Koopmanism for Attractors in Dynamical Systems

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Deutsche Zusammenfassung

In der vorliegenden Dissertation soll die Existenz und verschiedene Eigenschaften von Attraktoren in dynamischen Systemen untersucht werden. Dabei verwenden wir einen operatorentheoretischen Ansatz, den wir „Koopmanismus“ nennen. Dazu sei X ein Zustandsraum und $(\varphi_t)_{t\geq 0}$ ein Halbfluss auf X. Dann definieren wir auf einem geeigneten Funktionenraum $\mathcal{O}$ auf X, auch Observablenraum genannt, die zugehörige *Koopmanhalbgruppe* durch

$$T(t)f \coloneqq f \circ \varphi_t \quad \text{für } f \in \mathcal{O}\,,\ t \geq 0\,.$$

Diese Familie von linearen Operatoren $(T(t))_{t\geq 0}$ ist eine Operatorhalbgruppe auf $\mathcal{O}$, falls die Operatoren $T(t)$ den Raum $\mathcal{O}$ invariant lassen. Dies liefert eine globale Linearisierung des im Allgemeinen nichtlinearen Halbflusses $(\varphi_t)_{t\geq 0}$.

Diesen Übergang von einem dynamischen System $(X, (\varphi_t)_{t\geq 0})$ zum zugehörigen Koopmansystem $(\mathcal{O}, (T(t))_{t\geq 0})$ bezeichnen wir mit „Koopmanismus“. Die Eigenschaften des zugrundeliegenden Systems spiegeln sich, in einem gewissen Sinne, durch Eigenschaften des Koopmansystems wieder und umgekehrt.

Ein Attraktor eines dynamischen Systems $(X, (\varphi_t)_{t\geq 0})$ ist eine kompakte und $(\varphi_t)_{t\geq 0}$-invariante Teilmenge $\emptyset \neq M \subseteq X$ des Zustandsraums mit der Eigenschaft, dass alle Zustände $x \in X$ auf eine gewisse Art und Weise gegen M konvergieren, kurz „ $\varphi_t \to M$ “ für $t \to \infty$. Ein Attraktor sollte außerdem minimal mit dieser Eigenschaft sein. In der Literatur finden sich, motiviert durch wichtige physikalische Beispiele, viele Definitionen für Attraktoren. Wir behandeln gleichmäßige (asymptotisch stabile) Attraktoren, punktweise Attraktoren, Milnorattraktoren und Attraktionszentren, siehe hierzu Definition 3.22 auf Seite 30, sowie Lyapunovstabilität und Lyapunovfunktionen, siehe hierzu Definition 3.25 auf Seite 31 und Definition 3.29 auf Seite 32.

Unser Vorgehen lässt sich folgendermaßen zusammenfassen. Zu einer abgeschlossenen, $(\varphi_t)_{t\geq 0}$-invarianten Teilmenge $\emptyset \neq M \subseteq X$ betrachten wir den zugehörigen $(T(t))_{t\geq 0}$-invarianten Unterraum $I_M \subseteq \mathcal{O}$ aller Funktionen, die auf M verschwinden, also

$$I_M \coloneqq \{f \in \mathcal{O} \mid f|_M \equiv 0\}\,.$$

Wir charakterisieren Attraktivitätseigenschaften der abgeschlossenen, invarianten Teilmenge M durch Stabilität, d.h. Konvergenz gegen 0, der zugehörigen Koopmanhalbgruppe eingeschränkt auf I_M. Unser Leitmotiv lässt sich wie folgt darstellen:

$$„\varphi_t \to M“ \quad \Leftrightarrow \quad „T(t)\big|_{I_M} \to 0“ \ .$$

Im nächsten Schritt betrachten wir die maximalen Unterräume von $\mathcal{O}$, auf denen die Koopmanhalbgruppe stabil ist,

$$I := \{f \in \mathcal{O} \mid T(t)f \to 0 \text{ für } t \to \infty\}\,.$$

Dabei gibt es verschiedene Möglichkeiten die Konvergenz „$T(t)f \xrightarrow{t\to\infty} 0$“ zu verstehen. Es stellt sich heraus, dass alle diese Unterräume von der Form $I = I_M$ sind. Deshalb gehört zu jedem solchen Unterraum eine abgeschlossene, invariante, attraktive Teilmenge von X, die minimal mit dieser Eigenschaft ist. So stellen wir die Existenz verschiedener Attraktortypen sicher.

Der Aufbau der Arbeit ist wie folgt. Teil I besteht aus zwei vorbereitenden Kapiteln. Grundlegende und bekannte Definitionen und Resultate über Stabilität von Operatorhalbgruppen werden in Kapitel 2 wiederholt und diskutiert. In Kapitel 3 werden grundlegende Eigenschaften und Beispiele von topologischen dynamischen Systemen, invarianten Mengen und Attraktoren wiederholt.

Im zweiten Teil werden im Sinne des oben erläuterten Leitmotivs Attraktoren in dynamischen Systemen untersucht. In Kapitel 4 werden dynamische Systeme mit kompaktem Zustandsraum betrachtet. Dabei werden in Abschnitt 4.1 Eigenschaften absorbierender Mengen durch Eigenschaften der zugehörigen Koopmanhalbgruppe charakterisiert, siehe Proposition 4.7 auf Seite 38. Abschnitt 4.2 widmet sich der Charakterisierung von Attraktortypen durch Stabilität der Koopmanhalbgruppe. Dies ist in Theorem 4.9 auf Seite 40 aufgelistet und bildet das Hauptresultat des Kapitels. In Abschnitt 4.3 werden Lyapunovfunktionen mithilfe des Koopmanansatzes untersucht. Da wir nicht davon ausgehen, dass der zugrundeliegende Zustandsraum metrisch ist, verallgemeinern wir den Begriff einer Lyapunovfunktion auf eine Familie $(g_i)_{i\in I}$ von Funktionen, die auf dem Attraktor M verschwinden und $M = \cap_{i\in I} g_i^{-1}(\{0\})$ erfüllen. Wir zeigen, dass die Existenz einer solchen Lyapunovfamilie äquivalent zu starker Stabilität der eingeschränkten Koopmanhalbgruppe ist und verallgemeinern damit ein klassisches Resultat für asymptotisch stabile Attraktoren, siehe Theorem 4.11 auf Seite 44. Das Kapitel endet in Abschnitt

4.4 mit einer Diskussion über die Existenz und einer allgemeinen Charakterisierung aller genannten Attraktortypen. Besonders hervorzuheben ist die Charakterisierung des minimalen Attraktionszentrums durch den Abschluss der Vereinigung der Träger aller ergodischen Wahrscheinlichkeitsmaße auf X, siehe Proposition 4.24 auf Seite 52.

Anschließend werden in Kapitel 5 dynamische Systeme mit lokalkompaktem, nicht kompaktem, Zustandsraum betrachtet. In Abschnitt 5.1 werden grundlegende Eigenschaften von Koopmanoperatoren und Koopmanhalbgruppen in dieser Situation untersucht. Danach in den Abschnitten 5.2 und 5.3 werden erneut absorbierende und attraktive Mengen untersucht. Die Charakterisierung dieser durch die Koopmanhalbgruppe wird in Proposition 5.18 auf Seite 65 und Theorem 5.19 auf Seite 66 jeweils aufgelistet. Abschnitt 5.4 beschäftigt sich in diesem Kontext erneut mit Lyapunovfunktionen und -familien.

Abgeschlossen wird Kapitel 5 mit der Untersuchung der Existenz von Attraktoren. In Abschnitt 5.5 wird gezeigt, welche Ideale I_M zu kompakten Teilmengen des zugrundeliegenden Zustandsraums gehören, siehe Theorem 5.22 auf Seite 70. Außerdem werden die einzelnen Attraktortypen charakterisiert. Dies zusammen ermöglicht die Charakterisierung der Existenz von Attraktoren in dynamischen Systemen mit lokalkompaktem, nicht unbedingt kompaktem Zustandsraum durch die zugehörige Koopmanhalbgruppe. Dies ist Theorem 5.38 auf Seite 76.

Kapitel 6 gibt einen Ausblick darauf, wie Koopmanismus für Attraktoren in dynamischen Systemen auf vollständigen metrischen Räumen angewendet werden kann. In Abschnitt 6.1 werden allgemeine Eigenschaften von Koopmanoperatoren und -halbgruppen für diesen Fall besprochen. Im nächsten Abschnitt 6.2 werden erneut die verschiedenen Attraktortypen durch Stabilität der eingeschränkten Koopmanhalbgruppe charakterisiert, siehe Theorem 6.13 auf Seite 90.

Der dritte Teil der Arbeit befasst sich mit stark stetigen (Bi-)Markovverbandshalbgruppen auf L^p-Räumen. Wir charakterisieren solche Halbgruppen durch Eigenschaften ihres Generators, der sich wie eine Derivation auf dessen Definitionsbereich verhält. Das ist das Hauptresultat des Kapitels, siehe Theorem 7.12 auf Seite 98. Außerdem zeigen wir, dass auf einem Standardwahrscheinlichkeitsraum $\mathrm{X} = (X, \Sigma, \mu)$ die Markovverbandshalbgruppen auf $\mathrm{L}^1(X)$ genau den Koopmanhalbgruppen entsprechen. Abschließend konstruieren wir ein topologisches Modell für Koopmanhalbgruppen auf $\mathrm{L}^1(\mathrm{X})$ in Theorem 8.6 auf Seite 111.

1 Introduction

It was in the 1950s and 60s that “attractors” became popular objects in the field of dynamical systems. After the term first occurred in [ABS64, p. 55], cf. [Mil85, p. 177], many variations and modifications have been defined, each of them motivated by interesting examples.

One goal of this thesis is to establish a systematic hierarchy of different types of attractors appearing in the literature. We prove the existence of attractors, compare and characterize each while using operator theoretic tools.

We “translate” various concepts of attractors, such as asymptotically stable, pointwise or Milnor attractors and centers of attraction, into operator theoretic terms by globally linearizing the dynamical system. We call this process *“Koopmanism”*. Around 1930 this idea appeared in the papers [Koo31] by B. O. Koopman and [vNeu32b] by J. von Neumann and provided the precise mathematical framework to treat the so-called *ergodic hypothesis* from L. Boltzmann formulated in [Bol85]. It is based on the distinction between a *state space* X of a (physical) system and an associated *observable space* $\mathcal{O}$ being a (vector) space of real or complex valued functions on X. If the non-linear semiflow

$$\varphi_t \colon X \to X\,,\ t \geq 0\,,$$

describes the dynamics on the state space X, the maps

$$f \mapsto T(t)f := f \circ \varphi_t\,,\ f \in \mathcal{O}\,,$$

become linear operators and, if $\mathcal{O}$ remains invariant, $(T(t))_{t\geq 0}$ is a one-parameter semigroup of linear operators on $\mathcal{O}$, called *Koopman semigroup*.

This idea led to the proof of the classical ergodic theorems of J. von Neumann [vNeu32a] and G. D. Birkhoff [Bir31] and even gave rise to *ergodic theory* as a mathematical discipline.

The recent state of the art of this operator theoretic approach to ergodic theory as used in this thesis is presented in the monograph *“Operator Theoretic Aspects of Ergodic Theory”* by T. Eisner, B. Farkas, M. Haase und R. Nagel [EFHN15]. In addition, there is a vast spectrum of applications of “Koopmanism”, the recent book *"The Koopman Operator in Systems and Control"* by A. Mauroy, I. Mezić and Y. Susuki (Eds.), [MMS20], provides a broad overview.

The main focus of this thesis is the study and characterization of *attractors* using the Koopman approach. To this end, we consider dynamical systems with locally compact state space X and a continuous semiflow $(\varphi_t)_{t\geq 0}$ thereon, cf. Definition 3.1 on page 19. Given a closed $(\varphi_t)_{t\geq 0}$-invariant set $\emptyset \neq M \subset X$, our idea is to restrict the Koopman semigroup to $I_M \subset \mathcal{O}$, the corresponding subset of functions vanishing on M, i.e.,

$$I_M := \{f \in \mathcal{O} \mid f|_M \equiv 0\}.$$

Clearly, this set I_M is invariant under the Koopman semigroup $(T(t))_{t\geq 0}$. We characterize the long-term behavior of $(\varphi_t)_{t\geq 0}$ around M by asymptotic properties of the Koopman semigroup $(T(t))_{t\geq 0}$ restricted to I_M. So our leitmotif can be visualized as

$$\text{“}\varphi_t \to M\text{”} \quad \Leftrightarrow \quad \text{“}T(t)\big|_{I_M} \to 0\text{”} \ .$$

The idea to characterize attractivity properties of invariant sets of a flow by a stability property of the associated Koopman operators restricted to functions vanishing on the attractor appears, e.g., in A. Mauroy and I. Mezić, see [MM16, II.Prop. 1]. Their stability property corresponds to what we will later call weak stability.

For each Koopman semigroup $(T(t))_{t\geq 0}$ on a space $\mathcal{O}$ we define maximal subspaces I on which the semigroup is stable, i.e.,

$$I := \{f \in \mathcal{O} \mid T(t)f \to 0 \text{ as } t \to \infty\}.$$

Again, there are many ways how to interpret "$T(t)f \to 0$" (cf. Definition 2.1 on page 9 and Definition 2.3 on page 10). Each of these stability concepts yields an invariant subspace I that corresponds to a closed and invariant subset $\emptyset \neq M \subseteq X$, i.e., $I = I_M$. So we obtain a closed and invariant attractive subset of X that is minimal with this property. If the underlying state space X is compact, so is M and therefore is an attractor. If X is a non-compact, locally compact space we give conditions on I such that M is compact, i.e., that an attractor exists (cf. Theorem 5.38 on page 76).

The present thesis is composed of three parts. In Part I and Part II we consider topological dynamical systems, i.e., semiflows on a locally compact state space X to which we associate the corresponding Koopman semigroup. Part III addresses Koopman semigroups induced by measure preserving semiflows on probability spaces $\mathrm{X} = (X, \Sigma, \mu)$.

Part I starts with two preliminary chapters that summarize basic results on stability of operator semigroups (Chapter 2) and on topological dynamical systems (Chapter 3). Here, the monographs *"Stability of operators and operator semigroups"* by T. Eisner, [Eis10], and *"Stability Theory of Dynamical Systems"* by N. P. Bhatia and G. P. Szegö, [BS02], are our main references. In these two chapters we fix the notation, recall some results and definitions and, in Chapter 3, specify the term *attractor*, (cf. Definition 3.22 on page 30). The main results are contained in the following two parts.

Then we turn to Part II which contains the study of attractors using the Koopman approach and consists of the Chapters 4, 5 and 6. In Chapter 4 we study topological dynamical systems $(K, (\varphi_t)_{t\geq 0})$ with compact state space K and the corresponding Koopman semigroup $(T(t))_{t\geq 0}$ on the Banach space $\mathrm{C}(K)$ of continuous real-valued functions on K. We discuss strong continuity and basic properties of $(T(t))_{t\geq 0}$. Then we characterize absorbing and attractive invariant subsets of K (cf. Definition 3.13 on page 24, Definition 3.18 on page 27 and Definition 3.3 on page 20) by stability properties of the corresponding Koopman semigroup $(T(t))_{t\geq 0}$ in Sections 4.1 and 4.2. The main result of this chapter is the characterization of attractors via stability of the Koopman semigroup in Theorem 4.9 on page 40.

In Section 4.3 we use *Koopmanism* to study *Lyapunov functions* and *Lyapunov stability.* Classically, a Lyapunov function is an observable $g \in \mathcal{O}$ that vanishes on the attractor M and is strictly decreasing along the orbits, i.e.,

$$g(\varphi_t(x)) = T(t)g(x) < g(x) \quad \text{for all } x \in X \setminus M\,,\ t > 0\,.$$

This can be understood easily via the Koopman approach using our leitmotif. Not assuming the state space X to be metric, we generalize the notion of Lyapunov functions. We call a family $(g_i)_{i\in I}$ of functions vanishing on the attractor M *Lyapunov family* if $M = \cap_{i\in I} g_i^{-1}(\{0\})$ and they are strictly decreasing along the orbits outside their respective zero sets (cf. Definition 3.28 on page 31, Definition 3.29 on page 32). We prove that the Koopman semigroup restricted to I_M is strongly stable if and only if there exists a Lyapunov family for M, see Theorem 4.11 on page 44.

Another interesting property is *almost weak stability* on I_M, see Section 2.2. It translates into M being a *center of attraction.* It is known that on a compact metric space there always exists a unique minimal center of attraction, see for instance [NS60, Thm. 6.0.6], given by the closure of the union of the supports of ergodic measures, see [Man12, Ex. I.8.3] for a discrete version. In [Kre20, Thm. 4.7], H. Kreidler proves this for semigroup actions with a Følner sequence on a compact space K, not necessarily metric. We are able to give a simple proof for the existence of a unique minimal center of attraction in Proposition 4.24 on page 52 obtained as the closure of the union of the supports of ergodic measures. We utilize several equivalent characterizations of almost weak stability and the "translation" in Theorem 4.9 (IV) on page 40 while not assuming K to be metric. Furthermore, we use these results to examine an example of a minimal center of attraction that is not attractive, cf. Example 3.21 e) on page 29 and Example 4.25 on page 53.

Sections 4.1, 4.2 and 4.4 are based on the author's publication [Küh19] *What can Koopmanism do for attractors in dynamical systems?*, The Journal of Analysis (2019).

In Chapter 5 we are concerned with dynamical systems with locally compact, but non-compact state space Ω. We apply the results from Chapter 4 to the one-point compactification $\Omega \cup \{\infty\}$ of Ω. To a semiflow $(\varphi_t)_{t\geq 0}$ on Ω we associate a semigroup on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \cong \mathrm{C}(\Omega \cup \{\infty\})$ in the usual way by $T(t)f := f \circ \varphi_t$, $t \geq 0$, $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, with $\langle \mathbb{1} \rangle := \{c \cdot \mathbb{1} \mid c \in \mathbb{R}\}$. In Section 5.1 we elaborate on the fact that not every semiflow on Ω induces a semigroup leaving $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant and discuss the term "Koopman semigroup" for the associated semigroup. We show that Koopman operators on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ (cf. Definition 7.4 on page 96) are exactly those algebra, or equivalently lattice, homomorphisms T on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ with $T\mathbb{1} = \mathbb{1}$ that are τ_c-continuous, where τ_c denotes the compact-open topology on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. We then characterize Koopman semigroups on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ by means of their generator acting as a derivation on its domain.
This is followed by the Koopman characterization of absorbing and attractive sets in Sections 5.2 and 5.3. Then we discuss Lyapunov functions and Lyapunov stability in Section 5.4.

Chapter 5 is concluded by the discussion of the existence of attractors. In Section 5.5 we prove that an ideal I_M in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ corresponds to a *compact* set M if and only if I_M is τ_c-closed, see Theorem 5.22 on page 70. From this we conclude that an attractor exists if and only if the corresponding subspace $I := \{f \in \mathcal{O} \mid T(t)f \to 0 \text{ as } t \to \infty\}$ is τ_c-closed.

Thus, we obtain an operator theoretic characterization of *dissipative systems* (cf. [Hal10, Sect. 3.4], [Lad91, Chapt. 1, p.4], [Tem12, Chapt. I, Sect. 3, p.11], [SY13, Sect. 2.3.3] or [Chu15, Def. 2.2.1]).

Again, we are able to use our operator theoretic view-point to prove the existence of a minimal center of attraction in the locally-compact setting (Proposition 5.37 on page 75). It is given by the closure of the union of supports of invariant measures on the state space Ω.

We conclude Part II in Chapter 6 with an outlook on attractors for semiflows on a complete metric state space X. Given such a semiflow $(\varphi_t)_{t\geq 0}$, we associate it with the Koopman semigroup $(T(t))_{t\geq 0}$ defined on $\mathrm{C_b}(X)$, the space of all real-valued bounded continuous functions on X. Recall that $\mathrm{C_b}(X)$ is canonically isomorphic to $\mathrm{C}(\beta X)$ where βX denotes the Stone–Čech compactification of X. The goal ist to deduce attractivity properties of closed invariant subsets of X by means of the Koopman semigroup, using the results from the compact setting in Chapter 4 and, similarly to Chapter 5, by introducing the compact open topology τ_c on $\mathrm{C_b}(X)$. In Section 6.1, we show that Koopman operators are exactly those algebra, or lattice homomorphisms, T on $\mathrm{C_b}(X)$ with $T\mathbb{1} = \mathbb{1}$ that are τ_c-continuous. Furthermore, we prove that $I \subseteq \mathrm{C_b}(X)$ is a τ_c-closed ideal if and only if there exists a closed subset $M \subseteq X$ with $I = I_M$.

There are several problems arising in this setting. One of them is that Koopman semigroups on $\mathrm{C_b}(X)$ are generally not strongly continuous, even though the underlying semiflow $(\varphi_t)_{t\geq 0}$ is continuous. But a Koopman semigroup is strongly τ_c-continuous which suffices for the study of attractors.

We conclude this chapter with Section 6.2 by again characterizing how attractivity properties of a closed invariant subset of X are reflected by stability properties of the corresponding Koopman semigroup in this case, cf. Theorem 6.13 on page 90.

In Part III, we turn to measurable and measure preserving semiflows $(\varphi_t)_{t\geq 0}$, cf. Definition 8.1 on page 109, on a measure space $\mathrm{X} = (X, \Sigma, \mu)$. In Section 7.1, given a finite measure space $\mathrm{X} = (X, \Sigma, \mu)$ we characterize (bi-)Markov lattice semigroups on $\mathrm{L}^p(X)$ by means of their generator acting as a derivation on the dense subspace $\mathrm{L}^\infty(\mathrm{X}) \cap D(A)$, Theorem 7.12 on page 98, and give a similar result for lattice semigroups that are not necessarily Markov, Theorem 7.19 on page 103. Then we turn to a standard probability space $\mathrm{X} = (X, \Sigma, \mu)$ in Section 7.2 and apply these results to (bi-)Markov lattice semigroups $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$. In this case every operator $T(t)$, $t \geq 0$, is of the form $T(t)f = f \circ \varphi_t$, $f \in \mathrm{L}^p(\mathrm{X})$ for some (measure preserving) measurable map φ_t. We point out that the family $(\varphi_t)_{t\geq 0}$ of measurable maps does not form a semiflow as in Definition 8.1 on page 109. Furthermore, in Chapter 8, we discuss measurable and measure preserving semiflows and topological models of Markov lattice semigroups. Given a Markov lattice semigroup on $\mathrm{L}^1(\mathrm{X})$, we construct a compact space K and a regular Borel measure ν on K such that $\mathrm{L}^1(K,\nu) \cong \mathrm{L}^1(\mathrm{X})$ and the Markov semigroup on $\mathrm{L}^1(\mathrm{X})$ is isomorphic to a Koopman semigroup on $\mathrm{L}^1(K,\nu)$.

The results in Part III appear in [EGK19] Nikolai Edeko, Moritz Gerlach, Viktoria Kühner, *Measure preserving semiflows and one-parameter Koopman semigroups*, Semigroup Forum (2019), p.48-63, and have been rearranged for easier reading.

We conclude this introduction with some remarks on attractors in dynamical systems on complete metric spaces. The existence of attractors is, in contrast to the compact situation (Chapter 4), not guaranteed. To characterize those dynamical systems that allow for global attractors is quite difficult, cf. [EZM05], [SY13, Sect. 2.3] or [Chu15, Sect. 3]. Using the Koopman approach and the characterization in Theorem 6.13 on page 90 one might look for a necessary and sufficient algebraic or topological condition on the ideals

$$I = \{f \in \mathrm{C_b}(X) \mid T(t)f \to 0\}$$

such that the corresponding closed invariant subset $M \subseteq X$, with $I = I_M$, is a compact global attractor. This condition would include previously studied conditions as found in literature, e.g., dissipativity [SY13, Sect. 2.3.3, p. 32], asymptotically smooth systems [Hal10, Sect. 3.2, p. 36] or asymptotically compact systems [Lad91, Chapt. 3, p. 12] to name a few.

Part I

Preliminaries

2 Operator theoretical tools for topological dynamical systems

In this chapter we first review well-known stability concepts for strongly continuous operator semigroups on Banach spaces and introduce similar notions for spaces with locally convex topologies. Then in Section 2.2 the notion of "almost weak stability" and in Section 2.3 stability for positive strongly continuous operator semigroups on Banach lattices will be discussed in more detail.

2.1 Stability of operator semigroups on Banach spaces

Definition 2.1 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X. Then $(T(t))_{t\geq 0}$ is said to be

a) *nilpotent* if there exists $t_0 > 0$ such that

$$\|T(t_0)\| = 0\,,$$

b) *uniformly exponentially stable* if there exists $\delta > 0$ such that

$$\lim_{t\to\infty} e^{\delta t}\|T(t)\| = 0\,,$$

c) *uniformly stable* if

$$\lim_{t\to\infty} \|T(t)\| = 0\,,$$

d) *strongly stable* if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \text{ for all } x \in X\,,$$

e) *weakly stable* if

$$\lim_{t\to\infty} \langle T(t)x, x' \rangle = 0 \text{ for all } x \in X\,,\ x' \in X' \text{ and}$$

f) *almost weakly stable* if for all pairs $(x, x') \in X \times X'$ there exists a Lebesgue measurable subset $R \subseteq \mathbb{R}_+$ with density [1] 1 such that

$$\lim_{t\to\infty, t\in R} \langle T(t)x, x' \rangle = 0\,.$$

In the above definition the following chain of implications holds

$$a) \implies b) \implies c) \implies d) \implies e) \implies f)\,.$$

All implications are strict except b) $\iff$ c) which can be found in [EN00, Chapt. V, Sect. 1]. For examples we refer to [Eis10, Chapt. III], [Van12] and [EN00, Chapt. V, Sect. 1]. We remark the following fact.

Definition 2.2 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X. We define subsets corresponding to the properties d) and e) in Definition 2.1 above by

$$\begin{aligned} I_{\text{ss}} &:= \{x \in X \mid \|T(t)x\| \to 0 \text{ as } t \to \infty\} \text{ and} \\ I_{\text{ws}} &:= \{x \in X \mid \langle T(t)x, x' \rangle \to 0 \text{ as } t \to \infty \text{ for all } x' \in X'\}\,. \end{aligned}$$

Both are closed subspaces of X.

We generalize the concept of stability introduced in Definition 2.1 above to stability with respect to a locally convex topology. We will use this in Chapter 5. Let X be a Banach space and τ an additional locally convex Hausdorff topology on X (cf. [Sch71, §4,4.,p. 47]). We write $\mathcal{P}$ for a family of seminorms determining τ (cf. [Sch71, §4,4., p. 48]).

We distinguish between several stability notions with respect to τ as follows.

Definition 2.3 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X and τ an additional locally convex Hausdorff topology on X. Then $(T(t))_{t\geq 0}$ is said to be

[1] The density of a Lebesgue measurable subset $R \subset \mathbb{R}_+$ is

$$d(R) := \lim_{t\to\infty} \frac{1}{t} \lambda\left([0, t] \cap R\right),\ \lambda \text{ Lebesgue measure,}$$

if the limit exists.

a) *τ-nilpotent* if for every $p \in \mathcal{P}$ there exists $t_0 > 0$ with

$$p(T(t_0)x) = 0 \text{ for all } x \in X\,,$$

b) *τ-stable* if

$$\lim_{t\to\infty} p(T(t)x) = 0 \text{ for all } p \in \mathcal{P},\ x \in X\,.$$

c) *almost τ-stable* if for every $p \in \mathcal{P}$, $x \in X$ there exists a Lebesgue measurable subset $R \subseteq \mathbb{R}_+$ with density 1 such that

$$\lim_{t\to\infty, t\in R} p(T(t)x) = 0\,.$$

Proposition 2.4 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X and τ an additional locally convex Hausdorff topology on X. If $(T(t))_{t\geq 0}$ is also strongly τ-continuous in the sense that $t \mapsto p(T(t)x)$ is continuous for every $p \in \mathcal{P}$, $x \in X$, then assertion c) in Definition 2.3 on page 10 is equivalent to

c*) for all $p \in \mathcal{P}$, $x \in X$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t p(T(s)x)\,\mathrm{d}s = 0\,.$$

Proof. This follows by applying the so-called *Koopman-von-Neumann Lemma* [Eis10, Chapt. III, Lem. 5.2] to the positive and continuous function given by $t \mapsto p(T(t)x)$ for $p \in \mathcal{P}$, $x \in X$. □

Definition 2.5 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X and τ an additional locally convex Hausdorff topology on X. We define the following subsets of X.

$$I_\tau := \{x \in X \mid \lim_{t\to\infty} p(T(t)x) = 0 \text{ for all } p \in \mathcal{P}\}\,,$$

$$I_{\mathrm{a}\tau} := \{x \in X \mid \lim_{t\to\infty} \frac{1}{t} \int_0^t p(T(s)x)\,\mathrm{d}s = 0 \text{ for all } p \in \mathcal{P}\}\,.$$

These subsets are in general not τ-closed, see Example 5.24 on page 71.

We introduce an additional concept that will be useful later in the second part of this thesis.

Definition 2.6 Let $\mathcal{A}$ be a closed subspace of the space $C_b(X)$ of all bounded real-valued continuous functions on X, where X is a topological Hausdorff space, and let μ be a regular Borel measure on X (cf. [Bog07, Def. 7.1.1, Def. 7.1.5]). A strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on $\mathcal{A}$ is said to be (μ-)*almost everywhere pointwise stable* if for every $f \in \mathcal{A}$

$$T(t)f(x) \to 0 \text{ as } t \to \infty \text{ for } \mu\text{-almost all } x \in X\,.$$

We define a corresponding closed subspace by

$$I_{\text{aeps}} \coloneqq \{f \in \mathcal{A} \mid T(t)f(x) \to 0 \text{ as } t \to \infty \text{ for } \mu\text{-almost all } x \in X\}\,.$$

2.2 Almost weak stability

For a complete treatment of almost weak stability for strongly continuous semigroups on Banach spaces with relatively weakly compact orbits we refer to [Eis10, Chapt. III, Sect. 5]. The tools and ideas used in this section are based on [EFHN15, Chapt. 9].

The following proposition is a useful characterization of almost weak stability, it can be found in a more general setting in [Hia78, Thm. 2.2].

Proposition 2.7 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X. Then the following are equivalent.

a) $(T(t))_{t\geq 0}$ is almost weakly stable,

b)
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\langle T(t)x, x'\rangle| \,\mathrm{d}t = 0$$
for all $x \in X$, $x' \in X'$,

c)
$$\lim_{T\to\infty} \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x, x'\rangle| \,\mathrm{d}t = 0$$
for all $x \in X$.

Proof. The equivalence a) $\iff$ b) follows from the so called *Koopman-von Neumann Lemma*, see for example [Eis10, Chapt. III, Lem. 5.2]. The implication b) $\Longrightarrow$ c) in the time discrete analogue is due to *Jones and Lin*, cf. [JL76]. We adapt the proof given in [EFHN15, Prop. 9.17]. Every operator $T(t)$ as its adjoint $T(t)'$ is a contraction and the dual unit ball B' is compact with respect to the weak-* topology. Due to these facts we can define the Koopman system

$$(\mathrm{C}(B'), (\tilde{T}(t))_{t\geq 0})$$

where

$$\tilde{T}(t)f(x') := f(T(t)'x')$$

for $t \geq 0$, $f \in \mathrm{C}(B')$, $x' \in B'$. Fix $x \in X$ and define $g_x \in \mathrm{C}(B')$ by $g_x(x') := |\langle x, x'\rangle|$. By b)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{T}(t) g_x(x') \,\mathrm{d}t = 0$$

pointwise in x' and by Lebesgue's theorem of dominated convergence also weakly and thus in the norm of $\mathrm{C}(B')$, cf. [Eis10, Chapt. I, Thm. 2.25] or [EFHN15, Prop. 8.18]. This is property c). The implication c) to b) is clear.

□

Using this proposition we define a space I_{aws} as follows.

Definition 2.8 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X. We define

$$I_{\text{aws}} := \{x \in X \mid \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau |\langle T(t)x, x'\rangle| \,\mathrm{d}t = 0\}\,.$$

Proposition 2.9 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of contractions on a Banach space X. The subset I_{aws} is a closed, $(T(t))_{t\geq 0}$-invariant subspace of X.

Proof. That I_{aws} is a $(T(t))_{t\geq 0}$-invariant subspace is clear. To prove that it is in fact a closed subspace we use characterization c) of almost weak stability in Proposition 2.7 on page 12. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in I_{aws} with limit $x \in X$ and take $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\varepsilon}{2}$. By Proposition 2.7 c) on page 12 there exists $t(n) \geq 0$ such that

$$\sup_{x'\in X', \|x'\|\leq 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x'\rangle| \,\mathrm{d}t < \frac{\varepsilon}{2}$$

for all $T > t(n)$. This implies

$$
\begin{aligned}
&\sup_{x' \in X', \|x'\| \le 1} \frac{1}{T} \int_0^T |\langle T(t)x, x'\rangle| \, \mathrm{d}t \\
&\le \sup_{x' \in X', \|x'\| \le 1} \frac{1}{T} \int_0^T |\langle T(t)(x_n - x), x'\rangle| \, \mathrm{d}t + \sup_{x' \in X', \|x'\| \le 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x'\rangle| \, \mathrm{d}t \\
&\le \|x - x_n\| + \sup_{x' \in X', \|x'\| \le 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x'\rangle| \, \mathrm{d}t < \varepsilon \text{ for all } T \ge \max\{n, t(n)\} .
\end{aligned}
$$

This completes the proof by the equivalences of a) and c) in Proposition 2.7 on page 12. □

For strongly continuous semigroups of contractions with relatively weakly compact orbits there are other equivalent characterizations of almost weak stability, see [Eis10, Chapt. III, Sect. 5]. In the next proposition we discuss one of those properties and are able to prove that this property implies almost weak stability without assuming relatively weakly compact orbits.

Proposition 2.10 Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup of contractions on a Banach space X. If for all $x \in X$ there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ with $t_n \to \infty$ as $n \to \infty$ such that for all $x' \in X'$

$$\lim_{n\to\infty} \langle T(t_n)x, x'\rangle = 0 ,$$

then $(T(t))_{t\ge 0}$ is almost weakly stable.

Proof. Take $x \in X$ and $(t_n)_{n\in\mathbb{N}}$, $t_n \to \infty$ such that

$$\lim_{n\to\infty} \langle T(t_n)x, x'\rangle = 0$$

for all $x' \in X'$. As in the proof of Proposition 2.7 on page 12 we consider the induced Koopman system $(\mathrm{C}(B'), (\tilde{T}(t))_{t\ge 0})$ and the function

$$g_x(x') := |\langle x, x'\rangle| .$$

If $\mu \in \mathrm{C}(B')'$ vanishes on $\bigcup_{t\ge 0} (\mathrm{rg}(\mathrm{Id} - \tilde{T}(t)))$, then

$$\langle g_x, \mu\rangle = \langle \tilde{T}(t_n) g_x, \mu\rangle$$

for all $n \in \mathbb{N}$. We observe that

$$\begin{aligned}\langle \tilde{T}(t_n)g_x, \mu\rangle &= \int_{B'} \tilde{T}(t_n)g_x(x')\,\mathrm{d}\mu(x')\\ &= \int_{B'} |\langle x, T(t_n)'x'\rangle|\,\mathrm{d}\mu(x')\\ &= \int_{B'} |\langle T(t_n)x, x'\rangle|\,\mathrm{d}\mu(x')\,.\end{aligned}$$

By assumption $|\langle T(t_n)x, x'\rangle|$ converges to 0 for all $x' \in X'$. By Lebesgue's Theorem the integral $\int_{B'}|\langle T(t_n)x,' x'\rangle|\,\mathrm{d}\mu(x')$ goes to 0 as well, thus implying $\langle g_x, \mu\rangle = 0$. This implies $g_x \in \overline{\operatorname{lin}} \bigcup_{t\geq 0}(\operatorname{rg}(\mathrm{Id} - \tilde{T}(t)))$ by the theorem of Hahn-Banach. Thus,

$$\begin{aligned}&\frac{1}{T}\int_0^T |\langle T(t)x, x'\rangle|\,\mathrm{d}t\\ =&\frac{1}{T}\int_0^T (\tilde{T}(t)g_x)(x')\,\mathrm{d}t \xrightarrow{T\to\infty} 0\end{aligned}$$

for all $x' \in X'$, cf. [EN00, Chapt. V, Sect. 4] and in particular [EN00, Chapt. V, Lem. 4.4]. Since x was arbitrary, $(T(t))_{t\geq 0}$ is almost weakly stable. $\square$

2.3 Spectral conditions for stability of positive semigroups on Banach lattices

If $(T(t))_{t\geq 0}$ is a positive strongly continuous semigroup on a Banach lattice, some stability can be characterized by the spectrum of the generator A. We first recall the *uniform growth bound* ω_0 and the *growth bound* ω_1 of $(T(t))_{t\geq 0}$.

Definition 2.11 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. Then the *uniform growth bound* is

$$\omega_0 := \inf\{\omega \in \mathbb{R} \mid \|T(t)\| \leq Me^{-\omega t} \text{ for some } M > 0\}$$

and the *growth bound* is

$$\omega_1 := \inf\{\omega \in \mathbb{R} \mid \lim_{\tau\to\infty}\int_0^\tau e^{-\omega t}T(t)x\,\mathrm{d}t \text{ exists for all } x \in X\}\,.$$

Then

$$s(A) \leq \omega_1 \leq \omega_0 \,,$$

where $s(A)$ is the *spectral bound*

$$s(A) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(A)\} \,.$$

Theorem 2.12 [Nag+86, Sect. C-IV]
Let $(T(t))_{t\geq 0}$ be a positive strongly continuous semigroup with generator $(A, D(A))$ on a Banach lattice X. Then the following are equivalent.

a) $s(A) < 0$,

b) $\omega_1 < 0$,

c) $0 \in \rho(A)$ and $-A^{-1} = R(0, A) \geq 0$,

d) $\int_0^\infty T(s)y \,\mathrm{d}s$ exists for all $0 \leq y \in X$,

e) for all $0 \leq x \in X$ there exists a unique $0 \leq y \in D(A)$ with $Ay = -x$.

Corollary 2.13 If $X = \mathrm{C}(K)$, the space of all complex-valued continuous functions on a compact space K, then the assertions in Theorem 2.12 are equivalent to

e*) there exists $0 \leq g \in X$ such that

$$Ag = -\mathbb{1} \,.$$

f) $(T(t))_{t\geq 0}$ is weakly stable.

Proof. The equivalence of e*) and f) can be found in [Nag+86, Thm. 1.1, Sect. B-IV]. Since e) implies e*), it remains to show that e*) implies one of the other assertions. We show e*) $\Rightarrow$ c).
By [EN00, Chapt. 6, II.1.3], e*) implies

$$\begin{aligned}\int_0^t T(s)\mathbb{1} \,\mathrm{d}s &= \int_0^t -T(s)Ag \\ &= g - T(t)g \leq g\end{aligned}$$

since $T(t)$ is positive. Let $\varepsilon > 0$ and $\lambda > 0$, then

$$\|\int_r^t e^{-\lambda s} T(s)\mathbb{1}\,\mathrm{d}s\| \leq |e^{-\lambda r}|\|\int_r^t T(s)\mathbb{1}\,\mathrm{d}s\| \leq |e^{-\lambda r}|\|g\| < \varepsilon$$

holds for all t, r sufficiently large. This implies that $\int_0^\infty e^{-\lambda s} T(s) f\,\mathrm{d}s$ exists for all $f \in \mathrm{C}(K)$ and $\lambda > 0$, hence $R(\lambda, A)f = \int_0^\infty e^{-\lambda s} T(s) f\,\mathrm{d}s$ and $\lambda \in \rho(A)$ for all $\lambda > 0$, $f \in \mathrm{C}(K)$. For $\lambda > 0$ we obtain

$$\|R(\lambda, A)\| = \|R(\lambda, A)\mathbb{1}\| \leq \|g\| < \infty\,.$$

All in all, $R(\lambda, A)$ is uniformly bounded for all $\lambda > 0$ which implies $0 \in \rho(A)$, because if $s(A) = 0 \in \sigma(A)$ the resolvent would be unbounded for $\lambda \searrow 0$. $\square$

Corollary 2.14 In Corollary 2.13 on page 16 one can exchange $\mathbb{1}$ by any strictly positive $f \in \mathrm{C}(K)$ since such f is again an order unit.

3 Basic properties of topological dynamical systems

In this chapter we define topological dynamical systems and continuous semiflows and discuss basic concepts of those. This will lay the groundwork for our investigation of attractors and absorbing sets in topological dynamical systems using operator theoretic tools.

Definition 3.1 A (topological) dynamical system is a pair $(X, (\varphi_t)_{t\geq 0})$ consisting of a locally compact Hausdorff space X and a continuous semiflow $(\varphi_t)_{t\geq 0}$ on X. We call a family $(\varphi_t)_{t\geq 0}$ of continuous self-mappings on X, *semiflow* if $\varphi_0 = \mathrm{id}_X$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \geq 0$. We call a semiflow $(\varphi_t)_{t\geq 0}$ *continuous* if the mapping

$$\begin{aligned} \varphi\colon [0,\infty) \times X &\to X\,, \\ (t,x) &\mapsto \varphi_t(x) \end{aligned}$$

is continuous with respect to the product topology.

Throughout this thesis we will use the letter K for dynamical systems with a compact state space and Ω if the state space is non-compact and locally compact.

Example 3.2

1. Let X be a locally compact Hausdorff space and consider $\varphi_t := \mathrm{id}_X$ for all $t \geq 0$. Then $(X, (\varphi_t)_{t\geq 0})$ is a dynamical system.
2. Take $K := [0, 1]$. For $x \in K$ and $t \geq 0$, define $\varphi_t(x) := e^{-t}x$. This yields a dynamical system $(K, (\varphi_t)_{t\geq 0})$.
3. Another standard example is given by $\Omega := \mathbb{R}$ and the shift defined by $\varphi_t(x) := x + t$, $x \in \mathbb{R}$, $t \geq 0$ thereon.
4. We define a semiflow $(\psi_t)_{t\geq 0}$ on $[0,\infty]$, the one-point compactification of $[0,\infty)$, by

$$\psi_t(x) := e^{-t}x,\ x \in [0,\infty) \text{ and } \varphi_t(\infty) := \infty,\ t \geq 0.$$

5. Recall the homeomorphism θ defined by
$$\begin{aligned}\theta\colon [0,\infty) &\to [0,1)\\ x &\mapsto \frac{x}{x+1}\,.\end{aligned}$$
Starting from above Example 3.2, 4., we obtain a continuous semiflow $(\phi_t)_{t\geq 0}$ on $[0,1]$ by
$$\phi_t(y) := \theta \circ \psi_t \circ \theta^{-1}(y)$$
and $\phi_t(1) = 1$ for all $t \geq 0$. A short computation yields
$$\phi_t(y) = \frac{e^{-t}y}{e^{-t}y + 1 - y} \quad \text{for } y \in [0,1]\,.$$

For the rest of this chapter let $(X, (\varphi_t)_{t\geq 0})$ be a dynamical system. For a subset $M \subseteq X$ we denote the family of its neighborhoods by $\mathcal{U}(M)$.

3.1 Invariant sets

Of great interest are invariant subsets and fixed points of a dynamical system $(X, (\varphi_t)_{t\geq 0})$.

Definition 3.3 An element $x \in X$ (also called *state* or *point*) is called *fixed point* if
$$\varphi_t(x) = x \quad \text{for all } t \geq 0\,.$$
A subset $B \subseteq X$ is called *invariant* if
$$\varphi_t(B) \subseteq B \quad \text{for all } t \geq 0\,.$$

One important example of invariant subsets of X are “orbits”.

Definition 3.4 For $x \in X$ its *orbit* is
$$\operatorname{orb}(x) := \{\varphi_t(x) \mid t \geq 0\}\,,$$
and for a subset $A \subseteq X$ we define
$$\operatorname{orb}(A) := \{\varphi_t(A) \mid t \geq 0\}\,.$$
For the closure we write $\overline{\operatorname{orb}}(x) := \overline{\operatorname{orb}(x)}$ and $\overline{\operatorname{orb}}(A) := \overline{\operatorname{orb}(A)}$, respectively.

Other interesting closed invariant sets are so-called ω-limit sets, cf. [SY13, Sect. 2.1.2, p. 16], [BS02, Chapt. 3, Def. 3.1].

Definition 3.5 For $x \in X$ we define the ω-limit set of x by

$$\omega(x) := \bigcap_{\tau \geq 0} \overline{\{\varphi_t(x) \mid t \geq \tau\}}\,.$$

For a set $A \subseteq \Omega$ we define its ω-limit set by

$$\omega(A) := \bigcap_{\tau \geq 0} \overline{\{\varphi_t(A) \mid t \geq \tau\}}\,.$$

Remark 3.6 Remark that for $x \in X$

$$\omega(x) = \bigcap_{\tau \geq 0} \overline{\mathrm{orb}}(\varphi_\tau(x)) \subseteq \overline{\mathrm{orb}}(x)$$

and for $A \subseteq X$

$$\overline{\bigcup_{x \in A} \omega(x)} \subseteq \omega(A)\,.$$

In general $\overline{\bigcup_{x \in A} \omega(x)} \neq \omega(A)$ as the following example shows.

Example 3.7 Consider $K := \mathbb{R} \cup \{\infty\}$ the one-point compactification of $\mathbb{R}$ and the semiflow $(\varphi_t)_{t \geq 0}$ on K defined by

$$\varphi_t(x) := x + t,\ x \in \mathbb{R},\ t \geq 0 \text{ and } \varphi_t(\infty) = \infty \text{ for all } t \geq 0\,.$$

For the subset $L := (-\infty, 0] \cup \{\infty\}$ one obtains $\omega(L) = K$, but $\overline{\bigcup_{x \in L} \omega(x)} = \{\infty\}$.

The following useful characterization is well-known for dynamical systems with metric state space X. We repeat the arguments for sequences adjusted to our situation, cf. [BV13, Prop. 3.5].

Proposition 3.8 Let x, $y \in X$ and $A \subseteq X$.

1. Then $y \in \omega(x)$ if and only if there exists a net $(t_i)_{i \in I}$ in $[0, \infty)$, $t_i \to \infty$, with $\varphi_{t_i}(x) \xrightarrow{i \in I} y$.
2. Similarly, $y \in \omega(A)$ if and only if there exist nets $(t_i)_{i \in I}$ in $[0, \infty)$, $t_i \to \infty$, and $(x_i)_{i \in I}$ in A with $\varphi_{t_i}(x_i) \to y$.

Proof. Proof of 1.: Take $x \in X$ and $y \in \omega(x)$, i.e., $y \in \overline{\{\varphi_t(x) \mid t \geq \tau\}} = \overline{\mathrm{orb}}(\varphi_\tau(x))$ for all $\tau \geq 0$.

We distinguish between two cases. First, if there exists $\tau \geq 0$ with

$$y \in \overline{\mathrm{orb}(\varphi_\tau(x))} \setminus \mathrm{orb}(\varphi_\tau(x)),$$

there exists a net $(t_i)_{i \in I}$, $t_i \geq \tau$ with $\varphi_{t_i}(x) \to y$ for $i \in I$ and $t_i \to \infty$ since $y \notin \mathrm{orb}(\varphi_\tau(x))$.

If, on the other hand, $y \in \mathrm{orb}(\varphi_\tau(x))$ for all $\tau \geq 0$, then there exists $r > 0$ with $y = \varphi_r(x)$. Also, $y \in \mathrm{orb}(\varphi_\tau(x))$ for all $\tau > r$ which implies there exists $s > r$ with $\varphi_s(x) = y$. We write $y = \varphi_{s-r}(\varphi_r(x)) = \varphi_{s-r}(y)$ and obtain

$$y = \varphi_{n \cdot (s-r)}(y) \quad \text{for all } n \in \mathbb{N}.$$

The increasing sequence $t_n := n \cdot (s-r) + r$ satisfies $\varphi_{t_n}(x) = y$ and $t_n \to \infty$.

For the other implication let $(t_i)_{i \in I}$ be a net with $t_i \to \infty$ such that $\varphi_{t_i}(x)$ converges to y. For fixed $s \geq 0$ there exists $i_0 \in I$ such that $t_i \geq s$ for all $i \geq i_0$. Then $(\varphi_{t_i}(x))_{i \in I, i \geq i_0}$ is still a net converging to y. Therefore, $y \in \overline{\mathrm{orb}}(\varphi_s(x))$ for all $s \geq 0$.

Proof of 2.: Let $A \subseteq \Omega$ and $y \in \omega(A)$. We follow the same arguments as in the proof of 1.: If there exists $\tau \geq 0$ with

$$y \in \overline{\mathrm{orb}(\varphi_\tau(A))} \setminus \mathrm{orb}(\varphi_\tau(A)),$$

then there exists a net $(t_i)_{i \in I}$, $t_i \geq \tau$ and a net $(x_i)_{i \in I}$ in A with $\varphi_{t_i}(x_i) \to y$ for $i \in I$. Since $y \notin \mathrm{orb}(\varphi_\tau(A))$, $t_i \to \infty$.

Next, if $y \in \mathrm{orb}(\varphi_\tau(A))$ for all $\tau \geq 0$, there exists $t_0 \geq 0$ and $x_0 \in A$ such that $y = \varphi_{t_0}(x_0)$. Next, there exists $t_1 \geq t_0 + 1$ and $x_1 \in A$ such that $y = \varphi_{t_1}(x_1)$ since $y \in \mathrm{orb}(\varphi_{t_0+1}(A))$. Thus, one obtains a sequence $(t_n)_{n \in \mathbb{N}}$, $t_n \to \infty$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $y = \varphi_{t_n}(x_n)$.

To prove the converse implication, let $(t_i)_{i \in I}$ be a net with $t_i \to \infty$ and $(x_i)_{i \in I}$ a net in A such that $\varphi_{t_i}(x_i)$ converges to y. For fixed $s \geq 0$ there exists $i_0 \in I$

such that $t_i \geq s$ for all $i \geq i_0$. Then $(\varphi_{t_i}(x_i))_{i\in I, i\geq i_0}$ is still a net converging to y. Therefore, $y \in \overline{\mathrm{orb}}(\varphi_s(A))$ for all $s \geq 0$. □

Clearly, in the non-compact case, ω-limit sets may be empty.

Example 3.9 Consider $X := [0, \infty)$ and define a semiflow $(\varphi_t)_{t\geq 0}$ thereon by $\varphi_t(x) := x + t$. Then $\omega(x) = \omega(A) = \emptyset$ for all $x \in X$ and $A \subseteq X$.

To conclude this section we prove a connection between closed orbits and ω-limit sets. We modify the usual arguments, cf. [BV13, Prop. 3.6] for a locally-compact metric state space X to a general locally compact Hausdorff state space.

Proposition 3.10 For $L \subseteq X$ with $\overline{\mathrm{orb}}(L)$ compact,

(i) the set $\omega(L)$ is non-empty and compact, and

(ii) $\varphi_t(L) \to \omega(L)$ in the sense that for every neighborhood U of $\omega(L)$ there exists $t_0 \geq 0$ such that $\varphi_t(L) \subseteq U$ for all $t \geq t_0$.

Proof. For the proof of (i) remark that $\omega(L) \subset \overline{\mathrm{orb}}(L)$ and hence $\omega(L)$ is compact as a closed subset of a compact set. It is non-empty by the finite intersection property.
Now, to prove (ii), assume there is an open neighborhood U of $\omega(L)$ such that $\varphi_{t_i}(x_i) \in U^c$ for a net $(t_i)_{i\in I}$ in $[0, \infty)$ with $t_i \to \infty$ and a net $(x_i)_{i\in I}$ in L. But $\varphi_{t_i}(x_i) \in \overline{\mathrm{orb}}(L)$ for all $i \in I$ and therefore has a convergent subnet with limit in U^c which is a contradiction to Proposition 3.8, (1.) on page 21. □

Proposition 3.11 Let $L \subseteq X$ such that $\omega(L)$ is compact and non-empty, then

$$\varphi_t(L) \to \omega(L) \quad \text{as } t \to \infty$$

in the sense of Proposition 3.10 (ii).

Proof. Assume this is not true, then there is an open neighborhood U of $\omega(L)$ and a net $(t_i)_{i\in I}$ in $[0, \infty)$, $t_i \to \infty$ and a net $(x_i)_{i\in I}$ in L such that $\varphi_{t_i}(x_i) \in U^c$. But, U^c is compact in the one-point compactification $X \cup \{\infty\}$ of X, hence $(\varphi_{t_i}(x_i))_{i\in I}$ has a convergent subnet with limit in U^c, possibly ∞, which is a contradiction to Proposition 3.8, (1.), on page 21. □

Combining Proposition 3.10 and Proposition 3.11 we obtain the following.

Remark 3.12 For $L \subseteq X$ compact, $\overline{\mathrm{orb}}(L)$ is compact if and only if $\omega(L)$ is compact and non-empty.

Proof. The implication " $\Longrightarrow$ " has been proved in Proposition 3.10 (i) on page 23.
For the other implication remark that $\omega(L)$ being non-empty and compact already implies that $\varphi_t(L) \to \omega(L)$ by Proposition 3.11 on page 23. Now, let $(\varphi_{t_i}(x_i))_{i\in I}$ be a net in $\overline{\mathrm{orb}}(L)$. If $(t_i)_{i\in I}$ is bounded, it has a convergent subnet $(t_{i_j})_{j\in J}$ with limit $t \geq 0$ and thus $\varphi_{t_{i_j}}(x_{i_j}) \in \bigcup_{0\leq s\leq t} \varphi_s(L)$, which implies that $(\varphi_{t_{i_j}}(x_{i_j}))_{j\in J}$ has a convergent subnet because L is compact. Now assume that $(t_i)_{i\in I}$ is unbounded and take a compact neighborhood W of $\omega(L)$. Then there exists $i_0 \in I$ such that $\varphi_{t_i}(x_i) \in W$ for all $i \geq i_0$ by assumption. Since W is compact there exists a convergent subnet of $(\varphi_{t_i}(x_i))_{i\in I}$ with limit in W, thus every net in $\overline{\mathrm{orb}}(L)$ has a convergent subnet which implies that $\overline{\mathrm{orb}}(L)$ is compact. □

3.2 Asymptotic properties of dynamical systems

Now we introduce other types of invariant sets that will be the main focus of discussion throughout the second part of this thesis, namely *absorbing*, *attractive* and *Lyapunov stable* sets, (cf. [SY13, Sect. 2.3.3], [Tem12, Sect. 1.3, 1.4], [BS02, Chapt. V]).

3.2.1 Absorbing sets

Definition 3.13 A closed invariant set $\emptyset \neq M \subsetneq X$ is called

a) *absorbing* if there exists $t_0 > 0$ such that
$$\varphi_{t_0}(X) \subseteq M\,,$$

b) *compact absorbing* if for all $L \subseteq X$ compact there exists $t_0 > 0$ such that
$$\varphi_{t_0}(L) \subseteq M\,,$$

c) *pointwise absorbing* if for all $x \in X$ there exists $t_0 > 0$ such that
$$\varphi_{t_0}(x) \in M\,.$$

The three concepts do not coincide in general, as the following examples show.

Example 3.14 1. Let $\Omega := [0, \infty)$ and for $t \geq 0$, $x \in \Omega$ define

$$\varphi_t(x) := \max(0, x - t)\,.$$

This semiflow is continuous and the set $M := \{0\}$ is compact, invariant and compact absorbing but not absorbing.

Considering the same semiflow on a different space however gives an example for an absorbing set.

Let $\Omega := [0, 1)$ and for $t \geq 0$, $x \in \Omega$ define

$$\varphi_t(x) := \max(0, x - t)\,.$$

This semiflow is continuous and the set $M := \{0\}$ is absorbing.

2. We construct an example for a pointwise absorbing set that is not compact absorbing. Recall Example 3.2, 4., on page 19

On $\Omega := [0, \infty)$ we obtain a semiflow $(\varphi_t)_{t \geq 0}$ by

$$\varphi_t(x) := \begin{cases} \frac{e^{-t}x}{e^{-t}x + 1 - x} & x \in [0, 1) \\ e^{-t}(x - 1) + 1 & x \geq 1\,. \end{cases}$$

A short computation shows that $(\varphi_t)_{t \geq 0}$ is in fact continuous. The set $M := \left[0, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right]$ is pointwise absorbing, but not compact absorbing or absorbing. Take for example the compact set $L := [0, 1]$ then for $\frac{1}{2} < x < 1$, $\varphi_t(x) \in M$ for $t \geq -\ln\left(\frac{1}{x} - 1\right)$. But, $-\ln\left(\frac{1}{x} - 1\right) \to \infty$ for $x \nearrow 1$.

To give a condition under which compact absorbing and pointwise absorbing sets coincide we remark the following.

Remark 3.15 A locally compact space X is a Baire space, i.e., for a sequence of closed subsets X_m, $m \in \mathbb{N}$, with

$$X = \bigcup_{m \in \mathbb{N}} X_m$$

there exists $n \in \mathbb{N}$ such that X_n has non-empty interior $\operatorname{int}(X_n)$. If M is a pointwise absorbing subset of X, the sets $\varphi_n^{-1}(M)$, $n \in \mathbb{N}$, form a closed cover of X and $M \subseteq \varphi_n^{-1}(M)$ for all $n \in \mathbb{N}$ by invariance.

We use this fact to characterize under which condition a pointwise absorbing set is compact absorbing. First, consider the following example.

Example 3.16 Take $K := [0, \infty]$ and the semiflow defined by $\varphi_t(x) := e^{-t}x$ for $x \in [0, \infty)$, $t \geq 0$, and $\varphi_t(\infty) = \infty$ for all $t \geq 0$. The set $M := [0, 1] \cup \{\infty\}$ is pointwise absorbing but not compact absorbing. We observe that for $n \in \mathbb{N}$

$$\varphi_n^{-1}(M) = [0, e^n] \cup \{\infty\} \quad \text{and } \operatorname{int}\left(\varphi_n^{-1}(M)\right) = [0, e^n)\,.$$

Thus, M is not a subset of $\operatorname{int}(\varphi_n^{-1}(M))$.

This leads us to the following characterization.

Proposition 3.17 A compact invariant set $\emptyset \neq M \subsetneq X$ is compact absorbing if and only if it is pointwise absorbing and there exists $n \in \mathbb{N}$ such that

$$M \subseteq \operatorname{int}\left(\varphi_n^{-1}(M)\right)\,.$$

Proof. If M is compact absorbing it is pointwise absorbing. Take a compact neighborhood U of M, by assumption there exists $n \in \mathbb{N}$ with $\varphi_n(U) \subseteq M$ and thus $M \subseteq U \subseteq \varphi_n^{-1}(M)$. Since U is a neighborhood of M, so is $\varphi_n^{-1}(M)$ which implies M is in the interior of $\varphi_n^{-1}(M)$.

For the other implication let $L \subseteq X$ be compact and use that for every $x \in L$ there exists $t_x \geq 0$ such that

$$\varphi_{t_x}(x) \in \operatorname{int}\left(\varphi_n^{-1}(M)\right) \subset \varphi_n^{-1}(M)\,.$$

By continuity $\varphi_{t_x}^{-1}(\operatorname{int}(\varphi_n^{-1}(M)))$ is open for every $x \in L$ and

$$L \subseteq \bigcup_{x \in X} \varphi_{t_x}^{-1}(\operatorname{int}\left(\varphi_n^{-1}(M)\right))\,.$$

Since L is compact, there exist finitely many $x_1, \ldots, x_j$ for some $j \in \mathbb{N}$ such that

$$L \subseteq \bigcup_{k=1}^{j} \varphi_{t_{x_k}}^{-1}(\operatorname{int}\left(\varphi_n^{-1}(M)\right))\,.$$

This implies for $y \in L$ that

$$\varphi_{t_{x_k}}(y) \in \operatorname{int}\left(\varphi_n^{-1}(M)\right) \subset \varphi_n^{-1}(M)$$

for some $k \in \{1, \dots, j\}$ and therefore

$$\varphi_{t_{x_k}+n}(y) \in M\,.$$

Define $T := \max\{t_{x_k} \mid k \in \{1, \dots, j\}\}$, then

$$\varphi_{T+n}(y) \in M$$

by invariance of M. □

Now we turn to attractive subsets of X and attractors.

3.2.2 Attractive sets and attractors

Definition 3.18 A closed invariant set $\emptyset \neq M \subseteq X$ is called

a) *uniformly attractive* if for all $U \in \mathcal{U}(M)$ there exists $t_0 > 0$ such that

$$\varphi_t(X) \subseteq U \quad \text{for all } t \geq t_0\,,$$

b) *compact attractive* if for all $L \subseteq X$ compact and $U \in \mathcal{U}(M)$ there exists $t_0 > 0$ such that

$$\varphi_t(L) \subseteq U \quad \text{for all } t \geq t_0\,,$$

c) *(pointwise) attractive* if for all $x \in X$ and $U \in \mathcal{U}(M)$ there exists $t_0 > 0$ such that

$$\varphi_t(x) \in U \quad \text{for all } t \geq t_0\,,$$

d) *likely limit set (for μ)*, where μ is a quasi invariant regular Borel measure[1] on X, if for all $U \in \mathcal{U}(M)$ and μ-almost every $x \in K$ there exists $t_0 > 0$ with

$$\varphi_t(x) \in U \text{ for all } t \geq t_0\,,$$

[1]Given a semiflow $(\varphi_t)_{t\geq 0}$ on X, then a Borel measure μ on X is called *quasi invariant (with respect to $(\varphi_t)_{t\geq 0}$)* if for a Borel measurable set N, $\mu(N) = 0$ if and only if $\mu(\varphi_t^{-1}(N)) = 0$ for all $t \geq 0$.

e) *center of attraction* if for all $U \in \mathcal{U}(M)$

$$\lim_{t\to\infty} \frac{1}{t}\lambda\left(\{s \in [0,t] \mid \varphi_s(x) \in U\}\right) = 1$$

for all $x \in X$, where λ denotes the Lebesgue measure on $[0,\infty)$.

The concepts a), b) and c) in Definition 3.18 on page 27 have been established by A. M. Lyapunov in his dissertation ([Lya92]) in 1892 and have since been broadly applied and investigated for dynamical systems on metric spaces. See [BS02, Chapt. II] or [SY13, Sect. 2.3.3]. The property e) in Definition 3.18 on page 27 appears in G. D. Birkhoff's monograph "Dynamical Systems" [Bir66, Chapt. VII] as "central motion" and has been further investigated by H. Hilmy, see for example [Hil36], K. Sigmund, in [Sig77] and by H. Kreidler in [Kre20, Sect. 4] to name a few. Definition d) for semiflows on smooth compact manifolds is due to J. Milnor and can be found in [Mil85, Sect. 2].

Remark 3.19 If $(\Omega, (\varphi_t)_{t\geq 0})$ is a dynamical system with locally compact state space Ω that is metrizable and M is a compact subset of Ω, then there exists a μ-null set satisfying the assumptions in Definition 3.18 d) on page 27 that does not depend on $U \in \mathcal{U}(M)$ since the compact subset M has a countable neighborhood basis.

Remark 3.20 For the concepts defined in Definition 3.18 on page 27 the following implications hold.

$$\begin{array}{ccccccc} a) & \Longrightarrow & b) & \Longrightarrow & c) & \Longrightarrow & d) \\ & & & & & \searrow & \\ & & & & & & e) \end{array}$$

The opposite implications do not hold true in general as can be seen in the next example.

Example 3.21 a) Consider $K := \mathbb{R} \cup \{\infty\}$ the one-point compactification of $\mathbb{R}$ and the semiflow $(\varphi_t)_{t\geq 0}$ defined by

$$\varphi_t(x) := \begin{cases} x+t & x \in \mathbb{R} \\ \infty & x = \infty \end{cases}.$$

Then $M := \{\infty\}$ is attractive but not uniformly attractive.

b) Consider $\Omega := [0, \infty)$ and define $\varphi_t(x) := e^{-t}x$. Then $M := \{0\}$ is compact attractive but not uniformly attractive.

c) As in Example 3.14, 2., on page 25 we consider $\Omega := [0, \infty)$ and the semiflow $(\varphi_t)_{t\geq 0}$ on Ω, given by

$$\varphi_t(x) := \begin{cases} \frac{e^{-t}x}{e^{-t}x+1-x} & x \in [0,1) \\ e^{-t}(x-1)+1 & x \geq 1 \,. \end{cases}$$

The set $M := \{0\} \cup \{1\}$ is attractive but not compact attractive as can be seen by the same arguments used in Example 3.14, 2., on page 25.

d) Take $K := [0, \infty]$ and the semiflow $(\varphi_t)_{t\geq 0}$ on K, with

$$\varphi_t(x) := \begin{cases} e^{-t}x & x \in [0, \infty) \\ \infty & x = \infty \end{cases} .$$

Consider the standard Gaussian measure γ on $[0, \infty]$ which is a regular Borel measure on K that is quasi-invariant with respect to $(\varphi_t)_{t\geq 0}$ since it is equivalent to the Lebesgue measure λ. In particular, $\gamma(\{\infty\}) = 0$. Then $M := \{0\}$ is a likely limit set for γ since $\gamma([0, \infty)) = 1$ and $\varphi_t(x) \to 0$ for all $x \in [0, \infty)$ but it is neither attractive nor a center of attraction since $\varphi_t(\infty) = \infty$ for all $t \geq 0$.

e) In [Hil36, p. 287], H. Hilmy gave a concrete example for a center of attraction that is not attractive. We give a simplified version of this example. Take the following differential equation

$$\begin{cases} \dot{r} = -r\log(r)\left((1-r)^2 + \sin^2(\theta)\right) \\ \dot{\theta} = (1-r)^2 + \sin^2(\theta) \end{cases}$$

given in polar coordinates on $K := \{z = r \cdot e^{i\theta} \in \mathbb{C} \mid 1 \leq r \leq 2,\ 0 \leq \theta\}$. The solutions of above differential equation exist for all times and all initial values in K and form a semiflow $(\varphi_t)_{t\geq 0}$ thereon.

Since $\dot{r}(t) \leq 0$ for all $t \geq 0$ the radius r is monotonically decreasing and since $\dot{\theta}(t) \geq 0$ for all $t \geq 0$ the angle θ is monotonically increasing and unbounded, these facts clearly imply that the orbit of an initial state with radius $r > 1$ forms a spiral towards the unit circle. On the unit circle the radius is constant and the rate of change of θ is given by the differential equation

$$\dot{\theta} = \sin^2(\theta) \,.$$

This implies that the unit circle is $(\varphi_t)_{t\geq 0}$-invariant and attractive. Furthermore, $z_1 = 1$ and $z_2 = e^{i\pi}$ are fixed points, because $\sin^2(0) = \sin^2(\pi) = 0$. Thus, points on the unit circle converge to either z_1 or z_2. Remark that $M \coloneqq \{z_1\} \cup \{z_2\}$ is not attractive. Take $z \in K$ with $r > 1$ and a neighborhood U of M. Since θ is monotonically increasing and unbounded the set $\{t \in [0, \infty) \mid \varphi_t(z) \notin U\}$ is unbounded and hence M is not attractive. We claim that the set M is a center of attraction for $(X, (\varphi_t)_{t\geq 0})$ and is even minimal with this property. We prove these facts in Example 4.25 on page 53.

Concluding this section we give the following definition.

Definition 3.22 A compact invariant subset $\emptyset \neq M \subseteq X$ is called

- *(uniform/compact)* attractor if it is (uniformly/compact) attractive and minimal with this property,
- *Milnor attractor* for a quasi-invariant Borel measure μ on X if it is a likely limit set and minimal with this property.
- *Minimal center of attraction* if it is a center of attraction and minimal with this property.

We remark the following.

Remark 3.23 By Definition 3.18 on page 27 it becomes immediately clear that for the existence of an attractor it is necessary that ω-limit sets are non-empty. However, this is not a sufficient condition as the following example shows.

Example 3.24 On $\Omega \coloneqq \mathbb{C}$ using polar coordinates we define a rotation as follows

$$\varphi_t(r \cdot e^{i\theta}) \coloneqq r \cdot e^{i(\theta + 2\pi t)}\,,\ t \geq 0\,.$$

Then for $z = r \cdot e^{i\theta}$, $\omega(z) = \{y = \tilde{r} \cdot e^{i\tilde{\theta}} \in \mathbb{C} \mid \tilde{r} = r\}$ and for a compact subset $L \subseteq \mathbb{C}$ we obtain $\omega(L) = \bigcup\limits_{z \in L} \omega(z)$. Hence the ω-limit sets are non-empty and compact, but the only fixed point, $z = 0$, is not attractive in the sense of Definition 3.18 on page 27.

3.2.3 Lyapunov stable sets

One of the most prominent conditions in the context of attractors and invariant sets is *stability in the sense of Lyapunov*, as introduced in [Lya92] and later intensely studied, cf. [ABS64, Sect. 2] or [SY13, Sect. 2.3.3]. By this stability we mean that orbits that start sufficiently close to the attractor will remain close to the attractor.

Definition 3.25 A closed invariant set $\emptyset \neq M \subset X$ is called *stable in the sense of Lyapunov* (or Lyapunov stable) if for all $U \in \mathcal{U}(M)$ there exists $V \in \mathcal{U}(M)$, $V \subseteq U$ such that

$$\varphi_t(V) \subseteq U \text{ for all } t \geq 0\,.$$

Proposition 3.26 Let $\emptyset \neq M \subseteq X$ be closed and invariant. Then the following are equivalent.

a) The set M is stable in the sense of Lyapunov.

b) Every $U \in \mathcal{U}(M)$ contains an invariant $V \in \mathcal{U}(M)$. If U is closed, V can be chosen closed as well.

Proof. For the implication a)⇒b) take $U \in \mathcal{U}(M)$ closed and $V \in \mathcal{U}(M)$, $V \subseteq U$ such that $x \in V$ implies $\varphi_t(x) \in U$ for all $t \geq 0$. Consider $W := \overline{\bigcup_{t\geq 0} \varphi_t(V)}$. Then $V \subset W \subset U$. Therefore, W is still a closed neighborhood of M which is invariant. The implication b)⇒a) is trivial. □

Stability in the sense of Lyapunov does not imply any of the other attractivity properties defined in Definition 3.18 on page 27.

Example 3.27 Define a semiflow on $[0,1]$ by choosing $\varphi_t := \mathrm{id}_{[0,1]}$ for all $t \geq 0$. Every closed subset $M \subseteq [0,1]$ is invariant and stable in the sense of Lyapunov, but none of the properties in Definition 3.18 on page 27 apply.

The concept of *Lyapunov functions* is an often used tool in the context of attractors and Lyapunov stability. We give the following definition.

Definition 3.28 Let $\emptyset \neq M \subseteq X$ be closed and invariant. We call a family $(g_i)_{i\in I}$ of positive continuous functions $g_i \colon X \to \mathbb{R}$ *Lyapunov family (for M)* if $M = \cap_{i\in I} g_i^{-1}(\{0\})$ and for all $i \in I$ and $x \in X \setminus [g_i = 0]$ we have

$$g_i(\varphi_t(x)) < g_i(x) \text{ for all } t > 0\,.$$

If the state space is metric one can define the following.

Definition 3.29 Let $\emptyset \neq M \subseteq X$ be closed and invariant. We call a positive continuous function $g\colon X \to \mathbb{R}$ *Lyapunov function (for M)* if $M = g^{-1}(\{0\})$ and for all $x \in X \setminus M$ we have

$$g(\varphi_t(x)) < g(x) \text{ for all } t > 0\,.$$

The connection between Lyapunov functions and families and attractors using the Koopman approach shall be established in Sections 4.3 and 5.4.

Part II

Attractors and Koopman semigroups for topological dynamical systems on...

4 ... compact spaces

In this chapter[1] we consider dynamical systems (cf. Definition 3.1 on page 19) $(K, (\varphi_t)_{t\geq 0})$ consisting of a compact topological Hausdorff space K and a continuous semiflow $(\varphi_t)_{t\geq 0}$ on K. On the Banach space $(\mathrm{C}(K), \|\cdot\|_\infty)$ of all real-valued continuous functions on K endowed with the supremum norm $\|\cdot\|_\infty$, we associate a semigroup $(T(t))_{t\geq 0}$ of linear operators to the semiflow $(\varphi_t)_{t\geq 0}$ by defining

$$T(t)f := f \circ \varphi_t \text{ for } f \in \mathrm{C}(K)\,,\ t \geq 0\,.$$

We recall that $(\mathrm{C}(K), \|\cdot\|_\infty)$ is a Banach algebra and a Banach lattice for the usual pointwise operations and remark that each $T(t)$, $t \geq 0$, is an algebra and lattice homomorphism with $T\mathbb{1} = \mathbb{1}$.

An *attractor* is a closed, hence compact and $(\varphi_t)_{t\geq 0}$-invariant subset $\emptyset \neq M \subseteq K$ possessing a certain asymptotic property for which it is minimal, cf. Definition 3.22 on page 30. As addressed in the introduction every such subset of K corresponds to the closed and $(T(t))_{t\geq 0}$-invariant ideal

$$I_M := \{f \in \mathrm{C}(K) \mid f|_M \equiv 0\}$$

in the Banach algebra $\mathrm{C}(K)$. Essential to this matter is that all closed ideals in $\mathrm{C}(K)$ are of the form I_M where M is a closed subset of K (cf. [EFHN15, Theorem 4.8]). Also a subset M is $(\varphi_t)_{t\geq 0}$-invariant (cf. Definition 3.3 on page 20) if and only if the corresponding ideal I_M is $(T(t))_{t\geq 0}$-invariant, [EFHN15, Lem. 4.18]. Hence we have the following correspondence.

$$\begin{array}{lcl} M \subseteq K \text{ closed subset} & \Leftrightarrow & I_M \subseteq \mathrm{C}(K) \text{ closed ideal} \\[2ex] M\ (\varphi_t)_{t\geq 0}\text{-invariant} & \Leftrightarrow & I_M\ (T(t))_{t\geq 0}\text{-invariant.} \end{array}$$

[1] The results in Sections 4.1, 4.2 and 4.4 of this chapter are based on the publication [Küh19], Viktoria Kühner, *What can Koopmanism do for attractors in dynamical systems?*, The Journal of Analysis, (2019).

Given a function $f\colon K \to \mathbb{R}$ and $a \in \mathbb{R}$ we use the notation

$$[f < a] := f^{-1}((-\infty, a))\,,\ [f \leq a] := f^{-1}((-\infty, a])\,,\ [f = a] := f^{-1}(\{a\})$$

and, analoguously, $[f > a]$ and $[f \geq a]$. The sets $[|f| > 0]$, $f \in \mathrm{C}(K)$, form a basis for the topology on K since K is completely regular, [EFHN15, Appendix A.2] and [EFHN15, Proof of Lem. 4.12]. This is equivalent to the fact that the zero sets $[f = 0]$ for $f \in \mathrm{C}(K)$ form a basis of the closed subsets of K or that the topology on K coincides with the initial topology induced by $\mathrm{C}(K)$. Combining these facts, given a closed subset $M \subseteq K$ and U an open neighborhood of M, there exists $f \in \mathrm{C}(K)$ with $M \subseteq [f = 0]$ and $\varepsilon > 0$ such that $[|f| < \varepsilon] \subseteq U$, i.e. the sets of the form $U_{\varepsilon,f} := [|f| < \varepsilon]$, $f \in \mathrm{C}(K)$, $f(M) = \{0\}$ and $\varepsilon > 0$ form a basis for the system of neighborhoods of M.

In particular, for every closed ideal $I \subseteq \mathrm{C}(K)$ there exists a closed subset $M \subseteq K$ such that

$$I = I_M = \{f \in \mathrm{C}(K) \mid f|_M \equiv 0\} \text{ and } M = \bigcap_{f \in I} [f = 0]\,,$$

see [EFHN15, Thm. 4.8]. Furthermore, for $M \subseteq K$ closed, I_M is isomorphic to $\mathrm{C}_0(K \setminus M)$ by $f \mapsto f|_{K \setminus M}$, where $\mathrm{C}_0(K \setminus M)$ is the space of all real-valued continuous functions on $K \setminus M$ that vanish at infinity, cf. [Ped12, Sect. 1.7.6]. By the Riesz' representation theorem (cf. [EFHN15, Thm. 5.7 & Rem. 5.8]) we identify the dual spaces $\mathrm{C}(K)'$ and $I_M' \cong \mathrm{C}_0(K \setminus M)'$ of $\mathrm{C}(K)$ and I_M with the bounded regular Borel measures on K and $K \setminus M$, respectively.

As mentioned in the introduction, the idea to study dynamical systems by investigating the associated operator semigroup is due to Bernard Koopman and John v. Neumann, see [vNeu32b] and [Koo31]. This motivates the following terminology, cf. [EFHN15, Chapt. 4,p. 45].

Definition 4.1 Let K be a compact Hausdorff space and $T \in \mathcal{L}(\mathrm{C}(K))$. We call T *Koopman operator* if there exists a continuous mapping $\varphi\colon K \to K$ with

$$Tf = f \circ \varphi \quad \text{for all } f \in \mathrm{C}(K)\,.$$

Clearly, every continuous mapping $\varphi\colon K \to K$ *induces* a Koopman operator on $\mathrm{C}(K)$.

Koopman operators can be characterized as follows, see e.g., [Sch74, III, Prop. 9.1] or [DN79, Thm. 2.1].

Lemma 4.2 For an operator $T \in \mathcal{L}(\mathrm{C}(K))$ the following properties are equivalent.

a) T is an algebra homomorphism with $T\mathbb{1} = \mathbb{1}$.

b) T is a lattice homomorphisms with $T\mathbb{1} = \mathbb{1}$.

c) T is a Koopman operator.

Analoguously, we use the term *Koopman semigroup*.

Definition 4.3 Let K be a compact Hausdorff space. We call a semigroup $(T(t))_{t\geq 0}$ of linear operators on $\mathrm{C}(K)$ *Koopman semigroup* if every $T(t)$, $t \geq 0$, is a Koopman operator, i.e., there exists a semiflow $(\varphi_t)_{t\geq 0}$ on K (cf. Definition 3.1 on page 19) such that $T(t)f = f \circ \varphi_t$ for every $f \in \mathrm{C}(K)$, $t \geq 0$.

Clearly, every semiflow $(\varphi_t)_{t\geq 0}$ on K *induces* a Koopman semigroup on $\mathrm{C}(K)$.

Continuity of a semiflow $(\varphi_t)_{t\geq 0}$ (cf. Definition 3.1 on page 19) and strong continuity of the induced Koopman semigroup correspond in the following way.

Lemma 4.4 Let $(\varphi_t)_{t\geq 0}$ be a semiflow on K. Then the following are equivalent.

a) $(\varphi_t)_{t\geq 0}$ is continuous.

b) $(\varphi_t)_{t\geq 0}$ is separately continuous, i.e., $t \mapsto \varphi_t(x)$ is continuous for fixed $x \in K$ and $x \mapsto \varphi_t(x)$ is continuous for fixed $t \geq 0$.

c) The induced Koopman semigroup $(T(t))_{t\geq 0}$ is strongly continuous.

We refer to [Nag+86, B-II, Lem. 3.2] or [DN79, Lem. 2.4] for the proof.

Combining these facts one obtains a characterization of Koopman semigroups via their generators acting as derivations on their domain, see e.g., [DN79, Satz 2.4] for the proof.

Definition 4.5 An operator A on $\mathrm{C}(K)$ is called *derivation* if its domain $D(A)$ is a subalgebra of $\mathrm{C}(K)$ with $\mathbb{1} \in D(A)$ and the product rule

$$A(f \cdot g) = Af \cdot g + f \cdot Ag$$

holds for all $f, g \in D(A)$.

Theorem 4.6 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $\mathrm{C}(K)$ with generator $(A, D(A))$. Then the following assertions are equivalent.

a) $(T(t))_{t\geq 0}$ is a semigroup of algebra homomorphisms.

b) $(T(t))_{t\geq 0}$ is a semigroup of lattice homomorphisms with $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$.

c) $(T(t))_{t\geq 0}$ is a Koopman semigroup.

d) $(A, D(A))$ is a derivation.

In the following sections we first characterize absorbing and attractive subsets of a dynamical system by stability properties of the corresponding Koopman semigroup, then discuss stability in the sense of Lyapunov and Lyapunov functions and finally prove the existence of different types of attractors and characterize them.

From now on, $(K, (\varphi_t)_{t\geq 0})$ is always a dynamical system and $(T(t))_{t\geq 0}$ the corresponding Koopman semigroup on $\mathrm{C}(K)$.

4.1 Absorbing sets and nilpotency

The following section is dedicated to the characterization of *absorbing sets*, i.e., subsets of the state space that eventually contain every initial state, cf. Definition 3.13 on page 24.

Proposition 4.7 Let $\emptyset \neq M \subsetneq K$ be a closed invariant set and $(S(t))_{t\geq 0}$ the restricted Koopman semigroup, i.e. $S(t) := T(t)|_{I_M}$ for $t \geq 0$. Then all the assertions in (I) and all the assertions in (II) are equivalent.

(I)

a) $(S(t))_{t\geq 0}$ is nilpotent.

b) $(S(t))_{t\geq 0}$ is uniformly stable.

c) $\omega_0 = -\infty$, (cf. Definition 2.11 on page 15).

d) M is absorbing.

e) M is pointwise absorbing and $M \subset \operatorname{int}(\varphi_n^{-1}(M))$ for some $n \in \mathbb{N}$.

(II)

a) For all Dirac measures $\delta_x \in C(K)'$ there exists $t_0 > 0$ such that

$$S(t_0)'\delta_x = 0\,.$$

b) M is pointwise absorbing.

PROOF. We begin with the proof of (I). Clearly, a) $\implies$ b). For the implication b) $\implies$ d) assume M not to be absorbing and fix $t_0 > 0$, thus there is $x_0 \in K \setminus M$ with $\varphi_{t_0}(x_0) \in K \setminus M$. Since K is completely regular, there exists $f \in I_M$ with $\|f\| = 1$ and $f(\varphi_{t_0}(x_0)) = 1$. Therefore,

$$\|S(t_0)\| \geq \|S(t_0)f\| \geq S(t_0)f(x_0) = 1\,.$$

Since t_0 was arbitrary, $\|S(t)\| = 1$ for all $t \geq 0$ which contradicts b). The implication d) $\implies$ a) can be seen as follows. Let $t_0 > 0$ be such that $\varphi_{t_0}(K) \subseteq M$, thus $S(t_0)f(x) = f(\varphi_{t_0}(x)) = 0$ for every $f \in I_M$ and $x \in K$. This implies $\|S(t_0)\| = \sup\limits_{\|f\|\leq 1} \|S(t_0)f\| = 0$. Therefore, a), b) and d) are equivalent. Clearly a) implies c) which implies d). The equivalence of d) and e) has been proven in Proposition 3.17 on page 26.

Proof of (II): These equivalence is quite clear since a) implies that for all $x \in K$ there exists $t_0 > 0$ such that

$$\varphi_{t_0}(x) \in \bigcap_{f \in I_M} [f = 0] = M\,.$$

□

4.2 Asymptotics of dynamical systems

In this section we give an operator theoretic characterization of the attractivity properties of dynamical systems as defined in Definition 3.18 on page 27.

The following lemma will be useful for the characterization of centers of attractions via Koopman semigroups in the following theorem.

Lemma 4.8 Let $(T(t))_{t\geq 0}$ be a strongly continuous Koopman semigroup on $\mathrm{C}(K)$ and I a closed $(T(t))_{t\geq 0}$-invariant ideal in $\mathrm{C}(K)$. Then the restricted semigroup $(S(t))_{t\geq 0}$, with $S(t) \coloneqq T(t)|_I$, $t \geq 0$, is almost weakly stable if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S(s)|f|(x)\,\mathrm{d}s = 0 \text{ for all } x \in K\,,\ f \in I\,.$$

Proof. Note that I is a closed ideal and therefore of the form $I = I_M$ with $M \subseteq K$ closed, this implies $I \cong \mathrm{C}_0(K \setminus M)$. Hence, we identify the topological dual I' of I with the bounded regular Borel measures on $K \setminus M$. The implication "$\Rightarrow$" follows from Proposition 2.7 on page 12 since I' contains the point evaluations δ_x, $x \in K \setminus M$. The other implication is a direct consequence of Lebesgue's theorem of dominated convergence in the following way. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty)$ with $t_n \xrightarrow{n\to\infty} \infty$, $f \in I$ and $x \in K$. Remark that by the theorem of Fubini-Tonelli for $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{t_n}\int_0^{t_n} |S(s)f(x)| \,\mathrm{d}s &= \frac{1}{t_n}\int_0^{t_n} S(s)|f|(x)\,\mathrm{d}s \\ &= \langle \frac{1}{t_n}\int_0^{t_n} S(s)|f|\,\mathrm{d}s, \delta_x\rangle \,.\end{aligned}$$

Since $|S(t)f(x)| \leq \|f\|_\infty \mathbb{1}_K(x)$ for all $t \in [0,\infty)$, $x \in K$, Lebesgue's theorem of dominated convergence applies and thus for $\mu \in I'$

$$\begin{aligned}\frac{1}{t_n}\int_0^{t_n} |\langle S(s)f,\mu\rangle| \,\mathrm{d}s &\leq \frac{1}{t_n}\int_0^{t_n} \langle S(s)|f|, |\mu|\rangle\,\mathrm{d}s \\ &= \langle \frac{1}{t_n}\int_0^{t_n} S(s)|f|\,\mathrm{d}s, |\mu|\rangle \xrightarrow{n\to\infty} 0\,.\end{aligned}$$

This implies the implication "$\Leftarrow$". □

The following theorem characterizes all attractivity properties from Definition 3.18 on page 27 by means of the corresponding Koopman semigroup.

Theorem 4.9 Let $\emptyset \neq M \subseteq K$ be a closed invariant set, μ a quasi-invariant Borel measure on K and $(S(t))_{t\geq 0}$ the restricted Koopman semigroup, i.e. $S(t) := T(t)|_{I_M}$ for $t \geq 0$. Then the assertions in (I), (II), (III) and (IV), respectively, are equivalent.

(I) a) $(S(t))_{t\geq 0}$ is strongly stable.

b) M is uniformly attractive.

(II) a) $(S(t))_{t\geq 0}$ is weakly stable.

b) M is attractive.

(III) a) $(S(t))_{t\geq 0}$ is almost weakly stable.

b) M is a center of attraction.

(IV) a) $(S(t))_{t\geq 0}$ is μ-almost everywhere pointwise stable.

b) M is a likely limit set (for μ).

PROOF. Proof of (I): First we show a) $\Longrightarrow$ b). Take $U \in \mathcal{U}(M)$. Since K is completely regular, there is $f \in I_M$ and $\varepsilon > 0$ such that $U_{\varepsilon,f} = [|f| < \varepsilon] \subseteq U$. By assertion a) there is $t_0 > 0$ such that $\|S(t)f\| < \varepsilon$ for all $t \geq t_0$. This implies

$$|S(t)f(x)| = |f(\varphi_t(x))| < \varepsilon \text{ for all } x \in K\,,\ t \geq t_0\,.$$

Therefore, $\varphi_t(K) \subseteq U_{\varepsilon,f} \subseteq U$ for all $t \geq t_0$. Also, b) $\Longrightarrow$ a) since, for every $\varepsilon > 0$ and $f \in I_M$ there is a $t_0 > 0$ such that $\varphi_t(K) \subseteq U_{\varepsilon,f}$ for all $t \geq t_0$. This implies $|S(t)f(x)| < \varepsilon$ for all $t \geq t_0$ and $x \in K$ and therefore $\|S(t)f\| = \sup\limits_{x\in K} |S(t)f(x)| < \varepsilon$ for all $t \geq t_0$.

Proof of (II): To prove a) $\Longrightarrow$ b) take $U \in \mathcal{U}(M)$ and $x \in K$. Then there exist $\varepsilon > 0$ and $f \in I_M$ such that $U_{\varepsilon,f} \subseteq U$ and since $(S(t))_{t\geq 0}$ is weakly stable there exists $t_0 > 0$ such that

$$\langle S(t)f, \delta_x\rangle = f(\varphi_t(x)) < \varepsilon \text{ for all } t \geq t_0$$

which implies $\varphi_t(x) \in U_{\varepsilon,f} \subseteq U$ for all $t \geq t_0$. For the opposite implication let $\varepsilon > 0$, $f \in I_M$ and $x \in K$. By b) there exists $t_0 > 0$ such that $\langle S(t)f, \delta_x\rangle < \varepsilon$ for all $t \geq t_0$ and thus

$$\langle S(t)f, \delta_x\rangle \to 0 \text{ as } t \to \infty$$

for all Dirac measures δ_x. By Lebesgue's theorem of dominated convergence

$$\langle S(t)f, \mu\rangle \to 0 \text{ as } t \to \infty$$

for all $\mu \in I_M'$.

Proof of (III): To prove that b) implies a), assume that M is a center of attraction. Then for $U \in \mathcal{U}(M)$ open

$$\begin{aligned}
&\frac{1}{t}\,\lambda(\{s \in [0,t] \mid \varphi_s(x) \in U^c\}) \\
=&\frac{1}{t}\int_0^t \mathbb{1}_{U^c}(\varphi_s(x))\,\mathrm{d}s \xrightarrow{t\to\infty} 0 \text{ for all } x \in \Omega\,.
\end{aligned}$$

Now take $f \in I_M$ with $\|f\|_\infty = 1$ and $1 > \varepsilon > 0$. Then

$$\begin{aligned}
&\frac{1}{t}\int_0^t |S(s)f(x)|\,\mathrm{d}s \\
\le&\frac{1}{t}\int_0^t |S(s)f(x)|\mathbb{1}_{[|f|<\varepsilon]}(\varphi_s(x))\,\mathrm{d}s + \frac{1}{t}\int_0^t \|f\|_\infty \mathbb{1}_{[|f|\ge\varepsilon]}(\varphi_s(x))\,\mathrm{d}s \\
\le&\quad \varepsilon + \frac{1}{t}\int_0^t \|f\|_\infty \mathbb{1}_{[|f|\ge\varepsilon]}(\varphi_s(x))\,\mathrm{d}s < 2\varepsilon
\end{aligned}$$

for t sufficiently large since $[|f| \ge \varepsilon]$ is the complement of the open neighborhood $[|f| < \varepsilon]$ of M. Thus b) implies a) by Lemma 4.8 on page 39.

For the other implication take $x \in K$, $f \in I_M$, $f \ge 0$ and $\varepsilon > 0$. By assumption there exists a subset $R \subseteq [0,\infty)$ with density 1 and $t_0 > 0$ such that

$$\langle S(t)f, \delta_x\rangle < \varepsilon \quad \text{for all } t \ge t_0\,,\ t \in R\,.$$

Since $R \cap [t_0,\infty)$ still has density 1, we obtain

$$\frac{1}{t}\,\lambda\left(\{s \in [0,t] \mid \varphi_s(x) \in U_{\varepsilon,f}\}\right) \to 1\,.$$

This implies the assertion since the neighborhoods of the form $U_{\varepsilon,f}$, $\varepsilon > 0$, $f \in I_M$, form a neighborhood basis of M.

Proof of (IV): To prove a) implies b) take a neighborhood $U \in \mathcal{U}(M)$. Then there exist $f \in I_M$ and $\varepsilon > 0$ with $U_{\varepsilon,f} \subseteq U$. By assumption there is a quasi invariant Borel measure μ and a μ-null set N_f depending on f such that for every $x \in N_f^c$ there is $t_0 > 0$ such that

$$S(t)f(x) < \varepsilon$$

for all $t \ge t_0$. Clearly, this implies $\varphi_t(x) \in U_{\varepsilon,f} \subseteq U$ for all $t \ge t_0$. The other implication follows similarly. $\square$

4.3 Stability in the sense of Lyapunov and Lyapunov functions

In this section we address Lyapunov stability and Lyapunov functions. As pointed out in the introduction, not assuming the state space X to be metric, we generalize the notion of Lyapunov functions (cf. Definition 3.29 on page 32) to a family $(g_i)_{i\in I}$ of functions vanishing on the attractor M with $M = \cap_{i\in I}[g_i = 0]$, that are strictly decreasing along the orbits outside their respective zero sets, cf. Definition 3.28 on page 31. We prove that the existence of such a Lyapunov family is equivalent to strong stability of the restricted Koopman semigroup $(S(t))_{t\geq 0}$, thus extending the equivalent characterizations of strong stability and uniform attractivity in Theorem 4.9 (I) on page 40.

We remark that invariant subsets of dynamical systems that are attractive and stable in the sense of Lyapunov or equivalently uniformly attractive are often referred to as "asymptotically stable", cf. [BS02, Chapt. V, Def. 1.5] or [SY13, Sect. 2.3.3, p. 32].

First we examine Lyapunov stability further, cf. Definition 3.25 on page 31.

Proposition 4.10 Let $\emptyset \neq M \subseteq K$ be closed and invariant. M is Lyapunov stable if and only if

$$M = \bigcap_{\substack{V\in\mathcal{U}(M)\\ V \text{ inv.}}} V = \bigcap_{\substack{W\in\mathcal{U}(M)\\ W \text{ closed \& inv.}}} W\,.$$

Proof. The implication " $\Longrightarrow$ " is clear. To prove the converse, let U be an open neighborhood of M and assume there is no invariant neighborhood V of M with $V \subseteq U$. Then for all invariant neighborhoods V there exists $x_V \in V$ with $x_V \in U^c$. This defines a net $(x_V)_{V\in\mathcal{U}(M),\text{ inv.}}$ which has a convergent subnet since U^c is compact. We denote this convergent subnet by $(x_{V_i})_{i\in I}$ and its limit by $\overline{x}$. Now fix $W \in \mathcal{U}(M)$ closed and invariant. By cofinality of the index set I there exists $i_0 \in I$ such that $x_{v_i} \in W$ for all $i \geq i_0$. This implies $\overline{x} \in W$. Since W was arbitrary, it follows that $\overline{x} \in \bigcap_{\substack{W\in\mathcal{U}(M)\\ W \text{ closed \& inv.}}} W = M$ which contradicts $\overline{x} \in U^c$. This implies b) in Proposition 3.26 on page 31. □

In the next proposition we characterize uniform attractive invariant sets further, using stability in the sense of Lyapunov. By doing so we obtain additional equivalent properties for (I) in Theorem 4.9 on page 40.

Theorem 4.11 Let $\emptyset \neq M \subseteq K$ be closed and invariant and $(S(t))_{t\geq 0}$ the Koopman semigroup restricted to I_M. Then the following are equivalent.

a) $(S(t))_{t\geq 0}$ is strongly stable.

b) There exists a family $(g_i)_{i\in I}$ of positive functions in I_M such that $M = \cap_{i\in I}[g_i = 0]$ and for all $i \in I$ and $x \in K \setminus [g_i = 0]$ we have

$$S(t)g_i(x) < g_i(x) \text{ for all } t > 0\,.$$

c) M is attractive and Lyapunov stable.

d) M is uniformly attractive.

Proof. In Theorem 4.9 (I) on page 40 we have already proved the equivalence of a) and d).

Next we prove the equivalence of c) and d). Let M be uniformly attractive and assume that it is not stable in the sense of Lyapunov. Take $U \in \mathcal{U}(M)$ open. Then for every $V \in \mathcal{U}(M)$, $V \subset U$, there is $x_V \in V$ and $t_V > 0$ such that $\varphi_{t_V}(x_V) \in U^c$.

Because U^c is compact, the net $(\varphi_{t_V}(x_V))_{V\subset U}$ has a convergent subnet denoted by $\left(\varphi_{t_{V_i}}(x_{V_i})\right)_{i\in I}$ with limit $y \in U^c$. Since there exists $t_0 > 0$ such that $\varphi_t(U^c) \subseteq U$ for all $t \geq t_0$ by assumption, the net $(t_{V_i})_{i\in I}$ is bounded by t_0 and therefore has a convergent subnet, which we again denote by $(t_{V_i})_{i\in I}$, with limit $0 < t^* \leq t_0$. Furthermore, the net $(x_{V_i})_{i\in I}$ has a convergent subnet, again denoted by $(x_{V_i})_{i\in I}$, with limit $x \in M$. By continuity and invariance of M it follows that

$$\varphi_{t_{V_i}}(x_{V_i}) \to \varphi_{t^*}(x) \in M\,.$$

However, $\varphi_{t^*}(x) = y \in U^c$ which is a contradiction.

On the other hand, let M be (pointwise) attractive and stable in the sense of Lyapunov. Take $U \in \mathcal{U}(M)$ open and invariant. Then for every $x \in K$ there exists $t_0 = t_0(x, U)$ such that

$$\varphi_t(x) \in U \text{ for all } t \geq t_0\,.$$

Since φ_{t_0} is continuous and U is invariant, there exists an open neighborhood U_x of x such that

$$\varphi_t(U_x) \subseteq U \text{ for all } t \geq t_0\,.$$

Now since K is compact, there exist $x_1, \ldots, x_n \in K$ for some $n \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=1}^{n} U_{x_i}$. For $t \geq \max_{i=1,\ldots,n} t_0(x_i, U)$ we obtain

$$\varphi_t(K) \subseteq \varphi_t\left(\bigcup_{i=1}^{n} U_{x_i}\right) = \bigcup_{i=1}^{n} \varphi_t(U_{x_i}) \subseteq U\,.$$

This concludes the proof of the equivalence of c) and d).

Now we show that d) implies b). Take $f \in I_M$, $f \geq 0$, and consider the function

$$h_f(x) := \sup_{t\geq 0} f(\varphi_t(x))\,,\ x \in K\,.$$

First observe that

$$M \subseteq [h_f = 0] \subseteq [f = 0]$$

since $f \in I_M$ and M is invariant. Also,

$$h_f(\varphi_s(x)) \leq h_f(x) \text{ for all } x \in K\,,\ s \geq 0\,,$$

thus h_f is monotonically decreasing along the orbits and $h_f(x) = 0$ for all $x \in M$.

Now we show the continuity of h_f on $K \setminus M$. Take $x \in K \setminus M$ and $\varepsilon > 0$. Since M is Lyapunov stable by the fact that c) and d) are equivalent, we can assume the neighborhood $U_{\varepsilon,f}$ to be invariant. Furthermore, there exists a neighborhood W of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in W$. Since M is uniformly attractive by assumption, there exists $t_0 > 0$ such that $\varphi_t(W) \subseteq U_{\varepsilon,f}$ for all $t \geq t_0$. This implies

$$\begin{aligned}
&|h_f(x) - h_f(y)| \\
\leq &\left|\sup_{0\leq t\leq t_0} S(t)f(x) - \sup_{0\leq t\leq t_0} S(t)f(y)\right| + \left|\sup_{t\geq t_0} S(t)f(x) - \sup_{t\geq t_0} S(t)f(y)\right| \\
\leq &\left|\sup_{0\leq t\leq t_0} (S(t)f(x) - S(t)f(y))\right| + 2\varepsilon \\
\leq &3\varepsilon
\end{aligned}$$

for all

$$y \in \left\{z \in K \mid \left|\sup_{0\leq t\leq t_0} (S(t)f(x) - S(t)f(z))\right| < \varepsilon\right\} \subseteq W\,.$$

We used that $|\sup\{f, g\} - \sup\{f_1, g_1\}| \leq |f - f_1| + |g - g_1|$ for $f, g, f_1, g_1 \in \mathrm{C}(K)$, cf. [Sch74, Prop. 1.4 (6)].

Therefore, h_f is continuous. Now for $\lambda > 0$ define

$$g_f := \int_0^\infty e^{-\lambda s} S(s) h_f \, \mathrm{d}s \, .$$

The function g_f is an element of I_M. Also, $[g_f = 0] = [h_f = 0]$ and $S(t)g_f(x) < g_f(x)$ for all $t > 0$, $x \in K \setminus [g_f = 0]$ which can be seen as follows. Take $x \in K \setminus [g_f = 0]$ and $t > 0$. Then there exists $s_0 > 0$ such that

$$S(t+s)h_f(x) < S(s)h_f(x) \quad \text{for all} \quad s \geq s_0 \, .$$

If this is not the case, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to \infty$ as $n \to \infty$ such that

$$S(t+s_n)h_f(x) = S(s_n)h_f(x) > 0$$

for all $n \in \mathbb{N}$ which contradicts the fact that M is attractive. This implies

$$S(t)g_f(x) = \int_0^\infty e^{-s} S(t+s)h_f(x) \, \mathrm{d}s < \int_0^\infty e^{-s} S(s)h_f(x) \, \mathrm{d}s = g_f(x) \, .$$

Then the family $(g_f)_{f\in I_M}$ satisfies the assumptions in b).

Next, we prove that b) implies c). Let $(g_i)_{i\in I}$ be a family of functions satisfying the assumptions in b), fix $i \in I$ and $x \in K \setminus [g_i = 0]$. The net $(S(t)g_i(x))_{t\geq 0}$ has a convergent subnet $(S(t_j)g_i(x))_{j\in J}$ since its bounded. Denote the limit of $(S(t_j)g_i(x))_{j\in J}$ by $c \geq 0$. Assume $c > 0$. Now, consider a convergent subnet of $(\varphi_{t_j}(x))_{j\in J}$ with limit $y \in K$ and denote it again by $(\varphi_{t_j}(x))_{j\in J}$. Since by assumption $c > 0$, $y \in K \setminus [g_i = 0]$. Therefore, g_i is strongly decreasing on the orbit of y.

This implies the following for fixed $s > 0$

$$S(t_j)g_i(x) > \cdots > g_i(y) > S(s)g_i(y) \, .$$

Additionally, $S(t_j + s)g_i(x) \searrow S(s)g_i(y)$ by continuity of g_i and φ_s. Since $g_i(y) > S(s)g_i(y)$, there exists $j_0 \in J$ such that $g_i(y) > S(t_j + s)g_i(x)$ for all $j \geq j_0$. Thus for $t_k > t_j + s$ we obtain the following chain of inequalities

$$S(t_j)g_i(x) > \cdots > S(t_j)g_i(x) > \cdots > g_i(y) > S(t_j + s)g_i(x) > S(t_k)g_i(x)$$

which is a contradiction, thus $y \in [g_i = 0]$ which then implies $c = 0$. Therefore, $S(t)g_i(x) \to 0$ as $t \to \infty$ for all $x \in K$. Which implies $\varphi_t(x) \to [g_i = 0]$ for all $x \in K$ and since i was arbitrary $\varphi_t(x) \to \cap_{i\in I}[g_i = 0] = M$ for all $x \in K$. Also, the neighborhoods U_{ε,g_i} where $\varepsilon > 0$, $i \in I$ form a neighborhood basis of M and are invariant which implies M to be Lyapunov stable.

□

Example 4.12 Consider $K := \prod_{i\in\mathbb{R}} X_i$ where $X_i := [0,1]$ for all $i \in \mathbb{R}$ and $\varphi_t((x_i)_{i\in\mathbb{R}}) := (e^{-t}x_i)_{i\in\mathbb{R}}$, $t \geq 0$. Then $(\varphi_t)_{t\geq 0}$ is a continuous semiflow on K. The set $\{(0)_{i\in\mathbb{R}}\}$ is closed and invariant. For $i \in \mathbb{R}$ consider $f_i((x_i)_{i\in\mathbb{R}} := x_i$. Then the family $(f_i)_{i\in\mathbb{R}}$ is a Lyapunov family for $(0)_{i\in\mathbb{R}}$ which is therefore a uniformly attractive fixed point.

If the underlying state space K is a metric space, we obtain the following well-known fact, (cf. [BS02, Chapt. V, Thm. 2.2]).

Remark 4.13 If in the situation of Theorem 4.11 on page 44 the state space K is a metric space, then the assertions a)-d) in Theorem 4.11 on page 44 are equivalent to

b*) there exists $g \in I_M$, $g \geq 0$ with $M = [g = 0]$ and

$$S(t)g(x) < g(x) \quad \text{for all } t > 0\,,\ x \in K \setminus M\,.$$

Proof. First we prove d) of Theorem 4.11 on page 44 implies b*). If K is a metric space, then $\mathrm{C}(K)$ is separable. This implies I_M is separable since subspaces of metric separable spaces are again separable, cf. [Die11, Chapt. III, Sect. 10, 3.10.9]. Thus, there is a countable family of functions $(f_n)_{n\in\mathbb{N}}$, $f_n \geq 0$, $\|f_n\|_\infty \leq 1$ for $n \in \mathbb{N}$ in I_M with $\bigcap_{n\in N}[f_n = 0] = M$. As in the construction in the proof of Theorem 4.11 on page 44 d) $\Longrightarrow$ b) we consider $h_n := \sup_{t\geq 0} S(t)f_n$ and $g_n := \int_0^\infty e^{-\lambda t}S(t)h_n$ for some $\lambda > 0$ fixed, $n \in \mathbb{N}$. Now set

$$g := \sum_{n\in\mathbb{N}} \frac{1}{2^n} g_n\,.$$

The function g is continuous since $\|g_n\|_\infty \leq \|f_n\|_\infty \leq 1$. Furthermore, $g \geq 0$, $[g = 0] = \bigcap_{n\in\mathbb{N}} [g_n = 0] = M$ and g is strictly decreasing along the orbits for $y \in K \setminus M$, i.e., $S(t)g(y) < g(y)$ for all $t > 0$, $y \in K \setminus [g = 0]$.

On the other hand, take a function satisfying the assumptions in b*). Then by the arguments of the proof of Theorem 4.11 on page 44 b) $\Longrightarrow$ c) it follows directly that $\varphi_t(x) \to M$ for all $x \in K$. It remains to prove that M is Lyapunov stable. Take $U \in \mathcal{U}(M)$ and $c > 0$. Consider $V := U \cap [g \leq c]$. Then V is an invariant neighborhood of M. This concludes the proof. $\square$

4.4 Existence and characterization of global attractors

In this section we show that for a dynamical system $(K, (\varphi_t)_{t\geq 0})$ there always exist attractors in the sense of Definition 3.18, a),b),c) and d), on page 27. The subspaces I_{ss}, I_{ws}, I_{aeps} and I_{aws} (cf. Definition 2.2 on page 10, Definition 2.6 on page 12 and Proposition 2.9 on page 13) are all closed ideals of $\mathrm{C}(K)$ and are maximal with this property. We thus obtain corresponding closed invariant sets $\emptyset \neq M \subseteq K$ that are uniformly attractive, attractive, a likely limit set or a center of attraction, respectively, and are minimal with this property by construction. In this subsection we will discuss what the corresponding minimal attractor M looks like.

Proposition 4.14 The closed subspaces I_{ss}, I_{ws}, I_{aeps} and $I_{\mathrm{aws}} \subseteq \mathrm{C}(K)$ are lattice or equivalently algebra ideals in $\mathrm{C}(K)$.

Proof. We only compute this for I_{aws}, because I_{ws} and I_{aeps} follow analoguosly and I_{ss} is clearly a lattice ideal.
By Proposition 2.9 on page 13, I_{aws} is a closed subspace of $\mathrm{C}(K)$. It remains to show that it is an algebra or equivalently a lattice ideal. Take $f \in I_{\mathrm{aws}}$ we first show that $|f| \in I_{\mathrm{aws}}$. Take $x \in K$ and recall that for every $t \geq 0$, $|\langle T(t)f, \delta_x\rangle| = |T(t)f(x)| = T(t)|f|(x)$. We recall that

$$\begin{aligned} 0 \xleftarrow{T\to\infty} & \frac{1}{T}\int_0^T |\langle T(t)f, \delta_x\rangle| \,\mathrm{d}t \\ =& \frac{1}{T}\int_0^T \langle T(t)|f|, \delta_x\rangle \,\mathrm{d}t \\ =& \langle \frac{1}{T}\int_0^T T(t)|f| \,\mathrm{d}t, \delta_x\rangle\,, \end{aligned}$$

therefore the assertion follows by Lemma 4.8 on page 39. Additionally, if for $g \in \mathrm{C}(K)$, $|g| \leq f$ for some $f \in I_{\mathrm{aws}}$ it follows that $g \in I_{\mathrm{aws}}$. $\square$

Remark 4.15 Note that the ideals I_{ss}, I_{ws}, I_{aeps} and I_{aws} do not contain $\mathbb{1}$. This implies that each of these ideals corresponds to a unique closed invariant subset $\emptyset \neq M \subseteq K$. Applying the characterization Theorem 4.9 on page 40 we conclude that there always exist attractors in the sense of Definition 3.22 on page 30, but they might be equal to K.

4.4.1 Attractivity, ω-limit sets and Milnor attractors

We obtain the following characterization of uniform attractivity.

Proposition 4.16 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system and $\emptyset \neq M \subseteq K$ closed and invariant. Then the following are equivalent.

a) The set M is uniformly attractive.

b) $\bigcap\limits_{t\geq 0} \varphi_t(K) \subseteq M$.

Proof. If a) is true then for every $U \in \mathcal{U}(M)$, there is a $t_0 > 0$ such that

$$\bigcap_{t\geq 0} \varphi_t(K) \subseteq \bigcap_{t\geq t_0} \varphi_t(K) \subseteq U\,.$$

This implies

$$\bigcap_{t\geq 0} \varphi_t(K) \subseteq \bigcap_{U\in\mathcal{U}(M)} U = M\,.$$

The opposite implication is true since $\bigcap_{t\geq 0} \varphi_t(K)$ is itself uniformly attractive because $\varphi_r(K) \subseteq \varphi_s(K)$ for $r \geq s \geq 0$ and for $V \in \mathcal{U}(\bigcap_{t\geq 0} \varphi_t(K))$ open there exists $t_0 > 0$ such that

$$\bigcap_{t\geq 0} \varphi_t(K) \subseteq \varphi_{t_0}(K) \subseteq V\,.$$

This can be seen as follows. Assume this is not true, then there exists a subnet $(\varphi_{t_i}(x_i))_{i\in I}$ with $t_i \to \infty$, $x_i \in K$, such that $\varphi_{t_i}(x_i) \in V^c$. Since V^c is compact, there is a convergent subnet of this net with limit $y \in V^c$. This is a contradiction because $y \in \omega(K) = \bigcap\limits_{t\geq 0} \varphi_t(K)$ by Proposition 3.8, 2., on page 21. $\square$

As an immediate result we obtain the following.

Proposition 4.17 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system. Then there exists a unique minimal uniformly attractive subset of K given by

$$\bigcap_{t\geq 0} \varphi_t(K).$$

PROOF. The set $\bigcap_{t\geq 0} \varphi_t(K)$ is closed as an intersection of compact sets, non-empty by the finite intersection property of K, $(\varphi_t)_{t\geq 0}$-invariant and is uniformly attractive by Proposition 4.16 b) on the previous page and is minimal with this property by construction. □

Proposition 4.18 Combining Proposition 4.16 on page 49 and Proposition 4.17 we obtain the following characterization for the ideal of strong stability

$$I_{\text{ss}} = I_{\bigcap_{t\geq 0} \varphi_t(K)}.$$

Using ω-limit sets we obtain the following characterization of attractivity. The characterization of attractors via ω-limit sets is due to N.P. Bhatia and G.P. Szegö and can be found in [BS02, Chapt. V, Sect. 1]. It is important to note that ω-limit sets are non-empty by the finite intersection property of K.

Proposition 4.19 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system and $\emptyset \neq M \subseteq K$ closed and invariant. Then the following are equivalent.

a) The set M is attractive.

b) $\omega(x) \subseteq M$ for all $x \in K$.

PROOF. To prove a) $\Longrightarrow$ b) take $x \in K$. By a)

$$\omega(x) \subseteq \bigcap_{U\in\mathcal{U}(M)} U = M.$$

Consider $U \in \mathcal{U}(M)$ open and assume that a) does not hold, i.e., there exists $x \in K \setminus M$ with $\varphi_t(x) \in U^c$ for infinitely many $t > 0$. Since U^c is closed and hence compact there exists a convergent subnet $(t_i)_{i\in I}$, $t_i \to \infty$, such that $\varphi_{t_i}(x) \to z \in U^c$ which is a contradiction to b) by Proposition 3.8 on page 21.

□

Proposition 4.20 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system. Then there exists a unique minimal attractive subset of K given by

$$\overline{\bigcup_{x\in K} \omega(x)}\,.$$

Proof. In Proposition 4.19 b) on page 50 we have seen that $\omega(x)$ is contained in every closed, $(\varphi_t)_{t\geq 0}$-invariant and attractive subset $\emptyset \neq M \subseteq K$ therefore also

$$\bigcup_{x\in K} \omega(x) \subseteq M\,.$$

Also the closure $\overline{\bigcup_{x\in K} \omega(x)}$ is contained in every such M and $(\varphi_t)_{t\geq 0}$-invariant, attractive itself and minimal with this property by construction. □

Proposition 4.21 Combining Proposition 4.19 on page 50 and Proposition 4.20 we obtain the following characterization for the ideal of weak stability

$$I_{\mathrm{ws}} = I_{\overline{\bigcup_{x\in K} \omega(x)}}\,.$$

Proposition 4.22 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system with K metric, μ a quasi invariant regular Borel measure on K and $\emptyset \neq M \subseteq K$ closed and invariant. Then the following are equivalent.

a) The set M is a likely limit set.

b) $\omega(x) \subseteq M$ for μ-almost every $x \in K$.

Proof. We prove this similarly to Proposition 4.19 on page 50. Let M be a likely limit set for μ. Then there exists a μ-null set N such that for all $U \in \mathcal{U}(M)$ and $x \in N^c$ there exists $t_0 > 0$ such that $\varphi_t(x) \in U$ for all $t \geq t_0$. Remark that N can be chosen independently from U since K is metric. Hence, $\omega(x) \subseteq \cap_{U\in\mathcal{U}(M)} U = M$ for all $x \in N^c$.

Now assume there exists a μ-null set N such that $\omega(x) \subseteq M$ for all $x \in N^c$. Take $U \in \mathcal{U}(M)$ open. If a) does not hold there exists $x \in N^c$ such that $\varphi_t(x) \in U^c$ for infinitely many $t > 0$. Since U^c is compact there exists a convergent subsequence of $(\varphi_t(x))_{t\geq 0}$ with limit in U^c which is a contradiction to $\omega(x) \subseteq M$ by Proposition 3.8 on page 21. □

Proposition 4.23 Let $(K, (\varphi_t)_{t\geq 0})$ be a dynamical system with K metric and μ a quasi invariant regular Borel measure on K. By Proposition 4.22 there exists a μ-null set N such that

$$I_{\mathrm{aeps}} = I_{\overline{\bigcup_{x\in N^c} \omega(x)}}\,.$$

We thus obtain a characterization of the ideal of almost everywhere pointwise stability.

4.4.2 Minimal centers of attraction and ergodic measures

An interesting fact is that the minimal center of attraction is characterized by the ergodic measures on K. We recall that a regular Borel measure μ on K is called *invariant* if $\mu(\varphi_t^{-1}(A)) = \mu(A)$ for all Borel measurable sets A and $t \geq 0$. An invariant probability measure is called *ergodic* if the corresponding measure-preserving system $(K, (\varphi_t)_{t\geq 0}, \mu)$ is ergodic, i.e., if $A \subseteq K$ is Borel measurable and invariant then $\mu(A) \in \{0, 1\}$. In the following we write $\mathrm{M}^1(K)$ for the set of all regular Borel probability measures on K.

Proposition 4.24 The minimal center of attraction is given by the union of supports of ergodic measures, i.e.,

$$I_{\mathrm{aws}} = I_{M_{\mathrm{erg}}}$$

with $M_{\mathrm{erg}} := \overline{\bigcup_{\substack{\mu\in \mathrm{M}^1(K)\\ \mu \text{ ergodic}}} \operatorname{supp}(\mu)}$.

Proof. By [EFHN15, p.193, (10.1)] it suffices to show that $I_{\mathrm{aws}} = I_{M_{\mathrm{inv}}}$ where $M_{\mathrm{inv}} := \overline{\bigcup_{\substack{\mu\in \mathrm{M}^1(K)\\ \mu \text{ inv.}}} \operatorname{supp}(\mu)}$. First we show "$\subseteq$". Let $\mu \in \mathrm{M}^1(K)$ be invariant. For $f \in I_{\mathrm{aws}}$

$$\begin{aligned} \langle |f|, \mu\rangle &= \frac{1}{t}\int_0^t \langle |f|, \mu\rangle \,\mathrm{d}s \\ &\overset{\mu \text{ inv.}}{=} \frac{1}{t}\int_0^t \langle T(s)|f|, \mu\rangle \,\mathrm{d}s \to 0 \end{aligned}$$

by Lemma 4.8 on page 39 and Proposition 4.14 on page 48. Therefore, $f\big|_{\operatorname{supp}(\mu)} \equiv 0$ for all invariant $\mu \in \mathrm{M}^1(K)$. For the implication "$\supseteq$" let $x \in K$ and δ_x the corresponding Dirac measure and $f \in I_{M_{\mathrm{inv}}}$. We observe that by the Fubini-Tonelli Theorem

$$\begin{aligned} \frac{1}{t}\int_0^t |\langle T(s)f, \delta_x\rangle| \,\mathrm{d}s &= \frac{1}{t}\int_0^t \langle |f|, T(s)'\delta_x\rangle \,\mathrm{d}s \\ &= \langle |f|, \frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s\rangle \,. \end{aligned}$$

Since the dual unit ball $B' \subset \mathrm{C}(K)'$ is compact in the weak-*-topology and every $\frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s$ is bounded, every subnet of $\left(\frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s\right)_{t\geq 0}$ has a convergent subnet in B', i.e.,

$$\langle |f|, \frac{1}{t_i}\int_0^{t_i} T(s)'\delta_x \,\mathrm{d}s\rangle \to \langle |f|, \mu\rangle = 0$$

with μ an invariant probability measure. The fact that μ is invariant is true because for every $r \geq 0$ the subnet

$$\left(\frac{1}{t_i}\int_0^{t_i} T(r+s)'\delta_x \,\mathrm{d}s\right)_{i\in I} = \left(\frac{1}{t_i}\int_r^{t_i+r} T(s)'\delta_x \,\mathrm{d}s\right)_{i\in I}$$

has the same limit as $\left(\frac{1}{t_i}\int_0^{t_i} T(s)'\delta_x \,\mathrm{d}s\right)_{i\in I}$. Also, $\mu \neq 0$ since $\langle \mathbb{1}, \mu\rangle = 1$. Therefore, $\frac{1}{t}\int_0^t |T(s)f(x)| \,\mathrm{d}s \to 0$ as $t \to \infty$ for all $x \in K$. Thus, $f \in I_{\mathrm{aws}}$ by Lemma 4.8 on page 39. $\square$

Example 4.25 Continuation of Example 3.21 e) on page 29. As we have seen, the set $M = \{z_1\} \cup \{z_2\}$ is a closed and $(\varphi_t)_{t\geq 0}$-invariant subset of the unit circle $\mathbb{T}$ which is the minimal attractive subset. Furthermore, the point evaluations δ_{z_1} and δ_{z_2} are invariant measures, hence

$$M \subseteq M_{\mathrm{inv}} \subseteq \mathbb{T}.$$

However, by [EFHN15, Lem. 10.7] for every invariant measure μ, $\varphi_t(\mathrm{supp}(\mu)) = \mathrm{supp}(\mu)$ for all $t \geq 0$. Since θ is strongly increasing on $\mathbb{T} \setminus M$, $\varphi_t(L) \neq L$ for all sets $M \subsetneq L \subseteq \mathbb{T}$ and $t > 0$. Hence, there cannot be an invariant measure μ with $M \subsetneq \mathrm{supp}(\mu) \subseteq \mathbb{T}$ and therefore M is the minimal center of attraction by Proposition 4.24 on page 52.

Summarizing we have the following chain of inclusions

$$I_{\mathrm{ss}} \subseteq I_{\mathrm{ws}} \subseteq I_{\mathrm{aws}} \subsetneq \mathrm{C}(K)$$

and

$$\bigcap_{t\geq 0} \varphi_t(K) \supseteq \overline{\bigcup_{x\in K} \omega(x)} \supseteq \overline{\bigcup_{\substack{\mu\in \mathrm{M}^1(K)\\ \mu \text{ ergodic}}} \mathrm{supp}(\mu)} \supsetneq \emptyset.$$

5 ... locally compact spaces

In this chapter a dynamical system is a pair $(\Omega, (\varphi_t)_{t\geq 0})$ consisting of a non-compact, locally compact Hausdorff space Ω and a continuous semiflow $(\varphi_t)_{t\geq 0}$ on Ω, cf. Definition 3.1 on page 19.

We again associate a family of linear operators $(T(t))_{t\geq 0}$ to the semiflow $(\varphi_t)_{t\geq 0}$ by

$$T(t)f := f \circ \varphi_t \quad \text{for } t \geq 0\,,\ f \in \mathcal{A}\,,$$

where $\mathcal{A}$ is a suitable unital subalgebra of $\mathrm{C_b}(\Omega)$, the space of all real-valued bounded continuous functions on Ω endowed with the $\|\cdot\|_\infty$-norm. It is not immediately clear what the appropriate choice is such that $\mathcal{A}$ is $(T(t))_{t\geq 0}$-invariant and $(T(t))_{t\geq 0}$ becomes strongly continuous on $\mathcal{A}$.

Operator semigroups induced by continuous semiflows on a locally compact space Ω are treated e.g. in [Are82, Ex. 2.3] where the associated operators are defined on $\mathrm{C_0}(\Omega)$. However, in general these operators do not leave $\mathrm{C_0}(\Omega)$ invariant. As an example take $\Omega := (0, \infty]$ and the shift $\varphi_t(x) := x + t$ for $x \in (0, \infty)$ and $\varphi_t(\infty) := \infty$ thereon.

Our idea is to consider $\mathrm{C_0}(\Omega) \oplus \langle \mathbb{1} \rangle$, where $\langle \mathbb{1} \rangle := \{c\mathbb{1} \mid c \in \mathbb{R}\}$. Recall that $\mathrm{C_0}(\Omega) \oplus \langle \mathbb{1} \rangle$ is canonically isomorphic to $\mathrm{C}(\alpha\Omega)$ where $\alpha\Omega := \Omega \cup \{\infty\}$ denotes the *one-point compactification* of Ω. Since $\alpha\Omega$ is compact, we can use the results from the previous chapter.

There are two questions that need to be clarified. The first one is: given a semiflow $(\varphi_t)_{t\geq 0}$ on a locally compact, non-compact space Ω, when does the associated semigroup leave $\mathrm{C_0}(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant and is strongly continuous thereon? As it turns out, this is the case if and only if the underlying semiflow can be extended to a continuous semiflow on $\alpha\Omega$, cf. Proposition 6.10 on page 87.

The second question is which Koopman semigroups (cf. Definition 4.1 on page 36 and Definition 4.3 on page 37) on $\mathrm{C}(\alpha\Omega)$ induced by a semiflow $(\tilde{\varphi}_t)_{t\geq 0}$ on $\alpha\Omega$ correspond to a continuous semiflow on Ω, i.e., $\tilde{\varphi}_t(\Omega) \subseteq \Omega$, $t \geq 0$. We give a characterization for such Koopman semigroups in Lemma 5.13 on page 61 and Theorem 5.15 on page 62, where we also characterize such Koopman semigroups by their generator.

In Section 5.2 and 5.3 we treat absorbing and attractive subsets for semiflows $(\varphi_t)_{t\geq 0}$ on Ω. We will again "translate" attractivity properties of dynamical systems into stability properties of the corresponding Koopman semigroup. In Section 5.4, we discuss stability in the sense of Lyapunov and Lyapunov functions further.

Then, in Section 5.5, we give conditions for the existence of global attractors and characterize them in a similar manner to the previous chapter. Again, the closed ideals in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ will play an essential role in studying absorbing and attractive subsets of Ω. Since we want an attractor to be a compact and invariant subset of Ω, we characterize those norm-closed ideals in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ that are of the form I_M where $M \subset \Omega$ is compact, cf. Theorem 5.22 on page 70.

We continue to use the notation $[f < a] := f^{-1}((-\infty, a))$ for $a \in \mathbb{R}$, $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ and analoguosly for $[f \leq a]$, $[f > a]$, $[f \geq a]$ and $[f = a]$. For a closed subset $M \subseteq \Omega$ the sets of the form $U_{\varepsilon,f} := [|f| < \varepsilon]$ for $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, $f(M) = \{0\}$ and $\varepsilon > 0$ form a basis for the neighborhoods of M.

5.1 Strong continuity of Koopman semigroups and characterization via the generator

From now on let Ω be a non-compact, locally compact Hausdorff space. First we discuss how a single continuous mapping $\varphi \colon \Omega \to \Omega$ can be continuously extended to the one point compactification $\alpha\Omega$ of Ω by considering the associated operator T on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$.

Proposition 5.1 Let $\varphi \colon \Omega \to \Omega$ be a continuous mapping and T the associated bounded operator defined by $Tf := f \circ \varphi$ for $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then the following are equivalent.

a) The mapping φ can be continuously extended to $\alpha\Omega$.

b) The operator T leaves $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant.

Proof. The space $\mathrm{C}(\alpha\Omega)$ is canonically isomorphic to $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ by the isomorphism $R\colon \mathrm{C}(\alpha\Omega) \to \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ where

$$R\tilde{f} := \left(\tilde{f} - \tilde{f}(\infty)\mathbb{1}\right) + \tilde{f}(\infty)\mathbb{1} \quad \text{for } \tilde{f} \in \mathrm{C}(\alpha\Omega)\,.$$

First we prove that a) implies b). Assume that φ can be continuously extended to $\alpha\Omega$ with extension $\tilde{\varphi}$. Then $\tilde{\varphi}$ induces a Koopman operator $\tilde{T}\colon \mathrm{C}(\tilde{\Omega}) \to \mathrm{C}(\tilde{\Omega})$. Now consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle & \xrightarrow{T} & \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \\
\Big\downarrow{\scriptstyle R^{-1}} & & \Big\uparrow{\scriptstyle R} \\
\mathrm{C}(\alpha\Omega) & \xrightarrow{\quad\tilde{T}\quad} & \mathrm{C}(\alpha\Omega)\,.
\end{array}$$

In fact, for $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, $x \in \Omega$,

$$(R \circ \tilde{T} \circ R^{-1} f)(x) = (R(R^{-1} f(\tilde{\varphi}(x))) = f(\varphi(x)) = Tf(x)\,,$$

because $\tilde{\varphi}|_\Omega = \varphi$. Hence, $T = R \circ \tilde{T} \circ R^{-1} \in \mathcal{L}(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle)$.

On the other hand, if T leaves $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant, the operator $\tilde{T} := R^{-1} \circ T \circ R$ is an algebra homomorphism on $\mathrm{C}(\alpha\Omega)$ and is therefore induced by a continuous mapping $\tilde{\varphi}$ on $\alpha\Omega$ by Lemma 4.2 on page 37. For $x \in \Omega$ and $\tilde{f} \in \mathrm{C}(\alpha\Omega)$ this implies

$$\tilde{T}\tilde{f}(x) = \tilde{f}(\tilde{\varphi}(x))\,.$$

By definition of $\tilde{T}$ we have

$$\tilde{T}\tilde{f}(x) = (R^{-1} \circ T \circ R\tilde{f})(x) = \tilde{f}(\varphi(x))\,.$$

This implies $\tilde{\varphi}|_\Omega = \varphi$ since $\mathrm{C}(\alpha\Omega)$ separates the points of $\alpha\Omega$ and hence Ω. Thus $\tilde{\varphi}$ is a continuous extension of φ. □

This proposition justifies the following definition.

Definition 5.2 Let $\varphi\colon \Omega \to \Omega$ be a continuous mapping that can be continuously extended to $\alpha\Omega$. We call the associated operator $T \in \mathcal{L}(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle)$ defined by $Tf := f \circ \varphi$, $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, *Koopman operator induced by* φ.

Remark 5.3 Given a continuous self-mapping φ on Ω we call the operator induced by $Tf := f \circ \varphi$, $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, the "associated operator". If the mapping can be continuously extended to $\alpha\Omega$, hence the associated operator leaves $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant, then we write "Koopman operator" for T.

We further characterize when a continuous mapping on Ω can be continuously extended to $\alpha\Omega$

Proposition 5.4 Let $\varphi\colon \Omega \to \Omega$ be a continuous mapping and $a \in \Omega$. Then the following are equivalent.

a) The mapping φ can be continuously extended to $\tilde{\varphi}\colon \alpha\Omega \to \alpha\Omega$ with $\tilde{\varphi}(\infty) = a$.

b) The pre-image $\varphi^{-1}(A)$ is compact for every closed subset $A \subseteq \Omega$ that does not contain a.

Proof. We begin with the implication a) to b). Take $A \subseteq \Omega$ closed with $a \notin A$, then A^c is an open neighborhood of a. By continuity there exists an open neighborhood U of ∞ with

$$\tilde{\varphi}(U) \subseteq A^c\,.$$

Then $U = K^c$ for some $K \subseteq \Omega$ compact since the open neighborhoods of $\{\infty\}$ are exactly the complements of compact subsets of Ω. Therefore,

$$\tilde{\varphi}(K^c) \subseteq A^c$$

which is equivalent to

$$\tilde{\varphi}^{-1}(A) \subseteq K\,.$$

For the other implication let V be an open neighborhood of a. We have to show that there exists an open neighborhood U of ∞ with $\tilde{\varphi}(U) \subseteq V$. Now consider $\varphi^{-1}(V^c)$ which is compact by assumption and hence $\varphi^{-1}(V)$ is an open neighborhood of ∞ with $\tilde{\varphi}(\varphi^{-1}(V)) \subseteq V$. □

Combining Proposition 5.1 on page 56 and Proposition 5.4 we obtain the following.

Proposition 5.5 Let $\varphi\colon \Omega \to \Omega$ be a continuous mapping and T the associated bounded operator defined by $Tf := f \circ \varphi$ for $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then the following are equivalent.

a) The mapping φ can be continuously extended to $\alpha\Omega$ with $\tilde{\varphi}(\infty) = \infty$.

b) The pre-image $\varphi^{-1}(K)$ is compact for every $K \subset \Omega$ compact.

c) T leaves $\mathrm{C}_0(\Omega)$ invariant.

Proof. The equivalence of a) and c) follows from the fact that $\mathrm{C}_0(\Omega) \cong J_{\{\infty\}}$ and Proposition 5.1 on page 56. By the same arguments as in Proposition 5.4 on page 58 one obtains the equivalence of a) and b). $\square$

The equivalence of a) and b) in Proposition 5.5 above is well-known and can be found for example in [Bro88, Sect. 3.6] and the exercises therein.

Now we turn to continuous semiflows $(\varphi_t)_{t\geq 0}$ on Ω and the associated operator semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. We will see that $(T(t))_{t\geq 0}$ is strongly continuous if and only if the underlying semiflow $(\varphi_t)_{t\geq 0}$ on Ω can be continuously extended to the one-point compactification $\alpha\Omega$. In many cases, $(\varphi_t)_{t\geq 0}$ can be continuously extended by $\tilde{\varphi}_t(\infty) = \infty$, $t \geq 0$, e.g., if φ_t is invertible for all $t \geq 0$.

We define continuous extensions of continuous semiflows on Ω to $\alpha\Omega$ as follows.

Definition 5.6 Let $(\varphi_t)_{t\geq 0}$ be a continuous semiflow on Ω. We say $(\varphi_t)_{t\geq 0}$ can be *continuously extended* to $\alpha\Omega$ if there exists a continuous semiflow $(\tilde{\varphi}_t)_{t\geq 0}$ on $\alpha\Omega$ such that
$$\tilde{\varphi}_t|_\Omega = \varphi_t \quad \text{for all } t \geq 0\,.$$

The properties of a semiflow and the considerations in Proposition 5.4 on page 58 and Proposition 5.5 on page 58 imply the following.

Remark 5.7 Let $(\tilde{\varphi}_t)_{t\geq 0}$ be the continuous extension of $(\varphi_t)_{t\geq 0}$ on $\alpha\Omega$. There are two possibilities for $a_t := \tilde{\varphi}_t(\infty)$. Either $a_t = \infty$ for all $t \geq 0$ or $a_t \in \Omega$ for all $t > 0$ and $a_0 = \infty$.

We now give an example for a semiflow that cannot be continuously extended to $\alpha\Omega$ or equivalently where the associated operators do not leave $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant, cf. [Sie17, Bsp. 3.21].

Example 5.8 Consider $\Omega := [-1, 0) \cup (0, \infty]$ and the semiflow defined by
$$\varphi_t(x) := \begin{cases} e^{-t}x & x \in [-1, 0)\,, \\ x + t & x \in (0, \infty)\,, \\ \infty & x = \infty\,. \end{cases}$$

Then for $t > 0$
$$\lim_{x \searrow 0} \varphi_t(x) = t \quad \text{and} \quad \lim_{x \nearrow 0} \varphi_t(x) = 0.$$

This shows that φ_t cannot be continuously extended to $\alpha\Omega = [-1, \infty]$ for $t > 0$, hence the associated operators do not leave $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ invariant.

Given a continuous semiflow $(\varphi_t)_{t\geq 0}$ we obtain the following characterization of strong continuity for its associated operator family $(T(t))_{t\geq 0}$ on $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1} \rangle$.

Proposition 5.9 Let $(\Omega, (\varphi_t)_{t\geq 0})$ be a dynamical system and $(T(t))_{t\geq 0}$ the family of associated operators defined on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then the following are equivalent.

a) $(T(t))_{t\geq 0}$ is strongly continuous on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$.

b) The semiflow $(\varphi_t)_{t\geq 0}$ can be continuously extended to a continuous semiflow on $\alpha\Omega$.

Proof. By Proposition 5.1 on page 56 we know that every operator $T(t)$ leaves $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1} \rangle$ invariant if and only if the mapping φ_t can be continuously extended. The fact that the family $(\tilde{\varphi}_t)_{t\geq 0}$ of continuous extensions is a continuous semiflow follows from the fact that $\mathrm{C}(\alpha\Omega) \cong \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ and Lemma 4.4 on page 37. ☐

We give the following definition in analogy to Definition 7.4 on page 96.

Definition 5.10 Let $(\varphi_t)_{t\geq 0}$ be a continuous semiflow on Ω that can be continuously extended to $\alpha\Omega$. We call the associated strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1} \rangle$ with $T(t)f := f\circ\varphi_t$, $t \geq 0$, $f \in \mathrm{C}_0(\Omega)\oplus\langle \mathbb{1} \rangle$, *Koopman semigroup induced by* $(\varphi_t)_{t\geq 0}$.

Combining the previous considerations we obtain the following. See also [Sie17, Lem. 3.16].

Proposition 5.11 Let $(\Omega, (\varphi_t)_{t\geq 0})$ be a dynamical system and $(T(t))_{t\geq 0}$ the associated operators defined on $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1} \rangle$. Then the following are equivalent.

a) $(T(t))_{t\geq 0}$ leaves $\mathrm{C}_0(\Omega)$ invariant.

b) The semiflow $(\varphi_t)_{t\geq 0}$ can be continuously extended to $\alpha\Omega$ by $\tilde{\varphi}_t(\infty) := \infty$ for all $t \geq 0$.

Proof. By Proposition 5.5 on page 58 we know that the operators $T(t)$ leave $\mathrm{C}_0(\Omega)$ invariant if and only if every mapping φ_t can be extended by $\tilde{\varphi}_t(\infty) = \infty$. Then the family $(\tilde{\varphi}_t)_{t\geq 0}$ is a semiflow and it is continuous because it is separately continuous and hence continuous by Lemma 4.4 on page 37. ☐

Now we turn to the second question mentioned in the introduction of this chapter. Given a Koopman semigroup on $\mathrm{C}(\alpha\Omega)$ induced by a continuous semiflow $(\tilde{\varphi}_t)_{t\geq 0}$ on $\alpha\Omega$, under what additional assumption can this semiflow be restricted to a continuous semiflow on Ω, i.e., $\tilde{\varphi}_t(\Omega) \subseteq \Omega$ for all $t \geq 0$. Recall that by Lemma 4.2 on page 37 from the previous chapter we know that Koopman operators on $\mathrm{C}(\alpha\Omega)$ are exactly the algebra and lattice homomorphisms T on $\mathrm{C}(\alpha\Omega)$ with $T\mathbb{1} = \mathbb{1}$ and by isomorphy on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$.

To this end, we consider the compact-open topology τ_c, i.e., the topology of uniform convergence on compact subsets of Ω. It is induced by the seminorms defined by

$$p_K(f) := \|f|_K\|_\infty \quad \text{for } f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle,\ K \subseteq \Omega \text{ compact.}$$

The compact-open toplogy τ_c is coarser than the norm topology. Every norm bounded set is τ_c-bounded. In addition, $(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle, \tau_c)$ is sequentially complete for norm-bounded sequences [Jar81, Chapt. 3.H, Thm. 9].

We first observe the following fact.

Lemma 5.12 If $T \in \mathcal{L}(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle)$ is a Koopman operator induced by a continuous mapping φ on Ω, then T is τ_c-continuous.

Proof. Take a net $(f_i)_{i\in I}$ in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ with τ_c-limit f. For $K \subset \Omega$ compact

$$\| (Tf_i - Tf)\,\big|_K \|_\infty = \| (f_i - f)\,\big|_{\varphi(K)} \|_\infty \to 0 \quad \text{for } i \in I\,.$$

This implies the assertion. □

We can identify Koopman operators induced by a continuous mapping φ on Ω with those algebra and lattice homomorphisms on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ that are τ_c-continuous.

Lemma 5.13 Let $T \in \mathcal{L}(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle)$. Then the following are equivalent.

a) T is an algebra homomorphism, $T\mathbb{1} = \mathbb{1}$ and T is τ_c-continuous.

b) T is a lattice homomorphism, $T\mathbb{1} = \mathbb{1}$ and T is τ_c-continuous.

c) T is a Koopman operator induced by a continuous mapping φ on Ω.

Proof. We prove that b) implies c). Take $x \in \Omega$ and $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Since

$$\langle |f|, T'\delta_x \rangle = \langle |Tf|, \delta_x \rangle = |\langle f, T'\delta_x \rangle|\,,$$

the linear form $T'\delta_x$ is a lattice homomorphism. Now take a net $(f_i)_{i\in I}$ in $\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$ with $\|f_i\|_\infty \leq 1$ and $\tau_c - \lim\limits_{i\in I} f_i = \mathbb{1}$. Since T is a τ_c-continuous contraction and $T\mathbb{1} = \mathbb{1}$ it follows that

$$1 \geq \|T'\delta_x\| \geq |T'\delta_x(f_i)| \to 1 \text{ for } i \in I\,.$$

Since the normalized lattice homomorphisms are point evaluations in Ω, there is $y\in\Omega$ with $T'\delta_x = \delta_y$. We set $\varphi(x) \coloneqq y$. All in all,

$$\langle Tf, \delta_x\rangle = \langle f, T'\delta_x\rangle = \langle f, \delta_{\varphi(x)}\rangle = \langle f\circ\varphi, \delta_x\rangle$$

for all $f\in\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$. Therefore, $f\circ\varphi\in\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$ which implies that φ is continuous.

Clearly, every Koopman operator induced by a continuous mapping φ on Ω is an algebra homomorphism and lattice homomorphism with $T\mathbb{1} = \mathbb{1}$ and it is τ_c-continuous by Lemma 5.12 on page 61 which concludes the proof. □

Next, we characterize Koopman semigroups induced by a continuous semiflow on Ω by means of their generators acting as derivations on their domain.

Definition 5.14 Let A be a linear operator with domain $D(A)\subseteq\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$. Then A is called *derivation* if $D(A)$ is a subalgebra of $\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$, $\mathbb{1}\in D(A)$ and it satisfies the product rule

$$A(f\cdot g) = Af\cdot g + f\cdot Ag$$

for all $f, g\in D(A)$.

Theorem 5.15 Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $\mathrm{C}_0(\Omega)\oplus\langle\mathbb{1}\rangle$ with generator $(A, D(A))$. Then the following are equivalent.

a) $(T(t))_{t\geq 0}$ is a semigroup of lattice operators, $T(t)\mathbb{1} = \mathbb{1}$ and $T(t)$ is τ_c-continuous for all $t\geq 0$.

b) $(T(t))_{t\geq 0}$ is a semigroup of algebra homomorphisms and $T(t)$ is τ_c-continuous for all $t\geq 0$.

c) $(T(t))_{t\geq 0}$ is a Koopman semigroup induced by a continuous semiflow $(\varphi_t)_{t\geq 0}$ on Ω.

d) The generator $(A, D(A))$ is a derivation and for every net $(f_i)_{i\in I}$ with $\tau_c - \lim\limits_{i\in I} f_i = f$ and all $t \geq 0$ we have

$$\tau_c - \lim_{i\in I} A \int_0^t T(s) f_i \,\mathrm{d}s = A \int_0^t T(s) f \,\mathrm{d}s \,.$$

PROOF. The equivalences of a), b) and c) follow directly from above Lemma 5.13 on page 61 and Proposition 6.10 on page 87. Remark that a strongly continuous semigroup of algebra homomorphisms on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ always satisfies $T(t)\mathbb{1} = \mathbb{1}$. To prove that c) implies d) remark that, by Theorem 4.6 on page 38 and $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \cong \mathrm{C}(\alpha\Omega)$, the generator A is a derivation.

Let $(f_i)_{i\in I}$ be a net in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ with τ_c-limit $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then for $t > 0$, $K \subset \Omega$ compact we obtain

$$\left\| \left(A \int_0^t T(s) f_i \,\mathrm{d}s - A \int_0^t T(s) f \,\mathrm{d}s \right) \big|_K \right\|_\infty = \left\| (T(t)f_i - f_i - (T(t)f - f)) \big|_K \right\|_\infty$$

which converges to 0 since $T(t)$ is a Koopman operator and hence τ_c-continuous.

On the other hand, if A is a derivation, then $(T(t))_{t\geq 0}$ is a semigroup of algebra homomorphisms by Theorem 4.6 on page 38 and $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \cong \mathrm{C}(\alpha\Omega)$. Furthermore, for a τ_c-convergent net $(f_i)_{i\in I}$ with τ_c-limit f, $K \subseteq \Omega$ compact and $t \geq 0$

$$\begin{aligned}
\|(T(t)f_i \; &- T(t)f) \big|_K \|_\infty \\
&\leq \left\| (T(t)f_i - f_i - (T(t)f - f)) \big|_K \right\|_\infty + \left\| (f_i - f) \big|_K \right\|_\infty \\
&= \left\| \left(A \int_0^t T(s) f_i \,\mathrm{d}s - A \int_0^t T(s) f \,\mathrm{d}s \right)_K \right\|_\infty + \left\| (f_i - f) \big|_K \right\| \\
&\xrightarrow{i\in I} 0 \,.
\end{aligned}$$

This concludes the proof. $\square$

However, not every derivation on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ that is a generator induces a Koopman semigroup on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. The following example sheds light on this problem, hence the additional condition in above Theorem 5.15 d) is needed.

Example 5.16 Consider for example the space $\mathrm{C}_0([0,1)) \oplus \langle \mathbb{1} \rangle$ and the operator semigroup $(T(t))_{t\geq 0}$ defined by

$$T(t)f(x) := \begin{cases} f(x+t) & \text{for } x+t < 1 \\ 0 & \text{for } x+t \geq 1 \end{cases} \quad , \ f \in \mathrm{C}_0([0,1)) \,, \ t \geq 0$$

and $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. The semigroup is strongly continuous, leaves $\mathrm{C}_0([0,1)) \oplus \langle \mathbb{1} \rangle$ invariant and its generator is a derivation. However, $(T(t))_{t\geq 0}$ is not a Koopman semigroup induced by a semiflow on $[0,1)$ since the operators $T(t)$ are not τ_c-continuous. Take a net $(f_i)_{i\in I}$ in $\mathrm{C}_0([0,1))$ with $\tau_c - \lim\limits_{i\in I} f_i = \mathbb{1}$. Then, for $t > 1$, $\tau_c - \lim\limits_{i\in I} T(t) f_i = 0$ which shows that $T(t)$ is not τ_c-continuous.

5.2 Absorbing sets and nilpotency

From now on a dynamical system $(\Omega, (\varphi_t)_{t\geq 0})$ is a topological dynamical system with non-compact locally compact state space Ω such that the induced associated operator semigroup denoted by $(T(t))_{t\geq 0}$ is a Koopman semigroup on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$, see Definition 5.10 on page 60. Given a closed and invariant subset $\emptyset \neq M \subseteq \Omega$ we define

$$S(t) \coloneqq T(t)\big|_{I_M}, \ t \geq 0\,.$$

Recall that the topological dual of $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ is canonically isomorphic to $\mathrm{M}(\alpha\Omega)$ the space of all bounded regular Borel measures on $\alpha\Omega$. Since we do not want $\{\infty\}$ to be an attractor we consider the weak topology

$$\sigma_\Omega \coloneqq \sigma(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle, \mathrm{M}(\Omega))$$

induced by $\mathrm{M}(\Omega)$ the space of bounded regular Borel measures on Ω. This topology is given by the bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \times \mathrm{M}(\Omega) &\to \mathbb{C} \\ (f, \mu) &\mapsto \int_\Omega f \, \mathrm{d}\mu \end{aligned}$$

which yields a dual system $(\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle, \mathrm{M}(\Omega), \langle \cdot, \cdot \rangle)$, cf. [Sch71, Chapt. IV, §1, p. 123].

The following lemma will be useful in the context of almost σ_Ω-stability, similar to the corresponding result in Lemma 4.8 on page 39 in Chapter 4.

Lemma 5.17 Let $(T(t))_{t\geq 0}$ be a strongly continuous Koopman semigroup on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ and I a closed $(T(t))_{t\geq 0}$-invariant ideal in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then the restricted semigroup $(S(t))_{t\geq 0}$, with $S(t) \coloneqq T(t)|_I$, $t \geq 0$, is almost σ_Ω-stable, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\langle S(s)f, \mu\rangle| \, \mathrm{d}s = 0 \text{ for all } \mu \in \mathrm{M}(\Omega),\ f \in I,$$

if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S(s)|f|(x) \, \mathrm{d}s = 0 \text{ for all } x \in \Omega,\ f \in I\,.$$

We omit the proof since this fact can be shown in complete analogy to Lemma 4.8 on page 39 using Proposition 2.4 on page 11.

We recall the notion of absorbing sets introduced in Definition 3.13 on page 24 and give the following characterization via Koopman semigroups.

Proposition 5.18 Let $\emptyset \neq M \subsetneq \Omega$ be a closed invariant set and $(S(t))_{t\geq 0}$ the corresponding Koopman semigroup restricted to I_M for $t \geq 0$. Then the assertions in (I), (II) and (III), respectively, are equivalent.

(I)
- a) $(S(t))_{t\geq 0}$ is nilpotent.
- b) $(S(t))_{t\geq 0}$ is uniformly stable.
- c) $\omega_0 = -\infty$.
- d) M is absorbing.

(II)
- a) $(S(t))_{t\geq 0}$ is τ_c-nilpotent, (cf. Definition 2.3 a) on page 10).
- b) M is compact absorbing.
- c) M is pointwise absorbing and $M \subseteq \operatorname{int}(\varphi_n^{-1}(M))$ for some $n \in \mathbb{N}$.

(III)
- a) For all Dirac measures $\delta_x \in I_M'$ there exists $t_0 > 0$ such that

$$S(t_0)'\delta_x = 0\,.$$

- b) M is pointwise absorbing.

Proof. Proof of (I): This follows by the exact same arguments as in the proof of the equivalences a)-d) in Proposition 4.7 (I) on page 38. Use that Ω is completely regular and $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle \cong \mathrm{C}(\alpha\Omega)$.

Proof of (II): First we prove a) $\Longrightarrow$ b). Consider $M \subset \Omega$ closed and $(\varphi_t)_{t\geq 0}$ invariant. Assume $(S(t))_{t\geq 0}$ is τ_c-nilpotent. Take $K \subset \Omega$ compact, then there exists a t_0 such that $f(\varphi_t(K)) = 0$ for all $t \geq t_0$ and for all $f \in I_M$. This implies

$$\varphi_t(K) \subset \bigcap_{f\in I_M} [f = 0] = M\,.$$

The other implication is clear. The equivalence of b) and c) was proved in Proposition 3.17 on page 26.

Proof of (III): This follows directly by above (II) since singletons $\{x\}$, $x \in \Omega$ are compact and $S(t)f(x) = \langle f, S(t)'\delta_x\rangle$ for $f \in I_M$, $t \geq 0$. $\square$

5.3 Asymptotics of dynamical systems

This section investigates analoguous attractivity properties as in Definition 3.18 on page 27. For examples for all of these concepts, see Example 3.21 on page 28. The main focus is again the characterization of attractivity concepts by stability properties of the induced Koopman semigroup on $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1}\rangle$ which is Theorem 5.19 below. We refer to Definition 2.3 b) on page 10 for the definition of stability of an operator semigroup with respect to a locally convex topology.

Theorem 5.19 Let μ a quasi-invariant Borel measure on Ω, $\emptyset \neq M \subseteq \Omega$ closed and invariant and $(S(t))_{t\geq 0}$ the corresponding Koopman semigroup restricted to I_M. Then the assertions in (I), (II), (III), (IV) and (V), respectively, are equivalent.

(I) a) $(S(t))_{t\geq 0}$ is strongly stable.

b) M is uniformly attractive.

(II) a) $(S(t))_{t\geq 0}$ is τ_c-stable.

b) M is compact attractive.

(III) a) $(S(t))_{t\geq 0}$ is σ_Ω-stable.

b) M is attractive.

(IV) a) $(S(t))_{t\geq 0}$ is almost σ_Ω-stable.

b) M is a center of attraction.

(V) a) $(S(t))_{t\geq 0}$ is μ-almost everywhere pointwise stable.

b) M is a likely limit set (for μ).

PROOF. Proof of (I): cf. Theorem 4.9, (I) on page 40.

Proof of (II): To prove a) $\Longrightarrow$ b) take $U \in \mathcal{U}(M)$ and $K \in \mathcal{K}$. Then there exist $\varepsilon > 0$ and $f \in I_M$ such that $U_{\varepsilon,f} \subseteq U$ and since $(S(t))_{t\geq 0}$ is τ_c-stable there exists $t_0 > 0$ such that

$$\|(S(t)f)|_K\| \leq \varepsilon$$

for all $t \geq t_0$. Thus,

$$\varphi_t(K) \subseteq U_{\varepsilon,f} \subseteq U$$

for all $t \geq t_0$. The opposite implication follows similarly.

Proof of (III): This follows by (II) since singletons $\{x\}$, $x \in \Omega$ are compact and by Lebesgue's theorem of dominated convergence.

Proof of (IV): We repeat the arguments from Theorem 4.9 (III) on page 41. If M is a center of attraction, then for $U \in \mathcal{U}(M)$ open

$$\begin{aligned} &\frac{1}{t}\,\lambda(\{s \in [0,t] \mid \varphi_s(x) \in U^c\}) \\ =&\frac{1}{t}\int_0^t \mathbb{1}_{U^c}(\varphi_s(x))\,\mathrm{d}s \xrightarrow{t\to\infty} 0 \text{ for all } x \in \Omega\,. \end{aligned}$$

Hence for $f \in I_M$ with $\|f\|_\infty = 1$ and $1 > \varepsilon > 0$ we have

$$\begin{aligned} &\frac{1}{t}\int_0^t |S(s)f(x)|\,\mathrm{d}s \\ \leq&\frac{1}{t}\int_0^t |S(s)f(x)|\mathbb{1}_{[|f|<\varepsilon]}(\varphi_s(x))\,\mathrm{d}s + \frac{1}{t}\int_0^t \|f\|_\infty \mathbb{1}_{[|f|\geq\varepsilon]}(\varphi_s(x))\,\mathrm{d}s \\ \leq&\quad \varepsilon + \frac{1}{t}\int_0^t \|f\|_\infty \mathbb{1}_{[|f|\geq\varepsilon]}(\varphi_s(x))\,\mathrm{d}s < 2\varepsilon \end{aligned}$$

for t sufficiently large. Thus, b) implies a) by Lemma 5.17 on page 64.

Now take $x \in \Omega$, $f \in I_M$, $f \geq 0$ and $\varepsilon > 0$. If a) is true there exists a subset $R \subseteq [0,\infty)$ with density 1 and $t_0 > 0$ such that

$$\langle S(t)f, \delta_x\rangle < \varepsilon \quad \text{for all } t \geq t_0\,,\ t \in R\,.$$

Since $R \cap [t_0, \infty)$ still has density 1, we obtain

$$\frac{1}{t}\,\lambda\left(\{s \in [0,t] \mid \varphi_s(x) \in U_{\varepsilon,f}\}\right) \to 1\,.$$

The assertion now follows because the neighborhoods of the form $U_{\varepsilon,f}$, $\varepsilon > 0$, $f \in I_M$, form a neighborhood basis for M.

Proof of (V): This follows by the same arguments used in Theorem 4.9 (IV) on page 41. □

5.4 Stability in the sense of Lyapunov and Lyapunov functions

In this subsection we discuss *stability in the sense of Lyapunov* and *Lyapunov functions* for the locally compact setting, cf. Chapter 3, Section 3.3.

For the remainder of this chapter let $(\Omega, (\varphi_t)_{t\geq 0})$ be a dynamical system such that $\overline{\mathrm{orb}}(L)$ is compact for all $L \subseteq \Omega$ compact, which is necessary for the existence of attractors as discussed in Remark 3.23 on page 30.

Proposition 5.20 Let $\emptyset \neq M \subseteq \Omega$ be compact and invariant and $(S(t))_{t\geq 0}$ the corresponding Koopman semigroup restricted to I_M. Then the following are equivalent.

a) $(S(t))_{t\geq 0}$ is τ_c-stable.

b) There exists a family $(g_i)_{i\in I}$ of positive functions in I_M such that $M = \cap_{i\in I}[g_i = 0]$ and for all $i \in I$ and $x \in K \setminus [g_i = 0]$ we have

$$S(t)g_i(x) < g_i(x) \text{ for all } t > 0\,.$$

c) M is attractive and Lyapunov stable.

d) M is compact attractive.

Proof. The equivalence of a) and d) has already been proved in (II) of Theorem 5.19 on page 66. Next we show that d) $\Longrightarrow$ c). Let W be a compact neighborhood of M. Assume M is not Lyapunov stable, then for every $V \subseteq W$ there exists $t_V > 0$ and a $x_V \in V$ such that

$$\varphi_{t_V}(x_V) \in W^c\,.$$

Since the set $L := \varphi([0,t_v] \times \{x_v\})$ is connected, the union of the two disjoint open sets

$$L \cap \operatorname{int}(W) \text{ and } L \cap W^c .$$

is not all of L, i.e.,

$$L \neq (L \cap \operatorname{int}(W)) \cup (L \cap W^c) = L \cap (\operatorname{int}(W) \cup W^c) .$$

Therefore, there exists

$$y_V \in L \setminus (\operatorname{int}(W) \cup W^c) = L \cap \partial W .$$

This implies $y_V = \varphi_{t_V^*}(x_V)$ for some $t_V^* \in [0, t_V]$.

The boundary ∂W is compact which yields a convergent subnet $(y_{V_i})_{i \in I}$ with limit $y \in \partial W$. Since W is compact, the net $(x_{V_i})_{i \in I}$ has a convergent subnet which we will again denote by (x_{V_i}) converging to a point $x \in M$. In addition, since M is compact attractive, there is $t_0 > 0$ such that

$$\varphi_{t_0}(\partial W) \subseteq W .$$

This implies the above net $(t_{V_i}^*)_{i \in I}$ to be bounded by t_0, hence it has a convergent subnet, again denoted by $(t_{V_i}^*)$ with limit $t^* \geq 0$.

All in all, $x_{V_i} \to x \in M$, $t_{V_i}^* \to t^*$ and by continuity $\varphi_{t_{V_i}^*}(x_{V_i}) \to \varphi_{t^*}(x) \in M$ since M is invariant. This implies $y \in M$ which is a contradiction.

The other implication is clear by Theorem 4.9 (I) on page 40, c) $\Longrightarrow$ d), exchanging K by an arbitrary $L \subset \Omega$ compact.

The equivalence of b) and d) can be seen exactly as in Theorem 4.11 on page 44. $\square$

Remark 5.21 If in the situation of Proposition 5.20 on page 68 the state space Ω is a metric space, then the assertions a)-d) in Proposition 5.20 on page 68 are equivalent to

b*) there exists $g \in I_M$, $g \geq 0$ with $M = [g = 0]$ and

$$S(t)g(x) < g(x) \quad \text{for all } t > 0\,,\ x \in K \setminus M .$$

Proof. This can be proved by the same arguments used in Remark 4.13 on page 47. $\square$

5.5 Existence and characterization of global attractors

As mentioned before, given a dynamical system $(\Omega, (\varphi_t)_{t\geq 0})$ an *attractor* is a non-empty, invariant, attractive (cf. Definition 3.18 on page 27) and *compact* subset of Ω that is minimal with this property. To this end we first investigate how compact subsets of Ω can be characterized by algebraic properties of $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1}\rangle$. The next theorem characterizes the algebra or equivalently lattice ideals in $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1}\rangle$ that are of the form I_M where M is a *compact* subset of Ω.

Theorem 5.22 Let $I \subseteq \mathrm{C}_0(\Omega)\oplus\langle \mathbb{1}\rangle$ be a closed algebra or equivalently lattice ideal. Then the following are equivalent.

a) The ideal is of the form I_M with $M \subseteq \Omega$ compact.

b) There exists $f \in I$ that does not vanish at infinity.

c) The ideal I is τ_c-closed.

Proof. For a) implies b) let $I = I_M$ for $M \subseteq \Omega$ compact. This implies $M = \overline{M}^{\sim}$, where $\overline{M}^{\sim}$ denotes the closure of M in $\alpha\Omega$, in particular $\infty \notin \overline{M}^{\sim}$. By Urysohn's lemma there exists a continuous function $\tilde{f}\colon \alpha\Omega \to [0,1]$ with $\tilde{f}(\infty) = 1$ and $\tilde{f}|_M \equiv 0$. Then $f := \tilde{f}|_\Omega \in I$ and f does not vanish at infinity.

For the implication b) to a), let I be a closed algebra or equivalently lattice ideal in $\mathrm{C}_0(\Omega)\oplus\langle \mathbb{1}\rangle$. Then $I \cong J$ with $J \subseteq \mathrm{C}(\alpha\Omega)$ closed algebra or equivalently lattice ideal. Then $J = J_{\widetilde{M}}$ for a closed subset $\widetilde{M} \subseteq \alpha\Omega$. By assumption there exists $f \in I$ that does not vanish at infinity, hence its extension $\tilde{f}$ is an element of $J_{\widetilde{M}}$ and $\tilde{f}(\infty) \neq 0$ which implies that $\infty \notin \widetilde{M}$, hence $\widetilde{M}$ is already compact in Ω.

Next, we assume a) and prove c). Let $(f_i)_{i\in I}$ be a net in $I = I_M$ with τ_c-limit $f \in \mathrm{C}_0(\Omega)\oplus\langle \mathbb{1}\rangle$ and let $\varepsilon > 0$. Since M is compact

$$\|f|_M\|_\infty = \|(f_{i_0} - f)|_M\|_\infty < \varepsilon$$

for i_0 sufficiently large. This implies $f|_M \equiv 0$ since ε was arbitrary.

On the other hand, suppose c) is true and a) does not hold, i.e., $I \cong J_{\widetilde{M}} \subseteq \mathrm{C}(\alpha\Omega)$, with $\widetilde{M} \subseteq \alpha\Omega$ closed and $\infty \in \widetilde{M}$, because $\widetilde{M}$ cannot be a compact subset of Ω by assumption. Furthermore, there exists a net $(f_i)_{i\in I}$ in $\mathrm{C}_0(\alpha\Omega \setminus \widetilde{M}) \cong J_{\widetilde{M}}$ with

$$\|(f_i - \mathbb{1})_K\|_\infty \xrightarrow{i\in I} 0 \quad \text{for all } K \subseteq \alpha\Omega \setminus \widetilde{M} \text{ compact}.$$

Then $\mathbb{1} \in J_{\widetilde{M}} \cong \mathrm{C}_0(\alpha\Omega \setminus \widetilde{M})$ by a) which is a contradiction. Therefore, $\widetilde{M}$ is a compact subset of Ω and $I = I_{\widetilde{M}}$. □

Next we discuss some properties of the subsets I_{ss}, $I_{\sigma\Omega}$, $I_{\mathrm{a}\sigma\Omega}$, I_{τ_c} and I_{aeps} (cf. Definition 2.2 on page 10, Definition 2.5 on page 11 and Definition 2.6 on page 12) of $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$.

Proposition 5.23 The sets I_{ss}, $I_{\sigma\Omega}$, $I_{\mathrm{a}\sigma\Omega}$, I_{τ_c} and I_{aeps} are $\|\cdot\|_\infty$-closed algebra or equivalently lattice ideals in $\mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$.

Proof. We prove this for I_{τ_c}. Let $\varepsilon > 0$ and $(f_n)_{n\in\mathbb{N}}$ a $\|\cdot\|_\infty$-convergent sequence in I_{τ_c} with limit $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$. Then for $K \subseteq \Omega$ compact

$$\begin{aligned}\|(T(t)f)|_K\|_\infty &\le \|(T(t)(f_n - f))|_K\|_\infty + \|(T(t)f_n)|_K\|_\infty \\ &= \|(f_n - f)|_{\varphi_t(K)}\|_\infty + \|(T(t)f_n)|_K\|_\infty \\ &\le \|f_n - f\|_\infty + \|(T(t)f_n)|_K\|_\infty < \varepsilon\end{aligned}$$

for n and t sufficiently large since $f_n \xrightarrow{\|\cdot\|_\infty} f$ and $f_n \in I_{\tau_c}$s. Since $T(t)|f| = |T(t)f|$ for all $f \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ and $t \ge 0$ it follows that $|f| \in I_{\tau_c}$ for all $f \in I_{\tau_c}$ and for every $g \in \mathrm{C}_0(\Omega) \oplus \langle \mathbb{1} \rangle$ with $|g| \le f$ for some $f \in I_{\tau_c}$ it follows that $g \in I_{\tau_c}$. The proof for the other subspaces uses the same arguments as in Proposition 4.14 on page 48. □

Next we give an example where I_{τ_c} is not τ_c-closed.

Example 5.24 Consider the shift $\varphi_t(x) := x + t$ for $x \in \mathbb{R}$, $t \ge 0$. Then I_{τ_c} contains a net $(f_i)_{i\in I}$ with $f_i \in \mathrm{C}_0(\Omega)$ and $\tau_c - \lim_{i\in I} f_i = \mathbb{1}$, but clearly $\mathbb{1} \notin I_{\tau_c}$.

5.5.1 Attractivity, ω-limits sets and Milnor attractors

In this subsection we discuss the existence and characterization of attractors in the sense of Definition 3.22 on page 30 similar to Subsection 4.4.1 in Chapter 4 where the state space was compact.

The following proposition gives a characterization of uniform attractivity.

Proposition 5.25 Let $\emptyset \neq M \subseteq \Omega$ be closed and invariant. Then the following are equivalent.

a) The set M is uniformly attractive.

b) $\bigcap\limits_{t\geq 0} \overline{\varphi_t(\Omega)} \subseteq M$.

We omit the proof since this can be shown using the same arguments as in Proposition 4.16 on page 49. As an immediate result we obtain the following.

Proposition 5.26 There exists a unique minimal uniformly attractive subset of Ω given by

$$\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}\,.$$

PROOF. Note that $\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$ is non-empty, because for all $x \in \Omega$, $\omega(x) \neq \emptyset$ by assumption and $\omega(x) \subseteq \bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$. The set $\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$ is closed and $(\varphi_t)_{t\geq 0}$-invariant and is uniformly attractive by Proposition 4.16 b) on page 49 and is minimal with this property by construction. $\square$

Remark 5.27 Combining Proposition 5.25 and Proposition 5.26 one obtains

$$I_{\mathrm{ss}} = I_{\bigcap\limits_{t\geq 0} \overline{\varphi_t(\Omega)}}\,.$$

Next, we characterize compact attractivity further.

Proposition 5.28 Let $\emptyset \neq M \subseteq \Omega$ be closed and invariant. Then the following are equivalent.

a) M is compact attractive.

b) $\omega(L) \subseteq M$ for all $L \subseteq \Omega$ compact.

Proof. If M is compact attractive, then for a closed neighborhood $U \in \mathcal{U}(M)$ and a compact subset $L \subseteq \Omega$ we know there exists $t_0 \geq 0$ with $\varphi_t(L) \subseteq U$ for all $t \geq t_0$. This implies

$$\omega(L) = \bigcap_{t_0 \geq 0} \overline{\bigcup_{t \geq t_0} \varphi_t(L)} \subseteq \overline{\bigcup_{t \geq t_0} \varphi_t(L)} \subseteq U \,.$$

Since U was arbitrary, $\omega(L) \subseteq M$.

Now assume that $\omega(L) \subseteq M$ for all $L \subseteq \Omega$ compact and M is not compact attractive, i.e., there exists an open neighborhood $U \in \mathcal{U}(M)$, a compact subset $L \subseteq \Omega$ and nets $(t_i)_{i \in I}$, $t_i \to \infty$ and $(x_i)_{i \in I}$ in L with $\varphi_{t_i}(x_i) \in U^c$. The set $U^c \cup \{\infty\}$ is compact in $\alpha\Omega$ and therefore there is a convergent subnet of $(\varphi_{t_i}(x_i))_{i \in I}$ in $U^c \cup \{\infty\}$ with limit y. Since $\omega(L)$ is compact by assumption, $y \neq \infty$. Which implies $y \in \omega(L)$ by Proposition 3.10 on page 23 but $y \notin M$ which is a contradiction. $\square$

This leads us to the following conclusion.

Proposition 5.29 There exists a unique compact attractive subset of Ω that is minimal with this property given by

$$\overline{\bigcup_{\substack{L \subseteq \Omega \\ \text{compact}}} \omega(L)} \,.$$

All in all, we obtain the following.

Proposition 5.30 Combining Theorem 5.19 (II) on page 66 and above Proposition 5.29 we obtain

$$I_{\tau_c} = I_{\overline{\bigcup_{L \subseteq \Omega,\ \text{compact}} \omega(L)}} \,.$$

Similarly, we prove the existence of a unique (pointwise) attractive subset.

Proposition 5.31 Let $\emptyset \neq M \subset \Omega$ be closed and invariant. Then the following are equivalent.

a) The set M is attractive.

b) $\omega(x) \subseteq M$ for all $x \in \Omega$.

We omit the proof since this follows directly from above Proposition 5.28 on page 72.

Thus, we can conclude the following.

Proposition 5.32 There exists a unique minimal attractive subset of Ω given by

$$\overline{\bigcup_{x\in\Omega} \omega(x)}\,.$$

Again we omit the proof since it is similar to Proposition 5.29 on page 73.

And finally we obtain the following.

Proposition 5.33 Combining Theorem 5.19 (III) on page 66 and above Proposition 5.32 we obtain

$$I_{\sigma\Omega} = I_{\overline{\bigcup_{x\in\Omega}\omega(x)}}\,.$$

To conclude this subsection we turn to Milnor attractors, cf. Definition 3.22 on page 30.

Proposition 5.34 Assume that $(\Omega, (\varphi_t)_{t\geq 0})$ is a dynamical system where Ω is metric and separable, μ a quasi invariant regular Borel measure on Ω and $\emptyset \neq M \subseteq \Omega$ closed and invariant. Then the following are equivalent.

a) The set M is a likely limit set (for μ).

b) $\omega(x) \subseteq M$ for μ-almost every $x \in \Omega$.

Proof. We prove this similarly to Proposition 4.19 on page 50. Let M be a likely limit set for μ. Then there exists a μ-null set N such that for all $U \in \mathcal{U}(M)$ and $x \in N^c$ there exists $t_0 > 0$ such that $\varphi_t(x) \in U$ for all $t \geq t_0$. Remark that N can be chosen independently from U since Ω is metric and hence M admits a countable neighborhood basis. Hence, $\omega(x) \subseteq \cap_{U\in\mathcal{U}(M)} U = M$ for all $x \in N^c$.

Now assume there exists a μ-null set N such that $\omega(x) \subseteq M$ for all $x \in N^c$. Take $U \in \mathcal{U}(M)$ open. If a) does not hold there exists $x \in N^c$ such that $\varphi_t(x) \in U^c$ for infinitely many $t > 0$. Since $U^c \cup \{\infty\}$ is compact in $\alpha\Omega$ there exists a convergent subnet of $(\varphi_t(x))_{t\geq 0}$ with limit in $U^c \cup \{\infty\}$, possibly ∞, which is a contradiction to $\omega(x) \subseteq M$ by Proposition 3.8 on page 21. $\square$

This implies the following.

Proposition 5.35 Assume that $(\Omega, (\varphi_t)_{t\geq 0})$ is a dynamical system where Ω is metric and separable, μ a quasi invariant regular Borel measure on Ω and $\emptyset \neq M \subseteq \Omega$ closed and invariant. There exists a unique subset of Ω that is a likely limit set and minimal with property given by

$$\overline{\bigcup_{x\in N^c} \omega(x)}$$

for some μ-null set N.

And last we obtain the following correspondence.

Proposition 5.36 Let $(\Omega, (\varphi_t)_{t\geq 0})$ be a dynamical system with Ω metric and separable, μ a quasi invariant regular Borel measure on Ω. By Proposition 5.35 and Theorem 5.19 (V) on page 66 there exists a μ-null set N such that

$$I_{\text{aeps}} = I_{\overline{\bigcup_{x\in N^c} \omega(x)}}\,.$$

5.5.2 Minimal centers of attraction and ergodic measures

The minimal center of attraction is characterized by the invariant measures on Ω.

In the following we write $\mathrm{M}^1(\Omega)$ for the set of all regular Borel probability measures on Ω.

In contrast to the compact situation in Proposition 4.24 on page 52 of Chapter 4 the existence of an invariant probability Borel measure on Ω with compact support is not ensured.

Proposition 5.37 The minimal center of attraction is given by the union of supports of invariant measures, i.e.,

$$I_{\mathrm{a}\sigma_\Omega} = I_{M_{\text{inv}}}$$

with $M_{\text{inv}} := \overline{\bigcup_{\substack{\mu\in\mathrm{M}^1(\Omega)\\ \mu \text{ invariant}}} \operatorname{supp}(\mu)}$.

Proof. In this proof we use the same arguments as in Proposition 4.24. First we show "$\subseteq$". Take $\mu \in M_{\text{inv}}$ and $f \in I_{\mathrm{a}\sigma_\Omega}$, then $|f| \in I_{\mathrm{a}\sigma_\Omega}$ since it is a lattice ideal (cf. Proposition 5.23 on page 71) and

$$\begin{aligned} 0 \le \langle |f|, \mu\rangle &= \frac{1}{t}\int_0^t \langle |f|, \mu\rangle \,\mathrm{d}s \\ &\overset{\mu \text{ inv.}}{=} \frac{1}{t}\int_0^t \langle T(s)|f|, \mu\rangle \,\mathrm{d}s \to 0\,. \end{aligned}$$

Therefore, $f\big|_{\mathrm{supp}(\mu)} \equiv 0$ for all invariant $\mu \in M_{\mathrm{inv}}$, this implies the assertion.

For the implication "$\supseteq$" let $x \in \Omega$ and δ_x the corresponding Dirac measure and $f \in I_{M_{\mathrm{inv}}}$. We observe that

$$\begin{aligned} \frac{1}{t}\int_0^t |\langle T(s)f, \delta_x\rangle| \,\mathrm{d}s &= \frac{1}{t}\int_0^t \langle |f|, T(s)'\delta_x\rangle \,\mathrm{d}s \\ &= \langle |f|, \frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s\rangle\,. \end{aligned}$$

Recall that $T(s)'\delta_x = \delta_{\varphi_s(x)}$ is a point evaluation in Ω because $(T(t))_{t\ge 0}$ is a Koopman semigroup. This is crucial to ensure that the net $\left(\frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s\right)_{t\ge 0}$ is a net in $\mathrm{M}^1(\Omega)$.

Since $\mathrm{M}^1(\Omega)$ is compact in the weak-*-topology induced by $\mathrm{C}_0(\Omega)$, every subnet of $\left(\frac{1}{t}\int_0^t T(s)'\delta_x \,\mathrm{d}s\right)_{t\ge 0}$ has a convergent subnet $\left(\frac{1}{t_i}\int_0^{t_i} T(s)'\delta_x \,\mathrm{d}s\right)_{i\in I}$ with limit $\mu \in \mathrm{M}^1(\Omega)$. This implies

$$\langle |f|, \frac{1}{t_i}\int_0^{t_i} T(s)'\delta_x \,\mathrm{d}s\rangle \to \langle |f|, \mu\rangle = 0$$

because μ is invariant. Hence, $\frac{1}{t}\int_0^t |T(s)f(x)| \,\mathrm{d}s \to 0$ as $t \to \infty$ for all $x \in \Omega$. The assertion now follows by applying Lemma 5.17 on page 64. ☐

5.5.3 Existence of attractors

Now we combine the results from the previous subsections and give conditions for the existence of attractors in the sense of Definition 3.22 on page 30.

Theorem 5.38 All the assertions in (I), (II), (III), (IV) and (V) are equivalent respectively.

(I) a) There exists a uniform attractor.

b) The ideal I_{ss} is τ_c-closed.

c) There exists an absorbing set in Ω that is compact.

d) The set $\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$ is compact.

(II) a) There exists a compact attractor.

b) The ideal I_{τ_c} is τ_c-closed.

c) There exists a compact absorbing set in Ω that is compact.

d) The set $\overline{\bigcup_{\substack{L\subseteq\Omega \\ \text{compact}}} \omega(L)}$ is compact.

(III) a) There exists a (pointwise) attractor.

b) The ideal $I_{\sigma\Omega}$ is τ_c-closed.

c) There exists a pointwise absorbing set in Ω that is compact.

d) The set $\overline{\bigcup_{x\in\Omega} \omega(x)}$ is compact.

For the list of equivalences in (IV) we additionally assume that Ω is metric and separable and μ a quasi-invariant regular Borel measure on Ω.

(IV) a) There exists a Milnor attractor.

b) The ideal I_{aeps} is τ_c-closed.

c) There exists a compact subset of $L \subseteq \Omega$ and a μ-null set N such that for every $x \in N^c$ there exists $t_0 > 0$ with $\varphi_t(x) \in L$ for all $t \geq t_0$.

d) There exists a μ-null set N such that the set $\overline{\bigcup_{x\in N^c} \omega(x)}$ is compact.

(V) a) There exists a minimal center of attraction.

b) The ideal $I_{\text{a}\sigma\Omega}$ is τ_c-closed.

c) The set M_{inv} (cf. Proposition 5.37 on page 75) is compact.

Proof. Proof of (I): By Definition 3.22 on page 30 and Proposition 5.26 on page 72, if d) is true then there exists a uniform attractor, which is assertion a). Furthermore, a) implies b) by combining Theorem 5.22 on page 70 and Theorem 5.19 (I) on page 66. If b) is true, then by Theorem 5.22 on page 70 there exists a function $f \in I_{\text{ss}}$ that does not vanish at infinity, hence its zero set $[f = 0]$ is compact in Ω which implies that there exists a compact neighborhood W of $[f = 0]$. Furthermore, $[f = 0]$ contains $\bigcap\limits_{t\geq 0} \overline{\varphi_t(\Omega)}$ by

Remark 5.27 on page 72. Since the set $\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$ is uniformly attractive by Proposition 5.25 on page 72, there exists $t_0 > 0$ such that $\varphi_t(\Omega) \subseteq W$ for all $t \geq t_0$. This implies c). On the other hand, if there exists a absorbing set that is compact, then it must contain $\bigcap_{t\geq 0} \overline{\varphi_t(\Omega)}$ by Proposition 5.26 on page 72 which is therefore compact, hence d) is true.

Proof of (II): The implication d) $\Longrightarrow$ a) is true by Definition 3.22 on page 30 and Proposition 5.29 on page 73. Assertion a) in turn, combining Theorem 5.22 on page 70 and Theorem 5.19 (II) on page 66 implies b). The implication b) to c) is true by Theorem 5.22 on page 70 and Proposition 5.30 on page 73. Again, if there exists a function $f \in I_{\tau_c}$ with compact zero set, this zero set admits a compact neighborhood which is compact absorbing because $[f = 0]$ contains $\overline{\bigcup_{\substack{L\subseteq\Omega \\ \text{compact}}} \omega(L)}$. If c) is true, the given compact set that is compact absorbing must contain $\overline{\bigcup_{\substack{L\subseteq\Omega \\ \text{compact}}} \omega(L)}$ by Definition 3.22 on page 30 which is therefore compact, hence d).

Proof of (III): Using the same arguments as in (I) and (II) we obtain that d) implies a) by Definition 3.22 on page 30 and the characterization in Proposition 5.32 on page 74. By Theorem 5.19 (III) on page 66 and Theorem 5.22 on page 70 we obtain a) $\Longrightarrow$ b). If b) is true we obtain a function in I_{σ_Ω} with compact zero set by Theorem 5.22 on page 70 which has a compact neighborhood containing $\overline{\bigcup_{x\in\Omega} \omega(x)}$ by Proposition 5.33 on page 74. This compact neighborhood satisfies the assumptions in c) by Definition 3.18 c) on page 27. The implication c) $\Longrightarrow$ d) follows from Proposition 5.32 on page 74.

Proof of (IV): Using Definition 3.22 on page 30 and Proposition 5.35 on page 75 we obtain d) $\Longrightarrow$ a). Next, a) implies b) by Theorem 5.19 (IV) on page 66 and Theorem 5.22 on page 70. Again, the ideal I_{aeps} contains a function with compact zero set that admits a compact neighborhood satisfying the assumptions in c) by Theorem 5.22 on page 70 and Proposition 5.36 on page 75. The implication c) $\Longrightarrow$ d) follows from Proposition 5.35 on page 75.

Proof of (V): If the closure of the union of supports of invariant measures is compact, then, by Definition 3.22 on page 30 and Proposition 5.37 on page 75, it is the minimal center of attraction, hence c) implies a). Furthermore, by Theorem 5.19 (V) on page 66 and Theorem 5.22 on page 70 we conclude that a) implies b) and finally, b) implies a) by Proposition 5.37 on page 75.

6 ...metric spaces

In this chapter we consider semiflows on complete metric spaces and try to characterize attractive closed and invariant subsets again by properties of the corresponding Koopman semigroup. To this end, let (X, d) be a complete metric space and $(\varphi_t)_{t\geq 0}$ a continuous semiflow thereon, where continuity is defined as in Definition 3.1 on page 19. There are many examples for dynamical systems with complete metric but not locally compact state space that have a global attractor. Such dynamical systems appear for example as solutions of so-called retarded differential equations, see [Chu02, p. 13, Ex. 1.5] and [Hal10, Chapt. 4, Ex. 4.1.3]. Another often referenced example for such a dynamical system is given by the solutions of the so-called Cahn-Hilliard equation, see [SY13, Sect. 5.5, Thm. 55.8].

In contrast to the previous chapter, we cannot use the one-point compactification for a metric space that is not locally compact. Thus, we consider βX, the *Stone–Čech compactification* of X. Recall that $\mathrm{C}(\beta X) \cong \mathrm{C_b}(X)$ canonically, where $\mathrm{C_b}(X)$ denotes the Banach space of all bounded real-valued continuous functions on X endowed with the supremum norm $\|\cdot\|_\infty$, which is a Banach algebra and a Banach lattice with the usual pointwise operations.

We define the *Koopman semigroup* $(T(t))_{t\geq 0}$ corresponding to $(\varphi_t)_{t\geq 0}$ by

$$T(t)f \coloneqq f \circ \varphi_t \quad \text{for } f \in \mathrm{C_b}(X)\,,\ t \geq 0\,.$$

Then $\{T(t) \mid t \geq 0\} \subseteq \mathcal{L}(\mathrm{C_b}(X))$ and $\|T(t)\| = 1$ for all $t \geq 0$.

Similar to the previous chapter we use the compact-open topology τ_c on $\mathrm{C_b}(X)$. In Section 6.1, we discuss general facts about the τ_c topology needed for the study of attractors in the context of this chapter. We also relate continuity of the semiflow $(\varphi_t)_{t\geq 0}$ and strong τ_c-continuity of the associated Koopman semigroup $(T(t))_{t\geq 0}$. Note that such a Koopman semigroup is generally not strongly continuous but is *bi-continuous (with respect to τ_c)*. The generator acts as a derivation on its domain, similar to the previous chapters. For an overview on the concept of bi-continuous semigroups, we refer to the articles [Far04, Sect. 2] by B. Fárkas and [Küh03] by F. Kühnemund.

Section 6.2 is concerned with a characterization of attractivity properties of closed invariant subsets of dynamical systems via stability of the Koopman semigroup.

In contrast to the previous chapters compact subsets of X do not necessarily admit a neighborhood basis consisting of compact sets, but of bounded sets only. Recall that a subset $B \subseteq X$ is called *bounded* if there is a point $x \in X$ and a radius $r > 0$ such that B lies in the ball with radius r around x, i.e., $B \subseteq B_r(x)$.

Given a metric d one can always construct a topologically equivalent metric $\tilde{d}$ that is bounded, i.e., one can assume without loss of generality that the state space X is bounded. Thus, if a closed invariant subset $\emptyset \neq M \subseteq X$ attracts every bounded set, then it is already uniformly attractive. However, if M attracts only compact subsets of X, it may not be Lyapunov stable, i.e., the arguments discussed in Section 4.3 and 5.4. do not apply. It remains open to relate Lyapunov stability to a suitable stability notion of Koopman semigroups in this context.

Another problem is the existence of compact invariant subsets $\emptyset \neq M \subseteq X$. It is necessary that the corresponding ideal I_M is τ_c-closed, but it is open if this is a sufficient condition. There are several properties of dynamical systems, i.e., dissipativity [SY13, Sect. 2.3.3, p. 32], asymptotically smooth systems [Hal10, Sect. 3.2, p. 36] or asymptotically compact systems [Lad91, Chapt. 3, p. 12], that ensure the existence of a compact global attractor. It remains open to characterize these properties by means of the Koopman semigroup and to find conditions on I_M that ensure the compactness of M.

6.1 Characerization of Koopman operators on $\mathrm{C_b}(X)$-spaces and τ_c-closed ideals

Let τ_c denote the compact-open topology on $\mathrm{C_b}(X)$, i.e., the topology of uniform convergence on compact sets of X. Let $\mathcal{K}$ denote the family of compact subsets of X. Recall that the family $(p_K)_{K\in\mathcal{K}}$ of seminorms defined by

$$p_K(f) := \|f\big|_K\|_\infty = \sup_{x\in K} |f(x)| \quad \text{for } f \in \mathrm{C_b}(X)\,,\ K \in \mathcal{K}\,,$$

generates τ_c.

The compact-open topology τ_c is compatible with the algebra and lattice structure of $\mathrm{C_b}(X)$ since for every $K \in \mathcal{K}$ the corresponding seminorm p_K satisfies

$$p_K(fg) \le p_K(f)p_K(g) \text{ and } p_K(|f|) = p_K(f) \quad \text{for all } f,\, g \in \mathrm{C_b}(X)$$

and for $0 \le f \le g$ it follows that $p_K(f) \le p_K(g)$.

The following theorem is a *generalized Stone-Weierstraß Theorem*. This can be found in more generality in [Coo11, Chapt. II, Thm 1.13 & Cor. 1.14].

Theorem 6.1 (Generalized Stone-Weierstraß Theorem) Let $\mathcal{A}$ be a subalgebra of $\mathrm{C}_b(X)$ that separates the points of X and such that for every $x \in X$ there exists $f \in \mathcal{A}$ with $f(x) \neq 0$. Then $\mathcal{A}$ is τ_c-dense in $\mathrm{C}_b(X)$.

Proof. Let $\mathcal{A}$ be a subalgebra of $\mathrm{C_b}(X)$ satisfying the assumptions. Fix $K \in \mathcal{K}$ and consider the restricted algebra

$$\mathcal{A}|_K := \{f|_k \mid f \in \mathcal{A}\}\,.$$

Then $\mathcal{A}|_K$ is a subalgebra of $\mathrm{C}(K)$ that separates the points of K and for all $x \in K$ there exists $g \in \mathcal{A}|_K$ with $g(x) \neq 0$. By the usual Stone-Weierstraß theorem $\mathcal{A}|_K$ is dense in $\mathrm{C}(K)$ with respect to the supremum norm, i.e., for all $\varepsilon > 0$, $h \in \mathrm{C}(K)$ there exists $g \in \mathcal{A}|_K$ with $\|(h-g)|_K\|_\infty < \varepsilon$. This implies the assertion. $\square$

To follow the same leitmotif as in the previous chapters we have to ensure that the closed subsets of X coincide with certain ideals in $\mathrm{C_b}(X)$. The τ_c-closed ideals turn out to be the right choice.

Proposition 6.2 Let $I \subseteq \mathrm{C_b}(X)$ be a τ_c-closed ideal, then

$$I = I_M$$

where $M = \bigcap\limits_{f \in I} [f = 0]$ and $I_M := \{f \in \mathrm{C_b}(X) \mid f|_M = 0\}$.

Proof. We follow the arguments used in [Coo11, Chapt. II, Prop. 2.7]. Let I be a τ_c-closed ideal $I \subset \mathrm{C_b}(X)$. First note that $I \subseteq I_M$ is always true. Furthermore, I is $\|\cdot\|_\infty$-closed and therefore it is isomorphic to an ideal of the form $I_{\tilde{M}} = \{f \in \mathrm{C}(\beta X) \mid f|_{\tilde{M}} = 0\}$ where $\tilde{M} \subset \beta X$.

Assume $\tilde{M} \setminus \overline{M}^{\beta X} \neq \emptyset$, $M = \bigcap_{f \in I} [f = 0]$. This means there exists $x_0 \in \tilde{M}$ and $f_0 \in \mathrm{C}(\beta X)$ with $f_0(x_0) = 1$ and $f|_U \equiv 0$ for some neighborhood $U \in \mathcal{U}(\overline{M}^{\beta X})$.

Now fix $K \in \mathcal{K}$ and consider $I|_K := \{g|_K \mid g \in I\}$. This is an ideal in $\mathrm{C}(K)$ and therefore its closure is of the form

$$\overline{I|_K}^{p_K} = I_L \quad \text{for some compact } L \subset K\,.$$

In fact $L = M \cap K$. This implies $f_0\big|_K \in I_L$. Now take $\varepsilon > 0$. There exists $f_{K,\varepsilon} \in I\big|_K$ with

$$\|(f_0 - f_{K,\varepsilon})\big|_K\|_\infty = p_K(f_0 - f_{K,\varepsilon}) < \varepsilon\,.$$

Since $f_{K,\varepsilon} \in I|_K$, it is the restriction of a function $f_\varepsilon \in I$. Since K was arbitrary, $f_0 \in I$ which is a contradiction. Hence, $\tilde{M} = \overline{M}^{\beta X}$ which means that for every function $f \in \mathrm{C_b}(X)$ vanishing on M its extension to βX vanishes on $\tilde{M}$ and is therefore an element of I. This implies $I_M \subseteq I$ which concludes the proof. □

Recall that $\mathrm{C_b}(X)'$ is canonically isomorphic to the space $\mathrm{M}(\beta X)$ of all bounded regular Borel measures on βX. For our purpose we look for a subspace of measures with support in X. A suitable choice is the topological τ_c-dual of $\mathrm{C_b}(X)$ which we denote by $\mathrm{C_b}(X)'_{\tau_c} := (\mathrm{C_b}(X), \tau_c)'$. It is known that $\mathrm{C_b}(X)'_{\tau_c}$ is canonically isomorphic to $\mathrm{M}_c(X)$ the space of bounded regular Borel measures on X with compact support, cf. [Coo11, Chapt. II, Prop. 3.2].

We consider the dual system $(\mathrm{C_b}(X), \mathrm{M}_c(X), \langle\cdot,\cdot\rangle)$ (cf. [Sch71, Chapt. IV, §1, p. 123]) given by the bilinear form

$$\begin{aligned} \langle\cdot,\cdot\rangle \colon \mathrm{C_b}(X) \times \mathrm{M}_c(X) &\to \mathbb{C} \\ (f,\mu) &\mapsto \int_X f \,\mathrm{d}\mu\,, \end{aligned}$$

and write

$$\sigma_{\tau_c} := \sigma(\mathrm{C_b}(X), \mathrm{M}_c(X))$$

for the induced weak topology, [Sch71, Chapt. IV, §1, p. 124], and

$$\sigma^*_{\tau_c} := \sigma(\mathrm{M}_c(X), \mathrm{C_b}(X))$$

for the induced weak* topology respectively.

We define the subset $\Gamma(X)_{\tau_c}$ of multiplicative linear forms in $\mathrm{C_b}(X)'_{\tau_c}$, i.e.,

$$\Gamma(X)_{\tau_c} := \{\alpha \in \mathrm{C_b}(X)'_{\tau_c} \mid \|\alpha\| = 1\,,\ \alpha(f \cdot g) = \alpha f \cdot \alpha g \text{ for all } f, g \in \mathrm{C_b}(X)\}\,.$$

For every $x \in X$ the *point evaluation*

$$\delta_x \colon f \mapsto f(x)$$

is an element of $\Gamma(X)_{\tau_c}$. In fact, every multiplicative linear form in $\mathrm{C_b}(X)'_{\tau_c}$ is a point evaluation.

Proposition 6.3 The mapping

$$\begin{aligned} \delta \colon X &\to \Gamma(X)_{\tau_c} \\ x &\mapsto \delta_x \end{aligned}$$

is a homeomorphism where $\Gamma(X)_{\tau_c}$ is endowed with the $\sigma^*_{\tau_c}$-topology.

Proof. We follow the proof given in [Coo11, Chapt. II,Prop. 2.2]. First note that δ is injective since $\mathrm{C_b}(X)$ separates the points of X. To prove that δ is surjective take $\alpha \in \Gamma(X)_{\tau_c}$ and consider its kernel $\ker(\alpha)$. This is a τ_c-closed subalgebra of $\mathrm{C_b}(X)$ with codimension 1. Assume $\ker(\alpha)$ would not separate the points of X, then there exist $x \neq y \in X$ such that $\ker(\alpha)$ is in the kernel of the linear form given by $f \mapsto f(x) - f(y)$. Thus $\ker(\alpha)$ has codimension at least 2 which is a contradiction.

Furthermore, there exists an $x \in X$ such that $f(x) = 0$ for all $f \in \ker(\alpha)$ because otherwise $\ker(\alpha)$ satisfies the conditions of the generalized Stone-Weierstraß Theorem 6.1 on page 83. Hence, it is τ_c-dense in $C_b(X)$, i.e., $\alpha = 0$. This implies,

$$\ker(\alpha) \subseteq \{f \in C_b(X;\mathbb{R}) \mid f(x) = 0\} = \ker(\delta_x)\,.$$

If $\ker(\alpha) \subsetneq \ker(\delta_x)$, $\ker(\alpha)$ must have codimension at least 2 which is a contradiction. Hence, $\alpha = \delta_x$ for they both have codimension 1 and $\|\alpha\| = 1$.

Finally, we show that δ is in fact a homeomorphism, i.e., the inverse

$$\delta^{-1} \colon \Gamma(X)_{\tau_c} \to X$$

is continuous. Consider a convergent net $(\delta_{x_i})_{i\in I}$ with $\sigma^*_{\tau_c}$-limit δ_x. This means

$$\langle f, \delta_{x_i}\rangle = f(x_i) \xrightarrow[i\in I]{} f(x) = \langle f, \delta_x\rangle \quad \text{for all } f \in \mathrm{C_b}(X)\,.$$

Now consider an open neighborhood U of x and take $f \in \mathrm{C_b}(X)$ with $f(x) = 1$ and $f|_{X\setminus U} \equiv 0$, which exists since X is completely regular. We know that there exists $i_0 \in I$ such that

$$|f(x_i) - 1| < \frac{1}{2} \quad \text{for all } i \geq i_0 \,.$$

This implies $f(x_i) \geq \frac{1}{2}$ for all $i \geq i_0$ and therefore $x_i \in U$ for all $i \geq i_0$. □

Remark 6.4 Every τ_c-continuous normalized lattice homomorphism, i.e., a functional $\psi\colon \mathrm{C_b}(X) \to \mathbb{R}$, with $\psi(|f|) = |\psi(f)|$ for all $f \in \mathrm{C_b}(X)$ and $\psi(\mathbb{1}) = 1$, is a point evaluation.

Proof. Take a τ_c-continuous lattice homomorphism ψ with $\psi(\mathbb{1}) = 1$. This is also a $\|\cdot\|_\infty$-continuous lattice homomorphism and since $\mathrm{C_b}(X) \simeq \mathrm{C}(\beta X)$, it is a point evaluation by [Sch74, Chapt. III, Prop. 1.4]. The point evaluations on $\mathrm{C}(\beta X)$ coincide with the $\|\cdot\|_\infty$-continuous multiplicative linear forms on $\mathrm{C}(\beta X)$, cf. [EFHN15, Lem. 4.10]. Hence, ψ is a τ_c-continuous multiplicative linear form which is a point evaluation by Proposition 6.3 on page 85. □

Now we turn to the characterization of Koopman operators as algebra homomorphisms between $\mathrm{C_b}(X)$-spaces.

Definition 6.5 Let $\varphi\colon X \to X$ be a continuous map. We call the map $T_\varphi\colon \mathrm{C_b}(X) \to \mathrm{C_b}(X)$ defined via

$$T_\varphi f := f \circ \varphi \quad \text{for all } f \in \mathrm{C_b}(X)$$

Koopman operator (induced by φ).

Proposition 6.6 Let $\varphi\colon X \to X$ be a continuous map, and $T\colon \mathrm{C_b}(X) \to \mathrm{C_b}(X)$ the induced Koopman operator. Then T is a τ_c-continuous lattice and algebra homomorphism with $T\mathbb{1} = \mathbb{1}$.

Proof. It is clear that T is linear. Consider a convergent net $f_\alpha \xrightarrow{\tau_c} f$ in $\mathrm{C_b}(X)$. Remark that for $K \subset X$ compact the image $\varphi(K)$ is compact and thus,

$$p_K(Tf_\alpha) = p_{\varphi(K)}(f_\alpha) \to p_{\varphi(K)}(f) = p_K(Tf) \,.$$

Also, clearly $T\mathbb{1} = \mathbb{1}$, $T(fg) = Tf \cdot Tg$ and $T|f| = |Tf|$ for all $f, g \in \mathrm{C_b}(X)$.

□

Next, we use these facts to characterize homomorphisms between $\mathrm{C_b}(X)$-spaces.

Proposition 6.7 Let $T\colon \mathrm{C_b}(X) \to \mathrm{C_b}(X)$ be a τ_c-continuous algebra homomorphism with $T\mathbb{1} = \mathbb{1}$. Then there exists a continuous mapping $\varphi\colon X \to X$ such that

$$Tf = f \circ \varphi \text{ for all } f \in \mathrm{C_b}(X)\,.$$

PROOF. Take $x \in X$ then $T'\delta_x$ is a τ_c-continuous multiplicative linear form on $\mathrm{C_b}(X)$ with $T'\delta_x(\mathbb{1}) = 1$ which is a point evaluation by Proposition 6.3 on page 85. Set $\delta_{\varphi(x)} := T'\delta_x$. Thus by construction $Tf = f \circ \varphi \in \mathrm{C_b}(X)$ for every $f \in \mathrm{C_b}(X)$. This implies that φ is continuous because the sets of the form $[|f| > 0]$, $f \in \mathrm{C_b}(X)$, form a basis for the open sets in X. □

Proposition 6.8 Let $T\colon \mathrm{C_b}(X) \to \mathrm{C_b}(X)$ be a bounded operator. Then the following are equivalent.

a) T is a τ_c-continuous algebra homomorphism and $T\mathbb{1} = \mathbb{1}$,

b) T is a τ_c-continuous lattice homomorphism and $T\mathbb{1} = \mathbb{1}$,

c) T is a Koopman operator.

PROOF. The fact that a) $\iff$ b) is clear since the τ_c-continuous lattice homomorphisms and algebra homomorphisms coincide by Remark 6.4 on page 86. The rest is clear by above Propositions 6.6 and 6.7 on page 86 and on the current page.

Now we turn to semigroups of Koopman operators.

Definition 6.9 We call a semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on $\mathrm{C_b}(X)$ *Koopman semigroup* if $T(t)$ is a Koopman operator for every $t \geq 0$, i.e., there exists a semiflow $(\varphi_t)_{t\geq 0}$ on X with $T(t)f = f \circ \varphi_t$ for all $t \geq 0$, $f \in \mathrm{C_b}(X)$.

We have the following correspondence between continuity of a semiflow and the associated Koopman semigroup.

Proposition 6.10 Let $(T(t))_{t\geq 0}$ be a Koopman semigroup on $\mathrm{C_b}(X)$ induced by a semiflow $(\varphi_t)_{t\geq 0}$ on X. Then the following are equivalent.

a) The map

$$\begin{aligned} \varphi\colon [0,\infty) \times X &\to X\,, \\ (t,x) &\mapsto \varphi_t(x) \end{aligned}$$

is continuous.

b) The semigroup $(T(t))_{t\geq 0}$ is strongly τ_c-continuous, i.e., the mapping

$$\begin{aligned} [0,\infty) &\to (\mathrm{C_b}(X), \tau_c)\,, \\ t &\mapsto T(t)f \end{aligned}$$

is continuous for every $f \in \mathrm{C_b}(X)$.

Proof. To see that a) implies b), take $f \in \mathrm{C_b}(X)$ and $K \subseteq X$ compact. It suffices to check continuity at $t = 0$. The set $\varphi([0,1] \times K) =: L$ is compact as the continuous image of a compact set. Note that $K \subseteq L$ since $\varphi_0 = \mathrm{id}_X$. Therefore, $f|_L \in \mathrm{C}(L)$ and $T(t)f|_L \in \mathrm{C}(K)$ for every $t \in [0,1]$. Applying [EFHN15, Thm. 4.17], $t \mapsto T(t)f|_L \in \mathrm{C}(K)$ is continuous at 0, i.e.,

$$\|T(t)f|_L - f|_L\|_\infty = \|(T(t)f - f)|_K\|_\infty \to 0\,.$$

This implies b) since K was arbitrary.

On the other hand, if b) is true, using again [EFHN15, Thm. 4.17], we obtain that φ is continuous restricted to every set of the form $[0,t_0] \times K$, $t_0 > 0$, $K \subseteq X$ compact. Using the semiflow property, this implies φ is continuous on every compact subset of its domain. Hence, φ is continuous everywhere, because $[0,\infty) \times X$ and X are metric spaces. This completes the proof. □

Proposition 6.10 above can be found in [Küh03, Prop. 18]. Our proof is different from the proof given there.

6.2 Asymptotics of dynamical systems

Recall the absorbing and attractivity properties of closed invariant sets from Definition 3.13 on page 24 and Definition 3.18 on page 27 from Chapter 3. These notions are defined for complete metric spaces analoguously. Furthermore, recall the definition of stability of a semigroup with respect to a locally-convex topology in Definition 2.3 on page 10 and the result in Proposition 2.4 on page 11. We use these definitions and results analoguously for the situation in this chapter.

Proposition 6.11 Let $(X, (\varphi_t)_{t\geq 0})$ be a dynamical system, $\emptyset \neq M \subsetneq \Omega$ a closed invariant set and $(S(t))_{t\geq 0}$ the corresponding Koopman semigroup restricted to I_M for $t \geq 0$. Then the assertions in (I), (II) and (III) are equivalent, respectively.

(I) a) $(S(t))_{t\geq 0}$ is nilpotent.

b) $(S(t))_{t\geq 0}$ is uniformly stable.

c) M is absorbing.

(II) a) $(S(t))_{t\geq 0}$ is τ_c-nilpotent, (cf. Definition 2.3 a) on page 10).

b) M is compact absorbing.

(III) a) For all Dirac measures $\delta_x \in I_M'$ there exists $t_0 > 0$ such that

$$S(t_0)'\delta_x = 0$$

b) M is pointwise absorbing.

Proof. This can be shown by the arguments used in Proposition 4.7 on page 38 and Proposition 5.18 on page 65.

The following characterization of almost σ_{τ_c}-stability is similar to Lemma 4.8 on page 39 and Lemma 5.17 on page 64 from the previous chapters.

Lemma 6.12 Let $(T(t))_{t\geq 0}$ be a strongly continuous Koopman semigroup on $\mathrm{C_b}(X)$ and I a $\|\cdot\|_\infty$-closed $(T(t))_{t\geq 0}$-invariant ideal in $\mathrm{C_b}(X)$. Then the restricted semigroup $(S(t))_{t\geq 0}$, with $S(t) := T(t)|_I$, $t \geq 0$, is almost σ_{τ_c}-stable, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\langle S(s)f, \mu\rangle| \,\mathrm{d}s = 0 \text{ for all } \mu \in \mathrm{C_b}(X)'_{\tau_c},\ f \in I,$$

if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S(s)|f|(x) \,\mathrm{d}s = 0 \text{ for all } x \in X,\ f \in I\,.$$

Theorem 6.13 Let $(X, (\varphi_t)_{t\geq 0})$ be a dynamical system, μ a quasi-invariant Borel measure on X, $\emptyset \neq M \subset X$ closed and invariant and $(S(t))_{t\geq 0}$ the corresponding Koopman semigroup restricted to I_M. Then the assertions in (I), (II), (III), (IV) and (V), respectively, are equivalent.

(I) a) $(S(t))_{t\geq 0}$ is strongly stable.

b) M is uniformly attractive.

(II) a) $(S(t))_{t\geq 0}$ is τ_c-stable.

b) M is compact attractive.

(III) a) $(S(t))_{t\geq 0}$ is σ_{τ_c}-stable.

b) M is attractive.

(IV) a) $(S(t))_{t\geq 0}$ is almost σ_{τ_c}-stable.

b) M is a center of attraction.

(V) a) $(S(t))_{t\geq 0}$ is μ-almost everywhere pointwise stable.

b) M is a likely limit set (for μ).

Proof. The proofs of (I) and (II) are as for Theorem 4.9 (I) on page 40 and Theorem 5.19 (II) on page 66.

Proof of (III): If the semigroup is σ_{τ_c}-stable, for every $x \in X$ and $U \in \mathcal{U}(M)$ there exists $f \in I_M$, $\varepsilon > 0$ and $t_0 > 0$ with $\varphi_t(x) \in U_{\varepsilon,f} \subseteq U$ for all $t \geq t_0$. On the other hand if M is attractive, we obtain

$$\langle T(f)f, \delta_x\rangle \to 0 \text{ as } t \to \infty$$

for all $x \in X$, $f \in I_M$. Now take $\mu \in \mathrm{C_b}(X)'_{\tau_c}$ and note that $|T(t)f(x)| \leq \|f\|_\infty \cdot \mathbb{1}_X$ for all $t \geq 0$, $x \in X$. Since $\int_X \|f\|_\infty \cdot \mathbb{1}_X \,\mathrm{d}\mu < \infty$, we can apply Lebesgue's theorem of dominated convergence and therefore

$$\langle T(t)f, \mu\rangle \to 0 \text{ as } t \to \infty\,.$$

Proof of (IV): We prove first the implication b) $\Longrightarrow$ a). If M is a center of attraction and $U \in \mathcal{U}(M)$ open, then

$$\begin{aligned}&\frac{1}{t}\,\lambda(\{s \in [0,t] \mid \varphi_s(x) \in U^c\})\\ =&\frac{1}{t}\int_0^t \mathbb{1}_{U^c}(\varphi_s(x))\,\mathrm{d}s \xrightarrow{t\to\infty} 0 \text{ for all } x \in \Omega\,.\end{aligned}$$

Now take $f \in I_M$ with $\|f\|_\infty = 1$ and $1 > \varepsilon > 0$. Then

$$\begin{aligned}&\frac{1}{t}\int_0^t |S(s)f(x)|\,\mathrm{d}s\\ \le&\frac{1}{t}\int_0^t |S(s)f(x)|\mathbb{1}_{[|f|<\varepsilon]}(\varphi_s(x))\,\mathrm{d}s + \frac{1}{t}\int_0^t \|f\|_\infty\mathbb{1}_{[|f|\ge\varepsilon]}(\varphi_s(x))\,\mathrm{d}s\\ \le&\quad \varepsilon + \frac{1}{t}\int_0^t \|f\|_\infty\mathbb{1}_{[|f|\ge\varepsilon]}(\varphi_s(x))\,\mathrm{d}s < 2\varepsilon\end{aligned}$$

for t sufficiently large since $[|f| \ge \varepsilon]$ is the complement of the open neighborhood $[|f| < \varepsilon]$ of M. Thus b) implies a) by Lemma 5.17 on page 64.

For the other implication take $x \in X$, $f \in I_M$, $f \ge 0$ and $\varepsilon > 0$. By assumption there exists a subset $R \subseteq [0,\infty)$ with density 1 and $t_0 > 0$ such that

$$\langle S(t)f, \delta_x\rangle < \varepsilon \quad \text{for all } t \ge t_0\,,\ t \in R\,.$$

Since $R \cap [t_0,\infty)$ still has density 1, we obtain

$$\frac{1}{t}\,\lambda\left(\{s \in [0,t] \mid \varphi_s(x) \in U_{\varepsilon,f}\}\right) \to 1\,.$$

This implies the assertion since the neighborhoods of the form $[|f| < \varepsilon]$, $f \in I_M$, $\varepsilon > 0$, form a basis for the neighborhoods of M.

Proof of (V): This follows by the same arguments used in Theorem 4.9 (IV) on page 41. $\square$

With this result we conclude our outlook on how Koopmanism can be used for attractors in dynamical systems on metric spaces.

Part III

Measure Preserving Systems and Topological Models

7 Markov semigroups of lattice operators and their generators

This part is based on the article [EGK19] by Nikolai Edeko, Moritz Gerlach, Viktoria Kühner, *Measure preserving semiflows and one-parameter Koopman semigroups*, Semigroup Forum (2019), p.48-63. The author contributed approximately 30% of the scientific ideas and the writing of this article, except for the proofs of Theorem 7.20 on page 105 and Lemma 7.24 on page 107, ([EGK19, Thm. 2.1] and [EGK19, Lem. 2.5], respectively). These are due to the co-author N. Edeko.

In this chapter we address the following problems. First, in Section 7.1, we start with a finite measure space $\mathrm{X} = (X, \Sigma, \mu)$ and characterize strongly continuous Markov lattice semigroups $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$ by properties of their generators. The main result (Theorem 7.12 on page 98) is that a strongly continuous semigroup on $\mathrm{L}^p(\mathrm{X})$ is a Markov lattice semigroup if and only if its generator A acts as a derivation on $D(A) \cap \mathrm{L}^\infty(\mathrm{X})$, $\mathbb{1} \in D(A)$ and the semigroup is locally bounded on $\mathrm{L}^\infty(\mathrm{X})$. We have seen such characterizations previously in this thesis, cf. Theorem 4.6 on page 38 and Theorem 5.15 on page 62. In 2015, T. ter Elst and M. Lemańczyk proved a similar result in [tEL15] for unitary groups on $\mathrm{L}^2(\mathrm{X})$. Then we give a version of the main result in Theorem 7.19 on page 103 for lattice semigroups that are not necessarily Markov. In Section 7.2 we turn to standard probability spaces. We apply the results from Section 7.1 and prove that every Markov lattice semigroup is, in this case, a Koopman semigroup, cf. Definition 7.4 on the next page.

Now we recall basic concepts and fix the notation for this part of the thesis.

For a measure space $\mathrm{X} = (X, \Sigma, \mu)$ and $1 \leq p \leq \infty$ we denote by $\mathrm{L}^p(\mathrm{X}) \coloneqq \mathrm{L}^p(\mathrm{X}; \mathbb{C})$ the corresponding complex L^p-space. This is a complex Banach lattice in the sense of [EFHN15, Def. 7.2]. The lattice operations in $\mathrm{L}^p(\mathrm{X}; \mathbb{R})$ are denoted by $\vee$ and $\wedge$ and coincide with the respective pointwise operations. To be precise, the supremum and infimum is represented by the pointwise supremum and infimum of representatives of the equivalence classes f and g, cf. [EFHN15, Ex. 7.1, 2)].

Next we define (bi-)Markov lattice operators.

Definition 7.1 Let $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{Y} = (Y, \Sigma', \mu')$ be finite measure spaces and $T \in \mathcal{L}(\mathrm{L}^p(\mathrm{X}), \mathrm{L}^p(\mathrm{Y}))$, $1 \leq p < \infty$. The operator T is called

a) *positive* if it is real, i.e., $T\mathrm{L}^p(\mathrm{X}; \mathbb{R}) \subseteq \mathrm{L}^p(\mathrm{Y}; \mathbb{R})$, and its restriction to the Banach lattice $\mathrm{L}^p(\mathrm{X}; \mathbb{R})$ is positive, i.e., $Tf \geq 0$ for all $f \in \mathrm{L}^p(\mathrm{X}; \mathbb{R})$, $f \geq 0$.

b) *lattice homomorphism* (or *lattice operator*) if $|Tf| = T|f|$ for each $f \in \mathrm{L}^p(\mathrm{X})$,

c) *Markov operator* if it is positive and $T\mathbb{1}_X = \mathbb{1}_Y$, and *bi-Markov operator* if, additionally, $T'\mathbb{1}_Y = \mathbb{1}_X$.

We remark the following.

Remark 7.2 Every lattice homomorphism is positive and fulfills

$$T(f_+) = (Tf)_+ \quad \text{and} \quad T(f_-) = (Tf)_- \text{ for all } f \in \mathrm{L}^p(\mathrm{X}; \mathbb{R})\,.$$

For operator semigroups consisting of (bi-Markov) lattice operators we use the following notation.

Definition 7.3 Let $\mathrm{X} = (X, \Sigma, \mu)$ be a finite measure space and $(T(t))_{t \geq 0}$ a strongly continuous semigroup on $\mathrm{L}^p(\mathrm{X})$. The semigroup $(T(t))_{t \geq 0}$ is called

a) *lattice semigroup* if each operator $T(t)$ is a lattice homomorphism and

b) *(bi-)Markov lattice semigroup* if every operator $T(t)$ is a (bi-)Markov operator.

Now we turn to Koopman operators which clearly are Markov lattice operators.

Definition 7.4 We call an operator $T \in \mathcal{L}(\mathrm{L}^p(\mathrm{X}), \mathrm{L}^p(\mathrm{Y}))$ *Koopman operator* if there exists a measurable map $\varphi \colon Y \to X$ satisfying $\varphi^{-1}(N) = 0$ for every μ-null set $N \in \Sigma$ such that

$$Tf = f \circ \varphi \quad \text{for every } f \in \mathrm{L}^p(\mathrm{X})\,.$$

In this case we denote the operator by T_φ.

Remark 7.5 A Koopman operator $T_\varphi \in \mathcal{L}(\mathrm{L}^p(\mathrm{X}), \mathrm{L}^p(\mathrm{Y}))$, $1 \le p < \infty$, is a Markov lattice operator. If the corresponding map φ is even measure preserving[1], the Koopman operator T_φ is bi-Markov.

Definition 7.6 Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup on $\mathrm{L}^p(\mathrm{X})$ for some $1 \le p < \infty$. It is called a *Koopman semigroup* if for each $t \ge 0$ the operator $T(t)$ is a Koopman operator.

Furthermore, we use the following definition.

Definition 7.7 Given a finite measure space $\mathrm{X} = (X, \Sigma, \mu)$ and a linear operator δ on $\mathrm{L}^p(\mathrm{X})$, $1 \le p < \infty$, with domain $D(\delta)$. Then δ is called *derivation on $D(\delta) \cap \mathrm{L}^\infty(\mathrm{X})$* if $D(\delta) \cap \mathrm{L}^\infty(\mathrm{X})$ is an algebra (with respect to the pointwise almost everywhere multiplication) and

$$\delta(f \cdot g) = \delta f \cdot g + f \cdot \delta g \text{ for all } f, g \in D(\delta) \cap \mathrm{L}^\infty(\mathrm{X}) .$$

Next, we recall the so-called *measure algebra* $\Sigma(\mathrm{X})$. For further information on $\Sigma(\mathrm{X})$ we refer to [EFHN15, Sect. 6.1].

Definition 7.8 Consider the equivalence relation

$$M \sim N \quad \text{if} \quad \mathbb{1}_M = \mathbb{1}_N \ \mu\text{-almost everywhere}$$

on Σ. Then the set of equivalence classes

$$\Sigma(\mathrm{X}) \coloneqq \Sigma/{\sim}$$

is called the *measure algebra* of the measure space X.

It is a Boolean algebra with respect to the usual operations.

Remark 7.9 For simplicity, we do not distinguish notationally between elements of Σ and $\Sigma(\mathrm{X})$.

We define homomorphisms on $\Sigma(\mathrm{X})$ as follows.

Definition 7.10 A mapping $\theta \colon \Sigma(\mathrm{X}) \to \Sigma(\mathrm{X})$ is called *measure algebra homomorphism* if it is a Boolean algebra homomorphism and satisfies

$$\mu(\theta(A)) = \mu(A) \text{ for all } A \in \Sigma(\mathrm{X}) .$$

[1] Given a measure space $\mathrm{X} = (X, \Sigma, \mu)$ and a measurable mapping $\varphi \colon X \to X$, we call φ *measure preserving* if for every $A \in \Sigma$, $\mu(\varphi^{-1}(A)) = \mu(A)$.

An easy example is the following.

Example 7.11 Let $\mathrm{X} = (X, \Sigma, \mu)$ be a finite measure space and $\varphi\colon X \to X$ a measurable map such that $\mu(\varphi^{-1}(N)) = 0$ for every μ-null set N. Then φ induces a Boolean algebra homomorphism φ^* by

$$\begin{aligned} \varphi^*\colon \Sigma(\mathrm{X}) &\to \Sigma(\mathrm{X}) \\ A &\mapsto \varphi^{-1}(A)\,. \end{aligned}$$

If φ is measure preserving, φ^* is a measure algebra homomorphism.

7.1 Characterization of Markov lattice semigroups on L^p-spaces

This section is concerned with the characterization of strongly continuous Markov lattice semigroups on L^p-spaces by means of their generator. The following theorem is our main result in this section. As a corollary we obtain a characterization of strongly continuous bi-Markov lattice semigroups in Corollary 7.18 on page 103 and also prove a similar result for strongly continuous lattice semigroups that are not necessarily Markov, cf. Theorem 7.19 on page 103.

Throughout this section $\mathrm{X} = (X, \Sigma, \mu)$ is a finite measure space. Given a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A we write $D(A)$ for its domain.

Theorem 7.12 Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$. Then the following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is a Markov lattice semigroup.

(ii) For every $t \geq 0$ there exists a Boolean algebra homomorphism $\theta_t\colon \Sigma(\mathrm{X}) \to \Sigma(\mathrm{X})$ such that $T(t)\mathbb{1}_M = \mathbb{1}_{\theta_t(M)}$ for all $M \in \Sigma(\mathrm{X})$.

(iii) The space $\mathrm{L}^\infty(\mathrm{X})$ is invariant under $(T(t))_{t\geq 0}$, the map $t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$ is locally bounded, $\mathbb{1} \in D(A)$, and A is a derivation on $D(A) \cap \mathrm{L}^\infty(\mathrm{X})$.

We remark the following facts.

Remark 7.13 (i) Given a bounded operator S on $\mathrm{L}^p(\mathrm{X})$ such that $\mathrm{L}^\infty(\mathrm{X})$ is invariant under S, it follows from the closed graph theorem that the restriction $S|_{\mathrm{L}^\infty(\mathrm{X})}$ is a bounded operator. Therefore, the map

$$t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$$

is well-defined in (iii) of Theorem 7.12 on page 98. As will be shown in Lemma 7.24 on page 107, the local boundedness condition is automatically fulfilled for a strongly continuous operator group $(T(t))_{t\in\mathbb{R}}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \le p < \infty$.

(ii) A semigroup $(T(t))_{t\geq 0}$ satisfying (i)–(iii) in Theorem 7.12 on page 98 uniquely extends to a strongly continuous Markov lattice semigroup on $\mathrm{L}^q(\mathrm{X})$ for each $1 \le q < \infty$ with

$$\|T(t)\|_{\mathcal{L}(\mathrm{L}^q(\mathrm{X}))} = \|T(t)\|^{\frac{1}{q}}_{\mathcal{L}(\mathrm{L}^1(\mathrm{X}))},$$

use [EFHN15, Thm. 7.23]. In particular, it extends to $\mathrm{L}^1(\mathrm{X})$. Therefore, we only consider semigroups on $\mathrm{L}^1(\mathrm{X})$ in Chapter 8.

As a preparation for the proof of Theorem 7.12 on page 98, recall the following lemma relating the algebra and the lattice structure of $\mathrm{L}^\infty(\mathrm{X})$.

Lemma 7.14 Let $T\colon \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{X})$ be a bounded linear operator satisfying $T\mathbb{1} = \mathbb{1}$. Then the follwing assertions are equivalent.

(i) T is multiplicative.

(ii) T is a C^*-homomorphism.

(iii) T is a lattice homomorphism.

Proof. Obviously, (ii) implies (i). The equivalence of (ii) and (iii) can be found in [EFHN15, Thm. 7.23]. There, the operator is assumed to be conjugation-preserving but this assumption is superfluous since (ii) and (iii) directly imply positivity of the operator and (i) implies that characteristic functions are mapped to characteristic functions, hence (i) also implies positivity. The implication (i) $\implies$ (ii) follows from [EFHN15, Thm. 4.13], the analogous statement for spaces of continuous functions, by applying the Gelfand-Naimark theorem [EFHN15, Thm. 4.23]. □

The following continuity property will be essential for the proof of Theorem 7.12 on page 98.

Lemma 7.15 Let $B \subset \mathrm{L}^\infty(\mathrm{X})$ be bounded. Then the multiplication

$$\begin{aligned} \mathrm{L}^p(\mathrm{X}) \times B &\to \mathrm{L}^p(\mathrm{X}), \\ (f,g) &\mapsto fg \end{aligned}$$

is $\|\cdot\|_p$-continuous.

Proof. Let M be a bound for B. For $f, u \in \mathrm{L}^p(\mathrm{X})$, $g, v \in B$ and $c > 0$

$$\begin{aligned} fg - uv &= (f-u)g + u(g-v) \\ &= (f-u)g + u\mathbb{1}_{[|u|\leq c]}(g-v) + u\mathbb{1}_{[|u|>c]}(g-v) \end{aligned}$$

and thus

$$\limsup_{(f,g)\to(u,v)} \|fg - uv\|_p \leq 2M \left\| u\mathbb{1}_{[|u|>c]} \right\|_p \to 0 \quad \text{as } c \to \infty\,.$$

$\square$

Next, we apply Lemma 7.15 to a strongly continuous semigroup satisfying the assumptions in Theorem 7.12 on page 98.

Corollary 7.16 Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$. In addition, suppose that $\mathrm{L}^\infty(\mathrm{X})$ is invariant under $(T(t))_{t\geq 0}$ and that the map $t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$ is locally bounded. Then, for all $f \in \mathrm{L}^\infty(\mathrm{X})$ and all $g \in \mathrm{L}^p(\mathrm{X})$ the function

$$[0,\infty) \to (t \mapsto \mathrm{L}^p(\mathrm{X})), \quad t \mapsto T(t)f \cdot T(t)g$$

is continuous. Moreover, for $f, g \in D(A) \cap \mathrm{L}^\infty(\mathrm{X})$ this function is differentiable and satisfies the following product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(T(t)f \cdot T(t)g\right) = T(t)Af \cdot T(t)g + T(t)f \cdot T(t)Ag\,.$$

Proof. It suffices to prove the second part since the first is a consequence of Lemma 7.15. Let $f, g \in D(A) \cap \mathrm{L}^\infty(\mathrm{X})$ and $t \geq 0$. Use Lemma 7.15 on page 100 and differentiate to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(T(t)f \cdot T(t)g\right) &= \lim_{h\to 0}\frac{1}{h}\big([T(t+h)f - T(t)f]\, T(t)g \\ &\qquad + T(t+h)f\,[T(t+h)g - T(t)g]\big) \\ &= \lim_{h\to 0}\frac{1}{h}\big([T(t+h)f - T(t)f]\big)T(t)g \\ &\qquad + \lim_{h\to 0} T(t+h)f \cdot \lim_{h\to 0}\frac{1}{h}\,[T(t+h)g - T(t)g] \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}T(t)f\right)\cdot T(t)g + T(t)f \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}T(t)g\right)\end{aligned}$$

which proves the assertion. □

We are now able to prove Theorem 7.12 on page 98.

Proof (Proof of Theorem 7.12 on page 98.). The equivalence (i) $\Leftrightarrow$ (ii) is proved almost exactly as in the case of bi-Markov operators, see [EFHN15, Thm. 12.10].

To prove the implication (i) $\implies$ (iii), first note that every operator $T(t)$ is positive. Since $T(t)\mathbb{1} = \mathbb{1}$ for each $t \geq 0$, this already implies that the semigroup preserves the subspace $\mathrm{L}^\infty(\mathrm{X})$ and the restriction of each $T(t)$ to $\mathrm{L}^\infty(\mathrm{X})$ is a contraction. In particular, it follows from Lemma 7.14 on page 99 that every operator $T(t)$ is multiplicative on $\mathrm{L}^\infty(\mathrm{X})$. By Corollary 7.16 on page 100, for every $f, g \in D(A) \cap \mathrm{L}^\infty(\mathrm{X})$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}T(t)(f\cdot g) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(T(t)f \cdot T(t)g\big) \\ &= T(t)Af \cdot T(t)g + T(t)f \cdot T(t)Ag \\ &= T(t)\left[Af \cdot g + f \cdot Ag\right] .\end{aligned}$$

In particular, this shows $f \cdot g \in D(A)$ and $A(f\cdot g) = Af \cdot g + f \cdot Ag$, hence it proves that A is a derivation with $\mathbb{1} \in D(A)$.

We now prove that (iii) implies (i). Because of the local boundedness of $t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$, there exists a constant $C > 0$ such that

$$\left\|\frac{1}{t}\int_0^t T(s)f\,\mathrm{d}s\right\|_\infty \leq C\|f\|_\infty$$

for $0 < t \le 1$ and $f \in \mathrm{L}^\infty(\mathrm{X})$. This implies that $D := D(A) \cap \mathrm{L}^\infty(\mathrm{X})$ is a dense subspace of $\mathrm{L}^p(\mathrm{X})$. We use this fact to show that each $T(t)$ is multiplicative on $\mathrm{L}^\infty(\mathrm{X})$. For fixed $f, g \in D$ and $t > 0$ consider the mapping

$$s \mapsto \beta(s) := T(t-s)[T(s)f \cdot T(s)g]$$

on $[0,t]$. Since $\beta(0) = T(t)(f \cdot g)$ and $\beta(t) = T(t)f \cdot T(t)g$, it suffices to show that β is constant. To this end, consider the operator valued mappings $P, Q \colon [0,t] \to \mathcal{L}(\mathrm{L}^p(\mathrm{X}))$ given by $P(s) = T(t-s)$ and $Q(s) = \mathrm{M}_{T(s)f} \circ T(s)$, where $\mathrm{M}_{T(s)f}$ denotes the multiplication with the bounded function $T(s)f$. It follows from Corollary 7.16 on page 100 that Q is strongly continuous and that for each $h \in D$, $s \mapsto Q(s)h$ is differentiable with derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} Q(s)h = T(s)Af \cdot T(s)h + T(s)f \cdot T(s)Ah = A\big(T(s)f \cdot T(s)h\big)\,.$$

Here, the second equality follows from the fact that, by assumption, A is a derivation and D is invariant under each $T(t)$. In particular, D is invariant under Q. Since P is also strongly continuous and $s \mapsto P(s)h$ is differentiable for all $h \in D$, it follows from [EN00, Lem. B.16] that

$$\beta'(s) = -AT(t-s)[T(s)f \cdot T(s)g] + T(t-s)A[T(s)f \cdot T(s)g] = 0$$

for all $s \in [0,t]$. This shows that β is constant and thus that every $T(t)$ is multiplicative on D.

Since the multiplication with a fixed bounded function induces a bounded operator on $\mathrm{L}^p(\mathrm{X})$ and D is $\|\cdot\|_p$-dense in $\mathrm{L}^\infty(\mathrm{X})$, we can fix a function $g \in D$ and use a standard approximation argument to show that $T(f \cdot g) = T(t)f \cdot T(t)g$ for all $f \in \mathrm{L}^\infty(\mathrm{X})$ and $g \in D$. Fixing $f \in \mathrm{L}^\infty(\mathrm{X})$ and repeating the argument shows that $T(t)(f \cdot g) = T(t)f \cdot T(t)g$ for all $f, g \in \mathrm{L}^\infty(\mathrm{X})$, so $T(t)$ is multiplicative on all of $\mathrm{L}^\infty(\mathrm{X})$. Furthermore, $A\mathbb{1} = 0$ since A is a derivation, hence $T(t)\mathbb{1} = \mathbb{1}$ for all $t \ge 0$. Now Lemma 7.14 on page 99 yields that every $T(t)$ is a lattice homomorphism on $\mathrm{L}^\infty(\mathrm{X})$ and hence, by density and continuity, also on $\mathrm{L}^p(\mathrm{X})$. $\square$

Remark 7.17 Suppose that (iii) in Theorem 7.12 on page 98 is true without assuming $\mathbb{1} \in D(A)$. Then, as in the proof above, it still follows that each $T(t)$ is multiplicative on $\mathrm{L}^\infty(\mathrm{X})$ and hence maps characteristic functions to characteristic functions. From this, it follows that $T(t)\mathbb{1} = \mathbb{1}$ if $T(t)$ is isometric. In particular, then automatically $\mathbb{1} \in D(A)$ and $A\mathbb{1} = 0$.

As a corollary of Theorem 7.12 on page 98, we also obtain the following characterization of bi-Markov lattice semigroups.

Corollary 7.18 Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$. Then the following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is a bi-Markov lattice semigroup.

(ii) For every $t \geq 0$ there exists a measure algebra homomorphism $\theta_t \colon \Sigma(\mathrm{X}) \to \Sigma(\mathrm{X})$ such that $T(t)\mathbb{1}_M = \mathbb{1}_{\theta_t(M)}$ for all $M \in \Sigma(\mathrm{X})$.

(iii) The space $\mathrm{L}^\infty(\mathrm{X})$ is invariant under $(T(t))_{t\geq 0}$, the map $t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$ is locally bounded, A is a derivation on $D(A) \cap \mathrm{L}^\infty(\mathrm{X})$ and $A'\mathbb{1} = 0$.

Proof. For the equivalence of (i) and (ii), the reader is again referred to [EFHN15, Thm. 12.10]. For the equivalence of (i) and (iii), note that $A'\mathbb{1} = 0$ is equivalent to $T(t)'\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. Hence, (i) implies (iii) by Theorem 7.12 on page 98. For the converse, note as as in Remark 7.17 on page 102 that from (iii) it follows that each $T(t)$ maps characteristic functions to characteristic functions. Since $T(t)'\mathbb{1} = \mathbb{1}$, it follows that $\langle T(t)\mathbb{1}, \mathbb{1}\rangle = \langle \mathbb{1}, \mathbb{1}\rangle = \mu(X)$, and hence $T(t)\mathbb{1} = \mathbb{1}$ for each $t \geq 0$. Therefore, (iii) implies (i) by Theorem 7.12 on page 98. $\square$

Now we discuss strongly continuous lattice semigroups on $\mathrm{L}^p(\mathrm{X})$ that are not necessarily Markov. We show that their generator is a derivation perturbed by a bounded multiplication operator. If $f \in \mathrm{L}^\infty(\mathrm{X})$ is an essentially bounded function, M_f will denote its associated multiplication operator on $\mathrm{L}^p(\mathrm{X})$, $p \in [1, \infty]$.

Theorem 7.19 Let A be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$. Assume that $\mathbb{1} \in D(A)$ and $q := A\mathbb{1} \in \mathrm{L}^\infty(\mathrm{X})$. Then the following assertions are equivalent.

(i) $(S(t))_{t\geq 0}$ is a lattice semigroup.

(ii) The function q is real-valued and $A = B + q$ where B is the generator of a Markov lattice C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$.

If (ii) holds, then

$$S(t)f = \exp\left(\int_0^t T(s)q \,\mathrm{d}s\right) \cdot T(t)f \tag{7.1}$$

for all $t \geq 0$ and $f \in \mathrm{L}^p(\mathrm{X})$.

Proof. To show the equivalence of (i) and (ii), we first recall from [EN00, Thm. III.1.3] that $B \coloneqq A-q$ is a generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$ because B is a bounded perturbation of A. Since $B\mathbb{1} = 0$, $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. Now it follows from [Nag+86, Cor. C-II.5.8] (Kato's identity) that $(S(t))_{t\geq 0}$ is a lattice semigroup if and only if $D(A)$ is a sublattice of $\mathrm{L}^p(\mathrm{X})$ and

$$A|f| = \mathrm{Re}(\mathrm{sign}(f)Af)$$

for all $f \in D(A)$. Since (i) implies that q is real-valued, A satisfies this condition if and only if B does, which proves the equivalence of the assertions (i) and (ii). Now assume that (ii) holds. Since each $T(t)$ is multiplicative on $\mathrm{L}^\infty(\mathrm{X})$ by Lemma 7.14 on page 99, $T(t)\exp(g) = T(t)\sum_{n=0}^\infty \frac{g^n}{n!} = \exp(T(t)g)$ for each $g \in \mathrm{L}^\infty(\mathrm{X})$. Using this, one proves by induction that

$$\left(T(t)\mathrm{e}^{t\mathrm{M}_q}\right)^n = \mathrm{M}_{\exp\left(t\sum_{j=1}^n T(jt)q\right)} T(nt)$$

for each $n \geq 1$. Replace t by $\frac{t}{n}$ and note that

$$\sum_{j=1}^n \frac{t}{n} T\left(\frac{jt}{n}\right) q \xrightarrow[n\to\infty]{\|\cdot\|_p} \int_0^t T(s)q \,\mathrm{d}s.$$

By Lemma 7.15 on page 100, one obtains the convergence

$$\mathrm{M}_{\exp\left(\sum_{j=1}^n \frac{t}{n} T\left(\frac{jt}{n}\right)q\right)} \xrightarrow[n\to\infty]{} \mathrm{M}_{\exp\left(\int_0^t T(s)q\,\mathrm{d}s\right)}$$

of multiplication operators in the strong operator topology on $\mathcal{L}(\mathrm{L}^p(\mathrm{X}))$. By the Trotter product formula

$$S(t)f = \lim_{n\to\infty}\left[T\left(\frac{t}{n}\right)\exp\left(\frac{t}{n}\mathrm{M}_q\right)\right]^n f = \exp\left(\int_0^t T(s)q\,\mathrm{d}s\right)\cdot T(t)f$$

for all $f \in \mathrm{L}^p(\mathrm{X})$. $\square$

7.2 Markov lattice semigroups induced by measurable maps on standard probability spaces

As already noted, every Koopman semigroup on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$, is a Markov lattice semigroup but the converse is, in general, not true. This is due to the fact that mapping $\varphi \mapsto \varphi^*$ is not injective, cf. [EFHN15, Exm. 6.7]. However, it is true if X is a standard probability space (cf. [EFHN15, Def. 6.8]). For bi-Markov lattice homomorphisms, this is a classical theorem by von Neumann, cf. [EFHN15, Thm. 7.20]. Below, we extend this theorem to Markov lattice homomorphisms on L^p-spaces. We then relate this to results from the previous section.

We omit the proof of the following theorem which is due to our co-author N. Edeko and can be found in the joint article [EGK19, Thm. 2.1].

Theorem 7.20 Let $\mathrm{X} = (X, \Sigma_X, \mu_X)$ and $\mathrm{Y} = (Y, \Sigma_Y, \mu_Y)$ be standard probability spaces and $T\colon \mathrm{L}^p(\mathrm{X}) \to \mathrm{L}^p(\mathrm{Y})$, $1 \leq p \leq \infty$, a Markov lattice homomorphism (not necessarily bi-Markov). Then there is a measurable map $\varphi\colon Y \to X$ such that $T = T_\varphi$. If $\vartheta\colon Y \to X$ is another such map, then $\varphi = \vartheta$ μ_Y-almost everywhere.

Using this result we are able to extend Theorem 7.12 on page 98 in the following way.

Corollary 7.21 Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$, where $\mathrm{X} = (X, \Sigma, \mu)$ is a standard probability space. Then the equivalent assertions (i), (ii) and (iii) of Theorem 7.12 on page 98 are also equivalent to

(iv) There exists a family $(\varphi_t)_{t\geq 0}$ of measurable maps on X such that φ_t^{-1} maps null sets onto null sets and $T(t)f = f \circ \varphi_t$ for all $f \in \mathrm{L}^p(\mathrm{X})$ and $t \geq 0$, i.e., $(T(t))_{t\geq 0}$ is a Koopman semigroup.

Proof. If assertion (i) of Theorem 7.12 on page 98 holds, we obtain assertion (iv) by Theorem 7.20 above. Conversely, if (iv) holds, then every $T(t)$ is a Markov lattice homomorphism, thus assertion (i) holds. □

As a corollary we obtain the following for bi-Markov lattice semigroups.

Corollary 7.22 Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$, where $\mathrm{X} = (X, \Sigma, \mu)$ is a standard probability space. Then the equivalent assertions (i), (ii) and (iii) of Corollary 7.18 on page 103 are also equivalent to

(iv) There exists a family $(\varphi_t)_{t\geq 0}$ of measure-preserving maps on X such that $T(t)f = f \circ \varphi_t$ for all $f \in \mathrm{L}^p(\mathrm{X})$ and $t \geq 0$.

Proof. Assume (i) of Corollary 7.18 on page 103. Then Corollary 7.21 on page 105 shows that there are measurable maps $\varphi_t \colon X \to X$ such that $T(t)f = f \circ \varphi_t$. Moreover, for $M \in \Sigma$

$$\mu(\varphi_t^{-1}(M)) = \langle \mathbb{1}_{\varphi_t^{-1}(M)}, \mathbb{1}_X \rangle = \langle T(t)\mathbb{1}_M, \mathbb{1}_X \rangle = \langle \mathbb{1}_M, \mathbb{1}_X \rangle = \mu(M)$$

because each $T(t)$ is bi-Markov. Thus, each φ_t is measure-preserving. On the other hand, (iv) implies (i) because every Koopman operator induced by a measure-preserving map is bi-Markov. □

We have seen in Corollary 7.21 on page 105 that on a standard probability space $\mathrm{X} = (X, \Sigma, \mu)$ every strongly continuous Markov lattice semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathrm{X})$ is induced by a family $(\varphi_t)_{t\geq 0}$ of measurable maps on X such that each φ_t^{-1} maps null sets into null sets. Since $(T(t))_{t\geq 0}$ is a semigroup, one has $\varphi_0 = \mathrm{id}_X$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ almost everywhere, using the uniqueness in Theorem 7.20 on page 105. Note, however, that it can, in general, not be made into a semiflow (cf. Definition 8.1 on page 109 below) since the identity $\varphi_t \circ \varphi_s = \varphi_{t+s}$ might only be true outside of null-sets which depend on s and t.

We use Corollary 7.21 on page 105 to characterize lattice semigroups in the following way.

Corollary 7.23 Let A be the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on a space $\mathrm{L}^p(\mathrm{X})$, where X is a standard probability space and $1 \leq p < \infty$, such that $\mathbb{1} \in D(A)$ and $q := A\mathbb{1} \in \mathrm{L}^\infty(\mathrm{X})$. Then $(S(t))_{t\geq 0}$ is a lattice semigroup if and only if $q \in \mathrm{L}^\infty(\mathrm{X}; \mathbb{R})$ and there exists a family $(\varphi_t)_{t\geq 0}$ of measurable maps on X corresponding to a strongly continuous Koopman semigroup on $\mathrm{L}^p(\mathrm{X})$ such that

$$S(t)f = \exp\left(\int_0^t q \circ \varphi_s \,\mathrm{d}s\right) \cdot (f \circ \varphi_t) \tag{7.2}$$

for all $f \in \mathrm{L}^p(\mathrm{X})$ and $t \geq 0$.

Proof. If $(S(t))_{t\geq 0}$ is a lattice semigroup, it follows from Theorem 7.19 on page 103 that there exists a Markov lattice semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p(\mathrm{X})$ with generator $(A-q, D(A))$ such that (7.1) holds. The representation (7.2) hence follows from Corollary 7.21 on page 105. Conversely, every semigroup of the form (7.2) with real-valued q is a lattice semigroup. $\square$

The next lemma shows that for groups, the boundedness assumption in Theorem 7.12 on page 98 (iii) is superfluous. This allows us to recover [tEL15, Thm. 1.1] as Corollary 7.25. We omit the proof for Lemma 7.24 which again is due to N. Edeko and can be found in [EGK19, Lem. 2.5].

Lemma 7.24 Let X be a finite measure space and $(T(t))_{t>0}$ be a semigroup on $\mathrm{L}^\infty(\mathrm{X})$, strongly continuous with respect to $\|\cdot\|_p$ where $1 \leq p < \infty$. Then the mapping $t \mapsto \|T(t)\|_{\mathcal{L}(\mathrm{L}^\infty(\mathrm{X}))}$ is locally bounded.

Corollary 7.25 [tEL15, Thm. 1.1] Let A be the generator of a unitary C_0-group $(T(t))_{t\in\mathbb{R}}$ on $\mathrm{L}^2(\mathrm{X})$ where $\mathrm{X} = (X, \Sigma, \mu)$ is a standard probability space. Then the following assertions are equivalent.

(i) For every $t \in \mathbb{R}$ there exists an essentially invertible measurable and measure-preserving map $\varphi_t \colon X \to X$ such that $T(t)f = f \circ \varphi_t$ for all $f \in \mathrm{L}^2(\mathrm{X})$.

(ii) The space $\mathrm{L}^\infty(\mathrm{X})$ is invariant under $(T(t))_{t\geq 0}$ and A is a derivation on $D(A) \cap \mathrm{L}^\infty(\mathrm{X})$.

Proof. The implication (i) $\implies$ (ii) is a consequence of Corollary 7.18 on page 103. In order to prove the converse implication, we observe that it follows from Lemma 7.24 and Remark 7.17 on page 102 that A and $(T(t))_{t\geq 0}$ as well as $-A$ and $(T(-t))_{t\geq 0}$ fulfill condition (iii) in Theorem 7.12 on page 98. Corollary 7.21 on page 105 therefore shows that $T(t) = T_{\varphi_t}$ for measurable maps $\varphi_t \colon X \to X$ and $t \geq 0$. The essential invertibility of the maps φ_t follows from [EFHN15, Prop. 7.12] and [EFHN15, Cor. 7.21]. Also, since each $T(t)$ is unitary and a Markov operator, one shows as in Corollary 7.22 on page 106 that each φ_t is measure-preserving. $\square$

8 Topological model

In this chapter we show that every measurable and measure-preserving semiflow on a standard probability space is isomorphic to a continuous semiflow (cf. Definition 3.1 on page 19) on a compact space. More precisely, we show that a strongly continuous Markov lattice semigroup is always similar to a Koopman semigroup in the following way. We construct a compact space K and a Borel measure ν such that $\mathrm{L}^1(X, \Sigma, \mu)$ is isometrically Banach lattice isomorphic to $\mathrm{L}^1(K, \nu)$ and, via this isomorphism, the semigroup $(T(t))_{t \geq 0}$ is similar to a semigroup of Koopman operators on $\mathrm{L}^1(K, \nu)$ induced by a continuous semiflow $(\varphi_t)_{t \geq 0}$ on K. Furthermore, in case that the space $\mathrm{L}^1(X, \Sigma, \mu)$ is separable, we show that K can be chosen to be metrizable. Similar results have been already obtained for strongly continuous representations of locally compact groups on $\mathrm{L}^p(\mathrm{X})$ as bi-Markov embeddings, see [dJR17, Thm. 5.14].

We define the terms measurable and measure preserving semiflow as follows.

Definition 8.1 A family $(\varphi_t)_{t \geq 0}$ of measurable self-mappings on a finite measure space $\mathrm{X} = (X, \Sigma, \mu)$ is called *semiflow* if $\varphi_0 = \mathrm{id}_X$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \geq 0$ μ-almost everywhere. It is called *measurable semiflow* if the mapping

$$\begin{aligned} \varphi \colon [0, \infty) \times X &\to X\,, \\ (t, x) &\mapsto \varphi_t(x) \end{aligned}$$

is measurable.

A measurable semifow is called *measure-preserving* if φ_t is measure-preserving for all $t \geq 0$.

Remark 8.2 A measure preserving semiflow induces a Koopman semigroup (cf. Definition 7.6 on page 97) on each $\mathrm{L}^p(\mathrm{X})$, $1 \leq p < \infty$.

Example 8.3 Here, we give an example for a measurable semiflow that preserves null sets, but the induced Koopman semigroup is not strongly continuous on $\mathrm{L}^1(\mathrm{X})$. Consider the measure space

$$\mathrm{X} \coloneqq (\{0,1\}, \mathcal{P}(\{0,1\}), \frac{1}{2}(\delta_0 + \delta_1))$$

with

$$\varphi_0 = \mathrm{id}_{\{0,1\}} \text{ and } \varphi_t \equiv 0 \text{ for all } t > 0\,.$$

Then $(\varphi_t)_{t\geq 0}$ forms a measurable semiflow. The only $\mu \coloneqq \frac{1}{2}(\delta_0 + \delta_1)$-null set is $\emptyset$ and clearly $\mu(\varphi_t^{-1}(\emptyset)) = \mu(\emptyset) = 0$ for all $t \geq 0$.

The semiflow is not measure preserving since

$$\mu(\varphi_t^{-1}(\{0\})) = \mu(\{0,1\}) = 1\,,\ \text{for all } t > 0, \text{ but } \mu(\{0\}) = \frac{1}{2}\,.$$

Now for $f \in \mathrm{L}^1(\mathrm{X})$, $f \geq 0$, with $f(0) \neq f(1)$ and $t > 0$ we obtain

$$\begin{aligned}
\|T(t)f - f\|_1 &= \int |T(t)f - f| \mathrm{d}\mu \\
&= \frac{1}{2}\left(|f(\varphi_t(0)) - f(0)| + |f(\varphi_t(1)) - f(1)|\right) \\
&= \frac{1}{2}(f(0) - f(1)) \neq 0\,.
\end{aligned}$$

Hence, the induced Koopman semigroup is not strongly continuous on $\mathrm{L}^1(\mathrm{X})$.

Remark 8.4 If $(\varphi_t)_{t\geq 0}$ is measure preserving, then the associated Koopman semigroup is in fact strongly continuous on $\mathrm{L}^1(\mathrm{X})$. This result can be found even for measure preserving semiflows on σ-finite measure spaces in [Kre11, Chapt. 1, Thm. 6.13].

We now show that for measurable semiflows inducing a Koopman semigroup, one can construct a continuous semiflow on a compact metric space with a Borel probability measure such that the two semiflows are isomorphic in the sense of below Definition 8.5. This will be done by proving that every strongly continuous Markov lattice semigroup is similar to a Koopman semigroup induced by a continuous semiflow. We call a continuous semiflow (resp. Koopman semigroup) *topological model* for a measurable semiflow (resp. Markov lattice semigroup) if they are isomorphic (Markov similar). See also [EFHN15, Sect. 12.3] for this terminology and similar results in the time-discrete case.

Definition 8.5 Let $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{Y} = (Y, \Sigma', \mu')$ be finite measure spaces. We say that two measurable semiflows $(\varphi_t)_{t\geq 0}$ and $(\psi_t)_{t\geq 0}$ on X and Y are *isomorphic* if there is a measure-preserving and essentially invertible map $\rho\colon X \to Y$[1]such that

$$\psi_t \circ \rho = \rho \circ \varphi_t \text{ almost everywhere for all } t \geq 0\,.$$

We say that two Markov lattice semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathrm{X})$ and $\mathrm{L}^1(\mathrm{Y})$ are *Markov similar* if there is an invertible bi-Markov lattice homomorphism $\Phi\colon \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ such that

$$S(t) \circ \Phi = \Phi \circ T(t) \text{ for each } t \geq 0\,.$$

These notions are defined for flows and operator groups analogously.

The idea of the proof of the following result was kindly provided to us by Markus Haase. An analogous result was recently proved for bi-Markov lattice embedding representations of locally compact groups by de Jeu and Rozendaal, see [dJR17, Thm. 5.14]. For simplicity, we restrict ourselves to the case $p = 1$.

We call a measure space $\mathrm{X} = (X, \Sigma, \mu)$ *separable* if $\mathrm{L}^p(\mathrm{X})$ is separable for one (and hence all) $p \in [1, \infty)$. Sometimes we distinguish between a measurable function $f\colon X \to \mathbb{C}$ and its equivalence class with respect to a measure μ on X by writing $[f]_\mu$ for the equivalence class of f.

Theorem 8.6 (Topological Model) Let A be the generator of a strongly continuous Markov lattice semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathrm{X})$, where $\mathrm{X} = (X, \Sigma, \mu)$ is a finite measure space. Then there exist a compact space K, a continuous semiflow $(\psi_t)_{t\geq 0}$ on K and a strictly positive Borel probability measure ν such that the semiflow $(\psi_t)_{t\geq 0}$ induces a Koopman-semigroup on $\mathrm{L}^1(K, \nu)$ which is Markov similar to the semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathrm{X})$. The measure ν is $(\psi_t)_{t\geq 0}$-invariant if and only if $(T(t))_{t\geq 0}$ is a bi-Markov lattice semigroup.

Proof. Consider $\mathcal{A} := \{f \in \mathrm{L}^\infty(\mathrm{X})\colon s \mapsto T(s)f \text{ is } \|\cdot\|_\infty\text{-continuous}\}$. Since each operator $T(t)$ is contractive on $\mathrm{L}^\infty(\mathrm{X})$ and multiplicative by Lemma 7.14 on page 99, $\mathcal{A}$ is an algebra and clearly $\mathbb{1} \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is closed with respect to $\|\cdot\|_\infty$ and closed under conjugation. Therefore, $\mathcal{A}$ is a commutative C^*-algebra invariant under $(T(t))_{t\geq 0}$.

[1]Let $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{Y} = (Y, \Sigma', \mu')$ be finite measure spaces. A measure preserving map $\rho\colon X \to Y$ is called *essentially invertible* if there exists a measurable map $\theta\colon Y \to X$ such that $\theta \circ \rho = \mathrm{id}_X$ μ-a.e. and $\rho \circ \theta = \mathrm{id}_Y$ μ'-a.e.

We show that $\mathcal{A}$ is dense in $\mathrm{L}^1(\mathrm{X})$. The strong continuity of $(T(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathrm{X})$ implies that $\|\cdot\|_1$-$\lim_{t\searrow 0}\frac{1}{t}\int_0^t T(r)f\,\mathrm{d}r = f$ for each $f\in\mathrm{L}^\infty(\mathrm{X})$. Therefore, it suffices to show that $\int_0^t T(r)f\,\mathrm{d}r\in\mathcal{A}$ for $f\in\mathrm{L}^\infty(\mathrm{X})$. For all $0\leq s\leq t$ and $f\in\mathrm{L}^\infty(\mathrm{X})$

$$\begin{aligned}\left|T(s)\int_0^t T(r)f\,\mathrm{d}r-\int_0^t T(r)f\,\mathrm{d}r\right| &= \left|\int_0^t T(s+r)f\,\mathrm{d}r-\int_0^t T(r)f\,\mathrm{d}r\right|\\ &= \left|\int_s^{t+s} T(r)f\,\mathrm{d}r-\int_0^t T(r)f\,\mathrm{d}r\right|\\ &\leq \left|\int_t^{t+s} T(r)f\,\mathrm{d}r\right|+\left|\int_0^s T(r)f\,\mathrm{d}r\right|\\ &\leq 2s\|f\|_\infty \mathbb{1}\end{aligned}$$

since each $T(t)$ is $\|\cdot\|_\infty$-contractive. This shows that $s\mapsto T(s)\int_0^t T(r)f\,\mathrm{d}r$ is continuous at zero and hence on $[0,\infty)$ with respect to $\|\cdot\|_\infty$. Therefore, $\mathcal{A}$ is dense in $\mathrm{L}^1(\mathrm{X})$.

By a combination of the Gelfand-Naimark theorem and the Riesz representation theorem as in [EFHN15, Sect. 12.3] or [dJR17, Thm. 5.14] one obtains a compact space K, a $*$-isomorphism $\Phi\colon\mathcal{A}\to\mathrm{C}(K)$ with $\Phi\mathbb{1}=\mathbb{1}$, a unique probability measure ν on K such that

$$\int_X \Phi^{-1}g\,\mathrm{d}\mu=\int_K g\,\mathrm{d}\nu$$

for all $g\in\mathrm{C}(K)$ and a semiflow semiflow $(\psi_t)_{t\geq 0}$ on K such that $T(t)|_{\mathcal{A}}=\Phi^{-1}\circ T_{\psi_t}\circ\Phi$. By [EFHN15, Thm. 4.17], the semiflow ψ is continuous, cf. also [Nag+86, Thm. B-II.3.4].

Moreover, Φ extends to a bi-Markov lattice homomorphism $\Phi\colon\mathrm{L}^1(\mathrm{X})\to\mathrm{L}^1(K,\nu)$. Let $(S(t))_{t\geq 0}$ denote the semigroup $(T(t))_{t\geq 0}$ induces on $\mathrm{L}^1(K,\nu)$ via Φ. Then

$$S(t)[f]_\nu=[f\circ\psi_t]_\nu \tag{8.1}$$

for all continuous functions $f\in\mathrm{C}(K)$. By a standard approximation argument, this holds for all bounded, Baire-measurable functions, cf. [EFHN15, Thm. E.1]. Via monotone approximation, (8.1) extends to all positive integrable functions and is hence valid for all $[f]_\nu\in\mathrm{L}^1(K,\nu)$. Finally, $(T(t))_{t\geq 0}$

is bi-Markov if and only if $(S(t))_{t\geq 0}$ is, i.e.,

$$\int_K f\,\mathrm{d}\nu = \int_K S(t)f\,\mathrm{d}\nu = \int_K f\circ\psi_t\,\mathrm{d}\nu$$

for all $f\in \mathrm{L}^1(K,\nu)$. This holds if and only if $(\psi_t)_{t\geq 0}$ preserves ν. □

Proposition 8.7 If, in the situation of Theorem 8.6 on page 111, the measure space is separable, then the compact space K can be chosen to be metrizable.

PROOF. Let $\mathcal{A}$ be the algebra

$$\mathcal{A} \coloneqq \{f\in \mathrm{L}^\infty(\mathrm{X})\colon s\mapsto T(s)f \text{ is } \|\cdot\|_\infty\text{-continuous}\}$$

from the proof of Theorem 8.6 on page 111. Since $\mathrm{L}^1(\mathrm{X})$ is separable and $\mathcal{A}$ is dense in the former, there is a countable dense subset D_0 of $\mathcal{A}$. We set

$$D\coloneqq \{T(t)f\colon f\in D_0, t\in\mathbb{Q}_+\}\subset\mathcal{A}$$

and denote by $\mathcal{A}_0$ the C*-subalgebra of $\mathcal{A}$ generated by D. The algebra $\mathcal{A}_0$ is then separable, dense in $\mathrm{L}^1(\mathrm{X})$ and since $\mathcal{A}_0\subset\mathcal{A}$, $T(t)\mathcal{A}_0\subset\mathcal{A}_0$ for not only $t\in\mathbb{Q}_+$ but $t\in\mathbb{R}_+$. To complete the proof, one can now proceed as in the proof of Theorem 8.6 on page 111 with $\mathcal{A}$ replaced by $\mathcal{A}_0$, obtaining a compact representation space which is metrizable because $\mathrm{C}(K)$ is separable.

□

Remark 8.8 With slight notational adjustments, the proofs of the previous two results also work for Markov lattice groups and continuous flows.

Corollary 8.9 Let $\mathrm{X}=(X,\Sigma,\mu)$ be a standard probability space and $(\varphi_t)_{t\in\mathbb{R}}$ a measurable and measure-preserving flow on X. Then there are a compact metric space K, a continuous flow $(\psi_t)_{t\in\mathbb{R}}$ on K and a strictly positive $(\psi_t)_{t\in\mathbb{R}}$-invariant Borel probability measure ν on K so that the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ are isomorphic.

PROOF. By Remark 8.4 on page 110 and Theorem 7.12 on page 98, the flow $(\varphi_t)_{t\in\mathbb{R}}$ induces a bi-Markov group on $\mathrm{L}^1(\mathrm{X})$ that is strongly continuous and so Remark 8.8 shows that there are a compact metric space K, a continuous flow $(\psi_t)_{t\in\mathbb{R}}$ and a strictly positive $(\psi_t)_{t\in\mathbb{R}}$-invariant probability measure ν on K such that the groups $(T(t))_{t\in\mathbb{R}}$ and $(S(t))_{t\in\mathbb{R}}$ induced by the flows $(\varphi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ are Markov similar via an invertible bi-Markov lattice homomorphism Φ. Applying von Neumann's theorem shows that there is a measurable and measure-preserving map $\rho\colon Y \to X$ such that $\Phi = T_\rho$ and ρ is essentially invertible because Φ is invertible, see [EFHN15, Cor. 7.21]. The identity $\Phi \circ T(t) = S(t) \circ \Phi$ now shows that $\varphi_t \circ \rho = \rho \circ \psi_t$ ν-almost everywhere, see [EFHN15, Prop. 7.19]. $\square$

Remark 8.10 Corollary 8.9 on page 113 is similar to [AK42, Thm. 5] but there are two important differences: On the one hand, the authors of [AK42] work with a slightly stronger notion of isomorphism of flows. On the other hand, the models considered in [AK42] need not be compact.

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