Operators with dynamic boundary conditions and Dirichlet-to-Neumann operators

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Für meine Eltern

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[1] Das gemeinsame Preprint [BK20] ist nicht Teil dieser Arbeit.

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Summary

Deutsch

In der vorliegenden Arbeit wird eine Theorie für abstrakte Operatoren mit dynamischen Randbedingungen, welche in [Gre87], [CENN03], [Eng03], [EF05] entwickelt wurde, erweitert. Die Hauptidee ist es einen abstrakten Rahmen einzuführen, um dynamische Randbedingungen zu beschreiben. Dieser Ansatz erlaubt eine systematische Untersuchung dieser Operatoren.

Die Arbeit ist in zwei Teile unterteilt. Im ersten Teil studieren wir den abstrakten Rahmen und zeigen, dass er das perfekte Werkzeug ist, um Operatoren mit dynamischen Randbedingungen zu untersuchen. Im zweiten Teil konzentrieren wir uns auf konkrete Probleme mit dynamischen Randbedingungen.

Teil I basiert auf gemeinsamen Artikeln mit Klaus Engel und wird zusammengefasst in den Kapiteln III.1-III.7 wiedergegeben.
Unser erster Artikel [BE19] konzentriert sich Operatoren mit dynamischen Randbedingungen, welche analytische Halbgruppen erzeugen. Es wird gezeigt, dass unter sinnvollen Annahmen ein Problem mit dynamischen Randbedingungen in ein inneres und in ein Randproblem, welches durch den Dirichlet-zu-Neumann Operator beschrieben wird, entkoppelt werden kann. Dies wird verwendet, um eine Störungstheorie für Operatoren mit dynamischen Randbedingungen zu entwickeln.
Unser zweiter Artikel [BE20a] untersucht, wie abstrakte, homogene und inhomogene, elliptische und parabolische Problemen mittels Dirichlet-zu-Neumann Operatoren und Operatoren mit dynamischen Randbedingungen formuliert werden können. Außerdem werden Resultate für stark-stetige Halbgruppen analog zum Entkoppelungsresultat für analytische Halbgruppen bewiesen.
Unser letztes Manuskript [BE20b] untersucht, welche Eigenschaften bei der Entkoppelungsprozedur erhalten bleiben. Insbesondere untersuchen wir diesbezüglich Positivität, Stabilität und Spektraleigenschaften von Operatoren mit dynamischen Randbedingungen und Dirichlet-zu-Neumann Operatoren.

Die von mir allein geschriebenen Artikel [Bin19], [Bin20a], [Bin20b] beschäftigen sich mit konkreten Systemen von parabolischer Differentialgleichungen auf

Räumen stetiger Funktionen auf Mannigfaltigkeiten mit Rand.
Der erste Artikel [Bin20a] befasst sich mit elliptischen Operatoren mit Dirichlet Randbedingungen auf Mannigfaltigkeiten mit Rand. Es wird gezeigt, dass diese sektoriell mit optimalem Winkel $\frac{\pi}{2}$ sind und kompakte Resolventen haben. Dieses Resultat verallgemeinern die bekannten Resultate für beschränkte Gebiete auf Mannigfaltigkeiten mit Rand und spielt eine wichtige Rolle in meinen weiteren Artikeln [Bin19], [Bin20b].
In dem Artikel [Bin20b] werden elliptische Operatoren auf Mannigfaltigkeiten mit Rand mit dynamischen Randbedingungen mit einem zusätzlichen Driftterm betrachtet. Unter Verwendung unserer abstrakten Theorie wird gezeigt, dass auch solche Operatoren kompakte und analytische Halbgruppen mit Winkel $\frac{\pi}{2}$ erzeugen.
In dem dritten Manuskript [Bin19] werden elliptische Operatoren mit Wentzell Randbedingungen und Dirichlet-zu-Neumann Operatoren auf Mannigfaltigkeiten mit Rand betrachtet. Es wird zunächst gezeigt, dass der Dirichlet-zu-Neumann Operator eine kompakte und analytische Halbgruppe mit Winkel $\frac{\pi}{2}$ erzeugt. Weiter wird dieses Resultat mit der abstrakten Theorie kombiniert, um ein Generatorresultat für elliptische Operatoren mit Wentzell Randbedingungen zu zeigen. Insbesondere wird bewiesen, dass elliptische Operatoren mit Wentzell Randbedingungen auf glatten, beschränkten Gebieten kompakte und analytische Halbgruppen mit optimalen Winkel $\frac{\pi}{2}$ auf Räumen stetiger Funktionen erzeugen.

In der gemeinsamen Arbeit mit Tom ter Elst [BtE20] wird das Generatorresultat für elliptische Operatoren mit Wentzell Randbedingungen auf Operatoren mit weniger regulären Koeffizienten und auf weniger regulären Gebieten verallgemeinert. Zusätzlich wird unter diesen schwachen Voraussetzungen ein analoges Resultat für Operatoren auf L^p-Räumen bewiesen.

Das Manuskript mit Jonas Lampart [BL20] befasst sich mit inneren Randbedingungen. Diese Randbedingungen werden in der Quantenmechanik zur Beschreibung von Teilchenerzeugung und -vernichtung verwendet. Für diese wird ein abstrakter Rahmen entwickelt und Selbstadjungiertheit charakterisiert. Weiter werden Klassifikations- und Konvergenzresultate gezeigt.

English

In this thesis we extend the theory of abstract operators with dynamic boundary conditions proposed in [Gre87], [CENN03], [Eng03], [EF05]. The main idea is to introduce an abstract framework to study dynamic boundary conditions for partial differential operators. This approach allows a systematic investigation of these operators.

This thesis is divided into two parts. In the first part we study the abstract framework systematically and show that it is the perfect tool to examine operators with dynamic boundary conditions. In the second part we concentrate on concrete problems with dynamic boundary conditions.

The abstract part is based on articles of Klaus Engel and myself and summarized in Sections III.1-III.7.
Our first article [BE19] concentrates on analytic semigroups generated by operators with dynamic boundary conditions. It is shown that under sensible assumptions a problem with dynamic boundary conditions can be decoupled into a interior problem and a boundary problem for the so called Dirichlet-to-Neumann operator. Further, this result is used to develop a perturbation theory for operator with dynamic boundary conditions.
Our second article [BE20a] investigates the relationship between abstract, homogeneous and inhomogeneous, elliptic and parabolic problems and Dirichlet-to-Neumann operators and operators with dynamic boundary conditions. Moreover, we prove results for strongly continuous semigroups analogous to the decoupling theorem for analytic semigroups.
Our last manuscript [BE20b] studies which properties can be characterized by our decoupling procedure. In particular we investigate positivity, stability and spectral properties of operators with dynamic boundary conditions and of Dirichlet-to-Neumann operators.

The articles written by myself [Bin19], [Bin20a], [Bin20b] deal with concrete systems of parabolic differential equations on spaces of continuous functions on manifolds with boundary.
The first article [Bin20a] is concerned with elliptic operators with Dirichlet boundary conditions on manifolds with boundary. It shows that these operators are sectorial of optimal angle $\frac{\pi}{2}$ and have compact resolvents. This result extends the known results for bounded domains. It plays a crucial role in later articles [Bin19], [Bin20b].
In the second article [Bin20b] elliptic operators on manifolds with boundary with dynamic boundary conditions with an additional drift term at the boundary

are considered. By applying our abstract theory it is shown that such operators generate compact and analytic semigroups of angle $\frac{\pi}{2}$.
In the manuscript [Bin19] elliptic operators with Wentzell boundary conditions and Dirichlet-to-Neumann operators on manifolds with boundary are considered. We show generation of a compact and analytic semigroup of angle $\frac{\pi}{2}$ by the Dirichlet-to-Neumann operator. Further, this result is combined with the abstract theory to obtain a generation result for elliptic operators with Wentzell boundary conditions. In particular we prove that elliptic operators with Wentzell boundary conditions generate compact and analytic semigroup of optimal angle $\frac{\pi}{2}$ on spaces of continuous functions on smooth, bounded domains in $\mathbb{R}^n$.

In the joint work with Tom ter Elst [BtE20] the generation result for elliptic operators with Wentzell boundary conditions on domains is generalized to less regular coefficients of the operator and less regular domains. In addition an analogous result on L^p-spaces in proven.

The manuscript with Jonas Lampart [BL20] is concerned with interior boundary conditions. These boundary conditions are used to describe particle creation and annihilation in quantum mechanics. An abstract framework is developed and a self-adjointness theorem is proven. Further classification and convergence results are obtained.

List of Publications

Accepted Manuscripts

Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann operator by Tim Binz and Klaus-Jochen Engel, published in *Mathematische Nachrichten*, 2019, volume 292, pages 733–746 and cited as [BE19] in this thesis. The peer reviewed version accepted for publication is contained in Appendix A.1.1 and is available online at: `https://doi.org/10.1002/mana.201800064`

Strictly elliptic operators with Dirichlet boundary conditions on spaces of continuous functions on manifolds by Tim Binz, published in *Journal of Evolution Equations*, volume 20, pages 1005-1028 and cited as [Bin20a] in this thesis. The peer reviewed version accepted for publication is contained in Appendix A.1.2 and is available online at: `https://doi.org/10.1007/s00028-019-00548-y`

Strictly elliptic Operators with generalized Wentzell boundary conditions on spaces of continuous functions on manifolds by Tim Binz, published in *Archiv der Mathematik*, volume 115, pages 111-120 and cited as [Bin20b] in this thesis. The peer reviewed version accepted for publication is contained in Appendix A.1.3 and is available online at: `https://doi.org/10.1007/s00013-020-01457-0`

First order evolution equations with dynamic boundary conditions by Tim Binz and Klaus-Jochen Engel, will be published in *Philosophical Transactions A*, 2020, and cited as [BE20a] in this thesis. The final version accepted for publication is contained in Appendix A.1.4.

Submitted Manuscripts

Analytic Semigroups generated by Dirichlet-to-Neumann operators on manifolds by Tim Binz, submitted to *Semigroup Forum*, and cited as [Bin19] in this thesis. The submitted version is contained in Appendix A.2.1.

Dynamic boundary conditions for divergence form operators with Hölder coefficients by Tim Binz and Tom ter Elst, submitted to *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, and cited as [BtE20] in this thesis. The submitted version is contained in Appendix A.2.2.

Additional Manuscripts

Spectral theory, positivity and stability for operators with Wentzell boundary conditions and the associated Dirichlet-to-Neumann operators by Tim Binz and Klaus-Jochen Engel, in preparation, and cited as [BE20b] in this thesis. The preliminary version is contained in Appendix A.3.1.

An abstract framework for interior boundary conditions by Tim Binz and Jonas Lampart, in preparation, and cited as [BL20] in this thesis. The preliminary version is contained in Appendix A.3.2.

Personal Contribution

Accepted Manuscripts

The articles *Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann operator* [BE19] and *First order evolution equations with dynamic boundary conditions* [BE20a] are joint work with Klaus-Jochen Engel. All results were discussed, proven and formulated in cooperation.

For both articles I contributed 50 % of the scientific ideas and 50 % of the writing.

The articles *Strictly elliptic operators with Dirichlet boundary conditions on spaces of continuous functions on manifolds* [Bin20a] and *Strictly elliptic operators with generalized Wentzell boundary conditions on continuous functions on manifolds with boundary* [Bin20b] have been written by myself.

For both articles I contributed 100 % of the scientific ideas and 100 % of the writing.

Submitted Manuscripts

The article *Analytic Semigroups generated by Dirichlet-to-Neumann operators on manifolds* [Bin19] has been written by myself.

I contributed 100 % of the scientific ideas and 100 % of the writing of this article.

The article *Dynamic boundary conditions for divergence form operators with Hölder coefficients* [BtE20] is joint work with Tom ter Elst. All results were discussed, proven and formulated in cooperation.

I contributed 50 % of the scientific ideas and 50 % of the writing of this article.

Additional Manuscripts

The article *Spectral theory, positivity and stability for operators with Wentzell boundary conditions and the associated Dirichlet-to-Neumann operators* [BE20b] is joint work with Klaus-Jochen Engel. All results were discussed, proven and formulated in cooperation.

I contributed 50 % of the scientific ideas and 50 % of the writing of this article.

The article *An abstract framework for interior boundary conditions* is joint work with Jonas Lampart. The concept of this article was developed by Jonas Lampart and myself during two stays at Ludwig Maximilians University München. All results were discussed, proven and formulated in cooperation.

I contributed 50 % of the scientific ideas and 50 % of the writing of this article.

I Introduction

In his *Cours d' Analyse* in 1821 Augustin-Louis Cauchy posed the following problem[1]:

> Déterminer la fonction $\varphi(x)$ de manière qu'elle reste continue entre deux limites réelles quelconques de la variable x, et que l'on ait pour toutes les valeurs réelles des variables x et y
>
> $$\varphi(x+y) = \varphi(x)\varphi(y).^2$$
>
> A.-L. Cauchy, [Cau21, p. 100]

If we restrict ourself to functions $\varphi\colon \mathbb{R}_+ \to \mathbb{C}$ the exponential functions

$$t \mapsto \exp(ta)$$

for some $a \in \mathbb{C}$ satisfy the above functional equation and it turns out that there are no other continuous solutions[3]. Moreover the exponential functions are continuously differentiable and $u(t) = \exp(ta)u_0$ is the unique solution of the differential equation

$$\begin{cases} \dot{u}(t) = a \cdot u(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}$$

The same holds if we replace $\mathbb{C}$ by an arbitrary Banach space E and $a \in \mathbb{C}$ by a bounded linear operator $A \in \mathcal{L}(E)$. Indeed, the exponential functions defined as

$$\exp(tA) \coloneqq \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

[1]The following motivation is due to [EN00, Section I.1]. We refer to [EN00] for a more detailed historical introduction to semigroup theory.

[2]Determine the function $\varphi(x)$ in such a way that it remains continuous between two arbitrary real limits of the variable x, and that, for all real values of the variables x and y, one has

$$\varphi(x+y) = \varphi(x)\varphi(y).$$

[3]Considering $\mathbb{R}$ as a vector space over $\mathbb{Q}$ it is possible to find other solutions of this problem. We refer to [EN00, Comment I.1.5(iii)] and [Ham05].

are the unique solutions of Cauchy's Banach space valued problem and for $u_0 \in E$ the function $\exp(tA)u_0$ is again the unique solution of the initial value problem

$$\begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}$$

An interesting class of such problems occurs for differential operators on function spaces. Unfortunately, these operators are usually unbounded and hence the exponential function via the power series does not exist. Nevertheless the correspondence between Cauchy's problem and the initial value problem remains true for unbounded operators: For a closed, densely defined operator $A\colon D(A) \subset E \to E$ on a Banach space E every solution of the initial value problem

$$\begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}$$

for $u_0 \in D(A)$ satisfies Cauchy's functional equation. So the idea is to use this functional equation to define a family of bounded linear operators $(T(t))_{t\geq 0}$ yielding solutions of the initial value problem. However, besides this algebraic we also need some continuity property. The continuity of the map $\mathbb{R}_+ \to \mathcal{L}(E)\colon t \mapsto T(t)$ with respect to the uniform operator topology on $\mathcal{L}(E)$ is too strong for interesting examples. To find the „right" topological condition we should describe how the solutions $u(t) = T(t)u_0$ depend on its initial value $u_0 \in D(A)$.
In 1902 Jacques Hadamard[4] suggested that initial value problems modelling physical phenomena[5] should satisfy the following properties:

(i) a solution exists;

(ii) the solution is unique;

(iii) the solution depends continuously on the initial value.

He called such a problem *wellposed*. The third property corresponds to continuity of the maps $\mathbb{R}_+ \to \mathcal{L}(E)\colon t \mapsto T(t)u_0$ for initial values $u_0 \in E$. This leads to the theory of strongly continuous operator semigroups mainly established by Einar Hille [Hil42], [Hil48], [Hil50], [Hil65] and Kōsaku Yosida [Yos48], [Yos49], [Yos57], [Yos65].

[4] Had02, p. 49-52.

[5] This models *determinism*. For the relationship between semigroups, wellposedness and determinism we refer to [EN00, Epilogue].

Definition. A *strongly continuous semigroup* $(T(t))_{t\geq 0}$ on a Banach space E is a family of bounded linear operators satisfying

(i) $T(s+t) = T(s)T(t)$ for $s, t \geq 0$;

(ii) $T(0) = \mathrm{Id}$;

(iii) $\mathbb{R}_+ \to E\colon t \mapsto T(t)x$ is continuous for all $x \in E$.

In the next part we will see that such semigroups are a perfect tool to investigate wellposedness of initial value problems.

Abstract Cauchy problems

For a closed, densely defined operator $A\colon D(A) \subset X \to X$ on a Banach space X and an initial value $u_0 \in X$ the *abstract Cauchy problem* associated to A and u_0 is the initial value problem

$$\text{(ACP)} \quad \begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}$$

We call a continuously differentiable function $u\colon \mathbb{R}_+ \to X$ a *(classical) solution* of (ACP) if $u(t) \in D(A)$ for all $t \geq 0$ and it satisfies (ACP). Moreover we call a continuous function $u\colon \mathbb{R}_+ \to X$ a *mild solution* of (ACP) if $\int_0^t u(s)\,\mathrm{d}s \in D(A)$ for all $t \geq 0$ and it fulfils the integral equation

$$u(t) = u_0 + A\int_0^t u(s)\,\mathrm{d}s$$

for all $t \geq 0$. A mild solution is a classical solution if and only if it is continuously differentiable. The abstract Cauchy problem (ACP) is called *wellposed*[6] if for all $u_0 \in D(A)$ there exists a unique (classical) solution u of (ACP) which depends continuously on the initial value u_0, i. e., $u_0^n \to u_0$ implies $u^n(t) \to u(t)$ uniformly on compact intervals $[0, t_0]$. Moreover it is called *mildly wellposed* if for all $u_0 \in X$ there exists a unique mild solution u of (ACP). These two definitions are deeply connected to strongly continuous semigroups as the following theorem shows (see [ABHN11, Theorem 3.1.12] and [EN00, Theorem II.6.7]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be a closed, densely defined operator on a Banach space. Then the following statements are equivalent.*

[6] Our definition of wellposedness is not the only possible. For a discussion of wellposedness we refer to [EN00, Chapt. II.6] and the references therein.

(a) *the operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E.*

(b) *The abstract Cauchy problem* (ACP) *is mildly wellposed.*

(c) *The abstract Cauchy problem* (ACP) *is wellposed.*

Moreover, if one of the assertions holds, the mild solution is given by $u(t) = T(t)u_0$ for the initial value $u_0 \in E$. It is a classical solution if and only if $u_0 \in D(A)$.

Based on this theorem, strongly continuous semigroups are the perfect tool for the analysis of abstract and concrete Cauchy problems. Not only for uniqueness and existence of the solution but also, since the solution is governed by the semigroup, for their qualitative behaviour. For example, analyticity of the semigroup (see [EN00, Definition II.4.5]) reflects the analytic dependency of the solution u from its initial value u_0.

Now consider inhomogeneous abstract Cauchy problems. For a closed, densely defined operator $A\colon D(A) \subset E \to E$ on a Banach space E, an initial value $u_0 \in E$ and an inhomogeneity $f\colon [0,\tau] \to E$ for $\tau \in (0,\infty)$ or $\tau = \infty$ the *inhomogeneous abstract Cauchy problem* associated to A, u_0 and f is the problem

$$(\mathrm{ACP}_f) \quad \begin{cases} \dot{u}(t) = Au(t) + f(t) & \text{for } t \in [0,\tau], \\ u(0) = u_0. \end{cases}$$

If f is continuous and integrable, we call a continuously differentiable function $u\colon [0,\tau] \to E$ a *(classical) solution* of (ACP_f) if $u(t) \in D(A)$ for all $t \in [0,\tau]$ and it fulfils (ACP_f). Moreover, if f is integrable, we call a continuous function $u\colon [0,\tau] \to E$ a *mild solution* of (ACP_f) if $\int_0^t u(s)\,\mathrm{d}s \in D(A)$ for all $t \in [0,\tau]$ and it satisfies the *variation of the parameter formula*

$$u(t) = u_0 + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s$$

for all $t \in [0,\tau]$. Note that a mild solution is unique if it exists. If f is continuous and integrable, a mild solution is a classical solution if and only if it is continuously differentiable. If $f = 0$, the definitions for classical and mild solutions for the inhomogeneous abstract Cauchy problem coincide with the corresponding definitions for the abstract Cauchy problem. Again we can use semigroups to characterize solutions of the inhomogeneous abstract Cauchy problem (see [ABHN11, Proposition 3.1.16 and Corollary 3.1.17]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be a closed, densely defined operator on a Banach space E. Then the following properties are equivalent.*

(a) *The operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E.*

(b) *For every integrable function $f\colon [0,\tau] \to E$ and for every initial value $u_0 \in E$ the problem (ACP_f) has a unique mild solution.*

(c) *For every integrable $g\colon [0,\tau] \to E$, $f_0 \in E$ and every initial value $u_0 \in D(A)$ the problem (ACP_f) for f given by $f(t) := f_0 + \int_0^t g(s)\,ds$ has a unique (classical) solution.*

Moreover, if one of these assertions holds, the mild solution is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,\mathrm{d}s$$

for all $t \in [0,\tau]$.

Assertion (c) shows that for regular inhomogeneities and good initial values the solution becomes classic.

The next step is to consider semilinear equations. For a closed, densely defined operator $A\colon D(A) \subset E \to E$ on a Banach space X, an initial value $u_0 \in E$ and a nonlinearity $F\colon [t_0,\tau] \times E \to E$ for $t_0 < \tau \in (0,\infty)$ or $\tau = \infty$ the *semilinear abstract Cauchy problem* associated to A, u_0 and F is the problem

$$(\mathrm{sACP}_F) \quad \begin{cases} \dot{u}(t) = Au(t) + F(t,u(t)) & \text{for } t \in [t_0,\tau], \\ u(0) = u_0. \end{cases}$$

If F is continuous and integrable, we call a continuously differentiable function $u\colon [t_0,\tau] \to E$ a *(classical) solution* of (sACP_F) if $u(t) \in D(A)$ for all $t \in [t_0,\tau]$ and it fulfils (sACP_F). Moreover, if F is integrable, we call a continuous function $u\colon [t_0,\tau] \to X$ a *mild solution* of (ACP_f) if $\int_{t_0}^t u(s)\,\mathrm{d}s \in D(A)$ for all $t \in [t_0,\tau]$ and it satisfies the *variation of the parameter formula*

$$u(t) = u_0 + A\int_{t_0}^t u(s)\,\mathrm{d}s + \int_0^t F(s,u(s))\,\mathrm{d}s$$

for all $t \in [t_0,\tau]$. The result for (ACP_f) implies that a mild solution is unique if it exists. If F is continuous and integrable, a mild solution is a classical solution if and only if it is continuously differentiable. If the semilinearity F does not depend on u, these definitions coincide with the corresponding definitions for the inhomogeneous abstract Cauchy problem. Note that by $f(t) := F(t, u(t))$ we can formally write (sACP$_F$) as (ACP$_f$). Hence, combining the results for inhomogeneous abstract Cauchy problems with fixed point theorems, we obtain the following characterization (see [Paz83, Theorem 6.1.2 and Theorem 6.1.5]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be a closed, densely defined operator on a Banach space E. Then the following statements are equivalent.*

(a) *The operator A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E.*

(b) *For every function $F\colon [0,\tau] \times E \to E$ which is continuous in $[0,\tau]$ and uniformly Lipschitz continuous on E and for every initial value $u_0 \in E$ the problem* (sACP$_F$) *has a unique mild solution.*

(c) *For every continuously differentiable $F\colon [0,\tau] \times E \to E$ and every initial value $u_0 \in D(A)$ the problem* (sACP$_F$) *has a unique (classical) solution.*

Moreover, if one of the assertions holds, the mild solution satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s, u(s))\,\mathrm{d}s$$

for all $t \in [0, \tau]$.

Assertion (b) can be weakened to locally Lipschitz continuity yielding maximal mild solutions. For details see [Paz83, Theorem 6.1.4]. There are also other conditions on F that imply existence of a unique solution of (sACP$_F$). In particular, assuming additional properties of the semigroup only less assumptions for the semilinearity F are needed.
For compact semigroups we obtain the following local existence theorem (see [Paz83, Theorem 6.2.1]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be the generator of a compact, strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X. For every open subset $U \subset E$ and every continuous function $F\colon [0,\tau) \times U \to E$ and for every initial value $u_0 \in U$ there exists a constant $\tau_1 < \tau$ such that* (sACP$_F$) *has a mild solution $u \in \mathrm{C}([0,\tau_1], U)$ satisfying*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s, u(s))\,\mathrm{d}s$$

for all $t \in [0, \tau]$.

Further, there is a global existence result (see [Paz83, Corollary 6.2.3]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be the generator of a compact, strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E. Further, let $F\colon [0,\infty) \times E \to E$ be a continuous function which maps bounded set from $[0,\infty) \times E$ to bounded sets of E. Assume one of the following conditions.*

(a) *There exists a continuous function $K\colon [0,\infty) \to (0,\infty)$ such that the solution u satisfies $\|u(t)\| \leq K(t)$ for all $t \geq 0$.*

(b) *There exist two locally integrable functions $k_1\colon [0,\infty) \to [0,\infty)$ and $k_2\colon [0,\infty) \to [0,\infty)$ such that*

$$\|F(s,x)\| \leq k_1(s) \cdot \|x\| + k_2(s)$$

for all $s \geq 0$ and $x \in E$.

Then the initial value problem (sACP$_F$) *admits a global mild solution u given as*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s,u(s))\,\mathrm{d}s$$

for all $t \geq 0$.

Note that in both theorems we only obtain existence of a solution but no uniqueness. For analytic semigroups we make an additional assumption on F.

Assumptions (F). *Let $\alpha \in (0,1)$ and V an open subset of $\mathbb{R}_+ \times E_\alpha$. The function $F\colon V \to E$ satisfies the* assumption (F) *if for every pair $(t,x) \in V$ there exists a neighbourhood $W \subset V$ and constants $L \geq 0$ and $0 < \theta \leq 1$ such that*

$$\|F(t_1,x_1) - F(t_2,x_2)\| \leq L(|t_1 - t_2|^\theta + \|x_1 - x_2\|_\alpha)$$

for all $(t_i,x_i) \in W$.

This yields the following local existence theorem (see [Paz83, Theorem 6.3.1]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space E. By rescaling we assume without loss of generality that the semigroup is bounded and that A is invertible. If F satisfies assumption (F), then for every initial data $(t_0,x_0) \in V$ the initial value problem* (sACP$_F$) *has a unique local solution $u \in \mathrm{C}([t_0,\tau),E) \cap \mathrm{C}^1((0,\tau),E)$ where $\tau > t_0$ depends on t_0 and u_0 ans u is given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s,u(s))\,\mathrm{d}s$$

for all $t \geq t_0$.

Again we can conclude a global existence result (see [Paz83, Theorem 3.3]).

Theorem. *Let $A\colon D(A) \subset E \to E$ be the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space E. By rescaling we assume without loss of generality that the semigroup is bounded and that A is invertible. Further, assume that $F\colon [t_0,\infty) \times E_\alpha \to E$ satisfies assumption (F). If there is a continuous nondecreasing real valued function $K\colon [t_0,\infty) \to E$ such that*

$$\|F(t,x)\| \leq K(t)(1+\|x\|_\alpha)$$

for all $t \geq t_0$ and $x \in E_\alpha$, then for every initial value $u_0 \in E_\alpha$ the problem (sACP_F) *has a unique solution given as*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s,u(s))\,\mathrm{d}s$$

for all $t \geq t_0$.

For more details we refer to [Paz83, Section 6.2 & 6.3].

Finally, we come to quasilinear equations, i.e.,

$$(\mathrm{qACP}_F) \quad \begin{cases} \dot{u}(t) = \mathbf{A}(u(t))u(t) + F(t,u(t)) & \text{for } t \in [0,\tau], \\ u(0) = u_0 \end{cases}$$

for a semilinearity $F\colon [0,\tau]\times E \to E$ for $\tau \in (0,\infty)$ or $\tau = \infty$ and an initial value $u_0 \in E$ on a Banach space E. In the sequel we assume that the semilinearity F is continuous on $[0,\tau]$ and Lipschitz continuous on E. We give only a idea how to solve (qACP_F) using semigroup theory. The linearisation of (qACP_F) is

$$\begin{cases} \dot{w}(t) = \mathbf{A}(u(t))w(t) + F(t,u(t)) & \text{for } t \in [0,\tau], \\ w(0) = u_0 \end{cases}$$

for $w\colon [0,\tau] \to E$. Defining for fixed u the operator $A \coloneqq \mathbf{A}(u)$ and the inhomogeneity $f(t) \coloneqq F(t,u(t))$ we obtain

$$(\mathrm{qACP}_f\text{-lin}) \quad \begin{cases} \dot{w}(t) = Aw(t) + f(t) & \text{for } t \in [0,\tau], \\ w(0) = u_0. \end{cases}$$

Comparing the problems (sACP_F) and $(\mathrm{qACP}_f\text{-lin})$ we see that $(\mathrm{qACP}_f\text{-lin})$ admits a unique (mild) solution for $u_0 \in E$ and a unique classical solution for $u_0 \in D(A)$, hence more regular data yield more regular solutions. This idea leads to the definition of *trace space* associated to A and p. Take

$$I_p(A) \coloneqq \left\{ z \in E\colon \begin{array}{l} \text{there exists a } u \in \mathrm{W}^{1,p}((0,\tau),E) \cap \mathrm{L}^p((0,\tau),D(A)) \\ \text{with } u(0) = z \end{array} \right\}$$

equipped with the norm

$$\|z\|_{I_p(A)} \coloneqq \inf \left\{ \begin{array}{c} \|u\|_{\mathrm{W}^{1,p}((0,\tau),E)} \\ +\|Au\|_{\mathrm{L}^{1,p}((0,\tau),E)} \end{array} : \begin{array}{l} \text{there exists a function} \\ u \in \mathrm{W}^{1,p}((0,\tau),E) \text{ and} \\ u \in \mathrm{L}^p((0,\tau),D(A)) \\ \text{with } u(0) = z \end{array} \right\}.$$

Then $I_p(A)$ becomes a Banach space satisfying

$$D(A) \hookrightarrow I_p(A) \hookrightarrow E.$$

Of course, existence of a unique solution of (qACP$_f$-lin) does not suffice to solve (qACP$_F$). The solution should be as regular as the data. This leads to the following definition.

Definition. A closed operator $A\colon D(A) \subset E \to E$ has *maximal* L^p*-regularity* if for every pair $(f,u_0) \in \mathrm{L}^p((0,\tau),E) \times I_p(A)$ there exists a unique solution of (qACP$_f$-lin) which satisfies

$$\|\dot{u}\|_{\mathrm{L}^p((0,\tau),E)} + \|Au\|_{\mathrm{L}^p((0,\tau),E)} \leq C \cdot \left(\|f\|_{\mathrm{L}^p((0,\tau),E)} + \|u_0\|_{I_p(A)} \right)$$

for a constant $C = C_\tau > 0$.

Maximal L^p-regularity for some $p \in [1,\infty)$ implies maximal L^p-regularity for all $p \in (1,\infty)$. For a similar concept using the spaces of Hölder-continuous functions instead of Sobolev spaces we refer to [Lun95]. Note that maximal regularity is a quite strong condition and implies that A generates a bounded analytic semigroup of optimal angle $\frac{\pi}{2}$ on E. Maximal regularity implies the existence of the solution operator of (qACP$_f$-lin)

$$S_u\colon \mathrm{L}^p((0,\tau),E) \times I_p \to \mathrm{W}^{1,p}((0,\tau),E) \times \mathrm{L}^p((0,\tau),D(A)), \quad (f,u_0) \mapsto v.$$

Denote by $R(u) \coloneqq S_u(F(\cdot,u),u_0)$ the right hand side, then the maximal regularity of A and the Lipschitz continuity of F implies that R is a contraction on $\mathrm{W}^{1,p}((0,\tau),E) \times \mathrm{L}^p((0,\tau),D(A))$ for sufficiently small $\tau > 0$. Now Banach's fixed point theorem yields that the equation $R(u) = u$ has a unique fixed point and hence (qACP$_F$) is uniquely solvable.

The theory of strongly continuous semigroups and its relationship to wellposedness of abstract Cauchy problems as sketched above is well established and can be found in many excellent textbooks, e.g. [EN00], [Paz83], [Dav80], [Lun95], [Ama95] and [ABHN11].

As a concrete application we can rewrite (homogeneous, inhomogeneous, semilinear, quasilinear) parabolic initial value problems as (homogeneous, inhomogeneous, semilinear, quasilinear) abstract Cauchy problems for an appropriate operator on an appropriate Banach space and then apply the semigroup theory.

This is the leitmotif of our thesis and we give two concrete examples in the next sections.

Heat equation with dynamic boundary conditions

For linear parabolic partial differential equation on smooth bounded domains (or, more general, on compact Riemannian manifolds with boundary) one needs, beside an initial value, boundary conditions to guarantee uniqueness of the solution. These boundary conditions can be time-independent (or static) like Dirichlet, Neumann or Robin boundary conditions or time-dependent (or dynamic) as so called Wentzell boundary conditions. All these boundary conditions appear from different physical phenomena. For time-independent boundary conditions the functions in the domain of the elliptic operator need to satisfy the boundary condition. Then the wellposedness of the parabolic initial-value problem corresponds to the generator property of the operator defined on this domain. For dynamic boundary conditions the situation is more sophisticated.

As a simple but typical example we consider the heat equation on a smooth bounded domain $\Omega \subset \mathbb{R}$ with boundary $\partial\Omega$. On the Banach space $X := \mathrm{C}(\overline{\Omega})$ the *heat equation with dynamic (or Wentzell) boundary conditions* is modelled by

$$\text{(I.1)} \quad \begin{cases} \dot{u}(t) = \Delta u(t) & \text{for } t \geq 0, \\ \dot{u}|_{\partial\Omega}(t) = -\dfrac{\partial}{\partial n} u(t) & \text{for } t \geq 0, \\ u(0) = u_0 \end{cases}$$

on X, where $\frac{\partial}{\partial n}$ denotes the normal derivative. Our first goal is to find an operator A such that (I.1) becomes the abstract Cauchy problem (ACP) associated to A.

Consider the Laplace operator $A_m := \Delta$ with maximal domain

$D(A_m) := \left\{ f \in \mathrm{C}(\overline{\Omega}) \colon \Delta f \in \mathrm{C}(\overline{\Omega}) \right\}$, the trace operator $Lf = f|_{\partial\Omega}$ and the normal derivative $B := -\frac{\partial}{\partial n}$. Now (I.1) can be rewritten as

$$
\text{(I.2)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ \dot{x}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(t) = x(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}
$$

Since the trace operator is bounded, we obtain

$$
\dot{x}(t) = (\dot{Lu})(t) = L\dot{u}(t) = LA_m u(t)
$$

and (I.2) can be rewritten as

$$
\text{(I.3)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ LA_m u(t) = Bu(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}
$$

This initial value problem can be interpreted as an abstract Cauchy problem (ACP) for the *Laplace operator with Wentzell (or dynamic) boundary conditions*

$$
\text{(I.4)} \quad A^B f := \Delta f, \quad D(A^B) := \{ f \in D(A_m) \cap D(B) \colon LA_m f = Bf \}.
$$

Now the initial value problem (I.2) can be seen as a coupled system of two initial value problems: one for u and one for x. In order to decouple this system note that every continuous function on $\overline{\Omega}$ can be decomposed into a continuous function with zero trace and a harmonic function, i. e.,

$$
\text{(I.5)} \quad \mathrm{C}(\overline{\Omega}) = \mathrm{C}_0(\Omega) \oplus \ker(\Delta).
$$

Hence the initial value problem (I.2) induces an initial value problem on $\mathrm{C}_0(\Omega)$[7] given by

$$
\text{(I.6)} \quad \begin{cases} \dot{u}(t) = \Delta u(t), & \text{for } t \geq 0, \\ x(t) = 0, & \text{for } t \geq 0, \\ Lu(t) = x(t), & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}
$$

Here we only have a dynamics on the interior and no dynamic on the boundary.

[7] The Banach space $\mathrm{C}_0(\Omega) := \{ f \in \mathrm{C}(\overline{\Omega}) \colon f|_{\partial\Omega} = 0 \}$ equipped with the sup-norm.

By the continuity of the trace operator, this can be rewritten as the abstract Cauchy problem on the Banach space $\mathrm{C}_0(\Omega)$ for the Laplace operator with Dirichlet and so called *pure Wentzell boundary conditions* given by

$$A_0^0 f := \Delta f, \quad D(A_0^0) := \{f \in D(A_m) \colon Lf = 0, LA_m f = 0\}. \tag{I.7}$$

Moreover (I.2) induces an initial value problem on the space of harmonic functions[8] given by

$$\begin{cases} \Delta u(t) = 0 & \text{for } t \geq 0, \\ \dot{x}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(t) = x(t) & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases} \tag{I.8}$$

This system consists of the time-independent Laplace equation and a dynamic boundary condition. Hence it has dynamics on the boundary, whereas the interior is stationary. It corresponds to the *Dirichlet-to-Neumann operator N* given by the composition of the harmonic extension operator with the (negative) normal derivative, i.e., $N = BL_0$, where L_0 denotes the extension of a continuous function from $\partial\Omega$ to a harmonic function on $\overline{\Omega}$. Roughly speaking, L_0 is the solution operator of the Dirichlet problem and $L_0 x = f$ is equivalent to

$$\begin{cases} \Delta f = 0, \\ Lf = x, \end{cases} \tag{I.9}$$

while the Dirichlet-to-Neumann operator translates a Dirichlet boundary condition into a Neumann boundary condition.

It is well known that A_0^0 from (I.7) generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}_0(\Omega)$ and hence the initial value problem (I.6) is wellposed. Moreover Klaus Engel shows in [Eng03, Theorem 2.1] that the Dirichlet-to-Neumann operator generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$ and therefore the initial value problem (I.8) is wellposed. These facts imply that A^B given in (I.4) generates an analytic C_0-semigroup of angle $\frac{\pi}{2}$ and thus (I.2) is wellposed. This shows that we recover the coupled system (I.2) from the uncoupled subsystems (I.6) and (I.8).

[8] We call a function $f \in \mathrm{C}(\overline{\Omega})$ harmonic, if $\Delta f = 0$. The harmonic functions form a Banach space with the sup-norm.

Delay differential equations

Phenomena in population dynamics can be modelled by equations where the evolution of a state $x(t)$ at time t also depends on the history of the system. To obtain a deterministic dependence of such a system the state must contain information on the history of the system. To this end we consider the history segments $u(t)$ of x given by

$$u(t)\colon [-1,0] \to Y, \quad u(t)(s) \coloneqq x(t+s),$$

where Y is a suitable Banach space. As initial value we take a function $h\colon [-1,0] \to Y$ describing the prehistory of the system. Choosing the Banach space $X = \mathrm{C}([-1,0],Y)$ of continuous functions we obtain the *delay differential equation*

$$\text{(I.10)} \quad \begin{cases} \dot{x}(t) = Cx(t) + \Phi u(t) & \text{for } t \geq 0, \\ u(0) = h, \end{cases}$$

on X, where C is an operator on Y and $\Phi \in \mathcal{L}(X,Y)$ is the *delay operator*. Again we want to find an operator such that (I.10) becomes an abstract Cauchy problem of the form (ACP).

Consider the first derivative $A_m f \coloneqq \frac{\mathrm{d}}{\mathrm{d}s} f$ with maximal domain $D(A_m) \coloneqq \mathrm{C}^1([-1,0],Y)$, the trace operator $L \coloneqq \delta_0$ and $B \coloneqq C\delta_0 + \Phi$. Note that $Lu(t) = x(t)$. Hence (I.10) can be written as

$$\text{(I.11)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ \dot{x}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(t) = x(t) & \text{for } t \geq 0, \\ u(0) = h. \end{cases}$$

As above we obtain as an abstract Cauchy problem for the *delay operator*

$$A^B f \coloneqq \frac{\mathrm{d}}{\mathrm{d}s} f, \quad D(A^B) \coloneqq \{f \in D(A_m) \cap D(B) \colon LA_m f = Bf\}.$$

Using the decomposition

$$\text{(I.12)} \quad \mathrm{C}([-1,0],Y) = \mathrm{C}_0([-1,0),Y)^{8} \oplus (\langle \mathbb{1} \rangle \otimes Y),$$

[8] The Banach space $\mathrm{C}_0([-1,0),Y) \coloneqq \{f \in \mathrm{C}([-1,0],Y) \colon f(0) = 0\}$ equipped with the sup-norm.

we obtain the system on $C_0([-1,0),Y)$ given by

$$\text{(I.13)}\quad \begin{cases} \dot{u}(t) = \dfrac{\mathrm{d}}{\mathrm{d}s}u(t), & \text{for } t \geq 0,\\ x(t) = 0, & \text{for } t \geq 0,\\ Lu(t) = x(t), & \text{for } t \geq 0,\\ u(0) = u_0. \end{cases}$$

Again this system has no dynamics on the boundary and can be interpreted as the abstract Cauchy problem on the Banach space $C_0[-1,0)$ for the first derivative with Neumann and Dirichlet boundary conditions, i.e.,

$$A_0^0 f := \frac{\mathrm{d}}{\mathrm{d}s}f, \quad D(A_0^0) := \{f \in C_0^1[-1,0) \colon f'(0) = f(0) = 0\}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}s}u(t) = 0$, we obtain $u(t)(s) = u(t)(0) = x(t)$ and the initial value problem (I.13) becomes

$$\text{(I.14)}\quad \begin{cases} \dot{x}(t) = Bu(t) = Cx(t) + \Phi(x(t) \otimes 1) & \text{for } t \geq 0,\\ Lu(t) = x(t) & \text{for } t \geq 0,\\ x(0) = x_0 \end{cases}$$

on $\langle \mathbb{1} \rangle \otimes Y \cong Y$. The bounded perturbation theorem implies that this problem is wellposed if and only if C generates a strongly continuous semigroup on Y.

It is standard that A_0^0 generates a strongly continuous semigroup on $C_0[-1,0)$. Moreover the delay operator A^B generates a strongly continuous semigroup on $C[-1,0]$ if the operator B generates a strongly continuous semigroup on Y. Again it is possible to recover solutions of (I.11) from of the two uncoupled problems (I.13) and (I.14).

An abstract framework for dynamic boundary value problems

An abstract approach for operators with boundary conditions goes back to Greiner in [Gre87], who looked at time independent boundary conditions as perturbations of the domain of an operator. For dynamic boundary conditions an abstract framework has been developed by Engel in [Eng03] and [EF05].

We have seen that the equations in (I.1) and (I.10) have a similar structure yielding to analogous results. This indicates that there is a more general phenomenon leading to these results. Let us analyse the above situation from a more abstract point of view.

We start from two Banach spaces, the *state space* X and the *boundary space* ∂X. In the example of the heat equation the state spaces is $\mathrm{C}(\overline{\Omega})$ and the boundary spaces is $\mathrm{C}(\partial\Omega)$, whereas for the delay equation the state space is $\mathrm{C}([-1,0],Y)$ and the boundary space is Y.

Next we need a bounded *trace operator* $L\colon X \to \partial X$ connecting these two Banach spaces. For the heat equation this is the normal trace operator $Lf = f|_{\partial\Omega}$, for the delay equation it is $L = \delta_0$.

Third we have a densely defined *maximal operator* $A_m\colon D(A_m) \subset X \to X$ acting on the state space. Maximal here means maximal domain, i.e. without boundary conditions. For the heat equation this is the Laplace operator, and the first derivative for delay equations.

Finally, we need a *feedback operator* $B\colon D(B) \subset X \to \partial X$. It models the (time dependent) boundary condition. For the heat equation it is the normal derivative, whereas for delay equation it is given by $C\delta_0 + \Phi$.

Using these spaces and operators we formulate an abstract version of the initial value problems (I.2) and (I.11) as

$$\text{(I.15)}\qquad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ \dot{x}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(t) = x(t) & \text{for } t \geq 0, \\ u(0) = u_0 \end{cases}$$

for $u(t) \in X$. To rewrite this equation as an abstract Cauchy problem, we implement the conditions $Lu(t) = x(t)$ into the Banach space $\tilde{\mathcal{X}} := \left\{\binom{f}{x} \in X \times \partial X : Lf = x\right\}$ equipped with the norm $\left\|\binom{f}{x}\right\| := \|f\|_X + \|x\|_{\partial X}$. On this space we consider the *operator with dynamic boundary conditions* $\mathcal{A}^B\colon D(\mathcal{A}^B) \subset \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ given by

$$\text{(I.16)}\qquad \begin{aligned} \mathcal{A}^B \begin{pmatrix} f \\ x \end{pmatrix} &:= \begin{pmatrix} A_m f \\ Bf \end{pmatrix}, \\ D(\mathcal{A}^B) &:= \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in (D(A_m) \cap D(B)) \times \partial X : Lf = x, LA_m f = Bf \right\}. \end{aligned}$$

Now (I.15) is the abstract Cauchy problem (ACP) associated to the operator $\mathcal{A}^B$.

Since $L : X \to \partial X$ is bounded, the second line becomes

$$Bu(t) = \dot{x}(t) = \dot{\overline{Lu}}(t) = L\dot{u}(t) = LA_m u(t)$$

for $t \geq 0$ and hence the third line can be omitted. This yields the following initial value problem on X

$$\text{(I.17)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ LA_m u(t) = Bu(t) & \text{for } t \geq 0, \\ u(0) = u_0 \end{cases}$$

This is the abstract Cauchy problem for the *operator with Wentzell boundary conditions* $A^B \colon D(A^B) \subset X \to X$ given by

$$\text{(I.18)} \quad A^B f \coloneqq A_m f, \quad D(A^B) \coloneqq \{f \in D(A_m) \cap D(B) \colon LA_m f = Bf\}\,.$$

Note that the operator with dynamic boundary conditions (I.16) and the operator with Wentzell boundary conditions (I.18) are similar, i.e.,

$$\mathcal{A}^B = SA^B S^{-1}$$

for $S \colon X \to \tilde{\mathcal{X}} \colon f \mapsto \binom{f}{Lf}$.

We now look for an abstract analogue of the decompositions (I.5) and (I.12). For this purpose we introduce the abstract *Dirichlet operator* $L_0 \colon \partial X \to X$ given by the solution of the abstract Dirichlet problem, i.e.,

$$L_0 x = f \quad \iff \quad \begin{cases} A_m f = 0, \\ Lf = x. \end{cases}$$

Since L_0 is the left-inverse of L, the operator $L_0 L \in \mathcal{L}(X)$ is a projection onto $\ker(A_m)$ along $X_0 \coloneqq \ker(L)$ and we have the decomposition

$$\text{(I.19)} \quad X = X_0 \oplus \ker(A_m).$$

We also need the boundedness of the Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$. This assumption is equivalent to the closedness of the maximal operator A_m and hence is a natural condition. Since L_0 is an isomorphism from $\ker(A_m)$ to ∂X, it follows that

$$\ker(A_m) \cong \partial X.$$

Now the initial value-boundary problem (I.15) induces an initial value problem on X_0 given by

$$\text{(I.20)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ \dot{x}(t) = 0 & \text{for } t \geq 0, \\ x(t) = Lu(t) = 0 & \text{for } t \geq 0, \\ u(0) = u_0, \end{cases}$$

describing the interior dynamics of the coupled system. It can be rewritten as the abstract Cauchy problem of the operator $A_0^0 \colon D(A_0^0) \subset X_0 \to X_0$ given by

$$A_0^0 f \coloneqq A_m f, \quad D(A_0^0) \coloneqq \{f \in D(A_m) \cap X_0 \colon A_m f \in X_0\}.$$

On the other hand, we obtain an initial value problem on $\partial X \cong \ker(A_m)$ by

$$\text{(I.21)} \quad \begin{cases} A_m u(t) = 0 & \text{for } t \geq 0, \\ \dot{x}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(t) = x(t) & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases}$$

describing the boundary dynamics of the system. It can be seen as an *elliptic problem with dynamic boundary conditions* since it comes from the elliptic equation $A_m f = 0$ and a dynamics on the boundary space. It corresponds to the abstract Cauchy problem of the *Dirichlet-to-Neumann operator* $N \colon D(N) \subset \partial X \to \partial X$ obtained as the composition of the feedback operator B and the Dirichlet operator L_0, i.e.,

$$N = BL_0, \quad D(N) \coloneqq \{x \in \partial X \colon L_0 x \in D(B)\}.$$

Let us come back to the initial value problem (I.20) describing the interior dynamics of the system. Instead of using the decomposition (I.19) to characterize the interior dynamic we could also simply assume that there is no dynamics on the boundary, i.e., $\dot{x}(t) = 0$. This yields the initial value problem

$$\text{(I.22)} \quad \begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ \dot{x}(t) = 0 & \text{for } t \geq 0, \\ x(t) = Lu(t) = 0 & \text{for } t \geq 0, \\ u(0) = u_0, \end{cases}$$

on X and the *operator* $A^0 \colon D(A^0) \subset X \to X$ *with pure Wentzell boundary conditions* given by (I.18) for $B = 0$. Note that if A^0 generates a strongly continuous semigroup on X, we obtain that A_0^0 generates a strongly continuous semigroup on X_0 since A_0^0 is the restriction of A^0 to X_0.
Surprisingly the converse is also true. The semigroup $(T(t))_{t\geq 0}$ generated by A^0 can be constructed from the semigroup $(T_0(t))_{t\geq 0}$ generated by A_0^0 using the projection $\mathrm{Id} - L_0 L$ from X to X_0. More precisely, we have

$$T(t) = T_0(t)(\mathrm{Id} - L_0 L) + L_0 L.$$

Moreover there is a third possibility to describe the interior dynamics of the system. Instead of assuming that there is no dynamics on the boundary, we assume that the boundary values are equal to zero, i.e., $Lu(t) = 0$. This can be seen as an abstract form of the Dirichlet boundary conditions and yields the problem

$$\begin{cases} \dot{u}(t) = A_m u(t) & \text{for } t \geq 0, \\ Lu(t) = 0 & \text{for } t \geq 0, \\ u(0) = u_0, \end{cases}$$

on X. It corresponds to the abstract Cauchy problem (ACP) associated to the *operator* $A_0 \colon D(A_0) \subset X \to X$ *with Dirichlet boundary conditions* given by

$$A_0 f = A_m f, \quad D(A_0) := D(A_m) \cap X_0.$$

In concrete situation, e.g. for elliptic operators, this operator is quite well understood. However these results have some weaknesses from the semigroup generator point of view. Since $X_0 \subset X$ is a closed subset, the operator A_0 is not densely defined and therefore cannot be the generator of a strongly continuous semigroup. If we instead assume that A_0 is a Hille-Yosida operator (see [EN00, Definition II. 3.22]) on X, we obtain that A_0^0 and hence A^0 generate strongly continuous semigroups on X_0 and X, respectively. Indeed A_0^0 is the part (see [EN00, Section II.2.3]) of A_0 in X_0. However the converse is not true in general and we need the following definition.

Definition. Consider an operator $T \colon D(T) \subset X \to X$ on a Banach space X. We call T a *weak Hille-Yosida operator* on X if there exists constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega, \infty) \subset \rho(T)$ and

$$\|\lambda R(\lambda, T)\| \leq M \quad \text{for all } \lambda > \omega.$$

Note that this condition is the Hille-Yosida condition for $n = 1$ only. Assuming that A_0 is a weak Hille-Yosida operator on X, it follows that $X \subset \mathrm{Fav}_{-1}^{A_0^0}$ and $A_0 = (A_0^0)_{-1}|_X$, where we used the concept and notation of extrapolated Favard spaces from [EN00, Section II.5]. As a consequence, the generator property of A_0^0 implies that A_0 is a Hille-Yosida operator on X.

Before discussing the relationship between the problems (I.15) and (I.20) and (I.21) we summarize some general assumptions in order to pursue the idea of decoupling equations with dynamic boundary conditions.

First we need existence and boundedness of the Dirichlet operator L_0. Further, the operator A_0 must be a weak Hille-Yosida operator on X.

Moreover we have to control the feedback operator B. More precisely, using (I.19) the operator B splits into the operators $B_0, B_1 \subset B$ given by $B_0 \colon D(B_0) \to \partial X$ with $D(B_0) = D(B) \cap X_0$ and $B_1 \colon D(B_1) \to \partial X$ with $D(B_1) = D(B) \cap \partial X$. Note that $N = BL_0 = B_1 L_0$ and therefore B_1 is controlled by the Dirichlet-to-Neumann operator. Hence we assume that B_0 (and so B) is A_0-bounded of bound 0.

Our question now can be stated as follows: to what extend reflect the subsystems (I.20) and (I.21) the coupled system (I.15). We reformulate this in the language of semigroups: Which properties of A^B are reflected by properties of A_0^0 and N? In particular we are interested in the following:

Is the operator A^B with Wenzell boundary conditions generator of a strongly continuous semigroup if and only if the operators A_0^0 and the Dirichlet-to-Neumann operator N are?

Is the semigroup generated by the operator A^B with Wentzell boundary conditions, given in (I.18), analytic (of angle α)/ compact/ positive if (and only if) the semigroups generated by the operator A_0^0 and the Dirichlet-to-Neumann operator N are analytic (of angle α)/ compact/ positive?

How is the long time behaviour of the operator with Wentzell boundary conditions A^B given by (I.18) and of the operator A_0^0 and the Dirichlet-to-Neumann operator N related? Is the semigroup generated by A^B stable if and only if the semigroups generated by A_0^0 and N are?

How is the spectrum, and its fine structure, of the operator A^B with Wentzell boundary conditions, given in (I.18), reflected by the spectra of the operator A_0^0 and the Dirichlet-to-Neumann operator N?

These questions shall be answered in this thesis.

II Objectives

In [EF05] Klaus Engel and Genni Fragnelli introduced an abstract framework for operators with dynamic boundary conditions and related these operators to Dirichlet-to-Neumann operators and operators with Dirichlet boundary conditions. At the start of my research Klaus Engel and myself improved this result to an equivalence. This lead to the article [BE19].

Joachim Escher [Esc94] proved that Dirichlet-to-Neumann operators associated to elliptic operators with smooth coefficients on smooth domains generate analytic semigroups on the space of continuous functions without giving information about the angle of analyticity. On the other hand [Eng03] showed that for the Laplacian the Dirichlet-to-Neumann operator generates an analytic semigroup of optimal angle $\frac{\pi}{2}$ on the space of continuous functions. A deeper understanding of this problem in obtained in the articles [Bin19], [Bin20a] and [Bin20b].

During my master in mathematical physics I learned that operators with interior boundary conditions satisfy a decomposition similar to operators with dynamic boundary conditions. Jonas Lampart and myself recognized that this fact allows an abstract framework for such operators in [BL20].

Motivated by the result of Tom ter Elst and El Maati Ouhabaz [EO19a] about the optimal angle of analyticity for Dirichlet-to-Neumann operators associated to elliptic operators with Hölder continuous coefficients on $\mathrm{C}^{1,\kappa}$-domains, Tom ter Elst and myself extended my previous results on operators with dynamic boundary conditions to the case of less regular coefficients and domains. This leads to the article [BtE20].

Finally, Klaus Engel and myself wanted a better understanding of the equivalence proven in [BE19]. In particular, we were interested in properties which were respected by this equivalence. The results for strongly continuous semigroups leads to [BE20a], whereas the result concerning positivity, spectral theory and stability leads to [BE20b].

III Discussion of the Results

III.1 An abstract framework for dynamic boundary value problems

III.1.1 On spaces with bounded trace operator

We briefly recall the abstract setting from the introduction. It was introduced in [Nic02], [Eng03] and [EF05] and later developed in [BE04], [Nic04b] and [BEH05].

Abstract Setting III.1.1.1. Consider

(i) two Banach spaces X and ∂X

(ii) a densely defined maximal operator $A_m \colon D(A_m) \subset X \to X$;

(iii) a bounded trace operator $L \colon X \to \partial X$;

(iv) a feedback operator $B \colon D(B) \subset X \to \partial X$.

Using these spaces and operators we define the Banach spaces $X_0 := \ker(L)$ equipped with $\| \cdot \|_X$ and

$$\tilde{\mathcal{X}} := \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in X \times \partial X \colon Lf = x \right\}$$

equipped with the norm $\left\| \binom{f}{Lf} \right\| := \|f\|_X + \|x\|_{\partial X}$. On this space we consider the operator $\mathcal{A}^B \colon D(\mathcal{A}^B) \subset \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ with dynamic boundary conditions given by

$$\begin{aligned} \mathcal{A}^B \begin{pmatrix} f \\ x \end{pmatrix} &:= \begin{pmatrix} A_m f \\ Bf \end{pmatrix} \\ D(\mathcal{A}^B) &:= \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in (D(A_m) \cap D(B)) \times \partial X : Lf = x, LA_m f = Bf \right\}. \end{aligned} \tag{III.1}$$

Moreover we induces the operator $A^B\colon D(A^B) \subset X \to X$ with Wentzell boundary conditions given by

(III.2) $A^B f := A_m f, \quad D(A^B) := \{f \in D(A_m) \cap D(B) \colon LA_m f = Bf\}.$

In particular we have for $B = 0$ the operator $A^0\colon D(A^0) \subset X \to X$ with pure Wentzell boundary conditions. Note that the operator with dynamic boundary conditions and the operator with Wentzell boundary conditions are similar. Further we have the operator $A_0\colon D(A_0) \subset X \to X$ with Dirichlet boundary conditions given by

(III.3) $A_0 f := A_m f, \quad D(A_0) := \{f \in D(A_m) \colon Lf = 0\}$

and its restriction A_0^0 to X_0, i.e.

(III.4) $A_0^0 f := A_m f, \quad D(A_0^0) := \{f \in D(A_m) \colon LA_m f = Lf = 0\}.$

Moreover we define the Dirichlet operator $L_\lambda := (L|_{\ker(\lambda - A_m)})^{-1}\colon \partial X \to \ker(A_m)$ associated to $\lambda \in \rho(A_0)$ given by

(III.5) $L_\lambda x = f \quad \Longleftrightarrow \quad \begin{cases} A_m f = \lambda f, \\ Lf = x. \end{cases}$

and the Dirichlet-to-Neumann operator $N_\lambda\colon D(N_\lambda) \subset \partial X \to \partial X$ associated to $\lambda \in \rho(A_0)$ by

$$N_\lambda x := BL_\lambda x, \quad D(N_\lambda) := \{x \in \partial X \colon L_\lambda x \in D(B)\}.$$

Further we have seen that the following assumptions seems to be sensible.

Assumptions III.1.1.2.

(i) *The Dirichlet operator L_λ exists and is bounded for some $\lambda \in \rho(A_0)$;*

(ii) *the operator A_0 with Dirichlet boundary conditions is a weak Hille-Yosida operator on X;*

(iii) *the feedback operator B is relatively A_0-bounded of bound 0.*

Note that by [AE18, Lemma 3.2], in assumption (i) the existence of the Dirichlet operator L_λ is equivalent to the surjectivity of the trace operator L and the boundedness is equivalent to the closedness of the maximal operator A_m.

Moreover, the existence of the Dirichlet operator L_λ yields the decompositions

$$X = X_0 \oplus \ker(\lambda - A_m) \text{ and } D(A_m) = D(A_0) \oplus \ker(\lambda - A_m). \tag{III.6}$$

III.1.2 On spaces with unbounded trace operator

Instead of choosing spaces of continuous function to model the problems (I.1) and (I.10) we can also consider spaces of p-integrable functions. This yields a slightly different setting since the trace operator is not bounded on $L^p(\Omega)$. However we assume that the trace operator is defined on the domain of the maximal operator. Hence we obtain the following setting. It was introduced in [CENN03] and later used and developed e.g. in [Mug01], [Mug04], [Nic04a], [Nic04b], [CENP05] and [Mug11].

Abstract Setting III.1.2.1. (i) two Banach spaces X and ∂X;

(ii) a densely defined maximal operator $A_m\colon D(A_m) \subset X \to X$;

(iii) a trace operator $L\colon D(A_m) \subset X \to \partial X$;

(iv) a feedback operator $B\colon D(B) \subset X \to \partial X$.

Similar as above we consider the operator $\mathcal{A}^B\colon D(\mathcal{A}^B) \subset X \times \partial X \to X \times \partial X$ with dynamic boundary conditions given by

$$\begin{aligned} \mathcal{A}^B \begin{pmatrix} f \\ x \end{pmatrix} &\coloneqq \begin{pmatrix} A_m f \\ Bf \end{pmatrix}, \\ D(\mathcal{A}^B) &\coloneqq \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in (D(A_m) \cap D(B)) \times \partial X : Lf = x \right\}. \end{aligned} \tag{III.7}$$

Here it makes sense to consider the operator with dynamic boundary conditions on the product space $X \times \partial X$, since the trace operator is unbounded and hence the operator becomes densely defined. We cannot find an analogue of the operator A^B with Wentzell boundary conditions since the trace operator makes only sense on $D(A_m)$. Again we define the operator $A_0\colon D(A_0) \subset X \to X$ with Dirichlet boundary conditions by

$$A_0 f \coloneqq A_m f, \quad D(A_0) \coloneqq \{f \in D(A_m) : Lf = 0\} = \ker(L). \tag{III.8}$$

Note that here the operator A_0 with Dirichlet boundary conditions is densely defined and its domain is the kernel of the trace operator. Hence we can work with A_0 directly. This makes the theory much easier.

Moreover we define the Dirichlet operator $L_\lambda \coloneqq (L|_{\ker(\lambda - A_m)})^{-1} \colon \partial X \to \ker(\lambda - A_m)$ associated to $\lambda \in \rho(A_0)$ given by

$$L_\lambda x = f \quad \iff \quad \begin{cases} A_m f = \lambda f, \\ Lf = x. \end{cases}$$

and the Dirichlet-to-Neumann operator $N_\lambda \colon D(N_\lambda) \subset \partial X \to \partial X$ associated to $\lambda \in \rho(A_0)$ by

(III.9) $N_\lambda x \coloneqq BL_\lambda x, \quad D(N_\lambda) \coloneqq \{x \in \partial X \colon L_\lambda x \in D(B)\}.$

Again we need some general assumptions. Note that assumption (ii) in Assumptions III.1.1.2 is not needed (and does not make sense). This leads to

Assumptions III.1.2.2. (i) *The Dirichlet operator L_λ exists and is bounded for some $\lambda \in \rho(A_0)$;*

(ii) *the feedback operator B is relatively A_0-bounded of bound* 0.

Note that by [AE18, Lemma 3.2], the existence of the Dirichlet operator L_λ is equivalent to the surjectivity of the trace operator L and the boundedness is equivalent to the closedness of the operator $\binom{A_m}{L}$. In this sense assumption (i) replaces the boundedness of the trace operator L and the closedness of the maximal operator A_m in the other case. Moreover we obtain from the existence of the Dirichlet operator L_λ the decomposition

(III.10) $D(A_m) = D(A_0) \oplus \ker(\lambda - A_m).$

The decompositions of the whole space X cannot work, since we cannot define $L_\lambda L$ on the Banach space X.

Since the operator A_0 with Dirichlet boundary conditions is densely defined and the operator $\mathcal{A}^B$ with dynamic boundary conditions is defined on the product space $X \times \partial X$ the theory becomes much easier. Most of the results proven for the case of bounded trace operator hold verbatim (or with easier proofs) for the case of unbounded trace operator. We will not discuss this in detail and concentrate on situations with bounded trace operators.

III.2 Wellposedness of abstract parabolic dynamic boundary value problems

In this section we consider the homogeneous problem with dynamic boundary conditions (I.15). Recall that it is given by

$$\text{(III.11)}\quad \begin{cases} \dot{u}(t) = A_m u(t), & \text{for } t \geq 0, \\ L\dot{u}(t) = Bu(t), & \text{for } t \geq 0, \\ u(0) = u_0. \end{cases}$$

In the sequel we use Setting III.1.1.1 and make Assumptions III.1.1.2.

III.2.1 The homogeneous case

Using the boundedness of the trace operator L we show that (III.11) can be rewritten as the abstract Cauchy problem (ACP) of the operator A^B with Wentzell boundary conditions. More precisely we obtain the following result (see [BE20a, Theorem 5.1]).

Theorem III.2.1.1. *The following assertions are equivalent.*

(a) *The homogeneous problem* (III.11) *with dynamic boundary conditions is wellposed.*

(b) *The abstract Cauchy problem* (ACP) *associated to the operator A^B given in* (III.2) *is wellposed.*

(c) *The operator A^B defined in* (III.2) *generates a C_0-semigroup on X.*

This theorem shows that the initial value problem (III.11) is characterized by the operator A^B given in (III.2).

III.2.2 The inhomogeneous case

Next we study the *inhomogeneous parabolic problem with dynamic boundary conditions* given by

$$\text{(III.12)}\quad \begin{cases} \dot{u}(t) = A_m u(t) + f(t), & \text{for } t \in [0, \tau], \\ L\dot{u}(t) = Bu(t) + g(t), & \text{for } t \in [0, \tau], \\ u(0) = u_0 \end{cases}$$

for $\tau \in (0,\infty)$ or $\tau = \infty$, $f : [0,\tau] \to X$ and $g : [0,\tau] \to \partial X$. We call $u : \mathbb{R}_+ \to X$ a *classical solution* of (III.12) if $u : [0,\tau] \to X$ is continuously differentiable in X, $u(t) \in D(A_m) \cap D(B)$ for all $t \in [0,\tau]$ and (III.12) holds.

The solvability of (III.12) can now be characterized by the solvability of an inhomogeneous Cauchy problem for the operator A^B with generalized Wentzell boundary conditions (see [BE20a, Theorem 5.2]).

Theorem III.2.2.1. *Let $u_0 \in X$, $f \in \mathrm{L}^1(\mathbb{R}_+, X)$, $g \in \mathrm{L}^1(\mathbb{R}_+, \partial X)$ and assume that A^B generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Then* (III.12) *has at most one solution. Moreover, if $Lf = g$ then $u : \mathbb{R}_+ \to X$ defined by*

$$\text{(III.13)}\quad u(t) := T(t)x_0 + \int_0^t T(t-s)f(s)\,ds$$

is a classical solution of (III.12) *if it is a classical solution of the inhomogeneous abstract Cauchy problem* (ACP_f) *for A^B defined in* (III.2).

This theorem shows that the inhomogeneous initial value problem (III.12) is equivalent to the inhomogeneous abstract Cauchy problem (ACP_f) of the operator A^B given in (III.2).

III.3 Wellposedness of abstract elliptic dynamic boundary value problems

In this section we concentrate on the elliptic dynamic boundary value problem (I.21), but a slightly more general version. In the sequel we use Setting III.1.1.1. For $\lambda \in \mathbb{C}$ we consider the initial value problem

$$\text{(III.14)}\quad \begin{cases} A_m u(t) = \lambda u(t) & \text{for } t \geq 0, \\ (Lu)^{\cdot}(t) = Bu(t) & \text{for } t \geq 0, \\ Lu(0) = x_0 \end{cases}$$

on ∂X. Note that $X_1 := [D(A_m)]$ is a Banach space. In this section we need the following assumptions.

Assumptions III.3.0.1. (i) *the maximal operator $A_m \colon D(A_m) \subset X \to X$ is closed;*

(ii) *the trace operator $L \colon X \to \partial X$ is surjective;*

(iii) *the operator B is relatively A_0-bounded;*

(iv) *the operator* $B_1 := B|_{X_1} : D(A_m) \cap D(B) \subset X_1 \to \partial X$ *is closed.*

Note that the existence of L_λ implies the conditions (ii), and if L_λ exists, its boundedness is equivalent to condition (i).

III.3.1 Homogeneous problems

The coupled problem (III.14) is equivalent to two independent problems, the Dirichlet problem (III.5) and the abstract Cauchy problem (ACP) associated to the Dirichlet-to-Neumann operator N_λ. More precisely we obtain the following result (see [BE20a, Theorem 4.1]).

Theorem III.3.1.1. *The following statements are equivalent.*

(a) *The homogeneous problem* (III.14) *is (mildly) wellposed (see [BE20a, Definition 2.2]);*

(b) *The Dirichlet problem* (III.5) *admits a unique solution and the abstract Cauchy problem* (ACP) *for* N_λ *is wellposed on* ∂X*;*

(c) *The Dirichlet operator* L_λ *exists (and is bounded) and the Dirichlet-to-Neumann operator* N_λ *generates a strongly continuous semigroup on* ∂X*.*

This shows that the elliptic problem with dynamic boundary conditions (III.14) can be decoupled into a stationary Dirichlet problem (III.5) and a Cauchy problem for the Dirichlet-to-Neumann operator N_λ.

If we assume the existence of the Dirichlet operator, the following corollary from [BE20a, Theorem 4.1] characterizes the existence of classical solutions of (III.14).

Corollary III.3.1.2. *Assume that the Dirichlet operator* L_λ *exists (and is bounded). Then the following assertions are equivalent.*

(a) *For every* $x_0 \in L(D(A_m) \cap D(B))$ *the homogeneous problem* (III.14) *admits a unique classical solution (see [BE20a, Definition 2.1]).*

(b) *The abstract Cauchy problem* (ACP) *for* N_λ *is wellposed on* ∂X*.*

(c) *The Dirichlet-to-Neumann operator* N_λ *generates a strongly continuous semigroup on* ∂X*.*

III.3.2 Inhomogeneous problems

Having characterized the wellposedness of the homogeneous problem we consider now the inhomogeneous elliptic problem with dynamic boundary conditions

$$\text{(III.15)}\quad \begin{cases} A_m u(t) = \lambda u(t) + h(t) & \text{for } t \in [0,\tau], \\ (Lu)^{\cdot}(t) = Bu(t) + g(t) & \text{for } t \in [0,\tau], \\ Lu(0) = x_0 \end{cases}$$

on ∂X for $\lambda \in \mathbb{C}$, $\tau \in (0,\infty)$ or $\tau = \infty$, $h\colon [0,\tau] \to X$ and $g\colon [0,\tau] \to \partial X$. The following statement holds (cf. [BE20a, Theorem 4.2]).

Theorem III.3.2.1. *Let $\lambda \in \rho(A_0)$, $x_0 \in \partial X$, $h\colon [0,\tau] \to X$ and $g\colon [0,\tau] \to \partial X$. Moreover, assume that $N_\lambda = BL_\lambda$ generates a C_0-semigroup $(S_\lambda(t))_{t\geq 0}$ on ∂X. Then $u : \mathbb{R}_+ \to X$ defined by*

(III.16)
$$u(t) \coloneqq L_\lambda S_\lambda(t)x_0 + R(\lambda, A_0)h(t) + L_\lambda \int_0^t S_\lambda(t-s)\big(g(s) + BR(\lambda, A_0)h(s)\big)\,\mathrm{d}s$$

is a classical solution of (III.15) *if and only if the mild solution*

$$x(t) \coloneqq S_\lambda(t)x_0 + \int_0^t S_\lambda(t-s)h(s)\,\mathrm{d}s$$

of the inhomogeneous abstract Cauchy problem (ACP_f) *associated to N_λ and $f(t) \coloneqq g(t) + BR(\lambda, A_0)h(t)$ is a classical solution.*

We call a continuous function $u\colon [0,\tau] \to X$ a mild solution of (III.15) if it satisfies (III.16). This implies existence and uniqueness results for inhomogeneous and semilinear abstract Cauchy problems as discussed in the introduction.

III.4 Decoupling of dynamic boundary value problems

In the last sections we have seen that the coupled system (III.11) is described by the operator A^B with Wentzell boundary conditions given by (III.2).
Decoupling this problem, means to show its equivalence to a system consisting of the two independent Cauchy problems: the interior problem governed by A_0^0 given in (III.4) on X_0 and the elliptic problem with dynamic boundary conditions (III.14) governed by the Dirichlet-to-Neumann operator N given in

(III.9) on the boundary space ∂X. More precisely, we relate properties of the operator A^B on X with properties of the operators A_0^0 on X_0 and N on ∂X.
Our decoupling approach is based on similarity transformations, perturbation arguments and the theory of (one-side coupled) operator matrices developed by Nagel in [Nag89], [Nag90] and Engel in [Eng96], [Eng97b], [Eng98], [Eng99]. The state of art can be found in [Eng97a].

III.4.1 Analytic semigroups

We start with the case of analytic semigroups. This approach goes back to [EF05], where the direction (b) $\Rightarrow$ (a) of the next theorem is proven (see [EF05, Theorem 3.1]). Note that N_μ and N_λ for $\mu, \lambda \in \rho(A_0)$ just differ by a bounded perturbation. Since generation of analytic semigroups is stable under bounded perturbation we can restrict ourself for simplicity of the representation to the Dirichlet-to-Neumann operator $N = N_0$.

In the sequel we will need the following operator.

Notation III.4.1.1. *The operator $G_0 : D(G_0) \subset X \to X$ is defined by*

$$G_0 f \coloneqq (A_m f - L_0 B) f, \quad D(G_0) \coloneqq D(A_0) = D(A_m) \cap \ker(L).$$

We make Assumptions III.1.1.2 and obtain the following result (see [BE19, Theorem 3.1]).

Theorem III.4.1.2. *The following statements are equivalent.*

(a) *The operator A^B given by* (III.2) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *The operator A_0^0 and the Dirichlet-to-Neumann operator N generate analytic semigroups of angle $\alpha > 0$ on X_0 and ∂X, respectively.*

(c) *The operator $G_0^0 \coloneqq G_0|_{X_0}$ and the Dirichlet-to-Neumann operator N generate analytic semigroups of angle $\alpha > 0$ on X_0 and ∂X, respectively.*

This shows that, in the case of analytic semigroups, the problem (III.11) can be decomposed into the problems (III.14) and the interior problem governed by A_0^0 or G_0^0 on X_0. It indicated the following question: Can we replace the analytic semigroup property by other properties such that an analogous result holds true?
Our theorem has many variants which use different operators to describe the interior dynamics. We refer to [BE19] for more details.

III.4.2 Strongly continuous semigroups

Unfortunately we cannot prove an analogous result to Theorem III.4.1.2 for strongly continuous semigroups in general, since its proof based on perturbation techniques. Nevertheless there are partial analogues assuming stronger conditions on the feedback operator B. This can be interpreted in the following way: Since strongly continuous semigroups regularize less than analytic semigroups, the perturbation of the domain given by the feedback operator B need to be more bounded.

Assuming that the abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists, we obtain by (I.19) the decomposition $X = X_0 \oplus \ker(A_m)$. In this section we study the case where the feedback operator $B : D(B) \subset X \to \partial X$ is bounded on one subspace of this decomposition. This allows us to decouple the generator property of A^B as in the previous subsection without assuming analyticity. Note that N_μ and N_λ for $\mu, \lambda \in \rho(A_0)$ just differ by a bounded perturbation. Since generation of strongly continuous semigroups is stable under bounded perturbation, we can restrict ourself to the Dirichlet-to-Neumann operator $N = N_0$.

Feedback operator bounded on X_0

First we study the case where B is bounded on X_0. We work in Setting III.1.1.1 and instead of Assumptions III.1.1.2 we make the following assumptions.

Assumptions III.4.2.1. (i) *The operator A_0 with Dirichlet boundary conditions is invertible and hence the abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists and is bounded.*

(ii) *The operator A_0 with Dirichlet boundary conditions is a weak Hille–Yosida operator on X.*

(iii) *The operator $B_0 := B|_{X_0}$ is bounded, i.e., there exists $M \geq 0$ such that*

$$\|Bf\|_{\partial X} \leq M \cdot \|f\|_X \quad \textit{for all } f \in X_0.$$

Note that the invertibility of A_0 in (i) can be replaced by $\rho(A_0) \neq \emptyset$ by considering $A_0 - \lambda$ for $\lambda \in \rho(A_0)$.

If the operator A_0^0 and the Dirichlet-to-Neumann operator N generate strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively, then it follows from

[Eng99, Lemma 3.2] that for $t > 0$ the operators $R(t) : D(N) \subset \partial X \to X$ given by

$$\text{(III.17)}\quad R(t)x := A_m \int_0^t T(s) \cdot A_0^{-1} L_0 \cdot S(t-s) N x \, \mathrm{d}s$$

are well-defined. Under these assumptions the following holds (see [BE20a, Theorem 5.4]).

Theorem III.4.2.2. *The following statements are equivalent.*

(a) *The operator A^B defined in* (III.2) *generates a strongly continuous semigroup on X.*

(b) (i) *The operator A_0^0 and the Dirichlet-to-Neumann operator N generate strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on X_0 and ∂X, respectively, and*

(ii) *There exists $t_0 > 0$ and $M \geq 0$ such that*

$$\text{(III.18)}\quad \|R(t)x\|_{X_0} \leq M \cdot \|x\|_{\partial X}$$

for all $t \in (0, t_0]$ and $x \in D(N)$.

(c) (i) *The operator $G_0^0 := G_0|_{X_0}$ and the Dirichlet-to-Neumann operator N generate strongly continuous semigroups on X_0 and ∂X, respectively, and*

(ii) *There exists $t_0 > 0$ and $M \geq 0$ such that* (III.18) *holds.*

The previous result can also be interpreted in the following way: If the operator $R(t)$ in (III.17) remains norm bounded for $t \downarrow 0$, then the (coupled) problem (III.11) is wellposed if and only if the (independent) Cauchy problems for A_0^0 on X_0 and N on ∂X are. Coupled problems with bounded feedback operator B can be interpreted as an essentially uncoupled system of two equations. A similar result is shown in [Nic04b, Theorem 3.3.6].

Feedback operator bounded on $\ker(A_m)$

We now study the case where $B|_{\ker(A_m)}$ is bounded which, for A_0-bounded B, is equivalent to the fact that the Dirichlet-to-Neumann operator N becomes bounded on ∂X. We use Setting III.1.1.1. Our starting point are the following hypotheses.

Assumptions III.4.2.3. (i) *The abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists and is bounded.*

(ii) *The operator* A_0 *with Dirichlet boundary conditions is a weak Hille–Yosida operator on* X*.*

(iii) *The operator* B *is relatively* A_m*-bounded, i.e.,* $D(A_m) \subseteq D(B)$ *and there exist* $a, b \geq 0$ *such that*

$$\|Bf\|_{\partial X} \leq a \cdot \|A_m f\|_X + b \cdot \|f\|_X \quad \text{for all } f \in D(A_m).$$

Note that by the closed graph theorem $L_0 : \partial X \to [D(A_m)]$ is bounded, hence assumption (ii) above implies that $N = BL_0 \in \mathcal{L}(\partial X)$. In this situation the following holds (see [BE20a, Theorem 5.5]).

Theorem III.4.2.4. *The following statements are equivalent.*

(a) *The operator* A^B *defined in* (III.2) *generates a strongly continuous semigroup on* X*.*

(b) *the operator* $G_0^0 := G_0|_{X_0}$ *generates a strongly continuous semigroup on* X_0*.*

The previous result can be interpreted in the following way: Since the Dirichlet-to-Neumann operator N is bounded, there is essentially just an interior dynamics. Hence we only need assumptions on G_0^0.

III.4.3 Compactness

Having obtained generation properties of the operators A^B and A_0 and N, we concentrate on qualitative properties of these semigroups. Note that N_μ and N_λ for $\mu, \lambda \in \rho(A_0)$ just differ by a bounded perturbation. Since compactness of the resolvent is stable under bounded perturbation, we can restrict ourself to the Dirichlet-to-Neumann operator $N = N_0$.

Again we work in Setting III.1.1.1 and make Assumptions III.1.1.2.

By [EN00, Theorem II.4.29] an analytic[1] semigroup is compact if and only if its generator has compact resolvent. Hence the following result relates compactness of the semigroups governed by A_0, N and A^B in the case of analytic semigroups (see [BE19, Corollary 3.2]).

Corollary III.4.3.1. *The following statements are equivalent.*

(a) *The operator* A^B *defined in* (III.2) *has compact resolvent on* X*.*

[1] This works for a larger class of semigroups, see [EN00, Theorem II.4.29].

(b) *the operators A_0 and N have compact resolvents on X and ∂X, respectively.*

For more subtle criteria for compactness of the resolvents of A^B and N we refer to [BE20b, Corollary 3.9].

III.4.4 Positivity

In this section we consider the positivity of the semigroups generated by A_0^0, N_λ and A^B and the relationship between them. For this purpose we need to deal with all N_λ for $\lambda \in \rho(A_0)$ since positivity of the resolvent is not stable under arbitrary bounded perturbation.

We work with Setting III.1.1.1 and make Assumptions III.1.1.2. Further, we make the following additional assumptions.

Assumptions III.4.4.1.

(i) *The state space X and the boundary space ∂X are Banach lattices.*

(ii) *The trace operator $L\colon X \to \partial X$ is positive.*

Now we give a decoupling result for the positivity of the resolvent of A^B (cf. [BE20b, Theorem 5.10]).

Theorem III.4.4.2. *Assume that A_0 and A^B are weak Hille-Yosida operators on X. If L_λ is positive for $\lambda \geq \omega > s(A_0)$ and A_0 have positive resolvent, then following statements are equivalent.*

(a) *A^B is resolvent positive on X;*

(b) (i) *N_λ are resolvent positive on ∂X for all $\lambda > s(A_0)$;*

(ii) *$Bf \geq 0$ for all $f \in D(A_0)_+$.*

The typical applications of our theory are to spaces $X = \mathrm{C}(K)$ for a compact space K. In this situation we obtain a stronger result.

Corollary III.4.4.3. *Take the Banach lattice $X = \mathrm{C}(K)$ for some compact space K and $\partial X = \mathrm{C}(\partial K)$. Assume that A_0 has positive resolvent. Assume that A_0 and A^B are weak Hille-Yosida operators on X. If L_λ is positive for $\lambda \geq \omega > s(A_0)$ and A_0 have positive resolvent, then following statements are equivalent.*

(a) *A^B generates a strongly continuous semigroup of positive operators on X;*

(b) (i) *N_λ generate strongly continuous semigroups of positive operators on ∂X for all $\lambda > s(A_0)$;*

(ii) $Bf \geq 0$ *for all* $f \in D(A_0)_+$.

Note that our decoupling result for positivity is imperfect. The positivity of the resolvents of A_0 and N_λ for all $\lambda > s(A_0)$ imply the positivity of the resolvent of A^B, but conversely the positivity of the resolvent of A^B only implies the positivity of the resolvents of N_λ for all $\lambda > s(A_0)$. It is open, if the positivity of the resolvent of A^B also implies the positivity of the resolvent of A_0.

III.4.5 Maximal regularity

The concept of maximal regularity is related to the solvability of quasi linear equations as we briefly discussed in the introduction. For this property it is natural to work in so called *unconditional martingale differences* (UMD) Banach spaces (see [ABHN11, Page 198]). Note that $\mathrm{L}^p(\Omega)$ spaces for $p \in [1, \infty)$ are UMD, while $\mathrm{C}(\overline{\Omega})$ does not have the UMD property. Since Setting III.1.1.1 is tailored towards state spaces of continuous functions it does not make sense to work with this property. So it is more natural to work in Setting III.1.2.1 which is tailored to state spaces of p-integrable functions. We assume Assumptions III.1.2.2 and make the following additional assumptions.

Assumptions III.4.5.1. *The state space X and the boundary space ∂X are UMD spaces.*

We denote by $\omega(A)$ the *growth bound* of A, see [EN00, Definition I.5.6]. Note that maximal regularity implies generation of an analytic semigroup and hence by [EN00, Corollary IV.3.12] the spectral mapping theorem holds and the spectral bound equals the growth bound.

Maximal regularity is also stable under our decoupling. More precisely we obtain the following result.

Proposition III.4.5.2. *Let $r \in (1, \infty)$. The following statements are equivalent.*

(a) *The operator $\mathcal{A}^B - \omega(A^B)$ defined in* (III.7) *has maximal L^r-regularity on $X \times \partial X$.*

(b) *The operator $A_0 - \omega(A_0)$ has maximal L^r-regularity on X and the Dirichlet-to-Neumann operator $N - \omega(N)$ has maximal L^r-regularity on ∂X.*

The proof of (b) $\Rightarrow$ (a) is analogous to the proof of [BtE20, Theorem 4.5 (d)]. The other direction follows from similar arguments.

As mentioned in the introduction, (III.4.5.2) leads to the existence of solutions of quasi linear equations with dynamic boundary conditions on L^p-spaces.

III.5 Perturbation theory for operators with dynamic boundary conditions

In the sequel we concentrate on perturbations of analytic semigroups. In applications the verification of the assumptions of Theorem III.4.1.2 is difficult and cannot be performed directly for many important operators. However, using perturbation theory we can built complicated operators from simpler ones. There are two ways to perturb the operator with Wentzell boundary conditions, defined in (III.2): a perturbation on the domain and a perturbation at the boundary. More precisely let $B + CL : D(B + CL) \subset X \to \partial X$ be given by

$$(B + CL)f := Bf + CLf, \quad D(B + CL) := D(B) \cap D(CL),$$

where $D(CL) := \{f : Lf \in D(C)\}$ for $C : D(C) \subset \partial X \to \partial X$, and $P : D(P) \subset X \to X$ is a relatively A_m-bounded perturbation. Consider the operator $(A + P)^{B+CL} : D((A + P)^{B+CL}) \subseteq X \to X$ given by

$$\begin{aligned} &(A + P)^{B+CL} f := A_m f + P f, \\ &D\big((A + P)^{B+CL}\big) := \left\{ f \in D(A_m) \cap D(B) \cap D(CL) \colon \begin{array}{r} LA_m f + Pf \\ = Bf + CLf \end{array} \right\}. \end{aligned} \tag{III.19}$$

In the sequel we use Setting III.1.1.1 and make Assumptions III.1.1.2.

The following perturbation statement holds for the corresponding Dirichlet operators (cf. [BE19, Lemma 4.6]).

Lemma III.5.0.1. *Let $P : D(P) \subset X \to X$ be a relatively A_m-bounded perturbation. Then for $\lambda \in \rho(A_0) \cap \rho(A_0 + P)$ the perturbed Dirichlet-to-Neumann operator $L_\lambda^{A_m+P}$ exists and satisfies*

$$L_\lambda^{A_m+P} = L_\lambda^{A_m} + R(\lambda, A_0 + P) P L_\lambda^{A_m} = L_\lambda^{A_m} + R(\lambda, A_0) P L_\lambda^{A_m+P}.$$

In particular the difference $L_\lambda^{A_m+P} - L_\lambda^{A_m}$ is bounded from ∂X to $[D(A_0)]$.

Therefore we obtain that L_λ exists and is bounded for some $\lambda \in \rho(A_0)$ if and only if for all $\lambda \in \rho(A_0)$. This implies the following perturbation result for Dirichlet-to-Neumann operators (see [BE19, Proposition 4.7]).

Proposition III.5.0.2. *Let $P : D(P) \subset X \to X$ be a relatively A_m-bounded perturbation. Then for $\lambda \in \rho(A_0) \cap \rho(A_0 + P)$ the perturbed Dirichlet-to-Neumann* *operator $N_\lambda^{A_m+P}$ exists, satisfies $D(N_\lambda^{A_m}) = D(N_\lambda^{A_m+P})$ and*

$$N_\lambda^{A_m+P,B} = N_\lambda^{A_m,B} + BR(\lambda, A_0+P)PL_\lambda^{A_m} = N_\lambda^{A_m,B} + BR(\lambda, A_0)PL_\lambda^{A_m+P}.$$

In particular the difference $N_\lambda^{A_m} - N_\lambda^{A_m+P}$ is bounded.

Note that the domain of $D(N_\lambda)$ is independent of $\lambda \in \rho(A_0)$.

Combining this result with Theorem III.4.1.2 yields the following statements. For details we refer to [BE19, Section 4], in particular [BE19, Theorem 4.2].

Theorem III.5.0.3. *Let $P\colon D(P) \subset X \to X$ be relatively A_m-bounded with A_0-bound 0 and let $C\colon D(C) \subset \partial X \to \partial X$ be relatively N^{B_0}-bounded of bound 0. Then the following statements are equivalent.*

(a) *$(A+P)^{B+CL}$ in* (III.19) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^B defined in* (III.2) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(c) *A_0 is sectorial of angle $\alpha > 0$ on X and N^{B_0} generates an analytic semigroup of angle $\alpha > 0$ on ∂X.*

Further, we obtain a perturbation result, where the operator with dynamic boundary conditions is be perturbed by a generator on the boundary and a relatively bounded perturbation in the interior. We refer to [BE19, Theorem 4.3].

Theorem III.5.0.4. *Let $P\colon D(P) \subset X \to X$ be relatively A_m-bounded with A_0-bound 0 and let N^{B_0} be relatively C-bounded of bound 0 for some $C\colon D(C) \subset \partial X \to \partial X$. Then the following statements are equivalent.*

(a) *$(A+P)^{B+CL}$ in* (III.19) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^{CL} generates an analytic semigroup of angle $\alpha > 0$ on X.*

(c) *A_0 is sectorial of angle $\alpha > 0$ on X and C generates an analytic semigroup of angle $\alpha > 0$ on ∂X.*

III.6 Spectral theory of operators with dynamic boundary conditions

The decoupling of the operator A^B with Wentzell boundary conditions into the operator A_0 with Dirichlet boundary conditions and the Dirichlet-to-Neumann operator N respects many spectral properties.

For a closed, linear operator $A\colon D(A) \subseteq E \to E$ on a Banach space E one defines the spectrum and its fine structure by

$$\begin{aligned}
\rho(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \quad \text{is invertible}\\ \text{with bounded inverse}\end{matrix}\right\} && \text{the } \textit{resolvent set} \text{ of } A,\\
\sigma(A) &:= \mathbb{C} \setminus \rho(A) && \text{the } \textit{spectrum} \text{ of } A,\\
\sigma_p(A) &:= \{\lambda \in \mathbb{C}\colon \lambda - A \text{ is not injective}\} && \text{the } \textit{point spectrum} \text{ of } A,\\
\sigma_a(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \quad \text{is not injective}\\ \text{or has nonclosed range}\end{matrix}\right\} && \begin{matrix}\text{the } \textit{approximative}\\ \textit{point spectrum} \text{ of } A,\end{matrix}\\
\sigma_c(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \quad \text{is injective with}\\ \text{dense, nonclosed range}\end{matrix}\right\} && \begin{matrix}\text{the } \textit{continuous}\\ \textit{spectrum} \text{ of } A,\end{matrix}\\
\sigma_r(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \quad \text{is injective with}\\ \text{nonclosed range}\end{matrix}\right\} && \begin{matrix}\text{the } \textit{residual}\\ \textit{spectrum} \text{ of } A,\end{matrix}\\
\sigma_{ess}(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\operatorname{codim}(\operatorname{rg}(\lambda - A)) \ = \ \infty\\ \text{or } \dim(\ker(\lambda - A)) = \infty\end{matrix}\right\} && \begin{matrix}\text{the } \textit{essential}\\ \textit{spectrum} \text{ of } A,\end{matrix},\\
\sigma_d(A) &:= \sigma(A) \setminus \sigma_{ess}(A) && \begin{matrix}\text{the } \textit{discrete}\\ \textit{spectrum} \text{ of } A,\end{matrix}.
\end{aligned}$$

In the sequel we use Setting III.1.1.1 and make Assumptions III.1.1.2. Then we obtain the following result (see [BE20b, Theorem 3.7 & 3.8]).

Theorem III.6.0.1. *Assume that there exists $\lambda_0 \in \rho(A_0)$ such that N_{λ_0} is a weak Hille-Yosida operator on ∂X and take $\lambda \in \rho(A_0)$. Then*

(i) *$\lambda \in \rho(A^B)$ if and only if $\lambda \in \rho(N_\lambda)$. Moreover the following resolvent identity*

$$R(\lambda, A^B) = R(\lambda, A_0) + L_\lambda R(\lambda, N_\lambda)(BR(\lambda, A_0) + L)$$

holds.

(ii) *$\lambda \in \sigma_p(A^B)$ if and only if $\lambda \in \sigma_p(N_\lambda)$. In this case the dimensions of the* *eigenspaces match, i.e.* $\dim(\ker(\lambda - A^B)) = \dim(\ker(\lambda - N_\lambda))$.

(iii) $\lambda \in \sigma_a(A^B)$ *if and only if* $\lambda \in \sigma_a(N_\lambda)$.

(iv) $\lambda \in \sigma_c(A^B)$ *if and only if* $\lambda \in \sigma_c(N_\lambda)$.

(v) $\lambda \in \sigma_r(A^B)$ *if and only if* $\lambda \in \sigma_r(N_\lambda)$.

(vi) $\lambda \in \sigma_d(A^B)$ *if and only if* $\lambda \in \sigma_d(N_\lambda)$.

(vii) $\lambda \in \sigma_{ess}(A^B)$ *if and only if* $\lambda \in \sigma_{ess}(N_\lambda)$.

Hence the spectrum and its fine structure of A^B is characterized by the Dirichlet-to-Neumann operators N_λ. This result can be seen as an abstract analogue of the characteristic equation for the spectral values of delay operators. For more details we refer to [BE20b].

The point spectrum of the Dirichlet-to-Neumann operators N_λ is strongly connected to the point spectrum of operators with Robin boundary conditions. For $\mu \in \mathbb{C}$ define the *operator* $A_B^\mu \colon D(A_B^\mu) \subset X \to X$ *with Robin boundary conditions* associated to μ by

$$A_B^\mu f \coloneqq A_m f, \quad D(A_B^\mu) \coloneqq \{f \in D(A_m) \cap D(B) \colon Bf = \mu \cdot Lf\}.$$

Using these operators the following result holds (see [BE20b, Corollary 3.2]).

Proposition III.6.0.2. *For* $\lambda \in \rho(A_0)$ *and* $\mu \in \mathbb{C}$ *we have*

(i) $\mu \in \sigma_p(N_\lambda)$ *if and only if* $\lambda \in \sigma_p(A_B^\mu)$*;*

(ii) $\dim(\ker(\mu - N_\lambda)) = \dim(\ker(\lambda - A_B^\mu))$.

Assuming $\sigma_p(N_\lambda), \sigma_p(A_B^\mu) \subset \mathbb{R}$ for $\lambda \in \rho(A_0)$ and $\mu \in \mathbb{C}$ we obtain the following result (cf. [BE20b, Theorem 4.14]).

Theorem III.6.0.3. *Denote by* $\lambda_k(\mu)$ *the* k*-th eigenvalue of* A_B^μ*, by* $\lambda_k(\infty)$ *the* k*-th eigenvalue of* A_0 *and by* $\mu_n(\lambda)$ *the* n*-th eigenvalue of* N_λ*.*
Assume that the map $\lambda_k \colon \mathbb{R} \to \mathbb{R} \colon \mu \mapsto \lambda_k(\mu)$ *is strictly monotone decreasing. If* $k \in \mathbb{N}$ *such that* $\lambda_{k+1}(\infty) \neq \lambda_k(\infty)$*, the following statements are equivalent.*

(a) *The* n*-th eigenvalue* μ_n *of the Dirichlet-to-Neumann operator* N_λ *is positive, i. e.*

$$\mu_n(\lambda) > 0$$

for $\lambda \in (\lambda_{k+1}(\infty), \lambda_k(\infty))$.

(b) *The inequality*

$$\lambda_k(\infty) < \lambda_{k+n}(0)$$

holds.

This result can be seen as an abstract analogue of Friedlander's inequality (see [Fri91]). For details we refer to [BE20b].

III.7 Asymptotic behaviour of operators with dynamic boundary conditions

We now focus on the qualitative behaviour of operators with dynamic boundary conditions. In Section III.4 we already considered regularity and positivity properties of the corresponding semigroups. Now we investigate the relationship of the asymptotic behaviour of the semigroups generated by A^B, A_0^0 and N_λ.

We study this question only in the case of strongly continuous semigroups of positive operators. We use Setting III.1.1.1 and Assumptions III.1.1.2. Additional we assume as in Subsection III.4.4.

Assumptions III.7.0.1.

(i) *The state space X and the boundary space ∂X are Banach lattices.*

(ii) *The operator trace operator $L\colon X \to \partial X$ is positive.*

We denote by $s(A)$ the *spectral bound* of an operator A, see [EN00, Definition I.1.12]. By [EN00, Proposition VI.1.14] a strongly continuous semigroups of positive operators is *exponentially stable* (see [EN00, Definition V.1.5]) if and only if the spectral bound (see [EN00, Definition I.1.12]) of the generator A is negative, i.e. $s(A) < 0$. Hence the following theorem characterizes the exponential stability of the semigroups associated to A^B, A_0^0 and N (see [BE20b, Theorem 6.2]).

Theorem III.7.0.2. *Assume that $BR(\lambda, A_0)$ is positive and L_λ are positive operators for large λ. Moreover assume that A_0 have positive resolvent on X and that N_λ generate positive semigroups on ∂X for large λ. Further, let A^B generator of a C_0-semigroup on X. Then $s(A_0) \leq s(A^B)$ and for $\kappa \in \mathbb{R}$ we obtain*

$$s(A^B) < \kappa \quad \Longleftrightarrow \quad s(A_0) < \kappa \text{ and } s(N_\kappa) < \kappa.$$

Using $s(A_0) = s(A_0^0)$, [EN00, Proposition VI.1.14] implies the following (see [BE20b, Corollary 6.4]).

Corollary III.7.0.3. *Assume that $BR(\lambda, A_0)$ are positive and L_λ are positive operators for large λ. Moreover assume that A_0 has positive resolvent on X and that N_λ generate positive semigroups on ∂X for large λ. Further, let A^B generator of a C_0-semigroup on X. Then the semigroup generated by A^B is uniformly exponential stable if and only if the semigroups generated by A_0^0 and N are.*

III.8 Examples

III.8.1 A delay differential operator

In this subsection we apply our approach to operators related to delay differential equations, see [Hal77], [EN00, Section VI.6] and [BP05]. To a Banach space Y we associate the Banach space $X \coloneqq \mathrm{C}([-1,0], Y)$ of all continuous functions on $[-1,0]$ with values in Y equipped with the sup-norm. Moreover, we take a delay operator $\Phi \in \mathcal{L}(X, Y)$ and an operator $C\colon D(C) \subset Y \to Y$. With this notation we consider the *abstract delay differential operator* $A : D(A) \subset X \to X$ given by

$$\text{(III.20)}\quad Af \coloneqq f', \quad D(A) \coloneqq \left\{ f \in \mathrm{C}^1([-1,0], Y) : \begin{array}{l} f(0) \in D(C) \text{ and} \\ f'(0) = Cf(0) + \Phi f \end{array} \right\}$$

which governs a delay differential equation, see [EN00, Section VI.6] for details.

Generation of semigroups

We study the generation property of the operator A given in (III.20).

Choosing $X = \mathrm{C}([-1,0], Y)$, $\partial X = Y$, $A_m = \frac{\mathrm{d}}{\mathrm{d}r}$ with domain $D(A_m) = \mathrm{C}^1([-1,0], Y)$, $L = \delta_0$, $B = C\delta_0 + \Phi$, we obtain $A = A^B$ and the operator $B_0 \coloneqq B|_{X_0} = \Phi$ is bounded. Note that A_0^0 generates a strongly continuous semigroup on $\mathrm{C}_0([-1,0), Y)$ and that A_0 is a weak Hille-Yosida operator on $\mathbb{C}([-1,0], Y)$. Using the bounded perturbation theorem and Theorem III.4.2.2 we conclude the following result. For details we refer to [BE20a, Theorem 6.1].

Theorem III.8.1.1. *The operator A given by* (III.20) *generates a strongly continuous semigroup on* $\mathrm{C}([-1,0], Y)$ *if and only if C generates a strongly continuous semigroup on Y.*

For a different proof of this result see [EN00, Theorem VI.6.1].

Since $A_0 := \frac{\mathrm{d}}{\mathrm{d}r}, D(A_0) := \mathrm{C}_0^1([-1,0],Y) := \{f \in \mathrm{C}^1([0,1],Y) : f(0) = 0\}$ has compact resolvent, one obtains the following by bounded perturbation and Theorem III.6.0.1.

Corollary III.8.1.2. *The operator A given by* (III.20) *has compact resolvent on $X = \mathrm{C}([-1,0],Y)$ if and only if the operator C has compact resolvent on Y.*

Spectral theory

Once the wellposedness of delay equations is obtained, we now concentrate on its spectral properties.

Consider the operator $A_0 := \frac{\mathrm{d}}{\mathrm{d}r}$ with domain $D(A_0) := \mathrm{C}_0^1([-1,0],Y)$. Note that A_0 has empty spectrum and that its resolvent is given by

(III.21) $$(R(\lambda, A_0)f)(s) = \int_s^0 e^{\lambda(s-r)} f(r)\,\mathrm{d}r =: H_\lambda f(s).$$

Moreover the abstract Dirichlet operator is

(III.22) $$L_\lambda y = \varepsilon_\lambda \otimes y$$

where $\varepsilon_\lambda(s) := e^{\lambda s}$. Denote by $\Phi_\lambda := \Phi L_\lambda$ and see that $N_\lambda = C + \Phi_\lambda$. Theorem III.6.0.1 leads to the following.

Theorem III.8.1.3. *We obtain*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(C + \Phi_\lambda)$. *Moreover its resolvent can be expressed as*

$$R(\lambda, A^B)f = H_\lambda f + (\varepsilon_\lambda \otimes R(\lambda, \Phi_\lambda))(\Phi H_\lambda f + f(0))$$

for $f \in \mathrm{C}([-1,0],Y)$.

(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(C + \Phi_\lambda)$. *In this case the dimensions of the eigenspaces coincide, i.e.* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - C - \Phi_\lambda))$.

(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(C + \Phi_\lambda)$.

(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(C + \Phi_\lambda)$.

(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(C + \Phi_\lambda)$.

(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(C + \Phi_\lambda)$.

(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(C + \Phi_\lambda)$.

This result improves [EN00, Proposition VI.6.7] and [BP05, Proposition 3.19 & Lemma 3.20]. It can be seen as a generalized characteristic equation for delay equations. For more details see [BE20b, Corollary 7.1].
For the uncoupled case, i.e. $\Phi = 0$, we obtain the following corollary.

Corollary III.8.1.4. *Under above assumptions*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(C)$. *Moreover its resolvent can be expressed by*

$$R(\lambda, A^B)f = H_\lambda f + (\varepsilon_\lambda \otimes R(\lambda, C))f(0)$$

for $f \in \mathrm{C}([-1,0], Y)$.

(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(C)$. *In this case the dimensions of the eigenspaces are equal, i.e.* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - C))$.

(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(C)$.

(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(C)$.

(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(C)$.

(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(C)$.

(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(C)$.

Positivity and asymptotic behaviour

Now we study positivity of the semigroup generated by A and use this property to obtain uniformly exponential stability.

We assume that Y is a Banach lattice, hence $X = \mathrm{C}([-1,0], Y)$ is also a Banach lattice. By (III.21) the operator A_0 has positive resolvent. Further, by (III.22) the Dirichlet operator L_λ is positive for $\lambda \in \mathbb{R}$. Using $B|_{X_0} = \Phi$ and $N_\lambda = C + \Phi_\lambda$ we obtain from the positive perturbation theorem and Corollary III.4.4.3 this characterization.

Theorem III.8.1.5. *Assume that the delay operator* Φ *is positive and* C *generates a strongly continuous semigroup of positive operators on* Y. *Then the operator* A *given by* (III.20) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}([-1,0], Y)$.

For this statements see also [EN00, Theorem IV.6.11]. Now applying Corollary III.7.0.3 and using $s(A_0) < 0$ yields the following result.

Theorem III.8.1.6. *Assume that the delay operator Φ is positive and C generates a strongly continuous semigroup of positive operators on Y. Denote the semigroup on $\mathrm{C}\big([0,1],Y\big)$ generated by A by $(T_A(t))_{t\geq 0}$ and the semigroup on Y generated by N by $(T_N(t))_{t\geq 0}$. Then $(T_A(t))_{t\geq 0}$ is uniformly exponential stable on $\mathrm{C}\big([-1,0],Y\big)$ if and only if $(T_N(t))_{t\geq 0}$ is on Y.*

For this statement see also [EN00, Corollary IV.6.16].

Wellposedness of delay equations

After studying the semigroup generated by A given in (III.20) and its properties, we now concentrate on the associated Cauchy problem (ACP).

Assume that C is the generator of a strongly continuous semigroup on the Banach space Y. By Theorem III.8.1.1 the delay differential operator A given in (III.20) generates a C_0-semigroup on $\mathrm{C}([-1,0],Y)$. Now Theorem III.2.1.1 implies the following result.

Corollary III.8.1.7. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) & \text{for } t \geq 0,\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = Cu(t,0) + \Phi u(t,\cdot) & \text{for } t \geq 0, \\ \quad u(0,r) = u_0(r) & \text{for } r \in [-1,0], \end{cases}$$

with dynamic boundary conditions is wellposed[2] *on $\mathrm{C}([-1,0],Y)$. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$. Moreover, if the spectral bound $s(C) < 0$, then $\sup_{r\in[-1,0]} \|u(t,r)\|_Y \to 0$ for $t \to \infty$ for $u_0 \in D(A)$.*

Compare this statement to [EN00, Corollary VI.6.3]. Moreover we obtain from Theorem III.2.2.1 the following corollaries.

Corollary III.8.1.8. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}([-1,0],Y)$ and $\tilde{g}\colon [0,\tau] \to Y$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(s) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ such that $f(t)(0) =$*

[2]Recall from the introduction that a (classical solution) of an abstract Cauchy problem (ACP) is a continuously differentiable function $u\colon \mathbb{R}_+ \to X$ such that $u(t) \in D(A)$ for all $t \geq 0$ and (ACP) is fulfilled. Further, recall that (ACP) is wellposed if for all $u_0 \in D(A)$ there exists a unique (classical) solution u of (ACP) which depends continuously on the initial value u_0, i. e., $u_0^n \to u_0$ implies $u^n(t) \to u(t)$ uniformly on compact intervals $[0,t_0]$.

$g(t)$ for all $t \in [0,\tau]$. Then, for all $u_0 \in D(A)$ the problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) + f(t)(r), & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = Cu(t,0) + \Phi u(t,\cdot) + g(t), & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [-1,0], \end{cases}$$

has a unique solution[3] *on $\mathrm{C}([-1,0],Y)$. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial data u_0 and f are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.*

Corollary III.8.1.9. *Let $F\colon [0,\tau] \times \mathrm{C}([-1,0],Y) \to \mathrm{C}([-1,0],Y)$ and $G\colon [0,\tau] \times Y \to Y$ be continuously differentiable functions such that $F(t,u(t,\cdot))(0) = G(t,u(t,0))$. Then, for all $u_0 \in D(A)$ the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = Cu(t,0) + \Phi u(t,\cdot) + G(t,u(t,0)) & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [-1,0], \end{cases}$$

admits a unique solution on $\mathrm{C}([-1,0],Y)$. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial data u_0 and F are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Wellposedness of elliptic delay equations

We consider the elliptic problem with dynamic boundary conditions associated to A given by (III.20) and assume that C is the generator of a strongly continuous semigroup on the Banach space Y. Since by the bounded perturbation the Dirichlet-to-Neumann operator $N_\lambda = C + \Phi L_\lambda$ generates a C_0-semigroup for all $\lambda \in \mathbb{C}$. We conclude by Theorem III.3.1.1.

Corollary III.8.1.10. *For all $\lambda \in \mathbb{C}$ and $y_0 \in D(C)$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) & \text{for } t \geq 0,\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = Cu(t,0) + \Phi u(t,\cdot), & \text{for } t \geq 0, \\ u(0,0) = y_0 \end{cases}$$

with dynamic boundary conditions is wellposed on Y. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

[3] Here and in the sequel solution always means classical solution.

Moreover, Theorem III.3.2.1 implies the following results.

Corollary III.8.1.11. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}([-1,0],Y)$ and $\tilde{g}\colon [0,\tau] \to Y$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := f_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. Then, for all $\lambda \in \mathbb{C}$ and $y_0 \in D(C)$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r} u(t,r) + f(t)(r), & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = Cu(t,0) + \Phi u(t,\cdot) + g(t), & \text{for } t \in [0,\tau], \\ u(0,0) = y_0 \end{cases}$$

admits a unique solution on Y. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial data u_0 and f, g are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.1.12. *Let $F\colon [0,\tau] \times \mathrm{C}([-1,0],Y) \to \mathrm{C}([-1,0],Y)$ and $G\colon [0,\tau] \times Y \to Y$ continuously differentiable functions. Then, for all $\lambda \in \mathbb{C}$ and $y_0 \in D(C)$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r} u(t,r) + F(t,u(t,\cdot))(r), & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = Cu(t,0) + \Phi u(t,\cdot) + G(t,u(t,0)), & \text{for } t \in [0,\tau], \\ u(0,0) = y_0 \end{cases}$$

has a unique solution on Y. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial data u_0 and F, G are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

III.8.2 Banach space-valued second derivative

Instead of the first derivative with delay boundary conditions we now consider the second derivative with delay boundary conditions. We associate to an arbitrary Banach space Y the Banach space $X := \mathrm{C}([0,1],Y)$ of all continuous functions on $[0,1]$ with values in Y equipped with the sup-norm. Moreover, we take $\Phi \in \mathcal{L}(X,Y^2)$ and a weak Hille-Yosida operator $(\mathcal{C}, D(\mathcal{C}))$ on the Banach space $Y^2 := Y \times Y$. We consider the operator $A\colon D(A) \subset X \to X$ given by

$$Af := f'', \quad D(A) := \left\{ f \in \mathrm{C}^2([0,1],Y)\colon \begin{array}{l} \binom{f(0)}{f(1)} \in D(\mathcal{C}), \\ \binom{f''(0)}{f''(1)} = \Phi f + \mathcal{C}\binom{f(0)}{f(1)} \end{array} \right\}. \tag{III.23}$$

Generation of semigroups

We start with the investigation of the semigroup property of the operator A given by (III.23).

Choose $X = \mathrm{C}\big([0,1],Y\big)$, $\partial X = Y^2$, $A_m = \frac{d^2}{dr^2}$ with domain $D(A_m) = \mathrm{C}^2\big([0,1],Y\big)$, $L = \binom{\delta_0}{\delta_1}$, $B = \mathcal{C} \cdot \binom{\delta_0}{\delta_1} + \Phi$ leading to $A = A^B$. Moreover, $B_0 := B|_{X_0} = \Phi$ is bounded and A_0 is sectorial of angle $\frac{\pi}{2}$ on $\mathrm{C}\big([0,1],Y\big)$. Now Theorem III.4.1.2 and Theorem III.4.2.2 imply the following result. For details and generalizations see [BE19, Example 5.2].

Theorem III.8.2.1. *The operator A given by* (III.23) *generates a strongly continuous semigroup on* $\mathrm{C}\big([0,1],Y\big)$ *if and only if $\mathcal{C}$ does on Y^2. Moreover this semigroup can be extended to an analytic semigroup of angle $\alpha \in (0, \frac{\pi}{2}]$ on* $\mathrm{C}\big([0,1],Y\big)$ *if and only if the operator $\mathcal{C}$ generates an analytic semigroup of angle $\alpha \in (0, \frac{\pi}{2}]$ on Y^2.*

Since $A_0 := \frac{d^2}{dr^2}$ with domain $D(A_0) := \mathrm{C}_0^2\big((0,1),Y\big) := \{f \in \mathrm{C}^2\big([0,1],Y\big) \colon f(0) = f(1) = 0\}$ has compact resolvent, one obtains by bounded perturbation and Theorem III.6.0.1 the following result.

Corollary III.8.2.2. *The operator A given by* (III.23) *has compact resolvent on* $\mathrm{C}([-1,0],Y^2)$ *if and only if the operator $\mathcal{C}$ has compact resolvent on Y^2.*

Spectral theory

Note that A_0 has compact resolvent and hence $\sigma(A_0) = \sigma_p(A_0) = \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$. Theorem III.6.0.1 leads to the following.

Theorem III.8.2.3. *For $\lambda \in \mathbb{C} \setminus \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$ we obtain*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(\mathcal{C} + \Phi_\lambda)$.

(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(\mathcal{C} + \Phi_\lambda)$. *In this case the dimensions of the eigenspaces coincide, i.e.* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - \mathcal{C} - \Phi_\lambda))$.

(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(\mathcal{C} + \Phi_\lambda)$.

(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(\mathcal{C} + \Phi_\lambda)$.

(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(\mathcal{C} + \Phi_\lambda)$.

(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(\mathcal{C} + \Phi_\lambda)$.

(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(\mathcal{C} + \Phi_\lambda)$.

Positivity and asymptotic behaviour

Finally, we study positivity of the semigroup generated by A and use this to obtain a simple criterion for uniformly exponential stability.

We assume that Y is a Banach lattice. Then $X = \mathrm{C}([0,1], Y)$ and $Y^2 := Y \times Y$ are Banach lattices. By Hopf's maximum principle (see [GT01, Theorem 3.5]) it follows that the operator A_0 has positive resolvent. Further, it follows, by a direct calculation or the Hopf maximum principle, see [GT01, Theorem 3.5] that the Dirichlet operator L_λ is positive for $\lambda > 0$. Using $B|_{X_0} = \Phi$ and $N_\lambda = C + \Phi_\lambda$ it follows from the positive perturbation theorem and Corollary III.4.4.3

Theorem III.8.2.4. *Assume that the delay operator Φ is positive and $\mathcal{C}$ generates a strongly continuous semigroup of positive operators on Y^2. Then A given by* (III.23) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}([0,1], Y)$.

Now applying Corollary III.7.0.3 and using the fact that $s(A_0) < 0$ yields the following result.

Theorem III.8.2.5. *Assume that the delay operator Φ is positive and $\mathcal{C}$ generates a strongly continuous semigroup of positive operators on* $\mathrm{C}([0,1], Y)$. *Denote the semigroup on* $\mathrm{C}([0,1], Y)$ *generated by A by $(T_A(t))_{t\geq 0}$ and the semigroup on Y^2 generated by N by $(T_N(t))_{t\geq 0}$. Then $(T_A(t))_{t\geq 0}$ is uniformly exponentially stable on* $\mathrm{C}([0,1], Y)$ *if and only if $(T_N(t))_{t\geq 0}$ is on Y^2.*

Wellposedness of parabolic equations with dynamic boundary conditions

Combining Theorem III.2.1.1 with the results of the previous sections the following holds for every generator $\mathcal{C}$ of a C_0-semigroup on a Banach space Y^2 and a boundary functional $\Phi \in \mathcal{L}(C[0,1], Y), Y^2)$.

Corollary III.8.2.6. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2} u(t,r) & \text{for } t \geq 0,\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C} \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot) & \text{for } t \geq 0, \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

with dynamic boundary conditions is wellposed on $\mathrm{C}([0,1], Y)$. *Assume additionally that the semigroup generated by $\mathcal{C}$ is analytic, then the solution $t \to u(t,r)$*

is analytic for all $r \in [0,1]$ *and* $u(t,\cdot) \in \mathrm{C}^\infty([0,1],Y)$ *for* $t > 0$*. Moreover, if the semigroup generated by* $\mathcal{C}$ *is positive and the delay operator* Φ *is positive, then the solution* $u(t,\cdot)$ *is positive for* $t \geq 0$ *whenever the initial value* u_0 *is positive. Moreover, if* $s(\mathcal{C}) < 0$*, then* $\sup_{r\in[-1,0]} \|u(t,r)\|_Y \to 0$ *for* $t \to \infty$ *for* $u_0 \in D(A)$*.*

Moreover, Theorem III.2.2.1 implies the following corollaries.

Corollary III.8.2.7. *Let* $\tilde{f}\colon [0,\tau] \to \mathrm{C}([0,1],Y)$ *and* $\tilde{g}\colon [0,\tau] \to Y^2$ *integrable functions and* $f(t) \coloneqq f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$*,* $g(t) \coloneqq g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ *such that* $\binom{f(t)(0)}{f(t)(1)} = g(t)$ *for all* $t \in [0,\tau]$*. For all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2}u(t,r) + f(t)(r), & \text{for } t \in [0,\tau],\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot) + g(t), & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

admits a unique solution on $\mathrm{C}([-1,0],Y)$*. Assume the semigroup generated by* $\mathcal{C}$ *is positive and the delay operator* Φ *is positive. Then the solution* $u(t,\cdot)$ *to be positive for* $t \geq 0$ *whenever the initial data* u_0 *and* f *are positive.*

Corollary III.8.2.8. *Let* $F\colon [0,\tau] \times \mathrm{C}([0,1],Y) \to \mathrm{C}([0,1],Y)$ *and* $G\colon [0,\tau] \times Y^2 \to Y^2$ *continuously differentiable functions such that* $\binom{F(t,u(t,\cdot))(0)}{F(t,u(t,\cdot))(1)} = G\left(t, \binom{u(t,0)}{u(t,1)}\right)$*. Then, for all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \partial_t u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2}u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t \in [0,\tau],\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot) & \\ \qquad\qquad + G\left(t, \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}\right) & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1],Y)$*. Assume additionally that the semigroup generated by* $\mathcal{C}$ *is positive and the delay operator* Φ *is positive. Then, if the initial data* u_0 *and* F *are positive, the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$*.*

Wellposedness of elliptic equations with dynamic boundary conditions

Finally, we study the elliptic problem with dynamic boundary conditions associated to A given in (III.23).

Since A_0 has compact resolvent, the spectrum of A_0 is $\sigma(A_0) = \sigma_p(A_0) = \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$. By [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$ and we write $\Phi_\lambda := \Phi L_\lambda$ for $\lambda \in \rho(A_0)$. By the bounded perturbation theorem the Dirichlet-to-Neumann operator $N_\lambda = \mathcal{C} + \Phi L_\lambda$ generates for all $\lambda \in \rho(A_0)$ a strongly continuous semigroup on Y^2. So assertions (c) in Theorem III.3.1.1 is verified. Hence, the following holds for every generator $\mathcal{C}$ of a C_0-semigroup on a Banach space Y^2 and a boundary functional $\Phi \in \mathcal{L}(C[0,1], Y), Y^2)$.

Corollary III.8.2.9. *Let $\binom{y_0}{y_1} \in D(\mathcal{C})$. Then the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2} u(t,r), & \text{for } t \geq 0,\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot), & \text{for } t \geq 0, \\ u(0,0) = y_0, \\ u(0,1) = y_1, \end{cases}$$

with dynamic boundary conditions is wellposed if and only if $\lambda \in \mathbb{C} \backslash \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Further, we conclude from Theorem III.3.2.1 the following results.

Corollary III.8.2.10. *Let $\tilde{f} \colon [0,\tau] \to \mathrm{C}([-1,0], Y)$ and $\tilde{g} \colon [0,\tau] \to Y^2$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. Moreover, let $\binom{y_0}{y_1} \in D(C)$. Then the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2} u(t,r) + f(t)(r) & \text{for } t \in [0,\tau],\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot) + g(t) & \text{for } t \in [0,\tau], \\ u(0,0) = y_0, \\ u(0,1) = y_1, \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1], Y)$ if and only if $\lambda \in \mathbb{C} \setminus \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$. Assume additionally that the semigroup generated by C is positive and the delay operator Φ is positive. If the initial data u_0 and f, g are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.2.11. *Let $F \colon [0,\tau] \times \mathrm{C}([0,1], Y) \to \mathrm{C}([0,1], Y)$ and $G \colon [0,\tau] \times Y^2 \to Y^2$ continuously differentiable functions and $\binom{y_0}{y_1} \in D(C)$. Then the* *problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}^2}{\mathrm{d}r^2}u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t\in[0,\tau],\ r\in[0,1], \\ \partial_t \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = \mathcal{C}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} + \Phi u(t,\cdot) & \\ \qquad + G\left(t, \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}\right) & \text{for } t\in[0,\tau], \\ u(0,0) = y_0, & \\ u(0,1) = y_1, & \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1],Y)$ *if and only if* $\lambda \in \mathbb{C} \setminus \{-k^2\cdot\pi^2 \colon k\in\mathbb{N}\}$. *Assume additionally that the semigroup generated by* C *is positive and the delay operator* Φ *is positive. If the initial data* u_0 *and* F, G *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t\in[0,\tau]$.

III.8.3 Shift-Semigroup on $\mathrm{C}[-1,0]$

In the first subsection of this chapter we studied the first derivative $A^B \subseteq \frac{\mathrm{d}}{\mathrm{d}r}$ for a boundary operator $B : D(B) \subset X \to \partial X$ bounded on the first component of the decomposition $X = X_0 \oplus \ker(A_m)$ but unbounded on the second. Next we give an example where on the contrary B is unbounded on X_0, but bounded on $\ker(A_m)$. More precisely, we consider the Banach space $X := \mathrm{C}[-1,0]$ of all continuous, complex valued functions equipped with the sup-norm. Then, for some fixed $\alpha\in(0,1)$ we define the operator $A\colon D(A)\subset X\to X$ by

$$Af := f', \quad D(A) := \left\{ f\in \mathrm{C}^1[-1,0] \colon f'(0) = \int_{-1}^0 f'(r)\cdot(-r)^{-\alpha}\,\mathrm{d}r \right\}. \tag{III.24}$$

Generation of semigroups

We use our abstract theory to establish the wellposedness of the abstract Cauchy problem (ACP) associated with A given in (III.24).

Choosing $X = \mathrm{C}[-1,0]$, $\partial X = \mathbb{C}$, $A_m = \frac{\mathrm{d}}{\mathrm{d}r}$ with domain $D(A_m) = \mathrm{C}^1[-1,0]$, $L = \delta_0$ and

$$Bf := \int_{-1}^0 f'(r)\cdot(-r)^{-\alpha}\mathrm{d}r, \quad D(B) := \mathrm{W}^{1,1}(0,1)$$

we obtain $A = A^B$. Moreover, note that the Dirichlet-to-Neumann operator is zero, but the restriction $B_0 := B|_{X_0}$ is unbounded on $X_0 = \mathrm{C}_0[-1,0)$. In fact, if we define $k(r) := (-r)^\alpha$ for $r\in[-1,0]$, then $k \in X_0 \setminus D(B_0)$. Recall that

A_0^0 generates a strongly continuous semigroup on $\mathrm{C}_0[-1,0)$ and has compact resolvent. Using the Staffans Weiß perturbation theorem (see [ABE14, Theorem 10]), Theorem III.4.2.4 and Corollary III.4.3.1 we conclude the following result. For details, in particular for the verification that G_0^0 and A_0^0 differ by a Staffans Weiß perturbation (see [ABE14, Definition 9]), we refer to [BE20a, Theorem 6.4]. Moreover, by [BE20b, Corollary 7.11] it follows that the semigroup is not positive.

Theorem III.8.3.1. *The operator A given by* (III.24) *generates a C_0-semigroup on* $\mathrm{C}[-1,0]$*. Further, the operator A given by* (III.24) *has compact resolvent on* $\mathrm{C}[-1,0]$*. Moreover the semigroup is not positive on* $\mathrm{C}[-1,0]$*.*

Spectral theory

Now we consider the spectral theory of the operator A given by (III.24). First note that A_0 has empty spectrum and compact resolvent. Using (III.22), a short calculation shows

$$N_\lambda x = x \int_{-1}^{0} \lambda e^{\lambda r} \cdot (-r)^{-\alpha} \, \mathrm{d}r$$

for $\lambda, x \in \mathbb{C}$ and we conclude the following result by Theorem III.6.0.1. For more details see [BE20b, Corollary 7.10].

Theorem III.8.3.2. *We obtain $\lambda \in \sigma(A^B) = \sigma_p(A^B)$ if and only if* $\int\limits_{-1}^{0} e^{\lambda r} \cdot (-r)^{-\alpha} \, \mathrm{d}r = 1$*. Moreover, all eigenspaces are one-dimensional.*

Wellposedness of parabolic equations with dynamic boundary conditions

For our problems with dynamic boundary conditions Theorem III.2.1.1 yields the following.

Corollary III.8.3.3. *Let $\alpha \in (0,1)$. Then for all $u_0 \in D(A)$ the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r} u(t,r) & \text{for } t \geq 0,\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = \int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha} \, \mathrm{d}r & \text{for } t \geq 0, \\ \quad u(0,r) = u_0(r) & \text{for } r \in [-1,0], \end{cases}$$

with dynamic boundary conditions is wellposed on $\mathrm{C}[-1,0]$*.*

Moreover, Theorem III.2.2.1 implies the following corollaries about inhomogeneous and semilinear problems with dynamic boundary conditions.

Corollary III.8.3.4. *Let* $\alpha \in (0,1)$ *and* $\tilde{f}\colon [0,\tau] \to \mathrm{C}([-1,0],Y)$ *and* $\tilde{g}\colon [0,\tau] \to Y$ *integrable functions and* $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ *such that* $f(t)(0) = g(t)$ *for* $t \in [0,\tau]$. *Then for all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) + f(t)(r), & t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = \displaystyle\int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha}\,\mathrm{d}r + g(t), & t \in [0,\tau], \\ u(0,\cdot) = u_0 \end{cases}$$

admits a unique solution on $\mathrm{C}[-1,0]$.

Corollary III.8.3.5. *Let* $\alpha \in (0,1)$ *and* $F\colon [0,\tau] \times \mathrm{C}[-1,0] \to \mathrm{C}[-1,0]$ *and* $G\colon [0,\tau] \times \mathbb{C} \to \mathbb{C}$ *continuously differentiable functions such that* $F(t,u(t,\cdot))(0) = G(t,u(t,0))$. *Then for all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) + F(t,u(t,\cdot))(r) & \textit{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = \displaystyle\int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha}\,\mathrm{d}r & \\ \qquad\qquad + G(t,u(t,0)), & \textit{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \textit{for } r \in [-1,0], \end{cases}$$

admits a unique solution on $\mathrm{C}[-1,0]$.

Wellposedness of elliptic equations with dynamic boundary conditions

In this section we consider the corresponding elliptic problem with dynamic boundary conditions. Note that $\rho(A_0) = \mathbb{C}$ and that N_λ are bounded for all $\lambda \in \mathbb{C}$, and hence generate compact and analytic semigroups on $\mathbb{C}$. Now, for the elliptic problem with dynamic boundary conditions Theorem III.3.1.1 yields the following result.

Corollary III.8.3.6. *Let* $\alpha \in (0,1)$. *Then for all* $\lambda, x_0 \in \mathbb{C}$ *the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r}u(t,r) & \textit{for } t \geq 0,\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t,0) = \displaystyle\int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha}\,\mathrm{d}r & \textit{for } t \geq 0, \\ u(0,0) = x_0 \end{cases}$$

with dynamic boundary conditions is wellposed on $\mathrm{C}[-1,0]$.

Furthermore, from Theorem III.3.2.1 we obtain the following corollaries.

Corollary III.8.3.7. *Let $\alpha \in (0,1)$ and $\tilde{f}\colon [0,\tau] \to \mathrm{C}[-1,0]$ and $\tilde{g}\colon [0,\tau] \to \mathbb{C}$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. Then for all $\lambda, x_0 \in \mathbb{C}$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r} u(t,r) + f(t)(r) & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = \displaystyle\int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha}\,\mathrm{d}r + g(t) & \text{for } t \in [0,\tau], \\ u(0,0) = x_0, \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}[-1,0]$.

Corollary III.8.3.8. *Let $\alpha \in (0,1)$ and $F\colon [0,\tau] \times \mathrm{C}[-1,0] \to \mathrm{C}[-1,0]$ and $G\colon [0,\tau] \times \mathbb{C} \to \mathbb{C}$ continuous functions. Then for all $\lambda, x_0 \in \mathbb{C}$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{\mathrm{d}}{\mathrm{d}r} u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = \displaystyle\int_{-1}^{0} u'(t,r) \cdot (-r)^{-\alpha}\,\mathrm{d}r & \\ \qquad\qquad + G(t,u(t,0)) & \text{for } t \in [0,\tau], \\ u(0,0) = x_0, \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}[-1,0]$.

III.8.4 Degenerated elliptic second order differential operators on $\mathrm{C}([0,1],\mathbb{C}^n)$

For $n \in \mathbb{N}$ consider functions $a_i \in \mathrm{C}[0,1] \cap \mathrm{C}^1(0,1)$, $1 \le i \le n$, being strictly positive on $(0,1)$ such that $\frac{1}{a_i} \in L^1[0,1]$. Let $a := \operatorname{diag}(a_1,\dots,a_n)$ and $b, c \in \mathrm{C}([0,1],\mathrm{M}_n(\mathbb{C}))$. Moreover, define the maximal operator $A_m : D(A_m) \subset \mathrm{C}([0,1],\mathbb{C}^n) \to \mathrm{C}([0,1],\mathbb{C}^n)$ by

$$\begin{aligned} A_m f &:= af'' + bf' + cf, \\ D(A_m) &:= \{ f \in \mathrm{C}([0,1],\mathbb{C}^n) \cap \mathrm{C}^2((0,1),\mathbb{C}^n) \colon A_m f \in \mathrm{C}([0,1],\mathbb{C}^n) \} \end{aligned}$$

and take $B \in \mathcal{L}(C^1([0,1],\mathbb{C}^n),\mathbb{C}^{2n})$. Now we define the operator $A\colon D(A) \subset X \to X$ by

$$Af := A_m f, \quad D(A) := \left\{ f \in D(A_m)\colon \begin{pmatrix} (A_m f)(0) \\ (A_m f)(1) \end{pmatrix} = Bf \right\}. \tag{III.25}$$

Such second order differential operators on spaces of functions $f : [0,1] \to \mathbb{C}^n$ can be used to describe diffusion- and waves on networks. For some recent results in the L^p-context for operators with generalized Robin-type boundary conditions we refer to [EK19].

Generation of semigroups

In this subsection we investigate the generation property of the operator A given by (III.25).

Choose $X = \mathrm{C}([0,1],\mathbb{C}^n)$, $\partial X = \mathbb{C}^{2n}$, $Lf = \binom{f(0)}{f(1)}$ and A_m and B as above. Theorem III.4.1.2 and Corollary III.4.3.1 yields the following result. For details see [BE19, Example 5.1].

Theorem III.8.4.1. *We have* $D(A_m) \subset \mathrm{C}^1([0,1],\mathbb{C}^n) = D(B)$ *and* A *given by* (III.25) *generates a compact and analytic semigroup of angle* $\frac{\pi}{2}$ *on* $\mathrm{C}([0,1],\mathbb{C}^n)$.

This result generalizes [EF05, Example 4.1]. Now we make a particular choice for the operator B.

Corollary III.8.4.2. *For* $M_i, N_i \in \mathrm{M}_{2n\times n}(\mathbb{C})$, $i = 0,1$, *the operator* $A\colon D(A) \subset \mathrm{C}([0,1],\mathbb{C}^n) \to \mathrm{C}([0,1],\mathbb{C}^n)$ *given by*

$$\begin{aligned} Af &:= A_m f, \\ D(A) &:= \left\{ f \in D(A_m)\colon \begin{aligned} \begin{pmatrix} (A_m f)(0) \\ (A_m f)(1) \end{pmatrix} &= M_0 f'(0) + M_1 f'(1) \\ &\quad + N_0 f(0) + N_1 f(1) \end{aligned} \right\} \end{aligned}$$

generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ *on* $\mathrm{C}([0,1],\mathbb{C}^n)$.

Wellposedness of degenerated parabolic equations with dynamic boundary conditions

Combining Theorem III.2.1.1 with the results of the previous sections we conclude the following statement.

Corollary III.8.4.3. *For all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t,r) = A_m u(t,r) & \text{for } t \geq 0,\ r \in [0,1] \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = Bu(t,\cdot) & \text{for } t \geq 0, \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

with dynamic boundary conditions is wellposed on $\mathrm{C}([0,1],\mathbb{C}^n)$*. Further the solution* $t \to u(t,r)$ *is analytic for all* $r \in [0,1]$ *and* $u(t,\cdot) \in \mathrm{C}^\infty([0,1],\mathbb{C}^n)$ *for* $t > 0$.

Moreover, Theorem III.2.2.1 implies the following corollaries.

Corollary III.8.4.4. *Let* $\tilde{f}\colon [0,\tau] \to \mathrm{C}([0,1],\mathbb{C}^n)$ *and* $\tilde{g}\colon [0,\tau] \to \mathbb{C}^{2n}$ *integrable functions and* $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ *such that* $\binom{f(t)(0)}{f(t)(1)} = g(t)$ *for all* $t \in [0,\tau]$*. For all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = A_m u(t,r) + f(t)(r) & \text{for } t \in [0,\tau],\ r \in [0,1] \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = Bu(t,\cdot) + g(t) & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1],\mathbb{C}^n)$.

Corollary III.8.4.5. *Let* $F\colon [0,\tau] \times \mathrm{C}([0,1],\mathbb{C}^n) \to \mathrm{C}([0,1],\mathbb{C}^n)$ *and* $G\colon [0,\tau] \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ *continuously differentiable functions such that* $\binom{F(t,u(t,\cdot))(0)}{F(t,u(t,\cdot))(1)} = G\left(t, \binom{u(t,0)}{u(t,1)}\right)$*. Then, for all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,r) = A_m u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t \in [0,\tau],\ r \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = Bu(t,\cdot) + G\left(t, \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}\right) & \text{for } t \in [0,\tau], \\ u(0,r) = u_0(r) & \text{for } r \in [0,1], \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1],\mathbb{C}^n)$.

Wellposedness of degenerated elliptic equations with dynamic boundary conditions

Finally, we study the elliptic problem with dynamic boundary conditions associated to A given in (III.25).

Since A_0 has compact resolvent, $\sigma(A_0) = \sigma_p(A_0)$. By [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Further, since $\dim(\partial X) < \infty$, the Dirichlet-to-Neumann operators N_λ are bounded and hence generators of compact and analytic semigroups on $\mathbb{C}^n$ of angle $\frac{\pi}{2}$ for all $\lambda \in \rho(A_0)$. So assertions (c) in Theorem III.3.1.1 is verified. We conclude the following result.

Corollary III.8.4.6. *Let $x_0, x_1 \in \mathbb{C}^n$. Then the problem*

$$\begin{cases} \qquad \lambda u(t,r) = A_m u(t,r), & \text{for } t \geq 0,\ r \in [0,1] \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = Bu(t,\cdot), & \text{for } t \geq 0, \\ \quad u(0,0) = x_0, & \\ \quad u(0,1) = x_1, & \end{cases}$$

with dynamic boundary conditions is wellposed on $\mathrm{C}([0,1],\mathbb{C}^n)$ if and only if $\lambda \in \rho(A_0)$. Further, the solution $t \to u(t,r)$, if it exists, is analytic for all $r \in [0,1]$.

Further, we conclude from Theorem III.3.2.1 the following results.

Corollary III.8.4.7. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}([0,1],\mathbb{C}^n)$ and $\tilde{g}\colon [0,\tau] \to \mathbb{C}^{2n}$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. Moreover, let $y_0, y_1 \in \mathbb{C}^n$. Then the problem*

$$\begin{cases} \qquad \lambda u(t,r) = A_m u(t,r) + f(t)(r) & \text{for } t \in [0,\tau],\ r \in [0,1] \\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = Bu(t,\cdot) + g(t), & \text{for } t \in [0,\tau], \\ \quad u(0,0) = x_0, & \\ \quad u(0,1) = x_1, & \end{cases}$$

admits a unique solution on $\mathrm{C}([0,1],\mathbb{C}^n)$ if and only if $\lambda \in \rho(A_0)$.

Corollary III.8.4.8. *Let $F\colon [0,\tau] \times \mathrm{C}([0,1],\mathbb{C}^n) \to \mathrm{C}([0,1],\mathbb{C}^n)$ and $G\colon [0,\tau] \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ continuously differentiable functions. Then for all $x_0, x_1 \in \mathbb{C}^n$ the problem*

$$\begin{cases} \lambda u(t,r) = A_m u(t,r) + F(t,u(t,\cdot))(r) & \text{for } t \in [0,\tau],\ r \in [-1,0], \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t,0) = Bu(t,\cdot) + G\left(t, \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}\right) & \text{for } t \in [0,\tau], \\ u(0,0) = x_0, & \\ u(0,1) = x_1, & \end{cases}$$

with dynamic boundary conditions admits a solution on $\mathrm{C}([0,1],\mathbb{C}^n)$ if and only if $\lambda \in \rho(A_0)$.

III.8.5 Elliptic operators with Wentzell boundary conditions on domains

We consider a uniformly elliptic second-order differential operator with generalized Wentzell boundary conditions on $\mathrm{C}(\overline{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. On domains in $\mathbb{R}^n$ such operators have first been studied systematically by Wentzell [Wen59], Feller [Fel54] and Hintermann [Hin89]. To this end, we first take real-valued functions

$$a_{jk} = a_{kj} \in \mathrm{C}^\infty(\overline{\Omega}), \quad a_j \in \mathrm{C}^\infty(\overline{\Omega}), \quad a_0, b_0 \in \mathrm{C}^\infty(\overline{\Omega}), \quad 1 \leq j,k \leq n$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(r) \cdot \xi_j \xi_k \geq c \cdot \|\xi\|^2 \qquad \text{for all } r \in \overline{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$

and some fixed $c > 0$. Then we define the maximal operator $A_m : D(A_m) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ in divergence form by

$$\text{(III.26)} \quad \begin{aligned} A_m f &:= \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk} \partial_k f \Big) + \sum_{k=1}^{n} a_k \partial_k f + a_0 f, \\ D(A_m) &:= \Big\{ f \in \bigcap_{p \geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\overline{\Omega}) \colon A_m f \in \mathrm{C}(\overline{\Omega}) \Big\} \end{aligned}$$

and its (negative) conormal derivative $B : D(B) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$\text{(III.27)} \quad \begin{aligned} Bf &:= -\sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k f + b_0 L f, \\ D(B) &:= \Big\{ f \in \bigcap_{p \geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\overline{\Omega}) \colon Bf \in \mathrm{C}(\partial\Omega) \Big\}, \end{aligned}$$

where $L \in \mathcal{L}(\mathrm{C}(\overline{\Omega}), \mathrm{C}(\partial\Omega))$, $Lf := f|_{\partial\Omega}$ denotes the trace operator. We denote the conormal derivative by $\frac{\partial^a}{\partial n}$. Now one defines the operator $A : D(A) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ with Wentzell boundary conditions by

$$\text{(III.28)} \quad Af := A_m f, \quad D(A) := \{ f \in D(A_m) \cap D(B) \colon L A_m f = Bf \}.$$

Considering $X := \mathrm{C}(\overline{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$ and A_m, B and L as above, we obtain $A = A^B$, i.e. the abstract Wentzell boundary conditions coincide with the usual Wentzell boundary conditions.

Further, the abstract operator with Dirichlet boundary conditions A_0 becomes an elliptic operator with usual Dirichlet boundary conditions, i.e. $u|_{\partial\Omega} = 0$. On domains in $\mathbb{R}^n$ the generator property of differential operators with Dirichlet boundary conditions is quite well understood. From an operator theoretic point of view pioneer work was done by Browder [Bro61], Agmon [Agm62] and Stewart [Ste74]. For a modern collection of the result we refer to [Ama95] and [Lun95].

Moreover, the abstract Dirichlet problem (III.5) becomes an elliptic partial differential equation with non-homogeneous Dirichlet boundary conditions and L_0 is the corresponding solution operator. Finally, the Dirichlet-to-Neumann operator is the classic Dirichlet-to-Neumann operator

$$N = -\frac{\partial^a}{\partial n} L_0, \quad D(N) = \left\{ \varphi \in \mathrm{C}(\partial\Omega) \colon L_0\varphi \in D\left(\frac{\partial^a}{\partial n}\right) \right\}$$

see [US90], [LU01], [LTU03], [Tay81] and [Tay96, Appendix 12.C]. It is a pseudo differential operator of order 1, see [Tay96, Appendix 12.C]. For an application of Dirichlet-to-Neumann operators in stochastics see e.g. [GV18]. From the operator-theoretic point of view the Dirichlet-to-Neumann operator is studied by e.g. Amann and Escher [AE96], Arendt and ter Elst [AE11], [AEKS14] and [AE17] and ter Elst and Ouhabaz [EO14], [EO19a] and [EO19b]. In particular, on domains in $\mathbb{R}^n$ Escher [Esc94] has shown that such Dirichlet-to-Neumann operators generate analytic semigroups on the space of continuous functions, however without specifying the corresponding angle of analyticity. This result was improved by ter Elst and Ouhabaz [EO19a, Theorem 1.1] to the angle $\frac{\pi}{2}$ and extended to elliptic differential operators with merely Hölder continuous principal coefficients on $\mathrm{C}^{1,\kappa}$-domains.

Generation of semigroups

In this section we are interested in the generation of an analytic semigroup on $\mathrm{C}(\overline{\Omega})$ with the optimal angle of analyticity by the operator A given by (III.28).

Arendt et al. [AMPR03] proved that the Laplace operator with Wentzell boundary conditions generates a positive, contractive, strongly continuous semigroup on $\mathrm{C}(\overline{\Omega})$. Engel [Eng03] improves this result by showing that this semigroup is analytic with angle of analyticity $\frac{\pi}{2}$. Later Engel and Fragnelli [EF05] again generalize this result to uniformly elliptic operators, however without specifying the corresponding angle of analyticity. For related work see also [CT86], [CM98], [FGGR02a], [FGGR02b], [FGG+03], [CENN03], [VV03], [CENP05], [FGG+10], [War10] and the references therein.

Using our theory we can improve all these results. At the end of the last subsection we have seen how this problem fits into our abstract framework. First, using Theorem III.5.0.3 we assume without loss of generality $a_k = 0$ for $0 \leq k \leq n$. Now we verify Assumptions III.1.1.2. Regularity theory (see [GT01, Theorem 9.15]), Rellich's embedding theorem (see [Ada75, Theorem 6.2, Part III]) and Ehrling's lemma (cf. [RR04, Theorem 6.99]) imply that B is relatively A_0-bounded of bound 0. Moreover, from [GT01, Theorem 9.18] it follows that for every $x \in \mathrm{C}(\partial\Omega)$ the problem (III.5) has a unique solution $f \in D(A_m)$, and hence the Dirichlet operator L_0 exists. Further, by the maximum principle, see [GT01, Theorem 9.1], the solution operator L_0 is bounded. By [Lun95, Corollary 3.1.21.(ii)] that the operator A_0 is sectorial of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{\Omega})$ and has compact resolvent. In particular A_0 is a weak Hille-Yosida operator on $\mathrm{C}(\overline{\Omega})$. Moreover in [EO19a, Theorem 1.1] it is shown that the Dirichlet-to-Neumann operator generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$. We conclude by Theorem III.4.1.2 and Corollary III.4.3.1.

Theorem III.8.5.1. *The operator A given by* (III.28) *generates a compact and analytic semigroup on* $\mathrm{C}(\overline{\Omega})$ *of optimal angle* $\frac{\pi}{2}$.

For smooth domains and smooth coefficients without specifying the angle of analyticity this has been proven in [EF05, Corollary 4.5]. In [BE19, Example 5.4] it is shown that the angle of analyticity does not depend on the lower order terms. In [BE20a, Example 6.3] the result is proven for merely Lipschitz continuous principal coefficients and continuous lower order coefficients on $\mathrm{C}^{1,\kappa}$-domains. In [BtE20, Theorem 5.1] we generalize this result to merely Hölder continuous principal coefficients. Since the divergence theorem does not work in this situation, it is difficult to find an maximal operator and we cannot apply our theory directly. Nevertheless the same arguments work here.

Further, an analogous result holds on $\mathrm{L}^p(\Omega) \times \mathrm{L}^p(\partial\Omega)$ for $p \in (1, \infty)$, see [BtE20, Theorem 4.5(c)]. Indeed, in this situation, the operator with dynamic boundary conditions has maximal L^r-regularity for all $r \in (1, \infty)$, as proven in [BtE20, Theorem 4.5(d)]. For maximal regularity of the corresponding operator on $\mathrm{L}^p(\Omega) \times \mathrm{L}^p(\partial\Omega)$ see also [DPZ08] and [GGGR20]. We refer to [BtE20] for the precise formulation of the statements.

Positivity and asymptotic behaviour

In [AMPR03] and [Eng03] it is shown that the semigroup generated by the Laplacian with Wentzell boundary conditions is positive. In this section we generalize this result to arbitrary elliptic operators with Wentzell boundary conditions using our theory.

Hopf's maximum principle (cf. [GT01, Theorem 3.5]) implies that $R(\lambda, A_0)$ and L_λ are positive operators for large λ. Further, a small computation shows that $B|_{D(A_0)}$ is a positive operator and from [Esc94] it follows that the Dirichlet-to-Neumann operators N_λ are generators of strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$. Now Corollary III.4.4.3 implies the following result. For details see [BE20b, Corollary 7.14].

Theorem III.8.5.2. *The operator A given by* (III.28) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}(\overline{\Omega})$.

Wellposedness of parabolic equations with dynamic boundary conditions

Theorem III.8.5.1 and Theorem III.2.1.1 now give the following.

Corollary III.8.5.3. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum\limits_{j=1}^n \partial_j \Big(\sum\limits_{k=1}^n a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad\qquad + \sum\limits_{k=1}^n a_k(r) \partial_k u(t,r) + a_0(r) \cdot u(t,r) & \text{for } t \geq 0, r \in \overline{\Omega}, \\ \partial_t u(t,s) = - \sum\limits_{j,k=1}^n a_{jk}(s) \nu_j(s) \partial_k u(t,s) + b_0(s) \cdot u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega} \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{\Omega})$. *Further, if $u_0 \in D(A)$, the solution $t \to u(t,r)$ is analytic for all $r \in \overline{\Omega}$ and $u(t,\cdot) \in \mathrm{C}^\infty(\Omega) \cap \mathrm{C}(\overline{\Omega})$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.*

Finally, Theorem III.2.2.1 implies the following corollaries.

Corollary III.8.5.4. *Let $\tilde{f} \colon [0,\tau] \to \mathrm{C}(\bar{\Omega})$ and $\tilde{g} \colon [0,\tau] \to \mathrm{C}(\partial\Omega)$ integrable functions and $f(t) \coloneqq f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) \coloneqq g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ such that $f(t)(r) =$*

$g(t)(r)$ for all $t \in [0,\tau]$ and $r \in \partial\Omega$. For all $u_0 \in D(A)$ the problem

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad + b_0(s) \cdot u(t,s) + g(t)(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega}, \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{\Omega})$. If the initial data u_0 and f are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.5.5. *Let $F\colon [0,\tau] \times \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ and $G\colon [0,\tau] \times \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ continuously differentiable functions such that $F(t,u(t,\cdot))(s) = G(t,u(t,\cdot))(s)$ for all $t \in [0,\tau]$ and $s \in \partial\Omega$. For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + F(t,u(t,\cdot))(r) & \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad + b_0(s) \cdot u(t,s) + G(t,u(t,\cdot))(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega}, \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{\Omega})$. Moreover, if the initial data u_0 and F are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Wellposedness of elliptic equations with dynamic boundary conditions

Since the operator A_0 with Dirichlet boundary conditions has compact resolvent, by [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Moreover, Proposition III.5.0.2 implies that N_λ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$ for all $\lambda \in \rho(A_0)$. Theorem III.3.1.1 now give the following.

Corollary III.8.5.6. *For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) + a_0(r) \cdot u(t,r) & \text{for } t \geq 0, r \in \overline{\Omega}, \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) + b_0(s) \cdot u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ u(0,s) = x_0(s) & \text{for } s \in \partial\Omega \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{\Omega})$ if and only if $\lambda \in \rho(A_0)$. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{\Omega}$ and $u(t,\cdot) \in \mathrm{C}^\infty(\Omega)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Moreover, as a consequence of Theorem III.3.2.1 we obtain the following results.

Corollary III.8.5.7. *Let $\tilde{f} \colon [0,\tau] \to \mathrm{C}(\bar{\Omega})$ and $\tilde{g} \colon [0,\tau] \to \mathrm{C}(\partial\Omega)$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad + b_0(s) \cdot u(t,s) + g(t)(s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial\Omega \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{\Omega})$ if and only if $\lambda \in \rho(A_0)$. Moreover, if the initial data u_0 and f, g are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.5.8. *Let $F \colon [0,\tau] \times \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ and $G \colon [0,\tau] \times \mathrm{C}(\partial\Omega) \to$*

$\mathrm{C}(\partial\Omega)$ *continuously differentiable functions. For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + F(t, u(t,\cdot))(r) & \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad + b_0(s) \cdot u(t,s) + G(t, u(t,\cdot))(s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial\Omega \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{\Omega})$ if and only if $\lambda \in \rho(A_0)$. Moreover, if the initial data u_0 and F, G are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

III.8.6 Elliptic operator with generalized Wentzell boundary conditions on domains

In this section we study elliptic operators with a generalized Wentzell boundary conditions. Such operators were studied by Goldstein et. al, see e.g. [FGGR02a], [FGGR02b], [FGG+03], [FGG+10], [GGP17] and, using an abstract approach, in [Eng03], [EF05] and [BE19]. We consider as in the last section a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and an uniformly elliptic operator $A_m \colon D(A_m) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ in divergence form given (III.26). Further, we consider real-valued functions

$$c_{jk} = c_{kj} \in \mathrm{C}^\infty(\partial\Omega), \quad c_j \in \mathrm{C}^\infty(\partial\Omega), \quad c_0 \in \mathrm{C}^\infty(\partial\Omega), \quad 1 \le j,k \le n$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} c_{jk}(r) \cdot \xi_j \xi_k \ge \tilde{c} \cdot \|\xi\|^2 \quad \text{for all } r \in \partial\Omega,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

and some fixed $\tilde{c} > 0$. We define an operator $C : D(C) \subseteq \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ in divergence form on the boundary by

$$\begin{aligned} Cx &:= \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} c_{jk} \partial_k x \Big) + \sum_{k=1}^{n} c_k \partial_k x + c_0 x, \\ D(C) &:= \Big\{ x \in \bigcap_{p \geq 1} W^{2,p}(\partial\Omega) \cap \mathrm{C}(\partial\Omega) \colon Cx \in \mathrm{C}(\partial\Omega) \Big\}. \end{aligned} \tag{III.29}$$

Moreover we define the operator $B_0 \colon D(B_0) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ by (III.27) and finally $B \colon D(B) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$Bf := B_0 f + CLf, \quad D(B) := \{ f \in D(A_m) \cap D(B_0) \colon Lf \in D(C) \}.$$

Now we consider $A \colon D(A) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ given by

$$Af := A_m f, \ D(A) := \{ f \in D(A_m) \cap D(B) \colon LA_m f = B_0 f + CLf \}. \tag{III.30}$$

Generation of semigroups

In this subsection we concentrate to the generation of an analytic semigroup on $\mathrm{C}(\overline{\Omega})$ with optimal angle $\frac{\pi}{2}$ by A given in (III.30) and on the wellposedness of the associated abstract Cauchy problem (ACP).

Choosing $X := \mathrm{C}(\overline{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$ and A_m, B and L as above, we obtain $A = A^B$. Moreover, the operators A_0 and L_λ are the same as in the last section and the Dirichlet-to-Neumann operators satisfy

$$N_\lambda^B = C + N_\lambda^{B_0}. \tag{III.31}$$

In Subsection III.8.5 we already have seen that A_0 is sectorial of angle $\frac{\pi}{2}$ and has compact resolvent on $\mathrm{C}(\overline{\Omega})$ and L_0 exists and is bounded. Moreover, since $B|_{X_0} = B_0$ we conclude from the corresponding result in Subsection III.8.5 that B is relatively A_0-bounded of bound 0. Note, that C generates a compact and analytic semigroup with angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$. From (III.31) we conclude by perturbation (see [EN00, Theorem III.2.10]) that the Dirichlet-to-Neumann operator N^B generates a compact and analytic semigroup with angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$. Now Theorem III.4.1.2 and Corollary III.4.3.1 imply

Theorem III.8.6.1. *The operator A given by* (III.30) *generates a compact and analytic semigroup with angle $\frac{\pi}{2}$ on* $\mathrm{C}(\overline{\Omega})$.

The special case of this result, where A_m and C are Laplace operators can be found in [BE19, Example 5.3]. The general case follows from [Bin20b, Theorem 4.6].

Positivity and asymptotic behaviour

In this subsection we concentrate on the positivity and the asymptotic behaviour of the semigroup generated by (III.30).

We verify the conditions of Corollary III.4.4.3. Using the results from the last section it remains to show that N_λ^B generate strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$ for sufficient large λ. This follows from (III.31) and the positive minimum principle (cf. [Nag86, B-II, Theorem 1.6]) using the facts that $N_\lambda^{B_0}$ and C generate strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$ for sufficient large λ. Now Corollary III.4.4.3 implies

Theorem III.8.6.2. *The operator A given by* (III.30) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}(\overline{\Omega})$.

Wellposedness of parabolic equations with dynamic boundary conditions

Using the results from above, Theorem III.2.1.1 implies the following.

Corollary III.8.6.3. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n a_{jk}(r)\partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^n a_k(r)\partial_k u(t,r) + a_0(r)\cdot u(t,r) & \text{for } t \geq 0, r \in \overline{\Omega}, \\ \partial_t u(t,s) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n c_{jk}(s)\partial_k u(t,s) \Big) & \\ \qquad + \sum_{k=1}^n c_k(s)\partial_k u(t,s) + c_0(s)\cdot u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ \qquad - \sum_{j,k=1}^n a_{jk}(s)\nu_j(s)\partial_k u(t,s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega} \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{\Omega})$*. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{\Omega}$ and $u(t,\cdot) \in \mathrm{C}^\infty(\Omega)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.*

Moreover, we conclude from Theorem III.2.2.1 the following results.

Corollary III.8.6.4. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\bar{\Omega})$ and $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial\Omega)$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ such that $f(t)(r) = g(t)(r)$ for all $t \in [0,\tau]$ and $r \in \partial\Omega$. For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^n a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n c_{jk}(s) \partial_k u(t,s) \Big) & \\ \qquad + \sum_{k=1}^n c_k(s) \partial_k u(t,s) + c_0(s) \cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad - \sum_{j,k=1}^n a_{jk}(s) \nu_j(s) \partial_k u(t,s) + g(t)(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega} \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{\Omega})$. If the initial data u_0 and f are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.6.5. *Let $F\colon [0,\tau] \times \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ and $G\colon [0,\tau] \times \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ continuously differentiable functions such that $F(t,u(t,\cdot))(s) = G(t,u(t,\cdot))(s)$ for all $t \in [0,\tau]$ and $s \in \partial\Omega$. For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n a_{jk}(r) \partial_k u(t,r) \Big) & \\ \qquad + \sum_{k=1}^n a_k(r) \partial_k u(t,r) & \text{for } t \in [0,\tau], r \in \overline{\Omega}, \\ \qquad + a_0(r) \cdot u(t,r) + F(t,u(t,\cdot))(r) & \\ \partial_t u(t,s) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n c_{jk}(s) \partial_k u(t,s) \Big) & \\ \qquad + \sum_{k=1}^n c_k(s) \partial_k u(t,s) + c_0(s) \cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial\Omega, \\ \qquad - \sum_{j,k=1}^n a_{jk}(s) \nu_j(s) \partial_k u(t,s) & \\ \qquad + G(t,u(t,\cdot))(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{\Omega} \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{\Omega})$*. Moreover, if the initial data* u_0 *and* F *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$*.*

Wellposedness of elliptic equations with dynamic boundary conditions

Now we investigate elliptic differential equations with dynamic boundary conditions with an additional drift term at the boundary. This gives a system of differential equations, where the equation on the interior is elliptic and the equation on the boundary is parabolic. Note that such a problem is time dependent, since the boundary condition is.

Since the operator A_0 with Dirichlet boundary conditions has compact resolvent, by [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Moreover, Proposition III.5.0.2 implies that N_λ generate compact and analytic semigroups of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$ for all $\lambda \in \rho(A_0)$. Theorem III.3.1.1 now yields the following.

Corollary III.8.6.6. *For all* $x_0 \in D(C)$ *the problem*

$$\begin{cases} \lambda u(t,r) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n a_{jk}(r)\partial_k u(t,r)\Big) & \\ \qquad + \sum_{k=1}^n a_k(r)\partial_k u(t,r) + a_0(r)\cdot u(t,r) & \text{for } t \geq 0, r \in \overline{\Omega}, \\ \partial_t u(t,s) = \sum_{j=1}^n \partial_j \Big(\sum_{k=1}^n c_{jk}(s)\partial_k u(t,s)\Big) & \\ \qquad + \sum_{k=1}^n c_k(s)\partial_k u(t,s) + c_0(s)\cdot u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ \qquad - \sum_{j,k=1}^n a_{jk}(s)\nu_j(s)\partial_k u(t,s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial\Omega \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{\Omega})$ *if and only if* $\lambda \in \rho(A_0)$*. Further,* $t \to u(t,r)$ *is analytic for all* $r \in \overline{\Omega}$ *and* $u(t,\cdot) \in \mathrm{C}^\infty(\Omega)$*. If the initial value* u_0 *is positive, then the solution* $u(t,\cdot)$ *is positive for* $t \geq 0$*.*

Further Theorem III.3.2.1 yields the following corollaries.

Corollary III.8.6.7. *Let* $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\bar{\Omega})$ *and* $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial\Omega)$ *integrable functions and* $f(t) \coloneqq f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$*,* $g(t) \coloneqq g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$*. For all* $x_0 \in D(C)$

the problem

$$
\begin{cases}
\lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\
\qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) & \textit{for } t \in [0,\tau], r \in \overline{\Omega}, \\
\qquad + a_0(r) \cdot u(t,r) + f(t)(r) & \\
\partial_t u(t,s) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} c_{jk}(s) \partial_k u(t,s) \Big) & \\
\qquad + \sum_{k=1}^{n} c_k(s) \partial_k u(t,s) + c_0(s) \cdot u(t,s) & \textit{for } t \in [0,\tau], s \in \partial\Omega, \\
\qquad - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) + g(t)(s) & \\
u(0,s) = x_0(s) & \textit{for } s \in \partial\Omega
\end{cases}
$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{\Omega})$ *if and only if* $\lambda \in \rho(A_0)$*. Moreover, if the initial data* u_0 *and* f, g *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$*.*

Corollary III.8.6.8. *Let* $F\colon [0,\tau] \times \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ *and* $G\colon [0,\tau] \times \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ *continuously differentiable functions. For all* $x_0 \in D(C)$ *the problem*

$$
\begin{cases}
\lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) & \\
\qquad + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) + a_0(r) \cdot u(t,r) & \textit{for } t \in [0,\tau], r \in \overline{\Omega}, \\
\qquad + F(t, u(t,\cdot))(r) & \\
\partial_t u(t,s) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} c_{jk}(s) \partial_k u(t,s) \Big) & \\
\qquad + \sum_{k=1}^{n} c_k(s) \partial_k u(t,s) + c_0(s) \cdot u(t,s) & \textit{for } t \in [0,\tau], s \in \partial\Omega, \\
\qquad - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) + G(t, u(t,\cdot))(s) & \\
u(0,s) = x_0(s) & \textit{for } s \in \partial\Omega
\end{cases}
$$

admits a unique solution on $\mathrm{C}(\overline{\Omega})$ *if and only if* $\lambda \in \rho(A_0)$*. Moreover, if the initial data* u_0 *and* F, G *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$*.*

III.8.7 Elliptic operator with Wentzell boundary conditions on manifolds

In this section we generalize Subsection III.8.5 to smooth, compact, Riemannian manifolds $(\overline{M}, g)$ with smooth boundary ∂M. To this end, we take real-valued functions

$$a_j^k = a_k^j \in \mathrm{C}^\infty(\overline{M}), \quad b_j \in \mathrm{C}_c(M), \quad c \in \mathrm{C}(\overline{M}), \quad d \in \mathrm{C}(\partial M) \quad 1 \le j,k \le n,$$

satisfying the strict ellipticity condition

$$a_j^k(r) g^{jl}(r) X_k(r) X_l(r) > 0$$

for all co-vectorfields X_k, X_l on $\overline{M}$ with $(X_1(r), \dots, X_n(r)) \neq (0, \dots, 0)$. Replacing the divergence and the gradient by there analogues on Riemannian manifolds, we define the maximal operator in divergence form by

$$\begin{aligned} A_m f &:= \sqrt{|a|}\, \mathrm{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf, \\ D(A_m) &:= \Big\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon A_m f \in \mathrm{C}(\overline{M}) \Big\}, \end{aligned} \tag{III.32}$$

where $a = (a_j^k)$ and $b = (b_j)$. As feedback operator we take

$$\begin{aligned} Bf &:= -\frac{\partial^a}{\partial n^g} f + d \cdot Lf := -g(a \nabla_M^g f, \nu_g) + d \cdot Lf, \\ D(B) &:= \Big\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon Bf \in \mathrm{C}(\partial M) \Big\} \end{aligned} \tag{III.33}$$

where $Lf = f|_{\partial M}$ denotes the trace operator. Now we define the operator $A \colon D(A) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ by

$$Af := A_m f, \quad D(A) := \{ f \in D(A_m) \cap D(B) \colon LA_m f = Bf \}. \tag{III.34}$$

Generation of semigroups

Choosing $X = \mathrm{C}(\overline{M})$, $\partial X = \mathrm{C}(\partial M)$ and A_m, B, L as above the operator A given by (III.34) fits into our abstract framework $A = A^B$.

We induce a $(2,0)$-tensorfield on $\overline{M}$ given by

$$\tilde{g}^{kl} = a_i^k g^{il}.$$

Its inverse $\tilde{g}$ is a $(0,2)$-tensorfield on $\overline{M}$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $\overline{M}$. It turns out that the maximal operator A_m can be written as $\Delta_M^{\tilde{g}}$ plus lower order terms, whereas the feedback operator B is the (negative) normal derivative $-\frac{\partial}{\partial \nu_{\tilde{g}}}$ with respect to $\tilde{g}$. Applying perturbation theory (see Theorem III.5.0.3) it remains to show the result for the Laplace-Beltrami operator with Wentzell boundary conditions. The verification of Assumptions III.1.1.2 is similar as on domains. The sectoriality with optimal angle $\frac{\pi}{2}$ and compactness of the resolvent of the operator A_0 with Dirichlet boundary conditions is proven in [Bin20a]. Further, by [Tay96, Appendix 12.C (C.4)] it follows that

$$N = \sqrt{-\Delta_{\partial M}^{\tilde{g}}} + P,$$

where P denotes a pseudo differential operator of order 0. Following the proof of [Eng03, Theorem 2.1] we conclude that the Dirichlet-to-Neumann operator N generates a compact and analytic semigroup of angle $\frac{\pi}{2}$. For details we refer to [Bin19]. Now we conclude

Theorem III.8.7.1. *The operator A given by* (III.34) *generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$.*

Wellposedness of parabolic equations with dynamic boundary conditions

Theorem III.8.7.1 and Theorem III.2.1.1 now give the following.

Corollary III.8.7.2. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}} a(r) \nabla_M^g u(t,r)\right) & \text{for } t \geq 0, r \in \overline{M}, \\ \qquad\qquad + \langle b(r), \nabla_M^g u(t,r)\rangle + c(r)\cdot u(t,r) & \\ \partial_t u(t,s) = -\dfrac{\partial^a}{\partial n^g} u(t,s) + b_0(s)\cdot u(t,s) & \text{for } t \geq 0, s \in \partial M, \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{M})$. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{M}$ and $u(t,\cdot) \in \mathrm{C}^\infty(M)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Finally, Theorem III.2.2.1 implies the following corollaries.

Corollary III.8.7.3. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\overline{M})$ and $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial M)$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ such that $f(t)(r) =$*

$g(t)(r)$ for all $t \in [0,\tau]$ and $r \in \partial\Omega$. For all $u_0 \in D(A)$ the problem

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r)\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = -\dfrac{\partial^a}{\partial n^g} u(t,s) + b_0(s)\cdot u(t,s) + g(t)(s) & \text{for } t \in [0,\tau], s \in \partial M, \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{M})$. If the initial data u_0 and f are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.7.4. *Let $F\colon [0,\tau]\times \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ and $G\colon [0,\tau]\times \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ continuously differentiable functions such that $F(t,u(t,\cdot))(s) = G(t,u(t,\cdot))(s)$ for all $t \in [0,\tau]$ and $s \in \partial M$. For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r)\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + F(t,u(t,\cdot))(r) & \\ \partial_t u(t,s) = -\dfrac{\partial^a}{\partial n^g} u(t,s) + b_0(s)\cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial M, \\ \qquad + G(t,u(t,\cdot))(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{M})$. Moreover, if the initial data u_0 and F are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Wellposedness of elliptic equations with dynamic boundary conditions

Since the operator A_0 with Dirichlet boundary conditions has compact resolvent, by [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Moreover, Proposition III.5.0.2 implies that N_λ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$ for all $\lambda \in \rho(A_0)$. Theorem III.3.1.1 now give the following.

Corollary III.8.7.5. *For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|a(r)|}}a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle + c(r)\cdot u(t,r) & \text{for } t \geq 0, r \in \overline{M}, \\ \partial_t u(t,s) = -\frac{\partial^a}{\partial n^g}u(t,s) + d(s)\cdot u(t,s) & \text{for } t \geq 0, s \in \partial M, \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{M})$ if and only if $\lambda \in \rho(A_0)$. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{M}$ and $u(t,\cdot) \in \mathrm{C}^\infty(M)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Moreover, as a consequence of Theorem III.3.2.1 we obtain the following results.

Corollary III.8.7.6. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\overline{M})$ and $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial M)$ integrable functions and $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|a(r)|}}a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = -\frac{\partial^a}{\partial n^g}u(t,s) + d(s)\cdot u(t,s) + g(t)(s) & \text{for } t \in [0,\tau], s \in \partial M, \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{M})$ if and only if $\lambda \in \rho(A_0)$. Moreover, if the initial data u_0 and f, g are positive, then the solution $u(t,\cdot)$ is positive for $t \in [0,\tau]$.

Corollary III.8.7.7. *Let $F\colon [0,\tau] \times \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ and $G\colon [0,\tau] \times \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ continuously differentiable functions. For all $x_0 \in D(N)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|a(r)|}}a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle + c(r)\cdot u(t,r) & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + F(t,u(t,\cdot))(r) & \\ \partial_t u(t,s) = -\frac{\partial^a}{\partial n^g}u(t,s) + d(s)\cdot u(t,s) & \text{for } t \geq 0, s \in \partial M, \\ \qquad + G(t,u(t,\cdot))(s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{M})$ *if and only if* $\lambda \in \rho(A_0)$. *Moreover, if the initial data* u_0 *and* F, G *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$.

III.8.8 Elliptic operator with generalized Wentzell boundary conditions on manifolds

In this section we generalize Subsection III.8.6 to smooth, compact, Riemannian manifolds $(\overline{M}, g)$ with smooth boundary ∂M. We consider as in the last section a smooth, compact, Riemannian manifold $\overline{M}$ with smooth boundary ∂M and an uniformly elliptic operator $A_m\colon D(A_m) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ in divergence form given (III.33). Further, we consider real-valued functions

$$\alpha_{jk} = \alpha_{kj} \in \mathrm{C}^\infty(\partial M), \quad \beta_j \in \mathrm{C}^\infty(\partial M), \quad \gamma \in \mathrm{C}^\infty(\partial M), \quad 1 \le j,k \le n$$

satisfying the strict ellipticity condition

$$\alpha_j^k(s) g^{jl}(s) X_k(s) X_l(s) > 0$$

for all co-vectorfields X_k, X_l on ∂M with $(X_1(s), \dots, X_n(s)) \neq (0, \dots, 0)$. We define an operator $C : D(C) \subseteq \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ in divergence form on the boundary by

$$\text{(III.35)} \qquad \begin{aligned} Cx &\coloneqq \sqrt{|\alpha|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|\alpha|}}\alpha\nabla_M^g x\right) + \langle \beta, \nabla_M^g x\rangle + \gamma x, \\ D(C) &\coloneqq \Big\{x \in \bigcap_{p\ge 1} W^{2,p}(\partial M) \cap \mathrm{C}(\partial M) \colon Cx \in \mathrm{C}(\partial M)\Big\}, \end{aligned}$$

where $\alpha = (\alpha_j^k)$ and $\beta = (\beta_j)$. Moreover we define the operator $B_0\colon D(B_0) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial\Omega)$ by (III.33) and finally $B\colon D(B) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$ by

$$Bf \coloneqq B_0 f + CLf, \quad D(B) \coloneqq \{f \in D(A_m) \cap D(B_0) \colon Lf \in D(C)\}.$$

Now we consider $A\colon D(A) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ given by

$$\text{(III.36)} \quad Af \coloneqq A_m f,\ D(A) \coloneqq \{f \in D(A_m) \cap D(B) \colon LA_m f = B_0 f + CLf\}.$$

Generation of semigroups

Choosing $X = \mathrm{C}(\overline{\Omega})$, $\partial X = \mathrm{C}(\partial M)$ and A_m, B, L as above the operator A given by (III.36) fits into our abstract framework $A = A^B$.

Using the same arguments as in the last section we can assume without loss of generality that the maximal operator is the Laplace-Beltrami operator $\Delta_M^{\tilde{g}}$. It remains to show that the Dirichlet-to-Neumann operator N generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$. Note that

$$N_\lambda^B = C + N_\lambda^{B_0}, \quad D(N_\lambda^B) = D(C) \tag{III.37}$$

and hence by perturbation it is sufficient to prove the statement for C. Observe that C is a strictly elliptic operator on a smooth, closed, Riemannian manifold $(\partial M, \tilde{g})$.
Considering ∂M equipped with the new Riemannian metric induced by

$$\bar{g}^{kl} = \alpha_i^k \tilde{g}^{il}$$

it turns out that C can be rewritten as Laplace-Beltrami operator $\Delta_{\partial M}^{\bar{g}}$ plus lower order terms and it follows by perturbation that C generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$. For details see [Bin20a] and [Bin20b]. Now we conclude

Theorem III.8.8.1. *The operator A defined in* (III.36) *generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$.*

Wellposedness of parabolic equations with dynamic boundary conditions

Using the results from above, Theorem III.2.1.1 implies the following.

Corollary III.8.8.2. *For all $u_0 \in D(A)$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\operatorname{div}_g\left(\frac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle + c(r)\cdot u(t,r) & \text{for } t \geq 0, r \in \overline{M}, \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\operatorname{div}_g\left(\frac{1}{\sqrt{|\alpha(s)|}} \alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle + \gamma\cdot u(t,s) - \frac{\partial^a}{\partial n^g} u(t,s) & \text{for } t \geq 0, s \in \partial M, \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{M})$. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{M}$ and $u(t,\cdot) \in \mathrm{C}^\infty(M)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Moreover, we conclude from Theorem III.2.2.1 the following results.

Corollary III.8.8.3. *Let* $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\overline{M})$ *and* $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial M)$ *integrable functions and* $f(t) := f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) := g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$ *such that* $f(t)(r) = g(t)(r)$ *for all* $t \in [0,\tau]$ *and* $r \in \partial M$. *For all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|\alpha(s)|}} \alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle + \gamma\cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial M, \\ \qquad - \frac{\partial^a}{\partial n^g} u(t,s) + g(t)(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{M})$. *If the initial data* u_0 *and* f *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$.

Corollary III.8.8.4. *Let* $F\colon [0,\tau]\times \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ *and* $G\colon [0,\tau]\times\mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ *continuously differentiable functions such that* $F(t,u(t,\cdot))(s) = G(t,u(t,\cdot))(s)$ *for all* $t \in [0,\tau]$ *and* $s \in \partial M$. *For all* $u_0 \in D(A)$ *the problem*

$$\begin{cases} \partial_t u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + F(t,u(t,\cdot))(r) & \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|\alpha(s)|}} \alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle + \gamma\cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial M, \\ \qquad - \frac{\partial^a}{\partial n^g} u(t,s) + G(t,u(t,\cdot))(s) & \\ u(0,r) = u_0(r) & \text{for } r \in \overline{M} \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{M})$. *Moreover, if the initial data* u_0 *and* F *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$.

Wellposedness of elliptic equations with dynamic boundary conditions

Since the operator A_0 with Dirichlet boundary conditions has compact resolvent, by [BE20a, Lemma 3.2] it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Moreover, Proposition III.5.0.2 implies that N_λ generate compact and analytic semigroups of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$ for all $\lambda \in \rho(A_0)$. Theorem III.3.1.1 now give the following.

Corollary III.8.8.5. *For all $x_0 \in D(C)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}}a(r)\nabla_M^g u(t,r))\right) & \text{for } t \geq 0, r \in \overline{M}, \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle + c(r)\cdot u(t,r) & \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|\alpha(s)|}}\alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle & \text{for } t \geq 0, s \in \partial M, \\ \qquad + \gamma \cdot u(t,s) - \dfrac{\partial^a}{\partial n^g}u(t,s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

with dynamic boundary conditions is well-posed on $\mathrm{C}(\overline{M})$ if and only if $\lambda \in \rho(A_0)$. Further, $t \to u(t,r)$ is analytic for all $r \in \overline{M}$ and $u(t,\cdot) \in \mathrm{C}^\infty(M)$. If the initial value u_0 is positive, then the solution $u(t,\cdot)$ is positive for $t \geq 0$.

Further Theorem III.3.2.1 yields the following corollaries.

Corollary III.8.8.6. *Let $\tilde{f}\colon [0,\tau] \to \mathrm{C}(\overline{M})$ and $\tilde{g}\colon [0,\tau] \to \mathrm{C}(\partial M)$ integrable functions and $f(t) \coloneqq f_0 + \int_0^t \tilde{f}(s)\,\mathrm{d}s$, $g(t) \coloneqq g_0 + \int_0^t \tilde{g}(s)\,\mathrm{d}s$. For all $x_0 \in D(C)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}}a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + f(t)(r) & \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|\alpha(s)|}}\alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle + \gamma \cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial M, \\ \qquad - \dfrac{\partial^a}{\partial n^g}u(t,s) + g(t)(s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

with dynamic boundary conditions admits a unique solution on $\mathrm{C}(\overline{M})$ *if and only if* $\lambda \in \rho(A_0)$*. Moreover, if the initial data* u_0 *and* f, g *are positive, then the solution* $u(t, \cdot)$ *is positive for* $t \in [0, \tau]$*.*

Corollary III.8.8.7. *Let* $F\colon [0,\tau] \times \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ *and* $G\colon [0,\tau] \times \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ *continuously differentiable functions. For all* $x_0 \in D(C)$ *the problem*

$$\begin{cases} \lambda u(t,r) = \sqrt{|a(r)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|a(r)|}} a(r)\nabla_M^g u(t,r))\right) & \\ \qquad + \langle b(r), \nabla_M^g u(t,r)\rangle & \text{for } t \in [0,\tau], r \in \overline{M}, \\ \qquad + c(r)\cdot u(t,r) + F(t, u(t,\cdot))(r) & \\ \partial_t u(t,s) = \sqrt{|\alpha(s)|}\,\mathrm{div}_g\left(\dfrac{1}{\sqrt{|\alpha(s)|}}\alpha(s)\nabla_M^g u(t,s)\right) & \\ \qquad + \langle \beta(s), \nabla_M^g u(t,s)\rangle + \gamma\cdot u(t,s) & \text{for } t \in [0,\tau], s \in \partial M, \\ \qquad - \dfrac{\partial^a}{\partial n^g} u(t,s) + G(t, u(t,\cdot))(s) & \\ u(0,s) = x_0(s) & \text{for } s \in \partial M \end{cases}$$

admits a unique solution on $\mathrm{C}(\overline{M})$ *if and only if* $\lambda \in \rho(A_0)$*. Moreover, if* u_0 *and* F, G *are positive, then the solution* $u(t,\cdot)$ *is positive for* $t \in [0,\tau]$*.*

III.8.9 Interior boundary conditions

The concept of interior boundary conditions was introduced and developed by Teufel and Tumulka, see [TT15], [TT16], [ST19] [STT19], [DGT+19], [Tum20] and [HT20], to describe particle creation and annihilation in quantum dynamics.

For simplicity of the representation we first concentrate on a model of two particles, where one particle is fixed on the origin of our coordinate system. In the following we call the particle at the origin the x-particle and the other the y-particle. Further we assume that the x-particle can emit and absorb the y-particle.

Take as configuration space

$$Q \coloneqq Q_{(1)} \,\dot\cup\, Q_{(0)},$$

where $Q_{(1)} \coloneqq \mathbb{R}^3 \setminus \{0\}$ is the configuration space of y-particle and $Q_{(0)} \coloneqq \{0\}$ is the configuration space of the fixed x-particle. Note that we must exclude the origin in the configuration space of the y-particle, since if it reaches the origin it collides with the x-particle and will be absorbed. Hence as Hilbert space we choose the truncated fock space

$$\mathscr{H} := \mathrm{L}^2(Q) = \mathrm{L}^2(Q_{(1)}) \oplus \mathrm{L}^2(Q_{(0)}) \cong \mathrm{L}^2(\mathbb{R}^3) \oplus \mathbb{C}.$$

The maximal Hamiltonian $H_m \colon D(H_m) \subset \mathscr{H} \to \mathscr{H}$ is given by

$$\begin{aligned} &H_m(\psi_1 + \psi_0) := -\Delta\psi_1, \\ &D(H_m) := \{\psi_1 \in \mathrm{L}^2(Q_{(1)}) \colon \Delta\psi_1 \in \mathrm{L}^2(Q_{(1)})\} \oplus \mathbb{C}. \end{aligned} \tag{III.38}$$

The annihilation operator $B \colon D(B) \subset \mathscr{H} \to \mathbb{C}$ is defined as

$$\begin{aligned} &B\psi := \frac{1}{4\pi} \lim_{r\to 0} \partial_r \int_{\mathbb{S}^2} r\psi(r\omega)\,\mathrm{d}\omega, \\ &D(B) := \left\{\psi \in \mathscr{H} \colon \lim_{r\to 0} \partial_r \int_{\mathbb{S}^2} r\psi(r\omega)\,\mathrm{d}\omega \text{ exists}\right\} \end{aligned} \tag{III.39}$$

where (r, ω) are the spherical coordinates of $\mathbb{R}^3$. Moreover, the trace operator $L \colon D(H_m) \subset \mathscr{H} \to \mathbb{C}$ is the symmetrized evaluation

$$\begin{aligned} &L\psi := -4\pi \cdot \lim_{r\to 0} \int_{\mathbb{S}^2} r\psi(r\omega)\,\mathrm{d}\omega, \\ &D(L) := \left\{\psi \in \mathscr{H} \colon \lim_{r\to 0} \int_{\mathbb{S}^2} r\psi(r\omega)\,\mathrm{d}\omega \text{ exists}\right\}. \end{aligned} \tag{III.40}$$

Now the Hamiltonian $H_{\mathrm{IBC}} \colon D(H_{\mathrm{IBC}}) \subset \mathscr{H} \to \mathscr{H}$ is given by

$$\begin{aligned} H_{\mathrm{IBC}}\psi &:= H_m\psi + IB\psi, \\ D(H_{\mathrm{IBC}}) &:= \{\psi = \psi_1 + \psi_0 \colon \psi \in D(H_m) \cap D(B), \psi_0 \in \mathbb{C} \colon L\psi = \psi_0\}, \end{aligned}$$

where $I \colon \mathbb{C} \hookrightarrow \mathscr{H}$ denotes the embedding to the second component. In [LSTT18] it is shown that

$$D(H_m) = D(H_{\mathrm{free}}) \oplus \ker(\lambda - H_m) \tag{III.41}$$

for all $\lambda \in \rho(H_{\mathrm{free}})$, where $H_{\mathrm{free}} \colon D(H_{\mathrm{free}}) := \mathrm{W}^{2,2}(\mathbb{R}^3) \oplus \mathbb{C} \subset \mathscr{H} \to \mathscr{H} \colon \psi \mapsto H_m\psi$ denotes the free Hamiltonian. Moreover, we have

$$\ker(\lambda - H_m) = \mathrm{span}(g_\lambda) \cong \mathbb{C},$$

where $g_\lambda(x) := -\frac{e^{-\sqrt{|\lambda|}\cdot|x|}}{4\pi|x|}$.

Note that the interior boundary conditions given above are of Dirichlet type, since essentially it specifies the value of ψ_1 at the boundary. Replacing the evaluation $\psi|_{\partial Q}$ by $\alpha\frac{\partial}{\partial n}\psi + \beta\psi|_{\partial Q} = 0$ yields interior boundary conditions of Robin type. More precisely, we obtain the Hamiltonian $H^{\alpha,\beta}_{\mathrm{IBC}}$ with interior boundary conditions of Robin type (α, β) given by

$$\begin{aligned} H^{\alpha,\beta}_{\mathrm{IBC}}\psi &:= H_m\psi + \gamma IL\psi + \delta IB\psi, \\ D(H^{\alpha,\beta}_{\mathrm{IBC}}) &:= \left\{ \psi = \psi_1 + \psi_0 \colon \begin{array}{c} \psi \in D(H_m) \cap D(B), \psi_0 \in \mathbb{C} \\ \alpha B\psi + \beta L\psi = \psi_0 \end{array} \right\}. \end{aligned} \tag{III.42}$$

For more information about interior boundary conditions we refer to [TT15, Remark 3.3.5]. In [LSTT18] it is proven that all these operators satisfy (III.41).

Using (III.41) instead of (III.6) or (III.10) we developed in [BL20] an abstract framework which is tailored towards operators with interior boundary conditions.

Abstract Setting III.8.9.1. Consider

(i) two Hilbert spaces $\mathscr{H}$ and $\partial\mathscr{H}$;

(ii) a densely defined maximal operator $H_m \colon D(H_m) \subset \mathscr{H} \to \mathscr{H}$;

(iii) a trace operator $L \colon D(H_m) \subset \mathscr{H} \to \partial\mathscr{H}$;

(iv) a boundary operator $B \colon D(B) \subset \mathscr{H} \to \partial\mathscr{H}$;

(v) a identification operator $I \colon \partial\mathscr{H} \to H$.

Now an *abstract operator* $H^{\alpha,\beta}_{\mathrm{IBC}} \colon D(H^{\alpha,\beta}_{\mathrm{IBC}}) \subset \mathscr{H} \to \mathscr{H}$ *with interior boundary conditions* associated to $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ is given by

$$\begin{aligned} H^{\alpha,\beta}_{\mathrm{IBC}}\psi &:= H_m\psi + \gamma IL\psi + \delta IB\psi, \\ D(H^{\alpha,\beta}_{\mathrm{IBC}}) &:= \{\psi \in D(H_m) \cap D(B), \alpha B\psi + \beta L\psi = I^*\psi\}. \end{aligned}$$

Note that $H^{\alpha,\beta}_{\mathrm{IBC}}$ is symmetric if and only if $\beta\bar{\gamma} - \bar{\alpha}\delta = 1$ and $\bar{\alpha}\gamma \in \mathbb{R}$, $\bar{\beta}\delta \in \mathbb{R}$, see [BL20, Lemma 3.5]. Moreover let A_0, L_λ and N_λ defined as above.

Consider $\mathscr{H} := \mathrm{L}^2(Q)$, $\partial\mathscr{H} := \mathbb{C}$ and $I \colon \mathbb{C} \hookrightarrow \mathrm{L}^2(Q_1) \oplus \mathbb{C} \colon z \mapsto 0 + z$ the inclusion. Moreover, let H_m, B and L as in (III.38), (III.39) and (III.40). Then we obtain

$$\begin{aligned} H^{\alpha,\beta}_{\mathrm{IBC}}\psi &= H_m\psi + \gamma IL\psi + \delta IB\psi, \\ D(H^{\alpha,\beta}_{\mathrm{IBC}}) &= \{\psi = \psi_0 + \psi_1 \colon \psi_1 \in D(H_m) \cap D(B), \alpha B\psi + \beta L\psi = \psi_0\}, \end{aligned}$$

which coincides with (III.42). Further, we obtain $A_0 = H_{\text{free}}$ and N_λ is the corresponding Weyl function.

In the sequel we make the following assumptions.

Assumptions III.8.9.2. (i) *A_0 is self-adjoint on $\mathscr{H}$;*

(ii) *B is relatively A_0-bounded;*

(iii) $\operatorname{rg}((BR(\lambda, A_0))^*) \subset \ker(\bar{\lambda} - A_m)$ *for* $\lambda \in \rho(A_0)$*;*

(iv) $L((BR(\lambda, A_0))^*) = \operatorname{Id}_{\mathscr{H}}$ *for* $\lambda \in \rho(A_0)$*.*

Self-adjointness

In this subsection we study the self-adjointness of the operator with interior boundary conditions. Recall that by Stone's Theorem (see [EN00, Theorem II.3.24]) a densely defined operator $A\colon D(A) \subset E \to E$ on a Hilbert space E is self-adjoint if and only if iA generates a strongly continuous group of unitary operators. By [BL20, Lemma 3.5] the operator $H^{\alpha,\beta}_{\mathrm{IBC}}$ is symmetric if and only if $\beta\bar{\gamma} - \bar{\alpha}\delta = 1$ and $\bar{\alpha}\gamma \in \mathbb{R}$, $\bar{\beta}\delta \in \mathbb{R}$ and hence these conditions are necessary for the self-adjointness of $H^{\alpha,\beta}_{\mathrm{IBC}}$.

For $\alpha, \beta \in \mathbb{C}$ we denote by $A_{\alpha,\beta}\colon D(A_{\alpha,\beta}) \subset \mathscr{H} \to \mathscr{H}$ the *abstract operator with Robin boundary conditions*

$$\begin{aligned} A_{\alpha,\beta} f &:= H_m f, \\ D(A_{\alpha,\beta}) &:= \{f \in D(H_m) \cap D(B) \colon \alpha B f + \beta L f = 0\}. \end{aligned} \tag{III.43}$$

Now the operator $H^{\alpha,\beta}_{\mathrm{IBC}}$ can be seen as the a perturbed operator of $A_{\alpha,\beta}$ by two perturbations: one of the domain and one of the action. This fits perfectly into the theory of abstract boundary value problems, developed in [Gre87], [ABE14], [ABE17] and [AE18]. Note that the perturbation of the domain is I^* and therefore bounded and hence we can use similar arguments as in [Gre87]. Denoting the Dirichlet operator associated to $\alpha B + \beta L$ instead of L by $L^{\alpha,\beta}_\lambda$ and the corresponding Dirichlet-to-Neumann operator $N^{\alpha,\beta}_\lambda := (\gamma B + \delta L) L^{\alpha,\beta}_\lambda$ we obtain the following (see [BL20, Theorem 3.12]).

Theorem III.8.9.3. *Assume that $A_{\alpha,\beta}$ is self-adjoint and let $\lambda \in \rho(A_{\alpha,\beta}) \cap \rho(A_0)$. Assume also that $1 \in \rho(L^{\alpha,\beta}_\lambda I^*) \cap \rho(L^{\alpha,\beta}_{\bar{\lambda}} I^*)$ and*
$I N^{\alpha,\beta}_\lambda I^*$ is relatively $(\operatorname{Id} - L^{\alpha,\beta}_{\bar{\mu}} I^*)^* (A_{\alpha,\beta} - \mu)(\operatorname{Id} - L^{\alpha,\beta}_\mu I^*)$ bounded, with bound a; with $a < 1$ for $\mu \in \{\lambda, \overline{\lambda}\}$, then $H^{\alpha,\beta}_{\mathrm{IBC}}$ is self-adjoint on $\mathscr{H}$.

We point out that the identification operator I allows to compare the Dirichlet-to-Neumann operator N_λ with the operator with Dirichlet boundary conditions. Hence we do not need the self-adjointness of N_λ, since we can use a perturbation argument.
This theorem above has many variants and generalizations. We refer to [BL20, Theorem 3.14] and its corollaries.

Quasi-boundary triples

The theory of quasi-boundary triples was developed by Jussi Behrndt, see [BL07], [BL12], [BM14] and [BS19]. The state of art is summarized in [BHS20].

Definition III.8.9.4. A triple $(\partial\mathscr{H}, L, B)$ is called a *quasi-boundary triple* for an operator $A_m\colon D(A_m) \subset \mathscr{H} \to \mathscr{H}$ if $\partial\mathscr{H}$ is a Hilbert space and $L, B\colon D(A_m) \subset \mathscr{H} \to \partial\mathscr{H}$ are operators such that

(i) there exists a closed, densely defined operator $\hat{A}$ such that the closure of A_m is $\bar{A}_m = \hat{A}^*$;

(ii) the second Green identity holds

$$\langle A_m f, g\rangle_{\mathscr{H}} - \langle f, A_m g\rangle_{\mathscr{H}} = \langle Bf, Lg\rangle_{\partial\mathscr{H}} - \langle Lf, Bg\rangle_{\partial\mathscr{H}}$$

for all $f, g \in D(A_m)$.

(iii) The map $(L, B) : D(A_m) \to \partial\mathscr{H} \times \partial\mathscr{H}$ has dense range.

(iv) The restriction $A_0 := (A_m)|_{X_0}$ to $X_0 := \ker(L)$ is a self-adjoint operator on $\mathscr{H}$.

A quasi-boundary triple is called an ordinary boundary triple if $\operatorname{rg}(L, B) = \partial\mathscr{H} \times \partial\mathscr{H}$ and a generalized boundary triple if L is surjective. A quasi boundary triple for $\hat{A}^*$ exists if and only if the defect indices $n_\pm(\hat{A}) := \dim(\ker(\hat{A}^* \mp i))$ of A coincide. If the defect indices of A are finite the quasi-boundary triple for A is an ordinary boundary triple. Moreover, the operator $(L, B)\colon D(A_m) \subset \mathscr{H} \to \partial\mathscr{H} \times \partial\mathscr{H}$ is closable and by [BL07, Proposition 2.2] it follows that $\ker(L, B) = D(\hat{A})$ holds. By [BL07, Theorem 2.3] it follows that $A = \hat{A}^*$ if and only if $\operatorname{rg}(L, B) = \partial\mathscr{H} \times \partial\mathscr{H}$. In this case the restriction $A := \hat{A}^*|_{X_0}$ is self-adjoint and the quasi boundary triple $(\partial\mathscr{H}, L, B)$ is an ordinary boundary triple. For each $\lambda \in \rho(A_0)$ the definition of a quasi-boundary triple yields the decomposition

$$D(A_m) = D(A_0) \oplus \ker(\lambda - A_m).$$

Compare this decomposition of (III.6) and (III.10). Analogous as above we define the Dirichlet operator $L_\lambda := (L|_{\ker(\lambda - A_m)})^{-1}$ and the Dirichlet-to-Neumann operators $N_\lambda := BL_\lambda$. In the theory of quasi-boundary triples the Dirichlet operators are called γ-field and the Dirichlet-to-Neumann operators are called Weyl function. For more details about quasi-boundary triples see [BLL13], [BDGM18] and [BFK+17].

In comparison to our theory the theory of quasi-boundary triples has two mayor restrictions. First, it works only on Hilbert spaces, since its starting point is an abstract variant of Green's identity and hence a scalar product is necessary. The second one is that the feedback operator B has to be defined on $D(A_m)$ which implies that the Dirichlet-to-Neumann operators N_λ are bounded for all $\lambda \in \rho(A_0)$.

Consider the operator $\tilde{H}_m \colon D(\tilde{H}_m) \subset \mathscr{H} \to \mathscr{H}$ given by

$$\tilde{H}_m\psi := H_m\psi + II^*\psi + I^*(B - L)\psi, \quad D(\tilde{H}_m) := D(H_m) \cap D(B)$$

and the minimal operator $\tilde{H}_0 \subset \tilde{H}_m$ with $D(\tilde{H}_0) := \{\psi \in D(H_m) \cap D(B) \colon L\psi = B\psi = I^*\psi\}$. In the following we assume that $\tilde{H}_0$ is densely defined. In [BL20, Theorem 4.7] we show the following result about operators with interior boundary conditions of the Dirichlet type.

Theorem III.8.9.5. *Let $\alpha = 0$, $\beta = 1$. Assume that $0 \in \rho(A_0)$ and $1 \in \rho(L_0 I^*)$. Further, assume that INI^* is relatively $(\mathrm{Id} - L_0 I^*)^* A_0 (\mathrm{Id} - L_0 I^*)^*$-bounded of bound $a < 1$. If $(1 - I^* L_0)^{-1}$ leaves $D(N)$ invariant, then $(\partial\mathscr{H}, (L - I^*), (B - I^*))$ is a quasi-boundary triple for $\tilde{H}_m$.*

We point out that $0 \in \rho(A_0)$ can be replace by $\rho(A_0) \neq \emptyset$. For details see [BL20, Theorem 4.7].
In order to investigate this problem into more detail, we define the Dirichlet operator associated to $L - I^*$ and $\tilde{H}_m$ by F_λ and $M := (F_i^* F_i)^{\frac{1}{2}}$. Further, we need the associated Dirichlet-to-Neumann operator $S_\lambda := (B - I^*)F_\lambda, D(S_\lambda) := \{\varphi \in \partial\mathscr{H} \colon F_\lambda \varphi \in D(B)\}$. Using the classification of quasi-boundary triples, see [BM14, Section 3] this result yields a classification result of the self-adjoint extensions with interior boundary conditions. Let $\mathfrak{R}$ be a relation on $\partial\mathscr{H}$ and define the operator $H_{\mathfrak{R}}$ as the restriction of $\tilde{H}_m$ to

$$D(H_{\mathfrak{R}}) := \{f \in D(H_m) \cap D(B) \colon ((L - I^*)f, (B - I^*)f) \in \mathfrak{R}\}.$$

The following result classifies all self-adjoint extensions of $\tilde{H}_0$. For details we refer to [BL20, Theorem 4.9].

Theorem III.8.9.6. *Assume there exists $\lambda \in \mathbb{R} \cap \rho(H^{0,1}_{\mathrm{IBC}})$. Then $H_{\mathfrak{R}}$ is self-adjoint on $\mathscr{H}$ if and only if the relation*

$$M^{-1}(\mathfrak{R} - S_\lambda)M^{-1}$$

is self-adjoint and satisfies $D(\mathfrak{R}) \subset MD(S_\lambda)$.

Convergence

Next we are interested in the convergence of operators $H^{\alpha,1}_{\mathrm{IBC}}$ with interior boundary conditions associated to α and 1 for $\alpha \to 0$. Fix $\gamma = 1$, $\delta = 0$, then we obtain that $H^{\alpha,1}_{\mathrm{IBC}}$ are symmetric for all $\alpha \in \mathbb{R}$. Moreover, one has the following result, see [BL20, Theorem 4.11].

Theorem III.8.9.7. *Assume that there exists a $\lambda_0 \in \mathbb{R}$ such that $\lambda \in \rho(A_0)$, N_λ are self-adjoint and bounded from above, $1 \in \rho(I^*L_\lambda)$, $(\mathrm{Id} - I * L_\lambda)^{-1}$ leaves $D(N)$ invariant and $IN_\lambda I^*$ is relatively $(\mathrm{Id} - L_\lambda I^*)^*(A_0 - \lambda)(\mathrm{Id} - L_\lambda I^*)^*$-bounded of bound 0 for all $\lambda < \lambda_0$. Suppose that A_0 is bounded from below. Then the operators $H^{\alpha,1}_{IBC}$ converge to $H^{0,1}_{IBC}$ in the norm resolvent sense for $\alpha \downarrow 0$.*

In particular we conclude for $I = 0$ a convergence result for operators $A_{\alpha,\beta}$ with Robin boundary conditions to the operator with Dirichlet boundary conditions in Setting III.8.9.1. This result is analogous to [BE20b, Corollary 4.8] in Setting III.1.1.1.

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A Appendix

A.1 Accepted Publications

A.1.1 Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann operator

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ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann operator

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Abstract

In this paper we relate the generator property of an operator A with (abstract) generalized Wentzell boundary conditions on a Banach space X and its associated (abstract) Dirichlet-to-Neumann operator N acting on a "boundary" space ∂X. Our approach is based on similarity transformations and perturbation arguments and allows to split A into an operator A_{00} with Dirichlet-type boundary conditions on a space X_0 of states having "zero trace" and the operator N. If A_{00} generates an analytic semigroup, we obtain under a weak Hille–Yosida type condition that A generates an analytic semigroup on X if and only if N does so on ∂X. Here we assume that the (abstract) "trace" operator $L : X \to \partial X$ is bounded that is typically satisfied if X is a space of continuous functions. Concrete applications are made to various second order differential operators.

KEYWORDS

analytic semigroup, Dirichlet-to-Neumann operator, Wentzell boundary conditions

MSC (2010)

34G10, 47D06, 47E05, 47F05

1 | INTRODUCTION

The generation of analytic semigroups by differential operators with generalized Wentzell boundary conditions on spaces of continuous functions attracted the interest of many authors, and we refer, e.g., to [2–4,10,11]. For their derivation and physical interpretation we refer to [12]. The present paper is a continuation and improvement of [4] where we introduced a general abstract framework to deal with this problem. Before recalling this setting we consider the following typical example in order to explain the basic ideas and the goal of our approach.

Take a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Then consider on $\mathrm{C}(\overline{\Omega})$ the Laplacian Δ_m with "maximal" domain $D(\Delta_m) := \{f \in \mathrm{C}(\overline{\Omega}) : \Delta_m f \in \mathrm{C}(\overline{\Omega})\}$, where the derivatives are taken in the distributional sense. Finally, let $\frac{\partial}{\partial n} : D(\frac{\partial}{\partial n}) \subset \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ be the outer normal derivative, $\beta < 0$ and $\gamma \in \mathrm{C}(\partial\Omega)$. In this setting we define the Laplacian $A \subset \Delta_m$ with *generalized Wentzell boundary conditions* by requiring

$$f \in D(A) \quad :\Longleftrightarrow \quad \Delta_m f\Big|_{\partial\Omega} = \beta \cdot \tfrac{\partial}{\partial n} f + \gamma \cdot f\Big|_{\partial\Omega}. \tag{1.1}$$

Our approach decomposes a function $f \in \mathrm{C}(\overline{\Omega})$ into the (unique) sum $f = f_0 + h$ of a function f_0 vanishing at the boundary $\partial\Omega$ and a harmonic function h having the same trace as f. In other words, if $L : \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$, $Lf := f|_{\partial\Omega}$ denotes the trace operator, then $f_0 \in \ker L = \mathrm{C}_0(\Omega)$ while $h \in \ker(\Delta_m)$. Since h is uniquely determined by its trace, it can be identified with its boundary value $x := Lh$. Hence, every $f \in \mathrm{C}(\overline{\Omega})$ corresponds to a unique pair $\binom{f_0}{x} \in \mathrm{C}_0(\Omega) \times \mathrm{C}(\partial\Omega)$.

To formalize this decomposition we introduce an abstract "*Dirichlet operator*" $L_0 : \mathrm{C}(\partial\Omega) \to \mathrm{C}(\overline{\Omega})$. To this end we consider for a given "boundary function" $x \in \mathrm{C}(\partial\Omega)$ the Dirichlet problem

$$\begin{cases} \Delta_m f = 0, \\ f|_{\partial\Omega} = x. \end{cases} \tag{1.2}$$

This system admits a unique solution $f \in \mathrm{C}(\overline{\Omega})$, so by setting $L_0 x := f$ we obtain a bounded operator $L_0 \in \mathcal{L}\big(\mathrm{C}(\partial\Omega), \mathrm{C}(\overline{\Omega})\big)$. For $f \in \mathrm{C}(\overline{\Omega})$ we then have $f = f_0 + h$ where $f_0 := \big(\mathrm{Id} - L_0 L\big) f$ and $h = L_0 x$ for $x := Lf$. By (1.1) it then follows (for the details see Step 1 below in the proof of Theorem 3.1) that A on $\mathrm{C}(\overline{\Omega})$ transforms into an operator matrix $\mathcal{A}$ on $\mathrm{C}_0(\Omega) \times \mathrm{C}(\partial\Omega)$ of the form

$$\mathcal{A} := \begin{pmatrix} \Delta_m & 0 \\ 0 & N \end{pmatrix} + \mathcal{P} \tag{1.3}$$

with some appropriate "non-diagonal" domain $D(\mathcal{A}) \subset \mathrm{C}_0(\Omega) \times \mathrm{C}(\partial\Omega)$, see [5,6,14]. Here $\mathcal{P}$ denotes an unbounded perturbation while $N := \beta \cdot \frac{\partial}{\partial n} \cdot L_0$ is the so called *Dirichlet-to-Neumann operator* on $\mathrm{C}(\partial\Omega)$, see [9], [15, Sec. 12.C]. That is, Nx is obtained by applying the Neumann boundary operator to the solution f of the Dirichlet problem (1.2).

Using perturbation arguments one can show that $\mathcal{A}$, hence also A, generates an analytic semigroups if and only if the Dirichlet Laplacian Δ_{00} on $\mathrm{C}_0(\Omega)$ and the Dirichlet-to-Neumann operator N on $\mathrm{C}(\partial\Omega)$ do so. This means that we decoupled the operator $A \subset \Delta_m$ with generalized Wentzell boundary conditions on $X := \mathrm{C}(\overline{\Omega})$ into an operator $A_{00} := \Delta_{00}$ with Dirichlet boundary conditions on $X_0 := \mathrm{C}_0(\Omega)$ and the Dirichlet-to-Neumann operator $N := \beta \cdot \frac{\partial}{\partial n} \cdot L_0$ on the boundary space $\partial X := \mathrm{C}(\partial\Omega)$.

Since it is well-known that Δ_{00} generates an analytic semigroup, our main result applied to this example yields that A generates an analytic semigroup on $\mathrm{C}(\overline{\Omega})$ if and only if N generates an analytic semigroup on $\mathrm{C}(\partial\Omega)$. Since the latter is true, see [3, Sec. 2], we conclude that $A \subset \Delta_m$ with generalized Wentzell boundary condition (1.1) is the generator of an analytic semigroup. We mention that our approach also keeps track of the angle of analyticity and, in the above example, gives the optimal angle $\frac{\pi}{2}$.

This paper is organized as follows. In Section 2 we introduce our abstract setting and then state in Section 3 our main abstract generation result, Theorem 3.1. In the following Section 4 we show that the generator property of operators with generalized Wentzell boundary conditions is invariant under "small" perturbations with respect to the action as well as the domain, cf. Theorem 4.2 and Theorem 4.3. For these proofs we study in Lemma 4.6 and Proposition 4.7 how the Dirichlet- and Dirichlet-to-Neumann operator, respectively, behave under relatively bounded perturbations. Finally, in Section 5 we apply our abstract results to second order differential operators on $\mathrm{C}([0,1], \mathbb{C}^n)$, the Banach space-valued second-order derivative, a perturbed Laplacian with generalized Wentzell boundary conditions and uniformly elliptic operators on $\mathrm{C}(\overline{\Omega})$. Our notation closely follows the monograph [7].

2 | THE ABSTRACT SETTING

As in [4, Sec. 2], the starting point of our investigation is the following

Abstract Setting 2.1 Consider

- **(i)** two Banach spaces X and ∂X, called *state* and *boundary space*, respectively;
- **(ii)** a densely defined *maximal operator* $A_m : D(A_m) \subset X \to X$;
- **(iii)** a *boundary (or trace) operator* $L \in \mathcal{L}(X, \partial X)$;
- **(iv)** a *feedback operator* $B : D(B) \subseteq X \to \partial X$.

Using these spaces and operators we define the operator $A^B : D(A^B) \subset X \to X$ with abstract *generalized Wentzell boundary conditions* by

$$A^B \subseteq A_m, \quad D\big(A^B\big) := \big\{ f \in D\big(A_m\big) \cap D(B) : LA_m f = Bf \big\}. \tag{2.1}$$

If $B = 0$ the boundary conditions defined by (2.1) are called *pure Wentzell boundary conditions*. For an interpretation of Wentzell- as "dynamic boundary conditions" we refer to [4, Sec. 2].

To fit the example from the introduction into this setting it suffices to choose $X := \mathrm{C}(\overline{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$, $A_m := \Delta_m$, $Lf := f|_{\partial\Omega}$ and $B := \beta \cdot \frac{\partial}{\partial n} + \gamma \cdot L$.

In the sequel we need the (in general non-densely defined) operator $A_0 : D(A_0) \subset X \to X$ defined by

$$A_0 \subseteq A_m, \quad D(A_0) := D(A_m) \cap \ker(L).$$

In the example from the introduction A_0 is the Dirichlet Laplacian Δ_0 on $\mathrm{C}(\overline{\Omega})$ with non-dense domain $D(A_0) = D(\Delta_m) \cap \mathrm{C}_0(\Omega)$.

Assumptions 2.2. (i) The operator A_0 is a weak Hille–Yosida operator on X, i.e. there exist $\lambda_0 \in \mathbb{R}$ and $M > 0$ such that $[\lambda_0, \infty) \subset \rho(A_0)$ and

$$\left\| \lambda R(\lambda, A_0) \right\| \le M \quad \text{for all } \lambda \ge \lambda_0;$$

(ii) the operator B is relatively A_0-bounded with bound 0, i.e., $D(A_0) \subseteq D(B)$ and for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|Bf\|_{\partial X} \le \varepsilon \cdot \|A_0 f\|_X + M_\varepsilon \cdot \|f\|_X \quad \text{for all } f \in D(A_0);$$

(iii) the *abstract Dirichlet operator* $L_0 := \left(L|_{\ker(A_m)}\right)^{-1} : \partial X \to \ker(A_m) \subseteq X$ exists and is bounded, i.e., for every $x \in \partial X$ the abstract Dirichlet problem

$$\begin{cases} A_m f = 0, \\ Lf = x \end{cases}$$

admits a unique solution $f \in D(A_m)$ and $L_0 x := f$ defines an operator $L_0 \in \mathcal{L}(\partial X, X)$.

We note that by [13, Lem. 1.2] assumption (iii) is always satisfied if A_m is closed, $L : X \to \partial X$ is surjective and A_0 is invertible. Moreover, $L_0 L \in \mathcal{L}(X)$ is a projection onto the subspace $\ker(A_m)$ along $X_0 := \ker(L)$ which induces the decompositions as topological direct sums

$$X = X_0 \oplus \ker(A_m) \qquad \text{and} \qquad D(A_m) = D(A_0) \oplus \ker(A_m) \tag{2.2}$$

with respect to the norm on X and the graph norm on $D(A_m)$, respectively.

In the sequel we will need the following operators.

Notation 2.3. Define $G_m : D(G_m) \subset X \to X$ by

$$G_m f := A_m f - L_0 B \cdot (\mathrm{Id} - L_0 L) f, \quad D(G_m) := D(A_m).$$

Then for $* \in \{1, 0, 00\}$ we consider the restrictions $A_* \subset A_m$ and $G_* \subset G_m$ given by

$$\begin{aligned} A_0 &: D(A_0) \subset X \to X, & D(A_0) &:= \{f \in D(A_m) : Lf = 0\}, \\ A_1 &: D(A_1) \subset X \to X, & D(A_1) &:= \{f \in D(A_m) : LA_m f = 0\}, \\ A_{00} &: D(A_{00}) \subset X_0 \to X_0, & D(A_{00}) &:= \{f \in D(A_m) : Lf = 0,\ LA_m f = 0\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} G_0 &: D(G_0) \subset X \to X, & D(G_0) &:= D(A_0), \\ G_1 &: D(G_1) \subset X \to X, & D(G_1) &:= \{f \in D(G_m) : LG_m f = 0\}, \\ G_{00} &: D(G_{00}) \subset X_0 \to X_0, & D(G_{00}) &:= \{f \in D(G_m) : Lf = 0,\ LG_m f = 0\}. \end{aligned}$$

Observe that $G_{00} \subset G_0 = A_0 - L_0 B$. Moreover, note that D_* for $D \in \{A, G\}$ and $* \in \{0, 1, 00\}$ is a restriction of D_m. For $* = 0$ this restriction corresponds to abstract Dirichlet boundary conditions and for $* = 1$ to pure Wentzell boundary conditions on X, while D_{00} is the part of D_0 as well as of D_1 in X_0. While the operators A_* are quite natural, the operators G_* are needed for technical reasons. In fact, by Step 1 of the proof of Theorem 3.1 they are closely related to the first diagonal entry of the operator matrix $\mathcal{A}$ in (1.3). Using perturbation arguments they will be simplified to the corresponding operators A_*.

Finally, we define the abstract *Dirichlet-to-Neumann operator* $N : D(N) \subset \partial X \to \partial X$ by

$$Nx := BL_0x, \quad D(N) := \{x \in \partial X : L_0x \in D(B)\}. \tag{2.4}$$

This operator plays a crucial role in our approach.

3 | THE MAIN RESULT

The following is our main abstract result. In contrast to [4, Thm. 3.1] it proves (besides further generalizations) that (a) $\iff$ (b) and not only that (b)$\Rightarrow$(a) in case $D = A$.

Theorem 3.1. *Let $D \in \{A, G\}$. Then the following statements are equivalent*

- **(a)** A^B *given by* (2.1) *generates an analytic semigroup of angle* $\alpha > 0$ *on* X.
- **(b)** D_0 *is sectorial of angle* $\alpha > 0$ *on* X *and the Dirichlet-to-Neumann operator* N *generates an analytic semigroup of angle* $\alpha > 0$ *on* ∂X.
- **(c)** D_1 *and* N *generate analytic semigroups of angle* $\alpha > 0$ *on* X *and* ∂X*, respectively.*
- **(d)** D_{00} *and* N *generate analytic semigroups of angle* $\alpha > 0$ *on* X_0 *and* ∂X*, respectively.*

Proof. By [4, Thm. 3.1] we have that (b)$\Rightarrow$,(a) for $D_0 = A_0$. Since A_0 and G_0 only differ by a relatively bounded perturbation of bound 0, [7, Lem. III.2.6] implies that assumption (b) is equivalent for $D = A$ and $D = G$. This shows that (b)$\Rightarrow$,(a). The equivalences (b) $\iff$ (c) $\iff$ (d) for $D = A$ follow by [4, Lem. 3.3]. Now assume that $D = G$. Then by [7, Lem. III.2.5] there exists $\lambda \in \rho(G_0)$. Since L is surjective, [13, Lem. 1.2] implies that the Dirichlet operator for $G_m - \lambda$ exists. As before, [4, Lem. 3.3] now applied to $G_0 - \lambda$, $G_1 - \lambda$ and $G_{00} - \lambda$ gives the equivalence of (b), (c) and (d) for $D = G$.

To complete the proof it suffices to verify that (a)$\Rightarrow$(d) for $D_{00} = G_{00}$. We proceed in several steps where we put $\mathcal{X}_0 := X_0 \times \partial X$.

Step 1. The operator $A^B : D(A^B) \subset X \to X$ is similar to $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X}_0 \to \mathcal{X}_0$ given by

$$\mathcal{A} := \begin{pmatrix} G_0 & -L_0N \\ B & N \end{pmatrix}, \quad D(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_0) \times D(N) : G_0f - L_0Nx \in X_0 \right\}.$$

Proof. The operator

$$T : X \to \mathcal{X}_0, \quad Tf := \begin{pmatrix} f - L_0Lf \\ Lf \end{pmatrix}$$

is bounded and invertible with bounded inverse

$$T^{-1} : \mathcal{X}_0 \to X, \quad T^{-1}\begin{pmatrix} f \\ x \end{pmatrix} = f + L_0x.$$

We show that $\mathcal{A} = TA^BT^{-1}$. Using that $LL_0 = \mathrm{Id}_{\partial X}$, $X_0 = \ker(L)$, $A_mL_0 = 0$ and $D(A_0) = D(A_m) \cap X_0 \subseteq D(B)$ we have

$$\begin{aligned}
\begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A}) &\iff f \in D(A_0),\ x \in D(N) \text{ and } A_mf - L_0Bf - L_0Nx \in X_0 \\
&\iff f \in D(A_0),\ x \in D(N) \text{ and } LA_mf - Bf - Nx = 0 \\
&\iff f \in D(A_0),\ x \in D(N) \text{ and } LA_m(f + L_0x) = B(f + L_0x) \\
&\iff T^{-1}\begin{pmatrix} f \\ x \end{pmatrix} \in D(A^B) \iff \begin{pmatrix} f \\ x \end{pmatrix} \in TD(A^B).
\end{aligned}$$

Moreover, for $\binom{f}{x} \in TD(A^B) = D(\mathcal{A})$ we obtain using that $f + L_0 x \in D(A^B)$

$$
\begin{aligned}
TA^B T^{-1}\begin{pmatrix} f \\ x \end{pmatrix} &= TA_m(f + L_0x) \\
&= \begin{pmatrix} A_m(f + L_0x) - L_0 L A_m(f + L_0x) \\ LA_m(f + L_0x) \end{pmatrix} \\
&= \begin{pmatrix} A_0 f - L_0 B f - L_0 N x \\ Bf + Nx \end{pmatrix} \\
&= \begin{pmatrix} G_0 & -L_0 N \\ B & N \end{pmatrix}\begin{pmatrix} f \\ x \end{pmatrix}.
\end{aligned}
$$

□

Step 2. The operator $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathcal{X}_0 \to \mathcal{X}_0$ given by

$$\mathcal{A}_0 := \begin{pmatrix} G_0 & -L_0 N \\ 0 & N \end{pmatrix}, \quad D(\mathcal{A}_0) := D(\mathcal{A})$$

generates an analytic semigroup of angle $\alpha > 0$ on $\mathcal{X}_0$.

Proof. By assumption A^B generates an analytic semigroup of angle $\alpha > 0$ on X. Hence, by Step 1, $\mathcal{A}$ generates an analytic semigroup of angle $\alpha > 0$ on $\mathcal{X}_0$. Since B is relatively A_0-bounded with bound zero, a simple computation using the triangle inequality shows that $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ with domain $D(\mathcal{B}) := (D(B) \cap X_0) \times \partial X$ is relatively $\mathcal{A}$-bounded with bound zero. Hence, by [7, Lem. III.2.6] also $\mathcal{A}_0 = \mathcal{A} - \mathcal{B}$ generates an analytic semigroup with angle $\alpha > 0$ on $\mathcal{X}_0$. □

Step 3. There exists $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, +\infty) \subset \rho(G_0) \cap \rho(G_{00}) \cap \rho(N) \cap \rho(\mathcal{A}_0)$ and

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, G_{00}) & -R(\lambda, G_0) L_0 N R(\lambda, N) \\ 0 & R(\lambda, N) \end{pmatrix} \quad \text{for } \lambda \geq \lambda_0. \tag{3.1}$$

Proof. By assumption A_0 is a weak Hille–Yosida operator. Since A_0 and $G_0 = A_0 - L_0 B$ differ only by a relatively bounded perturbation of bound 0, by [7, Lem. III.2.5] also G_0 is a weak Hille–Yosida operator. In particular, there exists $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, +\infty) \subset \rho(G_0) \cap \rho(\mathcal{A}_0)$. Moreover, [7, Prop. IV.2.17] implies $\rho(G_0) = \rho(G_{00})$.

Next we claim that $\lambda - N$ is injective for $\lambda \geq \lambda_0$. If by contradiction we assume that there exists $0 \neq x \in \ker(\lambda - N)$, a simple computation shows that

$$0 \neq \begin{pmatrix} -R(\lambda, G_0) L_0 N x \\ x \end{pmatrix} \in \ker(\lambda - \mathcal{A}_0)$$

contradicting the fact $\lambda \in \rho(\mathcal{A}_0)$. Let now $R(\lambda, \mathcal{A}_0) = (R_{ij}(\lambda))_{2\times 2}$ and choose some arbitrary $\binom{g}{y} \in \mathcal{X}_0$. Then we have

$$
\begin{aligned}
R(\lambda, \mathcal{A}_0)\begin{pmatrix} g \\ y \end{pmatrix} = \begin{pmatrix} R_{11}(\lambda) g + R_{12}(\lambda) y \\ R_{21}(\lambda) g + R_{22}(\lambda) y \end{pmatrix} = \begin{pmatrix} f \\ x \end{pmatrix} &\iff (\lambda - \mathcal{A}_0)\begin{pmatrix} f \\ x \end{pmatrix} = \begin{pmatrix} g \\ y \end{pmatrix} \\
&\iff \begin{cases} (\lambda - G_0) f + L_0 N x & = g, \\ (\lambda - N) x & = y, \\ L G_0 f & = N x. \end{cases}
\end{aligned} \tag{3.2}
$$

For $y = 0$ it follows $(\lambda - N)x = 0$ and hence $x = 0$. This implies $R_{21}(\lambda) = 0$. Moreover, by (3.2) the operator $\lambda - N$ must be surjective, hence it is invertible with inverse $(\lambda - N)^{-1} = R_{22}(\lambda) \in \mathcal{L}(\partial X)$. Again by (3.2) this implies $R_{11}(\lambda) = R(\lambda, G_{00})$. On the other hand, choosing $g = 0$ we obtain $R_{21}(\lambda) = -R(\lambda, G_0) L_0 N R(\lambda, N)$ as claimed. □

Step 4. G_{00} and N generate analytic semigroups of angle $\alpha > 0$ on X_0 and ∂X, respectively.

Proof. Denote by $\big(\mathcal{T}_0(t)\big)_{t\geq 0}$ the semigroup generated by $\mathcal{A}_0$. Then by [7, Thm. II.1.10] for $\lambda \in \mathbb{R}$ sufficiently large $R\big(\lambda, \mathcal{A}_0\big)$ is given by the Laplace transform $\big(\mathcal{L}\mathcal{T}_0(\cdot)\big)(\lambda)$ of $\big(\mathcal{T}_0(t)\big)_{t\geq 0}$. Since $\mathcal{L}$ is injective, (3.1) implies that the semigroup generated by $\mathcal{A}_0$ is given by

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & * \\ 0 & S(t) \end{pmatrix},$$

where $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are semigroups on X_0 and ∂X generated by G_{00} and N, respectively. Since by assumption $\big(\mathcal{T}_0(t)\big)_{t\geq 0}$ is analytic of angle $\alpha > 0$, also the semigroups generated by G_{00} and N are analytic of angle α. □

This completes the proof of Theorem 3.1. □

Since by [7, Thm. II.4.29] an analytic semigroup is compact if and only if its generator has compact resolvent, the following result relates compactness of the semigroups generated by A and D_{00}, N.

Corollary 3.2. *Let $D \in \{A, G\}$. Then A has compact resolvent if and only if D_0 and N have compact resolvents on X and ∂X, respectively.*

Proof. By Step 1 in the proof of Theorem 3.1, A has compact resolvent if and only if $\mathcal{A}$ has. Since $\mathcal{A}$ and $\mathcal{A}_0$ differ only by the relatively bounded perturbation $\mathcal{B} := \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right)$ of bound 0, by [7, III-(2.5)] one of the operators $\mathcal{A}, \mathcal{A}_0$ has compact resolvent if and only if the other has. However, by (3.1) for $\lambda \in \rho\big(\mathcal{A}_0\big)$

$$\begin{aligned} R\big(\lambda, \mathcal{A}_0\big) \text{ is compact} \quad &\iff \quad R\big(\lambda, G_{00}\big),\ R(\lambda, N) \text{ and } R\big(\lambda, G_0\big)L_0 N R(\lambda, N) = \\ &\qquad \lambda R\big(\lambda, G_0\big)L_0 R(\lambda, N) - R\big(\lambda, G_0\big)L_0 \text{ are compact} \\ &\iff \quad R\big(\lambda, G_{00}\big),\ R(\lambda, N) \text{ and } R\big(\lambda, G_0\big)L_0 \text{ are compact} \\ &\iff \quad R\big(\lambda, G_0\big) = R\big(\lambda, G_{00}\big)\cdot\big(\operatorname{Id} - L_0 L\big) + R\big(\lambda, G_0\big)L_0 \cdot L, \text{ and } R(\lambda, N) \text{ are compact.} \end{aligned}$$

This completes the proof. □

4 | PERTURBATIONS OF OPERATORS WITH GENERALIZED WENTZELL BOUNDARY CONDITIONS

In many applications the feedback operator $B : D(B) \subset X \to \partial X$ which determines the boundary condition in (2.1) splits into a sum

$$B = B_0 + CL, \quad D(B) = D\big(B_0\big) \cap D(CL) \tag{4.1}$$

for some $C : D(C) \subset \partial X \to \partial X$. For example in (1.1) we could choose $B_0 = \beta\frac{\partial}{\partial n}$ (which determines the feedback from the interior of Ω to the boundary $\partial\Omega$) and the multiplication operator $C = M_\gamma \in \mathcal{L}(\partial X)$ (which governs the "free" evolution on $\partial\Omega$). Next we study this situation in more detail where we allow C to be unbounded. For a concrete example see [11, (1.2), (3.3)] and Subsection 5.3. Moreover, we will introduce a relatively bounded perturbation P of the operator A_m.

To this end we first have to generalize our notation concerning the Dirichlet- and Dirichlet-to-Neumann operators. For a closed operator $D_m : D\big(D_m\big) \subset X \to X$ let $D_0 \subset D_m$ with domain $D\big(D_0\big) := D\big(D_m\big) \cap \ker(L)$ on X. Then by [13, Lem. 1.2] for $\lambda \in \rho\big(D_0\big)$ the restriction $L|_{\ker(\lambda - D_m)} : \ker\big(\lambda - D_m\big) \to \partial X$ is invertible with bounded inverse

$$L_\lambda^{D_m} := \big(L|_{\ker(\lambda - D_m)}\big)^{-1} : \partial X \to \ker\big(\lambda - D_m\big) \subseteq X,$$

which we call the abstract Dirichlet operator associated to λ and D_m. Note that $L_\lambda^{D_m} = L_0^{D_m - \lambda}$, that is $L_\lambda^{D_m} x = f$ gives the unique solution of the abstract Dirichlet problem

$$\begin{cases} D_m f = \lambda f, \\ Lf = x. \end{cases}$$

If $D_m = A_m$ we will simply write $L_\lambda := L_\lambda^{A_m}$.

Next, for a relatively D_0-bounded feedback operator $F : D(F) \subset X \to \partial X$ we introduce the associated generalized abstract Dirichlet-to-Neumann operator $N_\lambda^{D_m,F} : D\left(N_\lambda^{D_m,F}\right) \subset \partial X \to \partial X$ defined by

$$N_\lambda^{D_m,F} x := F L_\lambda^{D_m} x, \quad D\left(N_\lambda^{D_m,F}\right) := \left\{x \in \partial X : L_\lambda^{D_m} x \in D(F)\right\}.$$

If $\lambda = 0$ we simply write $N^{D_m,F} := N_0^{D_m,F}$. If in addition $F = B$ we put $N^{D_m} := N_0^{D_m,B}$ and $N^F := N_0^{A_m,F}$ in case $D_m = A_m$. Finally, as before we set $N := N_0^{A_m,B}$.

To proceed we need the following domain inclusions where $B_0 : D(B) \subset X \to \partial X$ is relatively A_0-bounded and $C : D(C) \subset \partial X \to \partial X$.

Lemma 4.1. *The following assertions hold true.*

(i) *If C is relatively N^{B_0}-bounded, then $D(B_0) \subseteq D(CL)$.*

(ii) *If N^{B_0} is relatively C-bounded, then $D(A_m) \cap D(CL) \subseteq D(B_0)$.*

Proof. (i). Recall that $L_0 : \partial X \to \ker(A_m)$ is bijective with inverse L. Hence, using the first decomposition in (2.2) we conclude

$$LD(B_0) = L\left(\left(X_0 \oplus \ker(A_m)\right) \cap D(B_0)\right) = L\left(\ker(A_m) \cap D(B_0)\right) = L_0^{-1}\left(\ker(A_m) \cap D(B_0)\right) \subseteq D\left(N^{B_0}\right) \subseteq D(C).$$

This implies the claim.

(ii). By assumption, we have

$$L\, D(CL) \subseteq D(C) \subseteq D\left(N^{B_0}\right).$$

This implies

$$L_0 L\, D(CL) \subseteq L_0 D\left(N^{B_0}\right) \subseteq D(B_0).$$

On the other hand, $\left(\mathrm{Id} - L_0 L\right) D(A_m) = D(A_0) \subseteq D(B_0)$. Summing up this gives the desired inclusion. □

Note that in part (ii) of the previous result we cannot expect the inclusion $D(CL) \subset D(B_0)$ since always $X_0 = \ker(L) \subset D(CL)$ holds.

We now return to the decomposition $B = B_0 + CL$ from (4.1). Let $P : D(P) \subset X \to X$ be a relatively A_m-bounded perturbation with A_0-bound 0. That is, $D(A_m) \subseteq D(P)$, there exist $a, b \geq 0$ and for every $\varepsilon > 0$ a constant $M_\varepsilon \geq 0$ such that

$$\begin{aligned} \|Pf\| &\leq a \cdot \|A_m f\| + b \cdot \|f\| && \text{for all } f \in D(A_m), \\ \|Pf\| &\leq \varepsilon \cdot \|A_0 f\| + M_\varepsilon \cdot \|f\| && \text{for all } f \in D(A_0) = D(A_m) \cap X_0. \end{aligned}$$

Then we consider the operator $(A+P)^B : D\left((A+P)^B\right) \subseteq X \to X$ given by

$$\begin{aligned} (A+P)^B &\subseteq A_m + P, \\ D\left((A+P)^B\right) &:= \left\{f \in D(A_m) \cap D(B_0) \cap D(CL) : LA_m f + Pf = B_0 f + CLf\right\}. \end{aligned} \tag{4.2}$$

First we assume that C is relatively $N^{B_0} = B_0 L_0^{A_m}$-bounded of bound 0. Note that by the previous lemma part (i) this implies that $D(B) = D(B_0) \cap D(CL) = D(B_0)$.

Theorem 4.2. *Let $P : D(P) \subset X \to X$ be relatively A_m-bounded with A_0-bound 0 and let $C : D(C) \subset \partial X \to \partial X$ be relatively N^{B_0}-bounded of bound 0. Then for B given by* (4.1) *the following statements are equivalent.*

(a) *$(A+P)^B$ in* (4.2) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^{B_0} generates an analytic semigroup of angle $\alpha > 0$ on X.*

(c) *A_0 is sectorial of angle $\alpha > 0$ on X and N^{B_0} generates an analytic semigroup of angle $\alpha > 0$ on ∂X.*

Before giving the proof we state an analogous result where we interchange the roles of N^{B_0} and C. That is, we assume that N^{B_0} is relatively C-bounded of bound 0. Note that by Lemma 4.1.(ii) this implies that $D(A_m) \cap D(B) = D(A_m) \cap D(B_0) \cap D(CL) = D(A_m) \cap D(CL)$.

Theorem 4.3. *Let $P : D(P) \subset X \to X$ be relatively A_m-bounded with A_0-bound 0 and let N^{B_0} be relatively C-bounded of bound 0 for some $C : D(C) \subset \partial X \to \partial X$. Then for B given by* (4.1) *the following statements are equivalent.*

(a) *$(A + P)^B$ in* (4.2) *generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^{CL} generates an analytic semigroup of angle $\alpha > 0$ on X.*

(c) *A_0 is sectorial of angle $\alpha > 0$ on X and C generates an analytic semigroup of angle $\alpha > 0$ on ∂X.*

To prove the previous two theorems we use a series of auxiliary results. First we show the equivalences of (a) and (b) in case $P = 0$.

Lemma 4.4. *Let $C : D(C) \subset \partial X \to \partial X$ be relatively N^{B_0}-bounded of bound 0. Then the following statements are equivalent.*

(a) *A^{B_0} generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^B generates an analytic semigroup of angle $\alpha > 0$ on X.*

Proof. By Lemma 4.1.(i) the operator

$$B := B_0 + CL, \quad D(B) = D(B_0)$$

is well-defined. Since $D(A_0) \subset X_0$, the operators B and B_0 coincide on $D(A_0)$. Hence, B is relatively A_0-bounded if and only if B_0 is relatively A_0-bounded of bound 0. Moreover, we have

$$N^B = BL_0 = N^{B_0} + C, \quad D(N^B) = D(N^{B_0}).$$

By [7, Thm. III.2.10] it then follows that N^B generates an analytic semigroup of angle $\alpha > 0$ on ∂X if and only if N^{B_0} does. The claim now follows by Theorem 3.1. ☐

Lemma 4.5. *Let N^{B_0} be relatively C-bounded of bound 0 for some $C : D(C) \subset \partial X \to \partial X$. Then the following statements are equivalent.*

(a) *A^{CL} generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *A^B generates an analytic semigroup of angle $\alpha > 0$ on X.*

Proof. Let

$$B := B_0 + CL, \quad D(B) = D(B_0) \cap D(CL).$$

By the same reasoning as in the previous proof we conclude that B is relatively A_0-bounded of bound 0 if and only if B_0 is relatively A_0-bounded of bound 0. Moreover, by Lemma 4.1.(ii) we have

$$\begin{aligned} x \in D(N^B) &\iff L_0 x \in D(B) \\ &\iff L_0 x \in D(B_0) \cap D(CL) \cap D(A_m) \\ &\iff L_0 x \in D(CL) \cap D(A_m) \\ &\iff L_0 x \in D(CL) \\ &\iff x \in L\, D(CL) \subseteq D(C). \end{aligned}$$

This implies

$$N^B = BL_0 = N^{B_0} + C, \quad D(N^B) = D(C).$$

By [7, Thm. III.2.10] it follows that N^B generates an analytic semigroup of angle $\alpha > 0$ on ∂X if and only if C does. The claim then follows by Theorem 3.1. $\square$

Next we study how Dirichlet operators behave under perturbations.

Lemma 4.6. *Let $P : D(P) \subset X \to X$ be a relatively A_m-bounded perturbation. Then for $\lambda \in \rho(A_0) \cap \rho(A_0 + P)$ the Dirichlet operator $L_\lambda^{A_m+P} \in \mathcal{L}(\partial X, X)$ exists and satisfies*

$$L_\lambda^{A_m+P} - L_\lambda^{A_m} = R(\lambda, A_0 + P)PL_\lambda^{A_m} = R(\lambda, A_0)PL_\lambda^{A_m+P}. \tag{4.3}$$

Proof. Let $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ for the graph norm $\|\cdot\|_{A_m} := \|\cdot\|_X + \|A_m\cdot\|_X$. Then $P : [D(A_m)] \to X$ and $L_\lambda^{A_m} : \partial X \to [D(A_m)]$ are bounded, hence $PL_\lambda^{A_m} : \partial X \to X$ is bounded as well. This implies that

$$T := L_\lambda^{A_m} + R(\lambda, A_0 + P)PL_\lambda^{A_m} \in \mathcal{L}(\partial X, X).$$

Since

$$(A_m + P - \lambda)Tx = (A_m + P - \lambda)L_\lambda^{A_m}x + (A_m + P - \lambda)R(\lambda, A_0 + P)PL_\lambda^{A_m} = PL_\lambda^{A_m}x - PL_\lambda^{A_m}x = 0,$$

we have $\mathrm{rg}(T) \subseteq \ker(\lambda - A_m - P)$. Moreover, from

$$\mathrm{rg}\Big(R(\lambda, A_0 + P)PL_\lambda^{A_m}\Big) \subset D(A_0 + P) = D(A_0) \subset \ker(L)$$

it follows that $LTx = LL_\lambda^{A_m}x = x$. Hence, $L|_{\ker(\lambda - A_m - P)}$ is surjective with right-inverse T. Since $\ker(\lambda - A_m - P) \cap X_0 \subset \ker(\lambda - A_0 - P) = \{0\}$ we conclude that $L|_{\ker(\lambda - A_m - P)}$ is injective as well. This implies that it is invertible with inverse $L_\lambda^{A_m+P} = T$ and proves the first identity in (4.3). The second one follows by changing the roles of A_m and $A_m + P$. $\square$

Next we consider perturbations of Dirichlet-to-Neumann operators.

Proposition 4.7. *Let $P : D(P) \subset X \to X$ be a relatively A_m-bounded perturbation. Then for $\lambda \in \rho(A_0) \cap \rho(A_0 + P)$ the perturbed Dirichlet-to-Neumann operator $N_\lambda^{A_m+P}$ exists, $D(N_\lambda^{A_m}) = D(N_\lambda^{A_m+P})$ and the difference $N_\lambda^{A_m} - N_\lambda^{A_m+P}$ is bounded.*

Proof. Since

$$\mathrm{rg}\Big(R(\lambda, A_0)PL_\lambda^{A_m+P}\Big) \subset D(A_0) \subset D(B),$$

by Lemma 4.6 it follows that $D(N_\lambda^{A_m}) = D(N_\lambda^{A_m+P})$. Moreover, from (4.3) we conclude

$$N_\lambda^{A_m} - N_\lambda^{A_m+P} = BL_\lambda^{A_m} - BL_\lambda^{A_m+P} \supseteq -BR(\lambda, A_0)PL_\lambda^{A_m+P} \in \mathcal{L}(\partial X). \qquad \square$$

To conclude the proofs of Theorem 4.2 and Theorem 4.3, we need one further result. It shows that the assertion (a) in both results is stable under the perturbation P.

Lemma 4.8. *Let $P : D(P) \subset X \to X$ relatively A_m-bounded with A_0-bound 0. Then the following statements are equivalent.*

(a) *A^B generates an analytic semigroup of angle $\alpha > 0$ on X.*

(b) *$(A + P)^B$ generates an analytic semigroup of angle $\alpha > 0$ on X.*

Proof. Since A_0 is a weak Hille–Yosida operator and P is relatively A_0-bounded of bound 0, by [7, Lem. III.2.6] there exists a $\lambda \in \rho(A_0) \cap \rho(A_0 + P)$ and $A_0 - \lambda$, $A_0 + P - \lambda$ are again weak Hille–Yosida operators. Since B is relatively A_0-bounded of bound 0 a simple computation shows that it is also relatively $(A_0 - \lambda)$- and $(A_0 + P - \lambda)$-bounded of bound 0. Moreover, by Lemma 4.6 the operators $L_0^{A_m-\lambda}$ and $L_0^{A_m+P-\lambda}$ exist and are bounded. Hence, $A_0 - \lambda$ and $A_0 + P - \lambda$ both satisfy Assumptions 2.2.

Next we check the conditions in Theorem 3.1. By [7, Lem. III.2.6] the operator $A_0 - \lambda$ is sectorial of angle $\alpha > 0$ on X if and only if $A_0 + P - \lambda$ is. Moreover, by Proposition 4.7, $N^{A_m-\lambda}$ generates an analytic semigroup of angle $\alpha > 0$ if and only if $N^{A_m+P-\lambda}$ does. Applying Theorem 3.1 to $A_0 - \lambda$, $N^{A_m-\lambda}$ and $A_0 + P - \lambda$, $N^{A_m+P-\lambda}$, respectively, the claim follows. $\square$

Proof of Theorem 4.2 and Theorem 4.3. By Lemma 4.8 assertion (a) is independent of P while by Lemma 4.4 and Lemma 4.5, respectively, for $P = 0$ it is equivalent to (b). Since the equivalence of (b) and (c) follows Theorem 3.1 the proof is complete. $\square$

5 | EXAMPLES

5.1 | Second order differential operators on $\mathrm{C}([0,1],\mathbb{C}^n)$

For $n \in \mathbb{N}$ consider functions $a_i \in \mathrm{C}[0,1] \cap \mathrm{C}^1(0,1)$, $1 \le i \le n$, being strictly positive on (0,1) such that $\frac{1}{a_i} \in L^1[0,1]$. Let $a := \operatorname{diag}(a_1, \dots, a_n)$ and $b, c \in \mathrm{C}([0,1], \mathrm{M}_n(\mathbb{C}))$. Moreover, define the maximal operator $A_m : D(A_m) \subset \mathrm{C}([0,1],\mathbb{C}^n) \to \mathrm{C}([0,1],\mathbb{C}^n)$ by

$$A_m := af'' + bf' + cf, \quad D(A_m) := \left\{ f \in \mathrm{C}([0,1],\mathbb{C}^n) \cap \mathrm{C}^2((0,1),\mathbb{C}^n) : A_m f \in \mathrm{C}([0,1],\mathbb{C}^n) \right\}$$

and take $B \in \mathcal{L}\left(C^1([0,1],\mathbb{C}^n), \mathbb{C}^{2n}\right)$.

Corollary 5.1. *We have $D(A_m) \subset \mathrm{C}^1([0,1],\mathbb{C}^n) = D(B)$ and*

$$A \subseteq A_m, \quad D(A) = \left\{ f \in D(A_m) : \begin{pmatrix} (A_m f)(0) \\ (A_m f)(1) \end{pmatrix} = Bf \right\}$$

generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}([0,1],\mathbb{C}^n)$.

Proof. We consider $X := \mathrm{C}([0,1],\mathbb{C}^n) = \mathrm{C}[0,1] \times \cdots \times \mathrm{C}[0,1]$ equipped with the norm $\|f\|_{1,\infty} := \|f_1\|_\infty + \cdots + \|f_n\|_\infty$, $\partial X := \mathbb{C}^{2n}$ and define $L \in \mathcal{L}(X, \partial X)$ by $Lf := \binom{f(0)}{f(1)}$. Then as in [4, Cor. 4.1, Step (iii)] it follows that $D(A_m) \subset D(B)$, hence A coincides with the operator A^B defined in (2.1). Since

$$Pf := bf' + cf, \quad D(P) := \mathrm{C}^1([0,1],\mathbb{C}^n)$$

is a relatively A_m-bounded with A_0-bound 0 (see Step 4 below), we assume by Theorem 4.2 without loss of generality that $b = c = 0$.

Next we verify Assumptions 2.2 and the hypotheses of Theorem 3.1.

Step 1. The abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists.

Proof. We have $\ker(A_m) = \left\{ \varepsilon_0 \cdot x_0 + \varepsilon_1 \cdot x_1 : x_0, x_1 \in \mathbb{C}^n \right\}$ for

$$\varepsilon_0(s) := 1 - s \quad \text{and} \quad \varepsilon_1(s) := s, \quad s \in [0,1].$$

A simple calculation then shows that $L_0 := \left(L|_{\ker(A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X)$ is given by

$$L_0 \begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix} = \varepsilon_0 \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \varepsilon_1 \cdot \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \end{pmatrix}. \qquad \square$$

Step 2. The operator A_0 on X is sectorial of angle $\frac{\pi}{2}$ and has compact resolvent.

Proof. Let $A_i := a_i \cdot \frac{d^2}{ds^2}$ with domain $D(A_i) := \left\{ g \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : a_i \cdot g'' \in \mathrm{C}[0,1] \right\}$ for $1 \le i \le n$. Then

$$R(\lambda, A_0) = \operatorname{diag}\left(R(\lambda, A_1), \dots, R(\lambda, A_n)\right).$$

Since by [4, Cor. 4.1, Step (ii)] all A_i are sectorial of angle $\frac{\pi}{2}$ and have compact resolvents on $\mathrm{C}[0,1]$, the claim follows. $\square$

Step 3. The maximal operator A_m is densely defined and closed.

Proof. Since $\mathrm{C}^2([0,1],\mathbb{C}^n) \subset D(A_m)$, A_m is densely defined. By Step 1, Step 2 and [4, Lem. 3.2] it follows that A_m is closed. $\square$

Step 4. The feedback operator B is relatively A_0-bounded of bound 0.

Proof. Since $D(B) = \mathrm{C}^1([0,1],\mathbb{C}^n)$ it suffices to show that the first derivative with domain $\mathrm{C}^1([0,1],\mathbb{C}^n)$ is relatively A_0-bounded with bound 0. Let $f \in D(A_0)$. Then by [4, Cor. 4.1, Step (iii)] it follows that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned}\|f'\|_{1,\infty} &\leq \varepsilon \cdot \|A_1 f_1\|_\infty + \cdots + \varepsilon \cdot \|A_n f_n\|_\infty + C_\varepsilon \cdot \|f_1\|_\infty + \cdots + C_\varepsilon \cdot \|f_n\|_\infty \\ &= \varepsilon \cdot \|A_0 f\|_{1,\infty} + C_\varepsilon \cdot \|f\|_{1,\infty}.\end{aligned}$$ □

Step 5. The Dirichlet-to-Neumann operator N generates an analytic, compact semigroup of angle $\frac{\pi}{2}$ on ∂X.

Proof. Since the boundary space ∂X is finite dimensional, N is bounded. Hence N generates an analytic, compact semigroup of angle $\frac{\pi}{2}$ on ∂X. □

Now by Step 1–Step 5 all hypotheses of Theorem 3.1 and Corollary 3.2 are satisfied which imply the claim. This completes the proof of Corollary 5.1. □

Remark 5.2. Corollary 5.1 generalizes [4, Cor. 4.1] to arbitrary $n \in \mathbb{N}$.

We give a particular choice for the operator B.

Corollary 5.3. *For $M_i, N_i \in \mathrm{M}_{2n\times n}(\mathbb{C})$, $i = 0, 1$, the operator*

$$A \subseteq A_m,\ D(A) = \left\{ f \in D(A_m) : \begin{pmatrix}(A_m f)(0)\\(A_m f)(1)\end{pmatrix} = M_0 f'(0) + M_1 f'(1) + N_0 f(0) + N_1 f(1) \right\}$$

generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}([0,1],\mathbb{C}^n)$.

We remark that second order differential operators on spaces of functions $f : [0,1] \to \mathbb{C}^n$ can be used to describe diffusion- and waves on networks. For some recent results in the L^p-context for operators with generalized Robin-type boundary conditions we refer to [8].

5.2 | Banach space-valued second derivative

We associate to an arbitrary Banach space Y the Banach space $X := \mathrm{C}([0,1],Y)$ of all continuous functions on [0,1] with values in Y equipped with the sup-norm. Moreover, we take $P \in \mathcal{L}(\mathrm{C}^1([0,1],Y),X))$, $\Phi \in \mathcal{L}(X,Y^2)$ and an operator $(C, D(C))$ on Y^2. Then the following holds.

Corollary 5.4. *The operator C generates an analytic semigroups of angle $\alpha \in \left(0,\frac{\pi}{2}\right]$ on Y^2 if and only if the operator*

$$\begin{aligned} Af &:= f'' + Pf,\\ D(A) &:= \left\{ f \in \mathrm{C}^2([0,1],Y) : \begin{pmatrix}f(0)\\f(1)\end{pmatrix} \in D(C), \begin{pmatrix}f''(0)+Pf(0)\\f''(1)+Pf(1)\end{pmatrix} = \Phi f + C\begin{pmatrix}f(0)\\f(1)\end{pmatrix} \right\}\end{aligned}$$

generates an analytic semigroup of angle $\alpha \in \left(0,\frac{\pi}{2}\right]$ on X.

Proof. We consider $\partial X := Y^2$ and define $L \in \mathcal{L}(X,\partial X)$ by $Lf := \begin{pmatrix}f(0)\\f(1)\end{pmatrix}$. Moreover, define

$$A_m : D(A_m) \subseteq X \to X, \quad A_m f := f'' + Pf, \quad D(A_m) = \mathrm{C}^2([0,1],Y)$$

and

$$B : D(B) \subseteq X \to \partial X, \quad Bf := \Phi f + CLf, \quad D(B) := \left\{ f \in X : \begin{pmatrix}f(0)\\f(1)\end{pmatrix} \in D(C) \right\}.$$

Then A coincides with the operator A^B given by (2.1). Since P is a relatively A_m-bounded of A_m-bound 0 and $\Phi \in \mathcal{L}(X,\partial X)$, by Theorem 4.3 it suffices to verify the Assumptions 2.2 and that A_0 is sectorial of angle $\alpha > 0$.

Step 1. The abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists.

Proof. As in Step 1 of the proof of Corollary 5.1 we have $\ker\big(A_m\big) = \{\varepsilon_0 y_0 + \varepsilon_1 y_1 : y_0, y_1 \in Y\}$ for

$$\varepsilon_0(s) := 1 - s \quad \text{and} \quad \varepsilon_1(s) := s, \quad s \in [0,1].$$

Moreover, $L_0 := \big(L|_{\ker(A_m)}\big)^{-1} \in \mathcal{L}(\partial X, X)$ is given by

$$L_0 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \varepsilon_0 \cdot y_0 + \varepsilon_1 \cdot y_1. \qquad \square$$

Step 2. The operator A_0 on X is sectorial of angle $\frac{\pi}{2}$.

Proof. This follows as in the proof of [7, Thm. VI.4.1]. $\square$

Step 3. The maximal operator A_m is densely defined and closed.

Proof. Since $\mathrm{C}^2([0,1],Y) \subset D\big(A_m\big)$, A_m is densely defined. By Step 1, Step 2 and [4, Lem. 3.2] it follows that A_m is closed. $\square$

Step 4. The feedback operator B is relatively A_0-bounded of bound 0.

Proof. For $f \in D\big(A_0\big) \subset X_0$ we have $Bf = \Phi f$. Since Φ is bounded, this implies the claim. $\square$

Now by Step 1–Step 4 all hypotheses of Theorem 3.1 are satisfied. This implies the claim and completes the proof of Corollary 5.4. $\square$

5.3 | Perturbations of the Laplacian on $\mathrm{C}\big(\overline{\Omega}\big)$ with generalized Wentzell boundary conditions

In this subsection we complement the example from the introduction concerning the Laplacian on $\mathrm{C}\big(\overline{\Omega}\big)$ with generalized Wentzell boundary conditions, see also [3].

To this end we consider a bounded domain $\Omega \subset \mathbb{R}^n$ with C^∞-boundary $\partial\Omega$ and take an operator $P \in \mathcal{L}\big(\mathrm{C}^1\big(\overline{\Omega}\big), \mathrm{C}\big(\overline{\Omega}\big)\big)$ (e.g. a first-order differential operator). Then we define the perturbed Laplacian $A : D(A) \subset \mathrm{C}\big(\overline{\Omega}\big) \to \mathrm{C}\big(\overline{\Omega}\big)$ with generalized Wentzell boundary conditions by $Af := \Delta_m f + Pf$ for

$$f \in D(A) \quad :\iff \quad \begin{cases} f \in D\big(\Delta_m\big) \cap D\big(\frac{\partial}{\partial n}\big),\ f|_{\partial\Omega} \in D(\Delta_\Gamma) \quad \text{and} \\ \big(\Delta_m f + Pf\big)\big|_{\partial\Omega} = \beta \cdot \frac{\partial}{\partial n} f + \gamma \cdot f\Big|_{\partial\Omega} + q \cdot \Delta_\Gamma f|_{\partial\Omega}, \end{cases} \tag{5.1}$$

cf. also [11, (1.2), (3.3)]. Here $\beta < 0$, $\gamma \in \mathrm{C}(\partial\Omega)$, $q \geq 0$ and $\Delta_\Gamma : D(\Delta_\Gamma) \subset \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ denotes the Laplace–Beltrami operator. In case $P = 0$, $q = 0$ this just gives the operator A from the introduction. As we will see below for $q > 0$ the Laplace–Beltrami operator will dominate the dynamic on the boundary ∂X. However, in this case essentially the same generation result holds as for $q = 0$.

Corollary 5.5. *For all $q > 0$ the operator $A \subseteq \Delta_m + P$ with domain given in* (5.1) *generates a compact and analytic semigroup of angle $\frac{\pi}{2}$.*

Proof. Without loss of generality we assume that $\beta = -1$. To fit the operator A into our setting we define $X := \mathrm{C}\big(\overline{\Omega}\big)$, $\partial X := \mathrm{C}(\partial\Omega)$ and the trace $L \in \mathcal{L}(X, \partial X)$, $Lf := f|_{\partial\Omega}$. Then we consider $A_m := \Delta_m : D\big(\Delta_m\big) \subset X \to X$ and $B_0 := -\frac{\partial}{\partial n} : D\big(\frac{\partial}{\partial n}\big) \subset X \to \partial X$ as in [3] and put $C := q \cdot \Delta_\Gamma + M_\gamma : D(\Delta_\Gamma) \subset \partial X \to \partial X$ and $B := B_0 + CL$ as in (4.1). Then A coincides with the operator $(A+P)^B$ defined in (4.2).

By [1, Thm. 6.1.3], $A_0 = \Delta_0$ is sectorial of angle $\frac{\pi}{2}$ and by [3, (1.9)] and [7, Prop. II.4.25] has compact resolvent. Moreover, C generates a compact analytic semigroup of angle $\frac{\pi}{2}$. Let $W := (-\Delta_\Gamma)^{\frac{1}{2}}$. Then by the proof of [3, Thm. 2.1] there exists a relatively W-bounded perturbation $Q : D(Q) \subset \partial X \to \partial X$ such that $N^{B_0} = B_0 L_0^{A_m} = -W + Q$. This implies that N^{B_0} is relatively W-bounded and by [16, Thm. 6.10] it follows that N^{B_0} is relatively C-bounded of bound 0. Hence, by Theorem 4.3, $A = (A+P)^B$ generates an analytic semigroup of angle $\frac{\pi}{2}$. Compactness of this semigroup follows by Corollary 3.2. $\square$

We remark that Corollary 5.5 confirms the conjecture $\theta_\infty = \frac{\pi}{2}$ in [11, Sec. 5] for $a(x) \equiv \mathrm{Id}$ and constant $\beta < 0$.

5.4 | Uniformly elliptic operators on $\mathrm{C}(\overline{\Omega})$

We consider a uniformly elliptic second-order differential operator with generalized Wentzell boundary conditions on $\mathrm{C}(\overline{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with C^∞-boundary $\partial\Omega$. To this end, for $1 \leq j, k \leq n$ we first take real-valued functions

$$a_{jk} = a_{kj} \in \mathrm{C}^\infty(\overline{\Omega}), \quad a_j, a_0 \in \mathrm{C}(\overline{\Omega}), \quad b_0 \in \mathrm{C}(\partial\Omega)$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(x) \cdot \xi_j \xi_k \geq c \cdot \|\xi\|^2 \quad \text{for all } x \in \overline{\Omega},\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

and some fixed $c > 0$. Then we define the maximal operator $A_m : D(A_m) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ in divergence form by

$$A_m f := \sum_{j=1}^{n} \partial_j \left(\sum_{k=1}^{n} a_{jk} \partial_k f \right) + \sum_{k=1}^{n} a_k \partial_k f + a_0 f, \quad D(A_m) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\overline{\Omega}) : A_m f \in \mathrm{C}(\overline{\Omega}) \right\},$$

and the feedback operator $B : D(B) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$B := -\sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k + b_0 L, \quad D(B) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\overline{\Omega}) : Bf \in \mathrm{C}(\partial\Omega) \right\},$$

where $L \in \mathcal{L}(\mathrm{C}(\overline{\Omega}), \mathrm{C}(\partial\Omega))$, $Lf := f|_{\partial\Omega}$ denotes the trace operator.

Corollary 5.6. *The operator* $A : D(A) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\overline{\Omega})$ *given by*

$$A \subseteq A_m, \quad D(A) := \{ f \in D(A_m) \cap D(B) : LA_m f = Bf \}$$

generates a compact and analytic semigroup on $\mathrm{C}(\overline{\Omega})$.

Proof. Let $X := \mathrm{C}(\overline{\Omega})$ and $\partial X := \mathrm{C}(\partial\Omega)$. Define the maximal operator $\tilde{A}_m : D(\tilde{A}_m) \subseteq X \to X$ by

$$\tilde{A}_m := \sum_{j=1}^{n} \partial_j \left(\sum_{k=1}^{n} a_{jk} \partial_k \right), \quad D(\tilde{A}_m) := D(A_m)$$

and the feedback operator $\tilde{B}_0 : D(\tilde{B}_0) \subseteq \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$\tilde{B}_0 := -\sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k, \qquad D(\tilde{B}_0) := D(B).$$

Then by [4, Cor. 4.5] it follows that the operator $\tilde{A}^{\tilde{B}_0} : D(\tilde{A}^{\tilde{B}_0}) \subseteq X \to X$ with generalized Wentzell boundary conditions given by

$$\tilde{A}^{\tilde{B}_0} \subseteq \tilde{A}_m, \quad D\left(\tilde{A}^{\tilde{B}_0}\right) := \{ f \in D(\tilde{A}_m) \cap D(\tilde{B}_0) : L\tilde{A}_m f = \tilde{B}_0 f \}$$

generates a compact and analytic semigroup on X. Let $Pf := \sum_{j=1}^{n} a_j \partial_j f + a_0 f$ and $Cx := b_0 x$. Then P is relatively A_m-bounded with bound 0 and $C \in \mathcal{L}(\partial X)$. Since $A = (\tilde{A} + P)^{\tilde{B}_0 + CL}$ the claim follows from Theorem 4.2. □

Remark 5.7. This result generalizes [4, Cor. 4.5] and via Theorem 3.1 also the main theorem in [9]. Moreover, it shows that the angle of the analytic semigroup generated by A only depends on the matrix $(a_{jk})_{n\times n}$.

6 | CONCLUSION

Our abstract approach allows to decompose an operator A^B with generalized Wentzell boundary conditions (2.1) into an operator A_0 with (much simpler) abstract Dirichlet boundary conditions (2.3) and the associated abstract Dirichlet-to-Neumann operator N, cf. (2.4). In particular, we prove under the weak resolvent Assumptions 2.2.(i) on A_0 that

$$\left.\begin{array}{l} A^B \text{ generates an analytic semigroup} \\ \text{of angle } \alpha > 0 \end{array}\right\} \iff \begin{cases} A_0 \text{ is sectorial of angle } \alpha > 0, \text{ and} \\ N \text{ generates an analytic semigroup} \\ \text{of angle } \alpha > 0, \end{cases}$$

cf. Theorem 3.1. This equivalence is new and shows the sharpness of our approach. Moreover, while being very general, our theory applied to concrete examples (where typically A_0 is well-understood and sectorial of maximal angle $\frac{\pi}{2}$) gives new or improves known generation results, see Section 5.

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A.1.2 Strictly elliptic operators with Dirichlet boundary conditions on spaces of continuous functions on manifolds

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Strictly elliptic operators with Dirichlet boundary conditions on spaces of continuous functions on manifolds

Tim Binz

Abstract. We study strictly elliptic differential operators with Dirichlet boundary conditions on the space $\mathrm{C}(\overline{M})$ of continuous functions on a compact Riemannian manifold $\overline{M}$ with boundary and prove sectoriality with optimal angle $\frac{\pi}{2}$.

1. Introduction

Our starting point is a smooth compact Riemannian manifold $\overline{M}$ of dimension n with smooth boundary ∂M and Riemannian metric g and the initial value-boundary problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) &= \sqrt{|a|}\mathrm{div}_g\left(\frac{1}{\sqrt{|a|}}a\nabla_M^g u(t)\right) + \langle b, \nabla_M^g u(t)\rangle + cu(t) \quad \text{for } t > 0, \\ u(t)|_{\partial M} &= 0 \quad \text{for } t > 0, \\ u(0) &= u_0. \end{cases} \tag{IBP}$$

Here, a is a smooth $(1, 1)$-tensorfield, $b \in C(\overline{M}, \mathbb{R}^n)$ and $c \in C(\overline{M}, \mathbb{R})$. We are interested in existence, uniqueness and qualitative behaviour of the solution of this initial value-boundary problem. To study these properties systematically, the theory of operator semigroups (cf. [4,11,13,18]) can be used. We choose the Banach space $\mathrm{C}(\overline{M})$ and define the *differential operator with Dirichlet boundary condition*

$$A_0 f := \sqrt{|a|}\mathrm{div}_g\left(\frac{1}{\sqrt{|a|}}a\nabla_M^g f\right) + \langle b, \nabla_M^g u(t)\rangle + cf$$

with domain

$$D(A_0) := \left\{ f \in \bigcap_{p\geq 1} W^{2,p}(M) \cap \mathrm{C}_0(M) \colon A_0 f \in \mathrm{C}(\overline{M}) \right\}.$$

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Then, the initial value-boundary problem (IBP) is equivalent to the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = A_0 u(t) & \text{for } t > 0, \\ u(0) = u_0 \end{cases} \tag{ACP}$$

in $\mathrm{C}(\overline{M})$. In this paper, we show that the solution u of the above problems can be extended analytically in the time variable t to the open complex right half-plane. In operator theoretic terms this corresponds to the fact that A_0 is sectorial of angle $\frac{\pi}{2}$. Here is our main theorem.

Theorem 1.1. *The operator A_0 is sectorial of angle $\frac{\pi}{2}$ and has compact resolvent on* $\mathrm{C}(\overline{M})$.

For domains $\Omega \subset \mathbb{R}^n$, the generation of analytic semigroups by elliptic operators with Dirichlet boundary conditions on different spaces is well known. It was first shown by Browder in [8] for $L^2(\Omega)$, by Agmon in [3] for $L^p(\Omega)$ (see also [18, Chap. 3.1.1]) and by Stewart in [22] for $\mathrm{C}(\overline{\Omega})$ (see also [18, Chap. 3.1.5]). By Stewart's method, one even gets the angle of analyticity. Later Arendt proved in [5] (see also [1, Chap. III. 6]), using the Poisson operator, that the angle of the analytic semigroup generated by the Laplacian on the space $C(\overline{\Omega})$ is $\frac{\pi}{2}$. However, this method does not work on manifolds with boundary.

The angle $\frac{\pi}{2}$ of analyticity of A_0 plays an important role in the generation of analytic semigroups by elliptic differential operators with Wentzell boundary conditions on spaces of continuous functions. Many authors are interested in this topic, and we refer, e.g. to [9,10,12,14,15]. In this context, one starts from the "maximal" operator $A_m : D(A_m) \subseteq \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ in divergence form, given by

$$A_m f := \sqrt{|a|}\mathrm{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf$$

with domain

$$D(A_m) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) \colon A_m f \in \mathrm{C}(\overline{M}) \right\}.$$

Moreover, using the outer co-normal derivative $\frac{\partial^a}{\partial n} : D(\frac{\partial^a}{\partial n}) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$, a constant $\beta < 0$ and $\gamma \in \mathrm{C}(\partial M)$, one defines the differential operator A with *generalized Wentzell boundary conditions* by requiring

$$f \in D(A) \quad :\Longleftrightarrow \quad f \in D(A_m) \text{ and } A_m f\big|_{\partial M} = \beta \cdot \tfrac{\partial^a}{\partial n} f + \gamma \cdot f\big|_{\partial M}. \tag{1.1}$$

The main theorem in [6] shows that this operator A can be splitted into the operator A_0 with Dirichlet boundary conditions on $\mathrm{C}(\overline{M})$ and the *Dirichlet-to-Neumann operator* $N := \beta \cdot \frac{\partial^a}{\partial n} L_0$ on $\mathrm{C}(\partial M)$, where $L_0 \varphi = f$ denotes the unique solution of

$$\begin{cases} A_m f = 0, \\ f|_{\partial\Omega} = \varphi. \end{cases}$$

Using Theorem 1.1 and [6, Thm. 3.1 & Cor. 3.2], one obtains the following result.

Corollary 1.2. *The operator A with Wentzell boundary conditions generates a compact and analytic semigroup of angle $\theta > 0$ on $\mathrm{C}(\overline{M})$ if and only if the Dirichlet-to-Neumann operator N does so on $\mathrm{C}(\partial M)$.*

In an upcoming paper [7], we prove the latter statement with the optimal angle $\frac{\pi}{2}$ and conclude that elliptic differential operators with Wentzell boundary conditions generate compact and analytic semigroups of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$.

This paper is organized as follows.

In Sect. 2, we study the special case where A_0 is the Laplace–Beltrami operator with Dirichlet boundary conditions. We approximate its resolvents by modifying the Green's functions of the Laplace operator on $\mathbb{R}^n$, study the scaling of the error of the Laplace–Beltrami operator and prove estimates for the associated Green's functions. Finally, one obtains the sectoriality of angle $\frac{\pi}{2}$ for the Laplace–Beltrami operator with Dirichlet boundary conditions on $\mathrm{C}(\overline{M})$.

In Sect. 3, the main result from Sect. 2 is extended to arbitrary strictly elliptic operators. Introducing a new Riemannian metric, induced by the coefficients of the second-order part of the elliptic operator, the operator takes a simpler form: Up to a relatively bounded perturbation of bound 0, it is a Laplace–Beltrami operator for the new metric. Regularity and perturbation theory yield the main theorem in its full generality.

In this paper, the following notation is used. For a closed operator $T\colon\ D(T) \subset X \to X$ on a Banach space X, we denote by $[D(T)]$ the Banach space $D(T)$ equipped with the graph norm $\|\bullet\|_T := \|\bullet\|_X + \|T(\bullet)\|_X$ and indicate by $\hookrightarrow$ a continuous and by $\overset{c}{\hookrightarrow}$ a compact embedding. Moreover, we use Einstein's notation of sums, i.e.

$$x_k y^k := \sum_{k=1}^{n} x_k y^k$$

for $x := (x_1, \dots, x_n)$, $y := (y_1, \dots, y_n)$. Furthermore, we denote by $\mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}$ the positive real numbers and by $\mathbb{R}_- := \mathbb{R}\backslash\mathbb{R}_+$ the non-positive real numbers. Besides one defines the sector by $\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \theta\}$. Using the distance function d on $\overline{M}$, we denote by $B_R(x) := \{y \in \overline{M} : d(x, y) < R\}$.

2. Laplace–Beltrami operators with Dirichlet boundary conditions

In this section, we consider the special case where A_0 is the Laplace–Beltrami operator with Dirichlet boundary conditions, i.e.

$$\begin{aligned} \Delta_0^g f &:= \Delta^g f = \operatorname{div}_g(\nabla^g f) = g^{ij}\partial_{ij}^2 f - g^{ij}\,{}^g\Gamma_{ij}^k \partial_k f, \\ D(\Delta_0^g) &:= \left\{ f \in \bigcap_{p\geq 1} W^{2,p}(M) \cap \mathrm{C}_0(\overline{M})\colon \Delta^g f \in \mathrm{C}(\overline{M}) \right\} \end{aligned} \tag{2.1}$$

on the space $C(\overline{M})$ of continuous functions on $\overline{M}$. Here,

$$^{g}\Gamma_{ij}^{k} := \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

denote the Christoffel symbols of the Riemannian metric g.

Proposition 2.1. *For all $\lambda \in \mathbb{C}\backslash\mathbb{R}_-$ the operator $\lambda - \Delta_0^g$ is injective.*

Proof. Considering the equation

$$\begin{cases} \lambda f = \Delta^g f, \\ f|_{\partial M} = 0, \end{cases} \tag{2.2}$$

for $f \in C(\overline{M})$, one obtains that $\lambda - \Delta_0^g$ is injective if the only solution of (2.2) is zero.

Since $\overline{M}$ is compact, the domain $D(\Delta_0^g)$ is contained in $L^2(M)$ and $\Delta^g f \in L^2(M)$. Hence, Green's formula implies

$$\begin{aligned} \lambda \|f\|_{L^2(M)}^2 = \lambda \int_M f \overline{f} \ \mathrm{dvol}_M^g = \int_M \Delta^g f \overline{f} \\ \mathrm{dvol}_M^g = - \int_M g(\nabla^g f, \nabla^g \overline{f}) \ \mathrm{dvol}_M^g \in \mathbb{R}_-. \end{aligned}$$

Since $\lambda \in \mathbb{C}\backslash\mathbb{R}_-$, the term $\lambda \|f\|_{L^2(M)}^2$ can be in $\mathbb{R}_-$ only if $f = 0$. □

In the next step, we construct Green's functions such that the associated integral operators approximate the resolvent of A_0.

To this end, it is necessary to smooth the distance function d on $\overline{M}$. We consider a sufficiently small $\varepsilon > 0$ and define

$$\rho(x, y) := d(x, y) \chi \left(\frac{d(x, y)}{\varepsilon} \right) + 2\varepsilon \left(1 - \chi \left(\frac{d(x, y)}{\varepsilon} \right) \right),$$

where χ is a smooth cut-off function with $\chi(s) = 1$ if $s < 1$ and $\chi(s) = 0$ if $s > 2$. Then, $\rho \equiv d$ for $d(x, y) < \varepsilon$ and $\rho \in C^\infty((\overline{M} \times \overline{M})\backslash\{(x, x) : x \in \overline{M}\}, \mathbb{R})$.

Next, we extend the smoothed distance function ρ on $\overline{M}$ beyond the boundary ∂M. To this end, the set $S_{2\varepsilon} := \{x \in \overline{M} \colon d(x, \partial M) < 2\varepsilon\}$ is identified via the normal exponential map with $\partial M \times [0, 2\varepsilon)$. Considering $\overline{M} \cup (\partial M \times (-2\varepsilon, 0])$ and identifying ∂M with $\partial M \times \{0\}$ via $x \sim (x, 0)$, one obtains a smooth manifold $\widetilde{M}$. By Whitney's extension theorem (see [21]), the metric g can be extended to a smooth metric $\bar{g}$ on $\widetilde{M}$ and hence the smoothed distance function ρ can be extended to a smooth function $\bar{\rho}$ on $\widetilde{M} \times \widetilde{M}\backslash\{(x, x) : x \in \widetilde{M}\}$.

For $x \in S_{2\varepsilon}$, we consider the reflected point $x^* \in \widetilde{M}\backslash M$ with

$$\bar{\rho}(x, \partial M) = \bar{\rho}(x^*, \partial M)$$

such that the nearest neighbour of x on ∂M and the nearest neighbour of x^* on ∂M coincide.

Here and in the following, we denote by $n := \dim(\overline{M})$ the dimension of the manifold. The kernels are defined by

$$K_\lambda(x,y) := \begin{cases} \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\bar{\rho}(x,y))}{\bar{\rho}(x,y)^{\frac{n}{2}-1}} - \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\bar{\rho}(x^*,y))}{\bar{\rho}(x^*,y)^{\frac{n}{2}-1}}\right), & \text{if } \ d(x,\partial M) < \varepsilon, \\ \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\bar{\rho}(x,y))}{\bar{\rho}(x,y)^{\frac{n}{2}-1}} - \chi\left(\frac{\bar{\rho}(x,\partial M)}{\varepsilon}\right)\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\bar{\rho}(x^*,y))}{\bar{\rho}(x^*,y)^{\frac{n}{2}-1}}\right), & \text{if } \ d(x,\partial M) \in [\varepsilon, 2\varepsilon], \\ \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\bar{\rho}(x,y))}{\bar{\rho}(x,y)^{\frac{n}{2}-1}}, & \text{if } \ d(x,\partial M) > 2\varepsilon, \end{cases}$$

for $x \in \tilde{M}$, $y \in M$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}_-$, where $K_{\frac{n}{2}-1}$ is the modified Bessel function of the second kind (cf. Proposition A.1) of order $\frac{n}{2}-1$. Moreover, the associated integral operators are given by

$$(G_\lambda f)(x) := \int_M K_\lambda(x,y) f(y)\,\mathrm{d}y.$$

We now prove that the integral operators G_λ satisfy similar estimates as the resolvents of a sectorial operator.

Proposition 2.2. *Let $\eta > 0$. For $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$, the integral operators G_λ fulfil*

$$\|G_\lambda f\|_{L^\infty(M)} \leq \frac{C(\eta)}{|\lambda|}\|f\|_{L^\infty(M)}$$

for all $f \in \mathrm{C}(\overline{M})$ and $C(\eta) > 0$.

Proof. By Lemmas A.2 and A.3, we obtain

$$\begin{aligned} &\|G_\lambda f\|_{L^\infty(M\backslash S_{2\varepsilon})} \\ &\quad\leq C\sqrt{|\lambda|}^{\frac{n}{2}-1}\sup_{x\in M\backslash S_{2\varepsilon}}\int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\,\mathrm{d}y\cdot\|f\|_{L^\infty(M)} \\ &\quad\leq \frac{C'(\eta)}{|\lambda|}\|f\|_{L^\infty(M)} \end{aligned}\tag{2.3}$$

for $f \in \mathrm{C}(\overline{M})$. Moreover, Lemmas A.2, A.3 and Corollary A.4 imply

$$\begin{aligned} \|G_\lambda f\|_{L^\infty(S_\varepsilon)} \leq C\sqrt{|\lambda|}^{\frac{n}{2}-1}&\left(\sup_{x\in S_\varepsilon}\int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x,y))}{\overline{\rho}(x,y)^{\frac{n}{2}-1}}\,\mathrm{d}y\right. \\ &\left.\quad+\sup_{x\in S_\varepsilon}\int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\,\mathrm{d}y\right)\|f\|_{L^\infty(M)} \\ &\leq \frac{C'(\eta)}{|\lambda|}\|f\|_{L^\infty(M)} \end{aligned}\tag{2.4}$$

for $f \in \mathrm{C}(\overline{M})$. Furthermore, Lemmas A.2, A.3 and Corollary A.4 yield

$$\begin{aligned}
&\|G_\lambda f\|_{L^\infty(S_{2\varepsilon}\setminus S_\varepsilon)}\\
&\quad \leq C\sqrt{|\lambda|}^{\frac{n}{2}-1}\Bigg(\sup_{x\in S_{2\varepsilon}\setminus S_\varepsilon}\int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\,\mathrm{d}y\\
&\qquad + \underbrace{\sup_{x\in S_{2\varepsilon}\setminus S_\varepsilon}\chi\left(\frac{\overline{\rho}(x,\partial M)}{\varepsilon}\right)}_{=1}\\
&\quad \sup_{x\in S_{2\varepsilon}\setminus S_\varepsilon}\int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\,\mathrm{d}y\Bigg)\|f\|_{L^\infty(M)}\\
&\quad \leq \frac{C'(\eta)}{|\lambda|}\|f\|_{L^\infty(M)}
\end{aligned}\tag{2.5}$$

for $f \in \mathrm{C}(\overline{M})$. Summing up it follows that

$$\begin{aligned}
\|G_\lambda f\|_{L^\infty(M)} &= \|G_\lambda f\|_{L^\infty(M\setminus S_{2\varepsilon})} + \|G_\lambda f\|_{L^\infty(S_\varepsilon)} + \|G_\lambda f\|_{L^\infty(S_{2\varepsilon}\setminus S_\varepsilon)}\\
&\leq \frac{C(\eta)}{|\lambda|}\|f\|_{L^\infty(M)}
\end{aligned}$$

for $f \in \mathrm{C}(\overline{M})$ as claimed. ☐

To show that the kernel K_λ is approximately a Green's function for $\lambda - \Delta_0^g$, we need the following lemmata.

Lemma 2.3. *Let $\eta > 0$. For $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$, we have*

$$\begin{aligned}
&(\lambda - \Delta_x^g)\left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\right)\\
&\quad = \delta_x(y) + \mathcal{O}\Bigg(\sqrt{|\lambda|}^{\frac{n}{2}-1}\Bigg(\frac{\sqrt{|\lambda|}K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-2}}\\
&\qquad + \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\Bigg) + e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\Bigg)
\end{aligned}$$

for $x, y \in \overline{M}$.

Proof. Considering

$$K(r) := \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}-1}}\tag{2.6}$$

one obtains

$$K'(r) = \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{\sqrt{\lambda}K'_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}-1}} - \frac{(\frac{n}{2}-1)K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}}}\right)$$
$$= -\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{\sqrt{\lambda}K_{\frac{n}{2}}(\sqrt{\lambda}r)}{2r^{\frac{n}{2}-1}} + \frac{\sqrt{\lambda}K_{\frac{n}{2}-2}(\sqrt{\lambda}r)}{2r^{\frac{n}{2}-1}}\right.$$
$$\left. + \frac{\left(\frac{n}{2}-1\right)K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}}}\right) \tag{2.7}$$

and hence

$$K''(r) = \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{\lambda K''_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}-1}} - (n-2)\frac{\sqrt{\lambda}K'_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}}}\right.$$
$$\left. + \left(\frac{n^2}{4} - \frac{n}{2}\right)\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}+1}}\right). \tag{2.8}$$

These imply

$$K''(r) + \frac{n-1}{r}K'(r) - \lambda K(r)$$
$$= \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\left(\frac{\lambda K''_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}-1}} + \frac{\sqrt{\lambda}K'_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}}} + \left(\frac{n^2}{4} - \frac{n}{2}\right)\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}+1}}\right.$$
$$\left. - (n-1)\left(\frac{n}{2}-1\right)\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}+1}} - \lambda\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}r)}{r^{\frac{n}{2}-1}}\right)$$
$$= \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n r^{\frac{n}{2}+1}}\left(\lambda r^2 K''_{\frac{n}{2}-1}(\sqrt{\lambda}r) + \sqrt{\lambda}rK'_{\frac{n}{2}-1}(\sqrt{\lambda}r)\right.$$
$$\left. - \left(\left(\frac{n}{2}-1\right)^2 + \lambda r^2\right)K_{\frac{n}{2}-1}(\sqrt{\lambda}r)\right).$$

Remark that the kernel is rotation symmetric, and hence, the Laplacian is given by $\Delta_x K(|x|) = \partial_r^2 K(|x|) + \frac{n-1}{r}\partial_r K(|x|)$. Using (3.4), we conclude

$$\Delta_x K(|x|) - \lambda K(|x|) = -C \cdot \delta_0(x). \tag{2.9}$$

Next, we determine the constant C. For sufficient small $R > 0$ one has by Gauss Divergence Theorem

$$C = \int_{B_R(0)} C \cdot \delta_0(x)\ \mathrm{dvol}_{B_R(0)} = -\int_{B_R(0)} \Delta_x K(|x|) - \lambda K(|x|)\ \mathrm{dvol}_{B_R(0)}$$
$$= -\int_{\mathbb{S}_R^{n-1}} \frac{\partial}{\partial n}K(|x|)\ \mathrm{dvol}_{S_R^{n-1}} + \lambda\int_{B_R(0)} K(|x|)\ \mathrm{dvol}_{B_R(0)}$$

$$
\begin{aligned}
&= -\int_{\mathbb{S}_R^{n-1}} K'(R)\ \mathrm{dvol}_{S_R^{n-1}} + \lambda \int_{B_R(0)} K(|x|)\ \mathrm{dvol}_{B_R(0)} \\
&= -\mathrm{vol}(S^{n-1}) K'(R) R^{n-1} + \lambda \int_{B_R(0)} K(|x|)\ \mathrm{dvol}_{B_R(0)}\,.
\end{aligned}
$$

Using (2.7), we obtain

$$
\begin{aligned}
C = \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \mathrm{vol}(S^{n-1}) &\left(\frac{\sqrt{\lambda} K_{\frac{n}{2}}(\sqrt{\lambda}R)}{2R^{\frac{n}{2}-1}} + \frac{\sqrt{\lambda} K_{\frac{n}{2}-2}(\sqrt{\lambda}R)}{2R^{\frac{n}{2}-1}} \right. \\
&\left. + \frac{\left(\frac{n}{2}-1\right) K_{\frac{n}{2}-1}(\sqrt{\lambda}R)}{R^{\frac{n}{2}}} \right) R^{n-1} + \lambda \int_{B_R(0)} K(|x|)\ \mathrm{dvol}_{\,B_R(0)}.
\end{aligned}
$$

Since $K_\alpha(r) = \mathcal{O}(r^{-\alpha})$ for small $r \in \mathbb{R}_+$, the second and the fourth term vanishes by taking the limit $R \to 0$. Since

$$
\lim_{r\to 0} r^\alpha K_\alpha(r) = 2^{\alpha-1}\Gamma(\alpha)
$$

and

$$
\mathrm{vol}(\mathbb{S}^{n-1}) = n \cdot \frac{\sqrt{\pi}^n}{\Gamma\,(n/2+1)}
$$

the limit of the first term is given by

$$
\begin{aligned}
&\frac{\sqrt{\lambda}^{\frac{n}{2}}}{2^{\frac{n}{2}+1}\sqrt{\pi}^n} \cdot \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \lim_{R\to 0} K_{\frac{n}{2}}(\sqrt{\lambda}R) \cdot R^{\frac{n}{2}} \\
&\quad = \frac{\sqrt{\lambda}^{\frac{n}{2}}}{2^{\frac{n}{2}+1}} \cdot \frac{n}{\Gamma\,(n/2+1)} \lim_{R\to 0} K_{\frac{n}{2}}(\sqrt{\lambda}R) \cdot R^{\frac{n}{2}} \\
&\quad = \frac{1}{2^{\frac{n}{2}+1}} \cdot \frac{n}{\Gamma\left(\frac{n}{2}+1\right)} \lim_{R'\to 0} K_{\frac{n}{2}}(R') \cdot (R')^{\frac{n}{2}} \\
&\quad = \frac{1}{2^{\frac{n}{2}+1}} \cdot \frac{n}{\Gamma\left(\frac{n}{2}+1\right)} \cdot 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right) \\
&\quad = \frac{1}{2^{\frac{n}{2}+1}} \cdot \frac{2}{\Gamma\left(\frac{n}{2}\right)} \cdot 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right) = \frac{1}{2},
\end{aligned}
$$

where we used in the last line, that the Gamma function satisfies $\Gamma(x+1) = x\Gamma(x)$.

Similar the limit of the third term is

$$
\begin{aligned}
&\left(\frac{n}{2}-1\right) \cdot \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \cdot \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \lim_{R\to 0} K_{\frac{n}{2}-1}(\sqrt{\lambda}R) \cdot R^{\frac{n}{2}-1} \\
&= \left(\frac{n}{2}-1\right) \cdot \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2}^n} \cdot \frac{n}{\Gamma\left(\frac{n}{2}+1\right)} \cdot \lim_{R\to 0} K_{\frac{n}{2}-1}(\sqrt{\lambda}R) \cdot R^{\frac{n}{2}-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{n}{2}-1\right)\cdot\frac{1}{\sqrt{2}^n}\cdot\frac{n}{\Gamma\left(\frac{n}{2}+1\right)}\cdot\lim_{R'\to 0}K_{\frac{n}{2}-1}(R')\cdot(R')^{\frac{n}{2}-1}\\
&= \left(\frac{n}{2}-1\right)\cdot\frac{1}{\sqrt{2}^n}\cdot\frac{n}{\Gamma\left(\frac{n}{2}+1\right)}\cdot 2^{\frac{n}{2}-2}\Gamma\left(\frac{n}{2}-1\right)\\
&= \left(\frac{n}{2}-1\right)\cdot\frac{1}{\sqrt{2}^n}\cdot\frac{n}{\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\cdot\Gamma\left(\frac{n}{2}-1\right)}\cdot 2^{\frac{n}{2}-2}\Gamma\left(\frac{n}{2}-1\right)=\frac{1}{2}.
\end{aligned}
$$

Hence $C=\frac{1}{2}+\frac{1}{2}=1$. Moreover, we have

$$
(K\circ\rho)(x,y)=\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}
$$

for $x,y\in\overline{M}$. Using geodetic normal coordinates, the metric is given by

$$
g_{ij}(x)=\delta_{ij}+\mathcal{O}(\rho(x,y)^2).
$$

From $\Delta_x^\delta|x-y|^2=2n$, it follows

$$
\Delta_x^g(\rho(x,y)^2)=2n+\mathcal{O}(\rho(x,y)^2).
$$

Using

$$
\Delta_x^g(\rho(x,y)^2)=2|\nabla_x^g\rho(x,y)|_g^2+2\rho(x,y)\Delta_x^g\rho(x,y) \tag{2.10}
$$

First, we consider $y\in B_\varepsilon(x)$. Since $|\nabla_x^g\rho(x,y)|_g=1$, one obtains

$$
\Delta_x^g(\rho(x,y))=\frac{n-1}{\rho(x,y)}+\mathcal{O}(\rho(x,y)).
$$

Therefore, we obtain

$$
\begin{aligned}
&\Delta_x^g(K\circ\rho)(x,y)\\
&\quad= K''(\rho(x,y))|\nabla_x^g\rho(x,y)|_g^2+K'(\rho(x,y))\Delta_x^g\rho(x,y)\\
&\quad= K''(\rho(x,y))+K'(\rho(x,y))\Delta_x^g\rho(x,y)\\
&\quad= K''(\rho(x,y))+\frac{n-1}{\rho(x,y)}K'(\rho(x,y))+\mathcal{O}\left(\rho(x,y)|K'(\rho(x,y))|\right).
\end{aligned}
$$

Using (2.9) and Lemma A.2, it follows that

$$
\begin{aligned}
(\lambda-\Delta_x^g)(K\circ\rho)(x,y)&=\delta_x(y)+\mathcal{O}\left(\rho(x,y)|K'(\rho(x,y))|\right)\\
&=\delta_x(y)+\mathcal{O}\left(\sqrt{|\lambda|}^{\frac{n}{2}-1}\left(\frac{\sqrt{|\lambda|}K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-2}}\right.\right.\\
&\qquad\left.\left.+\frac{\left(\frac{n}{2}-1\right)\cdot K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\right)\right).
\end{aligned}
$$

Now let $y \in B_{2\varepsilon}(x) \backslash B_\varepsilon(x)$. Since ρ is smooth on $\overline{M} \backslash B_\varepsilon(x)$, we have $|\nabla_x^g \rho(x, y)|_g^2 \leq C$ and therefore by (2.10)

$$
\begin{aligned}
|\Delta_x^g (K \circ \rho)(x, y)| \leq C |K''(\rho(x, y))| + C(n) \frac{|K'(\rho(x, y))|}{\rho(x, y)} \\
+ \mathcal{O}(\rho(x, y) |K'(\rho(x, y))|).
\end{aligned}
$$

Moreover, one obtains by Lemma A.2

$$
\begin{aligned}
|K_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x, y))| &\leq K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\varepsilon) = \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}) \\
|K'_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x, y))| &\leq |K_{\frac{n}{2}}(\sqrt{\lambda}\rho(x, y))| + |K_{\frac{n}{2}}(\sqrt{\lambda}\rho(x, y))| \\
&\leq |K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\varepsilon)| + |K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
&= \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}) \\
|K'_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x, y))| &\leq |K_{\frac{n}{2}+1}(\sqrt{\lambda}\rho(x, y))| + |K_{\frac{n}{2}-1}(\sqrt{\lambda}\rho(x, y))| \\
&\quad + |K_{\frac{n}{2}-3}(\sqrt{\lambda}\rho(x, y))| \\
&= |K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\varepsilon)| + |K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
&\quad + |K_{\frac{n}{2}-3}(C(\eta)\sqrt{|\lambda|}\varepsilon)| \\
&= \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon})
\end{aligned}
$$

for $|\lambda|$and $\lambda \in \Sigma_{\frac{\pi}{2}-\eta}$. Since $\rho(x, y) \geq \varepsilon$, it follows

$$
(\lambda - \Delta_x^g)(K \circ \rho)(x, y) = \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon})
$$

for $|\lambda| \geq 1$.

Finally, we consider $y \in \overline{M} \backslash B_{2\varepsilon}(x)$. Since ρ is constant on $\overline{M} \backslash B_{2\varepsilon}(x)$, it follows that $\Delta_x^g (K \circ \rho) = 0$ and therefore as before

$$
(\lambda - \Delta_x^g)(K \circ \rho)(x, y) = \mathcal{O}(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon})
$$

for $|\lambda| \geq 1$. □

Lemma 2.4. *Let $\eta > 0$. For $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$, we have*

$$
\begin{aligned}
(\lambda - \Delta_x^g) &\left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \\
&= \delta_{x^*}(y) \\
&+ \mathcal{O}\Bigg(\sqrt{|\lambda|}^{\frac{n}{2}-1} \Bigg(\frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\
&+ \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}}
\end{aligned}
$$

$$
\begin{aligned}
&+ d(x,\partial m)\Bigg(\sqrt{|\lambda|}^{\frac{n}{2}-1}\Bigg(\frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}+1}}\\
&+ \sqrt{|\lambda|}^{\frac{n}{2}}\frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}}}\\
&+ \sqrt{|\lambda|}^{\frac{n}{2}+1}\Bigg(\frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\\
&+ e^{-C\sqrt{|\lambda|}\varepsilon}\Bigg)
\end{aligned}
$$

for $x \in S_{2\varepsilon}$ and $y \in \overline{M}$.

Proof. Considering the reflection $\sigma: S_{2\varepsilon} \to \tilde{M}: x \mapsto x^*$ and taking a point on the boundary $p \in \partial M$ every normal vector is an eigenvector for the eigenvalue -1 for the differential $D\sigma_p : T_pM \to T_pM$ and all tangential vectors on ∂M eigenvectors with eigenvalue 1. In particular, $D\sigma_p$ is a linear local isometry, i.e. $\sigma^*g = g$ for $p \in \partial M$. Since $\sigma^*g - g$ is smooth, we conclude that

$$\sigma^*g = g + \mathcal{O}(d(x,\partial M)).$$

Hence, one obtains $\nabla_x^\delta \sigma^* g = \nabla_x^\delta g + \mathcal{O}(1)$ and

$$\Delta^g \tilde{h} = \Delta^g(h \circ \sigma) = (\Delta^{\sigma^* g} h) \circ \sigma$$

for $\tilde{h}(x) := h(x^*)$. Therefore,

$$(\lambda - \Delta^g)\tilde{h} = ((\lambda - \Delta^{\sigma^* g})h) \circ \sigma + (\Delta^{\sigma^* g} h - \Delta^g h) \circ \sigma. \tag{2.11}$$

Using

$$\Delta^g f = g^{ij}(\partial_{ij}^2 f - \Gamma_{ij}^k \partial_k f)$$

we obtain

$$
\begin{aligned}
&|\Delta^g f - \Delta^{\sigma^* g} f|(x)\\
&\quad \le C \cdot |g - \sigma^* g|_g(x) \cdot \sum_{i,j=1}^n |\partial_{ij}^2 f|(x) + C \cdot \left|\nabla g - \nabla(\sigma^* g)\right|_g (x) \cdot |\nabla f|_g(x).
\end{aligned}
$$

Since $|g - \sigma^* g|_g(x) = \mathcal{O}(d(x,\partial M))$ and $|\nabla g - \nabla(\sigma^* g)|_g\,(x) = \mathcal{O}(1)$, we consider the derivatives of the kernel. Define K as in (2.6), we obtain

$$
\begin{aligned}
\partial_i(K \circ \overline{\rho})(x^*,y) &= K'(\overline{\rho}(x^*,y)) \cdot \partial_i \overline{\rho}(x^*,y)\\
\partial_{ij}^2(K \circ \overline{\rho})(x^*,y) &= K''(\overline{\rho}(x^*,y)) \cdot \partial_i \overline{\rho}(x^*,y) \cdot \partial_j \overline{\rho}(x^*,y)\\
&\quad + K'(\overline{\rho}(x^*,y)) \cdot \partial_{ij}^2 \overline{\rho}(x^*,y).
\end{aligned}
$$

Since $\partial_i \overline{\rho}(x^*, y) = \mathcal{O}(1)$ and (2.7), we obtain

$$\nabla_x (K \circ \overline{\rho})(x^*, y) = +\mathcal{O}\Bigg(\sqrt{|\lambda|}^{\frac{n}{2}-1} \Bigg(\frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}}} + \sqrt{|\lambda|}^{\frac{n}{2}} \Bigg(\frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}},$$

where we used that by Lemma A.2 and the monotonicity of Bessel functions

$$|K_{\frac{n}{2}-2}(\sqrt{\lambda}\overline{\rho}(x^*, y))| \leq K_{\frac{n}{2}-2}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y)) \leq K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))$$

holds. Similar we obtain from (2.8)

$$\begin{aligned} K''(\overline{\rho}(x^*, y)) = \mathcal{O}\Bigg(& \sqrt{|\lambda|}^{\frac{n}{2}-1} \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}} \\ & + \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\ & + \sqrt{|\lambda|}^{\frac{n}{2}+1} \frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \Bigg) \end{aligned}$$

Using

$$\partial^2_{ij}\overline{\rho}(x^*, y)^2 = 2\partial_i\overline{\rho}(x^*, y)\partial_j\overline{\rho}(x^*, y) + 2\overline{\rho}(x^*, y)\partial^2_{ij}\overline{\rho}(x^*, y)$$

and $\partial_i \overline{\rho}(x^*, y) = \mathcal{O}(1)$ and $\partial^2_{ij}\overline{\rho}(x^*, y)^2 = \mathcal{O}(1)$ one has

$$\partial^2_{ij}\overline{\rho}(x^*, y) = \mathcal{O}\left(\frac{1}{\overline{\rho}(x^*, y)} \right).$$

Hence,

$$\begin{aligned} \partial^2_{ij}(K \circ \overline{\rho})(x^*, y) = \mathcal{O}\Bigg(& \sqrt{|\lambda|}^{\frac{n}{2}-1} \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}} \\ & + \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\ & + \sqrt{|\lambda|}^{\frac{n}{2}+1} \frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \Bigg) \end{aligned}$$

Finally, we conclude

$$\begin{aligned} & \left((\Delta_x^{\sigma^* g} - \Delta_x^g) \left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(\cdot, y))}{\overline{\rho}(\cdot, y)^{\frac{n}{2}-1}} \right) \right) (x^*) \\ & = \mathcal{O}\Bigg(\sqrt{|\lambda|}^{\frac{n}{2}-1} \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \\
&+ d(x, \partial M) \left(\sqrt{|\lambda|}^{\frac{n}{2}-1} \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}+1}} \right. \\
&+ \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\
&\left. \left. + \sqrt{|\lambda|}^{\frac{n}{2}+1} \frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \right). \qquad (2.12)
\end{aligned}
$$

Now, the claim follows by Lemma 2.3 (for $\sigma^* g$ instead of g), using (2.11) and (2.12). $\square$

Lemma 2.5. *Let* $\eta > 0$. *We obtain*

$$
(\lambda - \Delta_x^g) \left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) = \mathcal{O}\left(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon} \right)
$$

for $y \in \overline{M}$, $x \in S_{2\varepsilon} \backslash S_\varepsilon$ *and for* $\lambda \in \Sigma_{\pi-\eta}$ *with* $|\lambda| \geq 1$,

Proof. By the product rule an easy calculation yields

$$
\begin{aligned}
&(\lambda - \Delta_x^g) \left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \\
&= \chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right) (\lambda - \Delta_x^g) \left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \\
&- \Delta_x^g \left(\chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right) \right) \frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \\
&- 2\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \left\langle \nabla_x^g \chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right), \nabla_x^g \left(\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \right\rangle.
\end{aligned}
$$

Using Lemmas 2.3 and 2.4, one obtains for the first term

$$
\begin{aligned}
&\chi \left(\frac{\rho(x, \partial M)}{\varepsilon} \right) (\lambda - \Delta_x^g) \left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n} \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \right) \\
&= \mathcal{O} \left(\sqrt{|\lambda|}^{\frac{n}{2}-1} \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \right. \\
&+ \sqrt{|\lambda|}^{\frac{n}{2}} \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}}
\end{aligned}
$$

$$
\begin{aligned}
&+ d(x,\partial M)\left(\sqrt{|\lambda|}^{\frac{n}{2}-1}\frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}+1}}\right.\\
&+ \sqrt{|\lambda|}^{\frac{n}{2}}\frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}}}\\
&+ \left.\sqrt{|\lambda|}^{\frac{n}{2}+1}\frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\right) + e^{-C\sqrt{|\lambda|}\varepsilon}\Bigg)
\end{aligned}
$$

Since $d(x,\partial M)\in[\varepsilon,2\varepsilon]$ is bounded away from 0, Lemma A.2 yields

$$
\frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}+1}} \le \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\varepsilon)}{\varepsilon^{\frac{n}{2}+1}}.
$$

Since $d(x,\partial M)<2\varepsilon$ and

$$
K_\alpha(\sqrt{|\lambda|}\varepsilon) = \mathcal{O}(e^{-\sqrt{|\lambda|\varepsilon}}),
$$

one concludes that

$$
\chi\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)(\lambda-\Delta_x^g)\left(\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\right) = \mathcal{O}\left(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\right)
$$

for $\lambda\in\Sigma_{\pi-\eta}$ with $|\lambda|\ge 1$. Since $|\nabla_x^g\rho|_g$ is bounded on $S_{2\varepsilon}\backslash S_\varepsilon$ and $|\Delta_x^g\rho|_g\le\frac{C}{\rho}$ on $S_{2\varepsilon}\backslash S_\varepsilon$, it follows that

$$
\nabla_x^g\left(\chi\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\right) = \chi'\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\frac{\nabla_x^g\rho(x,\partial M)}{\varepsilon} = \mathcal{O}(1)
$$

and

$$
\begin{aligned}
\Delta_x^g\left(\chi\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\right) &= \chi''\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\frac{|\nabla_x^g\rho(x,\partial M)|^2}{\varepsilon^2}\\
&\quad + \chi'\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\frac{\Delta_x^\delta\rho(x,\partial M)}{\varepsilon} = \mathcal{O}(1).
\end{aligned}
$$

Hence, the second term satisfies

$$
\Delta_x^g\left(\chi\left(\frac{\rho(x,\partial M)}{\varepsilon}\right)\right)\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^n}\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}} = \mathcal{O}\left(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\right)
$$

for $\lambda\in\Sigma_{\pi-\eta}$ with $|\lambda|\ge 1$. Since

$$
\begin{aligned}
\nabla_x^g\left(\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\right) &= \frac{\sqrt{\lambda}K'_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*,y))\nabla_x^g\overline{\rho}(x^*,y)}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\\
&\quad - \left(\frac{n}{2}-1\right)\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*,y))\nabla_x^g\overline{\rho}(x^*,y)}{\overline{\rho}(x^*,y)^{\frac{n}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\lambda} K_{\frac{n}{2}}(\sqrt{\lambda}\overline{\rho}(x^*, y))\nabla_x^g \overline{\rho}(x^*, y)}{2\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \\
&\quad + \frac{\sqrt{\lambda} K_{\frac{n}{2}-2}(\sqrt{\lambda}\overline{\rho}(x^*, y))\nabla_x^g \overline{\rho}(x^*, y)}{2\overline{\rho}(x^*, y)^{\frac{n}{2}-1}} \\
&\quad - \left(\frac{n}{2} - 1\right) \frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))\nabla_x^g \overline{\rho}(x^*, y)}{\overline{\rho}(x^*, y)^{\frac{n}{2}}} \\
&= \mathcal{O}\left(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\right),
\end{aligned}
$$

we conclude

$$
\frac{\sqrt{\lambda}^{\frac{n}{2}-1}}{\sqrt{2\pi}^{n}} \left\langle \nabla_x^g \chi\left(\frac{\rho(x, \partial M)}{\varepsilon}\right), \nabla_x^g \left(\frac{K_{\frac{n}{2}-1}(\sqrt{\lambda}\overline{\rho}(x^*, y))}{\overline{\rho}(x^*, y)^{\frac{n}{2}-1}}\right)\right\rangle = \mathcal{O}\left(e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\right)
$$

for $\lambda \in \Sigma_{\pi-\eta}$ for $|\lambda| \geq 1$. Summing up the claim follows. $\square$

Now, we are prepared to show that K_λ is approximately a Green's function for $\lambda - \Delta_x^g$.

Theorem 2.6. *The integral operators G_λ satisfy*

$$
\|(\lambda - \Delta_x^g)G_\lambda f - f\|_{L^\infty(M)} \leq \frac{C(\eta)}{\sqrt{|\lambda|}}\|f\|_{L^\infty(M)}
$$

for $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$, $\eta > 0$, and $f \in \mathrm{C}(\overline{M})$.

Proof. For $x \in \overline{M}\backslash S_{2\varepsilon}$ Lemma 2.3 yields

$$
\begin{aligned}
&\|(\lambda - \Delta_x^g)G_\lambda f - f\|_{L^\infty(M\backslash S_{2\varepsilon})} \\
&\quad \leq \sup_{x\in\overline{M}\backslash S_{2\varepsilon}} \left|\int_M \delta_x(y) f(y)\,\mathrm{d}y - f(x)\right| \\
&\qquad + \mathcal{O}\Bigg(\Bigg(\sup_{x\in\overline{M}\backslash S_{2\varepsilon}} \sqrt{|\lambda|}^{\frac{n}{2}} \int_M \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^{\frac{n}{2}-2}}\,\mathrm{d}y \\
&\qquad + \sup_{x\in\overline{M}\backslash S_{2\varepsilon}} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x, y))}{\rho(x, y)^{\frac{n}{2}-1}}\,\mathrm{d}y \\
&\qquad + \int_M e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\,\mathrm{d}y\Bigg)\|f\|_{L^\infty(M)}\Bigg)
\end{aligned}
$$

for $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$ and $f \in \mathrm{C}(\overline{M})$. Therefore, by Lemma A.3 it follows that

$$
\|(\lambda - \Delta_x^g)G_\lambda f - f\|_{L^\infty(M\backslash S_{2\varepsilon})} \leq \frac{C(\eta)}{\sqrt{|\lambda|}}\|f\|_{L^\infty(M)}
$$

for $\lambda \in \Sigma_{\pi-\eta}$ with $|\lambda| \geq 1$ and $f \in \mathrm{C}(\overline{M})$. For $x \in S_\varepsilon$ we obtain by Lemmas 2.3 and 2.4

$$
\begin{aligned}
&\|(\lambda - \Delta_x^g) G_\lambda f - f\|_{L^\infty(S_\varepsilon)} \\
&\quad \leq \sup_{x \in S_\varepsilon} \left| \int_M \delta_x(y) f(y)\, \mathrm{d}y - f(x) \right| \\
&\qquad + \mathcal{O}\Bigg(\Bigg(\sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}} \int_M \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-2}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}} \int_M \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-2}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}-1}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M d(x,\partial M) \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}+1}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}} \int_M d(x,\partial M) \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M d(x,\partial M) \frac{K_{\frac{n}{2}+1}(C(\eta)\sqrt{|\lambda|}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^{\frac{n}{2}+1}}\, \mathrm{d}y \\
&\qquad + \int_M e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}\, \mathrm{d}y \Bigg) \|f\|_{L^\infty(M)} \Bigg)
\end{aligned}
$$

for $f \in \mathrm{C}(\overline{M})$. Since $\overline{\rho}(x^*, y)$ only vanish if $x, y \in \partial M$ and $d(x, \partial M) = d(x^*, \partial M) \leq \overline{\rho}(x^*, y)$ for x, y near ∂M, Lemma A.3 and Corollary A.4 imply

$$
\|(\lambda - \Delta_x^g) G_\lambda f - f\|_{L^\infty(S_{2\varepsilon})} \leq \frac{C(\eta)}{\sqrt{|\lambda|}} \|f\|_{L^\infty(M)}
$$

for $|\lambda|$and $f \in \mathrm{C}(\overline{M})$. Moreover, we have for $x \in S_{2\varepsilon} \backslash S_\varepsilon$ by Lemmas 2.3 and 2.5

$$
\begin{aligned}
&\|(\lambda - \Delta_x^g) G_\lambda f - f\|_{L^\infty(S_{2\varepsilon} \backslash S_\varepsilon)} \\
&\quad \leq \sup_{x \in S_{2\varepsilon} \backslash S_\varepsilon} \left| \int_M \delta_x(y) f(y)\, \mathrm{d}y - f(x) \right| \\
&\qquad + \mathcal{O}\Bigg(\Bigg(\sup_{x \in S_{2\varepsilon} \backslash S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}} \int_M \frac{K_{\frac{n}{2}}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-2}}\, \mathrm{d}y \\
&\qquad + \sup_{x \in S_{2\varepsilon} \backslash S_\varepsilon} \sqrt{|\lambda|}^{\frac{n}{2}-1} \int_M \frac{K_{\frac{n}{2}-1}(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^{\frac{n}{2}-1}}\, \mathrm{d}y
\end{aligned}
$$

$$+ \int_M e^{-C\sqrt{|\lambda|}\varepsilon} \, dy \Big) \|f\|_{L^\infty(M)} \Big)$$

for $f \in C(\overline{M})$.

Since $\overline{M}$ is compact, it follows that

$$\int_M \frac{e^{-C(\eta)\sqrt{|\lambda|}\varepsilon}}{\varepsilon^{\frac{n}{2}+1}} \, dy \le \frac{\tilde{C}(\eta)}{\sqrt{|\lambda|}}$$

for $|\lambda| \ge 1$. Hence, as a consequence of Lemma A.3 one obtains

$$\|G_\lambda f - f\|_{L^\infty(S_{2\varepsilon} \setminus S_\varepsilon)} \le \frac{C(\eta)}{\sqrt{|\lambda|}} \|f\|_{L^\infty(M)}$$

for $|\lambda|$and $f \in C(\overline{M})$. Summing up we conclude that

$$\|(\lambda - \Delta_x^g) G_\lambda f - f\|_{L^\infty(M)} \le \frac{C(\eta)}{\sqrt{|\lambda|}} \|f\|_{L^\infty(M)}$$

for $|\lambda|$and $f \in C(\overline{M})$. □

Finally, we obtain the main theorem by combining the estimates from Proposition 2.2 and Theorem 2.6.

Theorem 2.7. *The operator Δ_0^g is sectorial of angle $\frac{\pi}{2}$ on* $C(\overline{M})$.

Proof. For $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$ Lemma 2.6 implies that

$$\|(\lambda - \Delta^g) G_\lambda - \mathrm{Id}\,\| \le \frac{C(\eta)}{\sqrt{|\lambda|}} < 1,$$

hence $(\lambda - \Delta^g) G_\lambda$ is invertible. Therefore,

$$\mathrm{Id} = (\lambda - \Delta^g) G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1}$$

and $(\lambda - \Delta^g)$ is right-invertible with right-inverse

$$(\lambda - \Delta^g)^{-1} = G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1}.$$

Hence, by Proposition 2.1 the operator $(\lambda - \Delta^g)$ is invertible and

$$(\lambda - \Delta^g)^{-1} = G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1}.$$

In particular, we obtain

$$\Delta^g G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1} f = \lambda G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1} f - f \in C(\overline{M})$$

for all $f \in C(\overline{M})$. Moreover $G_\lambda ((\lambda - \Delta^g) G_\lambda)^{-1} f$ is a solution of

$$\begin{cases} \Delta_x^g u = \lambda u - f, \\ u|_{\partial M} = 0 \end{cases}$$

for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. Since $f \in \mathrm{C}(M) \subset L^p(M)$ for every $p \geq 1$, elliptic regularity (cf. [16, Thm. 8.12]) implies $G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1} f \in \bigcap_{p\geq 1} W^{2,p}(M)$. Therefore $G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1} f \in D(A_0)$ and one concludes $R(\lambda, \Delta_0^g) = G_\lambda((\lambda - \Delta^g)G_\lambda)^{-1}$ for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. Thus by Proposition 2.2 it follows that

$$\|R(\lambda, \Delta_0^g)\| \leq \|G_\lambda\| \cdot \|((\lambda - \Delta^g)G_\lambda)^{-1}\| \leq \frac{C(\eta)}{|\lambda|}$$

for $\lambda \in \Sigma_{\pi-\eta}$ with sufficient large absolute value $|\lambda|$. By [1, Thm. 3.7.11] and [1, Cor. 3.7.17], Δ_0^g is sectorial of angle $\frac{\pi}{2}$. □

3. Strictly elliptic operators with Dirichlet boundary conditions

In this section, we consider strictly elliptic second-order differential operators with Dirichlet boundary conditions on the space $\mathrm{C}(\overline{M})$ of the continuous functions for a smooth, compact, Riemannian manifold $(\overline{M}, g)$ with smooth boundary ∂M. To this end, take real-valued functions

$$a_j^k = a_k^j \in \mathrm{C}^\infty(\overline{M}), \quad b_j, c \in \mathrm{C}(\overline{M}), \quad 1 \leq j, k \leq n.$$

satisfying the strict ellipticity condition

$$a_j^k(q) g^{jl}(q) X_k(q) X_l(q) > 0 \quad \text{for all } q \in \overline{M}$$

for all co-vector fields X_k, X_l on $\overline{M}$ with $(X_1(q), \ldots, X_n(q)) \neq (0, \ldots, 0)$ and define on $\mathrm{C}(\overline{M})$ the differential operator in divergence form with Dirichlet boundary conditions as

$$A_0 f := \sqrt{|a|}\mathrm{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf$$

with domain

$$D(A_0) := \left\{ f \in \bigcap_{p\geq 1} W^{2,p}(M) \cap C_0(\overline{M}) \colon A_0 f \in \mathrm{C}(\overline{M}) \right\}, \tag{3.1}$$

where $a = a_j^k$, $|a| = \det(a_j^k)$ and $b = (b_1, \ldots, b_n)$.

The key idea is to reduce the strictly elliptic operator on $\overline{M}$, equipped by g, to the Laplace–Beltrami operator on $\overline{M}$, corresponding to a new metric $\tilde{g}$.

For this purpose, we consider a $(2, 0)$-tensorfield on $\overline{M}$ given by

$$\tilde{g}^{kl} = a_i^k g^{il}.$$

Its inverse $\tilde{g}$ is a $(0, 2)$-tensorfield on $\overline{M}$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $\overline{M}$. We denote $\overline{M}$ with the old metric by $\overline{M}^g$ and with the new metric by $\overline{M}^{\tilde{g}}$ and remark that $\overline{M}^{\tilde{g}}$ is a smooth, compact, orientable Riemannian manifold with smooth boundary ∂M. Since the differentiable structures of $\overline{M}^{g}$ and $\overline{M}^{\tilde{g}}$ coincide, the identity

$$\mathrm{Id}\colon \overline{M}^{g} \longrightarrow \overline{M}^{\tilde{g}}$$

is a C^{∞}-diffeomorphism. Hence, the spaces

$$C(\overline{M}) := C(\overline{M}^{\tilde{g}}) = C(\overline{M}^{g})$$

coincide. Moreover, [17, Prop. 2.2] implies that the spaces

$$\begin{aligned} L^p(M) &:= L^p(M^{\tilde{g}}) = L^p(M^{g}), \\ W^{k,p}(M) &:= W^{k,p}(M^{\tilde{g}}) = W^{k,p}(M^{g}), \end{aligned} \tag{3.2}$$

for all $p \geq 1$ and $k \in \mathbb{N}$ coincide. We now denote by $\Delta_0^{\tilde{g}}$ the operator defined as in (2.1) respecting $\tilde{g}$. Moreover, we denote by $\tilde{A}_0$ the operator given in (3.1) for $b_k = c = 0$.

Lemma 3.1. *The operator A_0 and $\tilde{A}_0$ differ only by a relatively bounded perturbation of bound* 0.

Proof. Consider

$$Pf := g^{kl} b_k \partial_l f + cf$$

for $f \in D(A_0) \cap D(\tilde{A}_0)$. Since $D(\tilde{A}_0)$ is contained in $\bigcap_{p>1} W^{2,p}(M)$, Morreys embedding (cf. [2, Chap. V. and Rem. 5.5.2]) and the closed graph theorem imply

$$[D(\tilde{A}_0)] \overset{c}{\hookrightarrow} C^1(\overline{M}) \hookrightarrow \mathrm{C}(\overline{M}), \tag{3.3}$$

in particular $D(\tilde{A}_0)$ and $D(A_0)$ coincide. Since $P \in \mathcal{L}(C^1(\overline{M}), \mathrm{C}(\overline{M}))$ and it follows by (3.3) and Ehrling's Lemma (see [20, Thm. 6.99]) that P is relatively $\tilde{A}_0$-bounded with bound 0. □

Lemma 3.2. *The operator $\tilde{A}_0$ equals the Laplace–Beltrami operator $\Delta_0^{\tilde{g}}$ with respect to $\tilde{g}$.*

Proof. Using (3.2), we calculate in local coordinates

$$\begin{aligned} \tilde{A}_0 f &= \frac{1}{\sqrt{|g|}} \sqrt{|a|} \partial_j \left(\sqrt{|g|} \frac{1}{\sqrt{|a|}} a_l^j g^{kl} \partial_k f \right) \\ &= \frac{1}{\sqrt{|\tilde{g}|}} \partial_j \left(\sqrt{|\tilde{g}|} \tilde{g}^{kl} \partial_k f \right) \end{aligned}$$

for $f \in D(\tilde{A}_0)$, since $|g| = |a| \cdot |\tilde{g}|$. □

Theorem 3.3. *The operator A_0 is sectorial of angle $\frac{\pi}{2}$ on* $\mathrm{C}(\overline{M})$.

Proof. By Theorem 2.7 and Lemma 3.2 it follows that $\tilde{A}_0$ generates an analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$. Finally, Lemma 3.1 and [11, Thm. III. 2.10] implies the claim. □

Remark 3.4. This generalizes [18, Cor. 3.1.21.(ii)] to manifolds with boundary.

By Theorem 3.3, the abstract Cauchy problem (ACP) is well posed. This implies the existence and uniqueness of a continuous solution u of the initial value-boundary problem (IBP), having an analytic extension in a right half space in the time variable. Moreover, $u(t), A_0u(t) \in C^\infty(M) \cap \mathrm{C}(\overline{M})$ for all $t > 0$.

Corollary 3.5. *The resolvents $R(\lambda, A_0)$ are compact operators for all $\lambda \in \rho(A_0)$.*

Proof. This follows immediately by (3.3) and [11, Prop. II. 4.25]. □

We finish this section with the special case of closed manifolds, i.e. $\partial M = \emptyset$. Then, the Dirichlet boundary conditions gets an empty condition. Hence, the operator A_0 becomes

$$Af := \sqrt{|a|}\mathrm{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf,$$

with domain

$$D(A) := \left\{ f \in \bigcap_{p \geq 1} W^{2,p}(M) \colon A_0 f \in \mathrm{C}(M) \right\}.$$

Remark that then $d(x, \partial M) = d(x, \emptyset) = \infty$ and the kernel K_λ becomes much easier.

Corollary 3.6. *If the manifold M is closed, the operator A generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(M)$.*

Proof. Since $\mathrm{C}^2(M) \subset D(A)$ and $\mathrm{C}^2(M) \subset \mathrm{C}(M)$ dense, it follows that A is densely defined. Now Theorem 3.3 and [11, Thm. III.4.6] imply that A generates an analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(M)$. Finally, the compactness of the semigroup follows by Corollary 3.5 and [11, Thm. II.4.29]. □

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Appendix A. Bessel functions

The solutions of the ordinary differential equation

$$z^2 \frac{d^2}{dz^2} f(z) + z \frac{d}{dz} f(z) = (z^2 + \alpha^2) f(z) \tag{3.4}$$

for $z \in \mathbb{C}$ are called *modified Bessel functions of order* $\alpha \in \mathbb{R}$. In particular, we have the following.

Proposition A.1. *The modified Bessel functions of first kind of order* $\alpha \in \mathbb{R}$ *are given by*

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\alpha}}{\Gamma(k+\alpha+1)k!}$$

for $z \in \mathbb{C} \backslash \mathbb{R}_-$, *where* Γ *denotes the Gamma function. Moreover, we obtain the* modified Bessel function of second kind of order $\alpha \in \mathbb{R} \backslash \mathbb{Z}$ *by*

$$K_\alpha(z) = \frac{\pi}{2} \cdot \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\pi\alpha)}$$

for $z \in \mathbb{C} \backslash \mathbb{R}_-$. *If* $\alpha \in \mathbb{Z}$, *there exists a sequence* $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R} \backslash \mathbb{Z}$ *such that* $\alpha_n \to \alpha$ *and* K_α *is the limit*

$$K_\alpha(z) := \lim_{n \to \infty} K_{\alpha_n}(z)$$

for $z \in \mathbb{C} \backslash \mathbb{R}_-$.

First, we prove an estimate for the modified Bessel function of second kind.

Lemma A.2. *Let* $\alpha \in \mathbb{R}$ *and* $\eta > 0$. *Then, there exists a constant* $C(\eta) > 0$ *such that*

$$|K_\alpha(z)| \leq K_\alpha(C(\eta)|z|)$$

for all $z \in \Sigma_{\frac{\pi}{2}-\eta}$.

Proof. Since $\operatorname{Re}(z) > 0$ for all $z \in \Sigma_{\frac{\pi}{2}-\eta}$ and $\alpha \in \mathbb{R}$ it follows by [23, p. 181] that

$$|K_\alpha(z)| = \left| \int_0^\infty e^{-z\cosh(t)} \cosh(\alpha t)\, \mathrm{d}t \right| \leq \int_0^\infty e^{-\operatorname{Re}(z)\cosh(t)} \cosh(\alpha t)\, \mathrm{d}t.$$

Note that $z = |z|e^{i\varphi}$ with $|\varphi| \in [0, \pi/2 - \eta)$. The monotony of the cosinus implies

$$\frac{\operatorname{Re}(z)}{|z|} = \cos(\varphi) \geq \cos(\pi/2 - \eta) = \sin(\eta) =: C(\eta) > 0.$$

Using the monotony of the exponential function and the positivity of cosh, we conclude

$$\int_0^\infty e^{-\operatorname{Re}(z)\cosh(t)} \cosh(\alpha t)\, \mathrm{d}t \leq \int_0^\infty e^{-C(\varepsilon)|z|\cosh(t)} \cosh(\alpha t)\, \mathrm{d}t = K_\alpha(C(\eta)|z|)$$

for all $z \in \Sigma_{\frac{\pi}{2}-\eta}$. $\square$

Therefore, we obtain an estimate for the kernel.

Lemma A.3. *Let $\alpha \in \mathbb{R}$, $k \in [0, \infty)$ and $\lambda \in \Sigma_{\pi-\eta}$ for $\eta > 0$. If $k + \alpha < n$, we obtain*

$$\sup_{x\in M} \int_M \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k} \,\mathrm{d}y \leq C(\eta)\sqrt{|\lambda|}^{k-n}$$

for $|\lambda| \geq 1$.

Proof. Remark that

$$\int_M \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k} \,\mathrm{d}y = \int_{B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k} \,\mathrm{d}y + \int_{M\setminus B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k} \,\mathrm{d}y.$$

For the first term, one obtains

$$\begin{aligned}
\int_{B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k} \,\mathrm{d}y &\leq \tilde{C} \int_{\mathbb{R}^n} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}|y|)}{|y|^k} \,\mathrm{d}y \\
&= \hat{C}(\eta)\sqrt{|\lambda|}^k \frac{1}{\sqrt{|\lambda|}^n} \int_{\mathbb{R}^n} \frac{K_\alpha(|z|)}{|z|^k} \, dz \\
&= \hat{C}(\eta)\sqrt{|\lambda|}^{k-n} \int_0^\infty \int_{\mathbb{S}_r^{n-1}} \frac{K_\alpha(r)}{r^k} \,\mathrm{dvol}_{\mathbb{S}_r^{n-1}} \, dr \\
&= \check{C}(\eta)\sqrt{|\lambda|}^{k-n} \int_0^\infty K_\alpha(r) r^{n-1-k} \, dr.
\end{aligned}$$

Since

$$K_\alpha(r) = \mathcal{O}(r^{-\alpha})$$

for small $r \in \mathbb{R}_+$ and

$$K_\alpha(r) = \mathcal{O}\left(\frac{e^{-r}}{\sqrt{r}}\right)$$

for large $r \in \mathbb{R}_+$, we have

$$r^{n-1-k} K_\alpha(r) = \mathcal{O}(r^{n-1-k-\alpha})$$

for small $r \in \mathbb{R}_+$ and

$$r^{n-1-k} K_\alpha(r) = \mathcal{O}(r^{n-\frac{3}{2}-k} e^{-r})$$

for large $r \in \mathbb{R}_+$. Hence, there exists a constant $\bar{C} < \infty$ such that

$$\int_0^\infty K_\alpha(r) r^{n-1-k} \, dr < \bar{C}$$

and we conclude that

$$\int_{B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k}\,\mathrm{d}y \le C(\eta)\sqrt{|\lambda|}^{k-n}.$$

If $y \in \overline{M}\backslash B_R(x)$, we have $\rho(x,y) \ge R$ and therefore

$$\begin{aligned}\int_{M\backslash B_R(x)} \frac{K_\alpha(C(\eta)\sqrt{|\lambda|}\rho(x,y))}{\rho(x,y)^k}\,\mathrm{d}y &\le \frac{K_\alpha(C(\eta)R\sqrt{|\lambda|})}{R^k}\mathrm{vol}_g(M\backslash B_R(x))\\ &\le \hat{C}(\eta)e^{-\tilde{C}(\eta)\sqrt{|\lambda|}}\\ &\le \bar{C}(\eta)\sqrt{|\lambda|}^{k-n}\end{aligned}$$

for $|\lambda|$ since

$$K_\alpha(r) = \mathcal{O}\left(\frac{e^{-r}}{\sqrt{r}}\right)$$

for large $r \in \mathbb{R}_+$. □

Replacing x by x^* this yields an estimate for the reflected kernel.

Corollary A.4. *Let $\alpha \in \mathbb{R}$, $k \in [0,\infty)$ and $\lambda \in \Sigma_{\pi-\eta}$ for $\eta > 0$. Moreover, let $x \in S_{2\varepsilon}$. If $k+\alpha < n$, we obtain*

$$\sup_{x\in S_{2\varepsilon}} \int_M \frac{K_\alpha(C(\eta)\sqrt{\lambda}\overline{\rho}(x^*,y))}{\overline{\rho}(x^*,y)^k}\,\mathrm{d}y \le C\sqrt{|\lambda|}^{k-n}$$

for $|\lambda| \ge 1$.

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A.1.3 Strictly elliptic operators with generalized Wentzell boundary conditions on continuous functions on manifolds with boundary

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Strictly elliptic operators with generalized Wentzell boundary conditions on continuous functions on manifolds with boundary

Tim Binz

Abstract. We prove that strictly elliptic operators with generalized Wentzell boundary conditions generate analytic semigroups of angle $\frac{\pi}{2}$ on the space of continuous functions on a compact manifold with boundary.

Mathematics Subject Classification. 47D06, 34G10, 47E05, 47F05.

Keywords. Wentzell boundary conditions, Dirichlet-to-Neumann operator, Analytic semigroup, Riemannian manifolds.

1. Introduction. We start from a strictly elliptic differential operator A_m with domain $D(A_m)$ on the space $C(\overline{M})$ of continuous functions on a smooth, compact, orientable Riemannian manifold $(\overline{M}, g)$ with smooth boundary ∂M. Moreover, let C be a strictly elliptic differential operator on the boundary, take $\frac{\partial^a}{\partial \nu^g} : D(\frac{\partial^a}{\partial \nu^g}) \subset C(\overline{M}) \to C(\partial M)$ to be the outer conormal derivative, and functions $\eta, \gamma \in C(\partial M)$ with η strictly positive and a constant $q > 0$. In this setting, we define the operator $A^B \subset A_m$ with *generalized Wentzell boundary conditions* by requiring

$$
\begin{aligned}
f \in D(A^B) :\Longleftrightarrow\ & f \in D(A_m) \cap D\left(\frac{\partial^a}{\partial \gamma^g}\right),\ A_m f\big|_{\partial M} \\
&= q \cdot Cf|_{\partial M} - \eta \cdot \frac{\partial^a}{\partial \nu^g} f + \gamma \cdot f\big|_{\partial M}.
\end{aligned}
\tag{1.1}
$$

On a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, Favini, Goldstein, Goldstein, Obrecht, and Romanelli [8] showed that for $A_m = \Delta_\Omega$ and $C = \Delta_{\partial\Omega}$ the operator A^B generates an analytic semigroup of angle $\frac{\pi}{2}$ on $C(\overline{\Omega})$. In a preprint Goldstein, Goldstein, and Pierre [9] generalized this statement to arbitrary elliptic differential operators of the form $A_m f := \sum_{l,k=1}^n \partial_l (a^{kl} \partial_k f)$ and $C\varphi := \sum_{l,k=1}^n \partial_l (\alpha^{kl} \partial_k \varphi)$.

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Our main theorem (Theorem 4.6) generalizes these results to arbitrary strictly elliptic operators A_m and C on smooth, compact, orientable Riemannian manifolds with smooth boundary.

Consider a half-ball $B_1^+(0) := \{x \in \mathbb{R}^n : x_n \geq 0, |x| \leq 1\} \subset \mathbb{R}^n$. With the restriction g of the metric of $\mathbb{R}^n$ to $(B_1^+(0), g)$, we obtain a smooth, compact, orientable Riemannian manifold $B_1^+(0)$ with smooth boundary. It is not the closure of a domain in $\mathbb{R}^n$ since the boundary is only $\partial B_1^+(0) = \{x \in \mathbb{R}^n : x_n = 0, |x| \leq 1\}$.

The situation $q = 0$ on bounded, smooth domains in $\mathbb{R}^n$ was studied by Engel and Fragnelli [5] and on smooth, compact, orientable Riemannian manifolds in [3].

For $q = 0$, the boundary condition is a partial differential equation of first order whereas for $q > 0$ it is a partial differential equation of second order. Using the theory developed in [5] and [2], this yields two different abstract Dirichlet-to-Neumann operators: In the case $q = 0$, it is a pseudo differential operator of first order, in the case $q > 0$, it is an elliptic differential operator of second order perturbed by a pseudo differential operator of first order.

The paper is organized as follows. In the second section, we introduce the abstract setting from [5] and [2] for our problem. In the third section, we study the special case that A_m is the Laplace-Beltrami operator and B is the normal derivative. In the last section, we generalize to arbitrary strictly elliptic operators and their conormal derivatives.

Throughout the whole paper, we use the Einstein notation for sums and write $x_i y^i$ shortly for $\sum_{i=1}^n x_i y^i$. Moreover, we denote by $\hookrightarrow$ a continuous and by $\overset{c}{\hookrightarrow}$ a compact embedding.

2. The abstract setting. As in [5, Sect. 2], the basis of our investigation is the following.

Abstract setting 2.1. Consider

(i) two Banach spaces X and ∂X, called *state* and *boundary space*, respectively;
(ii) a densely defined *maximal operator* $A_m : D(A_m) \subset X \to X$;
(iii) a *boundary (or trace) operator* $L \in \mathcal{L}(X, \partial X)$;
(iv) a *feedback operator* $B : D(B) \subseteq X \to \partial X$.

Using these spaces and operators, we define the operator $A^B : D(A^B) \subset X \to X$ with abstract *generalized Wentzell boundary conditions* as

$$A^B f := A_m f, \quad D(A^B) := \big\{ f \in D(A_m) \cap D(B) : LA_m f = Bf \big\}. \tag{2.1}$$

For an interpretation of Wentzell boundary conditions as "dynamic boundary conditions", we refer to [5, Sect. 2].

In the sequel, we need the following operators.

Notation 2.2. The kernel of L is a closed subspace and we consider the restriction $A_0 \subset A_m$ given by

$$A_0 : D(A_0) \subset X \to X, \quad D(A_0) := \{f \in D(A_m) : Lf = 0\}.$$

The *abstract Dirichlet operator associated with* A_m is, if it exists,

$$L_0^{A_m} := (L|_{\ker(A_m)})^{-1} \colon \partial X \to \ker(A_m) \subseteq X,$$

i.e. $L_0^{A_m}\varphi = f$ is the unique solution of the abstract Dirichlet problem

$$\begin{cases} A_m f = 0, \\ Lf = \varphi. \end{cases} \tag{2.2}$$

If it is clear which operator A_m is meant, we simply write L_0.

Finally, we introduce the *abstract Dirichlet-to-Neumann operator associated with* (A_m, B), defined by

$$N^{A_m,B}\varphi := BL_0^{A_m}\varphi, \quad D(N^{A_m,B}) := \big\{\varphi \in \partial X : L_0^{A_m}\varphi \in D(B)\big\}.$$

If it is clear which operators A_m and B are meant, we write $N = N^{A_m,B}$ and call it the (abstract) Dirichlet-to-Neumann operator.

3. Laplace–Beltrami operator with generalized Wentzell boundary conditions. Take now as maximal operator $A_m \colon D(A_m) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ the Laplace-Beltrami operator Δ_M^g with domain $D(A_m) := \Big\{f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{loc}(M) \cap \mathrm{C}(\overline{M}) : A_m f \in \mathrm{C}(\overline{M})\Big\}$. Moreover, consider another strictly elliptic differential operator $C \colon D(C) \subset \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ in divergence form on the boundary space. To this end, take real valued functions

$$\alpha_j^k = \alpha_k^j \in \mathrm{C}^\infty(\partial M), \quad \beta_j \in \mathrm{C}(\partial M), \quad \gamma \in \mathrm{C}(\partial M), \quad 1 \le j,k \le n,$$

such that α_j^k are strictly elliptic, i.e.

$$\alpha_j^k(q) g^{jl}(q) X_k(q) X_l(q) > 0$$

for all co-vectorfields X_k, X_l on ∂M with $(X_1(q), \dots, X_n(q)) \neq (0, \dots, 0)$. Let $\alpha = (\alpha_j^k)_{j,k=1,\dots,n}$ denote the 1-1-tensorfield and $\beta = (\beta_j)_{j=1,\dots,n}$. Moreover, we denote by $|\alpha|$ the determinate of α and define $C \colon D(C) \subset \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ by

$$\begin{aligned} C\varphi &:= \sqrt{|\alpha|}\,\mathrm{div}_g\left(\frac{1}{\sqrt{|\alpha|}}\alpha\nabla_{\partial M}^g\varphi\right) + \langle\beta, \nabla_{\partial M}^g\varphi\rangle + \gamma\cdot\varphi, \\ D(C) &:= \left\{\varphi \in \bigcap_{p>1} \mathrm{W}^{2,p}(\partial M) \colon C\varphi \in \mathrm{C}(\partial M)\right\}. \end{aligned} \tag{3.1}$$

In order to define the feedback operator, we first consider $B_0 \colon D(B_0) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$ given by

$$B_0 f := -g(a\nabla_M^g f, \nu_g), \quad D(B_0) := \left\{f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{loc}(M) \cap \mathrm{C}(\overline{M}) \colon B_0 f \in \mathrm{C}(\partial M)\right\}.$$

This leads to the feedback operator $B \colon D(B) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$ given by

$$\begin{aligned} Bf &:= q \cdot CLf - \eta \cdot g(\nabla_M^g f, \nu_g), \\ D(B) &:= \{f \in D(A_m) \cap D(B_0) \colon Lf \in D(C)\}, \end{aligned}$$

where $L\colon \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M), f \mapsto f|_{\partial M}$ denotes the trace operator and $q > 0$ and $\eta \in \mathrm{C}(\overline{M})$ is positive. Using these operators A_m and B, we define the operator A^B with Wentzell boundary conditions on $\mathrm{C}(\overline{M})$ as in (2.1).

Note that the feedback operator B can be splitted into

$$B = q \cdot CL + \eta \cdot B_0.$$

The following proof is inspired by [7] and similar to [2, Ex. 5.3].

Lemma 3.1. *The operator B is relatively A_0-bounded of bound* 0.

Proof. Since $D(A_0) \subset \ker(L)$, the operators B and $\eta \cdot B_0$ coincide on $D(A_0)$. Hence it remains to prove the statement for the operator B_0. By [13, Chap. 5., Thm. 1.3] and the closed graph theorem, we obtain

$$[D(A_0)] \hookrightarrow \mathrm{W}^{2,p}(M).$$

Rellich's embedding (see [1, Thm. §3 2.10, Part III.]) implies

$$\mathrm{W}^{2,p}(M) \overset{c}{\hookrightarrow} \mathrm{C}^{1,\alpha}(M) \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{M})$$

for $p > \frac{n-1}{1-\alpha}$, where n denotes the dimension of $\overline{M}$. So we obtain

$$[D(A_0)] \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{M}) \hookrightarrow \mathrm{C}(\overline{M}).$$

Therefore, by Ehrling's lemma (cf. [12, Thm. 6.99]), for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_{\mathrm{C}^1(\overline{M})} \leq \varepsilon \|f\|_{A_0} + C_\varepsilon \|f\|_X$$

for every $f \in D(A_0)$. Since $B_0 \in \mathcal{L}(\mathrm{C}^1(\overline{M}), \partial X)$, this implies the claim. □

Lemma 3.2. *The operator N^{Δ_m, B_0} is relatively C-bounded of bound* 0.

Proof. Let $W := -(\Delta^g_{\partial M})^{\frac{1}{2}}$ and remark that by the proof of [3, Thm. 3.8], there exists a relatively W-bounded perturbation P of bound 0 such that

$$N^{\Delta_m, B_0} = W + P.$$

Therefore [11, Thm. 3.8] implies that N^{Δ_m, B_0} is relatively $\Delta^g_{\partial M}$-bounded of bound 0. Using the (uniform) ellipticity of C, there exists a constant $\Lambda > 0$ such that

$$\|\Delta^g_{\partial M} \varphi\|_{\mathrm{C}(\partial M)} \leq \Lambda \cdot \|C\varphi\|_{\mathrm{C}(\partial M)}$$

for $\varphi \in D(C) = D(\Delta^g_{\partial M})$. Hence N^{Δ_m, B_0} is relatively C-bounded of bound 0. □

Now the abstract results of [2] lead to the desired result.

Theorem 3.3. *The operator A^B with Wentzell boundary conditions associated to the Laplace-Beltrami operator $\Delta_m = \Delta^g_M$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$.*

Proof. We verify the assumptions of [2, Thm. 4.3]. Remark that by [3, Lem. 3.6] and Lemma 3.1 above, the Dirichlet operator $L_0 \in \mathcal{L}(\mathrm{C}(\partial M), \mathrm{C}(\overline{M}))$ exists and B is relatively A_0-bounded of bound 0. By multiplicative perturbation, we assume without loss of generality that $q = 1$. Now [4, Thm. 1.1] implies that A_0 is sectorial of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$ and has compact resolvent. Moreover, by [4, Cor. 3.6], the operator C generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$. Finally, the claim follows by [2, Thm. 4.3]. □

4. Elliptic operators with generalized Wentzell boundary conditions. Consider a strictly elliptic differential operator $A_m \colon D(A_m) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ in divergence form on the boundary space. To this end, let

$$a_j^k = a_k^j \in \mathrm{C}^\infty(\overline{M}), \quad b_j \in \mathrm{C}_c(M), \quad c \in \mathrm{C}(\overline{M}), \quad 1 \le j, k \le n,$$

be real-valued functions, such that a_j^k are strictly elliptic, i.e.

$$a_j^k(q) g^{jl}(q) X_k(q) X_l(q) > 0$$

for all co-vectorfields X_k, X_l on $\overline{M}$ with $(X_1(q), \dots, X_n(q)) \neq (0, \dots, 0)$. Let $a = (a_j^k)_{j,k=1,\dots,n}$ be the 1-1-tensorfield and $b = (b_j)_{j=1,\dots,n}$. Then we define $A_m \colon D(A_m) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$ by

$$\begin{aligned} A_m f &:= \sqrt{|a|}\,\mathrm{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + c \cdot f, \\ D(A_m) &:= \left\{ \varphi \in \bigcap_{p>1} \mathrm{W}^{2,p}_{loc}(M) \cap \mathrm{C}(\overline{M}) \colon A_m f \in \mathrm{C}(\overline{M}) \right\}. \end{aligned} \tag{4.1}$$

Note that, since $\overline{M}$ is compact, every strictly elliptic operator is uniformly elliptic (and of course vice versa).

We consider a $(2,0)$-tensorfield on $\overline{M}$ given by

$$\tilde{g}^{kl} = a_i^k g^{il}.$$

Its inverse $\tilde{g}$ is a $(0,2)$-tensorfield on $\overline{M}$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $\overline{M}$. We denote $\overline{M}$ with the old metric by $\overline{M}^g$ and with the new metric by $\overline{M}^{\tilde{g}}$ and remark that $\overline{M}^{\tilde{g}}$ is a smooth, compact, orientable Riemannian manifold with smooth boundary ∂M. Since the differentiable structures of $\overline{M}^g$ and $\overline{M}^{\tilde{g}}$ coincide, the identity

$$\mathrm{Id} \colon \overline{M}^g \longrightarrow \overline{M}^{\tilde{g}}$$

is a C^∞-diffeomorphism. Hence the spaces

$$\begin{aligned} X &:= \mathrm{C}(\overline{M}) := \mathrm{C}(\overline{M}^{\tilde{g}}) = \mathrm{C}(\overline{M}^g) \\ \text{and} \quad \partial X &:= \mathrm{C}(\partial M) := \mathrm{C}(\partial M^{\tilde{g}}) = \mathrm{C}(\partial M^g) \end{aligned}$$

coincide. Moreover, [10, Prop. 2.2] implies that the following spaces coincide

$$\begin{aligned}
&\mathrm{L}^p(M) := \mathrm{L}^p(M^{\tilde{g}}) = \mathrm{L}^p(M^g),\\
&\mathrm{W}^{k,p}(M) := \mathrm{W}^{k,p}(M^{\tilde{g}}) = \mathrm{W}^{k,p}(M^g),\\
&\mathrm{L}^p_{loc}(M) := \mathrm{L}^p_{loc}(M^{\tilde{g}}) = \mathrm{L}^p_{loc}(M^g),\\
&\mathrm{W}^{k,p}_{loc}(M) := \mathrm{W}^{k,p}_{loc}(M^{\tilde{g}}) = \mathrm{W}^{k,p}_{loc}(M^g),\\
&\mathrm{L}^p(\partial M) := \mathrm{L}^p(\partial M^{\tilde{g}}) = \mathrm{L}^p(\partial M^g),\\
&\mathrm{W}^{k,p}(\partial M) := \mathrm{W}^{k,p}(\partial M^{\tilde{g}}) = \mathrm{W}^{k,p}(\partial M^g),\\
&\mathrm{L}^p_{loc}(\partial M) := \mathrm{L}^p_{loc}(\partial M^{\tilde{g}}) = \mathrm{L}^p_{loc}(\partial M^g),\\
&\mathrm{W}^{k,p}_{loc}(\partial M) := \mathrm{W}^{k,p}_{loc}(\partial M^{\tilde{g}}) = \mathrm{W}^{k,p}_{loc}(\partial M^g)
\end{aligned} \tag{4.2}$$

for all $p > 1$ and $k \in \mathbb{N}$. Denote by $\hat{A}_m$ the maximal operator defined in (4.1) with $b_j = c = 0$ and by $\hat{C}$ the operator given in (3.1) for $\beta_j = \gamma = 0$. Moreover, denote the corresponding feedback operator by $\hat{B}$.

Next, we look at the operators A_m, B_0, and C with respect to the new metric $\tilde{g}$.

Lemma 4.1. *The operator $\hat{A}_m$ and the Laplace-Beltrami operator $\Delta^{\tilde{g}}_M$ coincide on $\mathrm{C}(\overline{M})$.*

Proof. Using local coordinates, we obtain

$$\begin{aligned}
\hat{A}_m f &= \frac{1}{\sqrt{|g|}}\sqrt{|a|}\partial_j\left(\sqrt{|g|}\frac{1}{\sqrt{|a|}}a^j_l g^{kl}\partial_k f\right)\\
&= \frac{1}{\sqrt{|\tilde{g}|}}\partial_j\left(\sqrt{|\tilde{g}|}\tilde{g}^{kl}\partial_k f\right) = \Delta^{\tilde{g}}_m f
\end{aligned}$$

for $f \in D(\hat{A}_m) = D(\Delta^{\tilde{g}}_m)$ since $|g| = |a| \cdot |\tilde{g}|$. ☐

Now we compare the maximal operators A_m and $\hat{A}_m$.

Lemma 4.2. *The operators A_m and $\hat{A}_m$ differ only by a relatively bounded perturbation of bound* 0.

Proof. Using (4.2), we define

$$P_1 f := b_l g^{kl}\partial_k f$$

for $f \in D(A_m) \cap D(\hat{A}_m)$. Since $b_l \in \mathrm{C}_c(M)$, there exist compact sets $K_l := \operatorname{supp}(b_l)$. Let $K := \bigcup_{l=1}^n K_l$ and note that it is a compact set and every b_l and hence $P_1 f$ vanishes outside of K. We define

$$\begin{aligned}
(\hat{A}_m)|_K f &:= \Delta^{\tilde{g}}_m f\\
D((\hat{A}_m)|_K) &:= \{f \in \mathrm{C}(K) :\ \text{there exists a function } \tilde{f} \in D(\hat{A}_m)\\
&\qquad \text{such that } \tilde{f}|_K = f\}.
\end{aligned}$$

Morreys embedding ([1, Thm. §3 2.10, Part III.]) implies

$$[D((\hat{A}_m)|_K)] \overset{c}{\hookrightarrow} \mathrm{C}^1(K) \hookrightarrow \mathrm{C}(K). \tag{4.3}$$

Moreover, we obtain

$$\begin{aligned}\|P_1 f\|_{\mathrm{C}(\overline{M})} &\le \sup_{q\in\overline{M}} |b_l(q) g^{kl}(q)(\partial_k f)(q)| \\ &= \sup_{q\in K} |b_l(q) g^{kl}(q)(\partial_k f)(q)| \\ &\le C\sum_{k=1}^n \|(\partial_k f)|_K\|_{\mathrm{C}(K)}\end{aligned}$$

and therefore $P_1 \in \mathcal{L}(\mathrm{C}^1(K), \mathrm{C}(\overline{M}))$. Hence $D(\hat{A}_m) = D(\tilde{A}_m)$. By (4.3), we conclude from Ehrling's lemma (see [12, Thm. 6.99]) that

$$\begin{aligned}\|P_1 f\|_{\mathrm{C}(\overline{M})} \le C\|f|_K\|_{\mathrm{C}^1(K)} &\le \varepsilon\|(\hat{A}_m)|_K f|_K\|_{\mathrm{C}(K)} \\ &\quad + \varepsilon\|f|_K\|_{\mathrm{C}(K)} + C(\varepsilon)\|f|_K\|_{\mathrm{C}(K)} \\ &\le \varepsilon\|\hat{A}_m f\|_{\mathrm{C}(\overline{M})} + \tilde{C}(\varepsilon)\|f\|_{\mathrm{C}(\overline{M})}\end{aligned}$$

for $f \in D(\hat{A}_m)$ and all $\varepsilon > 0$. Hence P_1 is relatively A_m-bounded of bound 0. Finally remark that

$$P_2 f := c\cdot f, \quad D(P_2) := \mathrm{C}(\overline{M})$$

is bounded and that

$$\tilde{A}_m f = \hat{A}_m f + P_1 f + P_2 f$$

for $f \in D(\hat{A}_m)$. □

Lemma 4.3. *The operators B_0 and the negative conormal derivative $-\frac{\partial}{\partial\nu^{\tilde{g}}}$ coincide.*

Proof. Since the Sobolev spaces coincide, we compute in local coordinates

$$\begin{aligned}B_0 f &= -g_{ij} g^{jl} a_l^k \partial_k f g^{im}\nu_m \\ &= -g_{ij}\tilde{g}^{jl}\partial_k f g^{im}\nu_m \\ &= -\tilde{g}_{ij}\tilde{g}^{jl}\partial_k f\tilde{g}^{im}\nu_m \\ &= -\frac{\partial}{\partial\nu^{\tilde{g}}} f\end{aligned}$$

for $f \in D(B) = D(\frac{\partial^{\tilde{g}}}{\partial\nu})$. □

Define $\tilde{C}\colon D(\tilde{C}) \subset \mathrm{C}(\partial M) \to \mathrm{C}(\partial M)$ by

$$\tilde{C}\varphi := \sqrt{|\tilde{\alpha}|}\,\mathrm{div}_{\tilde{g}}\left(\frac{1}{\sqrt{|\tilde{\alpha}|}}\tilde{\alpha}\nabla^{\tilde{g}}_{\partial M}\varphi\right), \quad D(C) := \{\varphi \in \mathrm{W}^{2,p}(\partial M)\colon C\varphi \in \mathrm{C}(\partial M)\},$$

where $\tilde{\alpha}(q) := a(q)^{-1}\cdot\alpha(q)$.

Lemma 4.4. *The operators $\hat{C}$ and $\tilde{C}$ coincide on* $\mathrm{C}(\partial M)$.

Proof. An easy calculation shows

$$\frac{|\tilde{g}|}{|\tilde{\alpha}|} = \frac{|g|}{|\alpha|},$$

$$\tilde{\alpha}_l^k \tilde{g}^{lj} = \alpha_l^k g^{lj}.$$

Hence we obtain in local coordinates

$$\begin{aligned}\tilde{C}\varphi &= \sqrt{\frac{|\tilde{\alpha}|}{|\tilde{g}|}}\partial_k\left(\sqrt{\frac{|\tilde{g}|}{|\tilde{\alpha}|}}\tilde{\alpha}_l^k\tilde{g}^{li}\partial_i\varphi\right)\\ &= \sqrt{\frac{|\alpha|}{|g|}}\partial_k\left(\sqrt{\frac{|g|}{|\alpha|}}\alpha_l^k g^{li}\partial_i\varphi\right)\\ &= \sqrt{|\alpha|}\mathrm{div}_g\left(\frac{1}{|\alpha|}\alpha\nabla^j\varphi\right) = \hat{C}\varphi\end{aligned}$$

for $\varphi \in D(\hat{C}) = D(\tilde{C})$. □

Next we compare the operators C and $\hat{C}$.

Lemma 4.5. *The operators C and $\hat{C}$ differ only by a relatively bounded perturbation of bound 0.*

Proof. Denote by

$$P\varphi := \langle \beta, \nabla^g_{\partial M}\rangle + \gamma\cdot\varphi \text{ for } f \in D(P) := \mathrm{C}^1(\partial M)$$

and note that $P \in \mathcal{L}(\mathrm{C}^1(\partial M), \mathrm{C}(\partial M))$. The Sobolev embeddings and the closed graph theorem imply

$$[D(C)] \overset{c}{\hookrightarrow} \mathrm{C}^1(\partial M) \hookrightarrow \mathrm{C}(\partial M).$$

Finally, the claim follows by Ehrling's lemma (cf. [12, Thm. 6.99]). □

Now we are prepared to prove our main theorem.

Theorem 4.6. *The operator A^B with Wentzell boundary conditions generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$.*

Proof. Since $\tilde{C}$ is a strictly elliptic differential operator in divergence form on $\mathrm{C}(\partial M)$, we obtain by Theorem 3.3 that the Laplace-Beltrami operator with Wentzell boundary conditions given by

$$(\Delta_M^{\tilde{g}} f)|_{\partial M} = q\cdot\tilde{C}f|_{\partial M} - \eta\frac{\partial^{\tilde{g}}}{\partial\nu}f$$

generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$. Now Lemma 4.1, Lemma 4.3, and Lemma 4.4 imply that the operator $\hat{A}^{\hat{B}}$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\overline{M})$. Note that A_m and $\hat{A}_m$ differ only by a relatively A_m-bounded perturbation of bound 0 by Lemma 4.2. By Lemma 4.5, one obtains that the perturbation on the boundary is relatively $\hat{C}$-bounded. Now the claim follows from [2, Thm. 4.2]. □

Remark 4.7. Theorem 4.6 generalizes the main theorem in [9] for the case $p = \infty$.

Corollary 4.8. *The initial-value boundary problem*

$$\begin{cases} \frac{d}{dt}u(t,q) &= A_m u(t,q), \; t \geq 0, \; q \in \overline{M}, \\ \frac{d}{dt}\varphi(t,q) &= Bu(t,q), \quad t \geq 0, \; q \in \partial M, \\ u(t,x) &= \varphi(t,x), \qquad t \geq 0, \; x \in \partial M, \\ u(0,q) &= u_0(q), \qquad\quad q \in \overline{M}, \end{cases}$$

on $\mathrm{C}(\overline{M})$ *is well-posed. Moreover, the solution* $\begin{pmatrix} u(t) \\ \varphi(t) \end{pmatrix} \in \mathrm{C}^\infty(M) \times \mathrm{C}^\infty(\partial M)$ *for* $t > 0$ *depends analytically on the initial value* $\begin{pmatrix} u_0 \\ u_0|_{\partial M} \end{pmatrix}$ *and is governed by a compact and analytic semigroup, which can be extended to the right half plane.*

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A.1.4 First order evolution equations with dynamic boundary conditions

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First order evolution equations with dynamic boundary conditions

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In this paper we introduce a general framework to study linear first order evolution equations on a Banach space X with dynamic boundary conditions, that is, with boundary conditions containing time derivatives. Our method is based on the existence of an abstract Dirichlet operator and yields finally to equivalent systems of two simpler independent equations. In particular, we are led to an abstract Cauchy problem governed by an abstract Dirichlet-to-Neumann operator on the boundary space ∂X . Our approach is illustrated by several examples and various generalizations are indicated.

1. Introduction

The study of the generator property of operators with Wentzell boundary conditions and of the Dirichlet-to-Neumann operator gained the interest of many authors. Recent developments and related references can be found in [7], [15], [22]. Starting in [12] we developed an abstract framework which was refined in [9] allowing to study these and related questions in a unified and systematic way. In the present paper we show how our abstract approach can be adapted to cover also first order evolution equations with dynamic boundary conditions.

More precisely, our starting point are the following two problems (P1) and (P2) with dynamic boundary conditions given by

$$\begin{cases} -\Delta u(t) = f(t) & \text{in } \Omega \times [0,+\infty), \\ \dot{u}(t) + \frac{\partial u}{\partial \nu}(t) = g(t) & \text{on } \partial\Omega \times [0,+\infty), \\ u(0) = u_0 & \text{in } \partial\Omega \end{cases} \tag{P1}$$

and

$$\begin{cases} \dot u(t) - \Delta u(t) = f(t) & \text{in } \Omega \times [0,+\infty), \\ \dot u(t) + \frac{\partial u}{\partial \nu}(t) = g(t) & \text{on } \partial\Omega \times [0,+\infty), \\ u(0) = u_0 & \text{in } \bar\Omega \end{cases} \tag{P2}$$

appearing both in [16], where $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative. The aim of this work is to develop an abstract framework to study the well-posedness of these and other similar problems in a systematic and unified way. In many cases this allows to "decouple" the original problem and show its equivalence to a system of two independent simpler problems.

This paper is organized as follows. In Section 2 we set up our abstract framework and reformulate (P1) and (P2) within this general setting, cf. (iaP1$_\lambda$), (iaP2). Section 3 provides the preliminaries needed in the sequel, in particular it introduces the abstract Dirichlet operator, the abstract Dirichet-to-Neumann operator and operators with abstract generalized Wentzell boundary conditions. After these preparations we characterize in Section 4 the well-posedness of the homogeneous problem (aP1$_\lambda$) in terms of two independent problems: an abstract Dirichlet problem (aDP$_\lambda$) and an abstract Cauchy problem (aCP$_{N_\lambda}$) for the Dirichlet-to-Neumann operator, see Theorem 4.3. Moreover, we relate solutions of the inhomogeneous problem (iaP1$_\lambda$) to the solutions of the inhomogeneous Cauchy problem (iaCP$_{N_\lambda}$) for the Dirichlet-to-Neumann operator, cf. Theorem 4.4. The following Section 5 is dedicated to the problem (iaP2). First we show that the homogeneous case (aP2) is equivalent to the abstract Cauchy problem (aCP$_{A^B}$) for the operator A^B with generalized Wentzell boundary conditions, see Theorem 5.1. Moreover, in Theorem 5.2 we show that, in the inhomogeneous case, the solutions of (iaP2) and (iaCP$_{A^B}$) are closely related. The last two subsections of Section 5 are devoted to the "decoupling" problem, that is we characterize the well-posedness of (aP2) in terms of the well-posedness of two simpler, independent Cauchy problems. This is done first in the analytic case and then for boundary operators B which are bounded on one part of the decomposition $X = X_0 \oplus \ker(A_m)$ of the state space X. In Section 6 we demonstrate in three examples how versatile our approach is. We show how it applies to delay-differential equations, to unbounded perturbations of the shift-semigroup and to (generalizations of) the above problems (P1) and (P2). Finally, in Section 7 we give a short conclusion and some final remarks on further developments and generalizations.

2. Abstract dynamic boundary value problems

In this section we embed the problems (P1) and (P2) from the introduction into an abstract framework. Before doing so we emphasize that our setting is tailored towards state spaces of continuous functions where the trace operator becomes bounded, cf. condition (iii) below. In Subsection 7.3 we mention related results in spaces of p-integrable functions where this condition obviously does not hold.

As in [9, Sect. 2] and [12, Sect. 2] we introduce the following setup.

General Setting 2.1. *Consider*

(i) two Banach spaces X and ∂X, called state *and* boundary space, *respectively;*
(ii) a densely defined and closed maximal[1] operator $A_m \colon D(A_m) \subset X \to X$;
(iii) a surjective trace operator $L \in \mathcal{L}(X, \partial X)$;
(iv) a boundary operator $B \colon D(B) \subseteq X \to \partial X$.

Note that due to the closedness of A_m its domain $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ equipped with the graph norm $\|x\|_{A_m} := \|x\| + \|A_m x\|$ for $x \in D(A_m)$ is a Banach space. Moreover, for every $\lambda \in \mathbb{C}$ the kernel $\ker(\lambda - A_m)$ is a closed subspace of X and of $[D(A_m)]$ and the restrictions of the corresponding norms to these kernels are equivalent.

Using the above spaces and operators we can now formulate abstract versions of the problems (P1) and (P2).

[1] "maximal" in the sense of "big", e.g., a differential operator without boundary conditions.

2.1 The homogeneous abstract boundary problem ($\mathrm{aP1}_\lambda$)

For $\lambda \in \mathbb{C}$ consider the *abstract elliptic problem with dynamic boundary conditions* on X given by

$$\begin{cases} \lambda u(t) = A_m u(t), & t \geq 0, \\ (\dot{Lu})(t) = Bu(t), & t \geq 0, \\ Lu(0) = x_0. \end{cases} \tag{$\mathrm{aP1}_\lambda$}$$

Our motivating problem (P1) fits (in the homogeneous case $f, g = 0$) into this setting if we choose $X := \mathrm{C}(\bar{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$, $\lambda = 0$, $A_m := \Delta$, $Lf := f|_{\partial\Omega}$ and $B := -\frac{d}{d\nu}$.

In order to study ($\mathrm{aP1}_\lambda$) we introduce some more terminology.

Definition 2.2. *A continuous function $u : \mathbb{R}_+ \to X$ is called a*

(i) classical solution *of* ($\mathrm{aP1}_\lambda$) *if $Lu : \mathbb{R}_+ \to \partial X$ is continuously differentiable in ∂X, $u(t) \in D(A_m) \cap D(B)$ for all $t \geq 0$ and* ($\mathrm{aP1}_\lambda$) *holds;*

(ii) mild solution *of* ($\mathrm{aP1}_\lambda$) *if $\int_0^t u(s)\, ds \in D(B)$, $u(t) \in \ker(\lambda - A_m)$, and*

$$Lu(t) - x_0 = B \int_0^t u(s)\, ds \quad \textit{for all } t \geq 0.$$

Moreover, we call ($\mathrm{aP1}_\lambda$) well-posed *if for every $x_0 \in \partial X$ it admits a unique mild solution.*

To indicate the dependence upon the initial value $x_0 \in \partial X$, in the sequel we also use the notation $u(\cdot, x_0)$ for solutions of ($\mathrm{aP1}_\lambda$). Moreover, we note that from Lemma A.1 it follows that a classical solution is always a mild solution while the contrary in general does not hold.

2.2 The homogeneous abstract boundary problem (aP2)

Consider the *abstract parabolic problem with dynamic boundary conditions* on X given by

$$\begin{cases} \dot{u}(t) = A_m u(t), & t \geq 0, \\ L\dot{u}(t) = Bu(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{aP2}$$

As before, the homogeneous case (P2) fits into this setting if we choose $X := \mathrm{C}(\bar{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$, $A_m := \Delta$, $Lf := f|_{\partial\Omega}$ and $B := -\frac{d}{d\nu}$.

Like above we need some more terminology.

Definition 2.3. *A continuous function $u : \mathbb{R}_+ \to X$ is called a*

(i) classical solution *of* (aP2) *if $u : \mathbb{R}_+ \to X$ is continuously differentiable in X, $u(t) \in D(A_m) \cap D(B)$ for all $t \geq 0$ and* (aP2) *holds;*

(ii) mild solution (aP2) *if $\int_0^t u(s)\, ds \in \ker(LA_m - B)$ and*

$$u(t) - u_0 = A_m \int_0^t u(s)\, ds \quad \textit{for all } t \geq 0.$$

Moreover, we call (aP2) well-posed *if for every $u_0 \in X$ it admits a unique mild solution.*

We also use the notation $u(\cdot, u_0)$ for solutions of (aP2) in order to indicate the dependence upon the initial value $u_0 \in X$. Note that a classical solution is always a mild solution while the contrary in general does not hold.

3. Preliminaries

In this section we introduce and study some operators needed in the sequel.

3.1 Operators with abstract Dirichlet boundary conditions and abstract Dirichlet operators

We define the, in general, *non*-densely defined, operator $A_0 : D(A_0) \subset X \to X$ with *abstract Dirichlet boundary conditions*

$$A_0 \subseteq A_m, \; D(A_0) := D(A_m) \cap \ker(L).$$

Moreover, for given $\lambda \in \mathbb{C}$ and $x \in \partial X$ we consider the *abstract Dirichlet problem*

$$\begin{cases} (\lambda - A_m) f = 0, \\ \qquad\quad Lf = x. \end{cases} \tag{aDP$_\lambda$}$$

If for every $x \in \partial X$ this problem has a unique solution $f \in \ker(\lambda - A_m)$, we can define the *abstract Dirichlet operator* $L_\lambda : \partial X \to \ker(\lambda - A_m) \subset X$ by $L_\lambda x := f$. Since A_m is closed, also $\ker(\lambda - A_m) \subset X$ is closed and since $L_\lambda = (L|_{\ker(\lambda - A_m)})^{-1}$, the closed graph theorem implies that $L_\lambda \in \mathcal{L}(\partial X, X)$. Conversely, if $L|_{\ker(\lambda - A_m)} : \ker(\lambda - A_m) \to \partial X$ is invertible with inverse L_λ, then for every $x \in \partial X$ the problem (aDP$_\lambda$) is uniquely solvable. Hence, we have the following.

Lemma 3.1. *The problem* (aDP$_\lambda$) *has for every* $x \in \partial X$ *a unique solution if and only if the Dirichlet operator* $L_\lambda = (L|_{\ker(\lambda - A_m)})^{-1} \in \mathcal{L}(\partial X, X)$ *exists.*

Now, by [3, Lem.3.1], the following relationship holds between A_0 and (aDP$_\lambda$).

Lemma 3.2. *We have* $\lambda \in \rho(A_0)$ *if and only if*

(i) the operator $\lambda - A_m : D(A_m) \to X$ *is surjective, and*
(ii) the Dirichlet operator L_λ *exists.*

In particular, if A_0 *has compact resolvent, then* $\lambda \in \rho(A_0)$ *if and only if* L_λ *exists.*

Finally, we note that $L_\lambda L \in \mathcal{L}(X)$ is a projection onto the subspace $\ker(\lambda - A_m) \subseteq X$ along the space $X_0 := \ker(L)$ which induces the decompositions

$$X = X_0 \oplus \ker(\lambda - A_m) \qquad \text{and} \qquad D(A_m) = D(A_0) \oplus \ker(\lambda - A_m). \tag{3.1}$$

3.2 The Dirichlet-to-Neumann operator

If the Dirichlet operator $L_\lambda \in \mathcal{L}(\partial X, X)$ exists, e.g. if $\lambda \in \rho(A_0)$ (use Lemma 3.2), we define the abstract *Dirichlet-to-Neumann operator* $N_\lambda : D(N_\lambda) \subset \partial X \to \partial X$ by

$$N_\lambda := BL_\lambda, \quad D(N_\lambda) := \big\{x \in \partial X : L_\lambda x \in D(B)\big\}.$$

If L_0 exists, we simply write $N := N_0 = BL_0$. These operators and the corresponding abstract Cauchy problems

$$\begin{cases} \dot{x}(t) = N_\lambda x(t), \quad t \geq 0, \\ x(0) = x_0 \end{cases} \tag{aCP$_{N_\lambda}$}$$

play a crucial role in our treatment of the problems (aP1$_\lambda$) and (aP2). We recall from [4, Sect. 3.1] that (aCP$_{N_\lambda}$) is called (mildly) *well-posed* if for every $x_0 \in \partial X$ there exists a unique mild solution. By [4, Thm. 3.1.12] this is equivalent to the fact that N_λ generates a C_0-semigroup on ∂X.

3.3 Operators with abstract generalized Wentzell boundary conditions

Finally, in order to study the problem (aP2) we need the operator $A^B : D(A^B) \subset X \to X$ with *abstract generalized Wentzell boundary conditions* given by

$$A^B \subseteq A_m, \quad D(A^B) := \{f \in D(A_m) \cap D(B) : LA_m f = Bf\}. \tag{3.2}$$

If $B = 0$, the boundary conditions defined in (3.2) are called *pure Wentzell boundary conditions*. This operator corresponds to the Cauchy problem

$$\begin{cases} \dot{u}(t) = A^B u(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{aCP$_{A^B}$}$$

In Theorem 5.1 we will see that the problems (aP2) and (aCP$_{A^B}$) are basically equivalent. In particular, this gives an interpretation of "Wentzell" as "dynamic boundary conditions". Moreover, as in [4, Sect. 3.1] we call (aCP$_{A^B}$) (mildly) *well-posed* if for every $u_0 \in X$ there exists a unique mild solution. By [4, Thm. 3.1.12] this is equivalent to the fact that A^B generates a C_0-semigroup on X.

4. The problems (aP1$_\lambda$) and (iaP1$_\lambda$)

The main aim of this section is to show that the homogeneous problem (aP1$_\lambda$) is equivalent to a system of two independent problems: an abstract Dirichlet problem (aDP$_\lambda$) for $\lambda - A_m$ and L and an abstract Cauchy problem (aCP$_{N_\lambda}$) for the Dirichlet-to-Neumann operator N_λ on ∂X. Moreover, we give an explicit formula for the solution of (iaP1$_\lambda$) in the inhomogeneous case.

Recall that $X_1 := [D(A_m)]$ is a Banach space. In addition to the conditions imposed in our General Setting 2.1, in this section we assume the following.

Assumptions 4.1. *Suppose that*

(i) B is relatively A_0-bounded, and
(ii) $B_1 := B|_{X_1} : D(A_m) \cap D(B) \subset X_1 \to \partial X$ is closed.

4.1 The homogeneous case

To show the aforementioned equivalence of (aP1$_\lambda$) on one side and of (aDP$_\lambda$) & (aCP$_{N_\lambda}$) on the other, we need the following result.

Lemma 4.2. *If for $\lambda \in \mathbb{C}$ the homogeneous problem (aP1$_\lambda$) is well-posed, then the abstract Dirichlet operator $L_\lambda \in \mathcal{L}(\partial X, X)$ exists.*

Proof. From the well-posedness of (aP1$_\lambda$) it immediately follows that the restriction $L|_{\ker(\lambda - A_m)} : \ker(\lambda - A_m) \to \partial X$ is surjective. Now assume that $Lf = 0$ for some $f \in \ker(\lambda - A_m)$. Then $f \in D(A_0) \subset D(B)$ and by assumption there exists a mild solution $u = u(\cdot, x_0)$ of (aP1$_\lambda$) for the initial value $Lu(0) = x_0 := Bf$. Define the continuous function $v : \mathbb{R}_+ \to X$ by

$$v(t) := f + \int_0^t u(s)\, ds, \quad t \geq 0.$$

Then $Lv : \mathbb{R}_+ \to \partial X$ is continuously differentiable and $v(t) \in D(A_m) \cap D(B)$ by the closedness of A_m and Lemma A.1. Moreover, for all $t \geq 0$ we have $(\lambda - A_m)v(t) = 0$, $Lv(0) = 0$ and

$$\begin{aligned} Bv(t) &= Bf + B\int_0^t u(s)\, ds = Bf + \int_0^t Bu(s)\, ds \\ &= Bf + \int_0^t (\dot{Lu})(s)\, ds = Bf + Lu(t) - Lu(0) \\ &= Lu(t) = (\dot{Lv})(t). \end{aligned}$$

This shows that v is a classical, hence also a mild solution of (aP1$_\lambda$) for the initial value $x_0 = 0$. By uniqueness, we obtain $v = u(\cdot, 0) = 0$ and therefore $0 = v(0) = f$. This proves that $L|_{\ker(\lambda - A_m)}$ is also injective, hence invertible with inverse L_λ. □

We can now prove the announced equivalence.

Theorem 4.3. *The following statements are equivalent.*

(a) The homogeneous problem (aP1$_\lambda$) *with dynamic boundary conditions is well-posed.*
(b) The Dirichlet problem (aDP$_\lambda$) *admits a unique solution and the Cauchy problem* (aCP$_{N_\lambda}$) *is wellposed.*
(c) The Dirichlet operator L_λ *exists and the Dirichlet-to-Neumann operator* $N_\lambda = BL_\lambda$ *generates a* C_0-*semigroup on* ∂X.

Proof. The equivalence of (b) and (c) follows immediately from Lemma 3.1 and [4, Thm. 3.1.12].

To show (a) $\Longleftrightarrow$ (b) we note that by Lemma 4.2 and Lemma 3.1, respectively, both assumptions (a) and (b) imply that L_λ exists. Now, if $u = u(\cdot, x_0)$ is a solution of (aP1$_\lambda$), then $x := Lu$ is a solution of (aCP$_{N_\lambda}$). Conversely, if $x = x(\cdot, x_0)$ is a solution of (aCP$_{N_\lambda}$), then $u := L_\lambda x$ is a solution of (aP1$_\lambda$). Hence, the problem (aP1$_\lambda$) admits a unique mild solution if and only if (aCP$_{N_\lambda}$) does. As pointed out in Subsection 3.2, the latter is equivalent to the fact that N_λ generates a C_0-semigroup on ∂X. This completes the proof. □

4.2 The inhomogeneous case

Having characterized the well-posedness of the homogeneous problem with dynamic boundary conditions we now study the *inhomogeneous* case

$$\begin{cases} \lambda u(t) = A_m u(t) + f(t), & t \geq 0, \\ (\dot{Lu})(t) = Bu(t) + g(t), & t \geq 0, \\ Lu(0) = x_0 \end{cases} \tag{iaP1$_\lambda$}$$

for $f : \mathbb{R}_+ \to X$ and $g : \mathbb{R}_+ \to \partial X$. We call $u : \mathbb{R}_+ \to X$ a *classical solution* of (iaP1$_\lambda$) if $Lu : \mathbb{R}_+ \to \partial X$ is continuously differentiable in ∂X, $u(t) \in D(A_m) \cap D(B)$ for all $t \geq 0$ and (iaP1$_\lambda$) holds.

As we will see next, the solvability of (iaP1$_\lambda$) can be characterized by the solvability of an inhomogeneous abstract Cauchy problem for the Dirichlet-to-Neumann operator N_λ.

Theorem 4.4. *Let* $\lambda \in \rho(A_0)$, $x_0 \in \partial X$, $f \in \mathrm{L}^1(\mathbb{R}_+, X)$ *and* $g \in \mathrm{L}^1(\mathbb{R}_+, \partial X)$. *Moreover, assume that* $N_\lambda = BL_\lambda$ *generates a* C_0-*semigroup* $(S_\lambda(t))_{t \geq 0}$ *on* ∂X. *Then* $u : \mathbb{R}_+ \to X$ *defined by*

$$u(t) := L_\lambda S_\lambda(t) x_0 + R(\lambda, A_0) f(t) + L_\lambda \int_0^t S_\lambda(t-s) \big(g(s) + BR(\lambda, A_0) f(s) \big)\, ds \tag{4.1}$$

is a classical solution of (iaP1$_\lambda$) *if and only if*

$$x(t) := S_\lambda(t) x_0 + \int_0^t S_\lambda(t-s) h(s)\, ds \tag{4.2}$$

is a classical solution of the inhomogeneous abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = N_\lambda x(t) + h(t), & t \geq 0, \\ x(0) = x_0 \end{cases} \tag{iaCP$_{N_\lambda}$}$$

where $h \in \mathrm{L}^1(\mathbb{R}_+, \partial X)$ *is given by* $h(t) := g(t) + BR(\lambda, A_0) f(t)$. *In particular,* (iaP1$_\lambda$) *has at most one solution which, if it exists, is given by* (4.1).

Proof. Note that $u(t) = L_\lambda x(t) + R(\lambda, A_0)f(t)$ and $x(t) = Lu(t)$ for all $t \geq 0$. Hence, $Lu(\cdot)$ is continuously differentiable if and only if $x(\cdot)$ is. Moreover, since $D(A_0) \subseteq D(B)$, we have

$$u(t) \in D(A_m) \cap D(B) \quad \Longleftrightarrow \quad L_\lambda x(t) \in D(B) \quad \Longleftrightarrow \quad x(t) \in D(N_\lambda).$$

Thus, it suffices to show that (iaP1$_\lambda$) holds for $u(\cdot)$ if and only if (iaCP$_{N_\lambda}$) holds for $x(\cdot)$. This, however, is easily verified by a simple and straightforward computation. The last affirmation follows since by [19, Cor. 4.2.2] the inhomogeneous Cauchy problem (iaCP$_{N_\lambda}$) has at most one solution which, if it exists, is given by (4.2). □

5. The problems (aP2) and (iaP2)

In this section we show equivalence of the homogeneous problem (aP2) and the abstract Cauchy problem (aCP$_{A^B}$) for the operator A^B with abstract generalized Wentzell boundary conditions. This fact is then used to characterize the well-posedness of the associated inhomogeneous problem (iaP2). In the remaining two subsections we then "decouple" (aP2) into two independent simpler abstract Cauchy problems on X_0 and ∂X, respectively.

5.1 The homogeneous case

Theorem 5.1. *The following assertions are equivalent*

(a) The homogeneous problem (aP2) *with dynamic boundary conditions is wellposed.*
(b) The abstract Cauchy problem (aCP$_{A^B}$) *is wellposed.*
(c) The operator A^B defined in (3.2) *generates a C_0-semigroup on X.*

Proof. Since (b) and (c) are equivalent by [4, Thm. 3.1.12], it is sufficient to prove the equivalence of (a) and (b).

Using that $L\dot{u}(t) = LA_m u(t)$, problem (aP2) is equivalent to the system

$$\begin{cases} \dot{u}(t) = A_m u(t), & t \geq 0, \\ LA_m u(t) = Bu(t), & t \geq 0, \\ u(0) = u_0. \end{cases}$$

By the definition of A^B this is equivalent to (aCP$_{A^B}$) and the claim follows. □

5.2 The inhomogeneous case

We now study the *inhomogeneous parabolic problem with dynamic boundary conditions* given by

$$\begin{cases} \dot{u}(t) = A_m u(t) + f(t), & t \geq 0, \\ L\dot{u}(t) = Bu(t) + g(t), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{iaP2}$$

for $f : \mathbb{R}_+ \to X$ and $g : \mathbb{R}_+ \to \partial X$. We call $u : \mathbb{R}_+ \to X$ a *classical solution* of (iaP2) if $u : \mathbb{R}_+ \to X$ is continuously differentiable in X, $u(t) \in D(A_m) \cap D(B)$ for all $t \geq 0$ and (iaP2) holds.

The solvability of (iaP1$_\lambda$) can be characterized by the solvability of an inhomogeneous Cauchy problem for the operator A^B with generalized Wentzell boundary conditions.

Theorem 5.2. *Let $u_0 \in X$, $f \in \mathrm{L}^1(\mathbb{R}_+, X)$, $g \in \mathrm{L}^1(\mathbb{R}_+, \partial X)$ and assume that A^B generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Then* (iaP2) *has at most one solution. Moreover, if $Lf = g$, then $u : \mathbb{R}_+ \to X$*

defined by

$$u(t) := T(t)x_0 + \int_0^t T(t-s)f(s)\,ds \tag{5.1}$$

is a classical solution of (iaP2) *if it is a classical solution of the inhomogeneous abstract Cauchy problem*

$$\begin{cases} \dot{u}(t) = A^B u(t) + f(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{iaCP$_{A^B}$}$$

Proof. Uniqueness of the solution follows since the difference of two solutions of (iaCP$_{A^B}$) solves the homogeneous problem (aCP$_{A^B}$) for the initial value $u_0 = 0$. In case $Lf = g$ the function u defined in (5.1) gives a solution of (iaCP$_{A^B}$). This follows immediately from the fact that $u(t) \in D(A^B)$ implies that $LA_m u(t) = Bu(t)$ for all $t \geq 0$. □

5.3 Decoupling for analytic semigroups

The problem (aP2) consists of two differential equations where the second one describes, by means of the boundary operator $B : D(B) \subset X \to \partial X$, an interaction between the state space X and the boundary space ∂X.

The aim of this section is to "decouple" this problem, i.e., we show its equivalence to a system consisting of two independent Cauchy problems: the first one on the space $X_0 = \ker(L)$ of functions having zero trace governed by the operator A_0 with abstract Dirichlet boundary conditions. The second one on the boundary space ∂X governed by a Dirichlet-to-Neumann operator N. Our approach is based on similarity transformations and perturbation arguments for analytic semigroups. For this reason we need to complement our General Setting 2.1 by some additional assumptions.

Assumptions 5.3. *(i) The operator A_0 is a weak Hille–Yosida operator on X, i.e. there exist $\lambda_0 \in \mathbb{R}$ and $M > 0$ such that $[\lambda_0, \infty) \subset \rho(A_0)$ and*

$$\|\lambda R(\lambda, A_0)\| \leq M \quad \text{for all } \lambda \geq \lambda_0; \tag{5.2}$$

(ii) the operator B is relatively A_0-bounded with bound 0, i.e., $D(A_0) \subseteq D(B)$ and for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|Bf\|_{\partial X} \leq \varepsilon \cdot \|A_0 f\|_X + M_\varepsilon \cdot \|f\|_X \quad \text{for all } f \in D(A_0);$$

(iii) the abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists.

In addition we define the operator $G_0 : D(G_0) \subset X \to X$ by

$$G_0 := A_m - L_0 B, \quad D(G_0) := D(A_0) = D(A_m) \cap \ker(L). \tag{5.3}$$

Theorem 5.4. *The following statements are equivalent*

(a) A^B given by (3.2) generates an analytic semigroup of angle $\alpha > 0$ on X.
(b) A_0 is sectorial of angle $\alpha > 0$ on X and the Dirichlet-to-Neumann operator N generates an analytic semigroup of angle $\alpha > 0$ on ∂X.
(c) $A_{00} := A_0|_{X_0}$ and the Dirichlet-to-Neumann operator $N := BL_0$ generate analytic semigroups of angle $\alpha > 0$ on X_0 and ∂X, respectively.

Moreover, A^B has compact resolvent if and only if A_0 and N have.

The proof of this result and various generalizations can be found in [9]. Here we only mention that the following lemma is a the key ingredient, see [9, Thm. 3.1].

Lemma 5.5. *The operator* $A^B : D(A^B) \subset X \to X$ *is similar to* $\mathcal{A} : D(\mathcal{A}) \subset X_0 \times \partial X \to X_0 \times \partial X$ *given by*

$$\mathcal{A} := \begin{pmatrix} G_0 & -L_0 N \\ B & N \end{pmatrix}, \quad D(\mathcal{A}) := \left\{ \binom{f}{x} \in D(A_0) \times D(N) \colon G_0 f + L_0 N x \in X_0 \right\}. \tag{5.4}$$

Remark 5.6. *By Theorem 5.4 it follows that in the analytic case the well-posedness of* (aP2) *remains unchanged if* A_m *is replaced by* $A_m - \lambda$ *for some* $\lambda \in \mathbb{C}$. *This is in strong contrast to the well-posedness of* ($\mathrm{aP1}_\lambda$) *where, by Theorem 4.3, we need that the Dirichlet operator* L_λ *exists, i.e.,* $\lambda \in \rho(A_0)$.

5.4 Decoupling for partially bounded boundary operators B

If the abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists, by (3.1) we obtain the decomposition $X = X_0 \oplus \ker(A_m)$ of the state space. In this section we study the case where the boundary operator $B : D(B) \subset X \to \partial X$ is bounded on one summand of this decomposition. This will allow us to decouple the generator property of A^B as in the previous subsection without assuming that the corresponding semigroup is analytic.

(i) B bounded on X_0

In order to proceed in the context of our General Setting 2.1 we need the following additional assumptions.

Assumptions 5.7. *(i) The operator* A_0 *is a weak Hille–Yosida operator on* X, *cf. Assumptions 5.3.(i);*
(ii) the operator $B_0 := B|_{X_0}$ *is bounded, i.e., there exists* $M \geq 0$ *such that*

$$\|Bf\|_{\partial X} \leq M \cdot \|f\|_X \quad \text{for all } f \in X_0;$$

(iii) A_0 *is invertible and hence the abstract Dirichlet operator* $L_0 \in \mathcal{L}(\partial X, X)$ *exists.*

If $A_{00} := A_0|_{X_0}$ and $N = BL_0$ generate C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively, then from [10, Lem. 3.2] it follows that for $t > 0$ the operator $R(t) : D(N) \subset \partial X \to X$ given by

$$R(t)x := A_m \int_0^t T(s) \cdot A_0^{-1} L_0 \cdot S(t-s) N x \, ds \tag{5.5}$$

is well-defined.

Theorem 5.8. *Under the above assumptions the following statements are equivalent.*
(a) The operator A^B *defined in* (3.2) *generates a* C_0*-semigroup on* X, *i.e.,* (aP2) *is well-posed.*
(b) (i) $A_{00} = A_0|_{X_0}$ *and* $N = BL_0$ *generate a* C_0*-semigroups* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$ *on* X_0 *and* ∂X, *respectively, and*
(ii) there exists $t_0 > 0$ *and* $M \geq 0$ *such that*

$$\|R(t)x\|_X \leq M \cdot \|x\|_{\partial X} \quad \text{for all } t \in (0, t_0] \text{ and } x \in D(N). \tag{5.6}$$

Proof. In a first step assume that $B|_{X_0} = 0$. Then by Lemma 5.5 the operator A^B is similar to

$$\mathcal{A} := \begin{pmatrix} A_0 & -L_0 N \\ 0 & N \end{pmatrix}, \quad D(\mathcal{A}) := \left\{ \binom{f}{x} \in D(A_0) \times D(N) \colon A_0 f + L_0 N x \in X_0 \right\} \tag{5.7}$$

and the claim follows from [10, Thm. 3.3]. If $B|_{X_0} \in \mathcal{L}(X_0, \partial X)$, the assertion follows from Lemma A.4. ☐

Remark 5.9. *The previous result can also be interpreted as follows: If the operator* $R(t)$ *in* (5.5) *remains norm bounded for* $t \downarrow 0$, *then the (coupled) problem* (aP2) *is well posed if and only if the (independent) Cauchy problems for* A_{00} *on* X_0 *and* N *on* ∂X *are.*

(ii) B bounded on $\ker(A_m)$

We now study the case where $B|_{\ker(A_m)}$ is bounded which, for A_0-bounded B, is equivalent to the fact that the Dirichlet-to-Neumann operator $N = BL_0$ gets bounded on ∂X. More precisely, our starting point are the following additional hypotheses complementing our General Setting 2.1. For the definition of the operator G_0 see (5.3).

Assumptions 5.10. *(i) The operator G_0 is a weak Hille–Yosida operator on X, cf. Assumptions 5.3.(i);*
(ii) the operator B is relatively A_m-bounded, i.e., $D(A_m) \subseteq D(B)$ and there exist $a, b \geq 0$ such that

$$\|Bf\|_{\partial X} \leq a \cdot \|A_m f\|_X + b \cdot \|f\|_X \quad \textit{for all } f \in D(A_m);$$

(iii) the abstract Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists.

Note that by the closed graph theorem $L_0 : \partial X \to [D(A_m)]$ is bounded, hence assumption (ii) above implies the boundedness of $N = BL_0 \in \mathcal{L}(\partial X)$.

Theorem 5.11. *Under the above assumptions the following statements are equivalent.*
(a) The operator A^B defined in (3.2) generates a C_0-semigroup on X, i.e., (aP2) is well-posed.
(b) $G_{00} := G_0|_{X_0} = (A_0 - L_0 B)|_{X_0}$ generates a C_0-semigroup on X_0.

Proof. By Lemma 5.5 the operator A^B is similar to $\mathcal{A}$ defined in (5.4). However, $\mathcal{A} = (\mathcal{Z} + \mathcal{P})|_{\mathcal{X}_0}$ for $D(\mathcal{Z}) := D(G_0) \times \partial X$ where

$$\mathcal{Z} := \begin{pmatrix} G_0 & 0 \\ B & 0 \end{pmatrix} : D(\mathcal{Z}) \subset \mathcal{X} \to \mathcal{X} \quad \text{and} \quad \mathcal{P} := \begin{pmatrix} 0 & -L_0 N \\ 0 & N \end{pmatrix} \in \mathcal{L}(\mathcal{X}_0, \mathcal{X}).$$

Since for every $\varepsilon > 0$ the matrix $\mathcal{Z}$ is similar to

$$\begin{pmatrix} G_0 & 0 \\ \varepsilon \cdot B & 0 \end{pmatrix} : D(\mathcal{Z}) \subset \mathcal{X} \to \mathcal{X},$$

it follows by Assumptions 5.10.(i) and [13, Lem. III.2.5] that $\mathcal{Z}$ is a weak Hille–Yosida operator. Moreover, since every generator is densely defined and the generator property is invariant under similarity transformations, both assumptions (a) and (b) imply that $\overline{D(\mathcal{Z})} = X_0 \times \partial X = \mathcal{X}_0$. Hence, by Lemma A.4 we conclude that A^B is a generator on X if and only if $\mathcal{Z}_0 := \mathcal{Z}|_{\mathcal{X}_0}$ is a generator on $\mathcal{X}_0$. However, $\mathcal{Z}_0$ is similar to $\operatorname{diag}(G_{00}, 0) : D(G_{00}) \times \partial X \subset \mathcal{X}_0 \to \mathcal{X}_0$ which is a generator if and only if G_{00} is. Summing up, this shows the equivalence (a) $\iff$ (b). $\square$

6. Examples

In this section we show how our approach can be applied in quite different situations. Here we concentrate on the homogeneous case and use Theorem 4.3 and Theorem 5.1 to show well-posedness of the corresponding problems (aP1$_\lambda$) and (aP2) with dynamic boundary conditions. To this end, we first study the operator A^B with generalized Wentzell boundary conditions and then indicate the consequences for the associated problems with dynamic boundary conditions. More examples can be found in [12, Sect. 4].

6.1 A delay differential operator

In this subsection we apply our approach to operators related to delay differential equations. More precisely, for a Banach space Y we define the Banach space $X := \mathrm{C}([-1, 0], Y)$ of all continuous functions on $[-1, 0]$ with values in Y equipped with the sup-norm. Moreover, we take a delay operator $\Phi \in \mathcal{L}(X, Y)$ and the generator C of a C_0-semigroup $(S(t))_{t \geq 0}$ on Y. With this notation we consider the *abstract delay differential operator* $A: D(A) \subset X \to X$ given by

$$Af := f', \quad D(A) := \left\{ f \in \mathrm{C}^1\big([-1,0], Y\big) : \begin{array}{l} f(0) \in D(C) \text{ and} \\ f'(0) = Cf(0) + \Phi f \end{array} \right\} \tag{6.1}$$

which governs a delay differential equations, see [13, Sect. VI.6] for details. The following result shows that this equation is well-posed, cf. [13, Thm. VI.6.1 and Cor. VI.6.3].

Theorem 6.1. *The operator A given by (6.1) generates a C_0-semigroup on $X = \mathrm{C}([-1,0], Y)$.*

Proof. The operator A fits in our general framework by choosing $\partial X := Y$, $A_m := \frac{d}{dr}$ with domain $D(A_m) := \mathrm{C}^1[-1,0]$, $L := \delta_0$ and $B := C\delta_0 + \Phi$. Then $A = A^B$ where $B_0 = B|_{X_0} = \Phi$ is bounded, i.e., we are in the situation of Subsection 5.4.(i).

First, we verify the Assumptions 5.7. In fact, $A_0 = \frac{d}{dr}$, $D(A_0) = \{f \in \mathrm{C}^1([-1,0], Y) : f(0) = 0\}$ has empty spectrum and its resolvent is given by

$$\big(R(\lambda, A_0)f\big)(r) = \int_r^0 e^{\lambda(r-s)} f(s)\, ds, \quad r \in [-1,0],\ \lambda \in \mathbb{C}.$$

For $\lambda \geq 0$ this implies

$$\lambda \cdot \big|\big(R(\lambda, A_0)f\big)(r)\big| \leq \int_r^0 \lambda e^{\lambda(r-s)}\, ds \cdot \|f\| \leq \|f\| \quad \text{for all } r \in [-1,0],$$

i.e., A_0 is a weak Hille–Yosida operator. Moreover, for all $\lambda \in \mathbb{C}$ the Dirichlet operator $L_\lambda \in \mathcal{L}(Y, X)$ exists and is given by $L_\lambda x = \varepsilon_\lambda \cdot x$ for $x \in \partial X = Y$ where $\varepsilon_\lambda(r) = e^{\lambda \cdot r}$ for $r \in [-1,0]$.

In order to apply Theorem 5.8 we note that $X_0 = \mathrm{C}_0([-1,0), Y)$ consists of all continuous functions on $[-1,0]$ vanishing in $r = 0$ and

$$A_{00} f := f', \quad D(A_{00}) := \left\{ f \in \mathrm{C}^1\big([-1,0], Y\big) : f(0) = f'(0) = 0 \right\}$$

generates the nilpotent left-shift semigroup $(T(t))_{t \geq 0}$ on X_0, cf. [13, Expl. II.4.31].

Now assume first that $\Phi = 0$. Then $N = BL_0 = C$ and by Theorem 5.8 we conclude that A generates a C_0-semigroup on X if (5.6) holds for $t_0 = 1$. To verify this condition we note that for $\mathbb{1} := \varepsilon_0$ we have $(A_0^{-1}\mathbb{1})(r) = r$, $r \in [-1,0]$, which implies

$$\big(T(s)A_0^{-1}\mathbb{1}\big)(r) = \begin{cases} (r+s) & \text{if } -1 \leq r \leq -s, \\ 0 & \text{if } -s < r \leq 0. \end{cases}$$

Using this we conclude for $x \in D(N) = D(C)$, $t \in (0, t_0] = (0,1]$ and $r \in [-1,0]$

$$\left(\int_0^t T(s) \cdot A_0^{-1} L_0 \cdot NS(t-s)x\, ds \right)(r) = \int_0^{\min\{-r,t\}} (r+s) \cdot CS(t-s)x\, ds$$

and further

$$\big(R(t)x\big)(r) = \begin{cases} \big(S(t) - S(t+r)\big)x & \text{if } r \in [-t, 0], \\ \big(S(t) - Id\big)x & \text{if } r \in [-1, -t). \end{cases}$$

This implies (5.6) for $\Phi = 0$. If $\Phi \in \mathcal{L}(X, Y)$, we obtain $N = C + \Phi L_0$, hence by Lemma 5.5 the operator A on X is similar to $\mathcal{A} = \mathcal{G}_0 := (\mathcal{Z} + \mathcal{P})|_{\mathcal{X}_0}$ on $\mathcal{X}_0 := X_0 \times \partial X$. Here we take the operator

$\mathcal{Z}$ on $\mathcal{X} := X \times \partial X$ and $\mathcal{P} \in \mathcal{L}(\mathcal{X}_0, \mathcal{X})$ as

$$\mathcal{Z} := \begin{pmatrix} A_0 & -L_0 C \\ 0 & C \end{pmatrix}, \quad D(\mathcal{A}) := \left\{ \binom{f}{x} \in D(A_0) \times D(C) : A_0 f - L_0 C x \in X_0 \right\} \quad \text{and}$$

$$\mathcal{P} := \begin{pmatrix} -L_0 \Phi & -L_0 \Phi L_0 \\ \Phi & \Phi L_0 \end{pmatrix}.$$

Now for $\lambda \in \rho(C) = \rho(\mathcal{Z})$ we have

$$R(\lambda, \mathcal{Z}) = \begin{pmatrix} R(\lambda, A_0) & -R(\lambda, A_0) L_0 C R(\lambda, C) \\ 0 & R(\lambda, C) \end{pmatrix}$$

where

$$\lambda \cdot \left\| R(\lambda, A_0) \cdot L_0 \cdot C R(\lambda, C) \right\| \le \left\| \lambda R(\lambda, A_0) \right\| \cdot \| L_0 \| \cdot \left\| \lambda R(\lambda, C) - Id \right\|$$

remains bounded for $\lambda \to +\infty$ since C is a generator. Hence, $\mathcal{Z}$ is a weak Hille–Yosida operator and the assertion follows by Lemma A.4. $\square$

While the operator A given by (6.1) is connected to delay differential equations (see [13, Sect. VI.6]) it is also related to our problems (aP1$_\lambda$) and (aP2). Since by the bounded perturbation theorem the Dirichlet-to-Neumann operator $N_\lambda = C + \Phi L_\lambda$ generates for all $\lambda \in \mathbb{C}$ a C_0-semigroup, by the above proof the conditions (c) in Theorem 4.3 and Theorem 5.1 are verified. Hence, the following holds for every generator C of a C_0-semigroup on a Banach space Y and a boundary functional $\Phi \in \mathcal{L}(C[-1,0], Y), Y)$.

Corollary 6.2. *For all $\lambda \in \mathbb{C}$ and $x_0 \in Y$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{d}{dr} u(t,r), & t \ge 0,\ r \in [-1,0] \\ \frac{d}{dt} u(t,0) = C u(t,0) + \Phi u(t,\cdot), & t \ge 0, \\ u(0,0) = x_0 \end{cases}$$

with dynamic boundary conditions is well-posed.

Corollary 6.3. *For all $u_0 \in X$ the problem*

$$\begin{cases} \frac{d}{dt} u(t,r) = \frac{d}{dr} u(t,r), & t \ge 0,\ r \in [-1,0] \\ \frac{d}{dt} u(t,0) = C u(t,0) + \Phi u(t,\cdot), & t \ge 0, \\ u(0,\cdot) = u_0 \end{cases}$$

with dynamic boundary conditions is well-posed.

6.2 The shift-semigroup on $\mathrm{C}[-1,0]$

In the previous subsection we studied the first derivative $A^B \subseteq \frac{d}{dr}$ for a boundary operator $B : D(B) \subset X \to \partial X$ bounded on the first component of the decomposition $X = X_0 \oplus \ker(A_m)$ but unbounded on the second. Next we will give an example where on the contrary B is unbounded on X_0 but bounded on $\ker(A_m)$. More precisely, we consider the Banach space $X := \mathrm{C}[-1,0]$ of all continuous, complex valued functions equipped with the sup-norm. Then, for some fixed $\alpha \in (0,1)$ we define the operator $A \colon D(A) \subset X \to X$ by

$$Af = f', \qquad D(A) := \left\{ f \in \mathrm{C}^1[-1,0] \colon f'(0) = \int_{-1}^{0} f'(r) \cdot (-r)^{-\alpha}\, dr \right\}. \tag{6.2}$$

Theorem 6.4. *The operator A given by (6.2) generates a C_0-semigroup on $X = \mathrm{C}[-1,0]$.*

Proof. Similarly as in the previous example the operator A fits in our setting if we choose $\partial X := \mathbb{C}$, $A_m := \frac{d}{dr}$ with domain $D(A_m) := \mathrm{C}^1[-1,0]$, $L := \delta_0$ and define $B : D(B) \subset X \to \partial X$ by

$$Bf := \int_{-1}^{0} f'(r) \cdot (-r)^{-\alpha}\, dr \quad \text{for} \quad f \in D(B) := \mathrm{W}^{1,1}(-1,0). \tag{6.3}$$

Then $A = A^B$. Moreover, B is relatively A_m-bounded and $N = 0$, i.e., we are in the situation of Subsection 5.4.(ii). We proceed by verifying the remaining Assumptions 5.10.

As in the former example $A_0 = \frac{d}{dr}$, $D(A_0) = \{f \in \mathrm{C}^1[-1,0] : f(0) = 0\}$ has empty spectrum and is a weak Hille–Yosida operator. Since $\partial X = \mathbb{C}$ is finite dimensional, $P := -L_0 B$ is relatively A_0-compact and hence $G_0 = A_0 + P$ is a weak Hille–Yosida operator by Lemma A.3. Moreover, L_λ exists for all $\lambda \in \mathbb{C}$ and is given by $L_\lambda x = \varepsilon_\lambda \cdot x$ for $x \in \mathbb{C}$.

By Theorem 5.11 it only remains to show that

$$G_{00} = (A_0 - L_0 B)|_{X_0}$$

generates a C_0-semigroup on $X_0 = \mathrm{C}_0[-1,0)$. To prove this assertion we verify that $P = -L_0 B$ is a Weiss–Staffans perturbation, cf. [2, Def. 9], of $A_{00} := A_0|_{X_0}$ generating the nilpotent left-shift semigroup $(T(t))_{t\geq 0}$ on X_0. To this end we first choose (here the subscript "$_{\mathrm{WS}}$" indicates the notation used in [2]) $X_{\mathrm{WS}} := X_0$, $Z_{\mathrm{WS}} := D(A_0) = \mathrm{C}_0^1[0,1)$, $U_{\mathrm{WS}} := \partial X = \mathbb{C}$ and the operators $A_{\mathrm{WS}} := A_{00}$, $B_{\mathrm{WS}} := -L_0$, $C_{\mathrm{WS}} := B$. Then by the proof of Lemma A.4 we have $X \subseteq (X_0)_{-1}$, hence $P := -L_0 B \in \mathcal{L}(Z, (X_0)_{-1})$.

Now we verify the conditions (i)-(v) of [2, Thm. 10] for $p = 1$ and a fixed $t_0 \in (0,1]$.

(i) By the reasoning in the first part of the proof of Lemma A.4, see (A 2), it follows that

$$\big(R(0,(A_{\mathrm{WS}})_{-1})B_{\mathrm{WS}}x\big)(r) = \big(A_0^{-1}L_0 x\big)(r) = r \cdot x, \quad r \in [-1,0],\ x \in \mathbb{C}$$

and therefore $\mathrm{rg}\big(R(0,(A_{\mathrm{WS}})_{-1})B_{\mathrm{WS}}\big) \subseteq D(A_0) = Z_{\mathrm{WS}}$ proving (i).

(ii) Let $u \in \mathrm{L}^1[0,t_0]$. Then

$$\begin{aligned}\left(\int_0^{t_0} T_{-1}(s)B_{\mathrm{WS}}u(t_0-s)\,ds\right)(r) &= \left(A_m\int_0^{t} T(t_0-s)A_0^{-1}L_0u(s)\,ds\right)(r)\\ &= \frac{d}{dr}\int_0^{\min\{-r,t_0\}} (r+s)u(t_0-s)\,ds\\ &= \int_0^{\min\{-r,t_0\}} u(t_0-s)\,ds\end{aligned}$$

and hence as needed

$$\int_0^{t_0} T_{-1}(s)B_{\mathrm{WS}}u(t_0-s)\,ds \in X_0 = X_{\mathrm{WS}}.$$

(iii) Let $0 < s < t_0 \leq 1$ and $f \in D(A_{\mathrm{WS}}) = D(A_{00})$. Since $f(0) = 0$ integration by parts yields

$$\begin{aligned}\big|C_{\mathrm{WS}}T(s)f\big| &= \left|\int_{-1}^{-s} f'(s+r)\cdot(-r)^{-\alpha}\,dr\right|\\ &= \left|-f(s-1) + \alpha\cdot\int_{-1}^{-s} f(s+r)\cdot(-r)^{-\alpha}\,dr\right|\\ &\leq \left(1 + \alpha\cdot\int_{-1}^{-s}(-r)^{-\alpha}\,dr\right)\cdot\|f\|_\infty\\ &= s^{-\alpha}\cdot\|f\|_\infty.\end{aligned}$$

This implies

$$\int_0^{t_0} \big|C_{\mathrm{WS}}T(s)f\big|\,ds \leq \tfrac{t_0^{1-\alpha}}{1-\alpha}\cdot\|f\|_\infty \leq \tfrac{1}{1-\alpha}\cdot\|f\|_\infty, \tag{6.4}$$

i.e., condition (iii) is satisfied.

(iv) Let $t \in (0, t_0)$ and $u \in \mathrm{L}^1[0, t_0]$. Then by (ii) we have

$$\left(\int_0^t T_{-1}(s) L_0 u(t-s)\, ds\right)(r) = \int_0^{\min\{-r,t\}} u(t-s)\, ds =: f_t(r)$$

with derivative

$$f_t'(r) := \begin{cases} -u(t+r) & \text{if } r \in [-t, 0], \\ 0 & \text{if } r \in [-1, -t). \end{cases}$$

Thus

$$\begin{aligned} \left| C_{\mathrm{WS}} \int_0^t T_{-1}(s) B_{\mathrm{WS}} u(t-s)\, ds \right| &= \left| \int_{-1}^0 f_t \cdot (-r)^{-\alpha}\, dr \right| = \left| \int_{-t}^0 u(t+r) \cdot (-r)^{-\alpha}\, dr \right| \\ &= \left| \int_0^t u(r) \cdot (t-r)^{-\alpha}\, dr \right| = (u * k_\alpha)(t) \end{aligned}$$

for $k_\alpha \in \mathrm{L}^1[0,1]$, $k_\alpha(r) := r^\alpha$. Young's inequality then implies

$$\int_0^{t_0} \left| C_{\mathrm{WS}} \int_0^t T_{-1}(s) B_{\mathrm{WS}} u(t-s)\, ds \right| dt \le \|u * k_\alpha\|_{\mathrm{L}^1[0,t_0]} \le \|u\|_1 \cdot \|k_\alpha\|_{\mathrm{L}^1[0,t_0]} \tag{6.5}$$

and hence condition (iv) holds true.

(v) From (6.5) it follows that

$$\|\mathcal{F}_{t_0}\| \le \|k_\alpha\|_{\mathrm{L}^1[0,t_0]} = \frac{t_0^{1-\alpha}}{1-\alpha} \to 0$$

as $t_0 \downarrow 0$ and therefore $1 \in \rho(\mathcal{F}_{t_0})$ for sufficient small $t_0 > 0$.

Now by [2, Theorem 10] the operator G_{00} generates a C_0-semigroup on X_0, hence the proof is complete. □

Note that for B defined in (6.3) the restriction $B_0 := B|_{X_0}$ is unbounded on $X_0 = \mathrm{C}_0[-1, 0)$. In fact, if we define $f(r) := (-r)^\alpha$ for $r \in [-1, 0]$, then $f \in X_0 \setminus D(B_0)$.

For our problems with dynamic boundary conditions Theorem 4.3 and Theorem 5.1 now give the following.

Corollary 6.5. *Let $\alpha \in (0, 1)$. Then for all $\lambda, x_0 \in \mathbb{C}$ the problem*

$$\begin{cases} \lambda u(t,r) = \frac{d}{dr} u(t,r), & t \ge 0,\ r \in [-1, 0], \\ \frac{d}{dt} u(t,0) = \int_{-1}^0 u'(t,r) \cdot (-r)^{-\alpha}\, dr, & t \ge 0, \\ u(0,0) = x_0 \end{cases}$$

with dynamic boundary conditions is well-posed.

Corollary 6.6. *Let $\alpha \in (0, 1)$. Then for all $u_0 \in \mathrm{C}[-1, 0]$ the problem*

$$\begin{cases} \frac{d}{dt} u(t,r) = \frac{d}{dr} u(t,r), & t \ge 0,\ r \in [-1, 0], \\ \frac{d}{dt} u(t,0) = \int_{-1}^0 u'(t,r) \cdot (-r)^{-\alpha}\, dr, & t \ge 0, \\ u(0,\cdot) = u_0 \end{cases}$$

with dynamic boundary conditions is well-posed.

6.3 The problems (P1) and (P2) revisited

We consider a uniformly elliptic second-order differential operator with generalized Wentzell boundary conditions on $\mathrm{C}(\bar{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with $\mathrm{C}^{1,\kappa}$-boundary $\partial\Omega$ for $\kappa > 0$.

To this end, we first take real-valued functions

$$a_{jk} = a_{kj} \in \mathrm{C}^{0,1}(\bar{\Omega}), \quad a_j \in \mathrm{C}_c(\Omega), \quad a_0, b_0 \in \mathrm{C}(\bar{\Omega}), \quad 1 \le j,k \le n$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(r) \cdot \xi_j \xi_k \ge c \cdot \|\xi\|^2 \quad \text{for all } r \in \bar{\Omega},\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

and some fixed $c > 0$. Then we define the maximal operator $A_m : D(A_m) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ in divergence form by

$$A_m f := \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk} \partial_k f \Big) + \sum_{k=1}^{n} a_k \partial_k f + a_0 f,$$
$$D(A_m) := \Big\{ f \in \bigcap_{p \ge 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\bar{\Omega}) \colon A_m f \in \mathrm{C}(\bar{\Omega}) \Big\}$$

and the boundary operator $B : D(B) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$Bf := -\sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k f + b_0 L f, \quad D(B) := \Big\{ f \in \bigcap_{p \ge 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\bar{\Omega}) \colon Bf \in \mathrm{C}(\partial\Omega) \Big\},$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the outer normal on $\partial\Omega$ and $L \in \mathcal{L}(\mathrm{C}(\bar{\Omega}), \mathrm{C}(\partial\Omega))$, $Lf := f|_{\partial\Omega}$ denotes the trace operator. Now we define the operator $A : D(A) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ with Wentzell boundary conditions by

$$A \subseteq A_m, \quad D(A) := \big\{ f \in D(A_m) \cap D(B) \colon L A_m f = Bf \big\}. \tag{6.6}$$

Theorem 6.7. *The operator A given by (6.6) generates a compact and analytic semigroup on $\mathrm{C}(\bar{\Omega})$ of optimal angle $\frac{\pi}{2}$.*

Proof. By [9, Thm. 4.2] we can assume without loss of generality $a_k = 0$ for $0 \le k \le n$.

Let $X := \mathrm{C}(\bar{\Omega})$, $\partial X := \mathrm{C}(\partial\Omega)$ and A_m and B as above. Then $A = A^B$. Now we verify the conditions from Assumptions 5.3. By [17, Cor. 3.1.21.(ii)] the operator A_0 is sectorial of angle $\frac{\pi}{2}$ on $\mathrm{C}(\bar{\Omega})$ and has compact resolvent. In particular A_0 is a weak Hille–Yosida operator on $\mathrm{C}(\bar{\Omega})$.

By [14, Thm. 9.15] and the closed graph theorem we obtain the continuous embedding

$$[D(A_0)] \hookrightarrow \mathrm{W}^{2,p}(\Omega).$$

for $p > 1$. Now Rellich's embedding theorem (see [1, Thm. 6.2, Part III]) implies

$$\mathrm{W}^{2,p}(\Omega) \overset{c}{\hookrightarrow} \mathrm{C}^{1,\alpha}(\bar{\Omega}) \hookrightarrow \mathrm{C}^1(\bar{\Omega})$$

for $p > \frac{n-1}{1-\alpha}$ where "$\overset{c}{\hookrightarrow}$" denotes a compact embedding. So we obtain

$$[D(A_0)] \overset{c}{\hookrightarrow} \mathrm{C}^1(\bar{\Omega}) \hookrightarrow \mathrm{C}(\bar{\Omega}).$$

Therefore, by Ehrling's lemma (cf. [20, Thm. 6.99]), for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_{\mathrm{C}^1(\bar{\Omega})} \le \varepsilon \|f\|_{A_0} + C_\varepsilon \|f\|_X$$

for every $f \in D(A_0)$. Since $B \in \mathcal{L}(\mathrm{C}^1(\bar{\Omega}), \mathrm{C}(\partial\Omega))$, this implies that B is relatively A_0-bounded of bound 0.

By [14, Thm. 9.18], for every $x \in \mathrm{C}(\partial\Omega)$ the problem (aDP$_\lambda$) has a unique solution $f \in D(A_m)$, hence L_0 exists. Further, by the maximum principle, cf. [14, Thm. 9.1], it is bounded.

Moreover, by [22, Thm. 1.1] the Dirichlet-to-Neumann operator $N = BL_0$ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$. Therefore, by Theorem 5.4 it follows that the operator A with Wentzell boundary conditions given by (6.6) generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\bar{\Omega})$ as claimed. ☐

Since the operator A_0 has compact resolvent, by Lemma 3.2 it follows that L_λ exists if and only if $\lambda \in \rho(A_0)$. Moreover, [9, Prop. 4.7] implies that N_λ generates a compact and analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial\Omega)$ for all $\lambda \in \rho(A_0)$. Theorem 4.3 and Theorem 5.1 now give the following.

Corollary 6.8. *For all $x_0 \in \mathrm{C}(\partial\Omega)$ the problem*

$$\begin{cases} \lambda u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) + a_0(r) \cdot u(t,r) & \text{for } t \geq 0, r \in \bar{\Omega}, \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) + b_0(s) u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ u(0,s) = x_0(s) & \text{for } s \in \partial\Omega \end{cases}$$

with dynamic boundary conditions is well-posed if and only if $\lambda \in \rho(A_0)$. In particular (P1) for $f, g = 0$ is wellposed.

Corollary 6.9. *For all $u_0 \in \mathrm{C}(\bar{\Omega})$ the problem*

$$\begin{cases} \partial_t u(t,r) = \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk}(r) \partial_k u(t,r) \Big) + \sum_{k=1}^{n} a_k(r) \partial_k u(t,r) + a_0(r) \cdot u(t,r) & \text{for } t \geq 0, r \in \bar{\Omega}, \\ \partial_t u(t,s) = - \sum_{j,k=1}^{n} a_{jk}(s) \nu_j(s) \partial_k u(t,s) + b_0(s) u(t,s) & \text{for } t \geq 0, s \in \partial\Omega, \\ u(0,r) = u_0(r) & \text{for } r \in \bar{\Omega} \end{cases}$$

with dynamic boundary conditions is well-posed. In particular (P2) for $f, g = 0$ is wellposed.

7. Conclusion and Further Remarks

In this paper we set up an abstract general framework to treat problems like (P1) and (P2) in a systematic and unified way. We showed that in many cases the dynamic boundary conditions in these problems linking the "interior" and the "boundary" dynamics can be decoupled leading to two simpler, independent problems. In case of (P1) to a "stationary" Dirichlet problem and a Cauchy problem for the Dirichlet-to-Neumann operator. For (P2) we obtain two independent Cauchy problems, one in the "interior" governed by an operator with Dirichlet boundary conditions and one on the "boundary" for the Dirichlet-to-Neumann operator.

The theory developed above can be elaborated and generalized in various ways. We close this work by indicating some recent results in this direction.

7.1 Perturbation theory for dynamic boundary conditions

In many applications the boundary operator $B : D(B) \subset X \to \partial X$ which determines the domain in (3.2) splits into a sum $B := B_0 + CL : D(B) \subset X \to \partial X$ where

$$Bf := B_0 f + CLf, \quad D(B) := D(B_0) \cap D(CL)$$

for $B_0 : D(B_0) \subset X \to \partial X$, $C\colon D(C) \subset \partial X \to \partial X$ and $D(CL) := \{f\colon Lf \in D(C)\}$. In order to perturb the action of A_m we then take a relatively A_m-bounded operator $P : D(P) \subset X \to X$ and consider $(A+P)^{B_0+CL} : D((A+P)^{B_0+CL}) \subseteq X \to X$ given by

$$\begin{aligned} (A+P)^{B_0+CL} &\subseteq A_m + P, \\ D\big((A+P)^{B_0+CL}\big) &:= \big\{ f \in D(A_m) \cap D(B_0) \cap D(CL) : L(A_m+P)f = (B_0+CL)f \big\}. \end{aligned} \tag{7.1}$$

By combining perturbation theorems for the Dirichlet- and Dirichlet-to-Neumann operators with Theorem 5.4 one can prove the following results where $N^{B_0} := B_0 L_0$, cf. [9, Sect. 4].

Theorem 7.1. *Let $P\colon D(P)\subset X\to X$ be relatively A_m-bounded with A_0-bound 0 and let $C\colon D(C)\subset \partial X\to\partial X$ be relatively N^{B_0}-bounded of bound 0. Then the following statements are equivalent.*
(a) $(A+P)^{B_0+CL}$ given in (7.1) generates an analytic semigroup of angle $\alpha>0$ on X.
(b) A^{B_0} generates an analytic semigroup of angle $\alpha>0$ on X.

Theorem 7.2. *Let $P\colon D(P)\subset X\to X$ be relatively A_m-bounded with A_0-bound 0 and let N^{B_0} be relatively C-bounded of bound 0. Then the following statements are equivalent.*
(a) $(A+P)^{B_0+CL}$ given in (7.1) generates an analytic semigroup of angle $\alpha>0$ on X.
(b) A^{CL} generates an analytic semigroup of angle $\alpha>0$ on X.

7.2 Spectral theory for dynamic boundary conditions

The decoupling of the operator A^B with Wentzell boundary conditions into the operator A_0 with Dirichlet boundary conditions and the Dirichlet-to-Neumann operator N preserves many spectral properties. For example, denoting by $\sigma(A)$ the spectrum, by $\sigma_p(A)$ the point spectrum, by $\sigma_a(A)$ the approximative point spectrum, by $\sigma_r(A)$ the residual spectrum, by $\sigma_c(A)$ the continuous spectrum and by $\sigma_{ess}(A)$ the essential spectrum of A, the following holds. Here for the definition of the various parts of the spectrum see, e.g., [3, Sect. A.3].

Theorem 7.3. *Assume that A_0 and N_{λ_0} for some $\lambda_0\in\rho(A_0)$ are weak Hille–Yosida operators. Then for $\lambda\in\rho(A_0)$ we have*
(i) $\lambda\in\rho(A^B)$ if and only if $\lambda\in\rho(N_\lambda)$. Moreover, in this case

$$R(\lambda,A^B)=R(\lambda,A_0)+L_\lambda R(\lambda,N_\lambda)\big(BR(\lambda,A_0)+L\big);$$

(ii) $\lambda\in\sigma_p(A^B)$ if and only if $\lambda\in\sigma_p(N_\lambda)$. Moreover, in this case $\dim(\lambda-A^B)=\dim(\lambda-N_\lambda)$;
(iii) $\lambda\in\sigma_a(A^B)$ if and only if $\lambda\in\sigma_a(N_\lambda)$;
(iv) $\lambda\in\sigma_r(A^B)$ if and only if $\lambda\in\sigma_r(N_\lambda)$;
(v) $\lambda\in\sigma_c(A^B)$ if and only if $\lambda\in\sigma_c(N_\lambda)$;
(vi) $\lambda\in\sigma_{ess}(A^B)$ if and only if $\lambda\in\sigma_{ess}(N_\lambda)$.

This result can be seen as an abstract characteristic equation for the spectral values of A^B. For the details we refer to [8].

7.3 Dynamic boundary conditions on L^p-spaces

The problems (P1) and (P2) from the introduction can also be treated in an abstract framework adapted towards state spaces of p-integrable functions. The main difference to state spaces of continuous functions is that the trace operator L becomes unbounded on L^p. To handle this case two modifications in our General Setting 2.1 are needed: we have to consider instead of (iii)

(iii') a surjective operator $L: D(A_m)\subset X\to\partial X$

and have to require in (ii) that

(ii') $\binom{A_m}{L}: D(A_m)\subset X\to X\times\partial X$ is closed and densely defined.

The operator $A_0: D(A_0)\subset X\to X$ with abstract Dirichlet boundary conditions is then defined as

$$A_0\subseteq A_m,\quad D(A_0)=\ker(L).$$

Note that, in contrast to our General Setting 2.1, A_0 is now densely defined which greatly simplifies the situation. The abstract Dirichlet and Dirichlet-to-Neumann operators L_λ and N_λ can be defined as in Section 2.

In this context results analogous to those in Section 4 and Section 5 hold true if in the latter the operator A^B with abstract Wentzell boundary conditions gets replaced by the operator

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$\mathcal{A}^B : D(\mathcal{A}^B) \subset X \times \partial X \to X \times \partial X$ with *dynamic boundary conditions* given by

$$\mathcal{A}^B \binom{f}{x} := \binom{A_m f}{Bf}, \quad D(\mathcal{A}^B) := \left\{ \binom{f}{x} : f \in D(A_m) \cap D(B), Lf = x \right\}.$$

Here the key observation in the study of the operator $\mathcal{A}^B$ is that it is similar to the operator matrix $\mathcal{A} : D(\mathcal{A}) \subset X \times \partial X \to X \times \partial X$ defined by

$$\mathcal{A} := \begin{pmatrix} A_0 - L_0 B & -L_0 N \\ B & N \end{pmatrix}, \quad D(\mathcal{A}) := D(A_0) \times D(N).$$

Due to its "diagonal" domain, $\mathcal{A}$ can be much easier "decoupled" as the corresponding operator $\mathcal{A}$ in Lemma 5.5. Summing up, the L^p-situation is much simpler to deal with than the one adapted for spaces of continuous functions we studied in Section 4 and Section 5.

7.4 Dynamic boundary conditions on manifolds with boundary

The example of Subsection 6.3 can be generalized to uniformly elliptic second-order differential operators on manifolds with boundary. More precisely, consider a compact, smooth, Riemannian manifold $(\bar{M}^n, g)$ with smooth boundary ∂M, embedded in $\mathbb{R}^{n+1}$. Further, consider a uniformly elliptic second-order differential operators A_m on $\mathrm{C}(\bar{M})$ and the corresponding conormal derivative B.

Using Theorem 7.1 and rewriting the operators A_m and B with respect to the Riemannian metric induced by $\tilde{g}^{ij} := a^i_k g^{kl}$ the situation becomes much simpler: It is sufficient to consider the Laplace-Beltrami operator and the normal derivative. Using a deep result of Taylor [21, App.C (C.4)] it follows similarly as in [11] that the Dirichlet-to-Neumann operator N generates a compact and analytic semigroup of optimal angle $\frac{\pi}{2}$ on $\mathrm{C}(\partial M)$. Furthermore, in [6] it is shown that A_0 is sectorial of angle $\frac{\pi}{2}$ on $\mathrm{C}(\bar{M})$. Applying Theorem 5.4 this implies that the operator $A^B : D(A^B) \subseteq \mathrm{C}(\bar{M}) \to \mathrm{C}(\bar{M})$ given by (3.2) generates a compact and analytic semigroup on $\mathrm{C}(\bar{M})$ of optimal angle $\frac{\pi}{2}$. For the details we refer to [5].

A. Appendix

Below we collect some results which were needed in the main part of this paper. First we recall that by [4, Prop. 1.1.6] the following holds for $X_1 := [D(A_m)]$.

Lemma A.1. *In addition to the General Setting 2.1, suppose that the Assumptions 4.1 are satisfied. If the function $u : \mathbb{R}_+ \to X_1$ is continuous, $u(s) \in D(B)$ for all $s \geq 0$ and $Bu : \mathbb{R}_+ \to \partial X$ is continuous then*

$$v := \int_0^t u(s)\, ds \in D(B) \quad \text{and} \quad Bv = \int_0^t Bu(s)\, ds.$$

Lemma A.2. *In the context of the General Setting 2.1, assume that B is relatively A_0-bounded and that for some $\lambda \in \rho(A_0)$ the Dirichlet-to-Neumann operator $N_\lambda = BL_\lambda$ is closed. Then the restriction $B_1 := B|_{X_1} : D(A_m) \cap D(B) \subset X_1 \to \partial X$ is closed.*

Proof. Without loss of generality assume that L_0 exists. Given $(f_n)_{n\in\mathbb{N}} \subset X_1 \cap D(B)$ such that $f_n \to f_0$ in X_1 and $Bf_n \to x_0$ in ∂X as $n \to +\infty$ we have to show that $f_0 \in D(B)$ and $Bf_0 = x_0$.

Since B is relatively A_0 bounded and $(\mathrm{Id} - L_0 L) f_n \to (\mathrm{Id} - L_0 L) f_0 \in D(A_0) \subset D(B)$ in $[D(A_0)]$ it follows that $B(\mathrm{Id} - L_0 L) f_n \to B(\mathrm{Id} - L_0 L) f_0$ as $n \to +\infty$. Hence, $L_0 L f_n \in D(B)$ for all $n \in \mathbb{N}$ and $BL_0 L f_n \to x_0 - B(\mathrm{Id} - L_0 L) f_0$ as $n \to +\infty$. Closedness of BL_0 then implies $L_0 L f_0 \in D(B)$ and $BL_0 L f_0 = x_0 - B(\mathrm{Id} - L_0 L) f_0$, hence $f_0 \in D(B)$ and $Bf_0 = x_0$. □

Lemma A.3. *Let Z be a weak Hille–Yosida operator (see Assumptions 5.3.(i)) on a Banach space X and let $P : D(Z) \subset X \to X$ be relatively Z-compact. Then also $G := Z + P$ with domain $D(G) = D(Z)$ is a weak Hille–Yosida operator.*

Proof. Let $\lambda \in \rho(Z)$ and assume without loss of generality that Z is invertible. Then for $T_\lambda := PR(\lambda, Z) \in \mathcal{L}(X)$ we obtain

$$\lambda - G = (1 - T_\lambda) \cdot (\lambda - Z). \tag{A 1}$$

Next we show that $\|T_\lambda^2\| \to 0$ as $\lambda \to +\infty$. In fact, there exist constants $\lambda_0 \in \mathbb{R}$ and $C > 0$ such that

$$\|T_\lambda^2\| \le \|PZ^{-1}\| \cdot \|ZR(\lambda, Z) \cdot PZ^{-1}\| \cdot \|ZR(\lambda, Z)\| \le C \cdot \|ZR(\lambda, Z) \cdot PZ^{-1}\| \quad \text{for all } \lambda \ge \lambda_0.$$

Now the operator family $(ZR(\lambda, Z))_{\lambda \ge \lambda_0} \subset \mathcal{L}(X)$ is bounded and converges pointwise to zero. Moreover, $PZ^{-1} \in \mathcal{L}(X)$ is compact, hence by [13, Prop. A.3] we conclude $\|ZR(\lambda, Z) \cdot PZ^{-1}\| \to 0$, hence $\|T_\lambda^2\| \to 0$ as $\lambda \to +\infty$. In particular, there exists $\mu_0 \ge \lambda_0$ such that $\|T_\lambda^2\| < \frac{1}{2}$ for $\lambda \ge \mu_0$ and by (A 1) this implies $[\mu_0, +\infty) \subset \rho(G)$ and

$$R(\lambda, G) = R(\lambda, Z) \cdot \sum_{n=0}^{+\infty} T_\lambda^n = R(\lambda, Z) \cdot \big(\mathrm{Id} + T_\lambda\big) \cdot \sum_{n=0}^{+\infty} T_\lambda^{2n}.$$

Thus, there exists a constant $K > 0$ such that

$$\big\|\lambda \cdot R(\lambda, G)\big\| \le \big\|\lambda \cdot R(\lambda, Z)\big\| \cdot (1 + \|T_\lambda\|) \cdot 2 \le K \quad \text{for all } \lambda \ge \mu_0.$$ □

Lemma A.4. *Let Z be a weak Hille–Yosida operator (see Assumptions 5.3.(i)) on a Banach space X and let $X_0 := \overline{D(Z)}$. If $P \in \mathcal{L}(X_0, X)$ then $Z_0 := Z|_{X_0}$ is a generator if and only if $G_0 := (Z + P)|_{X_0}$ is.*

Proof. Although this result basically follows from [18, Sect. 3], for completeness and since our situation is slightly different we give a complete proof. To this end we assume without loss of generality that Z is invertible, otherwise we replace Z by $Z - \lambda$ for some $\lambda \in \rho(Z)$.

We first show that for a weak Hille–Yosida operator Z the closure X_0 of its domain is dense in $(X, \|\cdot\|_{-1})$ where we define $\|x\|_{-1} := \|Z^{-1}x\|$. Let $x \in X$. Then the resolvent equation implies

$$\big\|x - \lambda R(\lambda, Z)x\big\|_{-1} = \big\|R(\lambda, Z)x\big\| \le \tfrac{M \cdot \|x\|}{\lambda} \to 0 \quad \text{as } \lambda \to +\infty.$$

Since $\lambda R(\lambda, Z)x \in X_0$ the claim follows. Hence, the completions $(X_0, \|\cdot\|_{-1})^\sim$ and $X_{-1} := (X, \|\cdot\|_{-1})^\sim$ coincide, and we obtain the continuous inclusions

$$X_0 \hookrightarrow X \hookrightarrow X_{-1}.$$

Next, $Z : D(Z) \subseteq X_0 \to X_{-1}$ is an isometry, hence admits a unique bounded extension $Z_{-1} : X_0 \to X_{-1}$. Applying [13, Lem. IV.1.15 & Prop. IV.2.17] we conclude $\rho(Z_0) = \rho(Z) = \rho(Z_{-1})$ and

$$R(\lambda, Z_0) \subseteq R(\lambda, Z) \subseteq R(\lambda, Z_{-1}) \quad \text{for } \lambda \in \rho(Z). \tag{A 2}$$

This implies for $x \in X$

$$\big\|\lambda Z_{-1} R(\lambda, Z)x\big\|_{-1} = \big\|\lambda R(\lambda, Z)x\big\| \le M \cdot \|x\| \quad \text{for } \lambda \ge \lambda_0.$$

Summing up, we proved that for a weak Hille–Yosida operator Z on X we have $X \subseteq F_{-1}$ where F_{-1} denotes the extrapolated Favard space of Z_0, cf. [13, Sect. II.5.b].

To prove the lemma we first assume that Z_0 is a generator on X_0. Then by [13, Cor. III.3.6] it follows that also $G_0 := (Z_{-1} + P)|_{X_0} = (Z + P)|_{X_0}$ is a generator on X.

For the converse implication we note that by [13, Lem. III.2.5] also $G := Z + P$ is a weak Hille–Yosida operator on X. Hence, $Z_0 = (G - P)|_{X_0}$ is a generator by the previous implication applied to G and the perturbation $-P$. □

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A.2 Submitted Manuscripts

A.2.1 Analytic semigroups generated by Dirichlet-to-Neumann operators on manifolds

ANALYTIC SEMIGROUPS GENERATED BY DIRICHLET-TO-NEUMANN OPERATORS ON MANIFOLDS

TIM BINZ

Abstract. We consider the Dirichlet-to-Neumann operator associated to a strictly elliptic operator on the space $\mathrm{C}(\partial M)$ of continuous functions on the boundary ∂M of a compact manifold $\overline{M}$ with boundary. We prove that it generates an analytic semigroup of angle $\pi/2$, generalizing and improving [Esc94] with a new proof. Our result fits with the main result in [EO19] in the case of domains with smooth boundary. Combined with [EF05, Thm. 3.1] and [Bin19] this yields that the corresponding strictly elliptic operator with Wentzell boundary conditions generates a compact and analytic semigroups of angle $\pi/2$ on the space $\mathrm{C}(\overline{M})$.

1. Introduction

Differential operators with dynamic boundary conditions on manifolds with boundary describe a system whose dynamics consisting of two parts: a dynamics on the manifold interacting with an additional dynamics on the boundary. This leads to differential operators with so called Wentzell boundary conditions, see [EF05, Sect. 2].

On spaces of continuous functions on domains in $\mathbb{R}^n$ such operators have first been studied systematically by Wentzell [Wen59] and Feller [Fel54]. Later Arendt et al. [AMPR03] proved that the Laplace operator with Wentzell boundary conditions generates a positive, contractive C_0-semigroup. Engel [Eng03] improves this by showing that this semigroup is analytic with angle of analyticity $\pi/2$. Later Engel and Fragnelli [EF05] generalize this result to uniformly elliptic operators, however without specifying the corresponding angle of analyticity. For related work see also [CT86], [CM98], [FGGR02], [CENN03], [VV03], [CENP05], [FGG$^+$10], [War10] and the references therein. Our interest in this context is the generation of an analytic semigroup with the optimal angle of analyticity.

As shown in [EF05] and [BE19] this problem is closely connected to the generation of an analytic semigroup by the Dirichlet-to-Neumann operator on the boundary space. More precisely, based on the abstract theory for boundary perturbation problems developed by Greiner in [Gre87], it has been shown in [EF05] and in [BE19] that the coupled dynamics can be decomposed into two independent parts: a dynamics on the interior and a dynamics on the boundary. The first one is described by the differential operator on the manifold with Dirichlet boundary conditions while the second is governed by the associated Dirichlet-to-Neumann operator.

On domains in $\mathbb{R}^n$ the generator property of differential operators with Dirichlet boundary conditions is quite well understood, see [Ama95] and [Lun95]. On compact Riemannian manifolds with boundary it has been shown in [Bin19] that strictly elliptic operators with Dirichlet boundary conditions are sectorial of angle $\pi/2$ and have compact resolvents on the space of continuous functions.

Dirichlet-to-Neumann operators have been studied e.g. by [US90], [LU01], [LTU03] and [Tay96, App. C]. For the operator-theoretic context see, e.g., the work of Amann and Escher

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[AE96] and Arendt and ter Elst [AE11], [AEKS14] and [AE17]. In particular, on domains in $\mathbb{R}^n$ Escher [Esc94] has shown that such Dirichlet-to-Neumann operators generate analytic semigroups on the space of continuous functions, however without specifying the corresponding angle of analyticity. Finally, ter Elst and Ouhabaz [EO19] proved that this angle is $\pi/2$ and extended the result of Escher [Esc94] to differential operators with less regular coefficients.

In this paper we study such Dirichlet-to-Neumann operators on the space of continuous functions on Riemannian manifolds and show that they generate compact and analytic semigroups of angle $\pi/2$ on the continuous functions.

We first explain our setting and terminology. Consider a strictly elliptic differential operator $A_m : D(A_m) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\overline{M})$, as given in (4.3), on the space $\mathrm{C}(\overline{M})$ of continuous functions on a smooth, compact, orientable Riemannian manifold $\overline{M}$ with smooth boundary ∂M. Moreover, let $\frac{\partial^a}{\partial \nu^g} : D(\frac{\partial^a}{\partial \nu^g}) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$ be the outer conormal derivative, $\beta > 0$ and $\gamma \in \mathrm{C}(\partial M)$. We consider $B := -\beta \cdot \frac{\partial^a}{\partial \nu^g} f + \gamma \cdot f\big|_{\partial M} : D(B) \subset \mathrm{C}(\overline{M}) \to \mathrm{C}(\partial M)$, as in (4.4), and define the operator $A^B f := A_m f$ with *Wentzell boundary conditions* by requiring

$$f \in D(A^B) \quad :\Longleftrightarrow \quad f \in D(A_m) \cap D(B) \text{ and } A_m f\big|_{\partial M} = Bf. \tag{1.1}$$

For a continuous function $\varphi \in \mathrm{C}(\partial M)$ on the boundary the corresponding Dirichlet problem

$$\begin{cases} A_m f = 0, \\ f|_{\partial M} = \varphi, \end{cases} \tag{1.2}$$

is uniquely solvable by [GT01, Cor. 9.18]. Moreover, by the maximum principle, see [GT01, Thm. 9.1], the associated solution operator $L_0 : \mathrm{C}(\partial M) \to \mathrm{C}(\overline{M})$ is bounded. Then the *Dirichlet-to-Neumann operator* is

$$N\varphi := -\beta \frac{\partial^a}{\partial \nu^g} \cdot L_0 \varphi \quad \text{ for } \varphi \in D(N) := \{\varphi \in \mathrm{C}(\partial M) \colon L_0\varphi \in D(B)\}. \tag{1.3}$$

That is, $N\varphi$ is obtained by applying the Neumann boundary operator $-\beta \frac{\partial^a}{\partial \nu^g}$ to the solution f of the Dirichlet problem (1.2).

Our main results are the following.

(i) The Dirichlet-to-Neumann operator N in (1.3) generates a compact and analytic semigroup of angle $\pi/2$ on $\mathrm{C}(\partial M)$;
(ii) the operator A^B with Wentzell boundary conditions (1.1) generates a compact and analytic semigroup of angle $\pi/2$ on $\mathrm{C}(\overline{M})$.

This extends the results from Escher [Esc94] and Engel-Fragnelli [EF05, Cor. 4.5] to elliptic operators on compact manifolds with boundaries and gives the maximal angle of analyticity $\pi/2$ in both cases. In the flat case the result for the Dirichlet-to-Neumann operator coincides with the result of ter Elst-Ouhabaz [EO19] in the smooth case. The techniques here are different and our proof is independent from theirs.

This paper is organized as follows. In Section 2 below we recall the abstract setting from [EF05] and [BE19] needed for our approach. Based on [Eng03, Sect. 2], we study in Section 3 the special case where A_m is the Laplace-Beltrami operator and B the normal derivative. In Section 4 we then generalize these results to arbitrary strictly elliptic operators and their conormal derivatives. Moreover, we use this to obtain uniqueness, existence and estimates for the solutions of the Robin-Problem. Here the main idea is to introduce a new Riemannian metric induced by the coefficients of the second order part of the elliptic operator. Then the operator takes a simpler form: Up to a relatively bounded perturbation of bound 0, it coincides with a Laplace-Beltrami operator for the new metric. Regularity and perturbation theory for operator semigroups as in [BE19, Sect. 4] then yield the first part of the main theorem in its full generality. The second part follows from [EF05, Thm. 3.1] and [Bin19, Thm. 1.1].

In this paper the following notation is used. For a closed operator $T\colon D(T)\subset X\to X$ on a Banach space X we denote by $[D(T)]$ the Banach space $D(T)$ equipped with the graph norm $\|\bullet\|_T:=\|\bullet\|_X+\|T(\bullet)\|_X$ and indicate by $\hookrightarrow$ a continuous and by $\overset{c}{\hookrightarrow}$ a compact embedding. Moreover, we use Einstein's notation of sums, i.e.,

$$x_ky^k:=\sum_{k=1}^n x_ky^k$$

for $x:=(x_1,\dots,x_n), y:=(y_1,\dots,y_n)$.

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2. The abstract Setting

The starting point of our investigation is the abstract setting proposed first in this form by [Gre87] and successfully used, e.g., in [CENN03], [CENP05] and [EF05] for the study of boundary perturbations.

Abstract Setting 2.1. Consider

(i) two Banach spaces X and ∂X, called *state* and *boundary space*, respectively;
(ii) a densely defined *maximal operator* $A_m\colon D(A_m)\subset X\to X$;
(iii) a *boundary (or trace) operator* $L\in\mathcal{L}(X,\partial X)$;
(iv) a *feedback operator* $B\colon D(B)\subseteq X\to\partial X$.

Using these spaces and operators we define the operator $A^B : D(A^B)\subset X\to X$ with *generalized Wentzell boundary conditions* by

$$A^Bf:=A_mf,\quad D(A^B):=\big\{f\in D(A_m)\cap D(B): LA_mf=Bf\big\}. \tag{2.1}$$

For our purpose we need some more operators.

Notation 2.2. We denote the (closed) kernel of L by $X_0:=\ker(L)$ and consider the restriction A_0 of A_m given by

$$A_0 : D(A_0)\subset X\to X,\quad D(A_0):=\big\{f\in D(A_m): Lf=0\big\}.$$

The *abstract Dirichlet operator associated with* A_m is, if it exists,

$$L_0^{A_m}:=\big(L|_{\ker(A_m)}\big)^{-1}\colon \partial X\to\ker(A_m)\subseteq X,$$

i.e., $L_0^{A_m}\varphi=f$ is equal to the solution of the abstract Dirichlet problem

$$\begin{cases} A_mf=0,\\ Lf=\varphi.\end{cases} \tag{2.2}$$

If it is clear which operator A_m is meant, we simply write L_0.
Moreover for $\lambda\in\mathbb{C}$ we define the *abstract Robin operator associated with* (λ,A_m,B) by

$$R_\lambda^{A_m,B}:=\big((B-\lambda L)|_{\ker(A_m)}\big)^{-1}\colon\partial X\to\ker(A_m)\cap D(B)\subseteq X,$$

i.e., $R_\lambda^{A_m,B}\varphi=f$ is equal to the solution of the abstract Robin problem

$$\begin{cases} A_mf=0,\\ Bf-\lambda Lf=\varphi.\end{cases} \tag{2.3}$$

If it is clear which operators A_m and B are meant, we simply write R_λ.

Furthermore, we introduce the *abstract Dirichlet-to-Neumann operator associated with* (A_m, B) defined by

$$N^{A_m,B}\varphi := BL_0^{A_m}\varphi, \quad D(N^{A_m,B}) := \{\varphi \in \partial X : L_0^{A_m}\varphi \in D(B)\}. \tag{2.4}$$

If it is clear which operators A_m and B are meant, we call N simply the (abstract) Dirichlet-to-Neumann operator. This Dirichlet-to-Neumann operator is an abstract version of the operators studied in many places, e.g., [Esc94], [Tay96, Sect. 7.11] and [Tay81, Sect. II.5.1].
The Dirichlet-to-Neumann and the Robin operator are connected in the following way.

Lemma 2.3. *If L_0 exists, we have $\lambda \in \rho(N^{A_m,B})$ if and only if $R_\lambda^{A_m,B} \in \mathcal{L}(\partial X, X)$ exists. If one of these conditions is satisfied, we obtain*

$$R_\lambda^{A_m,B} = -L_0 R(\lambda, N^{A_m,B}).$$

Proof. Assume that $R_\lambda \in \mathcal{L}(\partial X, X)$ exists. By the definition of N the equation

$$\lambda\varphi - N\varphi = \psi$$

for $\varphi, \psi \in \partial X$ is equivalent to

$$\lambda L L_0\varphi - BL_0\varphi = \psi \tag{2.5}$$

for $\varphi, \psi \in \partial X$. This again is equivalent to

$$-R_\lambda\psi = L_0\varphi.$$

Therefore, we have for $\varphi, \psi \in \partial X$ the equivalence

$$\mu\varphi - N\varphi = \psi \quad \iff \quad R_\lambda\psi = -L_0\varphi.$$

Since $R_{\lambda,\mu} : \partial X \to \ker(A_m) \cap D(B)$ exists and $L_0 : \partial X \to \ker(A_m)$ is an isomorphism, there exists a unique $\varphi \in D(N)$ for every $\psi \in \partial X$. Moreover its given by $\phi = -LR_{\lambda,\mu}\psi$ and therefore the boundedness of the inverse follows from the boundedness of L and R_λ. The formula for the resolvent of N follows, since $L|_{\ker(A_m)}$ is an isomorphism with inverse L_0 and the image of R_λ is contained in $\ker(A_m)$.

Conversely, we assume that $\mu \in \rho(N)$. Then (2.5) has a unique solution $\varphi \in D(N)$ for every $\psi \in \partial X$. Considering $f := -L_0\varphi$ we obtain a unique solution of (2.3) and hence R_λ exists. Boundedness follows from $R_\lambda = -L_0 R(\mu, N)$. ☐

3. Boundary problems for the Laplace-Beltrami operator

In order to obtain a concrete realization of the above abstract objects we consider a smooth, compact, orientable Riemannian manifold $(\overline{M}, g)$ with smooth boundary ∂M, where g denotes the Riemannian metric. Moreover, we take the Banach spaces $X := \mathrm{C}(\overline{M})$ and $\partial X = \mathrm{C}(\partial M)$ and as the maximal operator the Laplace-Beltrami operator

$$A_m f := \Delta_M^g f, \quad D(A_m) := \Big\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon \Delta_M^g f \in \mathrm{C}(\overline{M}) \Big\}. \tag{3.1}$$

As feedback operator we take the normal derivative

$$Bf := -g\big(\nabla_M^g f, \nu_g\big), \quad D(B) := \Big\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon Bf \in \mathrm{C}(\partial M) \Big\}, \tag{3.2}$$

where ∇_M^g denotes the gradient on M, which in local coordinates is given as

$$\big(\nabla_M^g f\big)^l = g^{kl}\partial_k f$$

for $f \in \bigcap_{p>1} W^{1,p}(M)$. Moreover, ν_g is the outer normal on ∂M given in local coordinates by

$$\nu_g^l = g^{kl}\nu_k.$$

Furthermore, we choose L as the trace operator, i.e.,

$$L\colon X \to \partial X,\ f \mapsto f|_{\partial M},$$

which is bounded with respect to the supremum norm. Later on we will also need the unique bounded extension of L to $\mathrm{W}^{1,2}(M)$, denoted by $\overline{L}\colon \mathrm{W}^{1,2}(M) \to \mathrm{L}^2(\partial M)$, and call it the (generalized) trace operator.

3.1. **The Laplace-Beltrami operator with Robin boundary conditions.**

In this setting we consider the Laplace-Beltrami operator with Robin boundary conditions and prove existence, uniqueness and regularity for the solution of (2.3). Moreover, we show that this solution satisfies a maximum principle.

For this purpose we need the concept of a weak solution of (2.3). If $f \in D(A_m) \cap D(B)$ is a solution of (2.3) we obtain by Green's Identity

$$\int_M g\big(\nabla_M^g f, \nabla_M^g \overline{\phi}\big)\ \mathrm{dvol}_M^g = -\int_{\partial M} Bf\overline{L\phi}\ \mathrm{dvol}_{\partial M}^g = -\int_{\partial M} \lambda\overline{L}f\overline{L\phi}\ \mathrm{dvol}_{\partial M}^g - \int_{\partial M} \varphi\overline{L\phi}\,\mathrm{dvol}_{\partial M}^g$$

for all $\phi \in \mathrm{W}^{1,2}(M)$. This motivates the following definition.

Definition 3.1 (Weak solution of the Robin Problem)**.** We call $f \in \mathrm{W}^{1,2}(M)$ a *weak solution of* (2.3) if it satisfies

$$\mathfrak{a}(f,\phi) := \int_M g\big(\nabla_M^g f, \nabla_M^g \overline{\phi}\big)\ \mathrm{dvol}_M^g + \int_{\partial M} \lambda\overline{L}f\overline{L\phi}\ \mathrm{dvol}_{\partial M}^g = -\int_{\partial M} \varphi\overline{L\phi}\,\mathrm{dvol}_{\partial M}^g =: F(\phi)$$

for all $\phi \in \mathrm{W}^{1,2}(M)$.

Definition 3.2. We call $f \in D(A_m) \cap D(B)$ a *strong solution of* (2.3) if it satisfies (2.3).

Next we prove the existence of such weak solutions.

Lemma 3.3 (Existence and Uniqueness of the weak solution of the Robin problem)**.** *For each* $\mathrm{Re}(\lambda) > 0$ *and each* $\varphi \in \mathrm{W}^{1/2,2}(\partial M)$ *the problem* (2.3) *has a unique weak solution.*

Proof. We consider $\mathfrak{a}$ and F as defined above. Obviously $\mathfrak{a}$ is sesquilinear and F is linear. By the Cauchy-Schwarz Inequality we have for $f, \phi \in \mathrm{W}^{1,2}(M)$ that

$$|\mathfrak{a}(f,\phi)| \le \|\nabla_M^g f\|_{\mathrm{L}^2(M)}\|\nabla_M^g \phi\|_{\mathrm{L}^2(M)} + |\lambda|\|\overline{L}f\|_{\mathrm{L}^2(\partial M)}\|\overline{L}\phi\|_{\mathrm{L}^2(\partial M)} \le C\|f\|_{\mathrm{W}^{1,2}(M)}\|\phi\|_{\mathrm{W}^{1,2}(M)},$$

hence $\mathfrak{a}\colon \mathrm{W}^{1,2}(M) \times \mathrm{W}^{1,2}(M) \to \mathbb{C}$ is bounded. Next we show that $\mathfrak{a}$ is coercive. If not, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}^{1,2}(M)$ such that

$$\|u_k\|_{\mathrm{W}^{1,2}(M)}^2 > k\,\mathrm{Re}\big(\mathfrak{a}(u_k,u_k)\big)$$

for all $k \in \mathbb{N}$. We consider

$$v_k := \frac{v_k}{\|v_k\|_{\mathrm{W}^{1,2}(M)}} \in \mathrm{W}^{1,2}(M)$$

and remark that $\|v_k\|_{\mathrm{W}^{1,2}(M)} = 1$ and therefore

$$\mathrm{Re}\big(\mathfrak{a}(v_k,v_k)\big) < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Since $(v_k)_{k\in\mathbb{N}}$ is bounded, by Rellich-Kondrachov (cf. [Heb96, Cor. 3.7]) there exists a subsequence $(v_{k_l})_{l\in\mathbb{N}}$ converging in $\mathrm{L}^2(M)$ to $v \in \mathrm{L}^2(M)$. On the other hand we have

$$\|\nabla_M^g v_{k_l}\|_{\mathrm{L}^2(M)} \le \mathrm{Re}\big(\mathfrak{a}(v_{k_l},v_{k_l})\big) < \frac{1}{k_l},$$

hence $(\nabla_M^g v_{k_l})_{l\in\mathbb{N}}$ converges to 0 in $\mathrm{L}^2(M)$. This shows $v \in \mathrm{W}^{1,2}(M)$ and $\nabla_M^g v = 0$. Moreover, we obtain

$$\|\nabla_M^g v_{k_l}\|_{\mathrm{L}^2(M)} = \int_M g_{ij}g^{ir}g^{js}\partial_r v_{k_l}\partial_s v_{k_l}\,\mathrm{dvol}_M^g = \int_M g^{rs}\partial_r v_{k_l}\partial_s v_{k_l}\,\mathrm{dvol}_M^g = \|\nabla v_{k_l}\|_{\mathrm{L}^2(M)},$$

where ∇v_{k_l} denotes the covariant derivative of v_{k_l}. Therefore, $(v_{k_l})_{l\in\mathbb{N}}$ converges in $\mathrm{W}^{1,2}(M)$ to v with $\|v\|_{\mathrm{W}^{1,2}(M)} = 1$. Moreover, we have

$$\|\overline{L}v_{k_l}\|_{\mathrm{L}^2(\partial M)} < \frac{1}{\mathrm{Re}(\lambda)k_l}$$

and therefore

$$\|\overline{L}v\|_{\mathrm{L}^2(\partial M)} \leq \|\overline{L}v - \overline{L}v_{k_l}\|_{\mathrm{L}^2(\partial M)} + \|\overline{L}v_{k_l}\|_{\mathrm{L}^2(\partial M)} < \frac{1}{\mathrm{Re}(\lambda)k_l} + C\|v - v_{k_l}\|_{\mathrm{W}^{1,2}(M)} \longrightarrow 0$$

and hence $\overline{L}v = 0$. Since $\nabla v = 0$, we conclude $v = 0$, which contradicts $\|v\|_{\mathrm{W}^{1,2}(M)} = 1$. Hence, $\mathfrak{a}$ is coercive. Since

$$|F(\phi)| \leq \|\varphi\|_{\mathrm{L}^2(\partial M)}\|\overline{L}\phi\|_{\mathrm{L}^2(\partial M)} \leq C\|\phi\|_{\mathrm{W}^{1,2}(\partial M)}$$

for all $\phi \in \mathrm{W}^{1,2}(M)$ we conclude that $F\colon \mathrm{W}^{1,2}(M) \to \mathbb{C}$ is bounded. By the Lax-Milgram and Fréchet-Riesz theorems it follows that $\alpha(f,\phi) = F(\phi)$ for all $\phi \in \mathrm{W}^{1,2}(M)$ has a unique solution $f \in \mathrm{W}^{1,2}(M)$. $\square$

Next we prove that every weak solution is even a strong solution.

Lemma 3.4 (Regularity of the Robin problem). *If $\varphi \in \mathrm{C}(\partial M)$, every weak solution of (2.3) is a strong solution.*

Proof. By [Tay96, Chap. 5., Prop. 1.6] we have $f \in \mathrm{C}^2(M) \subset \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M)$. Therefore, we obtain by the fundamental lemma of the calculus of variation that $\Delta^g_M f = 0$, in particular $\Delta^g_M f \in \mathrm{C}(\overline{M})$. Furthermore we have

$$Bf = \lambda Lf + \varphi \in \mathrm{C}(\partial M).$$

$\square$

Moreover we need a maximum principle for the Robin problem.

Lemma 3.5. *A solution $f \in D(A_m) \cap D(B) \subset X$ of (2.3) satisfies the maximum principle*

$$|\operatorname{Re}(\lambda)| \cdot \|f\|_X \leq \|\varphi\|_{\partial X}$$

for all $\mathrm{Re}(\lambda) \geq 0$ and $\varphi \in \partial X = \mathrm{C}(\partial M)$.

Proof. We consider a point $p \in \overline{M}$, where $|f|$ and therefore $|f|^2$ assumes its maximum. By the interior maximum principle (cf. [GT01, Thm. 9.1]) it follows that $p \in \partial M$. Hence, we have

$$g(p)\big(\nabla^g_M |f|^2(p), \nu_g(p)\big) \geq 0.$$

From

$$\begin{aligned} g\big(\nabla^g_M |f|^2, \nu_g\big) &= g\big(\nabla^g_M (f\overline{f}), \nu_g\big) = 2\operatorname{Re} g\big((\nabla^g_M f)\overline{f}, \nu_g\big) = 2\operatorname{Re}\Big(g\big((\nabla^g_M f), \nu_g\big)\overline{f}\Big) \\ &= -2\operatorname{Re}\big((Bf)\overline{f}\big) = -2\operatorname{Re}\big(\varphi\overline{f}\big) - 2\operatorname{Re}(\lambda)|f|^2, \end{aligned}$$

we obtain

$$\operatorname{Re}(\lambda)|f|^2(p) \leq -\operatorname{Re}\big(\varphi(p)\overline{f}(p)\big) \leq |\varphi|(p)|f|(p).$$

Since $\operatorname{Re}(\lambda) \geq 0$, this implies

$$|\operatorname{Re}(\lambda)| \cdot \|f\|_X = |\operatorname{Re}(\lambda)| \cdot |f|(p) \leq |\varphi|(p) \leq \|\varphi\|_{\partial X}.$$

$\square$

Summing up we obtain the following.

Corollary 3.6 (Existence and Uniqueness of the solution of the Robin problem). *For all $\mathrm{Re}(\lambda) > 0$ and $\varphi \in \mathrm{C}(\partial M)$ the problem (2.3) has a unique solution.*

Proof. If $\varphi \in \mathrm{W}^{1/2,2}(\partial M) \cap \mathrm{C}(\partial M)$ the claim follows by combining Lemma 3.3 and Lemma 3.4. For general $\varphi \in \mathrm{C}(\partial M)$, the claim follows by density of $\mathrm{W}^{1/2,2}(\partial M) \cap \mathrm{C}(\partial M) \subset \mathrm{C}(\partial M)$ and the maximum principle Lemma 3.5. $\square$

3.2. **Generator property for the Dirichlet-to-Neumann operator.**
Now we are able to prove our main result: The Dirichlet-to-Neumann operator generates a contractive and analytic semigroup of angle $\pi/2$ on $\partial X = \mathrm{C}(\partial M)$. To do so we represent the Dirichlet-to-Neumann operator as a relatively bounded perturbation of $-\sqrt{-\Delta^g_{\partial M}}$.

We first need the existence of the associated Dirichlet operator.

Lemma 3.7. *The Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists.*

Proof. This follows by [Tay96, Chap. 5. (2.26)], [GT01, Thm. 9.19] and [GT01, Thm 9.1]. □

Next we prove a first generation result for the Dirichlet-to-Neumann operator.

Proposition 3.8. *The Dirichlet-to-Neumann operator N defined in* (2.4) *generates a contraction semigroup on ∂X.*

Proof. By elliptic regularity theory (cf. [Tay96, Chap. 5.5. Ex. 2]), we have the inclusions

$$L_0\mathrm{C}^2(\partial M) \subset \mathrm{C}^1(\overline{M}) \subset D(B).$$

Since $\mathrm{C}^2(\partial M)$ is dense in ∂X, N is densely defined. By Lemma 2.3 and Corollary 3.6 it follows that the resolvent $R(\lambda, N)$ exists for all $\mathrm{Re}(\lambda) > 0$. By the interior maximum principle $L|_{\ker(A_m)} \colon \ker(A_m) \subset X \to \partial X$ is an isometry. Therefore, Lemma 2.3 and Lemma 3.5 imply

$$\left\|R(\lambda, N)\varphi\right\|_{\partial X} \leq \frac{1}{|\mathrm{Re}(\lambda)|}\|\varphi\|_{\partial X}$$

for all $\mathrm{Re}(\lambda) > 0$ and $\varphi \in \partial X$. Hence, the claim follows by the Hille-Yosida Theorem (cf. [EN00, Thm. II.3.5]). □

Now we prove the main result of this subsection.

Theorem 3.9. *The Dirichlet-to-Neumann operator N given by* (2.4) *for* (3.1) *and* (3.2) *generates an analytic semigroup of angle $\pi/2$ on ∂X.*

We proceed as in the proof of [Eng03, Thm. 2.1].Let $\overline{N}$ and $\overline{W}$ be the closure of N and W, respectively, in $Y := \mathrm{L}^2(\partial M)$. Moreover we need results from the theory of pseudo differential operators. We use the notation from [Tay81] and denote by $\mathrm{OPS}^k(\partial M)$ the pseudo differential operators of order $k \in \mathbb{Z}$ on ∂M.

Step 1. *Then the part $\overline{N}|_{\partial X}$ coincides with N.*

Proof. By Proposition 3.8 the Dirichlet-to-Neumann operator N is densely defined and $\lambda - N$, considered as an operator on Y, has dense range $\mathrm{rg}(\lambda - N) = \partial X \subset Y$ for all $\lambda > 0$. By Green's Identity we have

$$\int_M g\big(\nabla^g_M f, \nabla^g_M f\big)\,\mathrm{dvol}_M + \int_M f\Delta_M f\,\mathrm{dvol}_M = \int_{\partial M} g\big(\nabla^g_M f, \nu_g\big) Lf\,\mathrm{dvol}_{\partial M}.$$

Hence, for $f := L_0^{A_m}\varphi$ with $\varphi \in D(N)$ we obtain

$$0 \leq \int_M g\big(\nabla^g_M f, \nabla^g_M f\big)\,\mathrm{dvol}_M = -\int_{\partial M} \varphi N\varphi\,\mathrm{dvol}_{\partial M}$$

since $\Delta^g_M f = 0$. Hence, N as an operator on Y is dissipative. By the Lumer-Phillips theorem (see [EN00, Thm. II.3.15]) the closure $\overline{N}$ of N exists and generates a contraction semigroup on Y. This implies that on ∂X we have

$$(1 - N) \subseteq (1 - \overline{N})|_{\partial X},$$

where $1 - N$ is surjective and $1 - \overline{N}$ is injective on ∂X. This is possible only if the domains $D(1 - N)$ and $D(1 - \overline{N})$ coincide, i.e., $\overline{N}|_{\partial X} = N$. □

Step 2. *The operator $W := -\sqrt{-\Delta^g_{\partial M}}$ generates an analytic semigroup of angle $\pi/2$ on ∂X.*

Proof. The Laplace-Beltrami operator $\Delta^g_{\partial M}$ generates an analytic semigroup of angle $\pi/2$ on $\mathrm{C}(\partial M) = \partial X$. Hence, the assertion follows by [ABHN11, Thm. 3.8.3]. $\square$

Step 3. *The operator $\overline{W} := -\sqrt{-\overline{\Delta^g_{\partial M}}}$ satisfies $W = \overline{W}|_{\partial X}$.*

Proof. By [Tay81, Chap. 8, Prop. 2.4] the space $\mathrm{C}^\infty(\partial M)$ is a core for W and by [ABHN11, Prop. 3.8.2] the domain $D(\overline{\Delta^g_{\partial M}})$ is a core for $\overline{W}$. Hence, $\mathrm{C}^\infty(\partial M)$ is a core for $\overline{W}$ and since $\mathrm{C}^\infty(\partial M) \subset D(W)$ we obtain that $D(W)$ is a core for $\overline{W}$ on Y. This implies that $\overline{W}$ is indeed the closure of W in Y. Moreover, we obtain

$$(1 - W) \subseteq (1 - \overline{W})|_{\partial X},$$

where $1-W$ is surjective and $1-\overline{W}$ is injective on ∂X. This is possible only if for the domains we have

$$D(1 - W) = D(1 - \overline{W}),$$

i.e., $\overline{W}|_{\partial X} = W$. $\square$

Step 4. *The domain of W can be compactly embedded into the Hölder continuous functions, i.e., $[D(W)] \overset{c}{\hookrightarrow} \mathrm{C}^\alpha(M)$ for all $\alpha \in (0, 1)$.*

Proof. Consider $\overline{R} := (1 + \overline{W})^{-1}$. Then, by [Tay81, Chap. XII.1], $\overline{R} \in \mathrm{OPS}^{-1}(\partial M)$ and since $\varphi \in \partial X = \mathrm{C}(\partial M)$ we have by [Tay81, Chap. XI, Thm. 2.5] that $\overline{R}\varphi \in \mathrm{W}^{1,p}(\partial M)$ for all $p > 1$. Hence, $D(W) = \overline{R}\mathrm{C}(\partial M) \subset \mathrm{W}^{1,p}(\partial M)$. Moreover, by Sobolev embedding (see [Ada75, Chap. V. and Rem. 5.5.2])

$$\mathrm{W}^{1,p}(\partial M) \hookrightarrow \mathrm{C}(\partial M)$$

for $p > n - 1$. By the closed graph theorem we obtain

$$[D(W)] \hookrightarrow \mathrm{W}^{1,p}(\partial M)$$

for $p > n - 1$. Since Rellich's embedding (see [Ada75, Thm. 6.2, Part III.]) implies

$$\mathrm{W}^{1,p}(\partial M) \overset{c}{\hookrightarrow} \mathrm{C}^\alpha(\partial M)$$

for $p > \frac{n-1}{1-\alpha}$, the claim follows. $\square$

Step 5. *The difference $\overline{P} := \overline{N} - \overline{W} \in OPS^0(\partial M)$ is a pseudo differential operator of order 0. Moreover, $\overline{P}$ considered as an operator on Y is bounded.*

Proof. This follows from [Tay96, App. C, (C.4)] and [Tay81, Chap. XI, Thm. 2.2]. $\square$

Step 6. *The part $P := \overline{P}|_{\mathrm{C}^\alpha(\partial M)} \colon \mathrm{C}^\alpha(\partial M) \to \mathrm{C}^\alpha(\partial M)$ is bounded. Moreover, the operator P considered on ∂X is relatively W-bounded with bound 0.*

Proof. Form [Tay81, Chap. XI, Thm 2.2] it follows $P \in \mathcal{L}(\mathrm{C}^\alpha(\partial M))$. By **Step 4** we have

$$[D(W)] \overset{c}{\hookrightarrow} \mathrm{C}^\alpha(\partial M) \hookrightarrow \mathrm{C}(\partial M). \tag{3.3}$$

Therefore, by Ehrling's lemma (cf. [RR04, Thm. 6.99]), for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|\varphi\|_{\mathrm{C}^\alpha(\partial M)} \le \varepsilon\|\varphi\|_W + C_\varepsilon\|\varphi\|_\infty$$

for every $x \in D(W)$, i.e. P is relatively W-bounded with bound 0. $\square$

Step 7. *(Proof of Theorem 3.9)*

Proof. First we note that by **Step 5** we have

$$\overline{N} = \overline{W} - \overline{P},$$

and therefore using the **Steps 1, 3, 6** it follows that

$$N = \overline{N}|_{\partial X} = (\overline{W} - \overline{P})|_{\partial X} \supseteq \overline{W}|_{\partial X} - P = W - P. \tag{3.4}$$

On the other hand, by **Steps 2, 6** and [EN00, Lem. III.2.6], $W - P$ generates an analytic semigroup of angle $\pi/2$ on ∂X. Moreover, $\lambda \in \rho(N) \cap \rho(W - P)$ for λ large enough. This implies equality in (3.4) and hence the claim. □

Remark 3.10. After we finished this paper we have become in mind a different proof of Theorem 3.9 based on the work of ter Elst and Ouhabaz in [EO14].

First, note that by the remark at the end of [EO14, Sect. 1] all results in [EO14] still be true on Riemannian manifolds. Applying the same arguments as in the proof of [EO19, Prop. 2.3], using [EO14, Thm. 2.6] instead of [EO19, Thm. 2.1], the Dirichlet-to-Neumann operator generates a strongly continuous semigroup on $\mathrm{C}(\partial M)$. Using [EO14, Cor. 5. 14] one obtains in the same way as the proof of [EO19, Prop. 3.3] that the Dirichlet-to-Neumann operator generates a holomorphic semigroup of angle $\pi/2$ on $\mathrm{C}(\partial M)$. Combining these two results it follows that the Dirichlet-to-Neumann operator generates an analytic semigroup of angle $\pi/2$ on $\mathrm{C}(\partial M)$.

Corollary 3.11. *The Dirichlet-to-Neumann operator generates a compact semigroup on* $\mathrm{C}(\partial M)$.

Proof. By (3.3) the operator W has compact resolvent. Since the Dirichlet-to-Neumann operator N and W differ only by a relatively bounded perturbation of bound 0, it has compact resolvent by [EN00, III.-(2.5)]. Hence the claim follows by Theorem 3.9 and [EN00, Thm. II.4.29]. □

Remark 3.12. We can insert a strictly positive function $0 < \beta \in \mathrm{C}(\partial M)$ and consider $\tilde{B} := \beta \cdot B$. Then by multiplicative perturbation theory (cf. [Hol92, Sect. III.1]) the same generation result as above holds true.

3.3. The Laplace-Beltrami operator with Wentzell boundary conditions.

In this subsection we study the Laplace-Beltrami operator with Wentzell boundary conditions and prove that it generates an analytic semigroup of angle $\pi/2$ on $X = \mathrm{C}(\overline{M})$. To show this, we verify the assumptions of [BE19, Thm. 3.1].

Lemma 3.13. *The feedback operator B is relatively A_0-bounded with bound* 0.

Proof. By [Tay96, Chap. 5., Thm. 1.3] and the closed graph theorem we obtain

$$[D(A_0)] \hookrightarrow \mathrm{W}^{2,p}(M).$$

Rellich's embedding (see [Ada75, Thm. 6.2, Part III.]) implies

$$\mathrm{W}^{2,p}(M) \overset{c}{\hookrightarrow} \mathrm{C}^{1,\alpha}(M) \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{M})$$

for $p > \frac{m-1}{1-\alpha}$, so we obtain

$$[D(A_0)] \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{M}) \hookrightarrow \mathrm{C}(\overline{M}).$$

Therefore, by Ehrling's lemma (cf. [RR04, Thm. 6.99]), for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_{\mathrm{C}^1(\overline{M})} \le \varepsilon \|f\|_{A_0} + C_\varepsilon \|f\|_X$$

for every $f \in D(A_0)$. Since $B \in \mathcal{L}(\mathrm{C}^1(\overline{M}), \partial X)$, this implies the claim. □

Now we prove the generator result for the operator with Wentzell boundary conditions.

Theorem 3.14. *The operator A^B with Wentzell boundary conditions given by* (2.1) *for* (3.1) *and* (3.2) *generates a compact and analytic semigroup of angle $\pi/2$ on $X = \mathrm{C}(\overline{M})$.*

Proof. We verify the assumptions from [EF05, Thm. 3.1]. The operator A_0 with Dirichlet boundary conditions is sectorial of angle $\pi/2$ with compact resolvent by [Bin19, Thm. 2.8] and [Bin19, Cor. 3.4]. Moreover the Dirichlet operator L_0 exists by Lemma 3.7 and the feedback operator B is relatively A_0-bounded of bound 0 by Lemma 3.13. Lastly, the Dirichlet-to-Neumann operator N generates a compact and analytic semigroup of angle $\pi/2$ on $\mathrm{C}(\partial M)$ by Theorem 3.9 and Corollary 3.11. Now the claim follows from [EF05, Thm. 3.1]. □

Remark 3.15. As in Remark 3.12 we can insert a strictly positive, continuous function $\beta > 0$ and the same result as Theorem 3.14 becomes true.

4. Strictly elliptic operators on continuous functions on a compact manifold with boundary

In this section we consider strictly elliptic second-order differential operators with generalized Wentzell boundary conditions on $\tilde{X} := \mathrm{C}(\overline{M})$ for a smooth, compact, orientiable, Riemannian manifold $(\overline{M}, g)$ with smooth boundary ∂M. To this end, we take real-valued functions

$$a_j^k = a_k^j \in \mathrm{C}^\infty(\overline{M}), \quad b_j \in \mathrm{C}_c(M), \quad c, d \in \mathrm{C}(\overline{M}) \quad 1 \leq j, k \leq n, \tag{4.1}$$

satisfying the strict ellipticity condition

$$a_j^k(q) g^{jl}(q) X_k(q) X_l(q) > 0$$

for all co-vectorfields X_k, X_l on $\overline{M}$ with $(X_1(q), \ldots, X_n(q)) \neq (0, \ldots, 0)$. Then we define the maximal operator in divergence form as

$$\tilde{A}_m f := \sqrt{|a|}\operatorname{div}_g \left(\frac{1}{\sqrt{|a|}} a \nabla_M^g f \right) + \langle b, \nabla_M^g f \rangle + cf, \tag{4.2}$$

$$D(\tilde{A}_m) := \left\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon \tilde{A}_m f \in \mathrm{C}(\overline{M}) \right\}. \tag{4.3}$$

As feedback operator we take

$$\tilde{B} f := -g(a \nabla_M^g f, \nu_g) + dLf, \quad D(\tilde{B}) := \left\{ f \in \bigcap_{p>1} \mathrm{W}^{2,p}_{\mathrm{loc}}(M) \cap \mathrm{C}(\overline{M}) \colon \tilde{B} f \in \mathrm{C}(\partial M) \right\}. \tag{4.4}$$

Corresponding to L we choose $\partial \tilde{X} := \mathrm{C}(\partial M^g)$.

The key idea is to reduce the strictly elliptic operator and the conormal derivative on $\overline{M}$, equipped by g, to the Laplace-Beltrami operator and to the normal derivative on $\overline{M}$, endowed by a new metric $\tilde{g}$.

For this purpose we consider a $(2,0)$-tensorfield on $\overline{M}$ given by

$$\tilde{g}^{kl} = a_i^k g^{il}.$$

Its inverse $\tilde{g}$ is a $(0,2)$-tensorfield on $\overline{M}$, which is a Riemannian metric since $a_j^k g^{jl}$ is strictly elliptic on $\overline{M}$. We denote $\overline{M}$ with the old metric by $\overline{M}^g$ and with the new metric by $\overline{M}^{\tilde{g}}$ and remark that $\overline{M}^{\tilde{g}}$ is a smooth, compact, orientable Riemannian manifold with smooth boundary ∂M. Since the differentiable structures of $\overline{M}^g$ and $\overline{M}^{\tilde{g}}$ coincide, the identity

$$\mathrm{Id} \colon \overline{M}^g \longrightarrow \overline{M}^{\tilde{g}}$$

is a C^∞-diffeomorphism. Hence, the spaces

$$X := C(\overline{M}) := C(\overline{M}^{\tilde{g}}) = C(\overline{M}^{g}) = \tilde{X}$$
$$\text{and} \quad \partial X := C(\partial M) := C(\partial M^{\tilde{g}}) = C(\partial M^{g}) = \partial\tilde{X}$$

coincide. Moreover, [Heb00, Prop. 2.2] implies that the spaces

$$\begin{aligned} L^p(M) &:= L^p(M^{\tilde{g}}) = L^p(M^g), \\ W^{k,p}(M) &:= W^{k,p}(M^{\tilde{g}}) = W^{k,p}(M^g), \\ L^p_{loc}(M) &:= L^p_{loc}(M^{\tilde{g}}) = L^p_{loc}(M^g), \\ W^{k,p}_{loc}(M) &:= W^{k,p}_{loc}(M^{\tilde{g}}) = W^{k,p}_{loc}(M^g) \end{aligned} \tag{4.5}$$

for all $p > 1$ and $k \in \mathbb{N}$ coincide. We now denote by A_m and B the operators defined as in **Section 3** with respect to $\tilde{g}$. Moreover we denote $\hat{A}_m$ the operator defined in (4.3) for $b_k = c = 0$.

4.1. The associated Dirichlet-to-Neumann operator and the Robin problem.

In this subsection we study the Dirichlet-to-Neumann operator $N^{\tilde{A}_m,\tilde{B}}$ associated with $\tilde{A}_m$ and $\tilde{B}$. First we prove that the generator properties of the Dirichlet-to-Neumann operators associated with $(\tilde{A}_m, \tilde{B})$ and (A_m, B) are closely related.

Lemma 4.1. *The operators $\hat{A}_m$ and $\tilde{A}_m$ differ only by a relatively A_m-bounded perturbation of bound* 0.

Proof. From (4.5) we define

$$P_1 f := b_l g^{kl} \partial_k f$$

for $f \in D(A_m) \cap D(\hat{A}_m)$. Morreys embedding (cf. [Ada75, Chap. V. and Rem. 5.5.2]) implies

$$[D(\hat{A}_m)] \overset{c}{\hookrightarrow} C^1(M) \hookrightarrow C(M). \tag{4.6}$$

Since $b_l \in C_c(M)$ we obtain

$$\begin{aligned} \|P_1 f\|_{C(\overline{M})} &\leq \sup_{q \in \overline{M}} |b_l(q) g^{kl}(q)(\partial_k f)(q)| \\ &= \sup_{q \in M} |b_l(q) g^{kl}(q)(\partial_k f)(q)| \\ &\leq C \sum_{k=1}^{n} \|\partial_k f\|_{C(M)} \end{aligned}$$

and therefore $P_1 \in \mathcal{L}(C^1(M), C(\overline{M}))$. Hence $D(\hat{A}_m) = D(\tilde{A}_m)$. By (4.6) we conclude from Ehrling's Lemma (see [RR04, Thm. 6.99]) that

$$\begin{aligned} \|P_1 f\|_{C(\overline{M})} \leq C\|f\|_{C^1(M)} &\leq \varepsilon \|\hat{A}_m f\|_{C(\overline{M})} + \varepsilon\|f\|_{C(\overline{M})} + C(\varepsilon)\|f\|_{C(M)} \\ &\leq \varepsilon \|\hat{A}_m f\|_{C(\overline{M})} + \tilde{C}(\varepsilon)\|f\|_{C(\overline{M})} \end{aligned}$$

for $f \in D(\hat{A}_m)$ and all $\varepsilon > 0$ and hence P_1 is relatively A_m-bounded of bound 0. Finally, remark that

$$P_2 f := c \cdot f, \quad D(P_2) := C(\overline{M})$$

is bounded and that

$$\tilde{A}_m f = \hat{A}_m f + P_1 f + P_2 f$$

for $f \in D(\hat{A}_m)$. $\square$

Lemma 4.2. *The operator $\hat{A}_m$ equals to the Laplace-Beltrami operator $\Delta_m^{\tilde{g}}$.*

Proof. We calculate in local coordinates

$$\begin{aligned}\hat{A}_m f &= \frac{1}{\sqrt{|g|}}\sqrt{|a|}\partial_j\left(\sqrt{|g|}\frac{1}{\sqrt{|a|}}a_l^j g^{kl}\partial_k f\right)\\ &= \frac{1}{\sqrt{|\tilde{g}|}}\partial_j\left(\sqrt{|\tilde{g}|}\tilde{g}^{kl}\partial_k f\right) = \Delta_m^{\tilde{g}} f\end{aligned}$$

for $f \in D(\hat{A}_m) = D(\Delta_m^{\tilde{g}})$, since $|g| = |a| \cdot |\tilde{g}|$. □

Lemma 4.3. *The operators B and $\tilde{B}$ differ only by a bounded perturbation.*

Proof. Since the Sobolev spaces coincide, we compute in local coordinates

$$\begin{aligned}\tilde{B}f &= -g_{ij}g^{jl}a_l^k\partial_k f g^{im}\nu_m + dLf\\ &= -g_{ij}\tilde{g}^{jl}\partial_k f g^{im}\nu_m + b_0 Lf\\ &= -\tilde{g}_{ij}\tilde{g}^{jl}\partial_k f\tilde{g}^{im}\nu_m + dLf\\ &= Bf + dLf\end{aligned}$$

for $f \in D(B)$. Since $d \cdot Lf \in \mathrm{C}(\partial M)$ we obtain $D(B) = D(\tilde{B})$ and B and $\tilde{B}$ differ only by the bounded perturbation $d \cdot L$. □

Lemma 4.4. *The Dirichlet-to-Neumann operator $N^{\tilde{A}_m,\tilde{B}}$ associated with $\tilde{A}_m$ and $\tilde{B}$ generates a compact and analytic semigroup of angle $\alpha > 0$ on ∂X if and only if $N^{A_m,B}$ associated with A_m and B does so.*

Proof. Let P be the perturbation defined in the proof of Lemma 4.1. By Lemma 4.1 P is relatively A_m-bounded of bound 0. Moreover, $\tilde{B}$ and B only differ by a bounded perturbation by Lemma 4.3. Hence, the claim follows by [BE19, Prop. 4.7]. □

Theorem 4.5. *The Dirichlet-to-Neumann operator $N^{\tilde{A}_m,\tilde{B}}$ given by (2.4) for (4.3) and (4.4) generates a compact and analytic semigroup of angle $\pi/2$ on $X = C(\partial M)$.*

Proof. The claim follows by Theorem 3.9 and Lemma 4.4. □

Remark 4.6. As in Remark 3.12 we can insert a strictly positive, continuous function $\beta > 0$ and the same result as Theorem 3.9 becomes true.

Remark 4.7. Theorem 4.5 improves and generalizes the main result in [Esc94]. If we consider $M = \Omega \subset \mathbb{R}^n$ equipped with the euclidean metric $g = \delta$, we obtain the maximal angle $\pi/2$ of analyticity in this case. This is the main result in [EO19] for smooth coefficients.

Now we use Theorem 4.5 to obtain existence and uniqueness for the associated Robin problem (2.3). Moreover, we obtain a maximum principle for the solutions of these problems.

Corollary 4.8 (Existence, uniqueness and maximum principle for the general Robin problem)**.** *There exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C} \setminus (-\infty, \omega)$ the problem (2.3) has a unique solution $u \in D(A_m) \cap D(B)$. This solution satisfies the maximum principle*

$$|\lambda| \max_{p\in\overline{M}} |u(p)| \leq C|\lambda| \max_{p\in\partial M} |u(p)| = C|\lambda|\|Lu\|_{\partial X} \leq \tilde{C}\|\varphi\|_{\partial X} = \tilde{C} \max_{p\in\partial M} |\varphi(p)|.$$

Proof. The existence and uniqueness follows immediately by Theorem 4.5. The first inequality is the interior maximum principle. The second inequality is a direct consequence from Lemma 2.3 and Theorem 4.5. □

4.2. The associated operator $\tilde{A}^{\tilde{B}}$ with Wentzell boundary conditions.

Lemma 4.9. *The operator $\tilde{A}^{\tilde{B}}$ generates a compact and analytic semigroup of angle $\alpha > 0$ on X if and only if A^B does.*

Proof. As seen in the proof of Lemma 4.4, the operators A_m and $\tilde{A}_m$ differ only by a relatively A_m-bounded perturbation with bound 0 while B and $\tilde{B}$ differ only by a bounded perturbation. Therefore, the claim follows by [BE19, Thm. 4.2]. □

Theorem 4.10. *The operator $\tilde{A}^{\tilde{B}}$ given by (2.1) for (4.3) and (4.4) generates a compact and analytic semigroup of angle $\pi/2$ on $X = \mathrm{C}(\overline{M})$.*

Proof. The claim follows by Theorem 3.14 and Lemma 4.9. □

Remark 4.11. As in Remark 3.12 we can insert a strictly positive, continuous function $\beta > 0$ and the same result as Theorem 4.10 becomes true.

Remark 4.12. Theorem 4.10 improves and generalizes [EF05, Cor. 4.5]. If we consider $M = \Omega \subset \mathbb{R}^n$ equipped with the euclidean metric $g = \delta$, we obtain the maximal angle $\pi/2$ of analyticity.

Corollary 4.13. *By Theorem 4.10 the initial boundary problem*

$$\begin{cases} \frac{d}{dt}u(t,p) = \sqrt{|a(p)|}\mathrm{div}_g\left(\frac{1}{\sqrt{|a(p)|}}a(p)\nabla_M^g u(t,p)\right) & \\ \qquad\qquad + \langle b(p), \nabla_M^g u(t,p)\rangle + c(p)u(t,p) & \text{for } t \geq 0, p \in \overline{M}, \\ \frac{d}{dt}u(t,p) = -\beta g(a(p)\nabla_M^g u(t,p), \nu_g(p)) + d(p)u(t,p) & \text{for } t \geq 0, p \in \partial M, \\ \quad u(0,p) = u_0(p) & \text{for } p \in \overline{M} \end{cases}$$

for a, b, c, d as in (4.1), $\beta > 0$ and $u_0(p) \in D(A^B)$ has a unique solution on $\mathrm{C}(\overline{M})$. This solution is governed by an analytic semigroup in the right half-plane.

Finally, we consider the elliptic problem

$$\begin{cases} A_m f - \lambda f = h \\ LA_m f = Bf, \end{cases} \tag{4.7}$$

for $f \in D(A_m) \cap D(B)$ and $h \in X = \mathrm{C}(\overline{M})$. Then the following holds.

Corollary 4.14. *There exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C} \setminus (-\infty, \omega)$ the problem (4.7) has a unique solution $u \in D(A_m) \cap D(B)$. This solution satisfies the maximum principle*

$$|\lambda| \max_{p \in \overline{M}} |u(p)| = |\lambda| \|u\|_X \leq C\|h\|_X = C \max_{p \in \overline{M}} |h(p)|.$$

Proof. This follows immediately by Theorem 4.10. □

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A.2.2 Dynamic boundary conditions for divergence form operators with Hölder coefficients

Dynamic boundary conditions for divergence form operators with Hölder coefficients

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Abstract

We consider second-order elliptic operators in divergence form with merely Hölder continuous coefficients on bounded domains Ω with $C^{1,\kappa}$-boundary Γ with Wentzell boundary conditions of the type $\operatorname{Tr} Au = \beta\,\partial_\nu u + \alpha \operatorname{Tr} u$ on Γ. Under such weak assumptions the divergence theorem is not available and we cannot apply the usual theory. Nevertheless, even for strictly positive bounded measurable β we prove maximal regularity on $L_p(\Omega) \times L_p(\Gamma)$ for all $p \in (1, \infty)$, holomorphic semigroup with angle $\frac{\pi}{2}$ for all $p \in [1, \infty)$ and also an holomorphic semigroup with angle $\frac{\pi}{2}$ on $C(\overline{\Omega})$.

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Keywords: Dynamic boundary conditions, Dirichlet-to-Neumann operator, holomorphic semigroup, maximal regularity, Poisson bounds

1 Introduction

Recently there is a lot of interest in parabolic systems with dynamic (Wentzell) boundary conditions

$$\left[\begin{array}{ll} \frac{d}{dt}\, u(t,\cdot) = -B_m u(t,\cdot) & \text{on } \Omega, \\[2ex] \frac{d}{dt} \operatorname{Tr} u(t,\cdot) = -\beta\,\partial_\nu u(t,\cdot) - \alpha \operatorname{Tr} u(t,\cdot) & \text{on } \partial\Omega, \\[2ex] u(0,\cdot) = u_0 & \text{on } \Omega. \end{array}\right. \tag{1}$$

Here $\Omega \subset \mathbb{R}^d$ is an open, bounded and connected set, B_m is a second-order elliptic operator, $\alpha \in L_\infty(\Omega)$, the function $\beta\colon \Omega \to (0,\infty)$ is a bounded measurable with $\operatorname{ess\,inf} \beta > 0$ and ∂_ν is the outward co-normal derivative associated with the operator B_m. The system (1) can be rewritten on the product space $\Omega \oplus \partial\Omega$ in matrix form by

$$\frac{d}{dt} \begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix} = -\mathbb{A} \begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix}, \quad \begin{pmatrix} u(0,\cdot) \\ \varphi(0,\cdot) \end{pmatrix} = \begin{pmatrix} u_0 \\ \operatorname{Tr} u_0 \end{pmatrix}, \tag{2}$$

where

$$\mathbb{A} = \begin{pmatrix} B_m & 0 \\ \beta\,\partial_\nu & \alpha \end{pmatrix}$$

and $D(\mathbb{A}) \subset \{(v,\varphi) : \operatorname{Tr} v = \varphi\}$. Typical questions are whether $\mathbb{A}$ generates a C_0-semigroup, whether this semigroup is holomorphic and if so, what is the holomorphy angle. Another question is whether the operator $\mathbb{A}$ has maximal L_r-regularity for all $r \in (1,\infty)$.

Operators with Wentzell boundary conditions have been first studied by Wentzell [Ven] and Feller [Fel]. Hintermann [Hin] studied elliptic operators with dynamic or Wentzell boundary conditions on C^∞-domains and proved generation of strongly continuous semigroups. Amann–Escher [AmE] considered C^2-domains and operators in divergence form with uniformly continuous symmetric principal coefficients and $\beta = \mathbb{1}_\Gamma$, and proved on $C(\overline{\Omega})$ and for all $p \in [1,\infty)$ on $L_p(\Omega) \times L_p(\partial\Omega)$ the generation of a positive contraction semigroup. Since the operator is self-adjoint on L_2 they obtained by interpolation that the semigroup is holomorphic on L_p for all $p \in (1,\infty)$. In [FGGR] Favini et. al. studied degenerate operators of the form $\operatorname{div}(a\nabla\cdot)$ with Wentzell boundary conditions on C^2-domains with $a \in C^1(\overline{\Omega})$ and $\beta \in C^1(\partial\Omega)$, and proved similar results. The holomorphy on L_1 has been proved by Warma [War] for the Laplacian on C^∞-domains with $\beta \in C^1(\partial\Omega)$ and he also proved that the holomorphy angle is equal to $\frac{\pi}{2}$. In [FGG$^+$2] Favini et. al. extended these results on L_p for all $p \in [1,\infty)$ to arbitrary uniformly elliptic operators in divergence form on C^∞-domains with C^∞ principal coefficients and $\beta \in C^\infty(\Gamma)$ without proving the optimal angle of holomorphy.

Assuming Ω merely Lipschitz and β measurable, Arendt et. al. showed in [AMPR] that the Laplacian with Wentzell boundary conditions generates a strongly continuous semigroup on L_p for all $p \in [1,\infty)$ with holomorphy if $p \in (1,\infty)$. On $C(\overline{\Omega})$ they proved a C_0-semigroup if Ω is of class $C^{2,\kappa}$ with $\kappa > 0$ and β continuous. Moreover, they showed that the semigroup is ultracontractive. Engel [Eng] proved that the Laplacian with Wentzell boundary conditions on $C(\overline{\Omega})$ generates a holomorphic C_0-semigroup with angle $\frac{\pi}{2}$ if Ω is of class C^∞. Engel–Fragnelli [EF] extended this result to arbitrary uniformly elliptic operators in divergence form with C^∞ principal coefficients on C^∞-domains and $\beta = \mathbb{1}_\Gamma$ without proving the optimal angle of holomorphy. In [BE2] the authors generalized and proved that uniformly elliptic operators in divergence form with Lipschitz continuous principal coefficients on $C^{1,1}$-domains and $\beta = \mathbb{1}_\Gamma$ generate holomorphic semigroups of optimal angle $\frac{\pi}{2}$. Moreover in [Bin1], the same results were proved on smooth, compact, Riemannian manifolds with smooth boundary.

In [DPZ] Denk, Prüss and Zacher discussed the question of maximal L_r-regularity for uniformly elliptic operators in non-divergence form with continuous principal coefficients on $L_p(\Omega) \times L_p(\partial\Omega)$ on C^∞-domains with dynamic boundary conditions, with $p, r \in (1,\infty)$. Recently, in [GGGR] Goldstein et. al. proved maximal L_r-regularity for uniformly elliptic operators in non-divergence form with continuous principal coefficients on $L_p(\Omega) \times L_p(\partial\Omega)$ on C^2-domains with generalized Wentzell boundary conditions, with $p, r \in (1,\infty)$. Their boundary conditions are dynamic boundary conditions but with an additional elliptic second-order operator on the boundary. For such boundary conditions see also [FGG$^+$2] and [Bin2].

A major restriction in the above papers is the ability of the divergence theorem in order to obtain the Gauss–Green formula. For the main results in this paper we assume that Ω is of class $C^{1,\kappa}$, with $\kappa \in (0,1)$ and B_m is a second-order operator in divergence form with real uniform Hölder continuous coefficients plus a real valued bounded measurable potential. We define the co-normal derivative in a weak L_2-sense. Since the principal coefficients are in general not Lipschitz continuous, the divergence theorem is not applicable. For the functions α and β we require that $\beta \colon \Omega \to (0,\infty)$ is a bounded measurable function with $\operatorname{ess\,inf} \beta > 0$ and $\alpha \in L_\infty(\Omega)$. In this setting we shall prove in Section 2 via form methods as in [AMPR] and [AE1] that $-\mathbb{A}$ generates a C_0-semigroup S on $L_2(\Omega) \times L_2(\partial\Omega)$ which is holomorphic with (semi-)angle $\frac{\pi}{2}$. Moreover, we shall prove in Theorem 4.1 and Corollary 5.6 that S extends consistently to a C_0-semigroup on $L_p(\Omega) \times L_p(\partial\Omega)$ for all $p \in [1,\infty)$ and that the semigroup is holomorphic with optimal angle $\frac{\pi}{2}$. In addition we prove that the generator on $L_p(\Omega) \times L_p(\partial\Omega)$ has maximal L_r-regularity for all $p, r \in (1,\infty)$. We also prove that the part of $\mathbb{A}$ in $\{(u, u|_\Gamma) : u \in C(\overline{\Omega})\}$ generates a C_0-semigroup which is holomorphic with angle $\frac{\pi}{2}$. We emphasise that also on $C(\overline{\Omega})$ we do not require that β is continuous.

As in [CENN], [Eng], [EF] and [BE1] we use a similarity transformation to write the transformed image of the operator $\mathbb{A}$ as $\begin{pmatrix} B^D & 0 \\ 0 & \beta\mathcal{N} \end{pmatrix}$ plus a perturbation, where B^D is the elliptic operator with Dirichlet boundary conditions and $\mathcal{N}$ is the Dirichlet-to-Neumann operator, under the condition that the operator B^D is invertible. We shall show in Corollary 3.8 that $-\beta\mathcal{N}$ generates a C_0-semigroup on $L_2(\partial\Omega)$ which extends consistently to a C_0-semigroup on $L_p(\partial\Omega)$ and the latter semigroup is holomorphic with angle $\frac{\pi}{2}$ for all $p \in [1,\infty)$. Moreover, the semigroup extends to a holomorphic C_0-semigroup on $C(\partial\Omega)$ with angle $\frac{\pi}{2}$, see Corollary 3.11. In order to prove this we first show that the semigroup generated by $-\beta\mathcal{N}$ has Poisson kernel bounds on the right half-plane, using techniques developed in [EO1], [EO2] and [EO3]. Then Hieber–Prüss [HP] implies that $\beta\mathcal{N}$ has maximal L_r-regularity on $L_p(\partial\Omega)$ for all $p, r \in (1,\infty)$. A perturbation result of Kunstmann–Weis [KW] then gives maximal regularity of the operator $\mathbb{A}$ in $L_p(\Omega) \times L_p(\partial\Omega)$ for all $p \in (1,\infty)$.

2 The operator in L_2

In this section we introduce almost every notation that we need in this paper and construct the operator on L_2.

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected set with Lipschitz boundary. Write $\Gamma = \partial\Omega$. We provide Γ with the $(d-1)$-dimensional Hausdorff measure, denoted by σ.

For all $k, l \in \{1, \ldots, d\}$ let $c_{kl}, c_0 \colon \Omega \to \mathbb{R}$ be bounded measurable functions with $c_{kl} = c_{lk}$. Further, let $\alpha \colon \Gamma \to \mathbb{C}$ be a bounded measurable function and let $\beta \colon \Gamma \to (0,\infty)$ be a bounded measurable function such that $\operatorname{ess\,inf} \beta > 0$. We assume that there exists a

$\mu > 0$ such that $\operatorname{Re} \sum_{k,l=1}^d c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Note that $\frac{1}{\beta}$ is a bounded function. Define the form $\mathfrak{a} \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^d \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v} + \int_\Gamma \frac{\alpha}{\beta}\, (\operatorname{Tr} u)\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma.$$

It is well known that $\mathfrak{a}$ is a continuous elliptic form. Define

$$\mathbb{L}_2 := L_2(\Omega) \times L_2(\Gamma)$$

equipped with the norm

$$\|(u,\varphi)\|_{\mathbb{L}_2}^2 = \int_\Omega |u|^2 + \int_\Gamma |\varphi|^2\, \frac{d\sigma}{\beta},$$

where we recall that σ is the $(d-1)$-dimensional Hausdorff measure on Γ. Note the factor β in the norm.

Define $j \colon W^{1,2}(\Omega) \to \mathbb{L}_2$ by

$$j(u) = (u, \operatorname{Tr} u).$$

Then j is continuous and has dense range. Moreover, for all $\theta \in (0, \frac{\pi}{2})$ there exists an $\omega > 0$ such that

$$\mathfrak{a}(u,u) + \omega\, \|j(u)\|_{\mathbb{L}_2}^2 \in \Sigma_\theta,$$

where $\Sigma_\theta = \{re^{i\eta} : r \in [0,\infty) \text{ and } \eta \in [-\theta,\theta]\}$. We define the **variational operator** $\mathbb{A}$ to be the m-sectorial operator in $\mathbb{L}_2$ associated with $(\mathfrak{a}, j)$, see [AE1]. Then $-\mathbb{A}$ is the generator of a C_0-semigroup which is holomorphic in the right half-plane. By definition for all $(u,\varphi), (f,\eta) \in \mathbb{L}_2$ one has that $(u,\varphi) \in D(\mathbb{A})$ and $\mathbb{A}(u,\varphi) = (f,\eta)$ if and only if

$$\left[\begin{array}{l} u \in W^{1,2}(\Omega), \\ \varphi = \operatorname{Tr} u, \text{ and} \\ \mathfrak{a}(u,v) = \displaystyle\int_\Omega f\, \overline{v} + \int_\Gamma \eta\, \overline{\operatorname{Tr} v}\, \frac{\mathrm{d}\sigma}{\beta} \quad \text{for all } v \in W^{1,2}(\Omega). \end{array}\right. \tag{3}$$

In order to characterise the generator we introduce some more notation. Define the form $\mathfrak{b} \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{b}(u,v) = \sum_{k,l=1}^d \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v}. \tag{4}$$

Further define $\mathcal{B} \colon W^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \mathcal{B}u, \tau \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \mathfrak{b}(u,\tau).$$

We need the notion of a weak co-normal derivative. If $u \in W^{1,2}(\Omega)$ and $\psi \in L_2(\Gamma)$, then we say that $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u = \psi$ if $\mathcal{B}u \in L_2(\Omega)$ and

$$\mathfrak{b}(u,v) - \int_\Omega (\mathcal{B}u)\, \overline{v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma$$

for all $v \in W^{1,2}(\Omega)$. It follows from the Stone–Weierstraß theorem that the function ψ is indeed unique. We say that ψ is the **(weak) co-normal derivative** of u. Note that the co-normal derivative is independent of c_0 and that $D(\partial_\nu^C) \subset W^{1,2}(\Omega)$.

Lemma 2.1. *Let $(u, \varphi), (f, \eta) \in \mathbb{L}_2$. Then the following are equivalent.*

(i) $(u, \varphi) \in D(\mathbb{A})$ *and* $\mathbb{A}(u, \varphi) = (f, \eta)$.

(ii) $u \in D(\partial_\nu^C)$, $\varphi = \operatorname{Tr} u$, $f = \mathcal{B}u$ *and* $\eta = \beta\, \partial_\nu^C u + \alpha \operatorname{Tr} u$.

Proof. The proof is similar to the proof of [AE1] Proposition 4.17.

'(i)⇒(ii)'. Choosing $v \in C_c^\infty(\Omega)$ it follows from (3) that $\mathcal{B}u = f$. Then

$$\int_\Omega (\mathcal{B}u)\, \overline{v} + \int_\Gamma \eta\, \overline{\operatorname{Tr} v}\, \frac{\mathrm{d}\sigma}{\beta} = \mathfrak{a}(u, v) = \mathfrak{b}(u, v) + \int_\Gamma \frac{\alpha}{\beta} (\operatorname{Tr} u)\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma$$

for all $v \in W^{1,2}(\Omega)$. Therefore

$$\int_\Gamma \eta\, \overline{\operatorname{Tr} v}\, \frac{\mathrm{d}\sigma}{\beta} - \int_\Gamma \frac{\alpha}{\beta} (\operatorname{Tr} u)\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma = \mathfrak{b}(u, v) - \int_\Omega (\mathcal{B}u)\, \overline{v}$$

So u has a co-normal derivative and $\partial_\nu^C u = \frac{\eta}{\beta} - \frac{\alpha}{\beta} \operatorname{Tr} u$.

'(ii)⇒(i)'. The proof is similar. □

So $D(\mathbb{A}) = \{(u, \operatorname{Tr} u) : u \in D(\partial_\nu^C)\}$. Lemma 2.1 gives a precise meaning that $(t, x) \mapsto (e^{-t\mathbb{A}}(u_0, \operatorname{Tr} u_0))(x)$ satisfies (1) and (2) in the introduction.

The next perturbation result allows to restrict to the case that $c_0 \geq 0$. It follows immediately from Lemma 2.1.

Lemma 2.2. *Let $\lambda \in \mathbb{R}$ and let $\mathbb{A}^\sharp$ be the operator similar to $\mathbb{A}$, but with c_0 replaced by $c_0 + \lambda\, \mathbb{1}_\Omega$. Then $D(\mathbb{A}^\sharp) = D(\mathbb{A})$ and*

$$\mathbb{A}^\sharp(u, \varphi) = \mathbb{A}(u, \varphi) + (\lambda\, u, 0)$$

for all $(u, \varphi) \in D(\mathbb{A})$.

We next describe via a similarity transformation the operator $\mathbb{A}$ as an operator in $W^{1,2}(\Omega)$ with Wentzell boundary conditions. This was done in [FGG+1] Theorem 2.1 for the Laplacian and we adapt the argument given in [AE1] Proposition 4.19.

Proposition 2.3. *Define the operator A in the Hilbert space $W^{1,2}(\Omega)$ by*

$$D(A) = \{u \in D(\partial_\nu^C) : \mathcal{B}u \in W^{1,2}(\Omega) \text{ and } \beta\, \partial_\nu^C u = \operatorname{Tr} \mathcal{B}u - \alpha \operatorname{Tr} u\}$$

and $Au = \mathcal{B}u$. Then $-A$ generates a holomorphic C_0-semigroup on $W^{1,2}(\Omega)$.

Proof. Since j is injective, one can transfer the form $\mathfrak{a}$ on $W^{1,2}(\Omega)$ to a form $\widetilde{\mathfrak{a}}$ on $j(W^{1,2}(\Omega))$ by defining $D(\widetilde{\mathfrak{a}}) = j(W^{1,2}(\Omega))$ and $\widetilde{\mathfrak{a}}(j(u), j(v)) = \mathfrak{a}(u, v)$ for all $u, v \in W^{1,2}(\Omega)$. Then $\widetilde{\mathfrak{a}}$ is a densely defined closed sectorial form in $\mathbb{L}_2$ and $\mathbb{A}$ is the operator associated with $\widetilde{\mathfrak{a}}$. We provide $D(\widetilde{\mathfrak{a}})$ with the norm $\|j(u)\|_{D(\widetilde{\mathfrak{a}})} = \|u\|_{W^{1,2}(\Omega)}$ for all $u \in W^{1,2}(\Omega)$. Let $\widetilde{A}$ be the part of $\mathbb{A}$ in $D(\widetilde{\mathfrak{a}})$. So $D(\widetilde{A}) = \{F \in D(\widetilde{\mathfrak{a}}) : \mathbb{A}F \in D(\widetilde{\mathfrak{a}})\}$. Then $-\widetilde{A}$ is the generator of a holomorphic C_0-semigroup in $D(\widetilde{\mathfrak{a}})$. Define $J \colon W^{1,2}(\Omega) \to D(\widetilde{\mathfrak{a}})$ by $J(u) = j(u)$. Then J is an isomorphism, so $-J^{-1}\,\widetilde{A}\,J$ is the generator of a holomorphic C_0-semigroup on $W^{1,2}(\Omega)$. It remains to show that $A = J^{-1}\,\widetilde{A}\,J$.

Let $u \in D(A)$. Then $\mathcal{B}u \in W^{1,2}(\Omega)$ and $\beta\,\partial_\nu^C u = \operatorname{Tr}\mathcal{B}u - \alpha \operatorname{Tr} u$. So $j(u) = (u, \operatorname{Tr} u) \in D(\mathbb{A})$ and $\mathbb{A}j(u) = (\mathcal{B}u, \operatorname{Tr}\mathcal{B}u) = j(\mathcal{B}u) \in D(\widetilde{\mathfrak{a}})$ by Lemma 2.1 (ii)⇒(i). Hence $j(u) \in D(\widetilde{A})$ and $u \in D(J^{-1}\,\widetilde{A}\,J)$. Moreover, $J^{-1}\,\widetilde{A}\,Ju = \mathcal{B}u = Au$.

Conversely, let $u \in D(J^{-1}\,\widetilde{A}\,J)$. Then $j(u) \in D(\widetilde{\mathfrak{a}})$ and $\mathbb{A}j(u) \in D(\widetilde{\mathfrak{a}})$. So $u \in W^{1,2}(\Omega)$ and by Lemma 2.1 (i)⇒(ii) one deduces that $\mathcal{B}u \in W^{1,2}(\Omega)$ and $\beta\,\partial_\nu^C u = \operatorname{Tr}\mathcal{B}u - \alpha \operatorname{Tr} u$. So $u \in D(A)$. □

3 Multiplicative perturbation of the Dirichlet-to-Neumann operator

As an intermediate result, which is of independent interest, we study in this section a multiplicative perturbation of the Dirichlet-to-Neumann operator.

We adopt the notation and assumptions as in Section 2. Recall that Ω has a Lipschitz boundary in Section 2. Define the form $\mathfrak{b}_D \colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{C}$ by $\mathfrak{b}_D = \mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Then $\mathfrak{b}_D$ is a closed sectorial form in $L_2(\Omega)$. Let B_2^D be the operator associated to $\mathfrak{b}_D$. Throughout this section we assume in addition that $0 \notin \sigma(B_2^D)$.

We frequently need the notion of consistent operators and semigroups. Let X and Y be two vector spaces. If $T_0 \colon D(T_0) \to Y$ and $T_1 \colon D(T_1) \to Y$ are two operators with domains $D(T_0) \subset X$ and $D(T_1) \subset X$, then the operators T_0 and T_1 are called **consistent** if $T_0 x = T_1 x$ for all $x \in D(T_0) \cap D(T_1)$. Let X_0 and X_1 be two Banach spaces which are embedded in a vector space X. Let $S^{(0)}$ and $S^{(1)}$ be semigroups in X_0 and X_1. Then the semigroups $S^{(0)}$ and $S^{(1)}$ are called **consistent** if $S_t^{(0)}$ and $S_t^{(1)}$ are consistent for all $t > 0$. For more details we refer to [ER].

Recall that σ is the $(d-1)$-dimensional Hausdorff measure on Γ and the space $L_p(\Gamma)$ is with respect to the measure σ for all $p \in [1, \infty)$. We also need the Dirichlet-to-Neumann operator $\mathcal{N}_p$ on $L_p(\Gamma)$ for all $p \in [1, \infty)$. Here we use that $0 \notin \sigma(B_2^D)$. Let $\mathcal{N}$ be the self-adjoint operator associated with $(\mathfrak{b}, \operatorname{Tr})$ and let $T^{(2)}$ be the semigroup generated by $-\mathcal{N}$, see [AEKS] Theorem 4.5. If

(I) $c_0 \geq 0$ or,

(II) there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ, then the semigroup $T^{(2)}$ extends consistently to a semigroup $T^{(p)}$ on $L_p(\Gamma)$ for all $p \in [1,\infty]$ and $T^{(p)}$ is a C_0-semigroup if $p \in [1,\infty)$. This follows from [EO2] Theorem 2.2(b) in Case (I) and from [EO2] Lemma 8.1 in Case (II). We denote by $-\mathcal{N}_p$ the generator of $T^{(p)}$. So $\mathcal{N}_2 = \mathcal{N}$. In Case (II) the semigroup $T^{(p)}$ is holomorphic with angle $\frac{\pi}{2}$ for all $p \in [1,\infty)$ by [EO3] Proposition 3.3.

Recall that $\beta \colon \Gamma \to (0,\infty)$ is a measurable function with $\operatorname{ess\,inf} \beta > 0$. We denote by M_β the multiplication operator with β on $L_p(\Gamma)$.

The following proposition is inspired by [AE1] Proposition 4.10.

Proposition 3.1. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a* $\kappa > 0$ *such that* Ω *is of class* $C^{1,\kappa}$ *and the principal coefficients* c_{kl} *are uniformly Hölder continuous of order* κ.

Then one has the following.

(a) *The operator* $M_\beta \, \mathcal{N} \, M_\beta$ *is self-adjoint and lower bounded.*

(b) *The semigroup generated by* $-M_\beta \, \mathcal{N} \, M_\beta$ *extends consistently to a semigroup on* $L_p(\Gamma)$ *for all* $p \in [1,\infty]$, *which is a* C_0*-semigroup if* $p \in [1,\infty)$ *with generator* $-M_\beta \, \mathcal{N}_p \, M_\beta$.

(c) *There exist* $c, \omega > 0$ *such that*

$$\|e^{-t M_\beta \mathcal{N} M_\beta}\|_{L_p(\Gamma) \to L_q(\Gamma)} \leq c \, t^{-(d-1)(\frac{1}{p} - \frac{1}{q})} \, e^{\omega t} \tag{5}$$

for all $t > 0$ *and* $p, q \in [1,\infty]$ *with* $p \leq q$.

Proof. The proof is in several steps. We first assume prove the proposition in Case (I). Near the end we prove Case (II) via a perturbation argument.

Step 1 Clearly the operator $M_\beta \, \mathcal{N} \, M_\beta$ is self-adjoint and lower bounded, which is Statement (a). We describe it with form methods. Since $c_0 \geq 0$, the form $\mathfrak{b}$ is $\frac{1}{\beta} \operatorname{Tr}$-elliptic (see [AE1]). Let $\widetilde{\mathcal{N}}$ denote the operator associated with $(\mathfrak{b}, \frac{1}{\beta} \operatorname{Tr})$. If $\varphi \in D(\widetilde{\mathcal{N}})$ and $\psi = \widetilde{\mathcal{N}} \varphi$, then there there exists a $u \in W^{1,2}(\Omega)$ such that $\frac{1}{\beta} \operatorname{Tr} u = \varphi$ and $\mathfrak{b}(u,v) = (\psi, \frac{1}{\beta} \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. Then $\mathfrak{b}(u,v) = (\frac{1}{\beta} \, \psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. So $\operatorname{Tr} u \in D(\mathcal{N})$ and $\mathcal{N} \operatorname{Tr} u = \frac{1}{\beta} \, \psi$. Hence $\widetilde{\mathcal{N}} \subset M_\beta \, \mathcal{N} \, M_\beta$. The converse inclusion can be proved similarly. So $\widetilde{\mathcal{N}} = M_\beta \, \mathcal{N} \, M_\beta$.

Step 2 Let $C = \{\varphi \in L_2(\Gamma, \mathbb{R}) : \varphi \leq \frac{1}{\beta}\}$. We shall prove that C is invariant under the semigroup generated by $-M_\beta \, \mathcal{N} \, M_\beta$. The set C is closed and convex in $L_2(\Gamma)$. Define $P \colon L_2(\Gamma) \to C$ by $P\varphi = \frac{1}{\beta} \, \mathbb{1}_\Gamma \wedge \operatorname{Re} \varphi$. Then P is the orthogonal projection onto C. Let $u \in W^{1,2}(\Omega)$. Define $w = \mathbb{1}_\Omega \wedge \operatorname{Re} u \in W^{1,2}(\Omega)$. Then $P(\frac{1}{\beta} \operatorname{Tr} u) = \frac{1}{\beta} \operatorname{Tr} w$ and, moreover, $\operatorname{Re} \mathfrak{b}(u - w, w) = 0$. Also $\mathfrak{b}$ is accretive since $c_0 \geq 0$. Here we need the lower bound for c_0. Hence C is invariant under the semigroup generated by $-M_\beta \, \mathcal{N} \, M_\beta$ by [AE1] Proposition 2.9.

Step 3 Let $t > 0$ and $\varphi \in L_2(\Gamma, \mathbb{R})$ with $\varphi \leq 1$. Then $\varphi \leq \|\beta\|_\infty \frac{1}{\beta}$. Hence by the above $e^{-tM_\beta \mathcal{N} M_\beta}(\frac{1}{\|\beta\|_\infty} \varphi) \in C$ and $e^{-tM_\beta \mathcal{N} M_\beta} \varphi \leq \|\beta\|_\infty \frac{1}{\beta} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. So the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ extends to a bounded semigroup on $L_\infty(\Gamma)$ and $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{\infty \to \infty} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. By duality the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ extends to a bounded semigroup on $L_1(\Gamma)$ and $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{1 \to 1} \leq \frac{\|\beta\|_\infty}{\operatorname{ess\,inf} \beta}$. By interpolation the semigroup $(e^{-tM_\beta \mathcal{N} M_\beta})_{t>0}$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1, \infty]$.

Step 4 This step is inspired by the proof of Theorem 2.6 in [EO1]. First suppose that $d \geq 3$. By a compactness argument the norm on $W^{1,2}(\Omega)$ is equivalent to $u \mapsto (\mathfrak{b}(u, u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)})^{1/2}$. By Theorem 2.4.2 in [Neč], the trace Tr is a bounded operator from $W^{1,2}(\Omega)$ into $L_s(\Gamma)$, where $s = \frac{2(d-1)}{d-2}$. Hence there exists a $c > 0$ such that

$$\|\operatorname{Tr} u\|^2_{L_s(\Gamma)} \leq c \left(\mathfrak{b}(u, u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \right)$$

for all $u \in W^{1,2}(\Omega)$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Since $e^{-tM_\beta \mathcal{N} M_\beta} \varphi \in D(\widetilde{\mathcal{N}})$, there exists a $u \in W^{1,2}(\Omega)$ such that $\frac{1}{\beta} \operatorname{Tr} u = e^{-tM_\beta \mathcal{N} M_\beta} \varphi$ and $\mathfrak{b}(u, v) = (\widetilde{\mathcal{N}} e^{-tM_\beta \mathcal{N} M_\beta} \varphi, \frac{1}{\beta} \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. Choose $v = u$. Then

$$\begin{aligned}
\|e^{-tM_\beta \mathcal{N} M_\beta} \varphi\|^2_{L_s(\Gamma)} &= \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_s(\Gamma)} \leq \frac{1}{(\operatorname{ess\,inf} \beta)^2} \|\operatorname{Tr} u\|^2_{L_s(\Gamma)} \\
&\leq \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left(\mathfrak{b}(u, u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \right) \\
&= \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left((\widetilde{\mathcal{N}} e^{-t\widetilde{\mathcal{N}}} \varphi, e^{-t\widetilde{\mathcal{N}}} \varphi)_{L_2(\Gamma)} + \|e^{-t\widetilde{\mathcal{N}}} \varphi\|^2_{L_2(\Gamma)} \right) \\
&\leq \frac{c}{(\operatorname{ess\,inf} \beta)^2} \left(\frac{1}{e\,t} + 1 \right) \|\varphi\|^2_{L_2(\Gamma)}.
\end{aligned}$$

So $\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_2(\Gamma) \to L_s(\Gamma)} \leq \frac{2\sqrt{c}}{\operatorname{ess\,inf} \beta} t^{-1/2}$ if $t \in (0, 1]$. Since the semigroup is bounded on $L_\infty(\Gamma)$ and on $L_1(\Gamma)$, one can extrapolate using [Cou] to obtain a $c_1 > 0$ such that

$$\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_1(\Gamma) \to L_\infty(\Gamma)} \leq c_1 \, t^{-(d-1)}$$

for all $t \in (0, 1]$. Then by interpolation the bounds (5) follow.

Next suppose $d = 2$. Fix $s \in (2, \infty)$. Then it follows from (8) in [EO1] that there exists a $c_2 > 0$ such that

$$\|\varphi\|_{L_s(\Gamma)} \leq c_2 \, \|\varphi\|^{1-\theta}_{H^{1/2}(\Gamma)} \, \|\varphi\|^\theta_{L_2(\Gamma)}$$

for all $\varphi \in H^{1/2}(\Gamma)$, where $\theta = 2/s$. The trace is bounded from $W^{1,2}(\Omega)$ into $H^{1/2}(\Gamma)$ by [McL] Theorem 3.37. Hence there exists a $c_3 \geq 1$ such that $\|\operatorname{Tr} u\|^2_{H^{1/2}(\Gamma)} \leq c_3 \Big(\mathfrak{b}(u, u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \Big)$ for all $u \in W^{1,2}(\Omega)$. Then

$$\|\operatorname{Tr} u\|_{L_s(\Gamma)} \leq c_2 \, c_3 \left(\mathfrak{b}(u, u) + \|\frac{1}{\beta} \operatorname{Tr} u\|^2_{L_2(\Gamma)} \right)^{(1-\theta)/2} \|\operatorname{Tr} u\|^\theta_{L_2(\Gamma)}$$

for all $u \in W^{1,2}(\Omega)$. Arguing as above it follows that there exists a $c_4 > 0$ such that

$$\|e^{-tM_\beta \mathcal{N} M_\beta}\|_{L_2(\Gamma)\to L_s(\Gamma)} \leq c_4\, t^{-(\frac{1}{2}-\frac{1}{s})}$$

for all $t \in (0,1]$. Then the bounds (5) follow as before by extrapolation and interpolation.

Step 5 Now we consider Case (II), so we do not assume that $c_0 \geq 0$. Clearly the operator $M_\beta \mathcal{N} M_\beta$ is self-adjoint and lower bounded. There exists a $\lambda > 0$ such that $c_0 + \lambda \mathbb{1}_\Omega \geq \mathbb{1}_\Omega$. Let $\mathcal{N}_0$ be the Dirichlet-to-Neumann operator obtained with c_0 replaced by $c_0 + \lambda \mathbb{1}_\Omega$. By [EO2] Corollary 5.6 and Proposition 5.5(d) there exists a bounded self-adjoint operator $Q\colon L_2(\Gamma) \to L_2(\Gamma)$ such that $\mathcal{N} = \mathcal{N}_0 + Q$ and, moreover, for all $p \in [1,\infty]$ the operator Q is consistent with a bounded operator from $L_p(\Gamma)$ into $L_p(\Gamma)$. Then $M_\beta \mathcal{N} M_\beta = M_\beta \mathcal{N}_0 M_\beta + M_\beta Q M_\beta$. The operator $M_\beta Q M_\beta$ is consistent with a bounded operator from $L_p(\Gamma)$ into $L_p(\Gamma)$ for all $p \in [1,\infty]$. By standard perturbation theory the semigroup generated by $-M_\beta \mathcal{N} M_\beta$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1,\infty]$. By [AE2] Proposition 3.1(a) the semigroup $(e^{-tM_\beta \mathcal{N} M_\beta})_{t>0}$ is again ultracontractive, with the same ultracontractivity exponent. Then by interpolation the bounds (5) follow.

Step 6 It remains to identify in both cases the generator of the semigroup consistent with $(e^{-tM_\beta \mathcal{N} M_\beta})_{t>0}$ on $L_p(\Gamma)$. The semigroup is a C_0-semigroup if $p \in [1,\infty)$ and it is continuous with respect to the weak*-topology if $p = \infty$. Let A_p denote the generator for all $p \in [1,\infty]$. Then $(A_p)^* = A_{p'}$ for all $p \in [1,\infty]$, where p' is the dual exponent. Let $p \in [2,\infty]$. Since Γ has finite measure, it follows that $A_p \subset A_2$. Let $\varphi \in D(A_p)$. Then $\varphi \in D(A_2) \cap L_p(\Gamma)$ and $A_2\varphi = A_p\varphi \in L_p(\Gamma)$. Now $A_2 = M_\beta \mathcal{N} M_\beta$. Hence $\beta\,\varphi \in D(\mathcal{N}) \cap L_p(\Gamma)$ and $\mathcal{N}(\beta\,\varphi) = \beta^{-1} A_2\varphi \in L_p(\Gamma)$. Therefore $\beta\,\varphi \in D(\mathcal{N}_p)$ and $\mathcal{N}_p(\beta\,\varphi) = \mathcal{N}(\beta\,\varphi)$. Consequently $\varphi \in D(M_\beta \mathcal{N}_p M_\beta)$ and $M_\beta \mathcal{N}_p M_\beta\varphi = \beta\,\mathcal{N}_p(\beta\,\varphi) = \beta\,\mathcal{N}(\beta\,\varphi) = M_\beta \mathcal{N} M_\beta\varphi = A_2\varphi = A_p\varphi$. So $A_p \subset M_\beta \mathcal{N}_p M_\beta$. Similarly $M_\beta \mathcal{N}_p M_\beta \subset A_p$, so $A_p = M_\beta \mathcal{N}_p M_\beta$.

Finally, in $p \in [1,2)$, then $A_p = (A_{p'})^* = (M_\beta \mathcal{N}_{p'} M_\beta)^* = M_\beta \mathcal{N}_p M_\beta$. □

Now we consider the multiplicative perturbation of $\mathcal{N}$.

Proposition 3.2. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ.*

Then one has the following.

(a) *The operator $-\beta\mathcal{N}$ generates a holomorphic C_0-semigroup on $L_2(\Gamma)$ with angle $\frac{\pi}{2}$.*

(b) *The semigroup generated by $-\beta\mathcal{N}$ extends consistently to a semigroup on $L_p(\Gamma)$ for all $p \in [1,\infty]$, which is a C_0-semigroup if $p \in [1,\infty)$ with generator $-\beta\mathcal{N}_p$.*

(c) *There exist $c, \omega > 0$ such that*

$$\|e^{-t\beta\mathcal{N}}\|_{L_p(\Gamma)\to L_q(\Gamma)} \leq c\, t^{-(d-1)(\frac{1}{p}-\frac{1}{q})}\, e^{\omega t}$$

for all $t > 0$ and $p, q \in [1, \infty]$ with $p \leq q$.

Proof. Define $E \colon L_2(\Gamma) \to L_2(\Gamma)$ by $E\varphi = \beta\,\varphi$. Then E is a topological isomorphism. Consider the operator $M_\beta\,\mathcal{N}\,M_\beta$ in Proposition 3.1. Then $\beta^2\,\mathcal{N} = E\,M_\beta\,\mathcal{N}\,M_\beta\,E^{-1}$ is the minus-generator of a holomorphic C_0-semigroup S in $L_2(\Gamma)$. Since also E extends consistently to an topological isomorphism from $L_p(\Gamma)$ onto $L_p(\Gamma)$ for all $p \in [1, \infty]$, all properties for the operator $M_\beta\,\mathcal{N}\,M_\beta$ in Proposition 3.1 carry over to the operator $\beta^2\,\mathcal{N}$, with a different value for the constant c.

Finally replace β by $\sqrt{\beta}$. □

The semigroup generated by $-\beta\mathcal{N}$ is smoothing.

Proposition 3.3. *Suppose*

(I) $c_0 \geq 0$ *or,*

(II) *there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ.*

Then one has the following.

(a) *Let $p \in (d-1, \infty)$. Then $D(\mathcal{N}_p) \subset C(\Gamma)$.*

(b) *Let $p, q \in [1, \infty]$ with $p < q$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{d-1}$. Then $D(\mathcal{N}_p) \subset L_q(\Gamma)$.*

(c) *Let $p, q \in [1, \infty]$ with $p < q$. Let $\varphi \in D(\mathcal{N}_p)$ and suppose that $\mathcal{N}_p\varphi \in L_q(\Gamma)$. Then $\varphi \in L_q(\Gamma)$.*

(d) *If $t > 0$, then $e^{-t\beta\mathcal{N}} L_2(\Gamma) \subset C(\Gamma)$.*

Proof. '(a)'. By [EO2] Theorem 2.2(b) in Case (I) and [EO2] Lemma 8.1 in Case (II) there are $c_1, \omega_1 > 0$ such that $\|e^{-t\mathcal{N}}\|_{L_\infty(\Gamma)\to L_\infty(\Gamma)} \leq c_1\, e^{\omega_1 t}$ for all $t > 0$. By [EW] Theorem 5.5 and the remark following it, there exist $c_2, \omega_2 > 0$ and $\nu \in (0, 1)$ such that $e^{-t\mathcal{N}}$ maps $L_2(\Gamma)$ into $C^\nu(\Gamma)$ and

$$\|e^{-t\mathcal{N}}\|_{L_2(\Gamma)\to C^\nu(\Gamma)} \leq c_2\, t^{-\frac{(d-1)}{2}}\, t^{-2\nu}\, e^{\omega_2 t}$$

for all $t > 0$. (The exponent -2ν can be replaced by $-\nu$ if $d \geq 3$.) Let $p \in (2, \infty)$. Then by interpolation the operator $e^{-t\mathcal{N}}$ is bounded from $L_p(\Gamma)$ into $C^{2\nu/p}(\Gamma)$ with norm

$$\|e^{-t\mathcal{N}}\|_{L_p(\Gamma)\to C^{2\nu/p}(\Gamma)} \leq c_3\, t^{-\frac{(d-1)}{p}}\, t^{-\frac{4\nu}{p}}\, e^{\omega_3 t}$$

for all $t > 0$, where $c_3 = c_1 + c_2$ and $\omega_3 = \omega_1 + \omega_2$. Now choose $p = d - 1 + 5\nu$. Then

$$(\mathcal{N}_p + (\omega_3 + 1)\, I)^{-1} = \int_0^\infty e^{-\omega_3 t}\, e^{-t}\, e^{-t\mathcal{N}_p}\, dt$$

maps $L_p(\Gamma)$ into $C^{2\nu/p}(\Gamma)$. In particular $D(\mathcal{N}_p) \subset C^{2\nu/p}(\Gamma) \subset C(\Gamma)$.

'(b)'. The proof is similar, using the bounds of Proposition 3.2(c).

'(c)'. If $\frac{1}{p} - \frac{1}{q} < \frac{1}{d-1}$, then it follows from Statement (b) that $\varphi \in D(\mathcal{N}_p) \cap L_q(\Gamma)$ and $\mathcal{N}_p\varphi \in L_q(\Gamma)$. So $\varphi \in D(\mathcal{N}_q)$. Now use induction.

'(d)'. Choose $p = d$. Then ultracontractivity and holomorphy on $L_p(\Gamma)$ give

$$e^{-t\beta\mathcal{N}} L_2(\Gamma) \subset e^{-\frac{t}{2}\beta\mathcal{N}_p} L_p(\Gamma) \subset D(\beta\mathcal{N}_p) = D(\mathcal{N}_p) \subset C(\Gamma)$$

for all $t > 0$. □

For the remainder of this paper we assume Case (II), that is there exists a $\kappa > 0$ such that Ω is of class $C^{1,\kappa}$ and the principal coefficients c_{kl} are uniformly Hölder continuous of order κ.

Let $C^{0,1}(\Gamma)$ denote the space of Lipschitz continuous functions on Γ. It is endowed with the norm

$$\|g\|_{C^{0,1}(\Gamma)} = \|g\|_{L_\infty(\Gamma)} + \sup_{z,w\in\Gamma,\, z\neq w} \frac{|g(z) - g(w)|}{|z - w|}.$$

For all $g \in C^{0,1}(\Gamma)$ we use the notation $\mathrm{Lip}_\Gamma(g) = \sup_{z,w\in\Gamma,\, z\neq w} \frac{|g(z)-g(w)|}{|z-w|}$. It has been proved in [EO2] Theorem 7.3 that for all $p \in (1, \infty)$ there exists a $c > 0$ such that

$$\|[\mathcal{N}, M_g]\|_{L_p(\Gamma)\to L_p(\Gamma)} \le c\, \mathrm{Lip}_\Gamma(g)$$

for all $g \in C^{0,1}(\Gamma)$. These bounds carry over to commutator estimates for the operator $\beta\mathcal{N}$.

Proposition 3.4. *For all $p \in (1, \infty)$ there exists a $c > 0$ such that*

$$\|[\beta\mathcal{N}, M_g]\|_{L_p(\Gamma)\to L_p(\Gamma)} \le c\, \mathrm{Lip}_\Gamma(g)$$

for all $g \in C^{0,1}(\Gamma)$.

Proof. Let $g \in C^{0,1}(\Gamma)$. Then

$$[\beta\mathcal{N}, M_g] = M_\beta\, [\mathcal{N}, M_g].$$

So $\|[\beta\mathcal{N}, M_g]\|_{p\to p} \le \|\beta\|_\infty \, \|[\mathcal{N}, M_g]\|_{p\to p}$ and the result follows from [EO2] Theorem 7.3. □

Let $K_\mathcal{N}$ and $K_{\beta\mathcal{N}}$ denote the Schwartz kernel of $\mathcal{N}$ and $\beta\mathcal{N}$. Then $K_{\beta\mathcal{N}}(z, w) = \beta(z)\, K_\mathcal{N}(z, w)$ for all $z, w \in \Gamma$ with $z \neq w$. It follows from [EO2] Proposition 6.5 that there exists a $c > 0$ such that

$$|K_\mathcal{N}(z, w)| \le \frac{c}{|z - w|^d}$$

for all $z, w \in \Gamma$ with $z \neq w$. Consequently one has the next bounds.

Proposition 3.5. *There exists a $c > 0$ such that*

$$|K_{\beta\mathcal{N}}(z,w)| \le \frac{c}{|z-w|^d}$$

for all $z, w \in \Gamma$ with $z \neq w$.

It follows from Proposition 3.2 that for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ the operator $e^{-z\beta\mathcal{N}}$ on $L_2(\Gamma)$ has a kernel $K_z \in L_\infty(\Gamma \times \Gamma)$.

Proposition 3.6. *There exist $c, \omega > 0$ such that*

$$|K_t(w_1,w_2)| \le \frac{c\, t^{-(d-1)}\, e^{\omega t}}{\Big(1 + \dfrac{|w_1 - w_2|}{t}\Big)^d}$$

for all $t > 0$ and $w_1, w_2 \in \Gamma$.

Proof. This follows as in [EO2] Lemma 8.4 and the argument in Section 4 of [EO1]. For the latter, see also [EO2] pages 4270–4272. $\square$

Via an iteration argument the bounds can be improved to the right half of the complex plane.

Theorem 3.7. *Let $\kappa \in (0,1)$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected set of class $C^{1,\kappa}$. Write $\Gamma = \partial\Omega$. For all $k, l \in \{1, \ldots, d\}$ let $c_{kl} \in C^\kappa(\Omega, \mathbb{R})$ and let $c_0 \colon \Omega \to \mathbb{R}$ be a bounded measurable function. Suppose that $c_{kl} = c_{lk}$ for all $k, l \in \{1, \ldots, d\}$. Further, let $\beta \colon \Gamma \to (0,\infty)$ be a bounded measurable function such that $\operatorname{ess\,inf} \beta > 0$. We assume that there exists a $\mu > 0$ such that $\operatorname{Re} \sum_{k,l=1}^d c_{kl}(x)\, \xi_k\, \overline{\xi_l} \ge \mu\, |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Let $\mathfrak{b}$ be the elliptic form as in (4). Suppose that*

$$\{u \in W_0^{1,2}(\Omega) : \mathfrak{b}(u,v) = 0 \text{ for all } v \in W_0^{1,2}(\Omega)\} = \{0\}. \tag{6}$$

Let $\mathcal{N}$ be the Dirichlet-to-Neumann operator associated with $(\mathfrak{b}, \operatorname{Tr})$.

Then the operator $-\beta\mathcal{N}$ is the generator of a C_0-semigroup S which is holomorphic with angle $\frac{\pi}{2}$. Moreover, S has a kernel K and for all $\theta \in (0, \frac{\pi}{2})$ there are $c, \omega > 0$ such that

$$|K_z(w_1,w_2)| \le \frac{c\, |z|^{-(d-1)}\, e^{\omega |z|}}{\Big(1 + \dfrac{|w_1 - w_2|}{|z|}\Big)^d}$$

for all $z \in \mathbb{C} \setminus \{0\}$ and $w_1, w_2 \in \Gamma$ with $|\arg z| \le \theta$.

Proof. This follows from the previous three propositions as in [EO3] Lemma 3.1 and Theorem 3.2. The condition (6) is equivalent with $0 \notin \sigma(B_2^D)$. $\square$

Corollary 3.8. *Adopt the notation and assumptions as in Theorem 3.7. For all $p \in [1,\infty)$ the semigroup $(e^{-t\beta\mathcal{N}})_{t>0}$ extends consistently to a holomorphic semigroup on $L_p(\Gamma)$ with angle $\frac{\pi}{2}$.*

Corollary 3.9. *Adopt the notation and assumptions as in Theorem 3.7. For all $p, r \in (1, \infty)$ the operator $\beta\mathcal{N}_p$ has maximal L_r-regularity on $L_p(\Gamma)$.*

Proof. The operator $-\beta\mathcal{N}$ generates a holomorphic C_0-semigroup on $L_2(\Gamma)$ and this semigroup has a kernel with Poisson bounds by Proposition 3.6. Then the statement follows from Hieber–Prüss [HP] Theorem 3.1. □

Finally we consider the space $C(\Gamma)$. In order to obtain optimal results, without any continuity requirement on β we need the concept of sectorial operators and a maximal operator on $L_\infty(\Gamma)$.

In general, let A be an operator in a Banach space X and let $\alpha \in (0, \frac{\pi}{2}]$. Then we say that A **is sectorial of angle** α if for all $\theta \in (0, \alpha)$ there exist $M, \omega > 0$ such that $\sigma(A + \omega I) \subset \Sigma_\theta$ and

$$\|(A + (\omega + \lambda)I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $-\lambda \notin \Sigma_\theta$.

Define the operator $\mathcal{N}_{\infty m} \colon D(\mathcal{N}_{\infty m}) \to L_\infty(\Gamma)$ by

$$D(\mathcal{N}_{\infty m}) = \{\varphi \in D(\mathcal{N}) : \mathcal{N}\varphi \in L_\infty(\Gamma)\}$$

and $\mathcal{N}_{\infty m} = \mathcal{N}|_{D(\mathcal{N}_{\infty m})}$. It follows from Proposition 3.3(c) that $D(\mathcal{N}_{\infty m}) \subset L_\infty(\Gamma)$. We consider $\mathcal{N}_{\infty m}$ as a non-densely defined operator in $L_\infty(\Gamma)$, provided with the norm topology. It is easy to verify that $\mathcal{N}_{\infty m}$ is a closed operator. Moreover, Proposition 3.3(a) gives $D(\mathcal{N}_{\infty m}) \subset C(\Gamma)$.

Lemma 3.10. *Adopt the notation and assumptions as in Theorem 3.7.*

(a) *If $\lambda \in \rho(-\beta\mathcal{N})$, then $\lambda \in \rho(-\beta\mathcal{N}_{\infty m})$ and $(\beta\mathcal{N}_{\infty m} + \lambda I)^{-1} = (\beta\mathcal{N} + \lambda I)^{-1}|_{L_\infty(\Gamma)}$.*

(b) *The operator $\beta\mathcal{N}_{\infty m}$ is sectorial of angle $\frac{\pi}{2}$.*

(c) *$D(\mathcal{N}_{\infty m})$ is dense in $C(\Gamma)$.*

Proof. '(a)'. Easy.

'(b)'. This follows from the Poisson kernel bounds of Theorem 3.7.

'(c)'. Let $\mathcal{N}_c$ be the part of $\mathcal{N}$ in $C(\Gamma)$. Then $\mathcal{N}_c \subset \mathcal{N}_{\infty m}$. Moreover, $-\mathcal{N}_c$ is the generator of a holomorphic C_0-semigroup by [EO3] Proposition 3.3. Now let $\varphi \in C(\Gamma)$. Then $\lim_{t \downarrow 0} e^{-t\mathcal{N}_c}\varphi = \varphi$ in $C(\Gamma)$. But $e^{-t\mathcal{N}_c}\varphi \in D(\mathcal{N}_c) \subset D(\mathcal{N}_{\infty m})$ for all $t > 0$. □

Recall that we do not require β to be continuous. By Proposition 3.3(d) the semigroup $(e^{-t\beta\mathcal{N}})_{t>0}$ leaves the space $C(\Gamma)$ invariant.

Corollary 3.11. *Adopt the notation and assumptions as in Theorem 3.7. Define $T_t = e^{-t\beta\mathcal{N}}|_{C(\Gamma)} \colon C(\Gamma) \to C(\Gamma)$ for all $t > 0$. Then T is a C_0-semigroup which is holomorphic with angle $\frac{\pi}{2}$.*

Proof. This follows from Lemma 3.10(b), Lemma 3.10(c) and [ABHN] Remark 3.7.13. □

4 The operator in L_p

We return to the operator $\mathbb{A}$ with dynamical boundary conditions as introduced in Section 2. Under the smoothness assumptions as in Case (II) in Section 3 we show that for all $p \in (1,\infty)$ the operator $\mathbb{A}$ is consistent with an operator $\mathbb{A}_p$ on L_p such that $-\mathbb{A}$ is the generator of a C_0-semigroup which is holomorphic on the right half-plane and $\mathbb{A}_p$ has maximal L_r-regularity for all $r \in (1,\infty)$.

We extend the definition of $\mathbb{L}_2$. Define

$$\mathbb{L}_p = L_p(\Omega) \times L_p(\Gamma)$$

for all $p \in [1,\infty]$, with norm

$$\|(u,\varphi)\|_{\mathbb{L}_p}^p = \int_\Omega |u|^p + \int_\Gamma |\varphi|^p \,\frac{d\sigma}{\beta}$$

and obvious modification if $p = \infty$.

The main theorem of this section is as follows. In Corollary 5.6 we consider the case $p = 1$.

Theorem 4.1. *Let $\kappa \in (0,1)$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and connected set of class $C^{1,\kappa}$. Write $\Gamma = \partial\Omega$. For all $k,l \in \{1,\ldots,d\}$ let $c_{kl} \in C^\kappa(\Omega,\mathbb{R})$ and let $c_0 \colon \Omega \to \mathbb{R}$ be a bounded measurable function. Suppose that $c_{kl} = c_{lk}$ for all $k,l \in \{1,\ldots,d\}$. Further, let $\alpha \colon \Gamma \to \mathbb{C}$ be a bounded measurable function and let $\beta \colon \Gamma \to (0,\infty)$ be a bounded measurable function with $\operatorname{ess\,inf} \beta > 0$. We assume that there exists a $\mu > 0$ such that $\operatorname{Re} \sum_{k,l=1}^d c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Let $\mathbb{A}$ be the associated variational operator in $\mathbb{L}_2$ as in Section 2. Then for all $p \in (1,\infty)$ the semigroup generated by $-\mathbb{A}$ extends consistently to a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. Moreover, its generator has maximal L_r-regularity on $\mathbb{L}_p$ for all $r \in (1,\infty)$.*

The proof requires quite some preparation. The main problem to circumvent is that we cannot apply the divergence theorem since the principal coefficients are not Lipschitz continuous. Adopt the notation and assumptions as in Theorem 4.1. We use the notation as in Section 2. As in Section 3 define the form $\mathfrak{b}_D \colon W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{C}$ by $\mathfrak{b}_D = \mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Then $\mathfrak{b}_D$ is a closed sectorial form in $L_2(\Omega)$. Let B_2^D be the operator associated to $\mathfrak{b}_D$. By Lemma 2.2 we may assume that $0 \notin \sigma(B_2^D)$. The operator $-B_2^D$ generates a holomorphic C_0-semigroup $S^{(2)D}$ on $L_2(\Omega)$ with angle $\frac{\pi}{2}$. Moreover, for all $p \in [1,\infty]$ the semigroup $S^{(2)D}$ extends consistently to a semigroup $S^{(p)D}$ on $L_p(\Omega)$ with angle $\frac{\pi}{2}$ by [Ouh] Theorem 3.1(1). Moreover, $S^{(p)D}$ is a C_0-semigroup for all $p \in [1,\infty)$. We denote by $-B_p^D$ the generator of $S^{(p)D}$.

In addition we need a harmonic lifting (also called harmonic extension). Since $0 \notin \sigma(B_2^D)$ one can define $\gamma \colon \operatorname{Tr} W^{1,2}(\Gamma) \to W^{1,2}(\Omega)$ by $\gamma(\varphi) = u$, where $u \in W^{1,2}(\Omega)$ is such that $\operatorname{Tr} u = \varphi$ and $\mathcal{B}u = 0$. (So u is $\mathcal{B}$-harmonic.) Then by [EO2] Proposition 5.5 there exists an operator $\mathcal{H} \colon L_1(\Gamma) \to C(\Omega) \cap L_1(\Omega)$ such that

$$\mathcal{H}|_{\operatorname{Tr} W^{1,2}(\Omega)} = \gamma,$$
$$\mathcal{H}(L_p(\Gamma)) \subset L_p(\Omega) \text{ for all } p \in [1, \infty], \text{ and}$$
$$\mathcal{H}|_{L_p(\Gamma)} \colon L_p(\Gamma) \to L_p(\Omega) \text{ is continuous for all } p \in [1, \infty].$$

We write $\gamma_p = \mathcal{H}|_{L_p(\Gamma)} \colon L_p(\Gamma) \to L_p(\Omega)$ for all $p \in [1, \infty]$.

There is a remarkable relation between the elliptic operator B_p^D with Dirichlet boundary conditions, the harmonic lifting γ and the (weak) co-normal derivative. We denote by $(\nu_1, \ldots, \nu_d)$ the outer normal on Γ.

Proposition 4.2. *Let $p \in (d + 2\kappa, \infty)$. Then one has the following.*

(a) $D(B_p^D) \subset D(\partial_\nu^C) \cap C^{1+2\kappa/p}(\Omega)$.

(b) *If $u \in D(B_p^D)$, then*

$$\partial_\nu^C u = \sum_{k,l=1}^{d} \nu_k \, (c_{kl} \, \partial_l u)|_\Gamma.$$

In particular, $\partial_\nu^C u \in C(\Gamma)$.

(c) $\gamma_{p'}^* = -\partial_\nu^C \, (B_p^D)^{-1}$.

Proof. '(a)'. See [EO2] Propositions 5.3 and 4.3.

'(b)'. See [EO2] Proposition 5.3.

'(c)'. If $\varphi \in \operatorname{Tr} W^{1,2}(\Omega)$ and $u \in L_p(\Omega)$, then

$$(\varphi, \gamma_2^* u)_{L_2(\Gamma)} = (\gamma \varphi, u)_{L_2(\Omega)} = -(\varphi, \partial_\nu^C \, (B_p^D)^{-1} u)_{L_2(\Gamma)}$$

by [EO2] Lemma 5.4. Since $\operatorname{Tr} W^{1,2}(\Omega)$ is dense in $L_2(\Gamma)$ one deduces that $(\varphi, \gamma_2^* u)_{L_2(\Gamma)} = -(\varphi, \partial_\nu^C \, (B_p^D)^{-1} u)_{L_2(\Gamma)}$ for all $\varphi \in L_2(\Gamma)$. Now $\partial_\nu^C \, (B_p^D)^{-1} u \in C(\Gamma) \subset L_p(\Omega)$. Hence

$$\langle \gamma_{p'} \varphi, u \rangle_{L_{p'}(\Omega) \times L_p(\Omega)} = -\langle \varphi, \partial_\nu^C \, (B_p^D)^{-1} u \rangle_{L_{p'}(\Gamma) \times L_p(\Gamma)}$$

first for all $\varphi \in L_2(\Gamma)$ and then by density for all $\varphi \in L_{p'}(\Gamma)$. The statement follows. $\square$

Proposition 4.2(c) implies that $\partial_\nu^C u = -\gamma_{p'}^* \, B_p^D u$ for all $u \in D(B_p^D)$ if $p \in (d+2\kappa, \infty)$. It is unclear whether this is also valid for all $p \in (1, \infty)$, in particular if $p < 2$. We circumvent this problem by working with the operator $\gamma_{p'}^* \, B_p^D$ on $D(B_p^D)$.

For all $p \in [1, \infty)$ define $\widehat{\mathbb{A}}_p \colon D(B_p^D) \times D(\mathcal{N}_p) \to \mathbb{L}_p$ by

$$\widehat{\mathbb{A}}_p = \begin{pmatrix} B_p^D + \gamma_p \, M_\beta \, \gamma_{p'}^* \, B_p^D & -\gamma_p \, M_\beta \, \mathcal{N}_p - \gamma_p \, M_\alpha \\ -\beta \, \gamma_{p'}^* \, B_p^D & \beta \mathcal{N}_p + M_\alpha \end{pmatrix}.$$

We need a technical lemma.

Lemma 4.3. *Let $p \in (1,\infty)$. Then one has the following.*

(a) *The operator $\widehat{\mathbb{A}}_p$ is consistent with $\widehat{A}_2$.*

(b) *The operator $-\widehat{\mathbb{A}}_p$ is the generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$.*

(c) *For all $r \in (1,\infty)$ the operator $\widehat{\mathbb{A}}_p$ has maximal L_r-regularity on $\mathbb{L}_p$.*

For the proof of Lemma 4.3 we use an abstract lemma, which is contained in the proof of [EF] Lemma A.4.

Lemma 4.4. *Let A and B be operators in Banach spaces X and Y, respectively. Let $P_1 \colon D(A) \to X$ and $P_2 \colon D(A) \to Y$ be relatively A-bounded with A-bound zero. Let $P_3 \colon D(B) \to Y$ be relatively B-bounded with B-bound zero and $Q \colon D(B) \to X$ be a bounded operator. Define*

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} P_1 & Q \\ P_2 & P_3 \end{pmatrix}.$$

Then for all $\varepsilon > 0$ there exists an isomorphism $S \colon X \times Y \to X \times Y$ such that $S\,\mathcal{A}_0\,S^{-1} = \mathcal{A}_0$ and $S\,\mathcal{P}\,S^{-1}$ is relatively $\mathcal{A}_0$-bounded with $\mathcal{A}_0$-bound ε.

Proof of Lemma 4.3. Statement (a) is obvious.

'(b)'. First note that

$$\widehat{\mathbb{A}}_p - \begin{pmatrix} B_p^D & 0 \\ 0 & \beta\mathcal{N}_p \end{pmatrix} = \begin{pmatrix} \gamma_p\,M_\beta\,\gamma_{p'}^*\,B_p^D & -\gamma_p\,M_\beta\,\mathcal{N}_p - \gamma_p\,M_\alpha \\ -\beta\,\gamma_{p'}^*\,B_p^D & M_\alpha \end{pmatrix}. \tag{7}$$

The operator $-B_p^D$ generates a C_0-semigroup in $L_p(\Omega)$ which is holomorphic with angle $\frac{\pi}{2}$ and by Corollary 3.8 the operator $-\beta\mathcal{N}_p$ generates a C_0-semigroup in $L_p(\Gamma)$ which is holomorphic with angle $\frac{\pi}{2}$. It follows that

$$\begin{pmatrix} B_p^D & 0 \\ 0 & \beta\mathcal{N}_p \end{pmatrix}$$

is the minus-generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. We next show that $\gamma_{p'}^*\,B_p^D$ is B_p^D-bounded with relative bound zero.

Let $q = (p+1) \vee (d+3\kappa)$. It follows as in the proof of [EO2] Proposition 4.3 that there are $c, \omega > 0$ such that $\nabla\,e^{-tB_q^D} u \in C^{2\kappa/q}(\Omega) \subset C(\overline{\Omega})$ for all $u \in L_q(\Omega)$ and

$$\|\nabla\,e^{-tB_q^D}\|_{L_q(\Omega)\to C(\overline{\Omega})} \le c\,t^{-\frac{d}{2q}}\,t^{-\frac{1}{2}}\,e^{\omega t}$$

for all $t > 0$. Then a Laplace transform together with Proposition 4.2(b) gives

$$\begin{aligned}
\|\partial_\nu^C (B_q^D + \lambda\,I)^{-1} u\|_{L_q(\Gamma)} &\le (\sigma(\Gamma))^{1/q}\,\|C\|_\infty\,\|\nabla(B_q^D + \lambda\,I)^{-1} u\|_{C(\overline{\Omega})} \\
&\le c\,(\sigma(\Gamma))^{1/q}\,\|C\|_\infty \int_0^\infty t^{-\frac{d}{2q}}\,t^{-\frac{1}{2}}\,e^{-(\lambda-\omega)t}\,\|u\|_{L_q(\Omega)}\,dt \\
&= c_1\,(\lambda-\omega)^{-(\frac{1}{2}-\frac{d}{2q})}\,\|u\|_{L_q(\Omega)}
\end{aligned} \tag{8}$$

for all $\lambda > \omega$, where $c_1 = c\,(\sigma(\Gamma))^{1/q}\,\Gamma(\frac{1}{2} - \frac{d}{2q})$. Hence if $\lambda > \omega$ and $u \in D(B_q^D)$, then it follows from Proposition 4.2(c) that

$$\|\gamma_{q'}^* \, B_q^D u\|_{L_q(\Gamma)} = \|\partial_\nu^C u\|_{L_q(\Gamma)} \le c_1\,(\lambda - \omega)^{-(\frac{1}{2} - \frac{d}{2q})}\,\|(B_q^D + \lambda\,I)u\|_{L_q(\Omega)}. \tag{9}$$

Next, since $-B_1^D$ generates a C_0-semigroup there are $c_2, \omega_2 > 0$ such that $B_1^D + \lambda\,I$ is invertible and $\|B_1^D\,(B_1^D + \lambda\,I)^{-1}\|_{L_1(\Omega)\to L_1(\Omega)} \le c_2$ for all $\lambda \ge \omega_2$. Then

$$\|\gamma_{1'}^* \, B_1^D u\|_{L_1(\Gamma)} \le \|\gamma_\infty\|_{L_\infty(\Gamma)\to L_\infty(\Omega)}\, c_2\,\|(B_1^D + \lambda\,I)u\|_{L_1(\Omega)} = c_3\,\|(B_1^D + \lambda\,I)u\|_{L_1(\Omega)} \tag{10}$$

for all $\lambda \ge \omega_2$ and $u \in D(B_1^D)$, where $c_3 = c_2\,\|\gamma_\infty\|_{L_\infty(\Gamma)\to L_\infty(\Omega)}$. There exists a $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{1}$. Interpolation between (9) and (10) gives

$$\|\gamma_{p'}^* \, B_p^D u\|_{L_p(\Gamma)} \le c_1^\theta\, c_3^{1-\theta}\,(\lambda - \omega)^{-\delta}\,\|(B_p^D + \lambda\,I)u\|_{L_p(\Omega)}$$

for all $u \in D(B_p^D)$ and $\lambda > \omega \vee \omega_2$, where $\delta = \theta(\frac{1}{2} - \frac{d}{2q})$. Then

$$\|\gamma_{p'}^* \, B_p^D u\|_{L_p(\Gamma)} \le c_1^\theta\, c_3^{1-\theta}\,(\lambda - \omega)^{-\delta}\,\|B_p^D u\|_{L_p(\Omega)} + c_1^\theta\, c_3^{1-\theta}\,\lambda\,(\lambda - \omega)^{-\delta}\,\|u\|_{L_p(\Omega)}$$

for all $u \in D(B_p^D)$ and $\lambda > \omega \vee \omega_2$. So $\gamma_{p'}^* \, B_p^D$ and hence also $\beta\,\gamma_{p'}^* \, B_p^D$ and $\gamma_p\,M_\beta\,\gamma_{p'}^* \, B_p^D$ are B_p^D-bounded with relative bound zero.

Using (7) it now follows from Lemma 4.4 and a standard perturbation argument that $-\widehat{\mathbb{A}}_p$ is the generator of a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. This completes the proof of Statement (b).

'(c)'. We use again the perturbation (7). Write $\widehat{\mathbb{A}}_p^{(0)} = \begin{pmatrix} B_p^D & 0 \\ 0 & \beta\mathcal{N}_p \end{pmatrix}$. The operator B_p^D has maximal L_r-regularity by [HP] Example B and it is the minus-generator of a holomorphic semigroup. So together with Corollaries 3.8 and 3.9 the operator $\widehat{\mathbb{A}}_p^{(0)}$ has maximal L_r-regularity on $L_p(\Omega) \times L_p(\Gamma)$ and it is the minus-generator of a holomorphic semigroup. It follows from Lemma 4.4 that for all $\varepsilon > 0$ there exists an isomorphism $S\colon L_p(\Omega) \times L_p(\Gamma) \to L_p(\Omega) \times L_p(\Gamma)$ such that $S\,\widehat{\mathbb{A}}_p^{(0)}\,S^{-1} = \widehat{\mathbb{A}}_p^{(0)}$ and $S\,(\widehat{\mathbb{A}}_p - \widehat{\mathbb{A}}_p^{(0)})\,S^{-1}$ is relatively $\widehat{\mathbb{A}}_p^{(0)}$-bounded with $\widehat{\mathbb{A}}_p^{(0)}$-bound ε. If ε is small enough, then [KW] Corollary 2 implies that

$$S\,\widehat{\mathbb{A}}_p\,S^{-1} = \widehat{\mathbb{A}}_p^{(0)} + S\,(\widehat{\mathbb{A}}_p - \widehat{\mathbb{A}}_p^{(0)})\,S^{-1}$$

has maximal L_r-regularity on $L_p(\Omega) \times L_p(\Gamma)$. But then also $\widehat{\mathbb{A}}_p$ has maximal L_r-regularity first on $L_p(\Omega) \times L_p(\Gamma)$ and then also on $\mathbb{L}_p$. □

For all $p \in (1,\infty)$ the operator

$$\begin{pmatrix} I & -\gamma_p \\ 0 & I \end{pmatrix}$$

is an invertible operator from $\mathbb{L}_p$ onto $\mathbb{L}_p$ with inverse

$$\begin{pmatrix} I & \gamma_p \\ 0 & I \end{pmatrix}.$$

We define the operator $\mathbb{A}_p$ in $\mathbb{L}_p$ by

$$\mathbb{A}_p = \begin{pmatrix} I & \gamma_p \\ 0 & I \end{pmatrix} \widehat{\mathbb{A}}_p \begin{pmatrix} I & -\gamma_p \\ 0 & I \end{pmatrix}.$$

A reformulation and extension of Theorem 4.1 is the following theorem.

Theorem 4.5. *Adopt the notation and assumptions as in Theorem 4.1. Let $p \in (1,\infty)$. Then one has the following.*

(a) $\mathbb{A} = \mathbb{A}_2$.

(b) *The operator $\mathbb{A}_p$ is consistent with $\mathbb{A}_2$.*

(c) *The operator $-\mathbb{A}_p$ is the generator of a C_0-semigroup in $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$.*

(d) *For all $r \in (1,\infty)$ the operator $\mathbb{A}_p$ has maximal L_r-regularity on $\mathbb{L}_p$.*

Proof. The proofs of Statements (b), (c) and (d) are obvious.

'(a)'. Let $p \in (d + 2\kappa, \infty)$. We first prove that $\mathbb{A}_p \subset \mathbb{A}$. Let $(u, \varphi) \in D(\mathbb{A}_p)$. Write $(f, \eta) = \mathbb{A}_p(u, \varphi)$. Then $(u - \gamma_p\varphi, \varphi) \in D(\widehat{\mathbb{A}}_p)$ and $\widehat{\mathbb{A}}_p(u - \gamma_p\varphi, \varphi) = (f - \gamma_p\eta, \eta)$. So

$$u - \gamma_p\varphi \in D(B_p^D), \tag{11}$$

$$\varphi \in D(\mathcal{N}_p), \tag{12}$$

$$(B_p^D + \gamma_p\, M_\beta\, \gamma_{p'}^*\, B_p^D)(u - \gamma_p\varphi) - \gamma_p(\beta\, \mathcal{N}_p\varphi) - \gamma_p(\alpha\, \varphi) = f - \gamma_p\eta, \text{ and} \tag{13}$$

$$(-\beta\, \gamma_{p'}^*\, B_p^D)(u - \gamma_p\varphi) + \beta\, \mathcal{N}_p\varphi + \alpha\, \varphi = \eta. \tag{14}$$

If follows from (12) that $\varphi \in D(\mathcal{N}_2) \subset \operatorname{Tr} W^{1,2}(\Omega)$. Hence $\gamma_p\varphi = \gamma\varphi \in W^{1,2}(\Omega)$. Then (11) implies that $u - \gamma\varphi = u - \gamma_p\varphi \in D(B_p^D) \subset D(B_2^D) \subset W_0^{1,2}(\Omega)$. So $0 = \operatorname{Tr}(u - \gamma\varphi) = \operatorname{Tr} u - \varphi$ and $\operatorname{Tr} u = \varphi$. It follows from Proposition 4.2 that $(B_p^D)^{-1} v \in D(\partial_\nu^C)$ and $\partial_\nu^C\, (B_p^D)^{-1} v = -\gamma_{p'}^* v$ for all $v \in L_p(\Omega)$. Since $u - \gamma_p\varphi \in D(B_p^D)$ by (11) one can choose $v = B_p^D(u - \gamma_p\varphi)$ to deduce that $(u - \gamma_p\varphi) \in D(\partial_\nu^C)$ and $\partial_\nu^C(u - \gamma_p\varphi) = -\gamma_{p'}^*\, B_p^D(u - \gamma_p\varphi)$.

One obtains from (13) and (14) that

$$B_p^D(u - \gamma_p\varphi) - \gamma_p\, M_\beta\, \partial_\nu^C(u - \gamma_p\varphi) - \gamma_p(\beta\, \mathcal{N}_p\varphi) - \gamma_p(\alpha\, \varphi) = f - \gamma_p\eta \quad \text{and}$$

$$\beta\, \partial_\nu^C(u - \gamma_p\varphi) + \beta\, \mathcal{N}_p\varphi + \alpha\, \varphi = \eta.$$

Hence $B_2^D(u - \gamma\varphi) = B_p^D(u - \gamma_p\varphi) = f$. Taking the inner product with $\tau \in C_c^\infty(\Omega)$ gives $\mathfrak{b}(u, \tau) = \mathfrak{b}(u - \gamma\varphi, \tau) = (f, \tau)_{L_2(\Omega)}$. So $\mathcal{B}u = f$. Finally, since $\varphi \in D(\mathcal{N}_2)$ it follows that $\gamma\varphi$ has a co-normal derivative and $\partial_\nu^C\gamma\varphi = \mathcal{N}_2\varphi$. Hence u has a co-normal derivative and

$$\eta = \beta\, \partial_\nu^C(u - \gamma_p\varphi) + \beta\, \mathcal{N}_p\varphi + \alpha\, \varphi = \beta\, \partial_\nu^C u + \alpha \operatorname{Tr} u.$$

Now it follows from Lemma 2.1 (ii)$\Rightarrow$(i) that $(u, \varphi) \in D(\mathbb{A})$ and $\mathbb{A}(u, \varphi) = (f, \eta)$. So $\mathbb{A}_p \subset \mathbb{A}$.

Finally we shall prove that $\mathbb{A} = \mathbb{A}_2$. Take $p = d + 3\kappa$. We denote by $G(\mathbb{A}_p)$, $G(\mathbb{A}_2)$ and $G(\mathbb{A})$ the graphs of the operators $\mathbb{A}_p$, $\mathbb{A}_2$ and $\mathbb{A}$.

The operators $-\mathbb{A}_p$ and $-\mathbb{A}_2$ generate consistent semigroups by Statement (b). So $D(\mathbb{A}_p)$ is a core for $\mathbb{A}_2$ by [ER] Lemma 3.8, that is $\overline{G(\mathbb{A}_p)} = G(\mathbb{A}_2)$, where the closure is in $\mathbb{L}_2 \times \mathbb{L}_2$. We just showed that $G(\mathbb{A}_p) \subset G(\mathbb{A})$. The operator $\mathbb{A}$ is a closed operator, so the $G(\mathbb{A})$ is closed in $\mathbb{L}_2 \times \mathbb{L}_2$. Hence $G(\mathbb{A}_2) = \overline{G(\mathbb{A}_p)} \subset G(\mathbb{A})$. So $\mathbb{A}$ is an extension of $\mathbb{A}_2$. Next, both $-\mathbb{A}_2$ and $-\mathbb{A}$ generate a C_0-semigroup. Hence $\mathbb{A} = \mathbb{A}_2$. This completes the proof of the theorem and also of Theorem 4.1. □

5 The operator in the space of continuous functions

In Section 4 we proved that the semigroup generated by $-\mathbb{A}$ extends consistently to a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$. In this section we aim to prove a similar result on the space of continuous function which satisfy a trace condition. Define

$$X_c = \{(u, \varphi) \in C(\overline{\Omega}) \times C(\Gamma) : u|_\Gamma = \varphi\}.$$

Then X_c is naturally isomorphic with $C(\overline{\Omega})$. Let $\mathbb{A}_c$ be the part of $\mathbb{A}$ in X_c. The main theorem of this section is as follows. We emphasise that we do not assume that α or β are continuous.

Theorem 5.1. *Adopt the notation and assumptions as in Theorem 4.1. Then $-\mathbb{A}_c$ is the generator of a C_0-semigroup in X_c which is holomorphic with angle $\frac{\pi}{2}$.*

The proof requires again some preparation. Throughout the remainder of this section we adopt the notation and assumptions as in Theorem 4.1 and Section 4. Again by Lemma 2.2 we may assume that $0 \notin \sigma(B_2^D)$.

We need a maximal version for B^D on $L_\infty(\Omega)$ similar to $\mathcal{N}_{\infty m}$. Define the operator $B_{\infty m}^D \colon D(B_{\infty m}^D) \to L_\infty(\Omega)$ by

$$D(B_{\infty m}^D) = \{u \in D(B_2^D) : B_2^D u \in L_\infty(\Omega)\}$$

and $B_{\infty m}^D = B_2^D|_{D(B_{\infty m}^D)}$. Then $D(B_{\infty m}^D) \subset C_0(\Omega)$ by [AE3] Corollary 2.10. We consider $B_{\infty m}^D$ as a non-densely defined operator in $L_\infty(\Omega)$. Then $B_{\infty m}^D$ is a closed operator.

Lemma 5.2.

(a) *If $\lambda \in \rho(-B_2^D)$, then $\lambda \in \rho(-B_{\infty m}^D)$ and $(B_{\infty m}^D + \lambda I)^{-1} = (B_2^D + \lambda I)^{-1}|_{L_\infty(\Omega)}$.*

(b) *The operator $B_{\infty m}^D$ is sectorial of angle $\frac{\pi}{2}$.*

(c) *$D(B_{\infty m}^D)$ is dense in $C_0(\Omega)$.*

(d) *If $u \in D(B_{\infty m}^D)$, then $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u \in C(\Gamma)$.*

(e) *The operator ∂_ν^C is relatively $B_{\infty m}^D$-bounded with relative $B_{\infty m}^D$-bound zero.*

Proof. '(a)'. Easy.

'(b)'. This follows from the Gaussian kernel bounds for the semigroup generated by $-B_2^D$.

'(c)'. Let B_c^D be the part of B_2^D in $C_0(\Omega)$. Then $B_c^D \subset B_{\infty m}^D$. Moreover, $-B_c^D$ is the generator of a holomorphic C_0-semigroup by [AE3] Theorem 1.3. Now let $u \in C_0(\Omega)$. Then $\lim_{t\downarrow 0} e^{-tB_c^D} u = u$ in $C_0(\Omega)$. But $e^{-tB_c^D} u \in D(B_c^D) \subset D(B_{\infty m}^D)$ for all $t > 0$.

'(d)'. See Proposition 4.2.

'(e)'. Let $p = d + 3\kappa$. If $u \in D(B_{\infty m}^D)$, then $u \in C_0(\Omega) \cap D(B_2^D) \subset L_p(\Omega) \cap D(B_2^D)$ and $B_2^D u \in L_\infty(\Omega) \subset L_p(\Omega)$. Hence $u \in D(B_p^D)$. The Gaussian derivative bounds of [EO2] Theorem 3.1(a) give that there exist $c, \omega > 0$ such that

$$\|\nabla e^{-tB_p^D} u\|_{L_\infty(\Omega)} \le c\, t^{-1/2}\, e^{\omega t}\, \|u\|_{L_\infty(\Omega)}$$

for all $t > 0$ and $u \in L_\infty(\Omega)$. Arguing as in (8) one deduces that there is a $c' > 0$ such that

$$\|\partial_\nu^C (B_{\infty m}^D + \lambda I)^{-1} u\|_{C(\Gamma)} = \|\partial_\nu^C (B_p^D + \lambda I)^{-1} u\|_{C(\Gamma)} \le c' (\lambda - \omega)^{-1/2} \|u\|_{L_\infty(\Omega)}$$

for all $u \in L_\infty(\Omega)$ and $\lambda > \omega$. So

$$\begin{aligned}\|\partial_\nu^C u\|_{C(\Gamma)} &\le c' (\lambda - \omega)^{-1/2} \|(B_{\infty m}^D + \lambda I) u\|_{L_\infty(\Omega)} \\ &\le c' (\lambda - \omega)^{-1/2} \|B_{\infty m}^D u\|_{L_\infty(\Omega)} + c' (\lambda - \omega)^{-1/2} \lambda\, \|u\|_{L_\infty(\Omega)}\end{aligned}$$

and the statement follows. □

By Lemma 5.2(d) we can define $\widetilde{\mathbb{A}}_\infty \colon D(B_{\infty m}^D) \times D(\mathcal{N}_{\infty m}) \to L_\infty(\Omega) \times L_\infty(\Gamma)$ by

$$\widetilde{\mathbb{A}}_\infty = \begin{pmatrix} B_{\infty m}^D - \gamma_\infty M_\beta\, \partial_\nu^C & -\gamma_\infty M_\beta \mathcal{N}_{\infty m} - \gamma_\infty M_\alpha \\ \beta\, \partial_\nu^C & \beta \mathcal{N}_{\infty m} + M_\alpha \end{pmatrix}.$$

We consider $\widetilde{\mathbb{A}}_\infty$ as a non-densely defined closed operator in $L_\infty(\Omega) \times L_\infty(\Gamma)$.

Proposition 5.3.

(a) *The operator* $\widetilde{\mathbb{A}}_\infty$ *is sectorial of angle* $\frac{\pi}{2}$.

(b) $\widetilde{\mathbb{A}}_\infty \subset \widehat{\mathbb{A}}_p$ *for all* $p \in (1, \infty)$.

Proof. '(a)'. By Lemma 5.2(b) and Lemma 3.10(b) the operator $\begin{pmatrix} B_{\infty m}^D & 0 \\ 0 & \beta \mathcal{N}_{\infty m} \end{pmatrix}$ is sectorial of angle $\frac{\pi}{2}$. Then the statement follows from [EF] Lemma A.4 and Lemma 5.2(e).

'(b)'. By Lemma 4.3 it suffices to prove the statement for all $p \in (d + 2\kappa, \infty)$. Let $p \in (d + 2\kappa, \infty)$. Then $\partial_\nu^C u = -\gamma_{p'}^* B_p^D u$ for all $u \in D(B_p^D)$ by Proposition 4.2(c) and the inclusion follows. □

The domain of $\widetilde{\mathbb{A}}_\infty$ is not dense in $L_\infty(\Omega) \times L_\infty(\Gamma)$. We take a suitable restriction. Let $\widehat{\mathbb{A}}_c$ be the part of $\widetilde{\mathbb{A}}_c$ in $C_0(\Omega) \times C(\Gamma)$.

Proposition 5.4.

(a) $-\widehat{\mathbb{A}}_c$ *is the generator of a* C_0*-semigroup in* $C_0(\Omega) \times C(\Gamma)$ *which is holomorphic with angle* $\frac{\pi}{2}$.

(b) $\widehat{\mathbb{A}}_c \subset \widehat{\mathbb{A}}_p$ *for all* $p \in (1, \infty)$.

(c) $\widehat{\mathbb{A}}_c$ *is the part of* $\widehat{\mathbb{A}}_2$ *in* $C_0(\Omega) \times C(\Gamma)$.

Proof. '(a)'. It follows from Lemma 5.2(c) and Lemma 3.10(c) that $D(\widetilde{\mathbb{A}}_\infty)$ is dense in $C_0(\Omega) \times C(\Gamma)$. Also the operator $\widetilde{\mathbb{A}}_\infty$ is sectorial with angle $\frac{\pi}{2}$ by Proposition 5.3(a). Now the statement follows from [ABHN] Remark 3.7.13.

'(b) and (c)'. This follows from the definition of $\widehat{\mathbb{A}}_c$ and Proposition 5.3(b). □

Now we are able to proof the main theorem in this section.

Proof of Theorem 5.1. Define $\gamma_c \colon C(\Gamma) \to C(\overline{\Omega})$ by $\gamma_c = \gamma_\infty|_{C(\Gamma)}$. So $\gamma_c(\varphi)$ is the classical solution of the Dirichlet problem with boundary data φ. The operator $\begin{pmatrix} I & -\gamma_c \\ 0 & I \end{pmatrix}$ is maps X_c onto $C_0(\Omega) \times C(\Gamma)$. Therefore Proposition 5.4(c) implies that

$$\mathbb{A}_c = \begin{pmatrix} I & \gamma_c \\ 0 & I \end{pmatrix} \widehat{\mathbb{A}}_c \begin{pmatrix} I & -\gamma_c \\ 0 & I \end{pmatrix}$$

and the theorem follows from Proposition 5.4(a). □

Since $u \mapsto (u, u|_\Gamma)$ is an isomorphism from $C(\overline{\Omega})$ onto X_c one can reformulate Theorem 5.1. Recall once again that we do not require that α and β are continuous.

Theorem 5.5. *Adopt the notation and assumptions as in Theorem 4.1. Define the operator* A_c *in* $C(\overline{\Omega})$ *by*

$$D(A_c) = \{u \in C(\overline{\Omega}) \cap D(\partial_\nu^C) : \mathcal{B}u \in C(\overline{\Omega}) \text{ and } (\mathcal{B}u)|_\Gamma = \beta\, \partial_\nu^C u + \alpha\, u|_\Gamma \text{ a.e. on } \Gamma\}$$

and $A_c u = \mathcal{B}u$ *for all* $u \in D(A_c)$. *Then* $-A_c$ *is the generator of a* C_0*-semigroup in* $C(\overline{\Omega})$ *which is holomorphic with angle* $\frac{\pi}{2}$.

Using the arguments as in [War] we obtain a C_0-semigroup on $\mathbb{L}_1$ with optimal angle. For the convenience of the reader we give a direct proof.

Corollary 5.6. *Adopt the notation and assumptions as in Theorem 4.1. The semigroup generated by* $-\mathbb{A}$ *extends consistently to a* C_0*-semigroup on* $\mathbb{L}_1$ *which is holomorphic with angle* $\frac{\pi}{2}$.

Proof. Since the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ is of the same type as $\mathfrak{a}$ with α replaced by $\overline{\alpha}$, all the above is also valid for $\mathbb{A}^*$ instead of $\mathbb{A}$. Let $(\mathbb{A}^*)_c$ be the part of $\mathbb{A}^*$ in X_c.

Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Let $(u, \varphi) \in \mathbb{L}_2$ and $(v, \psi) \in X_c$. Then

$$\begin{aligned}
|(e^{-z\mathbb{A}}(u,\varphi),(v,\psi))| &= |((u,\varphi), e^{-\overline{z}\mathbb{A}^*}(v,\psi))| \\
&= |((u,\varphi), e^{-\overline{z}(\mathbb{A}^*)_c}(v,\psi))| \leq \|(u,\varphi)\|_{\mathbb{L}_1} \, \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c \to X_c} \, \|(v,\psi)\|_{\mathbb{L}_\infty}.
\end{aligned}$$

Hence $\|e^{-z\mathbb{A}}(u,\varphi)\|_{\mathbb{L}_1} \leq \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c \to X_c} \, \|(u,\varphi)\|_{\mathbb{L}_1}$. By density the operator $e^{-z\mathbb{A}}$ extends to a continuous operator $S^{(1)}_z$ from $\mathbb{L}_1$ into $\mathbb{L}_1$ with norm $\|S^{(1)}_z\|_{\mathbb{L}_1 \to \mathbb{L}_1} \leq \|e^{-\overline{z}(\mathbb{A}^*)_c}\|_{X_c \to X_c}$. It is easy to verify that $S^{(1)}_z \, S^{(1)}_w = S^{(1)}_{z+w}$ for all $z, w \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. Also for all $\theta \in (0, \frac{\pi}{2})$ there are $M, \omega > 0$ such that $\|S^{(1)}_z\|_{\mathbb{L}_1 \to \mathbb{L}_1} \leq M \, e^{\omega|z|}$ for all $z \in \Sigma_\theta$. Hence $z \mapsto \langle S^{(1)}_z(u,\varphi), (v,\psi)\rangle_{\mathbb{L}_1 \times \mathbb{L}_\infty}$ is holomorphic on Σ°_θ for all $(v,\psi) \in \mathbb{L}_\infty$ and $(u,\varphi) \in \mathbb{L}_2$, but then also for all $(u,\varphi) \in \mathbb{L}_1$. Since the measure on $\Omega \oplus \Gamma$ is finite and $(e^{-z\mathbb{A}})_{z \in \Sigma^\circ_\theta}$ is continuous it follows that $(e^{-z\mathbb{A}})_{z \in \Sigma^\circ_\theta}$ is (weakly) continuous for all $\theta \in (0, \frac{\pi}{2})$. $\square$

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A.3 Additional Manuscripts

A.3.1 Spectral theory, positivity and stability for operators with Wentzell boundary conditions and the associated Dirichlet-to-Neumann operators

SPECTRAL THEORY, POSITIVITY AND STABILITY FOR OPERATORS WITH WENTZELL BOUNDARY CONDITIONS AND THE ASSOCIATED DIRICHLET-TO-NEUMANN OPERATORS

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Abstract. We study the rich interplay between various spectral, positivity and stability properties of an operator A^B with generalized Wentzell boundary conditions, the associated Dirichlet-to-Neumann operators N_λ and the related operators with abstract Robin boundary conditions A^B_μ, cf. Definition 2.4 below. The results are then illustrated by various examples.

1. Introduction

The goal of this paper is to study the relations between an operator with abstract Wentzell-type boundary conditions and the associated Dirichlet-to-Neumann operators. To give a typical example, cf. Subsection 7.4 for a more general setting, we consider on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ the Laplacian $\Delta_W \subset \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ with *Wentzell boundary conditions* given by

$$\Delta_W := \Delta f,$$
$$D(\Delta_W) := \left\{ f \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\bar{\Omega}) \;\middle|\; \begin{array}{l} \Delta f \in \mathrm{C}(\bar{\Omega}), \\ \Delta f = a\frac{\partial}{\partial n} f + bf \text{ on } \partial\Omega \end{array} \right\}.$$

Here Δ denotes the Laplacian with "maximal" domain, i.e., without boundary conditions, $a, b \in \mathrm{C}(\partial\Omega)$ where a is strictly positive and $\frac{\partial}{\partial n}$ is the outer normal derivative. As we will see, this operator is closely connected to the associated *Dirichlet-to-Neumann Operators* $N_\lambda : D(N_\lambda) \subset \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ which is defined by

$$N_\lambda \varphi := -\frac{\partial}{\partial n} f_\varphi,$$

where f_φ is the (unique) solution of the Dirichlet problem

$$\begin{cases} \Delta f_\varphi = \lambda f_\varphi, \\ f_\varphi|_{\partial\Omega} = \varphi \end{cases}$$

and $\varphi \in D(N_\lambda)$ if and only if $f_\varphi \in D(\frac{\partial}{\partial n})$. Further, denote by Δ_D and Δ_N the Laplacian with *Dirichlet boundary conditions* and the Laplacian with *Neumann boundary conditions*,

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respectively, i.e

$$\Delta_D f := \Delta f, \quad \Delta_N f := \Delta f,$$
$$D(\Delta_D) := \left\{ f \in \bigcap_{p\geq 1} W^{2,p}_{\text{loc}}(\Omega) \cap \mathrm{C}(\bar\Omega) \,\middle|\, \begin{array}{l} \Delta f \in \mathrm{C}(\bar\Omega), \\ f = 0 \text{ on } \partial\Omega \end{array} \right\},$$
$$D(\Delta_N) := \left\{ f \in \bigcap_{p\geq 1} W^{2,p}_{\text{loc}}(\Omega) \cap \mathrm{C}(\bar\Omega) \,\middle|\, \begin{array}{l} \Delta f \in \mathrm{C}(\bar\Omega), \\ \frac{\partial}{\partial n} f = 0 \text{ on } \partial\Omega \end{array} \right\}.$$

Then their spectra are discrete and satisfy $\sigma(\Delta_D) = \sigma_p(\Delta_D) = \{0 > \lambda_1^D \geq \lambda_2^D \geq \cdots \to -\infty\}$ and $\sigma(\Delta_N) = \sigma_p(\Delta_N) = \{0 = \lambda_1^N \geq \lambda_2^N \geq \cdots \to -\infty\}$. By Friedlander's inequality (cf. [Fri91]) we have the relation

$$\lambda_k^D < \lambda_{k+1}^N$$

for all $k \in \mathbb{N}$. Friedlander's proof is based on the existence of a positive eigenvalue of the Dirichlet-to-Neumann operators N_λ for all $\lambda \in (-\infty, 0) \cap \rho(\Delta_D)$.

Moreover, in [AMPR03] it is proven that Δ_W generates a positive semigroup on $\mathrm{C}(\bar\Omega)$. Further in [Esc94] it is shown that N_λ generate positive semigroups on $\mathrm{C}(\partial\Omega)$ for large $\lambda \in \rho(\Delta_D)$. Both results are proven independently but as we will see they are basically equivalent.

The aim of this paper is to reformulate this example within an abstract general framework and then to formulate and prove the above and various other results relating properties of operators with Dirichlet, Neumann, Robin and Wentzell boundary conditions and the associated Dirichlet-to-Neumann operators within this setting.

To this end we first introduce the Banach spaces $X := \mathrm{C}(\bar\Omega)$ and $\partial X := \mathrm{C}(\partial\Omega)$, both equipped with the sup-norm. On X we consider the *"maximal" operator* $A_m := \Delta : D(A_m) \subset X \to X$. Then for the *trace operator* $L : X \to \partial X$, $Lf := f|_{\partial\Omega}$ and the *boundary operator* $B : D(B) \subset X \to \partial X$, $Bf := a\frac{\partial}{\partial n} f + bLf$ for $f \in D(B) := D(\frac{\partial}{\partial n})$ we have $\Delta_W = A^B : D(A^B) \subset X \to X$ where

$$A^B f := A_m f, \quad D(A^B) := \bigl\{ f \in D(A_m) \cap D(B) \colon LA_m f = Bf \bigr\}.$$

Moreover, if we define $L_0 : \partial X \to X$ by $L_0\varphi := f_\varphi$, then

$$N := BL_0, \quad D(N) := \bigl\{ \varphi \in \partial X \colon L_0\varphi \in D(B) \bigr\}.$$

In this way we succeeded to represent Δ_W and N within our general setting which we introduce in the following section.

This paper is organized as follows. In Section 2 we set up our abstract framework and introduce all relevant operators. In Section 3 we study the relationship of spectral properties between operators with Wentzell boundary conditions and its associated Dirichlet-to-Neumann operators. Many spectral properties of the operator with Wentzell boundary conditions are reflected by its Dirichlet-to-Neumann operator, cf. Theorem 3.7. Moreover, a resolvent formula for the operator with Wentzell boundary conditions is proven, see Theorem 3.8. The following Section 4 is dedicated to a deeper analysis of the spectral properties of Dirichlet-to-Neumann operators. We concentrate on the relationship to the operators with Robin boundary conditions. An abstract version of Friedlander's inequality is shown, cf. Theorem 4.14. In Section 5 we investigate positivity of the semigroups generated by the operators with Wentzell boundary conditions and its Dirichlet-to-Neumann operators. It turns out that these properties are deeply related, see Theorem 5.10. In Section 6 we combine our spectral and positivity results and obtain a stability result for operators with Wentzell boundary conditions, cf. Theorem 6.2. Finally, in Section 7 we demonstrate the benefits of our approach on some concrete examples.

2. The abstract Framework

We start by introducing our general

Abstract Setting 2.1. Consider

(i) two Banach spaces X and ∂X called *state* and *boundary space*, respectively;
(ii) a densely defined and closed *"maximal" operator* $A_m\colon D(A_m) \subset X \to X$;
(iii) an surjective *trace operator* $L\colon X \to \partial X$;
(iv) a *boundary operator* $B\colon D(B) \subset X \to \partial X$.

For our investigations we need to make the following hypotheses which are verified in all relevant examples, cf. Section 7.

Assumptions 2.2. (i) The operator $A_0 := A_m|_{\ker L}$ with abstract "Dirichlet" boundary conditions is a *weak Hille–Yosida operator* on X, i.e., there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A_0)$ and

$$\|\lambda R(\lambda, A_0)\| \leq C$$

for every $\lambda > \omega$;
(ii) the operator B is relatively A_0-bounded with bound 0, i.e., $D(A_0) \subseteq D(B)$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|Bf\|_{\partial X} \leq \varepsilon \cdot \|A_0 f\|_X + C_\varepsilon \cdot \|f\|_X \quad \text{for all } f \in D(A_0); \tag{2.1}$$

(iii) the abstract Dirichlet operator $L_{\lambda_0} := (L|_{\ker(\lambda_0 - A_m)})^{-1} \in \mathcal{L}(\partial X, X)$ exists for some $\lambda_0 \in \rho(A_0)$

Remark 2.3. Note that by [Gre87, Lem. 1.3] condition (iii) implies that $L_\lambda \in \mathcal{L}(\partial X, X)$ exists for all $\lambda \in \rho(A_0)$.

Definition 2.4. Using these spaces and operators we define the *operator with generalized Wentzell boundary conditions* $A^B\colon D(A^B) \subset X \to X$ by

$$A^B f := A_m f, \quad D(A^B) := \{f \in D(A_m) \cap D(B)\colon LA_m f = Bf\}.$$

Moreover, for $\lambda \in \rho(A_0)$ we introduce the *Dirichlet-to-Neumann operator* $N_\lambda : D(N_\lambda) \subset \partial X \to \partial X$ by

$$N_\lambda := BL_\lambda, \quad D(N_\lambda) := \{\varphi \in \partial X\colon L_\lambda \varphi \in D(B)\}.$$

Finally, for $\mu \in \mathbb{C}$ we denote by $A_B^\mu : D(A_B^\mu) \subset X \to X$ the *operator with abstract Robin boundary conditions*

$$A_B^\mu f := A_m f, \quad D(A_B^\mu) := \{f \in D(A_m) \cap D(B)\colon Bf = \mu Lf\}.$$

3. Spectral theory for operators with Wentzell boundary conditions

In the sequel we study the close relation between the point spectra of operators with Robin boundary conditions and Dirichlet-to-Neumann operators. For elliptic differential operators this relationship was observed by Arendt and Mazzeo in [AM12]. This section is inspired by their work.

Proposition 3.1. *The Dirichlet-operator L_λ maps* $\ker(\mu - N_\lambda)$ *onto* $\ker(\lambda - A_B^\mu)$. *Hence,* $L\colon \ker(\lambda - A_B^\mu) \to \ker(\mu - N_\lambda)$ *is an isomorphism.*

Proof. For $\varphi \in \ker(\mu - N_\lambda)$ we obtain

$$BL_\lambda \varphi = N_\lambda \varphi = \mu L L_\lambda \varphi$$

and hence $L_\lambda \varphi \in \ker(\lambda - A_B^\mu)$. Conversely, consider $f \in \ker(\lambda - A_B^\mu) \subset \ker(\lambda - A_m)$. Then exists $\varphi \in \partial X$ such that $L_\lambda \varphi = f$. Since $f \in D(A_B^\mu)$ one has

$$\mu\varphi = \mu L f = Bf = BL_\lambda\varphi = N_\lambda\varphi$$

and hence $\varphi \in \ker(\mu - N_\lambda)$. Thus, $L_\lambda \colon \ker(\mu - N_\lambda) \to \ker(\lambda - A_B^\mu)$ is onto.
The Dirichlet operator $L_\lambda \colon \partial X \to \ker(\lambda - A_m)$ is an isomorphism, in particular it is injective. Therefore, $L_\lambda \colon \ker(\mu - N_\lambda) \to \ker(\lambda - A_B^\mu)$ is bijective with inverse $L|_{\ker(\lambda - A_B^\mu)} \in \mathcal{L}(X, \partial X)$ and hence an isomorphism. □

Corollary 3.2. *For $\lambda \in \rho(A_0)$ and $\mu \in \mathbb{C}$ we have*
- (i) $\mu \in \sigma_p(N_\lambda) \iff \lambda \in \sigma_p(A_B^\mu)$;
- (ii) $\dim(\ker(\mu - N_\lambda)) = \dim(\ker(\lambda - A_B^\mu))$.

From the previous one obtains the following spectral relation between Dirichlet-to-Neumann operators and Neumann-to-Dirichlet operators.

Corollary 3.3. *Let $L \colon D(L) \subset X \to \partial X$ and $B \colon D(B) \subset X \to \partial X$ such that $D(A_m) \subset D(L), D(B) \subset X$ and that the abstract Dirichlet operators $L_\lambda, B_\lambda \in \mathcal{L}(\partial X, X)$ exist. Denote by $N_\lambda^{B,L} = BL_\lambda$ the Dirichlet-to-Neumann operator with respect to (B, L) and by $N_\lambda^{L,B} = LB_\lambda$ the Neumann-to Dirichlet- operator with respect to (L, B). If $\lambda \in \rho(A_0) \cap \rho(A_B)$ for $\mu \neq 0$ we have*

$$\mu \in \sigma_p\big(N_\lambda^{B,L}\big) \iff \tfrac{1}{\mu} \in \sigma_p\big(N_\lambda^{L,B}\big).$$

Proof. Denote by $A_{B,L}^\mu \subset A_m$ the operator with Robin boundary conditions given by $D(A_{B,L}^\mu) = \{f \in D(A_m) \colon Bf = \mu L f\}$. Note that $Bf = \mu Lf$ is equivalent to $Lf = \frac{1}{\mu} Bf$ and hence $A_{B,L}^\mu = A_{L,B}^{1/\mu}$. Therefore Corollary 3.2 implies

$$\mu \in \sigma_p(N_\lambda^{B,L}) \iff \lambda \in \sigma_p(A_{B,L}^\mu) \iff \lambda \in \sigma_p(A_{L,B}^{1/\mu}) \iff 1/\mu \in \sigma_p(N_\lambda^{L,B}). \qquad \square$$

Next we relate the spectra of the Dirichlet-to-Neumann operators N_λ and the operator A^B with Wentzell boundary conditions. To this end, we consider the Banach spaces $\tilde{\mathfrak{X}} := \{\binom{f}{x} \in X \times \partial X : Lf = x\}$ and $\mathfrak{X} := X \times \partial X$. Moreover, we introduce the operators $\mathcal{A}^B \colon D(\mathcal{A}^B) \subset \tilde{\mathfrak{X}} \to \tilde{\mathfrak{X}}$ given by

$$\mathcal{A}^B = \begin{pmatrix} A_m & 0 \\ B & 0 \end{pmatrix}, \quad D(\mathcal{A}^B) = \{\tbinom{f}{x} \in (D(A_m) \cap D(B)) \times \partial X \colon Lf = x, LA_m f = Bf\}$$

and $\hat{\mathcal{A}}_\lambda \colon D(\hat{\mathcal{A}}_\lambda) \subset \mathfrak{X} \to \mathfrak{X}$ given by

$$\lambda - \hat{\mathcal{A}}_\lambda = \begin{pmatrix} \lambda - A_0 & 0 \\ -B & \lambda - N_\lambda \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -L_\lambda \\ 0 & \mathrm{Id} \end{pmatrix}, \quad D(\hat{\mathcal{A}}_\lambda) = \{\tbinom{f}{x} \in X \times D(N_\lambda) \colon f - L_\lambda x \in D(A_0)\}$$

for $\lambda \in \rho(A_0)$. Note that $\mathcal{A}^B$ is similar to A^B and that $\hat{\mathcal{A}}$ can be rewritten as

$$\lambda - \hat{\mathcal{A}}_\lambda = \begin{pmatrix} \mathrm{Id} & 0 \\ -BR(\lambda, A_0) & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - N_\lambda \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -L_\lambda \\ 0 & \mathrm{Id} \end{pmatrix} \tag{3.1}$$

Lemma 3.4. *We have*

$$[D(\hat{\mathcal{A}})] \hookrightarrow \tilde{\mathfrak{X}} \hookrightarrow \mathfrak{X}.$$

Proof. From $f - L_\lambda x \in D(A_0) \subset \ker(L)$ and $LL_\lambda = \mathrm{Id}_{\partial X}$ it follows $Lf = x$ and hence the first inclusion. The second one follows is obvious. □

The following Lemma is analogous to [CENN03, Lem. 2.6 & Prop. 4.2]. It is essentially given in [EF05, Proof of Thm. 3.1, Step. 3]. For completeness we give the proof here.

Lemma 3.5. *For $\lambda \in \rho(A_0)$ we have*

$$\lambda - \mathcal{A}^B = (\lambda - \hat{\mathcal{A}}_\lambda)|_{\tilde{\mathfrak{X}}}.$$

Proof. First note that

$$\begin{aligned}
(\lambda - \hat{\mathcal{A}}_\lambda)\begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} \lambda - A_0 & 0 \\ -B & N_\lambda \end{pmatrix}\begin{pmatrix} \mathrm{Id} & -L_\lambda \\ 0 & Id \end{pmatrix}\begin{pmatrix} f \\ x \end{pmatrix} \\
&= \begin{pmatrix} \lambda - A_0 & 0 \\ -B & N_\lambda \end{pmatrix}\begin{pmatrix} f - L_\lambda x \\ x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_m)f \\ -Bf + \lambda x - N_\lambda x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_m)f \\ -Bf + N_\lambda x + \lambda x - N_\lambda x \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - A_m)f \\ Bf - \lambda x \end{pmatrix}.
\end{aligned}$$

Moreover, since $LL_\lambda = \mathrm{Id}_{\partial X}$, $L_\lambda D(N_\lambda) = \ker(\lambda - A_m) \cap D(B) \subset D(A_m) \cap D(B)$ and $D(A_0) \subset \ker(L)$ we have

$$\begin{aligned}
\begin{pmatrix} f \\ x \end{pmatrix} \in D(\hat{\mathcal{A}}_\lambda|_{\tilde{\mathfrak{X}}}) \quad &\iff \quad f \in X, x \in D(N_\lambda), f - L_\lambda x \in D(A_0) \text{ and} \\
&\qquad (\lambda - \hat{\mathcal{A}}_\lambda)\begin{pmatrix} f \\ x \end{pmatrix} = \begin{pmatrix} (\lambda - A_m)f \\ Bf - \lambda x \end{pmatrix} \in \tilde{\mathfrak{X}} \\
&\iff \quad f \in D(A_m) \cap D(B), Lf = x \text{ and } LA_m f = Bf \\
&\iff \quad \begin{pmatrix} f \\ x \end{pmatrix} \in D(\mathcal{A}^B).
\end{aligned}$$

□

Lemma 3.6. *Let $\lambda_0 \in \rho(A_0)$ such that N_{λ_0} is a weak Hille–Yosida operator on ∂X. Then there exists a $\lambda \in \rho(A_0) \cap (N_\lambda)$.*

Proof. Since N_{λ_0} is a weak Hille–Yosida operator on ∂X there exists a $\omega \in \mathbb{R}$ such that $[\omega, \infty) \subset \rho(N_{\lambda_0})$ and

$$\|R(\lambda, N_{\lambda_0})\| \le \frac{C}{|\lambda - \lambda_0|}.$$

Then

$$\lambda - N_\lambda = \lambda - N_{\lambda_0} - P = (\mathrm{Id} - PR(\lambda, N_{\lambda_0}))(\lambda - N_{\lambda_0}),$$

where $P := (\lambda_0 - \lambda)BR(\lambda, A_0)L_{\lambda_0}$. It remains to show that $1 \in \rho(PR(\lambda, N_{\lambda_0}))$. Since L_{λ_0} is bounded and B is relatively A_0-bounded of bound 0 we obtain

$$\begin{aligned}
\|PR(\lambda, N_{\lambda_0})\| &\le |\lambda - \lambda_0| \cdot \|BR(\lambda, A_0)\| \cdot \|R(\lambda, N_{\lambda_0})\| \\
&\le C \cdot \left(\varepsilon + \frac{C}{|\lambda|}\right) < 1
\end{aligned}$$

for sufficient small $\varepsilon > 0$ and sufficient large $|\lambda|$. Hence, the claim follows by using a Neumann series. □

After these preparations we can compare the spectra of A^B and N_λ.

Theorem 3.7. *Assume, that there exists a $\lambda_0 \in \rho(A_0)$ such that N_{λ_0} is a weak Hille–Yosida operator on ∂X. Then for $\lambda \in \rho(A_0)$ we have*

(i) $\lambda \in \rho(A^B)$ *if and only if* $\lambda \in \rho(N_\lambda)$*;*
(ii) $\lambda \in \sigma_p(A^B)$ *if and only if* $\lambda \in \sigma_p(N_\lambda)$*. In this case* $\dim(\ker(\lambda - A^B)) = \dim(\ker(\lambda - N_\lambda))$*;*
(iii) $\lambda \in \sigma_a(A^B)$ *if and only if* $\lambda \in \sigma_a(N_\lambda)$*;*
(iv) $\lambda \in \sigma_c(A^B)$ *if and only if* $\lambda \in \sigma_c(N_\lambda)$*;*
(v) $\lambda \in \sigma_r(A^B)$ *if and only if* $\lambda \in \sigma_r(N_\lambda)$*;*
(vi) $\lambda \in \sigma_d(A^B)$ *if and only if* $\lambda \in \sigma_d(N_\lambda)$*;*
(vii) $\lambda \in \sigma_{ess}(A^B)$ *if and only if* $\lambda \in \sigma_{ess}(N_\lambda)$*.*

Proof. For the assertion (ii) remark that

$$A^B f = \lambda f$$

implies

$$LA_m f = \lambda L f.$$

Hence, $\lambda \in \sigma_p(A^B)$ if and only if $\lambda \in \rho(A_B^\lambda)$ and the eigenspaces coincide. Now the claim follows by Corollary 3.2.

For assertions (i),(iii)–(v) and (vii) note that the first and the last operator matrix in (3.1) are invertible. Since $\lambda \in \rho(A_0)$ we obtain that

$$\lambda \in \sigma_*(\hat{\mathcal{A}}) \iff \lambda \in \sigma_*(N_\lambda)$$

for $* \in \{a, c, r, ess, \}$. By Lemma 3.4 and Lemma 3.6 the operator $\mathcal{A}^B = \hat{\mathcal{A}}|_{\tilde{\mathfrak{X}}}$ satisfies the assumptions in [AE18, Cor. A.9(vii)]. Now the claim follows by [AE18, Cor. A.9(vii)].

Finally, assertion (vi) follows from (i) and (vii). □

The previous result shows that the spectrum and its fine structure of A^B is characterized by the Dirichlet-to-Neumann operators N_λ. It can be seen as an abstract analogue of the characteristic equation for the spectral values of delay operators.

Moreover we obtain the following useful resolvent formula.

Theorem 3.8. *Let $\lambda \in \rho(A_0) \cap \rho(N_\lambda)$. Then $\lambda \in \rho(A^B)$ and*

$$R(\lambda, A^B) = R(\lambda, A_0) + L_\lambda R(\lambda, N_\lambda) \cdot \big(BR(\lambda, A_0) + L\big). \tag{3.2}$$

Proof. From (3.7) it follows that $\lambda \in \rho(A^B)$. Denote the right hand side of (3.2) by $R(\lambda)$. Since $\mathrm{rg}(L_\lambda) \subset D(A_m)$ it follows $\mathrm{rg}(R(\lambda)) \subset D(A_m)$. Since $D(A_0) \subset D(B)$ and $\mathrm{rg}(L_\lambda R(\lambda, N_\lambda)) \subset D(B)$ it follows $\mathrm{rg}(R(\lambda)) \subset D(B)$. It remains to prove the boundary condition. Note that $LA_m f = Bf$ is equivalent to $L(\lambda - A_m)f = \lambda Lf - Bf$. We obtain

$$L(\lambda - A_m)R(\lambda)f = Lf$$

for $f \in X$. Moreover we have, since $D(A_0) \subset \ker(L)$ and $LL_\lambda = \mathrm{Id}_{\partial X}$

$$\begin{aligned}(\lambda L - B)R(\lambda)f &= -BR(\lambda, A_0)f + (\lambda L - B)L_\lambda R(\lambda, N_\lambda)(BR(\lambda, A_0) + L)f \\ &= -BR(\lambda, A_0)f + (\lambda - N_\lambda)R(\lambda, N_\lambda)(BR(\lambda, A_0) + L)f \\ &= -BR(\lambda, A_0)f + BR(\lambda, A_0)f + Lf = Lf\end{aligned}$$

for $f \in X$. Hence we obtain $LA_mR(\lambda)f = BR(\lambda)f$ and therefore $R(\lambda)f \in D(A^B)$. Since $\mathrm{rg}(L_\lambda) \subset \ker(\lambda - A_m)$ it follows

$$(\lambda - A^B)R(\lambda) = (\lambda - A_m)R(\lambda) = (\lambda - A_m)R(\lambda, A_0) = \mathrm{Id}$$

and hence the claim. □

The resolvent formula (3.2) implies the following compactness results.

Corollary 3.9. *Assume that A_0 has compact resolvent and that one of the following assumptions is satisfied*

(i) *L_λ is compact for some $\lambda \in \rho(A_0)$;*
(ii) *N_λ has compact resolvent for some $\lambda \in \rho(A_0)$;*
(iii) *B is relatively A_0-compact and L is compact.*

Then

(i) *A_B has compact resolvent on X;*
(ii) *N_λ have compact resolvent on ∂X for all $\lambda \in \rho(A_0)$.*

Proof. The first claim follows immediately from Theorem 3.8, whereas the second claim follows from the first one using [BE18, Cor. 3.2]. □

4. Spectral theory for Dirichlet-to-Neumann operators

In the last section we have seen that the spectrum of A^B is deeply connected with the spectra of the Dirichlet-to-Neumann operators N_λ. Unfortunately, it is difficult to compute the spectrum or its parts of N_λ even in case of the point spectrum. For this reason we concentrate in the following on the location of the point spectrum of N_λ.

In Corollary 3.2 we have seen that the point spectra of the Dirichlet-to-Neumann operators are related to the point spectra of operators with Robin boundary conditions. Next we will use this fact to locate the point spectra of the Dirichlet-to-Neumann operators N_λ.
First we consider the spectral properties of the operators with Robin boundary conditions. To this end we need the following operator which can be seen as an analogue of the Dirichlet operator L_λ for Robin instead of Dirichlet boundary conditions.

Definition 4.1. For $\lambda \in \mathbb{C}$ we define the *abstract Robin operator associated with* (λ, A_m, B) by

$$R_{\lambda,\mu} := \left((B - \mu L)|_{\ker(\lambda - A_m)}\right)^{-1} : \partial X \to \ker(\lambda - A_m) \cap D(B) \subseteq X,$$

i.e., $R_{\lambda,\mu}\varphi = f$ gives the solution of the abstract Robin problem

$$\begin{cases} A_m f = \lambda f, \\ Bf - \mu Lf = \varphi. \end{cases} \tag{4.1}$$

The solution of the Robin problem is deeply connected to the resolvent set of the Dirichlet-to-Neumann operator.

Lemma 4.2. *If L_λ exists, we have $\mu \in \rho(N_\lambda)$ if and only if $R_{\lambda,\mu} \in \mathcal{L}(\partial X, X)$ exists. If one of these conditions is satisfied, we obtain*

$$L_\lambda R(\mu, N_\lambda) = -R_{\lambda,\mu}.$$

Proof. Assume that $R_{\lambda,\mu} \in \mathcal{L}(\partial X, X)$ exists. By the definition of N_λ the equation

$$\mu\varphi - N_\lambda\varphi = \psi$$

for $\varphi, \psi \in \partial X$ is equivalent to

$$\mu L L_\lambda \varphi - B L_\lambda \varphi = \psi \tag{4.2}$$

for $\varphi, \psi \in \partial X$. This again is equivalent to

$$-R_{\lambda,\mu}\psi = L_\lambda\varphi.$$

Therefore, we have for $\varphi, \psi \in \partial X$ the equivalence

$$\mu\varphi - N_\lambda\varphi = \psi \quad \iff \quad R_{\lambda,\mu}\psi = -L_\lambda\varphi.$$

Since $R_{\lambda,\mu} : \partial X \to \ker(\lambda - A_m) \cap D(B)$ exists and $L_\lambda : \partial X \to \ker(\lambda - A_m)$ is an isomorphism, there exists a unique $\varphi \in D(N_\lambda)$ for every $\psi \in \partial X$. Moreover its given by $\phi = -LR_{\lambda,\mu}\psi$ and therefore the boundedness of the inverse follows from the boundedness of L and $R_{\lambda,\mu}$. The formula for the resolvent of N_λ follows, since $L|_{\ker(\lambda - A_m)}$ is an isomorphism with inverse L_λ and the image of $R_{\lambda,\mu}$ is contained in $\ker(\lambda - A_m)$.

Conversely, we assume that $\mu \in \rho(N_\lambda)$. Then (4.2) has a unique solution $\varphi \in D(N_\lambda)$ for every $\psi \in \partial X$. Considering $f := L_\lambda\varphi$ we obtain a unique solution of (4.1) and hence $R_{\lambda,\mu}$ exists. Boundedness follows from $R_{\lambda,\mu} = -L_\lambda R(\mu, N_\lambda)$. □

Lemma 4.3. *Let $\lambda \in \rho(A_0)$. If $\mu_0 \in \rho(N_\lambda)$ and $\mu \in \mathbb{C}$ we have $f \in D(A_B^\mu)$ if and only if $(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}\, L)f \in D(A_B^{\mu_0})$ and*

$$A_B^\mu - \lambda = (A_B^{\mu_0} - \lambda)(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0} L). \tag{4.3}$$

Proof. Suppose $f \in D(A_B^\mu)$, then $f \in D(A_m) \cap D(B)$ and, since $\mathrm{rg}(R_{\lambda,\mu_0}) \subset D(A_m) \cap D(B)$ we conclude $f - (\mu - \mu_0)R_{\lambda,\mu_0}Lf \in D(A_m) \cap D(B)$. Using $(B - \mu_0 L)R_{\lambda,\mu_0} = \mathrm{Id}_{\partial X}$ we obtain

$$\begin{aligned}(B - \mu_0 L)(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}L)f &= (B - \mu_0 L)f - (\mu - \mu_0)(B - \mu_0 L)R_{\lambda,\mu_0}Lf \\ &= (B - \mu_0 L)f - (\mu - \mu_0)Lf \\ &= Bf - \mu Lf = 0\end{aligned}$$

and therefore $(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}L)f \in D(A_B^{\mu_0})$.
Conversely, assume $f \in D(A_B^{\mu_0}) \subset D(A_m) \cap D(B)$. Using $\mathrm{rg}(R_{\lambda,\mu_0}) \subset D(A_m) \cap D(B)$ we conclude that

$$f = (\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}L)f + R_{\lambda,\mu_0}(\mu - \mu_0)L \in D(A_m) \cap D(B).$$

Using $(B - \mu_0 L)R_{\lambda,\mu_0} = \mathrm{Id}_{\partial X}$ it follows

$$\begin{aligned}0 = (B - \mu_0 L)(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}L)f &= (B - \mu_0 L)f - (\mu - \mu_0)(B - \mu_0 L)R_{\lambda,\mu_0}Lf \\ &= (B - \mu_0 L)f - (\mu - \mu_0)Lf = Bf - \mu Lf\end{aligned}$$

and hence $f \in D(A_B^\mu)$. Finally, (4.3) follows from the fact that $\mathrm{rg}(R_{\lambda,\mu}) \subset \ker(\lambda - A_m)$. □

Lemma 4.4. *Let $\lambda \in \rho(A_0)$. For $\mu_0, \mu \in \rho(N_\lambda)$ is*

$$(\mathrm{Id} - (\mu - \mu_0)R_{\lambda,\mu_0}L)^{-1} = (Id + (\mu - \mu_0)R_{\lambda,\mu}L).$$

Proof. Using Lemma 4.2, $LL_\lambda = \mathrm{Id}_{\partial X}$ and the Resolvent Identity one obtains

$$\begin{aligned}
&(\mathrm{Id} - (\mu-\mu_0)R_{\lambda,\mu_0}L)(Id + (\mu-\mu_0)R_{\lambda,\mu}L) \\
&= \mathrm{Id} - (\mu-\mu_0)L_\lambda R(\mu_0,N_\lambda)L + (\mu-\mu_0)L_\lambda R(\mu,N_\lambda)L \\
&- (\mu-\mu_0)^2 L_\lambda R(\mu_0,N_\lambda)R(\mu,N_\lambda)L \\
&= \mathrm{Id} - (\mu-\mu_0)L_\lambda\Big(R(\mu_0,N_\lambda) - R(\mu,N_\lambda) + (\mu-\mu_0)R(\mu_0,N_\lambda)R(\mu,N_\lambda)\Big)L = \mathrm{Id}\,.
\end{aligned} \tag{4.4}$$

Since $R(\mu_0,N_\lambda)$ and $R(\mu,N_\lambda)$ commute, it follows by (4.4) that $(\mathrm{Id} - (\mu-\mu_0)R_{\lambda,\mu_0}L)$ and $(Id + (\mu-\mu_0)R_{\lambda,\mu}L)$ commute and hence the claim. □

Proposition 4.5. *Let $\lambda \in \rho(A_0)$. For $\mu_0, \mu_1 \in \rho(N_\lambda)$ we have*

$$R(\lambda, A_B^{\mu_1}) = \big(\mathrm{Id} + (\mu_1-\mu_0)R_{\lambda,\mu_1}L\big)\cdot R(\lambda, A_B^{\mu_0})$$

for $\lambda \in \rho(A_B^{\mu_1}) \cap \rho(A_B^{\mu_0})$.

Proof. By Lemma 4.4 it follows that the right hand side in (4.3) is invertible for $\lambda \in \rho(A_B^{\mu_0}) \cap \rho(A_B^{\mu_1})$. Using Lemma 4.4 we conclude

$$R(\lambda, A_B^{\mu_1}) = (\mathrm{Id} - (\mu_1-\mu_0)R_{\lambda,\mu_0}L)^{-1}R(\lambda, A_B^{\mu_0}) = \big(\mathrm{Id} + (\mu_1-\mu_0)R_{\lambda,\mu_1}L\big)\cdot R(\lambda, A_B^{\mu_0})$$

for $\lambda \in \rho(A_B^{\mu_0}) \cap \rho(A_B^{\mu_1})$. □

Corollary 4.6. *Let $\lambda \in \rho(A_0)$, then the map*

$$\rho(N_\lambda) \to \mathcal{L}(X)\colon \mu \mapsto R(\lambda, A_B^{\mu})$$

is holomorphic with derivative

$$\frac{d}{d\mu}\bigg|_{\mu=\mu_0} R(\lambda, A_B^{\mu}) = R_{\lambda,\mu_0}LR(\lambda, A_B^{\mu_0}).$$

In particular it is continuous in μ_0, i.e.,

$$\lim_{\mu\to\mu_0} \|R(\lambda, A_B^{\mu}) - R(\lambda, A_B^{\mu_0})\| = 0. \tag{4.5}$$

Lemma 4.7. *For $\mu_0 \in \rho(N_\lambda)$ we have*

$$R(\lambda, A_B^{\mu}) = \big(\mathrm{Id} - R_{\lambda,\mu}B\big)\cdot R(\lambda, A_0).$$

Proof. By Corollary 3.2 we conclude from $\mu_0 \notin \sigma_p(N_\lambda)$ that $\lambda \notin \sigma_p(A_B^{\mu})$ and hence $\lambda - A_B^{\mu}$ is injective. For $f \in X$ we have

$$(\mathrm{Id} - R_{\lambda,\mu}B)R(\lambda, A_0)f = R(\lambda, A_0)f - R_{\lambda,\mu}BR(\lambda, A_0)f \in D(A_m) \cap D(B).$$

Using $D(A_0) \subset \ker(L)$ and $(B - \mu_0 L)R_{\lambda,\mu} = \mathrm{Id}_{\partial X}$ one obtains

$$(B - \mu L)(R(\lambda, A_0)f - R_{\lambda,\mu_0}BR(\lambda, A_0)f) = BR(\lambda, A_0)f - BR(\lambda, A_0)f = 0$$

and therefore $(\mathrm{Id} - R_{\lambda,\mu}B)R(\lambda, A_0)f \in D(A_B^{\mu})$. It follows

$$\begin{aligned}
(\lambda - A_B^{\mu_0})(\mathrm{Id} - R_{\lambda,\mu}B)R(\lambda, A_0)f &= (\lambda - A_m)(\mathrm{Id} - R_{\lambda,\mu}B)R(\lambda, A_0)f \\
&= (\lambda - A_m)R(\lambda, A_0)f - (\lambda - A_m)R_{\lambda,\mu}BR(\lambda, A_0)f \\
&= (\lambda - A_m)R(\lambda, A_0)f = f.
\end{aligned}$$

Thus $\lambda - A_B^{\mu}$ is right-invertible (and hence invertible) with (right-)inverse $(\mathrm{Id} - R_{\lambda,\mu}B)R(\lambda, A_0)$. □

Corollary 4.8. *If $\lim_{\mu\to\infty}\|R(\mu,N_\lambda)\| = 0$, the operators A_B^μ converges to A_0 in the norm resolvent sense, i.e.*

$$\lim_{\mu\to\infty}\|R(\lambda,A_B^\mu)-R(\lambda,A_0)\| = 0.$$

Proof. Using Lemma 4.2 and Lemma 4.7 one obtains

$$\begin{aligned}\|R(\lambda,A_B^\mu)-R(\lambda,A_0)\| &= \|R_{\lambda,\mu}BR(\lambda,A_0)\| \\ &= \|L_\lambda R(\mu,N_\lambda)BR(\lambda,A_0)\| \\ &\le \|L_\lambda\|\cdot\|R(\mu,N_\lambda)\|\cdot\|BR(\lambda,A_0)\|,\end{aligned}$$

which converges to 0 for $\mu\to\infty$. ☐

Corollary 4.9. *Let $\lambda\in\rho(A_0)$ and $\lim_{\mu\to\infty}\|R(\mu,N_\lambda)\| = 0$. Moreover, let $\mu,\mu_0\in\rho(N_\lambda)$. Denote by $\lambda_k(\mu)$ the k-th eigenvalue of A_B^μ and by $\lambda_k(\infty)$ the k-th eigenvalue of A_0. Then*

$$\begin{aligned}\lim_{\mu\to\mu_0}\lambda_k(\mu) &= \lambda_k(\mu_0), \\ \lim_{\mu\to\infty}\lambda_k(\mu) &= \lambda_k(\infty).\end{aligned}$$

Proof. This follows from Corollary 4.6 and Corollary 4.8 using [Kat66, Thm. IV. 2.6, Thm. IV. 2.25 & Thm. IV. 3.5]. ☐

Corollary 4.10. *Assume that A_0, A_B^μ and N have compact resolvents for all $\mu\in\mathbb{C}$. Denote by $\lambda_k(\mu)$ the k-th eigenvalue of A_B^μ and by $\lambda_k(\infty)$ the k-th eigenvalue of A_0. Then*

$$\begin{aligned}\lim_{\mu\to\mu_0}\lambda_k(\mu) &= \lambda_k(\mu_0), \\ \lim_{\mu\to\infty}\lambda_k(\mu) &= \lambda_k(\infty).\end{aligned}$$

Proof. By Corollary 4.9 it remains to show that for every $\mu_0\in\mathbb{C}$ there exists a $\lambda\in\rho(A_0)$ such that $\mu_0\in\rho(N_\lambda)$.

Assume that there exists a $\mu_0\in\mathbb{C}$ such that $\mu_0\notin\rho(N_\lambda)$ for all $\lambda\in\rho(A_0)$. Since N and hence N_λ have compact resolvents it follows $\mu_0\in\sigma_p(N_\lambda)$ for all $\lambda\in\rho(A_0)$. By Corollary 3.2 we conclude $\rho(A_0)\subset\sigma_p(A_B^{\mu_0})$ which contradicts the fact that $\sigma_p(A_B^{\mu_0})$ is discrete, since $A_B^{\mu_0}$ has compact resolvent. ☐

Lemma 4.11. *Denote the k-th eigenvalue of A_B^μ by $\lambda_k(\mu)$ and by $\lambda_k(\infty)$ the k-th eigenvalue of A_0. Assume, that the map $\lambda_k\colon\mathbb{R}\to\mathbb{R}\colon\mu\mapsto\lambda_k(\mu)$ is monotone decreasing and $\sigma_p(N_\lambda)$ is discrete for all $\lambda\in\rho(A_0)$. If $\mu_1,\mu_2\in\mathbb{R}$ such that $\lambda_k(\mu_1)=\lambda_k(\mu_2)\in\rho(A_0)$, then $\mu_1=\mu_2$.*

Proof. Let $\lambda_k(\mu_1)=\lambda_k(\mu_2)=:\lambda\in\rho(A_0)$ and without loss of generality $-\infty<\mu_1\le\mu_2\le\infty$. By monotonicity and continuity of λ_k, one obtains that $\lambda_k(\mu)=\lambda$ for all $\mu\in[\mu_1,\mu_2]$. By Corollary 3.2 we conclude that $[\mu_1,\mu_2]\subset\sigma_p(N_\lambda)$. By discreteness of $\sigma_p(N_\lambda)$ it follows that $\mu_1=\mu_2$. ☐

To verify that $\sigma_p(N_\lambda)$ is discrete for all $\lambda\in\rho(A_0)$ we make the following expectation.

Lemma 4.12. *If N_λ has compact resolvent for some $\lambda\in\rho(A_0)$, then $\sigma_p(N_\lambda)$ is discrete for all $\lambda\in\rho(A_0)$.*

Proof. For $\eta\in\rho(A_0)$ we have by [Gre87, Lem. 1.3] that the difference

$$N_\lambda - N_\eta = (\eta-\lambda)BR(\eta,A_0)L_\lambda$$

is bounded. It follows from [EN00, Prop. III.1.12(ii)] that N_η has compact resolvent for all $\eta \in \rho(A_0)$. Now the claim follows by [EN00, Cor. IV.1.19]. □

Proposition 4.13. *Denote by $\lambda_k(\mu)$ the k-th eigenvalue of A_B^μ, by $\lambda_k(\infty)$ the k-th eigenvalue of A_0 and by $\mu_n(\lambda)$ the n-th eigenvalue of N_λ.*
Assume, that the map $\lambda_k\colon \mathbb{R} \to \mathbb{R}\colon \mu \mapsto \lambda_k(\mu)$ is strictly monotone decreasing and continuous. If $k \in \mathbb{N}$ such that $\lambda_{k+1}(\infty) \neq \lambda_k(\infty)$, the following statements are equivalent.

(a) *the n-th eigenvalue μ_n of the Dirichlet-to-Neumann operator N_λ is positive, i. e.*
$$\mu_n(\lambda) > 0$$
for $\lambda \in (\lambda_{k+1}(\infty), \lambda_k(\infty))$;

(b) *the inequality*
$$\lambda_k(\infty) < \lambda_{k+n}(0)$$
holds.

Proof. (a) $\Rightarrow$ (b): Assume $\lambda_k(\infty) \geq \lambda_{k+n}(0)$. Then there exists $\lambda_{k+n}(0) \leq \lambda \leq \lambda_k(\infty)$. Moreover we have

(i) $\lambda_k(\infty) < \lambda_k(\mu) \leq \lambda_l(\mu)$ for all $\mu < \infty$ and $l \leq k$;
(ii) $\lambda_l(\mu) \leq \lambda_{k+n}(\mu) < \lambda_{k+n}(0)$ for all $\mu > 0$ and $l \geq k+n$.

Hence $\lambda = \lambda_l(\mu_l)$ for $\mu_l \in (0,\infty)$ can only satisfied for $l \in \{k+1,\dots,k+n-1\}$. But $|\{k+1,\dots,k+n-1\}| < n$ and therefore by Corollary 3.2 it follows $\mu_n(\lambda) \notin (0,\infty)$.

(b) $\Rightarrow$ (a): By monotonicity and the inequality $\lambda_k(\infty) < \lambda_{k+n}(0)$ we obtain
$$\lambda_{k+n}(\infty) \leq \lambda_l(\infty) \leq \lambda_{k+1}(\infty) < \lambda < \lambda_k(\infty) < \lambda_{k+n}(0) \leq \lambda_l(0) \tag{4.6}$$
for $l \in \{k+1,\dots,k+n\}$. Therefore
$$\lambda_l(\infty) < \lambda < \lambda_l(0)$$
for $l \in \{k+1,\dots,k+n\}$. Using continuity and monotonicity of λ_l there exists a unique $m \in \mathbb{N}$ and unique $\mu_{l+m} \in (0,\infty)$ such that
$$\lambda = \lambda_l(\mu_{l+m}).$$
By Corollary 3.2 we conclude $\mu_{m+l} \in (0,\infty) \cap \sigma_p(N_\lambda)$. Since $|\{m+k+1,\dots,m+k+n\}| = n$ one obtains $|(0,\infty) \cap \sigma_p(N_\lambda)| \geq n$ for $\lambda \in (\lambda_{k+1}(\infty), \lambda_k(\infty))$. □

Now we are able to formulate the main theorem of this section.

Theorem 4.14. *Denote by $\lambda_k(\mu)$ the k-th eigenvalue of A_B^μ, by $\lambda_k(\infty)$ the k-th eigenvalue of A_0 and by $\mu_n(\lambda)$ the n-th eigenvalue of N_λ. Assume that A_0, A_B^μ and N have compact resolvents. Further, assume that the map $\lambda_k\colon \mathbb{R} \to \mathbb{R}\colon \mu \mapsto \lambda_k(\mu)$ is monotone decreasing. If $k \in \mathbb{N}$ such that $\lambda_{k+1}(\infty) \neq \lambda_k(\infty)$, the following statements are equivalent.*

(a) *the n-th eigenvalue μ_n of the Dirichlet-to-Neumann operator N_λ is positive, i. e.*
$$\mu_n(\lambda) > 0$$
for $\lambda \in (\lambda_{k+1}(\infty), \lambda_k(\infty))$;

(b) *the inequality*
$$\lambda_k(\infty) < \lambda_{k+n}(0)$$
holds.

Proof. We verify the assumptions of Proposition 4.13. From Corollary 4.10 it follows that the map $\lambda_k\colon \mathbb{R} \to \mathbb{R}\colon \mu \mapsto \lambda_k(\mu)$ is continuous. Moreover, from Lemma 4.11 and Lemma 4.12 the strict monotonicity follows. Now Proposition 4.13 implies the claim. $\square$

5. Positivity for operators with Wentzell boundary conditions

In the sequel we assume that $X, \partial X$ are Banach lattices and that $L \in \mathcal{L}(X, \partial X)$ is a positive operator. Note that the similarity transform between A^B and $\mathcal{A}^B$ is given by

$$S\colon X \to \tilde{\mathfrak{X}}\colon f \mapsto \begin{pmatrix} f \\ Lf \end{pmatrix},$$
$$S^{-1}\colon \tilde{\mathfrak{X}} \to X\colon \begin{pmatrix} f \\ x \end{pmatrix} \mapsto f.$$

Since L is positive it follows that S and S^{-1} are positive operators. The following result is analogous to [CENN03, Prop. 5.2].

Proposition 5.1. *Assume that $BR(\lambda, A_0)$ is positive and there exists a constant $\omega \in \mathbb{R}$ such that L_λ are positive operators for $\lambda \geq \omega$. Moreover assume that A_0 and N_λ have positive resolvents for all $\lambda \geq \omega$. Then A^B has positive resolvent for all $\lambda \geq \omega$.*

Proof. Since $\lambda \in \rho(A_0) \cap \rho(N_\lambda)$ it follows from (3.1) that

$$R(\lambda, \hat{\mathcal{A}}_\lambda) = \begin{pmatrix} \mathrm{Id} & L_\lambda \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} R(\lambda, A_0) & 0 \\ 0 & R(\lambda, N_\lambda) \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ BR(\lambda, A_0) & \mathrm{Id} \end{pmatrix}$$

Using the assumption it is easy to see that the three operators on the right hand side are positive. Hence $R(\lambda, \hat{\mathcal{A}}_\lambda)$ is positive. Now Lemma 3.5 and [EN00, Prop. IV.2.17] imply the claim. $\square$

The following corollary is useful in combination with [BE18, Thm.3.1] or [BE20, Thm. 5.3 & 5.4].

Corollary 5.2. *Assume there exists a constant $\omega \in \mathbb{R}$ such that L_λ and $BR(\lambda, A_0)$ are positive operators for $\lambda \geq \omega$. Moreover assume that A_0 and N_λ have positive resolvents for all $\lambda \geq \omega$. Further assume that A^B generates a strongly continuous semigroup on X. Then the semigroup generated by A^B is positive.*

In the typical situation where $X = \mathrm{C}(K)$ for some compact space K we obtain a stronger version of Proposition 5.1.

Corollary 5.3. *Let $X = \mathrm{C}(K)$ for some compact space K and $\partial X = \mathrm{C}(\partial K)$. Assume there exists a constant $\omega \in \mathbb{R}$ such that L_λ and $BR(\lambda, A_0)$ are positive operators for $\lambda \geq \omega$. Moreover assume that A_0 and N_λ have positive resolvents for all $\lambda \geq \omega$. Then A^B generates a positive strongly continuous semigroup on X.*

Note that the positivity of L_λ for sufficient large λ implies the positivity of the L_λ for all $\lambda > s(A_0)$.

Lemma 5.4. *If L_λ is positive for $\lambda \geq \omega > s(A_0)$. Then it follows that L_λ is positive for all $\lambda > s(A_0)$. Moreover, the map $(s(A_0), \infty) \to \mathcal{L}(X, \partial X)\colon \lambda \mapsto L_\lambda$ is monotonic decreasing.*

Proof. For $\lambda < \omega$ it follows

$$L_\lambda - L_\omega = (\omega - \lambda) R(\lambda, A_0) L_\omega > 0$$

which implies the claim. □

Similar we obtain the following results about the Dirichlet-to-Neumann operators.

Proposition 5.5. *Assume that $BR(\lambda, A_0)$ is positive and L_λ is positive for all $\lambda > s(A_0)$. Let N_η generates a positive semigroup on ∂X for $\eta > s(A_0)$. Then N_λ generate positive semigroups on ∂X for all $\eta \geq \lambda > s(A_0)$.*

Proof. Let $s(A_0) < \lambda < \eta$. Then

$$N_\lambda - N_\eta = (\eta - \lambda) BR(\lambda, A_0) L_\eta$$

and the right hand side is a positive and bounded operator. Hence the claim follows from [EN00, Cor. IV.1.11]. □

Next we assume that A^B is resolvent positive and then study the resulting consequences. To this end we first need the following lemmas.

Lemma 5.6. *We have* $\|BR(\lambda, A_0)\| \to 0 \quad$ *as* $\lambda \to +\infty$.

Proof. Let $\varepsilon > 0$. Then there exists C_ε such that (2.1) is satisfied. Using this we obtain for $f \in X$

$$\begin{aligned} \big\|BR(\lambda, A_0)f\big\| &\leq \varepsilon \cdot \big\|A_0 R(\lambda, A_0) f\big\| + \tfrac{C_\varepsilon \cdot C}{\lambda} \cdot \|f\| \\ &\leq \big(\varepsilon \cdot (C+1) + \tfrac{C_\varepsilon \cdot C}{\lambda}\big) \cdot \|f\| \\ &\leq 2(C+1) \cdot \varepsilon \cdot \|f\| \end{aligned}$$

if $\lambda > s(A_0)$ is sufficiently large. □

Lemma 5.7. *If A^B is a weak Hille-Yosida operator, then*

(i) $\|\lambda \cdot R(\lambda, N_\lambda)\| \leq M$ *as* $\lambda \to +\infty$ *for some* $M \geq 0$*;*
(ii) $\lambda \cdot R(\lambda, N_\lambda) x \to x$ *as* $\lambda \to +\infty$.

Proof. (i) By Theorem 3.8, for $f \in X$ and $\lambda > 0$ sufficiently large we have

$$\big\|LR(\lambda, A^B) f\big\| = \big\|R(\lambda, N_\lambda)\big(BR(\lambda, A_0) + L\big) f\big\| \leq \tfrac{K \cdot \|L\|}{\lambda} \cdot \|f\|$$

for a suitable constant $K \geq 0$. By choosing $f = L_0 x$, $x \in \partial X$ we obtain

$$\big\|R(\lambda, N_\lambda) x\big\| - \big\|R(\lambda, N_\lambda) \cdot BR(\lambda, A_0) L_0 x\big\| \leq \tfrac{K \cdot \|L\|}{\lambda} \cdot \|x\|$$

which implies

$$\big\|R(\lambda, N_\lambda)\big\| \leq \big\|R(\lambda, N_\lambda)\big\| \cdot \big\|BR(\lambda, A_0) L_0\big\| + \tfrac{K \cdot \|L\|}{\lambda}$$

and further

$$\big\|R(\lambda, N_\lambda)\big\| \cdot \big(1 - \big\|BR(\lambda, A_0) L_0\big\|\big) \leq \tfrac{K \cdot \|L\|}{\lambda}.$$

Hence, by Lemma 5.6 there exists $M \geq 0$ such that

$$\big\|\lambda \cdot R(\lambda, N_\lambda)\big\| \leq \tfrac{K \cdot \|L\|}{1 - \|BR(\lambda, A_0)\| \cdot \|L_0\|} \leq M$$

for $\lambda > 0$ sufficiently large.

(ii) We start by observing that by Theorem 3.8 for every $x \in \partial X$ we have

$$\begin{aligned} 0 \leftarrow LA_m R(\lambda, A^B)L_0 x &= \lambda LR(\lambda, A^B)L_0 x - x \\ &= \lambda R(\lambda, N_\lambda) \cdot \big(BR(\lambda, A_0)L_0 x + x\big) - x \\ &= \big(\lambda R(\lambda, N_\lambda)x - x\big) + \lambda R(\lambda, N_\lambda) \cdot BR(\lambda, A_0) \cdot L_0 x \end{aligned}$$

as $\lambda \to +\infty$. The claim now follows by (i) and Lemma 5.6.

□

Using the previous two lemmas we obtain the following.

Proposition 5.8. *Assume that A^B is a resolvent positive weak Hille-Yosida Operator. If there exists λ_0 such that L_{λ_0} exists and is positive then N_μ is resolvent positive for all $\mu > s(A_0)$.*

Proof. By replacing A_m by $A_m + \mu$ and B by $B + \mu L$, A^B rescales to $A^B + \mu$, A_0 to $A_0 + \mu$, L_μ to L_0 and hence $N_\mu = BL_\mu$ to $(B + \mu L)L_0 = BL_0 + \mu = N + \mu$. Note that this rescaling of A^B and N_μ by μ does not affect resolvent positivity of these operators. Hence, without loss of generality, we may assume that $\mu = 0$.
To prove the claim we use a perturbation argument. More precisely, for $v \geq 0$ we consider the perturbed operators

$$\begin{aligned} A_m^\nu f &\coloneqq A_m f + \nu \cdot L_0 L f, \quad && D(A_m^\nu) \coloneqq D(A_m), \\ B^\nu f &\coloneqq Bf + \nu \cdot Lf, && D(B^\nu) \coloneqq D(B). \end{aligned}$$

Then

$$\begin{aligned} LA_m^\nu f = B^\nu f \quad &\iff \quad LA_m f + \nu L L_0 L f = Bf + \nu Lf \\ &\iff \quad LA_m f = Bf \end{aligned}$$

and hence

$$A^\nu \coloneqq (A_m^v)^{B^\nu} = A^B + \nu \cdot L_0 L.$$

Since by assumption $\nu L_0 L \geq 0$, the Neumann expansion of $R(\lambda, A^\nu)$, see [EN00, III-(2.5)], shows that the perturbed operator A^ν remains resolvent positive.
Next observe that $(A_m^\nu)_0 = (A_m^\nu)|_{X_0} = A_0$ and thus $B^\nu R(\lambda + \nu, (A_m^\nu)_0) = BR(\lambda + \nu, A_0)$. Moreover, a simple computation shows that the Dirichlet operator with respect to A_m^ν is given by

$$L_{\lambda+\nu}^\nu = L_0 - \lambda R(\lambda + v, A_0)L_0.$$

This implies

$$N_{\lambda+\nu}^\nu \coloneqq B^\nu L_{\lambda+\nu}^\nu = BL_{\lambda+\nu}^\nu + \nu = N_0 - \lambda BR(\lambda + \nu, A_0)L_0 + \nu.$$

Now, using Lemma 5.6 we obtain for λ sufficiently large and $\nu \to +\infty$

$$\begin{aligned} &R(\lambda, N_0) - R\big(\lambda, N_0 - \lambda BR(\lambda + \nu, A_0)L_0\big) \\ &\qquad = R(\lambda, N_0) - R\big(\lambda + \nu, N_{\lambda+\nu}^\nu\big) \\ &\qquad = R\big(\lambda, N_0 - \lambda BR(\lambda + \nu, A_0)L_0\big) \cdot \lambda BR(\lambda + \nu, A_0)L_0 \cdot R(\lambda, N_0) \to 0. \end{aligned} \tag{5.1}$$

From Theorem 3.8 and Lemma 5.6 it therefore follows that

$$0 \leq LR(\lambda + \nu, A^\nu) = R(\lambda + \nu, N_{\lambda+\nu}^\nu) \cdot \big(BR(\lambda + \nu, A_0) + L\big) \to R(\lambda, N_0)L$$

as $\nu \to +\infty$. Hence, for λ large $R(\lambda, N_0) = R(\lambda, N_0)L \cdot L_{\lambda_0} \geq 0$ as claimed. □

Lemma 5.9. *If A^B and A^0 are both resolvent positive weak Hille–Yosida operators then $BR(\mu, A_0) \geq 0$ for all $\mu > s(A_0)$. Moreover, $Bf \geq 0$ for all $f \in D(A_0)_+$.*

Proof. By assumption we have

$$R(\lambda, A^B) > 0 \text{ and } R(\mu, A_0) > 0$$

for $\lambda > s(A^B)$ and $\mu > s(A_0)$. Now we obtain by Theorem 3.8 that

$$\begin{aligned} 0 \leq \lambda^2 LR(\lambda, A^B)R(\mu, A_0) &= \lambda^2 R(\lambda, N_\lambda)BR(\lambda, A_0)R(\mu, A_0) \\ &= \lambda R(\lambda, N_\lambda) \cdot BR(\mu, A_0) \cdot \lambda R(\lambda, A_0) \\ &=: T_\lambda \cdot R_\mu \cdot S_\lambda \end{aligned}$$

for all $\lambda > s(A^B)$ and $\mu > s(A_0)$. By Lemma 5.7 this gives

$$\left\|T_\lambda \cdot R_\mu \cdot S_\lambda f - R_\mu f\right\| \leq \|T_\lambda R_\mu\| \cdot \left\|(S_\lambda f - \mathrm{Id})f\right\| + \left\|(T_\lambda - \mathrm{Id}) \cdot R_\mu f\right\| \to 0$$

i.e.,

$$0 \leq T_\lambda \cdot R_\mu \cdot S_\lambda \to R(\mu, A_0).$$

as $\lambda \to +\infty$ showing the first claim.
To show the second assertion take $f \in D(A_0)_+$ and define $0 \leq f_n := nR(n, A_0)f \in D(A_0) \subseteq D(B)$ for $s(A_0) < n \in \mathbb{N}$. Then $f_n \to f$ in $[D(A_0)]$ and since $B|_{[D(A_0)]} \in \mathcal{L}([D(A_0)], \partial X)$ we conclude by the first part that

$$0 \leq nBR(n, A_0)f = Bf_n \to Bf$$

as claimed. □

Summing up the results above we conclude the main theorem of this section.

Theorem 5.10. *Assume that A_0 and A^B are weak Hille-Yosida operators on X. If L_λ is positive for $\lambda \geq \omega > s(A_0)$ and A_0 have positive resolvent, then following statements are equivalent.*

(a) *A^B is resolvent positive on X;*
(b) (i) *N_λ are resolvent positive on ∂X for all $\lambda > s(A_0)$;*
 (ii) *$Bf \geq 0$ for all $f \in D(A_0)_+$.*

6. Stability for operators with Wentzell boundary conditions

The following result is analogous to [CENN03, Lem. 5.3].

Proposition 6.1. *Assume that A_0 and A_B^μ have positive resolvents and that $BR(\lambda, A_0) > 0$ for $\lambda > s(A_0)$. Moreover assume that $L_\lambda, L > 0$ for $\lambda > s(A_0)$ and that N_λ generates a compact and positive semigroup on ∂X for $\lambda > s(A_0)$. Then the spectral bounds satisfy the following inequalities*

(a) *$s(N_\lambda) \leq s(N_\eta)$ for $s(A_0) \leq \eta \leq \lambda$;*
(b) *$s(A_0) \leq s(A_B^\mu) \leq s(A_B^\nu)$ for $s(A_0) \leq \nu \leq \mu$.*

Proof. By [Gre87, Lem. 1.3] we obtain

$$N_\lambda - N_\eta = (\eta - \lambda)BR(\lambda, A_0)L_\eta \geq 0$$

for $s(A_0) \leq \eta \leq \lambda$. It follows

$$R(\mu, N_\lambda) - R(\mu, N_\eta) = R(\mu, N_\lambda)PR(\mu, N_\eta) \geq 0$$

for $\max\{s(N_\lambda), s(N_\eta)\} \le \eta \le \lambda$. This implies $s(N_\lambda) \le s(N_\eta)$.
For $\lambda > \max\{s(A_0), s(A_B^\mu)\}$ or $\lambda > \max\{s(A_B^\mu), s(A_B^\nu)\}$ we obtain $\lambda \in \rho(A_B^\mu)$ and by Corollary 3.2, that $\mu \notin \sigma_p(N_\lambda) = \sigma(N_\lambda)$, since $R(\mu, N_\lambda)$ is compact. Using Lemma 4.2 and Lemma 4.7 we obtain

$$R(\lambda, A_0) \le (\mathrm{Id} + \underbrace{L_\lambda R(\mu, N_\lambda) B}_{\ge 0}) R(\lambda, A_B^\mu)$$

for $\lambda > \max\{s(A_0), s(A_B^\mu)\}$ and the first inequality follows. Moreover by Lemma 4.2 and Proposition 4.5 we conclude

$$R(\lambda, A_B^\mu) = (\mathrm{Id} - \underbrace{(\mu - \nu)}_{\ge 0} \underbrace{L_\lambda R(\mu, N_\lambda) L}_{\ge 0}) R(\lambda, A_B^\nu) \le R(\lambda, A_B^\nu)$$

for $\lambda > \max\{s(A_B^\mu), s(A_B^\nu)\}$ and the second inequality follows. □

Note that by [EN00, Prop. IV.1.14] for positive semigroups exponential stability is characterized in terms of the spectral bound. We finish this section with the following stability result.

Theorem 6.2. *Assume that $BR(\lambda, A_0)$ is positive and L_λ are positive operators for large λ. Moreover assume that A_0 have positive resolvent on X and that N_λ generate positive semigroups on ∂X for large λ. Further, let A^B generator of a C_0-semigroup on X. Then $s(A_0) \le s(A^B)$ and for $\kappa \in \mathbb{R}$ we obtain*

$$s(A^B) < \kappa \quad \Longleftrightarrow \quad s(A_0) < \kappa \text{ and } s(N_\kappa) < \kappa.$$

Proof. Note that by Lemma 5.4 and Proposition 5.5 the Dirichlet operators L_λ are positive and N_λ generate positive semigroups for all $\lambda > s(A_0)$.
From Theorem 3.8 we conclude

$$0 \le R(\lambda, A_0) \le R(\lambda, A^B)$$

for all $\lambda \ge \max\{s(A_0), s(A^B)\}$. Now it follows from the proof of [Nag86, Lem. 4.10] that $s(A_0) \le s(A^B)$.
Let $s(A^B) < \kappa$, then $s(A_0) < \kappa$ and by [Nag86, C-III. Thm. 1.1.(b)] $\kappa \in \rho(A^B) \cap \rho(A_0)$. Now Theorem 3.7(a) implies $\kappa \in \rho(N_\kappa)$. It follows by Theorem 3.8 that

$$0 \le R(\kappa, A^B) = R(\kappa, A_0) + L_\kappa R(\kappa, N_\kappa) L + L_\kappa R(\kappa, N_\kappa) B R(\kappa, A_0)$$

and therefore is $L_\kappa R(\kappa, N_\kappa) L$ positive. We conclude by [Nag86, C-III. Thm. 1.1.(b)]

$$s(L_\kappa N_\kappa L) < \kappa$$

Note that L_κ is the inverse of L from ∂X to $\ker(\kappa - A_m)$ and L, L_κ are positive. It follows by similarity

$$s(N_\kappa) = s(L_\kappa N_\kappa L) < \kappa.$$

Conversely assume $s(A_0), s(N_\kappa) < \kappa$. By Proposition 6.1 it follows

$$s(N_\lambda) \le s(N_\kappa) < \kappa \le \lambda$$

for all $\lambda \ge \kappa > s(A_0)$ and hence $\lambda \in \rho(N_\lambda) \cap \rho(A_0)$. Theorem 3.7(a) implies $\lambda \in \rho(A^B)$ for all $\mu \ge \kappa$. From Proposition 5.1 we obtain that A^B is generator a positive semigroup on X and by [Nag86, C-III. Thm. 1.1.(a)] we have $s(A^B) \in \sigma(A^B)$. We conclude $s(A^B) < \kappa$. □

On spaces of continuous functions we can omit the semigroup conditions for A^B.

Corollary 6.3. *Let $X = C(K)$ for some compact space K and $\partial X = C(\partial K)$. Assume that $BR(\lambda, A_0)$ is positive and L_λ are positive operators for large λ. Moreover assume that A_0 have positive resolvent on X and that N_λ generate positive semigroups on ∂X for large λ. Then $s(A_0) \le s(A^B)$ and for $\kappa \in \mathbb{R}$ we obtain*

$$s(A^B) < \kappa \quad \Longleftrightarrow \quad s(A_0) < \kappa \text{ and } s(N_\kappa) < \kappa.$$

Since $s(A_{00}) = s(A_0)$ we obtain $\kappa = 0$ the following result by [EN00, Prop. VI.1.14].

Corollary 6.4. *Assume that $BR(\lambda, A_0)$ is positive and L_λ are positive operators for large λ. Moreover assume that A_{00}, N_λ for large λ and A^B generate positive semigroups on X_0, ∂X and X, respectively. The semigroup $(T_{A^B}(t))_{t\ge 0}$ is uniformly exponential stable on X if and only if the semigroups $(T_{A_{00}}(t))_{t\ge 0}$ and $(T_N(t))_{t\ge 0}$ are uniformly exponential stable on X_0 and ∂X.*

7. Examples

In this section we show how our abstract approach applies in quite different situations.

7.1. **A Delay differential operator.** In this subsection we apply our approach to operators related to delay differential equations, see [EN00, Section VI.6]. For a Banach space Y we define the Banach space $X := \mathrm{C}([-1,0], Y)$ of all continuous functions on $[-1,0]$ with values in Y equipped with the sup-norm. Moreover, we take a delay operator $\Phi \in \mathcal{L}(X,Y)$ and an operator $C\colon D(C) \subset Y \to Y$. With this notation we consider the *abstract delay differential operator* $A : D(A) \subset X \to X$ given by

$$Af := f', \quad D(A) := \left\{ f \in \mathrm{C}^1\big([-1,0], Y\big) : \begin{matrix} f(0) \in D(C) \text{ and} \\ f'(0) = Cf(0) + \Phi f \end{matrix} \right\} \tag{7.1}$$

which governs a delay differential equations, see [EN00, Section VI.6] for details.

Consider the operator $A_0 := \frac{d}{dr}$ with domain $D(A_0) := \mathrm{C}^1_0([-1,0], Y)$. Note that A_0 has empty spectrum and that its resolvent is given by

$$(R(\lambda, A_0)f)(s) = \int_s^0 e^{\lambda(s-r)} f(r)\, \mathrm{d}r =: H_\lambda f(s). \tag{7.2}$$

Moreover the abstract Dirichlet operator is

$$L_\lambda x = \varepsilon_\lambda \otimes x \tag{7.3}$$

where $\varepsilon_\lambda(s) := e^{\lambda s}$. Moreover denote by $\Phi_\lambda := \Phi L_\lambda$ and see that $N_\lambda = C + \Phi_\lambda$. Now we conclude the following result.

Corollary 7.1. *We have*

(i) *$\lambda \in \rho(A)$ if and only if $\lambda \in \rho(C + \Phi_\lambda)$. Moreover, the resolvent of A can be expressed as*
$$R(\lambda, A)f = H_\lambda f + (\varepsilon_\lambda \otimes R(\lambda, \Phi_\lambda))(\Phi H_\lambda f + f(0))$$
for $f \in \mathrm{C}([-1,0], Y)$;

(ii) *$\lambda \in \sigma_p(A)$ if and only if $\lambda \in \sigma_p(C + \Phi_\lambda)$. In this case* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - C - \Phi_\lambda))$;

(iii) *$\lambda \in \sigma_a(A)$ if and only if $\lambda \in \sigma_a(C + \Phi_\lambda)$;*

(iv) *$\lambda \in \sigma_c(A)$ if and only if $\lambda \in \sigma_c(C + \Phi_\lambda)$;*

(v) *$\lambda \in \sigma_r(A)$ if and only if $\lambda \in \sigma_r(C + \Phi_\lambda)$;*

(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(C + \Phi_\lambda)$;
(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(C + \Phi_\lambda)$.

Proof. Let $L := \delta_0$, $A_m := \frac{d}{dr}$ with domain $D(A_m) := \mathrm{C}^1[-1, 0]$ and $B := C\delta_0 + \Phi$, then we obtain $A = A^B$. Next we verify Assumptions 2.2. For $\lambda \geq 0$ it follows from (7.2)

$$\lambda \cdot \|(R(\lambda, A_0)f)(s)\| \leq \int_s^0 \lambda e^{\lambda(s-r)}\, \mathrm{d}r \cdot \|f\| \leq \|f\| \quad \text{for all } s \in [-1, 0]$$

i.e. A_0 is a weak Hille–Yosida operator on $\mathrm{C}([-1, 0], Y)$. Further, it follows from $D(A_0) \subset \ker(\delta_0)$ that $B|_{D(A_0)} = \Phi|_{D(A_0)}$ is bounded and hence condition (ii). Finally, by (7.3) condition (iii) is satisfied. Since A_0 has empty spectrum the result follows by Theorem 3.7. □

Remark 7.2. This result improves [EN00, Prop. VI.6.7] and [BP05, Prop. 3.19 & Lem. 3.20]. It can be seen as a generalized characteristic equation for delay equations.

In particular for the uncoupled case, i.e. $\Phi = 0$, we obtain the following corollary, which shows that every set can be realized as spectrum of A^B.

Corollary 7.3. *We obtain*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(C)$. *Moreover its resolvent can be expressed by*

$$R(\lambda, A^B)f = H_\lambda f + (\varepsilon_\lambda \otimes R(\lambda, C))f(0)$$

for $f \in \mathrm{C}([-1, 0], Y)$.
(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(C)$. *In this case* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - C))$;
(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(C)$;
(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(C)$;
(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(C)$;
(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(C)$;
(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(C)$.

Now we study positivity of the semigroup generated by A and use this property to obtain uniformly exponential stability. We additionally assume that Y and hence $X = \mathrm{C}([0, 1], Y)$ is a Banach lattice. As discussed in [KN84] and [EN00, Sect. IV.6] the representation of B into Φ and C is not unique. Nevertheless there exists a emphasized splitting

$$B = \tilde{C}L + \tilde{\Phi},$$

where $\tilde{\Phi} := B(\mathrm{Id} - L_0 L) \in \mathcal{L}(X, Y)$. Now obtain the following result.

Corollary 7.4. *Assume that the delay operator* $\tilde{\Phi}$ *is positive. Then* $\tilde{C}$ *generates a strongly continuous semigroup of positive operators on* Y *if and only if* A *given by* (7.1) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}([-1, 0], Y)$.

Proof. By (7.2) the operator A_0 has positive resolvent. Further, by (7.3) the Dirichlet operator L_λ is positive for $\lambda \in \mathbb{R}$. Using $B|_{X_0} = \tilde{\Phi}$ and $N_\lambda = \tilde{C} + \tilde{\Phi}_\lambda$ it follows by positive perturbation that N_λ are generators of strongly continuous semigroups of positive operator for all $\lambda \in \mathbb{R}$. Now the claim follows from Corollary 5.3.

For the converse direction note that Corollary 5.3 implies that N_λ are generators of positive semigroups on Y for all $\lambda > s(A_0) = -\infty$. Further, for $\lambda \to +\infty$ we obtain

$$\begin{aligned}\|\tilde{\Phi}_\lambda\| &= \|\tilde{\Phi}L_\lambda\| = \|B(L_\lambda - L_0)\| = \|BR(\lambda, A_0)L_0\| \\ &= \|\tilde{\Phi}R(\lambda, A_0)L_0\| \le \|\tilde{\Phi}\| \cdot \|R(\lambda, A_0)\| \cdot \|L_0\| \le \frac{C}{|\lambda|} \to 0\end{aligned}$$

and therefore

$$N_\lambda x \to \tilde{C}x$$

for $x \in D(\tilde{C}) = D(N_\lambda)$. Since N_λ have positive resolvents $\tilde{C}$ has and the claim follows. □

This statements improves [EN00, Thm. IV.6.11] and [KN84, Thm. 3.4]. Note that $\tilde{\Phi}$ has no mass in 0. Now applying Theorem 6.2 by using the fact that $s(A_0) < 0$ yields the following result.

Corollary 7.5. *Assume that the delay operator Φ is positive and that C generates a strongly continuous semigroup of positive operators on Y. Denote the semigroup on $\mathrm{C}([0,1], Y)$ generated by A by $(T_A(t))_{t\ge 0}$ and the semigroup on Y generated by N by $(T_N(t))_{t\ge 0}$. Then $(T_A(t))_{t\ge 0}$ is uniformly exponential stable on $\mathrm{C}([0,1], Y)$ if and only if $(T_N(t))_{t\ge 0}$ is on Y.*

For this statement see also [EN00, Corollary IV.6.16].

7.2. **Banach space-valued second derivative.**
Instead of considering the first derivative with a delay boundary conditions we now consider the second derivative with a similar boundary conditions. We associate to an arbitrary Banach space Y the Banach space $X := \mathrm{C}([0,1], Y)$ of all continuous functions on $[0,1]$ with values in Y equipped with the sup-norm. Moreover, we take $\Phi \in \mathcal{L}(X, Y^2)$ and a weak Hille–Yosida operator $(\mathcal{C}, D(\mathcal{C}))$ on Y^2. We consider the operator $A\colon D(A) \subset X \to X$ given by

$$\begin{aligned} Af &:= f'', \\ D(A) &:= \left\{ f \in \mathrm{C}^2([0,1], Y)\colon \begin{pmatrix} f(0) \\ f(1)\end{pmatrix} \in D(\mathcal{C}), \begin{pmatrix} f''(0) \\ f''(1)\end{pmatrix} = \Phi f + \mathcal{C}\begin{pmatrix} f(0) \\ f(1)\end{pmatrix} \right\}. \end{aligned} \tag{7.4}$$

Now one obtains

Corollary 7.6. *For $\lambda \in \mathbb{C} \setminus \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$ we obtain*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(\mathcal{C} + \Phi_\lambda)$.
(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(\mathcal{C} + \Phi_\lambda)$. *In this case* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - \mathcal{C} - \Phi_\lambda))$*;*
(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(\mathcal{C} + \Phi_\lambda)$*;*
(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(\mathcal{C} + \Phi_\lambda)$*;*
(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(\mathcal{C} + \Phi_\lambda)$*;*
(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(\mathcal{C} + \Phi_\lambda)$*;*
(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(\mathcal{C} + \Phi_\lambda)$.

Proof. We consider $L := \binom{\delta_0}{\delta_1}$, $A_m := \frac{d^2}{dr^2}$ with domain $D(A_m) := \mathrm{C}^2[-1,0]$ and $B := C\delta_0 + \Phi$ and obtain $A = A^B$. Next we verify Assumptions 2.2. As in [EN00, Thm. IV.4.1] it follows that A_0 is a weak Hille–Yosida operator on $\mathrm{C}([0,1], Y^2)$. Further, it follows from $D(A_0) \subset \ker(L)$ that $B|_{D(A_0)} = \Phi|_{D(A_0)}$ is bounded and hence condition (ii). An easy calculation shows that

$$\left(L_0\begin{pmatrix} y_0 \\ y_1\end{pmatrix}\right)(s) = y_0 \cdot (1-s) + y_1 \cdot s \qquad \text{for } s \in [0,1],$$

i.e. L_0 exists and is bounded and hence condition (iii) is fulfilled. Note that A_0 has compact resolvent and $\sigma(A_0) = \sigma_p(A_0) = \{-k^2 \cdot \pi^2 \colon k \in \mathbb{N}\}$. Now the claim follows by Theorem 3.7. $\square$

In particular for the uncoupled case, i.e. $\Phi = 0$, we obtain the following corollary.

Corollary 7.7. *We obtain*

(i) $\lambda \in \rho(A)$ *if and only if* $\lambda \in \rho(\mathfrak{C})$.
(ii) $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \in \sigma_p(\mathfrak{C})$. *In this case* $\dim(\ker(\lambda - A)) = \dim(\ker(\lambda - \mathfrak{C}))$;
(iii) $\lambda \in \sigma_a(A)$ *if and only if* $\lambda \in \sigma_a(\mathfrak{C})$;
(iv) $\lambda \in \sigma_c(A)$ *if and only if* $\lambda \in \sigma_c(\mathfrak{C})$;
(v) $\lambda \in \sigma_r(A)$ *if and only if* $\lambda \in \sigma_r(\mathfrak{C})$;
(vi) $\lambda \in \sigma_d(A)$ *if and only if* $\lambda \in \sigma_d(\mathfrak{C})$;
(vii) $\lambda \in \sigma_{ess}(A)$ *if and only if* $\lambda \in \sigma_{ess}(\mathfrak{C})$.

Next, we study positivity of the semigroup generated by A and use this to obtain uniformly exponential stability. We additionally assume that Y is a Banach lattice. Then $X = \mathrm{C}([0,1], Y)$ and Y^2 are Banach lattices.

Corollary 7.8. *Assume that the delay operator Φ is positive and $\mathfrak{C}$ generates a strongly continuous semigroup of positive operators on Y^2, then A given by (7.4) generates a strongly continuous semigroup of positive operators on $\mathrm{C}\big([0,1], Y\big)$.*

Proof. By Hopf maximum principle, see [GT01, Thm. 3.5] the operator A_0 has positive resolvent. Further, it follows by a direct calculation or the Hopf maximum principle, see [GT01, Thm. 3.5] that the Dirichlet operator L_λ is positive for $\lambda > 0$. From $B|_{X_0} = \Phi$ it follows $BR(\lambda, A_0)$ are positive for $\lambda > 0$. By positive perturbation it follows that the Dirichlet-to-Neumann operators $N_\lambda = C + \Phi_\lambda$ generate strongly continuous semigroups of positive operators on Y^2 for $\lambda > 0$. Now the claim follows by Corollary 5.3. $\square$

Now applying Theorem 6.2 by using the fact that $s(A_0) < 0$ yields the following result.

Corollary 7.9. *Assume that the delay operator Φ is positive and $\mathfrak{C}$ generates a strongly continuous semigroup of positive operators on $\mathrm{C}\big([0,1], Y\big)$. Denote the semigroup on $\mathrm{C}\big([0,1], Y\big)$ generated by A by $(T_A(t))_{t\geq 0}$ and the semigroup on Y^2 generated by N by $(T_N(t))_{t\geq 0}$. Then $(T_A(t))_{t\geq 0}$ is uniformly exponential stable on $\mathrm{C}\big([0,1], Y\big)$ if and only if $(T_N(t))_{t\geq 0}$ is on Y^2.*

7.3. **Shift-Semigroup on** $\mathrm{C}[-1,0]$.

In this section, we consider the Banach space $X := \mathrm{C}[-1,0]$ of all continuous, complex valued functions equipped with the sup-norm and, for some fixed $\alpha \in (0,1)$ the operator $A \colon D(A) \subset X \to X$ by

$$Af := f', \quad D(A) := \left\{ f \in \mathrm{C}^1[-1,0] \colon f'(0) = \int_{-1}^{0} f'(r) \cdot (-r)^{-\alpha} \, \mathrm{d}r \right\}. \tag{7.5}$$

It follows

Corollary 7.10. *We obtain $\lambda \in \sigma(A^B) = \sigma_p(A^B)$ if and only if $1 = \int_{-1}^{0} e^{\lambda r} \cdot (-r)^{-\alpha} \, \mathrm{d}r$. Moreover, all eigenspaces are one-dimensional.*

Proof. Choosing $X = \mathrm{C}[-1,0]$, $\partial X = \mathbb{C}$, $A_m = \frac{d}{dr}$ with domain $D(A_m) = \mathrm{C}^1[-1,0]$, $L = \delta_0$ and

$$Bf := \int_{-1}^{0} f'(r) \cdot (-r)^{-\alpha} \mathrm{d}r, \quad D(B) := \mathrm{W}^{1,1}(0,1)$$

we obtain $A = A^B$. The verification of Assumptions 2.2 follows as in the proof of Corollary 7.1. Moreover, note that A_0 has empty spectrum and compact resolvent. Using (7.3), a short calculation shows

$$N_\lambda x = x \int_{-1}^{0} \lambda e^{\lambda r} \cdot (-r)^{-\alpha} \, \mathrm{d}r$$

for $\lambda, x \in \mathbb{C}$, in particular $N = 0$ and N has compact resolvent, and we conclude the claim by Theorem 3.7. ☐

Corollary 7.11. *The strongly continuous semigroup generated by A given by* (7.5) *is not positive on* $\mathrm{C}[-1,0]$.

Proof. In [BE20, Thm. 6.4] we prove that the operator A given by (7.5) generates a C_0-semigroup on $\mathrm{C}[-1,0]$. First of all, the spaces $X = \mathrm{C}[0,1]$ and $\partial X = \mathbb{C}$ are Banach lattices, by (7.2) the operator A_0 has positive resolvent and by (7.3) the Dirichlet operator L_λ is positive for $\lambda \in \mathbb{R}$. Consider now the function $f(r) := -re^r$. It is contained in $f \in D(A_0)_+$, but

$$Bf = -\int_{-1}^{0} r^2 e^r \cdot (-r)^{-\alpha} \, \mathrm{d}r < 0,$$

since the integrand is negative. Hence condition (b)(ii) in Theorem 5.10 is violated and the claim follows by Theorem 5.10. ☐

7.4. **Elliptic Operators with Wentzell boundary conditions.** We consider a uniformly elliptic second-order differential operator with Wentzell boundary conditions on $\mathrm{C}(\bar{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. To this end, we first take real-valued functions

$$a_{jk} = a_{kj}, \quad a_j, \quad a_0, \quad b_0 \in \mathrm{C}^\infty(\bar{\Omega}), \quad 1 \le j,k \le n$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(r) \cdot \xi_j \xi_k \ge c \cdot \|\xi\|^2 \quad \text{for all } r \in \bar{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$

and some fixed $c > 0$. Then we define the maximal operator $A_m : D(A_m) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ in divergence form by

$$\begin{aligned} A_m f &:= \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk} \partial_k f \Big) + \sum_{k=1}^{n} a_k \partial_k f + a_0 f, \\ D(A_m) &:= \Big\{ f \in \bigcap_{p \ge 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\bar{\Omega}) \colon A_m f \in \mathrm{C}(\bar{\Omega}) \Big\} \end{aligned} \tag{7.6}$$

and the boundary operator $B : D(B) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$Bf := -\sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k f + b_0 L f, \quad D(B) := \Big\{ f \in \bigcap_{p \ge 1} W^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{C}(\bar{\Omega}) \colon Bf \in \mathrm{C}(\partial\Omega) \Big\},$$

where $L \in \mathcal{L}(\mathrm{C}(\bar{\Omega}), \mathrm{C}(\partial\Omega))$, $Lf := f|_{\partial\Omega}$ denotes the trace operator. Now we define the operator $A : D(A) \subseteq \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ with Wentzell boundary conditions by

$$A \subseteq A_m, \quad D(A) := \big\{ f \in D(A_m) \cap D(B) \colon L A_m f = Bf \big\}. \tag{7.7}$$

We obtain

Corollary 7.12. *The operator A given by (7.7) generates a strongly continuous semigroup of positive operators on* $\mathrm{C}(\bar{\Omega})$.

Proof. Since the principle coefficients are smooth we can rewrite A_m as a uniformly elliptic operator in non-divergence form. By Hopf's maximum principle (cf. [GT01, Thm. 3.5 or Prob. 3.2]) it follows that L_λ is positive for all $\lambda \geq \|a_0\|_\infty$. Again, by Hopf's maximum principle the operator A_0 with Dirichlet boundary conditions has positive resolvent on $\mathrm{C}(\bar{\Omega})$ for $\lambda \geq \|a_0\|_\infty$. In [Esc94] it is shown that the Dirichlet-to-Neumann operators N_λ generate positive semigroups on $\mathrm{C}(\partial\Omega)$ for large $\lambda > \omega$ It follows by Proposition 5.5 that the Dirichlet-to-Neumann operators N_λ generate positive semigroups on $\mathrm{C}(\partial\Omega)$ for $\lambda > s(A_0)$. It remains to show that $BR(\lambda, A_0)$ is positive for large λ. To this end, we prove that $B|_{D(A_0)}$ is positive, which clearly implies the assertion. Let $f > 0$ and $f \in D(A_0)$, in particular $f(x) = 0$ for $x \in \partial\Omega$, then

$$
\begin{aligned}
(Bf)(x) &= -\sum_{j,k=1}^{n} a_{jk}(x)\nu_j(x)(\partial_k f)(x) + b_0(x)\cdot f(x) \\
&= -\lim_{s\downarrow 0} \frac{f(x - s\cdot(a(x)\cdot\nu(x))) - f(x)}{(-s)} + b_0(x)\cdot f(x) \\
&= \lim_{s\downarrow 0} \frac{f(x - s\cdot(a(x)\cdot\nu(x)))}{s} \geq 0
\end{aligned}
$$

for all $x \in \partial\Omega$, where $a(x) = (a_{ij}(x))_{n\times n}$ denotes the coefficient matrix. Now Corollary 5.3 yields the claim. ☐

Further, the spectrum of A can be characterized by

Corollary 7.13. *For $\lambda \in \rho(A_0)$ obtain*

$$
\lambda \in \sigma(A) = \sigma_p(A) \quad \iff \quad \lambda \in \sigma(N_\lambda) = \sigma_p(N_\lambda).
$$

Proof. By [Esc94] we obtain that N_λ have compact resolvents for all $\lambda \in \rho(A_0)$. Moreover by [EF05, Cor. 4.5] it follows that A^B has compact resolvent. Now the claim follows from Theorem 3.7. ☐

We finish this subsection by considering the special case of the Laplacian and the normal derivative, i.e. $a_{jk} = \delta_{jk}$, $a_k = a_0 = b_0 = 0$. One concludes

Corollary 7.14. *The semigroup generated by A given by (7.7) for $a_{jk} = \delta_{jk}$, $a_k = a_0 = b_0 = 0$ is not uniformly exponential stable on* $\mathrm{C}(\overline{\Omega})$.

Proof. Note that A_B^μ have compact resolvents for $\mu \in \mathbb{R}$ and from [Eng03, eq. 1.9] and [Esc94] that A_0 and N_λ have compact and positive resolvents for $\lambda > 0$. Moreover the min-max principle implies that the eigenvalues of A_B^μ are monotone decreasing in $\mu \in \mathbb{R}$. In [Fri91] it is shown that $\lambda_k(\infty) < \lambda_{k+1}(0)$ for all $k \in \mathbb{N}$. Hence Theorem 4.14 implies that there exists a positive eigenvalue of N_λ for all $\lambda > 0$. In particular $s(N_\lambda) > 0$ for all $\lambda > 0$ and by Proposition 5.1 for all $\lambda > s(A_0)$. Now the claim follows from Theorem 6.2. ☐

7.5. **Elliptic Operators with generalized Wentzell boundary conditions.** As in the last section we consider a uniformly elliptic operator in divergence form on $\mathrm{C}(\bar{\Omega})$ but with a different boundary condition. More precisely, let $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary $\partial\Omega$ and we consider the maximal operator $A_m\colon D(A_m) \subset \mathrm{C}(\bar{\Omega}) \to$

$\mathrm{C}(\bar{\Omega})$ given by (7.6). Moreover consider a uniformly elliptic differential operator $C\colon D(C) \subset \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ in divergence form on the boundary space. To this end, take real valued functions

$$ckj = c_{jk} \in \mathrm{C}^\infty(\partial\Omega), \quad c_j \in \mathrm{C}(\partial\Omega), \quad c_0 \in \mathrm{C}(\partial\Omega), \quad 1 \le j,k \le n,$$

such that c_{jk} are strictly elliptic, i.e.

$$\sum_{j,k=1}^{n} c_{jk}(s) \cdot \xi_j \xi_k > M \cdot \|\xi\|^2 \quad \text{for all } s \in \partial\Omega,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

and some fixed $M > 0$. We define the operator $C\colon D(C) \subset \mathrm{C}(\partial\Omega) \to \mathrm{C}(\partial\Omega)$ by

$$Cx := \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} c_{jk} \partial_k x \Big) + \sum_{k=1}^{n} c_k \partial_k x + c_0 x, \quad D(C) := \mathrm{W}^{2,2}(\partial\Omega)$$

and the feedback operator $B\colon D(B) \subset \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\partial\Omega)$ by

$$Bf = CLf - \frac{\partial^a}{\partial n} f, \quad D(B) = \left\{ f \in D(A_m) \cap D\left(\frac{\partial^a}{\partial n}\right) : Lf \in \mathrm{W}^{2,2}(\partial\Omega) \right\},$$

where $L \in \mathcal{L}(\mathrm{C}(\bar{\Omega}), \mathrm{C}(\partial\Omega))$, $Lf := f|_{\partial\Omega}$ denotes the trace operator. We define the operator $A\colon D(A) \subset \mathrm{C}(\bar{\Omega}) \to \mathrm{C}(\bar{\Omega})$ with generalized Wentzell boundary conditions by

$$A \subset A_m, \quad D(A) := \{ f \in D(A_m) \cap D(B) \colon LA_m f = Bf \}. \tag{7.8}$$

Corollary 7.15. *The operator A given by* (7.8) *generates a strongly continuous semigroup of positive operators on* $\mathrm{C}(\bar{\Omega})$.

Proof. Note that the maximal operator and the trace operator in this section coincide with the maximal operator and the trace operator of the last section. Hence, also the associated Dirichlet operators L_λ and the associated operators A_0 with Dirichlet boundary conditions coincide. Further, the feedback operator B restricted to X_0 coincides with feedback operator of the last section. Hence it follows from the proof of Corollary 7.12, that L_λ, $R(\lambda, A_0)$ and $BR(\lambda, A_0)$ are positive operators for $\lambda \ge \|a_0\|_\infty$. Therefore, it remains to show that the Dirichlet-to-Neumann operators N_λ generate strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$ for large λ. To this end note that

$$N_\lambda^B \varphi = C\varphi + N_\lambda^{-\frac{\partial^a}{\partial n}} \varphi, \quad D(N_\lambda^B) = D(C). \tag{7.9}$$

By [Bin20] it is generator of a strongly continuous semigroup on $\mathrm{C}(\partial\Omega)$. By [Nag86, B-II, Thm. 1.6] a semigroup on $\mathrm{C}(\partial\Omega)$ is positive if and only if its generator satisfies the *positive minimum principle*

$$\begin{cases} \text{for all } 0 \le \varphi \in D(A) \text{ and } x \in \partial\Omega \\ \varphi(x) = 0 \text{ implies } (A\varphi)(x) \ge 0 \end{cases} \tag{P}$$

It follows from the proof of Corollary 7.12 that the operators $N_\lambda^{-\frac{\partial^a}{\partial n}}$ generate strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$ for $\lambda > s(A_0)$. Now it follows from (7.9) and the positive minimum principle (P) that for every $0 \le \varphi \in D(N_\lambda) = D(C) \cap D(N_\lambda^{-\frac{\partial^a}{\partial n}})$ with

$\varphi(x) = 0$ implies

$$(N_\lambda\varphi)(x) = \underbrace{(C\varphi)(x)}_{\geq 0} + \underbrace{(N_\lambda^{-\frac{\partial^a}{\partial n}}\varphi)(x)}_{\geq 0} \geq 0$$

for $\lambda > s(A_0)$. Hence, applying the positive minimum principle (P) again yields that the Dirichlet-to-Neumann operators N_λ generate strongly continuous semigroups of positive operators on $\mathrm{C}(\partial\Omega)$. Now the claim follows from Corollary 5.3. □

Appendix A. Spectral theory

Notation A.1. For a closed, linear operator $A\colon D(A) \subseteq E \to E$ on a Banach space E one defines the spectrum and its fine structure by

$$\begin{aligned}
\rho(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \text{ is invertible}\\ \text{with bounded inverse}\end{matrix}\right\} && \text{the \textit{resolvent set} of } A,\\
\sigma(A) &:= \mathbb{C} \setminus \rho(A) && \text{the \textit{spectrum} of } A,\\
\sigma_p(A) &:= \{\lambda \in \mathbb{C}\colon \lambda - A \text{ is not injective}\} && \text{the \textit{point spectrum} of } A,\\
\sigma_a(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \text{ is not injective}\\ \text{or has non closed range}\end{matrix}\right\} && \text{the \textit{approximative point spectrum} of } A,\\
\sigma_c(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \text{ is injective with}\\ \text{dense, non closed range}\end{matrix}\right\} && \text{the \textit{continuous spectrum} of } A,\\
\sigma_r(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\lambda - A \text{ is injective with}\\ \text{non closed range}\end{matrix}\right\} && \text{the \textit{residual spectrum} of } A,\\
\sigma_{ess}(A) &:= \left\{\lambda \in \mathbb{C}\colon \begin{matrix}\operatorname{codim}(\operatorname{rg}(\lambda - A)) = \infty\\ \text{or } \dim(\ker(\lambda - A)) = \infty\end{matrix}\right\} && \text{the \textit{essential spectrum} of } A,\\
\sigma_d(A) &:= \sigma(A) \setminus \sigma_{ess}(A) && \text{the \textit{discrete spectrum} of } A.
\end{aligned}$$

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A.3.2 An abstract framework for interior boundary conditions

AN ABSTRACT FRAMEWORK FOR INTERIOR BOUNDARY CONDITIONS

TIM BINZ AND JONAS LAMPART

ABSTRACT. In a configuration space whose boundary can be identified with a subset of its interior, a boundary condition can relate the behaviour of a function on the boundary and in the interior. Additionally, boundary values can appear as additive perturbations. Such boundary conditions have recently provided insight into problems form quantum field theory. We discuss interior boundary conditions in an abstract setting, with a focus on self-adjoint operators, proving self-adjointness criteria, resolvent formulas, a classification theorem and a convergence result.

1. INTRODUCTION

Consider a differential operator on a configuration space consisting of a disjoint union of manifolds with boundary (or corners) of different dimensions,

$$M = \bigsqcup_{n=1}^{N} M_n. \tag{1.1}$$

If there is a map $\iota_n : \partial M_n \to M_{n-1}$, a boundary condition may relate boundary values of a function on M_n to to the values on $M_{n-1} \supset \iota_n(\partial M_n)$. We call such a boundary condition an interior boundary condition following Teufel and Tumulka [TT15; TT16; Tum20]. If, for example, ι_n is bijective, we may also add boundary value operators, such as $f|_{\partial M_n} \circ \iota_n^{-1}$ as perturbations to the differential operator. This gives rise to a coupled system of equations for functions $f_j = f|_{M_j}$.

A simple example of such a setup is obtained by taking $M_n = (\mathbb{R}_+)^n$ for $n = 0, 1$, i.e., $M_1 = \mathbb{R}_+$, $M_0 = \{0\} = \partial M_1$. As a differential operator L on $M = M_0 \sqcup M_1$ we may take the (negative) Laplacian on $\mathbb{R}_+$, extended to M by setting $Lf|_{M_0} = 0$. Taking a self-adjoint boundary condition for the Laplacian gives rise to decoupled equations for f_0, f_1. However, one can couple the two functions by a boundary condition such as $f_1'(0) = f_0$. The operator

$$H(f_0, f_1) = (f_1(0), -f_1'') = Lf + If_1(0), \tag{1.2}$$

subject to the boundary condition, is then symmetric with respect to the canonical scalar product on $L^2(M) \cong \mathbb{C} \oplus L^2(\mathbb{R}_+)$ (I denotes the inclusion of the first summand), since

$$\begin{aligned} \langle f, Hf \rangle &= \overline{f}_0 f_1(0) - \int_0^\infty \overline{f_1}(x) f_1''(x) dx \\ &= \int_0^\infty |f_1'(x)|^2 dx + \overline{f}_0 f_1(0) + \overline{f}_1(0) f_1'(0) \\ &= \int_0^\infty |f_1'(x)|^2 dx + 2\mathrm{Re}(\overline{f}_0 f_1(0)) \in \mathbb{R}. \end{aligned} \tag{1.3}$$

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We will work in a more abstract framework, where the configuration space actually plays no role. We rather consider directly two Hilbert spaces $\mathscr{H}$ and $\partial\mathscr{H}$, where the first would correspond to $L^2(M)$ and the second to $L^2(\partial M)$. Consider a densely defined "maximal" operator L_m on $\mathscr{H}$ and two "boundary value operators" A_m, B mapping (a subset of) $D(L_m)$ to $\partial\mathscr{H}$. This is a standard setup for abstract boundary value problems, see [Gre87], [ABE14], [ABE17], [AE18] and dynamical boundary conditions, see [CENN03], [EF05] and [BE19]. Further, on Hilbert spaces, such abstract boundary problems are related to the theory of quasi boundary triples, see [BL07], [BGN17], [BHS20] and Appendix A.2.
In addition to the usual ingredients, we assume that we are given a bounded operator

$$I : \partial\mathscr{H} \to \mathscr{H}.$$

This operator is the characteristic feature of the interior boundary conditions, since it allows for the formulation of conditions relating elements of $\mathscr{H}$ and $\partial\mathscr{H}$. Most of the theory we develop reduces to the usual theory of boundary conditions with the choice $I = 0$. Non-trivial examples where such a structure is relevant are hierarchies of boundary value problems. In this case $\mathscr{H}$ is a finite or countable direct sum of spaces $\mathscr{H}_n$, L_m is an operator on $\mathscr{H}_n$ for each n, A_m, B map $\mathscr{H}_n$ to $\partial\mathscr{H}_n$, and $I : \partial\mathscr{H}_n \to \mathscr{H}_{n-1}$ is an isomorphism. On this space we can consider operators such as $H = L_m + IA_m$ subject to boundary conditions, such as $Bf = I^*f$. Spelling out the equation $Hf = g$ on $\mathscr{H}_n$, it reads

$$L_m f_n + IA_m f_{n+1} = g_n, \tag{1.4}$$

with the boundary condition

$$Bf_n = I^* f_{n-1}. \tag{1.5}$$

The unknown f_n is thus coupled to f_{n-1} by the boundary condition and to f_{n+1} by the operator $IA : \mathscr{H}_{n+1} \to \mathscr{H}_n$.
Formulations of different models from quantum field theory (QFT) in terms of such hierarchies have been proposed by Landau and Peierls [LP30], Moshinsky and Lopez [MLL91], Yafaev [Yaf92], Teufel and Tumulka [TT15; TT16]. There, $\mathscr{H}_n$ is the kinematical Hilbert space for n (indistinguishable) particles and possibly some additional particles of a different type, e.g. n photons and a fixed number of electrons. The spaces with different numbers of particles are coupled since, in quantum field theory, particle numbers are not conserved. In this context A_m is closely related to the so-called annihilation operator (or a power thereof), an unbounded operator that reduces particle-number by one, and the boundary condition incorporates the process of particle creation. Similar (finite) hierarchies have also been studied as models for nuclear reactions [Mos51b; Mos51a; Mos51c; Tho84].
In quantum mechanics, the dynamics of a system are generated by a self-adjoint operator. However, constructing these and proving self-adjointness for quantum field theoretic models poses many difficulties. One of these is the problem of ultra-violet singularities that stem from the distributional nature of the interactions. Interior boundary conditions have proved to be an effective way of addressing these singularities. They provide an alternative to renormalisation techniques going back to Nelson [Nel64] and Eckmann [Eck70], with the benefit of giving a direct description of the domain of self-adjointness [KS16; LSTT18; LS19; Sch18; Sch19]. These ideas were later extended to more singular models by the second author [Lam19a; Lam19b]. Other aspects of specific models with interior-boundary conditions were investigated in [TG05; LN19; ST19].

These constructions of self-adjoint operators are related to singular number-preserving interactions that can be described by (generalised) boundary conditions and classified in terms of self-adjoint extensions of certain "minimal" operators (see e.g. [AGHKH88; CDF+15; BFK+17; Pos08], and references therein). Such methods were recently applied an abstract form of interior boundary conditions by Posilicano [Pos20] (see Remark 3.15 for comparison to our approach).
Our goal in this article is to develop a general theory of interior boundary conditions. In Section 2 we explain the abstract framework. In Section 3 we discuss operators with "Robin type" boundary conditions of the form $\alpha A_m f + \beta B f = I^* f$, their symmetry and self-adjointness. In Section 4 we consider more general boundary condtions. In particular, we construct a quasi boundary triple (see Appendix A.2) that allows us to relate operators with different boundary condtions and classify certain self-adjoint conditions. We also discuss the dependence of the operators on the paramteters in the boundary condition. In Section 5 we give a non-trivial example to which our theory can be applied.

2. Abstract Framework

In this section we introduce an abstract framework to formulate interior-boundary conditions and some notational conventions.

Notation 2.1. Let X and Y Banach spaces and $T\colon D(T) \subset X \to Y$ a densely defined operator. For $\lambda \in \rho(T)$, we denote the resolvent of T by

$$R(\lambda, T) := (\lambda - T)^{-1} \in \mathcal{L}(Y, X). \tag{2.1}$$

For $\lambda \in \mathbb{C} \setminus (\sigma_p(T) \cup \sigma_r(T))$, the algebraic inverse of $\lambda - T$ is a densely defined operator with $D((\lambda - T)^{-1}) = \operatorname{rg}(\lambda - T)$, for which we use the notation $(\lambda - T)^{-1}$.
If X and Y are Hilbert spaces, we denote the adjoint of T by $T^* : D(T^*) \subseteq Y \to X$.
Moreover let Z another Banach space and $S\colon D(S) \subset Y \to Z$ a densely defined operator. The composition $ST\colon D(ST) \subset X \to Z$ is the (not necessarily densely defined) operator given by

$$(ST)x := S(Tx), \qquad D(ST) := \{x \in D(T)\colon Tx \in D(S)\}. \tag{2.2}$$

Abstract Setting 2.2. As a starting point of our investigation, we assume that we are given the following objects:

(i) two Hilbert spaces $\mathscr{H}$ and $\partial\mathscr{H}$;
(ii) a "maximal" operator $L_m\colon D(L_m) \subset \mathscr{H} \to \mathscr{H}$;
(iii) a trace operator $B\colon D(L_m) \subset \mathscr{H} \to \partial\mathscr{H}$;
(iv) a boundary operator $A_m\colon D(A_m) \subset \mathscr{H} \to \partial\mathscr{H}$;
(v) a bounded "identification" operator $I\colon \partial\mathscr{H} \to \mathscr{H}$.

We then denote by L the restriction of L_m to the kernel of B

$$L := (L_m)|_{\ker(B)} \tag{2.3}$$

and also

$$A := (A_m)|_{\ker(B)}. \tag{2.4}$$

We assume these to have the following properties.

Assumptions 2.3. The operators L, A, B satisfy:

(a) L is self-adjoint
(b) the operator A is relatively L-bounded;

(c) for $\lambda \in \rho(L)$: $\operatorname{rg}\Big((AR(\lambda, L))^*\Big) \subseteq \ker(\overline{\lambda} - L_m)$;
(d) for $\lambda \in \rho(L)$: $B(AR(\lambda, L))^* = \mathrm{Id}_{\partial\mathscr{H}}$.

Remark 2.4. Note that Assumption 2.3 (d) implies that $(AR(\lambda, L))^*$ is injective and that B is surjective. Since, in general, $\operatorname{rg}(T)^\perp = \ker(T^*)$, $\operatorname{rg}(A) \subset \partial\mathscr{H}$ is dense.

Definition 2.5. For $\overline{\lambda} \in \rho(L)$ we define the *abstract Dirichlet operator* associated with λ as

$$G_\lambda := (AR(\overline{\lambda}, L))^*. \tag{2.5}$$

Moreover we define the *abstract Dirichlet-to-Neumann operator* associated with λ by

$$T_\lambda := A_m G_\lambda, \quad D(T_\lambda) = \{\varphi \in \partial\mathscr{H} \colon G_\lambda \varphi \in D(A_m)\}. \tag{2.6}$$

Remark 2.6. Note that by Assumption 2.3 (b) the abstract Dirichlet operator is bounded, $G_\lambda \in \mathcal{L}(\partial\mathscr{H}, \mathscr{H})$. Further by Assumption 2.3 (c) it satisfies $\operatorname{rg}(G_\lambda) \subseteq \ker(\lambda - L_m)$ and by Assumption 2.3 (d) it is the right-inverse of B. Our definition thus coincides with the common definition of a Dirichlet operator in the lierature, e.g. [Gre87].

We now collect some simple consequences of our general assumptions that will play an important role throughout.

Proposition 2.7. *Under Assumption 2.3 we have:*

(i) *The domain $D(L_m)$ of the maximal operator can be decomposed into*

$$D(L_m) = D(L) \oplus \ker(\lambda - L_m). \tag{2.7}$$

The projections are given by $G_\lambda B \colon D(L_m) \to \ker(\lambda - L_m)$ and $(\mathrm{Id}_{D(L_m)} - G_\lambda B) : D(L_m) \to D(L)$.

(ii) *The following identity holds*

$$G_\lambda^*(\overline{\lambda} - L)f = Af$$

for $f \in D(L)$.

(iii) *For $\lambda, \mu \in \rho(L)$ the domains of the Dirichlet-to-Neumann operators coincide, i. e. $D(T_\lambda) = D(T_\mu)$ (we will thus simply denote this domain by $D(T)$). Moreover their difference, given by*

$$T_\lambda - T_\mu = (\mu - \lambda) AR(\mu, L) G_\lambda,$$

is bounded.

Proof. (i) Since $BG_\lambda = \mathrm{Id}_{\partial\mathscr{H}}$, $G_\lambda B$ and $(\mathrm{Id}_{D(L_m)} - G_\lambda B)$ are projections on $D(L_m)$, and (algebraically) $D(L_m) = \operatorname{rg}(G_\lambda B) \oplus \operatorname{rg}(\mathrm{Id} - G_\lambda B)$. Assumption 2.3 (b) means that the image satisfies $\operatorname{rg}(G_\lambda B) \subseteq \ker(\lambda - L_m)$. Further, we clearly have

$$\operatorname{rg}(\mathrm{Id}_{D(L_m)} - G_\lambda B) \subseteq \ker(B) = D(L). \tag{2.8}$$

Therefore, using $\ker(G_\lambda B) = \operatorname{rg}(\mathrm{Id}_{D(L_m)} - G_\lambda B) \subseteq D(L)$, we obtain

$$D(L_m) = \operatorname{rg}(\mathrm{Id}_{D(L_m)} - G_\lambda B) \oplus \operatorname{rg}(G_\lambda B) \subseteq D(L) + \ker(\lambda - L_m) \subseteq D(L_m). \tag{2.9}$$

Since $\lambda \in \rho(L)$ by assumption, the latter sum is direct and we have

$$D(L_m) = D(L) \oplus \ker(\lambda - L_m). \tag{2.10}$$

(ii) The definition of G_λ implies for $f \in D(L), g \in \mathscr{H}$:

$$\begin{aligned}\langle(\overline{\lambda}-L)f, G_\lambda g\rangle_{\mathscr{H}} &= \langle(\overline{\lambda}-L)f, (AR(\overline{\lambda},L))^* g\rangle_{\mathscr{H}}\\ &= \langle(AR(\overline{\lambda},L))(\overline{\lambda}-L)f, g\rangle_{\mathscr{H}}\\ &= \langle Af, g\rangle_{\mathscr{H}}.\end{aligned} \tag{2.11}$$

(iii) From resolvent resolvent identity it follows

$$G_\lambda^* - G_\mu^* = A(R(\overline{\lambda},L) - R(\overline{\mu},L)) = (\overline{\mu}-\overline{\lambda})AR(\overline{\lambda},L)R(\overline{\mu},L).$$

Using the self-adjointness of L we conclude that

$$G_\lambda - G_\mu = (\mu-\lambda)R(\overline{\mu},L)^*(AR(\overline{\lambda},L))^* = (\mu-\lambda))R(\mu,L)G_\lambda. \tag{2.12}$$

Since A is relatively L-bounded, the term on the right hand side satisfies

$$\mathrm{rg}((\mu-\lambda)R(\mu,L)G_\lambda) \subseteq D(L) \subset D(A)$$

and the first claim follows. The identity for the difference follows from the definition of T_λ, T_μ and the boundedness from the fact that A is L-bounded.

□

We now give a construction procedure that leads to operators L_m, B, A_m etc. with the properties of Setting 2.2. This construction can be applied in many concrete cases, and we give a simple example below. The logic here is somewhat different than in the definitions, in that we start with the operators L, A, I and T_λ (for one, arbitrarily fixed $\lambda \in \rho(L)$). From these, we construct L_m, B and A_m as follows.

Construction 2.8. We are given thwo Hilbert spaces $\mathscr{H}$, $\partial\mathscr{H}$ and a bounded operator $I\colon \partial\mathscr{H} \to \mathscr{H}$. Further, we have a self-adjoint operator $L\colon D(L) \subset \mathscr{H} \to \mathscr{H}$, a relatively L-bounded operator $A\colon D(A) \to \mathscr{H}$ and a closed operator $T\colon D(T) \subset \partial\mathscr{H} \to \partial\mathscr{H}$. To construct the operators L_m, B and A_m we proceed the following steps:

Step 1. *Consider the "minimal" operator* $L_0\colon D(L_0) \subset \mathscr{H} \to \mathscr{H}$, *defined by*

$$L_0 f = Lf, \qquad D(L_0) = D(L) \cap \ker(A) = \ker(A). \tag{2.13}$$

Step 2. *Assume that* $\ker(A)$ *is dense, so the adjoint* L_0^* *is well defined. Let* $\lambda \in \rho(L)$, *and, since* L *is self-adjoint, also* $\overline{\lambda} \in \rho(L)$. *By Proposition 2.7 (ii), the operator* G_λ *given by* $G_\lambda := (AR(\overline{\lambda},L))^*$ *satisfies*

$$\langle(\lambda - L_0^*)G_\lambda f, \varphi\rangle_{\mathscr{H}} = \langle f, G_\lambda^*(\overline{\lambda}-L_0)\varphi\rangle_{\mathscr{H}} = \langle f, A\varphi\rangle_{\partial\mathscr{H}} = 0 \tag{2.14}$$

for $f \in \mathscr{H}$ *and* $\varphi \in D(L_0) \subset \ker(A)$. *So* $\mathrm{rg}(G_\lambda) \subseteq \ker(\lambda - L_0^*)$ *and, since* $\lambda \in \rho(L)$, *the operator*

$$L_m f := L_0^* f, \qquad D(L_m) := D(L) \oplus \mathrm{rg}(G_\lambda). \tag{2.15}$$

is well defined and Assumption 2.3 (c) is satisfied. Note that, since $\mathrm{rg}(G_\mu - G_\lambda) \subset D(L)$ *by (2.12), the right hand side is independent of* λ.

Step 3. *Now we define* $B\colon D(L_m) \to \partial\mathscr{H}$ *as the left-inverse of* G_λ, *i.e. using the unique decomposition* $f = f_0 + G_\lambda\varphi$, $f_0 \in D(L)$, $\varphi \in \mathrm{rg}(G_\lambda)$, *we set* $Bf = B(f_0 + G_\lambda\varphi) := \varphi$, *which satisfies Assumption 2.3 (d), and* $L = (L_m)|_{\ker(B)}$.

Step 4. *Let $\lambda \in \rho(L)$ and $T\colon D(T) \subset \partial\mathscr{H} \to \partial\mathscr{H}$ a fixed operator. We define the operator $A_m\colon D(A_m) \subset \mathscr{H} \to \partial\mathscr{H}$ by*

$$A_m f := Af_0 + T\varphi, \qquad D(A_m) = D(L) \oplus G_\lambda D(T) \subset D(L_m). \tag{2.16}$$

Hence $A := (A_m)|_{\ker(B)}$, so A_m extends A, and $T_\lambda = A_m G_\lambda = T$. The operators T_μ for $\mu \neq \lambda$ are then dertemined by Proposition 2.7 (iii).

The following example is essentially the model considered by Moshinsky [Mos51b; Mos51a; Mos51c] and Yafaev [Yaf92].

Example 2.9 (Moshinsky-Yafaev model). Set $\mathscr{H} = L^2(\mathbb{R}^3) \oplus \mathbb{C}$, $\partial\mathscr{H} = \mathbb{C}$, $Iz = (0, z)$. Define on $D(L) = H^2(\mathbb{R}^3) \oplus \mathbb{C}$

$$L(f, z) = (-\Delta f, 0). \tag{2.17}$$

Let $A : D(L) \to \partial\mathscr{H}$ be given by $A(f, z) = f(0)$.

Step 1. *The operator L_0 is the restriction of L to $D(L_0) = H_0^2(\mathbb{R}^3 \setminus \{0\}) \oplus \mathbb{C}$. Note that this operator is densely defined.*

Step 2. *The domain of the adjoint is given by*

$$D(L_0^*) = D(L) \oplus \operatorname{span}(g_\lambda, 0), \tag{2.18}$$

with, for any $\lambda \in \rho(L) = \mathbb{C} \setminus \mathbb{R}_+$ (taking the branch of the square root with positive real part)

$$g_\lambda(x) = -\frac{e^{-\sqrt{-\lambda}|x|}}{4\pi|x|}. \tag{2.19}$$

Moreover, we have $G_\lambda z = (z g_\lambda, 0)$. Hence we set[1] *$L_m := L_0^*$, which acts as $L_0^*(f, z) = \big(-\Delta_0^* f, 0\big)$, where $-\Delta_0^*$ is the adjoint of $-\Delta|_{H_0^2(\mathbb{R}^3\setminus\{0\})}$*

Step 3. *The operator B, defined as the left-inverse of G_λ, is given by the formula*

$$B(f, z) = -4\pi \lim_{x\to 0} |x| f(x). \tag{2.20}$$

Step 4. *Since g_λ is not continuous in $x = 0$ we cannot define A_m as the evaluation at $x = 0$. However, the following formula, which extends the evaluation, is well defined on $D(L_m)$*

$$A_m(f, z) = A_m f := \lim_{r\to 0} \partial_r r \frac{1}{4\pi} \int_{S^2} f(r\omega) d\omega. \tag{2.21}$$

This yields the formula for $T_\lambda : \mathbb{C} \to \mathbb{C}$

$$T_\lambda = A_m G_\lambda = \lim_{r\to 0} \partial_r \left(-\frac{e^{-\sqrt{-\lambda} r}}{4\pi} \right) = \frac{\sqrt{-\lambda}}{4\pi}. \tag{2.22}$$

With this framework in place, the operators with interior-boundary conditions take the form

$$H_{\mathrm{IBC}}^{\alpha,\beta}(f, z) = (-\Delta_0^* f, \gamma A_m f + \delta B f) \tag{2.23}$$

$$D(H_{\mathrm{IBC}}^{\alpha,\beta}) = \{(f, z) \in D(\Delta_0^*) \oplus \mathbb{C} : \alpha A_m f + \beta B f = z\}, \tag{2.24}$$

with complex numbers $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. One easily checks that these operators are symmetric iff $\bar{\alpha}\gamma, \bar{\beta}\delta \in \mathbb{R}$ and $\beta\bar{\gamma} - \bar{\alpha}\delta = 1$ (see also Lemma 3.4). It is also not difficult to show that these

[1] Note that in general the operator L_0^* is „too big“, in the sense that not all functions in $D(L_0^*)$ have boundary values in $\partial\mathscr{H}$, e.g. if $\partial\mathscr{H} = L^2(\partial M)$.

symmetric operators are self-adjoint, see [Yaf92]. Note that if instead of our choice of I we would have taken $I = 0$, we would have found the "Laplacian with δ-potential" [AGHKH88], which is a well known example in the theory of singular boundary value problems. We will use this example throughout the article to illustrate our results.

3. Interior-Boundary Conditions of Robin Type

In this section we will discuss a simple family of interior-boundary conditions in which the boundary operators A_m and B are related to the values in the interior simply by by some constants, exactly as in Example 2.9 (more general conditions are considered later, in Section 4). We then investigate symmetry and self-adjointness of these operators and prove various formulas for their resolvents.

Here, as always, we work within the framework introduced in Setting 2.2 and Assumption 2.3.

Definition 3.1. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The operators with *interior boundary conditions* (abbreviated IBCs) of type (α, β), denoted $H^{\alpha,\beta}_{\mathrm{IBC}} \colon D(H^{\alpha,\beta}_{\mathrm{IBC}}) \subset \mathscr{H} \to \mathscr{H}$ are defined by

$$\begin{aligned} H^{\alpha,\beta}_{\mathrm{IBC}} f &:= L_m f + \gamma I A_m f + \delta I B f, \\ D(H^{\alpha,\beta}_{\mathrm{IBC}}) &:= \{ f \in D(L_m) \cap D(A_m) \colon \alpha A_m f + \beta B f = I^* f \}. \end{aligned} \tag{3.1}$$

Note that $H^{\alpha,\beta}_{\mathrm{IBC}}$ is really a family of operators depending on γ, δ. However, since the values of γ, δ play only a minor role we suppress them in the notation. Note also that, up to a bounded perturbation, we can always assume that $\delta = 0$ (if $\beta \neq 0$) or $\gamma = 0$ (for $\alpha \neq 0$), since by the boundary condition (e.g. for $\beta \neq 0$)

$$\delta I B|_{D(H^{\alpha,\beta}_{\mathrm{IBC}})} = \delta \beta^{-1} \left(I I^* - \alpha I A_m \right)|_{D(H^{\alpha,\beta}_{\mathrm{IBC}})}.$$

3.1. Symmetry.

We start by investigating the elementary properties of $H^{\alpha,\beta}_{\mathrm{IBC}}$, in particular symmetry. For this we will make the following additional assumption for the remainder of the article.

Assumption 3.2. For all $\lambda \in \rho(L)$, we have $T_{\overline{\lambda}} \subset T^*_\lambda$. In particular, T_λ is symmetric on $\partial\mathscr{H}$ for $\lambda \in \mathbb{R} \cap \rho(L)$.

Note that, since L is self-adjoint by Assumption 2.3 a), $\lambda \in \rho(L)$ implies $\overline{\lambda} \in \rho(L)$, so the assumption makes sense. By Proposition 2.7 (iii), if $T_{\overline{\lambda}} \subset T^*_\lambda$ for one $\lambda \in \rho(L)$, then this automatically holds for all $\lambda \in \rho(L)$.

With this assumption, it is easy to show an *abstract Green's identity*, which essentially generalises integration-by-parts for Laplace-type operators to our abstract setting.

Lemma 3.3. *The following identity holds for all $f, g \in D(L_m) \cap D(A_m)$*

$$\langle L_m f, g \rangle_{\mathscr{H}} - \langle f, L_m g \rangle_{\mathscr{H}} = \langle B f, A_m g \rangle_{\partial\mathscr{H}} - \langle A_m f, B g \rangle_{\partial\mathscr{H}}.$$

Proof. Let $\lambda \in \rho(L)$ and $f, g \in D(L_m) \cap D(A_m)$. Note that $\overline{\lambda} \in \rho(L)$. Using $D(L_m) = D(L) \oplus \ker(\lambda - L_m)$, $\operatorname{rg}(G_\lambda) \subseteq \ker(\lambda - L_m)$ and Proposition 2.7 (ii) we obtain

$$\begin{aligned} \langle (L_m - \lambda) f, g \rangle_{\mathscr{H}} &= \langle (L_m - \lambda)(\mathrm{Id} - G_\lambda B) f, (\mathrm{Id} - G_{\overline{\lambda}} B) g + G_{\overline{\lambda}} B g \rangle_{\mathscr{H}} \\ &= \langle (L - \lambda)(\mathrm{Id} - G_\lambda B) f, (\mathrm{Id} - G_{\overline{\lambda}} B) g \rangle_{\mathscr{H}} + \langle (L - \lambda)(\mathrm{Id} - G_\lambda B) f, G_{\overline{\lambda}} B g \rangle_{\mathscr{H}} \\ &= \langle (L - \lambda)(\mathrm{Id} - G_\lambda B) f, (\mathrm{Id} - G_{\overline{\lambda}} B) g \rangle_{\mathscr{H}} + \langle (G_{\overline{\lambda}})^* (L - \lambda)(\mathrm{Id} - G_\lambda B) f, B g \rangle_{\partial\mathscr{H}} \\ &= \langle (L - \lambda)(\mathrm{Id} - G_\lambda B) f, (\mathrm{Id} - G_{\overline{\lambda}} B) g \rangle_{\mathscr{H}} - \langle A (\mathrm{Id} - G_\lambda B) f, B g \rangle_{\partial\mathscr{H}} \\ &= \langle (L - \lambda)(\mathrm{Id} - G_\lambda B) f, (\mathrm{Id} - G_{\overline{\lambda}} B) g \rangle_{\mathscr{H}} - \langle A_m f, B g \rangle_{\partial\mathscr{H}} + \langle T_\lambda B f, B g \rangle_{\partial\mathscr{H}}. \end{aligned} \tag{3.2}$$

By an analogous calculation we obtain

$$\langle f,(L_m-\overline{\lambda})g\rangle_{\mathscr{H}}=\langle(\mathrm{Id}-G_\lambda B)f,(L-\overline{\lambda})(\mathrm{Id}-G_{\overline{\lambda}}B)g\rangle_{\mathscr{H}}-\langle Bf,A_mg\rangle_{\partial\mathscr{H}}+\langle Bf,T_{\overline{\lambda}}Bg\rangle_{\partial\mathscr{H}}. \tag{3.3}$$

By the symmetry of L and $T_{\overline{\lambda}}\subset T_\lambda^*$, taking the difference of these two equations proves the claim. □

With this result we can easily determine when $H^{\alpha,\beta}_{\mathrm{IBC}}$ is symmetric. Conditions of this type where also given in [Tum20, eq. (8)-(10)]. The necessity of these conditions will be further addressed in the more general framework of Section 4.

Lemma 3.4. *The operators $H^{\alpha,\beta}_{\mathrm{IBC}}$ are symmetric on $\mathscr{H}$ if*

$$\bar\alpha\gamma,\bar\beta\delta\in\mathbb{R}\qquad\textit{and}\qquad\beta\bar\gamma-\bar\alpha\delta=1.$$

Proof. From Lemma 3.3 we conclude

$$\begin{aligned}\langle H^{\alpha,\beta}_{\mathrm{IBC}}f,g\rangle_{\mathscr{H}}-\langle f,H^{\alpha,\beta}_{\mathrm{IBC}}g\rangle_{\mathscr{H}}=&\langle Bf,A_mg\rangle_{\partial\mathscr{H}}-\langle A_mf,Bg\rangle_{\partial\mathscr{H}}\\&+\gamma\langle IA_mf,g\rangle_{\mathscr{H}}+\delta\langle IBf,g\rangle_{\mathscr{H}}\\&-\bar\gamma\langle f,IA_mg\rangle_{\mathscr{H}}-\bar\delta\langle f,IBg\rangle_{\mathscr{H}}\end{aligned}\tag{3.4}$$

for $f,g\in D(H^{\alpha,\beta}_{\mathrm{IBC}})$. The IBC $\alpha A_mf+\beta Bf=I^*f$ now implies

$$\begin{aligned}&\gamma\langle IA_mf,g\rangle_{\mathscr{H}}+\delta\langle IBf,g\rangle_{\mathscr{H}}-\bar\gamma\langle f,IA_mg\rangle_{\mathscr{H}}-\bar\delta\langle f,IBg\rangle_{\mathscr{H}}\\&=\bar\alpha\gamma\langle A_mf,A_mg\rangle_{\partial\mathscr{H}}+\bar\beta\gamma\langle A_mf,Bg\rangle_{\partial\mathscr{H}}+\bar\alpha\delta\langle Bf,A_mg\rangle_{\partial\mathscr{H}}+\bar\beta\delta\langle Bf,Bf\rangle_{\partial\mathscr{H}}\\&\quad-\alpha\bar\gamma\langle A_mf,A_mg\rangle_{\partial\mathscr{H}}-\beta\bar\gamma\langle Bf,A_mg\rangle_{\partial\mathscr{H}}-\alpha\bar\delta\langle A_mf,Bg\rangle_{\partial\mathscr{H}}-\beta\bar\delta\langle Bf,Bf\rangle_{\partial\mathscr{H}}\\&=(\bar\alpha\gamma-\alpha\bar\gamma)\langle A_mf,A_mg\rangle_{\partial\mathscr{H}}+(\bar\beta\delta-\beta\bar\delta)\langle Bf,Bg\rangle_{\partial\mathscr{H}}\\&\quad+\left(\bar\beta\gamma-\alpha\bar\delta\right)\langle A_mf,Bg\rangle_{\partial\mathscr{H}}-(\beta\bar\gamma-\bar\alpha\delta)\langle Bf,A_mg\rangle_{\partial\mathscr{H}}\end{aligned}\tag{3.5}$$

for $f,g\in D(H^{\alpha,\beta}_{\mathrm{IBC}})$. Combining (3.4) and (3.5) yields

$$\begin{aligned}\langle H^{\alpha,\beta}_{\mathrm{IBC}}f,g\rangle_{\partial\mathscr{H}}-\langle f,H^{\alpha,\beta}_{\mathrm{IBC}}g\rangle_{\partial\mathscr{H}}&=(\bar\alpha\gamma-\alpha\bar\gamma)\langle A_mf,A_mg\rangle_{\partial\mathscr{H}}+(\bar\beta\delta-\beta\bar\delta)\langle Bf,Bg\rangle_{\partial\mathscr{H}}\\&\quad+\left(\bar\beta\gamma-\alpha\bar\delta-1\right)\langle A_mf,Bg\rangle_{\partial\mathscr{H}}-(\beta\bar\gamma-\bar\alpha\delta-1)\langle Bf,A_mg\rangle_{\partial\mathscr{H}},\end{aligned}\tag{3.6}$$

so clearly $H^{\alpha,\beta}_{\mathrm{IBC}}$ is symmetric under the given conditions. □

Remark 3.5. The results of this section show that $(\partial\mathscr{H},B,A_m)$ is a *quasi boundary triple* for the restriction $(L_m)|_{D(A_m)}$ (see Definition A.1).
In this context, the identity (3.2) is called the *abstract Green's identity*. By Remark 2.4, $A=(A_m)|_{\ker(B)}$ has dense range and B is surjective. This implies that $(A_m,B)\colon D(A_m)\cap D(L_m)\to\partial\mathscr{H}\times\partial\mathscr{H}$ has dense range. Finally, $L=(L_m)|_{\ker(B)}$ is a self-adjoint operator on $\mathscr{H}$, by hypothesis.

3.2. **Self-adjointness.** In the framework of quasi boundary triples, the symmetric/self-adjoint boundary conditions for L_m have been studied extensively [DM91; DM95; BL07; BM14; Pos08]. In particular, this applies to the Robin-type conditions $\alpha Af+\beta Bf=0$, that correspond to the choice $I=0$ for $H^{\alpha,\beta}_{\mathrm{IBC}}$.

In this section we study the self-adjointness of $H_{\mathrm{IBC}}^{\alpha,\beta}$ in relation to these Robin-type operators and provide formulas for its resolvent. Throughout, we assume that the parameters $\alpha, \beta, \gamma, \delta$ satisfy the symmetry condition of Lemma 3.4

$$\bar{\alpha}\gamma,\ \bar{\beta}\delta \in \mathbb{R}, \text{ and } \beta\bar{\gamma} - \bar{\alpha}\delta = 1. \tag{3.7}$$

We begin by introducing the usual Robin-type operators.

Definition 3.6. For $\alpha, \beta \in \mathbb{C}$ we denote by $L_{\alpha,\beta} \colon D(L_{\alpha,\beta}) \subset \mathscr{H} \to \mathscr{H}$ the *abstract operator with Robin boundary conditions*

$$L_{\alpha,\beta} f := L_m f, \quad D(L_{\alpha,\beta}) := \{f \in D(L_m) \cap D(A_m) \colon \alpha A_m f + \beta B f = 0\}. \tag{3.8}$$

We will study the relationship between $L_{\alpha,\beta}$ and $H_{\mathrm{IBC}}^{\alpha,\beta}$. Expressing properties of $H_{\mathrm{IBC}}^{\alpha,\beta}$ by those of $L_{\alpha,\beta}$ is useful, since the latter are better understood. The operator $L_{0,1} = L$ is usually particularly simple.
The operators $L_{\alpha,\beta}$ are symmetric if $\alpha\overline{\beta} \in \mathbb{R}$ (note that is implied by the symmetry conditions for $H_{\mathrm{IBC}}^{\alpha,\beta}$), which follows from Lemma 3.4 with $I = 0$ or a simple calculation using Lemma 3.3. In the Moshinsky-Yafaev model (Example 2.9), the operators $L_{\alpha,\beta}$ correspond to the Laplacian in $\mathbb{R}^3$ with a δ-potential at $x = 0$ and coupling (scattering length) $\alpha\beta^{-1} = \alpha\bar{\beta}|\beta|^{-2} \in \mathbb{R} \cup \{\pm\infty\}$. The relationship between $L_{\alpha,\beta}$ and $H_{\mathrm{IBC}}^{\alpha,\beta}$ will be expressed using the following operators that generalise G_λ.

Definition 3.7. Let $\bar{\lambda} \in \rho(L_{\alpha,\beta})$. We define the *abstract Dirichlet-operators* associated with α, β and λ by

$$G_\lambda^{\alpha,\beta} = \Big((\gamma A_m + \delta B) R(\overline{\lambda}, L_{\alpha,\beta})\Big)^*, \qquad D(G_\lambda^{\alpha,\beta}) = \mathrm{rg}(\alpha T_{\bar{\lambda}}^* + \beta). \tag{3.9}$$

Moreover we define the *abstract Dirichlet-to-Neumann operator* associated with α, β and λ by

$$T_\lambda^{\alpha,\beta} := (\gamma A_m + \delta B) G_\lambda^{\alpha,\beta}, \qquad D(T_\lambda^{\alpha,\beta}) = \mathrm{rg}(\alpha T_{\bar{\lambda}}^* + \beta). \tag{3.10}$$

In order to investigate these operators, we need the following well-known resolvent formula for $L_{\alpha,\beta}$ (see e.g. [DM91; DM95; BL07]).

Lemma 3.8. *Let $(\alpha, \beta) \neq 0$ and $\lambda \in \rho(L)$. Then $\lambda \in \rho(L_{\alpha,\beta})$ if and only if $\alpha T_\lambda + \beta$ is one-to-one and $\mathrm{rg}(A) \subset \mathrm{rg}\,(\alpha T_\lambda + \beta)$. In this case the resolvent satisfies*

$$R(\lambda, L_{\alpha,\beta}) = \Big(1 - \alpha G_\lambda \left(\alpha T_\lambda + \beta\right)^{-1} A\Big) R(\lambda, L).$$

Proof. Since L is self-adjoint, we have $\lambda, \bar{\lambda} \in \rho(L)$ and we can write $f = f_0 + G_\lambda \varphi$ with $f_0 \in D(L)$. The equation $(\lambda - L_{\alpha,\beta})f = g$ then takes the form

$$\begin{aligned} (\lambda - L_m) f = (\lambda - L) f_0 &\overset{!}{=} g \\ (\alpha A + \beta B)\, f = \alpha A f_0 + (\alpha T_\lambda + \beta)\varphi &\overset{!}{=} 0. \end{aligned} \tag{3.11}$$

A solution ϕ to the second equation is clearly unique if and only if $\ker(\alpha T_\lambda + \beta) = \{0\}$, so $(\alpha T_\lambda + \beta)$ must be one-to-one.
Solving the first equation for $f_0 = R(\lambda, L)g$, we see that f_0 can be any element of $D(L)$, depending on g. Hence the solution to the system (3.11) exists for every $g \in \mathscr{H}$ exactly if $\mathrm{rg}(A) \subset \mathrm{rg}\,(\alpha T_\lambda + \beta)$. Under these hypothesis and using the algebraic inverse

$$(\alpha T_\lambda + \beta)^{-1} : \mathrm{rg}(\alpha T_\lambda + \beta) \to D(T_\lambda), \tag{3.12}$$

we obtain the solution to (3.11) as

$$f_0 = R(\lambda, L)g, \qquad \varphi = -\alpha\left(\alpha T_\lambda + \beta\right)^{-1} Af_0, \tag{3.13}$$

which gives the resolvent formula. □

Similar to Proposition 2.7 we now obtain the following statements.

Proposition 3.9. *For $\lambda \in \rho(L_{\alpha,\beta}) \cap \rho(L)$:*

(i) $G_{\bar\lambda}^{\alpha,\beta}$ *is densely defined and bounded;*

(ii) *The operator $\alpha T_\lambda^* + \beta$ has a densely defined inverse*

$$(\alpha T_\lambda^* + \beta)^{-1} : \mathrm{rg}(\alpha T_\lambda^* + \beta) \to \partial\mathscr{H}$$

and we have

$$G_{\bar\lambda}^{\alpha,\beta} = G_{\bar\lambda}\left(\alpha T_\lambda^* + \beta\right)^{-1};$$

(iii) *The image satisfies* $\mathrm{rg}(G_{\bar\lambda}^{\alpha,\beta}) \subset \ker(\bar\lambda - L_m) \cap D(A_m)$*;*

(iv) *The following identity holds*

$$(\alpha A_m + \beta B)G_{\bar\lambda}^{\alpha,\beta} = \mathrm{Id}_{D(G_{\bar\lambda}^{\alpha,\beta})}.$$

(v) *The operator $T_{\bar\lambda}^{\alpha,\beta}$ is densely defined and given by the formula*

$$T_{\bar\lambda}^{\alpha,\beta} = (\gamma T_{\bar\lambda} + \delta)(\alpha T_\lambda^* + \beta)^{-1}.$$

Proof. (i) By definition, $G_{\bar\lambda}^{\alpha,\beta}$ is (a restriction of) the adjoint of an everywhere-defined operator. It is thus sufficient to prove that $G_{\bar\lambda}^{\alpha,\beta}$ is densely defined, because this implies that it is the adjoint of a closable operator, and this is closed and bounded since it is everywhere defined. The claim will thus follow from (ii).

(ii) First note that for $\alpha\bar\beta \in \mathbb{R}$ (which follows from the assumed relations of $\alpha, \beta, \gamma, \delta$), we have $L_{\alpha,\beta} = L_{\bar\alpha,\bar\beta}$. With the resolvent formula of Lemma 3.8 we thus have

$$\begin{aligned}(\gamma A_m + \delta B)R(\lambda, L_{\alpha,\beta}) =& (\gamma A_m + \delta B)\left(1 - \bar\alpha G_\lambda\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1} A\right) R(\lambda, L)\\ =& \left(\gamma A - \bar\alpha(\gamma T_\lambda + \delta)\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1} A\right) R(\lambda, L).\end{aligned} \tag{3.14}$$

Now on $\mathrm{rg}(A)$

$$\bar\alpha(\gamma T_\lambda + \delta)\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1} = \gamma\,\mathrm{Id}_{\mathrm{rg}(A)} + \underbrace{(\bar\alpha\delta - \gamma\bar\beta)}_{=-1}\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1}, \tag{3.15}$$

so (3.14) simplifies to

$$(\gamma A_m + \delta B)R(\lambda, L_{\alpha,\beta}) = \left(\bar\alpha T_\lambda + \bar\beta\right)^{-1} AR(\lambda, L). \tag{3.16}$$

This shows that for all φ the domain of the adjoint of $\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1}$, which is well defined since $\mathrm{rg}(A) \subset D(\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1})$ is dense by Remark 2.4, we have

$$((\gamma A_m + \delta B)R(\lambda, L_{\alpha,\beta}))^*\varphi = G_{\bar\lambda}\left(\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1}\right)^*\varphi \tag{3.17}$$

We now need to show that $D(G_{\bar\lambda}^{\alpha,\beta}) = \mathrm{rg}(\alpha T_\lambda^* + \beta)$ is contained in the domain of this adjoint and dense. Density is an immediate consequence of Lemma 3.8, since

$$\mathrm{rg}(\alpha T_\lambda^* + \beta)^\perp = \ker(\bar\alpha T_\lambda + \bar\beta) = \{0\}. \tag{3.18}$$

For all $\psi \in D(T_\lambda^*)$, $\varphi \in \operatorname{rg}(\bar\alpha T_\lambda + \bar\beta)$ we have

$$\langle (\alpha T_\lambda^* + \beta)\psi, \left(\bar\alpha T_\lambda + \bar\beta\right)^{-1} \varphi\rangle_{\partial\mathscr{H}} = \langle \psi, \varphi\rangle_{\partial\mathscr{H}}, \tag{3.19}$$

so we clearly have

$$\left(\left(\bar\alpha T_\lambda + \bar\beta\right)^{-1}\right)^* (\alpha T_\lambda^* + \beta) = \mathrm{Id}_{D(T_\lambda^*)}\,. \tag{3.20}$$

This completes the proof of (ii) and thereby also (i).

(iii) The fact that $\operatorname{rg}(G_{\bar\lambda}^{\alpha,\beta}) \subset \ker(\bar\lambda - L_m)$ is immediate from (ii). Since the range of $(\alpha T_\lambda^* + \beta)^{-1}$ is contained in $D(T_\lambda^*)$, we also have $\operatorname{rg}(G_{\bar\lambda}^{\alpha,\beta}) \subset D(A_m)$.

(iv) Again using (ii) we find

$$(\alpha A_m + \beta B)G_{\bar\lambda}^{\alpha,\beta} = (\alpha A_m + \beta B)G_{\bar\lambda}\,(\alpha T_\lambda^* + \beta)^{-1} = (\alpha T_{\bar\lambda} + \beta)\,(\alpha T_\lambda^* + \beta)^{-1}.$$

Since $T_{\bar\lambda} \subset T_\lambda^*$ this proves the claim.

(v) This follows immediately from (i) and (ii). □

We can now go back to investigating the operator $H_{\mathrm{IBC}}^{\alpha,\beta}$. The following lemma provides a parametrisation of $D(H_{\mathrm{IBC}}^{\alpha,\beta})$ in terms of $D(L_{\alpha,\beta})$, under the condition that $\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*$ is invertible for some $\lambda \in \rho(L_{\alpha,\beta})$. This is certainly satisfied if $L_{\alpha,\beta}$ is self-adjoint, A_m is infinitesimally $L_{\alpha,\beta}$-bounded and the imaginary part of λ is large enough. It is also usually satisfied if there is a hierarchical structure of the form that we have in applications to quantum field theory, see Remark 5.5. In the Moshinsky-Yafaev model (Example 2.9), this is particularly obvious, since there $(G_\lambda^{\alpha,\beta} I^*)^2 = 0$, so the inverse is simply given by $(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)^{-1} = \mathrm{Id} + G_\lambda^{\alpha,\beta} I^*$. This parametrisation for the case $\alpha = 0$ appears already in the works [LS19; Lam19a; Sch19; Sch18], where it plays an important role.

Lemma 3.10. *Assume that* $\bar\lambda \in \rho(L) \cap \rho(L_{\alpha,\beta})$. *If* $1 \in \rho(G_\lambda^{\alpha,\beta} I^*)$, *we denote* $\Gamma_\lambda^{\alpha,\beta} := R(1, G_\lambda^{\alpha,\beta} I^*)$ *and the equality*

$$D(H_{\mathrm{IBC}}^{\alpha,\beta}) = (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)^{-1} D(L_{\alpha,\beta}) = \Gamma_\lambda^{\alpha,\beta} D(L_{\alpha,\beta}). \tag{3.21}$$

holds.

Proof. Since both sides are subsets of $D(L_m) \cap D(A_m)$ it is sufficient to verify the boundary conditions.

Assume first that $f = f_0 + G_{\bar\lambda}\varphi \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$, $f_0 \in D(L)$. Using the interior boundary condition and Lemma 3.8 we first find

$$I^* f = (\alpha T_{\bar\lambda} + \beta)\varphi + A f_0 \in \operatorname{rg}(\alpha T_{\bar\lambda} + \beta) \subset \operatorname{rg}(T_\lambda^* + \beta). \tag{3.22}$$

We thus have $f \in D(G_\lambda^{\alpha,\beta} I^*)$ and can use Proposition 3.9 (iv) to obtain

$$(\alpha A_m + \beta B)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f = (\alpha A_m + \beta B) f - (\alpha A_m + \beta B) G_\lambda^{\alpha,\beta} I^* f = I^* f - I^* f = 0. \tag{3.23}$$

Conversely, we assume that $\eta \in D(L_{\alpha,\beta})$. Since $(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)$ is invertible and hence surjective, there exists an $f \in D(G_\lambda^{\alpha,\beta} I^*)$ with $\eta = (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f$. Note that

$$f = \underbrace{\eta}_{\in D(A_m)\cap D(B)} - \underbrace{G_\lambda^{\alpha,\beta} I^* f}_{\in D(A_m)\cap D(B)} \in D(A_m) \cap D(B). \tag{3.24}$$

It follows from Proposition 3.9 (iv) that

$$(\alpha A_m + \beta B)f = (\alpha A_m + \beta B)G_\lambda^{\alpha,\beta} I^* f = I^* f \tag{3.25}$$

and hence $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$. □

The following lemma relates relative bounds of A_m and T_λ, using the L-boundedness of A and the decomposition of Proposition 2.7(i). Recall that by our convention

$$D(IT_\lambda^{\alpha,\beta} I^*) = \{f \in \mathscr{H} : I^* f \in D(T_\lambda^{\alpha,\beta})\}. \tag{3.26}$$

Lemma 3.11. *Assume that $\lambda, \bar\lambda \in \rho(L_{\alpha,\beta}) \cap \rho(L)$ and $1 \in \rho(G_\lambda^{\alpha,\beta} I^*)$. Further, assume that $IT_\lambda^{\alpha,\beta} I^*$ is relatively $(\mathrm{Id} - G_{\bar\lambda} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)$-bounded of bound $a < 1$.
Then*

(i) *$IT_\lambda^{\alpha,\beta} I^*$ is relatively $H_{\mathrm{IBC}}^{\alpha,\beta}$-bounded. If $a = 0$, i. e. the bound relative to $(\mathrm{Id} - G_{\bar\lambda} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)$ is infinitesimal, then the $H_{\mathrm{IBC}}^{\alpha,\beta}$-bound is also infinitesimal.*

(ii) *$\gamma I A_m$ is relatively $H_{\mathrm{IBC}}^{\alpha,\beta}$-bounded. If $a = 0$, then the $H_{\mathrm{IBC}}^{\alpha,\beta}$-bound is also infinitesimal.*

Proof. (i) From the definition of $G_{\bar\lambda}^{\alpha,\beta}$ we obtain

$$(G_{\bar\lambda}^{\alpha,\beta} I^*)^*(\lambda - L_{\alpha,\beta}) = I(G_{\bar\lambda}^{\alpha,\beta})^*(\lambda - L_{\alpha,\beta}) = \gamma I A_m + \delta I B. \tag{3.27}$$

Using $\mathrm{rg}(G_\lambda^{\alpha,\beta}) \subset \ker(\lambda - L_m)$ and Lemma 3.10 it follows for $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$

$$\begin{aligned}(H_{\mathrm{IBC}}^{\alpha,\beta} - \lambda)f =& (L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f + \gamma A_m f + \delta B f\\ =& (\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f\\ &+ (G_{\bar\lambda}^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f + \gamma I A_m f + \delta I B f.\end{aligned} \tag{3.28}$$

With (3.27) the last line becomes

$$(G_{\bar\lambda}^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f + \gamma I A_m f + \delta I B f = (\gamma I A_m + \delta I B)G_\lambda^{\alpha,\beta} I^* f = IT_\lambda^{\alpha,\beta} I^*, \tag{3.29}$$

and consequently

$$(H_{\mathrm{IBC}}^{\alpha,\beta} - \lambda)f = (\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f + IT_\lambda^{\alpha,\beta} I^*. \tag{3.30}$$

In particular we obtain that $D(H_{\mathrm{IBC}}^{\alpha,\beta}) \subset D(IT_\lambda^{\alpha,\beta} I^*)$. Since $I \in \mathcal{L}(\partial\mathscr{H}, \mathscr{H})$ we conclude

$$\begin{aligned}\|IT_\lambda^{\alpha,\beta} I^* f\| &\le a\|(\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)f\| + b\|f\|\\ &\le a\|H_{\mathrm{IBC}}^{\alpha,\beta} f\| + a\|IT_\lambda^{\alpha,\beta} I^* f\| + (b + |\lambda|)\|f\|\end{aligned} \tag{3.31}$$

for all $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$. Since $a < 1$, the claim follows by absorbing the $\|IT_\lambda^{\alpha,\beta} I^* f\|$-term on the left hand side.

(ii) The case $\gamma = 0$ is trivial so let $\gamma \neq 0$. By Proposition 3.9(i) we have that $G_{\bar\lambda}^{\alpha,\beta}$ is bounded and by the proof of Proposition 3.9(i) that $(\gamma A_m + \delta B)R(\lambda, L_{\alpha,\beta})$ is bounded. It follows from (3.16) that

$$(\bar\alpha T_\lambda + \bar\beta)^{-1} A R(\lambda, L) = (\gamma A_m + \delta B)R(\lambda, L_{\alpha,\beta}). \tag{3.32}$$

Hence Lemma 3.8 implies

$$\begin{aligned} BR(\lambda, L_{\alpha,\beta}) &= BR(\lambda.L_{\bar{\alpha},\bar{\beta}}) \\ &= B(\mathrm{Id} - \bar{\alpha} G_\lambda(\bar{\alpha} T_\lambda + \bar{\beta})^{-1} A) R(\lambda, L) \\ &= -\bar{\alpha}(\bar{\alpha} T_\lambda + \bar{\beta})^{-1} A R(\lambda, L) \\ &= -\bar{\alpha}(\gamma A_m + \delta B) R(\lambda, L_{\alpha,\beta}) \end{aligned} \tag{3.33}$$

is bounded. We conclude that $\gamma A_m R(\lambda, L_{\alpha,\beta})$ is bounded. In the following we consider the case $\beta \neq 0$. The case $\beta = 0$ works by the same arguments. By Lemma 3.10 we obtain, using the Robin boundary condition

$$\begin{aligned} \gamma I A_m f + \delta I B f &= I(\gamma A_m + \delta B) G_\lambda^{\alpha,\beta} I^* f + I(\gamma A_m + \delta B)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f \\ &= I T_\lambda^{\alpha,\beta} I^* f + I(\gamma A_m + \delta B)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f \\ &= I T_\lambda^{\alpha,\beta} I^* f + \frac{1}{\bar{\beta}} I A_m (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f \end{aligned} \tag{3.34}$$

for $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$. The first term is relatively $H_{\mathrm{IBC}}^{\alpha,\beta}$-bounded by (i). Since $A_m R(\lambda, L_{\alpha,\beta})$ is bounded and $1 \in \rho(G_\lambda^{\alpha,\beta} I^*)$, we obtain using (3.30) that

$$\begin{aligned} \| I A_m (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f \| &\leq \hat{a} \cdot \| (L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f \| + b \| f \| \\ &\leq \hat{a} C \cdot \| (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)^* (L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) \| + b \| f \| \\ &= \hat{a} C \cdot \| H_{\mathrm{IBC}}^{\alpha,\beta} f \| + \hat{a} C \cdot \| I T_\lambda^{\alpha,\beta} I^* f \| + b \| f \| \end{aligned} \tag{3.35}$$

for $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$, where $\hat{a} \coloneqq |\gamma|^{-1} \cdot \| A_m R(\lambda, L_{\alpha,\beta})$ and $C \coloneqq \| (\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)^{-1} \|$. Using (i) one concludes

$$\| \gamma I A_m f + \delta I B f \| \leq \tilde{a} \| H_{\mathrm{IBC}}^{\alpha,\beta} f \| + b \| f \| \tag{3.36}$$

for $f \in D(H_{\mathrm{IBC}}^{\alpha,\beta})$, i.e. the operator $(\gamma I A_m + \delta I B)$ is relatively $H_{\mathrm{IBC}}^{\alpha,\beta}$-bounded of bound $\tilde{a} \ \coloneqq a \cdot \left(1 + \frac{\hat{a} \cdot C}{|\beta|}\right)$. In particular the bound is infinitesimal if the bound a is infinitesimal. Now the claim follows since by the IBC

$$\gamma I A_m f + \delta I B f = \frac{1}{\bar{\beta}} A_m f - \frac{\delta}{\bar{\beta}} I I^* f \tag{3.37}$$

and the last term is bounded. ☐

Apart from the statement of Lemma 3.11, an important finding is the equation (3.30). It represents $H_{\mathrm{IBC}}^{\alpha,\beta}$ as a perturbation of an operator that is obtained by transforming $L_{\alpha,\beta}$. This leads to the main theorem of this section.

Theorem 3.12. *Assume that $L_{\alpha,\beta}$ is self-adjoint and let $\lambda \in \rho(L_{\alpha,\beta}) \cap \rho(L)$. Assume also that $1 \in \rho(G_\lambda^{\alpha,\beta} I^*) \cap \rho(G_{\bar{\lambda}}^{\alpha,\beta} I^*)$ and $I T_\mu^{\alpha,\beta} I^*$ is relatively $(\mathrm{Id} - G_{\bar{\mu}}^{\alpha,\beta} I^*)^* (L_{\alpha,\beta} - \mu)(\mathrm{Id} - G_\mu^{\alpha,\beta} I^*)$ bounded, with bound $a < 1$ for $\mu \in \{\lambda, \bar{\lambda}\}$, then $H_{\mathrm{IBC}}^{\alpha,\beta}$ is self-adjoint.*
Moreover, with $\Gamma_\lambda^{\alpha,\beta} = R(1, G_\lambda^{\alpha,\beta} I^*)$, we have

$$\lambda \in \rho(L_{\alpha,\beta}) \cap \rho(H_{\mathrm{IBC}}^{\alpha,\beta}) \quad \Longleftrightarrow \quad 1 \in \rho\Big((\Gamma_{\bar{\lambda}}^{\alpha,\beta})^* I T_\lambda^{\alpha,\beta} I^* \Gamma_\lambda^{\alpha,\beta} R(\lambda, L_{\alpha,\beta}) \Big)$$

and the resolvent is then given by

$$R(\lambda, H_{\mathrm{IBC}}^{\alpha,\beta}) = \Gamma_\lambda^{\alpha,\beta} R(\lambda, L_{\alpha,\beta})\Big(1 - (\Gamma_{\bar\lambda}^{\alpha,\beta})^* I T_\lambda^{\alpha,\beta} I^* \Gamma_\lambda^{\alpha,\beta} R(\lambda, L_{\alpha,\beta})\Big)^{-1} (\Gamma_{\bar\lambda}^{\alpha,\beta})^*.$$

Proof. As $L, L_{\alpha,\beta}$ are self-adjoint, we also have $\bar\lambda \in \rho(L_{\alpha,\beta}) \cap \rho(L)$.
Using (3.30) twice, we write

$$\begin{aligned} H_{\mathrm{IBC}}^{\alpha,\beta} =& \tfrac{1}{2}(H_{\mathrm{IBC}}^{\alpha,\beta} - \lambda) + \tfrac{1}{2}(H_{\mathrm{IBC}}^{\alpha,\beta} - \bar\lambda) + \mathrm{Re}(\lambda) \\ =& \tfrac{1}{2}(\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \lambda)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) f + \tfrac{1}{2}(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*)^*(L_{\alpha,\beta} - \bar\lambda)(\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*) f \\ &+ \tfrac{1}{2}(I T_{\bar\lambda}^{\alpha,\beta} I^* + I T_\lambda^{\alpha,\beta} I^*) + \mathrm{Re}(\lambda). \end{aligned} \tag{3.38}$$

Since both $H_{\mathrm{IBC}}^{\alpha,\beta}$ and the sum of the two expressions involving $L_{\alpha,\beta}$ are symmetric on $D(H_{\mathrm{IBC}}^{\alpha,\beta})$, so is the last line. Self-adjointness of $H_{\mathrm{IBC}}^{\alpha,\beta}$ thus follows from the Kato-Rellich theorem.
To show the resolvent formlula, we take (3.30) and use that $\lambda \in \rho(L_{\alpha,\beta})$ to write

$$\begin{aligned} \lambda - H_{\mathrm{IBC}}^{\alpha,\beta} =& (\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)^* \left(\lambda - L_{\alpha,\beta} - (\Gamma_{\bar\lambda}^{\alpha,\beta})^* I T_\lambda^{\alpha,\beta} I^* \Gamma_\lambda^{\alpha,\beta}\right)(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*) \\ =& (\mathrm{Id} - G_{\bar\lambda}^{\alpha,\beta} I^*)^* \left(1 - (\Gamma_{\bar\lambda}^{\alpha,\beta})^* I T_\lambda^{\alpha,\beta} I^* \Gamma_\lambda^{\alpha,\beta} R(\lambda, L_{\alpha,\beta})\right)(\lambda - L_{\alpha,\beta})(\mathrm{Id} - G_\lambda^{\alpha,\beta} I^*). \end{aligned} \tag{3.39}$$

Since $1 \in \rho(G_\lambda^{\alpha,\beta} I^*) \cap \rho(G_{\bar\lambda}^{\alpha,\beta} I^*)$ and $\lambda \in \rho(L_{\alpha,\beta})$, the right hand side is invertible if and only if $1 \in \rho((\Gamma_{\bar\lambda}^{\alpha,\beta})^* I T_\lambda^{\alpha,\beta} I^* \Gamma_\lambda^{\alpha,\beta} R(\lambda, L_{\alpha,\beta})$. Assuming this implies the formula as claimed. □

In Example 2.9, the hypothesis on $T_\lambda^{\alpha,\beta}$, are all trivially satisfied, since $\partial\mathscr{H}$ is one-dimensional. For the applications in [LS19; Lam19a; Sch19; Sch18] proving the relative bound for T_λ was the main technical difficulty. For the case $\alpha = 0$ relevant there, we can formulate the following corollary. A similar abstract formulation has appeared in [Pos20].

Corollary 3.13. *Let $\lambda \in \rho(L) \cap \mathbb{R}$ and assume that $1 \in \rho(G_\lambda I^*)$ and $I T_\lambda I^*$ is relatively $(\mathrm{Id} - G_\lambda I^*)^*(L - \lambda)(\mathrm{Id} - G_\lambda I^*)$-bounded of bound $a < |\beta|^{-2}$. Then $H_{\mathrm{IBC}}^{0,\beta}$ is self-adjoint for any γ, δ such that the symmetry conditions are satisfied.*

Proof. The symmetry condition $\beta\bar\gamma - \bar\alpha\delta = 1$ with $\alpha = 0$ implies that $\bar\gamma = \beta^{-1}$. Then with Proposition 3.9

$$T_\lambda^{0,\beta} = (\gamma A_m + B) G_\lambda^{0,\beta} = (\bar\beta^{-1} A_m + \delta) G_\lambda \beta^{-1} = |\beta|^{-2} T_\lambda + \delta\beta^{-1}.$$

The claim thus follows from our theorem. □

For $\alpha \neq 0$ we obtain the following corollary, which highlights a key difference, namely that for $\alpha \neq 0$ the boundary condition may be used to control A_m.

Corollary 3.14. *Let $\lambda \in \rho(L) \cap \mathbb{R}$. If $-\beta \in \rho(\alpha T_\lambda)$ then $\lambda \in \rho(L_{\alpha,\beta})$, and if additionally $1 \in \rho(G_\lambda^{\alpha,\beta} I^*)$ then $H_{\mathrm{IBC}}^{\alpha,\beta}$ is self-adjoint.*

Proof. By Lemma 3.8, $-\beta \in \rho(\alpha T_\lambda)$ implies that $\lambda \in \rho(L_{\alpha,\beta}) \cap \mathbb{R}$, so $L_{\alpha,\beta}$ is self-adjoint. Now assume that $1 \in \rho(G_\lambda^{\alpha,\beta} I^*)$ (note that this holds if A is infinitesimally L-bounded and

$\operatorname{dist}(\lambda, \sigma(L))$ is large enough by Proposition 3.9 (ii)). Since $\alpha T_\lambda + \beta)$ has a bounded inverse, we have $\operatorname{rg}(T_\lambda^* + \beta) = \partial\mathscr{H}$ and

$$\begin{aligned} \alpha T_\lambda^{\alpha,\beta} &= \alpha(\gamma T_\lambda + \delta)(\alpha T_\lambda^* + \beta)^{-1} \\ &= \gamma \cdot \mathrm{Id} + (\beta\gamma - \alpha\delta) R(-\beta, \alpha T_\lambda) \\ &= \gamma \cdot \mathrm{Id} + R(-\beta, \alpha T_\lambda). \end{aligned} \tag{3.40}$$

The operator $T_\lambda^{\alpha,\beta}$ is thus bounded, and the hypothesis of Theorem 3.12 are satisfied. □

Remark 3.15. Posilicano [Pos20] discusses self-adjointness of the operator $H_{\mathrm{IBC}}^{0,1}$ (with $\mathscr{H} = \partial\mathscr{H}$ and $I = \mathrm{Id}$), considering $T = T_{z_0}$ (for some fixed $z_0 \in \mathbb{C}$) as a parameter. The resolvent of $H_{\mathrm{IBC}}^{0,1}$ is constructed by first perturbing $L = L_{0,1}$ to obtain $L_{1,0}$ as in Lemma 3.8 and then obtaining $H_{\mathrm{IBC}}^{0,1}$ as an extension of the restriction of $L_m + IA_m$ to $D(L_{1,0}) \cap \ker(I^* - B)$, which is also a restriction of $L_{1,0}$.
In our notation, the formula fo the resolvent reads, with $\widehat{G}_z = \big((I^* - B)R(\bar{z}, L_{1,0})\big)^*$ (c.f. [Pos20, Thm.3.4])

$$\begin{aligned} R(\lambda, H_{\mathrm{IBC}}^{0,1}) =& R(\lambda, L_{1,0}) - \widehat{G}_\lambda\big((I^* - B)\widehat{G}_\lambda\big)^{-1}(I^* - B)R(\lambda, L_{1,0}) \\ =& \Big(1 - \widehat{G}_\lambda\big((I^* - B)\widehat{G}_\lambda\big)^{-1}(I^* - B)\Big)\big(1 - G_\lambda T_\lambda^{-1} A\big)R(\lambda, L). \end{aligned} \tag{3.41}$$

The validity of this formula requires somewhat stronger hypothesis than Corollary 3.13, such as invertibility of T_λ, though one can obtain a formula as in Theorem 3.12 by expanding (3.41) and thereby recover the weeker hypothesis (for $\alpha = 0$), see [Pos20, Thm.3.10].

4. Classification of interior-boundary conditions

In this section we will embed the IBC-operators studied in the previous sections into the extension theory of symmetric operators to obtain general criteria for self-adjointness and a classification of symmetric and self-adjoint IBCs. To achieve this, we take a family of self-adjoint IBC-operators that are all extensions of a common symmetric operator and thus all restrictions of one operator. We then construct a quasi boundary triple for such a "maximal" operator and thereby obtain conditions for a generalised IBC to be symmetric or self-adjoint. Consider for $0 \neq g \in \mathbb{R}$ the domain

$$D(H_0) = \{f \in D(L_m) \cap D(A_m) \colon gAf = gBf = I^* f\} = D(H_{\mathrm{IBC}}^{0,g}) \cap D(H_{\mathrm{IBC}}^{g,0}). \tag{4.1}$$

Clearly we have $D(H_0) \subset D(H_{\mathrm{IBC}}^{\alpha,\beta})$ if $\alpha + \beta = g$. Furthermore, we have

$$H_{\mathrm{IBC}}^{\alpha,\beta}|_{D(H_0)} = L_m + (\gamma + \delta) g I I^*, \tag{4.2}$$

so the actions of all $H_{\mathrm{IBC}}^{\alpha,\beta}$ with $(\gamma + \delta) = \text{const.}$ agree on $D(H_0)$ and all of these operators are symmetric/self-adjoint extensions of $H_0 := H_{\mathrm{IBC}}^{0,g}|_{D(H_0)}$. We consider only the case $\alpha + \beta = 1 = \gamma + \delta$. More general conditions can be reduced to this case by modifying the operator I, see Remark 5.7.

Definition 4.1. We define the operator $H_0 \colon D(H_0) \subset \mathscr{H} \to \mathscr{H}$ by

$$H_0 f = L_m f + II^* f, \qquad D(H_0) = \{f \in D(L_m) \cap D(A_m) \colon A_m f = Bf = I^* f\}. \tag{4.3}$$

Lemma 4.2. *The operator*

$$H_m := L_m + II^* + I(A_m - B), \qquad D(H_m) = D(L_m) \cap D(A_m) \tag{4.4}$$

is a restriction of H_0^*.

Proof. Take $f \in D(H_m)$ and $g \in D(H_0)$, then by Lemma 3.3

$$\begin{aligned} \langle f, H_0 g\rangle_{\mathscr{H}} &= \langle L_m f, g\rangle_{\mathscr{H}} - \langle Bf, A_m g\rangle_{\partial\mathscr{H}} + \langle A_m f, Bg\rangle_{\partial\mathscr{H}} + \langle II^* f, g\rangle_{\mathscr{H}} \\ &= \langle (L_m + I(A_m - B) + II^*)f, g\rangle. \end{aligned} \tag{4.5}$$

If $D(H_0)$ is dense, this proves that $H_m \subset H_0^*$ as operators. If $D(H_0)$ is not dense, the adjoint is not a well defined operator, but the equation shows that the graph of H_m is contained in the adjoint relation to the graph of H_m (see Appendix A.1), so $H_m \subset H_0^*$ in the sense of relations. □

Note that we avoid here the hypothesis that $D(H_0)$ is dense. Even though we expect this to be the case in relevant examples, it might be quite difficult to verify.

Lemma 4.3. *We have the abstract Green's identity*

$$\langle H_m f, g\rangle_{\mathscr{H}} - \langle f, H_m g\rangle_{\mathscr{H}} = \langle (B - I^*)f, (A_m - I^*)g\rangle_{\partial\mathscr{H}} - \langle (A_m - I^*)f, (B - I^*)g\rangle_{\partial\mathscr{H}} \tag{4.6}$$

for $f, g \in D(H_m)$.

Proof. Using the formula (3.2) for L_m and (4.4), we find

$$\begin{aligned} \langle H_m f, g\rangle_{\mathscr{H}} - \langle f, H_m g\rangle_{\mathscr{H}} =& \langle Bf, A_m g\rangle_{\partial\mathscr{H}} - \langle A_m f, Bg\rangle_{\partial\mathscr{H}} \\ &+ \langle (A_m - B + I^*)f, I^* g\rangle_{\partial\mathscr{H}} - \langle I^* f, (A_m - B + I^*)g\rangle_{\partial\mathscr{H}} \\ =& - \langle (B - I^*)f, I^* g\rangle_{\partial\mathscr{H}} + \langle Bf, A_m g\rangle_{\partial\mathscr{H}} - \langle I^* f, A_m g\rangle \\ &+ \langle A_m f, I^* g\rangle_{\partial\mathscr{H}} - \langle A_m f, Bg\rangle_{\partial\mathscr{H}} + \langle I^* f, (B - I^*)g\rangle_{\partial\mathscr{H}}, \end{aligned} \tag{4.7}$$

which yields the formula as claimed. □

We will obtain a classification of the extensions of H_0 by constructing a quasi boundary triple for H_m. To this end, we define the corresponding abstract Dirichlet operator.

Definition 4.4. Assume the hypothesis of Theorem 3.12 with $\alpha = 0, \beta = 1$ and let $\lambda \in \rho(H_{\mathrm{IBC}}^{0,1})$. We define

$$F_\lambda := ((A_m - I^*)R(\overline{\lambda}, H_{\mathrm{IBC}}^{0,1}))^*, \qquad D(F_\lambda) := D(T). \tag{4.8}$$

Lemma 4.5. *Let $1 \in \rho(I^* G_\lambda) \cap \rho(G_\lambda I^*)$. Then*

$$G_\lambda(\mathrm{Id} - G_\lambda I^*)^{-1} = (\mathrm{Id} - I^* G_\lambda)^{-1} G_\lambda. \tag{4.9}$$

Proof. We have

$$(\mathrm{Id} - G_\lambda I^*)G_\lambda = G_\lambda - G_\lambda I^* G_\lambda = G_\lambda(\mathrm{Id} - I^* G_\lambda). \tag{4.10}$$

Now the claim follows by multiplying with $(\mathrm{Id} - G_\lambda I^*)^{-1}$ from the right hand side and $(\mathrm{Id} - I^* G_\lambda)^{-1}$ from the left hand side. □

Proposition 4.6. *Assume the of Theorem 3.12 with $\alpha = 0, \beta = 1$ and that $(1 - I^* G_\lambda)^{-1}$ leaves $D(T)$ invariant. The operator F_λ for $\lambda \in \rho(H_{\mathrm{IBC}}^{0,1}) \cap \rho(L)$ satisfies*

(i) $\mathrm{rg}(F_\lambda) \subseteq \ker(\lambda - H_m)$;
(ii) $(B - I^*)F_\lambda = \mathrm{Id}_{D(T)}$.

Proof. We begin by proving that $\mathrm{rg}(F_\lambda) \subseteq D(H_m)$ and then check properties i), ii). From Theorem 3.12 we have, denoting $\Gamma_\lambda := \Gamma_\lambda^{0,1} = (\mathrm{Id} - G_\lambda I^*)^{-1}$,

$$\begin{aligned} R(\bar\lambda, H_{\mathrm{IBC}}^{0,1}) =& \Gamma_{\bar\lambda} R(\bar\lambda, L)(1 - \Gamma_\lambda^* I T_{\bar\lambda} I^* \Gamma_{\bar\lambda} R(\bar\lambda, L))^{-1} \Gamma_\lambda^* \\ =& \Gamma_{\bar\lambda} R(\bar\lambda, L) \left(1 + \Gamma_\lambda^* I T_{\bar\lambda} I^* \Gamma_{\bar\lambda} R(\bar\lambda, L)(1 - \Gamma_\lambda^* I T_{\bar\lambda} I^* \Gamma_{\bar\lambda} R(\bar\lambda, L))^{-1}\right) \Gamma_\lambda^* \\ =& \Gamma_{\bar\lambda} R(\bar\lambda, L) \Gamma_\lambda^* \left(1 + I T_{\bar\lambda} I^* R(\bar\lambda, H_{\mathrm{IBC}}^{0,1})\right). \end{aligned} \tag{4.11}$$

Denote $\Theta_\lambda = (A_m \Gamma_{\bar\lambda} R(\bar\lambda, L)\Gamma_\lambda^*)^*$. We have

$$A_m \Gamma_{\bar\lambda} R(\bar\lambda, L) = A_m R(\bar\lambda, L) + A_m G_{\bar\lambda} I^* \Gamma_{\bar\lambda} R(\bar\lambda, L) = G_\lambda^* + T_{\bar\lambda} I^* \Gamma_{\bar\lambda} R(\bar\lambda, L), \tag{4.12}$$

whence

$$\Theta_\lambda = \Gamma_\lambda G_\lambda + \Gamma_\lambda R(\lambda, L) \Gamma_{\bar\lambda}^* I T_{\bar\lambda}^*. \tag{4.13}$$

The first term, $\Gamma_\lambda G_\lambda = G_\lambda (1 - I^* G_\lambda)^{-1}$ maps $D(T)$ to $D(H_m)$ since by Lemma 4.5 the operator $(1 - I^* G_\lambda)^{-1}$ leaves $D(T)$ invariant and G_λ maps $D(T)$ to $D(A_m)$. The second term acts on $D(T)$ as $\Gamma_\lambda R(\lambda, L)\Gamma_{\bar\lambda}^* I T_\lambda$ because $T_\lambda \subset T_{\bar\lambda}^*$. By Lemma 3.10, $\Gamma_\lambda R(\lambda, L)\Gamma_{\bar\lambda}^* I$ is a bounded operator from $\partial\mathscr{H}$ to $D(H_{\mathrm{IBC}}^{0,1}) \subset D(H_m)$, so Θ_λ maps $D(T)$ to $D(H_m)$. Since I^* maps $D(H_{\mathrm{IBC}}^{0,1})$ to $D(T)$ and $I^* \Gamma_\lambda G_\lambda = ((1 - I^* G_\lambda)^{-1} - 1)$ leaves $D(T)$ invariant, by hypothesis, we see that $I^* \Theta_\lambda$ leaves $D(T)$ invariant. From (4.11) we then see that

$$F_\lambda = \Theta_\lambda + R(\lambda, H_{\mathrm{IBC}}^{0,1}) I T_\lambda I^* \Theta_\lambda - R(\lambda, H_{\mathrm{IBC}}^{0,1}) I, \tag{4.14}$$

and thus $\mathrm{rg}(F_\lambda) \subseteq D(H_m)$.

For (i) it is now sufficient to prove that $\mathrm{rg}(F_\lambda) \subseteq \ker(\lambda - H_0^*)$, by Lemma 4.2, which follows from

$$\langle F_\lambda \varphi, (H_0 - \bar\lambda) g\rangle_{\mathscr{H}} = \langle \varphi, F_\lambda^* (H_{\mathrm{IBC}}^{0,1} - \bar\lambda) g\rangle_{\partial\mathscr{H}} = -\langle \varphi, (A_m - I^*) g\rangle_{\partial\mathscr{H}} = 0, \tag{4.15}$$

for all $\varphi \in \partial\mathscr{H}$, $g \in D(H_0) \subset D(H_{\mathrm{IBC}}^{0,1})$.

To check (ii), notice that our previous analysis shows that for $\varphi \in D(T)$

$$F_\lambda \varphi = \Gamma_\lambda G_\lambda \varphi + f \tag{4.16}$$

with $f \in D(H_{\mathrm{IBC}}^{0,1}) \subset \ker(B - I^*)$. The claim thus follows from Lemma 4.5

$$(B - I^*)\Gamma_\lambda G_\lambda = (1 - I^* G_\lambda)^{-1} - I^* G_\lambda (1 - I^* G_\lambda)^{-1} = \mathrm{Id}_{D(T)}. \tag{4.17}$$

□

Theorem 4.7. *Assume the hypothesis of Proposition 4.6. Then the triple*

$$\Big(\partial\mathscr{H}, (B - I^*), (A_m - I^*)\Big)$$

is a quasi boundary triple for H_m. *Furthermore,* $\mathscr{G} := \mathrm{rg}(A_m - I^*)|_{\ker(B - I^*)}$ *is dense in* $\partial\mathscr{H}$.

Proof. We have already shown the abstract Green's identity (Lemma 4.3) and self-adjointness of $H_m|_{\ker(B-I^*)} = H_{\mathrm{IBC}}^{0,1}$ (Theorem 3.12), so it only remains to prove that $\mathrm{rg}(B - I^*, A_m - I^*)$ is dense in $\partial\mathscr{H} \times \partial\mathscr{H}$. To see this, first note that $\mathrm{rg}(B - I^*) = D(T)$ is dense. We can complete the argument by showing that $\mathscr{G}$ is dense, since then the affine space $\{(A_m - I^*) f : f \in D(H_m), (B - I^*) f = \varphi\}$ is also dense for all $\varphi \in \mathrm{rg}(B - I^*)$. To check this, it is sufficient to note that

$$\Big(\mathrm{rg}(A_m - I^*) R(\bar\lambda, H_{\mathrm{IBC}}^{0,1})\Big)^\perp = \ker(F_\lambda), \tag{4.18}$$

and that F_λ is injective since it has the left-inverse $B - I^*$, by Proposition 4.6. □

We can now use the theory of quasi boundary triples to obtain criteria for self-adjointness as well as a classification of interior-boundary conditions. We will formulate these in terms of linear realtion in $\partial\mathscr{H}$, i.e. linear subspaces of $\partial\mathscr{H} \oplus \partial\mathscr{H}$. This has the advantage of being able to deal with somewhat degenerate cases (e.g. where $\alpha = 0$ or $\beta = 0$) without distinction. We provide the relevant notions for calculating with relations in Appendix A.1.
We denote the Dirichlet-to-Neumann operator with respect to $A_m - I^*$ and $B - I^*$ by

$$S_\lambda := (A_m - I^*)F_\lambda,$$

where F_λ is defined in Definition 4.4. By Proposition 4.6 S_λ is well defined on $D(S_\lambda) = D(T)$ (since $D(A_m) \subset D(H_m)$). Following [BL07, Prop.2.4, Thm 2.8] we have (see Proposition 5.9 for an application):

Theorem 4.8. *Assume the hypothesis of Proposition 4.6. Let $\mathfrak{R}$ be a linear relation in $\partial\mathscr{H}$ and define*

$$H_\mathfrak{R} = H_m|_{D(H_\mathfrak{R})}$$
$$D(H_\mathfrak{R}) = \Big\{ f \in D(H_m) : \big((B - I^*)f, (A_m - I^*)f\big) \in \mathfrak{R} \Big\}.$$

If $\mathfrak{R}$ is symmetric, then $H_\mathfrak{R}$ is symmetric.
If moreover there exists $\lambda \in \mathbb{R} \cap \rho(H^{0,1}_{\mathrm{IBC}}) \cap \rho(L)$ such that the relation $\mathfrak{R} - S_\lambda$ is one-to-one and $\mathrm{rg}(F^*_\lambda) \subset \mathrm{rg}(\mathfrak{R} - S_\lambda)$, then $H_\mathfrak{R}$ is self-adjoint, $\lambda \in \rho(H_\mathfrak{R})$ and the resolvent is given by

$$R(\lambda, H_\mathfrak{R}) = (1 + F_\lambda(\mathfrak{R} - S_\lambda)^{-1}(A_m - I^*))R(\lambda, H^{0,1}_{\mathrm{IBC}}).$$

Proof. For $(f, g) \in \mathfrak{R} \subset \partial\mathscr{H} \times \partial\mathscr{H}$ and $(f^*, g^*) \in \mathfrak{R}^*$, then by definition of the adjoint relation (A.1),

$$\langle f, g^* \rangle_{\partial\mathscr{H}} - \langle g, f^* \rangle_{\partial\mathscr{H}} = 0.$$

Hence if $\mathfrak{R} \subset \mathfrak{R}^*$ the operator $H_\mathfrak{R}$ is symmetric by Lemma 4.3.
By Proposition 4.6 we can write any $f \in D(H_m)$ uniquely as $f = f_0 + F_\lambda\varphi$ with $f_0 \in D(H^{0,1}_{\mathrm{IBC}}) = \ker(B - I^*)$, $\varphi \in D(T)$. As in Lemma 3.8, solving $(\lambda - H_\mathfrak{R})f = g$ then amounts to solving

$$(\lambda - H^{0,1}_{\mathrm{IBC}})f_0 \overset{!}{=} g$$
$$\Big(\varphi, (A_m - I^*)f_0 + S_\lambda\varphi\Big) \overset{!}{\in} \mathfrak{R}.$$

The first equation and $\lambda \in \rho(H^{0,1}_{\mathrm{IBC}})$ imply

$$f_0 = R(\lambda, H^{0,1}_{\mathrm{IBC}})g.$$

With $(A_m - I^*)f_0 = F^*_\lambda g$, the inclusion is satisfied if and only if

$$(\varphi, F^*_\lambda g) \in \mathfrak{R} - S_\lambda.$$

Since $\mathrm{rg}(F^*_\lambda) \subset \mathrm{rg}(\mathfrak{R} - S_\lambda)$, such a φ exists and since $\mathfrak{R} - S_\lambda$ is one-to-one it is unique. Thus for every $g \in \partial\mathscr{H}$ we can uniquely solve for f_0 and φ, so $\lambda \in \rho(H_\mathfrak{R})$. Since

$$(\varphi, F^*_\lambda g) = (\varphi, (A_m - I^*)R(\lambda, H^{0,1}_{\mathrm{IBC}})g) \in \mathfrak{R} - S_\lambda,$$

we have

$$((A_m - I^*)R(\lambda, H^{0,1}_{\mathrm{IBC}})g, \varphi) \in (\mathfrak{R} - S_\lambda)^{-1},$$

which, sicne φ is unique, we write as $\varphi = (\mathfrak{R} - S_\lambda)^{-1}(A_m - I^*)R(\lambda, H_{\mathrm{IBC}}^{0,1})g$ and obtain the resolvent formula. Since $H_{\mathfrak{R}}$ is symmetric and ist resolvent set contains the real number λ, $H_{\mathfrak{R}}$ is self-adjoint. □

In order to obtain a classifiation of the self-adjoint restrictions of H_m, which are extensions of H_0, we need the additional hypothesis that $D(H_0)$ is dense. To formulate this result, define the non-negative, bounded operator (c.f. Appendix A.2)

$$M := (F_i^* F_i)^{1/2} \tag{4.19}$$

Applying Theorem A.4 to the boundary triple $(\partial\mathscr{H}, B - I^*, A_m - I^*)$ yields the following result.

Theorem 4.9 ([BM14, Sect. 3])**.** *Assume the hypothesis of Proposition 4.6, that $D(H_0)$ is dense and that there exists $\lambda \in \mathbb{R} \cap \rho(H_{\mathrm{IBC}}^{0,1})$. Let $\mathfrak{R}$ be a relation in $\partial\mathscr{H}$ and define $H_{\mathfrak{R}}$ as in Theorem 4.8. Then $H_{\mathfrak{R}}$ is self-adjoint if and only if the relation*

$$M^{-1}(\mathfrak{R} - S_\lambda)M_-^{-1} \tag{4.20}$$

is self-adjoint and satisfies $D(\mathfrak{R}) \subset M_- D(S)$.

This is a complete classification, since for any self-adjoint operator H with $H_0 \subset H \subset H_m$ there is a relation $\mathfrak{R}$ such that $H = H_{\mathfrak{R}}$, which is simply given by

$$\mathfrak{R} = \{((B - I^*)f, (A - I^*)f) \,|\, f \in D(H)\}. \tag{4.21}$$

In the Moshinsky-Yafaev model (Example 2.9), $\partial\mathscr{H} = \mathbb{C}$ and one easily checks that H_0 is densely defined with deficiency indices $(1, 1)$. The operators $H_{\mathrm{IBC}}^{\alpha,\beta}$ (with $\alpha + \beta = 1 = \gamma + \delta$) are all self-adjoint extensions of H_0. In [Yaf92], these are discussed as extensions of the (not densely defined!) restriction of L (and H_0) with boundary condtions $I^* f = 0 = Bf$. Our result clarifies their relation to the usual extension theory for symmetric operators.

4.1. **Continuity with respect to the boundary conditions.**

Using our results on the family $H_{\mathrm{IBC}}^{\alpha,\beta}$, we can now study the continuity with respect boundary parameters. For simplicity we consider in the sequel the operator $H_{\mathrm{IBC}}^{\alpha,1}$ with $\delta = 0$ and $\gamma = 1$. Note that these operators are symmetric for all $\alpha \in \mathbb{R}$ by Lemma 3.4. We show that the operators $H_{\mathrm{IBC}}^{\alpha,1}$ converge in the norm resolvent sense to $H_{\mathrm{IBC}}^{0,1}$.

Lemma 4.10. *Suppose that there exists a $\lambda_0 \in \mathbb{R}$ such that T_λ closed and $IT_\lambda I^*$ is relatively $(\mathrm{Id} - G_\lambda I^*)^*(L - \lambda)(\mathrm{Id} - G_\lambda I^*)$-bounded of infinitesimal bound for all $\lambda < \lambda_0$. Assume that L is bounded form below and the hypotheses of Proposition 4.6. Then the operators $T_\lambda(\mathrm{Id} - I^* G_\lambda)^{-1}$ and S_λ differ by a relatively bounded perturbation of bound $a < 1$ for sufficient negative $\lambda < \lambda_0$.*

Proof. From (4.14) we obtain

$$\begin{aligned} S_\lambda = {} & A_m \Theta_\lambda + A_m R(\lambda, H_{\mathrm{IBC}}^{0,1}) I T_\lambda^* I^* \Theta_\lambda - A_m R(\lambda, H_{\mathrm{IBC}}^{0,1}) I \\ & + I^* \Theta_\lambda + I^* R(\lambda, H_{\mathrm{IBC}}^{0,1}) I T_\lambda^* I^* \Theta_\lambda - I^* R(\lambda, H_{\mathrm{IBC}}^{0,1}) I. \end{aligned} \tag{4.22}$$

Note that the terms beginning with I^* are bounded by the same arguments as in the proof of Proposition 4.6. Lemma 3.11 implies that A_m is relatively $H_{\mathrm{IBC}}^{0,1}$-bounded with infinitesimal bound. Since L is bounded from below we have that $H_{\mathrm{IBC}}^{0,1}$ is bounded from below and the second term becomes sufficient small bound for sufficient large negative λ. From the proof of

Proposition 4.6 it follows that $T_\lambda^* I^* \Theta_\lambda$ is relatively T_λ-bounded. Hence we obtain that the second term of (4.22) is relatively T_λ bounded of bound 0, whereas the last term is bounded.

Finally, using Lemma 4.5 we obtain

$$\begin{aligned} A_m\Theta_\lambda &= A_m\Gamma_\lambda + A_m\Gamma_\lambda R(\lambda, L)\Gamma_\lambda^* I T_\lambda j \\ &= T_\lambda(\mathrm{Id} - I^* G_\lambda)^{-1} + A_m\Gamma_\lambda R(\lambda, L)\Gamma_\lambda^* I T_\lambda^*. \end{aligned} \tag{4.23}$$

As we have seen in the proof of Proposition 4.6 the operator $\Gamma_\lambda R(\lambda, L)\Gamma_\lambda^* I$ is bounded from $\partial\mathscr{H}$ to $D(H_{\mathrm{IBC}}^{0,1})$. Since by Lemma 3.11 A_m is relatively $H_{\mathrm{IBC}}^{0,1}$-bounded of bound 0 the second term in (4.23) is relatively T_λ-bounded with sufficient small bound for negative, sufficient large λ.
Since $(\mathrm{Id} - I^* G_\lambda)$ is bounded and invertible and leaves $D(T_\lambda)$ invariant the relatively T_λ-boundedness implies relatively $T_\lambda(\mathrm{Id} - I^* G_\lambda)^{-1}$-boundedness. □

Lemma 4.11. *Suppose that there exists a $\lambda_0 \in \mathbb{R}$ such that $\lambda \in \rho(H_{\mathrm{IBC}}^{0,1})$, T_λ are self-adjoint on $\partial\mathscr{H}$ and bounded from above and $I T_\lambda I^*$ is relatively $(\mathrm{Id} - G_\lambda I^*)^*(L-\lambda)(\mathrm{Id} - G_\lambda I^*)$-bounded of infinitesimal bound for all $\lambda < \lambda_0$. Assume that L is bounded from below and the hypotheses of Proposition 4.6 holds. Then for sufficient small $|\alpha|$, $\lambda \in \rho(H_{\mathrm{IBC}}^{\alpha,1})$ and resolvent can be written as*

$$R(\lambda, H_{\mathrm{IBC}}^{\alpha,1}) = \left(\mathrm{Id} + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m\right) R(\lambda, H_{\mathrm{IBC}}^{0,1}). \tag{4.24}$$

Proof. Since T_λ is self-adjoint and bounded from above there exists a constant $\omega \in \mathbb{R}$ such that $\|R(\mu, T_\lambda)\| \le \frac{C}{|\mu-\omega|}$ for $\mu \in \rho(T_\lambda)$. By Lemma 4.10 and [EN00, Lem. III. 2.6] there exists a constant $\tilde\omega \in \mathbb{R}$ that $\|R(\mu, S_\lambda)\| \le \frac{C}{|\mu-\tilde\omega|}$ for $\mu > \tilde\omega$. It follows that for sufficient small $|\alpha|$ the sum in the right hand side of (4.24) converges. Since

$$\mathrm{rg}(F_\lambda R(-1/\alpha, S_\lambda)) \subset F_\lambda D(S_\lambda) = F_\lambda\{\varphi \in \partial\mathscr{H} \colon F_\lambda\varphi \in D(A_m)\} \subset D(L_m) \cap D(A_m) \tag{4.25}$$

it follows

$$\begin{aligned} &\left(\mathrm{Id} + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m\right) R(\lambda, H_{\mathrm{IBC}}^{0,1}) f \\ &= R(\lambda, H_{\mathrm{IBC}}^{0,1}) f + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m R(\lambda, H_{\mathrm{IBC}}^{0,1}) f \\ &\in D(L_m) \cap D(A_m), \end{aligned} \tag{4.26}$$

for $f \in \mathscr{H}$. It remains to check the interior boundary conditions with respect to α. First note, that $S_\lambda = A_m F_\lambda - I^* F_\lambda$ and by Dyson-Phillips series

$$R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k = R(-1/\alpha, A_m F_\lambda). \tag{4.27}$$

Using Proposition 4.6 it follows

$$(\alpha A_m + B - I^*)\left(\operatorname{Id} + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m\right) R(\lambda, H^{0,1}_{\mathrm{IBC}})f \tag{4.28}$$
$$= \alpha A_m R(\lambda, H^{0,1}_{\mathrm{IBC}})f + (\alpha A_m F_\lambda + 1)R(-1/\alpha, A_m F_\lambda)A_m R(\lambda, H^{0,1}_{\mathrm{IBC}})f$$
$$= \alpha A_m R(\lambda, H^{0,1}_{\mathrm{IBC}})f - \alpha A_m R(\lambda, H^{0,1}_{\mathrm{IBC}})f = 0$$

for $f \in \mathscr{H}$. Hence the right hand side of (4.24) maps to $D(H^{\alpha,\beta}_{\mathrm{IBC}})$. Using Proposition 4.6 we conclude

$$(\lambda - H^{\alpha,1}_{\mathrm{IBC}})\left(\operatorname{Id} + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m\right) R(\lambda, H^{0,1}_{\mathrm{IBC}})f \tag{4.29}$$
$$= (\lambda - L_m - IA_m)\left(\operatorname{Id} + F_\lambda R(-1/\alpha, S_\lambda)\sum_{k=0}^{\infty}(I^* F_\lambda R(-1/\alpha, S_\lambda))^k A_m\right) R(\lambda, H^{0,1}_{\mathrm{IBC}})f$$
$$= (\lambda - L_m - IA_m)R(\lambda, H^{0,1}_{\mathrm{IBC}})f = f,$$

and hence the right hand side of (4.24) is right-inverse of $\lambda - H^{\alpha,1}_{\mathrm{IBC}}$. It remains to prove injectivity of $\lambda - H^{\alpha,1}_{\mathrm{IBC}}$. Consider $f \in \ker(\lambda - H^{\alpha,1}_{\mathrm{IBC}})$, i.e.

$$\begin{aligned} \lambda f - L_m f - IA_m f &= 0, \\ \alpha A_m f + Bf &= I^* f. \end{aligned} \tag{4.30}$$

Using the unique decomposition $f = f_0 + F_\lambda\varphi$ for $f_0 \in D(H^{0,1}_{\mathrm{IBC}})$ from the first equation follows using $\operatorname{rg}(F_\lambda) \subset \ker(\lambda - L_m - IA_m)$

$$(\lambda - H^{0,1}_{\mathrm{IBC}})f_0 = 0. \tag{4.31}$$

By $\lambda \in \rho(H^{0,1}_{\mathrm{IBC}})$ we conclude $f_0 = 0$. Hence the second equation in (4.30) becomes

$$\alpha A_m F_\lambda\varphi + BF_\lambda\varphi = I^* F_\lambda\varphi. \tag{4.32}$$

Using $(B - I^*)F_\lambda = \operatorname{Id}$ it follows

$$\alpha A_m F_\lambda\varphi = 0. \tag{4.33}$$

In (4.27) we have seen that $-1/\alpha \in \rho(A_m F_\lambda)$ and therefore we obtain $\varphi = 0$. All in all it follows $f = f_0 + F_\lambda\varphi = 0$ and $\lambda - H^{\alpha,1}_{\mathrm{IBC}}$ is injective. One concludes $\lambda \in \rho(H^{\alpha,1}_{\mathrm{IBC}})$. ☐

The following theorem is an easy consequence of Lemma 4.11.

Theorem 4.12. *Suppose that there exists a $\lambda_0 \in \mathbb{R}$ such that $\lambda \in \rho(H^{0,1}_{\mathrm{IBC}})$, T_λ are self-adjoint on $\partial\mathscr{H}$ and bounded from above and $IT_\lambda I^*$ is relatively $(\operatorname{Id} - G_\lambda I^*)^*(L-\lambda)(\operatorname{Id} - G_\lambda I^*)$-bounded of infinitesimal bound for all $\lambda < \lambda_0$. Assume that L is bounded from below and the hypotheses of Proposition 4.6 holds. Then the operators $H^{\alpha,1}_{\mathrm{IBC}}$ converge to $H^{0,1}_{\mathrm{IBC}}$ in the norm resolvent sense for $\alpha \downarrow 0$.*

Proof. By Lemma 3.11 the operator A_m is relatively $H^{0,1}_{\mathrm{IBC}}$-bounded. Using Lemma 4.11 we conclude

$$\begin{aligned} \|R(\lambda, H^{0,1}_{\mathrm{IBC}}) - R(\lambda, H^{\alpha,1}_{\mathrm{IBC}})\| &\le C \cdot \|R(-1/\alpha, S_\lambda)\| \cdot \|A_m R(\lambda, H^{0,1}_{\mathrm{IBC}})\| \\ &\le C \cdot |\alpha| \to 0 \end{aligned} \tag{4.34}$$

for $\lambda \in \rho(H^{0,1}_{\mathrm{IBC}}) \cap \rho(H^{\alpha,1}_{\mathrm{IBC}})$ and the claim is proven. □

Remark 4.13. Replacing T_λ by its Friedrich extension T^F_λ shows that in Theorem 4.12 the conditions T_λ can be weaken to T_λ symmetric and bounded from above or below. See Section 5 for more details. These assumptions can be shown in many concrete examples, see Section 5, in particular Proposition 5.3 and Lemma 5.8.

5. Applications

5.1. **A toy quantum-field theory.** Here we illustrate our results in a simple model that displays much of the structure relevant for applications in quantum field theory, without posing too many technical problems for the verification of key assumptions.
The physical picture behind this example is that of a particle, whose position we denote by $x \in \mathbb{R}$, moving in a one-dimensional space while creating/annihilation "particles". The latter can be thought of as elementary excitations of the background medium through which the first particle moves. We denote their positions by $y_1, y_2, \ldots$. Such models play an important role in condensed matter physics. In the specific case we will consider, the excitations would not move on their own, although they will display effective dynamics through repeated creation/annihilation at different positions. This is analogous to the well known Fröhlich polaron model [GW16] to which the arguments of this section should apply with minor modifications. Another similar model with contact interactions in a three-dimensional space, which leads to some subtle regularity issues, was treated in [Lam18].
Take

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^n) = \bigoplus_{n=0}^{\infty} L^2\left(\mathbb{R}, L^2_{\mathrm{sym}}(\mathbb{R}^n)\right) := \bigoplus_{n=0}^{\infty} \mathscr{H}^{(n)} \tag{5.1}$$

and $\partial\mathscr{H} = \mathscr{H}$ with $I = \mathrm{Id}$. Let N be the operator given by $(Nf)^{(n)} = nf^{(n)}$ (where $f^{(n)}$ is the projection of $f \in \mathscr{H}$ to $\mathscr{H}^{(n)}$), with the domain

$$D(N) = \{f \in \mathscr{H} : \|nf^{(n)}\|_{\mathscr{H}^{(n)}} \in \ell^2(\mathbb{N})\}. \tag{5.2}$$

Clearly $N, D(N)$ is self-adjoint. Let x denote the first of the $n+1$ arguments of a function $f \in \mathscr{H}^{(n)}$, then $-\Delta_x$ is a self-adjoint operator on the domain

$$D(-\Delta_x) = \bigoplus_{n=0}^{\infty} H^2\left(\mathbb{R}, L^2_{\mathrm{sym}}(\mathbb{R}^n)\right). \tag{5.3}$$

We set $L = -\Delta_x + N$ with $D(L) = D(-\Delta_x) \cap D(N)$.
We define $A : D(L) \cap \mathscr{H}^{(n)} \to \partial\mathscr{H}^{(n)} := \mathscr{H}^{(n-1)}$ as a symmetrised evaluation operator (A corresponds to the "annihilation operator" $a(x)$):
(5.4)

$$\left(Af^{(n)}\right)(x, y_1, \ldots, y_{n-1}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f^{(n)}(x, y_1, \ldots, y_n)|_{y_j = x} = \sqrt{n} f^{(n)}(x, y_1, \ldots, y_{n-1}, x).$$

One can check that A maps $D(A) := D(L)$ to $\mathscr{H}$ and that $\ker A$ is dense. The operator G_λ for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is then given by (denoting $Y = (y_1, \ldots, y_{n+1})$ and $\hat{Y}_j$ as Y without the entry

$y_j)$

$$\left(G_\lambda f^{(n)}\right)(x,Y) = \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}(\lambda - L)^{-1}\delta(x-y_j)f^{(n)}(x,\hat{Y}_j) = \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1} g_{n+1-\lambda}(x-y_j)f^{(n)}(y_j,\hat{Y}_j), \tag{5.5}$$

with the function

$$g_\mu(x) = -(\mu-\Delta)^{-1}\delta = -\frac{e^{-\sqrt{\mu}|x|}}{2\sqrt{\mu}}, \tag{5.6}$$

where the square root is the branch with $\mathrm{Re}(\sqrt{z}) \geq 0$. The operator L_m is now defined on $D(L_m) = D(L) \oplus G_\lambda(\partial\mathscr{H})$ by

$$L_m f = L_0^* f = L_0^*(f_0 + G_\lambda\varphi) = Lf_0 + \lambda G_\lambda \varphi. \tag{5.7}$$

The boundary operator B is defined as the left inverse to G_λ on $D(B) = D(L_m)$. In view of the fact that $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$ and $\lim_{r\to 0}(g'_\mu(r) - g'_\mu(-r)) = 1$, B is given by the following local formula

$$Bf^{(n)}(x,Y) = \sqrt{n}\lim_{r\to 0}\left(\left(\partial_x f^{(n)}\right)(x+r,Y,x) - \left(\partial_x f^{(n)}\right)(x-r,Y,x)\right), \tag{5.8}$$

where the limit is taken in $\mathscr{H}^{(n-1)}$.
Since g_μ is continuous we can extend A to $\operatorname{rg} G_\lambda$ canonically by using the same formula. This gives

$$\left(T_\lambda f^{(n)}\right)(x,Y) = \left(A_m G_\lambda f^{(n)}\right)(x,Y) = -\frac{f^{(n)}(x,Y)}{2\sqrt{n+1-\lambda}} + \sum_{j=1}^{n} g_{n+1-\lambda}(x-y_j)f^{(n)}(y_j,\hat{Y}_j,x). \tag{5.9}$$

Since g_μ is bounded, $T_\lambda : \mathscr{H}^{(n)} \to \mathscr{H}^{(n)}$ is a bounded operator. However, since the number of terms in the sum above is n, this does not give rise to a bounded operator on $\mathscr{H}$. We have the bound

$$\|T_\lambda f^{(n)}\|_{\mathscr{H}^{(n)}} \leq \frac{n+1}{2|\sqrt{n+1-\lambda}|}\|f^{(n)}\|_{\mathscr{H}^{(n)}}, \tag{5.10}$$

so on $D(T) = D(N^{1/2})$ we can define T as an unbounded operator on $\mathscr{H}$. This defines A_m with domain $D(A_m) = D(L) \oplus G_\lambda(D(N^{1/2}))$.

5.2. **Self-adjointness of Robin-type operators.**
The objects constructed above satisfy the hypothesis of our general setting, as explained in Construction 2.8. The operators with Robin type interior boundary conditions $H^{\alpha,\beta}_{\mathrm{IBC}}$ are thus well defined. The equation $H^{\alpha,\beta}_{\mathrm{IBC}} f = g$ (with the choice $\delta = 0$, $\gamma = \bar{\beta}^{-1}$, which is symmetric if $\alpha\bar{\beta} \in \mathbb{R}$) corresponds to the following hierarchy of boundary value problems

$$\begin{cases} (-\Delta_x + n)f^{(n)}(x,Y) + \bar{\beta}^{-1}\sqrt{n+1}f^{(n+1)}(x,Y,x) = g^{(n)}(x,Y) & x \neq y_j \\ \alpha\sqrt{n}f^{(n)}(x,Y) + \beta\sqrt{n}\left((\partial_x f^{(n)})_+ - (\partial_x f^{(n)})_-\right)(x,Y) = f^{(n-1)}(x,\hat{Y}_n) & x = y_n \end{cases} \tag{5.11}$$

where the subscript $\pm$ indicates that the right/left sided limit $x \to y_n$ is taken, as in (5.8), and $f^{(n)}$ is symmetric under permutation of $y_1, \dots, y_n$ which gives implies boundary conditions on the sets where $y_j = x$.
In order to establish self-adjointness of $H^{\alpha,\beta}_{\mathrm{IBC}}$ we need some properties of T_λ. We remark that non-positivity of T_λ is not generic in any way – in Example 2.9 the operator is non-negative instead, while in the more involved cases studied in [Tho84; LS19; Lam19b] both the positive and negative parts are generally unbounded.

Lemma 5.1. *For any real $\lambda < 0$, the operator T_λ is essentially self-adjoint and non-positive. Moreover,* $\mathrm{rg}(A) \subset \mathrm{rg}(z - T_\lambda)$ *for all $z \in \mathbb{C} \setminus \mathbb{R}_-$.*

Proof. By the Sobolev embedding theorem A has a natural extension to $D(L^{1/2}) \cap \mathscr{H}^{(n)} \supset C(\mathbb{R}, L^2_{\mathrm{sym}}(\mathbb{R}^n))$, which we denote by $\tilde{A}$. For $\lambda < 0$, $L - \lambda$ is a positive operator, and we can then write

$$T_\lambda|_{\mathscr{H}^{(n)}} = A_m(AR(\lambda, L))^* = -\left(\tilde{A}(L-\lambda)^{-1/2}\right)\left(\tilde{A}(L-\lambda)^{-1/2}\right)^*, \tag{5.12}$$

so T_λ is symmetric and non-positive.
Since $\mathscr{H}^{(n)}$ is T_λ-invariant, T_λ is an infinite direct sum of commuting bounded self-adjoint operators and thus essentially self-adjoint, since all vectors $f \in \mathscr{H}$ with only finitely many $f^{(n)} \neq 0$ are contained in $\mathrm{rg}(T_\lambda \pm i)$.
Now let $f \in \mathrm{rg}(A)$, i. e. $f = A(L-\lambda)^{-1} g$ for some $g \in \mathscr{H}$, and $z \in \mathbb{C} \setminus \mathbb{R}_- \subset \rho(\overline{T}_\lambda)$. By the formula for T_λ, the operator

$$R(z, \overline{T}_\lambda)\tilde{A}(L-\lambda)^{-1/2} \tag{5.13}$$

is bounded, since multiplying by its adjoint from the right yields

$$-R(z, \overline{T}_\lambda)T_\lambda R(\bar{z}, \overline{T}_\lambda). \tag{5.14}$$

Consequently

$$N^{1/2}R(z, \overline{T}_\lambda)A(L-\lambda)^{-1} = R(z, \overline{T}_\lambda)A(L-\lambda)^{-1}(N+1)^{1/2} \tag{5.15}$$

is also a bounded operator, and this shows that $(z - \overline{T}_\lambda)^{-1}\,\mathrm{rg}(A) \subset D(N^{1/2}) = D(T)$ and thus $\mathrm{rg}(A) \subset \mathrm{rg}(z - T_\lambda)$. □

In particular this lemma shows that $T_{\bar{\lambda}} \subset T_\lambda^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, as assumed from Section 3 on.
In view of Lemma 3.8, non-positivity of T_λ together with $\mathrm{rg}(A) \subset \mathrm{rg}(z - T_\lambda =$ implies that $L_{\alpha,\beta}$ is self-adjoint with $\sigma(L_{\alpha,\beta}) \subset [0, \infty)$ if $\alpha\bar{\beta} < 0$ (and of course for $\alpha = 0$, $\beta \neq 0$). These operators correspond to repulsive contact interactions between the first particle and the remaining ones.
In order to make conclusions on $H^{\alpha,\beta}_{\mathrm{IBC}}$, we need to verify the relevant hypothesis of Lemma 3.10, Lemma 3.11. The fact that $1 \in \rho(G_\lambda I^*) \cap \rho(I^* G_\lambda)$ is guaranteed by the following Lemma.

Lemma 5.2. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ the operator G_λ satisfies the bound*

$$\|G_\lambda f^{(n)}\|_{\mathscr{H}^{(n+1)}} \leq \frac{\sqrt{n+1}}{2|n+1-\lambda|^{1/2}\mathrm{Re}(\sqrt{n+1-\lambda})^{1/2}}\|f^{(n)}\|_{\mathscr{H}^{(n)}}$$

In particular for $\lambda < 0$ we have $\|G_\lambda\| < \frac{1}{2}$.

Proof. Since $\|g_\mu\|_{L^2} = \frac{1}{2\sqrt{|\mu|\mathrm{Re}(\sqrt{\mu})}}$ this follows from the triangle inequality. □

In view of Corollary 3.14 this yields the following, with boundedness from below being a consequence of the non-negativity of $L_{\alpha,\beta}$ and the use of Kato-Rellich in the proof.

Proposition 5.3. *Let $\alpha\bar\beta < 0$. For all δ, γ such that the symmetry condition of Lemma 3.4 is satisfied, $H_{\mathrm{IBC}}^{\alpha,\beta}$ is self-adjoint and bounded from below.*

Proof. Let $\lambda < 0$ and $\overline{T}_\lambda$ be the self-adjoint closure of T_λ. Then by Proposition 3.9 (v), $T_\lambda^{\alpha,\beta}$ is a restriction of the bounded operator $\gamma + R(-\beta, \overline{T}_\lambda)$, and this implies the relative bounds required in Theorem 3.12. □

To treat the case $\alpha = 0$ we also need:

Lemma 5.4. *For all $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$, the operators $G_\lambda I^*$, $\Gamma_\lambda = (1 - G_\lambda I^*)^{-1}$ and $(1 - I^* G_\lambda)^{-1}$ leave $D(N)$ as well as $D(N^{1/2})$ invariant.*

Proof. We give the proof only for $D(N)$, the proof for $D(N^{1/2})$ being essentially the same. For $G_\lambda I^*$ this is obvious, since

$$NG_\lambda I^* = G_\lambda I^*(N+1). \tag{5.16}$$

For Γ_λ, this follows by the same logic and Lemma 5.2, since $N(G_\lambda I^*)^4$ is bounded and

$$\Gamma_\lambda = \sum_{k=0}^{\infty} (G_\lambda I^*)^k.$$

The argument for $(1 - I^* G_\lambda)^{-1}$ is the same. □

Remark 5.5. The argument of the lemma shows that, in the case of a hierarchy, it is not necessary that $G_\lambda I^*$ to be small in norm for Γ_λ to exist and be given by the series. Rather, it is sufficient that

$$G_\lambda I^*|_{\mathscr{H}^{(n)}} : \mathscr{H}^{(n)} \to \mathscr{H}^{(n+1)}$$

has a norm that decreases with n, e.g., so that $\|G_\lambda I^*|_{\mathscr{H}^{(n)}}\|^n$ is summable.

Proposition 5.6. *For $\alpha = 0$, $\beta \neq 0$, and δ, γ such that the symmetry condition of Lemma 3.4 is satisfied, $H_{\mathrm{IBC}}^{0,\beta}$ is self-adjoint and bounded from below.*

Proof. We may pick any $0 > \lambda \in \rho(L)$, and in view of Corollary 3.13 it is sufficient to prove that $IT_\lambda I^*$ is infinitesimally $(\mathrm{Id} - G_\lambda I^*)^*(L-\lambda)(\mathrm{Id} - G_\lambda I^*)$-bounded .
By (5.10), $IT_\lambda I^*$ is $N^{1/2}$-bounded. Because N is L-bounded and the invariance of $D(N)$ established in Lemma 5.4, N is $(\mathrm{Id} - G_\lambda I^*)^*(L-\lambda)(\mathrm{Id} - G_\lambda I^*)$-bounded. This implies the required infinitesimal bound by interpolation and proves the claim. □

We see that for $\alpha\bar\beta \leq 0$ all of the theorems of Section 3 can be applied to this model. For the case $\alpha\bar\beta > 0$, which corresponds to attractive interactions between the first particle and the remaining ones (in addition to the interaction induced by creation/annihilation of particles) this is not obvious and we do not know whether it holds.

5.3. **A pointwise Robin condition.** We now discuss the applicability of our classification results of Section 4 and apply them to an example in which the coefficients α, β of the boundary condition are position-dependent. Such conditions were discussed in [Tum20], though without proving self-adjointness.
By Lemma 5.2, $(1-I^*G_\lambda)^{-1}$ leaves $D(T) = D(N^{1/2})$ invariant. Hence the hypothesis of Proposition 4.6 and Theorem 4.7 are satisfied and $\{\partial\mathscr{H}, (B-I^*), (A_m - I^*)\}$ is a quasi boundary triple for

$$H_m = L_m + \mathrm{Id} + I(A_m - B). \tag{5.17}$$

The self-adjoint restrictions of H_m are described in Theorem 4.8 and Theorem 4.9.
A relevant class of boundary condtions are local versions of the boundary condition $\alpha A f + \beta B f = I^* f$ where α, β are functions. Let $\alpha, \beta \in L^\infty(\mathbb{R})$ with $\alpha + \beta \equiv 1$ (see also Remark 5.7 below).
The corresponding boundary condition reads

$$\alpha(x)\left(A_m f^{(n+1)}\right)(x, Y) + \beta(x)\left(Bf^{(n+1)}\right)(x, Y) = f^{(n)}(x, Y), \tag{5.18}$$

for $n \in \mathbb{N}$ and $(x, Y) \in \mathbb{R} \times \mathbb{R}^n$.

Remark 5.7. The condition $\alpha + \beta \equiv 1$ is less restrictive than it might seem. Since our only requirement on I is boundedness, any pair of functions with $(\alpha+\beta)^{-1} \in L^\infty$ can be accommodated by modifying I. More precisely, set $\tilde{I} = (\bar{\alpha} + \bar{\beta})^{-1} I$, then the condition (5.19) becomes,

$$\frac{\alpha(x)}{\alpha(x)+\beta(x)} A_m f^{(n+1)} + \frac{\beta(x)}{\alpha(x)+\beta(x)} B f^{(n+1)} = \frac{1}{\alpha(x)+\beta(x)} f^{(n)} = (\tilde{I}^* f)^{(n)}, \tag{5.19}$$

where the coefficients $\tilde{\alpha} := \alpha(\alpha+\beta)^{-1}$, $\tilde{\beta} := \beta(\alpha+\beta)^{-1}$ now satisfy $\tilde{\alpha} + \tilde{\beta} \equiv 1$.

The relation corresponding to (5.19) is

$$\mathfrak{R}_{\alpha,\beta} = \{(\alpha f, -\beta f) | f \in \partial\mathscr{H}\}, \tag{5.20}$$

where α, β are the operators of multiplication by the respective function. To see this, note that $((B - I^*)f, (A_m - I^*)f) \in \mathfrak{R}_{\alpha,\beta}$ means that

$$\beta(B - I^*)f = -\alpha(A_m - I^*)f \iff \alpha A_m f + \beta B f = \underbrace{(\alpha+\beta)}_{=\mathrm{Id}} I^* f. \tag{5.21}$$

Recall that self-adjointness of $H_{\mathfrak{R}_{\alpha,\beta}}$ is related to the operator

$$S_\lambda = (A_m - I^*) F_\lambda \tag{5.22}$$

with $D(S_\lambda) = D(F_\lambda) = D(T)$, which has similar properties as T_λ.

Lemma 5.8. *Let $\lambda_0 := \inf \sigma(H^{0,1}_{\mathrm{IBC}})$. Then for $\lambda < \lambda_0$, S_λ is non-positive.*

Proof. For the form domains we have, similarly to Lemma 3.10 (to prove this, take the closure of $D(H^{0,1}_{\mathrm{IBC}})$ in the appropriate norm),

$$D\left(|H^{0,1}_{\mathrm{IBC}}|^{1/2}\right) = \Gamma D(L^{1/2}). \tag{5.23}$$

Since $g_\mu \in H^1(\mathbb{R})$, in our example we find $\Gamma D(L^{1/2}) \subset D(L^{1/2})$. Let $\tilde{A}$ be the extension of A to $D(L^{1/2}) \cap \mathscr{H}^{(n)}$ for arbitary n (note that $\tilde{A}(L-\lambda)^{-1/2}$ is not bounded, but, e.g., $N^{1/2}$-bounded, due to the prefactor $\sqrt{n}$ in the definition of A). For $\lambda < \lambda_0$, $-R(\lambda, H_{\mathrm{IBC}}^{0,1}) = (H_{\mathrm{IBC}}^{0,1} - \lambda)^{-1} > 0$, so we have

$$S_\lambda = -(\tilde{A} - I^*)(H_{\mathrm{IBC}}^{0,1} - \lambda)^{-1/2} \left((\tilde{A} - I^*)(H_{\mathrm{IBC}}^{0,1} - \lambda)^{-1/2} \right)^* \leq 0. \tag{5.24}$$

□

Proposition 5.9. *Let $\alpha, \beta \in L^\infty$ with $\alpha + \beta \equiv 1$ and assume that there exists $\delta > 0$ such that for all $x \in \mathbb{R}$*

$$\alpha(x)\bar{\beta}(x) \leq -\delta.$$

Then the operator $H_{\mathfrak{R}_{\alpha,\beta}}$ is self-adjoint.

Proof. We use the criterion of Theorem 4.8, i.e. we prove that $\mathfrak{R}_{\alpha,\beta} - S_\lambda$ is invertible on $\operatorname{rg} F_\lambda^*$ for $\lambda < \lambda_0$ (as above).
Using the properties of α and β, we see that $|\alpha| \cdot |1 - \alpha| \geq \delta$, so $\alpha^{-1} \in L^\infty$ and the relation can be rewritten as

$$\begin{aligned} \mathfrak{R}_{\alpha,\beta} - S_\lambda =& \{(\alpha\varphi, -\beta\varphi - S_\lambda \alpha\varphi) | \alpha\varphi \in D(T)\} \\ =& \{(\varphi, -\beta\alpha^{-1}\varphi - S_\lambda\varphi) | \varphi \in D(T)\}. \end{aligned} \tag{5.25}$$

Let $\lambda < \lambda_0$ as above, and let S_λ^F be the self-adjoint Friedrichs extension of S_λ. As $\beta\alpha^{-1} = \bar{\beta}\alpha(|\alpha|)^{-2} \leq -\delta\|\alpha\|_\infty^{-2}$, the operator

$$\beta\alpha^{-1} + S_\lambda^F \tag{5.26}$$

is strictly negative, self-adjoint and thus invertible. If $S_\lambda = S_\lambda^F$ we are finished since then the relation (5.25) is invertible everywhere.
If $S_\lambda \neq S_\lambda^F$ we can conlcude by showing that

$$(-\beta\alpha^{-1} - S_\lambda^F)^{-1} \operatorname{rg} F_\lambda^* \subset D(T) = D(N^{1/2}),$$

since then S_λ^F can again be replaced by S_λ. This follows from the representation (5.24) by the arguments of Lemma 5.1, since boundedness of

$$(\beta\alpha^{-1} + S_\lambda^F)^{-1}(\tilde{A} - I^*)(H_{\mathrm{IBC}}^{0,1} - \lambda)^{-1/2} \tag{5.27}$$

together with the equality of $D(L^{1/2})$ and $D(|H_{\mathrm{IBC}}^{0,1}|^{1/2})$ implies boundedness of

$$N^{1/2}(\beta\alpha^{-1} + S_\lambda^F)^{-1} F_\lambda^*. \tag{5.28}$$

□

Appendix A.

A.1. **Linear relations.** Here we briefly recall the relevant notions for linear relations in a Hilbert space $\mathscr{H}$. These generalise the corresponding notions for operators with the relation given by the graph. For a linear relation $\mathfrak{R}$ in $\mathscr{H}$ (i.e. a subspace of $\mathscr{H} \oplus \mathscr{H}$), the domain, range, kernel are defined by

$$\begin{aligned} D(\mathfrak{R}) &= \{\varphi \in \mathscr{H} | \exists \eta \in \mathscr{H} : (\varphi, \eta) \in \mathfrak{R}\} \\ \operatorname{rg}(\mathfrak{R}) &= \{\varphi \in \mathscr{H} | \exists \psi \in \mathscr{H} : (\psi, \varphi) \in \mathfrak{R}\} \\ \ker(\mathfrak{R}) &= \{\varphi \in \mathscr{H} | (\varphi, 0) \in \mathfrak{R}\}. \end{aligned}$$

The following operations are defined on the set of linear relations in a Hilbert space $\mathscr{H}$

$$
\begin{aligned}
\mathfrak{R}+\mathfrak{S} &= \{(\varphi,\xi+\eta)|(\varphi,\xi)\in\mathfrak{R},\ (\varphi,\eta)\in\mathfrak{S}\}\\
-\mathfrak{R} &= \{(\varphi,-\psi)|(\varphi,\psi)\in\mathfrak{R}\}\\
\mathfrak{R}\mathfrak{S} &= \{(\varphi,\eta)\in\mathscr{H}\oplus\mathscr{H}|\exists\xi\in\mathscr{H}:(\varphi,\xi)\in\mathfrak{S}\ \text{and}(\xi,\eta)\in\mathfrak{R}\}\\
\mathfrak{R}^{-1} &= \{(\varphi,\eta)\in\mathscr{H}\oplus\mathscr{H}|(\eta,\varphi)\in\mathfrak{R}\}.
\end{aligned}
$$

The adjoint relation is given by

$$
\mathfrak{R}^* := \{(\varphi,\eta)\in\mathscr{H}\oplus\mathscr{H}|\forall(\psi,\xi)\in\mathfrak{R}:\langle\varphi,\xi\rangle_{\mathscr{H}}=\langle\psi,\eta\rangle_{\mathscr{H}}\}. \tag{A.1}
$$

A relation is symmetric if $\mathfrak{R}\subset\mathfrak{R}^*$ (as sets) and self-adjoint if $\mathfrak{R}=\mathfrak{R}^*$. Clearly an operator is self-adjoint if and only if its graph is a self-adjoint relation.

A.2. **Quasi-boundary triples.** In this section we briefly recall the definition of quasi boundary triples. Moreover, we translate results for quasi boundary triples in the language of our abstract framework.

Definition A.1. A triple $(\partial\mathscr{H},\tilde{B},\tilde{A})$ is called a *quasi boundary triple* for an operator $\tilde{L}\colon D(\tilde{L})\subset\mathscr{H}\to\mathscr{H}$ if $\partial\mathscr{H}$ is a Hilbert space and $\tilde{A},\tilde{B}\colon D(\tilde{L})\subset\mathscr{H}\to\partial\mathscr{H}$ are operators such that

(i) the second Green identity holds

$$
\langle\tilde{L}f,g\rangle_{\mathscr{H}}-\langle f,\tilde{L}g\rangle_{\mathscr{H}}=\langle\tilde{A}f,\tilde{B}g\rangle_{\partial\mathscr{H}}-\langle\tilde{B}f,\tilde{A}g\rangle_{\partial\mathscr{H}}
$$

for all $f,g\in D(\tilde{L})$.

(ii) The map $(\tilde{A},\tilde{B}):D(\tilde{L})\to\partial\mathscr{H}\times\partial\mathscr{H}$ has dense range.

(iii) The restriction $L:=\tilde{L}|_{\ker(\tilde{B})}$ is a self-adjoint operator on $\mathscr{H}$.

This is equivalent to the definition given in [BL07, Def.2.1] in terms of relations, since by [BL07, Thm.2.3] the operator $\tilde{L}$ is always a restriction of (the relation) L_0^* for the closed, but not necessarily densely defined, operator $L_0=L|_{\ker A}$. In other works, the additional assumption that $D(L_0)$ is dense is made, but we explicitly avoid this.

By Remark 3.5, $(\partial\mathscr{H},A_m,B)$ is a quasi boundary triple for $\tilde{L}=L|_{D(A_m)}$. In Theorem 4.7 we show that $\big(\partial\mathscr{H},(B-I^*),(A_m-I^*)\big)$ is a quasi boundary triple for H_m.

A quasi boundary triple is called an ordinary boundary triple if $\operatorname{rg}(\tilde{A},\tilde{B})=\partial\mathscr{H}^2$ and a generalized boundary triple if $\operatorname{rg}(\tilde{B})=\partial\mathscr{H}$.

Note that a quasi boundary triple for $\hat{L}^*$ exists if and only if the defect indices $n_\pm(\hat{L}):=\dim(\ker(\hat{L}^*\mp i))$ of $\hat{L}^*$ coincide. Further, if the defect indices of $\hat{L}$ are finite the quasi boundary triple for $\hat{L}$ is an ordinary boundary triple. Moreover, the operator $(\tilde{A},\tilde{B})\colon D(\tilde{L})\subset\mathscr{H}\to\partial\mathscr{H}\times\partial\mathscr{H}$ is closable and by [BL07, Prop. 2.2] $\ker(\tilde{A},\tilde{B})=D(\hat{L})$ holds. By [BL07, Thm. 2.3] it follows that $\tilde{L}=\hat{L}^*$ if and only if $\operatorname{rg}(\tilde{A},\tilde{B})=\partial\mathscr{H}^2$. In this case the restriction $L:=\hat{L}^*|_{\ker(\tilde{B})}$ is self-adjoint and the quasi boundary triple $(\partial\mathscr{H},\tilde{B},\tilde{A})$ is an ordinary boundary triple.

For each $\lambda\in\rho(L)$ the definition of a quasi boundary triple yields the decomposition

$$
D(\tilde{L})=D(L)\oplus\ker(\lambda-\tilde{L}) \tag{A.2}
$$

For $\tilde{L}=L_m|_{D(A_m)}$, $\tilde{B}=B$ the decompositions (A.2) and (2.7) coincide. While we used the Dirichlet operator G_λ to obtain this decomposition, here the situation is contrariwise.

Starting with the decomposition one obtains that the restriction $\tilde{B}|_{\ker(\lambda-\tilde{L})}$ is injective and $\operatorname{rg}(\tilde{B}|_{\ker(\lambda-\tilde{L})}) = \operatorname{rg}(\tilde{B})$. This yields to the following definition.

Definition A.2. Let $(\partial\mathscr{H}, B, A)$ a quasi boundary triple for $\tilde{L} \subset \hat{L}^*$. The *$\gamma$-field corresponding to* $(\partial\mathscr{H}, \tilde{B}, \tilde{A})$ is given by

$$\tilde{G}\colon \rho(L) \to \mathcal{L}(\partial\mathscr{H}, \mathscr{H})\colon \lambda \mapsto (\tilde{B}|_{\ker(\lambda-\tilde{L})})^{-1}.$$

Moreover the *Weyl function associated to* $(\partial\mathscr{H}, \tilde{B}, \tilde{A})$ is given by

$$\lambda \mapsto \tilde{T}(\lambda) := A\tilde{G}(\lambda).$$

We point out that the γ-field $\tilde{G}(\lambda)$ at $\lambda \in \rho(L)$ equals to the abstract Dirichlet operator G_λ by Proposition 2.7 (i). Hence the Weyl function $\tilde{T}(\lambda)$ at $\lambda \in \rho(L)$ coincide with the abstract Dirichlet-to-Neumann operator T_λ.

Remark A.3. In [BL07] it is shown that for a boundary triple $(\partial\mathscr{H}, \tilde{B}, \tilde{A})$

$$\tilde{T}(\lambda)^* \subset \tilde{T}(\bar{\lambda}) \tag{A.3}$$

for all $\lambda \in \rho(L)$ holds. Further, if $(\partial\mathscr{H}, \tilde{B}, \tilde{A})$ is a generalized (in particular a ordinary) boundary triple, equality holds, i.e.

$$\tilde{T}(\lambda)^* = \tilde{T}(\bar{\lambda}) \tag{A.4}$$

for all $\lambda \in \rho(L)$.

Note that our approach is contrariwise. We start with Assumption 2.3 and show that $(\partial\mathscr{H}, B, A_m)$ is a quasi boundary triple for $L_m|_{D(A_m)}$. Whereas in [BL07] they start with a quasi boundary triple and see that (A.3) holds.

Consider a Hilbert space $\mathcal{H}$ and a closed, densely defined operator $U\colon D(U) \subset \mathcal{H} \to \mathcal{H}$ with $\rho(U) \neq \emptyset$. Replacing U by $U - \lambda$ for $\lambda \in \rho(U)$ we assume without loss of generality that $0 \in \rho(U)$. Now we define the norm $\|\phi\|_{-1} := \|U^{-1}\phi\|_{\mathcal{H}}$ and by $\mathcal{H}_{-1} := (\mathcal{H}, \|\cdot\|_{-1})^\sim$ the completion of $\mathcal{H}$ with respect to the $\|\cdot\|_{-1}$-norm. Now $\mathcal{H}_{-1}$ equipped with the $\|\cdot\|_{-1}$-norm is a Banach space and if U is symmetric a Hilbert space. Further, we define $\mathcal{H}_1$ by $D(U)$ equipped with the graph norm. Denote by U_{-1}^{-1} the unique extension of U^{-1} from $\mathcal{H}_{-1}$ to $\mathcal{H}$. We point out that U^{-1} is an isometry from $\mathcal{H}_1 \to \mathcal{H}$ and U_{-1}^{-1} is an isometry from $\mathscr{H} \to \mathscr{H}_{-1}$, i.e.,

$$\mathcal{H}_{-1} \xrightarrow{U_{-1}^{-1}} \mathcal{H} \xrightarrow{U^{-1}} \mathcal{H}_1.$$

If U is generator of a strongly continuous semigroup the space $\mathscr{H}_{-1}$ is the extrapolation space of order -1 associated to U (see [EN00, Def. II.5.4]). We refer to [EN00, Sect. II.5] for more details about extrapolation spaces.

Now consider the quasi boundary triple $(\partial\mathscr{H}, \tilde{A}, \tilde{B})$ for $\tilde{L}$. Let G_λ be the associated family of abstract Dirichlet operators (the γ-field). Assume that $\mathscr{G} := \operatorname{rg} \tilde{A}|_{D(L)}$ is dense. Then

$$M := (G_i^* G_i)^{1/2} \tag{A.5}$$

defines a positive, injective operator on $\partial\mathscr{H}$. By [BM14, Prop. 2.9] we have that $\mathscr{G} = \operatorname{rg} G_i^* = \operatorname{rg} M$. Note that $M^{-1}\colon \mathscr{G} \subset \partial\mathscr{H} \to \partial\mathscr{H}$ is an invertible, densely defined, closed operator.

We set $U := M^{-1}$ and $\mathscr{G}_+ := \mathscr{G}_1$, $\mathscr{G}_- := \mathscr{G}_{-1}$. Then the spaces are Hilbert spaces. Denote the unique extension $M_- := (M^{-1})^{-1}_{-1} \colon \mathscr{G}_- \to \partial\mathscr{H}$ we obtain the isometries

$$\mathscr{G}_- \xrightarrow{M_-} \partial\mathscr{H} \xrightarrow{M} \mathscr{G}_+.$$

Denote by $\bar{L}$ the closure of $\tilde{L}$ and by $\bar{B} \colon D(\bar{L}) \to \mathscr{G}_-$ the unique extension of $\tilde{B}$. Moreover (A.2) yields the decomposition

$$D(\bar{L}) = D(L) \oplus \ker(\lambda - \bar{L}) \tag{A.6}$$

for $\lambda \in \rho(L)$ and we denote the projection onto $D(L)$ by π. Define $\hat{B} := M_- \circ \bar{B} \colon D(\bar{L}) \to \partial\mathscr{H}$ and $\hat{A} := M^{-1} \circ \tilde{A} \circ \pi \colon D(\bar{L}) \to \partial\mathscr{H}$. By [BM14, Thm. 2.12] $(\partial\mathscr{H}, \hat{A}, \hat{B})$ is an ordinary boundary triple for $\bar{L}$. This allows to extend the classification of ordinary boundary triples (see [BM14, Thm. 3.1]) to quasi boundary triples (see [BM14, Thm. 3.4]). For more details we refer to [BM14, Sect. 3]. See also [DM95].

Theorem A.4 ([BM14, Cor. 3.5])**.** *Let $(\partial\mathscr{H}, \tilde{B}, \tilde{A})$ a quasi boundary triple for $\tilde{L}$. Assume that there exists a $\lambda \in \mathbb{R} \cap \rho(\tilde{L})$. Let $\mathfrak{R}$ be a relation in $\partial\mathscr{H}$. Then $L|_{\mathfrak{R}}$ given by*

$$L_{\mathfrak{R}} = H|_{D(L_{\mathfrak{R}})}$$
$$D(L_{\mathfrak{R}}) = \Big\{ f \in D(\tilde{L}) : \big(\tilde{B}f, \tilde{A}f\big) \in \mathfrak{R} \Big\}.$$

is self-adjoint if and only if the relation

$$M^{-1}(\mathfrak{R} - T(\lambda))M_-^{-1}$$

is self-adjoint and satisfies $D(\mathfrak{R}) \subset M_- D(T)$.

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