigenfunktionen von einfachen Eigenwerten, welche die Bedingungen des letzten Theorems erfüllen, in den soeben beschriebenen Fällen $\alpha = 1, 2$. Zunächst wird bewiesen, dass diese in den Fällen $\alpha = 2$, $\dim B \leq 3$ und $\alpha = 1$, $\dim B = 1$ gleichmäßig durch Eigenfunktionen von H_a approximiert werden. Anschließend wird dieses Ergebnis benutzt um die Knotenmengen der Eigenfunktionen zu untersuchen. Diese sind für eine Eigenfunktion φ definiert durch $\mathcal{N}(\varphi) = \overline{\varphi^{-1}(0) \cap M \setminus \partial M}$. Das Studium dieser Mengen ist ein klassisches Problem in der Theorie partieller Differentialgleichungen. So bewies Courant (siehe [64]) eine obere Schranke and die Anzahl der Knotengebiete, den Zusammenhangskomponenten von $M \setminus \mathcal{N}(\varphi)$, und Melas zeigte für beschränkte, konvexe Gebiete $D \subset \mathbb{R}^2$, dass $\mathcal{N}(\varphi_1) \cap \partial D \neq \varnothing$ für die erste angeregte Schwingung φ_1 [52]. Quantitative Varianten des letzteren Resultats von Jerison [35] und Grieser-Jerison [26] beschreiben die Lage von $\mathcal{N}(\varphi)$ für konvexe Gebiete großer Exzentrizität. Das verwandte Resultat von Freitas und Krejčiřík [18] beschreibt diese für dünne Streifen. Eine ausführlichere Diskussion dieser Literatur findet sich in Abschnitt 1.2.1.

In Abschnitt 3.3.2 wird gezeigt, dass die Knotenmengen sich im Wesentlichen nahe der Fasern (beziehungsweise des Urbildes unter π) über den Knotenmengen der zugehörigen Eigenfunktionen von H_a befinden müssen (siehe Sätze 3.24 und 3.26). Aus diesen Resultaten lassen sich einige Korollare über die Struktur der Knotenmengen und Knotengebiete ableiten.

Korollar. *Sei λ_0 konstant, M kompakt und $\dim B \leq 3$. Es existiert ein ε-unabhängiger, selbstadjungierter Operator auf $W^2(B) \subset L^2(B)$ mit der folgenden Eigenschaft: Ist μ ein einfacher Eigenwert von H_0 mit normierter Eigenfunktion ψ, so existiert ein einfacher Eigenwert $\varepsilon^2\lambda$ von H. Falls Null ein regulärer Wert von ψ ist und $0 \neq \varphi \in \ker(H - \varepsilon^2\lambda)$ so konvergiert für $\varepsilon \to 0$ die Menge $\mathcal{N}(\varphi)$ gegen $\pi^{-1}(\mathcal{N}(\psi))$ in der Hausdorff-Distanz* *kompakter Teilmengen. Ist zudem* $\partial M \neq \varnothing$*, so ist* $\mathcal{N}(\varphi) \cap \partial M \neq \varnothing$*.*

Der Operator H_0 ist hierbei explizit bestimmbar als der führende Teil von H_a in einer Störungsentwicklung (siehe Seite 96). Für den Fall eines quadratischen Minimums von λ_0 und $\dim B = 1$ sind alle niedrigen Eigenwerte einfach und die Form der Eigenfunktionen von H_a weitgehend bekannt, was zusätzliche Information über die Knotengebiete liefert.

Korollar. *Das Eigenband* λ_0 *habe ein eindeutiges quadratisches Minimum und es sei* $\dim B = 1$*. Es gibt für jedes* $J \in \mathbb{N}$ *ein* $\varepsilon_0 > 0$ *so dass für* $\varepsilon \leq \varepsilon_0$ *der Operator* H *mindestens* $J + 1$ *einfache Eigenwerte unterhalb des wesentlichen Spektrums besitzt. Die zugehörigen Eigenfunktionen* $\{\varphi_j : 0 \leq j \leq J\}$ *seien nach aufsteigenden Eigenwerten geordnet, dann ist für jedes* $1 < j \leq J$*:* $\mathcal{N}(\varphi_j) \cap \partial M \neq \varnothing$ *und für alle ungeraden* j *hat* φ_j *genau* $j + 1$ *Knotengebiete.*

Die Anzahl der Knotengebiete für ungerades j ist genau die obere Schranke aus Courants Theorem. Diese wird also zumindest für jede zweite der Eigenfunktion ϕ_j mit $j \leq J$ des Laplace-Operators auf (M, g_ε) angenommen. Für ein gegebenes Faserbündel über $\mathbb{R}$ oder S^1 mit kompakten, berandeten Fasern lassen sich stets Metriken angeben, welche die Voraussetzungen dieses Korollars erfüllen. Es gibt also eine große Menge von Beispielen in denen die Eigenfunktionen die dort genannten Eigenschaften haben.

Chapter 1

Introduction

In this thesis we study Schrödinger operators on 'thin' manifolds. These are spaces in which certain dimensions are of magnitude $\varepsilon \ll 1$ compared to others. The goal will be the derivation of asymptotics for various properties of the Schrödinger operator

$$H := -\Delta + V \,,$$

such as its spectrum and the dynamics it generates.

Problems of this type arise in different contexts in mathematics and physics. For example, models for the quantum mechanical motion of a particle confined to a small ε-neighbourhood of some submanifold of $\mathbb{R}^3$ take this form. More generally one may think of a submanifold in $\mathbb{R}^k$, that could represent configurations of nuclei corresponding to different molecules. The system is constrained to an ε-neighbourhood of this submanifold by the electronic forces that bind the molecules. Understanding the dynamics in these situations is the problem of constrained quantum mechanics. This problem was studied by purely modelling the constraining forces near the submanifold, rather than restricting to a small neighbourhood, by various authors [11, 20, 50, 53, 69]. We give a more detailed discussion of how methods and results of these works relate to ours in section 1.2.1.

A similar picture is of interest for the heat equation. The Laplacian with Dirichlet conditions on the boundary of a thin tube around a submanifold models the conduction of heat in this thin wire in an environment of fixed temperature. The relation of the heat equation in the wire to an effective equation on the lower dimensional submanifold was studied for one-dimensional submanifolds of Riemannian manifolds by Wittich [71].

For neighbourhoods of geodesics in surfaces the long-time behaviour was investigated by Krejčiřík and Kolb [40].

On the other hand, the spectral properties of the Laplacian on Riemannian manifolds, the case $V = 0$ for the Schrödinger operator, and their connection to the geometry is in itself an interesting subject. In this field the investigation of special classes, like thin manifolds, may often serve to construct examples with specific properties. For example one can find instances of non-compact manifolds with boundary for which the Laplacian has eigenvalues below the essential spectrum. This was demonstrated in [7, 9, 10, 16, 21, 24, 47] (and references therein) for a plethora of different geometries. All of these geometries have in common that they are built as some sort of tubular neighbourhood of a submanifold in $\mathbb{R}^k$, on the boundary of which Dirichlet conditions are imposed. In [43] a similar result is obtained for a curve on a surface. These quantum tubes may also represent constrained quantum systems, for which the results show existence of bound states. An important step in proving existence of eigenvalues below the essential spectrum is to show its stability under perturbations of the geometry, which is separately studied in [44]. The question whether the essential spectrum can be empty for non-compact manifolds was also studied for different geometries without reference to embeddings. In [1, 3, 5, 38] the authors give criteria for the discreteness of the spectrum on manifolds with a structure similar to the one we will consider.

Given that there are eigenvalues, which is always the case for compact manifolds, it is natural to ask about their asymptotic behaviour in ε. This was studied in [6, 14, 15, 16, 17, 19, 29, 42] (and references therein) for thin tubes. Finally it is possible to approximate eigenfunctions and obtain information on their set of zeros, the nodal set, as done in [18, 26, 36, 35]. We will give some perspective on these results in section 1.2.1, after introducing the necessary language.

We should also mention that there are many results for similar problems that will not be treated in this work. These include results for tubular neighbourhoods of graphs, boundary conditions of Neumann or Robin type and magnetic Laplacians (or non-trivial connections on a line bundle over the manifold). The literature on these subjects is covered in the review by Grieser [25] and the book of Post [59]. Foliated manifolds are discussed by Kordyukov [41]. There are also results concerning the limit of the Hodge Laplacian on differential forms [48, 49, 51] and Dirac operators (see [4, 12, 23] and references therein). The focus of these works are topological invariants and we will not obtain any results in this direction. Nevertheless we expect that the formulation of our results in a natural geometric language should lend itself to such generalisations.

The aim of this thesis will be to study the underlying principle of the approximations made in basically all the works on this subject. We aim at identifying natural conditions on the operator and the geometry for its validity and presenting our results in a form that provides insight into the many different aspects of this asymptotic limit.

This principle is the fact that a thin tube or manifold is an adiabatic problem. The different scale of the small dimensions compared to the remaining ones can be expressed by saying that objects associated with the small dimensions, like a Riemannian metric or an eigenfunction of the Laplacian, vary slowly when moving along the other directions. In the 'limit' $\varepsilon \to 0$ they do not vary at all, becoming adiabatic invariants. This can be exploited, say in a spectral problem, by using trial functions that are of a fixed form in the small dimensions. One is then left with a problem that mainly sees the remaining dimensions as well as some simplified effect of the presence of the small ones. This leads to the concept of an effective operator representing the original problem in a simplified way. Such an operator has the potential of capturing many different features of the original problem, including its spectrum and dynamics, so this is the formulation we will use.

We will not try to emulate or generalise all the details of the previous literature, but we will see that some of the reasoning behind these results may be applied to a very wide class of problems. Better understanding of the underlying structure of the arguments may also prove fruitful for those related problems we do not consider here.

1.1 The framework

In this section we give a short introduction to the framework we will work in. We begin by discussing fibre bundles and Riemannian submersions as well as their Laplacians. We then explain how certain shrinking families of embedded fibre bundles carry induced metrics that are close to Riemannian submersions. These metrics provide an important class of examples for the general theory we will develop in chapter 2. Finally we review the principles of adiabatic perturbation theory.

The reader familiar with these concepts may want to skip directly to the summary of the results and their relation to the existing literature in section 1.2.

1.1.1 Riemannian submersions

The space M on which all our analysis will take place is a smooth manifold with boundary that has the structure of a fibre bundle.

Definition 1.1. Let M, B be smooth manifolds (with boundary or not). A smooth map $\pi\colon M \to B$ is a *fibre bundle* over B with fibre F if π is onto and every x in B has an open neighbourhood U for which $\pi^{-1}(U)$ is diffeomorphic to $U \times F$ and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

commutes.

The fibre space $F \cong F_x := \pi^{-1}(x)$ will always be a compact manifold with smooth boundary of dimension n. The base space B need not be compact, its dimension will be called d. Of course the boundary of F might be empty, but in any case B shall not have a boundary. We will consider special Riemannian metrics on M and scale them in such a way that F is small compared to B.

Definition 1.2. Let g, g_B be Riemannian metrics on M and B respectively. A fibre bundle $\pi\colon M \to B$ is a *Riemannian submersion* if the differential π_* induces an isometry $TM/\ker\pi_* \to TB$.

In the usual terminology, a submersion would only require the differential of π to have maximal rank. Hermann [33] showed that if M is geodesically complete and π_* induces an isometry this already implies that $\pi\colon M \to B$ is a fibre bundle. Since we also want to consider manifolds with boundary we will have to explicitly add this assumption.

We now discuss some of the basic properties of Riemannian submersions. For a more detailed exposition see O'Neill [55] and the book of Lang [45]. Since π_* has maximal rank it splits the tangent bundle of M

$$TM = \ker \pi_* \oplus (\ker \pi_*)^\perp =: TF \oplus NF\,.$$

The vectors in the kernel of π_* are those tangent to the fibres. Their span TF is the *vertical* subbundle of TM. Its orthogonal complement with respect to g is the *horizontal* subbundle NF. It is isomorphic to π^*TB for any metric g. If $\pi\colon M \to B$ is a Riemannian submersion the metric can be written as

$$g = g_F + \pi^* g_B\,,$$

with $g_F(v, \cdot) = 0$ for any $v \in NF$. This means that lengths of horizontal vectors depend only on their projections. The fibre metric g_F is just the restriction of g to the fibres, but note that this may differ for fibres over distinct points of B. A metric g of this form is uniquely determined by the vertical part g_F, the metric on the base g_B and the horizontal subbundle NF.

Example 1.3.

- For any two Riemannian manifolds (F, g_F), (B, g_B) the Cartesian product with the natural product metric $(B \times F, g_B + g_F)$ is a Riemannian submersion. Given a smooth and positive function $f\colon B \to \mathbb{R}$ the *warped product* is the Riemannian manifold $(B \times F, g_B + f g_F)$. This is a Riemannian submersion with the projection to the first factor, but not to the second.

- Let F be a compact Lie group and M a smooth manifold (without boundary) with a smooth and free right action $\varrho\colon M \times F \to M$ of F. It can be shown (see [46]) that the space of F-orbits $M/F =: B$ is a smooth manifold and the natural projection $\pi\colon M \to B$ makes M a fibre bundle with fibres diffeomorphic to F. This is an *F-principal bundle* over B.

Let $\mathfrak{f}$ denote the Lie algebra of F. For every $\varphi \in \mathfrak{f}$ the map

$$M \to TM\,, \qquad y \mapsto X^\varphi := \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varrho\big(y, \exp(t\varphi)\big)$$

defines a vector field on M tangent to the fibres $X^\varphi \in \Gamma(TF)$. Let Ad be the adjoint representation of F on its Lie algebra. A one-form $\omega \in \Gamma(T^*M) \otimes \mathfrak{f}$ is called a *principal connection* on M if it is F-equivariant $(\varrho(\cdot, f)^*\omega = \mathrm{Ad}_{f^{-1}}\omega)$ and it satisfies $\omega(X^\varphi) = \varphi$ for every $\varphi \in \mathfrak{f}$. One easily checks that the second condition implies $TM = \ker\omega \oplus \ker\pi_*$. Given such a connection, a smooth map $g_F\colon B \to \mathfrak{f}^* \otimes \mathfrak{f}^*$ whose image is symmetric and positive definite for every $x \in B$, and a Riemannian metric g_B on B we can define a Riemannian metric on M by setting

$$g\big(X_y, X_y\big) := g_F|_{\pi(y)}(\omega(X_y), \omega(X_y)\big) + g_B(\pi_* X_y, \pi_* X_y\big)\,,$$

for $y \in M$ and $X \in \Gamma(TM)$. Any metric constructed in this way is a Riemannian submersion with $NF = \ker\omega$.

A vector field $\tilde{X} \in \Gamma(TM)$ is a *lift* of $X \in \Gamma(TB)$ if $\pi_*\tilde{X} = X$.

Lemma 1.4. *Let $X \in \Gamma(TB)$, $\tilde{X}$ a lift of X and $Y \in \Gamma(TM)$.*

1) If $\hat{X}$ is also a lift of X then $\tilde{X} - \hat{X}$ is vertical.

2) X has a unique horizontal lift X^*.

3) $\pi_*[\tilde{X}, Y] = [X, \pi_* Y]$.

4) If Y is vertical then so is $[\tilde{X}, Y]$.

Proof. The first claim follows immediately from $\pi_*(\tilde{X} - \hat{X}) = 0$ and $TF = \ker\pi_*$.

Given a metric, a horizontal lift may be constructed from $\tilde{X}$ by subtracting its projection to TF. For two distinct horizontal lifts the difference must be vertical by *1)*, but then it must be zero since the projections of both fields to TF vanish.

If $\partial M = \varnothing$ the third claim is a consequence of the fact that the flow of $\tilde{X}$ lifts the flow of X. On the other hand if the flow of $\tilde{X}$ does not exist for positive times in a point $p \in \partial M$ then the flow of $-\tilde{X}$ must exist in p and $[\tilde{X}, Y]_p$ is defined as the negative of the Lie derivative of Y along $-\tilde{X}$. The claim then follows from linearity of π_*.

The last statement is an immediate consequence of the third. □

For two fields $X, Y \in \Gamma(TB)$ the quantity

$$\Omega(X,Y) := [X^*, Y^*] - [X,Y]^*$$

is a vertical vector field. The tensor Ω is the *integrability tensor* of NF. Its vanishing implies the existence of submanifolds of M tangent to NF, hence integrability, by Frobenius' theorem.

Lemma 1.5. *Let $X, Y \in \Gamma(TB)$ and let ∇ denote the Levi-Cività connections of g and g_B where appropriate, then*

$$\nabla_{X^*} Y^* = (\nabla_X Y)^* + \tfrac{1}{2}\Omega(X,Y)\,.$$

Proof. This follows directly from the Koszul formula, see [55] for details. □

Now consider the rescaled metric

$$g_\varepsilon := g_F + \varepsilon^{-2}\pi^* g_B\,. \tag{1.1}$$

In this scaling F is not of size ε, but it is small compared to B, where lengths grow as ε^{-1}. The equivalent scaling $\varepsilon^2 g_F + \pi^* g_B$ makes the limit $\varepsilon \to 0$ appear more singular and we will not consider it here.

It is instructive to calculate the explicit ε-dependence on of some fundamental geometric quantities associated with g_ε.

Lemma 1.6. *Let η, η_ε be the mean curvature vector of the fibres with respect to g and g_ε. Then $\eta_\varepsilon = \varepsilon^2 \eta$.*

Proof. Let $U \subset M$ be open and chosen so that there are orthonormal frames $(Y_i)_{i\leq n}$ of $TF|_U$ and $(X_j)_{j\leq d}$ of $TB|_{\pi(U)}$. Then $(\varepsilon X_j^*)_{j\leq d}$ is orthonormal with respect to g_ε and by the Koszul formula

$$\begin{aligned}
\eta_\varepsilon &= \sum_{i,j} g_\varepsilon(\nabla^\varepsilon_{Y_i} Y_i, \varepsilon X_j^*)\varepsilon X_j^* \\
&= \varepsilon^2 \sum_{i,j} \tfrac{1}{2}\Big(2Y_i \underbrace{g_\varepsilon(Y_i, X_j^*)}_{=0} - \underbrace{X_j^* g_\varepsilon(Y_i, Y_i)}_{=0} + g_\varepsilon(X_j^*, \underbrace{[Y_i, Y_i]}_{=0}) \\
&\qquad\qquad + 2g_\varepsilon(Y_i, [X_j^*, Y_i])\Big) X_j^* \\
&= \varepsilon^2 \sum_{i,j} g_F(Y_i, [X_j^*, Y_i]) X_j^* = \varepsilon^2 \eta\,.
\end{aligned}$$

□

When $\varepsilon \to 0$ these quantities converge to those of the product $(F_x, g_{F_x}) \times (B, g_B)$ for every x, so M looks increasingly 'straight'.

1.1.2 The Laplacian

Let Δ_{g_ε} be the Laplace-Beltrami operator of the rescaled submersion metric g_ε. We calculate the splitting of this operator into horizontal and vertical parts as well as its dependence on ε following Lang [45].

Choose a local orthonormal frame of TM consisting of vertical fields $(Y_i)_{i\leq n}$ and horizontal lifts $(\varepsilon X_j^*)_{j\leq d}$ (the X_j have g_B-length one). We then have

$$\begin{aligned}
\Delta_{g_\varepsilon} = \operatorname{tr}_{g_\varepsilon} \nabla^2 &= \sum_{i\leq n} Y_i \circ Y_i - \nabla_{Y_i} Y_i + \varepsilon^2 \sum_{j\leq d} X_j^* \circ X_j^* - \nabla_{X_j^*} X_j^* \\
&= \sum_{i\leq n} Y_i \circ Y_i - \nabla^F_{Y_i} Y_i - \mathrm{II}\,(Y_i, Y_i) + \varepsilon^2 \sum_{j\leq d} X_j^* \circ X_j^* - (\nabla_{X_j} X_j)^* \\
&= \operatorname{tr}_{g_F} (\nabla^F)^2 + \varepsilon^2 (\operatorname{tr}_{\pi^* g_B} \nabla^2 - \eta) \\
&=: \Delta_F + \varepsilon^2 \Delta_h\,. \qquad (1.2)
\end{aligned}$$

Here η is the mean curvature vector of the fibres with $\varepsilon = 1$ as in lemma 1.6. The fibre Laplacian Δ_F is clearly just the Laplacian of the restricted metric on each fibre (F_x, g_{F_x}). If $\iota_x : F \to F_x$ is an embedding, it satisfies $\iota_x^* \Delta_F f = \Delta_{g_{F_x}} \iota_x^* f$.

The first term of the horizontal operator is a lift of the Laplacian on B in the sense that $\mathrm{tr}_{\pi^* g_B} \nabla^2(\pi^* f) = \pi^* \Delta_{g_B} f$. This is because $\nabla_{X_j^*} X_j^* = (\nabla_{X_j} X_j)^*$ by 1.5.

Example 1.7. Using spherical polar coordinates we have $\mathbb{R}^3 \setminus \{0\} \cong (0,\infty) \times S^2$. This fibre bundle over $(0,\infty)$ is a Riemannian submersion for the Euclidean metrics on $\mathbb{R}^3$ and $(0,\infty)$. The Laplacian in these coordinates is of course

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2}\Delta_{S^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\,,$$

while the mean curvature vector of the sphere of radius r is exactly $\eta = -\frac{2}{r}\partial_r$.

Remark 1.8. This formula for Δ_{g_ε} may also be obtained using integration by parts. Take $f \in \mathscr{C}_0^\infty(M \setminus \partial M)$ and let vol be the volume density of g_ε, then

$$\begin{aligned} -\int_M f \Delta_{g_\varepsilon} f \,\mathrm{vol} &= \int_M g_\varepsilon(\mathrm{d}f, \mathrm{d}f)\,\mathrm{vol} \\ &= \int_M g_F(\mathrm{d}f, \mathrm{d}f) + \varepsilon^2 \pi^* g_B(\mathrm{d}f, \mathrm{d}f)\,\mathrm{vol} \\ &=: -\int_M f(\Delta_F + \varepsilon^2 \Delta_h) f \,\mathrm{vol}\,. \end{aligned}$$

Noting that locally $\mathrm{vol} = \mathrm{vol}(g_{F_x}) \otimes \pi^* \mathrm{vol}(\varepsilon^{-2} g_B)$, and that the horizontal derivative of $\mathrm{vol}(g_{F_x})$ equals $-\eta$ by the variation of area formula, we obtain the same expressions for Δ_F and Δ_h, in agreement with the general formula.

In absence of a boundary the horizontal and vertical parts of the Laplacian commute if and only if the fibres are totally geodesic (see [8]). Given a complete horizontal vector field X one can calculate the Lie derivative

$\mathcal{L}_X \Delta_F$ to be

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{F_x} f\big((\Phi_X^{t*}\Delta_F)f\big)\,\mathrm{vol}_{F_x} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{F_x} f \Delta_{\Phi_X^{t*} g_F} f\,\mathrm{vol}_{F_x} \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{F_x} (\Phi_X^{t*} g_F)(\mathrm{d}f, \mathrm{d}f)\,\mathrm{vol}_{F_x} \\
&= -2 \int_{F_x} g_F(\mathrm{II}\,(\operatorname{grad} f, \operatorname{grad} f)\,, X)\,\mathrm{vol}_{F_x}\,. \qquad (1.3)
\end{aligned}$$

The last step holds because

$$\begin{aligned}
(\mathcal{L}_X g_F)(\mathrm{d}f, \mathrm{d}f) &= X g_F(\mathrm{d}f, \mathrm{d}f) - 2 g_F(\mathcal{L}_X \mathrm{d}f, \mathrm{d}f) \\
&= -X g_F(\operatorname{grad} f, \operatorname{grad} f) + 2 g_F(\operatorname{grad} f, [X, \operatorname{grad} f])\,,
\end{aligned}$$

which equals $2g_F(\nabla_{\operatorname{grad} f} \operatorname{grad} f, X) = 2g(\mathrm{II}\,(\operatorname{grad} f, \operatorname{grad} f)\,, X)$ by the Koszul formula.

Here we can note that if we take the Lie derivative along εX, which has g_ε-length of order one, the operator equals ε-times that for $\varepsilon = 1$. In this sense the fibre Laplacian is slowly varying in the horizontal direction in the metric g_ε.

In the presence of a boundary one needs to be more careful. For once the flow of X might not exist, and even if it does the meaning of $\mathcal{L}_X \Delta_F$ depends on the space of functions it is defined on (see example 2.2).

1.1.3 Embeddings and their induced metrics

Often M is given as an ε-neighbourhood of a submanifold of $\mathbb{R}^k$ rather than an abstract fibre bundle. Here we sketch how such a situation may be treated within our framework. The precise technical conditions are discussed in section 3.1.1 in a more general situation. A more detailed exposition will appear in [30].

The key observation is that the induced metric on a sufficiently 'thin' embedded fibre bundle is almost a Riemannian submersion. Its Laplacian can thus be treated as a perturbation of $\Delta_{g\varepsilon}$.

Let $\alpha\colon B \to \mathbb{R}^k$ be an embedding and NB the normal bundle. Let M be a fibre bundle over B as described above and $\beta\colon M \to NB$ an embedding that respects the projections, making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\beta} & NB \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{NB}} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

commute. We get a family of maps $\Psi_\varepsilon \colon M \to \mathbb{R}^k$ by identifying $\mathbb{R}^k$ with its tangent space and setting

$$\Psi_\varepsilon(x) := \alpha(\pi(x)) + \varepsilon\beta(x)\,.$$

In the setting we consider, we may assume without loss of generality that these maps are embeddings for $\varepsilon \leq 1$. With this construction both M and B naturally obtain metrics $G_\varepsilon := \varepsilon^{-2}\Psi_\varepsilon^*\delta$ and $g_B := \alpha^*\delta$, induced by the Euclidean metric δ on $\mathbb{R}^k$. In general $\pi\colon (M, G_\varepsilon) \to (B, g_B)$ will not be a Riemannian submersion.

The expression for G_ε can be derived in two steps. The first consists in calculating the metric on (an open subset of) NB induced by the map $\nu \mapsto \alpha(\pi(\nu)) + \nu$. By identification of $T_\nu(NB_x)$, the vertical subspace of TNB_x, with NB_x, the differential in the vertical direction is just the identity. Thus the vertical metric on NB is flat. To calculate the horizontal part we need to calculate the derivative of $\alpha(\pi(x)) + \nu$ in directions tangent to B, which yields $\alpha_* + \nabla\nu$. The projection of $\nabla\nu$ to TB equals the negative of the Weingarten map W, while the projection to NB is vertical. The horizontal part of the induced metric on NB is then given by

$$\tilde{G}(X^h, Y^h) = g_B\big((1 - W(\nu))X, (1 - W(\nu))Y\big)\,,$$

for horizontal lifts X^h, Y^h of $X, Y \in \Gamma(TB)$. The leading part is the *Sasaki metric* defined by g_B, δ and the given horizontal lift. It makes NB a Riemannian submersion with totally geodesic fibres.

Now if the codimension of M in $\mathbb{R}^k$ is zero, F is given by a compact domain of $\mathbb{R}^n$ and restricting G to $\beta(M)$ is trivial. The complete, rescaled metric is thus

$$G_\varepsilon = \varepsilon^{-2}\pi^* g_B\big((1 - \varepsilon\beta^* W)\,\cdot\,, (1 - \varepsilon\beta^* W)\,\cdot\,\big) + \delta\,.$$

In higher codimension similar arguments may be applied to β_*, see section 3.1.1. Again the metric takes the form

$$G_\varepsilon = g_\varepsilon + \mathcal{O}(\varepsilon)\,,$$

where g_ε is of the form (1.1) with $g_B = \alpha^*\delta$. The remainder is of order ε with respect to this metric, although it contains horizontal terms with a prefactor ε^{-1}, since it satisfies

$$|(G_\varepsilon - g_\varepsilon)(X,X)| \leq C\varepsilon g_\varepsilon(X,X) \tag{1.4}$$

for some constant $C > 0$. Because of its simple form we will try to use only geometric quantities related to the metric g_ε, while dealing with the remaining terms only in form of differential operators that perturb the Laplacian. The Dirichlet Laplacian of G_ε is given by the quadratic form

$$-\int_M f\Delta_{G_\varepsilon} f \,\mathrm{vol}_{G_\varepsilon} = \int G_\varepsilon(\mathrm{d}f,\mathrm{d}f)\,\mathrm{vol}_{G_\varepsilon}\,,$$

for smooth functions f that vanish on ∂M. In a first step we substitute the volume measure of G_ε by that of g_ε by applying the unitary transformation

$$U_\rho\colon L^2(M,\mathrm{vol}_{g_\varepsilon}) \to L^2(M,\mathrm{vol}_{G_\varepsilon}) \qquad f \mapsto \rho^{-1/2} f\,,$$

where ρ is the density $\rho := \mathrm{vol}_{G_\varepsilon}/\mathrm{vol}_{g_\varepsilon}$. This gives (see [69, 71] for details)

$$-\int_M fU_\rho^*\Delta_{G_\varepsilon}U_\rho f\,\mathrm{vol}_{g_\varepsilon} = \int_M G_\varepsilon(\mathrm{d}f,\mathrm{d}f) + V_\rho f^2\,\mathrm{vol}_{g_\varepsilon}$$

with

$$V_\rho = \tfrac{1}{4}G_\varepsilon(\mathrm{d}\log\rho,\mathrm{d}\log\rho) + \tfrac{1}{2}\,\mathrm{div}_{g_\varepsilon}\,\mathrm{grad}_{G_\varepsilon}\log\rho\,. \tag{1.5}$$

The potential V_ρ is often called the *geometric potential*, since it captures the geometry of the embedding of M into $\mathbb{R}^k$.

Example 1.9. Let B have dimension one, hence $B = \mathbb{R}$ or $B = S^1$, and be parametrised by arc length s. Let M have codimension zero and let $\kappa(s) = |\partial_s^2\alpha|$ denote the 'curvature'. Then a simple calculation gives

$$G_\varepsilon = \varepsilon^{-2}(1+\varepsilon\nu\kappa(s))^2\mathrm{d}s^2 + \mathrm{d}\nu^2$$

and consequently

$$\rho = \big(1 + \varepsilon\nu\kappa(s)\big)\,.$$

Then $\partial_\nu \log\rho = \varepsilon\kappa\rho^{-1}$ and $\partial_s \log\rho = \varepsilon(\partial_s\kappa)\nu\rho^{-1}$ and the potential is given by

$$\begin{aligned} V_\rho &= \tfrac{1}{4}(\partial_\nu \log\rho)^2 + \tfrac{1}{2}\partial_\nu^2 \log\rho + \tfrac{1}{4}\varepsilon^2\rho^{-2}(\partial_s \log\rho)^2 + \tfrac{1}{2}\varepsilon^2\partial_s\rho^{-2}\partial_s \log\rho \\ &= \rho^{-2}\big(-\tfrac{1}{4}\varepsilon^2\kappa^2 - \tfrac{5}{4}\varepsilon^2\rho^{-2}(\partial_s\rho)^2 + \tfrac{1}{2}\varepsilon^2\rho^{-1}\partial_s^2\rho\big) \\ &= -\tfrac{1}{4}\varepsilon^2\kappa^2 + \mathcal{O}(\varepsilon^3)\,. \end{aligned} \tag{1.6}$$

Take note that the leading contribution comes from the vertical derivatives only, since the horizontal derivatives all carry an additional prefactor ε from the metric.

If the codimension of M is zero we always have $V_\rho = \mathcal{O}(\varepsilon^2)$, while in general the expansion of G_ε only implies $V_\rho = \mathcal{O}(\varepsilon)$ (see also remark 3.5). Denote by $\varepsilon\tilde{G}_\varepsilon := G_\varepsilon - g_\varepsilon \in \Gamma(TM \otimes TM)$, then we get the formula

$$-U_\rho^*\Delta_{G_\varepsilon}U_\rho f = -\Delta_{g_\varepsilon} f - \varepsilon\,\mathrm{div}_{g_\varepsilon}\big(\tilde{G}_\varepsilon(\mathrm{d}f,\cdot)\big) + V_\rho f\,. \tag{1.7}$$

Note that $\mathrm{div}_{g_\varepsilon}\big(\tilde{G}_\varepsilon(\mathrm{d}f,\cdot)\big)$ is a differential operator that is bounded by Δ_{g_ε} because of the bound (1.4) on the coefficients. It is the operator $-U_\rho^*\Delta_{G_\varepsilon}U_\rho$ that we will analyse since it highlights the important contribution Δ_{g_ε}.

1.1.4 Fundamentals of the adiabatic approximation

Take the negative Laplacian $-\Delta_{g_\varepsilon}$ of a scaled family (1.1) of Riemannian submersions. It is an essentially self-adjoint non-negative operator on those smooth functions that vanish on ∂M. For fixed $x \in B$ the fibre operator $-\Delta_{F_x}$ has eigenvalues $0 \le \lambda_0(x) < \lambda_1(x) < \ldots$ of finite multiplicity accumulating at infinity. The operator $-\Delta_F$ can thus be expressed as a multiplication operator with the functions $\lambda_j(x)$, the *eigenbands*. These come with projections P, which are given for fixed λ and x by the projection onto the eigenspace of $-\Delta_{F_x}$ with eigenvalue $\lambda(x)$, hence satisfying

$$-\Delta_{F_x}P = \lambda(x)\,.$$

The adiabatic approximation (with respect to λ) consists in projecting to the image of P and considering the operator

$$H_{\mathrm{a}} := -P\Delta_{g_\varepsilon}P = -\varepsilon^2 P\Delta_h P + \lambda(x)\,.$$

Showing validity of this approximation amounts to proving that

$$-\Delta_{g_\varepsilon} - H_{\mathrm{a}} = \mathcal{O}(\varepsilon)\,,$$

in a suitable sense. Actually since $H_a(1-P) = 0$ such an estimate cannot be true in general but only on, or close to, the image of P. On the image of P the equation can equivalently be written as

$$(-\Delta_{g_\varepsilon} - H_{\mathrm{a}})P = -\Delta_{g_\varepsilon}P + P\Delta_{g_\varepsilon}P = [-\varepsilon^2\Delta_h, P]P = \mathcal{O}(\varepsilon)\,.$$

If λ is separated from the other bands (see chapter 2, condition 3), P is continuous in x (see lemma 2.9) and its image consists of sections of a vector bundle $\mathcal{E} \to B$, whose fibres are exactly the $\lambda(x)$-eigenspaces of $-\Delta_{F_x}$. In these circumstances we can locally express the projection as $P(x) = 1_{(\lambda-\delta,\lambda+\delta)}(\Delta_{F_x})$. The term we want to estimate depends on the commutator of P with vector fields of the form εX^*. This commutator may be written as a Lie derivative along εX^* and since $P(x)$ is a function of Δ_{F_x} its boundedness depends on that of (1.3).

If all the eigenbands of H_F are separated from each other, this procedure may be applied to every one of them. This leads to a total decomposition of the problem into separate equations for each band λ_j.

1.2 Overview of results

The object we analyse in this thesis is the Schrödinger operator

$$H := -\Delta_{g_\varepsilon} + \varepsilon H_1 + V$$

on a fibre bundle $F \to M \xrightarrow{\pi} B$ with a rescaled family of metrics g_ε (1.1). The potential V is assumed to be smooth and bounded and the correction H_1 is a smooth and symmetric differential operator relatively bounded by Δ_{g_ε} (see chapter 2, condition 2). In particular the operators (1.7) arising from a shrinking family of embeddings of M are of this form. If M has a boundary we impose Dirichlet boundary conditions, so the domain of H is the space

$$D(H) = \{\psi \in W^2(M) : \psi|_{\partial M} = 0\} := W^2(M) \cap W^1_0(M) \subset L^2(M, g_\varepsilon)\,.$$

For a separated eigenband λ of the fibre operator

$$H_F := -\Delta_F + V$$

we show validity of the adiabatic approximation under reasonable boundedness assumptions on the geometry of M (see chapter 2, condition 1). We improve this approximation by constructing a modified super-adiabatic projection P_ε that is close to the original P. In this way we obtain for any $N \in \mathbb{N}$ an effective operator H_{eff}, densely defined on $L^2(\mathcal{E})$, that is almost unitarily equivalent to H on the correspondent subspace:

$$(UH_{\mathrm{eff}}U^* - H)P_\varepsilon = \mathcal{O}(\varepsilon^N)\,.$$

This holds true for bounded energies of H. Subsequently we show approximation of the unitary group, proving in theorem 2.17 that

$$(\mathrm{e}^{-\mathrm{i}Ht} - U\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t}U^*)P_\varepsilon = \mathcal{O}(\varepsilon^N)$$

holds for bounded times and energies. Concerning the spectrum of H, we see in theorem 2.18 that for any $\mu \in \sigma(H_{\mathrm{eff}})$ there is $\tilde{\mu} \in \sigma(H)$ with $|\mu - \tilde{\mu}| = \mathcal{O}(\varepsilon^N)$. Since $\mathcal{E}$ is a vector bundle of finite rank over B this may greatly reduce the number of dimensions relevant to the solution of the problem. While the original problem is a differential equation in $n+d$ dimensions, the effective equation is a finite system of equations in only d dimensions.

The improvement on the adiabatic approximation is important for different reasons. On the one hand the scaling makes the base grow as $\varepsilon \to 0$, so the eigenvalues of both H and H_{eff} will tend to accumulate. Hence to make meaningful statements about the spectrum, the approximation needs to be better than the rate at which they converge. Additionally the growth of B means that dynamical effects take increasingly long times, $t \approx \varepsilon^{-1}$, to manifest themselves. On the other hand it is of general interest to see whether the approximation breaks down beyond a certain point.

The knowledge that it can accommodate all orders gives insight into the true nature of the errors. We exploit this in section 3.2, where we use the existence of the super-adiabatic projections to improve error bounds for the adiabatic approximation.

So far the eigenband λ on which the construction is based was arbitrary, under the sole condition of being separated from the rest of the spectrum of H_F. If one specialises to the ground state band $\lambda_0(x) := \min \sigma(H_F(x))$, and energies below the infimum of the next eigenband of H_F, more detailed results can be obtained. Since at these energies only one eigenband is relevant there is no need to project to the image of P_ε and the approximation is valid in the form

$$UH_{\text{eff}}U^* - H = \mathcal{O}(\varepsilon^N)\,,$$

as proved in theorem 2.20. Thus the correspondence of spectra between H and H_{eff} is reciprocal. This is in contrast to the situation at higher energies, where an eigenvalue μ of H might be associated with different eigenbands. Hence in that case the spectrum of H_{eff} approximates that of H, but not the other way around.

In section 3.2 we further analyse the adiabatic approximation at energies close to the bottom of the spectrum of H. In this regime the behaviour is dominated by that of the ground state band λ_0. If λ_0 is constant, all the terms of H_a carry the prefactor ε^2, so this is the characteristic scale at small energies. On this scale contributions by certain corrections in H_{eff}, like the geometric potential (1.5), are no longer small and determine the properties of H_a at leading order. If on the other hand λ has a unique, non-degenerate minimum, the leading part of H_{eff} resembles an harmonic oscillator. From this follows existence of eigenvalues with spacing of order ε close to the ground state. More generally we investigate energies of order ε^α above $\inf_{x\in B}\lambda_0(x)$ and show that the spectra of H and H_a approximate each other with an error of order $\varepsilon^{2+\alpha/2}$ (proposition 3.11). If there are simple eigenvalues with spacing of order ε^α they are even approximated to order $\varepsilon^{2+\alpha}$ (theorem 3.12).

At these low energies we also show approximation of eigenfunctions. In section 3.3 we apply this in special situations with low-dimensional base to study eigenfunctions of H in more detail. In particular we locate the regions where they may change sign, gaining information on their nodal sets in section 3.3.2. This relates to possible generalisations of the nodal line conjecture to manifolds with boundary and provides examples of manifolds where a large number of eigenfunctions have the maximal number of nodal domains (see also the discussion on page 33).

1.2.1 Comparison to existing literature

Here we give a brief review of existing literature on the topic and its relation to our results. Since the focus of these works is on diverse aspects of the adiabatic problem we will discuss some of these aspects separately.

Geometries

In the existing literature the problem is often formulated starting from some sort of tubular neighbourhood of an embedded submanifold of $\mathbb{R}^k$. This leads to a fibre bundle M with base diffeomorphic to that submanifold, as described in section 1.1.3.

The most commonly treated case is $B = \mathbb{R}$ or $B = I \subset \mathbb{R}$ an interval [6, 7, 10, 14, 15, 16, 19, 21, 24], which is usually referred to as a quantum waveguide. The fibre of such a tube is a compact domain whose dimension is the codimension of B. Topologically M is the product of a finite or infinite interval and a compact domain. Since we assume the base to be geodesically complete we will not cover the case of a finite interval, but additionally we may admit closed waveguides with $B = S^1$. This seemingly simple situation already allows for several different effects that depend on the codimension of B and are encoded in the metric of M. The authors of [7, 19] treat a tubular neighbourhood of varying width of the x-axis in $\mathbb{R}^2$. On $M = \mathbb{R} \times [0, 1]$ this can be represented by a metric in which the varying width is encoded by scaling the metric of each fibre and the choice of horizontal directions (see example 2.2). Conversely Goldstone and Jaffe [24] as well as Duclos and Exner ([16] and earlier works) consider curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ that are not straight lines. The tubular neighbourhood has a fixed cross-section, so the fibres, which are given by a compact domain, are all isometric. The horizontal part of the metric however acquires corrections due to the 'bending' of the curve (see example 1.9). The similar case with an embedding into a complete, non-compact surface, which leads to a slightly different metric on $M = \mathbb{R} \times [-1, 1]$ was treated by Krejčiřík [43] and, with a different question in mind, for neighbourhoods of geodesics by Krejčiřík and Kolb [40]. For the treatment of the same situation in $\mathbb{R}^3$ the additional condition that the cross-section does not twist against the curve is imposed. This can be expressed by requiring that for given $x \in B$, the plane in $\mathbb{R}^3$ containing the fibre F_x intersects the boundary of M normally, or equivalently that the horizontal bundle $NF \subset TM$ be tangent to ∂M. The analogous problem in arbitrary codimension is treated in [10]. The assumption that the fibre does not twist together with the fact that they are isometric leads to a situation in which the variation of the fibre eigenfunctions along the horizontal directions is trivial. Thus the only reason the problem does not decouple exactly into the vertical and horizontal equations are the corrections to the metric given in example 1.9. Without the assumption of no twisting the problem is analysed in [6, 14, 15, 21]. De Oliveira and Verri [15] and Gadyl'shin [21] simultaneously allow for fibres of varying size.

The results for bases of higher dimensions are far less detailed. In [9] and earlier works the authors study embeddings that make the base a complete and asymptotically flat two-dimensional surface in $\mathbb{R}^3$. Apart from these restrictions, B may have arbitrary topology and M is given by a neighbourhood of zero with fixed width in the normal bundle of B. Lin and Lu [47] consider special submanifolds of $\mathbb{R}^k$ of arbitrary dimension and codimension with asymptotically flat complete metrics. Here too, M is a neighbourhood of zero in NB and the fibre is an n-dimensional ball. Wittich [71] treats tubular neighbourhoods of compact manifolds in a Riemannian manifold (A, g_A) whose fibres are geodesic balls in the normal directions (see also remark 3.6). In all of these works there is no assumption forbidding twisting, but its effects are suppressed due to the (approximate) rotational symmetry of the fibre.

More general manifolds have been considered with metrics that are of a simpler structure than those arising from embeddings. In this context one is usually concerned with closed fibres. Baider [1] works with warped products (see example 1.3), Kleine [38] treats more general metrics on manifolds of the form $\mathbb{R}_+ \times F$ and the authors of [3, 5] study Riemannian submersions, especially with fibres whose mean curvature vector is a horizontal lift. The works [1, 5, 3, 38] derive conditions for the Laplacian on a non-compact M to have discrete spectrum. These will not be satisfied under our technical assumptions (chapter 2, condition 1), since they require non-uniform behaviour of the geometry.

Froese and Herbst [20] as well as Teufel and Wachsmuth [69] study localisation to submanifolds through potentials rather than boundary conditions. Because of the localisation close to $\alpha(B)$ they conveniently reformulate the problem on the normal bundle of B. In this sense the structure of the problem is very similar, with $M = NB$ and a potential $V \neq 0$ of a form that gives localised eigenfunctions of $H_F = -\Delta_F + V$. In [20] B is assumed to be a compact (without boundary) submanifold of $\mathbb{R}^k$, while in [69] the base and the ambient space in which it is embedded are, apart from technical assumptions, basically arbitrary complete Riemannian manifolds. The leading order of the metric on NB arising in this situation is the Sasaki metric, which is a Riemannian submersion with totally geodesic fibres.

Our approach considerably generalises the geometries that have been considered in the literature. This shows that a large class of problems have the sufficient structure for adiabatic techniques to be applicable. Our results also complement the previously studied quantum waveguides by allowing for generic deformations of the fibres, as opposed to scaling and twisting only. For example one may think of deforming a disk into an elliptic cross-section along the waveguide.

Effective operators

Many of the authors cited in above discussion explicitly derive an effective operator, mostly for the ground state band. The sense in which this operator is effective may be that it asymptotically describes the spectrum or dynamics of H, or that it is the $\varepsilon \to 0$ limit of H in the sense of quadratic forms or resolvents. It is important to understand that to have true convergence one needs a candidate limiting object independent of ε. Since the most simple form of the adiabatic operator is

$$H_\mathrm{a} = -\varepsilon^2 P \Delta_h P + \lambda$$

one can not hope to find such an object if λ does not have simple scaling properties. Hence in general the effective operator will depend on ε and approximate H in the asymptotic regime without having a limit as $\varepsilon \to 0$.

For quantum waveguides with isometric fibres, λ is of course constant and may be subtracted from the equation. Since the leading contribution from the potential (1.6) is $V_\rho = -\varepsilon^2\kappa^2/4 + \mathcal{O}(\varepsilon^3)$ we can rescale energies by ε^{-2} and obtain the candidate for a limiting operator

$$H_0 = -P\Delta_h P - \kappa^2/4 \,.$$

Note that this operator also accounts for the effects of possible twisting of the waveguide, which manifest themselves in the difference of $P\Delta_h P$ and the Laplacian on the base. For the ground state band, the authors of [6, 16] show convergence of the resolvent of $\varepsilon^{-2}H$ to that of the candidate operator H_0. De Oliveira [14] and Wittich [71] prove convergence of quadratic forms, which in general only implies strong convergence of resolvents. For a compact base the result of [14] reduces to that of [6].

If λ is not constant but the behaviour near its minima is known one can also determine the correct scaling α of the eigenvalues of H_a. Rescaling by $\varepsilon^{-\alpha}$ then gives a limiting operator. In [15, 19] convergence of resolvents in norm is shown for these operators. In [15] this scale is $\alpha = 1$, because λ has a unique non-degenerate minimum. The authors note that the effects of the embedding of B, such as twisting and bending, are suppressed on this energy scale. The limiting operator is, after rescaling $s \mapsto \varepsilon^{1/2}s$,

$$H_0 = -\partial_s^2 + a^2 s^2 \,.$$

Convergence of resolvents implies convergence of spectra and eigenfunctions. The resulting statements are thus closely related to our analysis of the spectrum at the appropriate energy scale in section 3.2. In particular [16, theorem 5.6] gives a statement for the eigenvalues. It derives mutual correspondence of discrete spectra (see proposition 3.11) and determines the correction to the eigenvalue asymptotics given by the adiabatic operator to be of order ε^4 in the rescaled energies. This statement is stronger than our general result 3.12, which only gives an error of $\mathcal{O}(\varepsilon^2)$. This is possible under the conditions of isometric fibres and no twisting because the fibre-eigenfunctions do not vary and the leading order of the error vanishes. Our result on the validity of the adiabatic approximation improves the asymptotics shown in [15, 19] by including the subleading terms, due to e.g. twisting and bending, as well as an additional differential operator due to the correction of the metric. These can be treated using standard perturbation theory for H_{a} to obtain expansions of eigenvalues and eigenfunctions beyond the leading order established by the resolvent limit.

Strong convergence of resolvents also implies the strong convergence of unitary groups (see [61, theorem VIII.21]). This holds uniformly in time for $t \leq T$, so convergence of $\varepsilon^{-\alpha} H$ to H_0 in this sense shows approximation of the dynamics for finite times. In our scaling this corresponds to times of order $\varepsilon^{-\alpha}$, but with energies of order ε^{α}. This is too short to see global effects because lengths in B still grow like ε^{-1}.

Froese and Herbst [20] derive an effective operator governing the time evolution starting from an harmonic confining potential of fixed form. Their scaling corresponds to the case $\alpha = 2$ discussed above, which again hinges on the fact that λ is constant. This operator contains a geometric potential depending on the scalar curvature of $(B, \alpha^*\delta)$ and the mean curvature of $\alpha(B)$. Just as the geometric potential for one-dimensional base it originates from an expansion of V_ρ (1.5) around B. The work of Teufel and Wachsmuth [69] considerably generalises this. For simple and separated eigenbands they derive a super-adiabatic projection and an effective operator satisfying

$$(H - U H_{\mathrm{eff}} U^*) P_\varepsilon = \mathcal{O}(\varepsilon^3)\,.$$

In order to achieve this, H_{eff} must contain the first super-adiabatic corrections as well as many terms originating from the correction to the metric. Many of these terms are not present in effective operators derived earlier because they are not relevant at the energy scales considered there.

In the spirit of [69] we derive effective operators for arbitrary separated eigenbands. We do not require them to be simple, as this would exclude many higher modes on fibres with symmetries. The operators can be constructed so that the approximation error is of arbitrary order in ε. For the ground state band we derive low energy asymptotics, extending known results for quantum waveguides.

Nodal geometry

For a real eigenfunction φ of a differential operator on M, the *nodal set* is defined as $\mathcal{N}(\varphi) := \overline{\varphi^{-1}(0) \cap (M \setminus \partial M)}$. The connected components of $M \setminus \mathcal{N}(\varphi)$ are called the *nodal domains* of φ. On each of these sets φ has a definite sign. In general they may form very complicated patterns but their number can be bounded by Courant's nodal domain theorem (see [64, chapter 6]).

Theorem (Courant's nodal domain theorem)**.** *Let (M, G) be a compact, connected Riemannian manifold with boundary. Let $0 \leq \lambda_0 < \lambda_1 \leq \ldots$ be the eigenvalues of $-\Delta_G$ with Dirichlet boundary conditions, repeated according to multiplicity. If φ_k is the eigenfunction corresponding to λ_k, the number of nodal domains of φ_k is at most $k + 1$.*

The eigenfunction φ_0 may be chosen positive everywhere and has exactly one nodal domain. Since φ_1 is orthogonal to φ_0 it must change sign, so the theorem implies it has exactly two nodal domains. There are known restrictions to attaining the bound of this theorem. For example Pleijel [58] proved for domains in $\mathbb{R}^2$ that only finitely many eigenfunctions of the Laplacian may have the maximum number of nodal domains.

In the same setting it was conjectured by Payne [56] that the nodal line of φ_1 cannot be closed. This is of course equivalent to the statement that it must meet the boundary. Because it must do so at a right angle it then joins two distinct points on the boundary. This conjecture has been proven by Melas [52] for convex domains. For general domains however a counter example was given by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and Nadirashvili [34]. For convex domains of large eccentricity, Jerison [36] proved that the nodal set of φ_1 touches the boundary also in higher dimensions. By comparing convex two dimensional domains with long and thin rectangles Jerison [35] and Grieser-Jerison [26] were able to obtain estimates on the location of the nodal set. The location is determined by the solution of an ordinary differential equation analogous to our effective operator. Similar ideas were used by the same authors to estimate the location and size of the maximum of φ_0 [27]. Using techniques from the study of quantum waveguides, Freitas and Krejčiřík [18] derived a similar result for thin domains that are given as embeddings of $[0, 1] \times \Omega$ into $\mathbb{R}^k$. The fibre $\Omega \subset \mathbb{R}^n$ is a bounded domain isometric to the embedded fibres, so the only variation of Ω along the base is due to twisting and bending, which ensures that λ_0 is constant as described for quantum waveguides above. In this asymptotic setting they find it natural to generalise the statement to a finite number of eigenfunctions, determined by the effective operator. The location of the nodal sets then also provides a lower bound on the number of nodal domains for these functions.

We compare the eigenfunctions of the adiabatic operator H_a and the Laplacian on M for two different classes of manifolds in section 3.3. The first of these classes are compact manifolds (with and without boundary) characterised by a constant lowest eigenvalue λ_0 of H_F and $\dim B \leq 3$. The other class are manifolds with non-empty boundary and $\dim B = 1$ for which λ_0 has a unique and non-degenerate minimum. This behaviour of λ_0 is of course the generic one. We prove uniform approximation of eigenfunctions, allowing us to locate the nodal sets of eigenfunctions of simple, low-lying eigenvalues in section 3.3.2. Using these estimates we describe, given some $k \in N$, metrics for which the $2k + 1$-th eigenfunction attains the bound of Courant's theorem in corollary 3.27. For sufficiently thin manifolds with non-empty boundary we also show that the nodal sets of the first eigenfunctions must reach the boundary, and give a lower bound on how often this happens, in corollary 3.28. This establishes results in the spirit of [18, 26, 36, 35] for a rather large class of thin manifolds and provides an answer to the question posed by Schoen and Yau [64, Problem 45] regarding such generalisations.

Chapter 2

Adiabatic theory on fibre bundles

In this chapter we develop a general adiabatic and super-adiabatic theory for Laplace-type and Schrödingeroperators on Riemannian submersions with a scaled family of metrics g_ε (1.1).

The first section is devoted to the construction of super-adiabatic projections and their properties. In the second part we use these projections to define an effective operator. We prove that this operator provides a good approximation of the original problem for both dynamical and spectral purposes under the right conditions.

Our method builds on the space-adiabatic perturbation theory, developed for the study of the Born-Opppenheimer approximation by Nenciu, Martinez, Sordoni and Panati, Spohn, Teufel and reviewed by Teufel in [67]. These methods were already used by Teufel and Wachsmuth [69] in the study of constrained quantum systems. We go beyond their treatment in applying the method to a wider class of geometries and showing that the approximation can be carried out to all orders.

Throughout this chapter let (M, g) denote a connected Riemannian manifold with boundary such that $F \to M \stackrel{\pi}{\to} B$ is a Riemannian submersion for a metric g_B on B. The study of global properties of differential equations on a non-compact Riemannian manifold (M, g) generally requires some uniformity of the geometry of M. The assumptions we make throughout are:

Condition 1.

- F is compact,
- (B, g_B) is of bounded geometry (definition A.1), in particular it is geodesically complete,

- $M \xrightarrow{\pi} B$ is uniformly locally trivial (definition A.3).

All the boundedness conditions are satisfied if M, and hence also B, is compact.

The details of these definitions are given and discussed in appendix A. Most importantly they imply the elliptic estimates given in theorem A.14 that we will use frequently. Denote by $\mathscr{H} := L^2(M, \mathrm{vol}_g)$ the Hilbert space of square-integrable, complex valued functions on $(M, g_{\varepsilon=1})$. In section A.2 we introduce Sobolev spaces W^k_ε adapted to the rescaled family of metrics g_ε (1.1). The norm on $W^{2k}_\varepsilon(M)$ is equivalent to the graph norm of $\Delta^k_{g_\varepsilon}$, with constants independent of ε, and a vector field εX^* with $X \in \Gamma(TB)$ defines an operator $W^1_\varepsilon \to \mathscr{H}$ with norm independent of ε, because the g_ε-length εX^* is just the g_B-length of X. All the Sobolev spaces we use are to be understood as their L^2 variants, although we do not make this explicit in the notation. Now set

$$H := -\Delta_{g_\varepsilon} + V + \varepsilon H_1\,,$$

with:

Condition 2.

- The potential $V \in \mathscr{C}^\infty_b(M)$ is smooth and bounded with all its derivatives.
- H_1 is a smooth differential operator of second order and symmetric on $D(H)$. It is bounded independently of ε as a map $W^{m+2}_\varepsilon \to W^m_\varepsilon$, for every $m \in N$ and satisfies $H_1 A \in \mathcal{A}^{k+2,l}$ for every $A \in \mathcal{A}^{k,l}$ (see definition 2.3).
- H is bounded from below.

Under these conditions H is self-adjoint on the Dirichlet domain

$$D(H) := W^2_\varepsilon(M) \cap W_{0,\varepsilon}(M)\,.$$

From now on H will always denote this self-adjoint operator, while expressions like the Laplacian Δ_{g_ε} or H_1 may also stand for a differential operator without reference to a specific domain.

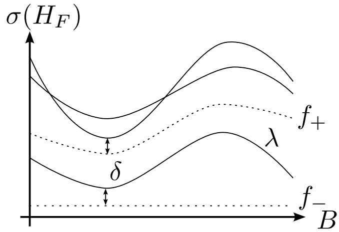
For $x \in B$ the fibre operator on F_x

$$H_F(x) := -\Delta_{F_x} + V|_{F_x}$$

is self-adjoint on the domain $W^2(F) \cap W_0^1(F)$. Because of compactness, the spectrum $\sigma(H_F(x))$ consists purely of eigenvalues of finite multiplicity accumulating at infinity. An eigenband is a map $\lambda\colon B \to \mathbb{R}$ with $\lambda(x) \in \sigma(H_F(x))$. We will only consider such bands with a *spectral gap*:

Condition 3. There exist $\delta > 0$ and bounded continuous functions $f_-, f_+ \in \mathscr{C}_b(B)$ with $\operatorname{dist}(f_\pm(x), \sigma(H_F(x))) \geq \delta$ such that

$$\forall x \in B : [f_-(x), f_+(x)] \cap \sigma(H_F(x)) = \lambda(x)\,.$$



This condition implies in particular that $\operatorname{dist}(\lambda, \sigma(H_F) \setminus \lambda) \geq 2\delta$.

Figure 2.1: An eigenband λ satisfying the gap condition.

This condition is automatically satisfied for the ground state band $\lambda_0(x) = \min \sigma(H_F(x))$ if $V = 0$ and F is connected (see lemma 3.7).

Remark 2.1. We could also consider an ε-dependent family V_ε with $\varepsilon \leq 1$ as long as conditions 2 and 3 are satisfied uniformly in ε. This does not change much in the end because the effective operator we obtain is ε-dependent in any case. However it makes the eigenfunctions and eigenvalues of H_F, and many derived objects, depend on ε, which makes the notation very cumbersome. For this reason we will not explicitly treat such dependence on ε although it should become clear from our proofs that uniformity of the conditions on V and λ is sufficient for the results we obtain.

The elements of $\mathscr{H}$ can be viewed as square-integrable sections of a vector bundle over B, whose fibres are $L^2(F)$. Although the norm of $L^2(F_x)$ depends on the metric g_{F_x}, the norms for different metrics are equivalent because F is compact. Hence as a topological vector space $L^2(F_x)$ is independent of x. The same holds for the domain of $H_F(x)$, so we can again think of this space as the fibre of a vector bundle (see appendix B.1 for a more detailed discussion), constructed as follows: Let $X(F)$ be some space of functions over F, $f \in X(F)$ and $\Phi : U \times F \to U \times F$ a transition function between two trivialisations of M over $U \subset B$. Mapping $(x, f) \mapsto f \circ \Phi(x, \cdot)$ defines a transition function on $U \times X(F)$. If $X(F)$ is a topological vector space for which all the transition functions induced by those of a cover $\mathfrak{U}$ are continuous this defines a topological vector bundle over B with fibre $X(F)$, that we denote by $X(F;\pi)$. If $X(F)$ is a Banach space this continuity is equivalent to the strong continuity of the map $x \mapsto f \circ \Phi(x, \cdot)$ by the uniform boundedness principle. We apply this construction to $X(F) = L^2(F)$ or equal to any Sobolev space $W^m(F)$ and fix the notation for

$$L^2(F;\pi) =: \mathscr{H}_F \qquad \text{and} \qquad W^2(F;\pi) \cap W^1_0(F;\pi) =: D(H_F)\,.$$

We treat these as hermitian vector bundles with their natural fibre-wise pairings. It is also helpful to picture H_F and its spectral projections as bundle maps on these vector bundles. Since they operate fibre-wisely they are sections of bundles, whose fibres consist of bounded linear maps, like $\mathscr{L}(\mathscr{H}_F)$ with fibre $\mathscr{L}(L^2(F))$. We can observe that H_F is a bounded section

$$H_F \in L^\infty\big(\mathscr{L}(D(H_F), \mathscr{H}_F)\big)\,,$$

as is the family of spectral projections P_0 associated with an eigenband λ

$$P_0 \in L^\infty\big(\mathscr{L}(\mathscr{H}_F)\big) \cap L^\infty\big(\mathscr{L}(D(H_F))\big)\,.$$

For fixed $x \in B$ the image of P_0 has finite dimension. If now P_0 as a section of $\mathscr{L}(\mathscr{H}_F)$ is continuous, $\operatorname{rank}(P_0) = \operatorname{tr} P_0$ must be constant and $\mathcal{E} := P_0\mathscr{H}_F$ defines a subbundle of $\mathscr{H}_F$ of finite rank. By identification $\mathscr{H} \cong L^2(\mathscr{H}_F)$ the operator P_0 defines a bounded operator on $\mathscr{H}$, whose image is $L^2(\mathcal{E})$.

2.1 Adiabatic and super-adiabatic projections

Studying the invariance of $P_0\mathscr{H}$ under Δ_{g_ε} amounts to the local study of $[-\Delta_{g_\varepsilon}, P_0] = [-\varepsilon^2\Delta_h, P_0]$. Since P_0 is a spectral projection of Δ_F it seems natural to approach this by first calculating $[-\varepsilon^2\Delta_h, \Delta_F]$, which locally (by formal calculation) consists of commutators of Δ_F with horizontal lifts (see (1.2)). It is important to warn here that, due to the presence of the boundary, this is very problematic. In fact '$[X^*, \Delta_F]$' does not make much sense since X^* need not be tangent to the boundary, so its application destroys the Dirichlet condition and this object has no sensible domain. To be more precise, a way to make sense of '$[X^*, \Delta_F]$' would be as a Lie derivative $\mathcal{L}_X H_F$ of the section $H_F \in L^\infty(\mathscr{L}(D(H_F), \mathscr{H}_F))$, as in (1.3). But the flow of X^*, even if it exists, might not leave $D(H_F)$ invariant. In this case one ends up calculating the derivative of the differential operator $-\Delta_F + V$ on $\mathscr{C}^\infty(F)$ instead of $(H_F, D(H_F))$. This may completely miss the point since it could vanish, even though the spectrum of $H_F(x)$ depends on x. To deal with this we need to work with objects that are adapted to the boundary. These are naturally found in trivialisations. We will use the fixed atlas $\mathfrak{U}$ of B introduced in the appendix (see page 131). The coordinate neighbourhoods $(U_\nu)_{\nu\in\mathbb{N}}$ of this atlas come with trivialisations Φ_ν of $\pi^{-1}(U_\nu)$, a partition of unity χ_ν and orthonormal frames $(X_i^\nu)_{i\leq d}$, all satisfying boundedness properties uniformly in ν.

For such a vector field on $U \in \mathfrak{U}$ we may split $X^* = \Phi^*X + Y$ with a vertical vector field Y, that is bounded by corollary A.6. Since Φ^*X is tangent to the boundary its flow exists for some positive time and preserves the Dirichlet condition, leaving $D(H_F)$ invariant. The contribution of the vertical part Y, which we treat separately, will be small since εX^* has length one in g_ε while εY goes to zero as $\varepsilon \to 0$. The horizontal vector fields on $\pi^{-1}(U)$ are thus increasingly well described by the trivialisation Φ, making it look more and more like a product.

Example 2.2. To illustrate the objects we have just discussed we calculate them in a simple example. Let $h \in \mathscr{C}_b^\infty(\mathbb{R})$ be a positive function and let $M = \mathbb{R} \times [0, 1+h] \subset \mathbb{R}^2$. Let $g_\varepsilon = \varepsilon^{-2}\mathrm{d}x^2 + \mathrm{d}y^2$ be the restriction of the rescaled metric on $\mathbb{R}^2$ and $H = -\Delta_{g_\varepsilon} = -\varepsilon^2\partial_x^2 - \partial_y^2$ on $D(H)$. The horizontal lift of $\partial_x \in \Gamma(T\mathbb{R})$ is trivial $\partial_x^* = \partial_x$, so on $\mathscr{C}^\infty(M)$ we have $[\partial_x, \partial_y^2] = 0$. A global trivialisation of M is given by

$$\Phi\colon M \to \mathbb{R} \times [0,1]\,; \qquad (x,y) \mapsto (x,z) = \big(x, y/(1+h(x))\big)\,.$$

For $f \in \mathscr{C}^\infty(M)$ one easily calculates

$$\Phi^*\partial_x f = \partial_x f\big(x,(1+h(x))z\big) = \partial_x f + h' z \partial_y f = \partial_x f + h' y/(1+h)\partial_y f\,,$$

so we can identify $Y = \partial_x^* - \Phi^*\partial_x = -\log(1+h)' y \partial_y$. Clearly $\Phi^*\partial_x$ is tangent to ∂M, so for any $f \in \mathscr{C}^\infty(M)$ that vanishes on ∂M, $\Phi^*\partial_x f$ is also zero on ∂M. On such functions we thus have

$$[\Phi^*\partial_x, \partial_y^2] = -[\Phi^*\partial_x, H_F] = [\log(1+h)' y\partial_y, \partial_y^2] = -2\log(1+h)'\partial_y^2\,.$$

We can observe here that $[\Phi^*\partial_x, H_F]$ is bounded relatively to H_F, which will hold in general.

2.1.1 The algebras $\mathcal{A}$ and $\mathcal{A}_H$

In order to keep track of the number of vertical and horizontal derivatives in a given expression and avoid tedious calculations in local coordinates we define special algebras of differential operators. These differential operators will have coefficients in $L^\infty(\mathscr{L}(\mathscr{H}_F))$, which are exactly the fibre-wise operators in $\mathscr{L}(\mathscr{H})$. We assume these coefficients to be smooth in the following sense: Take $U_\nu \in \mathfrak{U}$ and let $\mathcal{C}^\nu \subset L^\infty\left(\mathscr{L}(\mathscr{H}_F)|_{U_\nu}\right)$ be those linear operators A for which any commutator of the form

$$\big[\Phi_\nu^* X_{i_1}^\nu, \dots, [\Phi_\nu^* X_{i_k}^\nu, A]\cdots\big] \tag{2.1}$$

defines an element of $L^\infty\left(\mathscr{L}(\mathscr{H}_F)|_{U_\nu}\right)$, where $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{1, \dots, d\}$.

Let $\mathcal{C}_H^\nu \subset \mathcal{C}^\nu$ be the subset of $L^\infty\left(\mathscr{L}(\mathscr{H}_F, D(H_F))|_{U_\nu}\right)$ that is closed under commutators in the same way as $\mathcal{C}^\nu$. This is equivalent to saying that $A \in \mathcal{C}_H^\nu$ if and only if $H_F A \in \mathcal{C}^\nu$.

Definition 2.3. The algebras $\mathcal{A}$, $\mathcal{A}_H$ consist of those linear operators in $\mathscr{L}(W^\infty(M), \mathscr{H})$ satisfying $\pi(\operatorname{supp} Af) \subset \pi(\operatorname{supp} f)$ and

$$A|_{\pi^{-1}(U_\nu)} = \sum_{\alpha\in\mathbb{N}^d} A_\alpha^\nu \varepsilon^{|\alpha|} (\Phi^* X_1^\nu)^{\alpha_1} \cdots (\Phi^* X_d^\nu)^{\alpha_d}\,,$$

with $A^\nu_\alpha \in \mathcal{C}^\nu$, $\mathcal{C}^\nu_H$, for which there exist constants $C(\alpha, k) < \infty$ and $l \in \mathbb{N}$ (uniformly in ν and ε) such that the norm of the commutator (2.1) in $\mathscr{L}(\mathscr{H}_F)$ is bounded by $C(\alpha, k)$ and $A^\nu_\alpha = 0$ for $|\alpha| > l$.

From now on we write $\mathcal{C}^\nu_\bullet$ and $\mathcal{A}_\bullet$ in statements that hold with or without the subscript H. $\mathcal{A}_\bullet$ is an algebra because of the commutator condition (2.1) for $\mathcal{C}^\nu_\bullet$ and $[\Phi^* X_i, \Phi^* X_j] = \Phi^*[X_i, X_j]$, allowing us to arrange the vector fields in any order without producing vertical derivatives. $\mathcal{A}_H$ consists of those $A \in \mathcal{A}$ whose image consists of functions satisfying the Dirichlet condition and for which $H_F A \in \mathcal{A}$. Hence $\mathcal{A}_H \mathcal{A} \subset \mathcal{A}_H$ and $\mathcal{A}_H$ is a right ideal of $\mathcal{A}$.

$\mathcal{A}_\bullet$ is filtered by setting

$$\mathcal{A}^k_\bullet := \left\{ A \in \mathcal{A}_\bullet : \forall \nu \in \mathbb{N} \left(|\alpha| > k \Rightarrow A^\nu_\alpha = 0 \right) \right\} .$$

Clearly $\mathcal{A}^k \subset \mathscr{L}\left(W^k_\varepsilon, \mathscr{H}\right)$ so it inherits this operator norm, which we denote by $\|\cdot\|_k$. An additional filtration is given by the order in ε by defining $\mathcal{A}^{j,l}_\bullet$ to be those $A \in \mathcal{A}^j_\bullet$ for which the constants $C(\alpha, k)$ of definition 2.3 can be chosen of order ε^l for some $l \in \mathbb{N}$. This of course implies that $\|A\|_k = \mathcal{O}(\varepsilon^l)$. Note that a differential operator of order k is also one of order $k+1$, so $\mathcal{A}^k_\bullet \subset \mathcal{A}^{k+1}_\bullet$, while a norm of order $l+1$ is also of order l, so $\mathcal{A}^{k,l+1}_\bullet \subset \mathcal{A}^{k,l}_\bullet$. We may also observe that due to the commutation properties of the coefficients and vector fields we have for $A \in \mathcal{A}^k$, $B \in \mathcal{A}^l$

$$AB|_{\pi^{-1}(U_\nu)} = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} A^\nu_\alpha B^\nu_\beta \varepsilon^{k+l} (\Phi^* X^\nu_1)^{\alpha_1+\beta_1} \cdots (\Phi^* X^\nu_d)^{\alpha_d+\beta_d} + \mathcal{A}^{k+l-1} .$$

Note also that terms containing commutators of $\Phi^* X_i$ with other vector fields or the coefficients A_α, B_β produce terms of lower order in ε.

Remark 2.4. The condition $\pi(\operatorname{supp} Af) \subset \pi(\operatorname{supp} f)$ allows us to calculate the norms $\|\cdot\|_k$ locally with respect to the base since (see also remark A.13)

$$\begin{aligned}\|A\psi\|^2_{W^0_\varepsilon(M)} &= \sum_\nu \|\chi_\nu A\psi\|^2_{\mathscr{H}} = \sum_\nu \|\chi_\nu A \sum_\mu \chi_\mu \psi\|^2 \\ &\leq N(\mathfrak{U}) \sum_{\mu,\nu} \|\chi_\nu A \chi_\mu \psi\|^2 \\ &\leq N(\mathfrak{U})^2 \sum_\mu \sup_\nu \|\chi_\nu A\|^2_{\mathscr{L}(W^k_\varepsilon(\pi^{-1}U_\mu),\mathscr{H})} \|\chi_\mu \psi\|^2_{W^k_\varepsilon(\pi^{-1}U_\mu)} \\ &\leq N(\mathfrak{U})^2 \sup_\mu \|A\|^2_{\mathscr{L}(W^k_\varepsilon(\pi^{-1}U_\mu),\mathscr{H})} \|\psi\|^2_{W^k_\varepsilon(M)} \,.\end{aligned}$$

Thus for any $A \in \mathcal{A}^k$

$$\|A\|_k \leq N(\mathfrak{U})^{3/2} \sup_\mu \|A\|_{\mathscr{L}(W^k_\varepsilon(\pi^{-1}U_\mu),\mathscr{H})} \,,$$

where $W^k_\varepsilon(\pi^{-1}U_\nu)$ is defined in the trivialisation by Φ_ν, cf. (A.10).

The key properties for all later calculations with these algebras are content of the following lemma.

Lemma 2.5. *Let $A, B \in \mathcal{A}_H$ with $AB \in \mathcal{A}^{k,l}_H$, then*

$$[\Delta_{g_\varepsilon}, A]B \in \mathcal{A}^{k+1,l}$$

and

$$[\varepsilon^2 \Delta_h, A]B \in \mathcal{A}^{k+1,l+1} \,.$$

Proof. We split $\Delta_{g_\varepsilon} = \Delta_F + \varepsilon^2 \Delta_h$ and first observe that

$$[\Delta_F, A]B = \underbrace{\Delta_F A}_{\in \mathcal{A}} B - A \underbrace{\Delta_F B}_{\in \mathcal{A}} \in \mathcal{A}^{k,l} \,,$$

since $\Delta_F A_\alpha \in \mathcal{C}$ if $A_\alpha \in \mathcal{C}_H$. Using this, the second claim implies the first one.

Since the definition of $\mathcal{A}^k$ and its norm are local with respect to the base (cf. remark 2.4) it is sufficient to show the claim on $\pi^{-1}(U_\nu)$. We fix ν and split $X^*_i = \Phi^* X_i + Y_i$. In this frame we have $\varepsilon^2 \Delta_h = \varepsilon^2 \sum_{i \leq d} \Phi^* X_i \Phi^* X_i +$

$\varepsilon^2 D$, where D contains first order differential operators and second order parts that contain at least one vertical derivative. We have for every $j \in \{1, \dots, d\}$

$$[\Phi^* X_j, A]\,\big|_{\pi^{-1}(U)} = \sum_{\alpha \in \mathbb{N}^d} \varepsilon^{|\alpha|} \Big(\underbrace{[\Phi^* X_j, A_\alpha]}_{\in \mathcal{C}_H} (\Phi^* X_1)^{\alpha_1} \cdots (\Phi^* X_d)^{\alpha_d} + A_\alpha [\Phi^* X_j, (\Phi^* X_1)^{\alpha_1} \cdots (\Phi^* X_d)^{\alpha_d}] \Big) .$$

This is of the same order as A in $\mathcal{A}_H$ because $[\Phi^* X_j, \Phi^* X_i] = \Phi^*[X_j, X_i]$ and this Lie bracket is a bounded vector field. Hence

$$\chi[\sum_{i \le d} \varepsilon^2 \Phi^* X_i \Phi^* X_i, A] B \in \mathcal{A}_H^{k+1, l+1} .$$

Now for a bounded vertical field Y we have YAB and $AYB \in \mathcal{A}^{k,l}$. The commutator $[\Phi^* X_i, Y]$ is also vertical (see lemma 1.4), so by commuting all the $\Phi^* X_i$ to the right we see that

$$\chi[\varepsilon^2 D, A] B \in \mathcal{A}^{k+1, l+1} .$$

This proves the second claim and thus completes the proof. □

By commuting derivatives of the form $\Phi^* X_i^\nu$ to the right as in the precedent proof one sees that

$$\big\| [\varepsilon^2 \Delta_h, A] \big\|_{k+2} = \mathcal{O}(\varepsilon^{l+1}) \tag{2.2}$$

if $A \in \mathcal{A}_H^{k,l}$. For the same reason $\varepsilon^2 \Delta_h A$ and $\Delta_{g_\varepsilon} A$ are elements of $\mathcal{A}^{k+2,l}$. Functions in the image of A satisfy the Dirichlet condition and $HA \in \mathcal{A}^{k+2}$ by condition 2, hence $\mathcal{A}_H^k$ is contained in $\mathscr{L}\big(W_\varepsilon^{k+2}, D(H)\big)$.

Remark 2.6. In view of this discussion of $\mathcal{A}$ we can also see that the operator H_1 satisfies the final part (i.e. $H_1 A \in \mathcal{A}^{k+2,l}$) of condition 2 if it has the local form

$$H_1|_{\pi^{-1}(U_\nu)} = \sum_{|\alpha|=2} A_\alpha^\nu \varepsilon^2 (\Phi^* X_1^\nu)^{\alpha_1} \cdots (\Phi^* X_d^\nu)^{\alpha_d} + \sum_{i \le d} B_i^\nu \varepsilon \Phi^* X_i^\nu + C ,$$

with $A^\nu_\alpha \in \mathcal{C}^\nu$, $B^\nu_i \in L^\infty(\mathscr{L}(W^1(F;\pi), \mathscr{H}_F)|_{U_\nu})$ and $C \in L^\infty(\mathscr{L}(D(H_F), \mathscr{H}_F))$ satisfying commutator conditions analogous to those for $\mathcal{C}^\nu$. In typical examples H_1 will be exactly of this form (see chapter 3).

2.1.2 Construction of the projections

We are now set for the treatment of the projections. Form here on the capital letter C will be used to denote various constants independent of ε with no specified relation between them and possibly differing even within the same equation. Our method will heavily rely on functional calculus, so the resolvent is a central object.

Lemma 2.7. *Let* $z \in \mathbb{C}$ *with* $\operatorname{dist}(z, \sigma(H_F)) \geq C > 0$, *then*

$$R_F(z) := (H_F - z)^{-1} \in \mathcal{A}^{0,0}_H \,.$$

Proof. $R_F(z)$ is fibre-wise with norms

$$\begin{aligned} &\|R_F(z,x)\|^2_{\mathscr{L}(\mathscr{H}_F)} \leq C^{-2} \\ &\|R_F(z,x)\|^2_{\mathscr{L}(\mathscr{H}_F, D(H_F))} \leq 2 + (1 + 2\,|z|^2)C^{-2}\,, \end{aligned}$$

so $R_F(z) \in L^\infty\left(\mathscr{L}(\mathscr{H}_F)\right) \cap L^\infty\left(\mathscr{L}(\mathscr{H}_F, D(H_F)\right)$. The commutator condition (2.1) remains to be verified. Let $U \in \mathfrak{U}$ with corresponding Φ and $0 \neq X \in \Gamma_b(TU)$. We will do all the calculations on $U \times F$ and then map everything back to $\pi^{-1}(U)$. Endow $U \times F$ with the metric $\tilde{g} = \Phi_* g_F + g_B$ induced by Φ and choosing the canonical lift to the product as the horizontal direction. Then the map $W\colon L^2(U \times F, \tilde{g}) \to L^2(\pi^{-1}(U), g)$ given by $f \mapsto f \circ \Phi$ is unitary. Additionally $(WXW^*)f = (\Phi^* X)f$ and $W^* \Delta_F W = \Delta_{g_{F_x}}$, so

$$[\Phi^* X, R_F(z)] = W[X, (W^* H_F W - z)^{-1}]W^*\,,$$

with $W^* H_F W = -\Delta_{g_{F_x}} + \Phi_* V$.

Now choose $x_0 \in U$ and $U_F \subset F_{x_0}$ possessing an orthonormal frame of bounded vector fields $(Y_j)_{j \leq n}$. Extend these vertical vector fields by parallel transport along the integral curves of X starting at x_0. We claim that these extensions form an orthonormal frame of vertical vector fields wherever they are defined. To check this we calculate the derivative of the horizontal components of Y_j

$$X\tilde{g}(Y_j, X_i) = \tilde{g}(\underbrace{\tilde{\nabla}_X Y_j}_{=0}, X_i) + \tilde{g}(Y_j, \tilde{\nabla}_X X_i) = \tilde{g}(Y_j, \tilde{\nabla}_X X_i)\,.$$

This means the horizontal components of Y_j solve a first order differential equation along the integral curves of X. $\tilde{\nabla}_X X_i$ is horizontal because the integrability tensor Ω vanishes for the metric $\tilde{g}$ and lemma 1.5. The unique solution of the equation with the given initial value zero is the constant zero, so the extended Y_j must be vertical everywhere. They are orthonormal because parallel transport is an isometry, so they form an orthonormal frame of bounded vector fields of $TU_F \times \gamma$ on the integral curves γ of X. Thus $\Delta_{g_{F_x}} = \sum_{j \le n} Y_j \circ Y_j - \nabla_{Y_j} Y_j$ on $U_F \times \gamma$. Note that the flow of X preserves the Dirichlet condition (and obviously also differentiability), so it maps the domains of the fibre Laplacians to each other. Let $\mathcal{L}_X$ denote the Lie-derivative and $R(x,z) := (-\Delta_{g_{F_x}} + \Phi_* V - z)^{-1}$, then

$$\begin{aligned}
&[X, R(x,z)] \\
&\quad = \mathcal{L}_X \big(R(x,z)(-\Delta_{g_{F_x}} + \Phi_* V - z) R(x,z) \big) \\
&\quad = R(x,z)(\mathcal{L}_X \Delta_{g_{F_x}}) R(x,z) - R(x,z)\,(\Phi^* X V)\, R(x,z)\,.
\end{aligned} \tag{2.3}$$

Now on U_F

$$\mathcal{L}_X \Delta_{g_{F_x}} = \sum_{j \le n} [X, Y_j] Y_j + Y_j [X, Y_j] - [X, \nabla_{Y_j} Y_j]\,, \tag{2.4}$$

which is a second order vertical differential operator since $[X, Y_j]$ is vertical (cf. lemma 1.4). This field is bounded, so $\mathcal{L}_X \Delta_{g_{F_x}}$ defines a bounded operator from $W^2(F)$ to $L^2(F)$. Thus the composition (2.3) is a bounded operator from $L^2(F)$ to $W^* D(H_F)$.

The same reasoning applies to iterated commutators with (2.3) and (2.4), so together with $V \in \mathscr{C}_b^\infty(M)$ this proves the commutator condition and $R_F(z) \in \mathcal{A}_H^{0,0}$. □

Lemma 2.8. *The spectral gap of λ implies $P_0 \in \mathcal{A}_H^{0,0}$.*

Proof. Let $x_0 \in B$ and γ be the circle of radius δ around $\lambda(x_0)$ in $\mathbb{C}$. Due to the gap (condition 3) there is an open neighbourhood $W \subset B$ of x_0 such that $\operatorname{dist}(\gamma, \sigma(H_F(x))) > \delta/2$ for every $x \in W$. On W, P_0 is given by the Riesz formula

$$P_0 = \frac{\mathrm{i}}{2\pi} \int_\gamma R_F(z)\mathrm{d}z\,. \tag{2.5}$$

Thus the claim follows from lemma 2.7. □

In view of the lemma 2.5 and condition 2 this gives us

$$[H, P_0]P_0 = \underbrace{[-\varepsilon^2\Delta_h, P_0]P_0}_{\in\mathcal{A}^{1,1}} + \varepsilon[H_1, P_0]P_0 \in \mathcal{A}^{2,1}\,.$$

So P_0 is a projection in $\mathscr{L}(\mathscr{H})$ that commutes with H up to an operator that is of order ε from $D(H) \subset W^2_\varepsilon(M)$ to $\mathscr{H}$. In particular $P_0 \in \mathscr{L}(D(H))$.

We now show that $\mathcal{E}$ is a well defined subbundle of $\mathscr{H}_F$. In appendix B.2 we elaborate on the regularity of $\mathcal{E}$, showing that it is a smooth bundle of bounded geometry. Its differentiable structure is compatible with that of M in the sense that $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M, \mathbb{C})$.

Lemma 2.9. *$\mathcal{E} := P_0\mathscr{H}_F$ is a finite rank subbundle of $\mathscr{H}_F$ and $\lambda \in \mathscr{C}^\infty_b(B)$.*

Proof. To prove the first claim we need to show that for $U \in \mathfrak{U}$ the map

$$P_U : U \to \mathscr{L}(L^2(F))\,; \qquad x \mapsto \Phi(x, \cdot)_* P_0 \Phi(x, \cdot)^*$$

is strongly continuous, because that implies continuity of the projected transition maps of the bundle $\mathscr{H}_F$ (see appendix B.1). This amounts to showing continuity of the λ-eigenfunctions of H_F. In proposition B.7 we even show smoothness of these functions, but here we take a different route.

To start with, the map $x \mapsto \Delta_{g_{F_x}} + V \circ \Phi^{-1}$ is strongly continuous $U \to \mathscr{L}(W^2(F) \cap W^1_0(F), L^2(F))$ because g_F is smooth. By the resolvent formula

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1} \tag{2.6}$$

this implies that $R_F(z)$ is strongly continuous, which carries over to P_0 because of the Riesz formula (2.5). Since $P_0(x)$ is of finite rank for every x this proves in particular that $\operatorname{tr} P_0 = \dim \operatorname{ran}(P_0)$ is continuous and thus constant.

Let $r = \operatorname{rank}(\mathcal{E})$, then $\lambda = 1/r \operatorname{tr}(H_F P_0)$. Let $X \in \Gamma_b(TB)$ and calculate $X \cdot \lambda$ in the point $x \in U$ by lifting it to $\pi^{-1}(U)$:

$$
\begin{aligned}
\pi^*(X\lambda) &= [\Phi^* X, \pi^*\lambda] \\
&= 1/r \operatorname{tr}([\Phi^* X, H_F P_0]) \\
&= 1/r \operatorname{tr}([\Phi^* X, H_F] P_0 + H_F P_0 [\Phi^* X, P_0] + H_F[\Phi^* X, P_0] P_0)\,.
\end{aligned}
\tag{2.7}
$$

All of these terms are trace-class since they have finite rank. They are also separately bounded: The second and third terms are bounded because $P_0 \in \mathcal{A}_H^{0,0}$, by lemma 2.8. For the first term we additionally have $[\Phi^* X, H_F] \in L^\infty(\mathscr{L}(D(H_F), \mathscr{H}_F)|_U)$ as discussed in the proof of 2.7, so this is bounded as well. These terms are also continuous which can be seen in exactly the same way as the continuity of $\operatorname{tr} P_0$ above. Hence $X\lambda$ is bounded and continuous for every $X \in \Gamma_b(TB)$, therefore $\lambda \in \mathscr{C}_b^1(B)$. Repeating the same arguments for iterated commutators gives $\lambda \in \mathscr{C}_b^\infty(B)$. □

Example 2.10. We examine our recent results in the situation of $M = \mathbb{R} \times [0, 1+h]$ of example 2.2. The spectrum of $H_F(x)$ is given by the sequence of simple eigenvalues

$$\lambda_j(x) = \pi(j+1)/(1+h(x)) \qquad \text{with} \qquad j \in \mathbb{N}\,.$$

The corresponding eigenfunctions are

$$\phi_j(x,y) = \sqrt{\frac{2}{1+h(x)}} \sin\left(\frac{\pi(j+1)y}{1+h(x)}\right),$$

so all of the bands are simple and separated from each other by a gap. The unitary W from the proof of 2.7 is a map

$$W \colon L^2\big(\mathbb{R} \times [0,1], (1+h(x))\mathrm{d}x\mathrm{d}z\big) \to L^2(M, g)\,.$$

One can easily calculate

$$W^* H_F W = -(\Phi_* \partial_y)^2 = -(1+h(x))^{-2} \partial_z^2$$

and the image of P_0 for the j-th band

$$W^* P_0^j W = 2\sin((\pi(j+1)z) \langle \sin(\pi(j+1)z), \cdot \rangle_{L^2([0,1],\mathrm{d}z)} \,.$$

As in example 2.2 this gives $[\Phi^* \partial_x, H_F] = -2\log(1+h)' H_F$. Using (2.3) we obtain

$$\begin{aligned}
[\Phi^* \partial_x, P_0^j] &= \frac{\mathrm{i}}{2\pi} \int_{\gamma_j} [\Phi^* \partial_x, R_F(\zeta)] \mathrm{d}\zeta \\
&= -2\log(1+h)' H_F \frac{\mathrm{i}}{2\pi} \int_{\gamma_j} R_F(\zeta)^2 \mathrm{d}\zeta \\
&= 2\log(1+h)' H_F \frac{\mathrm{i}}{2\pi} \int_{\gamma_j} \partial_\zeta R_F(\zeta) \mathrm{d}\zeta = 0 \,,
\end{aligned}$$

which can of course be seen directly from $[\partial_x, W^* P_0^j W] = 0$. Inserting these objects into the formula (2.7) we correctly get

$$\partial_x \lambda_j = \mathrm{tr}([\Phi^* \partial_x, H_F] P_0^j) = -2\log(1+h)' H_F P_0^j = -2\log(1+h)' \lambda_j \,.$$

Observe that doing the same calculation formally for $[\partial_x^*, P_0^j]$ gives zero because $[\partial_x^*, \partial_y^2] = 0$, and this entails $\partial_x \lambda_j = 0$ which is of course not correct. The use of $\Phi^* \partial_x$ allows for a systematic treatment of commutators and may also simplify some explicit calculations, though it depends on the choice of trivialisation.

Corollary 2.11. $R_F(\lambda) := (H_F - \lambda)^{-1}(1 - P_0) \in \mathcal{A}_H^{0,0}$.

Proof. Follows directly from the lemmata 2.7, 2.8 and 2.9 together with the formula

$$R_F(\lambda) = (1-P_0) \frac{\mathrm{i}}{2\pi} \int_\gamma \frac{1}{\lambda - z} R_F(z) \mathrm{d}z (1 - P_0) \,.$$

□

Remark 2.12. Since P_0 is a projection it has the property that

$$[A, P_0] = [A, P_0^2] = P_0[A, P_0] + [A, P_0]P_0 \,,$$

and thus

$$P_0[A, P_0]P_0 = 2P_0[A, P_0]P_0 = 0 \,.$$

Hence the commutator is off-diagonal with respect to the splitting of $\mathscr{H} = P_0\mathscr{H} \oplus (1 - P_0)\mathscr{H}$ induced by P_0. This gives another perspective on why $[\Phi^*\partial_x, P_0]$ vanishes in the previous example: The commutator $[\Phi^*\partial_x, H_F]$ is proportional to H_F, so $[\Phi^*\partial_x, P_0]$ should be proportional to P_0 and hence must vanish because it is off-diagonal.

We will use this property of projections very frequently in the following construction of the super-adiabatic projections.

Proposition 2.13. *For all $N \in \mathbb{N}$ and $\Lambda > 0$ there exists an orthogonal projection $P_\varepsilon \in \mathscr{L}(\mathscr{H}) \cap \mathscr{L}(D(H))$ satisfying $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ in $\mathscr{L}(D(H))$ and*

$$\|[H, P_\varepsilon]\, \varrho(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{N+1}) \,,$$

for every Borel function $\varrho : \mathbb{R} \to [0, 1]$ with support in $(-\infty, \Lambda]$.

Before defining P_ε we construct a sequence of almost-projections P^N in $\mathcal{A}_H$ having the same asymptotic expansion in ε as P_ε. In this we follow the construction of [67, lemma 3.8], but instead of using the machinery of pseudodifferential calculus we give the expansion explicitly in form of commutators. This is possible because we are dealing with an eigenvalue with spectral gap, and not more general subsets of $\sigma(H_F)$.

Lemma 2.14. *For every $k \in \mathbb{N}$ there exists $P_k \in \mathcal{A}_H^{2^k,0}$, such that*

$$P^N = \sum_{k=0}^{N} \varepsilon^k P_k$$

satisfies

1) $(P^N)^2 - P^N \in \mathcal{A}_H^{2^{N+1},N+1}$,

2) $\big\| [H, P^N] \big\|_{2^N+2} = \mathcal{O}(\varepsilon^{N+1})$ *on* $D(H)$.

Proof. Take P_0 to be the projection on the eigenband λ as above. By lemma 2.8 we have $P_0 \in \mathcal{A}_H^{0,0} \subset \mathcal{A}_H^{1,0}$ and *1)* is trivially satisfied because it is a projection. For *2)* observe that by condition 2 we have $[H_1, P_0] = H_1P_0 - P_0H_1 = \mathcal{O}(1)$ and by (2.2) $\big\|[-\varepsilon^2\Delta_h, P_0]\big\|_2 = \mathcal{O}(\varepsilon)$, so *2)* holds.

We define P_{N+1} recursively by splitting it into diagonal and off-diagonal parts with respect to P_0 and prove *1)* and *2)* by induction. To shorten the notation we write $P_0^\perp := 1 - P_0$. Define

$$\varepsilon^{N+1}P_{N+1} := \underbrace{-P_0\left((P^N)^2 - P^N\right)P_0 + P_0^\perp\left((P^N)^2 - P^N\right)P_0^\perp}_{=:\varepsilon^{N+1}P^D_{N+1}} \underbrace{-P_0^\perp R_F(\lambda)\left[H, P^N\right]P_0 + P_0\left[H, P^N\right]R_F(\lambda)P_0^\perp}_{=:\varepsilon^{N+1}P^O_{N+1}} .$$

This is an element of $\mathcal{A}_H^{2^{N+1}}$ because of 2.5 and the fact that $\mathcal{A}_H$ is a right ideal, since $2^{N+1} \geq 2^N + 2$ for $N \geq 1$ and $P_1 \in \mathcal{A}_H^{2,0}$ because $P_0, R_F(\lambda) \in \mathcal{A}_H^{0,0}$ by 2.8, 2.11. P_{N+1} is of clearly order ε^0 by application of *1)* and *2)* to P^N, which is the induction hypothesis.

Proof of 1) We prove this for diagonal and off-diagonal parts separately. In both cases it is just a simple calculation using $P^N = P_0 + \mathcal{A}_H^{2^N,1} = P_0 + \mathcal{O}(\varepsilon)$.

- Diagonal:

$$\begin{aligned}
&P_0\left((P^{N+1})^2 - P^{N+1}\right)P_0 \\
&\quad= P_0\left((P^N + \varepsilon^{N+1}P_{N+1})^2 - P^N - \varepsilon^{N+1}P_{N+1}\right)P_0 \\
&\quad= P_0\left((P^N)^2 - P^N + \varepsilon^{N+1}\left(P^NP_{N+1} + P_{N+1}P^N - P_{N+1}\right)\right)P_0 \\
&\qquad\quad + \mathcal{A}_H^{2^{N+2},2N+2} \\
&\quad= \overbrace{P_0\left((P^N)^2 - P^N\right)P_0 + \varepsilon^{N+1}P_0P^D_{N+1}P_0}^{=0} + \mathcal{A}_H^{2^{N+2},N+2} \\
&\quad\in \mathcal{A}_H^{2^{N+2},N+2} .
\end{aligned}$$

- Off-diagonal:

$$
\begin{aligned}
&P_0^\perp \left((P^{N+1})^2 - P^{N+1}\right) P_0 \\
&\quad = P_0^\perp \left((P^N)^2 - P^N\right) P_0 + \varepsilon^{N+1} \overbrace{P_0^\perp \left(P_{N+1}P^N - P_{N+1}\right) P_0}^{\in \mathcal{A}_H^{2^{N+1}+2^N,1}} \\
&\qquad + \varepsilon^{N+1} \underbrace{P_0^\perp P^N P_{N+1} P_0}_{\in \mathcal{A}_H^{2^{N+1}+2^N,1}} + \mathcal{A}_H^{2^{N+2},2N+2} \\
&\quad = P_0^\perp \left((P^N)^2 - P^N\right) \left(P^N + P_0 - P^N\right) P_0 + \mathcal{A}_H^{2^{N+2},N+2} \\
&\quad = P_0^\perp \left((P^N)^2 - P^N\right) P^N P_0 + \mathcal{A}_H^{2^{N+2},N+2} \\
&\quad = P_0^\perp P^N \left((P^N)^2 - P^N\right) P_0 + \mathcal{A}_H^{2^{N+2},N+2} \\
&\quad \in \mathcal{A}_H^{2^{N+2},N+2} .
\end{aligned}
$$

The calculations for the $P_0^\perp$-$P_0^\perp$ and P_0-$P_0^\perp$ blocks are basically the same, so *1)* is verified.

Proof of 2)

- Diagonal: We will only do the calculation for the P_0-block. The one for $P_0^\perp$ is similar. One merely needs to use (2.2) instead of lemma 2.5, just as in proving $[H, P_0] = \mathcal{O}(\varepsilon)$ in the beginning. First we show $P_0[H, \varepsilon^{N+1} P^O_{N+1}]P_0 = \mathcal{O}(\varepsilon^{N+2})$:

$$
\begin{aligned}
&P_0 \left[H, \varepsilon^{N+1} P^O_{N+1}\right] P_0 \\
&\quad = \varepsilon^{N+1} P_0 \left(H P_0^\perp P^O_{N+1} - P^O_{N+1} P_0^\perp H\right) P_0 \\
&\quad = \varepsilon^{N+1} \Big(-P_0 \underbrace{[H, P_0] P^O_{N+1}}_{\in \mathcal{A}^{2^{N+1}+2,1}} P_0 - P_0 P^O_{N+1} P_0^\perp \underbrace{[H, P_0] P_0}_{\in \mathcal{A}^{2,1}}\Big) \\
&\quad = \mathcal{O}(\varepsilon^{N+2}) .
\end{aligned}
$$

Now by definition $P^{N+1} - \varepsilon^{N+1} P^O_{N+1} = P^N + \varepsilon^{N+1} P^D_{N+1}$, so we still have to calculate

$$
\begin{aligned}
&P_0 \left[H, P^N + \varepsilon^{N+1} P^D_{N+1}\right] P_0 \\
&\quad = P_0 \left[H, P^N - P_0((P^N)^2 - P^N)P_0\right] P_0 \\
&\quad = 2P_0 \left[H, P^N\right] P_0 - P_0 \left[H, (P^N)^2\right] P_0 \\
&\qquad + \underbrace{\left(P_0[H, P_0]((P^N)^2 - P^N)P_0 + P_0((P^N)^2 - P^N)[H, P_0]P_0\right)}_{\in \mathcal{A}_H^{2^{N+1}+2, N+2} \text{ by induction hypothesis and 2.5}} \\
&\quad = P_0 \left(2\left[H, P^N\right] - P^N \left[H, P^N\right] - \left[H, P^N\right] P^N\right) P_0 + \mathcal{O}(\varepsilon^{N+2}) \\
&\quad = \underbrace{P_0 \left((P_0 - P^N)\left[H, P^N\right] + \left[H, P^N\right](P_0 - P^N)\right) P_0}_{\in \mathcal{A}_H^{2^{N+1}+2, N+2}} + \mathcal{O}(\varepsilon^{N+2}) \\
&\quad = \mathcal{O}(\varepsilon^{N+2}) \,.
\end{aligned}
$$

- Off-diagonal:
  Here we make use of lemmas 2.9 and 2.5 to get

$$
[-\varepsilon^2 \Delta_h + \lambda, P_{N+1}] P_0 \in \mathcal{A}^{2^{N+1}+1,1} \,.
$$

This gives us

$$
[H, P_{N+1}]P_0 = [H_F - \lambda, P_{N+1}]P_0 + \underbrace{[-\varepsilon^2 \Delta_h + \lambda + \varepsilon H_1, P_{N+1}]P_0}_{\in \mathcal{A}^{2^{N+1}+2,1}} \,.
$$

We insert this into

$$
\begin{aligned}
&P_0^\perp \left[H, P^N + \varepsilon^{N+1} P_{N+1}\right] P_0 \\
&\quad = P_0^\perp \left(\left[H, P^N\right] + \varepsilon^{N+1} \left[H_F - \lambda, P_{N+1}\right]\right) P_0 + \mathcal{O}(\varepsilon^{N+2}) \\
&\quad = P_0^\perp \left(\left[H, P^N\right] + \varepsilon^{N+1}[H_F - \lambda, P_0^\perp P_{N+1} P_0]\right) P_0 + \mathcal{O}(\varepsilon^{N+2}) \\
&\quad = P_0^\perp(\left[H, P^N\right] - \underbrace{(H_F - \lambda) R_F(\lambda)}_{=1} \left[H, P^N\right]) P_0 + \mathcal{O}(\varepsilon^{N+2}) \\
&\quad = \mathcal{O}(\varepsilon^{N+2}) \,,
\end{aligned}
$$

which completes the proof for the $P_0^\perp$-P_0-block. The argument for the other off-diagonal block is the same, using (2.2).

□

Example 2.15. We continue the discussion of $M = \mathbb{R} \times [0, 1+h]$ started in examples 2.2 and 2.10. We explicitly construct the leading part of P_1 for this example. Since P_0 is a projection, in general $P_1^D = 0$. We first calculate

$$\begin{aligned}
[H, P_0]P_0 &= -\varepsilon^2 P_0^\perp [\partial_x^2, P_0] P_0 \\
&= -\varepsilon^2 P_0^\perp \big(\partial_x^* [\partial_x^*, P_0] + [\partial_x^*, P_0] \partial_x^* \big) P_0 \\
&= -\varepsilon^2 P_0^\perp \big(\underbrace{[\partial_x^*, [\partial_x^*, P_0]P_0]P_0}_{\in \mathcal{A}^{0,0}} + 2[\partial_x^*, P_0] P_0 \partial_x^* P_0 \big) \\
&= -\varepsilon^2 2 [\partial_x^*, P_0] P_0 \partial_x^* P_0 + \mathcal{O}(\varepsilon^2)\,.
\end{aligned}$$

So up to terms of order ε^2 we have

$$P_0^\perp P_1 P_0 = 2 P_0^\perp R_F(\lambda) [\partial_x^*, P_0] P_0 \varepsilon \partial_x^* P_0\,.$$

In order to evaluate this for the j-th eigenband we calculate

$$\begin{aligned}
[\partial_x^*, P_0] P_0 &= P_0^\perp \partial_x^* P_0 \\
&= P_0^\perp \Phi^* \partial_x P_0 - \log(1+h)' P_0^\perp y \partial_y P_0 \\
&= -\log(1+h)' \sum_{k \neq j} P_0^k y \partial_y P_0^j \\
&= -\log(1+h)' \sum_{k \neq j} \phi_k a_{jk} \langle \phi_j, \cdot \rangle_{L^2([0,1+h], \mathrm{d}y)}\,,
\end{aligned}$$

with

$$a_{jk} = 2\pi(j+1) \int_0^1 z \sin(\pi(k+1)z) \cos(\pi(j+1)z) \mathrm{d}z\,.$$

If we denote by $\nabla^B_{\partial_x} := P_0^j \partial_x P_0^j$ the Berry connection for this band, P_1 takes the form

$$P_1 P_0^j \psi = -2 \log(1+h)' \sum_{k \neq j} \phi_k \frac{a_{jk}}{\lambda_k - \lambda_j} \varepsilon \nabla^B_{\partial_x} P_0^j \psi + \mathcal{O}(\varepsilon)\,,$$

and $P_0^j P_1 = (P_1 P_0^j)^*$.

Now that we have constructed the approximate projections P^N, the super-adiabatic projections can be obtained by a regularisation procedure.

Proof of proposition 2.13. To prove the statement for $N \in \mathbb{N}$ and $\Lambda > 0$, take P^N from lemma 2.14 and let $\chi_1 \in \mathscr{C}_0^\infty(\mathbb{R}, [0,1])$ be a regular cut-off (see C.1), equal to one if $x \in [\inf \sigma(H) - 1, \Lambda + 1]$ and equal to zero if $x \notin (\inf \sigma(H) - 2, \Lambda + 2)$. Put $\tilde{P} := P^N - P_0 \in \mathcal{A}_H^{2^N,1}$ and define

$$P^\chi := P_0 + \tilde{P}\chi_1(H) + \chi_1(H)\tilde{P}\,(1 - \chi_1(H)) = P_0 + \mathcal{O}(\varepsilon)\,.$$

For every $m \in \mathbb{N}$ we have $\chi_1 \in \mathscr{L}(\mathscr{H}, D(H^m))$ and by elliptic regularity (see A.14) $D(H^m) \subset W_\varepsilon^{2m}$, so $\tilde{P}\chi_1 \in \mathscr{L}(\mathscr{H}) \cap \mathscr{L}(D(H))$. Therefore its adjoint is also a bounded operator and from the construction of P_N we can see that $\chi_1\tilde{P} = (\tilde{P}\chi_1)^*$ on $W_\varepsilon^{2^N}$, so they are equal in $\mathscr{L}(\mathscr{H})$ because $W_\varepsilon^{2^N}$ is a dense subspace of $\mathscr{H}$. Hence $P^\chi \in \mathscr{L}(\mathscr{H})$ is self-adjoint by construction.

We want to prove that also $P^\chi \in \mathscr{L}(D(H))$. To show $\chi_1\tilde{P} \in \mathscr{L}(D(H))$ we need to show $[H, \chi_1\tilde{P}] = \chi_1[H, \tilde{P}] \in \mathscr{L}(D(H), \mathscr{H})$. But actually, by the same argument as before, we have $\chi_1[H, \tilde{P}] = ([\tilde{P}, H]\chi_1)^*$ on $W_\varepsilon^{2^N+2} \cap D(H)$, and thus $\chi_1[H, \tilde{P}] \in \mathscr{L}(\mathscr{H})$. These norms are of order ε because $\tilde{P} \in \mathcal{A}_H^{2^N,1}$. Consequently $P^\chi - P_0 = \mathcal{O}(\varepsilon)$ in $\mathscr{L}(\mathscr{H})$ as well as $\mathscr{L}(D(H))$. We conclude that

$$\|[H, P^\chi]\|_{\mathscr{L}(D(H),\mathscr{H})} = \|[H, P_0]\|_{\mathscr{L}(D(H),\mathscr{H})} + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon)\,. \tag{2.8}$$

Now let $\chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}, [0,1])$ be another regular cut-off, equal to one on $[\inf \sigma(H), \Lambda]$ and equal to zero where $\chi_1 \neq 1$. Then we have $\chi_1\chi_2 = \chi_2$, $(1 - \chi_1)\chi_2 = 0$ and from lemma 2.14 we get

$$\|[H, P^\chi]\,\chi_2(H)\|_{\mathscr{L}(\mathscr{H})} = \|[H, P^N]\chi_2(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{N+1})\,. \tag{2.9}$$

Since P^χ is close to the projection P_0 we have for $m \in \{0, 1\}$:

$$\|(P^\chi)^2 - P^\chi\|_{\mathscr{L}(D(H^m))} = \mathcal{O}(\varepsilon)\,.$$

Thus there is a constant $C > 0$ such that the spectrum of P^χ (as an operator in $\mathscr{L}(\mathscr{H})$ as well as $\mathscr{L}(D(H))$) satisfies

$$\sigma(P^\chi) \subset [-C\varepsilon, C\varepsilon] \cup [1 - C\varepsilon, 1 + C\varepsilon]\,.$$

Take γ to be the circle of radius 1/2 around $z = 1$. Then for $\varepsilon < (4C)^{-1}$ the integral

$$P_\varepsilon := \frac{\mathrm{i}}{2\pi} \int_\gamma (P^\chi - z)^{-1} \,\mathrm{d}z$$

exists and is bounded by two in the norms of $\mathscr{L}(\mathscr{H})$ and $\mathscr{L}(D(H))$. It defines an orthogonal projection by the functional calculus.

To complete the proof of the proposition we will need to control the commutator of $R^\chi(z) := (P^\chi - z)^{-1}$ with $\chi_2(H)$. First of all $(1-\chi_1)\chi_2^r = 0$ for every $r > 0$, so (2.9) holds for every positive power of χ_2 and we can apply lemma C.2 with $T = P^\chi$ to get

$$\|[P^\chi, \chi_2]\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1})\,. \tag{2.10}$$

Then

$$\begin{aligned} &\|[R^\chi(z), \chi_2]\|_{\mathscr{L}(\mathscr{H}, D(H))} \\ &\qquad = \|R^\chi(z)[P^\chi, \chi_2]R^\chi(z)\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1})\,. \end{aligned} \tag{2.11}$$

Now since $\chi_2(H)\varrho(H) = \varrho(H)$ we have

$$\begin{aligned} &\|[H, P_\varepsilon]\varrho(H)\|_{\mathscr{L}(\mathscr{H})} \\ &\quad = \left\| \frac{\mathrm{i}}{2\pi} \int_\gamma R^\chi(z)\,[H, P^\chi]\,R^\chi(z)\chi_2(H)\varrho(H)\mathrm{d}z \right\| \\ &\quad = \Bigg\| \frac{\mathrm{i}}{2\pi} \int_\gamma R^\chi(z)\,[H, P^\chi]\,\chi_2(H)R^\chi(z)\varrho(H) \\ &\qquad\qquad + R^\chi(z)\,\underbrace{[H, P^\chi]}_{\overset{(2.8)}{=}\mathcal{O}(\varepsilon)}\,\underbrace{[R^\chi(z), \chi_2(H)]}_{\overset{(2.11)}{=}\mathcal{O}(\varepsilon^{N+1})}\,\varrho(H)\mathrm{d}z \Bigg\| \\ &\quad \le \left\| \frac{\mathrm{i}}{2\pi} \int_\gamma R^\chi(z)\,\underbrace{[H, P^\chi]\,\chi_2(H)}_{\overset{(2.9)}{=}\mathcal{O}(\varepsilon^{N+1})}\,R^\chi(z)\varrho(H)\mathrm{d}z \right\| + \mathcal{O}(\varepsilon^{N+2}) \\ &\quad = \mathcal{O}(\varepsilon^{N+1})\,. \end{aligned}$$

□

2.2 Main results

We are now ready to define the effective operator H_{eff} as a self-adjoint operator on $L^2(\mathcal{E})$ and prove that it approximates H in the sense of spectra and unitary groups. Using the results of the previous section the proofs rely mainly on standard techniques of perturbation theory. Results for the approximation of other quantities can be obtained in similar ways. For example the proof of the approximation for the unitary groups translates directly to an L^2-estimate for the heat semigroups and estimates in other norms can be derived from the same ideas.

We start by proving the approximate invariance of the image of P_ε, obtained from proposition 2.13 for fixed N and Λ, under the unitary group $\mathrm{e}^{-\mathrm{i}Ht}$.

Lemma 2.16. *Let P_ε be the projection of proposition 2.13, then there exist positive constants $C, \varepsilon_0 > 0$ such that*

$$\left\|[\mathrm{e}^{-\mathrm{i}Ht}, P_\varepsilon]1_{(-\infty,\Lambda]}(H)\right\|_{\mathscr{L}(\mathscr{H})} \leq C\varepsilon^{N+1}\,|t|$$

for every $\varepsilon \leq \varepsilon_0$.

Proof. We start by calculating

$$\begin{aligned}
&[\mathrm{e}^{-\mathrm{i}Ht}, P_\varepsilon]1_{(-\infty,\Lambda]}(H) \\
&\qquad = \mathrm{e}^{-\mathrm{i}Ht}\left(P_\varepsilon - \mathrm{e}^{\mathrm{i}Ht}P_\varepsilon \mathrm{e}^{-\mathrm{i}Ht}\right)1_{(-\infty,\Lambda]} \\
&\qquad = -\mathrm{i}\mathrm{e}^{-\mathrm{i}Ht}\int_0^t \mathrm{e}^{\mathrm{i}Hs}\left(HP_\varepsilon - P_\varepsilon H\right)\mathrm{e}^{-\mathrm{i}Hs}1_{(-\infty,\Lambda]}\,\mathrm{d}s \\
&\qquad = -\mathrm{i}\mathrm{e}^{-\mathrm{i}Ht}\int_0^t \mathrm{e}^{\mathrm{i}Hs}[H, P_\varepsilon]1_{(-\infty,\Lambda]}\mathrm{e}^{-\mathrm{i}Hs}\,\mathrm{d}s\,.
\end{aligned}$$

Now by 2.13 we have $\|[H, P_\varepsilon]1_{(-\infty,\Lambda]}(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{N+1})$, which is enough to prove the claim since $\|\mathrm{e}^{\mathrm{i}Hs}\|_{\mathscr{L}(\mathscr{H})} = 1$. □

Given that the image of P_ε is almost invariant under $\mathrm{e}^{-\mathrm{i}Ht}$ one may think to approximate the unitary group by $\mathrm{e}^{-\mathrm{i}P_\varepsilon HP_\varepsilon t}$. Not much is known however about this space, except that it is ε-close to the image of P_0. This property allows us to unitarily map $P_\varepsilon\mathscr{H}$ to $P_0\mathscr{H} = L^2(\mathcal{E})$, which has a much more accessible description.

Theorem 2.17. *Let $N \in \mathbb{N}$, $\Lambda > 0$ and P_ε be the projection constructed in proposition 2.13. There exists $\varepsilon_0 > 0$ such that there is a unitary $U_\varepsilon \in \mathscr{L}(\mathscr{H})$ with $U_\varepsilon P_0 = P_\varepsilon U_\varepsilon$. Furthermore $U_\varepsilon \in \mathscr{L}(D(H))$, the operator $H_{\text{eff}} := U_\varepsilon^* P_\varepsilon H P_\varepsilon U_\varepsilon$ is self-adjoint on $D_{\text{eff}} := U_\varepsilon^* P_\varepsilon D(H) \subset L^2(\mathcal{E})$ and satisfies*

$$\left\| \left(\mathrm{e}^{-\mathrm{i}Ht} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\text{eff}}t} U_\varepsilon^*\right) P_\varepsilon 1_{(-\infty,\Lambda]}(H) \right\|_{\mathscr{L}(\mathscr{H})} \leq C\varepsilon^{N+1} \, |t|$$

for every $\varepsilon \leq \varepsilon_0$.

Proof. Define $S := 1 - (P_\varepsilon - P_0)^2$. This is an invertible, positive operator since $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ (cf. the proof of 2.13). We can thus define U_ε by the Sz.-Nagy formula

$$U_\varepsilon := \big(P_\varepsilon P_0 + (1 - P_\varepsilon)\,(1 - P_0)\,\big) S^{-1/2} \tag{2.12}$$

and the required properties are easily checked after noting that S commutes with P_ε and P_0. The unitary is discussed in more detail in section 2.2.1.

As to self-adjointness of H_{eff} we can see that $H^D := P_\varepsilon H P_\varepsilon + P_\varepsilon^\perp H P_\varepsilon^\perp$ is self-adjoint on $D(H)$: The difference

$$H^D - H = -P_\varepsilon H(1 - P_\varepsilon) - (1 - P_\varepsilon) H P_\varepsilon = [H, P_\varepsilon](1 - 2P_\varepsilon)$$

is of order ε in $\mathscr{L}\left(D(H), \mathscr{H}\right)$. This means that for ε small enough $H^D - H$ is bounded relative to H with relative bound less than one, so H^D is self-adjoint on $D(H)$ by the Kato-Rellich theorem. Since H^D commutes with P_ε this shows self-adjointness of $(P_\varepsilon H P_\varepsilon, P_\varepsilon D(H))$ on $P_\varepsilon \mathscr{H}$, which is unitarily equivalent to $(H_{\text{eff}}, D_{\text{eff}})$ via U_ε.

To check the approximation of unitary groups we use Duhamel's formula

$$\mathrm{e}^{-\mathrm{i}Ht} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\text{eff}}t} U_\varepsilon^* = -\mathrm{i} \int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\text{eff}}(t-s)} U_\varepsilon^* (H - \underbrace{U_\varepsilon H_{\text{eff}} U_\varepsilon^*}_{= P_\varepsilon H P_\varepsilon}) \mathrm{e}^{-\mathrm{i}Hs} \, \mathrm{d}s$$

and since $U_\varepsilon H_{\mathrm{eff}} U_\varepsilon^* = P_\varepsilon H P_\varepsilon$ commutes with P_ε

$$\begin{aligned}
&\big(\mathrm{e}^{-\mathrm{i}Ht} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t} U_\varepsilon^*\big) P_\varepsilon 1_{(-\infty,\Lambda]}(H) \\
&\quad = \Big(P_\varepsilon \big(\mathrm{e}^{-\mathrm{i}Ht} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}t} U_\varepsilon^*\big) + [\mathrm{e}^{-\mathrm{i}Ht}, P_\varepsilon]\Big) 1_{(-\infty,\Lambda]}(H) \\
&\quad = \Big(- \mathrm{i}P_\varepsilon \int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}(t-s)} U_\varepsilon^* (H - P_\varepsilon H P_\varepsilon) \mathrm{e}^{-\mathrm{i}Hs}\,\mathrm{d}s \\
&\qquad\qquad + [\mathrm{e}^{-\mathrm{i}Ht}, P_\varepsilon]\Big) 1_{(-\infty,\Lambda]} \\
&\quad = -\mathrm{i} \int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}(t-s)} U_\varepsilon^* \underbrace{(P_\varepsilon H - P_\varepsilon H P_\varepsilon)}_{= -P_\varepsilon [H, P_\varepsilon]} 1_{(-\infty,\Lambda]} \mathrm{e}^{-\mathrm{i}Hs}\,\mathrm{d}s \\
&\qquad\qquad + [\mathrm{e}^{-\mathrm{i}Ht}, P_\varepsilon] 1_{(-\infty,\Lambda]}.
\end{aligned}$$

The first term is of order $\varepsilon^{N+1}\,|t|$ since the integrand is of order ε^{N+1} by 2.13. The second term is of the same order by lemma 2.16. □

What this theorem tells us is that if we start in the correct subspace $P_\varepsilon \mathscr{H}$ of $\mathscr{H}$ and with energy below Λ, then we stay in this subspace and the dynamics is determined by the unitary group of H_{eff} up to times of order ε^{-N}. The requirement of starting in the correct subspace is of course essential, because in a different subspace, say one constructed from a different eigenband $\tilde{\lambda}$, the dynamics will be quite different.

Theorem 2.18. *Let $N \in \mathbb{N}$, $\Lambda > 0$ and H_{eff} be the effective operator of theorem 2.17. Then for every $\delta > 0$ there exist constants C and $\varepsilon_0 > 0$ such that for every $\mu \in \sigma(H_{\mathrm{eff}})$ with $\mu \leq \Lambda - \delta$ and all $\varepsilon < \varepsilon_0$:*

$$\mathrm{dist}(\mu, \sigma(H)) \leq C\varepsilon^{N+1}.$$

Proof. Let $(\psi_k)_{k\in\mathbb{N}} \subset L^2(\mathcal{E})$ be a Weyl sequence for μ, i.e. for every $k \in \mathbb{N}$ $\|\psi_k\| = 1$ and $\lim_{k\to\infty} \|(H_{\mathrm{eff}} - \mu)\,\psi_k\| = 0$. We can even choose the ψ_k in the image of $1_{(-\infty,D]}(H_{\mathrm{eff}})$, with $D = \Lambda - \delta/2$, because μ is in the spectrum of H_{eff} restricted to this space. Then because $\psi_k \in P_0\mathscr{H}$

$$\begin{aligned}
&\|(H-\mu) U_\varepsilon \psi_k\|_{\mathscr{H}} \\
&\quad = \big\|(H-\mu)\, P_\varepsilon U_\varepsilon 1_{(-\infty,D]}(H_{\mathrm{eff}})\psi_k\big\| \\
&\quad \leq \|U_\varepsilon\,(H_{\mathrm{eff}} - \mu)\,\psi_k\| + \|P_\varepsilon^\perp H P_\varepsilon U_\varepsilon 1_{(-\infty,D]}(H_{\mathrm{eff}})\psi_k\|\,. \qquad (2.13)
\end{aligned}$$

Let χ be a regular cut-off with support in $(-\infty, \Lambda]$ that equals one on $(-\infty, D] \cap \sigma(H_{\mathrm{eff}})$. Then by lemma C.2

$$\begin{aligned}1_{(-\infty,D]}(H_{\mathrm{eff}}) &= \chi(H_{\mathrm{eff}})1_{(-\infty,D]}(H_{\mathrm{eff}}) \\ &= U_\varepsilon^* P_\varepsilon \chi(H) P_\varepsilon U_\varepsilon 1_{(-\infty,D]}(H_{\mathrm{eff}}) + \mathcal{O}(\varepsilon^{N+1}) .\end{aligned}$$

Using this and proposition 2.13 with $\varrho = \chi$ gives a bound on the second term

$$\begin{aligned}P_\varepsilon^\perp H P_\varepsilon U_\varepsilon 1_{(-\infty,D]}(H_{\mathrm{eff}}) &= [H, P_\varepsilon] P_\varepsilon \chi(H) P_\varepsilon U_\varepsilon 1_{(-\infty,D]}(H_{\mathrm{eff}}) + \mathcal{O}(\varepsilon^{N+1}) \\ &= \mathcal{O}(\varepsilon^{N+1}) .\end{aligned}$$

For the first one we can then simply choose k large enough for it to be smaller than the second term. This shows that for $\varphi = U_\varepsilon \psi_k$

$$\|(H-\mu)\varphi\|_{\mathscr{H}} \leq C\varepsilon^{N+1} .$$

So either $(H-\mu)\varphi = 0$ and μ is an eigenvalue of H, or the vector $\|(H-\mu)\varphi\|^{-1}(H-\mu)\varphi$ is normalised and

$$\begin{aligned}\operatorname{dist}(\mu, \sigma(H))^{-1} &= \|(H-\mu)^{-1}\|_{\mathscr{L}(\mathscr{H})} \\ &\geq \frac{1}{\|(H-\mu)\varphi\|} \|(H-\mu)^{-1}(H-\mu)\varphi\|_{\mathscr{H}} \\ &\geq \frac{1}{C\varepsilon^{N+1}} .\end{aligned}$$

□

Remark 2.19. We can observe that the proof of the previous theorem relies on the fact that the quasi-modes $U_\varepsilon \psi_k$ have unit norm. If we have $\mu \in \sigma(H)$ with Weyl sequence $(\varphi_k)_{k\in\mathbb{N}}$ the natural choice of quasi-modes for H_{eff} would be $U_\varepsilon^* P_\varepsilon \varphi_k$. If this sequence is bounded below in norm we can easily reproduce the proof to obtain $\operatorname{dist}(\mu, \sigma(H_{\mathrm{eff}})) \leq C\varepsilon^{N+1}$.

If the spectrum of H_F consists only of separated bands, the whole Hilbert space decomposes as $\mathscr{H} = \bigoplus_{j=0}^\infty P_0^j \mathscr{H}$. If ψ has energy below Λ only the projections onto bands with $\inf_{x\in B} \lambda_j(x) < \Lambda$ should give significant contributions on ψ, since

$$\Lambda \geq \langle \psi, H\psi \rangle = \sum_{j\in\mathbb{N}} \langle \psi, \left(-\varepsilon^2 \Delta_h + \varepsilon H_1 + \lambda_j(x)\right) P_0^j \psi \rangle .$$

But these are only finitely many, so $\Lambda > \mu \in \sigma(H)$ is expected to be associated with the effective operator of (at least) one band with $\inf_{x\in B} \lambda_j(x) < \Lambda$. In particular if $\Lambda \leq \inf_{x\in B} \lambda_1(x)$ this operator should be the effective operator for λ_0. To make this more precise, let $\lambda_0(x) = \min \sigma(H_F(x))$ be the smallest eigenvalue of $H_F(x)$. Suppose this satisfies the gap condition and that $-\varepsilon^2\Delta_h + \varepsilon H_1$ is bounded below by $-C\varepsilon$. Let P_ε be the super-adiabatic projection of some order N and put $\Lambda_1 := \inf_{x\in B} \inf \sigma(H_F(x))\backslash\lambda_0$, then given $c > 0$ there is ε_0 such that for every $\psi \in D(H)$ and $\varepsilon \leq \varepsilon_0$:

$$\begin{aligned}
&\langle \psi, P_\varepsilon^\perp H P_\varepsilon^\perp \psi\rangle \\
&\quad = \langle P_0^\perp \psi, H P_0^\perp \psi\rangle + \mathcal{O}(\varepsilon) \\
&\quad = \underbrace{\langle P_0^\perp \psi, (-\varepsilon^2\Delta_h + \varepsilon H_1) P_0^\perp \psi\rangle}_{\geq -C\varepsilon} + \underbrace{\langle P_0^\perp \psi, H_F P_0^\perp \psi\rangle}_{\geq \Lambda_1} + \mathcal{O}(\varepsilon) \\
&\quad \geq \Lambda_1 - c\,.
\end{aligned} \tag{2.14}$$

Of course $-\varepsilon^2\Delta_h + \varepsilon H_1$ has such a lower bound if $H_1 = 0$ or if it originates from an embedding that only corrects the horizontal part of the metric (see sections 1.1.3 and 3.1.1).

Theorem 2.20. *Let $-\varepsilon^2\Delta_h + \varepsilon H_1$ be bounded below by $-C\varepsilon$, λ_0 an eigenband with spectral gap (condition 3) and χ be a regular cut-off with* $\operatorname{supp}\chi \subset (-\infty, \Lambda_1)$. *Then, for ε small enough the effective operator of theorem 2.17 with energy cut-off χ, $H_{\mathrm{eff}}\chi(H_{\mathrm{eff}})$, is unitarily equivalent to $H\chi(H)$ up to errors of order ε^{N+1} in $\mathscr{L}(\mathscr{H})$.*

Proof. We have

$$\begin{aligned}
H\chi(H) &= \big(H^D + (1 - 2P_\varepsilon)[H, P_\varepsilon]\big)\,\chi(H) \\
&\overset{2.13}{=} U_\varepsilon H_{\mathrm{eff}} U_\varepsilon^* \chi(H) + P_\varepsilon^\perp H P_\varepsilon^\perp \chi(H) + \mathcal{O}(\varepsilon^{N+1})\,.
\end{aligned}$$

The proof may thus be completed by showing that $P_\varepsilon^\perp \chi(H) = \mathcal{O}(\varepsilon^{N+1})$ and $U_\varepsilon^* \chi(H) U_\varepsilon = \chi(H_{\mathrm{eff}}) + \mathcal{O}(\varepsilon^{N+1})$. Both these statements are implied by $\chi(H) - \chi(P_\varepsilon H P_\varepsilon) = \mathcal{O}(\varepsilon^{N+1})$, which we prove in a separate lemma. □

Lemma 2.21. *Let* $-\varepsilon^2\Delta_h + \varepsilon H_1 \geq -C\varepsilon$ *and* χ *be a regular cut-off with* $\operatorname{supp}\chi \subset (-\infty, \Lambda_1)$. *Then*

$$\|\chi(H) - \chi(P_\varepsilon H P_\varepsilon)\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1}) .$$

Proof. We first apply lemma C.2 with $T = P_\varepsilon^\perp$ to get

$$\left\|P_\varepsilon^\perp \chi(H) P_\varepsilon^\perp - \chi(P_\varepsilon^\perp H P_\varepsilon^\perp)\right\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1}) .$$

Then observe that for ε small enough $\operatorname{supp}\chi \cap \sigma(P_\varepsilon^\perp H P_\varepsilon^\perp) = \varnothing$ by (2.14), so $\chi(P_\varepsilon^\perp H P_\varepsilon^\perp) = 0$. Another application of lemma C.2 gives

$$\left\|P_\varepsilon^\perp \chi(H) P_\varepsilon\right\|_{\mathscr{L}(\mathscr{H}, D(H))} = \|[\chi(H), P_\varepsilon] P_\varepsilon\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1}) .$$

Together these statements imply

$$\left\|P_\varepsilon^\perp \chi(H)\right\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1}) = \left\|\chi(H) P_\varepsilon^\perp\right\|_{\mathscr{L}(\mathscr{H}, D(H))} ,$$

whereby

$$\|\chi(H) - P_\varepsilon \chi(H) P_\varepsilon\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^{N+1}) .$$

A final use of lemma C.2 with $T = P_\varepsilon$ concludes the proof. □

2.2.1 Expansion of the effective Hamiltonian

For a better understanding of the effective operator we expand it in powers of ε. We also identify how the terms appearing in the construction of P_ε in lemma 2.14 enter into the effective Hamiltonian. From lemma 2.14 and the proof of proposition 2.13 we already know a lot about the expansion of P_ε, so first we will need to take a closer look at U_ε. By definition we have

$$P_\varepsilon U_\varepsilon = P_\varepsilon P_0 \big(1 - (P_\varepsilon - P_0)^2\big)^{-1/2} .$$

Since $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ we can expand U_ε by using the power series

$$(1 - z^2)^{-1/2} = 1 + \sum_{k=1}^\infty a_k z^{2k}, \qquad \text{with} \qquad a_k = \frac{(2k-1)!}{2^{2k-1} k!(k-1)!} ,$$

which converges for $|z| < 1$. This gives

$$\begin{aligned} P_\varepsilon U_\varepsilon &= P_\varepsilon P_0 + \sum_{k=1}^{\infty} a_k P_\varepsilon P_0 (P_\varepsilon - P_0)^{2k} \\ &= P_0 + (P_\varepsilon - P_0)P_0 + \mathcal{O}(\varepsilon^2)\,. \end{aligned}$$

The corresponding expansion of $H_{\mathrm{eff}} = U_\varepsilon^* P_\varepsilon H P_\varepsilon U_\varepsilon =: P_0 H P_0 + H_{\mathrm{sa}}$ consists of the leading adiabatic part $H_{\mathrm{a}} := P_0 H P_0$ and additional super-adiabatic corrections H_{sa} originating from higher order terms in P_ε.

As to the adiabatic part we define a connection ∇^B on $\mathcal{E}$ using that $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M, \mathbb{C})$ (see appendix B.2) and setting

$$\nabla^B_X \psi := P_0 X^* P_0 \psi\,. \tag{2.15}$$

This is usually called the Berry connection. Since the volume of the fibres may vary this need not be a metric connection, as we will see in the following. Let $X \in \Gamma(TB)$ and $\psi, \varphi \in \Gamma(\mathcal{E})$. These functions vanish on the boundary of F, so

$$\begin{aligned} &X \langle \varphi, \psi \rangle_{\mathcal{E}} \\ &= \mathcal{L}_X \int_F P_0 \overline{\varphi}_x(y) P_0 \psi_x(y) \,\mathrm{vol}_{F_x}(\mathrm{d}y) \\ &= \left\langle \nabla^B_X \varphi, \psi \right\rangle_{\mathcal{E}} + \left\langle \varphi, \nabla^B_X \psi \right\rangle_{\mathcal{E}} + \int_F \overline{\varphi}_x(y) \psi_x(y) \left(\mathcal{L}_{X^*} \mathrm{vol}_{F_x}(\mathrm{d}y)\right) \\ &= \left\langle \nabla^B_X \phi, \psi \right\rangle_{\mathcal{E}} + \left\langle \phi, \nabla^B_X \psi \right\rangle_{\mathcal{E}} - \int_F \overline{\varphi}_x(y) \psi_x(y) g_B(X, \pi_* \eta) \,\mathrm{vol}_{F_x}(\mathrm{d}y) \\ &=: \left\langle \nabla^B_X \phi, \psi \right\rangle_{\mathcal{E}} + \left\langle \phi, \nabla^B_X \psi \right\rangle_{\mathcal{E}} - \langle \phi, \bar{\eta}(X) \psi \rangle_{\mathcal{E}}\ , \end{aligned} \tag{2.16}$$

by the variation of area formula $\mathcal{L}_{X^*}\mathrm{vol}_{F_x} = -g_B(X, \pi_*\eta)\mathrm{vol}_{F_x}$. This defines a form $\bar{\eta} \in \Gamma\left(T^*B\right) \otimes \mathrm{End}(\mathcal{E})$ by $\bar{\eta}(X) = P_0 g_B(X, \pi_*\eta) P_0$. It is basically given by the average of the mean curvature, weighted with the λ-eigenfunctions of H_F. That is, given a basis $(\phi_i)_{i \le k}$ of $\mathcal{E}_{x_0}$

$$\bar{\eta}_{ij}(X_{x_0}) = \int_F \overline{\phi}_i(y) \phi_j(y) g_B(X_{x_0}, \pi_*\eta) \,\mathrm{vol}_{F_{x_0}}(y)\,.$$

Let ψ be a smooth section of $\mathcal{E}$ with support in $U \in \mathfrak{U}$. We define the Laplacian Δ_B by the quadratic form

$$\begin{aligned}
&\int_B \langle \psi, -\Delta_B \psi\rangle_{\mathcal{E}} \operatorname{vol}_{g_B} \\
&\quad = \int_B \langle \nabla^B \psi, \nabla^B \psi\rangle_{T^*B\otimes\mathcal{E}} \operatorname{vol}_{g_B} \\
&\quad = \sum_{i\leq d} \int_B \langle \nabla^B_{X_i}\psi, \nabla^B_{X_i}\psi\rangle_{\mathcal{E}} \operatorname{vol}_{g_B} \\
&\quad = -\sum_{i\leq d} \Big(\int_B \langle \psi, \nabla^B_{X_i}\nabla^B_{X_i}\psi\rangle - \langle \psi, \bar\eta(X_i)\nabla^B_{X_i}\psi\rangle \operatorname{vol}_{g_B} \\
&\qquad\qquad + \int_B \langle \psi, \nabla^B_{X_i}\psi\rangle \mathcal{L}_{X_i}\operatorname{vol}_{g_B}\Big) \\
&\quad = -\sum_{i\leq d} \int_B \langle \psi, \big(\nabla^B_{X_i}\nabla^B_{X_i} - \nabla^B_{\nabla_{X_i}X_i}\big)\psi\rangle - \langle \psi, \bar\eta(X_i)\nabla^B_{X_i}\psi\rangle \operatorname{vol}_{g_B} \\
&\quad = -\int_B \langle \psi, \operatorname{tr}\big((\nabla^B)^2 - \bar\eta(\cdot)\nabla^B_{\cdot}\big)\psi\rangle \operatorname{vol}_{g_B}\,. \qquad (2.17)
\end{aligned}$$

The difference $\Delta_B - P_0\Delta_h P_0$ is a zeroth order operator usually called Born-Huang potential in adiabatic perturbation theory. It can easily be calculated from the quadratic forms and is given by

$$\begin{aligned}
V_{\mathrm{BH}} &:= \Delta_B - P_0\Delta_h P_0 \\
&= \operatorname{tr}_{NF}\Big(P_0\big(-[\cdot, P_0]^2 + [g_B(\cdot, \pi_*\eta), P_0][\cdot, P_0]\big)P_0\Big)\,.
\end{aligned}$$

The adiabatic operator can then be expressed as

$$H_{\mathrm{a}} = -\varepsilon^2\Delta_B + \lambda + \varepsilon P_0 H_1 P_0 + \varepsilon^2 V_{\mathrm{BH}}\,. \qquad (2.18)$$

Example 2.22. For $M = \mathbb{R}\times[0,1+h]$ as discussed in examples 2.2, 2.10 and 2.15 we have $\eta = 0$ and thus $\langle\phi_j, \partial_x\phi_j\rangle = \frac{1}{2}\partial_x\langle\phi_j,\phi_j\rangle = 0$ on the j-th band. This leads to

$$\nabla^B_{\partial_x}\phi_j(x,y)\psi(x) = P_0(\psi\partial_x\phi_j + \phi_j\partial_x\psi) = \phi_j\partial_x\psi\,.$$

Hence for every j

$$\Delta_B = \partial_x^2$$

and

$$V_{\mathrm{BH}} = \Delta_B - P_0 \Delta_h P_0 = P_0 \partial_x P_0 \partial_x P_0 - P_0 \partial_x^2 P_0 = -P_0 \partial_x P_0^\perp \partial_x P_0 \,.$$

For the j-th band this equals

$$\begin{aligned} V_{\mathrm{BH}} &= -\sum_{k \neq j} P_0^j \partial_x P_0^k \partial_x P_0^j = -\sum_{k \neq j} \langle \phi_j, \partial_x \phi_k \rangle \langle \phi_k, \partial_x \phi_j \rangle \\ &= \sum_{k \neq j} |\langle \phi_k, \partial_x \phi_j \rangle|^2 = |\langle \partial_x \phi_j, \partial_x \phi_j \rangle|^2 \,, \end{aligned}$$

which evaluates to

$$V_{\mathrm{BH}} = \big(\log(1+h)'\big)^2 \big(\tfrac{1}{6}(\pi(j+1))^2 + \tfrac{1}{4}\big) \,.$$

We discuss the adiabatic operator in more detail in the next chapter. Now we turn to the study of the super-adiabatic correction H_{sa}. First we need to establish some key properties of the expansion of U_ε. Note that

$$\begin{aligned} P_0 (P_\varepsilon - P_0)^2 &= P_0 P_\varepsilon^2 - P_0 P_\varepsilon P_0 - P_0^2 P_\varepsilon + P_0 \\ &= -P_0 (P_\varepsilon - P_0) P_0 = \mathcal{O}(\varepsilon^2) \,. \end{aligned} \tag{2.19}$$

Using this we can write the first terms of the expansion of $P_\varepsilon U_\varepsilon$ as

$$P_\varepsilon U_\varepsilon = P_0 + \underbrace{P_0^\perp (P_\varepsilon - P_0) P_0}_{=:\varepsilon U_1} \underbrace{- \tfrac{1}{2} P_0 (P_\varepsilon - P_0)^2 P_0}_{=:\varepsilon^2 U_2} + \mathcal{O}(\varepsilon^3) \,,$$

with $U_1, U_2 \in \mathscr{L}(\mathscr{H}) \cap \mathscr{L}(D(H))$. We then have

$$\begin{aligned} H_{\mathrm{sa}} &= H_{\mathrm{eff}} - H_{\mathrm{a}} \\ &= \varepsilon \left(U_1^* H P_0 + P_0 H U_1 \right) + \varepsilon^2 \left(U_1^* H U_1 + P_0 H U_2 + U_2 H P_0 \right) + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon \left(P_0 (P_\varepsilon - P_0)[H, P_0] P_0 + P_0 [P_0, H](P_\varepsilon - P_0) P_0 \right) && \text{(2.20a)} \\ &\quad + \varepsilon^2 \left(U_1^* H U_1 + P_0 H U_2 + U_2 H P_0 \right) + \mathcal{O}(\varepsilon^3) \,. && \text{(2.20b)} \end{aligned}$$

From this equation we conclude that $\|H_{\mathrm{sa}}\|_{\mathscr{L}(D(H),\mathscr{H})} = \mathcal{O}(\varepsilon^2)$. If we include only the leading super-adiabatic contribution we get a rather explicit expression for H_{eff}.

Proposition 2.23. *Let H_{eff} be the effective operator of theorem 2.17 and χ be a regular cut-off with support in $(-\infty, \Lambda]$, then*

$$\|H_{\text{eff}}\chi(H_{\text{eff}}) - \chi(H_{\text{eff}})\,(H_{\text{a}} + \mathcal{M})\,\chi(H_{\text{eff}})\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^3)\,,$$

with

$$\begin{aligned} H_{\text{a}} &= -\varepsilon^2\Delta_B + \lambda + \varepsilon P_0 H_1 P_0 + \varepsilon^2 V_{\text{BH}}\,,\\ \mathcal{M} &= P_0[H, P_0]R_F(\lambda)[H, P_0]P_0\,. \end{aligned} \tag{2.21}$$

Proof. The form of $H_{\text{a}} = P_0 H P_0$ has already been derived. Also

$$H_{\text{eff}}\chi(H_{\text{eff}}) - \chi(H_{\text{eff}})H_{\text{a}}\chi(H_{\text{eff}}) = \chi(H_{\text{eff}})H_{\text{sa}}\chi(H_{\text{eff}}) = \mathcal{O}(\varepsilon^2)\,.$$

By virtue of lemma C.2 we have $\|\chi(H_{\text{eff}}) - P_0\chi(H)\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon)$, so these cut-offs are exchangeable when dealing with H_{sa} in an expansion up to $\mathcal{O}(\varepsilon^3)$. Using this, the proof is completed in lemma 2.26. □

Example 2.24. In our example $M = \mathbb{R} \times [0, 1 + h]$ we have by the calculations of example 2.15

$$\begin{aligned} \mathcal{M} &= -P_0 P_1 [H, P_0] P_0 = -\sum_{k\neq j} P_0^j P_1 P_0^k [H, P_0^j] P_0^j \\ &= 4\varepsilon^4 \nabla^B_{\partial_x} [\partial_x^*, P_0] R_F(\lambda) [\partial_x^*, P_0] \nabla^B_{\partial_x} + \mathcal{O}(\varepsilon^3) \\ &= \Big(4\varepsilon^2 \big(\log(1+h)'\big)^2 \sum_{k\neq j} \frac{a_{jk}^2}{\lambda_k - \lambda_j}\Big) \varepsilon^2 \partial_x^2 + \mathcal{O}(\varepsilon^3)\,. \end{aligned}$$

The remainder of order ε^3 is a first order differential operator.

To get the explicit form of $\mathcal{M}$ we need to use information on the expansion of P_ε from its construction in lemma 2.14.

Lemma 2.25. *Let χ be a regular cut-off with support in $(-\infty, \Lambda]$ and let $P_k \in \mathcal{A}_H$ denote the operators constructed in lemma 2.14, then*

$$\Big\|(P_\varepsilon - P_0)\chi(H) - \sum_{k=1}^{N} \varepsilon^k P_k \chi(H)\Big\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^{N+1})\,.$$

Proof. We prove $P_\varepsilon\chi = P^\chi\chi + \mathcal{O}(\varepsilon^{N+1}) = P^N\chi + \mathcal{O}(\varepsilon^{N+1})$ (see the proof of proposition 2.13). By a result of Nenciu [54, proposition 3] we have

$$P_\varepsilon - P^\chi = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{(P^\chi - z)^{-1} + (P^\chi - 1 + z)^{-1}}{1-z}\mathrm{d}z\,((P^\chi)^2 - P^\chi)\,.$$

Thus calculating

$$\begin{aligned}((P^\chi)^2 - P^\chi)\chi(H) &\overset{(2.10)}{=} \chi^{1/2}(P^\chi - 1)P^\chi\chi^{1/2} + \mathcal{O}(\varepsilon^{N+1})\\ &= \chi^{1/2}((P^N)^2 - P^N)\chi^{1/2} + \mathcal{O}(\varepsilon^{N+1})\\ &= \mathcal{O}(\varepsilon^{N+1})\end{aligned}$$

proves the claim. □

If we apply this lemma together with equation (2.19) and $P_0P_1P_0 = P_0^\perp P_1 P_0^\perp = 0$ we get an explicit expansion of U_1

$$\begin{aligned}U_1\chi(H) &= \varepsilon^{-1}P_0^\perp\,(P_\varepsilon - P_0)\,P_0\chi\\ &= \varepsilon^{-1}P_0^\perp\,(P_\varepsilon - P_0)\,\chi + \mathcal{O}(\varepsilon)\\ &= P_0^\perp P_1\chi + \mathcal{O}(\varepsilon)\,.\end{aligned}$$

More precisely this means

$$\|(U_1 - P_0^\perp P_1)\chi(H)\|_{\mathscr{L}(\mathscr{H},D(H))} = \|(U_1 - P_1P_0)\,\chi(H)\| = \mathcal{O}(\varepsilon) \quad (2.22)$$

while a similar calculation with U_2 yields

$$\begin{aligned}&\left\|\chi(H)\left(U_2 + \tfrac{1}{2}P_1P_0^\perp P_1\right)\chi(H)\right\|_{\mathscr{L}(\mathscr{H},D(H))}\\ &= \left\|\chi(H)\left(U_2 + \tfrac{1}{2}P_0P_1P_1P_0\right)\chi(H)\right\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon)\,.\end{aligned}$$

With these we can express the leading order of H_{sa} in terms of P_0 and P_1, which is explicitly given in the construction 2.14. The energy cut-offs in the statement are clearly necessary because if H_1 is a second order differential operator $P_0P_1P_1P_0$ may already be of order four. In this case it will not be a bounded operator from $D(H)$ to $\mathscr{H}$.

Lemma 2.26. *Let χ be a regular cut-off with support in $(-\infty, \Lambda]$. The super-adiabatic operator satisfies*

$$\varepsilon^{-2}\big\|\chi(H)\,(H_{\mathrm{sa}} - P_0[H,P_0]R_F(\lambda)[H,P_0]P_0)\,\chi(H)\big\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon)\,.$$

Proof. We start by using (2.22) and lemma 2.14 on one of the terms in (2.20a)

$$\begin{aligned}\varepsilon P_0[P_0,H]U_1\chi(H) &= -P_0[H,P_0]P_0^{\perp}\varepsilon P_1 P_0\chi + \mathcal{O}(\varepsilon^3)\\ &= P_0[H,P_0]P_0^{\perp}R_F(\lambda)[H,P_0]P_0\chi + \mathcal{O}(\varepsilon^3)\,,\end{aligned}$$

The other term in (2.20a) is the adjoint of this one, which means they are equal since the result is clearly symmetric. Now because $U_2 = P_0U_2P_0$, inserting $H = (-\varepsilon^2\Delta_h + \lambda) + (H_F - \lambda) + \varepsilon H_1$ into (2.20b) gives

$$\begin{aligned}&U_1^*HU_1 + P_0HU_2 + U_2HP_0\\ &\quad= U_1^*(-\varepsilon^2\Delta_h+\lambda)U_1 + P_0(-\varepsilon^2\Delta_h+\lambda)U_2 + U_2(-\varepsilon^2\Delta_h+\lambda)P_0 \qquad (2.23a)\\ &\qquad + U_1^*(H_F-\lambda)U_1 + \mathcal{O}(\varepsilon)\,. \qquad (2.23b)\end{aligned}$$

Looking more closely at (2.23b) we find

$$\begin{aligned}&\chi(H)\,(U_1^*(H_F-\lambda)U_1)\,\chi(H)\\ &\quad= -\chi P_0[H,P_0]\underbrace{R_F(\lambda)(H_F-\lambda)}_{=1}R_F(\lambda)[H,P_0]P_0\chi + \mathcal{O}(\varepsilon)\\ &\quad= -\chi P_0[H,P_0]R_F(\lambda)[H,P_0]P_0\chi + \mathcal{O}(\varepsilon)\,.\end{aligned}$$

Thus after adding (2.20a) and (2.23b) we are left exactly with the correct expression (2.21).

To prove the claim we still need to show that the contribution of (2.23a) is of order ε. To see this first note that

$$\begin{aligned}\varepsilon^2(U_1^*U_1 + 2U_2) &= P_0(P_\varepsilon - P_0)P_0^{\perp}(P_\varepsilon - P_0)P_0 - P_0(P_\varepsilon - P_0)^2\\ &= \big(P_0(P_\varepsilon - P_0)P_0\big)^2 = \mathcal{O}(\varepsilon^4)\,.\end{aligned}$$

Therefore we can complete the proof by showing

$$\left\|\chi(H)U_1^*[-\varepsilon^2\Delta_h+\lambda,U_1]\chi(H)\right\|_{\mathscr{L}(\mathscr{H})}=\mathcal{O}(\varepsilon)\,, \tag{2.24a}$$

$$\left\|\chi(H)P_0[-\varepsilon^2\Delta_h+\lambda,U_2]\chi(H)\right\|_{\mathscr{L}(\mathscr{H})}=\mathcal{O}(\varepsilon)\,. \tag{2.24b}$$

Observe that since $[H,\chi(H)]=0$

$$\begin{aligned}
&\left\|P_0[-\varepsilon^2\Delta_h+\lambda,\chi^{1/2}(H)]\chi^{1/2}(H)\right\|_{\mathscr{L}(\mathscr{H})}\\
&\quad=\left\|P_0[H_F-\lambda,\chi^{1/2}]\chi^{1/2}+P_0[\varepsilon H_1,\chi^{1/2}]\chi^{1/2}\right\|_{\mathscr{L}(\mathscr{H})}\\
&\quad=\Big\|\underbrace{P_0(H_F-\lambda)}_{=0}\chi-P_0\chi^{1/2}(H_F-\lambda)\chi^{1/2}\Big\|_{\mathscr{L}(\mathscr{H})}+\mathcal{O}(\varepsilon)\\
&\quad=\left\|[P_0,\chi^{1/2}](H_F-\lambda)\chi^{1/2}\right\|_{\mathscr{L}(\mathscr{H})}+\mathcal{O}(\varepsilon)\\
&\quad=\mathcal{O}(\varepsilon)\,.
\end{aligned} \tag{2.25}$$

Since $U_j=U_jP_0$ for $j\in\{1,2\}$ this implies

$$\left\|[-\varepsilon^2\Delta_h+\lambda,U_j\chi^{1/2}]\chi^{1/2}-[-\varepsilon^2\Delta_h+\lambda,U_j]\chi\right\|_{\mathscr{L}(\mathscr{H})}=\mathcal{O}(\varepsilon)\,.$$

Using this together with (2.22) in (2.24a) gives

$$\begin{aligned}
&\left\|\chi(H)U_1^*[-\varepsilon^2\Delta_h+\lambda,U_1]\chi(H)\right\|_{\mathscr{L}(\mathscr{H})}\\
&\qquad=\left\|\chi U_1^*[-\varepsilon^2\Delta_h+\lambda,P_1P_0\chi^{1/2}]\chi^{1/2}\right\|+\mathcal{O}(\varepsilon)\\
&\qquad\leq\Big\|\chi U_1^*(\underbrace{[-\varepsilon^2\Delta_h,P_1P_0]\chi}_{\overset{(2.2)}{=}\mathcal{O}(\varepsilon)}+\underbrace{[\lambda,P_1]P_0}_{\in\mathcal{A}_H^{1,1}}\chi)\Big\|\\
&\qquad\qquad\quad+\left\|\chi U_1^*P_1P_0[-\varepsilon^2\Delta_h+\lambda,\chi^{1/2}]\chi^{1/2}\right\|+\mathcal{O}(\varepsilon)\\
&\qquad=\mathcal{O}(\varepsilon)\,.
\end{aligned}$$

The last step here is justified by (2.25) and $\chi U_1^*P_1P_0\in\mathscr{L}(\mathscr{H})$, which follows from an adjointness argument (cf. the proof of proposition 2.13). The calculation for (2.24b) is basically identical, so the proof is complete.
□

Chapter 3

Examples and applications

In this chapter we apply the general theory to more specific problems and analyse the adiabatic part of the effective operator in detail. First we show that the general theory of chapter 2 applies to the shrinking embeddings mentioned in section 1.1.3. Then we turn to the study of the ground state band $\lambda_0(x) = \min \sigma(H_F(x))$ and its corresponding effective operator. In particular we derive asymptotics of eigenvalues and eigenfunctions in terms of the adiabatic operator in section 3.2. In cases of low dimensional base we refine these asymptotics and relate them to questions concerning the nodal sets of eigenfunctions of the Laplacian in section 3.3.2.

3.1 Examples

3.1.1 Embeddings of fibre bundles

Here we analyse shrinking embeddings of fibre bundles, already touched upon in 1.1.3. We show that their Laplacians fit into the framework of chapter 2, so these embeddings provide a large class of examples for the general theory, with non-trivial perturbation H_1. The results here are based on the author's joint work with Haag and Teufel [30].

As discussed in section 1.1.3 let $\alpha\colon B \to \mathbb{R}^k$ and $\beta\colon M \to NB$ be embeddings such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \beta\ } & NB \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{NB}} \\ B & \xrightarrow{\ \mathrm{id}\ } & B \end{array}$$

commutes. We identify elements of $NB \subset T\mathbb{R}^k$ with vectors in $\mathbb{R}^k$ and define

$$\Psi_\varepsilon : M \to \mathbb{R}^k\,, \qquad x \mapsto \alpha(\pi(x)) + \varepsilon\beta(x)\,.$$

Let $\langle\cdot,\cdot\rangle$ denote the Euclidean metric on $\mathbb{R}^k$ and put $g_B := \alpha^*\langle\cdot,\cdot\rangle$, $G_\varepsilon := \varepsilon^{-2}\Psi_\varepsilon^*\langle\cdot,\cdot\rangle$. Whenever dealing with such embeddings we assume in the following:

Condition 4.

- $\alpha(B)$ has a tubular neighbourhood T of radius $r_T > 0$ and

$$\beta(M) \subset \{\nu \in NB : |\nu| < r_T\}\,.$$

- (B, g_B) is of bounded geometry and $(M, G_{\varepsilon=1}) \to B$ is uniformly locally trivial.
- The maps α and β are $\mathscr{C}^\infty$-bounded.

Under these conditions NB is a bundle of bounded geometry over B and the Weingarten map $W \in \Gamma(NB^*) \otimes \mathrm{End}(TB)$ of (B, α) is a bounded tensor. Let $\tilde{G}_\varepsilon$ be the metric on NB induced by the map $\nu \mapsto \alpha(\pi(\nu)) + \varepsilon\nu$ multiplied by ε^{-2}. We will give a short derivation for the key features of this metric, for a more detailed exposition see [30].

Choose local coordinates on $U \subset B$ with coordinate vector fields $(\partial_{x^i})_{i \leq d}$ and an orthonormal frame $\{e_j : j = 1, \ldots, \operatorname{rank} NB\}$ of $NB|_U$. This gives bundle coordinates $\nu = \nu^j e_j(x)$ on $NB|_U$ with coordinate vector fields $\{\partial_{x^i}, \partial_{\nu^j} : i \leq d, j \leq \operatorname{rank} NB\}$. For these one easily sees that

$$\begin{aligned} (\alpha \circ \pi + \varepsilon\nu)_*\partial_{x^i} &= \alpha_*\pi_*\partial_{x^i} + \varepsilon\partial_{x^i}\nu \\ (\alpha \circ \pi + \varepsilon\nu)_*\partial_{\nu^j} &= \varepsilon\partial_{\nu^j}\nu = \varepsilon e_j\,. \end{aligned} \tag{3.1}$$

With this we can determine the horizontal lift for the metric $\tilde{G}_\varepsilon$.

Lemma 3.1. *Let $\omega^N \in \Gamma(T^*U \otimes \mathrm{End}(NB))$ denote the connection form of the normal connection over U. The horizontal lift of ∂_{x^i} with respect to $\tilde{G}_\varepsilon$ is independent of ε and given by*

$$\partial^h_{x^i}\big|_\nu = \partial_{x^i} - \sum_{j=1}^{\operatorname{rank} NB} \langle \omega^N(\partial_{x^i})\nu, e_j\rangle \partial_{\nu^j}\,.$$

Proof. The vector field $\partial^h_{x^i}$ clearly projects to ∂_{x^i}, so we only need to check that it is horizontal for any ε. We have

$$\tilde{G}_\varepsilon(\partial_{x^i},\partial_{\nu^j}) = \varepsilon^{-2}\langle \alpha_*\pi_*\partial_{x^i} + \varepsilon\partial_{x^i}\nu, \varepsilon e_j\rangle = \langle \partial_{x^i}\nu, e_j\rangle = \langle \omega^N(\partial_{x^i})\nu, e_j\rangle$$

and

$$\tilde{G}_\varepsilon(\partial_{\nu^j},\partial_{\nu^l}) = \varepsilon^{-2}\langle \varepsilon e_j, \varepsilon e_l\rangle = \delta_{jl}\,.$$

Consequently for any $j \in \{1,\dots,\operatorname{rank} NB\}$:

$$\begin{aligned}\tilde{G}_\varepsilon(\partial^h_{x^i},\partial_{\nu^j}) &= \tilde{G}_\varepsilon\big(\partial_{x^i},\partial_{\nu^j}\big) - \tilde{G}_\varepsilon\Big(\sum_{l=1}^{\operatorname{rank} NB}\langle \omega^N(\partial_{x^i})\nu, e_l\rangle\partial_{\nu^l},\partial_{\nu^j}\Big)\\ &= \langle \omega^N(\partial_{x^i})\nu, e_j\rangle - \langle \omega^N(\partial_{x^i})\nu, e_l\rangle\delta_{jl} = 0\,.\end{aligned}$$

Since the fields $\{\partial_{\nu^j} : j = 1,\dots,\operatorname{rank} NB\}$ span the kernel of π_{NB*} this proves the claim. □

Since the Weingarten map is defined as minus the projection of $\nabla\nu$ to TB, equation (3.1) determines the horizontal part of $\tilde{G}_\varepsilon$ to be

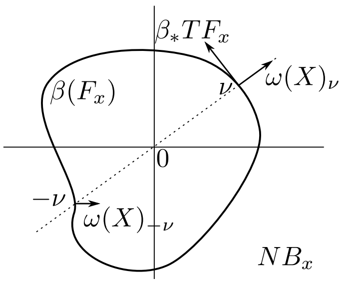
$$\tilde{G}_\varepsilon(X^h_\nu, Y^h_\nu) = \varepsilon^{-2}\pi^* g_B\big((1-\varepsilon W(\nu))X_{\pi(\nu)}, (1-\varepsilon W(\nu))Y_{\pi(\nu)}\big)\,,$$

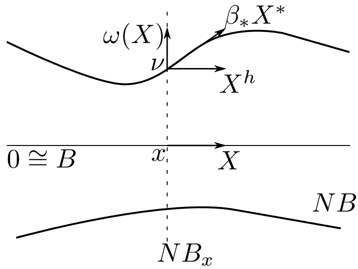
with $g_B = \alpha^*\langle\cdot,\cdot\rangle$. On vertical vectors, $\tilde{G}_\varepsilon$ is given by the flat metric δ induced by the bundle metric on NB. The leading part of this metric is a rescaled Riemannian submersion with totally geodesic fibres, defined by the horizontal lift X^h, g_B and $g_F = \delta$.

If the codimension of M in $\mathbb{R}^k$ is zero we have $G_\varepsilon = \tilde{G}_\varepsilon$ on $\beta(M)$. In the case of positive codimension the vertical metric g_F is simply obtained by the fibre-wise restriction of δ to $\beta(M)$. In particular g_F is independent of ε. In order to find a simple expression for G_ε we need to determine the horizontal lift, which will again turn out to be independent of ε. We begin by discussing the lift X^* for $G_{\varepsilon=1}$. For any $X \in \Gamma(TB)$ we have

$$\pi_{NB*}(\beta_* X^* - X^h) = \pi_* X^* - \pi_{NB*}X^h = 0$$

and thus $\beta_* X^* = X^h + \omega(X)$ with a vertical field $\omega(X) \in \Gamma(TNB)$.





By construction β_* is an isometry, so we have for every $Y \in \Gamma(TF)$

$$0 = G_{\varepsilon=1}(X^*, Y) = \tilde{G}_{\varepsilon=1}(X^h + \omega(X), \beta_* Y) = \delta(\omega(X), \beta_* Y)\,. \qquad (3.2)$$

This means the vertical field $\omega(X)$ is orthogonal to $\beta(F_x)$ in every fibre NB_x. Now let Q be the orthogonal projection to the normal bundle of $\beta(M)$ with respect to $\tilde{G}_{\varepsilon=1}$. Then $0 = Q\beta_* X^* = QX^h + Q\omega(X)$ and this equation uniquely determines ω as a tensor in $\Gamma(T^*B \otimes TNB)$ by virtue of the following lemma.

Lemma 3.2. *For any $y \in \beta(M)$ denote $x := \pi(y)$ and let $N\beta(F_x)_y \subset T_y(NB_x)$ be the fibre-wise normal space with respect to the fibre metric on NB. Then the restriction of Q to this space $Q\colon N\beta(F_x)_y \to NM_y$ is a bijection.*

Proof. Since β respects the projections, the space $N\beta(F_x)_y$ contains no tangent vectors to $\beta(M)$, so $\ker Q \cap N\beta(F_x)_y = \{0\}$ and the restriction is one-to-one. It also has to be onto for dimensional reasons. On the one hand we have

$$\dim(NB) = \operatorname{rank}(TNB) = \operatorname{rank}(\pi_{NB}^* TB) + \dim(NB_x)\,,$$

by the splitting of TNB into vertical and horizontal parts. On the other hand, applying the same idea to M,

$$\dim(NB) = \dim(M) + \operatorname{codim}(M) = \operatorname{rank}(\pi^* TB) + \dim(F) + \operatorname{codim}(M)\,.$$

This implies $\dim(NB_x) = \dim(F) + \operatorname{codim}(M)$, so the codimension of F_x in NB_x equals $\operatorname{codim}(M)$, which means exactly that $\dim(N\beta(F_x)_y) = \dim(NM_y)$ and proves the claim. $\square$

Given this lemma we have $\omega(X) = -Q^{-1}QX^h$, but for an explicit calculation the formula $\tilde{G}_{\varepsilon=1}(\omega(X), Y) = -\tilde{G}_{\varepsilon=1}(X^h, Y)$ for any $Y \in \Gamma(NM)$ may be more convenient. In particular if the codimension is one and the normal $v \in N\beta(M)_y$ has a vertical part v_{F_x}, which is then in $N\beta(F_x)$, and horizontal part v_h we have

$$\tilde{G}_{\varepsilon=1}(v_h, X^h) = -\tilde{G}_{\varepsilon=1}(v_{F_x}, \omega(X)) .$$

In this case $N\beta(F_x)$ is one-dimensional and by the argument of lemma 3.2 $v_{F_x} \neq 0$, so $\omega(X)_y$ is proportional to v_{F_x} and

$$\omega(X)_y = -\tilde{G}_{\varepsilon=1}(v_h, X^h)\tilde{G}_{\varepsilon=1}(v_{F_x}, v_{F_x})^{-1}v_{F_x} .$$

The field $\omega(X)$ is completely determined by the requirements that $X^h + \omega(X)$ be tangent to $\beta(M)$ and $\omega(X)_y \in N\beta(F_x)_y$. Both of these concepts are completely independent of ε, so the horizontal lift also has this property.

Lemma 3.3. *For any $X \in \Gamma(TB)$ the horizontal lift with respect to G_ε is independent of ε and given by*

$$\beta_* X^* = X^h + \omega(X) = (1 - Q^{-1}Q)X^h$$

in the notation of lemma 3.2.

Proof. The field $X^h + \omega(X)$ is a section of $T\beta(M)$ that projects to X, so we only need to check that it is orthogonal to $\beta_* TF$ for every G_ε, since then it must be the unique horizontal lift of X. For $Y \in \Gamma(TF)$ we have

$$G_\varepsilon(Y, X^*) = \tilde{G}_\varepsilon\big(\beta_* Y, X^h + \omega(X)\big) = \delta\big(\beta_* Y, \omega(X)\big) \overset{(3.2)}{=} 0 ,$$

because $\beta_* Y$ is vertical and the fibre metric does not depend on ε. □

This determines the horizontal component of G_ε to be

$$G_\varepsilon(X^*, Y^*) = \varepsilon^{-2}\Big(g_B\big((1-\varepsilon\beta^* W)X, (1-\varepsilon\beta^* W)Y\big) + \varepsilon^2\delta\big(\omega(X), \omega(Y)\big)\Big) .$$

The vertical metric is given by the restriction of δ to the fibres, $g_F = \beta^*\delta$. Because NB is a bundle of bounded geometry and β a bounded map, ω is a bounded tensor and this metric has the form

$$G_\varepsilon = g_\varepsilon + \mathcal{O}(\varepsilon) .$$

The leading part g_ε is a rescaled Riemannian submersion built from g_B, which is the restriction of the euclidean metric of $\mathbb{R}^k$ to $\alpha(B)$, $g_F = \beta^*\delta$ and the horizontal lift X^*. The remainder is of order ε in the sense that there exists $C > 0$ so that

$$\big|\big(G_\varepsilon - g_\varepsilon\big)(X,X)\big| \leq C\varepsilon g_\varepsilon(X,X)\,.$$

The Laplacian of G_ε fits into the framework of chapter 2 since the various terms of the difference $\Delta_{G_\varepsilon} - \Delta_{g_\varepsilon}$ may be collected in the perturbation εH_1.

Lemma 3.4. *The negative Laplacian $-\Delta_{G_\varepsilon}$ is unitarily equivalent to an operator H that satisfies condition 2. Let Δ_h be the horizontal Laplacian of g_ε, then there is $C > 0$ so that $-\varepsilon^2\Delta_h + \varepsilon H_1 \geq -C\varepsilon$.*

Proof. For the calculation of Δ_{g_ε} we need the metric on one-forms. The relation of G_ε and g_ε on tangent vectors $v \in TM$ implies

$$g_\varepsilon\big(G_\varepsilon(v,\cdot), G_\varepsilon(v,\cdot)\big) = g_\varepsilon\big(g_\varepsilon(v,\cdot), g_\varepsilon(v,\cdot)\big) + \mathcal{O}(\varepsilon) = g_\varepsilon(v,v) + \mathcal{O}(\varepsilon)\,,$$

which entails

$$\big(G_\varepsilon - g_\varepsilon\big)\big(G_\varepsilon(v,\cdot), G_\varepsilon(v,\cdot)\big) = G_\varepsilon(v,v) - g_\varepsilon(v,v) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon)\,.$$

This equation completely determines the difference and we have

$$\big|\big(G_\varepsilon - g_\varepsilon\big)(\xi,\xi)\big| \leq C\varepsilon g_\varepsilon(\xi,\xi)$$

for some $C > 0$ and every $\xi \in T^*M$. Now note that all the corrections in G_ε concern only the horizontal directions, so $G_\varepsilon - g_\varepsilon$ vanishes on vertical cotangent vectors and for $\xi \in T^*B$

$$\big|\big(G_\varepsilon - g_\varepsilon\big)(\pi^*\xi, \pi^*\xi)\big| \leq C\varepsilon^3 g_B(\xi,\xi)\,.$$

Rescaling the volume measure of (M, G_ε) with $\rho = \mathrm{vol}_{G_\varepsilon}/\mathrm{vol}_{g_\varepsilon}$ as discussed in section 1.1.3 we obtain the expression (1.7) for H:

$$H := -U_\rho^*\Delta_{G_\varepsilon}U_\rho\psi = -\Delta_{g_\varepsilon}\psi - \mathrm{div}_{g_\varepsilon}(G_\varepsilon - g_\varepsilon)(\mathrm{d}\psi,\cdot) + V_\rho\psi\,, \qquad (3.3)$$

with

$$V_\rho = \tfrac{1}{4}G_\varepsilon(\mathrm{d}\log\rho, \mathrm{d}\log\rho) + \tfrac{1}{2}\,\mathrm{div}_{g_\varepsilon}\,\mathrm{grad}_{G_\varepsilon}\log\rho\,.$$

Because α and β are $\mathscr{C}^\infty$-bounded, so are ω and β^*W. Thus we have $\rho = 1 + \mathcal{O}(\varepsilon)$ and $\varepsilon^{-1}V_\rho = \mathcal{O}(1)$ in $\mathscr{C}_b^\infty(M)$. Let H_1 be the operator

$$H_1\psi := -\varepsilon^{-1} \operatorname{div}_{g_\varepsilon}(G_\varepsilon - g_\varepsilon)(\mathrm{d}\psi, \cdot) + \varepsilon^{-1}V_\rho \,.$$

If we express this over $U \in \mathfrak{U}$ using the g_ε-orthonormal frame $(\varepsilon X_i^*)_{i\leq d}$ of $NF|_U$ (see equation (C.10)), we obtain an operator whose coefficients are given by a convergent power series depending on β^*W and ω since we basically need to compute the inverse of $G_\varepsilon(\varepsilon X_i^*, \varepsilon X_j^*)$ minus δ^{ij}. By condition 4 these objects are smooth, with derivatives bounded independently of ε. This ensures that $H_1 \in \mathscr{L}(W_\varepsilon^{m+2}, W_\varepsilon^m)$ for every $m \in \mathbb{N}$. We also have that $H_1 A \in \mathcal{A}^{k+2,l}$ for every $A \in \mathcal{A}_H^{k,l}$, since expressing H_1 by $\Phi^* X_i = X_i^* - Y_i$ we see that it has the form discussed in remark 2.6. Hence H satisfies condition 2.

The differential operator $\psi \mapsto -\varepsilon^2\Delta_h\psi - \operatorname{div}_{g_\varepsilon}(G_\varepsilon - g_\varepsilon)(\mathrm{d}\psi, \cdot)$ is the horizontal Laplacian of G_ε (with the volume measure of g_ε), hence it is non-negative and

$$-\varepsilon^2\Delta_h + \varepsilon H_1 \geq -\left\|V_\rho\right\|_\infty \geq -C\varepsilon \,.$$

□

Consequently, if condition 4 holds, all of the theorems of section 2.2 are valid for $-\Delta_{G_\varepsilon}$ and any eigenband with a gap. From now on we denote $S_\varepsilon := \varepsilon^{-3}\big(G_\varepsilon - g_\varepsilon\big) \in \Gamma(\pi^*T^*B \otimes \pi^*T^*B)$, which is a bounded tensor with respect to g_B because of the boundedness of β^*W, ω. For the differential operator in H_1 we adopt the notation

$$\operatorname{div} S_\varepsilon \colon W_\varepsilon^2(M) \to \mathscr{H} \qquad \psi \mapsto \operatorname{div}_{g_\varepsilon} S_\varepsilon(\mathrm{d}\psi, \cdot)\,. \tag{3.4}$$

We close the discussion of embeddings with some remarks.

Remark 3.5. In many cases the contribution of V_ρ to the adiabatic operator $H_\mathrm{a} = P_0 H P_0$ is of order ε^2. Because $\rho = 1 + \mathcal{O}(\varepsilon)$ it is clear that

$$G_\varepsilon(\mathrm{d}\log\rho, \mathrm{d}\log\rho) = \mathcal{O}(\varepsilon^2)\,.$$

The leading contribution of the second term of V_ρ is

$$\Delta_{g_\varepsilon} \log \rho = \Delta_F \log \rho + \varepsilon^2 \Delta_h \log \rho = \Delta_F \log \rho + \mathcal{O}(\varepsilon^3)\,.$$

Now if the codimension of M is zero, the leading term of $\log \rho$ is given by

$$\log \rho = \log \det(1 - \varepsilon W) = -\varepsilon \operatorname{tr} W + \mathcal{O}(\varepsilon^2)\,.$$

Since this leading term is linear in the fibre coordinate ν, the fibre Laplacian vanishes on it and $V_\rho = \mathcal{O}(\varepsilon^2)$.

If on the other hand $\partial M = \varnothing$ and we are considering the ground state band $\lambda \equiv 0$, with fibre eigenfunction $\phi_0 = \pi^* \mathrm{Vol}(F_x)^{-1/2}$, the potential V_ρ might be of order ε, but

$$P_0 \Delta_F \log \rho = \phi_0 \langle \phi_0, \Delta_F \log \rho \rangle_{\mathscr{H}_F} = \phi_0 \langle \Delta_F \phi_0, \log \rho \rangle = 0\,.$$

Hence in this case the contribution to H_a, $P_0 V_\rho P_0$, is again of order ε^2.

Remark 3.6. It is possible to treat embeddings of M into general Riemannian manifolds of bounded geometry (A, g_A). The maps Ψ_ε are then defined using the normal exponential map

$$\exp^N : NB \to A \qquad \nu \mapsto \exp_{\pi(\nu)} \nu\,.$$

The corresponding expansion of the metric $\tilde{G}_\varepsilon$ was derived by Wittich [71] using Jacobi fields. Though the general form of the metric is still the same, this introduces some new features.

- The vertical metric on NB is no longer flat. Hence its scaling behaviour under the rescaling $\nu \mapsto \varepsilon\nu$ is more complex and the vertical Laplacian acquires ε-dependent corrections. These may be treated as part of H_F or added to H_1. In that case $-\varepsilon^2 \Delta_h + \varepsilon H_1$ may be unbounded from below.

- The horizontal lift of $X \in \Gamma(TB)$ with respect to G_ε will depend on ε. Because this dependence is of lower order in ε we may define g_ε as the rescaled submersion metric obtained from $X^*_{\varepsilon=1}$, g_B and g_F and the general framework is still applicable.

3.1.2 The ground state band of the Laplacian

We now give a detailed derivation of the adiabatic operator associated with the ground state band of the Laplacian, that is the case $V \equiv 0$, $H_F = -\Delta_F$. This also indicates how the corresponding calculations for other bands or $V \neq 0$ can be performed. We now show that this eigenband satisfies the gap condition, because of the restrictions on the geometry, if the fibres are connected. Estimates on the spectral gap in terms of geometric quantities have been derived for many cases, see the book [64].

Lemma 3.7. *Let F be connected, $V = 0$ and $\lambda_0(x) = \min \sigma(-\Delta_{F_x})$ be the ground state band. If M satisfies condition 1 then λ_0 has a spectral gap of uniform size and satisfies condition 3.*

Proof. Let $\lambda_1(x) := \min(\sigma(-\Delta_{F_x}) \setminus \lambda_0)$. We first show that there is a constant $C > 0$ such that $\lambda_1 - \lambda_0 \geq C$. Assume this were not the case. Then there would be a sequence $(x_k)_{k\in\mathbb{N}}$ in B with $\lim_{k\to 0\infty} \lambda_1(x_k) - \lambda_0(x_k) = 0$. Now for every k take an open set $U_{\nu(k)} \in \mathfrak{U}$ containing x_k and denote by $g_k := (\Phi^{-1}_{\nu(k)})^* g_{F_{x_k}}$ the metric on F isometric to that of the fibre over x_k. Because of the bounds on $(\Phi^{-1}_{\nu(k)})^*$ that are required on M, for any $m \in \mathbb{N}$ the sequence $(g_k)_{k\in\mathbb{N}}$ is bounded in the $\mathscr{C}^{m+1}$-norm on $\Gamma(T^*F \otimes T^*F)$ with respect to g_0. Thus by the Arzelà-Ascoli theorem there is a subsequence converging to a symmetric bilinear form g_∞ of $\mathscr{C}^m$-regularity and by repeated extraction of subsequences and a diagonal argument g_∞ is a smooth tensor. Because of the bounds on the inverse $\Phi^*_{\nu(k)}$, the sequence of metrics is also positive definite in a uniform way and g_∞ is a Riemannian metric. Now it was shown by Bando and Urakawa that the eigenvalues of the Laplacian depend continuously on the metric (see [2], the proof is stated for manifolds without boundary but carries over to the Dirichlet Laplacian because the eigenvalues are determined by a maxi-min principle in a similar way). This means that the sequences $\lambda_1(x_k)$ and $\lambda_0(x_k)$ converge to the two smallest eigenvalues of (F, g_∞) and $\lambda_1(g_\infty) = \lambda_0(g_\infty)$. But this is impossible since λ_0 is always a simple eigenvalue (see [64]), thus a positive lower bound for $\lambda_1 - \lambda_0$ must exist.

The continuous dependence on the metric now shows that these eigenvalues may be separated by continuous functions, so condition 3 is satisfied. □

Now let F be connected, then for every $x \in B$ there is a unique real, positive and normalised solution $\phi_0(x,\cdot)$ to the equation $-\Delta_{F_x}\phi = \lambda_0(x)\phi$ (see e.g. [64]). This provides a trivialisation of the line bundle $\mathcal{E}$ and isomorphisms $\Gamma(\mathcal{E}) \cong \mathscr{C}^\infty(B)$, $L^2(\mathcal{E}) \cong L^2(B)$. We may calculate the Berry connection (2.15) in this trivialisation to be

$$\begin{aligned}
\nabla^B_X \phi_0\psi &= P_0 X^* \phi_0 \psi \\
&= \phi_0 (X\psi) + \phi_0\psi \left\langle \phi_0, X^*\phi_0 \right\rangle_{\mathscr{H}_F} \\
&=: \phi_0 \left(X + \omega^B(X)\right)\psi\,,
\end{aligned}$$

for $X \in \Gamma(TB)$ and $\psi \in W^1(B)$. Since ϕ_0 is real, the imaginary part of ω^B vanishes and we have

$$\begin{aligned}
2\omega^B(X) &= \omega^B(X) + \overline{\omega^B(X)} \\
&= \int_{F_x} \big(\phi_0 X^* \phi_0\big)(x,y) + \big((X^*\phi_0)\phi_0\big)(x,y)\,\mathrm{vol}_{g_{F_x}}(\mathrm{d}y) \\
&= -\int_{F_x} |\phi_0|^2 (x,y) \mathcal{L}_{X^*}\mathrm{vol}_{F_x}(\mathrm{d}y) \overset{(2.16)}{=} \bar\eta(X)\,.
\end{aligned}$$

Note that if $\partial M = \varnothing$ the ground state is explicitly given by $\phi_0(x,\cdot) = \pi^*\mathrm{Vol}(F_x)^{-1/2}$ and

$$\bar\eta \overset{\partial M = \varnothing}{=} -\mathrm{d}\left(\log \mathrm{Vol}(F_x)\right)\,. \tag{3.5}$$

From the formula $\nabla^B = \mathrm{d} + \frac{1}{2}\bar\eta$ in this trivialisation we may calculate the corresponding Laplacian (2.17)

$$\begin{aligned}
\Delta_B &= \mathrm{tr}\left((\nabla^B)^2 - \bar\eta(\cdot)\nabla^B_{\cdot}\right) \\
&= \sum_{i\le d} \left(X_i + \tfrac{1}{2}\bar\eta(X_i)\right)^2 - \left(\nabla_{X_i}X_i + \tfrac{1}{2}\bar\eta(\nabla_{X_i}X_i)\right) - \bar\eta(X_i)\nabla^B_{X_i} \\
&= \sum_{i\le d} \left(X_i - \tfrac{1}{2}\bar\eta(X_i)\right)\left(X_i + \tfrac{1}{2}\bar\eta(X_i)\right) - \left(\nabla_{X_i}X_i + \tfrac{1}{2}\bar\eta(\nabla_{X_i}X_i)\right) \\
&= \sum_{i\le d} \left(X_iX_i - \nabla_{X_i}X_i\right) + \tfrac{1}{2}\left(X_i\bar\eta(X_i)\right) - \tfrac{1}{2}\bar\eta(\nabla_{X_i}X_i) - \tfrac{1}{4}\bar\eta(X_i)^2 \\
&= \Delta_{g_B} + \tfrac{1}{2}\,\mathrm{tr}\left(\nabla_\cdot\bar\eta\right)(\cdot) - \tfrac{1}{4} g_B(\bar\eta,\bar\eta)\,.
\end{aligned}$$

In the particular case of a one-dimensional base we have that $\nabla_{X_1} X_1 = 0$, $X_1 = \partial_x$ may be chosen as a coordinate vector field and

$$\Delta_B \stackrel{d=1}{=} \partial_x^2 + \tfrac{1}{2}(\partial_x \bar{\eta}) - \tfrac{1}{4}\bar{\eta}^2 = \left(\partial_x - \tfrac{1}{2}\bar{\eta}\right)\left(\partial_x + \tfrac{1}{2}\bar{\eta}\right) .$$

We now calculate the Born-Huang potential from its local form

$$\begin{aligned} V_{\mathrm{BH}} &= \Delta_B - P_0 \Delta_h P_0 \\ &= -\sum_{i \le d} \left(P_0 X_i^* P_0^\perp X_i^* P_0 - P_0 g_B(X_i, \pi_* \eta) P_0^\perp X_i^* P_0 \right) . \end{aligned}$$

To express this in terms of ϕ_0 we first need

$$P_0^\perp X^* \phi_0 \psi = X^* \phi_0 \psi - \nabla_X^B \phi_0 \psi = \psi \left(X^* \phi_0 - \phi_0 \tfrac{1}{2} \bar{\eta}(X) \right) . \tag{3.6}$$

Applying $P_0 X^*$ to this expression gives

$$\begin{aligned} &P_0 X^* P_0^\perp X^* \phi_0 \psi \\ &\quad = \psi \left(P_0 X^* (X^* \phi_0) - \tfrac{1}{2} \nabla_X^B \phi_0 \bar{\eta}(X) \right) \\ &\quad = \psi \left(P_0 X^* (X^* \phi_0) - \phi_0 \left(\tfrac{1}{2} X \bar{\eta}(X) + \tfrac{1}{4} \bar{\eta}(X)^2 \right) \right) . \end{aligned} \tag{3.7}$$

By $\omega^B = \frac{1}{2}\bar{\eta}$ we have

$$\begin{aligned} \tfrac{1}{2} X \bar{\eta}(X) &= X \omega^B(X) = X \int_{F_x} \phi_0 (X^* \phi_0) \,\mathrm{vol}_{F_x} \\ &= \int_{F_x} |X^* \phi_0|^2 + \phi_0 X^* (X^* \phi_0) - \phi_0 (X^* \phi_0) g_B(X, \pi_* \eta) \,\mathrm{vol}_{F_x} , \end{aligned}$$

so the second term here will cancel the first term of (3.7) when calculating V_{BH}. Thus summation over i yields

$$\begin{aligned} &\sum_{i \le d} P_0 X_i^* P_0^\perp X_i^* \phi_0 \\ &= \sum_{i \le d} \int_{F_x} - |X_i^* \phi_0|^2 + \phi_0 g_B(\pi_* \eta, X_i) X_i^* \phi_0) \,\mathrm{vol}_{F_x} - \tfrac{1}{4} \bar{\eta}(X_i)^2 \\ &= - \int_{F_x} \pi^* g_B(\operatorname{grad} \phi_0, \operatorname{grad} \phi_0) \,\mathrm{vol}_{F_x} + \langle \phi_0, \eta \phi_0 \rangle - \tfrac{1}{4} g_B\left(\bar{\eta}, \bar{\eta}\right) . \end{aligned} \tag{3.8}$$

Applying $P_0 g_B(X_i, \pi^*\eta)$ to (3.6) and summing over i yields

$$\sum_{i\leq d} P_0 g_B(X_i, \pi^*\eta) P_0^\perp X_i^* \phi_0 = \langle \phi_0, \eta\phi_0 \rangle - \tfrac{1}{2} g_B\left(\bar{\eta}, \bar{\eta}\right) \,, \tag{3.9}$$

whereby

$$\begin{aligned} V_{\mathrm{BH}} &= -(3.8) + (3.9) \\ &= \int_{F_x} \pi^* g_B(\operatorname{grad}\phi_0, \operatorname{grad}\phi_0)\, \mathrm{vol}_{F_x} - \tfrac{1}{4} g_B\left(\bar{\eta}, \bar{\eta}\right) \,. \end{aligned}$$

Adding this to the negative of the Berry Laplacian we see that the term proportional to $g_B(\bar{\eta}, \bar{\eta})$ is cancelled and we have

$$\begin{aligned} H_{\mathrm{a}} &= -\varepsilon^2 \Delta_{g_B} + \lambda_0 + \varepsilon P_0 H_1 P_0 \\ &\qquad - \frac{\varepsilon^2}{2} \operatorname{tr}\left(\nabla.\bar{\eta}\right)(\cdot) + \varepsilon^2 \int_{F_x} \pi^* g_B(\operatorname{grad}\phi_0, \operatorname{grad}\phi_0)\, \mathrm{vol}_{F_x} \\ &= -\varepsilon^2 \Delta_{g_B} + \lambda_0 + \varepsilon P_0 H_1 P_0 + \varepsilon^2 V_0 \,. \end{aligned} \tag{3.10}$$

If $\partial M = \varnothing$ (3.5) we have $\lambda_0 \equiv 0$ and V_0 evaluates to

$$V_0 \overset{\partial M = \varnothing}{=} \tfrac{1}{2} \Delta(\log \mathrm{Vol}(F_x)) + \tfrac{1}{4} \left|\mathrm{d} \log \mathrm{Vol}(F_x)\right|^2_{g_B} \,. \tag{3.11}$$

3.2 Low energy asymptotics of the Laplacian

In this section we take a closer look at the asymptotic behaviour of the Laplacian at small energies. We will show that the adiabatic operator determines the spectrum of H with higher precision in this regime. In our earlier notation this means we consider connected fibres, $V \equiv 0$ and an operator H_1 of the type that arises from a perturbed metric G_ε as discussed in section 3.1.1. By small energies we mean energies whose distance to

$$\Lambda_0 := \inf_{x \in B} \min \sigma(\Delta_{F_x})$$

is of order ε^α, with $0 < \alpha \leq 2$. It is thus convenient to set

$$H := -\Delta_{g_\varepsilon} + \varepsilon H_1 - \Lambda_0 \,, \tag{3.12}$$

and

$$H_F := -\Delta_F - \Lambda_0 \,.$$

Clearly if $\partial M = \varnothing$, $\Lambda_0 = 0$ and $H = -U_\rho^* \Delta_{G_\varepsilon} U_\rho \geq 0$ (cf. equation (3.3)). If the metric G_ε arises from an embedding of M with codimension zero we have $H \geq -\varepsilon^2 \Delta_h + \varepsilon H_1 \geq - \|V_\rho\|_\infty \geq -C\varepsilon^2$ by remark 3.5.

The eigenband in question is the ground state band

$$\lambda_0(x) := \min \sigma\big(H_F(x)\big) \,,$$

which has spectral gap by lemma 3.7. For the following we fix the projection P_ε and the unitary U_ε constructed for λ_0, given Λ and $N \geq 3$. Analysing energies of order ε^α amounts to studying H only on the image of

$$\varrho_\alpha(H) := 1_{(-\infty, \varepsilon^\alpha \Lambda]}(H) \,.$$

Equivalently one may rescale the original problem by $\varepsilon^{-\alpha}$ and consider bounded energies. The most relevant energy scales are

- $\alpha = 1$: If λ_0 has a unique non-degenerate minimum on B, the smallest eigenvalues of $-\varepsilon^2 \Delta_B + \lambda_0$ behave like those of a d-dimensional harmonic oscillator (see [66]). In particular their difference is of order ε. We will show that this implies existence of eigenvalues of H with the same behaviour that are well approximated by those of H_a.

- $\alpha = 2$: Assume $\lambda_0 \equiv 0$, for example because $\partial M = \varnothing$ or the shape and volume of the fibres is fixed. Then, ignoring $\operatorname{div} S_\varepsilon$ for the moment, the adiabatic operator is $H_\mathrm{a} = \varepsilon^2(-\Delta_B + V_\mathrm{BH}) + P_0 V_\rho P_0$. The spectrum of this operator approximates that of H up to order ε^3 and vice versa (see proposition 3.11). If this operator has eigenvalues, which may happen also when B is not compact, they will typically scale like ε^2 since $P_0 V_\rho P_0 = \mathcal{O}(\varepsilon^2)$ (see remark 3.5). We will show that the simple eigenvalues of H_a are ε^4-close to those of H, with eigenfunctions approximated to order ε in $W^1_{\varepsilon=1}$.

For an embedding we know from lemma 3.4 that H_1 satisfies the conditions of theorem 2.20, so H is unitarily equivalent to H_eff at small energies.

Looking at the expansion of H_{eff} derived in proposition 2.23 we see that the super-adiabatic corrections essentially consist of horizontal differential operators. These corrections are of order ε^2 when $\varepsilon^2\Delta_h = \mathcal{O}(1)$, but when $\varepsilon^2\Delta_h = \mathcal{O}(\varepsilon^\alpha)$ we can expect them to be of order $\varepsilon^{2+\alpha k/2}$, if k is their order as a differential operator. Thus at this energy scale the adiabatic approximation should be more accurate, as long as H_1 does not contain potentials at leading order.

Remark 3.8. In the preceding section we saw that V_ρ may contain a term of order ε if M has codimension at least one. In remark 3.5 we noted that this term does not contribute to H_{a} if $\partial M = \varnothing$. One might thus expect that it appears in the leading order of the super-adiabatic corrections. Since it is a differential operator of order zero, the different energy scale will not help to make it smaller. This problem may however be circumvented by modifying H_F, setting (see also remark 2.1)

$$H_F := -\Delta_F + \tfrac{1}{4} g_F(\mathrm{d}\log\rho, \mathrm{d}\log\rho) + \tfrac{1}{2}\Delta_F \log\rho\,.$$

This operator includes in particular all the terms of V_ρ up to order ε^3. Its properties are also easily described in the following way: Consider on $\mathscr{H}_F$ the ε-dependent scalar product obtained by integrating against the density $\rho\,\mathrm{vol}_{F_x}$. Because $\rho = 1 + \mathcal{O}(\varepsilon)$ the norm obtained from this scalar product is equivalent to the original one with constants independent of ε. Let Δ_F^ρ be the Laplacian on F defined by the quadratic form using g_{F_x} and this product. We then have $H_F = -U_\rho^* \Delta_F^\rho U_\rho$ for the unitary $U_\rho : L^2(F, \mathrm{vol}_{F_x}) \to L^2(F, \rho\,\mathrm{vol}_{F_x})$ used in the definition of V_ρ. Hence the ground state of H_F is $\lambda_0 \equiv 0$ and the corresponding eigenfunction is proportional to $\sqrt{\rho}$. In this way we can absorb the ε-dependence of $\mathcal{E}$ into the bundle metric and that of P_0^ρ is easily controlled. Thus using the formula (3.10) for $P_0 \Delta_h P_0$ we get

$$P_0^\rho H P_0^\rho = -\varepsilon^2 P_0^\rho \Delta_h P_0^\rho + \varepsilon P_0^\rho H_1 P_0^\rho = -\varepsilon^2 P_0 \Delta_h P_0 + \varepsilon P_0 H_1 P_0 + \mathcal{O}(\varepsilon^3)\,,$$

and the adiabatic operator is the same as before up to order ε^3.

We now summarise the technical conditions on H_1 for the results of this section to hold.

Condition 5. The operator H_1 has the form

$$H_1 = -\varepsilon^2 \operatorname{div} S_\varepsilon + \varepsilon V_\varepsilon \,,$$

with $\operatorname{div} S_\varepsilon$ given by (3.4) for some $S_\varepsilon \in \Gamma_b(\pi^* T^* B \otimes \pi^* T^* B)$, satisfying $-\varepsilon^2 \Delta_h - \varepsilon^3 \operatorname{div} S_\varepsilon \geq 0$ and $V_\varepsilon \in \mathscr{C}_b^\infty(M)$ bounded uniformly in ε.

This is satisfied for embeddings of M satisfying condition 4 if $V_\rho = \mathcal{O}(\varepsilon^2)$, as in the case of quantum tubes (see remark 3.5), or if $\partial M = \varnothing$, by redefinition of H_F as discussed in remark 3.8. It also applies to any fibre bundle with a rescaled submersion metric and suitable perturbations thereof.

This condition implies that $H \geq -C\varepsilon^2$, so for $\alpha \leq 2$ we have $\|H \varrho_\alpha(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^\alpha)$. We now derive refined estimates for the operators appearing in the expansion of P_ε on the image of $\varrho_\alpha(H)$.

Lemma 3.9. *For $A \in \{H, H_\mathrm{a}\}$ we have*

$$\big\| [-\varepsilon^2 \Delta_h, P_0] P_0 \varrho_\alpha(A) \big\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{1+\alpha/2}) \,.$$

Proof. From lemma 2.5 we already know that $[\varepsilon^2 \Delta_h, P_0] P_0 \in \mathcal{A}^{1,1}$. We will now see that the part lying in $\mathcal{A}^0$ is actually of order ε^2, while first order horizontal differential operators may be bounded by $\sqrt{H} = \mathcal{O}(\varepsilon^{\alpha/2})$.

On $U \in \mathfrak{U}$ we use the corresponding orthonormal frame $(X_i)_{i \leq d}$ to calculate

$$\begin{aligned}
&\varepsilon^2 [\Delta_h, P_0] P_0 \\
&= \varepsilon^2 P_0^\perp \sum_{i \leq d} \Big([X_i^* X_i^*, P_0] - [(\nabla_{X_i} X_i)^*, P_0] - [g_B(\pi_* \eta, X_i) X_i^*, P_0] \Big) P_0 \\
&= \varepsilon^2 P_0^\perp \sum_{i \leq d} \Big(2 [X_i^* - g_B(\pi_* \eta, X_i), P_0] P_0 X_i^* \qquad \text{(3.13a)} \\
&\quad + \big[X_i^*, [X_i^*, P_0]\big] - [(\nabla_{X_i} X_i)^*, P_0] - g_B(\pi_* \eta, X_i) [X_i^*, P_0] \Big) P_0 \,. \qquad \text{(3.13b)}
\end{aligned}$$

The terms in (3.13a) and (3.13b) are traces of the maps

$$\begin{aligned}
&X, Y \mapsto 2 [X^* - g_B(\pi_* \eta, X), P_0] \nabla_Y^B \\
&X, Y \mapsto \big[X^*, [Y^*, P_0]\big] - [(\nabla_X Y)^*, P_0] - g_B(\pi_* \eta, X) [Y^*, P_0] \,.
\end{aligned}$$

One can check that these maps define sections of $T^*B^{\otimes 2} \otimes \mathscr{L}(D(H_F), \mathscr{H}_F)$ and $T^*B^{\otimes 2} \otimes \mathscr{L}(\mathscr{H}_F)$ respectively, by asserting that they are linear with respect to multiplication by $f \in \mathscr{C}^\infty(B)$. Consequently their g_B-traces are well defined sections of $\mathscr{L}(D(H_F), \mathscr{H}_F)$ and $\mathscr{L}(\mathscr{H}_F)$ that equal (3.13a), (3.13b). Because $P_0 \in \mathcal{A}_H^{0,0}$ the second line (3.13b) defines an element of $L^\infty(\mathscr{H}_F)$ and $\mathcal{A}^{0,2}$ whence it is of order ε^2.

The first line (3.13a) is an element of $\mathcal{A}^{1,1}$. To see that it is of order $\varepsilon^{1+\alpha/2}$ on the image of $\varrho_\alpha(A)$, we observe that $\varrho_\alpha(A)$ is a bounded operator from $\mathscr{H}$ to the domain of $\varepsilon^{-2\alpha}A^2$ with its graph norm, which we denote by $D^2_\alpha(A)$. The norm of (3.13a) as an operator in $\mathscr{L}(D^2_\alpha(A), \mathscr{H})$ can be estimated using the local expression (see remark 2.4) and

$$\|\varepsilon \nabla^B_X\|_{\mathscr{L}(D^2_\alpha(A),\mathscr{H})} = \mathcal{O}(\varepsilon^{\alpha/2}) ,$$

which is proved in lemma C.3 in the appendix. □

Lemma 3.10. *Let* $0 \le \alpha \le 2$ *and* $A \in \{H, H_\mathrm{a}\}$*, then*

$$\left\| P_0^\perp (P_\varepsilon - P_0)\varrho_\alpha(A) \right\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^{1+\alpha/2}) .$$

Proof. Let the regular cut-off $\chi \in \mathscr{C}_0^\infty(-\infty, \Lambda]$ be equal to one on the support of $\varrho_\alpha(x)$, so that $\varrho_\alpha(A) = \chi(A)\varrho_\alpha(A)$. By lemma C.2

$$\|\chi(H_\mathrm{a}) - P_0\chi(H)P_0\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon) ,$$

so by (2.19)

$$\|(P_\varepsilon - P_0)\varrho_\alpha(H_\mathrm{a})\|_{\mathscr{L}(\mathscr{H},D(H))} \le \left\| P_0^\perp (P_\varepsilon - P_0)\chi(H)\varrho_\alpha(H_\mathrm{a}) \right\| + \mathcal{O}(\varepsilon^2) .$$

Together with lemma 2.25 this implies

$$\left\| P_0^\perp (P_\varepsilon - P_0 - \varepsilon P_1)\varrho_\alpha(A) \right\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^2) .$$

Recall from the construction in lemma 2.14 that

$$P_0^\perp P_1 P_0 = -R_F(\lambda_0)[-\varepsilon\Delta_h + H_1, P_0]P_0 .$$

In view of this, lemma 3.9 together with the estimate $\|H_1\|_{\mathscr{L}(D^2_\alpha(A),\mathscr{H})} = \mathcal{O}(\varepsilon^{\alpha/2})$ from lemma C.3 proves the claim in the $\mathscr{L}(\mathscr{H})$-norm.

The image of $P_1 \in \mathcal{A}^2_H$ satisfies Dirichlet conditions, so by the elliptic estimates A.14 the result will also hold in the $\mathscr{L}(\mathscr{H}, D(H))$-norm if we can estimate $\Delta_{g_\varepsilon} P_0^\perp P_1 \varrho_\alpha(A)$. This estimate for the term

$$\Delta_{g_\varepsilon} R_F(\lambda_0)[H_1, P_0]P_0 = \mathcal{O}(\varepsilon^{\alpha/2})$$

is a consequence of lemma C.3. For the remaining term note that vertical derivatives acting from the left give bounded operators, while derivatives of the form $\Phi^* X$ may be commuted to the right since $R_F(\lambda_0) \in \mathcal{A}^0_H$ (see 2.11 and 2.5). In view of the proof of lemma 3.9 we can thus complete the proof by showing $\|P_0 X^*\|_{\mathscr{L}(D^2_\alpha(A), D(H))} = \mathcal{O}(\varepsilon^{\alpha/2})$, which again is lemma C.3. □

Proposition 3.11. *Let* $0 < \alpha \leq 2$ *and assume there are* $\varepsilon^\alpha \lambda \in \sigma(H)$ *and* $\varepsilon^\alpha \mu \in \sigma(H_\mathrm{a})$ *with* $\mu, \lambda < \Lambda$*. Then there exists a constant* C *independent of* μ *and* λ *for which*

1) $\operatorname{dist}\big(\varepsilon^\alpha \lambda, \sigma(H_\mathrm{a})\big) \leq C\varepsilon^{2+\alpha/2}$,

2) $\operatorname{dist}\big(\varepsilon^\alpha \mu, \sigma(H)\big) \leq C\varepsilon^{2+\alpha/2}$

Proof. 1) Let $(\varphi_k)_{k\in\mathbb{N}} \subset \operatorname{ran}(\varrho_\alpha(H))$ be a Weyl sequence for $\varepsilon^\alpha\lambda \in \sigma(H)$. For ε small enough $\operatorname{supp}\varrho_\alpha \subset (-\infty, \Lambda_1)$, so we can choose a regular cut-off $\chi \in \mathscr{C}^\infty_0(-\infty, \Lambda_1)$ satisfying $\varrho_\alpha(H) = \chi(H)\varrho_\alpha(H)$. The unitary equivalence shown in theorem 2.20 implies

$$\|(U_\varepsilon^* H U_\varepsilon - H_\mathrm{eff}) U_\varepsilon^* \varrho_\alpha(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{N+1}).$$

We now proceed to expand

$$(H_\mathrm{eff} - H_\mathrm{a})\, U_\varepsilon^* \varrho_\alpha(H) = U_\varepsilon^* \left(P_\varepsilon H P_\varepsilon - P_\varepsilon U_\varepsilon H U_\varepsilon^* P_\varepsilon\right) \varrho_\alpha ,$$

using

$$P_\varepsilon U_\varepsilon = P_\varepsilon + \underbrace{P_\varepsilon (P_0 - P_\varepsilon) P_\varepsilon^\perp}_{=:\varepsilon \tilde{U}_1} - \tfrac{1}{2} \underbrace{P_\varepsilon (P_0 - P_\varepsilon)^2 P_\varepsilon}_{\varepsilon^2 \tilde{U}_2} + \mathcal{O}(\varepsilon^3).$$

The terms in this expansion are exactly those appearing in the expansion of H_eff (2.20a), (2.20b) with reversed roles of P_0 and P_ε. We estimate these terms separately, starting with those of (2.20a):

$$\varepsilon\tilde{U}_1[H,P_\varepsilon]P_\varepsilon\varrho_\alpha(H) = \varepsilon\tilde{U}_1P_\varepsilon^\perp \underbrace{[H,P_\varepsilon]\varrho_\alpha}_{\overset{2.13}{=}\mathcal{O}(\varepsilon^{N+1})} = \mathcal{O}(\varepsilon^{N+2})\,,$$

$$\varepsilon P_\varepsilon[H,P_\varepsilon]\tilde{U}_1^*\varrho_\alpha(H) = P_\varepsilon \underbrace{[H,P_\varepsilon]}_{=\mathcal{O}(\varepsilon)} \underbrace{P_0^\perp(P_0-P_\varepsilon)\varrho_\alpha}_{\overset{3.10}{=}\mathcal{O}(\varepsilon^{1+\alpha/2})} + \mathcal{O}(\varepsilon^3)\,.$$

The terms corresponding to those of (2.20b) are:

$$\varepsilon^2\tilde{U}_1H\tilde{U}_1^*\varrho_\alpha(H) = \varepsilon\tilde{U}_1H \underbrace{P_0^\perp(P_0-P_\varepsilon)\varrho_\alpha}_{\overset{3.10}{=}\mathcal{O}(\varepsilon^{1+\alpha/2})} + \mathcal{O}(\varepsilon^3) = \mathcal{O}(\varepsilon^{2+\alpha/2})\,,$$

$$\varepsilon^2P_\varepsilon H\tilde{U}_2^*\varrho_\alpha(H) = P_\varepsilon HP_\varepsilon \underbrace{(P_0-P_\varepsilon)}_{=\mathcal{O}(\varepsilon)} \underbrace{P_0^\perp(P_0-P_\varepsilon)\varrho_\alpha}_{\overset{3.10}{=}\mathcal{O}(\varepsilon^{1+\alpha/2})} + \mathcal{O}(\varepsilon^3)\,,$$

$$\varepsilon^2\tilde{U}_2HP_\varepsilon\varrho_\alpha(H) = \varepsilon^2\tilde{U}_2 \underbrace{H\varrho_\alpha}_{=\mathcal{O}(\varepsilon^\alpha)} + \mathcal{O}(\varepsilon^{N+3}) = \mathcal{O}(\varepsilon^{2+\alpha})\,.$$

Altogether these estimates give (for k large enough):

$$\begin{aligned}\|(H_\mathrm{a}-\varepsilon^\alpha\lambda)\,U_\varepsilon^*\varphi_k\| &\leq \|(H_\mathrm{a}-U_\varepsilon^*HU_\varepsilon)U_\varepsilon^*\varphi_k\| + \|(H-\varepsilon^\alpha\lambda)\,\varphi_k\| \\ &= \|(H_\mathrm{a}-H_\mathrm{eff})U_\varepsilon^*\varphi_k\| + \mathcal{O}(\varepsilon^{N+1}) \\ &= \mathcal{O}(\varepsilon^{2+\alpha/2})\,,\end{aligned} \tag{3.14}$$

which proves $\mathrm{dist}(\varepsilon^\alpha\lambda,\sigma(H_\mathrm{a})) \leq C_1\varepsilon^{2+\alpha/2}$ (cf the proof of 2.18). The constant is independent of λ because we have only used estimates for the operators that hold uniformly at energies below Λ.

2) In order to prove the second claim we first show

$$\mathrm{dist}(\varepsilon^\alpha\mu,\sigma(H_\mathrm{eff})) = \mathcal{O}(\varepsilon^{2+\alpha/2})\,,$$

the claim will then follow from theorem 2.18. Let $(\psi_k)_{k\in\mathbb{N}} \subset \mathrm{ran}(\varrho_\alpha(H_\mathrm{a}))$ be a Weyl sequence for $\varepsilon^\alpha\mu$. As above we can apply lemmata 3.9 and 3.10 to the expansion of H_eff, written down in (2.20a) and (2.20b), to get

$$\|(H_\mathrm{eff}-H_\mathrm{a})\,\varrho_\alpha(H_\mathrm{a})\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^{2+\alpha/2})\,, \tag{3.15}$$

which gives $\|(H_{\text{eff}} - \varepsilon^\alpha \mu)\, \psi_k\| = \mathcal{O}(\varepsilon^{2+\alpha/2})$ for k large enough. From this we deduce existence of $\tilde{\mu} \in \sigma(H_{\text{eff}})$ with $|\varepsilon^\alpha \mu - \tilde{\mu}| = \mathcal{O}(\varepsilon^{2+\alpha/2})$. Theorem 2.18 states that $\operatorname{dist}(\tilde{\mu}, \sigma(H)) = \mathcal{O}(\varepsilon^{N+1})$, which proves the claim.

□

Theorem 3.12. *Let* $\varepsilon^\alpha \mu < \varepsilon^\alpha \Lambda$ *be a simple eigenvalue of* H_{a} *with* $\operatorname{dist}(\mu, \sigma(\varepsilon^{-\alpha} H_{\text{a}}) \setminus \mu) \geq C_\mu$ *for some* $C_\mu > 0$.

There is a unique simple eigenvalue $\varepsilon^\alpha \lambda$ *of* H *with* $|\lambda - \mu| = \mathcal{O}(\varepsilon^2)$ *and* $C_\lambda > 0$ *such that* λ *satisfies* $\operatorname{dist}(\lambda, \sigma(\varepsilon^{-\alpha} H) \setminus \lambda) \geq C_\lambda$.

Furthermore let $\psi \in \ker(H_{\text{a}} - \varepsilon^\alpha \mu)$ *be a normalised eigenfunction and* $P_\lambda(H)$ *the orthogonal projection to* $\ker(H - \varepsilon^\alpha \lambda)$. *Then*

$$\|\psi - P_\lambda \psi\|_{D(H)} = \mathcal{O}(\varepsilon^\beta) \qquad \text{and} \qquad \|\psi - P_\lambda \psi\|_{W^1_{\varepsilon=1}(M)} = \mathcal{O}(\varepsilon^{\alpha/2})\,,$$

with $\beta := \min\{2 - \alpha/2, 1 + \alpha/2\}$.

Proof. The existence of $\lambda \in \sigma(\varepsilon^{-\alpha} H)$ with $|\lambda - \mu| = \mathcal{O}(\varepsilon^{2-\alpha/2})$ is assured by proposition 3.11. Since the correspondence of spectra proved there is reciprocal, λ must be separated from the rest of the spectrum and simple, since this holds for μ.

To be more precise, let P_μ be the projection to $\ker(H_{\text{a}} - \varepsilon^\alpha \mu)$. Assume there exists $\varepsilon^\alpha \lambda \neq \varepsilon^\alpha \lambda' \in \sigma(H)$ with $|\mu - \lambda'| = \mathcal{O}(\varepsilon^{2-\alpha/2})$. We may choose Weyl sequences, $(\varphi_k)_{k\in\mathbb{N}}$ for $\varepsilon^\alpha \lambda$ and $(\phi_k)_{k\in\mathbb{N}}$ for $\varepsilon^\alpha \lambda'$, satisfying $\langle \varphi_j, \phi_k \rangle = 0$ for every j, k, by choosing both sequences in the image of spectral projections on disjoint intervals around λ and λ' respectively. We may choose an element $\varphi = \varphi_{k(\varepsilon)}$ of the first sequence satisfying (3.14), so since $|\mu - \lambda| = \mathcal{O}(\varepsilon^{2-\alpha/2})$

$$\begin{aligned}\|(1 - P_\mu)\, U_\varepsilon^* \varphi\|_{\mathscr{H}} &\leq \underbrace{\|(H_{\text{a}} - \varepsilon^\alpha \mu)^{-1}\, (1 - P_\mu)\|}_{\leq (C_\mu \varepsilon^\alpha)^{-1}} \underbrace{\|(H_{\text{a}} - \varepsilon^\alpha \mu)\, U_\varepsilon^* \varphi\|}_{\overset{(3.14)}{=} \mathcal{O}(\varepsilon^{2+\alpha/2})} \\ &= \mathcal{O}(\varepsilon^{2-\alpha/2})\,.\end{aligned}$$

This also holds for an element $\phi = \phi_{k(\varepsilon)}$ of the second sequence, so $P_\mu U_\varepsilon^* \varphi$ and $P_\mu U_\varepsilon^* \phi$ are two almost orthogonal ($|\langle P_\mu U_\varepsilon^* \varphi, P_\mu U_\varepsilon^* \phi \rangle| = \mathcal{O}(\varepsilon^{4-\alpha})$) vectors in $P_\mu \mathscr{H}$ with norm close to one ($1 - \|P_\mu U_\varepsilon^* \varphi\| = \mathcal{O}(\varepsilon^{2-\alpha/2})$), in contradiction to the simplicity of μ. Therefore no such λ' exists and $\varepsilon^\alpha \lambda$ is an isolated point in the spectrum of H, and thus an eigenvalue. It also has to be simple, since if there were two orthogonal eigenfunctions φ_1 and φ_2 of $\varepsilon^\alpha\lambda$ we could repeat the preceding argument with $\varphi := \varphi_1$ and $\phi := \varphi_2$ to arrive at a contradiction.

This also assures that λ is separated from the spectrum of $\varepsilon^{-\alpha}H$ at least by a quantity $\tilde{C}_\lambda = C_\mu - 2C\varepsilon^{2-\alpha/2}$, for if this were not the case proposition 3.11 would contradict the existence of C_μ. Hence we may take C to be the constant appearing there.

Before improving the estimate of $|\lambda-\mu|$ to $\mathcal{O}(\varepsilon^2)$ we must gain some information on the eigenfunctions. In proposition 3.11 we effectively proved that the eigenfunction ψ of H_a is a quasimode for H and thus

$$\begin{aligned}
&\|(1-P_\lambda)U_\varepsilon\psi\|_{D(H)} \\
&\quad\leq \underbrace{\|(H-\varepsilon^\alpha\lambda)^{-1}(1-P_\lambda)\|_{\mathscr{L}(\mathscr{H},D(H))}}_{\leq\sqrt{1+(1+2\varepsilon^{2\alpha}\lambda^2)}\tilde{C}_\lambda(\varepsilon^\alpha)^{-1}}\underbrace{\|(H-\varepsilon^\alpha\lambda)U_\varepsilon\psi\|_{\mathscr{H}}}_{\overset{3.11}{=}\mathcal{O}(\varepsilon^{2+\alpha/2})} \\
&\quad= \mathcal{O}(\varepsilon^{2-\alpha/2})\,.
\end{aligned}$$

Since $\psi = P_0\varrho_\alpha(H_\mathrm{a})\psi$ lemma 3.10 gives

$$\begin{aligned}
\|(U_\varepsilon-1)\psi\|_{D(H)} &= \left\|P_0^\perp(P_\varepsilon-P_0)\varrho_\alpha(H_\mathrm{a})\psi\right\|_{D(H)} + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{O}(\varepsilon^{1+\alpha/2})\,,
\end{aligned}$$

which implies

$$\|(1-P_\lambda)\psi\|_{D(H)} = \|(1-P_\lambda)U_\varepsilon\psi\| + \|(1-P_\lambda)(1-U_\varepsilon)\psi\| = \mathcal{O}(\varepsilon^\beta)\,.$$

Given this we can calculate

$$\begin{aligned}
\varepsilon^\alpha|\mu-\lambda| &= |\langle\psi,(H_\mathrm{a}-\varepsilon^\alpha\lambda)\psi\rangle| \\
&= |\langle P_0\psi,(H-\varepsilon^\alpha\lambda)P_0\psi\rangle| \qquad (3.16\mathrm{a})\\
&= |\langle(1-P_\lambda)\psi,(H-\varepsilon^\alpha\lambda)\psi\rangle| \\
&\leq \|(1-P_\lambda)\psi\|\varepsilon^\alpha|\mu-\lambda| \\
&\qquad + \left|\left\langle(1-P_\lambda)P_0\psi, P_0^\perp[H,P_0]P_0\rho_\alpha(H_\mathrm{a})\psi\right\rangle\right| \\
&\leq \left\|P_0^\perp P_\lambda\psi\right\|\underbrace{\|[H,P_0]P_0\rho_\alpha(H_\mathrm{a})\psi\|}_{\overset{3.9}{=}\mathcal{O}(\varepsilon^{1+\alpha/2})} + C\varepsilon^{\beta+\alpha}|\mu-\lambda|\,. \qquad (3.16\mathrm{b})
\end{aligned}$$

To estimate this we see that

$$\begin{aligned}\left\|P_0^\perp P_\lambda \psi\right\| &\overset{2.21}{=} \left\|P_0^\perp P_\varepsilon P_\lambda \psi\right\| + \mathcal{O}(\varepsilon^{N+1}) \\ &= \left\|P_0^\perp (P_\varepsilon - P_0)\rho_\alpha(H) P_\lambda \psi\right\| + \mathcal{O}(\varepsilon^{N+1}) \\ &\overset{3.10}{=} \mathcal{O}(\varepsilon^{1+\alpha/2}), \end{aligned} \tag{3.17}$$

giving (since $\beta \geq \alpha/2$)

$$|\mu - \lambda| \leq C_1 \varepsilon^\beta \, |\mu - \lambda| + C_2 \varepsilon^2 \overset{3.11}{=} \mathcal{O}(\varepsilon^2)\,.$$

We still need to prove the estimate for the $W^1_{\varepsilon=1}$-norm. For every $\phi \in D(H)$ we may estimate this norm by (see [62, proposition 3.25])

$$\begin{aligned} C^{-1} \|\phi\|^2_{W^1_{\varepsilon=1}} &\leq \|\phi\|^2_{\mathscr{H}} + \langle \phi, -\Delta_F - \Delta_h \phi\rangle \\ &\leq (1+\Lambda_0)\|\phi\|^2 + \varepsilon^{-2}\langle \phi, \underbrace{-\Delta_F - \Lambda_0}_{\geq 0} \phi\rangle + \langle \phi, -\Delta_h \phi\rangle \\ &\leq (1+\Lambda_0)\|\phi\|^2 + \varepsilon^{-2}\,|\langle \phi, H\phi\rangle| + \varepsilon^{-1}\,|\langle \phi, H_1 \phi\rangle| \\ &\leq \big(1 + \Lambda_0 + \varepsilon^{\alpha-2}\lambda\big)\|\phi\|^2 + \varepsilon^{-2}\,|\langle \phi, (H - \varepsilon^\alpha \lambda)\phi\rangle| \\ &\qquad + \varepsilon^{-1}\,|\langle \phi, H_1\phi\rangle|\,. \end{aligned}$$

If we apply this to $\phi := (1 - P_\lambda)\psi$ we can use the estimates leading from (3.16a) to (3.16b) to get $\varepsilon^{-2}\,|\langle \phi, (H - \varepsilon^\alpha\lambda)\phi\rangle| = \mathcal{O}(\varepsilon^\alpha)$. By condition 5 the term containing H_1 (and also additional terms arising from a redefinition of H_F as in remark 3.8) can be bounded by

$$\varepsilon^{-1}\,|\langle \phi, H_1 \phi\rangle| \leq \|V_\varepsilon\|_\infty \|\phi\|^2_{\mathscr{H}} + \varepsilon \|S_\varepsilon\|_\infty \|\phi\|^2_{W^1_{\varepsilon=1}}\,.$$

Already knowing that $\|\phi\|_{\mathscr{H}} = \mathcal{O}(\varepsilon^\beta)$ we conclude

$$\|(1 - P_\lambda)\,\psi\|_{W^1_{\varepsilon=1}} = \mathcal{O}(\varepsilon^{\alpha/2})\,.$$

□

Remark 3.13. The theorem is formulated to determine eigenvalues of H from those of H_a. This is relevant because H_a is simpler, or at least operates on a space of lower dimension. In principle though, the eigenvalues of H also determine those of H_a, by basically the same proof.

Moreover the first part of the proof naturally extends to a proof of the following statement (see also [37, section 4.3.5]):

Let $\Sigma_a \subset [0, \varepsilon^\alpha \Lambda]$ be a compact subset of $\sigma(H_a)$, that is separated from the rest of the spectrum by a gap of order ε^α and whose spectral projection $P_{\Sigma_a}(H_a)$ has finite rank. Then there exists a compact set $\Sigma \subset \sigma(H)$ with $\text{dist}(\Sigma_a, \Sigma) = \mathcal{O}(\varepsilon^{2+\alpha/2})$, separated from the rest of the spectrum of H and $\text{rank}(P_\Sigma(H)) = \text{rank}(P_{\Sigma_a}(H_a))$.

3.3 Eigenfunctions and their nodal sets

Theorem 3.12 tells us that if $\varepsilon^\alpha \mu$ is a simple eigenvalue of H_a with eigenfunction ψ this approximates the eigenfunction $\varphi := P_\lambda \psi / \|P_\lambda \psi\|$ of the corresponding eigenvalue $\varepsilon^\alpha \lambda$ of H. We use this as a starting point for a more detailed analysis of eigenfunctions in two different cases.

The first case is that of a constant ground state band $\lambda_0 \equiv 0$. We show that isolated eigenvalues of H_a, if they exist, are typically separated by gaps of order ε^2. Assuming that the base dimension is at most three we obtain uniform bounds on the difference $\varphi - \psi$.

The other case concerns an eigenband with a unique and non-degenerate minimum. It was shown by Simon [66] that H_a has eigenvalues below its essential spectrum and that they are separated by gaps of order $\alpha = 1$. We show uniform approximation for the eigenfunctions corresponding to these eigenvalues for the case $d = 1$.

Together, the low-energy effective operators and the uniform estimates for eigenfunctions provide a rather detailed description of the eigenfunctions of H. We use this to estimate the location of the nodal sets based on the analysis of the operator H_a for the given manifold. If appropriate this will imply that the nodal set touches the boundary and provide an estimate of the number of nodal domains.

3.3.1 Uniform approximation of eigenfunctions

We now derive uniform estimates on the difference $\psi - \varphi$ from the fact that it is of order $\varepsilon^{\alpha/2}$ in $W^1_{\varepsilon=1}$. The difference is small if the dimension of B is not too large ($d \leq 3$ for $\alpha = 2$ or $d = 1$ for $\alpha = 1$), but irrespective of the dimension of F. Like the earlier works [18, 26, 35, 36] our method relies on the generalised maximum principle [60, theorem 10].

Theorem (The generalised maximum principle)**.** *Let $\Omega \subset \mathbb{R}^k$ be a bounded domain. Let D be a uniformly elliptic (with negative principal symbol) operator of second order with coefficients in $\mathscr{C}^\infty(\overline{\Omega})$. If $\phi, \psi \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ satisfy the differential inequalities*

$$D\phi \geq 0$$
$$D\psi \leq 0$$

in Ω and $\phi > 0$ in $\overline{\Omega}$, then ψ/ϕ cannot attain a non-negative maximum in Ω, unless it is constant.

We will also make use of the following corollary:

Corollary 3.14. *Let $\Omega = B(r, 0)$ and let D^0 denote the operator D with Dirichlet boundary conditions. Assume D^0 is self-adjoint and that $g \in W^1(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ is strictly positive on $\partial\Omega$. Then if $\lambda < \min \sigma(D^0)$ the unique solution $\psi \in \mathscr{C}^\infty(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ of the boundary value problem*

$$\begin{aligned} D\psi = \lambda\psi \qquad & \text{in} \quad \Omega\,, \\ \psi = g \qquad & \text{on} \quad \partial\Omega \end{aligned}$$

is strictly positive.

Proof. Existence, uniqueness and regularity of the solution are proved in most textbooks on partial differential equations, e.g. [22]. The statement on positivity is also well known but we still give a short proof. let $R > r$, D_R be an extension of D to $B(R, 0)$ such that $\lambda < \mu := \min \sigma(D^0_R) < \min \sigma(D^0)$. Let ϕ be be the positive eigenfunction with eigenvalue μ of D_R. Now ϕ is strictly positive on $\overline{\Omega}$ and satisfies

$$(D - \lambda)\,\phi = (\mu - \lambda)\,\phi \geq 0\,.$$

Since $(D - \lambda)\psi = 0$, the maximum principle applies to $-\psi$ and $\phi > 0$. So if there were $x \in \Omega$ with $\psi(x) \leq 0$ we would get

$$0 \leq -\psi(x)/\phi(x) \leq \sup_{y \in \partial\Omega} -\psi(y)/\phi(y) = \sup_{y \in \partial\Omega} -g(y)/\phi(y) < 0\,.$$

This is a contradiction, so $\psi > 0$. $\square$

Now let $\delta := (1 - P_\lambda)\psi$ (in the notation of theorem 3.12) and take note that this is a smooth function vanishing on ∂M. Calculate

$$H\delta = H\psi - \varepsilon^\alpha \lambda P_\lambda \psi = \varepsilon^\alpha \lambda \delta + \underbrace{(H - H_\mathrm{a})\,\psi + \varepsilon^\alpha\,(\mu - \lambda)\,\psi}_{=:R_\varepsilon}\,. \tag{3.18}$$

Since $\delta \in \operatorname{ran}(1 - P_\lambda)$ vanishing of R_ε would imply $\delta = 0$. The eigenfunction $\psi \in L^2(\mathcal{E})$ of H_a can be written as a product $\psi = \phi_0 \tilde{\psi}$ where $\phi_0 \in \mathscr{C}^\infty(M)$ is the λ_0-eigenfunction of H_F and $\tilde{\psi} \in L^2(B)$ is an eigenfunction of H_a, as expressed it the trivialisation of $\mathcal{E}$ by ϕ_0 in (3.10). Since R_ε is given by the action of a horizontal differential operator on ψ we can hope to estimate it using elliptic regularity theory for ϕ_0 and $\tilde{\psi}$. Throughout this section we will commit abuse of notation by not distinguishing between ψ and $\tilde{\psi}$ or H_a and its form in the given trivialisation.

We now show that a bound on R_ε implies a pointwise bound on δ by a quantity related to the W^1_ε-norm of δ.

Lemma 3.15. *Assume there exist a constant $K > 0$ and $\gamma \in \mathbb{R}$ such that $\|R_\varepsilon\|_\infty < K\varepsilon^\gamma$. Then there are positive constants C and R, such that for every $x_0 \in M$*

$$|\delta(x_0)| \leq C\Bigg(\int\limits_{\Omega(x_0)} |\delta|^2 + g_\varepsilon\,(\mathrm{d}\delta, \mathrm{d}\delta)\ \mathrm{vol}_{g_\varepsilon}\Bigg)^{1/2} + 2\varepsilon^\gamma\,,$$

where $\Omega(x_0) = \{x \in M : \operatorname{dist}_{g_B}(\pi(x_0), \pi(x)) \leq \varepsilon R\}$.

Proof. We prove the statement for the positive part δ_+ of δ, the proof for the negative part being the same. First note that in the interior of $\Omega_+ := \operatorname{supp} \delta_+$ we have

$$(H - \varepsilon^\alpha \lambda - K)\,(\delta_+ + \varepsilon^\gamma) = \underbrace{R_\varepsilon - \varepsilon^\gamma K}_{\leq 0} - K\delta_+ - \varepsilon^{\alpha+\gamma}\lambda \leq 0\,. \tag{3.19}$$

We now aim at constructing a function f, defined on a neighbourhood of x_0, with $f \geq \delta_+$ but bounded by the integral in the statement of the theorem. This will be achieved by choosing f as the solution of an elliptic boundary value problem and then using the maximum principle.

First we must choose suitable neighbourhoods that will make the locally obtained estimates hold uniformly on M. Let $\mathfrak{U}^\varepsilon$ be a regular Atlas of (M, g_ε) as in lemma A.10, i.e. the coordinate maps κ_ν^ε are given either by g_ε-geodesic coordinates or normal collar maps with (ε independent) radius $r_\mathfrak{U}$. Since by proposition A.9 the injectivity radii have lower bounds independent of ε, there is $R > 0$ such that for every $x \in M$ the metric ball $B^\varepsilon(R, x) = \{y \in M : \operatorname{dist}(x, y) < R\}$ is completely contained in some coordinate neighbourhood $U_\nu^\varepsilon \in \mathfrak{U}^\varepsilon$ (cf. [62, lemma 3.19]). Recall that the image of U_ν^ε under the normal coordinate map is either given by the euclidean ball $B(r_\mathfrak{U}, 0)$ or (if $\nu < 0$) the cylinder $B(r_\mathfrak{U}, 0) \times [0, r_c/2)$.

The virtue of these ε-dependent coordinate systems is that they mitigate (the leading order of) the ε-dependence of g_ε since in geodesic coordinates this leading order is always given by the euclidean metric. By A.9 we have bounds on the coefficients of g_ε in these coordinates, that are uniform in ε and lead to mutual bounds of the distance functions (cf. (A.2)) of g_ε and the euclidean metric. Hence there is R_0, independent of x and ε, such that $B\big(R_0, \kappa_\nu^\varepsilon(x)\big)$ (or $B\big(R_0, \kappa_\nu^\varepsilon(x)\big) \cap \{y_{n+d} \geq 0\}$ if $\nu < 0$) is completely contained in $\kappa_\nu^\varepsilon\big(B^\varepsilon(R, x)\big)$.

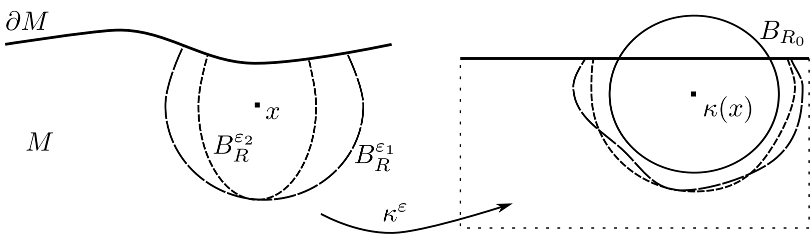


Figure 3.1: After a shift, the maps κ^ε send different metric balls to similar sets in $B(r_\mathfrak{U}, 0) \times [0, r_c/2)$.

Now shift the coordinate system so that $\kappa_\nu^\varepsilon(x) = 0$ and let $D_x^\varepsilon := (\kappa_\nu^\varepsilon)_* H$. If $\nu < 0$ extend this to an elliptic operator on $B(R_0, 0)$ (by extending the coefficients, which come from the metric G_ε). In this way we can obtain a family of elliptic operators D_x^ε on $B(R_0, 0)$ that have a common bound $\{e, c\}$ on their ellipticity constants and coefficients, because by A.9 the bounds on the geometry are uniform in ε (see also [62, proposition 5.14]).

Now because of these common bounds we can choose $r \leq R_0$, independent of x and ε, so that the Dirichlet energy

$$\inf_{0 \neq \phi \in W_0^1(B(r,0))} \frac{\langle \phi, D_x^\varepsilon \phi \rangle_{L^2(B(r,0))}}{\langle \phi, \phi \rangle} \geq 3K\,.$$

Then if $x_0 \in \Omega_+$ we can extend $(\kappa_\nu^\varepsilon)_* \delta_+$ by zero to a W^1-function $\delta_+^{x_0}$ on $B(r, 0)$ and the boundary value problem

$$D_{x_0}^\varepsilon f = 2Kf\,, \qquad f|_{\partial B(r,0)} = \delta_+^{x_0} + \varepsilon^\gamma$$

has a unique solution $f \in \mathscr{C}^\infty(B(r, 0))$(see [22, 8.6, 8.11]) that is strictly positive by corollary 3.14. For ε small enough this solution satisfies

$$\left(D_{x_0}^\varepsilon - \varepsilon^\alpha \lambda - K\right) f = (K - \varepsilon^\alpha \lambda)\, f > 0$$

in $B(r, 0)$, while $\delta_+^{x_0}$ satisfies (3.19)

$$\left(D_{x_0}^\varepsilon - \varepsilon^\alpha \lambda - K\right) \left(\delta_+^{x_0} + \varepsilon^\alpha\right) \leq 0$$

in $\kappa_\nu^\varepsilon\big(\Omega_+ \cap B(R, x_0)\big) \cap B(r, 0)$. Thus the maximum principle implies that that $(\delta_+^{x_0} + \varepsilon^\gamma)/f$ attains its maximum on the boundary of this set. On the boundary of $B(r, 0)$ the quotient equals one, while on the boundary of $\kappa_\nu^\varepsilon\big(\Omega_+ \cap B(R, x_0)\big)$ we know that $\delta_+^{x_0}$ equals zero, whereby

$$(\delta_+^{x_0} + \varepsilon^\gamma)/f \leq 1 + \varepsilon^\gamma / f\,.$$

In particular

$$\delta_+(x_0) \leq f(0) + 2\varepsilon^\gamma\,.$$

In order to complete the proof we need to bound $f(0)$. To begin with, we have the a priori estimate [22, 8.7] (with $C = C(K, \Lambda_0, r, e, c)$)

$$\|f\|_{W^1(B(r,0))} \leq C \|\delta_+^{x_0}\|_{W^1(B(r,0))}\,.$$

From this we can obtain bounds on higher Sobolev norms of f, using that in the interior of $B(r,0)$ it is an eigenfunction of $D^\varepsilon_{x_0}$. This relies on interior elliptic regularity [22, 8.10], giving for $r' < r$ (with $C = C(e,c,k,r-r')$)

$$\begin{aligned}\|f\|_{W^{k+2}(B(r',0))} &\le C\big(\|f\|_{W^1(B(r,0))} + \big\|D^\varepsilon_{x_0} f\big\|_{W^k(B(r,0))}\big)\\ &\le C\big(\|f\|_{W^1(B(r,0))} + \|(\Lambda_0 + 2K) f\|_{W^k(B(r,0))}\big)\big)\end{aligned}$$

Repeated application of this inequality (with successively smaller radii satisfying $r_i - r_{i+1} = r/2k$) gives an inequality

$$\|f\|_{W^{k+1}(B(r/2,0))} \le C(k,r,e,c,K,\Lambda_0)\, \|f\|_{W^1(B(r,0))}\ .$$

If we take $k > (n+d+1)/4$ the Sobolev embedding theorem gives a bound on $\sup_{y\in B(r/2,0)} f(y)$ and in particular on $f(0)$. Hence by mapping $\delta^{x_0}_+$ back to $B^\varepsilon(R,x_0) \subset \Omega(x_0)$ we get

$$\begin{aligned}\delta_+(x_0) &\le C_1 \|\delta^{x_0}_+\|_{W^1(B(r,0))} + 2\varepsilon^\gamma\\ &\le C_2 \Big(\int_{B^\varepsilon(R,x_0)} |\delta|^2 + g_\varepsilon(\mathrm{d}\delta,\mathrm{d}\delta)\,\mathrm{vol}_{g_\varepsilon}\Big)^{1/2} + 2\varepsilon^\gamma\\ &\le C_2 \Big(\int_{\Omega(x_0)} |\delta|^2 + g_\varepsilon\,(\mathrm{d}\delta,\mathrm{d}\delta)\ \mathrm{vol}_{g_\varepsilon}\Big)^{1/2} + 2\varepsilon^\gamma\ .\end{aligned}$$

□

The integral bounding $\delta(x_0)$ is basically the $W^1(\Omega(x_0), g_\varepsilon)$-norm of δ. Because of the different volume measures (in the definition of W^1_ε we used $\mathrm{vol}_{g_{\varepsilon=1}}$, see also lemma A.12) this is related to the $W^1_\varepsilon(\Omega(x_0))$-norm by $\|\phi\|_{W^1(\Omega(x_0),g_\varepsilon)} \le C\varepsilon^{-d}\, \|\phi\|_{W^1_\varepsilon(\Omega(x_0))}$. This may of course pose problems for large d. However we can still exploit the fact that $\Omega(x_0)$ is the lift of an ε-neighbourhood of x_0 and that we have control over the norm $W^1_{\varepsilon=1}$. The way in which this can be useful depends on the concentration behaviour of the eigenfunctions. If a large portion of the L^2-norm concentrates in $\Omega(x_0)$ for some x_0 the smallness of this set will not help much. This behaviour depends strongly on α, so we will have to treat the cases $\alpha = 2$ and $\alpha = 1$ separately.

Since δ is a solution of the differential equation (3.18), with Dirichlet boundary conditions, establishing uniform bounds on δ gives bounds on its derivatives.

Lemma 3.16. *There is a constant $C > 0$ such that δ satisfies*

$$\begin{aligned} \big\|\sqrt{g_\varepsilon(\mathrm{d}\delta, \mathrm{d}\delta)}\big\|_\infty &\leq C\big(\, \|\delta\|_\infty + \|R_\varepsilon\|_\infty \,\big) \\ \big\|\sqrt{g_\varepsilon(\nabla\mathrm{d}\delta, \nabla\mathrm{d}\delta)}\big\|_\infty &\leq C\big(\, \|\delta\|_\infty + \|R_\varepsilon\|_\infty \,\big) \end{aligned}$$

Proof. Write (3.18) as $(H - \varepsilon^\alpha\lambda)\delta = R_\varepsilon$. To prove the claim for fixed ε and $x \in M$ use the (unshifted) coordinate system $\kappa^\varepsilon_\nu(x)$ on the metric ball $B(R,x)$ introduced in the proof of lemma 3.15. Since the coordinate vector fields of these are uniformly bounded with respect to g_ε it is sufficient to prove $\mathscr{C}^2$-bounds for δ at $\kappa^\varepsilon_\nu(x)$ in these coordinates. These will follow from [22, lemma 6.4] once we have checked the relevant conditions. First of all, the operators $D^\varepsilon_x = (\kappa^\varepsilon_\nu)_* H$ are elliptic with uniform bounds on their coefficients and ellipticity constants, so it remains to give a bound on the distance to any boundary portion of $\kappa^\varepsilon_\nu(B(R,x))$ on which δ does not vanish. But we know that $B\big(R_0, \kappa^\varepsilon_\nu(x)\big)$ (or $B\big(R_0, \kappa^\varepsilon_\nu(x)\big) \cap \{y_{n+d} \geq 0\}$ if $\nu < 0$) is completely contained in $\kappa^\varepsilon_\nu\big(B^\varepsilon(R,x)\big)$. Hence the (euclidean) distance from $\kappa^\varepsilon_\nu(x)$ to any part of $\partial B(R_0, \kappa^\varepsilon_\nu(x))$ on which $\delta \neq 0$ is R_0, which completes the proof. □

The case of a constant eigenband

We now analyse the case in which λ_0 is constant, which means $\lambda_0 \equiv 0$ since we have subtracted its minimum from H. As discussed in the introduction this occurs for example if $\partial M = \varnothing$ or the fibres (F_x, g_{F_x}) are isometric. In this case we formally have

$$\varepsilon^{-2} H_\mathrm{a} = -\Delta_{g_B} + V_0 + P_0 V_\varepsilon P_0|_{\varepsilon=0} + \mathcal{O}(\varepsilon) =: H_0 + \mathcal{O}(\varepsilon)\,, \tag{3.20}$$

with the potential V_0 of equation (3.10) for H_a, and V_ε from condition 5. Apart from a potential, coming from the expansion of V_ε, the difference $H_\mathrm{a} - \varepsilon^2 H_0$ is given by $\varepsilon^3 P_0 \operatorname{div} S_\varepsilon P_0$. Since $S_\varepsilon \in \Gamma_b(\pi^* T^* B \otimes \pi^* T^* B)$ by condition 5 (or condition 4 if it arises from an embedding), the operator

$\psi \mapsto P_0 \operatorname{div} S_\varepsilon$ is a bounded operator from $W^2(\mathcal{E})$ to $L^2(\mathcal{E})$. Because $\lambda_0 \equiv 0$, the norm $\|\cdot\|_{W^2(\mathcal{E})}$ is equivalent to $\|\cdot\|_{D(\varepsilon^{-2}H_\mathrm{a})}$ and thus

$$\left\|\varepsilon^3 P_0 \operatorname{div} S_\varepsilon P_0\right\|_{\mathscr{L}(D(\varepsilon^{-2}H_\mathrm{a}),L^2(\mathcal{E}))} = \mathcal{O}(\varepsilon^3)\,.$$

Consequently, by standard perturbation theory, if H_0 has eigenvalues they approximate those of $\varepsilon^{-2}H_\mathrm{a}$ with errors of order ε. Since H_0 is completely independent of ε this also means that the eigenvalues of H_a have a spacing of order ε^2.

If $H = -U_\rho^* \Delta_{G_\varepsilon} U_\rho$ and G_ε is induced by an embedding as discussed in section 3.1.1 the operator H_0 can be given rather explicitly.

- For the case $\partial M = \varnothing$ we have $V_\varepsilon|_{\varepsilon=0} = 0$ (cf. remark 3.8), and V_0 is given by equation (3.11), hence

$$H_0 = -\Delta_{g_B} + \tfrac{1}{2}\Delta(\log \mathrm{Vol}(F_x)) + \tfrac{1}{4}\left|\mathrm{d} \log \mathrm{Vol}(F_x)\right|^2_{g_B}\,.$$

- If the embedding of M in $\mathbb{R}^k$ has codimension zero, the fibres carry a flat metric and are totally geodesic, hence $\bar{\eta} = 0$. The leading part of V_ε is the bending potential known from the study of quantum waveguides (see [30] for a more detailed discussion), $V_\varepsilon = V_\mathrm{bend} + \mathcal{O}(\varepsilon)$ with

$$V_\mathrm{bend} = \sum_{i=1}^n \tfrac{1}{4}\big(\operatorname{tr} W(e_i)\big)^2 - \tfrac{1}{2}(\operatorname{tr} W(e_i)^2)\,,$$

  where $(e_i)_{i\leq n}$ form a local orthonormal frame of the normal bundle NB. So in this situation

$$H_0 = -\Delta_{g_B} + \int_{F_x} \pi^* g_B(\operatorname{grad}\phi_0, \operatorname{grad}\phi_0)\,\mathrm{vol}_{F_x} + V_\mathrm{bend}\,. \quad (3.21)$$

Lemma 3.17. *Let $\lambda_0 \equiv 0$, $\alpha = 2$ and $\psi \in L^2(\mathcal{E})$ be a normalised eigenfunction of H_a with eigenvalue $\varepsilon^2\mu$. The remainder R_ε of equation* (3.18) *satisfies $\|R_\varepsilon\|_\infty = \mathcal{O}(\varepsilon^2)$.*

Proof. We will bound R_ε using the $\mathscr{C}_b^2(B)$-norm of ψ. This may be bounded using elliptic regularity, since $\mathcal{E}$ is a bundle of bounded geometry over (B, g_B). The $\mathscr{C}_b^2$-norm of ψ is bounded independently of ε because the norms $\|\cdot\|_{D(\varepsilon^{-2k}H_\mathrm{a}^k)}$ are equivalent to $\|\cdot\|_{W^{2k}(\mathcal{E})}$.

First of all, theorem 3.12 gives

$$|\varepsilon^\alpha (\mu - \lambda)\, \psi| \leq C\varepsilon^4 \|\psi\|_\infty \,,$$

while the other term is given by

$$(H - H_\mathrm{a})\, \psi = \big([-\varepsilon^2 \Delta_h, P_0]P_0 + P_0^\perp(-\varepsilon^3 \operatorname{div} S_\varepsilon + \varepsilon^2 V_\varepsilon)\big)P_0\psi\,. \quad (3.22)$$

Since we can commute all the derivatives to the right as in (3.13a) this is bounded by a constant times $\varepsilon^2 \|\psi\|_{\mathscr{C}_b^2(B)}$ (in fact it is bounded by $C(\varepsilon^2 \|\psi\|_{\mathscr{C}_b^1(B)} + \varepsilon^3 \|\psi\|_{\mathscr{C}_b^2(B)})$ since the only term with second derivatives is $\operatorname{div} S_\varepsilon$). □

Recall the notation $\delta = (1 - P_\lambda)\psi$ from the beginning of this section.

Proposition 3.18. *Assume $\lambda_0 \equiv 0$ and $d = \dim B \leq 3$. Let $\varepsilon^2\mu$ be a simple eigenvalue of H_a with normalised eigenfunction ψ, then*

$$\|\delta\|_\infty = \mathcal{O}\big(\varepsilon\sqrt{\theta_d(\varepsilon)}\big).$$

with $\theta_1(\varepsilon) = 1$, $\theta_2(\varepsilon) = -\log\varepsilon$ and $\theta_3(\varepsilon) = \varepsilon^{-1}$.

Proof. By lemma 3.17 we can apply lemma 3.15 with $\gamma = 2$. To prove the statement we now need to estimate the integral

$$\int_{\Omega(x)} |\delta|^2 + g_\varepsilon\,(\mathrm{d}\delta, \mathrm{d}\delta)\ \mathrm{vol}_{g_\varepsilon} = \int_{\Omega(x)} (1{+}\Lambda_0)\,|\delta|^2 + g_\varepsilon\,(\mathrm{d}\delta, \mathrm{d}\delta) - \Lambda_0\,|\delta|^2\ \mathrm{vol}_{g_\varepsilon}\,.$$

Since H_F is non-negative we may estimate the terms of the second line by integrating over the whole of M instead of $\Omega(x)$, which gives

$$\begin{aligned}\int_M g_\varepsilon(\mathrm{d}\delta, \mathrm{d}\delta) - \Lambda_0\,|\delta|^2\ \mathrm{vol}_{g_\varepsilon} &\overset{(3.12)}{=} \varepsilon^{-d}(\langle\delta, H\delta\rangle - \varepsilon\,\langle\delta, H_1\delta\rangle) \\ &\overset{(3.18)}{=} \varepsilon^{2-d}\lambda\,\|\delta\|_{\mathscr{H}}^2 + \varepsilon^{-d}\langle\delta, R_\varepsilon\rangle - \varepsilon^{1-d}\langle\delta, H_1\delta\rangle\,.\end{aligned}$$

Now by theorem 3.12 we have $\|\delta\|^2_{\mathscr{H}} = \mathcal{O}(\varepsilon^2)$, hence using condition 5

$$\begin{aligned}|\langle\delta, H_1\delta\rangle| &= |-\varepsilon^2\langle\delta, \operatorname{div} S_\varepsilon\delta\rangle + \varepsilon\langle\delta, V_\varepsilon\delta\rangle| \\ &\le C(\varepsilon^2 \|\delta\|^2_{W^1_{\varepsilon=1}} + \varepsilon \|\delta\|^2_{\mathscr{H}}) = \mathcal{O}(\varepsilon^3)\,,\end{aligned}$$

and

$$\begin{aligned}|\langle\delta, R_\varepsilon\rangle| &= |\langle\delta, \big(H - H_{\mathrm{a}} + \varepsilon^2(\mu-\lambda)\big)\,\psi\rangle| \\ &= |\langle\delta, \varepsilon^2 P_0^\perp[\Delta_h, P_0]P_0\psi\rangle| + \mathcal{O}(\varepsilon^4) \\ &\le \underbrace{\left\|P_0^\perp\delta\right\|_{\mathscr{H}}}_{\overset{(3.17)}{=}\mathcal{O}(\varepsilon^2)} \underbrace{\left\|\varepsilon^2[\Delta_h, P_0]\rho_\alpha(H_a)\psi\right\|_{\mathscr{H}}}_{\overset{3.9}{=}\mathcal{O}(\varepsilon^2)} + \mathcal{O}(\varepsilon^4) = \mathcal{O}(\varepsilon^4)\,.\end{aligned}$$

Altogether this gives a bound of $\mathcal{O}(\varepsilon^{4-d})$ on the integral. As to the integral of $|\delta|^2$, note that $\Omega(x)$ is the lift of a ball of radius εR in (B, g_B). When $d = 1$ the integral over such a ball is bounded by the W^1-norm (which controls the maximum) times the volume, thus eliminating the factor ε^{-d}. This idea can be extended to higher dimensions to some extent using the Fourier transform, as demonstrated in lemma C.4 in the appendix. For the problem at hand, first write the integral in a local trivialisation Φ

$$\int\limits_{\Omega(x)} \varepsilon^{-d}\,|\delta|^2\ \mathrm{vol}_{g_{\varepsilon=1}} = \int\limits_F \int\limits_{\pi(\Omega(x))} \varepsilon^{-d}\,|\Phi_*\delta|^2\ \mathrm{vol}_{g_B}\mathrm{vol}_{\Phi_* g_F}\,.$$

Now take g_B-geodesic coordinates κ on $B(r, \pi(x)) \supset \pi(\Omega(x))$ and a smooth cut-off $\chi \in \mathscr{C}_0^\infty(B(r, 0))$, equal to one on $(B(\varepsilon R, 0)) \subset B(r, 0)$. Lemma C.4 gives

$$\int\limits_{B(\varepsilon R,0)} \varepsilon^{-d}|\kappa_*\Phi_*\delta|^2(y)\mathrm{d}y \le C\theta_d(\varepsilon)\,\|\chi\kappa_*\Phi_*\delta\|^2_{W^1(B(r,0))}\,.$$

Now integrating this inequality over the fibres we obtain

$$\int_{\Omega(x)} \varepsilon^{-d}\,|\delta|^2\ \mathrm{vol}_{g_{\varepsilon=1}} \le \theta_d(\varepsilon)C\,\|\delta\|^2_{W^1_{\varepsilon=1}} \le C\varepsilon^2\theta_d(\varepsilon)\,,$$

with a constant $C(d, \Phi, \kappa, \chi)$ that may be chosen independent of x because (B, g_B) is of bounded geometry and $M \overset{\pi}{\to} B$ uniformly locally trivial. □

An eigenband with a non-degenerate minimum

Let λ_0 have a non-degenerate minimum at $x_0 \in B$ and assume that

$$\inf\{\lambda_0(x) : \mathrm{dist}(x, x_0) > 1\} > 0\,, \tag{3.23}$$

it was shown by Simon [66] that the typical distance between the eigenvalues of H_a is of order ε, so the relevant energy scaling is $\alpha = 1$.

For the discussion of eigenfunctions in this situation we will restrict ourselves to the case $d = 1$ as it allows for much simpler and stronger statements. In this case the only possible base manifolds are $B = \mathbb{R}$ and $B = S^1$. In both cases we take B to be parametrised by g_B-arc distance s from x_0. This parametrisation is a diffeomorphism $B \to \mathbb{R}$ or $B \setminus \{x_1\} \to I = (-L, L)$ that maps x_0 to $s = 0$. In this way we can view H_a either as an operator on $\mathbb{R}$ or as an operator on I with periodic boundary conditions. With this we get the explicit expression

$$H_\mathrm{a} = -\varepsilon^2\partial_s^2 + \lambda_0(s) - \varepsilon^3 P_0 \partial_s^* S_\varepsilon \partial_s^* P_0 + \varepsilon^2 V_\mathrm{eff}\,,$$

with $V_\mathrm{eff} = V_0 + V_\varepsilon$ for the potentials V_0 and V_ε of equation (3.10) and condition 5. If this operator arises from an embedding, necessarily with $\partial M \neq \varnothing$, these potentials take the same form as for $\alpha = 2$ (cf. equation (3.21)).

We now rephrase the main result of [66] for $d = 1$ in our notation. Let $a := (\frac{1}{2}\partial_s^2|_{s=0}\lambda_0)^{1/2}$ and denote by H_0 the operator

$$H_0 := -\partial_s^2 + as^2\,.$$

The spectrum of H_0 consists of simple eigenvalues

$$e_j = a(2j+1)$$

with eigenfunctions

$$f_j = c_j \mathrm{e}^{-as^2/2} h_j(\sqrt{a}s)\,,$$

for $j \in \mathbb{N}$, where h_j denotes the j-th Hermite polynomial and c_j a normalisation constant. Now [66, theorem 1.1] tells us that for given $J \in \mathbb{N}$

and ε small enough, H_{a} has at least $J+1$ eigenvalues below its continuous spectrum. By [66, theorem 4.1] they are simple and have an expansion

$$\varepsilon\mu_j = \varepsilon e_j + \varepsilon^2 e_j^1 + \mathcal{O}(\varepsilon^3)\,. \tag{3.24}$$

The corresponding eigenfunctions can be expanded in L^2 (and also in $W^2 \subset D(H_0)$). Their expansion up to order $\varepsilon^\beta = \varepsilon^{3/2}$ reads

$$D\psi_j = f_j + \sqrt{\varepsilon} f_j^1 + \varepsilon f_j^2 + \mathcal{O}(\varepsilon^{3/2})\,, \tag{3.25}$$

where the functions f_j^1, f_j^2 are given by a polynomial times $\mathrm{e}^{-as^2/2}$ and $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary dilation by $\sqrt{\varepsilon}$

$$(D\psi)\,(s) := \varepsilon^{1/4}\psi(\sqrt{\varepsilon}s)\,.$$

Such an expansion is also valid for simple eigenvalues of H_{a} in higher dimensions. The maximum value of the eigenfunction ψ behaves as $\varepsilon^{-d/4}$, so in general one should not expect the supremum of $(1-P_\lambda)\psi$ to be small.

Remark 3.19. Let us comment on some differences of the situation discussed here to that analysed in [66, theorem 4.1].

In the case $B \cong S^1$, the operator H_{a} is not densely defined on $L^2(\mathbb{R})$, so strictly speaking the theorem does not apply. However due to the exponential decay of the functions $D^* f_j$, $D^* f_j^1$, localising them in I using a suitable cut-off only produces exponentially small errors in ε. After doing this we may apply the technique of the proof on $L^2(I)$ and the statement still holds.

Compared with the set-up of Simon we also have an additional perturbation by the differential operator $\varepsilon^3 P_0 \partial_s^* S_\varepsilon \partial_s^*$. From the construction of the perturbation expansion one easily sees that this does not pose a problem. Since it is of order ε it does not enter the calculation of f_j^1, while for f_j^2 one can replace S_ε by the constant $S_\varepsilon(x_0)|_{\varepsilon=0}$.

Knowing the expansion of the eigenvalues (3.24) it is clear that one should look at the energy scale $\alpha = 1$. Since the eigenvalues are all simple, theorem 3.12 applies to all of the eigenvalues of H below $\varepsilon_0\Lambda < \Lambda_1$. This means:

Corollary 3.20. *Let $B = \mathbb{R}$ or $B = S^1$, assume that λ_0 has a unique, non-degenerate minimum and satisfies* (3.23)*. For every $J \in \mathbb{N}$ there exists ε_0 so that for $\varepsilon \leq \varepsilon_0$ the operator H has at least $J+1$ simple eigenvalues $\{\varepsilon\lambda_j : 0 \leq j \leq J\}$ below its essential spectrum, satisfying*

$$\lambda_j \overset{3.12}{=} \mu_j + \mathcal{O}(\varepsilon^2) = e_j + \varepsilon e_j^1 + \mathcal{O}(\varepsilon^2)\,.$$

Proof. If we choose $\Lambda = e_{J+1}$ for the construction of P_ε this is a direct consequence of [66, theorems 1.1, 4.1] and theorem 3.12. □

Theorem 3.12 also gives approximation of the corresponding eigenfunctions φ_j, $\|\varphi_j - \phi_0\pi^*\psi_j\|_{\mathscr{H}} = \mathcal{O}(\varepsilon^{3/2})$ and $\|\varphi_j - \phi_0\pi^*\psi_j\|_{W^1_{\varepsilon=1}} = \mathcal{O}(\sqrt{\varepsilon})$. We now use the expansion (3.25) of ψ_j together with lemma 3.15 to show validity of this approximation in the uniform norm.

Lemma 3.21. *Assume the conditions of corollary 3.20 hold. For every $j \in \{0, \dots, J\}$ the remainder R_ε of equation* (3.18) *satisfies $\|R_\varepsilon\|_\infty = \mathcal{O}(\varepsilon)$.*

Proof. The proof is the same for every j, so we drop that index for now. From (3.22) we see that

$$\|R_\varepsilon\|_\infty \leq C(\varepsilon^2 \|\psi\|_{\mathscr{C}^1_b(\mathbb{R})} + \varepsilon^3 \|\psi\|_{\mathscr{C}^2_b(\mathbb{R})})\,.$$

We estimate the $\mathscr{C}^1_b$-norm by comparing with D^*f, which satisfies

$$\|\varepsilon^{-1/4} f(\varepsilon^{-1/2}s)\|_{\mathscr{C}^1_b} = \varepsilon^{-1/2}\|f\|_\infty + \varepsilon^{-3/4}\|\partial_s f\|_\infty \leq \varepsilon^{-3/4}\|f\|_{\mathscr{C}^1_b}\,.$$

Since $d = 1$ we can bound the $\mathscr{C}^1_b$-norm of the difference $\psi - D^*f$ by its W^2-norm. From the perturbation expansion (3.25) and $D\partial_s^2 D^* = \varepsilon^{-1}\partial_s^2$ we see that

$$\|(\psi - D^*f)\|_{W^2} \leq \varepsilon^{-1}\|D\psi - f\|_{W^2} = \mathcal{O}(\varepsilon^{-1/2})\,. \tag{3.26}$$

Altogether we conclude

$$\|\psi\|_{\mathscr{C}^1_b} \leq \|D^*f\|_{\mathscr{C}^1_b} + C\|\psi - D^*f\|_{W^2} = \mathcal{O}(\varepsilon^{-3/4})\,, \tag{3.27}$$

so $\varepsilon^2\|\psi\|_{\mathscr{C}^1_b} = \mathcal{O}(\varepsilon^{5/4})$.

In order to gain control over its second derivatives we estimate the W^3-norm of ψ. From (3.26) and f we already have bound on the W^2-norm $\|\psi\|_{W^2} = \mathcal{O}(\varepsilon^{-1})$, so we only need to take care of

$$\int_B \left(\partial_s^3 \psi\right)^2 \mathrm{d}s = \int_B \overline{\partial_s^3 \psi} \partial_s \left(-\varepsilon^{-2} H_\mathrm{a} + \varepsilon^{-2} \lambda_0 - \varepsilon P_0 \partial_s S_\varepsilon \partial_s + V_\mathrm{eff}\right) \psi \, \mathrm{d}s \,.$$

Using the Cauchy-Schwarz inequality and

$$\|\partial_s^2 S_0 \partial_s \psi\|_{L^2} \leq \|S_0\|_{\mathscr{C}_b^2(M)} \|\psi\|_{W^3(B)}$$

we have

$$\begin{aligned} &C_1 \|\psi\|_{W^3} (1-\varepsilon) \\ &\quad \leq \varepsilon^{-1} \mu \|\partial_s \psi\|_{L^2} + \varepsilon^{-2} \|\partial_s \lambda_0 \psi\|_{L^2} + \|V_\mathrm{eff}\|_{\mathscr{C}_b^1} \|\psi\|_{W^1} + \mathcal{O}(\varepsilon^{-1}) \\ &\quad \leq \varepsilon^{-1} \mu \|\psi\|_{W^2} + \varepsilon^{-2} \|\lambda\|_{\mathscr{C}_b^1} \|\psi\|_{L^2} + \varepsilon^{-2} \|\lambda_0 \partial_s \psi\|_{L^2} + \mathcal{O}(\varepsilon^{-1}) \,. \end{aligned}$$

The first two terms have bounds of order ε^{-2}, which is sufficient for $R_\varepsilon = \mathcal{O}(\varepsilon)$. For the third term we again use the expansion 3.25 of ψ. The remainder of this expansion can be estimated by

$$\|\lambda_0 \partial_s (D^* f - \psi)\|_{L^2} \leq \varepsilon^{-1/2} \|\lambda_0\|_\infty \|(f - D\psi)\|_{W^1} = \mathcal{O}(1) \,.$$

Now since $\lambda_0(0) = (\partial_s \lambda_0)(0) = 0$ and the second derivatives of λ_0 are bounded, $\lambda_0(\sqrt{\varepsilon} s) \leq C \varepsilon s^2$. Using that f is a polynomial p times a Gaussian we have (extending λ_0 by zero outside of I)

$$\begin{aligned} \|D \lambda_0 \partial_s D^* f\|_{L^2(\mathbb{R})}^2 &= \int_\mathbb{R} \varepsilon^{-1} \left|\lambda_0(\sqrt{\varepsilon} s) \partial_s p(s) \mathrm{e}^{-as^2/2}\right|^2 \mathrm{d}s \\ &\leq \varepsilon^{-1} \int_\mathbb{R} \varepsilon^2 s^4 \left|\partial_s p(s) \mathrm{e}^{-as^2/2}\right|^2 \mathrm{d}s = \mathcal{O}(\varepsilon) \,. \end{aligned}$$

Hence $\|\psi\|_{W^3} = \mathcal{O}(\varepsilon^{-2}) = \|\psi\|_{\mathscr{C}_b^2}$ and $\|R_\varepsilon\|_\infty = \mathcal{O}(\varepsilon)$. □

Proposition 3.22. *Assume the conditions of corollary 3.20 hold. Then for any* $j \in \{0, \dots, J\}$, $\delta_j := (1 - P_{\lambda_j})\psi$ *satisfies* $\|\delta_j\|_\infty = \mathcal{O}(\varepsilon)$.

Proof. Lemma 3.21 shows that lemma 3.15 with $\gamma = 1$ may be applied. Hence we have to bound the integral over $\Omega(x)$ by $\mathcal{O}(\varepsilon)$. This is straight forward since $d = 1$ and

$$\int_M |\delta|^2 + g_\varepsilon(\mathrm{d}\delta, \mathrm{d}\delta)\, \mathrm{vol}_{g_\varepsilon} \leq \varepsilon^{-1} \|\delta\|^2_{W^2_\varepsilon} \overset{3.12}{\leq} C\varepsilon^{2\beta-1} = \mathcal{O}(\varepsilon^2)\,.$$

□

3.3.2 The location of nodal sets

In the previous section we showed that ψ and φ are uniformly close, so it is rather intuitive that their zeros should be located in roughly the same regions. Under some additional conditions we will determine the location of the nodal set $\mathcal{N}(\varphi)$ from that of ψ. The cases of empty or non-empty boundary and the different characteristic energy scales $\alpha = 1, 2$ display distinct behaviour of these eigenfunctions, so the precise statements differ for these cases, although they all capture this same basic idea. The fibre eigenfunction ϕ_0 will be of some importance here, so we will again distinguish the eigenfunction $\psi \in L^2(B)$ of H_a in the trivialisation by ϕ_0 and the section $\psi\phi_0 \in L^2(\mathcal{E})$ that we can compare to φ.

A constant eigenband

For this case we have proven in proposition 3.18 that $\|\psi\phi_0 - \varphi\|_\infty = \mathcal{O}(\varepsilon\sqrt{\theta_d(\varepsilon)})$. Now recall that for this case we defined an operator $H_0 = -\Delta_{g_B} + V_\mathrm{eff}$ in equation (3.20), with $H_0 - \varepsilon^{-2}H_\mathrm{a} = \mathcal{O}(\varepsilon)$. So to any simple eigenvalue μ_ε of $\varepsilon^{-2}H_\mathrm{a}$ there is also an eigenvalue μ_0 of H_0. Let $\psi_0 \in L^2(B)$ be the corresponding normalised eigenfunction. Then because of the equivalence of the norms of $D(\varepsilon^{-2}H_\mathrm{a})$ and $W^2(B, g_B)$ we have $\|\psi - \psi_0\|_\infty = \mathcal{O}(\varepsilon)$, hence ψ_0 may serve to approximate φ just as well as ψ. The fact that ψ_0 is independent of ε makes it a more convenient choice for the comparison with φ. The statements we obtain will be true for both ψ and ψ_0, although with different constants.

We now discuss the most simple case, $B = S^1$ and $\partial M = \varnothing$ with H_1 arising from an embedding, to illustrate the reasoning by which all the following theorems are true with the minimal technical difficulty at the individual steps. We again work in a fixed parametrisation by arc length of $B \setminus \{x_1\}$ and call the corresponding coordinate s. In these coordinates the operator H_0 has the explicit form

$$H_0 \overset{\partial M = \varnothing}{\underset{B=S^1}{=}} -\varepsilon^2 \partial_s^2 + \varepsilon^2 \left(\tfrac{1}{2} \partial_s^2 \log \mathrm{Vol}(F_x) + \tfrac{1}{4} \left| \partial_s \log \mathrm{Vol}(F_x) \right|^2 \right),$$

as an operator on an interval with periodic boundary conditions. Its spectrum is discrete and for every simple eigenvalue μ_0 there is μ_ε to which theorem 3.12 and proposition 3.18 apply. We now prove that

$$\mathcal{N}(\varphi) \subset \pi^{-1}\{x : \mathrm{dist}_{g_B}(x, \mathcal{N}(\psi_0)) \leq C\varepsilon\}.$$

Proposition 3.23. *Let μ_0 be a simple eigenvalue of H_0 with normalised eigenfunction ψ_0 and $\varphi \in \ker(H - \varepsilon^2 \lambda)$ the corresponding eigenfunction of H. There exists a constant $C > 0$ such that for $y \in M$ with*

$$\mathrm{dist}_{g_B}(\pi(y), \mathcal{N}(\psi_0)) \geq C\varepsilon$$

we have

$$\mathrm{sign}(\varphi(y)) = \mathrm{sign}(\psi_0(\pi(y)).$$

Proof. We carry out the proof in several small steps since the same structure will appear also in the proofs of more general statements.

Let $\mathcal{N}(\psi_0) = \{s_i : i \in I\}$ be the (finite) set of zeros of ψ_0.

1) There is $C_0 > 0$ such that $|\partial_s \psi_0|(s_i) \geq C_0$ for every $i \in I$:

ψ_0 solves the second order ordinary differential equation

$$\partial_s^2 \psi_0 = (V_{\mathrm{eff}} - \mu)\, \psi_0,$$

so if at any point $\psi_0(s) = \partial_s \psi_0(s) = 0$ it must vanish everywhere since zero is the unique solution of the equation with that initial condition. Thus the derivative of ψ_0 cannot vanish at any s_i. It is independent of ε since ψ_0 does not depend on ε at all. C_0 may now be chosen as the minimum of $|\partial_s \psi_0(s_i)|$ over the finite set I.

2) For any $C_1 > 0$ and $C_2 := 2C_1/C_0$ we have, if ε is small enough, $|\psi_0(s_i \pm C_2\varepsilon)| > C_1\varepsilon$:

By Taylor expansion

$$|\psi_0(s_i + 2C_2/C_0\varepsilon)| = 2C_1\varepsilon \, |\partial_s\psi_0| \, / C_0 + \mathcal{O}(\varepsilon^2) \geq C_1\varepsilon \, .$$

3) If $\operatorname{dist}(s, \mathcal{N}(\psi_0)) \geq C_2\varepsilon$, then $|\psi_0(s)| > C_1\varepsilon$ for ε small enough:

If in the interval $[s_i, s_j]$ between two consecutive zeros of ψ_0 there is no local minimum of $|\psi_0|$, then $|\psi_0|$ attains its minimum on the boundary of $[s_i + C_2\varepsilon, s_j - C_2\varepsilon]$, where it is larger than $C_1\varepsilon$ by step two.

If on the other hand there is a local minimum at $s^* \in [s_i, s_j]$ we just need ε to be small enough to ensure that $|\psi_0| \, (s^*) > C_1\varepsilon$.

4) Denote by C_3 the constant of proposition 3.18 with $d = 1$ and let

$$C_4 := \max\{\varepsilon^{-1} \, \|\psi - \psi_0\|_\infty \, \|\phi_0\|_\infty, C_3\} \, .$$

The proof is complete taking $C_1 = 2\|\phi_0^{-1}\|_\infty C_4$ and $C := C_2 = 2C_1/C_0$:

First note that since $\partial F = \varnothing$ we have $\phi_0 = \pi^*\mathrm{Vol}(F_x)^{-1/2}$ (or in case we redefined H_F as described in remark 3.8, ϕ_0 is proportional to $\sqrt{\rho}$ and equals $\pi^*\mathrm{Vol}(F_x)^{-1/2} + \mathcal{O}(\varepsilon)$), so $\|\phi_0^{-1}\|_\infty$ is a positive constant. Now let $y \in M$ and $x = \pi(y)$. By step three we have

$$|\psi_0\phi_0(y)| > 2C_4\varepsilon$$

for $\operatorname{dist}_{g_B}(x, \mathcal{N}(\psi_0)) \geq C\varepsilon$, and by proposition 3.18

$$\|\psi_0\phi_0 - \varphi\|_\infty \leq \|\psi - \psi_0\|_\infty \, \|\phi_0\|_\infty + \underbrace{\|\psi\phi_0 - \varphi\|_\infty}_{\leq C_3\varepsilon} \leq 2C_4\varepsilon \, ,$$

so φ must have the same sign as ψ_0.

□

We now generalise this statement to compact manifolds M for the case $\lambda_0 \equiv 0$ and $d \leq 3$ of proposition 3.18. This includes manifolds with non-empty boundary, in which case the behaviour of ϕ_0 near ∂M will require some attention. Also, in contrary to the one-dimensional case just proved, the nodal set of ψ_0 could already be complicated.

We say that $\psi_0 \in \ker(H_0 - \varepsilon^2\mu)$ has a *regular nodal set* if zero is a regular value of ψ_0. This implies of course that the nodal set is a $\mathscr{C}^\infty$-hypersurface in B. In fact the converse also holds, as ψ_0 can be shown to have a non-zero normal derivative on any smooth surface where it vanishes (see [22, lemma 3.4]).

Theorem 3.24. *Let B be compact and of dimension $d \leq 3$. Assume $\lambda_0 \equiv 0$ and let μ be a simple eigenvalue of H_0. If the normalised eigenfunction $\psi_0 \in \ker(H_0 - \mu)$ has a regular nodal set there is a constant $C > 0$ such that for every $y \in M \setminus \partial M$ with*

$$\mathrm{dist}_{g_B}(\pi(y), \mathcal{N}(\psi_0)) \geq C\varepsilon\sqrt{\theta_d(\varepsilon)}$$

we have

$$\mathrm{sign}(\varphi(y)) = \mathrm{sign}(\psi(\pi(y)) \,.$$

Proof. The proof follows the same steps as that of proposition 3.23.

1) By the assumption of regularity of $\mathcal{N}(\psi_0)$, the derivative in the direction of the unit normal ν of $\mathcal{N}(\psi_0)$ is nowhere zero. Because the nodal set is compact it is bounded below, $|\nu\psi_0| \geq C_0$, on $\mathcal{N}(\psi_0)$.

2) For ε small enough, $\varepsilon\sqrt{\theta_d(\varepsilon)}2C_1/C_0$ is smaller than the injectivity radius of (B, g_B). Then for every $x \in \mathcal{N}(\psi_0)$ the argument of the original step two applies on the geodesic $t \mapsto \exp_x(t\nu)$ and we have

$$\big|\psi_0(\exp_x(\varepsilon\sqrt{\theta_d(\varepsilon)}C_2\nu)\big| \geq C_1\varepsilon\sqrt{\theta_d(\varepsilon)}\,.$$

The expansion is uniform on $\mathcal{N}(\psi_0)$ since the second derivatives of ψ_0 are bounded independently of ε.

3) The original argument can be applied to the connected components of $B \setminus \mathcal{N}(\psi_0)$ and we obtain $|\psi_0(x)| > C_1\varepsilon\sqrt{\theta_d(\varepsilon)}$ for all $x \in B$ with $\mathrm{dist}_{g_B}(x, \mathcal{N}(\psi_0)) \geq C_2\varepsilon\sqrt{\theta_d(\varepsilon)}$.

4) If $\partial M = \varnothing$ the proof concludes just as in the case $d = 1$.

Otherwise we need some control of the fibre eigenfunction ϕ_0 to get sufficient estimates near the boundary, where both $\psi_0\phi_0$ and φ vanish.

5) Let $D(y) := \operatorname{dist}_{g_{F_{\pi(y)}}}\left(y, \partial F_{\pi(y)}\right)$. There exist constant $C_5, C_6 > 0$ such that

$$C_5 \geq \phi_0(y)/D(y) \geq C_6$$

for every $y \in M$:

The upper bound C_5 follows from the fact that the eigenfunctions for different fibres form a bounded set in $\mathscr{C}^1_b(F)$ (cf. section B.2).

If a lower bound C_6 does not exit, there is a sequence $(y_k)_{k\in\mathbb{N}} \subset M$ for which $\phi_0(y_k)/D(y_k)$ converges to zero as $k \to \infty$. Since M is compact we may extract a subsequence (from now on denoted y_k), converging to some $y \in M$. The limit point y cannot lie in the interior of M because ϕ_0 is strictly positive there. Hence in a trivialisation near $x = \pi(y)$ we have $\Phi(y_k) = (x_k, \tilde{y}_k)$, with x_k converging to x in B and $\tilde{y}_k$ converging to some $\tilde{y} \in \partial F$. Then since $\phi_0 \in \mathscr{C}^1(M)$ (see proposition B.7) the quotient converges to the derivative of ϕ_0 in the direction of the outward normal of F at $\tilde{y}$. But the boundary of F is smooth, so by a standard result (see e.g. [22, lemma 3.4]) this derivative is non-zero. This contradicts existence of the sequence y_k, so C_6 must exist.

In order to complete the proof as in step four we need to strengthen the result of 3.18 to an estimate of the form

$$\left\| \left(\psi\phi_0 - \varphi\right)/D\right\|_\infty \leq C_7 \varepsilon \sqrt{\theta_d(\varepsilon)}\,. \tag{3.28}$$

Since the g_ε-length of a fixed vertical vector field Y is independent of ε, lemma 3.16 together with proposition 3.18 implies $\|Y\delta\|_\infty = \mathcal{O}(\varepsilon\sqrt{\theta_d(\varepsilon)})$. In particular we have bounds on normal derivatives near the boundary, which imply (3.28).

The proof can now be completed setting

$$\tilde{C}_4 := \max\{\left(\varepsilon\sqrt{\theta_d(\varepsilon)}\right)^{-1} \|\psi - \psi_0\|_\infty C_5, C_7\}$$

and taking $C_1 = 2\tilde{C}_4/C_6$, $C := C_2 = 2C_1/C_0$, since

$$\|(\psi_0\phi_0 - \varphi)/D\|_\infty \leq \|\psi_0 - \psi\|_\infty C_5 + C_7\sqrt{\theta_d(\varepsilon)} \leq 2\tilde{C}_4\sqrt{\theta_d(\varepsilon)}$$

and

$$\left|\psi_0\phi_0/D\right| \geq |\psi_0|\, C_6 > 2\tilde{C}_4\varepsilon\sqrt{\theta_d(\varepsilon)}\,,$$

if $\mathrm{dist}_{g_B}\left(\pi(y), \mathcal{N}(\psi_0)\right) \geq C\varepsilon\sqrt{\theta_d(\varepsilon)}$ by step three.

□

An eigenband with a non-degenerate minimum

We now turn to the case of λ_0 having a unique minimum and $d = 1$ of proposition 3.22. We take up the notation used there. For every $j \leq J$ (chosen for corollary 3.20) we have the eigenfunction f_j of the harmonic oscillator H_0, the eigenfunction ψ_j of H_{a} and of course $\varphi_j = P_{\lambda_j}\psi_j/\|P_{\lambda_j}\psi_j\|$. f_j can be used to approximate ψ_j, which in turn approximates φ_j up to order ε. The second approximation is much more accurate, so it will provide sharper estimates on the location of $\mathcal{N}(\varphi_j)$. Since f_j is explicitly known, we can use it to establish some basic growth estimates for ψ_j near its zeros that effectively replace the estimates of step one to three in the proof of 3.23 or 3.24.

Lemma 3.25. *Assume the conditions of corollary 3.20 hold and let $L > 0$ such that $f_j^{-1}(0) \subset [-L, L] =: I_0$ for every $0 < j \leq J$. For ε small enough there exist positive constants c_0, c_1, c_2 such that for every $s \in \sqrt{\varepsilon} I_0$ and $0 < j \leq J$*

a) $\mathrm{dist}(s, \mathcal{N}(D^* f_j)) \geq c_2\varepsilon^{3/4} \implies |\psi(s)| > c_0,$

b) $\mathrm{dist}(s, \mathcal{N}(D^* f_j)) \leq c_2\varepsilon^{3/4} \implies |\partial_s\psi(s)| > c_1\varepsilon^{-3/4}.$

Proof. The steps of the proof do not depend on j, so we drop this index here.

a) In view of (3.25) we have $\|\psi - D^* f\|_{W^1} = \mathcal{O}(1)$. Since $d = 1$ this implies existence of c_0 with

$$\sup_{s\in\mathbb{R}} \left|\psi(s) - \varepsilon^{-1/4} f(\varepsilon^{-1/2}s)\right| \leq c_0\,. \tag{3.29}$$

Now we merely need to choose c_2 large enough to assure that

$$\varepsilon^{-1/4} f(s) > 2c_0$$

for $s \in I_0$ with $\operatorname{dist}(s, \mathcal{N}(f)) \geq c_2 \varepsilon^{1/4}$. Since $\partial_s f$ does not vanish on $\mathcal{N}(f)$ this is possible, as elaborated in proposition 3.23, steps one to three.

b) Equation (3.26) implies existence of a constant C with

$$\|\partial_s(\psi - D^* f)\|_\infty \leq C\varepsilon^{-1/2}.$$

Now since $\partial_s f$ does not vanish on the nodal set we have $|\partial_s f(s)| > c$ for $\operatorname{dist}(s, \mathcal{N}(f)) \leq c_2 \varepsilon^{1/4}$ for ε small enough and thus on this set

$$|\partial_s \psi(s)| \geq \varepsilon^{-3/4} c - C\varepsilon^{-1/2} \geq c_1 \varepsilon^{-3/4}.$$

□

Theorem 3.26. *Assume the conditions of corollary 3.20 hold and let I_0 be as in lemma 3.25. Then for every $0 < j \leq J$ and ε small enough the following statements hold*

a) There is a constant $C > 0$ such that for every $y \in (M \setminus \partial M) \cap \pi^{-1}(\sqrt{\varepsilon} I_0)$ with $\operatorname{dist}_{g_B}(\pi(y), \mathcal{N}(\psi_j)) \geq C\varepsilon^{7/4}$ we have

$$\operatorname{sign}(\varphi_j(y)) = \operatorname{sign}(\psi_j(\pi(y))).$$

b) $\mathcal{N}(\varphi_j) \cap \pi^{-1}(\sqrt{\varepsilon} I_0)$ is a smooth hypersurface in the interior of M.

Proof. In principle the proof of part a) follows the same steps as that of proposition 3.23 up to step five of theorem 3.24.

We drop j for the proof since the method does not depend on that index and denote $\{s_i : i \in I\} := \mathcal{N}(\psi) \cap \sqrt{\varepsilon} I_0$.

1) By part a) of lemma 3.25 we have $\operatorname{dist}(s_i, \mathcal{N}(D^* f)) \leq c_2 \varepsilon^{3/4}$ for every $i \in I$. Hence by part b), $|\partial_s \psi| \geq c_1 \varepsilon^{-3/4}$ on $\mathcal{N}(\psi) \cap \sqrt{\varepsilon} I_0$.

2) We can use the differential equation that ψ solves to obtain an improved bound on its second derivatives inside $\sqrt{\varepsilon} I_0$:

$$\partial_s^2 \psi = \varepsilon^{-2} \big(\lambda_0(s) - \varepsilon\mu - \varepsilon^3 P_0 \partial_s^* S_\varepsilon \partial_s^* P_0 + \varepsilon^2 V_{\text{eff}}\big)\psi$$

together with the bound $\lambda_0(s) = \mathcal{O}(|s|^2) = \mathcal{O}(\varepsilon)$ on $\sqrt{\varepsilon} I_0$ gives

$$\left|\partial_s^2 \psi\right| \leq c\big(\underbrace{\varepsilon^{-1} \|\psi\|_\infty}_{\overset{(3.29)}{=} \mathcal{O}(\varepsilon^{-5/4})} + \underbrace{\varepsilon \|\psi\|_{\mathscr{C}_b^1}}_{\overset{(3.27)}{=} \mathcal{O}(\varepsilon^{1/4})} \big) \leq c_3 \varepsilon^{-5/4} .$$

By Taylor expansion at s_i we get

$$\begin{aligned} |\psi(s)| &\geq c_1 \varepsilon^{-3/4} |s - s_i| - c_3 \varepsilon^{1/4} \big(\varepsilon^{-3/4} |s - s_i|\big)^2 \\ &\geq c_1 \varepsilon^{-3/4} |s - s_i| / 2 , \end{aligned} \tag{3.30}$$

if $|s - s_i| \leq 2c_2 \varepsilon^{3/4}$ and ε is small enough that $2c_2 c_3 \varepsilon^{1/4} \leq c_1/2$.

3) Again by part a) of lemma 3.25 we have $\operatorname{dist}(s_i, \mathcal{N}(D^* f)) \leq c_2 \varepsilon^{3/4}$ for every $i \in I$. So those $s \in \sqrt{\varepsilon} I_0$ with $\operatorname{dist}(s, \mathcal{N}(\psi)) \geq 2c_2 \varepsilon^{3/4}$ are also further than $c_2 \varepsilon^{3/4}$ from $\mathcal{N}(D^* f)$, which implies $|\psi(x)| \geq c_0$. We may thus deduce from (3.30) that if $s \in \sqrt{\varepsilon} I_0$ with $\operatorname{dist}(s, \mathcal{N}(\psi)) \geq C \varepsilon^{7/4}$, then

$$|\psi(x)| \geq C c_1 \varepsilon / 2 .$$

4) Exactly the same as step five of theorem 3.24.

In order to prove b) we show that $\partial_s^* \varphi \neq 0$ on the nodal set of φ. By lemma 3.25 and part a) we have $|\varepsilon \partial_s \psi| \geq c_1 \varepsilon^{1/4}$ on $\pi(\mathcal{N}(\varphi)) \cap \sqrt{\varepsilon} I_0$. We know that ϕ_0 is independent of ε, bounded in $\mathscr{C}^2$ (see proposition B.7) and that $\|\psi\|_\infty = \mathcal{O}(\varepsilon^{-1/4})$ by (3.29). Now let $D(y)$ be the distance to the boundary in $F_{\pi(y)}$ as in the proof of theorem 3.24, step five, and let C_6 be the lower bound for ϕ_0/D obtained there. Then on $\mathcal{N}(\varphi) \cap \pi^{-1}(\sqrt{\varepsilon} I_0)$

$$|\varepsilon(\partial_s^* \psi \phi_0)/D| \geq c_1 C_6 \varepsilon^{1/4} - c_5 \varepsilon^{3/4} > c_6 \varepsilon^{1/4} .$$

This lower bound for $\partial_s^* \psi \phi_0$ will imply a lower bound on $\partial_s^* \varphi$ if we can show that their difference is small. Since $g_\varepsilon(\varepsilon \partial_s^*, \varepsilon \partial_s^*) = 1$, application of lemma 3.16 gives $\|\varepsilon \partial_s^* \delta / D\|_\infty = \mathcal{O}(\varepsilon)$ (cf. proof of 3.24, step 5). Consequently

$$|\partial_s^* \varphi(y)| \geq c_7 \varepsilon^{-3/4} D(y)$$

on $\mathcal{N}(\varphi) \cap \pi^{-1}(\sqrt{\varepsilon} I_0)$, which proves the claim. □

The theorem locates parts of the nodal set of φ_j within a neighbourhood of size $\varepsilon^{7/4}$ of that of ψ_j. There may however be other parts located in regions at distance greater than $\sqrt{\varepsilon}L$ from the minimum x_0 of λ_0, where ψ_j is small. Because we also have control of the number of nodal domains close to x_0, Courant's theorem provides restrictions on the nodal set that is far from x_0.

Corollary 3.27. *Under the conditions of theorem 3.26 the set*

$$\{y \in M : \mathrm{dist}_{g_B}(\pi(y), x_0) \geq \sqrt{\varepsilon}L\} \setminus \mathcal{N}(\varphi_j)$$

has at most two connected components. If j is odd this set has exactly two components and φ_j attains the bound of Courant's theorem.

Proof. The eigenfunction f_j of the harmonic oscillator has $j+1$ nodal domains. All the zeros of $D^* f_j$ are contained in $\sqrt{\varepsilon}I_0$, so by (3.29) the set $\overline{B(\sqrt{\varepsilon}L, x_0)} \setminus \mathcal{N}(\psi_j)$ has at least $j+1$ connected components and by theorem 3.26 this also holds for $\overline{\pi^{-1}(B(\sqrt{\varepsilon}L, x_0))} \setminus \mathcal{N}(\varphi_j)$. Those components contained in the interior of this set are open and closed also in $M \setminus \mathcal{N}(\varphi_j)$. From this we deduce that $M \setminus \mathcal{N}(\varphi_j)$ has at least j connected components by inspecting the boundary. Let x_+, x_- be the two points in B that are mapped to $\pm\sqrt{\varepsilon}L$ by the coordinate map s. Since

$$\mathrm{dist}(\pm\sqrt{\varepsilon}L, \mathcal{N}(D^* f_j)) = \mathrm{dist}(\pm L, \mathcal{N}(f_j)) \geq C$$

it follows from (3.29) and theorem 3.26 that, for ε small enough,

$$\mathrm{sign}\big(f_j(\pm L)\big) = \mathrm{sign}\big(\psi_j(x_\pm)\big) = \mathrm{sign}\big(\varphi_j|_{F_{x_\pm}}\big) .$$

Thus the fibres F_{x_+} and F_{x_-} are each completely contained in one component of $M \setminus \mathcal{N}(\varphi_j)$. This means that there are at least $j-1$ connected components contained in the interior of $\overline{\pi^{-1}(B(\sqrt{\varepsilon}L, x_0))} \setminus \mathcal{N}(\varphi_j)$. Thus $\overline{(M \setminus \pi^{-1}(B(\sqrt{\varepsilon}L, x_0)))} \setminus \mathcal{N}(\varphi_j)$ cannot have more than two connected components without φ_j having more than $j+1$ nodal domains, which is impossible by Courant's nodal domain theorem.

If F_{x_+} and F_{x_-} belong to different components, $\overline{(M \setminus \pi^{-1}(B(\sqrt{\varepsilon}L, x_0)))} \setminus \mathcal{N}(\varphi_j)$ must have two connected components and $M \setminus \mathcal{N}(\varphi_j)$ has exactly $j+1$. This is the case if j is odd since then f_j is an odd function. □

For a given fibre bundle over $\mathbb{R}$ or S^1 with compact fibres and non-empty boundary one can always construct metrics satisfying the conditions of this corollary. One may choose any metric that makes the bundle a uniformly locally trivial Riemannian submersion and then find a positive function $f \in \mathscr{C}_b^\infty(B)$ so that the lowest eigenvalue $\lambda_0(x)$ of the metrics $f(x)g_{F_x}$ has a unique, non-degenerate minimum and is asymptotically larger than this minimum. Going to the adiabatic limit with this metric thus gives metrics on M for which any given number of the eigenfunctions φ_{2k+1} attains the bound of Courant's theorem.

The location of the nodal sets also tells us about its relation to the boundary, since we know that φ must change sign in certain cylinders over $\mathcal{N}(\psi)$. This is independent of λ_0 being constant or having a minimum. It forces $\mathcal{N}(\varphi)$ to touch the boundary, since otherwise one could find a curve joining two points at which φ has different signs without hitting the nodal set. We prove a more refined version of this statement, giving an estimate on 'how often' the boundary is reached.

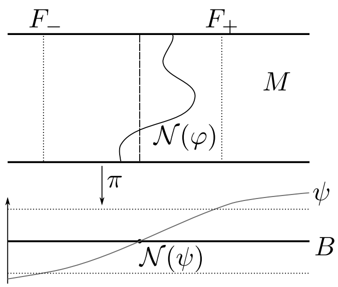


Figure 3.2: φ is positive on F_+ and negative on F_- because $|\psi|$ has reached a certain size there. $\mathcal{N}(\varphi)$ must separate these fibres and thus it must reach every connected component of ∂F.

Corollary 3.28. *Assume the conditions of theorem 3.24 or 3.26 hold and let ψ denote the function ψ_0 or ψ_j of the respective statement. If the nodal set of ψ has m connected components $K_1, \dots, K_m$, the set $\partial M \cap \mathcal{N}(\varphi)$ has at least as many connected components as*

$$D := \bigcup_{l \leq m} \partial M \cap \pi^{-1}(K_l)\,.$$

If B is one-dimensional and ∂F has k connected components, D has m times k components.

Proof. Let $K \subset \mathcal{N}(\psi)$ be a connected component. By theorems 3.24, 3.26 there is a small closed tubular neighbourhood $T(\varepsilon, K)$ of K in B such that φ changes sign in $\pi^{-1}(T)$. More precisely, let ν be the unique unit normal of K pointing into the region where ψ is positive. Then there are $t_+, t_- = \mathcal{O}(\varepsilon)$ such that if $x \in K$, $\varphi|_{\pi^{-1}(\exp_x(t_+\nu))} > 0$ and $\varphi|_{\pi^{-1}(\exp_x(t_-\nu))} < 0$, except on the boundary of these fibres. We have a projection $p\colon T \to K$ along the normal direction and $T \cong K \times [t_-, t_+]$. Since $[t_-, t_+]$ is contractible this implies $\pi^{-1}(T) \cong \pi^{-1}(K) \times [t_-, t_+]$ and p can be lifted to a projection $\overline{p}\colon \pi^{-1}(T) \to \pi^{-1}(K)$ (if T lies in a an open set U over which M is trivial $\overline{p}$ can be taken as the projection along $\Phi^*\nu$, otherwise such projections may be patched together, see [31, corollary 1.8]).

Because continuous images of connected sets are connected, the set $\overline{p}\,(\partial M \cap \mathcal{N}(\varphi)) \subset \partial M \cap \pi^{-1}(K)$ has at most as many connected components as $\partial M \cap \mathcal{N}(\varphi) \cap T$. We conclude the proof by showing that $\overline{p}\colon \partial M \cap \mathcal{N}(\varphi) \to \partial M \cap \pi^{-1}(K)$ is onto. Assume there were $y \in \partial M \cap \pi^{-1}(K)$ not contained in the image of $\overline{p}|_{\partial M \cap \mathcal{N}(\varphi)}$. Without the restriction to $\mathcal{N}(\varphi)$, $\overline{p}$ is clearly onto so the fibre of $\overline{p}$ over y is a curve in $\pi^{-1}(T)$, that projects to $\{\pi(y)\} \times [t_-, t_+]$. M is trivial over $\{\pi(y)\} \times [t_-, t_+]$, so $\overline{p}^{-1}(y)$ can be represented by a curve γ in $\partial F \times [t_-, t_+]$ with $\gamma(0) \in \partial F_{t_-}$, $\gamma(1) \in \partial F_{t_+}$ and $\gamma \cap \mathcal{N}(\varphi) = \varnothing$.

Since the nodal set is closed, there is an open neighbourhood U of γ in $F \times [t_-, t_+]$ that does not intersect $\mathcal{N}(\varphi)$ either. But then there is a curve $\tilde{\gamma}$ in the interior of $F \times [t_-, t_+]$, joining F_{t_-} and F_{t_+}, on which φ does not vanish. This contradicts the fact that $\varphi|_{F_{t_-}} < 0$ and $\varphi|_{F_{t_+}} > 0$.

If $d = 1$, $K = \{x\}$ is a point in B and $\partial M \cap \pi^{-1}(K) = \partial F_x$, so D clearly has m times k components. □

Now that we have located the nodal set of φ and established some of its properties, the question remains what it actually 'looks like'. We know that it is contained in cylinders around $\pi^{-1}(\mathcal{N}(\psi_0))$, or $\pi^{-1}(\mathcal{N}(D^*f_j))$ in the case $\alpha = 1$, that shrink as $\varepsilon \to 0$. So in the case $\lambda \equiv 0$, where the set $\pi^{-1}(\mathcal{N}(\psi_0))$ is independent of ε, we have $\lim_{\varepsilon \to 0} \mathcal{N}(\varphi) = \pi^{-1}(\mathcal{N}(\psi_0))$ in the Hausdorff metric on compact subsets of M. A natural question would be whether this limit is also correct in a more geometric sense. For example if for fixed $0 < \varepsilon \le \varepsilon_0$, $\mathcal{N}(\varphi)$ and $\pi^{-1}(\mathcal{N}(\psi_0))$ are diffeomorphic manifolds with boundary and, if this is the case, whether they are homotopic as submanifolds of M. Such properties are only established for very specific cases. For example if $M = S^1 \times S^1$ is a torus, the nodal set $\mathcal{N}(\varphi)$ consists of finitely many immersed circles (see [64]). If $\varphi = \varphi_1$ is the first excited state these must divide M into exactly two connected components. Now if the geometry is such that φ_1 corresponds to a simple eigenvalue of H_0, we know the locations of these circles. For this reason, and because we know the total nodal count, they must actually be embedded and cannot intersect each other. Thus $\mathcal{N}(\varphi) \cong S^1 \cup S^1 \cong \pi^{-1}(\mathcal{N}(\psi_0))$. Since each of these circles must cut the cylinder $[-C\varepsilon, C\varepsilon] \times S^1$ into two connected components, each of which contains either one of the fibres $\{\pm C\varepsilon\} \times S^1$, these circles must also have the same homotopy class in $\pi_1(M)$ as the fibres.

This argument relies on the fact that there is really just one compact one-dimensional manifold, so generalisations to higher dimensions are not obvious. A possible approach, at least for the case $\partial M = \varnothing$, would be to push $\mathcal{N}(\varphi)$ to that part of the boundary of the tube $T(\varepsilon)$ in M around $\pi^{-1}(\mathcal{N}(\psi_0))$ where it is positive, using the gradient flow of φ. This however requires a proof of the non-vanishing of $\operatorname{grad} \varphi$ on $T(\varepsilon)$, for which the derivative estimates obtained from theorem 3.24 and lemma 3.16 are not sufficient. Thus the topological behaviour of $\mathcal{N}(\varphi)$ in the limit remains an open question.

Appendix A

Bounded geometry

A.1 Manifolds with boundary and fibre bundles

In this section we discuss different concepts of bounded geometry for manifolds with and without boundary based on the expositions of Shubin [65] and Schick [62, 63]. We also introduce the concept of a uniformly locally trivial fibre bundle in definition A.3 and relate it to the established definitions in proposition A.4.

Definition A.1. A connected Riemannian manifold (M, g) is of *bounded geometry* if it has injectivity radius $r_i(M) > 0$ and for every $k \in \mathbb{N}$ there exists $C(k) > 0$ for which the curvature tensor R satisfies

$$\sup_M g(\nabla^k R, \nabla^k R) \leq C(k)\,.$$

Here g denotes the induced metric on the tensor-bundle $TM \otimes T^*M^{\otimes k+3}$ and ∇^k is the composition of the connections on the bundles $TM \otimes T^*M^{\otimes l+3}$ with $0 \leq l \leq k$, induced by the Levi-Cività connection of (M, g). In the following we will always denote induced metrics and connections on tensor bundles by the original symbols.

Bounded geometry is discussed in detail in [65, appendix A.1]. Most importantly it provides us with a geodesic coordinate system on the ball of radius $r < r_i(M)$ at any point $p \in M$. The coordinate vector fields of these coordinate systems have bounded covariant derivatives to any order and the transition functions are uniformly bounded in $\mathscr{C}^\infty(B(r, 0))$. This implies that a tensor T is represented by $\mathscr{C}^\infty$-bounded functions in these coordinates if and only if it satisfies

$$\sup_M g(\nabla^k T, \nabla^k T) \leq C(k)\,,$$

for every $k \in \mathbb{N}$. We denote spaces of $\mathscr{C}^\infty$-bounded functions or sections by $\mathscr{C}_b^\infty$ and Γ_b respectively.

A manifold with non-empty boundary is not of bounded geometry in this sense, even if it is compact, since it cannot have positive injectivity radius. An extension of the concept to manifolds with boundary was developed by Schick [62, 63].

Definition A.2 ([62, 63])**.** A Riemannian manifold (M, g) with boundary ∂M is a *∂-manifold of bounded geometry* if the following holds:

- *Normal collar*: Let ν be the inward pointing unit normal of ∂M. There exists $r_c > 0$ such that the map

  $$b\colon \partial M \times [0, r_c) \to M\,, \qquad (p, t) \mapsto \exp_p(t\nu)$$

  is a diffeomorphism onto its image.

- *Injectivity radius of the boundary*: The injectivity radius of ∂M with the induced metric is positive, $r_i(\partial M, g|_{\partial M}) > 0$.

- *Injectivity radius in the interior*: There is $r_i > 0$ such that for $p \in M$ with $\mathrm{dist}(p, \partial M) > r_c/3$ the exponential map is a diffeomorphism on $B(r_i, 0) \subset T_pM$.

- *Curvature bounds*: The curvature tensor of M and the second fundamental form S of ∂M are bounded tensors on M and ∂M respectively, $R \in \Gamma_b(T^*M^{\otimes 3} \otimes TM)$, $S \in \Gamma_b(T^*\partial M^{\otimes 2} \otimes N\partial M)$.

This definition also provides us with an atlas. In the interior the charts are again given by geodesic coordinates on $B(r_i, p)$, while for $p \in \partial M$ and $r < \min\{r_i(\partial M), r_c\}$ choosing an orthonormal basis of $T_p\partial M$ defines a coordinate map

$$b_p\colon B(r, 0) \times [0, r) \to M\,, \qquad (v, t) \mapsto \exp_{\exp_p^{\partial}(v)}(t\nu)\,. \tag{A.1}$$

The representation of g in these coordinate systems has bounded derivatives to any order [63, theorem 2.5], which in turn means that the coordinate vector fields are bounded, i.e. if $U \subset M$ is the image of one of these charts $\partial_k \in \Gamma_b(TM|_U)$. Both the geodesic coordinate systems in the interior and the boundary collar coordinates are referred to as *normal coordinates*.

Though the concept of a ∂-manifold of bounded geometry provides all the necessary tools for analysis of differential equations we will choose a different definition directly adapted to fibre bundles. We will then prove that this implies ∂-bounded geometry. Since we assume in chapters two and three that B is of bounded geometry in the usual sense and F is compact, the only source of non-uniform behaviour of geometric quantities in $F \to M \overset{\pi}{\to} B$ is the variation of the fibre. This suggests the following definition, which is very similar to that of a vector bundle of bounded geometry [62, 63, 65] except for the lack of a canonical metric on the model of the fibre.

Definition A.3. Let (B, g_B) be a manifold of bounded geometry. A Riemannian submersion $F \to (M, g) \overset{\pi}{\to} (B, g_B)$ is *uniformly locally trivial* if there exists a metric g_0 on F and for every $x \in B$ and metric ball $B(r, x)$ of radius $r < r_i(B)$ there is a trivialisation

$$\Phi\colon \left(\pi^{-1}(B(r,x)), g\right) \to \left(B(r,x) \times F, g_B \times g_0\right),$$

such that Φ_* and Φ^* are bounded with all their covariant derivatives, uniformly in x and r.

Boundedness of Φ_* of course means $\Phi_* \in \Gamma_b(T^*\pi^{-1}(B(r,p)) \otimes \Phi^* T(U \times F))$, with metric and connection induced by g and $g_B \times g_0$. Since both Φ_* and Φ^* are bounded the transition functions between two such trivialisations will be bounded, too.

Alternatively one could require M and B to be of bounded geometry, in the suitable sense, and π to be a bounded map. If there is no boundary this allows for the construction of trivialisations by lifting geodesics from B to horizontal geodesics in M. In this case one does not even need to require the fibre bundle property, since it follows from completeness (cf. [33]). However in the presence of a boundary it is not clear how to construct local trivialisations with sufficiently good properties, so the definition is quite natural. To clarify the relations between definitions A.1, A.2 and A.3 we formulate a proposition.

Proposition A.4. *Let $M \overset{\pi}{\to} B$ be uniformly locally trivial and F compact then*

- *if $\partial M = \varnothing$, M is of bounded geometry in the sense of definition A.1;*
- *if $\partial M \neq \varnothing$, M is of ∂-bounded geometry the sense of definition A.2.*

The statement also holds if (F, g_0) is not compact but has bounded geometry in the appropriate sense of definition A.1 or A.2. We will not be concerned with this case however and omit its discussion for the sake of coherence.

For the proof let $U = B(r, x) \subset B$ with $r < r_i(B)$ and note that the product $(U \times F, g_B \times g_0)$ has all the desired properties. The fibres in the product are totally geodesic, so the curvature bounds are given by the maximum of those for (F, g_0), which exist since F is compact, and those for (B, g_B), which is of bounded geometry. The width of the boundary collar is exactly that of (F, g_0). The injectivity radius in the interior is given by $\min\{r_i(F, g_0), r\}$ for all the points in F_x. Thus we need to assure that such estimates are conserved by diffeomorphisms that have bounded derivatives together with their inverse.

First take note of some bounds for elementary geometric quantities that arise from the requirements on M. Let $p, q \in \pi^{-1}(U)$ with $r < r_i(B)$. Then $\Phi(p)$ and $\Phi(q)$ are joined by a curve γ that is length-minimising for the product metric and

$$\begin{aligned}\mathrm{dist}_g(p,q) &\leq \int_0^1 \sqrt{g(\Phi^*\dot\gamma, \Phi^*\dot\gamma)}\,\mathrm{d}t \\ &\leq \int_0^1 C(\Phi)\sqrt{(g_B \times g_0)(\dot\gamma, \dot\gamma)}\,\mathrm{d}t \\ &= C(\Phi)\,\mathrm{dist}_{g_B \times g_0}(\Phi(p), \Phi(q))\,.\end{aligned}$$

The argument equally applies to $\mathrm{dist}_{g_B \times g_0}(\Phi(p))(\Phi(q))$, so we can choose $C > 0$ (depending on Φ) such that also

$$C^{-1}\,\mathrm{dist}_{g_B \times g_0}(\Phi(p), \Phi(q)) \leq \mathrm{dist}_g(p,q) \tag{A.2}$$

Let $p \in F_x$ and $r' < \min\{r_i(F, g_0), r\}$. Then there is another constant $C(\Phi) > 0$ with

$$C^{-1}\mathrm{Vol}_{g_B \times g_0}\big(B(r', \Phi(p))\big) \\ \leq \mathrm{Vol}_g\big(\Phi^{-1}B(r', \Phi(p))\big) \leq C\mathrm{Vol}_{g_B \times g_0}\big(B(r', \Phi(p))\big)\,,$$

because of the bounds on Φ_* and Φ^*. By compactness of F and bounded geometry of (B, g_B) the balls $B(r', \Phi(p))$ have upper and lower volume bounds in the product metric. These are independent of p, at least if $\Phi(p)$ lies not too close to the boundary, i.e. $\mathrm{dist}(\Phi(p), \partial F) > r_c/3$, because then we can estimate the volume of the ball $B(r', \Phi(p))$ using the exponential map. In case $\mathrm{dist}(\Phi(p), \partial F) \leq r_c/3$ the requirement $r' < r_i(F, g_0)$ implies $r' < r_c(F, g_0)/2$, so the ball can be mapped to $\mathbb{R}^m$ using the boundary collar coordinates (A.1), which gives similar bounds (cf [62, lemma 3.19]). Thus we get a constant $C(\Phi, g_B, g_0, r')$ with

$$C^{-1} \leq \mathrm{Vol}_g\big(\Phi^{-1}B(r', \Phi(p))\big) \leq C \tag{A.3}$$

We now prove the proposition A.4 over the course of several lemmata, also proving boundedness of some secondary quantities on the way. We begin by proving curvature bounds.

Lemma A.5. *Let $\Phi : (M, g) \to (N, h)$ be a diffeomorphism of Riemannian manifolds with boundary, for which Φ_*, Φ^* and all their covariant derivatives are bounded tensors. If the curvature tensor of (N, h) and all its derivatives are uniformly bounded this also holds for the curvature tensor of (M, g).*

Proof. Because Φ_* is bijective and bounded together with its inverse we have a one-to-one correspondence of bounded vector fields on M and N. We calculate explicitly

$$\begin{aligned} \Phi_*\nabla^M_X Y &= -\left(\nabla_X \Phi_*\right) Y + \nabla^{\Phi^*TN}_X \Phi_* Y \\ &= -\left(\nabla_X \Phi_*\right) Y + \nabla^N_{\Phi_* X} \Phi_* Y\,, \end{aligned} \tag{A.4}$$

where the second equality holds because Φ_* is an isomorphism and thus every section of Φ^*TN is a pullback. Insert this into the definition of the curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,,$$

by first calculating

$$\begin{aligned}
&\Phi_* \nabla^M_X \nabla^M_Y Z \\
&= -\left(\nabla_X \Phi_*\right)\left(\nabla^M_Y Z\right) + \nabla^N_{\Phi_* X} \Phi_* \nabla^M_Y Z \\
&= -\left(\nabla_X \Phi_*\right)\left(\nabla^M_Y Z\right) - \nabla^N_{\Phi_* X}\left((\nabla_Y \Phi_*) Z\right) + \nabla^N_{\Phi_* X} \nabla^N_{\Phi_* Y} \Phi_* Z
\end{aligned} \tag{A.5}$$

with

$$\begin{aligned}
&\nabla^N_{\Phi_* X}\left((\nabla_Y \Phi_*) Z\right) \\
&= \left(\nabla^2_{X,Y} \Phi_*\right) Z + \left(\nabla_{\nabla^M_X Y} \Phi_*\right) Z + \left(\nabla_Y \Phi_*\right)\left(\nabla^M_X Z\right)
\end{aligned} \tag{A.6}$$

and

$$\Phi_* \nabla^M_{[X,Y]} Z = -\left(\nabla_{[X,Y]} \Phi_*\right) Z + \nabla^N_{\Phi_* [X,Y]} \Phi_* Z \,. \tag{A.7}$$

Now in $\Phi_* R^M(X,Y)Z$ we have cancellations of

- the first term of (A.5) with the last one of (A.6) with X and Y interchanged,
- the second term of (A.6) with part of the first term of (A.7), since $[X,Y] = \nabla_X Y - \nabla_Y X$.

The last terms of (A.5) and (A.7) add up to the curvature tensor of N and the result reads

$$\begin{aligned}
&\Phi_* R^M(X,Y)Z \\
&= R^N(\Phi_* X, \Phi_* Y)\Phi_* Z - \left(\nabla^2_{X,Y} \Phi_*\right) Z + \left(\nabla^2_{Y,X} \Phi_*\right) Z \,.
\end{aligned} \tag{A.8}$$

The right hand side is bounded because Φ_* and R^N are, hence so is the left. For the treatment of the derivatives of the curvature note that, since $\nabla^k R$ is a tensor, its evaluation on vector fields $X, \dots$ at a point $p \in M$ depends only on the values $X_p, \dots$ of these fields in that point. We are therefore free to choose continuations $\tilde{X}, \dots$ of $X_p, \dots$ with $(\nabla^M \tilde{X})_p = 0$ for the calculations. For these we have

$$\left(\left(\nabla_{\tilde{X}} \Phi_*\right) \tilde{Y}\right)_{\Phi(p)} = \left(\nabla^N_{\Phi_* \tilde{X}} \Phi_* \tilde{Y}\right)_{\Phi(p)} ,$$

so any number of derivatives acting on (A.8) will only yield additional derivatives of R^N and Φ_*, which are bounded by hypothesis. □

Corollary A.6. *Let $F \to M \overset{\pi}{\to} B$ be uniformly locally trivial and F compact. Then the following geometric quantities are bounded with all their derivatives:*

1) The curvature tensor R of M.

2) The second fundamental form of ∂M.

3) The projection π_*.

4) The second fundamental form of the fibres.

5) The horizontal lift X^* of any $X \in \Gamma_b(TB)$.

6) The integrability tensor of the horizontal bundle NF.

Proof.

1) Let U be a normal coordinate chart of B, then A.5 proves the claim on $\pi^{-1}(U)$. Since the constants of the estimates for Φ and the curvature of $(U \times F, g_B \times g_0)$ are global so is the boundedness of R.

2) By (A.4) the second fundamental form equals $\nabla \iota_*$, where ι is the inclusion of the boundary. Its boundedness follows from the diagram

$$\begin{array}{ccc} \partial\pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times \partial F \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota_{\partial F}} \\ \pi^{-1}(U) & \xleftarrow{\ \Phi^{-1}\ } & U \times F \end{array}$$

because the inclusion $\iota_{\partial F}$ is bounded in the product metric.

3) Follows from the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

because the projection to the first factor is bounded.

4) Equation (A.4) applied to π_* gives $\pi_* \mathrm{II}\,(X,Y) = -(\nabla_X \pi_*)Y$ for vertical fields X, Y. Now 4) follows from 3) since π_* is an isometry on horizontal vectors.

5) The length of $\nabla_Y X^*$ only depends on the the lengths of $\nabla_{\pi_* Y} X$, X and on the norm of $\nabla \pi_*$. This can bee seen by splitting vertical and horizontal directions. Let Y, Z be horizontal and V, W vertical vector fields, then using (A.4) it is straightforward to calculate (cf. [55, lemma 3])

$$\begin{aligned}
&g\,(\nabla_Y X^*, Z) = g_B\,(\nabla_{\pi_* Y} X, \pi_* Z)\\
&g\,(\nabla_V X^*, Z) = g_B\,(-(\nabla_V \pi_*)X^*, \pi_* Z)\\
&g\,(\nabla_V X^*, W) = g_B\,((\nabla_V \pi_*)W, X) = g\left((\nabla_V \pi_*)^T X, W\right)\\
&g\,(\nabla_Y X^*, W) = g_B\,(X, -\pi_* \nabla_Y W) = g\left((\nabla_Y \pi_*)^T X, W\right)\ .
\end{aligned}$$

We can combine these equations to

$$\nabla_Y X^* = (\nabla_{\pi_* Y} X - (\nabla_Y \pi_*) X^*)^* + (\nabla_Y \pi_*)^T X$$

for arbitrary Y. To check this observe that the first bracket is horizontal by definition and produces the first two equations, while the second term is vertical since $g((\nabla_Y \pi_*)^T X, Z)$ vanishes if Z is a horizontal lift because of (A.4) and the first equation. Hence $\nabla^k X^*$ is bounded because π_* is and B is of bounded geometry.

6) Follows from 5).

□

Next we need to show that the injectivity radius has a lower bound at interior points as well as the existence of a normal collar. The arguments for this use results of Kodani [39] for compact manifolds with boundary to obtain local but uniform estimates on the injectivity radius. The reasoning of Kodani is based on a comparison theorem of Heintze and Karcher [32]. We give a brief review of the ideas to make the following proofs more transparent.

Given a compact, connected Riemannian manifold it is a classical result due to Klingenberg that the injectivity radius is the smaller of the conjugate radius and the length of the shortest closed and smooth geodesic in M [57, lemma 5.8.4]. If the maximum of the sectional curvature in M is K, the distance of two conjugate points in M is at least $\pi/\sqrt{K}$. Now assume M has volume V, diameter d and a closed geodesic γ of length L. Then the exponential map restricted to the normal bundle of γ is surjective onto M, since M is compact and every point $p \in M$ is connected to γ by a minimising curve, which is a geodesic that meets γ orthogonally. This remains true if we further restrict the exponential map to those normal vectors of γ with length less than d. By estimating the volume distortion of the exponential map and integrating over the vectors in $N\gamma$ of length less than d, Heintze and Karcher obtain an inequality [32, theorem 2.3] of the form

$$V \leq Lf(d, K)\,.$$

This shows that L cannot be arbitrarily small. Kodani applies these ideas to compact manifolds with boundary and extends them to obtain estimates for the length of geodesics that connect two points on the boundary. A crucial point is that surjectivity of the exponential map on $N\gamma$ is not always fulfilled. If however the boundary is convex, meaning that the second fundamental form is positive on the interior normal, surjectivity still holds. The intuition behind this is that positivity means curves in the boundary may be shortened by deforming them into the interior. Kodani manages to go beyond this case by estimating the volume of the region that is not reached by the exponential map using Jacobi fields.

Lemma A.7. *Let (M, g), (N, h) and Φ be as in lemma A.5 and let $q \in N$ have injectivity radius $r_i(q) \geq L$. There is a constant C, depending only on L, Φ and the curvature bounds of h, such that the g-injectivity radius of $p := \Phi^{-1}(q)$ is at least C.*

Proof. Choose an orthonormal basis of T_qN. We may modify this so that $g(p)$ is diagonal in the pulled back basis of T_pM since Φ_*g can be diagonalised by orthogonal transformations. Let $r \leq L$ and $x\colon B(r, q) \to B(r, 0)$ be geodesic coordinates for h using the basis chosen before. Then

x_*h is the euclidean metric up to errors of order $|x|^2$. The coordinate vector fields are bounded with their derivatives because of the curvature bounds of h. Thus the coordinate vector fields of

$$x \circ \Phi\colon \Phi^{-1}(B(r,q)) \to B(r,0)$$

are also bounded and $g_{kl} = g(p)_{kl} + \mathcal{O}(|x|)$, where the remainder depends on Φ and the curvature of h. Since $g(p)_{kl} = a_k^2 \delta_{kl}$ this means that the sets $\Phi^{-1}(B(r,q))$ are increasingly close to ellipsoids for g as their radius decreases. The second fundamental form scales like $1/r$ for small r, so the boundary of $\Phi^{-1}(B(r,q))$ will be convex eventually. More precisely, since $\min_k a_k^2$ and $\max_k a_k^2$ can be bounded in terms of Φ_*, Φ^*, there is a constant $c(\Phi)$ for which the second fundamental form of $\partial\Phi^{-1}(B(r,q))$ satisfies

$$(c(\Phi)/r + \mathcal{O}(1))\, g(X,X) \geq \mathrm{II}\,(X,X) \geq (c(\Phi)/r + \mathcal{O}(1))\, g(X,X)\,,$$

for any vector field X tangent to the boundary of $\Phi^{-1}(B(r,q))$. Consequently there exists $r_0(\Phi,h) > 0$, such that for $r \leq r_0$ the boundary of $\Phi^{-1}(B(r,p))$ is convex for the metric g.

Volume and diameter of $\Phi^{-1}(B(r_0,q))$ have lower and upper bounds in terms of Φ (cf. (A.2), (A.3)), and the sectional curvature of g is bounded by lemma A.5, whence C is given by Kodani's bound [39, proposition 6.1] for $\Phi^{-1}(B(r_0,q))$. □

Lemma A.8. *Let (M,g), (N,h) and Φ be as in lemma A.5. Assume the second fundamental form of ∂N is bounded and that ∂N has injectivity radius $r_i(\partial N)$ as well as a normal collar of width r_c. Then there is a constant C, depending on Φ, r_c, the curvature and second fundamental form of (N,h), such that (M,g) has a boundary collar of width at least C.*

Proof. First of all corollary A.6 shows that the second fundamental form of ∂M is bounded by $S(\Phi, \iota_{\partial N})$.

Now assume such a constant C does not exist. Then there is a sequence $(p_n, q_n, s_n) \in \partial M^2 \times (0,1)$ with $p_n \neq q_n$, $\exp_{p_n}(s_n\nu_g) = \exp_{q_n}(s_n\nu_g)$ and $s_n \to 0$, where ν_g is the inward pointing unit normal of ∂M. Choose an orthonormal basis of $T_{\Phi(p_n)}N$ for which g is diagonal in the pulled back basis as in the proof of A.7. For $r < \min\{r_i(\partial N), r_c\}$ we have normal coordinates (A.1) at $\Phi(p_n)$

$$b_n\colon B(r,0) \cap \{\langle x, \nu_h\rangle \geq 0\} \to U_n \subset N\,.$$

Fix r_0 and assume, without loss of generality, that the curves $\gamma_n^1(t) := \exp_{p_n}(ts_n\nu_g)$, $\gamma_n^2(t) := \exp_{q_n}(ts_n\nu_g)$, $t \in [0,1]$ are contained in $\Phi^{-1}U_n$. Let $\tilde{g}_n = b_n^*\Phi_*g$ be the metric on $B(r_0,0) \cap \{\langle x, \nu_h\rangle \geq 0\}$ induced by these coordinates. By the choice of coordinates and the bounds on Φ and b_n these metrics satisfy $(\tilde{g}_n)_{kl} = a(n)_k^2\delta_{kl} + \mathcal{O}(|x|)$, where the remainder and the numbers $a(n)$ are bounded uniformly in n. Now let $\lambda \leq 1$, restrict these metrics to $B(\lambda r_0,0) \cap \{\langle x, \nu_h\rangle \geq 0\}$ and then rescale lengths by λ^{-1} to obtain a family $\tilde{g}_{n,\lambda}$ of metrics on $B(r_0,0) \cap \{\langle x, \nu_h\rangle \geq 0\}$. These satisfy $(\tilde{g}_{n,\lambda})_{kl} = a(n)_k^2\delta_{kl} + \mathcal{O}(\lambda\,|x|)$ uniformly in n. Choose a compact manifold Ω with smooth, connected boundary

$$B(r_0/2,0) \cap \{\langle x, \nu_h\rangle \geq 0\} \subset \Omega \subset B(r_0,0) \cap \{\langle x, \nu_h\rangle \geq 0\}\,,$$

that is convex for the euclidean metric. Then there is a constant $c(\Phi,\Omega)$ for which the second fundamental form of $\partial\Omega$ with respect to $\tilde{g}_{n,\lambda}$ satisfies

$$c(\Phi,\Omega)\tilde{g}_{n,\lambda}(X,X) \geq \mathrm{II}\,(X,X)_{n,\lambda} \geq -\lambda\,(S + c(\Phi,\Omega))\,\tilde{g}_{n,\lambda}(X,X)\ ,$$

for any X tangent to the boundary of Ω. By the hypothesis on Φ and the boundary collar of (N,h) the manifolds $(\Omega, \tilde{g}_{n,\lambda})$ have diameter at most $d(r_0,\Phi)$, volume at least $V(r_0,\Phi,h)$ (cf. (A.2), (A.3)) and sectional curvatures bounded by $K(\Phi,h)$ (lemma A.5). By the result of Kodani [39, proposition 6.2] there exists $\lambda_-\,(d,V,K,c(\Phi,\Omega)) < 0$ satisfying the following: if the second fundamental form of the boundary is bounded below by λ_-, then there is $r_\partial > 0$ such that $(\Omega, \tilde{g}_{n,\lambda})$ has a boundary collar of width at least r_∂. Thus there is $\lambda_0 > 0$ such that $(\Omega, \tilde{g}_{n,\lambda})$ has a boundary collar of width $r_\partial > 0$ for all $\lambda \leq \lambda_0$. But the images of the curves γ_n^1, γ_n^2, under Φ, b_n^{-1} and rescaling by λ_0^{-1}, are normal geodesics contained in $(\Omega, \tilde{g}_{n,\lambda_0})$ for $s_n \leq \lambda_0 r_0/4$. They intersect at distance s_n/λ_0 to the boundary, which is a contradiction since s_n tends to zero and this is eventually less than r_∂. □

This completes the proof of proposition A.4. We consider the rescaled family (M, g_ε) so our analysis will require constants for the whole family and independent of ε.

Proposition A.9. *Let $M \xrightarrow{\pi} B$ be uniformly locally trivial and F compact. Then $(M, g_\varepsilon) = (M, g_F + \varepsilon^{-2}\pi^* g_B)$ satisfies definition A.1 or A.2 with the same constants $\{r_c, r_i(\partial M), r_i, C(k) : k \in \mathbb{N}\}$ as (M, g).*

Proof. The submersion $\pi\colon (M, g_\varepsilon) \to (B, \varepsilon^{-2} g_B)$ is uniformly locally trivial with the same trivialisations as for $\varepsilon = 1$. Thus (M, g_ε) is of bounded geometry in the appropriate sense by proposition A.4. It remains to show uniformity of the constants in ε. In view of the lemmata A.5, A.7 and A.8 we only need to control the curvature bounds and injectivity radii in the product metric $g_0 \times \varepsilon^{-2} g_B$ and the bounds on Φ_*, Φ^* as well as their g_ε-covariant derivatives uniformly in ε to prove the claim. The intuition behind the proof is that the constants can only become better with decreasing ε, since those pertaining to (F, g_0) stay unchanged, while those depending on horizontal quantities shrink (cf. lemma 1.6).

The normal collar of $(U \times F, g_0 \times \varepsilon^{-2} g_B)$ is the same for all ε since the metric on F is unchanged. The injectivity radius in the interior can only grow larger for smaller ε, since it equals the minimum of $r_i(F, g_0)$ and $r_i(B, \varepsilon^{-2} g_B) = \varepsilon^{-1} r_i(B, g_B)$. The curvature tensor of the product metric splits into that of (F, g_0) and that of $(B, \varepsilon^{-2} g_B)$. If $X, Y, Z \in \Gamma_b(U)$ are vector fields of g_B-length one, then $\varepsilon X, \varepsilon Y, \varepsilon Z$ have length one for $\varepsilon^{-2} g_B$ and

$$\varepsilon^{-2} g_B \left(R(\varepsilon X, \varepsilon Y)\varepsilon Z, R(\varepsilon X, \varepsilon Y)\varepsilon Z\right) = \varepsilon^4 g_B \left(R(X, Y)Z, R(X, Y)Z\right) .$$

Thus the norm of the curvature tensor and its derivatives scales like ε^2.

As to Φ we only prove the bounds for Φ^* as the proof for Φ_* is similar, and actually a bit simpler. We estimate the pointwise value of ${}^\varepsilon\nabla^k \Phi^*$ at $p \in M$ by inserting $k + 1$ vector fields, that have $g_0 \times \varepsilon^{-2} g_B$-length one and are parallel at p, and then estimating the g_ε-length of the resulting vector field T_k^ε.

First let T_k^ε be the vector field obtained from ${}^\varepsilon\nabla^k \Phi^*$ by inserting $k + 1$ vertical vector fields. The vertical part $(T_k^\varepsilon)^F$ can be calculated using only data of the fibre. The covariant derivatives can be expressed, by the Koszul formula, in terms of derivatives of lengths and Lie brackets of vertical vector fields. These Lie brackets are again vertical fields since TF is integrable. Thus no horizontal data and hence no dependence on ε

enters into the calculation, so $(T_k^\varepsilon)^F = T_k^F$ is completely independent of ε.

Now let Y be a vertical field and compute $({}^\varepsilon\nabla_Y T_k^\varepsilon)^H$, which equals $(T_{k+1}^\varepsilon)^H$ since we have chosen the vector fields to be parallel at p, by splitting $T_k^\varepsilon = (T_k^\varepsilon)^F + (T_k^\varepsilon)^H$ into its vertical and horizontal parts. Let $(X_i)_{i\leq d}$ be a g_B-orthonormal frame of U. By the Koszul formula we have

$$\begin{aligned}
&\varepsilon^{-2}(\pi^* g_B)({}^\varepsilon\nabla_Y (T_k^\varepsilon)^F, \varepsilon X_i^*) \\
&= -\frac{1}{2}\big(\varepsilon X_i^* g_F(T_k, Y) + g_F([Y, \varepsilon X_i^*], T_k) + g_F([T_k, \varepsilon X_i^*], Y)\big) \\
&= \varepsilon(\pi^* g_B)(\nabla_Y (T_k)^F, X_i^*)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^{-2}(\pi^* g_B)({}^\varepsilon\nabla_Y (T_k^\varepsilon)^H, \varepsilon X_i^*) \\
&= \varepsilon^{-2} Y(\pi^* g_B)(T_k^\varepsilon, \varepsilon X_i^*) - \varepsilon^{-2}(\pi^* g_B)(T_k^\varepsilon, {}^\varepsilon\nabla_Y \varepsilon X_i^*)\,.
\end{aligned}$$

Using

$${}^\varepsilon\nabla_Y X_i^* = \frac{\varepsilon^2}{2}\sum_{j\leq d} g_F(Y, [X_j^*, X_i^*])X_j^* = \frac{\varepsilon^2}{2}\sum_{j\leq d} g_F(Y, \Omega(X_j, X_i))X_j^*$$

we can combine these to get the expression for $(T_{k+1}^\varepsilon)^H$

$$\begin{aligned}
(T_{k+1}^\varepsilon)^H(p) = \sum_{i\leq d} &\varepsilon^2 g_B(\pi_*\nabla_Y T_k^F, X_i)X_i^* + (Y g_B(\pi_* T_k^\varepsilon, X_i))\, X_i^* \\
&- \varepsilon^2 \sum_{j\leq d}(\pi^* g_B)\big((T_k^\varepsilon)^H, X_j^*\big) g_F\big(Y, \Omega(X_j, X_i)\big) X_i^*\,.
\end{aligned}$$

Now note that $(T_0^\varepsilon)^H = T_0^H = \Phi^* Y = 0$ since $\pi_*\Phi^* = \mathrm{pr}_{1*}$. Hence this relation implies that $g_\varepsilon((T_k^\varepsilon)^H, (T_k^\varepsilon)^H) = \mathcal{O}(\varepsilon^2)$ by induction on k.

If we insert any horizontal field into ${}^\varepsilon\nabla^k\Phi^*$ we get an expression of the form

$$\varepsilon^2 g_\varepsilon({}^\varepsilon\nabla^k_{X,Y_1,\ldots,Y_{k-1}}\Phi^* Y_k, {}^\varepsilon\nabla^k_{X,Y_1,\ldots,Y_{k-1}}\Phi^* Y_k)\,,$$

that can easily be estimated by that for $\varepsilon = 1$, again using the Koszul formula. Therefore ${}^\varepsilon\nabla^k\Phi^*$ is bounded by the same constants for every $\varepsilon \leq 1$. □

A.2 Sobolev spaces and elliptic regularity

Among the technically most important properties of bounded geometry is the existence of an atlas of normal coordinates with a subordinate partition of unity that has convenient properties. This allows for a local definition of Sobolev norms that is consistent with global expressions, satisfying for example

$$\int_M |f|^2 + g(\mathrm{d}f, \mathrm{d}f)\,\mathrm{vol}_g \leq C\,\|f\|^2_{W^1}\ .$$

These Sobolev spaces will be particularly important for the study of elliptic differential operators on M, since the Sobolev norm of their order is essentially the graph-norm. On a manifold with boundary it is of course not sufficient to specify a differential operator, one must also give boundary conditions. Elliptic boundary value problems on a ∂-manifold of bounded geometry are discussed in depth by Schick [62]. Here we reformulate some of the results in our setting and give a short review of their derivation. We also introduce a slightly modified definition of the Sobolev norms, adapted to our ε-dependent family of uniformly locally trivial submersions, and relate it to the original definition.

Lemma A.10. *We may choose an atlas $\mathfrak{U} = \{(U_\nu, \kappa_\nu) : \nu \in \mathbb{Z}\}$ in the following way: Let $r_{\mathfrak{U}} < \frac{1}{3}\min\{r_i, r_i(\partial M), r_c\}$. For every $\nu \in \mathbb{Z}$, U_ν is either empty or*

- *if $\nu < 0$ there is a point $p_\nu \in \partial M$ so that $U_\nu = b_{p_\nu}\big(B(r_{\mathfrak{U}}, 0) \times [0, \frac{1}{2}r_c)\big)$ is the image of the normal collar map* (A.1) *and $\kappa_\nu = b^{-1}_{p_\nu}$;*
- *If $\nu \geq 0$ there is $p_\nu \in M$ with $\mathrm{dist}(p_\nu, \partial M) > \frac{2}{3}r_c$ and $U_\nu = B(r_{\mathfrak{U}}, p_\nu)$. The coordinates κ_ν on U_ν are given by geodesic coordinates at p_ν.*
- *There is $N(\mathfrak{U}) \in \mathbb{N}$, which is the maximum number of patches U_ν with non-empty intersection. That is, for every $\nu \in \mathbb{Z}$ the set $\{\mu \in \mathbb{Z} : U_\mu \cap U_\nu \neq \varnothing\}$ has at most $N(\mathfrak{U})$ elements.*

Subordinate to this atlas we have a partition of unity $\{\chi_\nu : \nu \in \mathbb{N}\}$ with uniformly bounded derivatives, i.e. the set $\{\kappa^*_\nu \chi_\nu : \nu \in \mathbb{Z}\} \subset \mathscr{C}^\infty_0(\mathbb{R}^m)$

is bounded in the Fréchet topology. The multiplicity of this partition is bounded by $N(\mathfrak{U})$*.*

The case without boundary is presented in [65, lemma A1.2, A1.3] and extended to manifolds with boundary in [62, lemma 3.22]. Using this atlas we may define the Sobolev norm of $\psi \in \mathscr{C}_0^\infty(M)$ by

$$\|\psi\|^2_{W^{k,2}(M,g)} := \sum_\nu \|\chi_\nu \kappa_\nu^* \psi\|^2_{W^{k,2}(\kappa_\nu^{-1} U_\nu)} \,. \tag{A.9}$$

The uniformity conditions assure that using a different atlas and partition of unity, satisfying the same conditions, gives equivalent norms with control of the constants [62, lemma 3.24]. The Sobolev space $W^k(M) := W^{k,2}(M)$ is then defined as the completion of $\mathscr{C}_0^\infty(M)$ under this norm and $W_0^k(M)$ is the completion of $\mathscr{C}_0^\infty(M \setminus \partial M)$.

We will use a slightly modified definition that makes the different and ε-dependent scaling of the vertical and horizontal directions in (M, g_ε) more transparent.

Let $\mathfrak{U}$ be an atlas of B with the properties listed above, with $U_\nu = \varnothing$ for $\nu < 0$ because $\partial B = \varnothing$. Let $\{\chi_\nu : \nu \in \mathbb{N}\}$ be the subordinate partition of unity and $\{\Phi_\nu : \nu \in \mathbb{N}\}$ a uniform family of trivialisations $\Phi_\nu \colon \pi^{-1}(U_\nu) \to U_\nu \times F$ in the sense of definition A.3. Then $\{\pi^* \chi_\nu : \nu \in \mathbb{N}\}$ is a partition of unity on M subordinate to the cover $\{\pi^{-1}(U_\nu) : \nu \in \mathbb{N}\}$ and has uniformly bounded derivatives. Let $\{X_i^\nu : i \in \{1, \dots, d\}\}$ be the orthonormal frame of U_ν obtained by parallel transport, of the basis defining the geodesic coordinates, along radial geodesics. This is a set of smooth sections, uniformly bounded in i and ν, because of the bounded geometry. We fix the data $\{U_\nu, \chi_\nu, \Phi_\nu, X_i^\nu : \nu \in \mathbb{N}, i \leq d\}$ for later use and simply refer to it as $\mathfrak{U}$.

Define the Sobolev spaces $W^k(F, g_0)$ as in equation A.9. Using the trivialisation Φ_ν we relate this norm to that on a fibre F_x and define the norms on $W^k(M)$ in form of a direct integral. In this way the ε-scaling may be introduced in a natural way.

Definition A.11. Let ρ_ν^2 be the density $(\Phi_{\nu *} \mathrm{vol}_{F_x})/\mathrm{vol}_{g_0}$ on F. For $\psi \in \mathscr{C}^\infty(F_x)$ and $k \in \mathbb{N}$ put

$$\|\psi\|_{W_\nu^k(F_x)} := \|(\Phi_* \psi) \rho_\nu\|_{W^k(F, g_0)} \,.$$

Let $\alpha \in \mathbb{N}^d$ be a multiindex. Define

$$\|\psi\|^2_{W^k_\varepsilon(M)} := \sum_\nu \sum_{|\alpha|\leq k} \int_{U_\nu} \left\| \varepsilon^{|\alpha|} \prod_{i\leq d} (\Phi_\nu^* X_i^\nu)^{\alpha_i} \chi_\nu \psi \right\|^2_{W_\nu^{k-|\alpha|}(F_x)} \mathrm{vol}_{g_B}(\mathrm{d}x) \tag{A.10}$$

and the *Sobolev space* W^k_ε as the completion of $\mathscr{C}_0^\infty(M)$ under this norm. Define $W^k_{0,\varepsilon}(M)$ as the closure of $\mathscr{C}_0^\infty(M \setminus \partial M)$ in $W^k_\varepsilon(M)$.

The virtue of this definition is that it uses the same coordinate maps for every ε. It is thus clear that these norms are equivalent for different values of ε, albeit with constants that depend on ε. The norms defined in this way are also equivalent to those of (M, g_ε), constructed directly from local coordinates. The constants are independent of ε up to a rescaling by ε^d, since in defining the norm (A.10) we used the volume measure of g_B rather than $\varepsilon^{-2} g_B$.

Lemma A.12. *For every $k \in \mathbb{N}$ there is a constant $C(k, \mathfrak{U}) > 0$ such that for every $\psi \in W^k_\varepsilon(M)$*

$$C^{-1} \|\psi\|_{W^k_\varepsilon(M)} \leq \varepsilon^d \|\psi\|_{W^k(M, g_\varepsilon)} \leq C \|\psi\|_{W^k_\varepsilon(M)} .$$

Proof. Choose coverings $\{(V_i, \tau_i) : i \in I\}$ of (F, g_0) and $\{(W_j^\varepsilon, \kappa_j^\varepsilon) : j \in J\}$ of (M, g_ε) by normal coordinate charts. The latter has data $\{r_c, r_i(\partial M), r_i, C(k) : k \in \mathbb{N}\}$ independent of ε by proposition A.9, so we can choose the subordinate partition of unity with derivatives and multiplicity bounded independently of ε. The coordinate changes $((\exp_{x_\nu}^{\varepsilon^{-2} g_B})^{-1}, \tau_i) \circ \Phi_\nu \circ \kappa_j^{-1}$ induce operators in $\mathscr{L}(W^k(\mathbb{R}^m), W^k(\mathbb{R}^m))$. The norms of these are uniformly bounded in ν, i, j by the definition of the coordinates and the prerequisites on Φ. They are bounded independently of ε since in (A.10) we used $\varepsilon \Phi^* X_i^\nu$, which extends to a g_ε-orthonormal basis at every $p_j \in \pi^{-1}(U_\nu)$, so the change of coordinates is an orthogonal map at first order and higher derivatives are bounded by A.9 and the construction of the coordinates.

Thus if we expand both norms into their local expressions, and rescale the volume measure properly, they are related by globally bounded maps. These local estimates can be patched together again using the calculations of [62, lemma 3.24], because of the bounded multiplicity. $\square$

Remark A.13. In particular $W^0_\varepsilon(M) = L^2(M, g_{\varepsilon=1}) = \mathscr{H}$ with ε-independent, equivalent norms, since

$$\begin{aligned}\|\psi\|^2_{W^0_\varepsilon} &= \sum_\nu \int_B \int_F |\Phi_{\nu *}\chi_\nu \psi|^2 \rho_\nu^2 \operatorname{vol}_{g_0} \operatorname{vol}_B = \sum_\nu \int_M |\chi_\nu \psi|^2 \operatorname{vol}_M \\ &\le \sum_{\mu,\nu} \int_M \chi_\mu \chi_\nu |\psi|^2 \operatorname{vol}_M = \|\psi\|^2_{L^2(M)} ,\end{aligned}$$

and

$$\begin{aligned}\|\psi\|^2_{L^2(M)} &= \sum_{\substack{\mu,\nu;\\ U_\nu \cap U_\mu \neq \varnothing}} \int_M |\psi(x)|^2 \chi_\nu \chi_\mu \operatorname{vol}_M(\mathrm{d}x) \\ &\le \sum_{\substack{\mu,\nu;\\ U_\nu \cap U_\mu \neq \varnothing}} \int_M |\psi(x)|^2 \frac{1}{2}\left(\chi_\nu^2 + \chi_\mu^2\right) \operatorname{vol}_M(\mathrm{d}x) \\ &\le N(\mathfrak{U}) \sum_\nu \int_M \chi_\nu^2 |\psi|^2 \operatorname{vol}_M(\mathrm{d}x) = N(\mathfrak{U}) \|\psi\|^2_{W^0_\varepsilon} .\end{aligned}$$

The following theorem is a reformulation of results of Schick [62] for the special case of the Laplacian and Dirichlet boundary conditions.

Theorem A.14 ([62])**.** *Let $\psi \in W^2_\varepsilon(M) \cap W^1_{\varepsilon,0}$ and $\Delta_{g_\varepsilon}\psi \in W^k_\varepsilon(M)$. Then $\psi \in W^{k+2}_\varepsilon(M)$ and there is a constant $C(k) > 0$ such that*

$$\|\psi\|^2_{W^{k+2}_\varepsilon} \le C(\|\Delta_{g_\varepsilon}\psi\|^2_{W^k_\varepsilon} + \|\psi\|^2_{\mathscr{H}}) .$$

The main statement of the theorem is given in [62, theorem 4.15]. The condition there is that Δ_{g_ε} be uniformly elliptic in the sense of an elliptic boundary value problem [62, definition 4.7]. The boundary operator for the Dirichlet Laplacian is given by

$$p : W^2_\varepsilon(M) \to W^{3/2}_\varepsilon(\partial M) , \qquad \psi \mapsto \psi|_{\partial M} .$$

The ellipticity condition requires existence of local fundamental solutions to the boundary value problem $\Delta_{g_\varepsilon}\psi = f$, $\psi|_{\partial M} = g$, that satisfy global bounds. For our application we also need these bounds to be independent of ε. Uniform ellipticity of this boundary value problem is discussed in [62, proposition 5.14]. The point is that, written in normal coordinates, the local problems give a family of boundary value problems on $B(r, 0)$ and $B(r, 0) \times [0, r_c/2)$. The coefficients of these differential operators depend only on the expressions g^{ij} in normal coordinates and form a bounded set in $\mathscr{C}^\infty(\mathbb{R}^m)$. Thus this family consists of uniformly elliptic boundary value problems that have common bounds on coefficients and ellipticity constants (because $g^{ij}(0) = \delta^{ij}$ in all of these systems). These bounds depend only on the local expressions, so by proposition A.9 they can be chosen independently of ε.

The regularity result is then obtained by proving regularity of the fundamental solutions [62, lemma 4.10], patching together local estimates and finally interpolation for Sobolev norms [62, theorem 4.15]. These steps all explicitly preserve the uniformity of the constants in ε.

Appendix B

Constructions of vector bundles

B.1 Vector bundles of functions associated with a fibration

We review some constructions of vector bundles of possibly infinite rank. For this we will need the notion of an infinite dimensional manifold, in which we mostly follow Lang [45]. However we will also allow manifolds modelled on normed spaces that are not complete and we use a slightly weaker definition of vector bundle. At this level of basic constructions the proofs easily carry over to this case and nothing is changed. Although we generally only work with smooth objects we need to make an exception here because in infinite dimensional vector spaces differentiability is too strong of a property.

Definition B.1. Let M be a topological (smooth) manifold of finite dimension and X a normed vector space. A *continuous (smooth) vector bundle over* M with fibre X is a manifold E and a continuous (smooth) map $\pi\colon E \to M$ with the following properties:

1) There exist an open covering $(U_i)_{i\in I}$ of M and homeo- (diffeo-) morphisms φ_i for which the diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\;\varphi_i\;} & U_i \times X \\ & {\scriptstyle \pi}\searrow \qquad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U_i & \end{array}$$

commutes.

2) For $p \in U_i \cap U_j$ the map $\varphi_i \circ \varphi_j^{-1}|_{\{p\}\times X}$ is a linear isomorphism of X.

Note that this does not fix a norm on the fibres E_p but only the topology. Different trivialisations yield equivalent but not identical norms.

This definition clearly implies continuity of the transition maps

$$\varphi_i \circ \varphi_j^{-1} \colon (U_i \cap U_j) \times X \to (U_i \cap U_j) \times X \,.$$

As a consequence, the map $p \mapsto (\varphi_i \varphi_j^{-1})(p, \cdot)$ is strongly continuous. If X is a Banach space the uniform boundedness principle makes these two notions of continuity equivalent. It is weaker however than the continuity of $p \mapsto (\varphi_i \varphi_j^{-1})(p)$ in the topology of the operator norm.

The transition functions uniquely determine a vector bundle:

Lemma B.2 ([45])**.** *Let $(U_i)_{i\in I}$ be an open covering of M. Suppose we have continuous (smooth) maps*

$$\varphi_{ij} \colon (U_i \cap U_j) \times X \to (U_i \cap U_j) \times X \tag{B.1}$$

that commute with the projection onto the first factor and satisfy the cocycle condition

$$\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \,. \tag{B.2}$$

Then there exists a unique vector bundle $\pi \colon E \to M$ that has a trivialising covering (U_i, φ_i) with these transition functions.

One can construct many vector bundles using operations on vector spaces such as forming duals or tensor products. This may be formalized by showing that functors on the category of vector spaces carry over to bundles over a fixed manifold. Let $\mathcal{F}$ be a functor of normed vector spaces. we say that $\mathcal{F}$ is continuous (smooth) if it preserves this property for morphisms. More precisely (for a functor of one variable) let U be a manifold, X, Y normed vector spaces and f a continuous (smooth) map

$$f \colon U \times X \to Y \tag{B.3}$$

with $f(p) \in \mathscr{L}(X, Y)$. $\mathcal{F}$ is continuous (smooth) if the map $\mathcal{F}(f)$, defined by $\mathcal{F}\big(f\big)(p) := \mathcal{F}\big(f(p)\big)$, is continuous (smooth).

Proposition B.3 ([45]). *Let $\mathcal{F}$ be a continuous (smooth) functor. Then for every manifold M there exists a unique functor $\mathcal{F}_M$ on vector bundles over M satisfying:*

1) $\mathcal{F}_M(E)_p = \mathcal{F}(E_p)$.

2) $\mathcal{F}_M\big(f\big)(p) = \mathcal{F}\big(f(p)\big)$.

3) If E is trivial, so is $\mathcal{F}_M(E)$.

4) If $\varphi\colon N \to M$ is a continuous (smooth) map, then

$$\mathcal{F}_N(\varphi^* E) = \varphi^* \mathcal{F}_M(E)\,. \tag{B.4}$$

For a bundle $\pi\colon E \to M$ and a continuous (smooth) bundle map f.

Of course this also holds for functors of severable variables.

The fibre bundle structure $M \xrightarrow{\pi} B$ induces a vector bundle $\tilde{\pi}\colon E \to B$ whose fibres are isomorphic to $\mathscr{C}^\infty(F)$. Take $U \subset B$ with a trivialisation $\Phi\colon \pi^{-1}(U) \to U \times F$ and define transition functions

$$f \mapsto f \circ \Phi_i \circ \Phi_j^{-1}\,. \tag{B.5}$$

Choosing a norm on $\mathscr{C}^\infty(F)$ for which these are continuous (smooth) uniquely determines E as a continuous (smooth) vector bundle. The bundle charts $\tilde{\Phi}\colon U \times \mathscr{C}^\infty(F) \to \tilde{\pi}^{-1}(U)$ are given by $(x, f) \mapsto f \circ \Phi|_{F_x}$. It is easy to see that these maps are continuous for a variety of norms (e.g. $\|\cdot\|_\infty$, $\|\cdot\|_{L^2}$). Their derivatives however include derivatives of f, so they do not define bounded operators on these spaces. Thus we obtain continuous vector bundles for these norms and denote them by $\mathscr{C}^\infty(F, \|\cdot\|\,;\pi)$, or just $\mathscr{C}^\infty(F;\pi)$ whenever the norm is obvious or irrelevant. We will want to complete $\mathscr{C}^\infty(F)$ with respect to various norms in order to obtain bundles with different function spaces as fibres.

Lemma B.4. *Let X be a separable normed vector space. The completion $X \mapsto \bar{X}$ defines a continuous functor.*

Proof. Completion is a functor since for normed vector spaces X, Y and a continuous linear map $f: X \to Y$ there is a unique continuous extension $g: \bar{X} \to \bar{Y}$ to the completions.

Let Y be a normed space and $f: U \times X \to Y$ continuous with $f(p) \in \mathscr{L}(X, Y)$. We need to show that the extension g of f to the completion $\bar{X}$ remains continuous. By the uniform boundedness principle this is equivalent to strong continuity of the map $g: U \to \mathscr{L}(\bar{X}, Y)$. Choose a sequence $p_n \in U$ converging to $p \in U$. Let $x \in \bar{X}$ be arbitrary. For every $\varepsilon > 0$ we find $\tilde{x} \in X$ with $\|x - \tilde{x}\|_X < \varepsilon$ and we have

$$\|(g(p_n) - g(p))\, x\|_Y \leq \|(g(p_n) - g(p))\,(x - \tilde{x})\|_Y + \|(f(p_n) - f(p))\, x\|_Y \,.$$

Since f is continuous we may choose n large enough for the second term to be less than ε. Thus it is sufficient to prove that $g(p_n)$ is bounded.

Suppose there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ with $\|g(p_{n_k})\|_{\mathscr{L}(\bar{X},Y)} > k^2$. Then for every k we can choose $x_k \in X$ with unit norm such that $\|f(p_{n_k}, x_k)\|_X = \|g(p_{n_k})x_k\|_X \geq k^2$. Now the sequence $z_k := k^{-1}x_k$ converges to zero, but $\|f(p_{n_k}, z_k)\|_X \geq k$, so this does not converge to $f(p, 0) = 0$ in contradiction to the continuity of f. □

This allows us to construct various vector bundles using B.3. The ones we will use most frequently are $L^2(F;\pi) =: \mathscr{H}_F$ and the completions with respect to the Sobolev norms, $W^m(F;\pi)$ and $W^m_0(F;\pi)$. Another important construction is that of continuous linear maps.

Lemma B.5. *Let X, Y be Banach spaces. The functor $(X, Y) \mapsto \mathscr{L}(X, Y)$ is continuous.*

Proof. Let Z, Z' be Banach spaces, $f: U \times Z \to X$ and $g: U \times Y \to Z'$ continuous functions with $f(p) \in \mathscr{L}(Z, X)$, $g(p) \in \mathscr{L}(Y, Z')$. The induced map $(f, g): U \times \mathscr{L}(X, Y) \to \mathscr{L}(Z, Z')$ is continuous by the equation

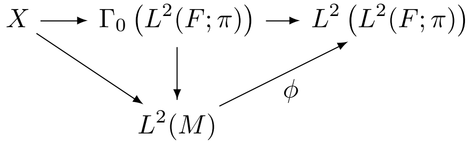
$$\begin{aligned} g(p_n)Tf(p_n) - g(p)Tf(p) = g(p_n)T\big((f(p_n) - f(p)\big) \\ - \big(g(p_n) - g(p)\big)Tf(p)\,, \end{aligned}$$

for fixed $T \in \mathscr{L}(X, Y)$, and the uniform boundedness principle. □

When the structure is induced by a fibre bundle we have the equivalent notations $\mathscr{L}(X, Y; \pi) = \mathscr{L}((X; \pi), (Y; \pi))$. Functions on M can alternatively be viewed as sections of vector bundles, the fibres of which are spaces of functions on F. For example:

Corollary B.6. *We have an isomorphism* $L^2\left(L^2(F; \pi)\right) \cong L^2(M)$.

Proof. Let $X = \Gamma_0(\mathscr{C}(F; \pi))$ be the space of continuous, compactly supported sections of $\mathscr{C}\left(F, \|\cdot\|_{L^2}; \pi\right)$. Obviously X and $\Gamma_0\left(L^2(F; \pi)\right)$ can be embedded into $L^2(M)$ as dense subspaces. Now these maps can be uniquely extended to maps on the completions $L^2(M)$ and $L^2\left(L^2(F; \pi)\right)$ of X and $\Gamma_0\left(L^2(F; \pi)\right)$, respectively. This gives us the diagram



and ϕ is an isomorphism because ϕ and ϕ^{-1} are given as the unique extensions to the completion of the different embeddings. □

B.2 The eigenspace bundle $\mathcal{E}$

Here we prove regularity of the eigenfunctions of H_F, as functions on $\pi^{-1}(U)$ for $U \subset B$, in the presence of a spectral gap (condition 3). This leads to regularity of the eigenspace bundle $\mathcal{E}$ spanned by these functions.

Proposition B.7. *If λ satisfies the gap condition the eigenspace bundle $\mathcal{E}$ has a differentiable structure such that $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M, \mathbb{C})$. With this structure it is a bundle of bounded geometry over B.*

Proof. We show that $\mathcal{E}$ has such a differentiable structure in lemma B.9, so for now take this as given.

Since $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M, \mathbb{C})$ we have a smooth metric connection given by $\nabla^{\mathcal{E}} := \nabla^B - \frac{1}{2}\bar{\eta}$ (cf. section 2.2.1). The curvature of this connection is is given by

$$R^{\mathcal{E}}(X, Y) = P_0 \Omega(X, Y) P_0 + P_0\big[[Y^*, P_0], [X^*, P_0]\big] P_0 \\ - \tfrac{1}{2}\left(\nabla_X \bar{\eta}\right)(Y) + \tfrac{1}{2}\left(\nabla_Y \bar{\eta}\right)(X).$$

By A.6 and 2.8 this is bounded with all its derivatives. This implies existence of trivialisations with bounded transition functions, that is bounded geometry of $\mathcal{E}$, as shown in [28, theorem 36]. □

We now need to show the existence of a differentiable structure with $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M, \mathbb{C})$. This amounts to proving that locally we can choose eigenfunctions ϕ_k of H_F with eigenvalue $\lambda(x)$ that are smooth functions on $\pi^{-1}(U)$. Such eigenfunctions are always elements of $\mathscr{C}^\infty(F_x)$ for fixed x by elliptic regularity, so we need to establish differentiability in the horizontal directions, using the regularity of P_0. As a tool we need the following, unpublished, lemma due to Wachsmuth.

Lemma B.8 ([70]). *Let $f \in L^2(\mathbb{R}^d \times \mathbb{R}^n)$ and*

$$f \in L^\infty(\mathbb{R}^d, \mathscr{C}^{0,1}(\mathbb{R}^n)) \cap \mathscr{C}^{0,1}(\mathbb{R}^d, L^2(\mathbb{R}^n)) .$$

Then there is $\tilde{f} \in \mathscr{C}(\mathbb{R}^d \times \mathbb{R}^n)$ which coincides with f almost everywhere.

Proof. Let $g_k := (k\sqrt{\pi})^{-1}\mathrm{e}^{-x^2/k^2} \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n)$, which converges weakly to the distribution δ_0 for $k \to \infty$. Let $f_k := f \star g_k$ be the convolution. Then it is well known that $f_k \in \mathscr{C}(\mathbb{R}^d \times \mathbb{R}^n)$ and $f_k \to f$ in L^2. This means that a subsequence, again denoted by f_k, converges pointwise almost everywhere. We prove that locally

$$\|f_k\|_{\mathscr{C}^{0,\alpha}(\mathbb{R}^d \times \mathbb{R}^n)} \leq C$$

with $\alpha = (1 + n/2)^{-1}$. Then the Arzelà-Ascoli theorem implies that f_k converges uniformly to $\tilde{f} \in \mathscr{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^n)$ and since the pointwise limit is unique, $f = \tilde{f}$ almost everywhere.

Now to show boundedness in the Hölder norm first note that

$$\|f_k\|_{L^\infty(\mathbb{R}^d, \mathscr{C}^{0,1}(\mathbb{R}^n))} \leq C_1 , \qquad \|f_k\|_{\mathscr{C}^{0,1}(\mathbb{R}^d, L^2(\mathbb{R}^n))} \leq C_2$$

uniformly, by standard estimates on the convolution. Let

$(x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}^n$ be arbitrary. Using the triangle inequality we obtain

$$\begin{aligned}
&\sup_{(x,y)} \frac{f_k(x,y) - f_k(x_0,y_0)}{(|x-x_0|^2 + |y-y_0|^2)^{2/\alpha}} \\
&\leq \sup_{(x,y)} \left(\frac{|f_k(x,y_0) - f_k(x,y_0)|}{(|x-x_0|^2 + |y-y_0|^2)^{2/\alpha}} + \frac{|f_k(x,y_0) - f_k(x_0,y_0)|}{(|x-x_0|^2 + |y-y_0|^2)^{2/\alpha}} \right) \\
&\leq \sup_{(x,y)} \frac{|f_k(x,y_0) - f_k(x,y_0)|}{|y-y_0|^\alpha} + \sup_x \left(\frac{|f_k(x,y_0) - f_k(x_0,y_0)|^{1/\alpha}}{|y-y_0|} \right)^\alpha .
\end{aligned}$$

The first term is bounded by the $\mathscr{C}^{0,\alpha}(\mathbb{R}^n)$-norm for fixed x, so it is bounded by C_1. The second term can be bounded by C_2^α, if we can control the numerator by the L^2-integral over $\mathbb{R}^n$.

Define $F(x,y) := |f_k(x,y) - f_k(x_0,y)|$ and $\delta = F(x,y_0)$. Then notice that F is uniformly Lipschitz, $|F(x,y) - F(x,y_0)| \leq 2C_1 |y-y_0|$. Thus $F(x,y) \geq \delta/2$ on the ball $B(\delta/(4C_1), y_0) \subset \mathbb{R}^n$. Then we have

$$\begin{aligned}
F(x,y_0)^{1/\alpha} = \delta^{1+n/2} &\leq C_3 \delta \operatorname{Vol}(B(\delta/(4C_1), y_0))^{1/2} \\
&\leq 2C_3 \left(\int_{B(\delta/(4C_1),y_0)} F(x,y)^2 \, \mathrm{d}y \right)^{1/2} \\
&\leq 2C_3 \|F(x,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&\leq 4C_3 C_2 \, ,
\end{aligned}$$

with C_3 depending only on C_1 and the volume of the unit ball in $\mathbb{R}^n$. This yields the claim, as explained above. □

We apply this lemma locally to an appropriate choice of eigenfunctions of H_F over $U \subset B$ to prove regularity of $\mathcal{E}$.

Lemma B.9. *For every $x_0 \in B$ there is $r > 0$ and functions $\{\phi_j \in \mathscr{C}^\infty(\pi^{-1}B(r,x_0)) : 1 \leq j \leq \operatorname{rank}(\mathcal{E})\}$ that span $\mathcal{E}_x$ for very $x \in B(r,x_0)$. These functions give a family of trivialisations that determine a differentiable structure of $\mathcal{E}$ for which $\Gamma(\mathcal{E}) \subset \mathscr{C}^\infty(M,\mathbb{C})$.*

Proof. Let $R < r_i(B)$, $x_0 \in B$, $U := B(R,x_0)$ and $\Phi\colon \pi^{-1}U \to U \times F$ be a trivialisation with bounded derivatives in the sense of A.3. As in lemma 2.7 let $W\colon L^2(U \times F) \to L^2(\pi^{-1}U)$ be the unitary given by composition with Φ. Now for fixed $x \in U$ the image of $P(x) = W^* P_0 W$ consists of eigenfunctions of $-\Delta_{g_{F_x}} + \Phi_* V$, with Dirichlet boundary conditions, with eigenvalue λ. These are smooth functions on F by elliptic regularity A.14. The x-dependent family of operators has uniform bounds on their coefficients and ellipticity constants, when expressed in a fixed system of normal coordinates $\{(V_i, \kappa_i) : i \in I\}$ for (F, g_0). Thus these eigenfunctions form a bounded set in $\mathscr{C}^\infty(F)$. If we can choose such an eigenfunction with $\phi \in \mathscr{C}^{0,1}(U, L^2(F))$, local application of lemma B.8 should give $\phi \in \mathscr{C}(U \times F)$.

Let $\Phi^*\phi \in \mathcal{E}_{x_0}$ have unit norm and introduce geodesic normal coordinates at x_0. Since $P(x)$ is continuous, as shown in lemma 2.9, we can adjust R to ensure that $\|(P(x) - P(x_0))\,\phi\| \leq 1/2$ for $\operatorname{dist}(x, x_0) \leq R$. Now define

$$\phi(x) = P(x)\phi\,\|P(x)\phi\|^{-1}_{L^2(F, g_{F_x})}\,,$$

which is a section of $\Phi_*\mathcal{E}_U$ of unit norm. Observe that because $P_0 \in \mathcal{A}^0$, by lemma 2.8, $\phi(x) \in \mathscr{C}^\infty(U, L^2(F))$.

Choose $r > 0$ small enough that $r < r_i(B)/2$, $\|(P(x) - P(x_0))\,\phi\| \leq 1/2$ for $\operatorname{dist}(x, x_0) \leq 2r$ and $B(r, y) \subset V_i$ for every $y \in F$ and some i for the chosen normal coordinate system (cf. [62, lemma 3.19] for this property of normal coordinates). For $x \in B(r, x_0)$ and any $y \in F$ we take coordinates $\tilde{\kappa} := (\exp_x^{-1}, \kappa_i)$ on $B(r, (x, y)) \subset U \times F$. Map ϕ to $\tilde{\kappa}_*\phi$ and if necessary extend it past past the boundary to a smooth function. Then choose a function $f \in \mathscr{C}_0^\infty(\mathbb{R}^{d+n})$ with support in the ball of radius r and equal to one in the ball of radius $r/2$. The product $f(v, w - \kappa_i(y))\tilde{\kappa}_*\phi(v, w)$ yields a function that satisfies the conditions of lemma B.8 and equals $\tilde{\kappa}_*\phi$ in the ball of radius $r/2$ around $(0, \kappa_i(y))$. Thus ϕ is continuous at (x, y). Since (x, y) was an arbitrary point we obtain $\phi \in \mathscr{C}(B(r, x_0) \times F)$.

In order to prove differentiability of ϕ we will need to obtain these properties for all the partial derivatives of ϕ, expressed in some coordinate system of $U \times F$. Let $1 \leq i \leq d$ and let ∂_{x^i} be a coordinate vector field of the normal coordinates on U. Then we have

$$(W^* H_F W - \lambda))\,\partial_{x^i}\phi(x) = [W^* H_F W, \partial_{x^i}]\phi(x) + (\partial_{x^i}\lambda)\phi(x)\,.$$

Note that in the proof of lemma 2.7 we see that $[W^* H_F W, \partial_{x^i}]$ is a smooth differential operator on F and that $\partial_{x^i}\phi(x) = 0$ on ∂F. $P_0 \in \mathcal{A}^0_H$ (lemma 2.8) implies that $\partial_{x^i}\phi \in W^2(F)$. Thus $\partial_{x^i}\phi(x)$ is a solution of the Dirichlet problem $\left(-\Delta_{g_{F_x}} + \Phi_* V - \lambda\right) f = g$ in F, $f = 0$ on ∂F, with a smooth function g. Consequently $\partial^i_x \phi \in L^\infty(U, \mathscr{C}^m(F))$ for every $m \in \mathbb{N}$, because we have bounds on $[W^* H_F W, \partial_{x^i}]$ and $\partial_{x^i}\lambda$ from 2.7, 2.9.

Now let $\alpha \in \mathbb{N}^d$ be a multiindex, then since $(W^* H_F W - \lambda)\phi = 0$ we have

$$(W^* H_F W - \lambda))\, \partial^\alpha_x \phi = [(W^* H_F W - \lambda))\, , \partial^\alpha_x]\phi\,, \tag{B.6}$$

where the right hand side may be expressed by derivatives of ϕ of order strictly less than $|\alpha|$, iterated commutators with $W^* H_F W$ and derivatives of λ. Thus by induction on the order of α we get $\partial^\alpha_x \phi \in L^\infty(U, \mathscr{C}^m(F))$ for every $m \in \mathbb{N}$.

Now assume we have $\phi \in \mathscr{C}^m(B(r, x_0) \times F)$ for some $m \in \mathbb{N}$. We will show that $\phi \in \mathscr{C}^{m+1}(B(r, x_0) \times F)$ by proving continuity of the partial derivatives of order $m + 1$.

First let $\alpha \in \mathbb{N}^d$ be a multiindex of order m and $1 \le i \le d$. Then $\partial^i_x \partial^\alpha_x \phi$ exists because $\phi \in \mathscr{C}^\infty(U, L^2(F))$ and defines an element of $L^\infty(U, \mathscr{C}^1(F))$ because it solves the differential equation (B.6). Thus we may argue as for continuity and obtain $\partial^i_x \partial^\alpha_x \phi \in \mathscr{C}(B(r, x_0) \times F)$.

Next take $y \in F$ and denote by ∂_{y^j} the coordinate vector fields of (V, κ) with $B(r, y) \subset V$. Let $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^n$ be multiindices with $|\alpha| + |\beta| = m$. Since $\phi \in \mathscr{C}^m(B(r, x_0) \times F)$ we may rearrange any derivative of order m into the form $\partial^\beta_y \partial^\alpha_x \phi$. Let $1 \le j \le n$, then $\partial_{y^j} \partial^\beta_y \partial^\alpha_x \phi \in L^\infty(U, \mathscr{C}^1(V))$, since $\partial^\alpha_x \phi \in L^\infty(U, \mathscr{C}^{2+|\beta|}(F))$. We now have to show that the expression is Lipschitz in x, in the L^2-sense. Let $x_1 \in U$ and calculate

$$\begin{aligned}
&\limsup_{x \to x_1} \frac{\left\| \partial_{y^j} \partial^\beta_y \partial^\alpha_x \phi(x) - \partial_{y^j} \partial^\beta_y \partial^\alpha_x \phi(x_1) \right\|_{L^2(V)}}{|x - x_1|} \\
&\le \left\| \partial_{y^j} \partial^\beta_y \right\|_{\mathscr{L}(W^{|\beta|+1}(V), L^2(V))} \limsup_{x \to x_1} \left\| \frac{\partial^\alpha_x \phi(x) - \partial^\alpha_x \phi(x_1)}{x - x_1} \right\|_{W^{|\beta|+1}(F)} .
\end{aligned}$$

This is bounded since $\partial^\alpha_x \phi$ is differentiable in x and its derivative is an element of $L^\infty(U, W^k(F))$ (for arbitrary $k \in \mathbb{N}$) by elliptic regularity A.14 and equation (B.6). Thus $\partial_{y^j} \partial^\beta_y \partial^\alpha_x \phi \in \mathscr{C}^{0,1}(U, L^2(V))$ and we can again apply lemma B.8 to show that this expression is continuous on $B(r, x_0) \times V$.

This is sufficient to show $\phi \in \mathscr{C}^{m+1}(B(r, x_0) \times F)$ by Schwarz' theorem, hence ϕ is smooth. Now choose an orthonormal basis $\{\phi_j : 1 \leq j \leq k\}$ of the range of $P(x_0)$. We can construct the functions $\phi_j \in \mathscr{C}^\infty(B(r, x_0) \times F)$ element by element. The resulting functions span the image of $P(x)$ since it can be mapped unitarily to that of $P(x_0)$ using the Sz.-Nagy construction (2.12). Finally, pulling the functions back to $\pi^{-1}B(r, x_0)$ with Φ preserves smoothness. $\square$

Appendix C

Technical lemmata

Energy cut-offs

Here we provide key tools for dealing with energy cut-offs. These need to sufficiently regular, in particular they should not have zeros of finite order.

Definition C.1. A function $f : \mathbb{R} \to [0, 1]$ is a *regular cut-off* if for every $s \in (0, \infty)$ the power $f^s \in \mathscr{C}_0^\infty(\mathbb{R})$.

Our treatment of these cut-offs relies on the Helffer-Sjöstrand formula for the functional calculus (see [13, chapter 2]). Let $f \in \mathscr{C}_0^\infty(\mathbb{R})$. We choose an extension $\tilde{f}$ of f to $\mathbb{C}$ with support in $\mathbb{R} \times [-\mathrm{i}, +\mathrm{i}]$, satisfying $|\partial_{\bar{z}} \tilde{f}| \le C(f)(\Im z)^3$ and call it an almost analytic extension. The functional calculus is then given by

$$f(H) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} (H - z)^{-1} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \, . \tag{C.1}$$

Lemma C.2. *Let H be self-adjoint on $D(H) \subset \mathscr{H}$. Let $T \in \mathscr{L}(\mathscr{H}) \cap \mathscr{L}(D(H))$ be self-adjoint on $\mathscr{H}$. If χ is a regular cut-off and*

$$\|[T, H]\|_{\mathscr{L}(D(H), \mathscr{H})} = \mathcal{O}(\varepsilon)$$

$$\|[T, H]\chi^s(H)\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^k) \, ,$$

for some $k \in \mathbb{N}$ and all $s \in (0, \infty)$, then

1) $\|[T, \chi(H)]\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^k)$*;*

2) If additionally T is a projection

$$\|\chi(THT) - T\chi(H)T\|_{\mathscr{L}(\mathscr{H}, D(H))} = \mathcal{O}(\varepsilon^k) \, .$$

Proof of 1. Using the second hypothesis and the Helffer-Sjöstrand formula (C.1) we get (see [69, lemma 3.5a] for details)

$$\|[T,\chi^s(H)]\chi^s(H)\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^k) \tag{C.2}$$

for every $s > 0$. Now

$$\begin{aligned}&\|[T,\chi^s(H)]\|_{\mathscr{L}(\mathscr{H})}\\ &\qquad = \|[T,\chi^{s/2}(H)]\chi^{s/2}(H) + \chi^{s/2}(H)[T,\chi^{s/2}(H)]\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^k)\,,\end{aligned}$$

since the first term is of this order by equation (C.2) and the second term equals minus its adjoint. This implies

$$\begin{aligned}&\|\chi^s(H)[T,\chi^s(H)]\|_{\mathscr{L}(\mathscr{H},D(H))}\\ &\qquad \leq \|\chi^s\|_{\mathscr{L}(\mathscr{H},D(H))}\,\|[T,\chi^s(H)]\|_{\mathscr{L}(\mathscr{H})} = \mathcal{O}(\varepsilon^k)\,.\end{aligned}$$

Setting $s = 1/2$ here and in (C.2) proves the claim. □

Proof of 2. Let $H^D = THT + (1-T)H(1-T)$. By the first condition $\|H^D - H\|_{\mathscr{L}(D(H),\mathscr{H})} \leq \alpha < 1$ for ε small enough. This implies that H^D is self-adjoint on $D(H)$ and the induced norms are equivalent (see [68, lemma 3]) with

$$c(\alpha)^{-1}\,\|\psi\|_{D(H)} \leq \|\psi\|_{D(H^D)} \leq c(\alpha)\,\|\psi\|_{D(H)}\,.$$

The Helffer-Sjöstrand formula (C.1) together with the resolvent formula (2.6) gives us

$$\begin{aligned}&\chi(THT) - T\chi(H)T\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde\chi\Big(T\left((THT-z)^{-1} - (H-z)^{-1}\right)T\Big)\,\mathrm{d}z\wedge\mathrm{d}\bar z\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde\chi\Big(\,(THT-z)^{-1}\,(H-THT)\,(H-z)^{-1}\,T\Big)\,\mathrm{d}z\wedge\mathrm{d}\bar z\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde\chi\Big(\,(THT-z)^{-1}\,T\,[T,H]\,(H-z)^{-1}\,T\Big)\,\mathrm{d}z\wedge\mathrm{d}\bar z\,.\end{aligned} \tag{C.3}$$

As an intermediate step we use this to deduce

$$\|\chi(THT) - T\chi(H)T\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^2)\,. \tag{C.4}$$

Since T is a projection, $T\,[H,T]\,T = 0$ (cf. 2.12), so
the operator under the integral equals

$$\begin{aligned} T\,[T,H]\,(H-z)^{-1}\,T &= T\,[T,H]\,[(H-z)^{-1}\,,T]T \\ &= T\,[T,H]\,(H-z)^{-1}\,[T,H]\,(H-z)^{-1}\,T \\ &= T\big(([T,H]\,(H-z)^{-1}\big)^2 T \end{aligned}$$

Because of the equivalence of norms induced by H and H^D we have

$$\begin{aligned} \|(THT-z)^{-1}\,T\|^2_{\mathscr{L}(\mathscr{H},D(H))} &\leq c(\alpha)^2 \|\big(H^D - z\big)^{-1}\,T\|^2_{\mathscr{L}(\mathscr{H},D(H^D))} \\ &\leq c(\alpha)^2\big(2 + (1+2\,|z|^2)/\Im z^2\big)\,. \end{aligned}$$

Thus

$$\begin{aligned} &\|(THT-z)^{-1}T\,[T,H]\,(H-z)^{-1}\,T\|_{\mathscr{L}(\mathscr{H},D(H))} \\ &\quad = \big\|\,(THT-z)^{-1}\,T\,\big([T,H]\,\big(H-z\big)^{-1}\,\big)^2\big\| \\ &\quad \leq \|(THT-z)^{-1}\,T\|^2_{\mathscr{L}(\mathscr{H},D(H))}\|[T,H](H-z)^{-1}\|^2_{\mathscr{L}(\mathscr{H})} \\ &\quad \leq C\varepsilon^2\big(2 + (1+2\,|z|^2)/\Im z^2\big)^{3/2}\,. \end{aligned}$$

So with the properties of $\tilde{\chi}$ the integrand is bounded by a constant times ε^2 and has compact support, which gives the a bound of $\mathcal{O}(\varepsilon^2)$ on the integral (C.3). In the presence of a cut-off χ we can use the same reasoning together with the second hypothesis to deduce

$$\|(\chi(THT) - T\chi(H)T)\,\chi^s(H)\|_{\mathscr{L}(\mathscr{H},D(H))} = \mathcal{O}(\varepsilon^k)\,, \tag{C.5}$$

for every $s > 0$. The same estimate also holds with the reverse order of the terms as $\chi^s(H)[T,H] = -\,([T,H]\chi^s(H)))^* = \mathcal{O}(\varepsilon^k)$.

Now take $m \geq k/2$ and let $A := \chi^{1/m}(THT)$, $B := T\chi^{1/m}(H)T$. Application of equations (C.4), (C.5) to these functions gives $A - B = \mathcal{O}(\varepsilon^2)$ and $(A-B)B = \mathcal{O}(\varepsilon^k)$. Now

$$\begin{aligned} (A-B)^m &= \mathcal{O}(\varepsilon^{2m}) \\ &= (A-B)^{m-1}\,A - \underbrace{(A-B)^{m-1}\,B}_{=\mathcal{O}(\varepsilon^k)} \\ &= (A-B)\,A^{m-1} + \mathcal{O}(\varepsilon^k)\,. \end{aligned}$$

So because $2m \geq k$ we have $A^m = BA^{m-1} + \mathcal{O}(\varepsilon^k)$ which means

$$\chi(THT) = T\chi^{1/m}(H)T\chi^{(m-1)/m}(THT) + \mathcal{O}(\varepsilon^k)\,.$$

Application of (C.5) with $s = 1/m$ to $\chi^{1/m}(H)\chi^{(m-1)/m}(THT)$ and part one of the lemma finally gives

$$\begin{aligned}\chi(THT) &= T\chi^{1/m}(H)T\chi^{(m-1)/m}(H)T + \mathcal{O}(\varepsilon^k)\\ &= T\chi(H)T + T\underbrace{\left[\chi^{1/m}(H),T\right]}_{=\mathcal{O}(\varepsilon^k)}\chi^{(m-1)/m}(H)T + \mathcal{O}(\varepsilon^k)\end{aligned}$$

□

Energy-dependent derivative estimates

This lemma proves the α-dependent estimates on derivatives needed in section 3.2. We use here the notation and conditions of that section.

Lemma C.3. *Let $X \in \Gamma_b(TB)$, $A \in \{H, H_a\}$ and denote by $D^2_\alpha(A)$ the domain of $\varepsilon^{-2\alpha}A^2$ with the graph-norm. There exists a constant $C > 0$ depending on the bounds on X and its derivatives such that*

$$\|\varepsilon P_0 X^*\|_{\mathscr{L}(D^2_\alpha(A),D(H))} \leq C\varepsilon^{\alpha/2}\,.$$

Additionally, the operator H_1 satisfies $\|H_1 P_0\|_{\mathscr{L}(D^2_\alpha(A),D(H))} = \mathcal{O}(\varepsilon^{\alpha/2})$.

Proof. 1) The case $A = H$:

Ignore the bounded operator P_0 for the moment. For $\psi \in D^2_\alpha(H)$ we have

$$\begin{aligned}\|\varepsilon X^*\psi\|^2_{\mathscr{H}} &= \int_M \varepsilon^2\,|X^*\psi|^2\,\mathrm{vol}_M = \int_M |\pi^* g_B(X^*, \mathrm{grad}\psi)|^2 \mathrm{vol}_M\\ &\leq \int_M g_B(X,X)\varepsilon^2\pi^* g_B(\mathrm{d}\bar{\psi},\mathrm{d}\psi)\mathrm{vol}_M\\ &\leq \|g_B(X,X)\|_\infty \int_M \varepsilon^2\pi^* g_B(\mathrm{d}\bar{\psi},\mathrm{d}\psi)\mathrm{vol}_M\\ &= \|g_B(X,X)\|_\infty \int_M \bar{\psi}(-\varepsilon^2\Delta_h\psi)\mathrm{vol}_M\,. \end{aligned} \tag{C.6}$$

Since $H_F \geq 0$ we also get

$$\left\langle \psi, -\varepsilon^2 \Delta_h \psi \right\rangle \leq |\langle \psi, H\psi \rangle| + \varepsilon \, |\langle \psi, H_1 \psi \rangle| \; .$$

We take ψ satisfying $\|\psi\|^2_{D^2_\alpha} = 1$, which implies $\|H\psi\|^2_{\mathscr{H}} = \mathcal{O}(\varepsilon^{2\alpha})$, and

$$\begin{aligned} \|\varepsilon X^* \psi\|^2_{\mathscr{H}} &\leq \|g_B(X,X)\|_\infty \left(|\langle \psi, H\psi \rangle| + \varepsilon \, |\langle \psi, H_1 \psi \rangle| \right) \\ &\leq C \, \|\psi\| \left(\|H\psi\|_{\mathscr{H}} + \varepsilon \, \|H_1 \psi\|_{\mathscr{H}} \right) . \end{aligned} \tag{C.7}$$

Now assume for the moment that we have $\|H_1 P_0\|_{\mathscr{L}(D^2_\alpha, \mathscr{H})} = \mathcal{O}(\varepsilon^{2\beta})$ for some $\beta \geq 0$. Then (C.7) and $P_0 \in \mathcal{A}^{0,0}_H$ imply

$$\|\varepsilon P_0 X^*\|_{\mathscr{L}(D^2_\alpha(A), \mathscr{H})} = \|\varepsilon X^* P_0\| + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon^{\min\{\alpha/2, 1/2+\beta\}}) . \tag{C.8}$$

We will see that $\beta = \alpha/4$ after looking at the estimate for the norm of $\mathscr{L}(D^2_\alpha, D(H))$. Let $\psi \in D^2_\alpha(H)$ and use A.14 to get

$$\begin{aligned} \|\varepsilon P_0 X^* \psi\|^2_{W^2_\varepsilon} &\leq C\varepsilon^2 \left(\|P_0 X^* \psi\|^2_{\mathscr{H}} + \|\Delta_{g_\varepsilon} P_0 X^* \psi\|^2_{\mathscr{H}} \right) \\ &\leq 2C\varepsilon^2 \big(\|X^*\psi\|^2_{\mathscr{H}} + \|X^* \Delta_{g_\varepsilon} \psi\|^2_{\mathscr{H}} \\ &\qquad + \|([\Delta_{g_\varepsilon}, P_0] X^* + P_0 [\Delta_{g_\varepsilon}, X^*])\psi\|^2_{\mathscr{H}} \big) . \end{aligned}$$

The operator $[\Delta_{g_\varepsilon}, P_0] X^*$ is a second order differential operator whose norm in $\mathscr{L}(W^2_\varepsilon, \mathscr{H}_F)$ is bounded independently of ε because $P_0 \in \mathcal{A}^{0,0}$ and $[P_0, H_F] = 0$ (see also (2.2)). The same holds for $[\Delta_{g_\varepsilon}, X^*]$ by direct calculation, because commutators of X^* with vertical fields are vertical (see 1.4). Thus the second line is bounded by a constant times $\|\psi\|_{D(H)} \leq \|\psi\|_{D_\alpha(H)}$ by theorem A.14. Concerning the second term of the first line we have

$$\begin{aligned} \|X^* \Delta_{g_\varepsilon} \psi\|^2_{\mathscr{H}} &= \|X^*(H + \Lambda_0 + \varepsilon H_1)\psi\|^2_{\mathscr{H}} \\ &\leq 3 \left(\|X^* H\psi\|^2_{\mathscr{H}} + \Lambda_0^2 \, \|X^*\psi\|^2_{\mathscr{H}} + \|X^* \varepsilon H_1 \psi\|^2_{\mathscr{H}} \right) . \end{aligned}$$

Now take ψ with $\|\psi\|_{D^2_\alpha} = 1$, then we get inequalities for the individual terms

$$\begin{aligned} \|X^* H\psi\|^2_{\mathscr{H}} &\leq C \, \|X^*\|^2_{\mathscr{L}(W^2_\varepsilon, \mathscr{H})} \, \|H\psi\|^2_{D(H)} \leq C \, \|X^*\|^2_{\mathscr{L}(W^2_\varepsilon, \mathscr{H})} \, \varepsilon^{2\alpha} , \\ \|X^* \varepsilon H_1 \psi\|^2_{\mathscr{H}} &\leq C\varepsilon^2 \, \|X^*\| \, \|H_1\|_{\mathscr{L}(D(H^2), W^2_\varepsilon)} \, \|\psi\|^2_{D(H^2)} \; . \end{aligned}$$

From (C.6) it is clear that $\|\varepsilon X^*\|^2_{\mathscr{L}(W^2_\varepsilon,\mathscr{H})} = \mathcal{O}(1)$, so putting all of these terms together with (C.8) we get

$$\|\varepsilon P_0 X^* \psi\|^2_{W^2_\varepsilon} \leq C\varepsilon^{\min\{\alpha,1+2\beta\}} + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^{\min\{\alpha,1+2\beta\}})\,. \tag{C.9}$$

We now use this to show that $\beta = \alpha/4$. Since we are assuming that condition 5 holds, H_1 has the form $H_1 = -\varepsilon^2 \operatorname{div} S_\varepsilon + \mathcal{O}(\varepsilon)$. By the argument of remark 2.4 we can estimate $\left\|\varepsilon^2 \operatorname{div} S_\varepsilon P_0\right\|_{\mathscr{L}(D_\alpha,\mathscr{H})}$ by the local norms $\left\|\varepsilon^2 \operatorname{div} S_\varepsilon P_0|_{\pi^{-1}(U_\nu)}\right\|_{\mathscr{L}(D_\alpha,\mathscr{H})}$. On $U_\nu \in \mathfrak{U}$ we express $\operatorname{div} S_\varepsilon$, denoting by ξ^i the dual one-form to X_i and suppressing ν,

$$\begin{aligned} &\varepsilon^2 \operatorname{div} S_\varepsilon|_{\pi^{-1}(U)} \\ &\quad= \sum_{i,j\leq d} \Big(\varepsilon X^*_i \pi^* g_B\big(S(\xi^j,\cdot), X^*_i\big) - \varepsilon\pi^* g_B\big(S(\xi^j,\cdot),(\nabla_{X_i}X_i)^*\big)\Big)\varepsilon X^*_j\,. \end{aligned} \tag{C.10}$$

By boundedness of S_ε the term in the bracket defines a bounded operator from W^1_ε to $\mathscr{H}$ while our intermediate result (C.9) gives

$$\left\|\varepsilon X^*_j P_0\right\|^2_{\mathscr{L}(D^2_\alpha, W^2_\varepsilon)} = \mathcal{O}(\varepsilon^{\min\{\alpha,1+2\beta\}})\,.$$

Consequently β satisfies the equation $4\beta = \min\{\alpha, 1+2\beta, 2\}$. Since $\alpha \leq 2$ this simplifies to $\beta = \alpha/4$ and the proof for $A = H$ is complete. This estimate can be improved to an estimate of $H_1 P_0$ in the norm of $\mathscr{L}(D^2_\alpha, D(H))$ using the same procedure as for $X^* P_0$.

2) The case $A = H_{\mathrm{a}}$:

The proof for in this case is similar. Instead of the elliptic estimates A.14 on M it uses the corresponding estimates for the operator $-\Delta_B$ on $\mathcal{E}$, which is a bundle of bounded geometry by proposition B.7. With this in mind the basic calculations are identical to those above, only now we have to bound the individual terms by H_{a} instead of H. The only term for which this makes any difference is

$$\begin{aligned} [\Delta_{g_\varepsilon}, P_0 X^*]P_0 &= [\Delta_{g_\varepsilon}, P_0]X^* P_0 + P_0[\Delta_{g_\varepsilon}, X^*]P_0 \\ &= [\Delta_{g_\varepsilon}, P_0]\nabla^B_X + P_0\big([\Delta_{g_\varepsilon}, P_0][X^*, P_0] + [\Delta_{g_\varepsilon}, X^*]\big)P_0\,. \end{aligned}$$

The term in the bracket defines a second order differential operator on $\mathcal{E}$ whose coefficients are bounded since $P_0 \in \mathcal{A}^0_H$. Hence it is bounded by the graph norm of the elliptic operator $-\varepsilon^2\Delta_B$ and is an element of $\mathscr{L}(D(H), L^2(\mathcal{E}))$. By (3.13a), (3.13b) the first term can be expressed as a second order operator on $\mathcal{E}$ with coefficients in $\mathscr{L}(\mathscr{H})$ that are off-diagonal with respect to P_0. Thus by the same argument it is an element of $\mathscr{L}(D(H), \mathscr{H})$. With this in mind the proof can be carried out following the steps of the case $A = H$.

□

A concentration estimate using the Sobolev norm

Lemma C.4. *Let $d \in \mathbb{N}$ and $f \in W^1(\mathbb{R}^d)$. There is a constant $C(d) > 0$ such that*

$$\varepsilon^{-d} \int\limits_{B(\varepsilon,0)} |f(x)|^2 \,\mathrm{d}x \leq C\theta_d(\varepsilon) \, \|f\|^2_{W^1} \, ,$$

with $\theta_1(\varepsilon) = 1$, $\theta_2(\varepsilon) = -\log \varepsilon$ and $\theta_d(\varepsilon) = \varepsilon^{2-d}$ for $d \geq 3$.

Proof. Let $\hat{f}$ denote the Fourier transform of f. Then for any $R \geq 0$

$$\begin{aligned}
\int\limits_{B(\varepsilon,0)} |f(x)|^2 \,\mathrm{d}x &= \frac{1}{(2\pi)^d} \int\limits_{B(\varepsilon,0)} \Big| \int\limits_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k) \mathrm{d}k \Big|^2 \mathrm{d}x \\
&\leq \frac{2}{(2\pi)^d} \int\limits_{B(\varepsilon,0)} \Big| \int\limits_{|k|\leq R} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k) \,\mathrm{d}k \Big|^2 + \Big| \int\limits_{|k|>R} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k) \,\mathrm{d}k \Big|^2 \mathrm{d}x \, .
\end{aligned}$$

The the integral over $k \leq R$ satisfies

$$\begin{aligned}
\Big| \int\limits_{|k|\leq R} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k) \,\mathrm{d}k \Big|^2 &\leq \Bigg(\int\limits_{|k|\leq R} \frac{1}{\sqrt{1+k^2}} \sqrt{1+k^2} |\hat{f}| \,\mathrm{d}k \Bigg)^2 \\
&\leq \|f\|^2_{W^{1,2}} \int\limits_{|k|\leq R} \frac{1}{1+k^2} \,\mathrm{d}k \, .
\end{aligned}$$

For the second term we use Parseval's identity on the function

$1_{(R,\infty)}(|k|)\hat{f}(k) \in L^2(\mathbb{R}^d)$

$$\frac{1}{(2\pi)^d} \int\limits_{B(\varepsilon,0)} \Big| \int\limits_{|k|>R} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k)\,\mathrm{d}k \Big|^2 \mathrm{d}x \le \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \Big| \int\limits_{|k|>R} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k)\,\mathrm{d}k \Big|^2 \mathrm{d}x$$
$$\le \int\limits_{|k|>R} |\hat{f}(k)|^2\,\mathrm{d}k$$
$$\le \int\limits_{|k|>R} \frac{k^2}{R^2} |\hat{f}(k)|^2\,\mathrm{d}k$$
$$\le R^{-2} \, \|f\|^2_{W^1} \, .$$

Combining these terms again we get

$$\int\limits_{B(\varepsilon,0)} |f(x)|^2\,\mathrm{d}x \le C\Big(\mathrm{Vol}(B(\varepsilon,0)) \int\limits_{|k|\le R} \frac{1}{1+k^2}\,\mathrm{d}k + R^{-2}\Big) \, \|f\|^2_{W^1} \, .$$

If $d = 1$ we may let $R \to \infty$ to obtain the result. For $d > 1$ we put $R = \varepsilon^{-1}$ and estimate

$$\int\limits_{|k|\le\varepsilon^{-1}} \frac{1}{1+k^2}\,\mathrm{d}k \le C\Big(1 + \int\limits_1^{\varepsilon^{-1}} r^{d-3}\mathrm{d}r\Big) \, ,$$

which yields the correct behaviour. □

Index of Symbols

Symbol	Explanation	Page
$\mathbb{N}$	Natural numbers including zero.	
(B, g_B)	A complete, connected d-dimensional Riemannian manifold of bounded geometry.	14, 35
$M \xrightarrow{\pi} B$	A fibre bundle over B.	14, 35
F	The fibre of π, a compact n-dimensional manifold with boundary.	14, 35
TF	Vertical subbundle of TM.	15
NF	Horizontal subbundle of TM.	15
g_ε	Rescaled Riemannian submersion metric on M.	17
g_F	Restriction of g_ε to TF.	15
X^*	Horizontal lift of $X \in \Gamma(TB)$.	16
$\Omega(X, Y)$	Integrability tensor of the horizontal subbundle.	17
φ_*	Differential of the map φ.	14
$\mathcal{L}_X$	Lie-derivative along the vector field X.	20
$\mathscr{H}$	Hilbert space of square integrable, complex valued functions on (M, g)	36
W^k	Sobolev space of functions with weak derivatives of order up to k in L^2.	131
W_0^k	Subspace of W^k of functions vanishing on the boundary.	131
W^∞	Sobolev space of functions with infinitely many weak derivatives.	
$W_\varepsilon^k(M)$	Sobolev Space adapted to (M, g_ε).	36, 131
$\Gamma(E)$	Space of smooth sections of the vector bundle E.	16

$\Gamma_b(E)$	Fréchet space of smooth and bounded sections of E.	118
$\mathscr{C}^k$	Fréchet space of k-times differentiable functions.	
$\mathscr{C}^k_b$	Banach space of $\mathscr{C}^k$-bounded functions.	
$\mathscr{C}^\infty_0$	Fréchet space of smooth functions with compact support.	
$\mathscr{C}^\infty$, $\mathscr{C}^\infty_b$	Fréchet spaces of smooth and $\mathscr{C}^\infty$-bounded functions.	
$\mathscr{H}_F$	Hermitian vector bundle over B with fibre $L^2(F)$.	38, 138
H_F	The fibre Hamiltonian.	37, 81
$D(H_F)$	Hermitian vector bundle over B with fibre $W^2(F) \cap W^1_0(F)$.	38
$\mathcal{E}$	Eigenspace bundle of H_F.	38, 139
$\mathscr{L}(X, Y)$	Space of continuous linear maps between vector spaces X and Y.	
$\mathscr{L}(E, E')$	Bundle of continuous bundle maps between vector bundles E and E'.	
$\mathcal{A}$	An algebra of differential operators on the vector bundle $\mathscr{H}_F$.	40
$\mathcal{A}_H$	Elements of $\mathcal{A}$ with image in $D(H_F)$.	40
$D^2_\alpha(A)$	Domain of $\varepsilon^{-2\alpha}A^2$ with the graph-norm.	84
P_0	Projection to an eigenband of H_F.	38, 45
P_ε	Super-adiabatic projection for an eigenband of H_F.	49
U_ε	Unitary map on $\mathscr{H}$ intertwining P_0 and P_ε.	57
Δ_g	Laplace-Beltrami operator of the metric g.	
$\varepsilon^2\Delta_h$	Horizontal part of Δ_{g_ε}.	18
∇^B	Berry connection on $\mathcal{E}$.	62
Δ_B	Laplacian of the Berry connection.	63
H	Hamiltonian defined on $D(H) \subset \mathscr{H}$.	36, 80

H_1	Lower order perturbation in H.	36, 83
H_{eff}	Effective operator.	57
H_{a}	Adiabatic operator $P_0 H P_0$.	63, 80
H_0	Leading order approximation of H_{a}	96, 100
$\mathfrak{U}$	Atlas of normal coordinate neighbourhoods of B.	39, 131
$N(\mathfrak{U})$	Local multiplicity of the cover $\mathfrak{U}$.	130
$B(r, p)$	Metric ball of radius r around a point p.	
$\mathrm{Vol}(S)$	Volume of the set S.	
vol_g	Volume measure associated with the Riemannian metric g.	
$\mathrm{II}\,(X, Y)$	Second fundamental form of a submanifold.	
η	Mean curvature vector of F.	17
$\bar{\eta}$	Projection of η to $\mathcal{E}$.	62
$\mathcal{N}(\varphi)$	Nodal set of the eigenfunction φ.	32

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