On Fano Arrangement Varieties

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CONTENTS

INTRODUCTION

This thesis contributes to the study of projective varieties with torus action. In particular, we present explicit descriptions and classification results for torus actions of complexity two and contribute to the classification of singular Fano 3-folds.

The presence of a torus action on a variety brings combinatorial aspects into the game, as becomes most evident in the case of *toric varieties*, that means normal varieties Z containing an algebraic torus $\mathbb{T}$ as an open subset, such that the group structure of $\mathbb{T}$ extends to an action on Z. The theory of toric varieties has its origin in 1970, when Demazure observed the fundamental correspondence between toric varieties and fans [26], and became a well established and active field of research [22, 24, 26, 34, 65, 66].

A natural step beyond the toric case is to consider $\mathbb{T}$-*varieties of complexity one*, i.e. normal varieties X with an effective action of an algebraic torus $\mathbb{T} \times X \to X$ such that the complexity $\dim(X) - \dim(\mathbb{T})$ equals one. $\mathbb{T}$-varieties of complexity one are studied as well since the 1970s. Here, we mention the work on $\mathbb{K}^*$-surfaces [31, 32, 33, 67, 68, 69, 70], the combinatorial approaches [1, 2, 56, 76] and the more algebraic point of view, based on trinomials [40, 41, 47, 64]. Our work builds up on the description of rational $\mathbb{T}$-varieties of complexity one via their Cox rings [40].

In this thesis, we consider $\mathbb{T}$-varieties of arbitrary complexity. More precisely, we are interested in $\mathbb{T}$-varieties that are *Mori dream spaces*, meaning normal projective varieties X with finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated Cox ring. The $\mathbb{T}$-action on X gives rise to a rational quotient $\pi\colon X \dashrightarrow Y$, the so-called *maximal orbit quotient*. Note, that for rational $\mathbb{T}$-varieties of complexity one, the target space Y of the maximal orbit quotient equals $\mathbb{P}_1$ and the critical values of π form a point configuration. In general, the variety Y is again a Mori dream space suitably representing the field of rational invariants $\mathbb{K}(X)^{\mathbb{T}} = \mathbb{K}(Y)$, the dimension of Y equals the complexity of the torus action and π is defined on an open subset consisting of points with finite $\mathbb{T}$-isotropy, see Definition 1.2.13. In Chapter 1, we present a method to systematically produce all Mori dream spaces X with torus action having a prescribed maximal orbit quotient $X \dashrightarrow Y$, see Construction 1.2.5.

Our major example class, the *(general/special) arrangement varieties*, directly extends the class of rational $\mathbb{T}$-varieties of complexity one in the following sense: the target space of the maximal orbit quotient $\mathbb{P}_1$ is replaced with a higher dimensional projective space $\mathbb{P}_c$ and the point configuration building the critical values of π is replaced with a hyperplane arrangement in (general/special) position. Some already known examples are the *intrinsic quadrics*. These are Mori dream spaces having a Cox ring generated by homogeneous generators such that the ideal of relations is generated by a single homogeneous quadratic relation. Intrinsic quadrics were introduced in [15] as an example class for the bunched ring approach to Mori dream spaces and were for example used by Bourqui in [17] as a testing ground for Manin's conjecture. Moreover, in [29] Fujita's freeness conjecture was verified for smooth intrinsic quadrics of Picard number at most two.

Chapter 2 is dedicated to the study of general arrangement varieties. We explicitly describe their Cox rings as complete intersection rings very similar to the case of rational $\mathbb{T}$-varieties of complexity one. This allows for instance an explicit description of their anticanonical divisor classes, see Proposition 2.2.4. Moreover, we present in Corollary 2.2.16 a smoothness criterion for general arrangement varieties using toric embeddings constructed via their Cox rings.

Let us turn to smooth general arrangement varieties of small Picard number. Recall, that in the toric case, the only smooth examples of Picard number one are the projective spaces $\mathbb{P}_n$. In Picard number two, Kleinschmidt described all smooth projective toric varieties as projectivized split vector bundles [57]. In the case of smooth rational $\mathbb{T}$-varieties of complexity one, the classification in Picard number one is due to [59]. Here, the only varieties are the smooth projective quadrics in dimensions three and four. In Picard number two, the description of all smooth rational $\mathbb{T}$-varieties of complexity one is presented in [30] and consists of 13 different families. Moreover, in [29] all smooth intrinsic quadrics of Picard number at most two are determined.

For smooth projective general arrangement varieties of complexity two, one retrieves in Picard number one precisely the smooth projective quadrics, see Proposition 2.2.23. Similar to the case of complexity one, the situation in Picard number two is much more ample. Theorem 3.1.1 presents the full description of all smooth projective general arrangement varieties of complexity and Picard number two in 14 different families. We prove in Section 3.2 that all of them are of *true complexity* two, meaning that they do not admit a torus action of lower complexity. In contrast to the case of complexity one, where being of true complexity one is simply characterized by a singular total coordinate space, in complexity two this turns out to be a serious case-by-case work introducing and comparing various invariants of the $\mathrm{Cl}(X)$-graded Cox rings.

Using the explicit description of the anticanonical divisor class for general arrangement varieties, one extracts the Fano varieties from Theorem 3.1.1 which leads to the complete classification of smooth Fano general arrangement varieties of true complexity two and Picard number two in any dimension. We prove in Section 3.4, that all the Fano varieties in Theorem 3.1.3 arise from a finite set of smooth projective general arrangement varieties of complexity two and Picard number two having dimensions 5 to 8 via iterated *duplication of a free weight*, i.e. given a variable that does not show up in the defining relations of the Cox ring $\mathcal{R}(X)$, one adds a further free variable of the same degree, see Construction 3.4.1. Geometrically a duplication of a free weight corresponds to two elementary contractions and a series of isomorphisms in codimension one. This establishes a similar finiteness feature as observed in [30] in the case of complexity one: Here the smooth rational Fano $\mathbb{T}$-varieties of complexity one and Picard number two arise via iterated duplication of a free weight from a finite list of smooth rational projective $\mathbb{T}$-varieties of complexity one of dimensions 4 to 7.

Theorem (Compare Cor. 3.4.6)**.** *Every smooth Fano general arrangement variety of true complexity two and Picard number two arises via iterated duplication of a free weight from a finite set of smooth projective general arrangement varieties of true complexity two and Picard number two of dimensions* 5 *to* 8*.*

Let us enter the field of singular $\mathbb{T}$-varieties. We concentrate on the singularity types arising naturally in the context of the Minimal Model Program.

The model case are toric varieties. Toric Fano varieties Z are in one-to-one correspondence to the so called *Fano polytopes* $\mathcal{A}_Z$. The boundary of the Fano polytope $\partial\mathcal{A}_Z$ is determined by the property that it encodes the discrepancies of any toric resolution of singularities:

$$\partial\mathcal{A}_Z = \left\{ \frac{v_\varrho}{1+\operatorname{discr}(D_\varrho)};\ v_\varrho \text{ primitive} \right\},$$

where $\operatorname{discr}(D_\varrho)$ is the discrepancy of the torus invariant prime divisor D_ϱ defined by the ray $\varrho = \operatorname{cone}(v_\varrho)$. This turns the Fano polytope into a combinatorial tool characterizing singularity types of toric Fano varieties in terms of lattice points. For instance, Z has at most canonical singularities if and only if the origin is the only lattice point in the interior of $\mathcal{A}_Z$. Moreover, Z has at most terminal singularities if and only if the origin and the primitive ray generators of the defining fan of Z are the only lattice points of $\mathcal{A}_Z$. The Fano polytope was used i.a. by Kasprzyk to classify all three-dimensional toric Fano varieties with at most canonical singularities [54, 55].

Now consider rational Fano $\mathbb{T}$-varieties of complexity one. These varieties allow a natural $\mathbb{T}$-equivariant embedding into a toric variety $X \subseteq Z$, where the defining fan Σ of Z is constructed via the Cox ring of X, see [6]. Intersecting X with the Torus $T \subseteq Z$, we assign a tropical variety $\operatorname{trop}(X)$ to X and a *weakly tropical resolution* $X' \to X$, where X' is the proper transform of X with respect to the toric morphism $Z' \to Z$ given by the common refinement of Σ and $\operatorname{trop}(X)$. In this situation, the *anticanonical complex*, a polyhedral complex supported inside $\operatorname{trop}(X)$, can be defined in analogy to the toric Fano polytope, see [13]. The anticanonical complex $\mathcal{A}_X$ is bounded if and only if X is log terminal. In this situation its boundary $\partial\mathcal{A}_X$ is determined by the property that it encodes the discrepancies of any toric ambient resolution of singularities in full analogy to the Fano polytope. Here, a *toric ambient resolution of singularities* is a resolution of singularities $X'' \to X$, induced by a toric resolution of singularities $Z'' \to Z$ factoring over $Z' \to Z$. In particular canonicity and terminality of X can be read off $\mathcal{A}_X$ in the same manner as for the toric Fano polytope. The anticanonical complex was used in [13] to classify all three-dimensional $\mathbb{Q}$-factorial rational Fano non-toric $\mathbb{T}$-varieties of complexity and Picard number one, having at most terminal singularities.

Building up on [13], we enlarge the area of application of the anticanonical complex to Mori dream spaces with torus action of higher complexity. Similar to the rational $\mathbb{T}$-varieties of complexity one, these varieties admit a specific equivariant embedding $X \subseteq Z_X$ into a toric variety, adapted to the geometry of Z_X, see Theorem 1.2.10. Using this toric embedding $X \subseteq Z_X$ and the associated weakly tropical resolution $X' \to X$, the central question is whether a toric ambient resolution of singularities exists. In this situation we say that X *admits an anticanonical complex* if there exists a polyhedral complex $\mathcal{A}_X$ supported inside $\operatorname{trop}(X)$ encoding the discrepancies of any toric ambient resolution of singularities via its boundary as indicated above; see 4.2.1 for the precise definition. Our main result reduces the question of toric ambient resolvability and therefore the existence of an anticanonical complex to an *explicit maximal orbit quotient*, meaning a maximal orbit quotient $X \dashrightarrow Y$ fitting into the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z_X \\ {\scriptstyle /\mathbb{T}} \downarrow & & \downarrow {\scriptstyle /\mathbb{T}} \\ Y & \longrightarrow & Z_Y, \end{array}$$

where $Z_X \dashrightarrow Z_Y$ is defined on the union over all toric orbits of codimension at most one and there yields a categorical quotient for the $\mathbb{T}$-action; see Construction 4.3.3 for the details. The necessary property of the quotient space $Y \subseteq Z_Y$ is *semi-locally toric weakly tropical resolvability*, meaning that the weakly tropical resolution $Y' \subseteq Z_{Y'}$ is locally toric in a strong sense, reflecting properties of its ambient toric variety; see Definition 4.1.3.

Theorem (Compare Cor. 4.3.9)**.** *Let X be a $\mathbb{Q}$-Gorenstein Mori dream space with torus action having an explicit maximal orbit quotient $X \dashrightarrow Y$, where Y is complete and admits a semi-locally toric weakly tropical resolution. Then X admits an anticanonical complex $\mathcal{A}_X$ and the following statements hold:*

(i) *X has at most log terminal singularities if and only if the anticanonical complex $\mathcal{A}_X$ is bounded.*

(ii) *X has at most canonical singularities if and only if* 0 *is the only lattice point in the relative interior of $\mathcal{A}_X$.*

(iii) *X has at most terminal singularities if and only if* 0 *and the primitive generators of the rays of the defining fan of Z_X are the only lattice points of $\mathcal{A}_X$.*

Note, that in this theorem X is not assumed to be Fano. In fact, the Fano property reflects in certain convexity properties of the anticanonical complex, see Corollary 4.6.3 and Example 4.6.4.

Now, the idea is to apply the anticanonical complex to arrangement varieties. As a first result we obtain anticanonical complexes for general arrangement varieties:

Theorem (Compare Thm. 4.4.1)**.** *Every $\mathbb{Q}$-Gorenstein general arrangement variety admits an anticanonical complex.*

In Sections 4.5 and 4.6 we provide explicit descriptions for the anticanonical complex of a general arrangement variety. As an application, we characterize log-terminality for general arrangement varieties of complexity two in terms of exponents in the defining relations of their Cox rings, see Corollary 4.5.16.

In Chapter 5 we use the anticanonical complex to obtain classification results for three-dimensional Fano intrinsic quadrics having at most canonical singularities:

Theorem (Compare Thm. 5.1.1)**.** *Every three-dimensional $\mathbb{Q}$-factorial Fano intrinsic quadric having Picard number one and at most canonical singularities is isomorphic to precisely one of the following varieties X defined by its $\mathrm{Cl}(X)$-graded Cox ring $\mathcal{R}(X)$, its matrix of generator degrees $Q = [w_1, \ldots, w_r]$ and its anticanonical divisor class $-\mathcal{K}_X \in \mathrm{Ample}(X)$. Moreover, we list their Fano-index $q(X)$ and their anticanonical self-intersection number $-\mathcal{K}_X^3$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
1	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \end{bmatrix}$	3	54
2	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 6 \end{bmatrix}$	6	36
3	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 6 \end{bmatrix}$	6	48
4	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 5 & 3 \end{bmatrix}$	$\begin{bmatrix} 9 \end{bmatrix}$	9	$\frac{729}{20}$
5	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 6 & 3 & 5 & 4 \end{bmatrix}$	$\begin{bmatrix} 12 \end{bmatrix}$	12	$\frac{96}{5}$
6	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 5 & 1 & 7 & 4 \end{bmatrix}$	$\begin{bmatrix} 12 \end{bmatrix}$	12	$\frac{1152}{35}$
7	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 7 & 2 & 8 & 5 \end{bmatrix}$	$\begin{bmatrix} 15 \end{bmatrix}$	15	$\frac{1125}{56}$
8	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{1} \end{bmatrix}$	1	$\frac{32}{3}$
9	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{1} \end{bmatrix}$	3	$\frac{36}{5}$
10	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 6 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{1} \end{bmatrix}$	3	12
11	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{1} \end{bmatrix}$	3	24
12	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{1} \end{bmatrix}$	3	27
13	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	54
14	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	32

15	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 5 \\ \bar{1} \end{bmatrix}$	5	$\frac{125}{6}$
16	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	18
17	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	18
18	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	24
19	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 7 \\ \bar{1} \end{bmatrix}$	7	$\frac{343}{48}$
20	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
21	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
22	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 1 & 3 & 3 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{256}{15}$
23	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{64}{3}$
24	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 3 & 4 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 9 \\ \bar{1} \end{bmatrix}$	9	$\frac{243}{20}$
25	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 3 & 5 & 4 & 4 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{25}{3}$
26	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5 & 1 & 3 & 3 & 4 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{50}{3}$
27	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
28	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
29	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7 & 3 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{288}{35}$
30	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 6 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
31	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 6 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
32	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7 & 3 & 5 & 5 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 14 \\ \bar{0} \end{bmatrix}$	14	$\frac{98}{15}$
33	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18
34	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18
35	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{0} & \bar{2} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
36	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{2} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
37	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{0} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	16
38	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{1} \end{bmatrix}$	1	19

39	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{3} & \bar{1} & \bar{0} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{2} \end{bmatrix}$	2	$\frac{16}{3}$
40	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{2} \end{bmatrix}$	2	16
41	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{27}{2}$
42	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{3} \end{bmatrix}$	3	$\frac{27}{2}$
43	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{27}{2}$
44	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	16
45	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 5 \\ \bar{0} \end{bmatrix}$	5	$\frac{49}{6}$
46	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
47	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
48	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{16}{3}$
49	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_5$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{54}{5}$
50	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{5} & \bar{2} & \bar{4} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
51	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
52	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{4} & \bar{0} & \bar{2} & \bar{5} & \bar{5} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
53	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{4} & \bar{0} & \bar{5} & \bar{2} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{2} \end{bmatrix}$	4	$\frac{32}{3}$
54	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{1} \end{bmatrix}$	3	$\frac{27}{4}$
55	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	8
56	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{6} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{4} \end{bmatrix}$	4	8
57	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_9$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{4} & \bar{5} & \bar{3} & \bar{6} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	6
58	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{7} & \bar{1} & \bar{4} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{9}{2}$
59	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{1} \end{bmatrix}$	1	$\frac{19}{2}$
60	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{1} \\ \bar{1} \end{bmatrix}$	1	16
61	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{1} \\ \bar{1} \end{bmatrix}$	3	$\frac{45}{4}$

62	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{0}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{0}\end{bmatrix}$	3	$\frac{27}{2}$
63	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{1}\end{bmatrix}$	3	$\frac{27}{2}$
64	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{1}\end{bmatrix}$	3	$\frac{27}{2}$
65	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{0}\end{bmatrix}$	4	16
66	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 3&3&3&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	6
67	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2&2&2&1&3\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
68	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 3&1&2&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
69	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&3&2&2&2\\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	9
70	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1&1&1&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\\ \bar{0}\end{bmatrix}$	6	18
71	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2&4&3&3&2\\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0}\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}8\\ \bar{0}\\ \bar{0}\end{bmatrix}$	8	$\frac{16}{3}$
72	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 2&2&2&1&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\\ \bar{1}&\bar{3}&\bar{3}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{2}\end{bmatrix}$	2	4
73	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 1&1&1&2&1\\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\\ \bar{1}&\bar{3}&\bar{3}&\bar{1}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{2}\end{bmatrix}$	2	8
74	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1}\\ \bar{2}&\bar{0}&\bar{1}&\bar{3}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{0}\end{bmatrix}$	4	8
75	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\\ \bar{4}&\bar{1}&\bar{1}&\bar{5}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{3}\end{bmatrix}$	3	$\frac{9}{2}$
76	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\\ \bar{2}&\bar{0}&\bar{4}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{0}\end{bmatrix}$	3	$\frac{9}{2}$
77	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\\ \bar{2}&\bar{4}&\bar{3}&\bar{3}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}$	3	$\frac{9}{2}$
78	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_3)^2$	$\begin{bmatrix} 1&1&1&1&1\\ \bar{1}&\bar{2}&\bar{1}&\bar{2}&\bar{0}\\ \bar{2}&\bar{1}&\bar{1}&\bar{2}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\\ \bar{0}\end{bmatrix}$	3	6
79	$\frac{\mathbb{K}[T_1,\ldots,T_4,S_1]}{\langle T_1^2+T_2^2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} 1&1&1&1&2\\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\\ \bar{1}&\bar{0}&\bar{0}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\\ \bar{0}\\ \bar{0}\end{bmatrix}$	4	8

Variety No. 1 *is smooth and varieties Nos.* 4, 19 *and* 49 *are terminal. Moreover, varieties Nos.* 64, 69, 77 *and* 79 *are of true complexity two. All the others are of true complexity one.*

In the case of $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three and complexity two that have at most canonical singularities, we are even able to classify all varieties without restrictions on the Picard number.

Theorem (Compare Thm. 5.1.2)**.** *Every three-dimensional $\mathbb{Q}$-factorial Fano intrinsic quadric of true complexity two having at most canonical singularities is isomorphic to precisely one of the varieties X, specified by its $\mathrm{Cl}(X)$-graded Cox ring $\mathcal{R}(X)$, its matrix of generator degrees $Q = [w_1, \ldots, w_r]$ and its anticanonical divisor class $-\mathcal{K}_X \in \mathrm{Ample}(X)$ as follows:*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$
1	$\frac{\mathbb{K}[T_1,\ldots,T_4,S_1]}{\langle T_1^2+T_2^2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
2	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
3	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
4	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$
5	$\frac{\mathbb{K}[T_1,\ldots,T_4,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
6	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
7	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
8	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ -2 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
9	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,S_2]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}^3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 2 \\ \bar{0} \\ \bar{1} \end{bmatrix}$

We turn to special arrangement varieties. Recall that a special arrangement variety is a Mori dream space X with torus action having $X \dashrightarrow \mathbb{P}_c$ as maximal orbit quotient, such that the critical values form a hyperplane arrangement in special position. Note that not all special arrangement varieties are *honestly special*, meaning that they do not admit a torus action of lower complexity, turning them into a general arrangement variety, see Example 6.3.3. As we have already treated the general arrangement case, we therefore restrict ourselves to the case of honestly special arrangement varieties.

A first observation is that there are no smooth honestly special arrangement varieties with Picard number at most two, see Theorem 6.3.4.

For the further study, we establish in Sections 6.4 and 6.5 the existence of anticanonical complexes for special arrangement varieties.

Theorem (Compare Thm. 6.5.1)**.** *Every $\mathbb{Q}$-Gorenstein special arrangement variety admits an anticanonical complex.*

In contrast to the general arrangement case, where the weakly tropical resolution is again a general arrangement variety, this is no longer true for special arrangement varieties. Under the assumption that the weakly tropical resolution is a toric ambient modification in the sense of [6], we provide in Section 6.5 an explicit description of the anticanonical complexes for these varieties.

As an application, we obtain classification results for complexity two torus actions in dimension three. Here, we consider the simplest honestly special arrangement varieties that appear: these have five lines in $\mathbb{P}_2$ as critical divisors of the maximal orbit quotient. Moreover, we say that a $\mathbb{T}$-variety X is of *finite isotropy order at most k*, if there is an open subset $U \subseteq X$ with complement $X \setminus U$ of codimension at least two, such that the isotropy group $\mathbb{T}_x$ is either infinite or of order at most k for all $x \in U$.

Theorem (Compare Thm. 6.6.2)**.** *Every three-dimensional Fano honestly special arrangement variety of complexity two, having a divisor class group of rank at most two, at most canonical singularities, five critical lines as the critical values of the maximal orbit quotient and finite isotropy order at most two is isomorphic to one of the following Fano varieties X, specified by its $\mathrm{Cl}(X)$-graded Cox ring $\mathcal{R}(X)$, its matrix $Q = [w_1, \ldots, w_r]$ of generator degrees and its anticanonical divisor class $-\mathcal{K}_X \in \mathrm{Ample}(X)$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$
1	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,S_1]}{\left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4^2, \\ T_2^2+aT_3^2+T_5^2 \end{array} \right\rangle}$ $a \neq 0,1$	$\mathbb{Z} \times (\mathbb{Z}_2)^4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
2	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,S_1]}{\left\langle \begin{array}{c} T_1^2+T_2^2+T_4^2, \\ T_1^2+T_3^2+T_5^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
3	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5^2, \\ T_3^2+aT_4^2+T_6^2 \end{array} \right\rangle}$ $a \neq 0,1$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
4	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1T_2+T_3^2+T_5^2, \\ T_1T_2+T_4^2+T_6^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{3} & \bar{1} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
5	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1^2+T_2T_3+T_5^2, \\ T_1^2+T_4^2+T_6^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{2} & \bar{1} & \bar{3} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
6	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6^2 \end{array} \right\rangle}$ $a \neq 0,1$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{2} & \bar{3} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
7	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7]}{\left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5T_6, \\ T_3^2+aT_4^2+T_7^2 \end{array} \right\rangle}$ $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$

8	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5T_6, \\ T_3^2+aT_4^2+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
9	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4T_5+T_6^2, \\ T_2T_3+aT_4T_5+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
10	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4T_5+T_6^2, \\ T_2T_3+aT_4T_5+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 2 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
11	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
12	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
13	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
14	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
15	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
16	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
17	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 3 & 2 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
18	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
19	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
20	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
21	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
22	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 2 & 1 & -1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
23	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -2 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
24	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
25	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$

26	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5^2, \\ T_3^2+aT_4^2+T_6^2 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
27	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6^2 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
28	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3^2+T_5^2, \\ T_1T_2+T_4^2+T_6^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
29	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5^2, \\ T_1^2+T_4^2+T_6^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
30	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4^2, \\ T_2^2+aT_3^2+T_5T_6 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{0} \\ \bar{1} \end{bmatrix}$
31	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
32	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 & 1 & -1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
33	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
34	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a\neq 0,1 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
35	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
36	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
37	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
38	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
39	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 2 & -2 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
40	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array}\right\rangle$	$\mathbb{Z}^2\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$

The last chapter is an outlook beyond arrangement varieties. Here, we consider *arrangement-product varieties*, i.e. Mori dream spaces X with torus action and maximal orbit quotient $X \dashrightarrow Y$, where Y is a product of projective spaces $\mathbb{P}_{c_1} \times \ldots \times \mathbb{P}_{c_t}$ and the critical divisors form a collection of hyperplanes compatible with the product structure of Y, see Definition 7.1.1. For these varieties we provide explicit descriptions of their

Cox rings and obtain classification results in the smooth case. Similar to the special arrangement case, there are no smooth arrangement-product varieties of Picard number one, see Proposition 7.1.8. In Picard number two, we obtain the following result:

Theorem (Compare Thm. 7.1.9 and Cor. 7.1.10)**.** *Every smooth projective arrangement-product variety of Picard number two is isomorphic to a variety X specified by its Cox ring*

$$\mathcal{R}(X) = \mathbb{K}[T_{11}, \ldots, T_{1k_1}, T_{21}, \ldots T_{2k_2}]/\langle g_1, g_2 \rangle,$$

where

$$g_i = \begin{cases} T_{i1}T_{i2} + \ldots + T_{ik_i-1}T_{ik_i}, & k_i \geq 6 \quad \textit{even}, \\ T_{i1}T_{i2} + \ldots + T_{ik_i-2}T_{ik_i-1} + T_{ik_i}^2, & k_i \geq 5 \quad \textit{odd}, \end{cases}$$

the matrix Q of generator degrees and an ample class $u \in \mathrm{Cl}(X) = \mathbb{Z}^2$

$$Q = \left[\begin{array}{ccc|cccc} 1 & \ldots & 1 & a_1 & a_2 & \ldots & a_{k_2} \\ 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \end{array}\right], \qquad u = [a_1 + 1, 1],$$

where we have $a_i \geq a_{i+2} \geq 0$ and $a_i + a_{i+1} = 0$ for i odd and $a_{k_2} = 0$ if k_2 is odd. Moreover, X is Fano if and only if $0 \leq a_1 \leq \frac{k_1-2}{k_2-2}$ holds.

CHAPTER

ONE

EXPLICIT $\mathbb{T}$-VARIETIES

In this chapter we develop an approach to systematically produce algebraic varieties with torus action by constructing them as suitably embedded subvarieties of toric varieties. The resulting varieties admit an explicit treatment in terms of toric geometry and graded ring theory. Our approach extends existing constructions of rational varieties with torus action of complexity one and delivers all Mori dream spaces with torus action. The results of this chapter are published in the joint work [42].

1.1 Background on toric varieties and Cox rings

In this section we provide the necessary background and fix our notation on toric geometry and Cox rings. Throughout the whole thesis, the ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. Moreover, the word variety refers to an integral separated scheme of finite type over $\mathbb{K}$. In particular, we assume varieties to be irreducible. By a point we mean a closed point.

When we speak of an *action* of an algebraic group G on a variety X, then we always assume the action map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ to be a morphism of varieties. A *torus* is an algebraic group $\mathbb{T}$ isomorphic to a *standard torus* $\mathbb{T}^n = (\mathbb{K}^*)^n$ and a $\mathbb{T}$-*variety* is a normal variety X with an effective torus action, where effective means that only the neutral element $\mathbb{1} \in \mathbb{T}$ acts trivially. The *complexity* $c(X)$ of a $\mathbb{T}$-variety X is the difference $\dim(X) - \dim(\mathbb{T})$.

Toric geometry treats the case of complexity zero. More precisely, a *toric variety* is a $\mathbb{T}$-variety Z with a base point $z_0 \in Z$ such that the orbit map $t \mapsto t \cdot z_0$ yields an open embedding $\mathbb{T} \to Z$; we call $\mathbb{T}_Z = \mathbb{T}$ the *acting torus* of Z and write $\mathbb{T}_Z \subseteq Z$, identifying $\mathbb{1} \in \mathbb{T}$ with $z_0 \in Z$ and $\mathbb{T}_Z$ with its orbit $\mathbb{T}_Z \cdot z_0$. Toric geometry originates in Demazure's work [26] in the 1970s and connects combinatorics, represented by fans, with algebraic geometry, represented by toric varieties. As introductory references, we mention [25, 66, 34, 22]. Here comes the fundamental construction, which at the end yields a covariant equivalence between the categories of fans and toric varieties.

Construction 1.1.1. A *fan* in $\mathbb{Z}^n$ is a finite collection Σ of pointed, convex, polyhedral cones living in $\mathbb{Q}^n$ such that for any $\sigma \in \Sigma$ also every face of σ belongs to Σ and for any two $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face of both, σ and σ'. Given a fan Σ in $\mathbb{Z}^n$, the *associated toric variety* Z is built by equivariantly gluing the spectra Z_σ of the monoid algebras $\mathbb{K}[M_\sigma]$ of the monoids $M_\sigma := \sigma^\vee \cap \mathbb{Z}^n$ of lattice points inside the dual cones:

$$Z \ = \ Z_\Sigma \ = \ \bigcup_{\sigma \in \Sigma} Z_\sigma, \qquad Z_\sigma \ = \ \operatorname{Spec} \mathbb{K}[M_\sigma], \qquad \mathbb{K}[M_\sigma] \ = \ \bigoplus_{u \in M_\sigma} \mathbb{K}\chi^u.$$

The acting torus $\mathbb{T}_Z = \mathbb{T}^n = \operatorname{Spec} \mathbb{K}[\mathbb{Z}^n]$ embeds via $\mathbb{K}[M_\sigma] \subseteq \mathbb{K}[\mathbb{Z}^n]$ canonically into each of the $Z_\sigma \subseteq Z$ and one takes the neutral element $\mathbb{1}_n \in \mathbb{T}_Z = \mathbb{T}^n$ as base point $z_0 \in Z$. The action of $\mathbb{T}_Z$ on Z then just extends the group structure of $\mathbb{T}_Z \subseteq Z$. Locally, on the affine open subsets $Z_\sigma \subseteq Z$, the $\mathbb{T}_Z$-action is given by its comorphism $\chi^u \mapsto \chi^u \otimes \chi^u$.

Remark 1.1.2. Let Σ be a fan in $\mathbb{Z}^n$ and Z the associated toric variety. The cones of Σ are in bijection with the $\mathbb{T}_Z$-orbits via $\sigma \mapsto \mathbb{T}_Z \cdot z_\sigma$, where z_σ denotes the common limit point for $t \to 0$ of all one-parameter groups $t \mapsto (t^{v_1}, \ldots, t^{v_n})$ of $\mathbb{T}_Z$ with $v \in \mathbb{Z}^n$ taken from the relative interior $\sigma^\circ \subseteq \sigma$. The dimension of $\mathbb{T}_Z \cdot z_\sigma$ equals $n - \dim(\sigma)$. In particular, the rays $\varrho_1, \ldots, \varrho_r$ of Σ, that means the one-dimensional cones, define the $\mathbb{T}_Z$-invariant prime divisors $D_i := \overline{\mathbb{T}_Z \cdot z_{\varrho_i}}$ of Z.

Cox's quotient presentation generalizes the classical construction of the projective space $\mathbb{P}_n$ as the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ by $\mathbb{K}^*$ acting via scalar multiplication. It delivers, for instance, any complete toric variety as a quotient of an open toric subset of some affine space by a *quasitorus*, that means an algebraic group isomorphic to a direct product of a torus and a finite abelian group. Below and later, we write $\tau \preccurlyeq \sigma$ if $\tau \subseteq \sigma$ is a face of the convex, polyhedral cone σ.

Construction 1.1.3. See [21], also [22, Sec. 5] and [6, Sec. 2.1.3]. Consider a fan Σ in $\mathbb{Z}^n$ and let $\varrho_1, \ldots, \varrho_r$ denote its rays. In each ϱ_i sits a unique primitive lattice vector v_i, the generator of the monoid $\varrho_i \cap \mathbb{Z}^n$. The *generator matrix of* Σ is the $(n \times r)$-matrix

$$P \ = \ [v_1, \ldots, v_r]$$

having $v_1, \ldots, v_r$ as its columns, numbered accordingly to $\varrho_1, \ldots, \varrho_r$. We use the letter P as well to denote the associated linear maps $\mathbb{Z}^r \to \mathbb{Z}^n$ and $\mathbb{Q}^r \to \mathbb{Q}^n$. As any integral $n \times r$ matrix, P defines a homomorphism of tori

$$p\colon \mathbb{T}^r \ \to \ \mathbb{T}^n, \qquad t \ \mapsto \ (t^{P_{1*}}, \ldots, t^{P_{n*}})$$

where $t^{P_{i*}} = t_1^{p_{i1}} \cdots t_r^{p_{ir}}$ has the i-th row of $P = (p_{ij})$ as its exponent vector. Now assume that $v_1, \ldots, v_r$ generate $\mathbb{Q}^n$ as a vector space, meaning that the associated toric variety

Z has no torus factor. Consider the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ and the set

$$\hat{\Sigma} \ := \ \{\tau \preccurlyeq \gamma;\ P(\tau) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

Then $\hat{\Sigma}$ is a subfan of the fan $\bar{\Sigma}$ of faces of the orthant $\gamma \subseteq \mathbb{Q}^r$. Moreover, P sends cones from $\hat{\Sigma}$ into cones of Σ. Thus, $p\colon \mathbb{T}^r \to \mathbb{T}^n$ extends to a morphism $p\colon \hat{Z} \to Z$ of the associated toric varieties. We arrive at the following picture

$$\begin{array}{ccccc} \hat{Z} & \subseteq & \bar{Z} & := & \mathbb{K}^r \\ {\scriptstyle p}\big\downarrow {\scriptstyle /\!\!/ H} & & & & \\ Z & & & & \end{array}$$

where $\hat{Z} \subseteq \bar{Z}$ is an open $\mathbb{T}^r$-invariant subvariety and $H \subseteq \mathbb{T}^r$ is the kernel of the homomorphism $p\colon \mathbb{T}^r \to \mathbb{T}^n$ of the acting tori. Being a closed subgroup of a torus, H is a quasitorus. For any cone $\sigma \in \Sigma$, we have

$$p^{-1}(Z_\sigma) \ = \ \hat{Z}_{\hat{\sigma}}, \quad \hat{\sigma} \ := \ \mathrm{cone}(e_i;\ v_i = P(e_i) \in \sigma), \qquad p^*\mathcal{O}(Z_\sigma) \ = \ \mathcal{O}(\hat{Z}_{\hat{\sigma}})^H,$$

where $e_i \in \mathbb{Z}^r$ is the i-th canonical basis vector. Thus, $p\colon \hat{Z} \to Z$ is an affine morphism and the pull back functions are precisely the H-invariants. In other words, p is a *good quotient* for the H-action, as indicated by "$/\!\!/ H$".

Generalizing the idea of homogeneous coordinates on the projective space, one uses Cox's quotient presentation to obtain global coordinates on toric varieties.

Remark 1.1.4. Let Z be a toric variety with quotient presentation $p\colon \hat{Z} \to Z$ as in 1.1.3. Then every p-fiber contains a unique closed H-orbit. The presentation in *Cox coordinates* of a point $x \in Z$ is

$$x \ = \ [z_1, \dots, z_r], \qquad \text{where } z = (z_1, \dots, z_r) \in p^{-1}(x) \text{ with } H \cdot z \subseteq \hat{Z} \text{ closed.}$$

Thus, $[z]$ and $[z']$ represent the same point $x \in Z$ if and only if z and z' lie in the same closed H-orbit of $\hat{Z}$. For instance, the points $z_\sigma \in Z$, where $\sigma \in \Sigma$, are given in Cox coordinates as

$$z_\sigma \ = \ [\varepsilon_1, \dots, \varepsilon_r], \qquad \varepsilon_i \ = \ \begin{cases} 0, & P(e_i) \in \sigma, \\ 1, & P(e_i) \notin \sigma. \end{cases}$$

Remark 1.1.5. Let Z be a toric variety with quotient presentation $p\colon \hat{Z} \to Z$ as in 1.1.3. Then we obtain an injection from the closed subvarieties $X \subseteq Z$ to the H-invariant closed subvarieties of $\bar{Z} = \mathbb{K}^r$ via

$$X \ \mapsto \ \bar{X} \ := \ \overline{p^{-1}(X)} \subseteq \bar{Z}.$$

The vanishing ideal $I(\bar{X}) \subseteq \mathbb{K}[T_1, \dots, T_r]$ is generated by polynomials $g_1, \dots, g_s$ being H-homogeneous in the sense that $g_j(h \cdot z) = \chi_j(h) g_j(z)$ holds with characters $\chi_j \in \mathbb{X}(H)$. We call $g_1, \dots, g_s$ *defining equations in Cox coordinates* for $X \subseteq Z$.

We turn to Cox rings. Their history starts in the 1970s in a geometric setting, when Colliot-Thélène and Sansuc introduced the universal torsors presenting smooth varieties in a universal way as quotients [20]. In toric geometry, the quotient presentation and Cox rings popped up in the 1990s in work of Audin [7], Cox [21] and others. In 2000, Hu and Keel observed fundamental connections between Cox rings, Mori theory and geometric invariant theory [51]. As a general introductory reference on Cox rings, we mention [6].

We enter the subject. Consider a normal variety X with only constant invertible global functions and finitely generated divisor class group $\mathrm{Cl}(X)$. For a Weil divisor D on X, denote by $\mathcal{O}(D)$ the associated sheaf of sections. Then the *Cox sheaf* $\mathcal{R}$ and the *Cox ring* $\mathcal{R}(X)$ of X are defined as

$$\mathcal{R} \ := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}(D), \qquad \mathcal{R}(X) \ := \ \Gamma(X,\mathcal{R}) \ = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X,\mathcal{O}(D)).$$

Observe that we grade $\mathcal{R}$ and $\mathcal{R}(X)$ by divisor classes whereas the homogeneous components are defined by divisors. If $\mathrm{Cl}(X)$ is torsion free, then this problem of well-definedness is solved by just regarding $\mathcal{R}$ as the sheaf of multi-section algebras: fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ of the Weil divisor group mapping isomorphically onto $\mathrm{Cl}(X)$ and work with

$$\mathcal{R} \ := \ \mathcal{S} \ = \ \bigoplus_{D\in K} \mathcal{O}(D).$$

The case of torsion in $\mathrm{Cl}(X)$ requires more care: fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ mapping onto $\mathrm{Cl}(X)$, denote by $K^0 \subseteq K$ the subgroup consisting of all principal divisors of K and choose functions $\chi^E \in \mathbb{K}(X)$, where $E \in K^0$, satisfying

$$\mathrm{div}(\chi^E) \ = \ E, \qquad \chi^E\chi^{E'} \ = \ \chi^{E+E'}.$$

Consider the sheaf $\mathcal{S}$ of multi-section algebras associated with K, the subsheaf $\mathcal{I} \subseteq \mathcal{S}$ of ideals generated by $1-\chi^E$, where $E \in K^0$ and define $\mathcal{R} := \mathcal{S}/\mathcal{I}$. Then $\mathcal{R}$ is graded by $K/K^0 = \mathrm{Cl}(X)$ via

$$\mathcal{R}_{[D]} \ := \ \pi\left(\bigoplus_{D+K^0} \mathcal{O}(D)\right),$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. Then, up to isomorphy, this construction turns out not to depend on any of the choices made; we refer to [14, 39] and [6, Sec. 1.1.4] for the details.

Remark 1.1.6. See [21], also [22, Sec. 5] and [6, Sec. 2.1.3]. Let Z be a toric variety without torus factor and let $D_1, \ldots, D_r$ be the $\mathbb{T}_Z$-invariant prime divisors of Z. Then the Cox ring of Z and its $\mathrm{Cl}(Z)$-grading are given as

$$\mathcal{R}(Z) \ = \ \mathbb{K}[T_1,\ldots,T_r], \qquad \deg(T_i) \ = [D_i] \ \in \ \mathrm{Cl}(Z).$$

For a more explicit picture, let Z arise from a fan Σ in $\mathbb{Z}^n$. Then $D_i = \overline{\mathbb{T}_Z \cdot z_{\varrho_i}} \subseteq Z$ holds with the rays $\varrho_1, \ldots, \varrho_r$ of Σ. The divisor class group of Z and the divisor classes of the D_i are described by

$$\mathrm{Cl}(Z) \ = \ K \ := \ \mathbb{Z}^r/\mathrm{im}(P^*), \qquad\qquad [D_i] \ = \ w_i \ := \ Q(e_i) \ \in \ K,$$

where we denote by P^* the transpose of the generator matrix P of Σ, by $Q\colon \mathbb{Z}^r \to K$ the projection and by $e_i \in \mathbb{Z}^r$ the i-th canonical basis vector.

Remark 1.1.7. In Construction 1.1.3, the toric variety Z is represented as a quotient of $\hat{Z} \subseteq \mathbb{K}^r$ by the quasitorus $H = \ker(p) \subseteq \mathbb{T}^r$. With $K = \mathrm{Cl}(Z) = \mathbb{Z}^r/\mathrm{im}(P^*)$ from Remark 1.1.6, we can view H also as the spectrum of the associated group algebra:

$$H \ = \ \mathrm{Spec}\, \mathbb{K}[K], \qquad\qquad \mathbb{K}[K] \ = \ \bigoplus_{w \in K} \mathbb{K}\chi^w.$$

Here, the elements $\chi^w \in \mathbb{K}[K]$ are the characters of H and $w \mapsto \chi^w$ defines an isomorphism between K and the character group $\mathbb{X}(H)$. Setting $\chi_i := \chi^{w_i}$, we retrieve the H-action from the $\mathrm{Cl}(Z)$-grading of the Cox ring $\mathcal{R}(Z)$ as

$$h \cdot z \ = \ (\chi_1(h) z_1, \ldots, \chi_r(h) z_r).$$

In general, the Cox ring $\mathcal{R}(X)$ is normal, integral and, as its main algebraic feature, it is $\mathrm{Cl}(X)$-factorial [4, 39]. Let us recall the meaning. A ring $R = \oplus_K R_w$ graded by an abelian group K is *K-integral* if it has no homogeneous zero divisors. A nonzero homogeneous non-unit $f \in R$ is *K-prime* if, whenever f divides a product gh of two homogeneous $g, h \in R$, then it divides g or h. The ring R is called *K-factorial* if it is K-integral and every nonzero homogeneous non-unit of R is a product of K-primes. If $\mathrm{Cl}(X)$ is torsion free, then the Cox ring admits unique factorization in the usual sense, see [14, Prop. 8.4] and also [28, Cor. 1.2].

The bunched ring approach presented in [15, 38, 6] uses Cox rings to encode algebraic varieties. The central construction starts with a given K-factorial ring R and produces varieties X having divisor class group K and Cox ring R. In this thesis, we will work with the following variant being closer to toric geometry in the sense that it uses fans instead of the bunches of cones of [15, 38, 6]. Let us fix the necessary notation. By an *affine algebra* we mean a finitely generated reduced $\mathbb{K}$-algebra. If K is an abelian group, then we denote by $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ the associated rational vector space. Given $w \in K$, we write as well w for the element $w \otimes 1 \in K_{\mathbb{Q}}$. Moreover, if $Q\colon K \to K'$ is a homomorphism, we denote the associated linear map $K_{\mathbb{Q}} \to K'_{\mathbb{Q}}$ as well by Q.

Construction 1.1.8. Let K be a finitely generated abelian group and $R = \oplus_K R_w$ a K-factorial, normal, integral, affine $\mathbb{K}$-algebra with only constant homogeneous units. Suppose that $f_1, \ldots, f_r$ are pairwise non-associated K-prime generators of R such that any $r-1$ of the degrees $w_i := \deg(f_i)$ generate K as a group and for $\tau_i := \mathrm{cone}(w_j;\ j \neq i)$, the intersection $\tau_1 \cap \ldots \cap \tau_r$ is of full dimension in $K_{\mathbb{Q}}$. Consider the closed embedding

$$\mathrm{Spec}\, R =: \bar{X} \ \xrightarrow{x \mapsto (f_1(x), \ldots, f_r(x))} \ \bar{Z} := \mathbb{K}^r.$$

The quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ acts on $\bar{Z}$ via $h \cdot z = (\chi_1(h)z_1, \dots, \chi_r(h)z_r)$, where $\chi_i \in \mathbb{X}(H)$ is the character corresponding to $w_i \in K$. This action leaves the subvariety $\bar{X} \subseteq \bar{Z}$ invariant. Now, consider the degree map

$$Q\colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ w_i.$$

Let P be an integral $(n \times r)$-matrix, the rows of which generate $\ker(Q) \subseteq \mathbb{Z}^r$. Then the assumptions on $f_1, \dots, f_r$ ensure that the columns of P are pairwise different and primitive, see [6, Thm. 2.2.2.6 and Lemma 2.1.4.1]. Fix any fan Σ having P as generator matrix. The associated toric variety Z and $p\colon \hat{Z} \to Z$ from Construction 1.1.3 fit into a commutative diagram

$$\begin{array}{ccc} \hat{X} & \subseteq & \hat{Z} \\ {\scriptstyle /\!\!/ H} \big\downarrow {\scriptstyle p} & & {\scriptstyle p} \big\downarrow {\scriptstyle /\!\!/ H} \\ X & \subseteq & Z \end{array}$$

where we set $\hat{X} := \bar{X} \cap \hat{Z}$ and $X := \hat{X} /\!\!/ H$ is a normal, closed subvariety of Z. We speak of $X \subseteq Z$ as an *explicit variety*, refer to $\alpha = (f_1, \dots, f_r)$ as the *embedding system* of $X \subseteq Z$ and call any system of K-homogeneous generators $g_1, \dots, g_s$ of the vanishing ideal $I(\bar{X}) \subseteq \mathbb{K}[T_1, \dots, T_r]$ *defining equations* for X.

Remark 1.1.9. If, in Construction 1.1.8, the ambient toric variety Z is affine (complete, projective), then the resulting X is affine (complete, projective).

The interface from explicit varieties to bunched rings relies on linear Gale duality. Here comes how this concretely works in our situation.

Remark 1.1.10. Notation as in 1.1.3, 1.1.6 and 1.1.8. To any explicit variety $X \subseteq Z$, we can apply the machinery of bunched rings [6, Chap. 3]. The translation into the latter setting runs as follows. Consider the homomorphisms

$$P\colon \mathbb{Z}^r \to \mathbb{Z}^n, \ e_i \mapsto v_i, \qquad\qquad Q\colon \mathbb{Z}^r \to K, \ e_i \mapsto w_i,$$

where $K = \mathbb{Z}^r/\operatorname{im}(P^*)$. For $\sigma \in \Sigma$, the face $\hat{\sigma} \preccurlyeq \gamma \subseteq \mathbb{Q}^r$ of the orthant is generated by the e_i with $v_i \in \sigma$. The *complementary face* $\hat{\sigma}^* \preccurlyeq \gamma$ of $\hat{\sigma} \preccurlyeq \gamma$ is generated by the e_i with $e_i \notin \hat{\sigma}$. Set

$$\Phi \ := \ \{Q(\hat{\sigma}^*); \ \sigma \in \Sigma \text{ with } X \cap \mathbb{T}_z \cdot z_\sigma \neq \emptyset\}.$$

Then Φ is a collection of cones in $K_\mathbb{Q}$ with pairwise intersecting relative interiors. With the system of generators $\mathfrak{F} := (f_1, \dots, f_r)$, we obtain a *bunched ring* $(R, \mathfrak{F}, \Phi)$ in the sense of [6, Def. 3.2.1.2]. We have an open inclusion

$$X \ \subseteq \ X(R, \mathfrak{F}, \Phi)$$

into the variety associated with the bunched ring [6, Def. 3.2.1.3] such that the complement of X in $X(R, \mathfrak{F}, \Phi)$ is of codimension at least two. If X is affine or complete, then the above inclusion is even an equality.

Based on this translation, we will import several statements on the geometry of explicit ($\mathbb{T}$-)varieties in Section 1.4. For the moment, we just mention the following.

Remark 1.1.11. See [6, Thm. 3.2.1.4]. For every explicit variety $X \subseteq Z$, the divisor class group and the Cox ring of X are given as

$$\mathrm{Cl}(X) \ = \ K \ = \ \mathrm{Cl}(Z), \qquad\qquad \mathcal{R}(X) \ = \ R \ = \ \mathcal{R}(Z)/I(\bar{X}).$$

Moreover, $\hat{X}$ is the relative spectrum of the Cox sheaf on X, which in turn is given as the direct image $\mathcal{R} = p_*\mathcal{O}_{\hat{X}}$. Finally, we have the prime divisors

$$D_i^X \ = \ X \cap D_i^Z \ \subseteq \ X$$

induced by the toric prime divisors $D_1^Z, \ldots, D_r^Z$. Here each D_i^X is determined by the property $p^* D_i^X = V_{\bar{X}}(T_i)$.

Coming embedded into a toric variety, every explicit variety $X \subseteq Z$ inherits the A_2-*property*: any two points of X admit a common affine neighborhood. The normal A_2-varieties are precisely the normal varieties that are embeddable into a toric variety, see [77]. An A_2-variety Y is A_2*-maximal* if it does not allow open embeddings into A_2-varieties Y' such that $Y' \setminus Y$ is non-empty of codimension at least two. For example, affine and projective varieties are A_2-maximal.

Remark 1.1.12. See [6, Thm. 3.2.1.9]. Every A_2-maximal variety with only constant invertible global functions, finitely generated divisor class group and finitely generated Cox ring can be represented as an explicit variety.

In [51], Hu and Keel introduced the *Mori dream spaces* as $\mathbb{Q}$-factorial projective varieties with a Mori chamber decomposition satisfying suitable finiteness properties which in particular guarantee an optimal behavior with respect to the minimal model programme.

Remark 1.1.13. According to [51, Prop. 2.9], the Mori dream spaces are precisely the $\mathbb{Q}$-factorial projective varieties with a finitely generated Cox ring. In particular, every Mori dream space can be represented as an explicit variety.

1.2 Constructing explicit $\mathbb{T}$-varieties

We present our method of producing systematically explicit $\mathbb{T}$-varieties, see Construction 1.2.5, and formulate basic properties, see Proposition 1.2.7, Theorem 1.2.10 and Proposition 1.2.17. The proofs of the latter results are given in the subsequent section. We begin by indicating the ideas behind Construction 1.2.5. First, take a glance at the following naive way to produce varieties with torus action sitting inside a given toric variety.

Recipe 1.2.1. Let Z be a toric variety, $\mathbb{T}_Z = \mathbb{T}' \times \mathbb{T}$ a splitting of the acting torus into closed subtori and $Y \subseteq \mathbb{T}'$ a closed subvariety. Consider the closure

$$X \;:=\; \overline{Y \times \mathbb{T}} \;\subseteq\; Z.$$

Then the variety $X \subseteq Z$ is invariant under the action of $\mathbb{T}$ on Z and thus we obtain an effective algebraic torus action $\mathbb{T} \times X \to X$. By construction, we have

$$X \cap \mathbb{T}_Z \;=\; Y \times \mathbb{T}, \qquad\qquad \mathbb{K}(X)^{\mathbb{T}} \;=\; \mathbb{K}(Y).$$

In particular, Y represents the field of $\mathbb{T}$-invariant rational functions of X and thus the projection $\mathbb{T}_Z \to \mathbb{T}'$ defines a *rational quotient* $X \dashrightarrow Y$ for the $\mathbb{T}$-action on X.

So far, Recipe 1.2.1 provides no specifically close relations between the geometry of X and that of its ambient toric variety Z. Nevertheless, we know in advance the rational quotient Y and, stemming from a subtorus action on Z, the $\mathbb{T}$-action on X can be studied by toric methods. Moreover, Recipe 1.2.1 produces for instance all projective $\mathbb{T}$-varieties, as we infer from the following.

Remark 1.2.2. Any $\mathbb{T}$-variety X that admits an equivariant embedding into a toric variety Z with $\mathbb{T}$ acting as a subtorus of $\mathbb{T}_Z$ can be represented as in Recipe 1.2.1. The techniques from [37, 38] yield such equivariant embeddings for $\mathbb{T}$-varieties X with the A_2-property provided they are $\mathbb{Q}$-factorial or, more generally, divisorial in the sense of [16], or have a Cox sheaf of locally finite type.

Our aim is to bring together the features of Recipe 1.2.1 with those of Construction 1.1.8. Let us first look at a concrete example, indicating the main rules of the subsequent construction game and illustrating the notation used there. The example we are going to treat is a well known $\mathbb{K}^*$-surface, occurring as an important step in resolving the E_6-singular cubic surface; see [36, Sec. 4] and, for links to various other aspects, also [6, p. 522].

Example 1.2.3. Our initial data is a projective line $Y \subseteq \mathbb{P}_2$ given in homogeneous coordinates by the following equation:

$$Y \;=\; V(T_0 + T_1 + T_2) \;\subseteq\; \mathbb{P}_2.$$

We regard $\mathbb{P}_2$ as the toric variety defined by the complete fan Δ with the generator matrix

$$B \;=\; [u_0, u_1, u_2] \;=\; \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Now we start the game that builds up the generator matrix P of the fan of the prospective ambient toric variety Z of our final X. First produce a matrix

$$P_0 \;=\; [u_{01,}, u_{02}, u_{11}, u_{21}] \;=\; \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \end{bmatrix},$$

the columns u_{ij} of which are positive multiples of the columns u_i of B. Then append a zero column to P_0, a block d below P_0 and a block d' below the zero column:

$$P = [v_{01,}, v_{02}, v_{11}, v_{21}, v_1] = \begin{bmatrix} -3 & -1 & 3 & 0 & 0 \\ -3 & -1 & 0 & 2 & 0 \\ -2 & -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} P_0 & 0 \\ d & d' \end{bmatrix}.$$

Let Σ be any complete fan in $\mathbb{Z}^3$ having P as its generator matrix and let Z be the associated toric variety. Then the acting torus $\mathbb{T}_Z$ of Z splits as

$$\mathbb{T}_Z = \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{K}^*.$$

Moreover, $Y \cap \mathbb{T}^2 \subseteq \mathbb{T}^2$ is the zero set of $1 + T_1/T_0 + T_2/T_0$. Proceeding exactly as in Recipe 1.2.1 yields a surface X coming with an effective $\mathbb{K}^*$-action:

$$X := \overline{(Y \cap \mathbb{T}^2) \times \mathbb{K}^*} \subseteq Z.$$

We have $X \cap \mathbb{T}^3 = V(1 + T_1/T_0 + T_2/T_0)$. Pulling back that equation via the homomorphism $p\colon \mathbb{T}^5 \to \mathbb{T}^3$ given by P leads to the equation for X in Cox coordinates:

$$\bar{X} = V(T_{01}^3 T_{02} + T_{11}^3 + T_{21}^2) \subseteq \bar{Z} = \mathbb{K}^5,$$

where the variables T_{ij} represent columns of P_0 and the index ij tells us that we have the j-th repetition of the i-th column of B, scaled by the exponent l_{ij} of T_{ij}.

Remark 1.2.4. In Example 1.2.3, we encountered two explicit varieties in the sense of Construction 1.1.8: first, the projective line $Y \subseteq \mathbb{P}_2$ and second, the $\mathbb{K}^*$-surface $X \subseteq Z$. In particular, divisor class group and Cox ring of X are given as

$$\begin{aligned} \mathrm{Cl}(X) &= K = \mathbb{Z}^5/\mathrm{im}(P^*) = \mathbb{Z}^2, \\ \mathcal{R}(X) &= \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_1]/\langle T_{01}^3 T_{02} + T_{11}^3 + T_{21}^2 \rangle. \end{aligned}$$

Observe that the manipulations on the matrix B turned the redundant defining relation $T_0 + T_1 + T_2$ of Y into the serious relation $T_{01}^3 T_{02} + T_{11}^3 + T_{21}^2$, defining the resulting X.

We come to the general construction of explicit $\mathbb{T}$-varieties. It starts with a given explicit variety $Y \subseteq Z_\Delta$ provided by Construction 1.1.8 and delivers an explicit variety $X \subseteq Z_\Sigma$ which is invariant under a direct factor $\mathbb{T} \subseteq \mathbb{T}_\Sigma$ of the acting torus $\mathbb{T}_\Sigma \subseteq Z_\Sigma$.

Construction 1.2.5. Let $Y \subseteq Z_\Delta$ be an explicit variety with embedding system $\alpha = (f_0, \ldots, f_r)$. The defining fan Δ of Z_Δ lives in some $\mathbb{Z}^t$ and has a $t \times (r+1)$ generator matrix

$$B = [u_0, \ldots, u_r].$$

In particular, $\mathrm{Cl}(Y) = \mathrm{Cl}(Z_\Delta)$ equals $K_B := \mathbb{Z}^{r+1}/\mathrm{im}(B^*)$ and the Cox ring of Y equals the K_B-factorial input ring R_Y of Construction 1.1.8. We build up a new matrix from B and the following data

- positive integers $n_0, \dots, n_r$ and non-negative integers m, s with $t + s \le n + m$, where $n := n_0 + \dots + n_r$,
- for any two i, j, where $i = 0, \dots, r$ and $j = 1, \dots, n_i$, a positive integer l_{ij} and a vector $d_{ij} \in \mathbb{Z}^s$,
- for any k, where $1 \le k \le m$, a vector $d'_k \in \mathbb{Z}^s$,

where, with the multiples $u_{ij} := l_{ij} u_i \in \mathbb{Z}^t$ of the columns of B, we require that the vectors

$$v_{ij} \;=\; (u_{ij}, d_{ij}) \;\in\; \mathbb{Z}^{t+s}, \qquad\qquad v_k \;=\; (0, d'_k) \;\in\; \mathbb{Z}^{t+s}$$

are all primitive, any two of them are distinct and altogether they generate $\mathbb{Q}^{t+s}$ as a vector space. Store the v_{ij} and v_k as columns in a $(t + s) \times (n + m)$ matrix

$$P \;=\; [v_{ij}, v_k] \;=\; \begin{bmatrix} u_{01} & \dots & u_{0n_0} & \dots & u_{r1} & \dots & u_{rn_r} & 0 & \dots & 0 \\ d_{01} & \dots & d_{0n_0} & \dots & d_{r1} & \dots & d_{rn_r} & d'_1 & \dots & d'_m \end{bmatrix}.$$

Now, let Σ be any fan in $\mathbb{Z}^{t+s}$ having P as its generator matrix and denote by Z_Σ the associated toric variety. Then we obtain a commutative diagram

$$\begin{array}{ccc} X & \subseteq & Z_\Sigma \\ \downarrow & & \downarrow \\ Y & \subseteq & Z_\Delta \end{array}$$

where the rational map $Z_\Sigma \dashrightarrow Z_\Delta$ is given by the projection $\mathbb{T}^t \times \mathbb{T}^s \to \mathbb{T}^t$ of the respective acting tori $\mathbb{T}_\Sigma = \mathbb{T}^t \times \mathbb{T}^s$ and $\mathbb{T}_\Delta = \mathbb{T}^t$ and we define

$$X \;:=\; X(\alpha, P, \Sigma) \;:=\; \overline{(Y \cap \mathbb{T}^t) \times \mathbb{T}^s} \;\subseteq\; Z_\Sigma.$$

Then $X \subseteq Z_\Sigma$ is invariant under the action of the subtorus $\mathbb{T} = \{\mathbb{1}_t\} \times \mathbb{T}^s$ of the acting torus $\mathbb{T}_\Sigma = \mathbb{T}^t \times \mathbb{T}^s$ of Z_Σ. Moreover, set

$$T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{K}[T_{ij}, S_k], \qquad K_P := \mathbb{Z}^{n+m}/\mathrm{im}(P^*) = \mathrm{Cl}(Z_\Sigma).$$

Let $h_1, \dots, h_q$ be defining equations of Y in Cox coordinates, that means K_B-homogeneous generators for the ideal of relations between $f_0, \dots, f_r$. Consider the factor ring

$$R(\alpha, P) \;:=\; \mathbb{K}[T_{ij}, S_k]/\langle h_1(T_0^{l_0}, \dots, T_r^{l_r}), \dots, h_q(T_0^{l_0}, \dots, T_r^{l_r})\rangle$$

and denote by $Q_P \colon \mathbb{Z}^{n+m} \to K_P$ the projection. We turn $R(\alpha, P)$ into a K_P-graded algebra via

$$\deg(T_{ij}) \;:=\; w_{ij} \;:=\; Q_P(e_{ij}), \qquad \deg(T_k) \;:=\; w_k \;:=\; Q_P(e_k),$$

where $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ are the canonical basis vectors. Observe that we have a unique homomorphism of graded algebras $R_Y \to R(\alpha, P)$ sending f_i to $T_i^{l_i}$.

Remark 1.2.6. For $t = n = 0$, the above construction yields the usual construction of a toric variety from a fan. Moreover, for $s = m = 0$ and $n = r + 1$, we arrive at Construction 1.1.8.

We come to the first basic property of Construction 1.2.5. Note that in concrete cases the assumptions of this proposition on $R(\alpha, P)$ and the T_{ij} made below can be checked algorithmically via absolute factorization; see [44, Rem. 3.8].

Proposition 1.2.7. *Let $X = X(\alpha, P, \Sigma)$ arise from Construction 1.2.5. If $R(\alpha, P)$ is a K_P-integral affine algebra with only constant homogeneous units and the T_{ij} define pairwise non-associated K_P-primes in $R(\alpha, P)$, then $X \subseteq Z_\Sigma$ is an explicit variety.*

Definition 1.2.8. By an *explicit $\mathbb{T}$-variety $X \subseteq Z$* we mean a variety $X = X(\alpha, P, \Sigma)$ in $Z = Z_\Sigma$ together with the action of $\mathbb{T} = \{\mathbb{1}_t\} \times \mathbb{T}^s$ arising from Construction 1.2.5 such that the assumptions of Proposition 1.2.7 are satisfied.

Corollary 1.2.9. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Then X is a normal variety with only constant invertible global functions. Moreover, dimension, complexity, divisor class group and Cox ring of X are given by*

$$\dim(X) = s + \dim(Y), \quad c(X) = \dim(Y), \quad \mathrm{Cl}(X) = K_P, \quad \mathcal{R}(X) = R(\alpha, P).$$

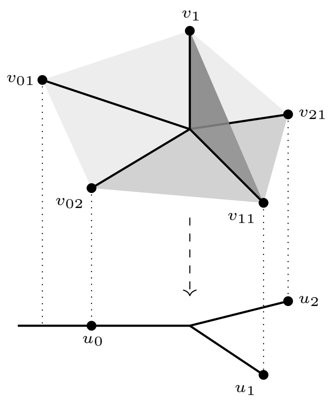
We say that a $\mathbb{T}$-variety X' admits a *presentation as an explicit $\mathbb{T}$-variety* if there is a $\mathbb{T}$-equivariant isomorphism $X' \to X$ with some explicit $\mathbb{T}$-variety $X \subseteq Z$.

Theorem 1.2.10. *Let X be an A_2-maximal $\mathbb{T}$-variety having only constant invertible global functions, finitely generated divisor class group and finitely generated Cox ring. Then X admits a presentation as an explicit $\mathbb{T}$-variety.*

Corollary 1.2.11. *Every Mori dream space with an effective torus action admits a presentation as an explicit $\mathbb{T}$-variety.*

In the rest of the section, we discuss the geometry of the torus action of an explicit $\mathbb{T}$-variety $X \subseteq Z$, aiming for a suitable quotient. First, we continue Example 1.2.3.

Example 1.2.12. Consider again the explicit $\mathbb{K}^*$-surface $X \subseteq Z$ from 1.2.3. An important source of information is the location of the columns of P over those of B with respect to the projection $\mathrm{pr}\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ onto the first two coordinates:



Each column v_{ij} projects into the ray through u_i and v_1 lies in the kernel of the projection. The rays ϱ_{ij} through v_{ij} and ϱ_1 through v_1 define prime divisors D_{ij}^Z and E_1^Z of Z, respectively. Cutting down to X gives us prime divisors

$$D_{ij}^X \ := \ X \cap D_{ij}^Z \ \subseteq \ X, \qquad\qquad E_1^X \ := \ X \cap E_1^Z \ \subseteq \ X,$$

where the basic reason for primality is that the divisors are given in Cox coordinates by K-prime ideals; for instance D_{01} is defined by $\langle T_{01}, T_{11}^3 + T_{21}^2 \rangle$. We are interested in the isotropy groups. Recall that $\mathbb{K}^*$ acts on X as the subtorus

$$\mathbb{T} \ := \ \{\mathbb{1}_2\} \times \mathbb{K}^* \ \subseteq \ \mathbb{T}^3 \ = \ \mathbb{T}_Z.$$

In particular, the isotropy groups of the $\mathbb{T}$-action are constant along the $\mathbb{T}_Z$-orbits. Consider the kernel $L = \{0\} \times \mathbb{Z}$ of $\mathrm{pr}\colon \mathbb{Z}^3 \to \mathbb{Z}^2$. Then [6, Prop. 2.1.4.2] yields for any $\sigma \in \Sigma$ that the isotropy group of $\mathbb{T}$ at $z_\sigma \in Z$ has character group

$$\mathbb{X}(\mathbb{T}_{z_\sigma}) \ \cong \ (L \cap \mathrm{lin}(\sigma)) \oplus (\mathrm{pr}(\mathrm{lin}(\sigma)) \cap \mathbb{Z}^2)/(\mathrm{pr}(\mathrm{lin}(\sigma) \cap \mathbb{Z}^3),$$

where $\mathrm{lin}(\sigma) \subseteq \mathbb{Q}^3$ denotes the $\mathbb{Q}$-linear hull. Looking at $\sigma = \varrho_{ij}$, we see that the isotropy group $\mathbb{T}_x$ of the general point $x \in D_{ij}^X$ is cyclic of order l_{ij}, where l_{ij} is the exponent of T_{ij} in the defining relation of X, that means

$$l_{01} = 3, \qquad l_{02} = 1, \qquad l_{11} = 3, \qquad l_{21} = 2.$$

Moreover, the curve E_1^X consists of fixed points of the $\mathbb{T}$-action and there are two isolated fixed points, forming the intersections of X with the toric orbits $\mathbb{T}_Z \cdot z_\sigma$ for $\sigma = \mathrm{cone}(v_{01}, v_{02})$ and $\sigma = \mathrm{cone}(v_{02}, v_{11}, v_{21})$, respectively. In particular, we see that

$$X_0 \ = \ X \cap (\mathbb{T}_Z \cup \mathbb{T}_Z \cdot z_{\varrho_{01}} \cup \mathbb{T}_Z \cdot z_{\varrho_{02}} \cup \mathbb{T}_Z \cdot z_{\varrho_{11}} \cup \mathbb{T}_Z \cdot z_{\varrho_{21}}) \ \subseteq \ X$$

is the open subset of X consisting of all points $x \in X$ having finite isotropy group $\mathbb{T}_x$. The projection $\mathrm{pr}\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ defines a rational quotient $Z \dashrightarrow \mathbb{P}_2$ for the $\mathbb{T}$-action inducing a rational quotient $X \dashrightarrow Y$ which in turn is defined on $X_0 \subseteq X$ and gives a surjective morphism $X_0 \to Y$, where $Y = \mathbb{P}_1$.

Before entering the general case, let us give the precise definitions of the necessary concepts of quotients. For the moment, X may be any variety with an action of an algebraic group G. As already indicated, a *rational quotient* for the G-variety X is a dominant rational map $\pi\colon X \dashrightarrow Y$ such that $\pi^*\mathbb{K}(Y) = \mathbb{K}(X)^G$ holds. A *representative* of a rational quotient $\pi\colon X \dashrightarrow Y$ is a surjective morphism $W \to V$ representing π on a non-empty open G-invariant subset $W \subseteq X$ and an open subset $V \subseteq Y$. By results of Rosenlicht, rational quotients always exist and admit a representative having G-orbits as its fibers [73].

Behind Construction 1.2.5 there is a specific rational quotient, the maximal orbit quotient. Recall that a *geometric quotient* of a $\mathbb{T}$-variety X is a good quotient $X \to Y$ having precisely the $\mathbb{T}$-orbits as its fibers. Moreover, for any $\mathbb{T}$-variety X, we denote by $X_0 \subseteq X$ the open subset consisting of all points $x \in X$ with finite isotropy group.

Definition 1.2.13. A *maximal orbit quotient* for a $\mathbb{T}$-variety X is a rational quotient $\pi\colon X \dashrightarrow Y$ admitting a representative $\psi\colon W \to V$ and prime divisors $C_0, \ldots, C_r$ on Y such that the following properties are satisfied:

(i) one has $W \subseteq X_0$ and the complements $X_0 \setminus W \subseteq X_0$ and $Y \setminus V \subseteq Y$, both are of codimension at least two,

(ii) for every $i = 0, \ldots, r$, the inverse image $\psi^{-1}(C_i) \subseteq W$ is a union of prime divisors $D_{i1}, \ldots, D_{in_i} \subseteq W$,

(iii) all $\mathbb{T}$-invariant prime divisors of X_0 with non-trivial generic isotropy group occur among the D_{ij},

(iv) every sequence $J = (j_0, \ldots, j_r)$ with $1 \le j_i \le n_i$ defines a geometric quotient $\psi\colon W_J \to V$ for the $\mathbb{T}$-action, where $W_J := W \setminus \cup_{j \neq j_i} D_{ij}$.

We call $C_0, \ldots, C_r \subseteq Y$ a collection of *doubling divisors* for $\pi\colon X \dashrightarrow Y$. The closure of any D_{ij} in X is a $\mathbb{T}$-invariant prime divisor of X, again denoted by D_{ij} and called a *multiple divisor*. Moreover, we denote by $E_1, \ldots, E_m$ the prime divisors in the complement $X \setminus X_0$ and call them the *boundary divisors*. Finally, we call $\psi\colon W \to V$ a *big representative* for $\pi\colon X \dashrightarrow Y$.

Example 1.2.14. We continue 1.2.3 and 1.2.12. The rational quotient $X \dashrightarrow Y$ arising from the projection $\mathbb{T}^3 \to \mathbb{T}^2$ of tori is a maximal orbit quotient. The intersection points c_i of $Y \subseteq \mathbb{P}_2$ with the coordinate axes $V(T_i)$ yield a collection of doubling divisors, the multiple divisors over c_i are the D_{ij}^X and the (only) boundary divisor is E_1^X.

Remark 1.2.15. Observe that Definition 1.2.13 leaves some freedom for choosing the doubling divisors $C_0, \ldots, C_r$. Some divisors necessarily appear: the images of divisors with non-trivial finite generic isotropy group and the images of invariant divisors which cannot be separated by ψ. Beyond those, we are free to choose further doubling divisors C_i, which then means to insert D_{i1} accordingly.

Remark 1.2.16. Let $\pi\colon X \dashrightarrow Y$ and $\pi'\colon X \dashrightarrow Y'$ be maximal orbit quotients for a $\mathbb{T}$-variety X. Then there are open subsets $U \subseteq Y$ and $U' \subseteq Y'$ having complements of codimension at least two and an isomorphism $U \to U'$ which sends any collection of doubling divisors for π to a collection of doubling divisors of π'.

Proposition 1.2.17. *Let $X \subseteq Z_\Sigma$ be an explicit $\mathbb{T}$-variety. Let $Z_\Sigma^1 \subseteq Z_\Sigma$ be the union of $\mathbb{T}_\Sigma$ and all toric orbits $\mathbb{T}_\Sigma \cdot z_{\varrho_{ij}}$ and $Z_\Delta^1 \subseteq Z_\Delta$ the union of all toric orbits of codimension at most one. Then, for $X_1 := X \cap Z_\Sigma^1$ and $Y_1 := Y \cap Z_\Delta^1$, we have a commutative diagram*

$$\begin{array}{ccc} X_1 & \subseteq & Z_\Sigma^1 \\ \downarrow & & \downarrow \\ Y_1 & \subseteq & Z_\Delta^1 \end{array}$$

where the downwards maps are maximal orbit quotients for the action of $\mathbb{T}$. Denoting by D_{ij}^Σ and D_k^Σ the toric prime divisors of Z_Σ corresponding to the rays $\varrho_{ij} = \operatorname{cone}(v_{ij})$ and $\varrho_k = \operatorname{cone}(v_k)$, we obtain the multiple divisors and the boundary divisors of X as

$$D_{ij}^X \;=\; X \cap D_{ij}^\Sigma, \qquad\qquad E_k^X \;=\; X \cap D_k^\Sigma.$$

The generic isotropy group of E_k^X is a one-dimensional torus and the generic isotropy group of D_{ij}^X is finite of order l_{ij}. The doubling divisors are the intersections of $C_i = Y \cap D_i^\Delta$ with the toric prime divisors of $D_i^\Delta \subseteq Z_\Delta$.

We briefly discuss relations to polyhedral divisors [1, 2]. First we have the following recipe to convert explicit $\mathbb{T}$-varieties into the setting of polyhedral divisors.

Remark 1.2.18. Given an explicit $\mathbb{T}$-variety $X \subseteq Z_\Sigma$, we indicate how to obtain a describing divisorial fan in the sense of [1, 2]. First follow [1, Sec. 11]. For every $\sigma \in \Sigma$, let Δ_σ be the fan in $\mathbb{Z}^t$ obtained as the coarsest common refinement of the projections $\operatorname{pr}(\tau) \subseteq \mathbb{Q}^t$ of all faces $\tau \preccurlyeq \sigma \subseteq \mathbb{Q}^{s+t}$. The toric variety associated with Δ_σ is the normalized Chow quotient $Z_\sigma /\!\!/\!\!/ \mathbb{T}$, see [53, 23]. Let Y_σ' be the normalization of the closure of the image of $X \cap \mathbb{T}_Z$ in $Z_\sigma /\!\!/\!\!/ \mathbb{T}$ and write $D_{\sigma,\varrho}$ for the pull back of the toric prime divisor D_ϱ of $Z_\sigma /\!\!/\!\!/ \mathbb{T}$ to Y_σ'. Then

$$\mathcal{D}_\sigma \;:=\; \sum A_\varrho \otimes D_{\sigma,\varrho}, \qquad \mathrm{A}_\varrho \;:=\; \sigma \cap \operatorname{pr}^{-1}(v_\varrho) \;\subseteq\; \mathbb{Q}^{s+t}$$

defines a polyhedral divisor on Y_σ' describing the $\mathbb{T}$-action on $X_\sigma := X \cap Z_\sigma$. Now, follow the proof of [2, Thm. 5.6] to bring the local pictures together. Choose projective closures $Y_\sigma' \subseteq Y_\sigma''$ and, via resolving indeterminancies of the birational maps between the Y_σ'' induced by those between the X_σ, construct a normal projective variety Y'' dominating birationally all the Y_σ''. Pulling back the $\mathcal{D}_\sigma$ to Y'' yields the desired divisorial fan describing the $\mathbb{T}$-action on X.

Remark 1.2.19. In general, maximal orbit quotient and Chow quotient of a $\mathbb{T}$-variety differ from each other. For example, let $\mathbb{T} = \mathbb{K}^*$ act on $X = \mathbb{K}^4$ via

$$t \cdot z \;=\; (t^{-1}z_1, t^{-1}z_2, tz_3, tz_4).$$

Working for instance in terms of fans we see that in this particular case we obtain a maximal orbit quotient just by taking the good quotient

$$\pi\colon X \;\to\; X/\!\!/\mathbb{T}, \qquad\qquad z \;\mapsto\; (z_1z_3, z_1z_4, z_2z_3, z_2z_4),$$

where $X/\!\!/\mathbb{T} = \{w \in \mathbb{K}^4;\ w_1w_4 = w_2w_3\}$, and the canonical map $X/\!\!/\!\!/\mathbb{T} \to X/\!\!/\mathbb{T}$ from the Chow quotient onto the good quotient resolves the singularity $0 \in X/\!\!/\mathbb{T}$.

1.3 Proofs to Section 1.2

Here we prove the statements made in Construction 1.2.5, Proposition 1.2.7, Theorem 1.2.10 and Proposition 1.2.17. We will make use of Bechtold's normality criterion [12, Cor. 6]; for convenience we give a direct proof here.

Proposition 1.3.1. *Let K be a finitely generated abelian group, R a K-factorial affine $\mathbb{K}$-algebra with only constant K-homogeneous units and $f_1, \ldots, f_r$ a system of pairwise non-associated K-prime generators for R. If any $r-1$ of the* $\deg(f_i)$ *generate K as a group, then R is integral and normal.*

Proof. The K-grading of R defines an action of the quasitorus $H := \operatorname{Spec}\mathbb{K}[K]$ on $\bar{X} := \operatorname{Spec} R$ such that the homogeneous elements $f \in R$ of degree $w \in K$ are precisely the functions on $\bar{X}$ which are homogeneous with respect to $\chi^w \in \mathbb{X}(H)$. Set $g_i := \prod_{j \neq i} f_j$ and consider the H-invariant open subset

$$\hat{X} \;:=\; \bar{X}_{g_1} \cup \ldots \cup \bar{X}_{g_r} \;\subseteq\; \bar{X}.$$

Since the f_i are pairwise non-associated K-primes, $\hat{X}$ has complement of codimension at least two in $\bar{X}$. According to [11, Thm. 1.3], each $\bar{X}_{g_i}/H$ is factorial and hence normal. By [6, Prop. 1.2.2.8], the H-action on $\bar{X}_{g_i}$ is free. Thus, Luna's slice theorem [60, Thm. III.1] tells us that the quotient map $\bar{X}_{g_i} \to \bar{X}_{g_i}/H$ is an étale H-principal bundle. As étale morphisms preserve normality, see [63, Prop. 8.1], we conclude that each $\bar{X}_{g_i}$ and hence $\hat{X}$ is normal. Now, observe

$$R \;=\; \mathcal{O}(\bar{X}) \;\subseteq\; \mathcal{O}(\hat{X}).$$

We claim that the last inclusion is in fact an equality. Let $g \in \mathcal{O}(\hat{X})$ be an H-homogeneous function. Since g is a regular homogeneous function on $\bar{X}_{g_1}$, we have $g = g'/g_1^l$ with a homogeneous function $g' \in R$. Using K-factoriality, we find pairwise non-associated K-primes $p_i, f_j \in R$, where $f_2, \ldots, f_r$ are the generators fixed before, such that

$$g \;=\; \frac{p_1^{\nu_1} \cdots p_s^{\nu_s}}{f_2^{\mu_2} \cdots f_r^{\mu_r}}.$$

Since g is regular on the normal variety $\hat{X}$ and $\bar{X} \setminus \hat{X}$ is of codimension at least two in $\bar{X}$, we must have $\mu_2 = \ldots = \mu_r = 0$. Consequently, $g \in R$ holds. Now, every regular function on $\hat{X}$ is a sum of K-homogeneous ones and thus extends to a regular function on $\bar{X}$. In particular, $R = \mathcal{O}(\hat{X})$ is normal.

To see that R is integral, we have to show that $\bar{X} = \operatorname{Spec} R$ is irreducible. Due to normality, the irreducible components of $\bar{X}$ coincide with its connected components $\bar{X}_1, \ldots, \bar{X}_k$. Indeed, if two distinct irreducible components have a common point, then the corresponding local ring has zero divisors, contradicting normality. The assumption that R is K-integral means on the geometric side that H permutes transitively the $\bar{X}_i$. So, we can choose $h_i \in H$ with $\bar{X}_i = h_i \bar{X}_1$ and a non-trivial character $\chi \in \mathbb{X}(H)$ vanishing along the stabilizer of $\bar{X}_1$. Then, setting $f(z) := \chi(h_i)$ for $z \in \bar{X}_i$ defines a homogeneous unit on $\bar{X}$, which is non-constant as soon as $k > 1$ holds. We conclude $k = 1$ and thus $\bar{X}$ is irreducible. □

Proof of Construction 1.2.5, Proposition 1.2.7 and Proposition 1.2.17. The generator matrix B of the fan Δ and the generator matrix P of the fan Σ fit into the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+m} & \xrightarrow{\;P\;} & \mathbb{Z}^{t+s} \\ {\scriptstyle A}\downarrow & & \downarrow \\ \mathbb{Z}^{r+1} & \xrightarrow[\;B\;]{} & \mathbb{Z}^{t} \end{array}$$

where the lifting $A\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{r+1}$ of the projection $\mathbb{Z}^{t+s} \to \mathbb{Z}^t$ sends the canonical basis vectors $e_{ij} \in \mathbb{Z}^{n+m}$ to $l_{ij} e_i \in \mathbb{Z}^{r+1}$ and $e_k \in \mathbb{Z}^{n+m}$ to $0 \in \mathbb{Z}^{r+1}$. Dualizing leads to a commutative ladder of abelian groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longleftarrow & K_P & \xleftarrow{\;Q_P\;} & \mathbb{Z}^{n+m} & \xleftarrow{\;P^*\;} & \mathbb{Z}^{t+s} & \longleftarrow & 0 \\ & & {\scriptstyle \imath}\uparrow & & {\scriptstyle A^*}\uparrow & & \uparrow & & \\ 0 & \longleftarrow & K_B & \xleftarrow[\;Q_B\;]{} & \mathbb{Z}^{r+1} & \xleftarrow[\;B^*\;]{} & \mathbb{Z}^{t} & \longleftarrow & 0 \end{array}$$

We validate Construction 1.2.5. According to Construction 1.1.8, the canonical basis vector $e_i \in \mathbb{Z}^{r+1}$ is sent by Q_B to $\deg(f_i) \in K_B$. Thus, the induced map $\imath\colon K_B \to K_P$ sends $\deg(f_i) \in K_B$ $Q_P(l_{i1}e_{i1} + \ldots + l_{in_i}e_{in_i}) \in K_P$. Define a K_B-grading on the polynomial ring $\mathbb{K}[F_0, \ldots, F_r]$ by $\deg(F_i) = \deg(f_i)$ and a K_B-grading on $\mathbb{K}[T_{ij}, S_k]$ by $\deg(T_{ij}) = Q_P(e_{ij})$ and $\deg(S_k) = Q_P(e_k)$. Then the homomorphism

$$\mathbb{K}[F_0, \ldots, F_r] \;\to\; \mathbb{K}[T_{ij}, S_k], \qquad f_i \;\mapsto\; T_i^{l_i}$$

sends homogeneous elements of degree $w \in K_B$ to homogeneous elements of degree $\imath(w) \in K_P$. In particular, the defining relations $h_j(T_0^{l_0}, \dots, T_r^{l_r})$ are K_P-homogeneous, the K_P-grading of $R(\alpha, P)$ is well defined and, moreover, we have the induced homomorphism of the graded algebras $R_Y \to R(\alpha, P)$ sending f_i to $T_i^{l_i}$ as desired.

We turn to Proposition 1.2.7. Let $\bar{Y} \subseteq \mathbb{K}^{r+1}$ and $\bar{X} \subseteq \mathbb{K}^{n+m}$ denote the closures of the inverse images of $Y \cap \mathbb{T}^t$ and $X \cap \mathbb{T}^{t+s}$ under the homomorphisms of tori $b\colon \mathbb{T}^{r+1} \to \mathbb{T}^t$ and $p\colon \mathbb{T}^{n+m} \to \mathbb{T}^{t+s}$ defined by B and P respectively. Observe that $\bar{Y} = \operatorname{Spec} R_Y$ holds. With the quasitori $H_Y := \operatorname{Spec} \mathbb{K}[K_B]$ and $H_X := \operatorname{Spec} \mathbb{K}[K_P]$ and the homomorphism of tori $a\colon \mathbb{T}^{n+m} \to \mathbb{T}^{r+1}$ defined by A, we have a commutative diagram

$$
\begin{array}{ccc}
\bar{X} \cap \mathbb{T}^{n+m} & \xrightarrow[p]{/H_X} & X \cap \mathbb{T}^{t+s} \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle /\mathbb{T}^s} \\
\bar{Y} \cap \mathbb{T}^{r+1} & \xrightarrow[b]{/H_Y} & Y \cap \mathbb{T}^t.
\end{array}
$$

Consider the product $f \in R_Y$ over all the generators f_i of R_Y and the product $g \in R(\alpha, P)$ over all the generators T_{ij} and S_k of $R(\alpha, P)$. Then, using the above diagram, we see

$$((R_Y)_f)^{H_Y} \;\cong\; a^*\left((R_Y)_f\right)^{H_Y} \;=\; \left((R(\alpha, P)_g)^{H_X}\right)^{\mathbb{T}^s}.$$

Since the left hand side ring is factorial, also the right hand side ring is so. By assumption, $R(\alpha, P)$ is K_P-integral and the generators T_{ij} are K_P-prime. Using [11, Thm. 1.3], we see that $R(\alpha, P)$ is factorially K_P-graded and Proposition 1.3.1 shows that $R(\alpha, P)$ is integral and normal. Consequently, we are in the setting of Construction 1.1.8 which establishes Proposition 1.2.7.

Finally, we show Proposition 1.2.17. First note that $Z_\Sigma^1 \to Z_\Delta^1$ defines a maximal orbit quotient of the $\mathbb{T}^s$-action on Z_Σ. The toric prime divisors of Z_Σ^1 cut down to the prime divisors D_{ij}^X and D_k^X of X_1 and those of Z_Δ^1 to the prime divisors C_i of Y_1. Thus, we can infer the statements on the isotropy groups from [6, Prop. 2.1.4.2] and conclude that $X_1 \to Y_1$ is a big representative of a maximal orbit quotient of the $\mathbb{T}^s$-variety X. □

We come to the proof of Theorem 1.2.10. The task is to provide for any abstractly given A_2-maximal $\mathbb{T}$-variety X with only constant invertible global functions, finitely generated divisor class group $\operatorname{Cl}(X)$ and finitely Cox ring $\mathcal{R}(X)$ a presentation as an explicit $\mathbb{T}$-variety $X \subseteq Z$. This runs via general Cox ring theory. Let us recall the necessary background. Mimicking Cox's quotient presentation 1.1.3, one looks at

$$\bar{X} \;=\; \operatorname{Spec} \mathcal{R}(X), \qquad\qquad H \;=\; \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)],$$

the *total coordinate space* and the *characteristic quasitorus* of X. Then H acts on $\bar{X}$, where this action is defined via its comorphism, sending a homogeneous element $f \in \mathcal{R}(X)$ of degree $[D]$ to the element $\chi^{[D]} \otimes f$ of $\mathbb{K}[\operatorname{Cl}(X)] \otimes \mathcal{R}(X)$. Moreover, X can be reconstructed as a good quotient

$$\begin{array}{ccccccc} \operatorname{Spec}_X \mathcal{R} & = & \hat{X} & \subseteq & \bar{X} & = & \operatorname{Spec} \mathcal{R}(X) \\ & & {\scriptstyle /\!\!/ H} \big\downarrow {\scriptstyle p} & & & & \\ & & X & & & & \end{array}$$

Here the relative spectrum $\hat{X}$ of the Cox sheaf $\mathcal{R}$ is called the *characteristic space* over X. It is an open H-invariant subset of $\bar{X}$ and the complement $\bar{X} \setminus \hat{X}$ is small in the sense that it is of codimension at least two in $\bar{X}$; see [6, Sec. 1.6.1].

We will deal with canonical sections, which in the context of Cox rings means the following. For any effective representative D of a class $[D] \in \operatorname{Cl}(X)$, there is, up to scalars, a unique $f \in \mathcal{R}(X)_{[D]}$ with $\operatorname{div}(f) = p^*D$ on $\hat{X}$. In this situation, we call f a *canonical section* of D and write $f = 1_D$. A canonical section 1_D is a $\operatorname{Cl}(X)$-prime element of $\mathcal{R}(X)$ if and only if D is a prime divisor on X. See [6, Prop. 1.5.3.5 and Lemma 1.5.3.6] for the full details.

Proof of Theorem 1.2.10. Write for short $K := \operatorname{Cl}(X)$ and $R := \mathcal{R}(X)$. In a first step, we lift the action of the torus $\mathbb{T}$ to the total coordinate space $\bar{X}$. Consider the characteristic space $p\colon \hat{X} \to X$ over X. By [6, Thm. 4.2.3.2], there are a $\mathbb{T}$-action on $\hat{X}$ and a positive integer b such that for all $t \in \mathbb{T}$, $h \in H$ and $x \in \hat{X}$, we have

$$t \cdot h \cdot x \ = h \cdot t \cdot x, \qquad\qquad p(t \cdot x) \ = \ t^b \cdot p(x).$$

Since $\hat{X} \subseteq \bar{X}$ has a small complement and $\bar{X}$ is normal, the $\mathbb{T}$-action on $\hat{X}$ extends to $\bar{X}$. The fact that the actions of $\mathbb{T}$ and H commute means that we have an action of $\mathbb{T} \times H$ on $\bar{X}$. Thus, the K-grading of R refines to a $(M \times K)$-grading for $M = \mathbb{X}(\mathbb{T})$. As M is torsion free, [11, Thm. 1.5] yields that R is $(M \times K)$-factorial and $(M \times K)$-primality coincides with K-primality in R.

Now, let $\mathfrak{F} = (f_1, \ldots, f_q)$ be a system of pairwise non-associated K-prime generators of R such that every $\mathbb{T}$-invariant prime divisor of X having non-trivial generic isotropy group has a canonical section among the f_i. Then, as mentioned before, the f_i are $(M \times K)$-prime, and thus in particular $(M \times K)$-homogeneous. Similarly as in Construction 1.1.8, we obtain a $(\mathbb{T} \times H)$-equivariant closed embedding

$$\operatorname{Spec} R =: \bar{X} \xrightarrow{x \mapsto (f_1(x), \ldots, f_q(x))} \bar{Z} := \mathbb{K}^q.$$

Let $Q\colon \mathbb{Z}^q \to K$, $e_i \mapsto \deg(f_i)$ be the degree map of the K-grading. Then, in the language of [6, Thm. 3.1.4.4], we have a maximal bunch of orbit cones

$$\Phi \ = \ \{Q(\gamma_x);\ x \in \hat{X} \text{ with } H \cdot x \subseteq \hat{X} \text{ closed}\}, \qquad \gamma_x \ = \ \operatorname{cone}(e_i;\ f_i(x) \neq 0).$$

Moreover, [6, Props. 3.2.2.2, 3.2.2.5] ensure that we obtain a bunched ring $(R, \mathfrak{F}, \Phi)$ in the sense of [6, Def. 3.2.1.1]. Now we reverse the translation performed in Remark 1.1.10.

Fix a $q' \times q$ matrix P, the rows of which form a lattice basis for $\ker(Q)$. For a face $\gamma_0 \preccurlyeq \gamma$ of the orthant $\gamma = \mathbb{Q}^q_{\geq 0}$, let $\gamma_0^* \preccurlyeq \gamma$ be the complementary face. Then

$$\Sigma_X \ := \ \{P(\gamma_x^*); \ x \in \hat{X} \text{ with } H \cdot x \subseteq \hat{X} \text{ closed}\}$$

is a set of cones intersecting in common faces; see [6, Thm. 2.2.1.14]. Let Σ be any fan in $\mathbb{Z}^{q'}$ such that $\Sigma_X \subseteq \Sigma$ holds. Consider the associated toric variety $Z = Z_\Sigma$ and Cox's quotient presentation $\hat{Z} \to Z$. We will build up the following commutative diagram

$$\begin{array}{ccc} \hat{X} & \subseteq & \hat{Z} \\ {\scriptstyle /\!\!/H}\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle /\!\!/H} \\ X & \longrightarrow & Z \\ {\scriptstyle \pi}\,\vdots & & \vdots \\ Y & \longrightarrow & Z_\Delta. \end{array}$$

By A_2-maximality of X and the choice of Σ, we have $\hat{X} = \bar{X} \cap \hat{Z}$ and the induced morphism $X \to Z$ of quotient spaces is a closed embedding. Moreover, $X \to Z$ is $\mathbb{T}$-equivariant, where $\mathbb{T}$ acts on Z as a subtorus of $\mathbb{T}_Z \subseteq Z$. Choose a splitting $\mathbb{T}_Z = \mathbb{T}^t \times \mathbb{T}$. Accordingly, the lattice hosting the fan Σ splits as $\mathbb{Z}^{q'} = \mathbb{Z}^t \times \mathbb{Z}^s$. Let Δ be the fan in $\mathbb{Z}^t$ consisting of the zero cone and the projections of the rays of Σ. Then the projection $\mathbb{T}_Z \to \mathbb{T}^t$ of acting tori defines the rational map $Z \dashrightarrow Z_\Delta$. Defining $Y \subseteq Z_\Delta$ to be the closure of the image of $X \cap \mathbb{T}_Z$, we complete the commutative diagram.

We investigate the shape of the generator matrices B of Δ and P of Σ. Numbering its columns as $u_0, \ldots, u_r$, we turn B into a $t \times (r+1)$ matrix. For every $i = 0, \ldots, r$, denote by $v_{i1}, \ldots, v_{in_i}$ the columns of P such that the ray $\varrho_{ij} = \operatorname{cone}(v_{ij})$ projects onto $\operatorname{cone}(u_i)$. Moreover, denote by $v_1, \ldots, v_m$ the columns of P such that the ray $\varrho_k = \operatorname{cone}(v_k)$ lies in the kernel of the projection $\mathbb{Q}^t \times \mathbb{Q}^s \to \mathbb{Q}^t$. Then P is a $(n+m) \times (t+s)$ matrix, where $n = n_0 + \ldots + n_r$. Consider the toric prime divisors $D^Z_{ij} \subseteq Z$ and $E^Z_k \subseteq Z$ corresponding to the rays ϱ_{ij} and ϱ_k respectively. Computing the generic isotropy groups $\mathbb{T}_x$ of these divisors according to [6, Prop. 2.1.4.2], we see that the E^Z_k are the boundary divisors of the $\mathbb{T}$-action and that the v_{ij} have a non-trivial $\mathbb{Z}^t$-part being the l_{ij}-fold multiple of the primitive generator $u_i \in \mathbb{Z}^t$. Thus, B and P look as in Construction 1.2.5.

We claim that the dashed arrows are maximal orbit quotients for the $\mathbb{T}$-actions on Z and X respectively. Consider the union $Z^1 \subseteq Z$ of $\mathbb{T}_Z$ and all toric orbits $\mathbb{T}_Z \cdot z_{\varrho_{ij}}$. Then $Z^1 \subseteq Z_0$ is an open subset with complement of codimension at least two in the set $Z_0 \subseteq Z$ consisting of all points $z \in Z$ with finite isotropy group $\mathbb{T}_z$. Let C_i be the prime divisor of Z_Δ corresponding to $\operatorname{cone}(u_i)$, where $i = 0, \ldots, r$. Then $C_0, \ldots, C_r$ serve as doubling divisors and $Z^1 \to Z_\Delta$ is a big representative for the rational quotient $Z \dashrightarrow Z_\Delta$, where Property 1.2.13 (iv) is due to [6, Cor. 2.3.1.7]. Cutting down to X gives an open subset $X^1 = X \cap Z^1$ of X_0 and a morphism $\psi \colon X^1 \to Y$, which inherits the properties of a big representative from $Z^1 \to Z_\Delta$. In particular, Y is normal. A collection of doubling divisors is given by $C_i = Y \cap D_i$, where $D_0, \ldots, D_r \subseteq Z_\Delta$ are the invariant prime divisors of Z_Δ. Observe that each C_i is prime, because it is the image of $X \cap D$ for any $\mathbb{T}_Z$-invariant prime divisor $D \subseteq Z$ lying over $D_i \subseteq Z_\Delta$.

To conclude the proof, we still have to show that $Y \subseteq Z_\Delta$ is an explicit variety, that means that $1_{C_0}, \ldots, 1_{C_r}$ generate the Cox ring $\mathcal{R}(Y)$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+m} & \xrightarrow{P} & \mathbb{Z}^{t+s} \\ {\scriptstyle A}\downarrow & & \downarrow \\ \mathbb{Z}^{r+1} & \xrightarrow[B]{} & \mathbb{Z}^{t} \end{array}$$

where the matrix $A\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{r+1}$ defines the homomorphism of $a\colon \mathbb{T}^{n+m} \to \mathbb{T}^{r+1}$ which in turn uniquely extends to the monomial map

$$a\colon \mathbb{K}^{n+m} \ \to \ \mathbb{K}^{r+1}, \qquad\qquad (z,w) \ \mapsto \ (z_0^{l_0}, \ldots, z_r^{l_r}).$$

Note that a is the good quotient for the action of the quasitorus $\ker(a)$ on $\mathbb{K}^{n+m}$. The total coordinate space $\bar{X} \subseteq \mathbb{K}^{n+m}$ is invariant and thus maps onto a closed normal subvariety $\bar{Y} \subseteq \mathbb{K}^{r+1}$. Moreover, $\bar{Y}$ inherits from X the property that the coordinate functions of $\mathbb{K}^{r+1}$ define pairwise non-associated elements on $\mathcal{O}(\bar{Y})$. By construction, $\bar{Y} \cap \mathbb{T}^{r+1}$ dominates $Y \subseteq Z_\Delta$. Thus, using [6, Lemmas 3.4.1.7, 3.4.1.9 and Cor. 3.4.1.6], we see that $\mathcal{O}(\bar{Y})$ is the Cox ring of Y. □

As a consequence of the above proofs we retrieve [46, Thm. 1.2] for the special case of $\mathbb{T}$-varieties with finitely generated Cox ring.

Corollary 1.3.2. *Let X be a $\mathbb{T}$-variety with finitely generated Cox ring $\mathcal{R}(X)$. Then X admits a maximal orbit quotient $\pi\colon X \dashrightarrow Y$ and a collection $C_0, \ldots, C_r$ of doubling divisors such that we have an isomorphism of* $\mathrm{Cl}(X)$*-graded rings*

$$\mathcal{R}(X) \ \cong \ \mathcal{R}(Y)[T_{ij}, S_k]/\langle T_i^{l_i} - U_i;\ i = 0, \ldots, r\rangle,$$

where $T_{ij}, S_k \in \mathcal{R}(X)$ and $U_i \in \mathcal{R}(Y)$ are canonical sections of the multiple divisors D_{ij}, boundary divisors E_k and doubling divisors C_i respectively and the $\mathrm{Cl}(X)$*-grading on the right hand side is given by*

$$\deg(U_i) = [l_{i1}D_{i1} + \ldots + l_{in_i}D_{in_i}], \qquad \deg(D_{ij}) = [D_{ij}], \qquad \deg(S_k) = [E_k].$$

Moreover, we have $\bar{Y} = \bar{X} /\!\!/ H_{X,Y}$, where the quasitorus $H_{X,Y} \subseteq \mathbb{T}^{n+m}$ is the kernel of the homomorphism of tori $\mathbb{T}^{n+m} \to \mathbb{T}^{r+1}$ sending (t,s) to $(t_0^{l_0}, \ldots, t_r^{l_r})$.

1.4 First properties and examples

We discuss basic geometric properties of explicit $\mathbb{T}$-varieties. First we provide a collection of general statements directly imported from [6, Chap. 3], concerning singularities, the Picard group and various cones of divisor classes. Then we present more specific statements involving the $\mathbb{T}$-action. The second part of the section is devoted to examples. We indicate how to apply the results in practice by means of a concrete (new) example, we show how the construction of rational $\mathbb{T}$-varieties of complexity one from [40, 47] fits into the framework of explicit $\mathbb{T}$-varieties and finally, we present the Grassmannian $\mathrm{Gr}(2, n)$ with its maximal torus action as an explicit $\mathbb{T}$-variety.

When we speak about an explicit $\mathbb{T}$-variety $X \subseteq Z$ or, more specifically, about an explicit $\mathbb{T}$-variety $X(\alpha, P, \Sigma)$ in Z_Σ, then we allow ourselves to make free use of the notation introduced in Construction 1.2.5. Recall from Remark 1.2.6 that the case of a trivial $\mathbb{T}$-action, that means the explicit varieties from Construction 1.1.8, is included via $s = m = 0$ and $n = r + 1$.

Remark 1.4.1. Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. The total coordinate spaces $\bar{X}$ and $\bar{Z}$, that means the spectra of the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Z)$, are given as

$$\bar{X} \ := \ \bar{X}(\alpha, P) \ := \ V(h_1(T_0^{l_0}, \ldots, T_r^{l_r}), \ldots, h_q(T_0^{l_0}, \ldots, T_r^{l_r})) \ \subseteq \ \mathbb{K}^{n+m} \ =: \ \bar{Z}.$$

The embedding $\bar{X} \subseteq \bar{Z}$ is equivariant with respect to the actions of the characteristic quasitorus $H = \operatorname{Spec} \mathbb{K}[K_P]$ defined by the gradings of $\mathcal{R}(X)$ and $\mathcal{R}(Z)$ by $K_P = \mathrm{Cl}(X) = \mathrm{Cl}(Z)$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \hat{X} & \subseteq & \hat{Z} \\ {\scriptstyle /\!\!/ H}\big\downarrow & & \big\downarrow{\scriptstyle /\!\!/ H} \\ X & \subseteq & Z \end{array}$$

where $\hat{Z} \to Z$ is Cox's quotient presentation 1.1.3 and $\hat{X} = \bar{X} \cap \hat{Z}$ holds. The good quotients $\hat{X} \to X$ and $\hat{Z} \to Z$ are the characteristic spaces over X and Z, respectively.

Every explicit $\mathbb{T}$-variety $X \subseteq Z$ inherits a decomposition into locally closed subsets by cutting down the toric orbit decomposition of Z. Generalizing well-known basic facts of toric geometry, one can express several geometric properties of X in terms of this inherited decomposition. Let us introduce the necessary notation for precise statements.

Definition 1.4.2. Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Set $\gamma := \mathbb{Q}^{n+m}_{\geq 0}$. An *$\bar{X}$-face* is a face $\gamma_0 \preccurlyeq \mathbb{Q}^{n+m}$ such that the complementary face $\gamma_0^* \preccurlyeq \gamma$ satisfies

$$\mathbb{K}^{n+m} \ \supseteq \ \bar{X}(\gamma_0) \ := \ \bar{X} \ \cap \ \mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \ \neq \ \emptyset.$$

For $\sigma \in \Sigma$ and the *corresponding* face $\gamma_0 \preccurlyeq \gamma$, that means the face with $P(\gamma_0^*) = \sigma$, consider the intersection of X and the associated toric orbit of $Z = Z_\Sigma$:

$$X(\gamma_0) \ := \ X(\sigma) \ := \ X \cap \mathbb{T}^{t+s} \cdot z_\sigma \ \subseteq \ Z.$$

We call $\sigma \in \Sigma$ an *X-cone* and $\gamma_0 \preccurlyeq \gamma$ an *X-face* if $X(\gamma_0) = X(\sigma)$ is non-empty. Moreover, we denote

$$\mathrm{rlv}(X) \ := \ \{\gamma_0 \preccurlyeq \gamma; \ \gamma \text{ is an } X\text{-face}\}.$$

Finally, we call the subsets $X(\gamma_0) \subseteq X$, where γ_0 is an X-face, the *pieces* of the explicit $\mathbb{T}$-variety $X \subseteq Z$.

Remark 1.4.3. Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Then every piece $X(\gamma_0) \subseteq X$ is locally closed and X is the disjoint union of its pieces:

$$X = \bigsqcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0).$$

Moreover, $\gamma_0 \preccurlyeq \gamma$ is an X-face if and only if it is an $\bar{X}$-face and we have $P(\gamma_0^*) \in \Sigma$. If $\gamma_0 \preccurlyeq \gamma$ is an X-face, then $\bar{X}(\gamma_0)$ maps onto $X(\gamma_0)$.

We describe basic local properties in terms of the pieces. Consider for the moment any normal variety X. A point $x \in X$ is *factorial* if every Weil divisor of X is Cartier near x. Moreover, $x \in X$ is *$\mathbb{Q}$-factorial* if for every Weil divisor of X some nonzero multiple is Cartier near x.

Proposition 1.4.4. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Consider an X-face $\gamma_0 \preccurlyeq \gamma$ and $\sigma = P(\gamma_0^*) \in \Sigma$. Then the following statements are equivalent.*

(i) *The piece $X(\sigma)$ consists of $\mathbb{Q}$-factorial points of X.*

(ii) *The cone σ is simplicial.*

(iii) *The cone $Q(\gamma_0) \subseteq K_{\mathbb{Q}}$ is of full dimension.*

Proof. Translate via Remark 1.1.10 and apply [6, 3.3.1.8 and 3.3.1.12]. □

Proposition 1.4.5. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Consider an X-face $\gamma_0 \preccurlyeq \gamma$ and $\sigma = P(\gamma_0^*) \in \Sigma$. Then the following statements are equivalent.*

(i) *The piece $X(\sigma)$ consists of factorial points of X.*

(ii) *The cone σ is regular.*

(iii) *The set $Q(\gamma_0 \cap \mathbb{Z}^{n+m})$ generates K as a group.*

Moreover, $X(\sigma)$ consists of smooth points of X if and only if one of the above statements holds and $\bar{X}(\gamma_0)$ consists of smooth points of $\bar{X}$.

Proof. Translate via Remark 1.1.10 and apply [6, 3.3.1.8, 3.3.1.9 and 3.3.1.12]. □

We turn to the Picard group and the various cones of divisor classes. For a subset $A \subseteq V$ of a $\mathbb{Q}$-vector space, we denote by $\mathrm{lin}(A) \subseteq V$ its $\mathbb{Q}$-linear hull.

Proposition 1.4.6. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Then, in $K_P = \mathrm{Cl}(X)$, the Picard group of X is given by*

$$\mathrm{Pic}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^{n+m}).$$

Moreover, in $(K_P)_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$, the cones of effective, movable, semiample and ample divisor classes are given by

$$\mathrm{Eff}(X) \;=\; Q(\gamma), \qquad \mathrm{Mov}(X) \;=\; \bigcap_{\gamma_0 \preccurlyeq \gamma \text{ facet}} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0), \qquad \mathrm{Ample}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0)^\circ.$$

Proof. Translate via Remark 1.1.10 and apply [6, Cor. 3.3.1.6 and Prop. 3.3.2.9]. □

Remark 1.4.7. Let $X = X(\alpha, P, \Sigma)$ in $Z = Z_\Sigma$ be an explicit $\mathbb{T}$-variety. If the fan Σ is the normal fan of a polytope in $\mathbb{Q}^{t+s}$, then Z and hence X are projective. Conversely, if X is projective, choose any class $u \in \mathrm{Ample}(X)$, an element $e \in \mathbb{Q}^{n+m}$ with $Q(e) = u$ and consider the polytope

$$B(u) \;=\; (P^*)^{-1}(Q^{-1}(u) \cap \gamma) - e) \;\subseteq\; \mathbb{Q}^{t+s}.$$

Then, with the normal fan $\Sigma(u)$ of $B(u)$, we have $X = X(\alpha, P, \Sigma(u))$, whereas the toric ambient variety $Z(u)$ associated with $\Sigma(u)$ may differ from the original $Z = Z_\Sigma$. Note that in terms of the faces $\gamma_0 \preccurlyeq \gamma$, the normal fan is given as

$$\Sigma(u) \;=\; \{P(\gamma_0^*);\; \gamma_0 \preccurlyeq \gamma \text{ with } u \in Q(\gamma_0)^\circ\}.$$

We indicate, in our setting, the fundamental connection between geometric invariant theory and Mori theory found by Hu and Keel [51, Thm. 2.3]; we refer to [6, Sections 3.1.2 and 3.3.4] for additional background.

Remark 1.4.8. Let $X = X(\alpha, P, \Sigma(u))$ in $Z = Z_{\Sigma(u)}$ be a projective explicit $\mathbb{T}$-variety. For every $u \in \mathrm{Eff}(X)$, denote by $\Gamma(u)$ the collection of $\bar{X}$-faces $\gamma_0 \preceq \gamma$ with $u \in Q(\gamma_0)$ and define a convex polyhedral cone

$$\lambda_u \;=\; \bigcap_{\gamma_0 \in \Gamma(u)} Q(\gamma_0) \;\subseteq\; K_{\mathbb{Q}} \;=\; \mathrm{Cl}_{\mathbb{Q}}(X).$$

The cones λ_u form a fan Λ subdividing $\mathrm{Eff}(X)$. The fan Λ maintains the geometric invariant theory of the characteristic quasitorus action on $\bar{X}$ in the sense that the sets $X^{ss}(u)$ of semistable points associated with the characters $\chi^u \in \mathbb{X}(H)$ satisfy

$$\bar{X}^{ss}(u) \;\subseteq\; \bar{X}^{ss}(u') \quad\Longleftrightarrow\quad \lambda_u \succcurlyeq \lambda_{u'}.$$

Observe that if one of the conditions holds, then we have an induced morphism $X_u \to X_{u'}$ of the associated quotients by H. Now, look at the $u \in \mathrm{Mov}(X)^\circ$. These define projective explicit $\mathbb{T}$-varieties. More precisely, we have

$$X_u \;=\; X(\alpha, P, \Sigma(u)), \qquad\qquad \hat{X}_u \;=\; \bar{X}^{ss}(u).$$

Each X_u has λ_u as its semiample cone and is $\mathbb{Q}$-factorial if and only if $\dim(\lambda_u)$ equals $\dim(K_{\mathbb{Q}})$. Moreover, Λ reflects the Mori equivalence: the birational map $X_u \dashrightarrow X_u'$ is an isomorphism if and only if $u, u' \in \lambda^\circ$ holds for some $\lambda \in \Lambda$.

Now we discuss more specific properties of explicit $\mathbb{T}$-varieties $X \subseteq Z$ involving in particular the torus action. Recall that $\mathbb{T} = \mathbb{T}^s$, being a factor of $\mathbb{T}_Z = \mathbb{T}^t \times \mathbb{T}^s$, acts on Z and leaves $X \subseteq Z$ invariant. Moreover, the projection $\mathbb{T}_Z \to \mathbb{T}^t$ defines the maximal orbit quotient $Z \dashrightarrow Z_\Delta$ for the $\mathbb{T}$-action on Z and by restricting we obtain a maximal orbit quotient $\pi\colon X \dashrightarrow Y$ for the $\mathbb{T}$-action on X.

Proposition 1.4.9. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Let $L \subseteq \mathbb{Z}^{t+s}$ be the kernel of the projection $\mathrm{pr}\colon \mathbb{Z}^{t+s} \to \mathbb{Z}^t$. Then, for every X-cone $\sigma \in \Sigma$ and every $x \in X(\sigma)$, the isotropy group $\mathbb{T}_x$ satisfies*

$$\mathbb{X}(\mathbb{T}_x) \;\cong\; (L \cap \mathrm{lin}(\sigma)) \,\oplus\, (\mathrm{pr}(\mathrm{lin}(\sigma)) \cap \mathbb{Z}^t)/(\mathrm{pr}(\mathrm{lin}(\sigma) \cap \mathbb{Z}^{t+s}).$$

Proof. From [6, Prop. 2.1.4.2] we infer the formula for the isotropy group of $\mathbb{T} \subseteq \mathbb{T}_Z$ at the point $z_\sigma \in Z$. Since the isotropy groups of the $\mathbb{T}$-action are constant along the toric orbits, this is all we need. □

Proposition 1.4.10. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. Suppose that the Cox ring presentation $\mathcal{R}(Y) = \mathbb{K}[f_1, \ldots, f_r]/\langle h_1, \ldots, h_q\rangle$ is a complete intersection. Then, with $h_u' := h_u(T_0^{l_0}, \ldots, T_r^{l_r})$, also the Cox ring presentation*

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{ij}, S_k]/\langle h_1', \ldots, h_q'\rangle$$

is a complete intersection. Moreover, in the latter case, the canonical divisor class of X is given by

$$\mathcal{K}_X \;=\; -\sum_{i=0}^{r}\sum_{j=1}^{n_i} \deg(T_{ij}) - \sum_{k=1}^{m} \deg(S_k) + \sum_{u=1}^{q} \deg(h_u') \;\in\; K_P \;=\; \mathrm{Cl}(X).$$

In particular, with the canonical divisor class $\mathcal{K}_Y \in K_B = \mathrm{Cl}(Y)$ and the maximal orbit quotient $\pi\colon X \dashrightarrow Y$, we have

$$\mathcal{K}_X - \pi^*(\mathcal{K}_Y) \;=\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} (l_{ij} - 1)\deg(T_{ij}) - \sum_{k=1}^{m} \deg(S_k).$$

Proof. The second and third statement follow from [6, Prop. 3.3.3.2]. The first one is seen via a simple dimension computation:

$$\begin{aligned}\dim(\bar{X}) &= \dim(X) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\ &= s + \dim(Y) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\ &= s + \dim(\bar{Y}) - \mathrm{rk}\,(\mathrm{Cl}(Y)) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\ &= s + (r+1-q) - (r+1-t) + (n+m-t-s) \\ &= n+m-q.\end{aligned}$$

□

For the next observation, note that in Construction 1.2.5, we may remove successively all maximal cones from the fan Σ that are not X-cones. The result is a minimal fan Σ defining still the initial X. We call $Z = Z_\Sigma$ in this case the *minimal ambient toric variety* of X.

Proposition 1.4.11. *Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety and assume that Z is the minimal toric ambient variety of X. Let $L \subseteq \mathbb{Z}^{t+s}$ be the kernel of the projection $\mathbb{Z}^{t+s} \to \mathbb{Z}^t$.*

(i) *The normalization of the general $\mathbb{T}$-orbit closure of X is the toric variety defined by the fan Σ_L in L, where*

$$\Sigma_L \;:=\; \{\tau;\ \tau \preccurlyeq (\sigma \cap L_\mathbb{Q}),\ \sigma \in \Sigma\}.$$

(ii) *If the maximal orbit quotient $\pi\colon X \dashrightarrow Y$ is a morphism, then Σ_L is a subfan of Σ.*

Proof. As Z is the minimal toric embedding, the general $\mathbb{T}$-orbit closure of X equals the general $\mathbb{T}$-orbit closure of Z. This reduces the problem to standard toric geometry. □

Corollary 1.4.12. *Assumptions as in Proposition 1.4.11. If X is complete and Σ_L is a subfan of Σ, then we have*

$$\mathrm{rk}\,(\mathrm{Cl}(X)) - \mathrm{rk}\,(\mathrm{Cl}(Y)) \;>\; n - r - 1.$$

Proof. According to Proposition 1.4.11, the general $\mathbb{T}$-orbit closure of X has divisor class group of rank $m - s > 0$. Thus, the assertion follows from

$$\mathrm{rk}\,(\mathrm{Cl}(X)) \;=\; n+m-t-s, \qquad \mathrm{rk}\,(\mathrm{Cl}(Y)) \;=\; r+1-t.$$

□

We come to the announced example discussions. First, we use Construction 1.2.5 to produce a concrete example of a $\mathbb{Q}$-factorial Fano variety with torus action of complexity two and maximal orbit quotient $X \dashrightarrow \mathbb{P}_1 \times \mathbb{P}_1$.

Example 1.4.13. Consider the surface $Y := \mathbb{P}_1 \times \mathbb{P}_1$. Then we have $\mathrm{Cl}(Y) = \mathbb{Z}^2$ and the Cox ring of Y is the polynomial ring $\mathbb{K}[T_0, T_1, T_2, T_3]$, where the $\mathbb{Z}^2$-grading is given by

$$\deg(T_0) = \deg(T_1) = (1,0), \qquad \deg(T_2) = \deg(T_3) = (0,1).$$

Consider the redundant system $\alpha = (f_0, \ldots, f_5)$ of generators for $\mathcal{R}(Y)$ consisting of $f_i := T_i$ for $i = 0, \ldots, 3$ and the defining equations of the diagonals

$$f_4 \ := \ T_0T_3 - T_1T_2, \qquad f_5 \ := \ T_0T_2 - T_1T_3,$$

both being of degree $(1,1)$. A matrix B of relations between the degrees of generators $f_0, \ldots, f_5$ is given by

$$B \ := \ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Then Y is embedded into the toric variety Z_Δ, the fan Δ of which lives in $\mathbb{Z}^4$ and has the following four maximal cones

$$\mathrm{cone}(v_1, v_3, v_4),\ \mathrm{cone}(v_1, v_2, v_5),\ \mathrm{cone}(v_0, v_3, v_5),\ \mathrm{cone}(v_0, v_2, v_4),$$

where v_i denotes the i-th column of B. Note that Y is given in Cox coordinates by the equation $f_4 = f_0f_3 - f_1f_2$ and $f_5 = f_0f_2 - f_1f_3$. To build the variety X, consider the matrix

$$P \ := \ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\ -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & -1 \end{bmatrix}$$

obtained from B by firstly doubling the last column, then multiplying its last and third last columns with 2, adding a zero column and, after that, adding two new rows as d, d' part. We gain polynomials by modifying the variables of the describing relations of $Y \subseteq Z_\Delta$ accordingly to the column modifications:

$$g_1 := T_{41}^2 - T_{01}T_{31} + T_{11}T_{21}, \qquad g_2 := T_{51}T_{52}^2 - T_{01}T_{21} + T_{11}T_{31}.$$

By construction, the polynomials g_i are homogeneous with respect to the grading of $\mathbb{K}[T_{ij}, S_1]$ given by

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K, \qquad \deg(S_1) \ := \ Q(e_1) \ \in \ K,$$

where $Q\colon \mathbb{Z}^8 \to K := \mathbb{Z}^8/\mathrm{im}(P^*) \cong \mathbb{Z}^2$, is the projection and $e_{ij}, e_1 \in \mathbb{Z}^8$ are the canonical basis vectors, numbered according to the variables T_{ij} and S_1. Let $\Sigma = \Sigma(u)$ in $\mathbb{Z}^6$ be the normal fan of the polytope

$$(P^*)^{-1}(Q^{-1}(u) \cap \gamma) - e) \ \subseteq \ \mathbb{Q}^6,$$

where $u := (8, -4) \in K$ and $e \in \mathbb{Z}^8$ is any point with $Q(e) = u$. Then Σ has the columns of P as its primitive generators. Moreover, the projection $\mathbb{Z}^8 \to \mathbb{Z}^6$ onto the first six coordinates sends the rays of Σ into the rays of Δ. This gives a rational toric map $\pi\colon Z_\Sigma \dashrightarrow Z_\Delta$. Now, define a variety

$$X \ = \ X(\alpha, P, \Sigma) \ := \ \overline{\pi^{-1}(Y \cap \mathbb{T}^4)} \ \subseteq \ Z_\Sigma.$$

Then X is invariant under the action of the subtorus $\mathbb{T} := \{\mathbb{1}_4\} \times \mathbb{T}^2$ of the acting torus $\mathbb{T}^6$ of Z. The $\mathbb{T}$-variety X is normal, of dimension four with divisor class group and Cox ring given by

$$\mathrm{Cl}(X) \ = \ \mathbb{Z}^2, \qquad\qquad \mathcal{R}(X) \ = \ \mathbb{K}[T_{ij}, S_1]/\langle g_1, g_2\rangle,$$

where the grading of the Cox ring is the one given above. This involves application of Proposition 1.2.7; the necessary assumptions are directly verified. Now, applying Propositions 1.4.4, 1.4.6, 1.4.10 and their implementation in [43], we see that X is a $\mathbb{Q}$-factorial Fano variety of Gorenstein index 30.

Remark 1.4.14. The Cox ring based approach of [40, 47] produces all A_2-maximal rational $\mathbb{T}$-varieties X of complexity one with only constant invertible global functions via a construction having a pair of matrices and a bunch of cones as input data. Let us see how to retrieve these X via Construction 1.2.5. Two types of $\mathbb{T}$-varieties are distinguished: the first admits non-constant $\mathbb{T}$-invariant functions, the second does not.

Type 1. The starting variety is $Y = \mathbb{K}$ in $Z_\Delta = \mathbb{K}^{r+1}$, where the generator matrix of Δ is $B = \mathbb{E}_{r+1}$, we embed via the system $\alpha = (f_0, \ldots, f_r)$ given by $f_i = T - a_i$ with pairwise different $a_i \in \mathbb{K}$ and the defining relations for Y are

$$h_i \ = \ U_i - U_{i+1} - (a_i + a_{i+1}) \ \in \ \mathbb{K}[U_0, \ldots, U_r], \quad i = 0, \ldots, r-1.$$

Type 2. The starting variety is $Y = \mathbb{P}_1$ in $Z_\Delta = \mathbb{P}_r$, where Δ has generator matrix $B = [-\mathbb{1}_r, \mathbb{E}_r]$, the embedding system $\alpha = (f_0, \ldots, f_r)$ with $f_i := a_{i,1}T_1 + a_{i,2}T_2$ such that $[a_{i,1}, a_{i,2}] \in \mathbb{P}_1$ are pairwise different and the defining relations for Y are

$$h_i \ := \ \det \begin{bmatrix} a_{i,1} & a_{i+1,1} & a_{i+2,1} \\ a_{i,2} & a_{i+1,2} & a_{i+2,2} \\ U_i & U_{i+1} & U_{i+2} \end{bmatrix} \ \in \ \mathbb{K}[U_0, \ldots, U_r], \quad i = 0, \ldots, r-1.$$

Now run Construction 1.2.5 for both types. The assumptions of Proposition 1.2.7 are satisfied by [40, Thm. 10.4] and [47, Thm. 1.5]. Thus, Theorem 1.2.10 gives the desired result. The input data A, P and Φ of [40, 47] are recovered as follows: the matrix A is $[a_0, \ldots, a_r]$, where for Type 1 the a_i are as above and for Type 2 we set $a_i = (a_{i,1}, a_{i,2})$, the matrix P is the one produced by Construction 1.2.5 and the bunch of cones Φ is related to the fan Σ via Gale duality as outlined in Remark 1.1.10.

Example 1.4.15. Fix $n \geq 5$ and let $X_{2,n} = \mathrm{Gr}(2, n)$ be the Grassmannian of two-dimensional vector subspaces of $\mathbb{K}^n$. Then $X_{2,n}$ has the projective linear group $\mathrm{PGL}(n)$ as its automorphism group [19]. We will pick a maximal torus $\mathbb{T} \subseteq \mathrm{PGL}(n)$ and show how to obtain the $\mathbb{T}$-variety $X_{2,n}$ via Construction 1.2.5. Set

$$r \ := \ \binom{n}{2}.$$

Identify the Plücker coordinate space $\mathbb{K}^n \wedge \mathbb{K}^n$ with $\mathbb{K}^r$ such that the basis $(e_i \wedge e_j)$ of $\mathbb{K}^n \wedge \mathbb{K}^n$ corresponds to the basis (e_{ij}) of $\mathbb{K}^r$, where $e_{ij} \in \mathbb{K}^r$ has ij-th Plücker coordinate equal to one and all others zero. We order the bases $(e_i \wedge e_j)$ and (e_{ij}) lexicographically. Accordingly, we have the Plücker ideal and the affine cone

$$I_{2,n} \ \subseteq \ \mathbb{K}[T_{ij};\ 1 \leq i < j \leq n], \qquad\qquad \bar{X}_{2,n} \ = \ V(I_{2,n}) \ \subseteq \ \mathbb{K}^r.$$

Look at the largest diagonal torus $\bar{\mathbb{T}} \subseteq \mathrm{GL}(r)$ leaving $\bar{X}_{2,n}$ invariant. With $t := r - n$, we obtain $\bar{\mathbb{T}}$ as the kernel of the homomorphism $b\colon \mathbb{T}^r \to \mathbb{T}^t$ defined by the following $t \times r$ matrix, the first n columns of which are defined by the remaining ones as indicated:

$$B = [v_{12}, \ldots, v_{n-1\,n}] = [v_{12}, \ldots, v_{1n}, \mathbb{E}_t, -\mathbb{1}_t]\,, \quad v_{1i} := \textstyle\sum_{\{j,k\} \cap \{1,i\} = \emptyset} v_{jk}.$$

Observe that $\bar{\mathbb{T}} \subseteq \mathrm{GL}(r)$ is of dimension n. The corresponding torus $\mathbb{T} \in \mathrm{PGL}(r)$ is of dimension $n-1$. Moreover, $\mathbb{T}$ acts effectively on $X_{2,n} \subseteq \mathbb{P}_{r-1}$ and thus $\mathbb{T}$ defines a maximal torus of the automorphism group $\mathrm{PGL}(n)$. Now, look at the $(r-1) \times r$ stack matrix

$$P \ := \ \begin{bmatrix} B \\ d \end{bmatrix} = \begin{bmatrix} v_{12}, \ldots, v_{1n} & \mathbb{E}_t & -\mathbb{1}_t \\ \mathbb{E}_{n-1} & 0 & -\mathbb{1}_{n-1} \end{bmatrix}.$$

Observe that the kernel of P is generated by the vector $\mathbb{1}_r \in \mathbb{Z}^r$. In particular, P differs from $[\mathbb{E}_r, -\mathbb{1}_r]$ by multiplication with a unimodular matrix from the left. Let Σ be the unique complete fan in $\mathbb{Z}^r$ having P as generator matrix and let Δ be the fan in $\mathbb{Z}^t$ having the rays through the columns of B as its maximal cones. Then we obtain a commutative diagram

$$\begin{array}{ccc} \bar{X}_{2,n} & \subseteq & \mathbb{K}^r \\ \downarrow p & & \downarrow p \\ X_{2,n} & \subseteq & Z_\Sigma \\ \downarrow /\!\!/\mathbb{T} & & \downarrow /\!\!/\mathbb{T} \\ Y_{2,n} & \subseteq & Z_\Delta \end{array} \qquad b\colon \bar{X}_{2,n} \dashrightarrow Y_{2,n}, \quad b\colon \mathbb{K}^r \dashrightarrow Z_\Delta$$

where $Y_{2,n} \subseteq Z_\Delta$ is the closure of the image $b(\bar{X}_{2,n} \cap \mathbb{T}^r)$. We have $Z_\Sigma = \mathbb{P}_{r-1}$ and $X_{2,n} \subseteq Z_\Sigma$ is the Plücker embedding. Moreover, $R_{2,n} = \mathbb{K}[T_{ij}]/I_{2,n}$ is a unique factorization domain [74, Prop. 8.5] and the variables T_{ij} define prime elements. Thus, $Y_{2,n} \subseteq Z_\Delta$ is an explicit variety and the $\mathbb{T}$-variety $X_{2,n} \subseteq Z_\Sigma$ is an explicit $\mathbb{T}$-variety. Moreover, the $\mathbb{T}$-action has maximal orbit quotient

$$\pi\colon X_{2,n} \dashrightarrow Y_{2,n}.$$

We claim that, up to codimension two, $Y_{2,n}$ equals the blowing up $\mathrm{Bl}_{n-1}(\mathbb{P}_{n-3})$ of $\mathbb{P}_{n-3}$ at $n-1$ points in general position. Indeed, we may first blow up the $n-2$ toric fixed points of $\mathbb{P}_{n-3}$ and then the base point of the resulting toric variety. Doing the latter via [44, Alg. 5.7], one directly checks that the procedure terminates after the first step and delivers $R_{2,n}$ as Cox ring of $\mathrm{Bl}_{n-1}(\mathbb{P}_{n-3})$. Thus, we may also take $X_{2,n} \dashrightarrow \mathrm{Bl}_{n-1}(\mathbb{P}_{n-3})$ as a maximal orbit quotient for the $\mathbb{T}$-action.

Remark 1.4.16. In order to describe a Mori dream space with torus action via divisorial fans [1, 2], it happens that one has to start with a non Mori dream space as prospective Chow quotient. For example, the maximal torus action on the Grassmannian $\mathrm{Gr}(2,n)$ has the moduli space $\overline{M}_{0,n}$ as its Chow quotient [52] and for $n \geq 10$, it is known that $\overline{M}_{0,n}$ and hence all its blow ups have a non-finitely generated Cox ring [18, 35, 45]. Note that the Chow quotient $\overline{M}_{0,n}$ starts differing at $n = 6$ from the maximal orbit quotient discussed just before. Altmann and Hein gave in [3] a description of the maximal torus action on $\mathrm{Gr}(2,n)$ by means of a divisorial fan living on $\overline{M}_{0,n}$.

CHAPTER

TWO

GENERAL ARRANGEMENT VARIETIES

In this chapter we introduce our first example class, the general arrangement varieties. These are certain $\mathbb{T}$-varieties having projective spaces as the target spaces of their maximal orbit quotients, naturally generalizing the rational projective $\mathbb{T}$-varieties of complexity one. We show that all A_2-maximal general arrangement varieties can be presented as explicit $\mathbb{T}$-varieties. Moreover, we use the methods on explicit $\mathbb{T}$-varieties from Chapter 1 to investigate the geometry of general arrangement varieties. In particular, we give an explicit description of their anticanonical divisor class and characterize smoothness of general arrangement varieties. The results of this chapter are published in the joint work [42].

2.1 General arrangement varieties

Let us recall the necessary notions on projective hyperplane arrangements. A *hyperplane* H in the projective space $\mathbb{P}_n$ is the zero set of a nonzero homogeneous polynomial of degree one. A *hyperplane arrangement* in $\mathbb{P}_n$ is a finite collection $H_1, \ldots, H_r$ of hyperplanes in $\mathbb{P}_n$. A hyperplane arrangement in $\mathbb{P}_n$ is called *general* if for any $1 \leq i_1 < \ldots < i_k \leq r$, the intersection $H_{i_1} \cap \ldots \cap H_{i_k}$ is of dimension $(n-k)$.

Definition 2.1.1. A *(general) arrangement variety* of complexity c is a $\mathbb{T}$-variety X with maximal orbit quotient $X \dashrightarrow \mathbb{P}_c$ such that the doubling divisors $C_0, \ldots, C_r$ form a (general) hyperplane arrangement in $\mathbb{P}_c$.

Remark 2.1.2. The projective general arrangement varieties of complexity $c = 1$ are precisely the rational projective $\mathbb{T}$-varieties of complexity one. Indeed, any rational projective $\mathbb{T}$-variety X of complexity one has maximal orbit quotient $\pi\colon X \dashrightarrow \mathbb{P}_1$. The doubling divisors form a point configuration in $\mathbb{P}_1$, which trivially satisfies the conditions of a general hyperplane arrangement.

We enter the construction of general arrangement varieties. As in the case of complexity one [41, 40, 6], we first write down the prospective Cox rings in terms of generators and relations, then investigate their algebraic properties and after all that construct the varieties we are aiming for.

Construction 2.1.3. Fix integers $r \geq c > 0$ and $n_0, \dots, n_r > 0$ as well as $m \geq 0$. Set $n := n_0 + \dots + n_r$. The input data is a pair (A, P_0), where

- A is a $(c+1) \times (r+1)$ matrix over $\mathbb{K}$ such that any $c+1$ of its columns $a_0, \dots, a_r$ are linearly independent,
- P_0 is an integral $r \times (n+m)$ matrix built from tuples of positive integers $l_i = (l_{i1}, \dots, l_{in_i})$, where $i = 0, \dots, r$, as follows

$$P_0 \ := \ \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \dots & 0 \end{bmatrix}.$$

Write $\mathbb{K}[T_{ij}, S_k]$ for the polynomial ring in the variables T_{ij}, where $i = 0, \dots, r$, $j = 1, \dots, n_i$, and S_k, where $k = 1, \dots, m$. Every l_i defines a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Moreover, for every $t = 1, \dots, r-c$, we obtain a polynomial g_t by computing the following $(c+2) \times (c+2)$ determinant

$$g_t \ := \ \det \begin{bmatrix} a_0 & \dots & a_c & a_{c+t} \\ T_0^{l_0} & \dots & T_c^{l_c} & T_{c+t}^{l_{c+t}} \end{bmatrix} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Now, let $e_{ij} \in \mathbb{Z}^n$ and $e_k \in \mathbb{Z}^m$ denote the canonical basis vectors and consider the projection

$$Q_0 \colon \mathbb{Z}^{n+m} \ \to \ K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$$

onto the factor group by the row lattice of P_0. Then the K_0-*graded* $\mathbb{K}$-*algebra associated with* (A, P_0) is defined by

$$R(A, P_0) \ := \ \mathbb{K}[T_{ij}, S_k]/\langle g_1, \dots, g_{r-c} \rangle,$$

$$\deg(T_{ij}) := Q_0(e_{ij}), \qquad \deg(S_k) := Q_0(e_k).$$

Example 2.1.4. Let us take $c = 2$ and $r = 3$. Thus, we will work with a 3×4 matrix A. Moreover, let $n_0 = 2$ and $n_1 = n_2 = n_3 = 1$ and fix $m = 0$. This amounts to $n = 5$ and a 3×5 matrix P_0. We choose

$$A \ = \ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad P_0 \ = \ \begin{bmatrix} -1 & -2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 & 4 \end{bmatrix}.$$

So, the exponent vectors l_i are $l_0 = (1,2)$, $l_1 = l_2 = (2)$ and $l_3 = (4)$. Accordingly, we obtain the four monomials

$$T_0^{l_0} \;=\; T_{01}T_{02}^2, \qquad T_1^{l_1} \;=\; T_{11}^2, \qquad T_2^{l_2} \;=\; T_{21}^2, \qquad T_3^{l_3} \;=\; T_{31}^4.$$

We arrive at $r - c = 1$ relation $g_1 \in \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}]$, obtained by computing the following 4×4 determinant

$$g_1 \;=\; \det \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ T_{01}T_{02}^2 & T_{11}^2 & T_{21}^2 & T_{31}^4 \end{bmatrix} \;=\; T_{01}T_{02}^2 + T_{11}^2 + T_{21}^2 + T_{31}^4.$$

The canonical basis vectors of the row space of P_0 are indexed in accordance with the variables T_{ij}, that means that we write

$$e_{01}, e_{02}, e_{11}, e_{21}, e_{31} \;\in\; \mathbb{Z}^5 \;=\; \mathbb{Z}^{n_0+n_1+n_2+n_3}.$$

We have $K_0 = \mathbb{Z}^5/\mathrm{im}(P_0^*) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The projection $Q_0\colon \mathbb{Z}^5 \to K_0$ sending e_{ij} to its class in K_0 is made concrete by the degree matrix

$$Q_0 \;=\; [Q_0(e_{ij})] \;=\; \begin{bmatrix} 2 & 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}.$$

Consequently, for the initial data A and P_0 of this example, the resulting K_0-graded algebra $R(A, P_0)$ is given by

$$\mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}] \;/\; \langle T_{01}T_{02}^2 + T_{11}^2 + T_{21}^2 + T_{31}^4 \rangle, \qquad \deg(T_{ij}) \;=\; Q_0(e_{ij}).$$

We present the basic properties of the graded algebra $R(A, P_0)$. Recall that a grading of a $\mathbb{K}$-algebra $R = \oplus_K R_w$ by a finitely generated abelian group is *effective* if the weights $w \in K$ with $R_w \neq \{0\}$ generate K as a group and *pointed*, if $R_0 = \mathbb{K}$ holds and $R_w \neq \{0\} \neq R_{-w}$ is only possible for torsion elements $w \in K$. Finally, we say that an effective grading is of *complexity* c if $\dim(R) - \mathrm{rk}\,(K) = c$ holds.

Theorem 2.1.5. *Let $R(A, P_0)$ be a K_0-graded $\mathbb{K}$-algebra arising from Construction 2.1.3. Then $R(A, P_0)$ is an integral, normal, complete intersection ring satisfying*

$$\dim(R(A, P_0)) \;=\; n + m - r + c, \qquad R(A, P_0)^* \;=\; \mathbb{K}^*.$$

The K_0-grading of $R(A, P_0)$ is effective, pointed, factorial and of complexity c. The variables T_{ij}, S_k define pairwise non-associated K_0-primes in $R(A, P_0)$, and for $c \geq 2$, they define even primes.

The following auxiliary statements for the proof of this theorem are also used later. We begin with discussing the specific nature of the matrix A and its impact on the ideal of relations of $R(A, P)$.

Remark 2.1.6. Situation as in Construction 2.1.3. For any tuple $I = (i_1, \ldots, i_{c+2})$ of strictly increasing integers from $[0, r]$, consider the matrix

$$A(I) \ := \ [a_{i_1}, \ldots, a_{i_{c+2}}],$$

Let $w(I) \in \mathbb{K}^{c+2}$ denote the cross product of the rows of $A(I)$ and define a vector $v(I) \in \mathbb{K}^{r+1}$ by putting the entries of $w(I)$ at the right places:

$$v(I)_i \ := \ \begin{cases} w(I)_j, & i = i_j \text{ occurs in } I = (i_1, \ldots, i_{c+2}), \\ 0, & \text{else.} \end{cases}$$

Then any linearly independent choice of vectors $v(I_1), \ldots, v(I_{r-c})$ is a basis for $\ker(A)$. Note that any nonzero $v \in \ker(A)$ has at least $c + 2$ nonzero coordinates.

Remark 2.1.7. Situation as in Construction 2.1.3. Every vector $v \in \ker(A) \subseteq \mathbb{K}^{r+1}$ defines a polynomial

$$g_v \ := \ v_0 T_0^{l_0} + \ldots + v_r T_r^{l_r} \ \in \ \langle g_1, \ldots, g_{r-c} \rangle.$$

Moreover, if a subset $B \subseteq \ker(A)$ generates $\ker(A)$ as a vector space, then the polynomials g_v, $v \in B$, generate the ideal $\langle g_1, \ldots, g_{r-c} \rangle$. In particular, we have

$$\langle g_1, \ldots, g_{r-c} \rangle \ = \ \langle g_{v(I)};\ I = (i_1, \ldots, i_{c+2}),\ 0 \le i_1 < \ldots < i_{c+2} \le r \rangle,$$

with the tuples I from Remark 2.1.7. Observe that each g_v, $0 \neq v \in \ker(A)$, has at least $c + 2$ of the monomials $T_i^{l_i}$ and all the g_v share the same K_0-degree.

Lemma 2.1.8. *Let $R(A, P_0)$ be a graded algebra arising from Construction 2.1.3.*

(i) *If we have $l_{i1} + \ldots + l_{in_i} = 1$ for some i, then $R(A, P_0)$ is isomorphic to a ring $R(A', P_0')$ with data $r' = r - 1$ and $c' = c$.*

(ii) *If we have $c \ge 2$, then for any generator T_{ij}, the factor ring $R(A, P_0)/\langle T_{ij} \rangle$ is isomorphic to a ring $R(A', P_0')$ with data $r' = r - 1$ and $c' = c - 1$.*

Proof. To obtain (i), let A' be the matrix obtained by deleting the i-th column from A. Then the respective ideals defined by A and A' produce isomorphic rings. Adapting the matrix P_0 accordingly, gives the desired P_0'.

We show (ii). As elementary row operations on A neither change the required properties of A nor the defining ideal of $R(A, P)$, we may assume that $a_{i1} \neq 0$ holds and all other entries of the i-th column of A equal zero. Then the matrix A' obtained by deleting the first row and the i-th column from A satisfies the assumptions of Construction 2.1.3 with $r' = r - 1$ and $c' = c - 1$. Using Remarks 2.1.6 and 2.1.7, we see that the ideal defined by A' corresponds to the defining ideal of $R(A, P_0)/\langle T_{ij} \rangle$. Again, adapting the matrix P_0 accordingly, gives the desired P_0'. ☐

Definition 2.1.9. Situation as in Construction 2.1.3. We say that a point $z \in \mathbb{K}^{n+m}$ with coordinates z_{ij}, z_k is of

(i) *big type*, if for every $i = 0, \ldots, r$, there is an index $1 \leq j_i \leq n_i$ such that $z_{ij_i} = 0$ holds,

(ii) *leaf type*, if there is a set $I_z = \{i_1, \ldots, i_c\}$ of indices $0 \leq i_1 < \ldots < i_c \leq r$, such that for all i, j, we have $z_{ij} = 0 \Rightarrow i \in I_z$.

Remark 2.1.10. Situation as in Construction 2.1.3. Consider $\gamma = \mathbb{Q}^{n+m}$, a face $\gamma_0 \preccurlyeq \gamma$ and the complementary face $\gamma_0^* \preccurlyeq \gamma$. Then any coordinate z_{ij}, z_k of $z = z_{\gamma_0^*} \in \mathbb{K}^{n+m}$ equals zero or one and we have

$$z_{ij} = 0 \iff e_{ij} \in \gamma_0, \qquad z_k = 0 \iff e_k \in \gamma_0.$$

In particular, there is a point of big (leaf) type in $\mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \subseteq \mathbb{K}^{n+m}$ if and only if all points of this toric orbit are of big (leaf) type. Moreover, in terms of the cone $P_0(\gamma_0^*)$ with P_0 from Construction 2.1.3, we obtain the following characterizations:

(i) $z_{\gamma_0^*}$ is of big type if and only if $P_0(\gamma_0^*) = \mathbb{Q}^r$ holds,

(ii) $z_{\gamma_0^*}$ is of leaf type if and only if $P_0(\gamma_0^*) \neq \mathbb{Q}^r$ and $\dim(P_0(\gamma_0^*)) \leq c$.

Observe that if one of the conditions of (ii) holds, then the image cone $P_0(\gamma_0^*)$ is generated by at most r vectors from $-\mathbb{1}_r, e_1 \ldots, e_r \in \mathbb{Z}^r$ and thus is pointed.

Lemma 2.1.11. *For $\bar{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ from Construction 2.1.3, we have the following statements.*

(i) *Every point $z \in \bar{X}$ is either of big type or it is of leaf type.*

(ii) *Every $z \in \mathbb{K}^{n+m}$ of big type is contained in $\bar{X}$.*

(iii) *For every $z \in \mathbb{K}^{n+m}$ of leaf type, there is a $t \in \mathbb{T}^{n+m}$ with $t \cdot z \in \bar{X}$.*

Proof. To obtain (i), we have to show that any $z \in \bar{X}$ which is not of big type must be of leaf type. Otherwise, there are indices $i_1 < \ldots < i_{c+1}$ and associated j_q with $z_{i_q j_q} = 0$. As z is not of big type, there is at least one index i_0 with $z_{i_0 j} \neq 0$ for all $j = 1, \ldots, n_{i_0}$. Remarks 2.1.6 and 2.1.7 provide us with a relation $g \in \langle g_1, \ldots, g_{r-c} \rangle$ involving precisely the monomials $T_i^{l_i}$ for $i = i_0, i_1, \ldots, i_{c+1}$. Then $g(z) = 0$ implies $z_{i_0 j} = 0$ for some $j = 1, \ldots, n_{i_0}$; a contradiction.

We verify (ii) and (iii). Let $z \in \mathbb{K}^{n+m}$. If z is of big type, then we obviously have $g_i(z) = 0$ for $i = 1, \ldots, r-c$. Thus, $z \in \bar{X}$. Now, assume that z is of leaf type. First consider the case $I_z = \{1, \ldots, c\}$. Then, suitably scaling $z_{c+1,1}$, we achieve $g_1(z) = 0$. Next we scale $z_{c+2,1}$ to ensure $g_2(z) = 0$, and so on, until we have also $g_{r-c}(z) = 0$. Then we have found our $t \in \mathbb{T}^{n+m}$ with $t \cdot z \in \bar{X}$. Given an arbitrary I_z, Remarks 2.1.6 and 2.1.7 yield a suitable system $g_1', \ldots, g_{r-c}'$ of ideal generators that allows us to argue analogously. $\square$

Lemma 2.1.12. *Situation as in Construction 2.1.3. Let* $\bar{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ *and denote by* J *the Jacobian of* $g_1, \ldots, g_{r-c}$*. Then, for any* $z \in \bar{X}$*, the following statements are equivalent:*

(i) *The Jacobian* $J(z)$ *is not of full rank, i.e., we have* $\mathrm{rk}\,(J(z)) < r - c$.

(ii) *The point* $z \in \bar{X}$ *is of big type and there are* $i_1 < \ldots < i_{c+2}$ *such that each of these* i_q *fulfills one of the subsequent two conditions:*

- $z_{i_q j_q} = 0$ *and* $l_{i_q j_q} \geq 2$ *hold for at least one* $1 \leq j_q \leq n_{i_q}$,
- $z_{i_q j} = 0$ *and* $l_{i_q j} = 1$ *hold for at least two* $1 \leq j \leq n_{i_q}$.

In particular, the set of points $z \in \bar{X}$ *with* $J(z)$ *not of full rank is of codimension at least* $c+1$ *in* $\bar{X}$.

Proof. Assertion (ii) directly implies the supplement and, by a simple computation, also (i). We are left with proving "(i)⇒(ii)". So, let $z \in \bar{X}$ be a point such that $J(z)$ is not of full rank. Then there is a non-trivial linear combination annulating the lines of $J(z)$:

$$\eta_1 \mathrm{grad}(g_1)(z) + \ldots + \eta_{r-c} \mathrm{grad}(g_{r-c})(z) \;=\; 0.$$

The corresponding $g := \eta_1 g_1 + \ldots + \eta_{r-c} g_{r-c}$ satisfies $\mathrm{grad}(g)(z) = 0$ and is of the form $g = g_v$ with a nonzero $v \in \ker(A)$ as in Remark 2.1.7. The condition $\mathrm{grad}(g)(z) = 0$ implies $z_{ij_i} = 0$ for some $1 \leq j_i \leq n_i$ whenever the monomial $T_i^{l_i}$ shows up in g. As observed in Remark 2.1.7, the polynomial g has at least $c+2$ monomials. Thus, we have $z_{ij_i} = 0$ for at least $c+2$ different i. By Lemma 2.1.11, the point $z \in \bar{X}$ is of big type. Moreover, the two conditions of (ii) reflect the fact $\mathrm{grad}(g)(z) = 0$. □

Proof of Theorem 2.1.5. For $c = 1$, the statement is proven in [40, Thm. 10.1 and Prop. 10.7]. So, assume $c \geq 2$. First we show that $\bar{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ is connected. By construction, the quasitorus $H_0 \subseteq \mathbb{T}^{n+m}$ is the kernel of the homomorphism $\mathbb{T}^{n+m} \to \mathbb{T}^r$ defined by P_0. Consider the multiplicative one-parameter subgroup $\mathbb{K}^* \to H_0$, $t \mapsto (t^\zeta, t^\xi)$, where

$$\zeta \;=\; \left(\frac{n_0 \cdots n_r l_{01} \cdots l_{rn_r}}{n_0 l_{01}}, \ldots, \frac{n_0 \cdots n_r l_{01} \cdots l_{rn_r}}{n_r l_{rn_r}}\right) \;\in\; \mathbb{T}^n, \quad \xi \;=\; (1, \ldots, 1) \;\in\; \mathbb{T}^m.$$

This gives rise to a $\mathbb{K}^*$-action on $\bar{X}$ having the origin as an attractive fixed point. Consequently, $\bar{X}$ is connected. Moreover, we can conclude that all invertible functions as well as all H_0-invariant functions are constant on $\bar{X}$.

Now, Lemma 2.1.12 allows us to apply Serre's criterion and thus we obtain that $R(A, P_0)$ is an integral, normal, complete intersection. By construction, the K_0-grading is effective and as seen above, it is pointed. To obtain factoriality of the K_0-grading, localize $R(A, P_0)$ by the product over all generators T_{ij}, S_k, observe that the degree zero part of the resulting ring is a polynomial ring and apply [11, Thm. 1.1]. Finally, primality of the generators T_{ij} follows from Lemma 2.1.8 (ii). □

Now we use the algebras $R(A, P_0)$ obtained by Construction 2.1.3 to produce general arrangement varieties. The basic idea is to turn $R(A, P_0)$ into a prospective Cox ring via coarsening the grading by $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$ to a grading by $K = \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$, where P arises from P_0 by adding suitable further rows.

Construction 2.1.13. Let A and P_0 be input data as in Construction 2.1.3. Moreover, fix $1 \le s \le n+m-r$ and let d be an integral $s \times (n+m)$ matrix such that the columns v_{ij}, v_k of the $(r+s) \times (n+m)$ stack matrix

$$P \;:=\; \begin{bmatrix} P_0 \\ d \end{bmatrix}$$

are pairwise different, primitive and generate $\mathbb{Q}^{r+s}$ as a vector space. Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$. Then the projection $Q\colon \mathbb{Z}^{n+m} \to K$ factors through Q_0 and we obtain the *K-graded $\mathbb{K}$-algebra associated with* (A, P):

$$R(A,P) \;:=\; \mathbb{K}[T_{ij}, S_k]/\langle g_1, \ldots, g_{r-c}\rangle,$$

$$\deg(T_{ij}) := w_{ij} := Q(e_{ij}), \qquad \deg(S_k) := w_k \;:=\; Q(e_k).$$

Now, let Σ be any fan in $\mathbb{Z}^{r+s}$ having precisely the rays through the columns of P as its one-dimensional cones and let Z be the associated toric variety. Then we have a commutative diagram

$$\begin{array}{ccccccc}
V(g_1, \ldots, g_{r-c}) & = & \bar{X} & \subseteq & \bar{Z} & = & \mathbb{Z}^{n+m} \\
 & & \cup\!\shortmid & & \cup\!\shortmid & & \\
 & & \hat{X} & \subseteq & \hat{Z} & & \\
 & & {\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H} & & \\
 & & X & \subseteq & Z & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{P}_c & \longrightarrow & \mathbb{P}_r & &
\end{array}$$

with the quasitorus $H = \mathrm{Spec}\,\mathbb{K}[K]$, Cox's quotient presentation $\hat{Z} \to Z$ and the induced quotient $\hat{X} \to X$, where $\hat{X} := \bar{X} \cap \hat{Z}$. The resulting variety $X = X(A, P, \Sigma)$ is normal with dimension, invertible functions, divisor class group and Cox ring given by

$$\dim(X) = s + c, \qquad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*, \qquad \mathrm{Cl}(X) = K, \qquad \mathcal{R}(X) = R(A,P).$$

The acting torus $\mathbb{T}_Z \subseteq Z$ splits as $\mathbb{T}_Z = \mathbb{T}^r \times \mathbb{T}^s$ and the factor $\mathbb{T} = \{\mathbb{1}_r\} \times \mathbb{T}^s$ leaves $X \subseteq Z$ invariant. The induced $\mathbb{T}$-action on X is effective and of complexity

$$c(X) \;=\; c.$$

Finally, the dashed arrows indicate the maximal orbit quotients for the $\mathbb{T}$-actions on X and Z respectively and $\mathbb{P}_c \subseteq \mathbb{P}_r$ is the linear subspace given by

$$\mathbb{P}_c \;=\; V(h_1,\ldots,h_{r-c}), \qquad h_t \;:=\; \det \begin{bmatrix} a_0 & \ldots & a_c & a_{c+t} \\ U_0 & \ldots & U_c & U_{c+t} \end{bmatrix} \;\in\; \mathbb{K}[U_0,\ldots,U_r].$$

A collection of doubling divisors for the maximal orbit quotient $X \dashrightarrow \mathbb{P}_c$ is given by the intersections of $\mathbb{P}_c$ with the coordinate hyperplanes of $\mathbb{P}_r$ which form the general hyperplane arrangement

$$H_0,\ldots,H_r \;\subseteq\; \mathbb{P}_c, \qquad H_i \;:=\; \{z \in \mathbb{P}_c;\; a_{i0}z_0 + \ldots + a_{ic}z_c = 0\}.$$

Proposition 2.1.14. *Let $X = X(A,P,\Sigma)$ arise from Construction 2.1.13. Consider the explicit variety $Y = \mathbb{P}_c$ in $Z_\Delta = \mathbb{P}_r$ with embedding system $\alpha = (f_0,\ldots,f_r)$, where Δ is the complete fan in $\mathbb{Z}^r$ with generator matrix $B = [-\mathbb{1}_r, \mathbb{E}_r]$ and*

$$f_i \;=\; a_{i0}U_0 + \ldots + a_{ic}U_c \;\in\; \mathbb{K}[U_0,\ldots,U_c] \;=\; \mathcal{R}(\mathbb{P}_c).$$

Then the variety $X(A,P,\Sigma) \subseteq Z_\Sigma$ equals the variety $X(\alpha,P,\Sigma) \subseteq Z_\Sigma$ arising from Construction 1.2.5. In particular, $X(A,P,\Sigma) \subseteq Z_\Sigma$ is an explicit $\mathbb{T}$-variety.

Proof of Construction 2.1.13 and Proposition 2.1.14. The fact that $X(A,P,\Sigma) \subseteq Z_\Sigma$ equals $X(\alpha,P,\Sigma) \subseteq Z_\Sigma$ is clear by construction. Observe that, forgetting for the moment about the gradings, we have $R(A,P) = R(A,P_0)$. Thus, Theorem 2.1.5 ensures that $R(A,P)$ is normal, integral with only constant homogeneous units. Moreover, for $c \geq 2$, the generators T_{ij} and S_k of $R(A,P)$ are pairwise non-associated. Being prime and K-homogeneous, they are also K-prime. For $c = 1$, we infer K-primality of the generators from [40, Thm. 10.4]. So, $X(A,P,\Sigma) \subseteq Z_\Sigma$ satisfies the conditions of Definition 1.2.8 and hence is an explicit $\mathbb{T}$-variety. This yields in particular the statements on the divisor class group and the Cox ring. The statement on the maximal orbit quotient is due to Proposition 1.2.17. □

Remark 2.1.15. According to Lemma 2.1.8 (i), we may always assume that the defining data P of Construction 2.1.13 is *irredundant* in the sense that $l_{i0} + \ldots + l_{in_i} \geq 2$ holds for every $i = 0,\ldots,r$. In this case, we also say that $X(A,P,\Sigma)$ is *irredundant.*

Definition 2.1.16. By an *explicit general arrangement variety* we mean a $\mathbb{T}$-variety $X = X(A,P,\Sigma)$ in $Z = Z_\Sigma$ arising from Construction 2.1.13.

Example 2.1.17. Let A and P_0 be as in Example 2.1.4. We enhance P_0 by an 1×5 block as follows

$$P \;=\; \begin{bmatrix} P_0 \\ d \end{bmatrix} \;=\; \begin{bmatrix} -1 & -2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 & 4 \\ -1 & -3 & 1 & 1 & 1 \end{bmatrix}.$$

So, we chose $s = 1$. We have $K = \mathbb{Z}^5/\mathrm{im}(P^*) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $Q\colon \mathbb{Z}^5 \to K$ is represented by the degree matrix, having $w_{ij} = Q(e_{ij}) \in K$ as its columns:

$$Q \;=\; [w_{01}, w_{02}, w_{11}, w_{21}, w_{31}] \;=\; \begin{bmatrix} 2 & 1 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}.$$

Let Σ be the unique complete fan in $\mathbb{Z}^4$ with P as its generator matrix. Then we arrive at a projective explicit arrangement variety $X = X(A, P, \Sigma)$ in $Z = Z_\Sigma$ with

$$\dim(X) \;=\; 3, \qquad c(X) \;=\; 2, \qquad \mathrm{Cl}(X) \;=\; \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Moreover, assigning to each generator T_{ij} the divisor class $Q(e_{ij})$, we obtain a representation of the Cox ring by homogeneous generators and relations:

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}] \;/\; \langle T_{01}T_{02}^2 + T_{11}^2 + T_{21}^2 + T_{31}^4 \rangle.$$

As a maximal orbit quotient, we have $\pi\colon X \dashrightarrow \mathbb{P}_2$ and the doubling divisors form the general line configuration in $\mathbb{P}_2$ given by

$$V(T_0), \quad V(T_1), \quad V(T_2), \quad V(T_0 + T_1 + T_2).$$

Theorem 2.1.18. *Let X be an A_2-maximal general arrangement variety. Then X admits a presentation as an explicit general arrangement variety.*

Proof. According to Definition 2.1.1, there is a maximal orbit quotient $\pi\colon X \dashrightarrow Y$ with $Y = \mathbb{P}_c$ admitting a general hyperplane arrangement $C_0, \ldots, C_r$ as a collection of doubling divisors. Then the canonical sections $1_{C_0}, \ldots, 1_{C_r}$ are of degree one in the Cox ring $\mathcal{R}(Y) = \mathbb{K}[U_0, \ldots, U_c]$. Suitably enhancing the general hyperplane arrangement $C_0, \ldots, C_r$, we achieve that $1_{C_0}, \ldots, 1_{C_r}$ generate $\mathcal{R}(Y)$. Regard $\mathcal{R}(Y)$ as a graded subalgebra of $\mathcal{R}(X)$ as in Corollary 1.3.2 and let $\alpha = (f_0, \ldots, f_q)$ be pairwise non-associated $\mathrm{Cl}(X)$-prime generators of the Cox ring $\mathcal{R}(X)$ such that the f_i lying in $\mathcal{R}(Y)$ are precisely $1_{C_0}, \ldots, 1_{C_r}$. Then, following the lines of the proof of Theorem 1.2.10, one reproduces X as an explicit $\mathbb{T}$-variety $X(\alpha, P, \Sigma)$ in $Z = Z_\Sigma$. Thus, Proposition 2.1.14 gives the assertion. □

Remark 2.1.19. Let X be a general arrangement variety of complexity c. Then the torus action of X has $\mathbb{P}_c$ as Chow quotient; use [9, Props. 2.4 and 2.5] for a proof. Using the conversions Remark 1.2.18 and [46, Thm. 4.8], we see that the general arrangement varieties are precisely the $\mathbb{T}$-varieties arising from a divisorial fan Ξ on a projective space $\mathbb{P}_c$ in the sense of [2] such that the prime divisors $D \subseteq \mathbb{P}_c$ with non-trivial slices Ξ_D form a general hyperplane arrangement in $\mathbb{P}_c$.

2.2 Examples and first properties

We discuss examples and study basic structural properties of general arrangement varieties. For instance, we investigate torsion in the divisor class group, describe the canonical class and give a combinatorial characterization of the X-cones which in turn leads to the combinatorial smoothness criterion provided by Corollary 2.2.16. Moreover, we specify constraints on the defining data of an explicit general arrangement variety imposed by conditions on the singularities, preparing the classification presented in Chapter 3. As a first concrete application, we prove at the end of this section that the smooth projective general arrangement varieties of Picard number one are just the classical smooth projective quadrics; see Proposition 2.2.23.

We begin with the examples. The first one shows how to realize intrinsic quadrics as explicit general arrangement varieties. Recall from [15] that an *intrinsic quadric* is a normal projective variety with a Cox ring defined by a single quadratic relation. The intrinsic quadrics form a playground immediately adjacent to the one given by the projective toric varieties, which have a polynomial ring as Cox ring. We mention Bourqui's work [17] proving Manin's conjecture for the full intrinsic quadrics and the classification results on smooth (Fano) intrinsic quadrics of low Picard number in [29] as examples for research in this field.

Example 2.2.1. The normal form for graded quadrics provided by [29, Prop. 2.1] shows that we can represent every intrinsic quadric as an explicit general arrangement variety $X \subseteq Z$ with defining matrix P having left upper block

$$\begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \qquad l_0 = \ldots = l_q = (1,1), \quad l_{q+1} = \ldots = l_r = (2),$$

where $-1 \le q \le r$ and the variables T_{i1} with $i = q+1, \ldots, r$ have pairwise distinct K-degrees. Moreover, for the dimension of X, the rank of the divisor class group and the complexity of the torus action on X, we have

$$\dim(X) \ = \ r-1+s, \qquad \operatorname{rk}(\operatorname{Cl}(X)) \ = \ m+q+2-s, \qquad c(X) \ = \ r-1.$$

In the second example we exhibit a series of general arrangement varieties producing many Fano examples. We pick up these varieties again in Example 2.2.18, when the necessary methods are available to figure out the smooth Fano varieties.

Example 2.2.2. Fix integers $r > c \ge 1$. Consider the product $Z = \mathbb{P}_r \times \mathbb{P}_r$ and the intersection $X = V(g_1) \cap \ldots \cap V(g_{r-c}) \subseteq Z$ of the $r-c$ divisors of bidegree (a,b) in Z given by

$$\begin{aligned} g_1 \ &= \ \lambda_{1,0} T_{01}^a T_{02}^b + \lambda_{1,1} T_{11}^a T_{12}^b + \ldots + \lambda_{1,c} T_{c1}^a T_{c2}^b + T_{c+1,1}^a T_{c+1,2}^b, \\ &\ \ \vdots \\ g_{r-c} \ &= \ \lambda_{r-c,0} T_{01}^a T_{02}^b + \lambda_{r-c,1} T_{11}^a T_{12}^b + \ldots + \lambda_{r-c,c} T_{c1}^a T_{c2}^b + T_{r1}^a T_{r2}^b, \end{aligned}$$

where $a, b > 0$ are coprime integers and any $c+1$ of the vectors $\lambda_i = (\lambda_{i,0}, \ldots, \lambda_{i,c})$ are linearly independent. Observe that for $r > c+1$, the divisors $V(g_i) \subseteq Z$ are singular. We realize $X \subseteq Z$ as an explicit general arrangement variety. Let P be the stack matrix with upper and lower blocks

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \qquad l_0 = \ldots = l_r = (a, b),$$

$$d \;=\; \begin{bmatrix} -d_0 & d_1 & & 0 \\ \vdots & & \ddots & \\ -d_0 & 0 & & d_r \end{bmatrix}, \qquad d_0 = \ldots = d_r = (v, u),$$

where u and v are integers with $ua - vb = 1$. We claim that there is precisely one complete fan Σ with generator matrix P and the associated toric variety $Z = Z_\Sigma$ is the product $\mathbb{P}_r \times \mathbb{P}_r$. Indeed, consider the matrices

$$\begin{bmatrix} u \cdot \mathbb{E}_r & -b \cdot \mathbb{E}_r \\ -v \cdot \mathbb{E}_r & a \cdot \mathbb{E}_r \end{bmatrix}, \qquad \begin{bmatrix} -\mathbb{1}_r & \mathbb{E}_r & 0 & 0 \\ 0 & 0 & -\mathbb{1}_r & \mathbb{E}_r \end{bmatrix}.$$

The first one is unimodular and multiplying it from the left to P yields, after suitably renumbering columns, the second one. Now, choosing a suitable $(c+1) \times (r+1)$ matrix A, we obtain the above relations as the output of Construction 2.1.13. Thus, $X = X(A, P, \Sigma)$ is of dimension $r + c$ and comes with an effective r-torus action.

We enter the study of structural properties of explicit general arrangement varieties $X \subseteq Z$ as provided by Construction 2.1.13. We will freely use the notation fixed there. Our first observation is that there may occur unavoidable torsion in the divisor class group.

Proposition 2.2.3. *Let $X \subseteq Z$ be an explicit general arrangement variety. Then $\mathbb{Z}^r/\mathrm{im}(P_0)$ is a finite subgroup of the divisor class group $\mathrm{Cl}(X)$.*

Proof. The divisor class group of X equals $K = \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$. Moreover, $\mathbb{Z}^r/\mathrm{im}(P_0)$ is the torsion part K_0^{tors} of the factor group $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$. Applying the snake Lemma to the exact sequences arising from P_0^* and P^* yields that the kernel of $K_0 \to K$ injects into $\mathbb{Z}^s$. Consequently, the torsion part K_0^{tors} maps injectively into K. □

In Remark 2.1.7, we observed that $R(A, P)$ is a complete intersection ring. Thus, we can apply Proposition 1.4.10 and obtain the following description of the canonical class.

Proposition 2.2.4. *Let $X \subseteq Z$ be an explicit general arrangement variety of complexity $c(X) = c$. Then the canonical class of X is given in terms of the generator degrees $w_{ij} = \deg(T_{ij})$ and $w_k = \deg(S_k)$ as*

$$\mathcal{K}_X \;=\; -\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} \;-\; \sum_{k=0}^{r} w_k \;+\; (r-c)\sum_{j=1}^{n_0} l_{0j} w_{0j} \;\in\; K \;=\; \mathrm{Cl}(X).$$

Example 2.2.5. Consider again the explicit general arrangement varieties $X \subseteq Z$ discussed in Example 2.2.2. The degree matrix Q is

$$Q \;=\; [w_{01}, w_{02}, \ldots, w_{r1}, w_{r2}] \;=\; [\mathbb{E}_2, \ldots, \mathbb{E}_2].$$

Thus, Proposition 1.4.6 tells us $\mathrm{Eff}(X) = \mathrm{SAmple}(X) = \mathrm{cone}(e_1, e_2)$. Moreover, the anticanonical class of X is given by

$$-\mathcal{K}_X \;=\; (r + 1 - (r - c)a,\, r + 1 - (r - c)b) \;\in\; \mathrm{Cl}(X) \;=\; \mathbb{Z}^2.$$

as we infer from Proposition 2.2.4. In particular, X is a Fano variety if and only if the following two conditions are satisfied

$$a \;<\; \frac{r+1}{r-c}, \qquad\qquad b \;<\; \frac{r+1}{r-c}.$$

Recall that in Definition 1.4.2 we introduced for any explicit variety $X \subseteq Z$ the X-cones as those cones $\sigma \in \Sigma$ of the defining fan of Z such that X intersects the corresponding orbit $\mathbb{T}_Z \cdot z_\sigma$ non-trivially. For explicit general arrangement varieties $X \subseteq Z$, we may determine the X-cones in a simple purely combinatorial way.

Definition 2.2.6. Consider the setting of Construction 2.1.13 and let $\sigma \in \Sigma$. We say that the cone σ is

(i) *big (elementary big)* if σ contains at least (precisely) one column v_{ij} of P for every $i = 0, \ldots, r$,

(ii) a *leaf cone* if there is a set $I_\sigma = \{i_1, \ldots, i_c\}$ of indices $0 \le i_1 < \ldots < i_c \le r$ such that for any i, we have $v_{ij} \in \sigma \Rightarrow i \in I_\sigma$.

Remark 2.2.7. Situation as in Construction 2.1.13. Given $\sigma \in \Sigma$, let $\gamma_0 \preccurlyeq \gamma$ be the corresponding face, that means that $\sigma = P(\gamma_0^*)$ holds. Then σ is a big (leaf) cone if and only if the toric orbit $\mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \subseteq \mathbb{K}^{n+m}$ consists of points of big (leaf) type in the sense of Definition 2.1.9.

Proposition 2.2.8. *Let $X \subseteq Z$ be an explicit general arrangement variety. Then, for every $\sigma \in \Sigma$, the following statements are equivalent.*

(i) *The cone σ is an X-cone.*

(ii) *The cone σ is big or a leaf cone.*

Proof. Consider the face $\gamma_0 \preccurlyeq \gamma$ with $P(\gamma_0^*) = \sigma$. By Remark 2.2.7, our σ is a big (leaf) cone if and only if $\bar{X}(\gamma_0)$ consists of points of big (leaf) type. The assertion thus follows from Lemma 2.1.11. □

Example 2.2.9. We look again at $X = X(A, P, \Sigma)$ in $Z = Z_\Sigma$ from Examples 2.1.4 and 2.1.17. Recall that we have

$$P = [v_{01}, v_{02}, v_{11}, v_{21}, v_{31}] = \begin{bmatrix} -1 & -2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 & 4 \\ -1 & -3 & 1 & 1 & 1 \end{bmatrix}.$$

Except $\operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31})$, every cone generated by some of the v_{ij} occurs in the fan Σ. In particular, Σ has two big cones

$$\sigma_1 = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}), \qquad \sigma_2 = \operatorname{cone}(v_{02}, v_{11}, v_{21}, v_{31}),$$

and six maximal leaf cones

$$\tau_1 = \operatorname{cone}(v_{01}, v_{02}, v_{11}), \quad \tau_2 = \operatorname{cone}(v_{01}, v_{02}, v_{21}), \quad \tau_3 = \operatorname{cone}(v_{01}, v_{02}, v_{31}),$$

$$\tau_4 = \operatorname{cone}(v_{11}, v_{21}), \quad \tau_5 = \operatorname{cone}(v_{11}, v_{31}), \quad \tau_6 = \operatorname{cone}(v_{21}, v_{31}).$$

Thus, by Proposition 2.2.8 the X-cones of Σ are σ_1, σ_2 and the faces of $\tau_1, \ldots, \tau_6$. This allows us to determine the Picard group $\operatorname{Pic}(X)$. Recall the degree matrix

$$Q = [w_{01}, w_{02}, w_{11}, w_{21}, w_{31}] = \begin{bmatrix} 2 & 1 & 2 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix},$$

having the generator degrees $w_{ij} = Q(e_{ij}) \in K = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as its columns. The X-faces corresponding to the X-cones $\sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3$ are

$$\gamma_1 = \operatorname{cone}(e_{02}), \qquad \gamma_2 = \operatorname{cone}(e_{02}),$$

$$\eta_1 = \operatorname{cone}(e_{11}, e_{21}), \quad \eta_2 = \operatorname{cone}(e_{11}, e_{21}), \quad \eta_3 = \operatorname{cone}(e_{21}, e_{31}).$$

Observe that these are precisely the minimal ones among all X-faces $\gamma_0 \preccurlyeq \gamma = \mathbb{Q}^5_{\ge 0}$. Thus, Proposition 1.4.6 yields

$$\operatorname{Pic}(X) = \bigcap_{i=1}^{2} Q(\operatorname{lin}_{\mathbb{Q}}(\gamma_i) \cap \mathbb{Z}^4) \cap \bigcap_{i=1}^{3} Q(\operatorname{lin}_{\mathbb{Q}}(\eta_i) \cap \mathbb{Z}^4) = \mathbb{Z} \cdot (4, \bar{0}, \bar{0}) \subseteq \operatorname{Cl}(X).$$

Using Proposition 2.2.4, we see that $(4, \bar{0}, \bar{0})$ equals the anticanonical class $-\mathcal{K}_X$. In particular, X is a Gorenstein Fano threefold.

Big and leaf cones admit also simple characterizations in terms of the geometry of the defining fan of the ambient toric variety.

Remark 2.2.10. Consider the setting of Construction 2.1.13 and let $L \subseteq \mathbb{Z}^{r+s}$ be the kernel of the projection $\operatorname{pr}\colon \mathbb{Z}^{r+s} \to \mathbb{Z}^s$. Then, for any $\sigma \in \Sigma$, the following statements are equivalent.

(i) The cone σ is big.

(ii) We have $\mathrm{pr}(\sigma) = \mathbb{Q}^r$.

(iii) We have $\sigma \not\subseteq L_\mathbb{Q}$ and $\sigma^\circ \cap L_\mathbb{Q} \neq \emptyset$.

Moreover, $\sigma \in \Sigma$ is a leaf cone if and only if its image $\mathrm{pr}(\sigma) \subseteq \mathbb{Q}^r$ is a pointed cone of dimension at most c.

Proposition 2.2.11. *Situation as in Construction 2.1.13. Let $L \subseteq \mathbb{Z}^{r+s}$ be the kernel of the projection $\mathrm{pr}\colon \mathbb{Z}^{r+s} \to \mathbb{Z}^s$ and Σ_L the fan in $\mathbb{Z}^{r+s}$ consisting of all the faces of the cones $\sigma \cap L_\mathbb{Q}$, where $\sigma \in \Sigma$. Then the following statements are equivalent.*

(i) *Σ_L is a subfan of Σ.*

(ii) *Σ contains no big cone.*

(iii) *Σ consists of leaf cones.*

Proof. The equivalence of (ii) and (iii) is clear. We prove "(i)⇒(ii)". Assume that there is a big cone $\sigma \in \Sigma$. Then $\sigma \cap L_\mathbb{Q}$ belongs to Σ_L but not to Σ according to 2.2.10 (iii); a contradiction. We turn to "(ii)⇒(i)". The task is to show that for every cone $\sigma \in \Sigma$, the intersection $\sigma \cap L_\mathbb{Q}$ is a face of σ. Let $\tau \preccurlyeq \sigma$ be the minimal face containing $\sigma \cap L_\mathbb{Q}$. Then $\tau^\circ \cap L_\mathbb{Q}$ is non-empty. Since $\tau \in \Sigma$ is not big, we can use 2.2.10 (iii) to conclude $\tau \subseteq L_\mathbb{Q}$. This means $\sigma \cap L_\mathbb{Q} = \tau$. □

We use the the concrete description of X-cones as big cones and leaf cones to study (quasi)smoothness properties of explicit general arrangement varieties $X \subseteq Z$. First, let us define quasismoothness.

Definition 2.2.12. Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety. We say that $x \in X$ is a *quasismooth* point of X if the fiber $p^{-1}(x) \subseteq \hat{X}$ consists of smooth points of $\bar{X}$.

Remark 2.2.13. Let $X \subseteq Z$ be an explicit $\mathbb{T}$-variety, $\sigma \in \Sigma$ an X-cone and $\gamma_0 \preccurlyeq \gamma$ the face with $P(\gamma_0^*) = \sigma$. Then, for $x \in X(\sigma)$, the intersection $p^{-1}(x) \cap \bar{X}(\gamma_0)$ equals the closed orbit $H \cdot z$ of $p^{-1}(x)$. In particular, $x \in X$ is quasismooth if and only if $z \in \bar{X}$ is smooth. Moreover, $X(\sigma)$ consists of quasismooth points of X if and only if $\bar{X}(\gamma_0)$ consists of smooth points of $\bar{X}$.

Proposition 2.2.14. *Let $X \subseteq Z$ be an explicit general arrangement variety.*

(i) *For every big cone $\sigma \in \Sigma$, the following statements are equivalent.*

   (a) *There is a quasismooth point of X in the piece $X(\sigma) \subseteq X$.*

   (b) *The piece $X(\sigma) \subseteq X$ consists of quasismooth points of X.*

(c) *Every sequence* $0 \le i_1 < \ldots < i_{c+2} \le r$ *admits* $1 \le q \le c+2$ *and* $1 \le j \le n_{i_q}$ *such that* $v_{i_q j} \in \sigma$, $l_{i_q j} = 1$ *and* $v_{i_q k} \notin \sigma$ *for all* $k \neq j$.

(ii) *For every leaf cone* $\sigma \in \Sigma$, *the piece* $X(\sigma) \subseteq X$ *consists of quasismooth points of* X.

Proof. According to Remark 2.2.13, we just have to care about smoothness of the points of $\bar{X}(\gamma_0)$. By Remark 2.1.7, a point $z \in \bar{X}(\gamma_0)$ is smooth if and only if the Jacobian $J(z)$ of $g_1, \ldots, g_{r-c}$ is of full rank. The latter is characterized via Lemma 2.1.12 (ii). In particular, we see that in the case of a leaf cone σ, all points of $\bar{X}(\gamma_0)$ are smooth, proving (ii). To show (i), let σ be big. By the nature of Condition 2.1.12 (ii), there is a smooth point of $\bar{X}$ in $\bar{X}(\gamma_0)$ if and only if every point of $\bar{X}(\gamma_0)$ is smooth in $\bar{X}$. This establishes the equivalence of (a) and (b). The equivalence of (a) and (c) is obtained by negating Condition 2.1.12 (ii) for a point z of big type. □

Corollary 2.2.15. *Let* $X \subseteq Z$ *be a quasismooth explicit general arrangement variety. Assume that* P *is irredundant and let* $\sigma = \mathrm{cone}(v_{0j_0} + \ldots + v_{rj_r})$ *be an elementary big cone of* Σ.

(i) *We have* $l_{ij_i} \ge 2$ *for at most* $c+1$ *different* $i = 0, \ldots, r$.

(ii) *We have* $n_i = 1$ *for at most* $c+1$ *different* $i = 0, \ldots, r$.

Combining Proposition 2.2.14, Remark 2.2.13 and Proposition 1.4.5 leads to the following purely combinatorial smoothness criterion for explicit general arrangement varieties.

Corollary 2.2.16. *Let* $X \subseteq Z$ *be an explicit general arrangement variety.*

(i) *Let* $\sigma \in \Sigma$ *be a big cone and* $\gamma_0 \preccurlyeq \gamma$ *the corresponding* X*-face. Then the following statement are equivalent.*

  (a) *There is a smooth point of* X *in the piece* $X(\sigma) = X(\gamma_0) \subseteq X$.
  (b) *The piece* $X(\sigma) = X(\gamma_0) \subseteq X$ *consists of smooth points of* X.
  (c) *The cone* σ *is regular and 2.2.14 (i) (c) holds.*
  (d) *We have* $K = Q(\mathrm{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ *and 2.2.14 (i) (c) holds.*

(ii) *Let* $\sigma \in \Sigma$ *be a leaf cone and* $\gamma_0 \preccurlyeq \gamma$ *the corresponding* X*-face. Then the following statements are equivalent.*

  (a) *There is a smooth point of* X *in the piece* $X(\sigma) = X(\gamma_0) \subseteq X$.
  (b) *The piece* $X(\sigma) = X(\gamma_0) \subseteq X$ *consists of smooth points of* X.
  (c) *The cone* σ *is regular.*
  (d) *We have* $K = Q(\mathrm{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$.

Example 2.2.17. Let us continue the discussion of $X \subseteq Z$ from 2.1.4 and 2.1.17. In 2.2.9, we determined the maximal X-cones: there are two big cones σ_1, σ_2 and six maximal leaf cones $\tau_1, \dots, \tau_6$. Using Corollary 2.2.16, we see that the associated pieces are precisely those consisting of singular points of X. Note that

$$\overline{X(\tau_i)} \;=\; X(\sigma_1)\; \cup X(\tau_i) \cup\; X(\sigma_2), \qquad i = 1, 2, 3,$$

are curves, each being the closure of the $\mathbb{T}$-orbit $X(\tau_i)$; use Proposition 1.4.9. The union over these $\overline{X(\tau_i)}$ is a connected component of the singular locus of X. In addition, we have $X(\tau_i)$ for $i = 4, 5, 6$, each consisting of an isolated singularity.

Example 2.2.18. We continue Example 2.2.2. Suitably renumbering the variables we achieve $a \ge b$. We claim that X is smooth if and only if one of the following conditions is satisfied:

(i) $r = c + 1$, $a \ge 1$ and $b = 1$,

(ii) $r = c + 2$ and $a = b = 1$.

Indeed, $r \le c + 2$ holds, because otherwise the big cone $\sigma \in \Sigma$ generated by all v_{ij} with $i \le r - 2$ and $v_{r-1\,1}$, v_{r2} yields a singular point in $X(\sigma)$. Now, the elementary big cones of Σ are precisely $\mathrm{cone}(v_{0j_0}, \dots, v_{rj_r})$, where $\{j_0, \dots, j_r\}$ equals $\{1, 2\}$, and the claim follows from Corollaries 2.2.15 and 2.2.16. In particular we obtain smooth Fano varieties in the cases

(iii) $r = c + 1$, $1 \le a \le r$ and $b = 1$,

(iv) $r = c + 2$ and $a = b = 1$.

Finally, we observe constraints on the defining data of an explicit general arrangement variety arising from local factoriality and $\mathbb{Q}$-factoriality.

Proposition 2.2.19. *Let $X \subseteq Z$ be a locally factorial explicit general arrangement variety, where P is irredundant. Assume that Σ consists of leaf cones and each of the sets cone$(v_{i1}) + L_{\mathbb{Q}}$ is covered by cones of Σ. Then $n_i \ge 2$ holds for $i = 0, \dots, r$.*

Proof. Assume that $n_i = 1$ holds for some i. Let ϱ denote the ray through v_{i1} and consider the cone $\tau := \varrho + L_{\mathbb{Q}}$. We claim that for every $\sigma \in \Sigma$, the intersection $\tau \cap \sigma$ is a face of σ. Indeed, as Σ consists of leaf cones, the image of $\mathrm{pr}(\sigma)$ under the projection $\mathrm{pr}\colon \mathbb{Q}^{r+s} \to \mathbb{Q}^r$ is a pointed cone, having $\mathrm{pr}(\varrho)$ as an extremal ray. Thus, $\tau = \mathrm{pr}^{-1}(\mathrm{pr}(\varrho))$ cuts out a face from σ.

By our assumptions, the above claim implies that $\tau = \varrho + L_{\mathbb{Q}}$ is a union of cones of Σ. Any cone of $\Sigma \setminus \Sigma_L$ contained in τ is necessarily of the form $\varrho + \sigma_L \in \Sigma$ with $\sigma_L \in \Sigma_L$. We conclude that in particular all the cones $\sigma = \varrho + \sigma_L$, where $\dim(\sigma_L) = s$, must belong to Σ. As σ and σ_L are leaf cones, they are X-cones by Proposition 2.2.8. Thus, Proposition 1.4.5 yields that σ and σ_L are regular. This implies $l_{i1} = 1$; a contradiction to the assumption that P is irredundant. $\square$

Corollary 2.2.20. *Let $X \subseteq Z$ be a non-toric, projective, locally factorial explicit general arrangement variety. If Σ consists of leaf cones, then the Picard number and the complexity of X satisfy*

$$\rho(X) \;\geq\; r+3 \;\geq\; c(X)+4.$$

Proof. Since X is non-toric, we may assume that P is irredundant with $r > c$. Moreover, as X is projective, we may assume that Σ is complete. Thus, Proposition 2.2.19 applies and we obtain $n \geq 2r+2$. Then Corollary 1.4.12 yields the desired estimate. □

Proposition 2.2.21. *Let $X \subseteq Z$ be a $\mathbb{Q}$-factorial explicit general arrangement variety. If Σ admits a big cone, then it admits an elementary big cone.*

Proof. Let $\sigma \in \Sigma$ be a big cone. Then σ is an X-cone according to Proposition 2.2.8. Proposition 1.4.4 tells us that σ is simplicial. Now, any elementary big face of σ is as wanted. □

Corollary 2.2.22. *Let $X \subseteq Z$ be a non-toric, projective, locally factorial explicit general arrangement variety. If X is of Picard number $\rho(X) \leq c+3$, then Σ admits an elementary big cone.*

We conclude the section with a closer look at smooth projective general arrangement varieties of Picard number one.

Proposition 2.2.23. *Let X be a non-toric, smooth, projective general arrangement variety of Picard number one. Then X is a quadric $V(T_0^2 + \ldots + T_r^2) \subseteq \mathbb{P}_r$.*

Proof. According to Theorem 2.1.18, it suffices to consider the explicit general arrangement varieties $X \subseteq Z$. Moreover, we may assume that P is irredundant and $n_0 \geq \ldots \geq n_r$ holds. Finally, we have $K_{\mathbb{Q}} = \mathbb{Q}$ and may assume that the effective cone of X is $\mathbb{Q}_{\geq 0}$.

First we show that the are no variables of type S_k in $R(A,P)$. Otherwise, $\gamma_1 = \operatorname{cone}(e_1) \preccurlyeq \gamma$ is an X-face and thus we find a point $x \in \hat{X}$ having x_1 as its only nonzero coordinate. By smoothness of X, the Jacobian of $g_1, \ldots, g_{r-c}$ does not vanish at x; see Proposition 1.4.5. This implies $l_{i1} + \ldots + l_{in_i} = 1$ for some i; a contradiction to irredundance of P.

According to Corollary 2.2.22, the fan Σ admits an elementary big cone. Proposition 2.2.15 tells us $n_0 \geq 2$. Thus $\gamma_{0j} = \operatorname{cone}(e_{0j}) \preccurlyeq \gamma$ is an X-face. Proposition 1.4.5 yields that $\deg(T_{0j})$ generates K. We conclude $K = \mathbb{Z}$ and $\deg(T_{0j}) = 1$. Additionally, smoothness of $X(\gamma_{01})$ implies that $\operatorname{grad}(g_1)(x) \neq 0$ holds for every point $x \in \bar{X}(\gamma_{01})$. We conclude $n_0 = 2$ and $\deg(g_1) = 2$. This implies $\deg(T_{ij}) = 1$ and for all i, j, we obtain $l_{ij} = 1$ or $l_{ij} = 2$ according to $n_i = 2$ or $n_i = 1$.

Finally, observe that $c = r-1$ holds, i.e., that there is only one defining relation. Indeed, otherwise, we find generators $g'_1, \ldots, g'_{r-c}$, each involving precisely $c+2$ monomials and

g'_{r-c} all different from $T_0^{l_0}$. Then the corresponding Jacobian vanishes at any $x \in \bar{X}(\gamma_{01})$, showing that $X(\gamma_{01})$ is singular. A contradiction. $\square$

Remark 2.2.24. Consider a smooth, projective explicit general arrangement variety $X \subseteq Z$ of Picard number one with P being irredundant. By Proposition 2.2.23, the divisor class group $\mathrm{Cl}(X)$ is torsion free. Thus, Proposition 2.2.3 yields

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \quad l_0 = \ldots = l_{r-1} = (1,1), \quad l_r \;=\; \begin{cases} (1,1), & n \text{ even}, \\ (2), & n \text{ odd}, \end{cases}$$

where $n = n_0 + \ldots + n_r$. Moreover, the torus action on X is the action of the maximal torus of $\mathrm{Aut}(X) = \mathrm{O}(n)$. In particular, the torus action on X is of complexity

$$c \;=\; \begin{cases} \frac{n}{2} - 2, & n \text{ even}, \\ \frac{n-1}{2} - 1, & n \text{ odd}. \end{cases}$$

CHAPTER

THREE

SMOOTH GENERAL ARRANGEMENT VARIETIES

Extending recent classification work in complexity one [30], this chapter is dedicated to the study of smooth projective general arrangement varieties of true complexity two and Picard number two. Here, *true complexity two* means that the torus action is of complexity two and the variety does not admit a torus action of lower complexity. In Section 3.1 we recall the classification results from [42]. In the subsequent Section 3.2, we prove that all of the varieties from Theorem 3.1.1 are of true complexity two. In Section 3.3, we turn to the Fano case. Here, we present an elaborated version of the results from [42, 78]. Finally, in Section 3.1 we show that all Fano varieties from Theorem 3.1.3 can be constructed from a finite list of smooth general arrangement varieties of complexity two and Picard number two via iterated duplication of a free weight. Section 3.1 and parts of Section 3.3 are published in the joint work [42].

3.1 Classification in Picard number two

In this section we recall the classification results already presented in [42]. We begin with the description of all smooth general arrangement varieties of true complexity two and Picard number two. Note that being Mori dream spaces, the varieties listed below are indeed determined by their Cox ring together with an ample class; see Remark 1.4.7 and Remark 3.1.2 for the defining data as explicit general arrangement varieties.

Theorem 3.1.1. *Every smooth projective general arrangement variety of true complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, the matrix $[w_1, \ldots, w_r]$ of generator degrees and an ample class $u \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1,\dots,w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1&1 \\ 1&1&0&a_1&a_1'&a_2&a_2'&a_3&a_3' \end{bmatrix}$ $1\le a_1\le a_2\le a_3,\ a_i'=2-a_i$	$\begin{bmatrix} 1 \\ a_3+1 \end{bmatrix}$	6
2	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0&1&0 \\ 1&1&0&1&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	6
3	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1 \\ 1&1&0&a_1&a_1'&a_2&a_2'&1 \end{bmatrix}$ $1\le a_1\le a_2,\ a_i'=2-a_i$	$\begin{bmatrix} 1 \\ a_2+1 \end{bmatrix}$	5
4	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}+T_7T_8^{l_8}\rangle}$ $m\ge 0$	$\begin{bmatrix} 0&1&a_1&1&a_2&1&a_3&1&\vert&d_1&\dots&d_m \\ 1&0&1&0&1&0&1&0&\vert&1&\dots&1 \end{bmatrix}$ $0\le a_1\le a_2\le a_3,\ d_1\le\dots\le d_m,$ $l_2=a_1+l_4=a_2+l_6=a_3+l_8$	$\begin{bmatrix} d \\ 1 \end{bmatrix}$ *d max of* a_3, d_m	$m+5$
5	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\ge 0$	$\begin{bmatrix} 0&2a+1&a&1&a&1&a&1&\vert&1&\dots&1 \\ 1&1&1&0&1&0&1&0&\vert&0&\dots&0 \end{bmatrix}$ $a\ge 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+5$
6	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\ge 0$	$\begin{bmatrix} 0&2a_3+1&a_1&a_2&a_3&1&a_3&1&\vert&1&\dots&1 \\ 1&1&1&1&1&0&1&0&\vert&0&\dots&0 \end{bmatrix}$ $2a_3+1=a_1+a_2,\ 0\le a_1\le a_2$	$\begin{bmatrix} 2a_3+2 \\ 1 \end{bmatrix}$	$m+5$
7	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m\ge 1$	$\begin{bmatrix} 0&2a_5+1&a_1&a_2&a_3&a_4&a_5&1&\vert&1&\dots&1 \\ 1&1&1&1&1&1&1&0&\vert&0&\dots&0 \end{bmatrix}$ $2a_5+1=a_1+a_2=a_3+a_4,\ a_i\ge 0$	$\begin{bmatrix} 2a_5+2 \\ 1 \end{bmatrix}$	$m+5$
8	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\ge 1$	$\begin{bmatrix} 0&0&0&0&0&0&-1&1&\vert&1&\dots&1 \\ 1&1&1&1&1&1&1&1&\vert&0&\dots&0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+5$
9	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\ge 2$	$\begin{bmatrix} 0&a_1&a_2&a_3&a_4&a_5&a_6&a_7&\vert&1&\dots&1 \\ 1&1&1&1&1&1&1&1&\vert&0&\dots&0 \end{bmatrix}$ $a_1=a_2+a_3=a_4+a_5=a_6+a_7$ $a_i\ge 0$	$\begin{bmatrix} a_1+1 \\ 1 \end{bmatrix}$	$m+5$
10	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\ge 2$	$\begin{bmatrix} 0&0&0&0&0&0&0&0&\vert&1&1&\dots&1 \\ 1&1&1&1&1&1&1&1&\vert&0&d_2&\dots&d_m \end{bmatrix}$ $0\le d_2\le\dots\le d_m,\ d_m>0$	$\begin{bmatrix} 1 \\ d_m+1 \end{bmatrix}$	$m+5$
11	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\ge 1$	$\begin{bmatrix} -1&1&0&0&0&0&0&\vert&1&\dots&1 \\ 1&1&1&1&1&1&1&\vert&0&\dots&0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+4$
12	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\ge 2$	$\begin{bmatrix} 0&2a_5&a_1&a_2&a_3&a_4&a_5&\vert&1&\dots&1 \\ 1&1&1&1&1&1&1&\vert&0&\dots&0 \end{bmatrix}$ $a_1+a_2=a_3+a_4=2a_5,\ a_i\ge 0$	$\begin{bmatrix} 2a_5+1 \\ 1 \end{bmatrix}$	$m+4$
13	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\ge 2$	$\begin{bmatrix} 0&0&0&0&0&0&0&\vert&1&1&\dots&1 \\ 1&1&1&1&1&1&1&\vert&0&d_2&\dots&d_m \end{bmatrix}$ $0\le d_2\le\dots\le d_m,\ d_m>0$	$\begin{bmatrix} 1 \\ d_m+1 \end{bmatrix}$	$m+4$
14	$\frac{\mathbb{K}[T_1,\dots,T_{10}]}{\left\langle \begin{matrix} T_1T_2+T_3T_4+T_5T_6+T_7T_8, \\ \lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10} \end{matrix} \right\rangle}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	6

Moreover, each of the listed data defines a smooth projective general arrangement variety of true complexity two and Picard number two.

Remark 3.1.2. According to Theorem 2.1.18, the varieties from Theorem 3.1.1 can be represented as explicit general arrangement varieties $X(A, P, \Sigma)$ in the sense of Definition 2.1.16. The following table provides the defining data P and Σ, where we denote the columns of P by $v_1, \dots, v_{n+m}$ and set $\sigma_{i,j} := \operatorname{cone}(v_k;\ k \neq i,\ k \neq j)$.

<table>
<tr><th>No.</th><th>P</th><th>maximal cones of Σ</th></tr>
<tr><td>1</td><td>$\left[\begin{array}{ccccccccc} -1 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u_9 \end{array}\right]$
$u_i \in \mathbb{Z}^4,\ u_3 = -u_5 - u_7 - u_9,$
$u_1 = -u_2 - a'_1 u_5 - a'_2 u_7 - a'_3 u_9$</td><td>$\sigma_{1,i}, \sigma_{2,i},\ i = 3, \ldots, 9$</td></tr>
<tr><td>2</td><td>$\left[\begin{array}{ccccccccc} -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right]$</td><td>$\sigma_{1,i}, \sigma_{2,i}, \sigma_{5,i}, \sigma_{7,i}, \sigma_{9,i},$
$i = 3, 4, 6, 8$</td></tr>
<tr><td>3</td><td>$\left[\begin{array}{cccccccc} -1 & -1 & -2 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -2 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 & 0 & 0 & 2 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & u_8 \end{array}\right]$
$u_i \in \mathbb{Z}^3,\ u_3 = -u_5 - u_7 - u_8,$
$u_1 = -u_2 - a'_1 u_5 - a'_2 u_7 - u_8$</td><td>$\sigma_{1,i}, \sigma_{2,i}\ i = 3, \ldots, 7$</td></tr>
<tr><td>4</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -l_2 & 1 & l_4 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -l_2 & 0 & 0 & 1 & l_6 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -l_2 & 0 & 0 & 0 & 0 & 1 & l_8 & 0 & \ldots & 0 \\ u_1 & u_2 & 0 & u_4 & 0 & u_6 & 0 & u_8 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u_1 = -u'_1 - \ldots - u'_m$
$u_2 = -u_4 - u_6 - u_8 - d_1 u'_1 - \ldots - d_m u'_m$</td><td>$\sigma_{2,3}, \sigma_{2,5}, \sigma_{2,7}, \sigma_{2,8+i},$
$\sigma_{1,4}, \sigma_{4,5}, \sigma_{4,7}, \sigma_{4,8+i},$
$\sigma_{1,6}, \sigma_{3,6}, \sigma_{6,7}, \sigma_{6,8+i},$
$\sigma_{1,8}, \sigma_{3,8}, \sigma_{5,8}, \sigma_{8,8+i},$
$i = 1, \ldots, m$</td></tr>
<tr><td>5</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u_1 = -u_2 - u_3 - u_5 - u_7,$
$(2a+1)u_2 = -a(u_3 + u_5 + u_7) - u'_1 - \ldots - u'_m$</td><td>$\sigma_{1,4}, \sigma_{2,4}, \sigma_{4,5}, \sigma_{4,7},$
$\sigma_{1,6}, \sigma_{2,6}, \sigma_{3,6}, \sigma_{6,7},$
$\sigma_{1,8}, \sigma_{2,8}, \sigma_{3,8}, \sigma_{5,8},$
$\sigma_{j,(8+i)},\ i = 1, \ldots, m,$
$j = 1, 2, 3, 5, 7$</td></tr>
<tr><td>6</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u_1 = -u_2 - u_3 - u_5 - u_7,$
$(2a_3+1)u_2 = -a_1 u_3 - a_3 u_5 - a_3 u_7 - u'_1 - \ldots - u'_m$</td><td>$\sigma_{1,6}, \sigma_{2,6}, \sigma_{3,6}, \sigma_{4,6},$
$\sigma_{6,7}, \sigma_{1,8}, \sigma_{2,8}, \sigma_{3,8},$
$\sigma_{4,8}, \sigma_{5,8}, \sigma_{j,8+i},$
$i = 1, \ldots, m,$
$j = 1, 2, 3, 4, 5, 7$</td></tr>
<tr><td>7</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u_1 = -u_2 - u_3 - u_5 - u_7,$
$(2a_5+1)u_2 = -a_1 u_3 - a_3 u_5 - a_5 u_7 - u'_1 - \ldots - u'_m$</td><td>$\sigma_{j,8}, \sigma_{j,8+i}, \sigma_{7,8+i}$
$i = 1, \ldots, m,$
$j = 1, \ldots, 6$</td></tr>
<tr><td>8</td><td>$\left[\begin{array}{cccccccc|cc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{1}_{m-1} & \mathbb{E}_{m-1} \end{array}\right]$</td><td>$\sigma_{j,8}, \sigma_{j,8+i}, \sigma_{7,8+i}$
$i = 1, \ldots, m,$
$j = 1, \ldots, 6$</td></tr>
<tr><td>9</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u_1 = -u_2 - u_3 - u_5 - u_7,$
$a_1 u_2 = -a_2 u_3 - a_4 u_5 - a_6 u_7 - u'_1 - \ldots - u'_m$</td><td>$\sigma_{8+i,j},$
$i = 1, \ldots, m,$
$j = 1, \ldots, 8$</td></tr>
<tr><td>10</td><td>$\left[\begin{array}{cccccccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & 0 & u'_1 & \ldots & u'_m \end{array}\right]$
$u_i, u'_i \in \mathbb{Z}^{m+3},\ u'_1 = -u'_2 - \ldots - u'_m,$
$u_1 = -u_2 - u_3 - u_5 - u_7 - d_2 u'_2 - \ldots - d_m u'_m$</td><td>$\sigma_{8+i,j},$
$i = 1, \ldots, m,$
$j = 1, \ldots, 8$</td></tr>
</table>

11	$\left[\begin{array}{rrrrrrr\|cc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{1}_{m-1} & \mathbb{E}_{m-1} \end{array}\right]$	$\sigma_{2,j}, \sigma_{j,7+i}$, $i = 1, \ldots, m$, $j = 1, 3, 4, 5, 6$
12	$\left[\begin{array}{ccccccc\|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & u_1' & \ldots & u_m' \end{array}\right]$ $u_i, u_i' \in \mathbb{Z}^{m+2},\ u_1 = -u_2 - u_3 - u_5 - u_7,$ $2a_5u_2 = -a_1u_3 - a_3u_5 - a_5u_7 - u_1' - \ldots - u_m'$	$\sigma_{7+i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, 6$
13	$\left[\begin{array}{ccccccc\|ccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & \ldots & 0 \\ u_1 & u_2 & u_3 & 0 & u_5 & 0 & u_7 & u_1' & \ldots & u_m' \end{array}\right]$ $u_i, u_i' \in \mathbb{Z}^{m+2},\ u_1' = -u_2' - \ldots - u_m',$ $u_1 = -u_2 - u_3 - u_5 - u_7 - d_2u_2' - \ldots - d_mu_m'$	$\sigma_{7+i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, 6$
14	$\left[\begin{array}{rrrrrrrrrr} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right]$	$\sigma_{1,4}, \sigma_{1,6}, \sigma_{1,8}, \sigma_{1,10}$, $\sigma_{2,3}, \sigma_{3,6}, \sigma_{3,8}, \sigma_{3,10}$, $\sigma_{2,5}, \sigma_{4,5}, \sigma_{5,8}, \sigma_{5,10}$, $\sigma_{2,7}, \sigma_{4,7}, \sigma_{6,7}, \sigma_{7,10}$, $\sigma_{2,9}, \sigma_{4,9}, \sigma_{6,9}, \sigma_{8,9}$

We proceed with the Fano varieties contained in Theorem 3.1.1. The key to obtain their full classification is the description of the anticanonical class of an explicit general arrangement variety $X \subseteq Z$ given by

$$-\mathcal{K}_X \;=\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} + \sum_{k=0}^{r} w_k \;-\; (r-c)\sum_{j=1}^{n_0} l_{0j}w_{0j} \;\in\; K \;=\; \mathrm{Cl}(X),$$

where $c = c(X)$ is the complexity and $w_{ij} = \deg(T_{ij})$ as well as $w_k = \deg(T_k)$ are the Cox ring generator degrees, see Proposition 2.2.4. Going through the list of Theorem 3.1.1 and picking the cases with $-\mathcal{K}_X$ lying in the ample cone, we obtain the following.

Theorem 3.1.3. *Every smooth Fano general arrangement variety of true complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and the matrix $[w_1, \ldots, w_r]$ of generator degrees $w_i \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\left[\begin{smallmatrix} 0&0&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1&1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right]$	6
2	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\left[\begin{smallmatrix} 0&0&1&1&0&1&0&1&0 \\ 1&1&0&1&1&1&1&1&1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right]$	6
3	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\left[\begin{smallmatrix} 0&0&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right]$	5
4A	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m \geq 0$	$\left[\begin{smallmatrix} 0&1&2&1&2&1&2&1&2&\ldots&2 \\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 7+2m \\ 3+m \end{smallmatrix}\right]$	$m+5$
4B	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m \geq 0$	$\left[\begin{smallmatrix} 0&1&1&1&1&1&1&1&1&\ldots&1 \\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 4+m \\ 3+m \end{smallmatrix}\right]$	$m+5$

4C	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 4+m\\ 3+m \end{bmatrix}$	$m+5$
4D	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&d_1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $d_1 \in \{0,1\}$	$\begin{bmatrix} 5+m-1+d_1\\ 3+m \end{bmatrix}$	$m+5$
4E	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3\\ 3+m \end{bmatrix}$	$m+5$
4F	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&0&1&0&1&d_1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $d_1 \in \{-1,0\}$	$\begin{bmatrix} 2+d_1\\ 3+m \end{bmatrix}$	$m+5$
4G	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccccc} 0&1&0&1&0&1&0&1&d_1&d_2&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&1&1&\dots&1 \end{array}\right]$ $d_1, d_2 \leq 0,\ d_1+d_2 \geq -2$	$\begin{bmatrix} 3+d_1+d_2\\ 3+m \end{bmatrix}$	$m+5$
5	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0&2a+1&a&1&a&1&a&1&1&\dots&1\\ 1&1&1&0&1&0&1&0&0&\dots&0 \end{array}\right]$ $a \geq 0,\ m > 3a$	$\begin{bmatrix} 3a+3+m\\ 3 \end{bmatrix}$	$m+5$
6	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0&2a_3+1&a_1&a_2&a_3&1&a_3&1&1&\dots&1\\ 1&1&1&1&1&0&1&0&0&\dots&0 \end{array}\right]$ $0 \leq a_1 \leq a_2,\ a_1+a_2 = 2a_3+1$ $m > 4a_3+1$	$\begin{bmatrix} 4a_3+3+m\\ 4 \end{bmatrix}$	$m+5$
7	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0&2a_5+1&a_1&a_2&a_3&a_4&a_5&1&1&\dots&1\\ 1&1&1&1&1&1&1&0&0&\dots&0 \end{array}\right]$ $a_i \geq 0,\ m > 5a_5+2,$ $a_1+a_2 = a_3+a_4 = 2a_5+1$	$\begin{bmatrix} 5a_5+3+m\\ 5 \end{bmatrix}$	$m+5$
8	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $1\leq m\leq 5$	$\left[\begin{array}{cccccccc\|ccc} 0&0&0&0&0&0&-1&1&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$	$\begin{bmatrix} m\\ 6 \end{bmatrix}$	$m+5$
9	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|ccc} 0&a_1&a_2&a_3&a_4&a_5&a_6&a_7&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$ $a_i \geq 0,\ m > 3a_1,$ $a_1 = a_2+a_3 = a_4+a_5 = a_6+a_7$	$\begin{bmatrix} 3a_1+m\\ 6 \end{bmatrix}$	$m+5$
10	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|cccc} 0&0&0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&d_2&\dots&d_m \end{array}\right]$ $0 \leq d_2 \leq \dots \leq d_m,\ 0 < d_m \leq 5,$ $md_m < 6+d_2+\dots+d_m$	$\begin{bmatrix} 6+\sum^{m} d_k \end{bmatrix}$	$m+5$
11	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $1\leq m\leq 4$	$\left[\begin{array}{ccccccc\|ccc} -1&1&0&0&0&0&0&1&\dots&1\\ 1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$	$\begin{bmatrix} m\\ 5 \end{bmatrix}$	$m+4$
12	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|ccc} 0&2a_5&a_1&a_2&a_3&a_4&a_5&1&\dots&1\\ 1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$ $a_1+a_2 = a_3+a_4 = 2a_5,$ $a_i \geq 0,\ m > 5a_5$	$\begin{bmatrix} m+5a_5\\ 5 \end{bmatrix}$	$m+4$
13	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|cccc} 0&0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&1&0&d_2&\dots&d_m \end{array}\right]$ $0 \leq d_2 \leq \dots \leq d_m,\ 0 < d_m,$ $md_m < 5+d_2+\dots+d_m$	$\begin{bmatrix} 5+\sum^{m} d_k \end{bmatrix}$	$m+4$
14	$\frac{\mathbb{K}[T_1,\dots,T_{10}]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6+T_7T_8, \\ \lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10} \end{array} \right\rangle}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 3\\ 3 \end{bmatrix}$	6

Moreover, each of the listed data defines a smooth Fano general arrangement variety of true complexity two and Picard number two.

Remark 3.1.4. Some of the above Fano varieties are intrinsic quadrics. Here is the overlap of Theorem 3.1.3 with [29, Cor. 1.2]:

(i) Cases 10 and 13 are intrinsic quadrics of Type 1,

(ii) Cases 9 and 12 are intrinsic quadrics of Type 2,

(iii) Cases 8 and 11 are intrinsic quadrics of Type 3,

(iv) Case 4.G is an intrinsic quadric of Type 4.

Finally, we turn to the almost Fano case. As done above, we go through the list of Theorem 3.1.1 and pick the cases with $-\mathcal{K}_X$ lying on the boundary of the semi-ample cone.

Theorem 3.1.5. *Every smooth projective truly almost Fano general arrangement variety of true complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, the matrix $[w_1, \ldots, w_r]$ of generator degrees and an ample class $u \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	u	$\dim(X)$
4A	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^4+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&3&1&3&1&3&1&3&\ldots&3\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 4\\1 \end{bmatrix}$	$m+5$
4B	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&2&1&2&1&2&1&1&2&\ldots&2\\ 1&0&1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 3\\1 \end{bmatrix}$	$m+5$
4C	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&2&1&2&1&2&\ldots&2\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 3\\1 \end{bmatrix}$	$m+5$
4D	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^4+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&2&1&2&1&2&1&2&\ldots&2\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 3\\1 \end{bmatrix}$	$m+5$
4E	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 2\\1 \end{bmatrix}$	$m+5$
4F	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccccc} 0&1&1&1&1&1&1&1&0&0&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 2\\1 \end{bmatrix}$	$m+5$
4G	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&-1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 2\\1 \end{bmatrix}$	$m+5$
4H	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&1&1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 2\\1 \end{bmatrix}$	$m+5$
4I	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&0&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 2\\1 \end{bmatrix}$	$m+5$

4J	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&1&1&1&1&0&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}2\\1\end{bmatrix}$	$m+5$
4K	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}2\\1\end{bmatrix}$	$m+5$
4L	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}2\\1\end{bmatrix}$	$m+5$
4M	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&d_1&\dots&d_m\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$ $d_k\leq 0,\ \sum d_k=-3$	$\begin{bmatrix}1\\1\end{bmatrix}$	$m+5$
4N	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&d_1&\dots&d_m\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$ $d_k\leq 0,\ \sum d_k=-2$	$\begin{bmatrix}1\\1\end{bmatrix}$	$m+5$
4O	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&0&1&0&1&-1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}1\\1\end{bmatrix}$	$m+5$
4P	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4^4+T_5T_6^4+T_7T_8^4\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}1\\1\end{bmatrix}$	$m+5$
5	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&2a+1&a&1&a&1&a&1&1&\dots&1\\ 1&1&1&0&1&0&1&0&0&\dots&0 \end{array}\right]$ $a\geq 0, m=3a$	$\begin{bmatrix}2a+2\\1\end{bmatrix}$	$m+5$
6	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&2a_3+1&a_1&a_2&a_3&1&a_3&1&1&\dots&1\\ 1&1&1&1&1&0&1&0&0&\dots&0 \end{array}\right]$ $0\leq a_1\leq a_2,\ 0\leq a_3,$ $a_1+a_2=2a_3+1,\ m=4a_3+1$	$\begin{bmatrix}2a_3+2\\1\end{bmatrix}$	$m+5$
7	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0&2a_5+1&a_1&a_2&a_3&a_4&a_5&1&1&\dots&1\\ 1&1&1&1&1&1&1&0&0&\dots&0 \end{array}\right]$ $a_i\geq 0,\ m=5a_5+2,$ $a_1+a_2=a_3+a_4=2a_5+1$	$\begin{bmatrix}2a_5+2\\1\end{bmatrix}$	$m+5$
8	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m=6$	$\left[\begin{array}{cccccccc\|ccc} 0&0&0&0&0&0&-1&1&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+5$
9	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|ccc} 0&a_1&a_2&a_3&a_4&a_5&a_6&a_7&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&\dots&0 \end{array}\right]$ $a_i\geq 0,\ m=3a_1,$ $a_1=a_2+a_3=a_4+a_5=a_6+a_7$	$\begin{bmatrix}a_1+1\\1\end{bmatrix}$	$m+5$
10	$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|cccc} 0&0&0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&1&1&0&d_2&\dots&d_m \end{array}\right]$ $0\leq d_2\leq\dots\leq d_m, d_m\leq 6$ $ma_m=6+d_2+\ldots+d_m$	$\begin{bmatrix}1\\d_m+1\end{bmatrix}$	$m+5$
11	$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m=5$	$\left[\begin{array}{ccccccc\|ccc} 1&1&1&1&1&1&1&0&\dots&0\\ -1&1&0&0&0&0&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+4$
12	$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|ccc} 1&1&1&1&1&1&1&0&\dots&0\\ 0&2a_5&a_1&a_2&a_3&a_4&a_5&1&\dots&1 \end{array}\right]$ $2a_5=a_1+a_2=a_3+a_4,$ $a_i\geq 0,\ m=5a_5$	$\begin{bmatrix}2a_5+1\\1\end{bmatrix}$	$m+4$
13	$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|cccc} 1&1&1&1&1&1&1&0&d_2&\dots&d_m\\ 0&0&0&0&0&0&0&1&1&\dots&1 \end{array}\right]$ $d_2\leq\ldots\leq d_m,$ $md_m=5+d_2+\ldots+d_m$	$\begin{bmatrix}1\\d_m+1\end{bmatrix}$	$m+4$

Moreover, each of the listed data defines a smooth truly almost Fano general arrangement variety of true complexity two and Picard number two.

3.2 True complexity

In this section we prove that all of the varieties listed in Theorem 3.1.1 are of true complexity two. We proceed as follows: In a first step we show that the varieties in our list are non-toric. We go on by comparing the varieties in Theorem 3.1.1 with those from [30, Thm 1.1] and prove that they are pairwise non-isomorphic. This will be done by comparing the geometric data encoded in their respective Cox rings.

Let us recall the basic facts and observations concerning invariants of graded rings. Let R be a finitely generated integral algebra with an effective pointed grading of a finitely generated abelian group K. As usual we denote by $S(R)$ the *weight monoid* of R, i.e. the monoid of all $w \in K$ with non-trivial homogeneous component R_w.

The first invariant we consider is the *set of primitive elements* of $S(R)$ which we will denote by $S_p(R)$. If $(\varphi, \tilde{\varphi})\colon R \to R'$ is an isomorphism of graded rings then $\tilde{\varphi}$ maps $S_p(R)$ onto $S_p(R')$.

A second important invariant is the *set of generator degrees*

$$\Omega_R := \{w \in K;\ R_w \not\subseteq R_{<w}\}.$$

As before graded isomorphisms map sets of generator degrees onto sets of generator degrees. In particular, if $(\varphi, \mathrm{id})\colon R \to R'$ is a graded isomorphism, then $\Omega_R = \Omega_{R'}$ holds. The two invariants are connected via the following inclusions: if $f_1, \ldots, f_r$ are K-homogeneous generators for R, then we have

$$S_p(R) \subseteq \Omega_R \subseteq \{\deg(f_1), \ldots, \deg(f_r)\}.$$

A way to compute the set of generator degrees is to consider *minimal presentations of* R, i.e. graded epimorphisms $(\varphi, \tilde{\varphi})$ from a K-graded polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$, with homogeneous variables, to R, such that $\tilde{\varphi}\colon K \to K$ is an isomorphism and $\ker(\varphi) \subseteq \langle T_1, \ldots, T_r\rangle^2$ holds. If $(\varphi, \mathrm{id})\colon \mathbb{K}[T_1, \ldots, T_r] \to R$ is a minimal presentation then the K-grading on $\mathbb{K}[T_1, \ldots, T_r]$ is effective and pointed as well and we have

$$\Omega_R = \Omega_{\mathbb{K}[T_1,\ldots,T_r]} = \{\deg(T_1), \ldots, \deg(T_r)\}.$$

Remark 3.2.1. Let $X := X(A, P, \Sigma)$ be a general arrangement variety with Cox ring $R(A, P)$. If P is irredundant, then the canonical projection $\mathbb{K}[T_{ij}, S_k] \to R(A, P)$ is a minimal presentation. Moreover let $w \in K_{\mathbb{Q}}$ be a primitive ray generator of an extremal ray of the effective cone $\mathrm{Eff}(X)$ and let $n \in \mathbb{Z}_{\geq 0}$ be the minimal multiplicity such that the graded component $R(A, P)_{nw}$ is not trivial. Then the number $\mu_{nw} := \dim(R_{nw})$ is an invariant under graded isomorphy on $R(A, P)$ and each set of K-prime generators of $R(A, P)$ contains at least μ_{nw} generators of degree nw.

Proposition 3.2.2. *Each of the varieties listed in Theorem 3.1.1 is of true complexity two, i.e., it does not admit torus actions of lower complexity.*

Proof. First observe that each of the varieties listed in Theorem 3.1.1 has a singular total coordinate space and hence is not toric. Thus, we have to show that none of them is isomorphic to a smooth non-toric variety of Picard number two with torus action of complexity one, which in turn are all given in [30, Thm. 1.1]. Let X be a variety listed in Theorem 3.1.1 and assume X has a complexity one torus action. Then, comparing the dimension of the total coordinate space $\overline{X}$ in combination with the dimension of its singular locus to the respective data in [30, Thm. 1.1], we see that X can only be one of the varieties from Nos. $3, 4, 5$ and 6. We now go through the cases and show that none of them admits a torus action of complexity one.

No. 3: Recall that Cox ring, degree matrix and an ample class of X are given as

$$\begin{aligned} R &= \mathbb{K}[T_1,\dots,T_8]/\langle T_1T_2T_3^2 + T_4T_5 + T_6T_7 + T_8^2\rangle, \\ Q &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & b_1 & 2-b_1 & b_2 & 2-b_2 & 1 \end{bmatrix}, \quad 1 \le b_1 \le b_2 \\ u &= (1, b_2+1). \end{aligned}$$

The total coordinate space $\mathrm{Spec}(R)$ of X is of dimension 7 with singular locus of codimension 5. Computing these data also for the varieties X' from [30, Thm. 1.1], we see that X can be isomorphic at most to one of the varieties X' defined via the data in Nos. 4, 7, 8 or 9 in this list. We now go through these cases.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 4]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2^{l_2} + T_3T_4^{l_4} + T_5T_6^{l_6}\rangle, \quad m' \ge 0, \\ Q' &= \left[\begin{array}{cccccc|ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & c_1 & \dots & c_{m'} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0 \le a_1 \le a_2, \\ c_1 \le \dots \le c_{m'}, \\ 1 \le l_2 = a_1 + l_4 = a_2 + l_6, \end{array} \\ u' &= (\max(a_2, c'_m) + 1, 1). \end{aligned}$$

The total coordinate space $\mathrm{Spec}(R')$ of X' is of dimension $m'+5$ and the codimension of its singular locus equals 5 minus the number of i with $l_i \ge 2$. Consequently, we obtain

$$m' = 2, \quad l_2 = l_4 = l_6 = 1, \quad a_1 = a_2 = 0.$$

We write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q. Analogously, we define w'_i and μ'_i. Then we have

$$\mu_1 = 2, \quad \mu_3 \in \{1,2,3\}, \quad \mu'_1 \in \{3,4,5\}, \quad \mu'_2 = 3$$

for the primitive generators of the effective cones of X and X'. Observe that $\mu_1 = 2$ but $\mu'_1, \mu'_2 \ge 3$ hold; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 7]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 1,\\ Q' &= \left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right]\\ u' &= (1,2). \end{aligned}$$

As above we write w_i for the i-th column of Q, denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. The semiample cone of X' is generated by the primitive generators w_1' and w_6'. Both of them lie in the interior of the effective cone of X'. This is in contrast to w_1 which is a semiample primitive generator of the effective cone of X; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 8]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 2,\\ Q' &= \left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_{m'} \end{array}\right], \quad \begin{array}{l} 0\leq a_2\leq\dots\leq a_{m'}\\ a_{m'}>0 \end{array}\\ u' &= (1,a_{m'}+1). \end{aligned}$$

As above we write w_i for the i-th column of Q, denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. Then we have

$$\mu_1 = 2, \qquad \mu_1' = 6$$

for the only semiample primitive generators of the effective cones of X and X'; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 9]. After a unimodular transformation, the Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 2,\\ Q' &= \left[\begin{array}{cccc|ccc} 1 & 1 & \dots & 1 & 0 & \dots & 0\\ 0 & a_2 & \dots & a_6 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2\\ a_2=a_3+a_4=a_5+a_6 \end{array}\\ u' &= (1,a_2+1). \end{aligned}$$

The total coordinate space $\mathrm{Spec}(R')$ of X' is of dimension $m'+5$ and hence $m'=2$. We write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q. Analogously, we define w_i' and μ_i'. We distinguish between two cases.

Case $b_2 \leq 2$*:* In this case we have $\mathrm{Eff}(X) = \mathrm{cone}(e_1,e_2) = \mathrm{Eff}(X')$. Observe that w_1 and w_7' are the only semiample primitive generators of the effective cones of X and X'.

This implies $\Omega_R = \Omega_{R'}$ and we conclude $a_2 = b_2$. We are left with $b_1, b_2 \in \{1,2\}$ and claim that all possible choices lead to a contradiction. Assume $b_1 = b_2 = 2$ holds. Due to the multiplicity $\mu_3 = 3$ of the primitive extremal generator degree w_3, we may assume without loss of generality that $a_4 = a_6 = 0$. Therefore homogeneity of the relation implies $a_3 = a_5 = 2$. This is a contradiction to $\Omega_R = \Omega_{R'}$ as $w_8 = (1,1) \neq w'_i$ for all $i = 1, \dots, r$. The other cases can be excluded with analogous arguments.

Case $b_2 > 2$: We apply a unimodular change of coordinates on $\mathrm{Eff}(X)$ and obtain the Cox ring, the degree matrix and an ample class of X as

$$\begin{aligned} R &= \mathbb{K}[T_1, \dots, T_8]/\langle T_1T_2T_3^2 + T_4T_5 + T_6T_7 + T_8^2 \rangle, \\ Q &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & b_2-2 & b_1+b_2-2 & b_2-b_1 & 2b_2-2 & 0 & b_2-1 \end{bmatrix}, \quad 1 \le b_1 \le b_2 \\ u &= (1, 2b_2-1). \end{aligned}$$

As above we have $\mathrm{Eff}(X) = \mathrm{cone}(e_1, e_2) = \mathrm{Eff}(X')$ and as w_1 and w'_7 are the only semiample primitive generators for the effective cones of X and X' we have $\Omega_R = \Omega_{R'}$. As we have $b_2 > 2$ and $a_2 \geq a_i$ holds for all $i \geq 3$, we obtain

$$a_2 = 2b_2 - 2 = a_3 + a_4 = a_5 + a_6 \quad \text{and} \quad b_2 - 2,\ b_2 - 1 \in \{a_3, a_4, a_5, a_6\}.$$

In particular, using $a_2 - (b_2 - 2) = b_2$ and $a_2 - (b_2 - 1) = b_2 - 1$ we conclude $a_3 = b_2 - 2$, $a_4 = b_2$ and $a_5 = b_2 - 1 = a_6$ due to homogeneity of the relations. Using $\Omega_R = \Omega_{R'}$ and $b_1 \geq 1$, we conclude $b_1 = 2$. In order to show, that this configuration of weights does not give rise to an isomorphism of graded rings, we show that the graded components R_w and R'_w are not of the same dimension for $w = (1, b_2 - 2)$. This is a contradiction as $\varphi(R_w) = R'_w$ holds. Note, that the graded components R_w and R'_w are generated by

$$T_3,\ T_5,\ T_7T_1^{\alpha}T_2^{\beta}, \quad \text{resp. } T_3,\ T_1S_1^{\alpha}, S_2^{\beta}, \quad \text{where } \alpha + \beta = b_2 - 2.$$

We claim that these generators are linearly independent. Assume not. Then we obtain two linear combinations

$$\sum \mu_{\alpha,\beta} T_7 T_1^{\alpha} T_2^{\beta} + \mu_3 T_3 + \mu_5 T_5 \in \langle g \rangle \text{ and } \sum \lambda_{\alpha,\beta} T_1 S_1^{\alpha} S_2^{\beta} + \lambda_3 T_3 \in \langle g' \rangle.$$

As $\deg(g) = (2, 2b_2 - 2) = \deg(g')$ holds, this implies that each term with non vanishing coefficient is of degree $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_1 \geq 2$ and $\kappa_2 \geq 2b_2 - 2$. This is impossible as all occurring terms are by construction of degree $(1, b_2 - 2)$. This proves the claim.

No. 4 in Theorem 3.1.1: Recall that Cox ring, degree matrix and an ample class of X are

$$\begin{aligned} R &= \mathbb{K}[T_1, \dots, T_8, S_1, \dots, S_m]/\langle T_1T_2^{l_2} + T_3T_4^{l_4} + T_5T_6^{l_6} + T_7T_8^{l_8} \rangle, \quad m \ge 0 \\ Q &= \left[\begin{array}{cccccccc|ccc} 0 & 1 & b_1 & 1 & b_2 & 1 & b_3 & 1 & d_1 & \dots & d_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{c} 0 \le b_1 \le b_2 \le b_3 \\ d_1 \le \dots \le d_m \\ l_2 = b_1 + l_4 = b_2 + l_6 = b_3 + l_8 \end{array} \\ u &= (\max(b_3, d_m), 1). \end{aligned}$$

The total coordinate space Spec(R) of X is of dimension $7 + m$ with singular locus of codimension 7 minus the number of i with $l_i \geq 2$. Computing these data also for the varieties X' from [30, Thm. 1.1], we see that X can be isomorphic at most to one of the varieties X' defined via the data of Nos. 4, 5, 6, 7, 8, 9, 10, 11 or 12 from this list. We now go through these cases.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 4]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2^{l_2'} + T_3T_4^{l_4'} + T_5T_6^{l_6'}\rangle, \quad m' \geq 0,\\
Q' &= \left[\begin{array}{cccccc|ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & c_1 & \dots & c_{m'} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0 \leq a_1 \leq a_2, \\ c_1 \leq \dots \leq c_{m'}, \\ 1 \leq l_2' = a_1 + l_4' = a_2 + l_6', \end{array}\\
u' &= (\max(a_2, c_m') + 1, 1).
\end{aligned}
$$

We write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q. Analogously, we define w_i' and μ_i'. We obtain $\mu_2 = 4$ and $\mu_2' = 3$ for the only non semiample primitive generators of the effective cones of X and X'; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 5]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2 + T_3^2T_4 + T_5^2T_6\rangle, \quad m' \geq 0,\\
Q' &= \left[\begin{array}{cccccc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad a \geq 0\\
u' &= (2a+2, 1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. We obtain $\mu_1 = 4$ and $\mu_1' \in \{1, 3\}$ for the only non semiample primitive generators of the effective cones of X and X'; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 6]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2 + T_3T_4 + T_5^2T_6\rangle, \quad m' \geq 1,\\
Q' &= \left[\begin{array}{cccccc|ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{l} a_1, a_2, a_3 \geq 0 \\ a_1 < a_2 \\ a_1 + a_2 = 2a_3 + 1 \end{array}\\
u' &= (2a_3+2, 1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. We obtain $\mu_1 = 4$ and

$\mu_1' \in \{1,2,3\}$ for the only non semiample primitive generators of the effective cones of X and X'; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 7]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 1,\\
Q' &= \left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]\\
u' &= (1,2).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and define analogously w_i'. We obtain w_1 as a semiample primitive generator of the effecitve cone of X. But the variety X' has no semiample divisor on the boundary of the cone of effective divisors; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 8]. After a suitable change of coordinates, the Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 2,\\
Q' &= \left[\begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m'\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right], \quad \begin{array}{l} 0\leq a_2\leq\ldots\leq a_m' \\ a_m'>0 \end{array}\\
u' &= (a_m+1,1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and define analogously w_i'. We obtain w_2 and w_1' as the only semiample primitive generators of the effective cones of X and X'. Looking at the homogeneous component of R and R' of degree $2w_2$ resp. $2w_1'$ we obtain a contradiction:

$$10 = \dim(R_{2w_2}) = \dim(R'_{2w_1'}) = 21.$$

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 9]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 2,\\
Q' &= \left[\begin{array}{cccccc|c} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1\\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right], \quad \begin{array}{l} 0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2 \\ a_2=a_3+a_4=a_5+a_6 \end{array}\\
u' &= (a_2+1,1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. We obtain $\mu_2 = 4$ and $\mu_1' \in \{1,2,3,6\}$ for the only non semiample primitive generators of the effective cones of X and X'; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 10]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2\rangle, \quad m'\geq 1,\\
Q' &= \left[\begin{array}{ccccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0\\ -1 & 1 & 0 & 0 & 0 & 1 & \dots & 1 \end{array}\right]\\
u' &= (2,1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and define analogously w_i'. We obtain w_2 as a semiample primitive generator of the effecitve cone of X. The variety X' has no semiample divisor on the boundary of the cone of effective divisors; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 11]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2\rangle, \quad m'\geq 1,\\
Q' &= \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m'\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 \end{array}\right], \quad \begin{array}{l} 0\leq a_2\leq\dots\leq a_m' \\ a_m'>0 \end{array}\\
u' &= (a_m+1,1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and define analogously w_i'. We obtain w_2 and w_1' as the only semiample primitive generators of the effective cones of X and X'. Looking at the homogeneous component of R and R' of degree $2w_2$ resp. $2w_1'$ we obtain a contradiction:

$$10=\dim(R_{2w_2})=\dim(R'_{2w_1'})=15.$$

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 12]. The Cox ring, the degree matrix and an ample class of X' are given as

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2\rangle, \quad m'\geq 2,\\
Q' &= \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0\\ 0 & 2a_3 & a_2 & a_2 & a_3 & 1 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0\leq a_1\leq a_3\leq a_2 \\ a_1+a_2=2a_3 \end{array}\\
u' &= (1,2a_3+1).
\end{aligned}
$$

As above, we write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. We obtain $\mu_1=4$ and $\mu_1'\in\{1,2,5\}$ for the only non semiample primitive generators of the effective cones of X and X'; a contradiction.

No. 5 in Theorem 3.1.1: Recall that Cox ring, degree matrix and an ample class of X are

$$
\begin{aligned}
R &= \mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]/\langle T_1T_2 + T_3^2T_4 + T_5^2T_6 + T_7^2T_8\rangle, \quad m \geq 0,\\
Q &= \left[\begin{array}{cccccccc|ccc} 0 & 2b+1 & b & 1 & b & 1 & b & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad b \geq 0,\\
u &= (2b+2,1).
\end{aligned}
$$

The total coordinate space $\mathrm{Spec}(R)$ of X is of dimension $m+7$ with singular locus of codimension 4. Computing these data also for the varieties from [30, Thm. 1.1], we see that X can be isomorphic at most to one of the varieties X' defined via the data of Nos. 4, 6, 10, 11 or 12 of [30, Thm. 1.1]. We now go through these cases.

Assume that X is isomorphic to the variety [30, Thm. 1.1, No. 4], which we denote by X'. Cox ring, degree matrix and an ample class of X' are given by

$$
\begin{aligned}
R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2^{l_2} + T_3T_4^{l_4} + T_5T_6^{l_6}\rangle, \quad m' \geq 0,\\
Q' &= \left[\begin{array}{cccccc|ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & c_1 & \dots & c_{m'}\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0 \leq a_1 \leq a_2,\\ c_1 \leq \dots \leq c_{m'},\\ 1 \leq l_2 = a_1 + l_4 = a_2 + l_6 b, \end{array}\\
u' &= (\max(a_2, c'_m)+1,1).
\end{aligned}
$$

The total coordinate space $\mathrm{Spec}(R')$ of X' is of dimension $m'+5$ and the codimension of its singular locus equals 5 minus the number of i with $l_i \geq 2$. Consequently, we obtain

$$m' = m+2, \quad l_4 = l_6 = 1, \quad l_2 = a_1 + 1 = a_2 + 1 \geq 2, \quad a_1 = a_2.$$

We write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q. Analogously, we define w'_i and μ'_i. Then we have

$$\mu_1 \in \{1,4\}, \quad \mu_4 = 3+m, \qquad \mu'_2 = 3, \quad \mu'_1 \leq 1+m'.$$

Observe that w_1, w_4 are the primitive generators of the extremal rays of the effective cone of X and w_4 is a semiample class, whereas w_1 is not semiample. Moreover, w'_2 is a semiample primitive generator of the effective cone of X'. We conclude

$$3+m = \mu_4 = \dim(R_{w_4}) = \dim(R'_{w'_2}) = \mu'_2 = 3.$$

Thus, $m = 0$ and $m' = 2$ hold. Comparing the multiplicities $\dim(R_w)$ and $\dim(R'_{w'})$ for w and w' being the primitive generators differing from $(1,0)$ of the respective effective, moving and semiample cones of X and X', we obtain

$$b, c_1, c_2 > 0, \quad \mu'_1 = \mu_1 = 1 \quad b = a_1 = a_2 = c_1 < c_2 = 2b+1.$$

But then the anticanonical class $-\mathcal{K}_X = (3a+3,3)$ is divisible by 3, whereas $-\mathcal{K}_{X'} = (4b+3,3)$ is not; a contradiction.

Assume that X is isomorphic to a variety X' as in [30, Thm. 1.1, No. 6]. Here, Cox ring, the degree matrix and an ample class look as follows:

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2T_6\rangle, \quad m'\geq 1, \\ Q' &= \left[\begin{array}{cccccc|ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{l} a_1,a_2,a_3\geq 0, \\ a_1<a_2, \\ a_1+a_2=2a_3+1, \end{array} \\ u' &= (2a_3+2,1). \end{aligned}$$

The dimension of the total coordinate space $\mathrm{Spec}(R')$ of X' equals $m'+5$ and hence $m'=m+2$ must hold. As before, let w_i be the i-th column of Q and μ_i the number of times it shows up as a column of Q. Define w_i' and μ_i' analogously. We obtain

$$\mu_1\in\{1,4\}, \qquad \mu_4=\mu_6'=m+3, \qquad \mu_1'\in\{1,2,3\}.$$

For X as well as for X', we find precisely one semiample primitive generator of the effective cone, namely w_4 and w_6'. Consequently we obtain

$$1=\mu_1=\mu_1', \qquad a_1,a_2,a_3>0.$$

Comparing the multiplicities $\dim(R_w)$ and $\dim(R'_{w'})$ for w and w' being the primitive generators differing from $(1,0)$ of the effective, movable and semiample cones of X and X', we arrive at $a_1=a_2=a_3=b$, which contradicts, for instance, $a_1<a_2$.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 10]. In this case, the Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2\rangle, \quad m'\geq 1, \\ Q' &= \left[\begin{array}{ccccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & \dots & 1 \end{array}\right], \\ u' &= (2,1). \end{aligned}$$

Let w_i, w_i' and μ_i, μ_i' be as before. Then w_4 and w_1' are the only semiample primitive generators of the effective cones of X and X', respectively. Thus, we obtain $1=\mu_1'=\mu_4=3+m$; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 11]. The Cox ring, the degree matrix and an ample class of X' are then given by

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5^2\rangle, \quad m'\geq 2, \\ Q' &= \left[\begin{array}{ccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_{m'} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0\leq a_2\leq\dots\leq a_{m'}, \\ a_{m'}>0, \end{array} \\ u' &= (a_{m'}+1,1). \end{aligned}$$

With w_i, w_i' and μ_i, μ_i' as before, we see that, again, w_4, w_1' are the only semiample primitive generators of the effective cones of X, X', respectively, and conclude

$$5=\mu_1'=\mu_4=3+m.$$

This implies $m = 2$. Thus, $\mathrm{Spec}(R)$ is of dimension $7 + m = 9$. Consequently, $\mathrm{Spec}(R')$ is of dimension $9 = 4 + m'$, showing $m' = 5$. Looking for R and R' at the homogeneous components of degrees $2w_4$ and $2w'_1$ respectively, we arrive at a contradiction:

$$15 \;=\; \dim(R_{2w_4}) \;=\; \dim(R'_{2w'_1}) \;=\; 14.$$

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 12]. The Cox ring, the degree matrix and an ample class of the latter are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_{m'}]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle, \quad m' \geq 2, \\ Q' &= \left[\begin{array}{ccccc|cccc} 0 & 2a_3 & a_1 & a_2 & a_3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{l} 0 \leq a_1 \leq a_3 \leq a_2, \\ a_1 + a_2 = 2a_3, \end{array} \\ u' &= (2a_3 + 1, 1). \end{aligned}$$

Comparing the primitive generators w, w' and the corresponding multiplicities $\dim(R_w)$, $\dim(R'_w)$ of the effective, moving and semiample cones of X and X', we arrive at

$$m' = 3 + m, \qquad 0 < b, \qquad 0 < a_1 = a_2 = a_3.$$

Now, comparing the determinants of the Mori chambers of X and X' leads to a contradiction: we obtain

$$b = \det(w_3, w_1) = \det(w'_5, w_2) = a_3, \qquad b + 1 = \det(w_2, w_3) = \det(w'_2, w'_3) = a_3.$$

No. 6 in Theorem 3.1.1: Recall that Cox ring, degree matrix and an ample class of X are

$$\begin{aligned} R &= \mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]/\langle T_1T_2 + T_3T_4 + T_5^2T_6 + T_7^2T_8\rangle, \quad m \geq 0 \\ Q &= \left[\begin{array}{cccccccc|ccc} 0 & 2b_3+1 & b_1 & b_2 & b_3 & 1 & b_3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right], \quad \begin{array}{c} 2b_3 + 1 = b_1 + b_2 \\ 0 \leq b_1 \leq b_2 \end{array} \\ u &= (2b_3 + 2, 1). \end{aligned}$$

The total coordinate space $\mathrm{Spec}(R)$ of X is of dimension $7 + m$ with singular locus of codimension 5. Computing these data also for the varieties X' from [30, Thm 1.1], we see that X can be isomorphic at most to one of the varieties X' defined via the data of Nos. 4, 7, 8 or 9 from this list. We now go through these cases.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 4]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_{m'}]/\langle T_1T_2^{l'_2} + T_3T_4^{l'_4} + T_5T_6^{l'_6}\rangle, \quad m' \geq 0, \\ Q' &= \left[\begin{array}{cccccc|ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & c_1 & \dots & c_{m'} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right], \quad \begin{array}{l} 0 \leq a_1 \leq a_2, \\ c_1 \leq \dots \leq c_{m'}, \\ 1 \leq l'_2 = a_1 + l'_4 = a_2 + l'_6, \end{array} \\ u' &= (\max(a_2, c'_m) + 1, 1). \end{aligned}$$

The total coordinate space $\operatorname{Spec}(R')$ of X is of dimension $5 + m'$ with singular locus of dimension 5 minus the number of i with $l'_i \geq 2$. Consequently we obtain $m' = m+2$ and $l'_i = 1$ for all i. In particular we obtain $a_1 = a_2 = 0$ since $l'_2 = a_1 + l'_4 = a_2 + l'_6$ holds. We write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q. Analogously, we define w'_i and μ'_i. We obtain

$$\mu_1 \in \{1, 2, 4, 5\}, \quad \mu_6 = 2 + m, \quad \mu'_1 = 3 + m' \quad \mu'_2 = 3$$

for the primitive generators of the effective cones of X and X'. Observe that w_6 and w'_2 are the only semiample among these. This shows $m = 1$ which implies $m' = 3$. For the non semiample generator w'_1 of the effective cone X' this means $\mu'_1 = 6$; a contradiction to $\mu_1 \in \{1, 2, 4, 5\}$.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 7]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1, \ldots, T_6, S_1, \ldots, S_{m'}]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle, \quad m' \geq 1, \\ Q' &= \left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right] \\ u' &= (1, 2). \end{aligned}$$

As above, we write w_i for the i-th column of Q and define analogously w'_i. We obtain w_6 as a semiample primitive generator of the effective cone of X. But the variety X' has no semiample divisor on the boundary of the cone of effective divisors; a contradiction.

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 8]. After a suitable change of coordinates, the Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{aligned} R' &= \mathbb{K}[T_1, \ldots, T_6, S_1, \ldots, S_{m'}]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle, \quad m' \geq 2, \\ Q' &= \left[\begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_{m'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right], \quad \begin{array}{l} 0 \leq a_2 \leq \ldots \leq a_{m'} \\ a_{m'} > 0 \end{array} \\ u' &= (a_{m'} + 1, 1). \end{aligned}$$

The total coordinate space $\operatorname{Spec}(R')$ of X is of dimension $5+m'$. Consequently we obtain $m' = m + 2$. As above we write w_i for the i-th column of Q, denote by μ_i the number of times it shows up as a column of Q and define w'_i and μ'_i analogously. We obtain

$$\mu_1 \in \{1, 2, 4, 5\}, \quad \mu_6 = 2 + m, \quad \mu'_1 = 6, \quad 1 \leq \mu_7 \leq m' - 1$$

for the primitive generators of the effective cones of X and X'. Observe that w_6 and w'_1 are the only semiample among these. We obtain $m = 4$ and $m' = 6$. Looking at the homogeneous component of R and R' of degree $2w_6$ resp. $2w'_1$ we obtain a contradiction:

$$6 = \dim(R_{2w_6}) = \dim(R'_{2w'_1}) = 15.$$

Assume that X is isomorphic to the variety X' as in [30, Thm. 1.1, No. 9]. The Cox ring, the degree matrix and an ample class of X' are given as

$$\begin{array}{rcl} R' & = & \mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_{m'}]/\langle T_1T_2+T_3T_4+T_5T_6\rangle, \quad m'\geq 2, \\ Q' & = & \left[\begin{array}{cccc|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right], \quad \begin{array}{l} 0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2 \\ a_2=a_3+a_4=a_5+a_6 \end{array} \\ u' & = & (a_2+1,1). \end{array}$$

As above, we write w_i for the i-th column of Q and denote by μ_i the number of times it shows up as a column of Q and define w_i' and μ_i' analogously. We obtain

$$\mu_1\in\{1,2,4,5\},\quad \mu_6=2+m,\quad \mu_1'\in\{1,2,3,6\},\quad \mu_7=m'$$

for the primitive generators of the effective cones of X and X'. Note that w_1 and w_1' are the only non semiample among these. Consequently we obtain $\mu_1=\mu_1'\in\{1,2\}$.

Now note that w_2 and w_2' are semiample primitive generators in the interior of the effective cones of X resp. X'. We obtain

$$2b_3+1=\det(w_2,w_1)=\det(w_2',w_1')=a_2$$

for the volume of the complementary cone of the semiample cones of X and X'. Moreover we obtain

$$b_1\neq b_2,\quad a_3\neq a_4,\quad a_5\neq a_6$$

since $2b_3+1=a_2$ is odd. Now assume $b_2=b_3$. Since $b_1+b_2=2b_3+1$ we obtain $b_1=b_3+1>b_2$; a contradiction to $b_1\leq b_2$.

We distinguish between the cases $\mu_1=1$ and $\mu_1=2$.

Assume $\mu_1=1$: In this case we have $a_3,b_1>0$ and we obtain

$$b_2<2b_3+1,\qquad a_4<a_2.$$

Assume $b_1=b_3$ then w_3 is a primitive generator of the moving cone of X with $\mu_3=3$. So we are left with the cases $a_3=a_5=a_6$ or $a_4=a_5=a_6$ which both contradicts $a_5\neq a_6$. Assume $b_1\neq b_3$ then w_5 is a primitive generator of the moving cone of X with $\mu_5=2$. Observe that the number of Mori chambers of R and R' coincide. Hence the cases $a_3=a_5$ and $a_6=a_4$ lead to a contradiction and we are left with $a_5=a_6$; a contradiction.

Assume $\mu_2=2$. Then $a_3=b_1=0$ and $b_3,a_5>0$ hold. Since $b_3<b_2$ we obtain w_5 as a primitive generator of the moving cone of X with $\mu_5=2$. Observe that the Mori chamber decomposition of $\mathrm{Mov}(X)$ has 3 chambers. So we obtain $a_5=a_6$; a contradiction. □

3.3 Geometry in the Fano case

In this section we discuss some aspects of the geometry of the Fano varieties listed in Theorem 3.1.3. We take a look at *elementary contractions*, i.e., the morphisms obtained by passing to facets of the ample cone with respect to the Mori chamber decomposition, which is directly computable in terms of the data listed in Theorem 3.1.3; use Proposition 1.4.6 and Remark 1.4.8. Moreover, we look at *small degenerations*, that means degenerations with fibers all sharing the same divisor class group. In fact, degenerating the quadrinomial equations of the Cox ring into trinomial ones, reflects a degeneration of Cox rings inducing a small degeneration of the underlying Fano variety into a possibly singular variety with a torus action of complexity one.

We now explicitly go through the cases of Theorem 3.1.3 and list the basic information in the subsequent table which are discussed in more detail afterwards. Let us explain how to read the table. By Q_k, we denote the smooth projective quadric of dimension k and by $Q_{k,l} \subseteq \mathbb{P}_l$ the projective quadric of rank k in $\mathbb{P}_l$. We write $Y_{a;1^k,d^l}$ for a hypersurface of degree a in the weighted projective space $\mathbb{P}(1^k, d^l)$, where we do not specify the equation, and we set

$$
\begin{aligned}
Y_{4B} &= V(T_0^3 + T_1T_2^2 + T_3T_4^2 + T_5T_6^2) \subseteq \mathbb{P}_{m+6},\\
Y_{4F} &= V(T_0T_1^2 + T_2T_3^2 + T_4T_5^2 + T_6T_7^2) \subseteq \mathbb{P}_{m+6}.
\end{aligned}
$$

As we consider smooth Fano varieties of Picard number two, there will be at most two elementary contractions for each. If we have a birational elementary contraction, then a prime divisor gets contracted. In this case we write $X \sim Y$ and denote by $C \subseteq Y$ the center of this contraction. The other possibility is that we have a Mori fiber space. Then we write $X \to Y$ and denote by F_{gen} the general fiber. If there are no special fibers, then we write just F for the fiber. Moreover, when we say that a variety is Gorenstein, terminal, etc. then we mean that it is singular but has at most Gorenstein, terminal, etc. singularities. We computed small degenerations for every case in the lowest dimensions. The resulting varieties are always normal and Fano with a torus action of complexity one. The properties of being Gorenstein, terminal etc. have been checked using [13, 43]. If we say two, three, etc. degenerations, then this means that we found small degenerations into two, three, etc. non-isomorphic Fano $\mathbb{T}$-varieties of complexity one.

Remark 3.3.1. The following table lists the elementary contractions and small degenerations obtained via degenerating the Cox ring for the Fano varieties of Theorem 3.1.3.

No.	dim(X)	Contraction 1	Contraction 2	Small Degenerations
1	6	$X \sim Q_6$ $C = Q_4$	$X \to \mathbb{P}_1$ $F_{\text{gen}} = Q_5$	two Gorenstein, terminal locally factorial
2	6	$X \sim Q_6$ $C = \mathbb{P}_2$	$X \to \mathbb{P}_4$ $F = \mathbb{P}_2$	two Gorenstein, terminal locally factorial
3	5	$X \sim Q_5$ $C = Q_3$	$X \to \mathbb{P}_1$ $F_{\text{gen}} = Q_4$	three Gorenstein, terminal locally factorial

4A	$m+5$	$X \sim Y_{3;1^4,2^{m+3}}$ $C = \mathbb{P}_{m+2}$	$X \to \mathbb{P}_3$ $F = \mathbb{P}_{m+2}$	$\dim(X) \leq 6$: two Gorenstein, terminal, locally factorial
4B	$m+5$	$X \sim Y_{4\mathrm{B}}$ $C = \mathbb{P}_{m+2}$	$X \to \mathbb{P}_3$ $F = \mathbb{P}_{m+2}$	$\dim(X) \leq 6$: two Gorenstein, log terminal, locally factorial
4C	$m+5$	—	$X \to \mathbb{P}_3$ $F = \mathbb{P}_{m+2}$	$\dim(X) \leq 6$: two Gorenstein, terminal, locally factorial
4D	$m+5$	if $d_1{=}1$ or $m{=}0$: $X \sim Q_{7,m+6}$ $C = \mathbb{P}_{m+2}$	$X \to \mathbb{P}_3$ $F = \mathbb{P}_{m+2}$	$\dim(X) \leq 6$: two Gorenstein, terminal, locally factorial
4E	$m+5$	$X \to \mathbb{P}_{m+3}$ $F_{\mathrm{gen}} = Y_{3;1^4}$	$X \to \mathbb{P}_3$ $F = \mathbb{P}_{m+2}$	$\dim(X) \leq 6$: one Gorenstein, locally factorial

4F	$m+5$	if $d_1{=}0$ or $m{=}0$:
		$X \to \mathbb{P}_{m+3}$
		$F_{\mathrm{gen}} = \mathbb{P}_1 \times \mathbb{P}_1$
		if $d_1{=}{-1}$:
		$X \sim Y_{4F}$
		$C = \mathbb{P}_{m+2}$
			$X \to \mathbb{P}_3$
			$F = \mathbb{P}_{m+2}$
				$\dim(X) \leq 6$: one Gorenstein, log terminal, locally factorial

4G	$m+5$	if $d_i{=}0$ or $m{=}0$:
		$X \to \mathbb{P}_{m+3}$
		$F_{\mathrm{gen}} = \mathbb{P}_2$
		if $d_1{=}{-1}$ and $d_2{=}0$:
		$X \sim Q_{7,m+6}$
		if $d_1{=}{-2}$ and $d_2{=}0$:
		$X \sim Y_{3;1^4,2^{m+3}}$
			$X \to \mathbb{P}_3$
			$F = \mathbb{P}_{m+2}$
				$\dim(X) \leq 6$: one Gorenstein, terminal, locally factorial

5	$m+5$	$X \to \mathbb{P}_{m+2}$ $F_{\mathrm{gen}} = Q_3$	—	$\dim(X) = 6$: one Gorenstein, terminal, locally factorial; one Gorenstein, log terminal locally factorial
6	$m+5$	$X \to \mathbb{P}_{m+1}$ $F_{\mathrm{gen}} = Q_4$	—	$\dim(X) = 7$: two Gorenstein, terminal, locally factorial
7	$m+5$	$X \to \mathbb{P}_m$ $F_{\mathrm{gen}} = Q_5$	—	$\dim(X) = 8$: two Gorenstein, terminal, locally factorial
8	$m+5$	$X \sim \mathbb{P}_{m+5}$ $C = Q_4$	if $m{=}1$: $X \sim Q_6$ $C = \{\mathrm{pt}\}$	$\dim(X) = 6$: one Gorenstein, terminal, locally factorial; one of Gorenstein index 2, terminal, $\mathbb{Q}$-factorial
9	$m+5$	$X \to \mathbb{P}_{m-1}$ $F_{\mathrm{gen}} = Q_6$	if $a_1{=}\ldots{=}a_7{=}0$: $X \to Q_6$ $F = \mathbb{P}_{m-1}$	$\dim(X) = 7$: one Gorenstein, terminal, locally factorial
10	$m+5$	$X \to Q_6$ $F_{\mathrm{gen}} = \mathbb{P}_{m-1}$	if $0{<}d_2{=}\ldots{=}d_m$: $X \sim Y_{2;1^8,d_2^{m-1}}$ $C = \mathbb{P}_{m-2}$	$\dim(X) = 7$: one Gorenstein, terminal, locally factorial

11	$m+4$	$X \sim \mathbb{P}_{m+4}$ $C = Q_3$	if $m=1$: $X \sim Q_5$ $C = \{\mathrm{pt}\}$	$\dim(X) = 5$: two Gorenstein, terminal, locally factorial; one Gorenstein, terminal $\mathbb{Q}$-factorial. $\dim(X) = 6$: two Gorenstein, terminal, locally factorial; one of Gorenstein index 2, terminal, $\mathbb{Q}$-factorial
12	$m+4$	$X \to \mathbb{P}_{m-1}$ $F_{\mathrm{gen}} = Q_5$	if $a_1=\ldots=a_5=0$: $X \to Q_5$ $F = \mathbb{P}_{m-1}$	$\dim(X) = 6$: two Gorenstein, terminal, locally factorial
13	$m+4$	$X \to Q_5$ $F_{\mathrm{gen}} = \mathbb{P}_{m-1}$	if $0<d_2=\ldots=d_m$: $X \sim Y_{2;1^7,d_2^{m-1}}$ $C = \mathbb{P}_{m-2}$	$\dim(X) = 6$: two Gorenstein, terminal, locally factorial
14	6	$X \to \mathbb{P}_4$ $F_{\mathrm{gen}} = \mathbb{P}_2$	$X \to \mathbb{P}_4$ $F_{\mathrm{gen}} = \mathbb{P}_2$	one Gorenstein, terminal locally factorial

No. 1: The variety X is of dimension 6 and admits two elementary contractions $Q_6 \leftarrow X \to \mathbb{P}_1$. Here, $X \to Q_6$ is birational with center Q_4 and $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber Q_5 and special fibers over $[0,1]$ and $[1,0]$, both isomorphic to the singular quadric $V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}_6$. Moreover, we obtain small degenerations of X into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 2. The variety X is of dimension 6 and admits two elementary contractions $Q_6 \leftarrow X \to \mathbb{P}_4$. The morphism $X \to Q_6$ is birational with center $\mathbb{P}_2$ and $X \to \mathbb{P}_4$ is a Mori fiber space with general fiber $\mathbb{P}_2$. Moreover, we obtain small degenerations of X into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 3. The variety X is of dimension 5 and admits two elementary contractions $Q_5 \leftarrow X \to \mathbb{P}_1$. The morphism $X \to Q_5$ is birational with center Q_3 and $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber Q_4 and singular fibers over $[0,1]$ and $[1,0]$, both isomorphic to the singular quadric $V(T_1T_2 + T_3T_4 + T_5^2) \subseteq \mathbb{P}_5$. Moreover, we obtain small degenerations on X into three different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 4A. The variety X is of dimension $m+5$ and admits two elementary contractions $Y \leftarrow X \to \mathbb{P}_3$. Here, Y is a hypersurface of degree 3 in $\mathbb{P}(1^4, 2^{m+3})$. The morphism $X \to Y$ is birational with center isomorphic $\mathbb{P}_{m+2}$ and the morphism $X \to \mathbb{P}_3$ is a Mori fiber space with fibers $\mathbb{P}_{m+2}$. Moreover, for $\dim(X) \leq 6$, we obtain small degenerations of X into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 4B. The variety X is of dimension $m+5$ and admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_3$, where

$$Y = V(T_0^3 + T_1T_2^2 + T_3T_4^2 + T_5T_6^2) \subseteq \mathbb{P}_{m+6}.$$

The morphism $X \rightarrow \mathbb{P}_3$ is a Mori fiber space with fibers $\mathbb{P}_{m+2}$. Moreover, for $\dim(X) \leq 6$, we obtain small degenerations into two different log terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 4C. The variety X is of dimension $m+5$ and admits a Mori fiber space $X \rightarrow \mathbb{P}_3$ with fibers $\mathbb{P}_{m+2}$. Moreover, for $\dim(X) \leq 6$, we obtain small degenerations int two different into terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 4D. In both cases $d_1 = 0, 1$ the corresponding variety X is of dimension $m+5$ and admits a Mori fiber space $X \rightarrow \mathbb{P}_3$ with fibers $\mathbb{P}_{m+2}$. In case $d_1 = 1$ or $m = 0$ we obtain a birational elementary contraction $X \rightarrow Y$, where

$$Y = V(T_0^2 + T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}_{m+6}.$$

Moreover, for $\dim(X) \leq 6$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 4E. The variety X is of dimension $m+5$ and admits two Mori fiber spaces $\mathbb{P}_{m+3} \leftarrow X \rightarrow \mathbb{P}_3$. The morphism $X \rightarrow \mathbb{P}_3$ has fibers $\mathbb{P}_{m+2}$. To describe the fibers of $\varphi \colon X \rightarrow \mathbb{P}_{m+3}$ set

$$Y_c := \{[z_0, \ldots, z_{m+3}] \in \mathbb{P}_{m+3};\ z_i = 0 \text{ for exactly } c \text{ entries } i \in \{0,1,2,3\}\}.$$

Then we obtain

$$\varphi^{-1}(z) \cong \begin{cases} \mathbb{P}_3 & \text{if } z \in Y_4 \\ \mathbb{P}_2 & \text{if } z \in Y_3 \\ \mathrm{V}_{\mathbb{P}_3}(T_0^3 + T_1^3) & \text{if } z \in Y_2 \\ \mathrm{V}_{\mathbb{P}_3}(T_0^3 + T_1^3 + T_2^3) & \text{if } z \in Y_1 \\ \mathrm{V}_{\mathbb{P}_3}(T_0^3 + T_1^3 + T_2^3 + T_3^3) & \text{otherwise.} \end{cases}$$

Note that $Y_4 = \emptyset$ in case $m = 0$. Moreover, for $\dim(X) \leq 6$, we obtain a small degeneration into a Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one with singularities worse than log terminal.

No. 4F. Case $d_1 = 0$ or $m = 0$: The variety X is of dimension $m+5$ and admits two Mori fiber spaces $\mathbb{P}_{m+3} \leftarrow X \rightarrow \mathbb{P}_3$. The morphism $X \rightarrow \mathbb{P}_3$ has fibers $\mathbb{P}_{m+2}$. To describe the fibers of $\varphi \colon X \rightarrow \mathbb{P}_{m+3}$, set as above

$$Y_c := \{[z_0, \ldots, z_{m+3}] \in \mathbb{P}_{m+3};\ z_i = 0 \text{ for exactly } c \text{ entries } i \in \{0,1,2,3\}\}.$$

Note that $Y_4 = \emptyset$ in case $m = 0$. We obtain

$$\varphi^{-1}(z) \cong \begin{cases} \mathbb{P}_3 & \text{if } z \in Y_4 \\ \mathbb{P}_2 & \text{if } z \in Y_3 \\ \mathrm{V}_{\mathbb{P}_3}(T_0T_1) & \text{if } z \in Y_2 \\ \mathrm{V}_{\mathbb{P}_3}(T_0T_1 + T_2^2) & \text{if } z \in Y_1 \\ \mathbb{P}_1 \times \mathbb{P}_1 & \text{otherwise.} \end{cases}$$

Case $d_1 = -1$*:* The variety X is of dimension $m + 5$ and admits two elementary contractions. One of them is birational

$$X \to V(T_0T_1^2 + T_2T_3^2 + T_4T_5^2 + T_6T_7^2) \subseteq \mathbb{P}_{m+6}$$

with center $\mathbb{P}_{m+2}$. The other one is a Mori fiber space $X \to \mathbb{P}_3$ with fibres $\mathbb{P}_{m+2}$.

In both cases, for $\dim(X) \leq 6$, we obtain a small degeneration into a log terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

No. 4G. The variety X is of dimension $m + 5$. and admits a Mori fiber space $X \to \mathbb{P}_3$ with fibers $\mathbb{P}_{m+2}$. If $d_i = 0$ or $m = 0$ holds, then X admits another Mori fiber space $X \to \mathbb{P}_{m+3}$ with general fiber $\mathbb{P}_2$ and special fiber $\mathbb{P}_3$ over $\mathrm{V}(T_0, T_1, T_2, T_3)$. If $d_1 = -1$ and $d_2 = 0$ holds, then we obtain a birational elementary contraction

$$X \;\to\; V(T_0T_1 + T_2T_3 + T_4T_5 + T_6T_7) \;\subseteq\; \mathbb{P}_{m+6}.$$

In case $d_1 = -2$ and $d_2 = 0$ we obtain a birational contraction $X \to Y$ onto a hypersurface Y of degree 3 in $\mathbb{P}(1^4, 2^{m+3})$. Moreover, for $\dim(X) \leq 6$, we obtain a small degeneration into a terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

No. 5. The variety X is of dimension $m + 5$ and admits a Mori fiber space $X \to \mathbb{P}_{m+2}$. As earlier, set

$$Y_c \;:=\; \{[z_0, \ldots, z_{m+2}] \in \mathbb{P}_{m+2};\; z_i = 0 \text{ for exactly } c \text{ entries } i \in \{0, 1, 2\}\}.$$

Then we obtain

$$\varphi^{-1}(z) \cong \begin{cases} \mathbb{P}_4 & \text{if } z \in Y_3 \\ \mathrm{V}_{\mathbb{P}_4}(T_1T_2) & \text{if } z \in Y_2 \\ \mathrm{V}_{\mathbb{P}_4}(T_0T_1 + T_2T_3) & \text{if } z \in Y_1 \\ Q_3 & \text{otherwise.} \end{cases}$$

Moreover, for $\dim(X) = 6$, we obtain a small degeneration into a terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one and another one into a log terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

No. 6. The variety X is of dimension $m+5 \geq 7$ and admits a Mori fiber space $X \to \mathbb{P}_{m+1}$. Set

$$Y_c \;:=\; \{[z_0, \ldots, z_{m+1}] \in \mathbb{P}_{m+1};\; z_i = 0 \text{ for exactly } c \text{ entries } i \in \{0,1\}\}.$$

Then we obtain

$$\varphi^{-1}(z) \cong \begin{cases} \mathrm{V}_{\mathbb{P}_5}(T_1T_2 + T_3T_4) & \text{if } z \in Y_2 \\ \mathrm{V}_{\mathbb{P}_5}(T_0T_1 + T_2T_3 + T_4^2) & \text{if } z \in Y_1 \\ Q_4 & \text{otherwise.} \end{cases}$$

Moreover, for $\dim(X) = 7$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 7. The variety X is of dimension $m+5$ and admits a Mori fiber space $X \to \mathbb{P}_m$ with general fiber Q_5 and fibers isomorphic to the singular quadric $\mathrm{V}(T_0T_1+T_2T_3+T_4T_5) \subseteq \mathbb{P}_6$ over $[0, z_1, \ldots, z_m]$. Moreover, for $\dim(X) = 8$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 8. The variety X is of dimension $m+5$ and admits a birational elementary contraction $X \to \mathbb{P}_{m+5}$ with center Q_4. If $m = 1$ holds, then X is of dimension 6 and admits a birational elementary contraction $X \to Q_6$ sending a $\mathbb{P}_5$ to a point. Moreover, for $\dim(X) = 6$, we obtain a small degeneration into a terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one and another one into a terminal, $\mathbb{Q}$-factorial, Fano $\mathbb{T}$-variety of Gorenstein index two and complexity one.

No. 9. The variety X is of dimension $m+5$ and admits a Mori fiber space $X \to \mathbb{P}_{m-1}$ with general fiber Q_6. If $a_i = 0$ holds for all i, then X admits moreover a Mori fiber space $X \to Q_6$ with fibers $\mathbb{P}_{m-1}$. Moreover, for $\dim(X) = 7$, we obtain a small degeneration into terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

No. 10. The variety X is of dimension $m+5$ and admits a Mori fiber space $X \to Q_6$ with general fiber isomorphic to $\mathbb{P}_{m-1}$. In the case that $0 < d_2 = \ldots = d_m$ holds the variety X admits moreover a birational contraction $X \to Y$, where

$$Y \;:=\; V(T_0T_1 + T_2T_3 + T_4T_5 + T_6T_7) \;\subseteq\; \mathbb{P}(1^8, d_2^{m-1}),$$

with center $\mathbb{P}_{m-2}$. Moreover, for $\dim(X) = 7$, we obtain a small degeneration into a terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

No. 11. The variety X is of dimension $m+4$ and admits a birational elementary contraction $X \to \mathbb{P}_{m+4}$. If $m = 1$ holds X is of dimension 5 and admits a birational elementary contraction $X \to Q_5$ sending a $\mathbb{P}_4$ to a point. Moreover, for $\dim(X) = 5$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one and another one into a terminal, Gorenstein, $\mathbb{Q}$-factorial, locally factorial, Fano $\mathbb{T}$-variety of complexity one. For $\dim(X) = 6$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one and another one into a terminal, $\mathbb{Q}$-factorial, locally factorial, Fano $\mathbb{T}$-variety of Gorenstein index 2 and complexity one.

No. 12. The variety X is of dimension $m+4$ and admits a Mori fiber space $X \to \mathbb{P}_{m-1}$ with general fiber Q_5. In the case that $a_i = 0$ holds for all i the variety X admits moreover a Mori fiber space $X \to Q_5$ with fibers $\mathbb{P}_{m-1}$. Moreover, for $\dim(X) = 6$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 13. The variety X is of dimension $m+4$ and admits a Mori fiber space $X \to Q_5$ with general fiber $\mathbb{P}_{m-1}$. If $0 < d_2 = \ldots = d_m$ holds, then X admits moreover a birational contraction $X \to Y$, where

$$Y \ := \ V(T_0T_1 + T_2T_3 + T_4T_5 + T_6^2) \ \subseteq \ \mathbb{P}(1^7, d_2^{m-1}),$$

with center $\mathbb{P}_{m-2}$. Moreover, for $\dim(X) = 6$, we obtain small degenerations into two different terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-varieties of complexity one.

No. 14. The variety X is of dimension 6 and admits two Mori fiber spaces $\mathbb{P}_4 \leftarrow X \to \mathbb{P}_4$. In both cases we have general fibers isomorphic to $\mathbb{P}_2$ and special fibers $\mathbb{P}_3$ over the points $[0,0,0,0,1], [0,0,0,1,0], [0,0,1,0,0], [0,1,0,0,0]$ and $[1,0,0,0,0]$. Moreover X admits a small degeneration into terminal, Gorenstein, locally factorial, Fano $\mathbb{T}$-variety of complexity one.

3.4 Duplication of free weights

In this section we present the finite set of starting varieties from which one can construct all varieties of Theorem 3.1.3 via iterated *duplication of a free weight* as introduced in [30, Constr. 5.1]. In this procedure, one takes a Cox ring generator S_k of X not occurring in the defining relations and constructs a new Cox ring by adding a further free generator S_k' of the same degree as S_k. The resulting variety X' is of one dimension higher. In terms of birational geometry, the duplication of a free weight means taking an elementary contraction $\widetilde{X}_1 \to X$ with fiber $\mathbb{P}_1$, passing via a series of small quasimodifications to $\widetilde{X}_t$ and then performing a contraction of a prime divisor $\widetilde{X}_t \to X'$, see [30, Prop. 5.3]. We start by adapting the methods from [30, Sec. 5] to explicit $\mathbb{T}$-varieties.

Construction 3.4.1. Let $X := X(\alpha, P, \Sigma)$ be a projective explicit $\mathbb{T}$-variety with Cox ring $R(\alpha, P) = \mathbb{K}[T_{ij}, S_k]/\langle g_1, \ldots, g_q\rangle$, let $Q\colon \mathbb{Z}^{n+m} \to K_P$ be the corresponding degree map, $u \in \mathrm{Cl}(X) = K_P$ an ample class and fix an index $1 \le j \le m$. Now, consider the block matrix

$$P' := \begin{bmatrix} P & 0 \\ e_j & -1 \end{bmatrix},$$

where $e_j \in \mathbb{Z}^{n+m}$ is interpreted as a row vector and let

$$Q'\colon \mathbb{Z}^{n+m+1} \to \mathbb{Z}^{n+m+1}/\mathrm{im}(P')^* = K_P$$

be the linear projection extending $Q\colon \mathbb{Z}^{n+m} \to K_P$ and sending e_{m+1} to $Q(e_j)$. This defines an explicit $\mathbb{T}$-variety $X' := X(\alpha, P', \Sigma'(u))$, where

$$\Sigma'(u) := \{P'(\gamma_0^*);\ \gamma_0 \preccurlyeq \gamma \subseteq \mathbb{Z}^{n+m+1} \text{ with } u \in Q'(\gamma_0)^\circ\}.$$

Note that Construction 3.4.1 is the Gale dual version of [30, Constr. 5.1.]. Therefore, in the situation of Construction 3.4.1, we say that X' arises from X via *duplication of the free weight* $\deg(S_j)$.

Proposition 3.4.2 (See [30, Prop. 5.2]). *Let $X' := X(\alpha, P', \Sigma')$ arises from an explicit $\mathbb{T}$-variety $X := X(\alpha, P, \Sigma)$ via duplication of the free weight $\deg(S_j)$ as in Construction 3.4.1. Then the following statements hold:*

(i) *We have* $\dim(X') = \dim(X) + 1$.

(ii) *The cones of semi-ample divisor classes satisfy* $\mathrm{SAmple}(X') = \mathrm{SAmple}(X)$.

(iii) *The variety X' is smooth if and only if X is smooth.*

(iv) *The ring $R(\alpha, P')$ is a complete intersection if and only if $R(\alpha, P)$ is so.*

(v) *If $R(\alpha, P)$ is a complete intersection,* $\deg(S_j)$ *semi-ample and X Fano, then X' is so.*

We turn to the description of the Fano varieties from Theorem 3.1.3 via iterated duplication of free weights. For this, we will refer to a variety as a *starting variety*, if it is defined via a datum from the following list.

No.	$\mathcal{R}(X)$	$[w_1, \dots, w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$	6
2	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0&1&0 \\ 1&1&0&1&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 6 \end{bmatrix}$	6
3	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1 \\ 1&1&0&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 5 \end{bmatrix}$	5
4.A.1	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\begin{bmatrix} 0&1&2&1&2&1&2&1 \\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 3 \end{bmatrix}$	5
4.A.2	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&2&1&2&1&2&1&2 \\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 9 \\ 4 \end{bmatrix}$	6
4.B.1	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\begin{bmatrix} 0&1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	5
4.B.2	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 5 \\ 4 \end{bmatrix}$	6
4.C.1	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$	$\begin{bmatrix} 0&1&0&1&1&1&1&1 \\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	5

4.C.2	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&1&1&1&1&1\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}5\\4\end{bmatrix}$	6
4.D.1	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\begin{bmatrix} 0&1&1&1&1&1&1&1\\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix}4\\3\end{bmatrix}$	5
4.D.2	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&1&1&1&1&1&1&0\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}5\\4\end{bmatrix}$	6
4.D.3	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&1&1&1&1&1&1&1\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}6\\4\end{bmatrix}$	6
4.D.4	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,S_2]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&1&1&1&1&1&1&1&0&1\\ 1&0&1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix}6\\5\end{bmatrix}$	7
4.E.1	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	5
4.E.2	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}3\\4\end{bmatrix}$	6
4.F.1	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	5
4.F.2	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&-1\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}1\\4\end{bmatrix}$	6
4.F.3	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}2\\4\end{bmatrix}$	6
4.F.4	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,S_2]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&1&0&1&0&1&0&1&-1&0\\ 1&0&1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix}1\\5\end{bmatrix}$	7
4.G.1	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	5
4.G.2	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&-1\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}2\\4\end{bmatrix}$	6
4.G.3	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&-2\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}1\\4\end{bmatrix}$	6
4.G.4	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&1&0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix}3\\4\end{bmatrix}$	6
4.G.5	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&1&0&1&0&1&0&1&-1&0\\ 1&0&1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix}2\\5\end{bmatrix}$	7
4.G.6	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&1&0&1&0&1&0&1&-2&0\\ 1&0&1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix}1\\5\end{bmatrix}$	7
5	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&2a+1&a&1&a&1&a&1&1\\ 1&1&1&0&1&0&1&0&0 \end{array}\right]$ $a \geq 0$	$\begin{bmatrix}3a+4\\3\end{bmatrix}$	6
6	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&2a_3+1&a_1&a_2&a_3&1&a_3&1&1\\ 1&1&1&1&1&0&1&0&0 \end{array}\right]$ $0 \leq a_1 \leq a_2,\ a_1+a_2=2a_3+1$	$\begin{bmatrix}4a_3+4\\4\end{bmatrix}$	6
7	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&2a_5+1&a_1&a_2&a_3&a_4&a_5&1&1\\ 1&1&1&1&1&1&1&0&0 \end{array}\right]$ $a_i \geq 0,$ $a_1+a_2=a_3+a_4=2a_5+1,$	$\begin{bmatrix}5a_5+4\\5\end{bmatrix}$	6
8	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|c} 0&0&0&0&0&0&-1&1&1\\ 1&1&1&1&1&1&1&1&0 \end{array}\right]$	$\begin{bmatrix}1\\6\end{bmatrix}$	6
9	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&a_1&a_2&a_3&a_4&a_5&a_6&a_7&1&1\\ 1&1&1&1&1&1&1&1&0&0 \end{array}\right]$ $a_i \geq 0,$ $a_1=a_2+a_3=a_4+a_5=a_6+a_7,$	$\begin{bmatrix}3a_1+2\\6\end{bmatrix}$	7

10.1	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|cc} 0&0&0&0&0&0&0&0&1&1\\ 1&1&1&1&1&1&1&1&0&d \end{array}\right]$ $1 \le d \le 5$	$\begin{bmatrix} 2 \\ 6+d \end{bmatrix}$	7
10.2	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|ccc} 0&0&0&0&0&0&0&0&1&1&1\\ 1&1&1&1&1&1&1&1&0&d-1&d \end{array}\right]$ $2 \le d \le 4$	$\begin{bmatrix} 3 \\ 5+2d \end{bmatrix}$	8
10.3	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$	$\left[\begin{array}{cccccccc\|ccc} 0&0&0&0&0&0&0&0&1&1&1\\ 1&1&1&1&1&1&1&1&0&1&3 \end{array}\right]$	$\begin{bmatrix} 3 \\ 10 \end{bmatrix}$	8
11	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$	$\left[\begin{array}{ccccccc\|c} -1&1&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$	5
12	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$	$\left[\begin{array}{ccccccc\|cc} 0&2a_5&a_1&a_2&a_3&a_4&a_5&1&1\\ 1&1&1&1&1&1&1&0&0 \end{array}\right]$ $a_1+a_2=a_3+a_4=2a_5$ $a_i \ge 0$	$\begin{bmatrix} 2+5a_5 \\ 5 \end{bmatrix}$	6
13.1	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$	$\left[\begin{array}{ccccccc\|cc} 0&0&0&0&0&0&0&1&1\\ 1&1&1&1&1&1&1&0&d \end{array}\right]$ $1 \le d \le 4$	$\begin{bmatrix} 2 \\ 5+d \end{bmatrix}$	6
13.2	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$	$\left[\begin{array}{ccccccc\|ccc} 0&0&0&0&0&0&0&1&1&1\\ 1&1&1&1&1&1&1&0&d-1&d \end{array}\right]$ $2 \le d \le 3$	$\begin{bmatrix} m \\ 4+2d \end{bmatrix}$	7
14	$\frac{\mathbb{K}[T_1,\dots,T_{10}]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6+T_7T_8, \\ \lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10} \end{array}\right\rangle}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$	6

Moreover, we say that a starting variety is of *Type a* or *Type b* according to the following list:

Type a: $1, 2, 3, 4.A.1, 4.B.1, 4.C.1, 4.D.1, 4.D.2, 4.E.1, 4.F.1, 4.F.2, 4.G.1, 4.G.2, 4.G.3, 14,$

Type b: $4.A.2, 4.B.2, 4.C.2, 4.D.3, 4.D.4, 4.E.2, 4.F.3, 4.F.4, 4.G.4, 4.G.5, 4.G.6, 5, 6, 7, 8, 9, 10.1, 10.2, 10.3, 11, 12, 13.1, 13.2.$

We say that a variety X arises via *iterated duplication of free weights from one of the starting varieties*, if there is a sequence $X = X_r, \dots, X_0$, where X_0 is a starting variety and for $i > 0$, the variety X_i arises via duplication of a free weight from X_{i-1}.

Remark 3.4.3. All starting varieties are smooth projective general arrangement varieties of true complexity and Picard number two. Moreover, any variety X arising via iterated duplication of free weights from one of the starting varieties is as well smooth and projective and has true complexity and Picard number two.

Proposition 3.4.4. *All starting varieties of* Type a *are Fano varieties. None of them allows duplication of a free weight in the sense that the resulting variety is Fano.*

Proposition 3.4.5. *Let X arise via iterated duplication of free weights from a starting variety X_0 of* Type b *and denote by $\mu_i \in \mathbb{Z}_{\geq 0}$ the number of duplications of the free weight* $\deg(S_i)$. *Then X and X_0 are Fano if and only if they fulfill the requirements listed in the following table:*

No.	X_0 *Fano*	X *Fano*
4.A.2	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.B.2	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.C.2	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.D.3	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.D.4	*Yes*	$\mu_1 = 0, \mu_2 \in \mathbb{Z}_{\geq 0}$
4.E.2	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.F.3	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.F.4	*Yes*	$\mu_1 = 0, \mu_2 \in \mathbb{Z}_{\geq 0}$
4.G.4	*Yes*	$\mu_1 \in \mathbb{Z}_{\geq 0}$
4.G.5	*Yes*	$\mu_1 \in \{0,1\}, \mu_2 \in \mathbb{Z}_{\geq 0}$
4.G.6	*Yes*	$\mu_1 = 0, \mu_2 \in \mathbb{Z}_{\geq 0}$
5	$a = 0$	$\mu_1 \geq 3a$
6	*No*	$\mu_1 \geq 4a_3 + 1$
7	*No*	$\mu_1 \geq 5a_5 + 2$
8	*Yes*	$0 \leq \mu_1 \leq 4$
9	$a_1 = 0$	$\mu_1 \geq 3a_1$
10.1	*Yes*	$\mu_1 = 0, \mu_2 \in \mathbb{Z}_{\geq 0}$ *or* $\mu_1 = 1, \mu_2 \in \mathbb{Z}_{\geq 0}, d = 2$ *or* $1 \leq \mu_1 \leq 4, \mu_2 \in \mathbb{Z}_{\geq 0}, d = 1$
10.2	*Yes*	$\mu_1 = \mu_2 = 0, \mu_3 \in \mathbb{Z}_{\geq 0}$ *or* $\mu_1 = 0, \mu_2 = 1, \mu_3 \in \mathbb{Z}_{\geq 0}, 2 \leq d \leq 3$ *or* $\mu_1 = 1, \mu_2 = 0, \mu_3 \in \mathbb{Z}_{\geq 0}, d = 2$
10.3	*Yes*	$\mu_1 = \mu_2 = 0, \mu_3 \in \mathbb{Z}_{\geq 0}$
11	*Yes*	$0 \leq \mu_1 \leq 3$
12	$a_5 = 0$	$\mu_1 \geq 5a_5$
13.1	*Yes*	$\mu_1 = 0, \mu_2 \in \mathbb{Z}_{\geq 0}$ *or* $\mu_1 = 1, \mu_2 \in \mathbb{Z}_{\geq 0}, 1 \leq d \leq 2$ *or* $\mu_1 = 3, \mu_2 \in \mathbb{Z}_{\geq 0}, d = 1$
13.2	*Yes*	$\mu_1 = \mu_2 = 0, \mu_3 \in \mathbb{Z}_{\geq 0}$ *or* $\mu_1 = 0, \mu_2 = 1, \mu_3 \in \mathbb{Z}_{\geq 0}, 1 \leq d \leq 2$

Corollary 3.4.6. *Every smooth Fano general arrangement variety of true complexity and Picard number two is either a starting variety or arises via iterated duplication of free weights from precisely one of the starting varieties of* Type b. *In particular every such variety arises from a not necessary Fano one of dimension* 5 *to* 8.

Question 3.4.7. It follows from [30, Prop. 5.4, Thm. 5.5] that every smooth Fano variety of true complexity one and Picard number two is of dimension 4 to 7 or arises via iterated duplications of free weights from a finite set of smooth projective varieties of true complexity one and Picard number two of dimension 4 to 7. Corollary 3.4.6 establishes the analogous statement with a finite set of starting varieties of dimensions 5 to 8 for the Fano general arrangement varieties of true complexity two listed in Theorem 3.1.3. It would be interesting to see if the smooth Fano general arrangement varieties of Picard number two but higher complexity behave similarly.

CHAPTER

FOUR

THE ANTICANONICAL COMPLEX

The anticanonical complex has been introduced in [13] as a natural generalization of the toric Fano polytope and so far has been successfully used for the study of singular Fano varieties with a torus action of complexity one. In this chapter, we extend the area of application to (not necessarily Fano) varieties, suitably realized inside toric varieties, for example explicit $\mathbb{T}$-varieties. In this situation, the central question is, whether ambient toric resolutions provide enough discrepancies to define anticanonical complexes. Our main result reduces this question to an *explicit maximal orbit quotient*, see Construction 4.3.3 and Theorem 4.3.6. Applying this result to general arrangement varieties, we give an explicit construction for their anticanonical complexes. Note that being Fano is reflected in a certain convexity property of the anticanonical complex. This allows us to give a second construction in this situation. As an application we characterize log-terminality in terms of exponents for general arrangement varieties of complexity two. The results of this chapter are published in the joint work [49].

4.1 Toric ambient resolutions of singularities

In toric geometry resolution of singularities can be performed in a purely combinatorial manner. The idea of toric ambient resolutions of singularities is to make this methods accessible for closed subvarieties of toric varieties. The aim of this section is to give a sufficient criterion on an embedded variety for the existence of a toric ambient resolution of singularities, see Proposition 4.1.6.

Let us fix our terminology: Consider a toric variety Z with acting torus T and a normal closed subvariety $X \subseteq Z$. Let $\varphi\colon Z' \to Z$ be a birational toric morphism. The *proper transform* of $X \subseteq Z$ is the closure $X' \subseteq Z'$ of $\varphi^{-1}(X \cap T)$. We call $\varphi\colon Z' \to Z$ a *toric ambient modification* if it maps X' properly onto X. If furthermore the proper transform X' is smooth, we call φ a *toric ambient resolution of singularities* of X.

Our approach to toric ambient resolutions of singularities is the following two-step procedure: at first we use methods from tropical geometry to prepare the embedded variety for resolving its singularities in a second step with methods from toric geometry.

Let us recall the basic notions on tropical varieties. For a closed subvariety $X \subseteq Z$ intersecting the torus non trivially consider the vanishing ideal $I(X \cap T)$ in the Laurent polynomial ring $\mathcal{O}(T)$. For every $f \in I(X \cap T)$ let $|\Sigma(f)|$ denote the support of the codimension one skeleton of the normal quasifan of its Newton polytope, where a quasifan is a fan, where we allow the cones to be non-pointed. Then the *tropical variety* $\mathrm{trop}(X)$ *of* X is defined as follows, see [61, Def. 3.2.1]:

$$\mathrm{trop}(X) := \bigcap_{f \in I(X \cap T)} |\Sigma(f)| \subseteq \mathbb{Q}^{\dim(Z)}.$$

A closed subvariety $X \subseteq Z$ is called *weakly tropical* if the fan Σ corresponding to Z is supported on $\mathrm{trop}(X)$. In the following we will always assume $\mathrm{trop}(X)$ to be endowed with a fixed quasifan structure. If $X \subseteq Z$ is weakly tropical then by sufficiently refining the quasifan structure fixed on $\mathrm{trop}(X)$ we achieve that Σ is a subfan of $\mathrm{trop}(X)$.

Construction 4.1.1. Let $X \subseteq Z$ be a closed subvariety intersecting the torus non-trivially, consider the defining fan Σ of Z and the coarsest common refinement

$$\Sigma' := \Sigma \sqcap \mathrm{trop}(X) := \{\sigma \cap \tau;\ \sigma \in \Sigma,\ \tau \in \mathrm{trop}(X)\}.$$

Let $\varphi\colon Z' \to Z$ be the toric morphism arising from the refinement of fans $\Sigma' \to \Sigma$ and let X' be the proper transform of X under φ. We call $Z' \to Z$ a *weakly tropical resolution of* X.

Let $Z' \to Z$ be a weakly tropical resolution of X. Then the embedding $X' \subseteq Z'$ is weakly tropical as by construction $|\mathrm{trop}(X)| = |\mathrm{trop}(X')|$ holds. Note that Z' and thus X' depend on the choice of the quasifan structure fixed on $\mathrm{trop}(X)$.

For a toric variety Z we denote by $Z_\sigma \subseteq Z$ the affine toric chart corresponding to the cone $\sigma \in \Sigma$ in the lattice N. We will make use of the local product structure of toric varieties:

Construction 4.1.2. Let $X \subseteq Z$ be weakly tropical and let $\sigma \in \Sigma$ be any cone. Choose a maximal cone $\tau \in \mathrm{trop}(X)$ with $\sigma \preceq \tau$, set $N(\tau) := N \cap \mathrm{lin}_{\mathbb{Q}}(\tau)$ and fix a decomposition $N = N(\tau) \oplus \tilde{N}$. Accordingly, we obtain a product decomposition

$$Z_\sigma \cong U(\sigma) \times \tilde{\mathbb{T}},$$

where $U(\sigma) := U(\sigma, \tau)$ is the affine toric variety corresponding to the cone σ in the lattice $N(\tau)$ and $\tilde{\mathbb{T}}$ is a torus. We write $\pi_\sigma := \pi_{\sigma,\tau,\tilde{N}}$ for the projection $Z_\sigma \to U(\sigma)$.

Due to the structure theorem for tropical varieties, the maximal cones $\tau \in \mathrm{trop}(X)$ are of dimension $\dim(X)$. In particular, in the situation of the above construction, $U(\sigma)$ does up to isomorphism not depend on the choices made. Moreover, if $X \subseteq Z$ is complete then due to [61, Prop. 6.4.7] we have $|\Sigma| = |\mathrm{trop}(X)|$ and for any maximal cone $\sigma \in \Sigma$ the cone $\tau \in \mathrm{trop}(X)$ as chosen above equals σ.

Definition 4.1.3. Let $X \subseteq Z$ be weakly tropical. We call $X \subseteq Z$ *semi-locally toric* if for every maximal cone $\sigma \in \Sigma$ there exists a projection π_σ as in Construction 4.1.2 that maps $X_\sigma := X \cap Z_\sigma$ isomorphically onto its image $\pi_\sigma(X_\sigma)$ and the latter is an open subvariety of $U(\sigma)$.

Remark 4.1.4. Let X be a rational $\mathbb{T}$-variety of complexity one. Then X allows an equivariant embedding into a toric variety Z such that any weakly tropical resolution is semi-locally toric, see [6, Prop. 3.4.4.6].

Note that given a weakly tropical embedding $X \subseteq Z$ the notion of being semi-locally toric is preserved when passing over to another embedding $X' \subseteq Z'$ provided by a toric isomorphism $Z \to Z'$ as made precise in the following remark:

Remark 4.1.5. Let $A\colon N_1 \to N_2$ be an isomorphism of lattices (we use the letter A as well to denote the induced linear map $N_1 \otimes \mathbb{Q} \to N_2 \otimes \mathbb{Q}$) defining an isomorphism of affine toric varieties $\varphi_A\colon Z_1 \to Z_2$, with defining cones σ_1 in N_1 and $\sigma_2 := A(\sigma_1)$ in N_2. Let $X \subseteq Z_1$ be a semi-locally toric closed subvariety and $N_1 = N_1(\sigma) \oplus \tilde{N}_1$ be a decomposition of N_1 such that the corresponding projection $\pi_\sigma\colon Z_1 \to U(\sigma_1)$ maps X isomorphically onto an open subset of $U(\sigma_1)$. Then $A(\mathrm{trop}(X)) = \mathrm{trop}(\varphi_A(X))$ holds and the decomposition $N_2 = A(N_1(\sigma)) \oplus A(\tilde{N}_1)$ corresponds to a projection $\pi_{\sigma_2}\colon Z_2 \to U(\sigma_2)$ mapping $\varphi_A(X_{\sigma_1})$ isomorphically onto an open subset of $U(\sigma_2)$.

Proposition 4.1.6. *Let $X \subseteq Z$ be a closed subvariety admitting a semi-locally toric weakly tropical resolution, meaning $X' \subseteq Z'$ is semi-locally toric. Then $X \subseteq Z$ admits a toric ambient resolution of singularities.*

The rest of this section is dedicated to the proof of Proposition 4.1.6. Below (and in the rest of this article) we will make frequent use of the following criterion, to which we will refer to as *Tevelev's criterion*, see [75, Lem. 2.2] and [61, Thm. 6.3.4]:

Remark 4.1.7. Let $X \subseteq Z$ be a closed embedding. Then X intersects the torus orbit $T \cdot z_\sigma$ corresponding to the cone $\sigma \in \Sigma$ non-trivially if and only if the relative interior σ° intersects the tropical variety $\mathrm{trop}(X)$ non-trivially. Moreover, if $X \subseteq Z$ is weakly tropical, then the intersection $T \cdot z_\sigma \cap X$ is pure of dimension $\dim(X) - \dim(\sigma)$.

Lemma 4.1.8. *Let $X \subseteq Z$ be a closed embedding. Then any weakly tropical resolution $\varphi\colon Z' \to Z$ is a toric ambient modification.*

Proof. We have to show that φ maps X' properly onto X. Consider any completion $\mathrm{trop}(X)^c$ of the quasifan $\mathrm{trop}(X)$, i.e. a quasifan $\mathrm{trop}(X)^c$ with support $|\mathrm{trop}(X)^c| = \mathbb{Q}^{\dim(Z)}$ such that $\mathrm{trop}(X)$ is a subfan of $\mathrm{trop}(X)^c$. Then the morphism of fans $\Sigma \sqcap \mathrm{trop}(X)^c \to \Sigma$ defines a proper morphism of toric varieties $\tilde{\varphi}\colon Z'' \to Z$ with $\tilde{\varphi}|_{Z'} = \varphi$, where we regard Z' as an open subset of Z''. We conclude that $\varphi\colon X' \to X$ is proper as Tevelev's criterion implies

$$X' = \overline{\varphi^{-1}(X \cap T)}^{Z'} = \overline{\tilde{\varphi}^{-1}(X \cap T)}^{Z''}.$$

□

Let Σ be a fan. We say that a fan Σ' in the same lattice is a *subdivision* of Σ if any cone $\sigma' \in \Sigma'$ is contained in a cone $\sigma \in \Sigma$ and $|\Sigma| = |\Sigma'|$ holds. Any subdivision of fans $\Sigma' \to \Sigma$ defines a proper birational morphism of the corresponding toric varieties $Z' \to Z$.

Lemma 4.1.9. *Let $X \subseteq Z$ be a semi-locally toric weakly tropical embedding. Consider a proper birational toric morphism $\psi\colon Z' \to Z$ defined by a subdivision of fans $\Sigma' \to \Sigma$. For $\sigma \in \Sigma$ denote by $\pi_\sigma\colon Z_\sigma \to U(\sigma)$ the projection mapping X_σ isomorphically onto an open subvariety of $U(\sigma)$ and consider the morphism of toric varieties $\psi(\sigma)\colon V(\sigma) \to U(\sigma)$ arising via the subdivision of the cone σ in $N(\sigma)$. Then there is a commutative diagram*

$$\begin{array}{ccccc} Z'_\sigma & \cong & V(\sigma) \times \tilde{\mathbb{T}} & \xrightarrow{\pi'_\sigma} & V(\sigma) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi(\sigma)\times \mathrm{id}} & & \downarrow{\scriptstyle \psi(\sigma)} \\ Z_\sigma & \cong & U(\sigma) \times \tilde{\mathbb{T}} & \xrightarrow{\pi_\sigma} & U(\sigma), \end{array}$$

where we set $Z'_\sigma := \psi^{-1}(Z_\sigma)$, and π'_σ maps $X' \cap Z'_\sigma$ isomorphically onto an open subvariety of $V(\sigma)$. Moreover, the following statements hold:

(i) *The proper transform X' with respect to ψ equals $\psi^{-1}(X)$.*

(ii) *The subvariety $X' \subseteq Z'$ is weakly tropical and semi-locally toric.*

Proof. Let $N = N(\sigma) \oplus \tilde{N}$ be the decomposition of N giving rise to the isomorphism $Z_\sigma \cong U(\sigma) \times \tilde{\mathbb{T}}$ and the corresponding projection π_σ. Then by construction the defining fan of Z'_σ is supported in $N(\sigma) \otimes \mathbb{Q}$. Using the same decomposition of N as above we thus obtain an isomorphism $Z'_\sigma \cong V(\sigma) \times \tilde{\mathbb{T}}$, the corresponding projection π'_σ and a commutative diagram as claimed. As π_σ maps X_σ isomorphically onto its image we conclude that the projection π'_σ restricts to an isomorphism

$$\pi'_\sigma\colon \psi^{-1}(X_\sigma) \to \psi(\sigma)^{-1}(\pi_\sigma(X_\sigma)).$$

We show that $\psi^{-1}(X_\sigma) = X' \cap Z'_\sigma$ holds: As $\psi(\sigma)^{-1}(\pi_\sigma(X_\sigma))$ is irreducible, so is $\psi^{-1}(X_\sigma)$. Thus using that $X' \cap Z'_\sigma \subseteq \psi^{-1}(X_\sigma)$ is a closed irreducible subvariety of the same dimension we obtain equality as claimed. This proves Supplement (i) and the assertion as $\psi(\sigma)^{-1}(\pi_\sigma(X_\sigma))$ is an open subvariety of $V(\sigma)$. Supplement (ii) follows by restricting the toric projection π'_σ to the affine toric charts. □

Proof of Proposition 4.1.6. Let $\varphi\colon Z' \to Z$ be a semi-locally toric weakly tropical resolution. Then due to Lemma 4.1.8 the morphism φ maps X' properly onto X. Now let $\psi\colon Z'' \to Z'$ be any toric resolution of singularities of Z' arising via a regular subdivision of its defining fan Σ' and denote by X'' the proper transform of X' with respect to ψ. Then $\varphi \circ \psi\colon X'' \to X$ is the composition of proper morphisms and hence is proper. Moreover, as Z'' is smooth, the toric varieties $U(\sigma'')$, where $\sigma'' \in \Sigma''$, are smooth. In particular, as X'' is semi-locally toric due to Lemma 4.1.9 (ii), it is smooth as well and we conclude that $\varphi \circ \psi$ is a toric ambient resolution of singularities. □

4.2 Singularities and the anticanonical complex

Let Z be a toric variety and $X \subseteq Z$ be a closed subvariety. Denote by $Z_0 \subseteq Z$ the (open) union of all T-orbits of codimension at most one in Z. Assume that X intersects $T \subseteq Z$ and that $X_0 := X \cap Z_0$ has a complement of codimension at least two in X. Then we obtain a pullback homomorphism

$$\mathrm{WDiv}^T(Z) \;\to\; \mathrm{WDiv}(X), \qquad D \;\mapsto\; D|_X,$$

which, given a T-invariant Weil divisor on Z, first restricts to the smooth Z_0, then pulls back to X_0 and finally extends to X by closing components. In this situation, we call $X \subseteq Z$ *adapted* if for every T-invariant prime divisor D on Z, the pullback $D|_X$ is a prime divisor on X.

Assume $X \subseteq Z$ to be adapted. Let $\varphi\colon Z' \to Z$ be a toric ambient modification, arising from a refinement of fans $\Sigma' \to \Sigma$ in a lattice N, meaning that every $\sigma' \in \Sigma'$ is contained in some $\sigma \in \Sigma$. We call φ an *adapted toric ambient modification* if besides $X \subseteq Z$ also $X' \subseteq Z'$ is adapted (requiring in particular X' to be normal).

Let X be a normal $\mathbb{Q}$-Gorenstein variety. Recall that given any proper birational morphism $\varphi\colon X' \to X$ with a normal variety X' and a canonical divisor $k_{X'}$ on X', we have the ramification formula

$$k_{X'} - \varphi^*\varphi_* k_{X'} \;=\; \sum a_E E,$$

where E runs through the exceptional prime divisors of $X' \to X$. The number $\mathrm{discr}_X(E) := a_E \in \mathbb{Q}$ is called the *discrepancy* of X with respect to E; it doesn't depend on the choice of $k_{X'}$ and, identifying E with the local ring $\mathcal{O}_{X,E} \subseteq \mathbb{K}(X)$, it depends not even on the choice of $\varphi\colon X' \to X$.

Definition 4.2.1. Let X be $\mathbb{Q}$-Gorenstein and $X \subseteq Z$ be an adapted embedding. Assume that the following conditions hold:

(i) The weakly tropical resolution $\varphi\colon Z' \to Z$ is adapted.

(ii) Every proper birational toric morphism $Z'' \to Z'$ is adapted.

(iii) There exists at least one toric ambient resolution of singularities $Z'' \to Z'$.

Then for every ray $\varrho \subseteq |\Sigma'|$ there exists a proper toric morphism $\psi\colon Z'' \to Z'$ with $\varrho \in \Sigma''$. Denote by $D_{Z''}^{\varrho}$ the corresponding toric divisor and set

$$a_\varrho := \mathrm{disc}_X(D_{Z''}^{\varrho}|_{X''}).$$

Let v_ϱ denote the primitive ray generator of ϱ and for $a_\varrho > -1$ set $v'_\varrho := \frac{1}{a_\varrho+1} v_\varrho$. The *anticanonical region* of $X \subseteq Z$ is the set

$$\mathcal{A} := \bigcup_{\varrho \subseteq |\Sigma'|} \mathcal{A}_\varrho, \qquad \mathcal{A}_\varrho := \begin{cases} \mathrm{conv}(0, v'_\varrho), & \text{if } a_\varrho > -1 \\ \varrho, & \text{else.} \end{cases}$$

In the situation of Definition 4.2.1 let $Z'' \to Z$ be a toric ambient resolution of singularities factorizing over the weakly tropical resolution $Z' \to Z$. Then the exceptional divisors of $X'' \to X$ are precisely the pullbacks of the exceptional divisors of $Z'' \to Z$ and we obtain the following:

Remark 4.2.2. Let $X \subseteq Z$ be as in Definition 4.2.1 and let $\mathcal{A}$ be the anticanonical region. Then the following statements hold:

(i) X has at most log terminal singularities if and only if the anticanonical region $\mathcal{A}$ contains no ray.

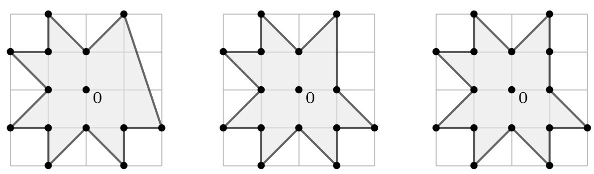
(ii) X has at most canonical singularities if and only if for every ray $\varrho \subseteq |\Sigma'|$ we have $\varrho \cap \mathcal{A} \subseteq \mathrm{conv}(0, v_\varrho)$, i.e. $||v'_\varrho|| \le ||v_\varrho||$.

(iii) X has at most terminal singularities if and only if for every ray $\varrho \subseteq |\Sigma'|$ with $\varrho \notin \Sigma$ we have $\varrho \cap \mathcal{A} \subsetneq \mathrm{conv}(0, v_\varrho)$, i.e. $||v'_\varrho|| < ||v_\varrho||$.

Remark 4.2.3. If Z is a $\mathbb{Q}$-Gorenstein toric variety arising from a fan Σ, then for each $\sigma \in \Sigma$ there is a rational linear form u_σ such that the anticanonical divisor of Z is given on the affine chart $Z_\sigma \subseteq Z$ by $\mathrm{div}(\chi^{u_\sigma}) := m^{-1}\mathrm{div}(\chi^{mu})$, where $m > 0$ is any integer such that mu lies in the dual lattice M of N. In particular, for the anticanonical region $\mathcal{A}$ of Z we have

$$\mathcal{A} \cap \sigma \;=\; \{v \in \sigma;\ \langle u_\sigma, v\rangle \ge -1\}.$$

This turns $\mathcal{A}$ into a polyhedral complex and properties of Z being log terminal, canonical or terminal become questions on boundedness and behaviour with respect to lattice points of this polyhedral complex.



In the pictures above, we look at the complete fan Σ having the bullets different from the origin as its primitive ray generators. Then the shadowed areas indicate the anticanonical regions of a log terminal, a canonical and a terminal (hence smooth) projective toric surface Z defined by Σ. Note that the polyhedral complexes drawn above are not convex, which implies that the corresponding toric variety is not Fano. We will investigate this correlation in more generality in Corollary 4.6.3.

Definition 4.2.4. Let $X \subseteq Z$ be as in Definition 4.2.1 and assume that the anticanonical region $\mathcal{A}$ can be endowed with the structure of a polyhedral complex. In this situation we refer to the anticanonical region as the *anticanonical complex* and say that $X \subseteq Z$ *admits an anticanonical complex.*

Remark 4.2.5. Let $X \subseteq Z$ admits an anticanonical complex. Then Remark 4.2.2 specializes to the following:

(i') X has at most log terminal singularities if and only if $\mathcal{A}$ is bounded.

(ii') X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of $\mathcal{A}$.

(iii') X has at most terminal singularities if and only if 0 and the primitive generators of the rays of the fan of Z are the only lattice points of $\mathcal{A}$.

Our main result of this section is a sufficient criterion on when an embedding $X \subseteq Z$ admits an anticanonical complex. Let $X \subseteq Z$ be adapted. Then the pullback homomorphism on the level of Weil divisors described above induces a pullback homomorphism $\mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ on the level of divisor class groups. We call the embedding $X \subseteq Z$ *neat* if this homomorphism is an isomorphism.

Proposition 4.2.6. *Let $X \subseteq Z$ be a neat embedding admitting a semi-locally toric weakly tropical resolution and assume there exists a T-invariant $\mathbb{Q}$-Cartier divisor D on Z whose pullback $D|_X$ is a canonical divisor on X. Then $X \subseteq Z$ admits an anticanonical complex.*

The rest of this section is dedicated to the proof of the above result. Starting with an adapted embedding $X \subseteq Z$ in a first step we explicitly construct a polyhedral complex out of its weakly tropical resolution. We then show under which conditions one can read discrepancies of X off this complex. In the second step we show that under the assumptions of Proposition 4.2.6 this complex is indeed the anticanonical complex of $X \subseteq Z$.

Remark 4.2.7. Let $X \subseteq Z$ be adapted and consider an adapted toric ambient modification $Z' \to Z$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{WDiv}^{T'}(Z') & \longrightarrow & \mathrm{WDiv}(X') \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{WDiv}^{T}(Z) & \longrightarrow & \mathrm{WDiv}(X), \end{array}$$

where $T' \subseteq Z'$ is the acting torus of Z', the horizontal arrows are the pullback homomorphisms defined above and the φ_* are the usual birational transforms of Weil divisors via φ, i.e. for any prime divisor D we have $\varphi_*(D) := \overline{\varphi(D)}$ if $\mathrm{codim}(\varphi(D)) = 1$ holds, and 0 otherwise.

For any toric variety Z, we denote by k_Z the toric canonical divisor on Z given as minus the sum over all toric prime divisors. Here comes our main technical tool for the construction of the anticanonical complex:

Definition 4.2.8. Let $X \subseteq Z$ be adapted and $\varphi\colon Z' \to Z$ an adapted toric ambient modification. A *toric canonical φ-family* is a family $(U_i, D_i)_{i\in I}$, where the $U_i \subseteq Z'$ are toric open subsets covering Z' and the D_i are T'-invariant Weil divisors on Z' such that for every $i \in I$ the following holds:

(i) $D_i|_{X'}$ is a canonical divisor on X',

(ii) on U_i we have $D_i = k_{Z'}$,

(iii) the T-invariant divisor $\varphi_*(D_i)$ is $\mathbb{Q}$-Cartier.

Remark 4.2.9. Let $X \subseteq Z$ be adapted, $\varphi\colon Z' \to Z$ an adapted toric modification and $(U_i, D_i)_{i\in I}$ a toric canonical φ-family. Then, by refining, we can achieve that $I = \Sigma'$ holds and the $U_i = Z'_{\sigma'}$ are the affine toric charts of Z'.

Let $u \in M_{\mathbb{Q}}$ be a rational character. Then the multiplicity of $\operatorname{div}(\chi^u)$ along the divisor D_Z^{ϱ} corresponding to a ray $\varrho \in \Sigma$ is given as $\langle u, v_\varrho\rangle$, where, as usual, $v_\varrho \in \varrho$ denotes the primitive lattice vector inside ϱ.

Construction 4.2.10. Let $X \subseteq Z$ be adapted, $\varphi\colon Z' \to Z$ an adapted weakly tropical resolution and $(Z'_{\sigma'}, D_{\sigma'})_{\sigma'\in\Sigma'}$ a toric canonical φ-family. For every $\sigma' \in \Sigma'$ choose a $\sigma \in \Sigma$ with $\sigma' \subseteq \sigma$ and a $u_{\sigma'} \in M_{\mathbb{Q}}$ with $\varphi_* D_{\sigma'} = \operatorname{div}(\chi^{u_{\sigma'}})$ on $Z_\sigma \subseteq Z$. Set

$$\mathcal{A} \ := \ \bigcup_{\sigma'\in\Sigma'} A_{\sigma'}, \qquad\qquad A_{\sigma'} \ := \ \sigma' \cap \{v \in N_{\mathbb{Q}};\ \langle u_{\sigma'}, v\rangle \ge -1\}.$$

Then $\mathcal{A}$ admits the structure of a polyhedral complex in $N_{\mathbb{Q}}$ by defining the cells to be the faces of the polyhedra $A_{\sigma'} \subseteq N_{\mathbb{Q}}$.

Remark 4.2.11. In the situation of Construction 4.2.10, consider a cone $\sigma' \in \Sigma'$, a ray $\varrho \preceq \sigma'$, the corresponding toric prime divisor $D_{Z'}^{\varrho}$ and $D_{X'}^{\varrho} := D_{Z'}^{\varrho}|_{X'}$. Then we have

$$\varphi^*\varphi_*(D_{\sigma'}|_{X'}) = \varphi^*((\varphi_* D_{\sigma'})|_X) = (\varphi^*\varphi_* D_{\sigma'})|_{X'},$$

where the first equality is due to the commutative diagram given in Remark 4.2.7 and the second follows by direct calculation in charts. In particular, the discrepancy of $D_{X'}^{\varrho}$ with respect to X is given by

$$\operatorname{discr}_X(D_{X'}^{\varrho}) \ = \ -1 - \langle u_{\sigma'}, v_\varrho\rangle,$$

as the r.h.s. is the multiplicity of $D_{\sigma'} - \varphi^*\varphi_* D_{\sigma'}$ along $D_{Z'}^{\varrho}$ for any $\sigma' \in \Sigma'$ with $\varrho \preceq \sigma'$. In particular, we conclude that the defining inequalities $u_{\sigma'} \ge -1$ of $\mathcal{A}$ and thus the whole set $\mathcal{A}$ do not depend on the choice of the toric canonical φ-family.

Definition 4.2.12. Let $X \subseteq Z$ be adapted, $\varphi\colon Z' \to Z$ an adapted weakly tropical resolution and consider a proper adapted toric ambient modification $\psi\colon Z'' \to Z'$. A *toric canonical ψ-family over Z* is a toric canonical $(\varphi \circ \psi)$-family $(V_i, C_i)_{i\in I}$ such that $V_i = \psi^{-1}(U_i)$ holds with toric open subsets $U_i \subseteq Z'$ and $(U_i, \psi_* C_i)_{i\in I}$ is a toric canonical φ-family.

Proposition 4.2.13. *Situation as in Construction 4.2.10. Let $\psi\colon Z'' \to Z'$ be a proper adapted toric ambient modification admitting a toric canonical ψ-family over Z and denote by $X'' \subseteq Z''$ the proper transform of $X \subseteq Z$. Let $\varrho \in \Sigma''$ be a ray and, provided ϱ intersects the boundary of $\mathcal{A}$, denote by v'_ϱ the intersection point. Then the discrepancy a_ϱ of X with respect to the divisor $D^\varrho_{X''} = D^\varrho_{Z''}|_{X''}$ satisfies*

$$a_\varrho \;=\; \frac{||v_\varrho||}{||v'_\varrho||} - 1, \quad \text{if } \varrho \not\subseteq \mathcal{A}, \qquad\qquad a_\varrho \le -1, \quad \text{if } \varrho \subseteq \mathcal{A}.$$

Proof. Denote the toric canonical ψ-family over Z by $(V_i, C_i)_{i\in I}$. Refining if necessary, we achieve $I = \Sigma'$ and $V_{\sigma'} = Z''_{\sigma'} := \psi^{-1}(Z'_{\sigma'})$. By Remark 4.2.11, we may assume $D_{\sigma'} = \psi_* C_{\sigma'}$ for constructing the polyhedral complex $\mathcal{A}$ according to 4.2.10. Now choose $\sigma' \in \Sigma'$ and $\sigma \in \Sigma$ with $\varrho \subseteq \sigma' \subseteq \sigma$. Moreover, let $u_{\sigma'} \in M_{\mathbb{Q}}$ with $\varphi_* D_{\sigma'} = \mathrm{div}(\chi^{u_{\sigma'}})$ on $Z_\sigma \subseteq Z$. Set $\pi := \varphi \circ \psi$. Then, on $Z''_{\sigma'}$, we have

$$C_{\sigma'} - \pi^*\pi_* C_{\sigma'} \;=\; \sum_{\eta\subseteq\sigma'} -D^\eta_{Z''} - \pi^*\mathrm{div}(\chi^{u_{\sigma'}}) \;=\; \sum_{\eta\subseteq\sigma'} (-1 - \langle u_{\sigma'}, v_\eta\rangle) D^\eta_{Z''},$$

where η runs over the rays of Σ'' that lie in the cone σ'. Thus, in particular, our ϱ occurs among the η. Now, applying the pullback homomorphism $D \mapsto D|_{X''}$ to these identities gives the ramification formula for a canonical divisor on X''. Thus, if $\varrho \not\subseteq \mathcal{A}$ holds, then we obtain

$$a_\varrho \;=\; -1 - \langle u_{\sigma'}, v_\varrho\rangle \;=\; -1 - \frac{||v_\varrho||}{||v'_\varrho||}\langle u_{\sigma'}, v'_\varrho\rangle \;=\; -1 + \frac{||v_\varrho||}{||v'_\varrho||},$$

using $\langle u_{\sigma'}, v'_\varrho\rangle = -1$, which just rephrases that v'_ϱ lies on the bounding hyperplane $u_{\sigma'} = -1$ of $\mathcal{A}$. If $\varrho \subseteq \mathcal{A}$ holds, then we have $\langle u_{\sigma'}, v\rangle \ge -1$ even for all $v \in \varrho$. □

Let $X \subseteq Z$ be adapted with adapted weakly tropical resolution $Z' \to Z$. The above result shows that the polyhedral complex $\mathcal{A}$ as in Construction 4.2.10 is the anticanonical complex of $X \subseteq Z$ if every proper birational toric morphism $\psi\colon Z'' \to Z'$ is an adapted toric ambient modification and admits a canonical toric ψ-family over Z. We show that any subvariety $X \subseteq Z$ meeting the assumptions of Proposition 4.2.6 fulfills this condition.

Lemma 4.2.14. *Let $X \subseteq Z$ be an adapted embedding admitting a semi-locally toric weakly tropical resolution $\varphi\colon Z' \to Z$. Then φ is an adapted toric ambient modification. Moreover, if $X \subseteq Z$ is a neat embedding, then $X' \subseteq Z'$ is neat.*

Proof. Note that the complement $X' \setminus (X' \cap Z'_0)$ lies in the union of all T'-orbits of Z' of codimension at least two. As $X' \subseteq Z'$ is weakly tropical, Tevelev's criterion implies that $X' \cap Z'_0$ is of codimension at least two in X'. Thus we have a well defined pullback homomorphism $\mathrm{WDiv}^{T'}(Z') \to \mathrm{WDiv}(X')$.

Now let $D^{\varrho}_{Z'}$ be a T'-invariant prime divisor on Z' corresponding to a ray $\varrho \in \Sigma'$. We claim that the pullback of this divisor is a prime divisor on X'. In order to prove this we may restrict $D^{\varrho}_{Z'}$ to the toric chart $Z'_{\varrho} \cong U(\varrho) \times \tilde{\mathbb{T}}$ as the complement of $X' \cap Z'_0$ is of codimension at least two in X'. Note that the restriction $D^{\varrho}_{Z'}|_{Z'_{\varrho}}$ equals the pullback of the T'-invariant divisor $D^{\varrho}_{U(\varrho)}$ on $U(\varrho)$ with respect to the projection. As $U(\varrho)$ and its preimage under the projection $U(\varrho) \times \tilde{\mathbb{T}}$ are smooth, the pullback of $D^{\varrho}_{Z'}$ to X' equals the intersection of $D^{\varrho}_{U(\varrho)}$ with X_{ϱ} inside $U(\varrho)$ and thus is a prime divisor.

For the supplement let $X \subseteq Z$ be neat. In order to prove that the pullback induces an isomorphism of divisor class groups, we may assume $Z = Z_0$ and $Z' = Z'_0$ due to the adaptedness of $X \subseteq Z$ and $X' \subseteq Z'$. In particular, we have a proper toric morphism $Z' \to Z$. Let $E_1, \ldots, E_r$ be the T'-invariant prime divisors in the exceptional locus of $Z' \to Z$. As the embedding $X' \subseteq Z'$ is weakly tropical and adapted we conclude that the prime divisors in the exceptional locus of $X' \to X$ are exactly the pullbacks $E_1|_{X'}, \ldots, E_r|_{X'}$, where we use Tevelev's criterion to show that these are indeed all. Note that these divisors generate free subgroups of rank r in $\mathrm{Cl}(Z')$ and $\mathrm{Cl}(X')$ respectively. Thus we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus \mathbb{Z} \cdot [E_i] & \longrightarrow & \mathrm{Cl}(Z') & \xrightarrow{\tilde{\varphi}_*} & \mathrm{Cl}(Z) & \longrightarrow & 0 \\
 & & \cong \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & \bigoplus \mathbb{Z} \cdot [E_i|_{X'}] & \longrightarrow & \mathrm{Cl}(X') & \xrightarrow[\varphi_*]{} & \mathrm{Cl}(X) & \longrightarrow & 0,
\end{array}$$

where the downward arrows are the pullback homomorphisms, and $\tilde{\varphi}_*$ and φ_* denote the canonical push forward homomorphisms. Applying the Five Lemma we obtain that the pullback homomorphisms $\mathrm{Cl}(Z') \to \mathrm{Cl}(X')$ induced by the embedding $X' \subseteq Z'$ is an isomorphism. □

Lemma 4.2.15. *Let $X \subseteq Z$ be a neat, weakly tropical embedding and let $U \subseteq Z$ be an open T-invariant subvariety. Then the pullback homomorphism $\mathrm{Cl}(U) \to \mathrm{Cl}(X \cap U)$ induced by the embedding $X \cap U \subseteq U$ is an isomorphism.*

Proof. As U is a T-invariant subset of Z and the embedding $X \subseteq Z$ is neat, we have

$$Z \setminus U = D^{\varrho_1}_Z \cup \ldots \cup D^{\varrho_r}_Z \cup B \quad \text{and} \quad X \setminus (X \cap U) = D^{\varrho_1}_X \cup \ldots \cup D^{\varrho_r}_X \cup (B \cap X)$$

with T-invariant prime divisors $D^{\varrho_i}_Z \subseteq Z$ and a T-invariant closed subset $B \subseteq Z$ of codimension at least two. Moreover, as $X \subseteq Z$ is weakly tropical, we have $\mathrm{codim}_X(B \cap X) \geq 2$. Thus the following commutative diagram with exact rows gives the assertion:

$$\begin{array}{ccccccc}
\mathbb{Z}^r & \xrightarrow{e_i \mapsto [D^{\varrho_i}_Z]} & \mathrm{Cl}(Z) & \longrightarrow & \mathrm{Cl}(U) & \longrightarrow & 0 \\
\| & & \downarrow \cong & & \downarrow & & \\
\mathbb{Z}^r & \xrightarrow[e_i \mapsto [D^{\varrho_i}_X]]{} & \mathrm{Cl}(X) & \longrightarrow & \mathrm{Cl}(X \cap U) & \longrightarrow & 0.
\end{array}$$

□

Lemma 4.2.16. *Let $X \subseteq Z$ be a neat embedding and let $\varphi\colon Z' \to Z$ be a semi-locally toric weakly tropical resolution such that there exists a T-invariant $\mathbb{Q}$-Cartier divisor D with $D|_X$ a canonical divisor on X. Consider a proper toric morphism $\psi\colon Z'' \to Z'$ defined by a subdivision of fans $\Sigma'' \to \Sigma'$. Then the following statements hold:*

(i) *The embedding $X'' \subseteq Z''$ is neat.*

(ii) *$\psi\colon Z'' \to Z'$ is an adapted toric ambient modification and there exists a toric canonical ψ-family over Z.*

Proof. We prove (i). Note that by sufficiently refining the quasifan structure on $\mathrm{trop}(X)$ we may assume that the toric morphism $\varphi \circ \psi\colon Z'' \to Z$ arises from a refinement of fans $\Sigma \sqcap \mathrm{trop}(X) \to \Sigma$. It thus defines a weakly tropical resolution $Z'' \to Z$ of X which is semi-locally toric according to Lemma 4.1.9. Applying Lemma 4.2.14 we conclude that $X'' \subseteq Z''$ is a neat embedding.

We come to (ii). As $\psi\colon Z'' \to Z'$ is a proper morphism, so is its restriction $X'' \to X'$. In particular, using (i) we obtain that ψ is an adapted toric ambient modification and there exists a T''-invariant divisor D on Z'' whose pullback $D|_{X''}$ is a canonical divisor on X''. We proceed the proof by constructing a toric canonical ψ-family over Z. Let $\sigma' \in \Sigma'$ be any cone. Then, in the notation of Lemma 4.1.9, the projection of $X'' \cap Z''_{\sigma'}$ into $V(\sigma')$ is an open subset of $V(\sigma')$. Thus there exists a canonical divisor on X'' that equals $k_{Z''}|_{X''}$ on $X'' \cap Z''_{\sigma'}$. Applying Lemma 4.2.15 we obtain $[k_{Z''}|_{Z''_{\sigma'}}] = [D|_{Z''_{\sigma'}}] \in \mathrm{Cl}(Z''_{\sigma'})$ and thus on $Z''_{\sigma'}$ we have

$$D = k_{Z''} + \mathrm{div}(\chi^u)$$

with a character χ^u of T''. Setting $C_{\sigma'} := D - \mathrm{div}(\chi^u)$ we obtain a toric canonical $(\varphi \circ \psi)$-family $(Z''_{\sigma'}, C_{\sigma'})_{\sigma' \in \Sigma'}$. Due to Lemma 4.2.14 the morphism φ is an adapted toric ambient modification. Moreover, by construction $Z''_{\sigma'} = \psi^{-1}(Z'_{\sigma'})$ holds and the family $(Z'_{\sigma'}, \psi_* C_{\sigma'})$ is a toric canonical φ-family. This proves that $(Z''_{\sigma'}, C_{\sigma'})_{\sigma' \in \Sigma'}$ is a toric canonical ψ-family over Z. □

Proof of Proposition 4.2.6. Let $\varphi\colon Z' \to Z$ be any semi-locally toric weakly tropical resolution. Then φ is an adapted toric ambient modification due to Lemma 4.2.14. Now let $\psi\colon Z'' \to Z'$ be any proper birational toric morphism. Then due to Lemma 4.2.16 (ii) it is an adapted toric ambient modification and admits a canonical toric ψ-family over Z. Moreover, Proposition 4.1.6 ensures that there exists at least one proper birational toric morphism that induces a resolution of singularities. Now, using Proposition 4.2.13 we conclude that the support of the polyhedral complex $\mathcal{A}$ as constructed in 4.2.10 is the anticanonical region of $X \subseteq Z$. This completes the proof. □

4.3 The quotient criterion for explicit $\mathbb{T}$-varieties

In this section we apply our results from Section 4.2 to explicit varieties $X \subseteq Z$. As a first result we obtain a criterion for the existence of anticanonical complexes in the $\mathbb{Q}$-Gorenstein case, see Corollary 4.3.2. Our main result concerns explicit $\mathbb{T}$-varieties $X(\alpha, P, \Sigma) \subseteq Z$. For these, we can deduce the existence of an anticanonical complex from an explicit maximal orbit quotient, see Theorem 4.3.6 and Construction 4.3.3.

Remark 4.3.1. Let $X \subseteq Z$ be an explicit variety. Then the embedding $X \subseteq Z$ is neat, divisor class group, Picard group and Cox ring of Z are given as

$$\mathrm{Cl}(Z) \cong \mathrm{Cl}(X), \qquad \mathrm{Pic}(Z) \cong \mathrm{Pic}(X), \qquad \mathcal{R}(Z) = \mathbb{K}[T_\varrho;\ \varrho \in \Sigma^{(1)}],$$

where $\Sigma^{(1)}$ denotes the set of rays of the fan Σ defining the toric variety Z. Moreover, the ample divisor classes of X and Z coincide under the isomorphism.

Using the neat embedding of an explicit variety $X \subseteq Z$ we can directly deduce the following corollary from Proposition 4.2.6:

Corollary 4.3.2. *Let $X \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit variety and assume there exists a semi-locally toric weakly tropical resolution. Then $X \subseteq Z$ admits an anticanonical complex.*

Now let us furthermore assume that the explicit variety $X \subseteq Z$ under consideration is endowed with an effective action of an algebraic torus $\mathbb{T}$. In particular, we work with explicit $\mathbb{T}$-varieties $X(\alpha, P, \Sigma) \subseteq Z$ from Chapter 1.

Construction 4.3.3. Let $X(\alpha, P, \Sigma) \subseteq Z$ be an explicit $\mathbb{T}$-variety. Denote by N the lattice of one-parameter subgroups of the acting torus T on Z and let Σ be the defining fan of Z. Denote by $N_{\mathbb{T}}$ the sublattice in N corresponding to $\mathbb{T} \subseteq T$. Set $N' := N/N_{\mathbb{T}}$, let $P_1 \colon N \to N'$ be the projection and $\pi_1 \colon T \to T'$ the associated homomorphism of tori. Set

$$\Delta_0 := \left\{P_1(\varrho);\ \varrho \in \Sigma^{(1)}\right\} \cup \{0\}$$

and let Y_0 be the closure of $\pi_1(X \cap T)$ in the toric variety Z_{Δ_0} corresponding to the fan Δ_0. Then π_1 defines a rational quotient $X \dashrightarrow Y_0$, i.e. a dominant rational map such that $\pi_1^*\mathbb{K}(Y_0) = \mathbb{K}(X)^{\mathbb{T}}$ holds. Now let Δ be any fan having the same rays as Δ_0 and let $Y \subseteq Z_\Delta$ be the closure of Y_0. We call the rational map $X \dashrightarrow Y$ an *explicit maximal orbit quotient* for the explicit $\mathbb{T}$-variety $X(\alpha, P, \Sigma) \subseteq Z$.

Remark 4.3.4. Explicit maximal orbit quotients are indeed maximal orbit quotients as in Definition 1.2.13, compare Proposition 1.2.17.

Remark 4.3.5. In the situation of Construction 4.3.3, the linear map $P_1 \colon N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$ maps $\mathrm{trop}(X)$ onto $\mathrm{trop}(Y)$. In particular, we have $\mathrm{trop}(X) = \mathrm{trop}(Y) \oplus \ker(P_1)$, see [61, Cor. 6.2.15].

We now come to the main result of this Section. Starting with an explicit $\mathbb{T}$-variety $X := X(\alpha, P, \Sigma) \subseteq Z$ explicit maximal orbit quotients $X \dashrightarrow Y$ as in Construction 4.3.3 provide the possibility to check, if the variety $X \subseteq Z$ admits an anticanonical complex by studying the lower dimensional variety Y:

Theorem 4.3.6. *In the notation of Construction 4.3.3, let $X := X(\alpha, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit $\mathbb{T}$-variety and $X \dashrightarrow Y$ an explicit maximal orbit quotient such that*

(i) *$Y \subseteq Z_\Delta$ admits a semi-locally toric weakly tropical resolution,*

(ii) *P_1 maps $|\Sigma \sqcap \mathrm{trop}(X)|$ into $|\Delta \sqcap \mathrm{trop}(Y)|$.*

Then $X \subseteq Z$ admits an anticanonical complex.

Proof. Let $Z_{\Delta'} \to Z_\Delta$ be a semi-locally toric weakly tropical resolution of Y. In particular, we have $\Delta' = \Delta \sqcap \mathrm{trop}(Y)$ for a fixed quasifan structure on $\mathrm{trop}(Y)$ and by refining we achieve that Δ' is a subfan of $\mathrm{trop}(Y)$. Consider the quasifan structure $\left\{P_1^{-1}(\sigma);\ \sigma \in \mathrm{trop}(Y)\right\}$ on $\mathrm{trop}(X)$ and let $\sigma' \in \Sigma' = \Sigma \sqcap \mathrm{trop}(X)$ be any cone. We claim that $X'_{\sigma'}$ is semi-locally toric.

Assume $\delta' := P_1(\sigma') \in \Delta'$ holds. Note that due to Remark 4.1.5 we can identify N' with a sublattice of N and Y_0 with its image in Z under the toric morphism defined by the morphism $N' \to N$. As Y' is semi-locally toric, there exists a maximal cone $\tau \in \mathrm{trop}(Y)$ and a decomposition $N' = N'(\delta') \oplus \tilde{N}'$ such that the corresponding projection $\pi_{\delta'}$ maps $Y_{\delta'}$ isomorphically onto and open subset of $U(\delta')$. As $P_1^{-1}(\tau)$ is a maximal cone in $\mathrm{trop}(X)$ containing σ', any maximal cone of a refined quasifan structure on $\mathrm{trop}(X)$ spans the same linear subspace. Therefore, we can choose $N(\sigma') = N_{\mathbb{T}} \oplus N'(\delta')$. In particular, choosing the decomposition $N = N(\sigma') \oplus \tilde{N}'$ we obtain a commutative diagram

$$\begin{array}{ccccc} Z_{\sigma'} & \cong & U(\sigma') \times \tilde{\mathbb{T}} & \xrightarrow{\pi_{\sigma'}} & U(\sigma') \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \psi \times \mathrm{id}} & & \big\downarrow{\scriptstyle \psi} \\ Z_{\delta'} & \cong & U(\delta') \times \tilde{\mathbb{T}} & \xrightarrow{\pi_{\delta'}} & U(\delta'), \end{array}$$

where ψ is the morphism of affine toric varieties arising via the projection of lattices $N(\sigma') \to N'(\delta')$ mapping σ' onto δ'. As $\pi_{\delta'}$ maps $Y_{\delta'}$ isomorphically onto its image we conclude that the projection $\pi_{\sigma'}$ maps $\pi_1^{-1}(Y_{\delta'})$ isomorphically onto the open subset $\psi^{-1}(\pi_{\delta'}(Y_{\delta'})) \subseteq U(\sigma')$. We claim that $X'_{\sigma'}$ equals $\pi_1^{-1}(Y_{\delta'})$: As $\psi^{-1}(\pi_{\delta'}(Y_{\delta'}))$ is irreducible, so is $\pi_1^{-1}(Y_{\delta'})$. Thus $X'_{\sigma'} \subseteq \pi_1^{-1}(Y_{\delta'})$ is a closed irreducible subvariety of the same dimension and thus equality holds.

In order to conclude the proof it is only left to show that for any $\sigma' \in \Sigma'$ we can achieve $P_1(\sigma') \in \Delta'$ by sufficiently refining the quasifan structure on $\mathrm{trop}(Y)$. By construction of Σ' we have $P_1(\sigma') \subseteq \delta$ for some $\delta \in \Delta'$. Consider any complete fan Δ^c with $P_1(\sigma') \in \Delta^c$.

Then $\mathrm{trop}(Y) \sqcap \Delta^c$ defines a refined fan structure on $\mathrm{trop}(Y)$ that contains $P_1(\sigma')$ and we set $\Delta'' := \mathrm{trop}(Y) \sqcap \Delta^c \sqcap \Delta'$. Now using Lemma 4.1.9 we conclude that the proper transform Y'' with respect to the morphism $Z_{\Delta''} \to Z_{\Delta'}$ corresponding to the refinement $\Delta'' \to \Delta'$ is semi-locally toric as Y' is so. Thus by the above considerations we obtain that $X'_{\sigma'}$ is semi-locally toric and the assertion follows with Corollary 4.3.2 □

The proof of Theorem 4.3.6 provides indeed the following explicit way to construct a semi-locally toric weakly tropical resolution.

Remark 4.3.7. Let $X(\alpha, P, \Sigma) \subseteq Z$ and $Y \subseteq Z_\Delta$ be as in Theorem 4.3.6 and let $Z_{\Delta'} \to Z_\Delta$ be the semi-locally toric weakly tropical resolution of Y. Fix any quasifan structure on $\mathrm{trop}(Y)$ having Δ' as a subfan and endow $\mathrm{trop}(X)$ with the quasifan structure defined by the cones $P_1^{-1}(\tau)$ with $\tau \in \mathrm{trop}(Y)$. Then the refinement of fans $\mathrm{trop}(X) \sqcap \Sigma \to \Sigma$ defines a semi-locally toric weakly tropical resolution $Z' \to Z$ of X.

Corollary 4.3.8. *Let $X \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit $\mathbb{T}$-variety and let $X \dashrightarrow Y$ be an explicit maximal orbit quotient, such that Y is complete. Then $X \subseteq Z$ admits an anticanonical complex if $Y \subseteq Z_\Delta$ admits a semi-locally toric weakly tropical resolution.*

Proof. By construction Y is the closure of Y_0 in a toric variety Z_Δ, where Δ contains Δ_0 as a subfan. Moreover, as Y is complete, the support of the defining fan Δ contains $|\mathrm{trop}(Y)|$. As P_1 maps $|\mathrm{trop}(X)|$ into $|\mathrm{trop}(Y)|$, and Y admits by assumption a semi-locally toric weakly tropical resolution, we meet the conditions of Theorem 4.3.6 and the assertion follows. □

Note that in the case that $X(\alpha, P, \Sigma) \subseteq Z$ is a complete explicit $\mathbb{T}$-variety we have $|\mathrm{trop}(X)| \subseteq |\Sigma|$. In particular, in this situation every variety Y fulfilling the conditions of Theorem 4.3.6 has to be complete as this is equivalent to $|\mathrm{trop}(Y)| \subseteq |\Delta|$.

We end this section by finally summarizing the results for explicit $\mathbb{T}$-varieties that are Mori dream spaces:

Corollary 4.3.9. *Let X be a $\mathbb{Q}$-Gorenstein Mori dream space with torus action having an explicit maximal orbit quotient $X \dashrightarrow Y$, where Y is complete and admits a semi-locally toric weakly tropical resolution. Then X admits an anticanonical complex $\mathcal{A}$ and the following statements hold:*

(i) *X has at most log terminal singularities if and only if the anticanonical complex $\mathcal{A}$ is bounded.*

(ii) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of $\mathcal{A}$.*

(iii) *X has at most terminal singularities if and only if 0 and the primitive generators of the rays of the defining fan of Z_X are the only lattice points of $\mathcal{A}$.*

Proof. We are in the situation of Corollary 4.3.8. Therefore, $X \subseteq Z$ admits an anticanonical complex and Remark 4.2.5 gives the characterizations of the singularity types. □

4.4 Application to general arrangement varieties

In this section we apply the quotient criterion from Section 4.3 to explicit general arrangement varieties. Our main result is the following theorem:

Theorem 4.4.1. *Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety. Then the weakly tropical resolution $Z' \to Z$ of X is semi-locally toric. In particular, if X is $\mathbb{Q}$-Gorenstein, then $X \subseteq Z$ admits an anticanonical complex.*

We begin by investigating the explicit maximal orbit quotient $X \dashrightarrow Y$ from Construction 4.3.3 for general arrangement varieties:

Construction 4.4.2. Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety. Then X is invariant under the subtorus action of $\mathbb{T}^s \subseteq \mathbb{T}^{r+s}$ on Z. Using Construction 4.3.3, the projection of lattices $P_1 \colon \mathbb{Z}^{r+s} \to \mathbb{Z}^r$ with the corresponding projection of tori $\pi_1 \colon \mathbb{T}^{r+s} \to \mathbb{T}^r$ give rise to a fan Δ_0 defining a toric variety Z_{Δ_0} and an explicit variety $Y_0 \subseteq Z_{\Delta_0}$:

$$\Delta_0 := \{P_1(\varrho);\ \varrho \in \Sigma^{(1)}\}, \qquad Y_0 := \overline{\pi_1(X \cap \mathbb{T}^{r+s})} \subseteq Z_{\Delta_0}.$$

Moreover, we obtain a commutative diagram of rational quotients:

$$\begin{array}{ccc} X & \hookrightarrow & Z \\ \downarrow & & \downarrow \\ Y_0 & \hookrightarrow & Z_{\Delta_0}. \end{array}$$

As above, let $a_0, \ldots, a_r$ denote the columns of A. Then the variety $Y_0 \cap \mathbb{T}^r$ is given as the vanishing set of the linear equations $h_1, \ldots, h_{r-c}$, where

$$h_t := \det \begin{bmatrix} a_0 & a_1 & \ldots & a_c & a_{c+t} \\ 1 & U_1 & \ldots & U_c & U_{c+t} \end{bmatrix} \in \mathbb{K}[U_1^{\pm}, \ldots, U_r^{\pm}].$$

Note that Δ_0 is the one-skeleton of the defining fan of $\mathbb{P}_r$. Moreover, the closure of $Y_0 \cap \mathbb{T}^r$ inside $\mathbb{P}_r$ is a linear subspace $\mathbb{P}_c \subseteq \mathbb{P}_r$ and the equations of this embedding are given via the kernel of the matrix $A = (a_0, \ldots, a_r)$. Note that $X \dashrightarrow \mathbb{P}_c$ defines an explicit maximal orbit quotient as in Construction 4.3.3.

Remark 4.4.3. In the situation of Construction 4.4.2 the tropical variety of Y_0 is the c-skeleton of the fan of $\mathbb{P}_r$, i.e.

$$\operatorname{trop}(Y_0 \cap \mathbb{T}^r) = \Sigma_{\mathbb{P}^r}^{\leq c} := \{\sigma \in \Sigma_{\mathbb{P}^r};\ \dim(\sigma) \leq c\}.$$

Using Remark 4.3.5 we conclude $|\operatorname{trop}(X)| = |\Sigma_{\mathbb{P}_r}^{\leq c}| \times \mathbb{Q}^s$. In the following, if not specified otherwise, we will always assume $\operatorname{trop}(X)$ to be endowed with the quasifan structure defined by the product $\Sigma_{\mathbb{P}_r}^{\leq c} \times \mathbb{Q}^s$.

Example 4.4.4. Consider the explicit general arrangement variety $X := X(A, P, \Sigma) \subseteq Z$ from Example 2.1.17. For completeness we recall the defining data, i.e. the matrices A and P and the maximal cones of Σ:

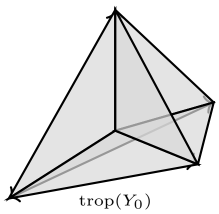
$$A := \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad P := \begin{bmatrix} -1 & -2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 & 4 \\ -1 & -3 & 1 & 1 & 1 \end{bmatrix},$$

$$\Sigma^{\max} := \left\{ \begin{array}{c} \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_{31}), \quad \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_{31}), \quad \mathrm{cone}(v_{01}, v_{02}, v_{11}), \\ \mathrm{cone}(v_{01}, v_{02}, v_{21}), \quad \mathrm{cone}(v_{01}, v_{02}, v_{31}) \end{array} \right\}$$

In particular, the variety X is a Gorenstein Fano variety, having dimension three and Picard number one. The $\mathbb{T}$-action on X is of complexity two and arises as a subtorus action $\mathbb{K}^* \subseteq \mathbb{T}^4$ acting on Z. Using the projection of tori $\mathbb{T}^{3+1} \to \mathbb{T}^3$, we obtain a rational quotient $X \dashrightarrow \mathbb{P}_2$, where

$$\mathbb{P}_2 \cap \mathbb{T}^3 \cong \mathrm{V}_{\mathbb{T}_3}(1 + U_1 + U_2 + U_3) \subseteq \mathbb{T}^3$$

and the tropical variety of X is given as $\mathrm{trop}(\mathbb{P}_2 \cap \mathbb{T}^3) \times \mathbb{Q} = \Sigma_{\mathbb{P}_3}^{\leq 2} \times \mathbb{Q}$.



Note that $\mathrm{trop}(\mathbb{P}_2) = \Sigma_{\mathbb{P}_3}^{\leq 2}$ is a subfan of $\Sigma_{\mathbb{P}_3}$. Thus the weakly tropical resolution of $\mathbb{P}_2 \subseteq \mathbb{P}_3$ is the identity. We look at the affine toric chart defined by the cone $\sigma = \mathrm{cone}(e_1, e_2) \subseteq \mathbb{Q}^3$. We obtain

$$(\mathbb{P}_2)_\sigma \subseteq \mathbb{K}^2 \times \mathbb{K}^* \qquad \text{and} \qquad (\mathbb{P}_2)_\sigma \cap \mathbb{T}^3 = V(1 + U_1 + U_2 + U_3).$$

Now, any point $x = (x_1, x_2, x_3) \in (\mathbb{P}_2)_\sigma$ is defined by its first two coordinates, as $x_3 = -x_1 - x_2 - 1 \neq 0$ holds. Thus the projection $\mathbb{K}^2 \times \mathbb{K}^* \to \mathbb{K}^2$ maps $(\mathbb{P}_2)_\sigma$ onto the open set $\mathbb{K}^2 \setminus V(-x_1 - x_2 - 1)$. Computing this for the other cones in $\Sigma_{\mathbb{P}_3}^{\leq 2}$ shows that $\mathbb{P}_2 \subseteq \mathbb{P}_3$ is indeed semi-locally toric. Using the quotient criterion Theorem 4.3.6 yields that $X \subseteq Z$ admits an anticanonical complex. We will investigate the structure of this complex in the next section.

Let us now turn to the proof of Theorem 4.4.1.

Lemma 4.4.5. *Let $A = (a_0, \ldots, a_r)$ be a matrix as in Construction 2.1.3 and consider the linear subspace $\mathbb{P}_c \subseteq \mathbb{P}_r$ defined via the kernel of A, i.e. the vanishing set of the relations $f_1, \ldots, f_{r-c}$, where*

$$f_t := \det \begin{bmatrix} a_0 & a_1 & \ldots & a_c & a_{c+t} \\ U_0 & U_1 & \ldots & U_c & U_{c+t} \end{bmatrix} \in \mathbb{K}[U_0, \ldots, U_r].$$

Fix the fan structure $\Delta := \Sigma_{\mathbb{P}_r}^{\leq c}$ *on* $\mathrm{trop}(\mathbb{P}_c)$*. Then the weakly tropical resolution of* $\mathbb{P}_c \subseteq \mathbb{P}_r$ *is semi-locally toric.*

Proof. As the tropical variety of $\mathbb{P}_c \subseteq \mathbb{P}_r$ is a subfan of the fan of $\mathbb{P}_r$, Tevelev's criterion implies that the weakly tropical resolution is the identity on $\mathbb{P}_c$. Therefore, we only have to show that $\mathbb{P}_c \subseteq Z_\Delta$ is semi-locally toric. Let $\delta \in \Delta$ be a maximal cone. We consider the situation exemplarily for $\delta = \mathrm{cone}(e_1, \ldots, e_c)$, i.e. we have $(\mathbb{P}_c)_\delta \subseteq \mathbb{K}^c \times (\mathbb{K}^*)^{r-c}$. By construction $\mathbb{P}_c \cap \mathbb{T}^r$ is given as the vanishing set of the linear equations $h_1, \ldots, h_{r-c}$, where

$$h_t := \det \begin{bmatrix} a_0 & a_1 & \ldots & a_c & a_{c+t} \\ 1 & U_1 & \ldots & U_c & U_{c+t} \end{bmatrix} \in \mathbb{K}[U_1^{\pm}, \ldots, U_r^{\pm}].$$

Therefore, any point in $(\mathbb{P}_c)_\delta$ can be written as $(t, \eta_1(t), \ldots, \eta_{r-c}(t))$ where $t \in \mathbb{K}^c$ and the η_i are affine linear forms. This implies that $\mathbb{P}_c \subseteq \mathbb{P}_r$ is semi-locally toric as the projection π_δ maps $(\mathbb{P}_c)_\delta$ isomorphically onto the following open subset of $\mathbb{K}^c$:

$$\pi_\delta(Y_\delta) = \{t \in \mathbb{K}^c;\ \eta_i(t) \neq 0 \text{ for } 1 \leq i \leq r-c\}\,.$$

□

Proof of Theorem 4.4.1. We prove that any explicit general arrangement variety $X(A, P, \Sigma) \subseteq Z$ fulfills the conditions of Theorem 4.3.6. Due to Construction 4.4.2 the fan Δ_0 is a subfan of the defining fan of $\mathbb{P}_r$. In particular, in the notation of Theorem 4.3.6 we may choose $\Delta := \Sigma_{\mathbb{P}_r}$ and obtain $Y = \mathbb{P}_c$ as the closure of Y_0 in $\mathbb{P}_r$. As this embedding is defined via the kernel of A we can apply Lemma 4.4.5 and obtain that the embedding $\mathbb{P}_c \subseteq \mathbb{P}_r$ is weakly tropical and semi-locally toric. Thus it is only left to show that in this situation we meet condition (ii) of Theorem 4.3.6. This follows as $\mathbb{P}_c$ is complete and thus $|\Delta \sqcap \mathrm{trop}(Y)| = |\mathrm{trop}(Y)|$ holds. □

4.5 Explicit description for general arrangement varieties

In this section we give an explicit description of anticanonical complexes for general arrangement varieties $X := X(A, P, \Sigma) \subseteq Z$, see Proposition 4.5.4 and Corollary 4.5.5. After fixing a quasifan structure on $\mathrm{trop}(X)$ we investigate the fan of the weakly tropical resolution $\mathrm{trop}(X) \sqcap \Sigma$, see Proposition 4.5.9. In particular, we obtain in Corollary 4.5.12 that the weakly tropical resolution of an explicit general arrangement variety is again an explicit general arrangement variety. Applying our description of the anticanonical complexes and our characterization of the several singularity types, we prove Theorem 4.5.14, which gives first bounding conditions on the exponents l_{ij} occurring in the defining relations of the Cox ring of X. Specializing to torus actions of complexity two, we obtain concrete bounds for the exponents in the defining equations in the log terminal case, see Corollary 4.5.16.

In this section let $X := X(A,P,\Sigma) \subseteq Z$ always be an explicit general arrangement variety of complexity c and let $\mathrm{trop}(X) \subseteq \mathbb{Q}^{r+s}$ be its tropical variety endowed with the quasifan structure $\Sigma_{\mathbb{P}_r}^{\leq c} \times \mathbb{Q}^s$, given in Remark 4.4.3.

Construction 4.5.1. Denote by $e_1, \ldots, e_{r+s}$ the canonical basis vectors of $\mathbb{Q}^{r+s}$ and set $e_0 := -\sum e_i$. For any subset $I \subseteq \{0, \ldots, r\}$ of k indices we set

$$\lambda_I := \mathrm{cone}(e_i;\ i \in I) + \mathrm{lin}(e_{r+1}, \ldots, e_{r+s}).$$

If $1 \leq k \leq c$ holds, then we have $\lambda_I \in \mathrm{trop}(X)$ and we call λ_I a *k-leaf* or, not specifying k, a *leaf* of $\mathrm{trop}(X)$. Moreover, the collection of all leafs of $\mathrm{trop}(X)$ determines the *lineality space of* $\mathrm{trop}(X)$:

$$\lambda_{\mathrm{lin}} := \bigcap_{\substack{I \subseteq \{0,\ldots,r\}, \\ |I| \leq c}} \lambda_I.$$

Definition 4.5.2. In the notation of Construction 4.5.1 we say that

(i) a cone $\sigma \in \Sigma$ is a *leaf cone*, if $\sigma \subseteq \lambda_I$ holds for a leaf λ_I of $\mathrm{trop}(X)$.

(ii) a cone $\sigma \in \Sigma$ is called *big*, if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for all 1-leaves λ_i of $\mathrm{trop}(X)$.

Note that any cone $\sigma \in \Sigma$ is either a big or a leaf cone, see Proposition 2.2.8. In particular, an explicit general arrangement variety $X(A,P,\Sigma) \subseteq Z$ is weakly tropical if and only if Σ consists of leaf cones.

Construction 4.5.3. Denote by $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ the columns of P. Consider a pointed cone of the form

$$\sigma \ = \ \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r}) \ \subseteq \ \mathbb{Q}^{r+s},$$

that means that σ contains exactly one vector v_{ij} for every $i = 0, \ldots, r$. We call such a cone σ a *P-elementary cone* and associate to it the following numbers

$$\ell_{\sigma,i} \ := \ \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \ldots, r, \qquad \ell_\sigma \ := \ (c-r) l_{0j_0} \cdots l_{rj_r} + \sum_{i=0}^{r} \ell_{\sigma,i}$$

Moreover, we set

$$v_\sigma \ := \ \ell_{\sigma,0} v_{0j_0} + \ldots + \ell_{\sigma,r} v_{rj_r} \ \in \ \mathbb{Z}^{r+s}, \qquad \varrho_\sigma \ := \ \mathbb{Q}_{\geq 0} \cdot v_\sigma \ \in \ \mathbb{Q}^{r+s},$$

and denote by c_σ the greatest common divisor of the entries of v_σ.

Recall that, if X is $\mathbb{Q}$-Gorenstein with weakly tropical resolution $Z' \to Z$, then $X' \subseteq Z'$ is semi-locally toric due to Theorem 4.4.1. Therefore, X admits an anticanonical complex

$\mathcal{A}$ as provided by Construction 4.2.10, which is locally defined by linear forms $u_{\sigma'} \in M_{\mathbb{Q}}$. More precisely we have

$$A_{\sigma'} = \mathcal{A} \cap \sigma' = \sigma' \cap \{v \in N_{\mathbb{Q}};\ \langle u_{\sigma'}, v\rangle \geq -1\}.$$

We call any $u \in M_{\mathbb{Q}}$ fulfilling the above equation a *defining linear form* for $A_{\sigma'}$. In the following we fix the polyhedral complex structure defined by the polyhedra $A_{\sigma'}$ and call a point $x \in \mathcal{A}$ a *vertex of* $\mathcal{A}$ if it is a vertex of one of the polyhedra $A_{\sigma'}$.

The following proposition gives a description of the linear forms $u_{\sigma'}$ and thus the anti-canonical complex $\mathcal{A}$ in terms of the numbers defined above.

Proposition 4.5.4. *Let $X(A,P,\Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit general arrangement variety. Then any $u \in M_{\mathbb{Q}}$ is a defining linear form for $A_{\sigma'}$ if and only if it fulfills the following conditions:*

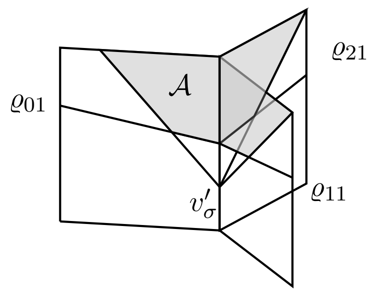
$$\langle u_{\sigma'}, v\rangle = \begin{cases} -1, & \text{if } v = v_\varrho, \text{ where } \varrho \in (\sigma')^{(1)} \cap \Sigma^{(1)}. \\ -\ell_\sigma, & \text{if } v = v_\sigma, \text{ where } \sigma \in \Sigma \text{ is a } P\text{-elementary cone with } \varrho_\sigma \preceq \sigma'. \end{cases}$$

Corollary 4.5.5. *Let $X := X(A,P,\Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein general arrangement variety. Then the vertices of the anticanonical complex of $X \subseteq Z$ are the origin, the primitive ray generators of Σ and the points $v'_\sigma := \ell_\sigma^{-1} v_\sigma$, where $\sigma \in \Sigma$ is a P-elementary cone and $\ell_\sigma > 0$ holds. Moreover, if $\ell_\sigma > 0$ holds for all P-elementary cones $\sigma \in \Sigma$, then X is log terminal and each polyhedron $A_{\sigma'}$ is a polytope and therefore determined by the above vertices.*

Example 4.5.6. Consider the affine explicit general arrangement variety $X := X(A,P,\Sigma) \subseteq Z$ where A and P are as follows and Σ is defined by the maximal cone σ:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} -3 & 4 & 0 \\ -3 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \sigma = \operatorname{cone}(v_{01}, v_{11}, v_{21}).$$

Then $\ell_\sigma = -8$ holds, we have $v'_\sigma = (0,0,-5)$ holds and the anticanonical complex is not bounded. In particular, X is not log terminal:



Example 4.5.7. Consider the variety $X := X(A, P, \Sigma)$ from Example 4.4.4. We use Proposition 4.5.4 and Corollary 4.5.5 to compute the anticanonical complex $\mathcal{A}$ of X: Its vertices are given by the columns $v_{01}, v_{02}, v_{11}, v_{21}, v_{31}$ of P and the points in the lineality space

$$v_{\mathrm{lin}1} = [0, 0, 0, 1/5], \qquad v_{\mathrm{lin}2} = [0, 0, 0, -1/3].$$

The anticanonical complex $\mathcal{A}$ of X has the following 15 maximal polytopes:

$$\mathrm{conv}(0, v_{01}, v_{02}, v_{i1}),\ \mathrm{conv}(0, v_{01}, v_{i1}, v_{\mathrm{lin}1}),\ \mathrm{conv}(0, v_{02}, v_{i1}, v_{\mathrm{lin}2}),\ 1 \le i \le 3,$$

$$\mathrm{conv}(0, v_{i1}, v_{j1}, v_{\mathrm{lin}1}),\ \mathrm{conv}(0, v_{i1}, v_{j1}, v_{\mathrm{lin}2}),\ 1 \le i < j \le 3.$$

Besides the origin and the primitive ray generators of Σ the anticanonical complex $\mathcal{A}$ of X contains precisely the following lattice points:

$$[0, 0, 1, 0], \quad [1, 1, 0, 1], \quad [1, 0, 2, 1], \quad [0, 1, 2, 1],$$

$$[0, -1, -1, -1], \quad [-1, 0, -1, -1], \quad [-1, -1, 0, -1], \quad [-1, -1, 1, -1].$$

It turns out that $[0, 0, 0, 0]$ is the only lattice point in the relative interior of $\mathcal{A}$ and therefore X is a canonical Gorenstein Fano explicit general arrangement variety of dimension three, complexity two and Picard number one.

Remark 4.5.8. Consider two fans Σ_1 and Σ_2 in $\mathbb{Q}^n$. Then the common refinement $\Sigma_1 \sqcap \Sigma_2$ consists of the cones $\sigma_1 \cap \sigma_2$ with $\sigma_i \in \Sigma_i$ for $i = 1, 2$. Let $\tau \preceq \sigma_1 \cap \sigma_2$ be any face. Then there exist faces $\tau_i \preceq \sigma_i$ such that $\tau = \tau_1 \cap \tau_2$ holds.

Proposition 4.5.9. *Let $X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety. Then the set of rays of $\Sigma \sqcap \mathrm{trop}(X)$ is given by:*

$$(\Sigma \sqcap \mathrm{trop}(X))^{(1)} = \Sigma^{(1)} \cup \{\varrho_\sigma;\ \sigma \in \Sigma \text{ is } P\text{-elementary}\}.$$

Lemma 4.5.10. *Let $\sigma \in \Sigma$ be a big cone.*

(i) *If $\sigma_1 \subseteq \sigma$ is a P-elementary cone, then σ_1 is simplicial, we have $v_{\sigma_1} \in \sigma_1^\circ$ and $\varrho_{\sigma_1} = \sigma_1 \cap \lambda_{\mathrm{lin}}$ holds.*

(ii) *If $\varrho_{\sigma_1} = \varrho_{\sigma_2}$ holds for any two P-elementary cones $\sigma_1, \sigma_2 \subseteq \sigma$, then σ is P-elementary. In particular, we have $\sigma_1 = \sigma_2 = \sigma$.*

Proof. As the definition of a P-elementary cone does just depend on the special structure of the matrix P, these statements can be deduced from the proof of [5, Prop. 3.8 (iii),(iv)]. □

Lemma 4.5.11. *Let $\sigma \in \Sigma$ be a big cone, $\tau \in \mathrm{trop}(X)$ and let $\varrho \in (\sigma \cap \tau)^{(1)}$ be any ray. Then one of the following statements hold:*

(i) *We have $\varrho \in \sigma^{(1)}$.*

(ii) *We have* $\varrho = \varrho_{\sigma_1}$*, where* $\sigma_1 \preceq \sigma$ *is a* P*-elementary face. In particular,* $\varrho \subseteq \lambda_{\mathrm{lin}}$ *holds.*

Proof. Due to Remark 4.5.8 there exists $\sigma_\varrho \preceq \sigma$ and $\tau_\varrho \preceq \tau$ such that $\sigma_\varrho \cap \tau_\varrho = \varrho$ holds and we may assume these cones to be minimal with this property. We distinguish between the following two cases.

Case 1: We have $\tau_\varrho = \lambda_{\mathrm{lin}}$, i.e. $\varrho = \sigma_\varrho \cap \lambda_{\mathrm{lin}}$. If we have $\sigma_\varrho \subseteq \lambda_{\mathrm{lin}}$, then $\varrho = \sigma_\varrho \in \sigma^{(1)}$ holds. So, assume not. Then with $\sigma_\varrho^\circ \cap \lambda_{\mathrm{lin}} \neq \emptyset$, we conclude that σ_ϱ is big and there exists a P-elementary cone $\sigma_1 \subseteq \sigma_\varrho$. We obtain

$$\varrho_{\sigma_1} = \sigma_1 \cap \lambda_{\mathrm{lin}} \subseteq \sigma_\varrho \cap \lambda_{\mathrm{lin}} = \varrho$$

and therefore $\varrho = \varrho_{\sigma_1}$. As this does not depend on the choice of the P-elementary cone σ_1 we conclude that σ_ϱ is P-elementary due to Lemma 4.5.10 (ii).

Case 2: We have $\varrho = \sigma_\varrho \cap \tau_\varrho$ with $\varrho \subseteq \tau_\varrho^\circ$ and $\tau_\varrho \neq \lambda_{\mathrm{lin}}$. Assume $\sigma_\varrho \subseteq \lambda_\varrho$ holds. Then $\varrho = \sigma_\varrho \in \sigma^{(1)}$ holds. So assume $\sigma_\varrho \not\subseteq \lambda_\varrho$. If σ_ϱ is a leave cone, i.e. $\sigma \subseteq \lambda_I \in \mathrm{trop}(X)$ holds, then due to minimality of τ_ϱ we have $\tau \preceq \lambda_I$. We conclude $\varrho \in (\sigma_\varrho \cap \lambda_I)^{(1)} = \sigma^{(1)}$. So assume σ_ϱ is a big cone. In this case there exists a P-elementary cone $\sigma_1 \subseteq \sigma_\varrho$ with

$$\varrho_{\sigma_1} = \sigma_1 \cap \lambda_{\mathrm{lin}} \subseteq \sigma_\varrho \cap \tau_\varrho = \varrho$$

and therefore $\varrho = \varrho_{\sigma_1}$. As this does not depend on the choice of the P-elementary cone σ_1 we conclude that σ_ϱ is P-elementary due to Lemma 4.5.10 (ii). □

Proof of Proposition 4.5.9. We show "$\subseteq$". Let ϱ be any ray of $\Sigma \sqcap \mathrm{trop}(X)$. Then due to Remark 4.5.8 we have $\varrho = \sigma \cap \tau$ with minimal cones $\sigma \in \Sigma$ and $\tau \in \mathrm{trop}(X)$. Assume σ is a leaf cone, i.e. $\sigma \subseteq \lambda_I \in \mathrm{trop}(X)$ holds. Then due to minimality of τ we have $\tau \preceq \lambda_I$. We conclude $\varrho \in (\sigma \cap \lambda_I)^{(1)} = \sigma^{(1)}$. If σ is a big cone, Lemma 4.5.11 gives the assertion.

We prove "$\supseteq$". Due to construction, the rays of Σ are supported on the tropical variety. Thus it is only left to show that ϱ_σ is a ray of $\Sigma \sqcap \mathrm{trop}(X)$ for a P-elementary cone $\sigma \in \Sigma$. This follows using Lemma 4.5.10 (ii). □

As a consequence of Proposition 4.5.9 we obtain the following corollary:

Corollary 4.5.12. *The weakly tropical resolution of an explicit general arrangement variety is again an explicit general arrangement variety.*

Proof. Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety and consider its weakly tropical resolution $Z' \to Z$. Due to Proposition 4.5.9 the rays of $\Sigma' = \Sigma \sqcap \mathrm{trop}(X)$ which are not rays of Σ are contained in the lineality space of $\mathrm{trop}(X)$. In particular, the fans Δ_0 and Δ_0' as in Construction 4.3.3 coincide and therefore the explicit maximal orbit quotients $X \dashrightarrow Y_0$ and $X' \dashrightarrow Y_0'$ coincide up to small birational modifications. We conclude that $X' = X(A, P', \Sigma')$ holds, where A is the same matrix as for X and P' contains the primitive ray generators of the fan Σ', see [42, Sec. 6]. □

Due to Theorem 2.1.5, the Cox ring $R(A,P)$ of an explicit general arrangement variety $X := X(A,P,\Sigma) \subseteq Z$ is a complete intersection ring. Therefore, we can apply [6, Prop. 3.3.3.2] and obtain the canonical class of X via the following formula:

$$\mathcal{K}_X = -\sum_{\varrho \in \Sigma^{(1)}} \deg(T_\varrho) + \sum_{i=1}^{r-c} \deg(g_i) \in \mathrm{Cl}(X) \cong \mathbb{Z}^{n+m}/\mathrm{im}(P^*).$$

Proposition 4.5.13. *Let $X := X(A,P,\Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit general arrangement variety with weakly tropical resolution $Z' \to Z$ and let $\sigma \in \Sigma$ be a P-elementary cone. Then the following statements hold:*

(i) *The discrepancy along the prime divisor of $X' \subseteq Z'$ corresponding to ϱ_σ equals $c_\sigma^{-1}\ell_\sigma - 1$.*

(ii) *The ray ϱ_σ is not contained in the anticanonical complex $\mathcal{A}$, if and only if $\ell_\sigma > 0$ holds; in this case, ϱ_σ leaves $\mathcal{A}$ at $v'_\sigma = \ell_\sigma^{-1} v_\sigma$.*

Proof. We prove (i). Due to Theorem 4.4.1 the variety $X \subseteq Z$ admits a semi-locally toric weakly tropical resolution. Applying Lemma 4.2.16 we conclude that there exists a toric canonical φ-family. Therefore, explicitly constructing a pair $(Z'_{\varrho_\sigma}, D_{\varrho_\sigma})$ as in Definition 4.2.8 we can use Remark 4.2.11 to calculate the discrepancy along $D_{X'}^{\varrho_\sigma}$:

Consider the ray $\varrho_\sigma \in \Sigma'$. Then $\varrho_\sigma \subseteq \lambda_I$ holds for every maximal leaf of $\mathrm{trop}(X)$. In particular, we may choose $I := \{1, \dots, c\}$ and consider the divisor

$$D_{\varrho_\sigma} := \sum_{j=1}^{n_0} (r-c) l_{0j} D_{\varrho_{0j}} - \sum_{\varrho' \in (\Sigma')^{(1)}} D_{\varrho'}.$$

Then, as $X' \subseteq Z'$ is an explicit general arrangement variety due to Corollary 4.5.12, the pullback $D_{\varrho_\sigma}|_{X'}$ is a canonical divisor on X'. Moreover, the push forward $\varphi_*(D_{\varrho_\sigma})$ is $\mathbb{Q}$-Cartier and by construction we have $D_{\varrho_\sigma} = k_{Z'}$ on Z'_{ϱ_σ}. In particular, we have constructed a tuple $(Z'_{\varrho_\sigma}, D_{\varrho_\sigma})$ as claimed. Now, let $u \in \mathbb{Q}^{r+s}$ be an element such that $\mathrm{div}(\chi^u) = \varphi_*(D_{\varrho_\sigma})$ holds on Z_σ. Then due to Remark 4.2.11 we have

$$\mathrm{discr}_X(D_{X'}^{\varrho_\sigma}) = -1 - \langle u, v_{\varrho_\sigma} \rangle.$$

Therefore, using $v_\sigma = v_{\varrho_\sigma} \cdot c_\sigma$, we obtain the assertion with

$$\langle u, v_\sigma \rangle = \langle u, \sum_{i=0}^{r} \ell_{\sigma,i} v_{ij_i} \rangle = \sum_{i=0}^{r} \ell_{\sigma,i} \langle u, v_{ij_i} \rangle = -\ell_\sigma.$$

Using (i) assertion (ii) follows from the definition of the anticanonical complex. ☐

Proof of Proposition 4.5.4. Let $\sigma' \in \Sigma'$ be any cone. Then a linear form $u \in M_{\mathbb{Q}}$ is defining for $A_{\sigma'}$ if and only if for all rays $\varrho \in \sigma'$ we have

$$\operatorname{disc}_X(D_{X'}^{\varrho}) = -1 - \langle u, v_\varrho \rangle.$$

Due to Proposition 4.5.9 the rays of σ' are either rays of Σ, then $\langle u, v_\varrho \rangle = -1$ holds, or they are of the form ϱ_σ where $\sigma \in \Sigma$ is a P-elementary cone. In this case the assertion follows from Proposition 4.5.13 (ii). □

Proof of Corollary 4.5.5. By definition the vertices of $\mathcal{A}$ are the vertices of $A_{\sigma'}$ where σ' runs over all cones of Σ'. In particular, they arise as the intersection of the hyperplane $u_{\sigma'} = -1$ with the rays of σ'. Therefore, Proposition 4.5.4 gives the assertion. The supplement follows using the characterization of log terminality as given in Remark 4.2.5 (i'). □

Theorem 4.5.14. *Let $X \subseteq Z_X$ be a $\mathbb{Q}$-Gorenstein general arrangement variety of complexity c and consider a cone $\sigma = \varrho_{0j_0} + \ldots + \varrho_{rj_r} \in \Sigma$. If the singularity defined by σ is*

(i) *log terminal, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} > r - c$ holds,*

(ii) *canonical, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} \geq r - c + c_\sigma \prod_{\varrho \in \sigma^{(1)}} l_\varrho^{-1}$ holds,*

(iii) *terminal, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} > r - c + c_\sigma \prod_{\varrho \in \sigma^{(1)}} l_\varrho^{-1}$ holds,*

where c_σ is the greatest common divisor of the entries of the vector v_σ built up from the primitive generators $v_{ij_i} \in \varrho_{ij_i}$ as follows:

$$v_\sigma \;:=\; \ell_{\sigma,0} v_{0j_0} + \ldots + \ell_{\sigma,r} v_{rj_r} \;\in\; \mathbb{Z}^{r+s}, \qquad \ell_{\sigma,i} \;:=\; \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \;\in\; \mathbb{Z}.$$

Proof. The cone $\sigma \in \Sigma_X$ is by definition P-elementary and big. Thus, the assertion follows via direct calculation from Proposition 4.5.13. □

Remark 4.5.15. Consider a P-elementary cone $\sigma = \varrho_0 + \cdots + \varrho_r \in \Sigma$ defining a log terminal singularity and assume $l_{\varrho_0} \geq \cdots \geq l_{\varrho_r}$ holds. Then the condition in Theorem 4.5.14 (i) implies that $\sum_{i=0}^{c+1} l_{\varrho_i}^{-1} > 1$ and $l_{\varrho_{c+2}} = \cdots = l_{\varrho_r} = 1$ holds.

Corollary 4.5.16. *Let $X \subseteq Z_X$ be a $\mathbb{Q}$-Gorenstein general arrangement variety of complexity two. Then X is log terminal if and only if for any cone $\sigma = \varrho_{0j_0} + \ldots + \varrho_{rj_r} \in \Sigma$ we achieve by suitably renumbering the involved rays that $l_{4j_4} = \ldots = l_{rj_r} = 1$ holds and the tuple $(l_{0j_0}, l_{1j_1}, l_{2j_2}, l_{3j_3})$ is one of the following:*

(i) $(1, x, y, z)$,

(ii) $(2, 2, x, y)$,

(iii) $(2, 3, \leq 5, x)$, $(2, 3, 7, \leq 41)$, $(2, 3, 8, \leq 23)$, $(2, 3, 9, \leq 17)$, $(2, 3, 10, \leq 14)$, $(2, 3, 11 \leq 13)$,

(iv) $(2, 4, 4, x)$, $(2, 4, 5, \leq 19)$, $(2, 4, 6, \leq 11)$, $(2, 4, 7, \leq 8)$,

(v) $(2, 5, 5, \leq 9)$, $(2, 5, 6, \leq 6)$,

(vi) $(3, 3, 3, x)$, $(3, 3, 4, \leq 11)$, $(3, 3, 5, \leq 6)$,

(vii) $(3, 4, 4, 5)$, $(3, 4, 4, 4)$.

Proof. Using Remark 4.5.15 the claim follows from Theorem 4.5.14 via a direct calculation in the complexity two case. □

4.6 An alternative construction

In this section we consider explicit general arrangement varieties $X := X(A, P, \Sigma) \subseteq Z$ with ample anticanonical divisor and give an alternative description of the anticanonical complex in this setting. Following the same steps as done in [13], we explicitly construct a polyhedral complex and show that it is indeed the anticanonical complex of X, see Theorem 4.6.2. In particular, we make the construction developed in [13] applicable in a broader setting: Besides leaving the Fano case by dropping the condition on X to be projective, Example 4.6.8 shows, that in general the varieties $X(A, P, \Sigma) \subseteq Z$ are not treatable with the methods developed there.

In this section let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety with Cox ring $R(A, P)$ given by generators T_{ij}, S_k and relations $g_1, \ldots, g_{r-c}$ as in Construction 2.1.13. Moreover let $Z' \to Z$ be its weakly tropical resolution defined by the fan $\Sigma' = \Sigma \sqcap \operatorname{trop}(X)$ in $\mathbb{Q}^{r+s}$.

Construction 4.6.1. Let $\gamma_{n+m} \subseteq \mathbb{Q}^{n+m}$ be the positive orthant and let $e_\Sigma \in \mathbb{Z}^{n+m}$ be any representative of the canonical class $\mathcal{K}_Z$ of Z. Define polytopes

$$B(-\mathcal{K}_X) := Q^{-1}(-\mathcal{K}_X) \cap \gamma_{n+m} \subseteq \mathbb{Q}^{n+m}$$

and $B := B(g_1) + \ldots + B(g_{r-c})$ as the Minkowski sum of the Newton polytopes $B(g_i)$ of the relations g_i. The *anticanonical polyhedron* $A_X \subseteq \mathbb{Q}^{r+s}$ of X is the dual polyhedron of the polyhedron

$$B_X \ := \ (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \ \subseteq \ \mathbb{Q}^{r+s}, \qquad A_X := B_X{}^\circ.$$

Theorem 4.6.2. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit general arrangement variety with ample anticanonical class. Then the anticanonical complex $\mathcal{A}$ of X is the polyhedral complex*

$$\operatorname{faces}(A_X) \sqcap \Sigma \sqcap \operatorname{trop}(X) = \operatorname{faces}(A_X) \sqcap \Sigma'.$$

Corollary 4.6.3. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a Fano general arrangement variety. Then its anticanonical complex $\mathcal{A}$ is piecewise convex, i.e.*

$$\mathrm{conv}(|\mathcal{A}|) \cap |\mathrm{trop}(X)| = |\mathcal{A}|.$$

Example 4.6.4. Consider the variety $X := X(A, P, \Sigma) \subseteq Z$ from Examples 4.4.4 and 4.5.7. By construction, X is a Fano general arrangement variety and its anticanonical complex is given as

$$|\mathcal{A}| = \mathrm{conv}(v_{01}, v_{02}, v_{11}, v_{21}, v_{31}, v_{\mathrm{lin}1}, v_{\mathrm{lin}2}) \cap |\mathrm{trop}(X)|.$$

Lemma 4.6.5. *Let $X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety with $A = (a_0, \dots, a_r)$ and consider the linear subspace $\mathbb{P}_c \subseteq \mathbb{P}_r$ defined via the kernel of A, i.e. the vanishing set of the relations $f_1, \dots, f_{r-c}$, where*

$$f_t := \det \begin{bmatrix} a_0 & a_1 & \dots & a_c & a_{c+t} \\ U_0 & U_1 & \dots & U_c & U_{c+t} \end{bmatrix} \in \mathbb{K}[U_0, \dots, U_r].$$

Then $\mathrm{trop}(\mathbb{P}_c \cap \mathbb{T}^r) = \Sigma_{\mathbb{P}_r}^{\leq c}$ *is a subfan of the normal fan of*

$$\tilde{B} := B(h_1) + \dots + B(h_{r-c}) \subseteq \mathbb{Q}^r, \qquad \textit{with} \quad h_i := f_i(1, U_1, \dots, U_r).$$

In particular, the tropical variety $\mathrm{trop}(X)$ *is a subfan of the normal quasifan of $\tilde{B}$ considered as a a polytope in $\mathbb{Q}^{r+s}$.*

Proof. Let $e_1, \dots, e_r$ denote the standard basis vectors of $\mathbb{Q}^r$ and set $e_0 := -\sum e_i$. As $\mathrm{trop}(\mathbb{P}_c \cap \mathbb{T}^r)$ is by definition a refinement of a subfan of $\mathcal{N}(\tilde{B})$, it suffices to show that $\mathrm{cone}(e_k)$ is a ray of $\mathcal{N}(\tilde{B})$ for every $k = 0, \dots, r$. For this set

$$J_t := \{j;\ U_j \text{ is a monomial of } f_t\} = \{0, \dots, c, c+t\}.$$

Then the lineality space of $\mathcal{N}(B(h_t))$, i.e. the maximal linear subspace contained in $\mathcal{N}(B(h_t))$, is

$$\sigma_t^{\mathrm{lin}} := \mathrm{lin}(e_j;\ j \in \{0, \dots, r\} \setminus J_t).$$

Now let $k \leq c$. Then $\mathrm{cone}(e_k) \times \sigma_t^{\mathrm{lin}} \in \mathcal{N}(B(h_t))$ holds for every $t = 1, \dots, r-c$ and we claim

$$\mathrm{cone}(e_k) = \bigcap_{t=1}^{r-c} \left(\mathrm{cone}(e_k) \times \sigma_t^{\mathrm{lin}}\right) =: \sigma \in \mathcal{N}(\tilde{B}),$$

i.e. we have to show the inclusion "$\supseteq$". Let $a \in \sigma$ be any point. Then for every $t = 1, \dots, r-c$ we have a description

$$a = \sum_{j>c,\ j\neq t} a_{tj} e_j + b_{tk} e_k, \quad \text{with} \quad a_{tj} \in \mathbb{Q},\ b_{tj} \in \mathbb{Q}_{\geq 0}.$$

Usig that $\{e_k, e_c, \ldots, e_r\}$ are linearly independent, we conclude $a_{tj} = 0$ for all t, j and therefore $a \in \mathrm{cone}(e_k)$. We come to the case $k > c$. Here we have $\mathrm{cone}(e_k) \times \sigma_k^{\mathrm{lin}} \in \mathcal{N}(B(h_k))$ and we claim

$$\mathrm{cone}(e_k) = \left(\mathrm{cone}(e_k) \times \sigma_k^{\mathrm{lin}}\right) \cap \bigcap_{t=1, t\neq k}^{r-c} \sigma_t^{\mathrm{lin}} \in \mathcal{N}(\tilde{B}).$$

Analogously to the first case this can be verified by a direct calculation. □

Lemma 4.6.6. *Let $B \subseteq \mathbb{Q}^m$ be any polyhedron and denote by $\mathcal{N}(B)$ its normal quasifan. Let further $P \colon \mathbb{Q}^n \to \mathbb{Q}^m$ be a surjective linear map. Then $\mathcal{N}(P^*(B)) = P^{-1}(\mathcal{N}(B))$ holds.*

Proof. Let B be any polyhedron. Then $\mathcal{N}(B) = \{C_F^\vee;\ F \preceq B \text{ face}\}$ holds, where $C_F := \mathrm{cone}(u - v;\ u \in B, v \in F)$. Thus we have

$$C_F^\vee = \{y;\ \langle y, u - v\rangle \geq 0 \text{ for all } u \in B, v \in F\}.$$

Note that due to injectivity of P^* the faces of $P^*(B)$ are precisely the images $P^*(F)$ of the faces $F \preceq B$. We conclude

$$\begin{aligned} C_{P^*(F)}^\vee &= \{x;\ \langle x, P^*(u) - P^*(v)\rangle \geq 0 \text{ for all } u \in B, v \in F\} \\ &= \{x;\ \langle P(x), u - v\rangle \geq 0 \text{ for all } u \in B, v \in F\} \\ &= P^{-1}(C_F^\vee). \end{aligned}$$

□

To a T-invariant Weil divisor $D = \sum a_\varrho D_\varrho$ on Z we assign a polyhedron:

$$B_D := \{u \in M_\mathbb{Q};\ \langle u, v_\varrho\rangle \geq -a_\varrho\} \subseteq M_\mathbb{Q}.$$

Proposition 4.6.7. *Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit general arrangement variety with ample anticanonical class, fix a T-invariant divisor $-k_X$ on Z such that $-k_X|_X$ is an anticanonical divisor on X. Let B_{-k_X} denote the polyhedron corresponding to $-k_X$ and let $B \subseteq \mathbb{Q}^{r+s}$ and $\tilde{B} \subseteq \mathbb{Q}^{r+s}$ be as in Construction 4.6.1 and Lemma 4.6.5. Then we have the following equalities:*

(i) $P^*(B_{-k_X}) - k_X = B(-\mathcal{K}_X)$.

(ii) $P^*(\tilde{B}) = B - (r - c) \cdot l_0$, *where l_0 is identified with $(l_0, 0, \ldots, 0) \in \mathbb{K}^{n+m}$*

In particular, the fan $\Sigma \sqcap \mathrm{trop}(X)$ is a subfan of the normal fan of B_X.

Proof. The two equalities follow by direct calculation. We prove the supplement. Using Lemma 4.6.6 we obtain

$$P^{-1}(\mathcal{N}(B_{-k_X})) = \mathcal{N}(B(-\mathcal{K}_X)) \quad \text{and} \quad P^{-1}(\mathcal{N}(\tilde{B})) = \mathcal{N}(B).$$

As $-k_X$ is ample due to Remark 4.3.1, the fan Σ is a subfan of the normal fan $\mathcal{N}(B_{-k_X})$, and Lemma 4.6.5 shows that $\operatorname{trop}(X)$ is a subfan of $\mathcal{N}(\tilde{B})$. It follows that $P^{-1}(\Sigma) \sqcap P^{-1}(\operatorname{trop}(X))$ is a subfan of the normal fan of $B(-\mathcal{K}_X) + B$. Projecting the involved fans via P to $\mathbb{Q}^{r+s}$ gives the assertion. □

Proof of Theorem 4.6.2. Let $\sigma' \in \Sigma'$ be any cone. Then due to Proposition 4.6.7 we have $\sigma' \in \mathcal{N}(B_X)$. Fix any maximal cone $\tau \in \mathcal{N}(B_X)$ with $\sigma' \preceq \tau$. Then we have $\sigma' \preceq \sigma \cap \lambda_I \preceq \tau$, where λ_I is a maximal leaf of $\operatorname{trop}(X) = \operatorname{trop}(X')$ and $\sigma \in \Sigma$ holds. Denote by $u \in B_X$ the vertex corresponding to τ. Then we have a decomposition $P^*(u) = \mu + \nu - e_\Sigma$, with $\mu \in B(-\mathcal{K}_X)$ and $\nu \in B$. We claim that the family

$$(Z'_{\sigma'}, D_{\sigma'})_{\sigma' \in \Sigma'}, \qquad \text{with} \quad D_{\sigma'} = \sum_{\varrho \in (\Sigma')^{(1)} \cap \Sigma^{(1)}} \langle \nu, e_\varrho \rangle D_\varrho - \sum_{\varrho \in (\Sigma')^{(1)}} D_\varrho$$

is a toric canonical φ-family and $\operatorname{div}(\chi^u) = \varphi_*(D_{\sigma'})$ holds on X_σ.

In order to verify the claim we first show that $\langle \nu, e_\varrho \rangle = 0$ holds for all $\varrho \in (\sigma')^{(1)} \cap \sigma^{(1)}$. Denote by $\tilde{\nu}$ the vertex of $\tilde{B}$ corresponding to ν. Then $\tilde{\nu}$ defines the maximal cone

$$\left\{ x \in \mathbb{Q}^{r+s};\ \langle x, u - \tilde{\nu} \rangle \geq 0 \text{ for all } u \in \tilde{B} \right\},$$

which contains λ_I. After suitably renumbering we may assume $I = \{1, \ldots, c\}$ and thus $\tilde{\nu}_1 = \ldots = \tilde{\nu}_c = 0$. With $P^*(\tilde{B}) = B - (r-c) \cdot l_0$ we obtain that

$$\langle \nu, e_\varrho \rangle = \langle P^*(\tilde{\nu}) - (r-c) l_0, e_\varrho \rangle = \langle \tilde{\nu}, v_\varrho \rangle - \langle (r-c) l_0, e_\varrho \rangle = 0$$

holds for all $\varrho \in \sigma^{(1)} \cap (\sigma')^{(1)}$. This shows that $(Z'_{\sigma'}, D_{\sigma'})_{\sigma' \in \Sigma'}$ is a toric canonical φ-family.

It is only left to show that $\operatorname{div}(\chi^u) = \varphi_*(D_{\sigma'})$ holds on X_σ. We fix a T-invariant divisor $-k_X = \sum_{\varrho \in \Sigma^{(1)}} a_\varrho D_\varrho$ whose pullback $-k_X|_X$ is an anticanonical divisor on X and denote by $\tilde{\mu}$ the vertex in B_{-k_X} corresponding to μ. Then, as $-k_X|_X$ and therefore $-k_X$ is ample, we have $\langle \tilde{\mu}, v_\varrho \rangle = -a_\varrho$ for all rays $\varrho \in \sigma^{(1)}$, see [22, Prop. 6.2.5]. We conclude

$$\langle \mu, e_\varrho \rangle = \langle P^*(\tilde{\mu}) - k_X, e_\varrho \rangle = \langle \tilde{\mu}, v_\varrho \rangle + \langle -k_X, e_\varrho \rangle = 0.$$

Since $\langle u, v_\varrho \rangle = \langle \mu + \nu - e_\Sigma, e_\varrho \rangle$ holds, this completes the proof. □

Example 4.6.8. Consider the variety $X := X(A, P, \Sigma) \subseteq Z$ from Examples 4.4.4, 4.5.7 and 4.6.4. As before, we denote the primitive ray generators of Σ by

$$[v_{01}, v_{02}, v_{11}, v_{21}, v_{31}] = \begin{bmatrix} -1 & -2 & 2 & 0 & 0 \\ -1 & -2 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 & 4 \\ -1 & -3 & 1 & 1 & 1 \end{bmatrix}$$

and the defining relation of the Cox ring of X by $g = T_{01}T_{02}^2 + T_{11}^2 + T_{21}^2 + T_{31}^4$. The common refinement $\Sigma' = \Sigma \sqcap \operatorname{trop}(X)$ is pure of dimension 3 and we have

$$(\Sigma')^{(1)} = \Sigma^{(1)} \cup \{\operatorname{cone}(e_4),\ \operatorname{cone}(-e_4)\}\,.$$

Further refining this fan we obtain a smooth toric variety Z'' whose fan has 72 maximal cones and primitive ray generators given by the columns of the matrix

$$P_2 := \begin{bmatrix} -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ -2 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 \\ -3 & -3 & -2 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We show that this example leaves the framework in which [13, Thm. 1.4] can be applied as $X \subseteq Z$ is not *tropical resolvable* in their sense: Note that we have $\operatorname{cone}(v_{02}, v_{31}),\ \operatorname{cone}(v_{21}, v_{31}) \in \Sigma'$. In particular, any regular refinement of Σ' contains the following rays:

$$\varrho_1 := \operatorname{cone}([-1,-1,1,-1]) \quad \text{and} \quad \varrho_2 := \operatorname{cone}([1,1,0,1]).$$

Let P'' be a matrix whose columns are the primitive generators of the rays of a regular refinement Σ'' of Σ'. Then the *shift* of g with respect to P'' and P is the unique polynomial $\tilde{g} \in \mathbb{K}[T_\varrho;\ \varrho \in (\Sigma'')^{(1)}]$ without monomial factors satisfying that its push with respect to P'' equals the push of g with respect to P. In particular, we have $\tilde{g} = m_1 + m_2 + m_3 + m_4$, where the m_i satisfy

$$T_{\varrho_1} | m_i \Leftrightarrow i = 1,2 \quad \text{and} \quad T_{\varrho_2} | m_i \Leftrightarrow i = 3,4$$

In the example $P'' = P_2$ we have

$$\begin{aligned} m_1 &:= T_{24}T_{25}T_{26}T_{27}T_{28}T_{29}{T_{30}}^2{T_{31}}^2T_9 \\ m_2 &:= T_{17}T_{18}T_{19}T_{20}T_{21}{T_{22}}^2{T_{23}}^2T_{29}T_8 \\ m_3 &:= {T_7}^2T_{12}T_{13}{T_{14}}^2{T_{15}}^3{T_{16}}^4T_{19}T_{20}{T_{21}}^2T_{23}T_{26}T_{27}{T_{28}}^2T_{31}T_2T_5T_6 \\ m_4 &:= {T_1}^2{T_2}^2T_3T_4T_5T_6T_7T_8T_9, \end{aligned}$$

where $T_7 = T_{\varrho_1}$ and $T_{29} = T_{\varrho_2}$. Now, assume X is tropical resolvable in the sense of [13, Def. 2.2]. Then there exists a regular refinement Σ'' of Σ' giving rise to a toric variety Z'' such that the Cox ring of the proper transform X'' inside Z'' equals the freely graded ring $\mathbb{K}[T_\varrho;\ \varrho \in (\Sigma'')^{(1)}]/\langle \tilde{g} \rangle$. This is a contradiction since T_{ϱ_1} is not prime.

CHAPTER

FIVE

CANONICAL FANO INTRINSIC QUADRICS OF DIMENSION THREE

In this chapter we treat the example class of *intrinsic quadrics*, i.e. projective varieties X that admit a presentation of their Cox rings $\mathcal{R}(X)$ by $\mathrm{Cl}(X)$-homogeneous generators, such that the ideal of relations is generated by a single quadratic polynomial. These varieties were introduced in [15] as an example class for the bunched ring approach to Mori dream spaces and described there in case of smooth full intrinsic quadrics of Picard number at most two, where full means, that every generator of the Cox ring shows up in the defining relation. In [17] Bourqui used the intrinsic quadrics as a testing ground for Manin's conjecture. The description of smooth intrinsic quadrics of Picard number at most two is due to [29], where in addition in this case Fujita's freeness conjecture is verified. In the singular case, the terminal Fano intrinsic quadrics of dimension three having true complexity one and Picard number one are known due to [13]. We will extend these results by treating the three-dimensional Fano intrinsic quadrics having at most canonical singularities. Parts of Sections 5.1, 5.3 and 5.5 are published in [48] and parts of Sections 5.1, 5.2 and 5.4 are published in the joint work [49].

5.1 Statement of the main results

Due to [29, Prop 2.1] any intrinsic quadric can be realized as an explicit general arrangement variety $X = X(A, P, \Sigma) \subseteq Z$. More precisely the defining relation of its Cox ring $R(A, P)$ is a quadric of the following form:

$$g = T_{01}T_{02} + \ldots + T_{(q-1)1}T_{(q-1)2} + T_{q1}^2 + \ldots + T_{r1}^2 \qquad \text{with} \quad 0 \leq q \leq r+1$$

This allows us to work in the language of explicit general arrangement varieties.

Our focus is on Fano intrinsic quadrics of dimension three. Here, the case of $\mathbb{Q}$-factorial Fano intrinsic quadrics of complexity and Picard number one having at most terminal singularities is known [13]. Our first result extends this classification in two directions: On the one hand, we leave the terminal case and consider canonical intrinsic quadrics. On the other hand, using our results from Chapter 4, we no longer restrict our study to torus actions of complexity one.

Theorem 5.1.1. *Every three-dimensional $\mathbb{Q}$-factorial Fano intrinsic quadric having Picard number one and at most canonical singularities is isomorphic to precisely one of the following varieties X defined by its $\mathrm{Cl}(X)$-graded Cox ring $\mathcal{R}(X)$, its matrix of generator degrees $Q = [w_1, \ldots, w_r]$ and its anticanonical class $-\mathcal{K}_X \in \mathrm{Ample}(X)$. Moreover, we list their Fano-index $q(X)$ and their anticanonical self-intersection number $-\mathcal{K}_X^3$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$q(X)$	$-\mathcal{K}_X^3$
1	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \end{bmatrix}$	3	54
2	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 6 \end{bmatrix}$	6	36
3	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 6 \end{bmatrix}$	6	48
4	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 5 & 3 \end{bmatrix}$	$\begin{bmatrix} 9 \end{bmatrix}$	9	$\frac{729}{20}$
5	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 6 & 3 & 5 & 4 \end{bmatrix}$	$\begin{bmatrix} 12 \end{bmatrix}$	12	$\frac{96}{5}$
6	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 5 & 1 & 7 & 4 \end{bmatrix}$	$\begin{bmatrix} 12 \end{bmatrix}$	12	$\frac{1152}{35}$
7	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 7 & 2 & 8 & 5 \end{bmatrix}$	$\begin{bmatrix} 15 \end{bmatrix}$	15	$\frac{1125}{56}$
8	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{1} \end{bmatrix}$	1	$\frac{32}{3}$
9	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 6 & 4 & 5 & 5 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{1} \end{bmatrix}$	3	$\frac{36}{5}$
10	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 4 & 2 & 3 & 3 & 6 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{1} \end{bmatrix}$	3	12
11	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{1} \end{bmatrix}$	3	24
12	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{1} \end{bmatrix}$	3	27
13	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	54
14	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	32
15	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 5 \\ \bar{1} \end{bmatrix}$	5	$\frac{125}{6}$
16	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	18
17	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\begin{bmatrix} 2 & 2 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	18

18	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1&3&1&3&2 \\ \bar{1}&\bar{1}&\bar{0}&\bar{0}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	24
19	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4&2&3&3&1 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 7 \\ \bar{1} \end{bmatrix}$	7	$\frac{343}{48}$
20	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4&2&3&3&2 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
21	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 4&2&3&3&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{32}{3}$
22	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5&1&3&3&2 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{256}{15}$
23	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 1&3&2&2&4 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{64}{3}$
24	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5&3&4&4&1 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 9 \\ \bar{1} \end{bmatrix}$	9	$\frac{243}{20}$
25	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 3&5&4&4&2 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{25}{3}$
26	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 5&1&3&3&4 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 10 \\ \bar{0} \end{bmatrix}$	10	$\frac{50}{3}$
27	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6&4&5&5&2 \\ \bar{1}&\bar{1}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
28	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 6&4&5&5&2 \\ \bar{0}&\bar{0}&\bar{0}&\bar{1}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{36}{5}$
29	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7&3&5&5&2 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	$\frac{288}{35}$
30	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2&4&3&3&6 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
31	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 2&4&3&3&6 \\ \bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	12	12
32	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\begin{bmatrix} 7&3&5&5&4 \\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 14 \\ \bar{0} \end{bmatrix}$	14	$\frac{98}{15}$
33	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{1}&\bar{2}&\bar{0}&\bar{0}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18
34	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{1}&\bar{2}&\bar{1}&\bar{2}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	18
35	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 2&2&1&3&2 \\ \bar{0}&\bar{2}&\bar{0}&\bar{2}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
36	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 2&2&1&3&2 \\ \bar{0}&\bar{2}&\bar{2}&\bar{0}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	12
37	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_3$	$\begin{bmatrix} 1&3&1&3&2 \\ \bar{0}&\bar{2}&\bar{1}&\bar{1}&\bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	16
38	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{1}&\bar{3}&\bar{2}&\bar{0}&\bar{3} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{1} \end{bmatrix}$	1	19
39	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 2&4&3&3&2 \\ \bar{3}&\bar{1}&\bar{0}&\bar{2}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{2} \end{bmatrix}$	2	$\frac{16}{3}$
40	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1&1&1&1&2 \\ \bar{2}&\bar{0}&\bar{1}&\bar{3}&\bar{2} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{2} \end{bmatrix}$	2	16
41	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1&1&1&1&1 \\ \bar{2}&\bar{2}&\bar{1}&\bar{3}&\bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{27}{2}$

42	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{3} \end{bmatrix}$	3	$\frac{27}{2}$
43	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{27}{2}$
44	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	16
45	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 5 \\ \bar{0} \end{bmatrix}$	5	$\frac{49}{6}$
46	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
47	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 2 \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \end{bmatrix}$	6	9
48	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_4$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	8	$\frac{16}{3}$
49	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_5$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{54}{5}$
50	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{5} & \bar{2} & \bar{4} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
51	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
52	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{4} & \bar{0} & \bar{2} & \bar{5} & \bar{5} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	9
53	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{4} & \bar{0} & \bar{5} & \bar{2} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{2} \end{bmatrix}$	4	$\frac{32}{3}$
54	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{3} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{1} \end{bmatrix}$	3	$\frac{27}{4}$
55	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \end{bmatrix}$	4	8
56	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{2} & \bar{0} & \bar{5} & \bar{1} & \bar{6} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{4} \end{bmatrix}$	4	8
57	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_9$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{4} & \bar{5} & \bar{3} & \bar{6} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	6
58	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{0} & \bar{7} & \bar{1} & \bar{4} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \end{bmatrix}$	3	$\frac{9}{2}$
59	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{1} \end{bmatrix}$	1	$\frac{19}{2}$
60	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{1} \\ \bar{1} \end{bmatrix}$	1	16
61	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{1} \\ \bar{1} \end{bmatrix}$	3	$\frac{45}{4}$
62	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	3	$\frac{27}{2}$
63	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$	3	$\frac{27}{2}$
64	$\frac{\mathbb{K}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$	3	$\frac{27}{2}$

65	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	4	16
66	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 3 & 3 & 3 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	6	6
67	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 3 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	6	9
68	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 3 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	6	9
69	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	6	9
70	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	6	18
71	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	8	$\frac{16}{3}$
72	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{3} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{2} \end{bmatrix}$	2	4
73	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{3} & \bar{3} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{2} \end{bmatrix}$	2	8
74	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \\ \bar{2} & \bar{0} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	4	8
75	$\frac{\mathbb{K}[T_1,T_2,T_3,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{4} & \bar{1} & \bar{1} & \bar{5} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{3} \end{bmatrix}$	3	$\frac{9}{2}$
76	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,S_1]}{\langle T_1T_2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{2} & \bar{0} & \bar{4} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	3	$\frac{9}{2}$
77	$\frac{\mathbb{K}[T_1,\ldots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2\rangle}$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$	3	$\frac{9}{2}$
78	$\frac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_3)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{0} \\ \bar{2} & \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{0} \end{bmatrix}$	3	6
79	$\frac{\mathbb{K}[T_1,\ldots,T_4,S_1]}{\langle T_1^2+T_2^2+T_3^2+T_4^2\rangle}$	$\mathbb{Z}\times(\mathbb{Z}_2)^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$	4	8

Variety No. 1 *is smooth and varieties Nos.* 4, 19 *and* 49 *are terminal. Moreover, varieties Nos.* 64, 69, 77 *and* 79 *are of true complexity two. All the others are of true complexity one.*

The proof of the above theorem is split into three parts: In Section 5.3 we establish the list of all intrinsic quadrics of complexity one. The case of complexity two torus actions is treated in Section 5.4, and the remaining assertions are proven in Section 5.5.

We leave the case of Picard number one. Here, we were able to classify all $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three and true complexity two having at most canonical singularities:

Theorem 5.1.2. *Every three-dimensional $\mathbb{Q}$-factorial Fano intrinsic quadric of true complexity two having at most canonical singularities is isomorphic to precisely one of the varieties X, specified by its $\mathrm{Cl}(X)$-graded Cox ring $\mathcal{R}(X)$, its matrix of generator degrees $Q = [w_1, \dots, w_r]$ and its anticanonical class $-\mathcal{K}_X \in \mathrm{Ample}(X)$ as follows:*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \dots, w_r]$	$-\mathcal{K}_X$
1	$\frac{\mathbb{K}[T_1,\dots,T_4,S_1]}{\langle T_1^2+T_2^2+T_3^2+T_4^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
2	$\frac{\mathbb{K}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
3	$\frac{\mathbb{K}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
4	$\frac{\mathbb{K}[T_1,\dots,T_5]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & \bar{4} & \bar{3} & \bar{3} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$
5	$\frac{\mathbb{K}[T_1,\dots,T_4,S_1,S_2]}{\langle T_1^2+T_2^2+T_3^2+T_4^2 \rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
6	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
7	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
8	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 4 \\ -2 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
9	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2]}{\langle T_1T_2+T_3^2+T_4^2+T_5^2 \rangle}$	$\mathbb{Z}^3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 2 \\ \bar{0} \\ \bar{1} \end{bmatrix}$

In order to prove the above theorem, in Section 5.2 we give effective bounds for the Picard number of a $\mathbb{Q}$-factorial Fano intrinsic quadric:

Proposition 5.1.3. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial, Fano intrinsic quadric. Then $\varrho(X) \leq 3 + m$ holds.*

Corollary 5.1.4. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial, Fano intrinsic quadric of complexity $c = \dim(X) - 1$. Then $\varrho(X) \leq 5$ holds.*

As an application of our classification Theorem 5.1.2, we finally give a negative result for terminality in arbitrary dimensions for one-dimensional torus actions:

Proposition 5.1.5. *Let $X = X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano intrinsic quadric of complexity $c = \dim(X) - 1$. Then X is not terminal.*

5.2 Structural results

In this section we prove structural results on Fano intrinsic quadrics that we will use in the subsequent sections to prove our classification results. We start by investigating the X-faces of a Fano intrinsic quadric X. Then we go on proving our bounds on the Picard number given in Proposition 5.1.3 and Corollary 5.1.4. As an application we give in Proposition 5.2.2 an effective bound for the Picard number in the three-dimensional case with torus action of complexity two.

Lemma 5.2.1. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a Fano intrinsic quadric. Then after suitably renumbering we may assume $n_0 \geq \ldots \geq n_r$ and we are in one of the following situations:*

(i) *We have $n_{r-1} = n_r = 1$ and $\operatorname{cone}(e_{(r-1)1}, e_{r1}, e_1, \ldots, e_m)$ is an X-face.*

(ii) *We have $n_0 = \ldots = n_{r-1} = 2 > n_r = 1$ and $\operatorname{cone}(e_{01}, e_{02}, e_{r1}, e_1, \ldots, e_m)$ is an X-face.*

(iii) *We have $n_0 = \ldots = n_r = 2$ and $\operatorname{cone}(e_{01}, e_{02}, e_{11}, e_{12}, e_1, \ldots, e_m)$ is an X-face.*

Proof. First note that in any of the above cases the cones under consideration are $\bar{X}$-faces. To prove that they are indeed X-faces we show that their images in $K_{\mathbb{Q}}$ contain $-\mathcal{K}_X$ in their relative interior. Since all of the cones are pointed it suffices to show that $-\mathcal{K}_X$ can be written as a strictly positive combination over all extremal rays of the respective cone. For this let $n^{(1)}$ be the number of indices i with $n_i = 1$. Then we have

$$-\mathcal{K}_X = \frac{2r - n^{(1)}}{2} \deg(g) + \sum w_k \quad \text{where} \quad 0 < \frac{2r - n^{(1)}}{2} \leq r.$$

In case (i) we have $\deg(g) = 2w_{(r-1)1} = 2w_{r1}$ and in case (ii) we have $\deg(g) = w_{01} + w_{02} = 2w_{r1}$, which proves the assertion in these cases. Finally, in case (iii) we have $n^{(1)} = 0$ and thus obtain

$$-\mathcal{K}_X = r \deg(g) + \sum w_i = (w_{01} + w_{02}) + (r-1)(w_{11} + w_{12}) + \sum w_k.$$

□

Proof of Proposition 5.1.3. We distinguish between the three cases treated in Lemma 5.2.1 and show that the dimension of the X-faces occurring there is at most $m + 3$. Then using $\mathbb{Q}$-factoriality of X, we obtain the bound on $\varrho(X)$ as claimed. In Case (i) of Lemma 5.2.1 we obtain an X-face of dimension at most $1 + m$ as $w_{r1} = w_{(r-1)1}$ holds due to homogeneity of the defining relation g. Similar, in Case (ii) we obtain an X-face of dimension at most $2 + m$ as $w_{r1} \in \operatorname{cone}(w_{01}, w_{02})$ holds. Consider Case (iii). Here we have

$$w_{01} + w_{02} - w_{11} - w_{12} = \deg(g) - \deg(g) = 0.$$

In particular, the cone $\operatorname{cone}(w_{01}, w_{02}, w_{11}, w_{12}, w_1, \ldots, w_m)$ is of dimension at most $3+m$. This completes the proof. □

Proof of Corollary 5.1.4. In this situation we have $s = 1$ for the lower part of the matrix P. This implies $m \leq 2$, as the columns of P are assumed to be pairwise different and primitive, which directly gives the assertion. □

Proposition 5.2.2. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a three-dimensional $\mathbb{Q}$-factorial Fano intrinsic quadric of complexity $c = 2$. Then $\varrho(X) \leq 3$ holds. Moreover, if X is a full intrinsic quadric, then $\varrho(X) = 1$ holds.*

Proof. Due to Corollary 5.1.4 we have $\varrho(X) \leq 5$. Assume $\varrho(X) \geq 4$. We go through the possible configurations of $n = n_0 + \ldots + n_3$ and $0 \leq m \leq 2$. After renumbering the columns of P we arrive at one of the following cases:

(i) $n_0 = \ldots = n_2 = 2 > n_3 = 1,\ m = 1$

(ii) $n_0 = n_1 = 2 > n_2 = n_3 = 1,\ m = 2$

(iii) $n_0 = \ldots = n_3 = 2,\ m = 1$

(iv) $n_0 = \ldots = n_2 = 2 > n_3 = 1,\ m = 2$

In the Cases (i) and (ii) we have $\varrho(X) = 4$ and in the Cases (iii) and (iv), we have $\varrho(X) = 5$. Applying Lemma 5.2.1 (ii) in the Cases (i) and (iv) we obtain a three-dimensional X-face which contradicts $\mathbb{Q}$-factoriality of X. Similar, applying Lemma 5.2.1 (i) in the Cases (ii) and Lemma 5.2.1 (iii) in the Case (iii) we obtain a three-dimensional X-face and thus a contradiction to $\mathbb{Q}$-factoriality as well. For the supplement let X be a full intrinsic quadric and assume $\varrho(X) > 1$. Due to Proposition 5.1.3 we have $\varrho(X) \leq 3$. Thus renumbering the columns of P we are left with the following situations:

(i) $n_0 = n_1 = 2 > n_2 = n_3 = 1$

(ii) $n_0 = n_1 = n_2 = 2 > n_3 = 1$

In Case (i) we have $\varrho(X) = 2$ and in Case (ii) we have $\varrho(X) = 3$. Using the same argument as before we exclude Case (i) using Lemma 5.2.1 (i) and Case (ii) using Lemma 5.2.1 (ii). □

5.3 Classification in the complexity one case

The aim of this section is to provide the list of all intrinsic quadrics in Theorem 5.1.1 having a torus action of complexity one. We work in the language of explicit general arrangement varieties $X(A, P, \Sigma) \subseteq Z$. In particular, we use results from Sections 1.4 and 2.2 concerning the combinatorial data encoding the geometry of these varieties and the explicit description of their anticanonical complexes from Section 4.5. In a first step we distinguish the possible configurations for the parameters n and m. Then, we proceed by investigating these configurations case by case, see 5.3.2, 5.3.5 and 5.3.9. In each configuration we give effective bounds on the entries of P to finally test all varieties $X(A, P, \Sigma) \subseteq Z$ for canonicity and obtain our classification result.

Remark 5.3.1. Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial explicit intrinsic quadric of dimension three and Picard number one. Then $n + m = 5$ holds and we are in one of the following situations.

(i) $n = 3$ and $m = 2$.

(ii) $n = 4$ and $m = 1$.

(iii) $n = 5$ and $m = 0$.

Setting 5.3.2. Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano explicit intrinsic quadric of dimension three and Picard number and complexity one, having at most canonical singularities with $n = 3$ and $m = 2$. Then the matrix P has the following form:

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix}$$

Remark 5.3.3. Situation as in 5.3.2. As X has Picard number one, we obtain a big cone and an associated vertex of the anticanonical complex of X:

$$\sigma = \operatorname{cone}(v_{01}, v_{11}, v_{21}) \in \Sigma, \qquad v'_\sigma = [0, 0, x_1 + x_2 + x_3, y_1 + y_2 + y_3].$$

In particular, forgetting about the first two coordinates, the anticanonical complex of X intersected with the lineality space is a lattice polytope

$$\Delta := \operatorname{conv}([x_4, y_4], [x_5, y_5], [z_1, z_2]), \qquad [z_1, z_2] := [x_1 + x_2 + x_3, y_1 + y_2 + y_3].$$

As X has at most canonical singularities, the origin is the only interior lattice point of Δ. Thus, by applying admissible operations on the last two rows of P, we may assume that Δ is one of the 16 two-dimensional reflexive polytopes [10, 71, 58]. In particular, as Δ has three vertices, we may assume that it is one of the following.

$$\operatorname{conv}([1,0],[0,1],[-1,-1]),\ \operatorname{conv}([1,1],[-1,1],[0,-1]),\ \operatorname{conv}([1,1],[-1,1],[-1,-2]),$$

$$\operatorname{conv}([1,1],[-1,1],[-1,-3]),\ \operatorname{conv}([2,1],[-1,1],[-1,-2]).$$

Remark 5.3.4. Situation as in 5.3.3. Then, the vertices of Δ are invariant under adding a multiple of the first two rows of P to one of the last two rows of P. Thus we may assume in addition, that we have $x_2, x_3, y_2, y_3 \in \{0, 1\}$. Note that any such choice fixes all entries of P, due to the definition of the vertex $[z_1, z_2]$.

Setting 5.3.5. Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano explicit intrinsic quadric of dimension three and Picard number and complexity one, having at most canonical singularities with $n = 4$ and $m = 1$. By applying admissible operations we may assume to be in the following situation:

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & y_3 & y_4 & y_5 \end{bmatrix}, \qquad \begin{array}{c} x_2 > 0, \\ 0 < x_5 \le |y_5|, \\ x_4, y_4 \in \{0, 1\}. \end{array}$$

Moreover, by multiplying the last row with (-1), if necessary, we may assume that we have positive weights:

$$\begin{aligned} w_{01} &= 4x_2y_5 + 2x_3y_5 - 2x_5y_3 + 2x_4y_5 - 2x_5y_4, \\ w_{02} &= -2x_3y_5 + 2x_5y_3 - 2x_4y_5 + 2x_5y_4, \\ w_{11} &= 2x_2y_5, \\ w_{21} &= 2x_2y_5, \\ w_1 &= -2x_2y_3 - 2x_2y_4. \end{aligned}$$

Note that the last row operation possibly changes the sign of y_4. Thus we may only assume that $y_4 \in \{-1, 0, 1\}$ holds.

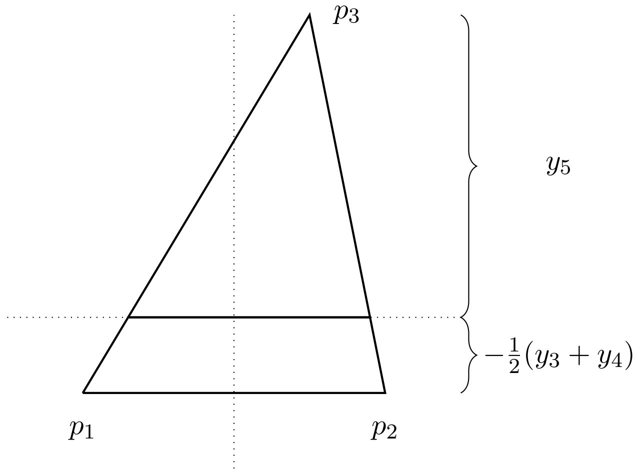
Remark 5.3.6. Situation as in 5.3.5. As X has Picard number one, we obtain two big cones with associated vertices of the anticanonical complex of X:

$$\begin{aligned} \sigma_1 &= \mathrm{cone}(v_{01}, v_{11}, v_{21}), & v'_{\sigma_1} &= [0, 0, \tfrac{1}{2}(x_3 + x_4), \tfrac{1}{2}(y_3 + y_4)]) \\ \sigma_2 &= \mathrm{cone}(v_{02}, v_{11}, v_{21}), & v'_{\sigma_2} &= [0, 0, \tfrac{1}{2}(x_3 + x_4) + x_2, \tfrac{1}{2}(y_3 + y_4)]. \end{aligned}$$

In particular, forgetting about the first coordinates, the anticanonical complex of X intersected with the lineality space, is a triangle $\Delta = \mathrm{conv}(p_1, p_2, p_3)$, with

$$p_1 = [\tfrac{1}{2}(x_3 + x_4), \tfrac{1}{2}(y_3 + y_4)], \quad p_2 = [\tfrac{1}{2}(x_3 + x_4) + x_2, \tfrac{1}{2}(y_3 + y_4)], \quad p_3 = [x_5, y_5].$$

Remark 5.3.7. Situation as in 5.3.6. We investigate the polytope Δ. First note, that by assumption $x_5 > 0$ holds and we obtain $y_5 > 0$, as x_2 and $w_{11} = 2x_2y_5$ are positive. In particular, the vertex p_3 is contained in the positive orthant. Moreover, as w_1 is positive, we conclude $y_3 + y_4 < 0$ and thus the points p_1 and p_2 are contained in the lower half plane. Note that the line segment $\overline{p_1p_2}$ is parallel to the x-axis. As X is Fano, we have $0 \in \Delta^\circ$ and conclude $x_3 + x_4 < 0$, as x_2 is positive. We sketch the situation:



Note, that we can not determine the position of p_2 with respect to the y-axis.

Proposition 5.3.8. *Situation as in 5.3.5. Then we obtain the following estimates for the entries of P:*

$$0 < x_2 \leq 2 - (y_3 + y_4), \qquad 0 \leq x_4 \leq 1, \qquad 0 < x_5 \leq |y_5|,$$

$$-18 - y_4 \leq y_3 < -y_4, \qquad -1 \leq y_4 \leq 1$$

$$\frac{2x_5y_3 + 2x_5y_4 - 2x_4y_5 - 4x_2y_5}{2y_5} < x_3 < -x_4, \qquad 0 < y_5 \leq \begin{cases} 9 & x_2 = 1 \\ \frac{x_2 - \frac{1}{2}(y_3+y_4)}{x_2 - 1} & \textit{else.} \end{cases}$$

Proof. Note that by assumption $x_2 > 0$, $x_4 \in \{0,1\}$, $y_4 \in \{-1,0,1\}$ and $0 < x_5 \leq |y_5|$ hold. Now, positivity of the weights w_{01}, w_{11} and w_1 imply

$$\frac{2x_5y_3 + 2x_5y_4 - 2x_4y_5 - 4x_2y_5}{2y_5} < x_3, \qquad 0 < y_5 \qquad \text{and} \qquad y_3 < -y_4.$$

Moreover, similar as in Remark 5.3.7, we have $\frac{1}{2}(x_3 + x_4) < 0$ and conclude $x_3 < -x_4$. We investigate slices of the polytope Δ: Due to the singularity type of X, we have

$$|\Delta \cap \{y = 0\}| = x_2 + x_2 \frac{\frac{1}{2}(y_3 + y_4)}{y_5 - \frac{1}{2}(y_3 + y_4)} \leq 2.$$

Thus, reordering suitably and using $y_5 > 0$ yields

$$x_2 \leq 2 - \frac{(y_3 + y_4)}{y_5} \leq 2 - (y_3 + y_4).$$

Similarly, we have

$$\Delta \cap \{y = 1\} = x_2 - x_2 \frac{1 + \frac{1}{2}(y_3 + y_4)}{y_5 - \frac{1}{2}(y_3 + y_4)} \leq 1.$$

In particular, if $x_2 \neq 1$ holds, this implies

$$y_5 \leq \frac{x_2 - \frac{1}{2}(y_3 + y_4)}{x_2 - 1}.$$

We proceed by investigating the tetrahedron Δ' defined by the following vertices:

$$[0, 0, x_5, y_5], \qquad [-1, -1, x_2, 0],$$

$$[0, 0, \tfrac{1}{2}(x_3 + x_4), \tfrac{1}{2}(y_3 + y_4)], \qquad [0, 0, \tfrac{1}{2}(x_3 + x_4) + x_2, \tfrac{1}{2}(y_3 + y_4)].$$

Note that by construction Δ' is contained in the anticanonical complex of X and thus has the origin as its unique interior lattice point. The polytope Δ' is living inside the linear space spanned by $[1, 1, 0, 0], [0, 0, 1, 0]$ and $[0, 0, 0, 1]$. In particular, we may regard Δ' as a polytope in $\mathbb{Q}^3$ by forgetting about the first coordinate. Now, Δ' is contained in the lattice polytope Δ'' defined by the following vertices:

$$[-1, x_2, 0], \quad [1, x_3 + x_4 - x_2], \quad [1, x_3 + x_4 + x_2, y_3 + y_4], \quad [1, 2x_5 - x_2, 2y_5].$$

Note that by construction Δ'' is a lattice polytope having the origin as its unique interior lattice point. Thus, due to [8, Thm 2.2], its standard $\mathbb{Q}^3$-volume is bounded by 12 which gives

$$\frac{2}{3}x_3(y_3 + y_4) - \frac{4}{3}x_3 y_5 \leq 12 \tag{5.3.1}$$

Now, reordering yields

$$\frac{18}{x_3} + 2y_5 - y_4 \leq y_3$$

and as $\frac{1}{x_3} \geq -1$ and $y_5 > 0$ hold, we obtain at $-18 - y_4 \leq y_3$. Moreover, reordering 5.3.1 once more, we arrive at

$$-\frac{4}{3}x_3 y_5 \leq 12 - \frac{2}{3}x_3(y_3 + y_4).$$

Using positivity of $x_3(y_3 + y_4)$ and $-1 \leq \frac{1}{x_3} < 0$, we conclude $y_5 \leq 9$. □

Setting 5.3.9. Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano explicit intrinsic quadric of dimension three and Picard number and complexity one, having at most canonical singularities with $n = 5$ and $m = 0$. By applying admissible operations on P, we may assume to be in the following situation:

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & x_2 & x_3 & 0 & x_5 \\ 0 & 0 & y_3 & 0 & y_5 \end{bmatrix}, \qquad \begin{array}{c} x_2 > 0, \\ 0 < x_5 \leq |y_5|. \end{array}$$

Moreover, by multiplying the last row with (-1), if necessary, we may assume, that we have positive weights:

$$\begin{aligned} w_{01} &= 2x_2y_3 - x_3y_5 + x_5y_3, \\ w_{02} &= x_3y_5 - x_5y_3, \\ w_{11} &= -x_2y_5, \\ w_{12} &= 2x_2y_3 + x_2y_5, \\ w_{21} &= x_2y_3. \end{aligned}$$

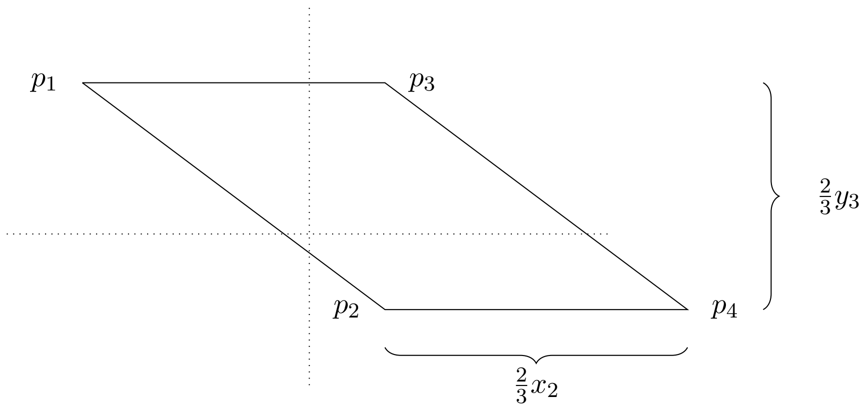
Remark 5.3.10. Situation as in 5.3.9. As X has Picard number one, we obtain four big cones with associated vertices of the anticanonical complex

$$\begin{array}{lllllll} \sigma_1 & = & \operatorname{cone}(v_{01}, v_{11}, v_{21}), & & v'_{\sigma_1} & = & [0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_3, \frac{1}{3}y_5 + \frac{2}{3}y_3], \\ \sigma_2 & = & \operatorname{cone}(v_{01}, v_{12}, v_{21}), & & v'_{\sigma_2} & = & [0, 0, \frac{1}{3}x_5, \frac{1}{3}y_5], \\ \sigma_3 & = & \operatorname{cone}(v_{02}, v_{11}, v_{21}), & & v'_{\sigma_3} & = & [0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_3 + \frac{2}{3}x_2, \frac{1}{3}y_5 + \frac{2}{3}y_3], \\ \sigma_4 & = & \operatorname{cone}(v_{02}, v_{12}, v_{21}), & & v'_{\sigma_4} & = & [0, 0, \frac{1}{3}x_5 + \frac{2}{3}x_2, \frac{1}{3}y_5]. \end{array}$$

In particular, forgetting about the first coordinates, the anticanonical complex of X intersected with the lineality space is a trapezoid $\Delta = \operatorname{conv}(p_1, p_2, p_3, p_4)$, with

$$\begin{array}{llllll} p_1 & = & [\frac{1}{3}x_5 + \frac{2}{3}x_3, \frac{1}{3}y_5 + \frac{2}{3}y_3], & p_2 & = & [\frac{1}{3}x_5, \frac{1}{3}y_5], \\ p_3 & = & [\frac{1}{3}x_5 + \frac{2}{3}x_3 + \frac{2}{3}x_2, \frac{1}{3}y_5 + \frac{2}{3}y_3], & p_4 & = & [\frac{1}{3}x_5 + \frac{2}{3}x_2, \frac{1}{3}y_5]. \end{array}$$

Remark 5.3.11. Situation as in 5.3.10. We investigate the polytope Δ. In a first step, we determine the position of its vertices relative to the x- and y-axis. First note that by assumption x_2 and x_5 are positive and by positivity of the weight $w_{11} = -x_2y_5$ we obtain $y_5 < 0$. In particular, we have $(p_2)_1, (p_4)_1 > 0$ and $(p_2)_2, (p_4)_2 < 0$. Moreover, as X is Fano, we obtain $0 \in \Delta^\circ$ and thus $(p_1)_1 < 0$ and $(p_1)_2 > 0$ due to the positivity of x_2. We sketch the situation:



Note that we can not determine the position of p_3 with respect to the y-axis.

Proposition 5.3.12. *Situation as in 5.3.9. Then we obtain the following estimates for the entries of P:*

$$0 < x_2 \leq 3, \qquad \frac{2x_2y_3 + x_5y_3}{y_5} < x_3 < \frac{x_5y_3}{y_5}, \qquad 0 < x_5 \leq |y_5|,$$
$$0 < y_3 \leq \frac{72}{3(x_2+1)}, \qquad -2y_3 < y_5 < 0.$$

Proof. Note that by assumption $x_2 > 0$ and $0 < x_5 \leq |y_5|$ holds. Now, positivity of the weights w_{21}, w_{11} and w_{12} imply

$$y_3 > 0, \qquad y_5 < 0 \qquad \text{and} \qquad -2y_3 < y_5.$$

Thus, using positivity of w_{01} and w_{02}, we conclude

$$x_3 > \frac{2x_2y_3 + x_5y_3}{y_5} \qquad \text{and} \qquad x_3 < \frac{x_5y_3}{y_5}.$$

Now, due to the singularity type of X, the $y = 0$ slice of Δ implies

$$|\Delta \cap \{y = 0\}| = \frac{2}{3}x_2 \leq 2$$

and thus $x_2 \leq 3$ holds. We proceed by investigating the pyramid

$$\Delta' := \operatorname{conv}([0, 2, x_5, y_5], v'_{\sigma_1}, \ldots, v'_{\sigma_4}),$$

By construction Δ' is contained in the anticanonical complex of X and by deleting the first coordinate, we may regard Δ' as a polytope inside $\mathbb{Q}^3$ having the origin as its unique interior lattice point, due to the singularity type of X. We proceed by modifying $\Delta' \subseteq \mathbb{Q}^3$. By extending the edges starting in $[2, x_5, y_5]$, we enlarge Δ' to the lattice polytope Δ'' having the following vertices:

$$[2, x_5, y_5], \qquad [-1, x_3, y_3], \qquad [-1, 0, 0], \qquad [-1, x_2 + x_3, y_3], \qquad [-1, x_2, 0].$$

Note that by construction Δ'' still has the origin as its unique interior lattice point. Thus, due to [8, Thm. 2.2], its standard $\mathbb{Q}^3$-volume is bounded by 12 and we conclude

$$y_3 \leq \frac{72}{3(x_2+1)}.$$

□

Proof of Theorem 5.1.1 (complexity one). Due to Remark 5.3.4 and Propositions 5.3.8 and 5.3.12 we only have finitely many possible Fano varieties $X(A, P, \Sigma)$ to check. Computing the anticanonical complex for all possible configurations the resulting canonical Fano varieties are listed in Theorem 5.1.1 with Nos. 1 - 63, 65 - 68, 70 - 76 and 78. Note that none of them is toric as their total coordinate spaces have a singularity at the origin. □

5.4 Classification in the complexity two case

This section is dedicated to the proof of Theorem 5.1.2, meaning that we classify all $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three and true complexity two having at most canonical singularities. In particular, we obtain all varieties having Picard number one and appear as Nos. 64, 69, 77 and 79 in Theorem 5.1.1, which completes the classification list. As an application we prove Proposition 5.1.5. As before, we work in the language of explicit general arrangement varieties.

Remark 5.4.1. Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano intrinsic quadric of dimension three with torus action of complexity two. Then the dimension of the total coordinate space and the Picard number are given as

$$\dim(\overline{X}) = n + m - 1, \quad \varrho(X) = n + m - 4, \quad 4 \leq n \leq 8, \quad 0 \leq m \leq 2.$$

Remark 5.4.2. Let $X := X(A, P, \Sigma) \subseteq Z$ be a Fano explicit general arrangement variety with anticanonical complex $\mathcal{A}$. Then every convex combination of vertices of $\mathcal{A}$ that lie inside the tropical variety $\mathrm{trop}(X) \subseteq \mathbb{Q}^{r+s}$ is contained in $\mathcal{A}$. In particular, if X is of dimension three, has a torus action of complexity two and at most canonical singularities, then for any such point $v = (v_1, \ldots, v_{r+1})$ that lies inside the lineality space of the tropical variety $\mathrm{trop}(X)$, we have $-1 \leq v_{r+1} \leq 1$.

Let X be a Mori dream space. In order to detect a maximal torus action on X we will make use of the procedure of *lifting automorphisms*. Assume there is a torus action $\mathbb{T} \times X \to X$. Then due to [6, Thm. 4.2.3.2] there is a lifted action $\mathbb{T} \times \hat{X} \to \hat{X}$ with

$$t \cdot (h \cdot \hat{x}) = h \cdot (t \cdot \hat{x}) \qquad \text{for all } t \in \mathbb{T},\ h \in H_X,\ \hat{x} \in \hat{X}.$$

Thus $\mathbb{T}$ as well as the product $\mathbb{T} \times H_X$ act on $\hat{X}$ and therefore on $\bar{X}$ as $\hat{X} \subseteq \bar{X}$ is of codimension two. We will identify both groups with the corresponding subgroups of translations inside the automorphism group $\mathrm{Aut}(\bar{X})$.

We will show that in our situation this action is diagonal in the following sense: Let $X \subseteq \mathbb{K}^n$ be an affine variety endowed with an effective quasitorus action $H \times X \to X$. We say that H *acts diagonally* on X if there are characters $\chi^{w_1}, \ldots, \chi^{w_n} \in \mathbb{X}(H)$ such that $h \cdot (x_1, \ldots, x_n) = (\chi^{w_1}(h)x_1, \ldots, \chi^{w_n}(h)x_n)$ holds for all $h \in H$ and $(x_1, \ldots, x_n) \in X$. Note that this is equivalent to homogeneity of the coordinate functions $T_i \in \mathcal{O}(X)$, where we endow $\mathcal{O}(X)$ with the grading corresponding to the action of H on X.

Lemma 5.4.3. *Let X be a Mori dream space with torus action $\mathbb{T} \times X \to X$ and Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle$. If all homogeneous components $\mathcal{R}(X)_{w_i}$ with $w_i = \deg(T_i)$ are one-dimensional then $\mathbb{T} \times H_X \subseteq \mathrm{Aut}(\bar{X})$ acts diagonally.*

Proof. Consider the grading on $\mathcal{R}(X)$ defined by the action $\mathbb{T} \times H_X$ on $\bar{X}$. Then the K_X-grading is a coarsening of this grading and thus, as the homogeneous components $\mathcal{R}(X)_{w_i}$ are one-dimensional, they are homogeneous components in this refined grading as well. Thus the assertion follows. □

Proof of Theorem 5.1.2. According to Proposition 5.2.2 we have $\varrho(X) \le 3$. We first go through the cases sorted by the Picard number and then prove that none of the varieties in the list are isomorphic. Finally, we prove that they are of true complexity two by determining the dimension of the maximal tori in their automorphism groups.

Case (I) ($\varrho(X) = 1$): Due to Remark 5.4.1 we are left with the following configurations:

(a) $n = 4$ and $m = 1$

(b) $n = 5$ and $m = 0$

Case (I)(a): As in the case of Picard number one any $\bar{X}$-face is an X-face, we obtain a big cone $\sigma = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}) \in \Sigma$. After applying suitable row operations on the matrix P, we may assume

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ x & 1 & 1 & 1 & -1 \end{bmatrix}, \qquad v'_\sigma = \left[0, 0, 0, \frac{x+3}{2}\right].$$

As X has at most canonical singularities, we conclude $0 < (x+3)/2 \le 1$ and thus $x = -1$ as the columns of P are primitive. The resulting variety $X(A, P, \Sigma)$ is canonical and appears as No. 1 in our list.

Case (I)(b): After suitably renumbering we may assume $n_0 = 2$ and with $\varrho(X) = 1$ we obtain the following two big cones in Σ:

$$\sigma_j := \operatorname{cone}(v_{0j}, v_{11}, v_{21}, v_{31}), \qquad \text{where } 1 \le j \le 2.$$

Moreover, after applying suitable row operations, the matrix P and the vectors v'_{σ_1} and v'_{σ_2} are of the following form:

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ x & y & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{aligned} v'_{\sigma_1} &= \left[0, 0, 0, \tfrac{2x+3}{3}\right], \\ v'_{\sigma_2} &= \left[0, 0, 0, \tfrac{2y+3}{3}\right], \end{aligned}$$

where $x < y$ holds. As X has at most canonical singularities, we conclude $0 < (2x + 3)/3 \le 1$ and $-1 \le (2y + 3)/3 < 0$. This implies $x \in \{-1, 0\}$ and $y \in \{-3, -2\}$. Computing their anticanonical complexes shows that all of the possible varieties $X(A, P, \Sigma)$ are canonical. Note that for $x = -1$, $y = -3$ and $x = 0$, $y = -2$ the resulting rings $R(A, P)$ are isomorphic. All in all this gives the varieties Nos. 2 to 4 in our list.

Case (II) ($\varrho(X) = 2$): Due to Remark 5.4.1 we are in one of the following situations:

(a) $n = 4$ and $m = 2$

(b) $n = 5$ and $m = 1$

(c) $n = 6$ and $m = 0$

Case (II)(a): After applying suitable row operations, we arrive at

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ x & 1 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

We obtain a point

$$(0, 0, 0, \frac{x+3}{4}) = \frac{1}{4}(v_{01} + v_{11} + v_{21} + v_{31}) \in \mathrm{conv}(v_{01}, v_{11}, v_{21}, v_{31}) \cap |\mathrm{trop}(X)|.$$

Remark 5.4.2 implies $-1 \leq 1/4(x+3) \leq 1$ and thus $-7 \leq x \leq 1$. Assume x is even. Then the first column of P is not primitive; a contradiction. For $x \in \{-7, -5, -1, 1\}$, calculating the anticanonical divisor class shows that the resulting varieties are not Fano. Thus, the only possible case left is $x = -3$. In this situation, computing the anticanonical complex shows that the resulting variety is a canonical Fano variety, which appears as No. 5 in our list.

Case (II)(b): We may assume n_0=2 and after applying suitable row operations we arrive at

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ x & y & 1 & 1 & 1 & -1 \end{bmatrix},$$

where we may assume $x \leq y$. Note that due to completeness of X, we have $|\mathrm{trop}(X)| \subseteq |\Sigma|$. Therefore, we obtain a big cone σ containing $[0, 0, 0, 1]$ and a vertex v'_σ of $\mathcal{A}$

$$\sigma = \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_{31}), \qquad v'_\sigma = [0, 0, 0, 1 + \frac{2}{3}y].$$

Due to canonicity of X we conclude $0 \leq 1 + (2/3)y \leq 1$ and thus $-1 \leq y \leq 0$. Now consider the point

$$[0, 0, 0, \frac{2x+3}{5}] = \frac{1}{5}(2v_{01} + v_{11} + v_{21} + v_{31}) \in \mathrm{conv}(v_{01}, v_{11}, v_{21}, v_{31}) \cap |\mathrm{trop}(X)|.$$

Using Remark 5.4.2 we obtain $-1 \leq 1/5(2x+3) \leq 1$ and thus $-4 \leq x \leq 1$. Computing the anticanonical complex in these cases gives Nos. 6 to 8 in our list.

Case (II)(c): In this case X is a full intrinsic quadric and therefore Proposition 5.2.2 implies $\varrho(X) = 1$, a contradiction to $\varrho(X) = 2$.

Case (III) ($\varrho(X) = 3$): Due to Remark 5.4.1 and Lemma 5.2.1 (i) we may assume $2 = n_0 > n_1 = \ldots = n_3 = 1$ and $m = 2$. After applying suitable row operations we arrive at

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ x & y & 1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

where we may assume $x \le y$. Consider the point

$$[0, 0, 0, \frac{2x+3}{5}] = \frac{1}{5}(2v_{01} + v_{11} + v_{21} + v_{31}) \in \operatorname{conv}(v_{01}, v_{11}, v_{21}, v_{31}) \cap |\operatorname{trop}(X)|.$$

Using Remark 5.4.2 we obtain $-1 \le 1/5(2x+3) \le 1$ and thus $-4 \le x \le 1$. Replacing v_{01} with v_{02} in the above calculation, we obtain $-4 \le y \le 1$ as well. Computing the anticanonical complex in these cases gives No. 9 in our list.

We proceed by proving that the varieties defined by the data in our list are pairwise non-isomorphic.

Considering the divisor class groups, the only possible combinations to compare are Nos. 2 and 3 and Nos. 6, 7 and 8. The Fano index of a Fano explicit general arrangement variety $X = X(A, P, \Sigma)$ is the largest integer $q(X)$ such that $-\mathcal{K}_X = q(X)w$ holds with some $w \in \operatorname{Cl}(X)$. If X is isomorphic to another general arrangement variety X' then their Fano indices coincide. Denote by X_i the explicit general arrangement variety defined by the i-th datum in our list. Then the varieties X_2 and X_3, X_6 and X_8 and X_7 and X_8 are not isomorphic due to the following table:

X_i	X_2	X_3	X_6	X_7	X_8
$q(X_i)$	3	6	1	1	2

Thus, we are left with comparing Nos. 6 and 7. The effective cone of an explicit general arrangement variety $X = X(A, P, \Sigma)$ is the cone

$$\operatorname{Eff}(X) = \operatorname{cone}(w_{ij}, w_k,\ 0 \le i \le r, 1 \le j \le n_i, 1 \le k \le m).$$

If X is isomorphic to another general arrangement variety X' then there is a lattice isomorphism mapping the extremal primitive ray generators of $\operatorname{Eff}(X)$ onto that of $\operatorname{Eff}(X')$. Considering the varieties X_6 and X_7 their effective cones are $\operatorname{Eff}(X_6) = \operatorname{cone}([1, -1], [0, 1])$ and $\operatorname{Eff}(X_7) = \operatorname{cone}([-1, 2], [1, 0])$. Thus the varieties are not isomorphic as $\operatorname{Eff}(X_6)$ is a smooth cone whereas $\operatorname{Eff}(X_7)$ is not.

We finish the proof by showing that the varieties in our list are of true complexity two. We treat all varieties except the one encoded by the 2nd datum of our list at once. Let X be any of these varieties and assume $\mathbb{T} \times X \to X$ is a maximal torus action on X. As all the homogeneous components $\mathcal{R}(X)_{w_i}$ are one-dimensional Lemma 5.4.3 applies and we conclude that in these cases the generators T_i resp. S_k are homogeneous with respect to the grading defined via the $(\mathbb{T} \times H_X)$-action on $\bar{X}$. As the ideal defining $\bar{X}$ is principle, the relation is homogeneous as well and thus $\mathbb{T} \times H_X$ acts as a sub-quasitorus of the maximal quasitorus defined via the maximal diagonal grading, see [6, Constr. 3.2.4.2]. Modding out the H_X-action yields that $\mathbb{T}$ is indeed one-dimensional.

Now, let X be the variety encoded by the 2nd datum in our list. Here, the homogeneous components $\mathcal{R}(X)_{w_i}$ are one-dimensional for $i \geq 3$. In particular, the variables T_3, T_4 and T_5 are homogeneous with respect to the $(\mathbb{Z}^t \times K_X)$-grading defined via the $(\mathbb{T} \times H_X)$-action on $\bar{X}$. Considering the 2-dimensional graded component $\mathcal{R}(X)_{w_1} = \mathcal{R}(X)_{w_2}$, one concludes that there exists a $(\mathbb{Z}^t \times K_X)$-homogeneous set of generators of the form $T_1 + \lambda T_2,\ \mu T_1 + T_2,\ T_3,\ T_4$ and $T_5 \in \mathcal{R}(X)$, where $\lambda, \mu \in \mathbb{K}$. We obtain a graded isomorphism between $\mathcal{R}(X)$ and

$$R := \mathbb{K}[f_1, f_2, f_3, f_4, f_5]/\langle \mu f_1^2 - (\lambda\mu + 1) f_1 f_2 + \lambda f_2^2 - (\lambda\mu - 1)^2 (T_3^2 + T_4^2 + T_5^2) \rangle,$$

where the variables and the relation of the latter ring are $(\mathbb{Z}^t \times K_X)$-homogeneous. We conclude that the $(\mathbb{Z}^t \times K_X)$-grading on R is a coarsening of its maximal diagonal grading. Modding out K_X, we conclude $t \leq 1$ and $\mathbb{T}$ is indeed one-dimensional. □

Proof of Proposition 5.1.5. Denote by $n^{(1)}$ resp. $n^{(2)}$ the number of terms of g with $n_i = 1$ resp. $n_i = 2$, where $0 \leq i \leq r$. Then, as X is $\mathbb{Q}$-factorial and of complexity $c = \dim(X) - 1$, the dimension and the Picard number of X are given as

$$\dim(X) = c + 1 = n^{(1)} + n^{(2)} - 1, \qquad \varrho(X) = n + m - r - 1 = n^{(2)} + m.$$

In particular, using Proposition 5.1.3 we conclude $\dim(X) = \varrho(X) - m + n^{(1)} - 1 \leq n^{(1)} + 2$. In case that X is of dimension two, terminality means smoothness and the assertion follows due to the classification of smooth Del Pezzo surfaces, see [27, 62]. In case that X is of dimension 3, Theorem 5.1.2 shows that there exist no terminal varieties. Assume $\dim(X) \geq 4$. Then $n^{(1)} \geq 2$ holds and after reordering and applying admissible row operations we may assume that P contains the following two columns:

$$v_{(r-1)1} = (0, \ldots, 0, 2, 0, 1), \qquad v_{r1} = (0, \ldots, 0, 2, 1).$$

As X is complete and $c \geq 2$ holds we have $\operatorname{cone}(v_{(r-1)1}, v_{r1}) \in \Sigma$ and Remark 5.4.2 implies

$$(0, \ldots, 0, 1, 1, 1) \in \operatorname{conv}(v_{(r-1)1}, v_{r1}) \subseteq \mathcal{A}.$$

In particular, there is a lattice point in $\mathcal{A}$ which is neither the origin nor a primitive ray generator of Σ and therefore X can not be terminal. □

5.5 Proof of Theorem 5.1.1

In this section we prove the remaining assertions stated in Theorem 5.1.1. We begin by proving that all $\mathbb{Q}$-factorial Fano intrinsic quadrics of dimension three and Picard number one are $\mathbb{T}$-varieties of complexity one or two. This proves that Sections 5.3 and 5.4 provide indeed the full classification list.

Proposition 5.5.1. *Let $X = X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial Fano explicit intrinsic quadric of dimension three and Picard number one. Then X is either of complexity one or two.*

Proof. We consider the Cox ring $R(A, P)$ of X. By assumption we have $n + m = 5$ for the number of variables in $R(A, P)$. Thus, by renaming the variables, we may assume that $R(A, P) = \mathbb{K}[T_1, \ldots, T_5]/\langle g \rangle$ holds, where g is a quadratic polynomial contained in the following list:

(i) T_1^2, T_1T_2 or $T_1^2 + T_2^2$,

(ii) $T_1T_2 + T_3^2$ or $T_1T_2 + T_3T_4$,

(iii) any quadratic polynomial with three or four terms,

(iv) $T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2$.

If g is one of the polynomials in *(i)*, then $R(A, P)$ is not integral; a contradiction. Now assume g is one of the polynomials in *(ii)*. Then the finest possible grading on $R(A, P)$ leaving the variables $T_1, \ldots, T_5$ and the relation g homogeneous turns the total coordinate space of X into a toric variety. Thus X is toric, implying that its Cox ring is a polynomial ring. This contradicts the fact that its total coordinate space has a singularity at the origin. Finally, assume $g = T_1^2 + T_2^2 + T_3^2 + T_4^2 + T_5^2$ holds. Then we obtain

$$P_0 = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Therefore in order to make the columns of P primitive, P has to be quadratic, contradicting $\varrho(X) = 1$. Thus the only case left is *(iii)* which proves the assertion. □

Now we turn to the irredundancy of the classification list.

Remark 5.5.2. Let $X = X(A, P, \Sigma) \subseteq Z$ be an n-dimensional explicit intrinsic quadric. Then the following numbers are invariants of X:

(i) The *anticanonical self-intersection number* $-\mathcal{K}_X^n$, which can be directly computed via [6, Constr. 3.3.3.4].

(ii) The *Fano index* $q(X)$, which is defined as the largest integer $q(X)$, such that $-\mathcal{K}_X = q(X) \cdot w$ holds with some $w \in \mathrm{Cl}(X)$.

(iii) The *Picard index* $p(X)$, which is defined as the index of the Picard group inside the divisor class group. Note, that in our situation, the Picard group is given as

$$\mathrm{Pic}(X) = \bigcap_{\substack{\gamma_0 \preceq \gamma \\ X\text{-face}}} Q(\gamma_0 \cap \mathbb{Z}^{n+m}) \subseteq \mathrm{Cl}(X).$$

(iv) The dimension of the automorphism group dim(Aut(X)).

Moreover, if X is isomorphic to another intrinsic quadric $X' = X(A', P', \Sigma')$, then $\mathcal{R}(X)$ and $\mathcal{R}(X')$ are isomorphic as graded rings. In this case, the following holds:

(i) We have $\dim(\overline{X}^{\text{sing}}) = \dim(\overline{X'}^{\text{sing}})$.

(ii) There is a bijection between the set of generator degrees Ω_X and $\Omega_{X'}$.

(iii) The sets $\Omega_X^{\dim} := \{\dim(\mathcal{R}(X)_w);\ w \in \Omega_X\}$ and $\Omega_{X'}^{\dim}$ coincide.

Proposition 5.5.3. *The varieties defined by the data in Theorem 5.1.1 are pairwise non-isomorphic.*

Proof. We denote by X_i the Fano variety defined by the i-th datum in Theorem 5.1.1, by $\mathcal{R}_i$ its Cox ring, by $\overline{X_i}$ its total coordinate space and by $\Omega_i = \{w_1, \ldots, w_r\}$ its set of generator degrees. As the divisor class group, the Fano index and the anticanonical self-intersection number presented in Theorem 5.1.1 are invariants, we only need to compare those varieties X_i and X_j, where all these data coincide. The next table presents invariants of these varieties, where the cases to compare are divided via horizontal lines:

i	$p(X_i)$	dim(Aut(X$_i$))	$\dim(\overline{X_i}^{\text{sing}})$
16	24	2	1
17	24	2	0
20	48	2	1
21	24	2	1
27	240	2	1
28	120	2	1
30	24	2	1
31	48	2	1
33	9	2	0
34	9	2	0
35	54	2	0
36	18	2	0
41	16	2	0
42	16	2	1
43	8	2	1
46	48	2	1
47	48	2	1
50	36	2	0
51	36	2	0

52	36	2	1
55	64	2	1
56	64	2	1
62	8	2	2
63	8	2	1
64	8	1	0
67	48	2	2
68	48	2	1
69	48	1	0
75	72	2	2
76	72	2	1
77	72	1	0

There are only 4 cases left, that can not be distinguished via the table above. We treat them in the following paragraphs:

X_{33} and X_{34}. In this case the homogeneous component of $\mathcal{R}_{33}$ of degree $(1,0) \in \Omega_{33}$ has dimension three. This is in contrast to to R_{34}, where the maximal dimension of the homogeneous components with respect to the generator degrees in Ω_{34} is two.

X_{46} and X_{47}. In this situation, all homogeneous components of $\mathcal{R}_{46}$ with respect to the weights in Ω_{46} are one-dimensional which is in contrast to the two-dimensional homogeneous component of $\mathcal{R}_{47}$ of degree $(2,0) \in \Omega_{47}$.

X_{50} and X_{51}. Note, that due to Remark 5.5.2, we have a bijection $\Omega_{50} \to \Omega_{51}$. Now $|\Omega_{50}| = 5$ which is in contrast to $|\Omega_{51}| = 4$.

X_{55} and X_{56}. Assume there is a graded isomorphism $\mathcal{R}_{55} \to \mathcal{R}_{56}$. Then we have an isomorphism $\mathrm{Cl}(X_{55}) \to \mathrm{Cl}(X_{56})$ mapping Ω_{55} onto Ω_{56}. We go through the possible images of $(1,\bar{1}) \in \Omega_{55}$: Assume that $(1,\bar{1})$ is mapped on either $(1,\bar{1})$ or $(1,\bar{5})$. Then $(2,\bar{2}) \in \Omega_{55}$ is mapped on $(2,\bar{2})$ which is not contained in Ω_{56}; a contradiction. Now assume $(1,\bar{1})$ is mapped on $(1,\bar{0})$ or $(1,\bar{2})$ then $(2,\bar{2})$ is mapped on either $(2,\bar{0})$ or $(2,\bar{4})$ which are not contained in Ω_{56}; a contradiction; Finally assume that $(1,\bar{1})$ is mapped on $(2,\bar{6})$. Then $(2,\bar{2})$ is mapped on $(4,\bar{4})$ which is again not contained in Ω_{56}; a contradiction. This implies that there is no graded isomorphism $\mathcal{R}_{55} \to \mathcal{R}_{56}$ and thus X_{55} and X_{56} can not be isomorphic. □

CHAPTER

SIX

SPECIAL ARRANGEMENT VARIETIES

In this chapter we investigate the example class of *special arrangement varieties*, i.e. arrangement varieties, where the collection of doubling divisors forms a hyperplane arrangement in special position. In Section 6.1, we give explicit descriptions of their Cox rings and investigate their realizations as explicit $\mathbb{T}$-varieties. As it turns out, some special arrangement varieties admit a torus action turning them into a general arrangement variety. We refer to the others as *honestly special arrangement varieties.* As a first result, we show in Theorem 6.3.4 that honestly special arrangement varieties of Picard number at most two are never smooth. Proceeding with the singular case, we prove that all arrangement varieties admit anticanonical complexes, see Theorem 6.5.1. In Section 6.5, we obtain an explicit description of the anticanonical complex for special arrangement varieties in case that the weakly tropical resolution is a toric ambient modification as defined in [6]. As an application we obtain classification results in the case of three-dimensional canonical Fano honestly special arrangement varieties of complexity two and divisor class group of rank at most two, see Theorem 6.6.2. The results of this chapter are published in the joint work [50].

6.1 Arrangement varieties and their Cox rings

In Chapter 2 we have introduced the general arrangement varieties as an example class for explicit $\mathbb{T}$-varieties. In this section we extend the description to *special arrangement varieties*, i.e. arrangement varieties having a hyperplane arrangement in special position as their doubling divisors. In the first part of this section we recall and adapt the notions from Chapter 2 to arbitrary arrangement varieties. In particular, we explicitly describe their Cox rings and their realization as explicit $\mathbb{T}$-varieties. Accompanying the reader, we have the running example 6.1.1, 6.1.5, 6.1.8, 6.1.13, 6.1.16 and 6.1.21. The proofs of the statements in this section are presented in the subsequent section.

Recall, that a *projective hyperplane arrangement* $H_0, \ldots, H_r$ in $\mathbb{P}_n$ is called *in general position*, if for every choice $0 \le i_1 < \ldots < i_k \le r$, the intersection $H_{i_1} \cap \ldots \cap H_{i_k}$ is of codimension k and otherwise, it is called in *special position*.

Example 6.1.1. Consider the following collection of lines in $\mathbb{P}_2$:

$$H_0 := V(T_0), \qquad H_1 := V(T_1), \qquad H_2 := V(T_2)$$

$$H_3 := V(T_0 + T_1), \qquad H_4 := V(T_0 + T_2).$$

Then $H_0, \ldots, H_r$ is a projective hyperplane arrangement in special position.

Definition 6.1.2 (See Def. 2.1.1)**.** A *(general / special) arrangement variety* is a variety X with an effective torus action $\mathbb{T} \times X \to X$ having $\pi \colon X \dashrightarrow \mathbb{P}_c$ as a maximal orbit quotient and the critical values form a projective hyperplane arrangement in general (special) position.

Remark 6.1.3. Every arrangement variety has a finitely generated Cox ring, due to [46, Thm. 1.2].

Having a finitely generated Cox ring, we turn to the description of arrangement varieties as explicit $\mathbb{T}$-varieties as in Chapter 1. We work in a similar manner as done in the case of general arrangement varieties and begin with the description of their Cox rings.

Construction 6.1.4. Fix integers $r \ge c > 0$, $n_0, \ldots, n_r > 0$ and $m \ge 0$ and set $n := n_0 + \ldots + n_r$. The input datum is a tuple (A, P_0) as follows:

- $A = (a_0, \ldots, a_r)$ is a $(c+1) \times (r+1)$-matrix of full rank with pairwise linearly independent columns a_i.
- P_0 is a $r \times (n+m)$-matrix build up from tuples $l_i = (l_{i1}, \ldots, l_{in_i})$ of positive integers

$$P_0 = \begin{bmatrix} -l_0 & l_1 & & & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ -l_0 & & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Write $\mathbb{K}[T_{ij}, S_k]$ for the polynomial ring in the variables T_{ij}, where $i = 0, \ldots, r$, $j = 1, \ldots, n_i$, and S_k, where $k = 1, \ldots, m$. For every l_i we define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij}, S_k]$$

and to any $v \in \mathbb{K}^{r+1}$ we assign the polynomial

$$g_v := v_0 T_0^{l_0} + \ldots + v_r T_r^{l_r} \in \mathbb{K}[T_{ij}, S_k].$$

Then any tuple (A, P_0) defines a $\mathbb{K}$-*algebra*

$$\mathbb{K}[T_{ij}, S_k] / \langle g_v;\ v \in \ker(A) \rangle.$$

Now let $e_{ij} \in \mathbb{Z}^n$ and $e_k \in \mathbb{Z}^m$ denote the canonical basis vectors and let

$$Q_0 \colon \mathbb{Z}^{n+m} \ \to \ K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$$

be the projection onto the factor group by the row lattice of P_0. This defines a K_0-graded $\mathbb{K}$-algebra

$$R(A, P_0) := \mathbb{K}[T_{ij}, S_k]/\langle g_v;\ v \in \ker(A)\rangle,$$
$$\deg(T_{ij}) := Q_0(e_{ij}), \qquad \deg(S_k) := Q_0(e_k).$$

Example 6.1.5. Consider the projective hyperplane arrangement $H_0, \ldots, H_4$ from Example 6.1.1. We store the coefficients of the defining linear forms as the columns of a matrix A and set

$$A := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad P_0 := \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

Using this as input data in Construction 6.1.4, we obtain the following $\mathbb{Z}^3 \times (\mathbb{Z}_2)^3$ graded $\mathbb{K}$-algebra, where we store the degrees of the generators T_{ij} and S_k as the columns of a matrix Q_0:

$$R(A, P_0) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}, T_{41}, S_1]/\langle T_{01}T_{02} + T_{11}^2 + T_{31}^2, T_{01}T_{02} + T_{21}^2 + T_{41}^2\rangle,$$

$$Q_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$$

A direct computation shows that the ring $R(A, P_0)$ is an integral, normal, complete intersection ring of dimension 5 having $R(A, P_0)^* = \mathbb{K}^*$.

Theorem 6.1.6. *Let $R(A, P_0)$ be a $\mathbb{K}$-algebra arising from Construction 6.1.4. Then $R(A, P_0)$ is an integral, normal, complete intersection ring satisfying*

$$\dim(R(A, P_0)) \ = \ n + m - r + c, \qquad R(A, P_0)^* \ = \ \mathbb{K}^*.$$

The K_0-grading is effective, pointed, factorial and of complexity c.

Construction 6.1.7. Let $R(A, P_0)$ be as in Construction 6.1.4. We build up a new $(r + s) \times (n + m)$ matrix

$$P := \begin{bmatrix} P_0 \\ d \end{bmatrix},$$

where we require the columns of P to be pairwise different and primitive and generate $\mathbb{Q}^{r+s}$ as a vectorspace. Let $Q \colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\mathrm{im}(P^*) =: K$ be the canonical projection.

Then we obtain a new graded ring $R(A,P)$ by defining a K-grading on the ring $R(A,P_0)$ by setting

$$R(A,P) := \mathbb{K}[T_{ij}, S_k]/\langle g_v;\ v \in \ker(A)\rangle,$$
$$\deg(T_{ij}) := Q(e_{ij}), \qquad \deg(S_k) := Q(e_k).$$

Example 6.1.8. We continue Example 6.1.5 and choose

$$d := \begin{bmatrix} -2 & -3 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This defines a $\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$ grading on the resulting algebra

$$R(A,P) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}, T_{41}, S_1]/\langle T_{01}T_{02} + T_{11}^2 + T_{31}^2, T_{01}T_{02} + T_{21}^2 + T_{41}^2\rangle,$$

where we store the degrees of the variables as the columns of the following matrix Q:

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}.$$

Corollary 6.1.9. *The K-grading on $R(A,P)$ is effective, factorial and of complexity c. Moreover, if the columns of P generate $\mathbb{Q}^{r+s}$ as a cone, then the K-grading is pointed.*

We obtain the following proposition as a direct consequence of the results from Chapter 1.

Proposition 6.1.10. *The Cox ring of an arrangement variety is isomorphic to a ring $R(A,P)$ as in Construction 6.1.7.*

Now we turn to the realization of arrangement varieties as explicit $\mathbb{T}$-varieties using the rings $R(A,P)$. Note, that the subsequent statements are adapted versions from Chapter 2.

Recall that $R(A,P)$ is an irreducible, normal complete intersection ring of dimension $n+m-(r-c)$. In particular, it defines an affine variety $V(g_1,\dots,g_{r-c})$, where $g_1,\dots,g_{r-c}$ are generators for the ideal $\langle g_v, v \in \mathrm{Ker}\,(A)\rangle$ as in Construction 6.1.7.

Construction 6.1.11. Let $R(A,P)$ be a K-graded ring as in Construction 6.1.7 and assume the variables T_{ij}, S_k to be K-prime. Choose any fan Σ in $\mathbb{Z}^{r+s}$ having precisely the columns of P as its primitive ray generators and denote by Z the corresponding toric variety. Then we obtain the following diagram

$$\begin{array}{ccccccc} V(g_1,\dots,g_{r-c}) & =: & \bar{X} & \hookrightarrow & \bar{Z} & := & \mathbb{K}^{n+m} \\ & & \cup\!\!\mid & & \cup\!\!\mid & & \\ \bar{X} \cap \hat{Z} & =: & \hat{X} & \hookrightarrow & \hat{Z} & & \\ & & \big\downarrow /\!\!/H & & \big\downarrow /\!\!/H & & \\ & & X & \hookrightarrow & Z & & \end{array}$$

where $H := \operatorname{Spec} \mathbb{K}[K]$ is the characteristic quasitorus of Z, acting on the characteristic space $\hat{Z} \to Z$ and $X := X(A, P, \Sigma)$ is the image of $\hat{X}$ under the latter morphism. The torus T acting on Z splits as a product $T^r \times T^s$ and the $\mathbb{T} := T^s$-factor leaves $X \subseteq Z$ invariant.

Remark 6.1.12. The varieties $X := X(A, P, \Sigma)$ as in Construction 6.1.11 are irreducible and normal with dimension, invertible functions, divisor class group and Cox ring given by

$$\dim(X) = s + c, \quad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*, \quad \operatorname{Cl}(X) = K, \quad \mathcal{R}(X) = R(A, P).$$

Moreover the $\mathbb{T}$-action on X is effective and of complexity c.

Example 6.1.13. We continue Example 6.1.8. Note that the variables T_{ij} and S_1 of $R(A, P)$ are K-prime. Denoting the columns of P with v_{ij} and v_1 with respect to the variables T_{ij} and S_1 we choose the fan Σ with maximal cones

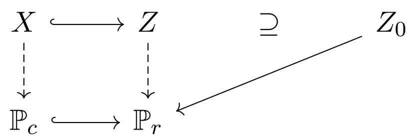
$$\operatorname{cone}(v_{02}, v_{11}, v_{21}, v_{31}, v_{41}),\ \operatorname{cone}(v_{01}, v_{21}, v_{41}, v_1),$$

$$\operatorname{cone}(v_{01}, v_{11}, v_{31}, v_1),\ \operatorname{cone}(v_{01}, v_{02}, v_{21}, v_{41}),\ \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{31}),$$

$$\operatorname{cone}(v_{31}, v_{41}, v_1),\ \operatorname{cone}(v_{21}, v_{31}, v_1),\ \operatorname{cone}(v_{11}, v_{41}, v_1),\ \operatorname{cone}(v_{11}, v_{21}, v_1).$$

The resulting variety $X(A, P, \Sigma)$ has dimension three, only constant invertible global functions, divisor class group $\operatorname{Cl}(X) \cong \mathbb{Z}^2 \times (\mathbb{Z}_2)^3$ and Cox ring $\mathcal{R}(X) = R(A, P)$. Moreover, the $\mathbb{T}$-action is of complexity two.

Construction 6.1.14. Let $X := X(A, P, \Sigma) \subseteq Z$ be as in Construction 6.1.11. Then X fits into the following diagram:



where Z_0 is the (open) union of the torus and all orbits of codimension one in Z, the morphism $Z_0 \to \mathbb{P}_r$ is a toric morphism induced by the projection of tori $T^{r+s} \to T^r$, the downward rational maps are defined via this morphism and $\mathbb{P}_c$ is linearly embedded into $\mathbb{P}_r$ via $[x] \mapsto [A^t x]$.

Remark 6.1.15. The rational map $X \dashrightarrow \mathbb{P}_c$ is a maximal orbit quotient for the $\mathbb{T}$-action of X, where the critical values form the hyperplane arrangement

$$H_0, \ldots, H_r \subseteq \mathbb{P}_c, \qquad H_i := \{[x] \in \mathbb{P}_c; \langle a_i, x \rangle = 0\}.$$

In particular, any variety $X(A, P, \Sigma)$ as in Construction 6.1.11 is an arrangement variety.

Example 6.1.16. We continue Example 6.1.13. The variety $X = X(A, P, \Sigma)$ is an arrangement variety having $X \dashrightarrow \mathbb{P}_2$ as a maximal orbit quotient. In this case $\mathbb{P}_2$ is realized inside $\mathbb{P}_4$ as

$$\mathbb{P}_2 = V(T_0 + T_1 + T_3, T_0 + T_2 + T_4) \subseteq \mathbb{P}_4$$

and the critical values form the line arrangement in special position from Example 6.1.1:

$$H_0 = V(T_0), \qquad H_1 = V(T_1), \qquad H_2 = V(T_2)$$

$$H_3 = V(T_0 + T_1), \qquad H_4 = V(T_0 + T_2).$$

In particular X is an arrangement variety and as we will see later it is one of the three-dimensional Fano canonical complexity two varieties in Theorem 6.6.2.

Definition 6.1.17. We call a variety $X(A, P, \Sigma) \subseteq Z$ as in Construction 6.1.11 an explicit arrangement variety.

Remark 6.1.18. Every arrangement variety is equivariantly isomorphic to an explicit arrangement variety.

Let us recall the basic notions on tropical varieties. Let Z be a toric variety with acting torus T. For a closed subvariety $X \subseteq Z$ intersecting the torus non trivially consider the vanishing ideal $I(X \cap T)$ in the Laurent polynomial ring $\mathcal{O}(T)$. For every $f \in I(X \cap T)$ let $|\Sigma(f)|$ denote the support of the codimension one skeleton of the normal quasifan of its Newton polytope, where a quasifan is a fan, where we allow the cones to be non-pointed. Then the *tropical variety* trop(X) *of* X is defined as follows, see [61, Def. 3.2.1]:

$$\operatorname{trop}(X) := \bigcap_{f \in I(X \cap T)} |\Sigma(f)| \subseteq \mathbb{Q}^{\dim(Z)}.$$

Definition 6.1.19. Let $X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety. We denote the columns of P with v_{ij} and v_k according to the variables T_{ij} and S_k. A *P-cone* is a cone $\sigma \subseteq \mathbb{Q}^{r+s}$ such that its set of primitive ray generators is a subset of the columns of P, i.e.

$$\sigma = \operatorname{cone}(v_{ij_i}, v_k;\ i \in I \subseteq \{0, \ldots, r\},\ j_i \in J_i \subseteq \{1, \ldots, n_i\}, k \in K \subseteq \{1, \ldots, m\}).$$

We call a P-cone $\sigma \subseteq \mathbb{Q}^{r+s}$

(i) a *leaf cone*, if $\sigma \subseteq |\operatorname{trop}(X)|$ holds.

(ii) a *big cone*, if $\sigma^\circ \cap (\{0\} \times \mathbb{Q}^s) \neq \emptyset$ holds.

(iii) a *special cone*, if it is neither big nor leaf but $\sigma^\circ \cap |\operatorname{trop}(X)| \neq \emptyset$ holds.

Applying [75, Lem. 2.2], we obtain the following remark.

Remark 6.1.20. Let $X(A,P,\Sigma) \subseteq Z$ be an explicit arrangement variety. Then the cones in Σ are of leaf, special or big type.

Example 6.1.21. We continue Example 6.1.16 and investigate the fan Σ. To describe the tropical variety of X denote by $e_1,\dots,e_4$ the canonical basis vectors of $\mathbb{Q}^4$ and set

$$e_0 := -e_1 - \ldots - e_4, \qquad e_5 := e_0 + e_2 + e_4, \qquad e_6 := e_0 + e_1 + e_3.$$

We define a fan Δ with maximal cones $\mathrm{cone}(e_i, e_j)$, where (i,j) is one of the following tuples:

$$(0,5),\ (0,6),\ (1,2),\ (1,4),\ (1,6),\ (2,3),\ (2,5),\ (3,4),\ (3,6),\ (4,5).$$

Then $\mathrm{trop}(X) = |\Delta \times \mathbb{Q}|$ holds. Checking the items in Definition 6.1.19 for the cones in Σ, we obtain one big cone

$$\mathrm{cone}(v_{02}, v_{11}, v_{21}, v_{31}, v_{41}),$$

four special cones

$$\mathrm{cone}(v_{01}, v_{21}, v_{41}, v_1),\ \mathrm{cone}(v_{01}, v_{11}, v_{31}, v_1),$$

$$\mathrm{cone}(v_{01}, v_{02}, v_{21}, v_{41}),\ \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{31}),$$

and four leaf cones

$$\mathrm{cone}(v_{31}, v_{41}, v_1),\ \mathrm{cone}(v_{21}, v_{31}, v_1),$$

$$\mathrm{cone}(v_{11}, v_{41}, v_1),\ \mathrm{cone}(v_{11}, v_{21}, v_1).$$

6.2 Proofs to Section 6.1

This section is dedicated to the proofs of the statements in Section 6.1. In a first step we investigate product structures on the rings $R(A,P_0)$. Then we turn to the proof of Theorem 6.1.6.

Definition 6.2.1. Let $R(A,P_0)$ be a ring as in Construction 6.1.4.

(i) We call the matrix A *indecomposable* if for any subset $I \subseteq \{1,\dots,r+1\}$ we have $\{0\} \neq \mathrm{Lin}(a_i;\ i \in I) \cap \mathrm{Lin}(a_j;\ j \notin I)$.

(ii) We call a ring $R(A,P_0)$ *indecomposable* if A is indecomposable and $l_{ij}n_i > 1$ holds for all i.

Proposition 6.2.2. *Every $\mathbb{K}$-algebra $R(A,P_0)$ from Construction 6.1.4 is isomorphic as a $\mathbb{K}$-algebra (forgetting the K_0-grading) to a product*

$$\bigotimes_{i=1}^{t} R(A^{(i)}, P_0^{(i)}) \otimes \mathbb{K}[S_1,\dots,S_{m'}], \tag{6.2.1}$$

where the algebras $R(A^{(i)}, P_0^{(i)})$ are indecomposable with $m^{(i)} = 0$ for all $i = 1,\dots,t$ and $m' \geq m$ holds.

Remark 6.2.3. Note that these algebras are in general not isomorphic as graded algebras concerning their natural gradings: The K_0-grading defined on the product via the isomorphism is in general coarsening the grading defined via the product $K_0^{(1)} \times \ldots \times K_0^{(t)} \times \mathbb{Z}^{m'}$. We will investigate this fact in Chapter 7, where we consider arrangement-product varieties.

Remark 6.2.4. The following list of *admissible operations* does not effect the isomorphy type of a ring $R(A, P_0)$:

(i) any elementary row operation on A.

(ii) swap columns in A and accordingly columns in P_0.

(iii) swap any column in P_0 inside a block l_i.

In particular, without loss of generality we may always assume the matrix A to be in reduced row echelon form $A = (E_{c+1}, a_{c+1}, \ldots, a_r)$ and the polynomials g_i generating $\mathrm{Ker}(A)$ to be of the following form:

$$g_i := \lambda_{0,i} T_0^{l_0} + \ldots + \lambda_{c,i} T_c^{l_c} + \lambda_{(c+i),i} T_{(c+i)}^{l_{(c+i)}}, \qquad 1 \leq i \leq r - c. \tag{6.2.2}$$

Note that we have $g_i = g_{v_i}$ for $v_i = (a_{c+1+i}, -e_i)$, where e_i denotes the i-th canonical basis vector of $\mathbb{K}^{r-c}$.

We turn to the proof of Proposition 6.2.2. The following lemma is straightforward but for the convenience of the reader we will prove it here:

Lemma 6.2.5. *Let $A = (a_0, \ldots, a_r)$ be a matrix as in Construction 6.1.4. Then there exists a unique decomposition of $\mathbb{K}^{c+1}$ into vectorsubspaces $V_1 \oplus \ldots \oplus V_t$ such that the following holds:*

(i) *For each $0 \leq i \leq r$ there exists $j(i) \in \{1, \ldots, t\}$ with $a_i \in V_{j(i)}$.*

(ii) *If $V_1' \oplus \ldots \oplus V_s'$ is any other decomposition fulfilling* (i)*, then for every $1 \leq i \leq t$ there exists $1 \leq j \leq s$ with $V_i \subseteq V_j'$.*

Proof. Let $V_1 \oplus \cdots \oplus V_t$ be any decomposition of $\mathbb{K}^{c+1}$ fulfilling (i). We construct a decomposition fulfilling (ii) by successively refining this given decomposition. For this let $V_1' \oplus \ldots \oplus V_s'$ be any other decomposition fulfilling (i) and assume our given decomposition does not fulfill (ii). Then there exists $1 \leq i \leq s$ such that $V_i \not\subseteq V_j'$ for all $1 \leq j \leq s$. Set $A_i := \{k;\ a_k \in V_i\}$. Then for every $k \in A_i$ there exists $j(k)$ with $a_k \in V_{j(k)}'$. In particular, we obtain a decomposition

$$V_i = V_i \cap (\bigoplus_{k \in A_i} V_{j(k)}') = \bigoplus_{k \in A_i} V_i \cap V_{j(k)}',$$

and $\dim(V_i) > \dim(V_i \cap V_{j(k)})$ holds for any $k \in A_i$. Iterating this step we end up with a decomposition fulfilling (ii). $\square$

Proposition 6.2.2 is a direct consequence of the following more technical Lemma.

Lemma 6.2.6. *Let $R(A, P_0)$ be a ring as in Construction* 6.1.4. *Then the following statements hold:*

(i) *Let $V_1 \oplus V_2 = \mathbb{K}^{c+1}$ be a decomposition fulfilling Assertion (i) of Lemma 6.2.5 and assume* $\dim(V_1) = 1$. *Then $R(A, P_0) \cong R(A', P_0')$ holds, for a tuple (A', P_0') with* $\mathrm{rk}(A') < \mathrm{rk}(A)$ *and $m' > m$.*

(ii) *Let $V_1 \oplus V_2 = \mathbb{K}^{c+1}$ be a decomposition fulfilling Assertion (i) of Lemma 6.2.5 and assume* $\dim(V_i) > 1$ *for $i = 1, 2$. Then we have*

$$R(A, P_0) \cong R(A^{(1)}, P_0^{(1)}) \otimes R(A^{(2)}, P_0^{(2)}),$$

for suitably chosen data $(A^{(i)}, P_0^{(i)})$ with $\mathrm{rk}(A^{(1)}) + \mathrm{rk}(A^{(2)}) = \mathrm{rk}(A)$ *holds.*

Proof. Let $V_1 \oplus V_2 = \mathbb{K}^{c+1}$ be a decomposition fulfilling Assertion (i) of Lemma 6.2.5. Then, by applying Remark 6.2.4 (ii) we may assume $V_1 = \mathrm{Lin}(a_0, \ldots, a_t)$ and $V_2 = \mathrm{Lin}(a_{t+1}, \ldots, a_r)$. Furthermore, as elementary row operation do note effect the isomorphy type of $R(A, P_0)$, we may assume $V_1 = \mathrm{Lin}(e_1, \ldots, e_s)$ and $V_2 = \mathrm{Lin}(e_{s+1}, \ldots, e_{c+1})$.

We prove (i). For this let $\dim(V_1) = 1$, i.e. we have $t = 0$ and $s = 1$. Then $a_0 = \lambda e_1$ holds and all entries a_{1j} with $j \geq 1$ equal zero. We conclude $R(A, P_0) \cong R(A', P_0')$, with $m' = m + n_0$, A' is the matrix obtained by deleting the first row and the first column of A and P_0' is build up from the tuples $l_1, \ldots, l_r$.

We turn to (ii). Assume we have $\dim(V_i) > 1$ for $i = 1, 2$. Then A is a block matrix of the form

$$A = \begin{bmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{bmatrix}$$

and we conclude $R(A, P_0) \cong R(A^{(1)}, P_0^{(1)}) \otimes R(A^{(2)}, P_0^{(2)})$, where $P_0^{(1)}$ is build up from $l_0, \ldots, l_t$, $P_0^{(2)}$ is build up from $l_{t+1}, \ldots, l_r$ and m_1, m_2 are positive integers with $m_1 + m_2 = m$. ☐

We turn to the proof of Theorem 6.1.6. Let (A, P_0) be as in Construction 6.1.4. Then the defining relations g_v of the ring $R(A, P_0)$ can be obtained in the following way: For any $v \in \mathrm{Ker}\,(A)$ write

$$f_v := v_0 T_0 + \ldots v_r T_r \in \mathbb{K}[T_0, \ldots, T_r]$$

for the corresponding linear form. Then $g_v = f_v(T_0^{l_0}, \ldots, T_r^{l_r})$ holds. We will use this observation to prove in Lemma 6.2.8 connectedness of the affine variety $X := V(g_v;\ v \in \mathrm{Ker}\,(A))$. Moreover, in Proposition 6.2.9 we deduce the dimension of X from that of $Y := V(f_v;\ v \in \mathrm{Ker}\,(A))$.

Remark 6.2.7. Consider the polynomial ring $\mathbb{K}[T_0, \ldots, T_r]$ endowed with an effective pointed $\mathbb{Z}$-grading $\deg(T_i) = w_i \in \mathbb{Z}_{>0}$ and let $f \in \mathbb{K}[T_0, \ldots, T_r]$ be any homogeneous polynomial. Then the polynomial

$$g := f(T_0^{l_0}, \ldots, T_r^{l_r}) \in \mathbb{K}[T_{ij}, S_k]$$

is homogeneous with respect to the grading defined by:

$$\deg(T_{ij}) = \frac{n_0 \ldots n_r l_{01} \ldots l_{rn_r}}{n_i l_{ij}} \cdot w_i \in \mathbb{Z}_{>0}. \tag{6.2.3}$$

Lemma 6.2.8. *In the situation of Remark 6.2.7 let $f_1, \ldots, f_s \in \mathbb{K}[T_0, \ldots, T_r]$ be homogeneous polynomials and set $g_i := f_i(T_0^{l_0}, \ldots, T_r^{l_r}) \in \mathbb{K}[T_{ij}, S_k]$. Then the affine variety $X := \mathrm{V}(g_1, \ldots, g_s)$ is connected.*

Proof. Consider the acting torus $(\mathbb{K}^*)^{n+m}$ of $\mathbb{K}^{n+m}$ and the multiplicative one-parameter subgroup

$$\lambda \colon \mathbb{K}^* \to (\mathbb{K}^*)^{n+m}, \qquad t \mapsto (t^{\zeta_{01}}, \ldots, t^{\zeta_{rn_r}}, t, \ldots, t),$$

where $\zeta_{ij} := \deg(T_{ij})$ is as in (6.2.3). Then by construction the image $\lambda(\mathbb{K}^*)$ acts on X and has 0 as an attractive fixed point. This gives the assertion. □

Proposition 6.2.9. *Let $Y = V(f_1, \ldots, f_s) \subseteq \mathbb{K}^{r+1}$ be irreducible of dimension $r + 1 - s$ and set*

$$X := V(g_1, \ldots, g_s) \quad \text{with} \quad g_i := f_i(T_0^{l_0}, \ldots, T_r^{l_r}) \in \mathbb{K}[T_{ij}, S_k].$$

Then X is pure of dimension $n + m - s$.

Proof. Let $X = X_1 \cup \ldots \cup X_t$ be the decomposition of X into irreducible components. Note that we have $\dim(X_j) \geq n + m - s$ for $1 \leq j \leq t$, as $X = V(g_1, \ldots, g_s)$ holds. Consider the surjective morphism

$$\varphi \colon \mathbb{K}^{n+m} \to \mathbb{K}^{r+1} \qquad (x_{01}, \ldots, x_{rn_r}, x_1, \ldots x_m) \mapsto (x_0^{l_0}, \ldots, x_r^{l_r}).$$

Then by construction of X the restriction $\varphi|_X \colon X \to Y$ is again surjective and we conclude that

$$\varphi_j := \varphi|_{X_j} \colon X_j \to \overline{\varphi(X_j)} =: Y_j \subseteq Y$$

is dominant for every irreducible component X_j. Applying [72, Thm 1.1], we obtain an open subset $U \subseteq Y_j$ such that for all $y \in U$ we have

$$\dim(\varphi_j^{-1}(y)) = \dim(X_j) - \dim(Y_j) = \dim(X_j) - \dim(Y) + k$$

with $k \geq 0$. Note that the latter equality holds as $Y_j \subseteq Y$ is a closed subvariety. Now for any $y \in \mathbb{K}^{r+1}$ we have

$$\varphi^{-1}(y) = V(T_0^{l_0} - y_0, \ldots, T_r^{l_r} - y_r) \subseteq \mathbb{K}^{n+m}$$

and thus $\dim(\varphi^{-1}(y)) = n + m - (r + 1)$ holds. We conclude

$$\dim(X_j) - \dim(Y) + k = \dim(\varphi_j^{-1}(y)) \leq \dim(\varphi^{-1}(y)) = n + m - (r + 1)$$

and therefore $\dim(X_j) \leq n + m - s + k$ holds, which gives the assertion. □

Proof of Theorem 6.1.6. In order to prove this statement it suffices to consider indecomposable algebras $R(A, P_0)$. Let $\mathcal{B}$ be a basis for $\ker(A)$ and fix $v \in \mathcal{B}$. Then the linear forms

$$f_v := v_0 T_0 + \ldots + v_r T_r \in \mathbb{K}[T_0, \ldots, T_r]$$

are $\mathbb{Z}$-homogeneous with respect to the standard $\mathbb{Z}$-grading on $\mathbb{K}[T_0, \ldots, T_r]$. In particular, applying Lemma 6.2.8 we conclude that $X := \mathrm{V}(g_v;\ v \in \mathcal{B})$ is connected.

We want to use Serre's criterion to show that X is normal and $\mathrm{I}(X) = \langle g_1, \ldots, g_{r-c} \rangle$ holds. In particular, as X is connected this implies that $R(A, P_0)$ is integral. Assume A to be in reduced row echelon form as in Remark 6.2.4 and set $A' := (a'_{ij})_{i,j} := (a_{c+1}, \ldots, a_r)$. Recall that the relations $g_1, \ldots, g_{r-c}$ are of the form $g_{v_1}, \ldots, g_{v_{r-c}}$, where v_i denotes the i-th row of the following block-matrix:

$$\left[\begin{array}{c|c} (A')^t & -E_{r-c} \end{array}\right].$$

Now, set $\delta_i := \mathrm{grad}(T_i^{l_i})$ and $J_1 := (a'_{ji} \cdot \delta_i)_{i,j}$. Then the Jacobian of $g_1, \ldots, g_{r-c}$ is of the form

$$J = \left[\begin{array}{c|ccc} & -\delta_{c+1} & & \\ J_1 & & \ddots & \\ & & & -\delta_r \end{array}\right].$$

Now assume that $J(x)$ is not of full rank. Then there exist at least two indices $c+1 \le i_1 < i_2 \le r$ such that $\delta_{i_k}(x) = 0$ holds. Moreover, as the columns of A' are pairwise linearly independent, we have $\delta_{i_3}(x) = 0$ for at least one more index $0 \le i_3 \le c$. In particular, this implies that there exist $1 \le j_k \le n_{i_k}$ such that $x_{i_1,j_1} = x_{i_2,j_2} = x_{i_3,j_3} = 0$ holds. We conclude that any x with $J(x)$ not of full rank is contained in one of the finitely many affine subvarieties of X of the following form:

$$\mathrm{V}(\tilde{f}_1(T_0^{l_0}, \ldots, T_r^{l_r}), \ldots, \tilde{f}_{r-c}(T_0^{l_0}, \ldots, T_r^{l_r}), T_{i_1,j_1}, T_{i_2,j_2}, T_{i_3,j_3}),$$

where $\tilde{f}_i := f_i(\tilde{T}_0, \ldots, \tilde{T}_r)$ with $\tilde{T}_{i_k} := 0$ for $k = 1,2,3$ and $\tilde{T}_i := T_i$ else. We claim that these subvarieties are of codimension at least 2 in X. By Proposition 6.2.9 it suffices to show that $\tilde{Y} := \mathrm{V}(\tilde{f}_1, \ldots, \tilde{f}_{r-c}, T_{i_1}, T_{i_2}, T_{i_3}) \subseteq \mathbb{K}^{r+1}$ is of codimension at least $r - c + 2$. For this note that

$$Y' := \mathrm{V}(\tilde{f}_i;\ i \notin \{i_1 - c, i_2 - c\}) \cap \mathrm{V}(T_{i_1}, T_{i_2}, T_{i_3}) \subseteq \mathbb{K}^{r+1}$$

is irreducible and of codimension $r - c + 1$. Consider the matrix B arising out of A' by replacing its i_3-th row with a zero row. Then by construction for $k = 1, 2$ we have $\tilde{f}_{i_k - c} = f_v$, where $v = (b_{1,i_k}, \ldots, b_{c+1,i_k}, 0, \ldots, 0) \in \mathbb{K}^{r+1}$. In particular $\tilde{f}_{i_k - c} \in I(Y')$ for $k \in \{1, 2\}$ if any only if $\tilde{f}_{i_k - c} = 0$. We conclude that $\tilde{Y} \subseteq \mathbb{K}^{r+1}$ is of codimension at most $r - c + 1$ if and only if $\tilde{f}_{i_1 - c} = \tilde{f}_{i_2 - c} = 0$ holds. This contradicts the fact that the columns of A' are pairwise linearly independent.

In order to complete the proof we have to show that the K_0-grading has the desired properties. By construction, the K_0-grading is effective. Moreover, using Remark 6.2.7 we obtain a one parameter subgroup of $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ via

$$\mathbb{K}^* \to H_0, \qquad t \mapsto (t^{\zeta_{01}}, \ldots, t^{\zeta_{rn_r}}, t, \ldots, t),$$

where $\zeta_{ij} = \deg(T_{ij})$ is as in (6.2.3) with $w_0 = \ldots = w_r = 1$. As $\zeta_{ij} > 0$ holds for all $0 \le i \le r$, and $1 \le j \le n_i$ we conclude that the grading is pointed. To obtain factoriality of the K_0-grading, we localize $R(A, P_0)$ by the product over all generators T_{ij}, S_k, and observe that the degree zero part of the resulting ring is a polynomial ring. Now applying [11, Thm. 1.1] completes the proof. □

6.3 (No) Smooth special arrangement varieties of small Picard number

In this section we prove that in contrast to the general arrangement case, see Chapter 3, there are no smooth honestly special arrangement varieties of Picard number at most two.

Definition 6.3.1. An *honestly special arrangement variety* is a special arrangement variety X with *honestly special arrangement Cox ring* $R(A, P)$, i.e. we have $l_{ij} n_i > 1$ for all $i = 0, \ldots, r$ and $j = 1, \ldots, n_i$ and the graded ring $R(A, P)$ is not isomorphic to a Cox ring $R(A', P')$ of a general arrangement variety.

Remark 6.3.2. A special arrangement variety is honestly special if and only if it does not admit a torus action that turn it into a general arrangement variety.

Example 6.3.3. Consider the ring $R(A, P)$ defined by the following data

$$A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P := \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Then any variety $X(A, P, \Sigma)$ is a special arrangement variety, that is not honestly special: Consider the matrix

$$A' := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the ring $R(A, P)$ is isomorphic as a graded ring to the ring $R(A', P)$, which in turn is the Cox ring of a complexity one $\mathbb{T}$-variety.

Theorem 6.3.4. *Let X be a projective honestly special arrangement variety of Picard number at most two. Then X is singular.*

The rest of this section is dedicated to the proof of Theorem 6.3.4. We work in the language of explicit $\mathbb{T}$-varieties from Chapter 1.

Remark 6.3.5. Let $X(A, P, \Sigma)$ be an honestly special arrangement variety. Then the ideal of relations between the generators T_{ij}, S_k of $R(A, P)$ is generated by at least two relations g_v.

Remark 6.3.6. Let $X := X(A, P, \Sigma)$ be a projective honestly special arrangement variety. After suitably renumbering we may assume that for the defining relations

$$g_t = \lambda_{t0} T_0^{l_0} + \ldots + \lambda_{tc} T_c^{l_c} + T_{t+c}^{l_{t+c}}, \quad \text{where } 1 \leq t \leq r - c \tag{6.3.1}$$

of $R(A, P)$ there exists an index $k \in \{0, 1, \ldots, c\}$ such that $\lambda_{1k} = 0$ and $\lambda_{2k} \neq 0$ holds. In particular the face

$$\gamma_0 := \mathrm{cone}(e_{k1}, \lambda_{tk} e_{(t+c)1};\ 2 \leq t \leq r - 2) \preccurlyeq \gamma$$

is an $\overline{X}$-face. Moreover the corresponding stratum in $\overline{X}$ is singular due to the number of relations defining $R(A, P)$.

Remark 6.3.7. Let $X := X(A, P, \Sigma)$ be a projective arrangement variety of Picard number one. Then every $\overline{X}$-face $\{0\} \neq \gamma_0 \preccurlyeq \gamma$ is an X-face.

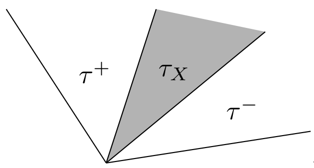
Proof of Theorem 6.3.4 for $\varrho(X) = 1$. Assume there is a smooth explicit honestly special arrangement variety $X := X(A, P, \Sigma)$ with $\varrho(X) = 1$. Then, using Remark 6.3.6 we obtain a $\overline{X}$-face with singular stratum in $\overline{X}$ which is an X-face due to Remark 6.3.7. This contradicts smoothness of X due to Proposition 1.4.5. □

We consider the case of Picard number two. In a first step, we adapt techniques from [30, 42] to treat these varieties. We proceed with Lemma 6.3.10, where we obtain first constraints on the defining data A, P and Σ. Then we go on proving Theorem 6.3.4 for $\varrho(X) = 2$.

Remark 6.3.8. Let $X := X(A, P, \Sigma)$ be an explicit arrangement variety with divisor class group $K = \mathrm{Cl}(X)$ of rank two. Then, inside the rational divisor class group $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathbb{Q}^2$, the effective cone of X is of dimension two and decomposes as

$$\mathrm{Eff}(X) \;=\; \tau^+ \cup \tau_X \cup \tau^-,$$

where $\tau_X \subseteq \mathrm{Eff}(X)$ is the ample cone, τ^+, τ^- are closed cones not intersecting τ_X and $\tau^+ \cap \tau^-$ consists of the origin:



.

Due to $\tau_X \subseteq \mathrm{Mov}(X)$, each of the cones τ^+ and τ^- contains at least two of the rational weights

$$w_{ij} := (x_{ij}, y_{ij}) := \deg_{\mathbb{Q}}(T_{ij}), \qquad w_k := (x_k, y_k) := \deg_{\mathbb{Q}}(S_k).$$

Moreover, for every $\overline{X}$-face $\{0\} \neq \gamma_0 \preccurlyeq \gamma$ precisely one of the following inclusions holds:

$$Q(\gamma_0) \ \subseteq \ \tau^+, \qquad \tau_X \ \subseteq \ Q(\gamma_0)^\circ, \qquad Q(\gamma_0) \ \subseteq \ \tau^-.$$

The X-faces are precisely those $\overline{X}$-faces $\gamma_0 \preceq \gamma$ with $\tau_X \subseteq Q(\gamma_0)^\circ$.

Remark 6.3.9. In the situation of Remark 6.3.8 consider a positively oriented pair $w, w' \in \mathbb{Q}^2$. If, for instance, $w \in \tau^-$ and $w' \in \tau^+$ hold, then $\det(w, w')$ is positive. Moreover, if the variety X is smooth and w, w' are the weights stemming from a two-dimensional X-face $\gamma_0 \preccurlyeq \gamma$, then we have $\det(w, w') = 1$ due to Proposition 1.4.5. In this case, we can achieve

$$w \ = \ (1, 0), \qquad\qquad w' \ = \ (0, 1)$$

by a suitable unimodular coordinate change on $\mathbb{Z}^2 \subseteq \mathbb{Q}^2$. Then $w'' = (x'', 1)$ holds whenever w, w'' are the weights stemming from a two-dimensional X-face and, similary, $w'' = (1, y'')$ holds whenever w'', w' are these weights.

Recall, that an explicit arrangement variety $X := X(A, P, \Sigma)$ is called *quasismooth*, if for every X-face, the corresponding stratum in $\bar{X}$ is smooth.

Lemma 6.3.10. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-factorial quasismooth projective honestly special arrangement variety of Picard number two. Then the following assertions hold:*

(i) *If $m > 0$ holds, then all weights w_k lie either in τ^+ or in τ^-.*

(ii) *If $n_i \geq 2$ holds for at least one index $0 \leq i \leq r$, then $m = 0$ holds.*

(iii) *$n_i \leq 2$ holds for all $0 \leq i \leq r$.*

Proof. We prove (i). Let $m \geq 2$. As $\mathrm{cone}(e_k)$ is an $\overline{X}$-face for $1 \leq k \leq m$, Remark 6.3.9 implies $w_k \notin \tau_X$ due to $\mathbb{Q}$-factoriality of X. So assume we have $w_{k_1} \in \tau^+$ and $w_{k_2} \in \tau^-$. Then $\mathrm{cone}(e_{k_1}, e_{k_2})$ is an X-face with singular stratum in $\overline{X}$; a contradiction to quasismoothness of X.

We prove (ii). Let $m > 0$. Then we may assume that $w_k \in \tau^+$ holds for all $1 \leq k \leq m$. Assume there exists a weight $w_{ij} \in \tau^-$ with $n_i \geq 2$. Then $\mathrm{cone}(e_1, e_{i1})$ is an X-face with singular $\overline{X}$-stratum, which contradicts quasismoothness of X. Thus $w_{ij} \in \tau^+$ holds for all i with $n_i \geq 2$. Due to homogeneity of the relations we conclude $w_{ij} \in \tau^+$ for all i with $n_i = 1$ and there are no weights left to lie in τ^-; a contradiction due to Remark 6.3.9.

We prove (iii). Assume there exists an index i with $n_i \geq 3$. Then after suitably renumbering we may assume $i = 0$. We claim that all w_{0j} lie either in τ^+ or in τ^-. Assume that this is not true and $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$ holds. Then the cone $\mathrm{cone}(e_{01}, e_{02})$ is an X-face with singular $\overline{X}$-stratum as there are at least two relations defining $R(A,P)$; a contradiction to quasismoothness of X. Thus we may assume $w_{0j} \in \tau^+$ for $j = 1,2,3$ and homogeneity of the relations implies that all weights w_{kj} with $n_k = 1$ or $n_k \geq 3$ lie in τ^+. Using Part (ii) and Remark 6.3.9 we conclude that there exist at least two indices i_1, i_2 with $n_{i_1} = n_{i_2} = 2$ and after suitably renumbering we may assume that $w_{i_1 1}, w_{i_2 1} \in \tau^-$ holds. In particular, the $\overline{X}$-face $\mathrm{cone}(e_{i_1 1}, e_{01})$ is an X-face. As we have at least two relations defining $R(A,P)$, the corresponding stratum in $\overline{X}$ is singular; a contradiction to quasismoothness of X. □

Proof of Theorem 6.3.4 for $\varrho(X) = 2$. We show that the existence of a smooth variety $X(A,P,\Sigma)$ as in the theorem leads to a contradiction in all possible cases.

Assume $n_i = 1$ holds for all $0 \leq i \leq r$. Then due to homogeneity of the relations we may assume that all weights w_{i1} lie in τ^+. Thus due to Remark 6.3.9 there exist at least two weights $w_1, w_2 \in \tau^-$. Due to Remark 6.3.6 there exists an index $k \in \{0,1,\ldots,c\}$ such that $\mathrm{cone}(e_1, e_{k1}, \lambda_{tk} e_{(t+c)1};\ 2 \leq t \leq r-2)$ is an X-face with singular stratum in $\overline{X}$; a contradiction to smoothness of X.

Now, due to Lemma 6.3.10 we may assume that $2 = n_0 \geq \ldots \geq n_r \geq 1$ and $m = 0$ holds. Due to homogeneity of the relations we may assume that all weights w_{i1} with $n_i = 1$ lie in τ^+ and thus due to Remark 6.3.9 $n_0 = n_1 = 2$ holds with $w_{01}, w_{11} \in \tau^+$ and $w_{02}, w_{12} \in \tau^-$. Considering the X-faces $\mathrm{cone}(e_{01}, e_{12}), \mathrm{cone}(e_{02}, e_{11})$, quasismoothness of X implies $l_{01} = l_{02} = l_{11} = l_{12} = 1$. After suitably renumbering we may moreover assume $w_{11} \in \mathrm{cone}(w_{01}, w_{02})$. And thus applying Remark 6.3.9 to the X-face $\mathrm{cone}(e_{01}, e_{12})$ turns the degree matrix Q into the shape

$$Q = \left[\begin{array}{cc|cc|c} 1 & x_{02} & x_{11} & 0 & \ldots \\ 0 & y_{02} & y_{11} & 1 & \ldots \end{array}\right],$$

where $x_{11}, y_{11} \geq 0$. Applying Remark 6.3.9 to the X-face $\mathrm{cone}(e_{11}, e_{02})$ we obtain $1 = \det(w_{11}, w_{02}) = x_{11}y_{02} - x_{02}y_{11}$. Using homogeneity of the relations we obtain

$$y_{02} = l_{02}y_{02} = l_{11}y_{11} + l_{12} = y_{11} + 1, \quad 1 + x_{02} = l_{01} + l_{02}x_{02} = l_{11}x_{11} = x_{11}.$$

This implies $x_{02} = -y_{11}$ and $y_{02} = 2 - x_{11}$, hence $1 - y_{11} = x_{11} \geq 0$ and thus $0 \leq y_{11} \leq 1$. Assume $y_{11} = 0$. This turns the degree matrix Q into the shape

$$Q = \left[\begin{array}{cc|cc|c} 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & \ldots \end{array}\right].$$

In particular, the degree of the relations is $(1,1)$. This implies $n_i \neq 1$ for all $0 \leq i \leq r$ due to the honesty of $R(A,P)$. Assume there exists an index k with $w_{k1}, w_{k2} \in \tau^+$ then $\mathrm{cone}(e_{k1}, e_{02}), \mathrm{cone}(e_{k2}, e_{02})$ are X-faces and applying Remark 6.3.9 gives $w_{k1} = (1, y_{k1})$ and $w_{k2} = (1, y_{k2})$ in contradiction to homogeneity of the relations. Similar arguments hold for $w_{k1}, w_{k2} \in \tau^-$. Thus we may assume $w_{i1} \in \tau^+$ and $w_{i2} \in \tau^-$ for all $0 \leq i \leq r$.

Due to Remark 6.3.6 there exists an index $k \in \{0, 1, \dots, c\}$ such that $\mathrm{cone}(e_{k1}, e_{(c+2)2})$ is an X-face with singular stratum in $\overline{X}$. This contradicts smoothness of X. Thus we may assume $y_{11} = 1$ this gives $w_{11} = (0,1) = w_{12}$ which in turn is a contradiction to $w_{11} \in \tau^+$ and $w_{12} \in \tau^-$. □

6.4 Toric ambient resolutions of singularities

The purpose of this section is to prove that explicit arrangement varieties admit a toric ambient resolution of singularities as introduced in Section 4.1.

Theorem 6.4.1. *Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety. Then $X \subseteq Z$ admits a toric ambient resolution of singularities.*

In order to prove the above result, we make use of the weakly tropical resolution of an explicit arrangement variety $X = X(A, P, \Sigma) \subseteq Z$. As the weakly tropical resolution of X depends on the choice of a quasifan structure on $\mathrm{trop}(X)$, in the following remark we will have a closer look at two possible choices.

Remark 6.4.2. Let $X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety. Then, due to Construction 6.1.14, the matrix A gives rise to a linear embedding $\mathbb{P}_c \subseteq \mathbb{P}_r$. Moreover, the projection $P_1 \colon \mathbb{Q}^{r+s} \to \mathbb{Q}^r$ onto the first r coordinates maps $\mathrm{trop}(X)$ onto $\mathrm{trop}(Y)$ and we obtain

$$|\mathrm{trop}(X)| = |\mathrm{trop}(\mathbb{P}_c \cap \mathbb{T}^r)| \times \mathbb{Q}^s.$$

In the following we will construct a fan structure on $\mathrm{trop}(Y)$ and will endow $\mathrm{trop}(X)$ with the corresponding quasifan structure, i.e.

$$\mathrm{trop}(X) = \left\{ P_1^{-1}(\lambda);\ \lambda \in \mathrm{trop}(Y) \right\}.$$

Let A be as above and denote by $\mathcal{A}$ the set of columns of A. The *lattice of flats* $\mathcal{L}(A)$ is the partially ordered set of all subspaces of $\mathbb{K}^{r+1}$ spanned by subsets of $\mathcal{A}$. Note that all maximal chains in $\mathcal{L}(A)$ have length $c+1$. For any $S \in \mathcal{L}(A)$ denote by $I(S) \subseteq \{0, \dots, r\}$ the indices with $a_i \in S$ and set $e_S := \sum_{i \in I(S)} e_i$, where $e_0 := -\sum_{i=1}^{r} e_i$. For any maximal chain $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_c \subseteq \mathbb{K}^{r+1}$ we define a cone $\mathrm{cone}(e_{S_1}, \dots, e_{S_c})$ and denote with $\Delta(\mathcal{A})$ the fan having these cones as maximal ones. Then due to [61, Thm. 4.3.7] this defines a fan structure on the tropical variety $\mathrm{trop}(\mathbb{P}_c)$.

Note that the tropical variety of a variety $Y \subseteq \mathbb{T}^r$ defined by linear relations can be endowed with a unique coarsest fan structure, the so called Bergman-fan, see [61, Chap. 4].

Lemma 6.4.3. *Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety. Then $X \subseteq Z$ admits a semi-locally toric weakly tropical resolution.*

Proof. Due to Theorem 4.3.6 it suffices to show that the embedding $\mathbb{P}_c \subseteq \mathbb{P}_r$ defined by the matrix A admits a semi-locally weakly tropical resolution. For our purposes we will endow $\mathrm{trop}(\mathbb{P}_c)$ with the fan structure $\Delta(\mathcal{A})$ defined in Remark 6.4.2. Denote the corresponding fan of $\mathbb{P}_r$ with Δ. Then $\mathrm{trop}(\mathbb{P}_c) \sqcap \Delta = \mathrm{trop}(\mathbb{P}_c)$ holds. Let $\tau \in \mathrm{trop}(Y)$ be a maximal cone. Then $\tau \subseteq \delta$ holds for a cone $\delta \in \Delta$ and after a coordinate change we may assume $\delta = \mathbb{Q}^r_{\geq 0}$. Denote by $v_1, \ldots, v_c$ the primitive generators of the rays of τ. Then by definition of the fan structure on $\mathrm{trop}(Y)$ and after suitably renumbering we achieve

$$v_i = \sum_{j=1}^{k_i} e_j, \quad \text{where} \quad 1 = k_1 < k_2 < \ldots < k_c.$$

We complement the set $v_1, \ldots, v_c$ to a lattice basis of $\mathbb{Z}^r$ by successively adding canonical basis vectors in the following way: Whenever $k_{i+1} > k_i + 1$ we add $e_{k_i+2}, \ldots, e_{k_{i+1}}$. Moreover, we add the vectors $e_{k_c+1}, \ldots e_r$. This gives rise to a decomposition $\mathbb{Z}^r =: N = N(\tau) \oplus \tilde{N}$ as in Construction 4.1.2 and we obtain an isomorphism $Z_\tau \cong U(\tau) \times \mathbb{T}^{r-c} \cong \mathbb{K}^c \times \mathbb{T}^{r-c}$. On the torus this isomorphism is given by the homomorphism $\varphi_B \colon \mathbb{T}^r \to \mathbb{T}^r$ defined by the matrix B whose columns are the above lattice basis.

Let $I := \langle f_1, \ldots, f_{r-c} \rangle$ be the ideal corresponding to $\mathbb{T}^r \cap Y$. For any $f \in I \cap \mathbb{K}[T_1, \ldots, T_r]$ denote by $\tilde{f}$ the push-down of f with respect to φ_B, i.e., the unique $(\varphi_B)_*(f) \in \mathbb{K}[T_1, \ldots T_r]$ without monomial factors such that $T^\mu \varphi_B^*((\varphi_B)_*(f)) = f$ for a $\mu \in \mathbb{Z}^r_{\geq 0}$. Then by construction we have

$$Y_\tau \cong \tilde{Y}_\tau := \mathrm{V}(\tilde{f};\ f \in I \cap \mathbb{K}[T_1, \ldots, T_r]) \subseteq \mathbb{K}^c \times \mathbb{T}^{r-c} \cong Z_\tau,$$

where the isomorphism on the left hand side is the restriction of the isomorphism on the right hand side. In order to complete the proof we need to show that the restriction of the projection onto the first c coordinates to $\tilde{Y}_\tau$ is an isomorphism onto its image and the latter is an open subset of $\mathbb{K}^c$. We show this by proving that for any $k > c$ there exists a push-down $\tilde{f}$ of an equation $f \in I$ of the form $\tilde{f} = h + \lambda_k T_k$, where $h \in \mathbb{K}[T_1, \ldots, T_{k-1}]$ and $\lambda_k \neq 0$.

Our proof is by induction. Let $k = c + 1$. Recall that the $(c+1)$-th column of B is a canonical basis vector e_i for some $1 \leq i \leq r$. We distinguish between the following two cases:

Case 1: There exists a ray generator $v_{j(i)}$ with $k_{j(i)} > i$. Let $j(i)$ be minimal with this property. By construction of A we have $i = k_{j(i)-1} + 2$. Moreover, due to the construction of the fan structure on $\mathrm{trop}(Y)$ the columns $a_1, \ldots, a_i$ of A are linearly dependant and there exists a push-down $\tilde{f} = h + \lambda_k T_k$ as claimed.

Case 2: We have $i = k_c + 1$. Then the columns $a_1, \ldots, a_{k_c}, a_i, a_0$ are linearly dependant as τ was chosen maximal. In particular there exists a push down $\tilde{f} = h + \lambda_0 + \lambda_i T_i$ with $\lambda_i \neq 0$ and $h \in \mathbb{K}[T_1, \ldots, T_{k_c}]$.

Now assume we have proven the above for all $c < k \leq n$. Consider the case $k = n + 1$.

As above the $(n+1)$-th column of B is a canonical basis vector e_i for some $1 \leq i \leq r$ and we follow the same lines as in the induction basis:

Case 1: There exists a ray generator $v_{j(i)}$ with $k_{j(i)} > i$. Let $j(i)$ be minimal with this property, i.e. we have $k_{j(i)-1} < i < k_{j(i)}$ and there exists $\alpha \geq 2$ with $i = k_{j(i)-1} + \alpha$. We conclude that the columns $a_1, \ldots, a_i$ of A are linearly dependant and there exists a push-down $\tilde{f} = h + \lambda_k T_k$ with $h \in \mathbb{K}[T_1, \ldots, T_{k-1}]$ as claimed.

Case 2: We have $i > k_c$ and the existence of a push-down $\tilde{f}$ follows with exactly the same arguments as in the induction basis. □

Proof of Theorem 6.4.1. Due to Lemma 6.4.3 the embedding $X \subseteq Z$ admits a semi-locally weakly tropical resolution. Therefore the assertion follows using [49, Prop. 2.6]. □

6.5 The anticanonical complex for arrangement varieties

In this section we investigate the anticanonical complex for arrangement varieties. As a direct consequence of Theorem 6.4.1 and Lemma 6.4.3 we obtain the existence of anticanonical complexes for arrangement varieties:

Theorem 6.5.1. *Every $\mathbb{Q}$-Gorenstein explicit arrangement variety $X(A, P, \Sigma) \subseteq Z$ admit an anticanonical complex.*

Note that for general arrangement varieties an explicit description of the anticanonical complex is given in the Sections 4.5 and 4.6.

We consider the case, where we have an explicit description of canonical divisors via the theory of Cox rings. We begin by constructing candidates for the Cox ring of the weakly tropical resolution of an explicit arrangement variety $X := X(A, P, \Sigma) \subseteq Z$. We use the concept of toric ambient modifications presented in [44]. Let $\operatorname{trop}(X)$ be endowed with a fixed quasifan structure and consider the toric morphism $Z_{\Sigma'} \to Z_\Sigma$ defined via the subdivision $\Sigma' = \operatorname{trop}(X) \sqcap \Sigma \to \Sigma$ of fans. Denote by P and P' the matrices whose columns are the primitive ray generators of Σ and Σ'. Then the corresponding maps $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$ and $P'\colon \mathbb{Z}^{r'} \to \mathbb{Z}^n$ define homomorphisms of tori

$$\mathbb{T}^{r'} \xrightarrow{p'} \mathbb{T}^n \xleftarrow{p} \mathbb{T}^r .$$

Let $g_i \in \mathbb{K}[T_1, \ldots, T_r]$ be one of the defining polynomials of $\mathcal{R}(X) = R(A, P)$. The push-down of g_i is the unique $p_*(g_i) \in \mathbb{K}[T_1, \ldots, T_n]$ without monomial factors such that $T^\mu p^*(p_*(g_i)) = g_i$ holds for some Laurent monomial $T^\mu \in \mathbb{K}[T_1^{\pm 1}, \ldots, T_r^{\pm}]$. The *shift* of g_i is the unique $g_i' \in \mathbb{K}[T_1, \ldots, T_{r'}]$ without monomial factors satisfying $p'_*(g_i') = p_*(g_i)$.

Definition 6.5.2. Let $X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety with Cox ring

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_\varrho;\ \varrho \in \Sigma^{(1)}]/\langle g_1, \ldots, g_s \rangle.$$

We call the weakly tropical resolution $X' \to X$ arising from a subdivision $\Sigma \sqcap \mathrm{trop}(X) \to \Sigma$ *explicit* if X' has a complete intersection Cox ring defined by the shifts g_i' of g_i:

$$\mathcal{R}(X') \ = \ \mathbb{K}[T_{\varrho'};\ \varrho' \in \Sigma'^{(1)}]/\langle g_1', \ldots, g_s' \rangle.$$

From now on let $\mathrm{trop}(X)$ be endowed with any coarsening of the quasifan structure defined in Remark 6.4.2. In this situation, if the weakly tropical resolution of X is semi-locally toric, it suffices to compute the discrepancies along the divisors corresponding to the rays of Σ' to describe the whole anticanonical complex. This motivates the subsequent study of the rays of Σ'. We work in the notation of Definition 6.1.19 and denote by e_{ij} resp. e_k the canonical basis vectors of $\mathbb{Q}^{n+m}$. We set

$$v_{ij} := P(e_{ij}), \qquad v_k := P(e_k).$$

Moreover for a fan Σ we denote by $\Sigma^{(1)}$ its set of rays.

Definition 6.5.3. Let $X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety and $\sigma \subseteq \mathbb{Q}^{r+s}$ be a P-cone of special or big type. We call σ *elementary*, if the following statements hold:

(i) For all $0 \le i \le r$ there exists at most one index $1 \le j_i \le n_i$ such that v_{ij_i} is a primitive ray generator of σ.

(ii) There is a ray ϱ in $\sigma \sqcap \mathrm{trop}(X)$ with $\varrho \cap \sigma^\circ \neq \emptyset$.

Construction 6.5.4. Let $X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety and let $\sigma \subseteq \mathbb{Q}^{r+s}$ be an elementary P-cone. Denote by I the set of indices i such that v_{ij_i} is a primitive ray generator of σ and define

$$\ell_{\sigma,i} := \frac{\prod_{k \in I} l_{kj_k}}{l_{ij_i}} \text{ for } i \in I, \qquad v_\sigma := \sum_{i \in I} \ell_{\sigma,i} v_{ij_i}, \qquad \varrho_\sigma := \mathbb{Q}_{\ge 0} \cdot v_\sigma.$$

Proposition 6.5.5. *Let $X := X(A, P, \Sigma) \subseteq Z$ be an explicit arrangement variety. Then the set of rays of $\Sigma \sqcap \mathrm{trop}(X)$ is given as*

$$(\Sigma \sqcap \mathrm{trop}(X))^{(1)} = \Sigma^{(1)} \cup \{\varrho_\sigma;\ \sigma \in \Sigma \text{ is elementary}\}.$$

Using the above result and applying the methods developed in [49] we obtain the following description of the discrepancies along the divisors corresponding to the rays of Σ' which leads to a full description of the anticanonical complex.

Proposition 6.5.6. *Let $X := X(A, P, \Sigma) \subseteq Z$ be a $\mathbb{Q}$-Gorenstein explicit arrangement variety admitting a semi-locally toric explicit weakly tropical resolution $Z' \to Z$ and let $\sigma = \mathrm{cone}(v_{ij_i};\ i \in I) \in \Sigma$ be an elementary cone. Then the following statements hold:*

(i) *The discrepancy along the prime divisor of $X' \subseteq Z'$ corresponding to ϱ_σ equals $c_\sigma^{-1}\ell_\sigma - 1$, where*
$$\ell_\sigma := \sum_{i\in I} \ell_{\sigma,i} - k \cdot \prod_{i\in I} l_{ij_i}$$
and k is the number of the defining equations g_i of the Cox ring $R(A,P)$ of X with $g_i(x) = 0$ for all $x \in \mathrm{V}(T_{ij_i};\ i \in I)$.

(ii) *The ray ϱ_σ is not contained in the anticanonical complex $\mathcal{A}$, if and only if $\ell_\sigma > 0$ holds; in this case, ϱ_σ leaves $\mathcal{A}$ at $v'_\sigma = \ell_\sigma^{-1} v_\sigma$.*

The rest of this section is dedicated to the proofs of Propositions 6.5.5 and 6.5.6. In the following let $X(A,P,\Sigma) \subseteq Z$ be an explicit special arrangement variety with an $(c+1)\times(r+1)$-matrix A. Let $P_1\colon \mathbb{Q}^{r+s} \to \mathbb{Q}^r$ denote the projection onto the first r coordinates and $\Delta := \Sigma_{\mathbb{P}_r}$ the fan corresponding to $\mathbb{P}_r$. Note that in this situation P_1 maps the rays of Σ onto the rays of Δ.

Lemma 6.5.7. *Let $X(A,P,\Sigma) \subseteq Z$ be an explicit arrangement variety with maximal orbit quotient $\mathbb{P}_c \subseteq \mathbb{P}_r$ as in Construction 6.1.14. Then any ray of $\Sigma \sqcap \mathrm{trop}(X)$ is either projected onto the origin or onto a ray of $\Delta \sqcap \mathrm{trop}(\mathbb{P}_c)$.*

Proof. Let ϱ be any ray of $\Sigma \sqcap \mathrm{trop}(X)$. Then there exist cones $\sigma \in \Sigma$ and $\tau \in \mathrm{trop}(X)$ such that $\varrho = \sigma \cap \tau$. By construction of the quasifan structure on $\mathrm{trop}(X)$ we have $\tau = P_1(\tau) \times \mathbb{Q}^s$ with a cone $P_1(\tau) \in \mathrm{trop}(Y)$. Therefore, we have
$$P_1(\varrho) = P_1(\sigma \cap \tau) = P_1(\sigma) \cap P_1(\tau) = \cup_{\delta\in D}(\delta \cap P_1(\tau)),$$
where D is a subset of Δ and the last equality follows as Δ is complete. As $P_1(\varrho)$ is of dimension at most one, we conclude that there exists a $\delta \in D$ with $P_1(\varrho) = \delta \cap P_1(\tau)$ and the assertion follows. □

Let B denote the matrix whose columns are the primitive ray generators of $\mathbb{P}_r$. Then we have $\mathbb{P}_c = X(A,B,\Delta) \subseteq \mathbb{P}_r$ and we may use the notions of Definitions 6.1.19 and 6.5.3.

Lemma 6.5.8. *Let $\sigma \in \Sigma$ be any special cone. Then the following statements hold:*

(i) *If $\delta := P_1(\sigma)$ is an elementary B-cone, then we have $P_1(\varrho) = \varrho_\delta$ for all $\varrho \in (\sigma \sqcap \mathrm{trop}(X))^{(1)} \setminus \sigma^{(1)}$.*

(ii) *If σ is elementary and ϱ is a ray in $\sigma \sqcap \mathrm{trop}(X)$ with $\varrho \cap \sigma^\circ \neq \emptyset$, then $\varrho = \varrho_\sigma$ holds.*

Proof. We prove (i). Let $\varrho \in \sigma \sqcap \mathrm{trop}(X)$ be any ray with $\varrho \cap \sigma^\circ \neq \emptyset$. Then there exists $\tau \in \mathrm{trop}(X)$ such that $\varrho = \sigma \cap \tau$ holds. As σ is special we have $\varrho \not\subseteq \lambda_{\mathrm{lin}}$. Therefore, applying Lemma 6.5.7 yields
$$P_1(\sigma \cap \tau) = P_1(\varrho) \in (\Delta')^{(1)}.$$

Using $\emptyset \neq \varrho \cap \sigma^\circ = \varrho \setminus \{0\}$ we conclude $P_1(\sigma)^\circ \cap P_1(\varrho) \neq \emptyset$. As the quasifan structure fixed on $\mathrm{trop}(Y)$ coarsens the matroid fan structure defined in Remark 6.4.2 we obtain that for every cone of $\delta \in \Delta$ there exists at most one ray $\varrho' \in \Delta'$ such that $\varrho' \cap \delta^\circ \neq \emptyset$ and this ray equals ϱ_δ.

We prove (ii). As σ is elementary, the projection $P_1(\sigma)$ is an elementary B-cone. As $\varrho \cap \sigma^\circ \neq \emptyset$ we conclude $\varrho \notin \sigma^{(1)}$. Therefore we may apply (i) and obtain $P_1(\varrho) = \varrho_\delta$. Due to the structure of σ there is exactly one ray in $\varrho_\delta \times \mathbb{Q}^s$ and this ray equals ϱ_σ. This completes the proof. ☐

Lemma 6.5.9. *Let $\sigma \in \Sigma$ be any special or big cone. If $\varrho_{\sigma_1} = \varrho_{\sigma_2}$ holds for any two elementary P-cones $\sigma_1, \sigma_2 \subseteq \sigma$, then σ is elementary.*

Proof. Assume σ is not elementary and denote by I the set of indices i such that there exists at least one index $1 \leq k \leq n_i$ with $\mathrm{cone}(v_{ik}) \in \sigma^{(1)}$. Then there exists $t \in I$ and cones

$$\tau \;=\; \mathrm{cone}(v_{ij_i};\ i \in I) \;\subseteq\; \sigma_0, \qquad \tau' \;=\; \mathrm{cone}(v_{ij'_i};\ i \in I) \;\subseteq\; \sigma_0$$

with $j_t \neq j'_t$ and $j_i = j'_i$ for all $i \neq t$. In particular we have $\tau \neq \tau'$. Consider v_τ and $v_{\tau'}$ and denote by c_τ and $c_{\tau'}$ the respective greatest common divisors of their entries. Here, we may assume that $c_\tau^{-1} l_{tj_t} \geq c_{\tau'}^{-1} l_{tj'_t}$ holds. Moreover, as $\varrho_\tau = \varrho_{\tau'}$ holds, we have $c_\tau^{-1} v_\tau = c_{\tau'}^{-1} v_{\tau'}$. We conclude

$$c_{\tau'}^{-1} \ell_{\tau',t} v_{tj'_t} \;=\; c_\tau^{-1} \ell_{\tau,k} v_{tj_t} + \sum_{i \in I, i \neq t} (c_\tau^{-1} \ell_{\tau,i} - c_{\tau'}^{-1} \ell_{\tau',i}) v_{ij_i}$$

and $c_\tau^{-1} \ell_{\tau,i} \geq c_{\tau'}^{-1} \ell_{\tau',i}$ holds for all $1 \leq i \leq r$. This implies $v_{tj'_t} \in \tau$. But as $\mathrm{cone}(v_{tj'_t})$ is an extremal ray of σ_0 and $\tau' \subseteq \sigma_0$ holds, $\mathrm{cone}(v_{tj'_t})$ is also an extremal ray of τ. This contradicts the choice of j'_t. ☐

Proof of Proposition 6.5.6. We show "$\subseteq$". Let ϱ be any ray of $\Sigma \sqcap \mathrm{trop}(X)$. Then there exist $\sigma \in \Sigma$ and $\lambda \in \mathrm{trop}(X)$ with $\sigma \cap \lambda = \varrho$ and we will always assume σ and λ to be minimal with this property. Note that if σ is a leaf cone, then we have $\sigma \subseteq |\mathrm{trop}(X)|$ and due to the quasifan structure fixed on $\mathrm{trop}(X)$ we obtain $\sigma = \varrho$ for a ray ϱ of Σ. So assume σ is not a leaf cone. We distinguish between the following two cases:

Case 1: We have $\lambda = \lambda_{\mathrm{lin}}$ and with $\sigma \cap \lambda \neq \emptyset$ we conclude that σ is big. In particular, there exists a P-elementary cone $\sigma_1 \subseteq \sigma$ with

$$\varrho_{\sigma_1} = \sigma_1 \cap \lambda_{\mathrm{lin}} = \sigma \cap \lambda_{\mathrm{lin}} = \varrho.$$

As this equality holds for any P-elementary cone $\sigma_1 \subseteq \sigma$ we can apply Lemma 6.5.9 and conclude that σ is P-elementary and $\varrho = \varrho_\sigma$.

Case 2: We have $\lambda \neq \lambda_{\mathrm{lin}}$. Due to minimality of σ and λ we have $\varrho \not\subseteq \lambda_{\mathrm{lin}}$ which implies that σ is special and $\sigma^\circ \cap \varrho = \sigma^\circ \cap \lambda \neq \emptyset$. In particular, either σ fulfills already condition

(i) of Definition 6.5.3 and is elementary, or it contains a P-elementary cone σ_1. In the latter case we conclude

$$\varrho_{\sigma_1} = \sigma_1 \cap \lambda = \sigma \cap \lambda = \varrho.$$

As the above equality does not depend on the choice of $\sigma_1 \subseteq \sigma$ we conclude that σ is elementary due to Lemma 6.5.9 and $\varrho = \varrho_\sigma$.

We show "$\supseteq$". By construction the rays of Σ are supported on the tropical variety. Therefore it is only left to show that any ray ϱ_σ lies in $(\Sigma \sqcap \operatorname{trop}(X))^{(1)}$. This follows by definition of “elementary” and Lemma 6.5.8 (ii). □

Due to Proposition 6.1.10 the Cox ring $R(A, P)$ of an explicit special arrangement variety $X := X(A, P, \Sigma) \subseteq Z$ is a complete intersection ring and therefore so is the Cox ring of any explicit weakly tropical resolution X'. Applying [6, Prop. 3.3.3.2] we obtain the canonical class of X' via the following formula:

$$\mathcal{K}_{X'} = -\sum_{\varrho \in (\Sigma')^{(1)}} \deg(T_\varrho) + \sum_{i=1}^{r-c} \deg(g_i') \in \operatorname{Cl}(X) \cong \mathbb{Z}^{n+m}/\operatorname{im}((P')^*). \tag{6.5.1}$$

In order to prove Proposition 6.5.6 we directly import the notion of a toric canonical φ-family from Section 4.2.

Proof of Proposition 6.5.6. We prove (i). By assumption $X' \subseteq Z'$ is semi-locally toric and by Lemma 4.2.16 there exists a toric canonical φ-family. Therefore, explicitly constructing a pair $(Z'_{\varrho_\sigma}, D_{\varrho_\sigma})$ as in Definition 4.2.8 we can calculate the discrepancy along $D_{X'}^{\varrho_\sigma}$:

Consider the ray $\varrho_\sigma \in \Sigma'$ and let $g_1', \ldots, g_r'$ be the defining relations of $\mathcal{R}(X')$. Then in each g_t' we can choose a monomial not divisible by the variable T_{ϱ_σ}. Let us denote this monomial with $T_t^{l_t} = T_{t1}^{l_{t1}} \cdots T_{tn_t}^{l_{tn_t}}$. Then we may choose

$$D_{\varrho_\sigma} := \sum_{t=1}^{r} \sum_{j=1}^{n_t} l_{tj} D_{\varrho'_{tj}} - \sum_{\varrho' \in (\Sigma')^{(1)}} D_{\varrho'},$$

where $D_{\varrho'_{tj}}$ denotes the divisor corresponding to the variable T_{tj} in $\mathcal{R}(X')$. As X' has complete intersection Cox ring, the pullback $D_{\varrho_\sigma}|_{X'}$ is a canonical divisor on X'. Moreover, the push forward $\varphi_*(D_{\varrho_\sigma})$ is $\mathbb{Q}$-Cartier and by construction we have $D_{\varrho_\sigma} = k_{Z'}$ on Z'_{ϱ_σ}. In particular, we have constructed a tuple $(Z'_{\varrho_\sigma}, D_{\varrho_\sigma})$ as claimed. Now, let $u \in \mathbb{Q}^{r+s}$ be an element such that $\operatorname{div}(\chi^u) = \varphi_*(D_{\varrho_\sigma})$ holds on Z_σ. Then, due to Remark 4.2.11, we have

$$\operatorname{discr}_X(D_{X'}^{\varrho_\sigma}) = -1 - \langle u, v_{\varrho_\sigma} \rangle.$$

Therefore, using $v_\sigma = v_{\varrho_\sigma} \cdot c_\sigma$, we obtain the assertion with

$$\langle u, v_\sigma \rangle = \langle u, \sum_{i=0}^{r} \ell_{\sigma,i} v_{ij_i} \rangle = \sum_{i=0}^{r} \ell_{\sigma,i} \langle u, v_{ij_i} \rangle = -\ell_\sigma.$$

Here the last equality holds, as D_{ij_i} occurs in $\varphi_*(D_{\varrho_\sigma})$ if and only if there exists a term $T^\nu \cdot T_i^{l_i}$ with $\nu \in \mathbb{Z}_{\geq 0}^{r+s}$ which is not divisible by T_{σ_ϱ} in one of the shifts. Using (i) assertion (ii) follows from the definition of the anticanonical complex. □

6.6 Classification results for special arrangement varieties

In this section we present classification results for three-dimensional special arrangement varieties of complexity two, having at most canonical singularities. In a first step we investigate the doubling divisors for an honestly special arrangement variety of complexity two:

Proposition 6.6.1. *Let X be a projective honestly special arrangement variety of complexity two. Then the maximal orbit quotient $X \dashrightarrow \mathbb{P}_2$ has at least five lines as its critical values.*

We consider the simplest honestly special arrangement varieties, which have five lines in $\mathbb{P}_2$ as their doubling divisors. In this situation we obtain classification results for three-dimensional $\mathbb{Q}$-Gorenstein Fano honestly special arrangement varieties X of complexity two, having a divisor class group of rank at most two, at most canonical singularities and *finite isotropy order at most two*. Here, the latter means that there is an open subset $U \subseteq X$ with complement $X \setminus U$ of codimension at least two, such that the isotropy group $\mathbb{T}_x$ is either infinite or of order at most k for all $x \in U$.

Theorem 6.6.2. *Every three-dimensional Fano honestly special arrangement variety of complexity two, having a divisor class group of rank at most two, at most canonical singularities, five critical lines as the critical values of the maximal orbit quotient and finite isotropy order at most two is isomorphic to one of the following Fano varieties X, specified by its* $\mathrm{Cl}(X)$*-graded Cox ring $\mathcal{R}(X)$, its matrix $Q = [w_1, \dots, w_r]$ of generator degrees and its anticanonical class $-\mathcal{K}_X \in$* Ample(X)*.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$Q = [w_1, \dots, w_r]$	$-\mathcal{K}_X$
1	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,S_1]}{\left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4^2, \\ T_2^2+aT_3^2+T_5^2 \\ a \neq 0,1 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
2	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,S_1]}{\left\langle \begin{array}{c} T_1^2+T_2^2+T_4^2, \\ T_1^2+T_3^2+T_5^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}$
3	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5^2, \\ T_3^2+aT_4^2+T_6^2 \\ a \neq 0,1 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
4	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1T_2+T_3^2+T_5^2, \\ T_1T_2+T_4^2+T_6^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{3} & \bar{1} & \bar{2} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
5	$\dfrac{\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6]}{\left\langle \begin{array}{c} T_1^2+T_2T_3+T_5^2, \\ T_1^2+T_4^2+T_6^2 \end{array} \right\rangle}$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{2} & \bar{1} & \bar{3} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$

6	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z} \times (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{2} & \bar{3} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 \\ \bar{0} \\ \bar{0} \\ \bar{2} \end{bmatrix}$
7	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5T_6, \\ T_3^2+aT_4^2+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
8	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5T_6, \\ T_3^2+aT_4^2+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
9	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4T_5+T_6^2, \\ T_2T_3+aT_4T_5+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
10	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4T_5+T_6^2, \\ T_2T_3+aT_4T_5+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 2 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
11	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
12	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
13	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 3 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
14	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
15	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
16	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
17	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2T_3+T_6^2, \\ T_1^2+T_4T_5+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 3 & 2 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
18	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
19	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
20	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
21	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5^2+T_6^2, \\ T_3T_4+aT_5^2+T_7^2 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
22	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 2 & 1 & -1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
23	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \big/ \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4T_5, \\ T_2^2+aT_3^2+T_6T_7 \end{array} \right\rangle$, $a \neq 0,1$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -2 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$

24	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} -1 & 1 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
25	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3T_4+T_6^2, \\ T_1T_2+T_5^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
26	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3^2+T_4^2+T_5^2, \\ T_3^2+aT_4^2+T_6^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
27	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6^2 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
28	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1T_2+T_3^2+T_5^2, \\ T_1T_2+T_4^2+T_6^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
29	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5^2, \\ T_1^2+T_4^2+T_6^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{bmatrix}$
30	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,S_1] \,/\, \left\langle \begin{array}{c} T_1^2+T_2^2+T_3^2+T_4^2, \\ T_2^2+aT_3^2+T_5T_6 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^3$	$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \\ \bar{1} \\ \bar{0} \\ \bar{1} \end{bmatrix}$
31	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{0} \end{bmatrix}$
32	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 & 1 & -1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
33	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
34	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_4^2+T_5^2, \\ T_2T_3+aT_4^2+T_6T_7 \\ a \neq 0,1 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
35	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
36	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
37	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
38	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
39	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & 2 & -2 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \\ \bar{1} \end{bmatrix}$
40	$\mathbb{K}[T_1,T_2,T_3,T_4,T_5,T_6,T_7] \,/\, \left\langle \begin{array}{c} T_1^2+T_2T_3+T_5T_6, \\ T_1^2+T_4^2+T_7^2 \end{array} \right\rangle$	$\mathbb{Z}^2 \times (\mathbb{Z}_2)^2$	$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 3 & 2 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \\ \bar{0} \\ \bar{0} \end{bmatrix}$

We turn to the proofs of the above results.

Proof of Proposition 6.6.1. We realize X as an explicit $\mathbb{T}$-variety $X(A,P,\Sigma) \subseteq Z$. For one up to three lines, the Cox ring $R(A,P)$ is a polynomial ring and thus X is toric. In case of four lines, X admits either a torus action of complexity one, having one hidden free variable as in Example 6.3.3 or is a general arrangement variety. Thus the assertion follows. □

Remark 6.6.3. Let $X := X(A,P,\Sigma) \subseteq Z$ be an explicit honestly special arrangement variety of complexity two and assume the maximal orbit quotient $X \dashrightarrow \mathbb{P}_2$ has a line arrangement of five lines as its critical values. Then we obtain the following two types of relations, as the remaining cases are either general or of complexity one:

$$\begin{array}{rll} \text{(I)} & g_1 = T_0^{l_0} + T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, & g_2 = T_1^{l_1} + aT_2^{l_2} + T_4^{l_4}, \\ \text{(II)} & g_1 = T_0^{l_0} + T_1^{l_1} + T_3^{l_3}, & g_2 = T_0^{l_0} + T_2^{l_2} + T_4^{l_4}. \end{array}$$

Now specializing to dimension three and divisor class group of rank at most two, we obtain $n_i \le 2$, $n_i = 2$ for at most two indices $i = 0, \ldots, 4$ and $m \le 2$. Assuming that X is of finite isotropy order at most two, we obtain $l_i = 2$ for $n_i = 1$ and $l_i = (1,1)$ for $n_i = 2$. In particular the polynomials g_1, g_2 are quadratic.

Note that in question of isomorphy, the distribution of the n_i is important: Two rings $R(A,P)$ and $R(A',P')$ of the same type (I) or (II), where the vector $(n_0, \ldots, n_r)$ is a permutation of the vector $(n'_0, \ldots, n'_r)$ do not need to be isomorphic, see i.a. Nos. 7 and 9 in our list.

Remark 6.6.4. Let $X(A,P,\Sigma) \subseteq Z$ be an explicit arrangement variety with divisor class group of rank two and consider the weight matrix Q whose columns consist of the free part of the weights $w_{ij} := \deg(T_{ij})$ resp. $w_k := \deg(T_k)$. We write a $\mathbb{Q}$-basis for the kernel of P in the rows of a matrix $\tilde{Q}$ and define a vector $-w_X$:

$$\tilde{Q} = [\tilde{w}_{01}, \ldots, \tilde{w}_{rn_r}], \qquad -w_X := \sum \tilde{w}_{ij} - (r-c) \sum_j l_{0j} \tilde{w}_{0j}.$$

Then there is a $\mathbb{Q}$-linear isomorphism mapping the columns of Q on the columns of $\tilde{Q}$ and thus the canonical class $-\mathcal{K}_x$ on $-w_X$. This isomorphism is either orientation preserving or reversing. Therefore, in question of the position of weights inside $\mathrm{Eff}(X)$ as in Remark 6.3.8, it suffices to look at *rational weight matrices* $\tilde{Q}$ with *rational anticanonical vectors* $-w_X$.

Remark 6.6.5. The following list of admissible operations on P do not effect the isomorphy type of the rings $R(A,P)$:

(i) Swapping two columns inside a block $v_{ij_1}, \ldots, v_{ij_{n_i}}$.

(ii) Adding multiples of the upper r rows to one of the last s rows.

(iii) Any elementary row operations among the last s rows.

(iv) Swapping two columns inside the last s rows lying under the 0-block of the matrix P_0.

Proof of Theorem 6.6.2. Let $X := X(A, P, \Sigma)$ be an explicit special arrangement variety as in Theorem 6.6.2. Then the Cox ring of X is given as a ring $R(A, P)$ as in Remark 6.6.3. In a first step, we bound the entries of P to obtain a list of candidates. Note that our computations are independent of the type of the relations and the distribution of the n_i.

Case $\operatorname{rk}(\operatorname{Cl}(X)) = 1$*:* Due to Remark 6.6.3 we are left with the following three cases:

(a) $n = 5$ and $m = 1$

(b) $n = 6$ and $m = 0$

Note that in case of a divisor class group of rank one, every $\bar{X}$-face is an X-face.

Case (a): After applying suitable admissible operations on P we may assume that we are in the following situation

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ x & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now the big cone σ gives rise to a vertex v'_σ of the anticanonical complex

$$\sigma = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_{4,1}), \qquad v'_\sigma = [0, 0, 0, 0, 4 + x].$$

Thus $x = -5$ holds due to the singularity type of X.

Case (b): After applying suitable admissible operations on P we may assume that we are in the following situation

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ x & y & 1 & 1 & 1 & 1 \end{bmatrix},$$

where we may assume $x > y$. Now, the two big cones

$$\sigma_1 = \operatorname{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_{41}), \qquad \sigma_2 = \operatorname{cone}(v_{02}, v_{12}, v_{21}, v_{31}, v_{41})$$

give the vertices v'_{σ_1} and v'_{σ_2} of the anticanonical complex

$$v'_{\sigma_1} = [0, 0, 0, 0, 2 + x], \qquad v'_{\sigma_2} = [0, 0, 0, 0, 2 + y].$$

We conclude $x = -1$ and $y = -3$ due to the singularity type of X.

Case $\mathrm{rk}\,(\mathrm{Cl}(X)) = 2$*:* Due to Remark 6.6.3 we are left with the following three cases:

(a) $n = 5$ and $m = 2$,

(b) $n = 6$ and $m = 1$,

(c) $n = 7$ and $m = 0$.

Case (a): After applying suitable admissible operations on P we may assume that we are in the following situation

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ x & 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Now, a rational weight matrix $\tilde{Q}$ and the corresponding rational vector $-w_X$ is given as

$$\tilde{Q} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -x-4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad -w_X = \begin{bmatrix} -x-3 \\ 2 \end{bmatrix}.$$

In case $-x - 4 \leq 0$ we have $\mathrm{SAmple}(X) = \mathrm{cone}([1,0],[0,1])$ and thus $-x - 3 > 0$ due to the Fano property of X. This implies $-4 \leq x < -3$ and thus $x = -4$; a contradiction to the primality of the columns of P. In case $-x - 4 > 0$, we have $\mathrm{SAmple}(X) = \mathrm{cone}([1,0],[-x-4,1])$. This implies

$$-w_X = \begin{bmatrix} -x-3 \\ 2 \end{bmatrix} = (x+5) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -x-4 \\ 1 \end{bmatrix}$$

with $x + 5 > 0$ due to the Fano property of X. Thus, we obtain $-5 < x < -4$; a contradiction.

Case (b): After applying suitable admissible operations on P we may assume that we are in the following situation

$$P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & x & y & 1 & 1 & 1 & 1 \end{bmatrix},$$

where we may assume $x > y$. Due to completeness of X, we obtain an elementary big cone $\sigma \in \Sigma$ defining a vertex v'_σ of the anticanonical complex:

$$\sigma = \mathrm{cone}(v_{01}, v_{12}, v_{21}, v_{31}, v_{41}), \qquad v'_\sigma = [0,0,0,0,y+2].$$

Thus, we conclude $y = -3$ due to the singularity type of X. Now a rational weight matrix $\tilde{Q}$ and the corresponding rational vector $-w_X$ are given as

$$\tilde{Q} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 1 & -2x-4 \\ 1 & 0 & 2 & 1 & 1 & 1 & 2 \end{bmatrix}, \qquad -w_X = \begin{bmatrix} -2x-4 \\ 2 \end{bmatrix}.$$

In particular, we obtain $\mathrm{SAmple}(X) \subseteq \mathbb{Q}^2_{\geq 0}$. As X is Fano, this implies $-2x - 2 \geq 0$ and thus $x \leq -1$.

Case (c): After applying suitable admissible operations on P we may assume that we are in the following situation

$$P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & x & y & z & 0 & 1 & 1 \end{bmatrix},$$

where we may assume $x > y$ and $z > 0$. Now, due to completeness of X we obtain two elementary big cones

$$\sigma_1 = \mathrm{cone}(v_{01}, v_{11}, v_{21}, v_{31}, v_{41}) \quad \text{and} \quad \sigma_2 = \mathrm{cone}(v_{01}, v_{12}, v_{22}, v_{31}, v_{41}),$$

defining the following vertices of the anticanonical complex:

$$v'_{\sigma_1} = [0, 0, 0, 0, 1 + (2/3)x + (2/3)z], \qquad v'_{\sigma_2} = [0, 0, 0, 0, 1 + (2/3)y].$$

Thus we conclude

$$-3 \leq y \leq -2 \qquad \text{and} \qquad 0 < 1 + (2/3)x + (2/3)z \leq 1$$

due to the singularity type of X. This implies $-2 \leq x < 0$ and thus $0 < z \leq 2$.

Now, any of the configurations above gives a ring $R(A, P)$. A direct computation shows that in all cases, the generators T_{ij} are K-prime. To obtain our list, we computed the anticanonical complexes for all configurations and checked for canonicity using the characterization in Remark 4.2.5. After removing some redundancy, we obtain the varieties in our list. □

CHAPTER

SEVEN

OUTLOOK

Our approach to Mori dream spaces with torus action described in Chapter 1 opens the possibility to systematically produce all $\mathbb{T}$-varieties X with prescribed maximal orbit quotient $X \dashrightarrow Y$. In this thesis we treated so far the case where Y is a projective space and the doubling divisors form a hyperplane arrangement. In order to investigate new example classes, one can for instance modify this setting in the following two directions. On the one hand one could stay with $Y = \mathbb{P}_c$ and consider doubling divisors of higher degree. On the other hand, one could replace $\mathbb{P}_c$ with any other Mori dream space. We take a glimpse in the second direction and consider in this last chapter Mori dream spaces with torus action where the maximal orbit quotient decomposes as a product of arrangements. Note that their Cox rings (without the grading) already appeared in Section 6.1: As rings they are isomorphic to decomposable rings $R(A, P_0)$. We follow the ideas of Section 6.2 by constructing Cox rings of these varieties and realize them as explicit $\mathbb{T}$-varieties. As an application we obtain in Proposition 7.1.7 a criterion to determine the true complexity of a special arrangement variety. Finally, we give a full classification in the smooth case for projective varieties up to Picard number two and characterize the Fano property of these varieties. The results of this chapter are published in the joint work [50].

7.1 Beyond arrangement varieties

Definition 7.1.1. An *arrangement-product variety* is a variety X with an effective action of an algebraic torus $\mathbb{T} \times X \to X$ having $X \dashrightarrow \mathbb{P}_{c_1} \times \ldots \times \mathbb{P}_{c_t}$ with $t > 1$ as a maximal orbit quotient and the doubling divisors are a collection of products

$$\mathbb{P}_{c_1} \times \ldots \mathbb{P}_{c_{i-1}} \times D_k^{(i)} \times \mathbb{P}_{c_{i+1}} \times \ldots \times \mathbb{P}_t,$$

where $i = 1, \ldots, t$ and the collection $D_k^{(i)}$ is a hyperplane arrangement in $\mathbb{P}_{c_i}$.

Remark 7.1.2. Arrangement-product varieties have finitely generated Cox rings due to [46, Thm 1.2].

We go on by constructing Cox rings of arrangement-product varieties. We will make use of the notion of (in-)decomposability of rings $R(A, P)$ as in Definition 6.2.1.

Construction 7.1.3. Consider indecomposable rings $R(A^{(i)}, P_0^{(i)})$ for $i = 1, \dots, t$ from Construction 6.1.4 with $m_i' = 0$. Choose integers $s > 0$, $m \geq 0$ and set

$$c := c^{(1)} + \dots + c^{(t)}, \ n := n^{(1)} + \dots + n^{(t)} \text{ and } r := r^{(1)} + \dots + r^{(t)}.$$

We build up a new $(c+t) \times (r+t)$ resp. $(r+s) \times (n+m)$ matrices

$$A := \begin{bmatrix} A^{(1)} & & \\ & \ddots & \\ & & A^{(t)} \end{bmatrix}, \qquad P := \begin{bmatrix} P_0 \\ \hline d \end{bmatrix} := \begin{bmatrix} P_0^{(1)} & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & P_0^{(t)} & 0 & \dots & 0 \\ \hline & & d & & & \end{bmatrix},$$

where we require the columns of P to be pairwise different and primitive, generating $\mathbb{Q}^{r+s}$ as a vector space. Denote by $e_{kl}^{(i)}$ resp. $e_k^{(i)}$ the canonical basis vectors of $\mathbb{Q}^{n+m}$ accordingly to the decomposition $n = n^{(1)} + \dots + n^{(t)}$ and let $Q_0 \colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*) := K_0$ be the projection onto the factor group. We define a $\mathbb{K}$-algebra

$$R_{\mathrm{prod}}(A, P_0) := \bigotimes R(A^{(i)}, P_0^{(i)})$$

and endow it with a K_0-grading by setting

$$\deg(T_{kl}^{(i)}) := Q(e_{kl}^{(i)}), \qquad \deg(S_k) := Q(e_k).$$

Moreover, by considering the projection $Q \colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\mathrm{im}(P^*) := K$, we define analogously a K-graded $\mathbb{K}$-algebra $R_{\mathrm{prod}}(A, P)$.

Remark 7.1.4. The K_0-graded $\mathbb{K}$-algebras $R_{\mathrm{prod}}(A, P_0)$ are integral, normal, complete intersection rings satisfying

$$\dim(R_{\mathrm{prod}}(A, P_0)) \ = \ n + m - r + c, \qquad R_{\mathrm{prod}}(A, P_0)^* \ = \ \mathbb{K}^*$$

and the K_0-grading is the finest possible grading on $R_{\mathrm{prod}}(A, P_0)$, leaving the variables and the relations homogeneous. Moreover, it is effective, pointed, factorial and of complexity c. Considering the rings $R_{\mathrm{prod}}(A, P)$, the K-grading is effective, factorial and of complexity c and, if the columns of P generate $\mathbb{Q}^{r+s}$ as a cone, it is pointed as well.

Remark 7.1.5. Note that any arrangement-product variety has a K-graded $\mathbb{K}$-algebra $R_{\mathrm{prod}}(A, P)$ as in Construction 7.1.3 as its Cox ring.

As done in Section 6.1, we use the rings $R_{\text{prod}}(A,P)$ to construct explicit $\mathbb{T}$-varieties following precisely the same steps as in Construction 6.1.11. We will denote the resulting explicit $\mathbb{T}$-varieties with $X_{\text{prod}}(A,P,\Sigma) \subseteq Z$.

Remark 7.1.6. Let $X := X_{\text{prod}}(A,P,\Sigma) \subseteq Z$ be an explicit arrangement-product variety. Then the subtorus action of $\mathbb{T}^s \subseteq \mathbb{T}^{r+s}$ on Z leaves X invariant. This turns X into a $\mathbb{T}^s$-variety of complexity $c = c_1 + \ldots + c_t$.

Proposition 7.1.7. *Let $X := X(A,P,\Sigma) \subseteq Z$ be an explicit special arrangement variety of complexity c with a decomposable ring $R(A,P)$. Then X is not of true complexity c.*

Proof. If $R(A,P)$ is decomposable, then X can be regained as an explicit $\mathbb{T}'$-variety out of its Cox ring $R_{\text{prod}}(A,P)$, where the $\mathbb{T}'$-action is of lower complexity by Remark 7.1.6. □

We now turn to our main results concerning smoothness of arrangement-product varieties. We will without further explanation use the language of explicit $\mathbb{T}$-varieties as done in Chapter 1. In particular, the smoothness criteria from Proposition 1.4.5 can be applied in our situation.

Proposition 7.1.8. *Let X be a projective arrangement-product variety of Picard number one. Then X is singular.*

Proof. By definition the Cox ring $R_{\text{prod}}(A,P)$ is decomposable into $t>1$ indecomposable rings $R^{(i)}$. Therefore the cone

$$\gamma^{(1)} := \operatorname{cone}(e_{kl}^{(1)};\ 0 \le k \le r^{(1)}, 1 \le l \le n_k^{(1)})$$

is an $\overline{X}$-face whose corresponding $\overline{X}$-stratum is singular. As X is of Picard number one, any $\overline{X}$-face is an X-face and we conclude that X is singular. □

Theorem 7.1.9. *Every smooth projective arrangement-product variety of Picard number two is isomorphic to a variety X specified by its Cox ring*

$$\mathcal{R}(X) = \mathbb{K}[T_{11},\ldots,T_{1k_1},T_{21},\ldots T_{2k_2}]/\langle g_1,g_2\rangle$$

where

$$g_i = \begin{cases} T_{i1}T_{i2} + \ldots + T_{ik_i-1}T_{ik_i}, & k_i \ge 6 \quad \textit{even} \\ T_{i1}T_{i2} + \ldots + T_{ik_i-2}T_{ik_i-1} + T_{ik_i}^2, & k_i \ge 5 \quad \textit{odd}, \end{cases}$$

the matrix Q of generator degrees and an ample class $u \in \operatorname{Cl}(X) = \mathbb{Z}^2$

$$Q = \left[\begin{array}{ccc|cccc} 1 & \ldots & 1 & a_1 & a_2 & \ldots & a_{k_2} \\ 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \end{array}\right], \qquad u = [a_1+1, 1],$$

where we have $a_i \ge a_{i+2} \ge 0$ and $a_i + a_{i+1} = 0$ for i odd and $a_{k_2} = 0$ if k_2 is odd.

Corollary 7.1.10. *A smooth projective arrangement-product variety of Picard number two as in Theorem 7.1.9 is Fano if and only if* $0 \leq a_1 \leq \frac{k_1-2}{k_2-2}$ *holds.*

Corollary 7.1.11. *If X is a smooth projective arrangement-product variety of Picard number two, then the dimension of X is at least* 6.

The rest of this section is dedicated to the proofs of the previous statements.

Lemma 7.1.12. *Let* $X := X_{\mathrm{prod}}(A, P, \Sigma)$ *be a $\mathbb{Q}$-factorial, quasismooth projective arrangement-product variety of Picard number two with Cox ring* $R_{\mathrm{prod}}(A, P)$ *decomposing into* $t > 1$ *indecomposable rings* $R^{(i)} := R(A^{(i)}, P^{(i)})$. *Then the following statements hold:*

(i) *Let* $w_{kl}^{(i)}$ *denote the weights corresponding to the variables of the ring* $R^{(i)}$. *Then the* $w_{kl}^{(i)}$ *lie either all in* τ^- *or in* τ^+.

(ii) *For all* $0 \leq \alpha \leq r^{(i)}$ *where* $1 \leq i \leq t$ *the number of variables per term* $n_\alpha^{(i)}$ *is at most* 2.

(iii) *We have* $m^{(i)} = 0$ *for all* $1 \leq i \leq t$, *and* $t = 2$ *holds.*

(iv) *If* $n_\alpha^{(i)} = 2$ *holds for one index* $0 \leq \alpha \leq r^{(i)}$ *then the corresponding ring* $R^{(i)}$ *has exactly one defining relation.*

(v) *If* $n_\alpha^{(1)} = 2 = n_\beta^{(2)}$ *holds for two indices* $0 \leq \alpha \leq r^{(1)}$ *and* $0 \leq \beta \leq r^{(2)}$ *then we have*

$$l_{\alpha 1}^{(1)} = l_{\alpha 2}^{(1)} = l_{\beta 1}^{(2)} = l_{\beta 2}^{(2)} = 1.$$

Proof. We prove (i). By construction the cone

$$\gamma^{(i)} := \mathrm{cone}(e_{kl}^{(i)};\ 0 \leq k \leq r^{(i)}, 1 \leq l \leq n_k^{(i)})$$

is an $\overline{X}$-face. As $R_{\mathrm{prod}}(A, P)$ is decomposable with $t > 1$ the corresponding $\overline{X}$-stratum is singular. Now assume that not all weights $w_{kl}^{(i)}$ lie in the same cone τ^+ or τ^-. Then $\gamma^{(i)}$ is X-relevant which contradicts quasismoothness.

We turn to (ii). Assume there exists a ring $R^{(i)}$ such that $n_\alpha^{(i)} > 2$ holds for one index $0 \leq \alpha \leq r^{(i)}$. Due to (i) we may assume that all weights $w_{kl}^{(i)}$ lie in τ^-. As there have to be at least two weights in τ^+ there exists an index $1 \leq j \leq t$ such that all weights $w_{kl}^{(j)}$ lie in τ^+. We obtain an X-face $\mathrm{cone}(e_{\alpha 1}^{(i)}, \gamma^{(j)})$ whose corresponding $\overline{X}$-stratum is singular as $n_\alpha^{(i)} > 2$ holds; a contradiction.

We prove (iii). Assume $m^{(j)} > 0$ holds for at least one $1 \leq j \leq t$. Suitably renumbering we may assume $j = 1$. Moreover, with the same arguments as in the proof of Lemma 6.3.10 (i) we may assume that $w_k^{(1)} \in \tau^+$ holds for all $1 \leq k \leq m^{(1)}$. Due to

Remark 6.3.9 there are at least two weights that lie in τ^- and using (i) we conclude that there exists $1 \le i \le t$ such that all weights $w_{kl}^{(i)}$ lie in τ^-. This gives an X-face $\mathrm{cone}(\gamma^{(i)}, e_k^{(1)})$ with singular stratum as $t > 1$ holds; a contradiction to quasismoothness.

Now assume $t \ge 3$ holds. As each of τ^+ and τ^- have to contain at least two weights we may assume that all weights $w_{kl}^{(1)}$ lie in τ^- and all weights $w_{kl}^{(2)}$ lie in τ^+. This gives an X-face $\mathrm{cone}(\gamma^{(1)}, \gamma^{(2)})$ which is singular as $t \ge 3$ holds. This contradicts quasismoothness.

We turn to (iv). By renumbering we may assume $i = 1$. Let $n_\alpha^{(1)} = 2$. Then $\mathrm{cone}(e_{\alpha 1}^{(1)}, \gamma^{(2)})$ is an X-face whose stratum is singular if there is another defining relation in $R^{(1)}$. This proves the assertion.

We prove (v). Due to (iii) and Remark 6.3.9 we may assume that all weights $w_{kl}^{(1)}$ lie in τ^+ and all weights $w_{kl}^{(2)}$ lie in τ^-. This implies that the cones $\mathrm{cone}(e_{\alpha l}^{(1)}, e_{\beta l'}^{(2)})$ with $l, l' \in \{1, 2\}$ are X-faces. As the corresponding $\overline{X}$-strata have to be smooth, the assertion follows. □

Proof of Theorem 7.1.9. By construction, $R_{\mathrm{prod}}(A, P)$ admits a decomposition into indecomposable rings $R^{(i)} := R(A^{(i)}, P^{(i)})$, where $1 \le i \le t$ and $t > 1$ holds. Applying Lemma 7.1.12 (iii) we obtain $t = 2$ and $m = 0$. Moreover, due to Lemma 7.1.12 (i) we may assume that all weights $w_{kl}^{(1)}$ of the variables of the ring $R^{(1)}$ lie in τ^- and all weights $w_{kl}^{(2)}$ of the variables of the ring $R^{(2)}$ lie in τ^+. Since X is projective and $m = 0$ holds, Σ contains at least one big cone $\sigma = P(\gamma^*)$ with an X-face γ. Recall that in our situation an $\overline{X}$-face is an X-face if it contains at least one ray corresponding to a variable of $R^{(1)}$ and another one corresponding to a variable of $R^{(2)}$. We conclude that there exists $0 \le \alpha \le r^{(1)}$ and $0 \le \beta \le r^{(2)}$ with $n_\alpha^{(1)} = 2$ and $n_\beta^{(2)} = 2$, where equality holds due to Lemma 7.1.12 (ii). Moreover, Lemma 7.1.12 (v) implies

$$l_{\alpha 1}^{(1)} = l_{\alpha 2}^{(1)} = l_{\beta 1}^{(2)} = l_{\beta 2}^{(2)} = 1$$

for all $0 \le \alpha \le r^{(1)}$ and $0 \le \beta \le r^{(2)}$ with $n_\alpha^{(1)} = 2$ and $n_\beta^{(2)} = 2$. Applying Lemma 7.1.12 (iv) we are left with one homogeneous defining relation g_1 for $R^{(1)}$ and another homogeneous defining relation g_2 for $R^{(2)}$ whose weights $\deg(g_1) = w^{(1)}$ and $\deg(g_2) = w^{(2)}$ lie in τ^- and τ^+ respectively. In particular, by a suitable unimodular coordinate change on $\mathbb{Z}^2$ we can achieve $w^{(1)} = (w_1, 0)$ and $w^{(2)} = (0, w_2)$ with positive integers w_1, w_2. We conclude that for any α and β with $n_\alpha^{(1)} = 2$ and $n_\beta^{(2)} = 2$ as above there exists integers a and b such that

$$w_{\alpha 1}^{(1)} = (w_1/2, b), \quad w_{\alpha 2}^{(1)} = (w_1/2, -b), \quad w_{\beta 1}^{(2)} = (a, w_2/2). \quad w_{\beta 1}^{(2)} = (-a, w_2/2).$$

As the cones $\mathrm{cone}(e_{\alpha 1}^{(1)}, e_{\beta 1}^{(2)})$ and $\mathrm{cone}(e_{\alpha 2}^{(1)}, e_{\beta 1}^{(2)})$ are X-faces, we conclude

$$\frac{1}{4} w_1 w_2 + ab. = \det(w_{\alpha 1}^{(1)}, w_{\beta 1}^{(2)}) = 1 = \det(w_{\alpha 2}^{(1)}, w_{\beta 1}^{(2)}) = \frac{1}{4} w_1 w_2 + ab. \tag{7.1.1}$$

This implies $a = 0$ or $b = 0$ and we may assume the latter holds. Moreover, we obtain $w_1 = w_2 = 2$ and homogeneity of the relations implies that the relations g_1 and g_2 are quadratic. As equation (7.1.1) holds for any choice of α and β we are left with the following configuration of weights:

$$Q = \left[\begin{array}{ccc|cccc} 1 & \dots & 1 & a_1 & a_2 & \dots & a_{k_2} \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{array}\right].$$

We show that there is at most one term in each of g_1 and g_2 with only one variable. Assume there is more than one. Then the divisor class group contains torsion, see Proposition 2.2.3. This is a contradiction, as $\operatorname{cone}(e_{\alpha 1}^{(1)}, e_{\beta 1}^{(2)})$ is an X-face and therefore the divisor class group is isomorphic to $\mathbb{Z}^2$. Now, the conditions on the a_i follow due to homogeneity of the relations and by suitably renumbering. In order to complete the proof it is only left to show, that the varieties in this class are indeed smooth. This follows directly by checking the criterion of Proposition 1.4.5. □

Proof of Corollary 7.1.10. In order to prove the statement we consider the varieties of Theorem 7.1.9 and check under which condition the anticanonical class lies in the ample cone. As $\mathcal{R}(X)$ is a complete intersection ring the anticanonical class is given as $-\mathcal{K}_X = (k_1 - 2, k_2 - 2)$. Moreover, due to Lemma 7.1.12 we have $\tau^- = \operatorname{cone}((1,0))$ and $\tau^+ = \operatorname{cone}((a_1, 1), (-a_1, 1))$. We conclude that $-\mathcal{K}_X$ lies in the ample cone if and only if $0 \le a_1 \le \frac{k_1-2}{k_2-2}$ holds. □

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