General Proof Theory and Definitional Reflection

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To Dag Prawitz on His 80th Birthday

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For a long time we have been planning to write a survey of definitional reflection. Definitional reflection was originally conceived as a powerful inversion principle applicable both within and beyond logic, but can be viewed as a general method to exhibit the structure of reasoning and proof. This abstract of what we see as connected with this notion we dedicated to Dag Prawitz, our mentor and doctoral supervisor/examiner, on the occasion of his 80th birthday on 16 May 2016. As our project has not materialized so far, we make the abstract available here as an online resource, as readers may get some inspiration from our remarks.

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GENERAL PROOF THEORY AND DEFINITIONAL REFLECTION

by Lars Hallnäs and Peter Schroeder-Heister

To Dag Prawitz on his 80th birthday

ABSTRACT

General proof theory, as introduced by Dag Prawitz in 1971, studies the notion of proof in its own right, and not just as a vehicle to establish provability in certain formal systems. Definitional reflection is a generalisation of general proof theory, basing the study of proofs on elementary principles of symmetry and duality.

If we view higher-level (and also higher-order) rules as definitional rules, or definitional clauses, over a universe of atoms of some sort, then it is natural to introduce a principle of derivation/derivability dual to the introduction of atoms. This principle of elimination is what has been called *definitional reflection*.

To make this formally explicit it is natural to focus on a system directly capturing the notion of consequence, that is,. a calculus of sequents, since systems of natural deduction rest on too strong assumptions which limit the potential of definitional reflection. The most significant such assumption is the idea that the validity of hypothetical proofs is defined by the transmission of the validity of categorical proofs, which can be indicated by the schema

(1) $A \vdash B$ iff (if $\vdash A$, then $\vdash B$)

This schema underlies standard approaches to proof-theoretic semantics for example in the intuitionistic tradition (BHK interpretation, Lorenzen's admissibility interpretation, realisability semantics, the Prawitz/Dummett definition of validity) and enters formally the Curry-Howard interpretation of deductions and with it powerful theories such as Martin-Löf's constructive theory of types. In fact, it is justified in the "well-behaved" cases, for example, of logical constants and monotone inductive definitions, namely in all cases where we find well-foundedness. However, from the general perspective of definitional reflection, where we do not want to make this presupposition and where, even more importantly, it is not decidable of whether well-foundedness can be obtained in a particular case, we do not want to assume (1).

In a sequent system of the envisaged kind the principle of introduction of atoms – *definitional closure* – introduces atoms to the right (in the succedent of the sequent), while the principle of definitional reflection introduces atoms to the left (in the antecedent of the sequent). This makes the relation between closure and reflection a duality between "or" and "and", or between existential and universal quantification. The rules for the standard logical constants are covered by this framework, which goes, however, way beyond these cases.

The principle of definitional reflection is in some sense a principle of completion since it covers *all* defining clauses of a given atom. The idea of definitional reflection corresponds to the extremal clause sometimes stated at the end of an an inductive

definition: “Nothing else defines A”. This relates the theory of definitional reflection to the theory of inductive definitions, which any decent theory dealing with mathematical reasoning must cover.

The resulting sequent system is also definitionally complete in the sense that all atoms are given a local definitional reading. This local reading does not necessarily imply far-reaching global properties such as (1). Whether the logic of definitional reflection entails a property like (1) is an “accidental” property of the given definition, that is, something that, as a matter of mathematical fact, may hold of the definition, but nothing that is forced upon it. This is analogous to the situation in the theory of partial recursive functions, where a partial recursive function may be total, but where being total is not a decidable property of the definition.

It is a fundamental principle concerning the definitions considered here that no restrictions are imposed on them. We call this the principle of *definitional freedom*. It is perfectly legitimate, for example, that we define an atom A by its own negation not-A. That in such a case the property (1) is invalidated, is not unexpected (and not unwanted).

In the theory of logic programming the idea of definitional freedom has always been present. There a logic program consisting solely of the clause “A if not-A” is considered a standard example to study the behaviour and semantics of negation. It is thus not surprising that the theory of definitional reflection received much inspiration from logic programming. However, here it is developed as a much broader approach towards logical foundations, whereas the computational aspects, which are closely connected to the problem of variables and algorithms for finding appropriate substitutions are not in focus. A fully comprehensive theory of definitional reflection would also deal with these questions (for which there is already some substantial literature available).

The system of definitional reflection can be used as a foundational system in general proof theory in various ways, of which we mention just a few.

- (i) To provide a general foundation for the treatment of proof theoretic semantics, i.e. notions of validity,
- (ii) To open up for more general discussions on the fine structure of proofs, i.e. the inversion principle,
- (iii) To provide an interpretation of the idea of introduction rules as “definitions”,
- (iv) To provide a more abstract foundation of general proof theory by introducing “variable-free” proof objects,
- (v) To develop a theory of paradoxical reasoning covering the standard mathematical and semantical paradoxes
- (vi) To lay out a theory of denial and negation yielding an appropriate treatment of “direct” negation versus “negation by failure”
- (vii) To exhibit dualities in reasoning beyond the dualities found in classical logic
- (viii) To provide an inntensional notion of harmony as an alternative to common extensional notions
- (ix) To link reasoning in “standard” general proof theory to categorial proof theory

We give short indication of what is meant by these points.

(i) *Notions of validity*
Given a natural deduction system the Prawitz-style definition of validity for the system can be seen as a higher-order-rule definition of that notion over a universe of natural deduction derivations. The study of these definitions is then itself a matter of general proof theory where various logical and structural reflections are useful as methodological tool, but are also interesting to study in their own right. While definitional reflection is not based on such a notion of validity, it provides nevertheless means to study this notion.

(ii) *The inversion principle*
Starting with Lorenzen's first formulation of this concept, various definitions of inversion have been proposed. In fact, definitional reflection as a local principle can be viewed as based on the idea of inversion. In fact, the different versions of it may lead to alternative variants of definitional reflection, in particular when it comes to atoms containing variables. This is particularly significant when generalised notions of quantification are to be framed using definitional reflection, which is, for example, important in natural language applications.

(iii) *Introduction rules as definitions*
The introduction rule in a system of natural deduction can be seen as a defining clause (Gentzen) whereas the elimination rule express global closure properties of the definition. This means that there is a certain asymmetry in the relation between introduction and elimination rules in systems of natural deduction. The introduction rule can be read as a *local* definitional clause, while the elimination rule is the *global* reflection of this definition, which is a very strong assumption on the given definition as a whole. It actually implies principle (1). In contradistinction to this idea the principle of definitional reflection is local in nature and makes no assumption on the definition as a whole. Definitional reflection is what characterises a rule as a definitional clause, whereas the elimination rule in a system of natural deduction says much more than that.

(iv) *Abstract foundation of general proof theory*
One basic issue in general proof theory is that the objects of study, i.e. proofs in systems of natural deduction, are very complex combinatorial objects due to problems with the notion of a closed assumption. This issue is related to the fact that the definition of derivations in a natural deduction system in some sense is more elementary than the objects themselves making it necessary with certain side-conditions. The consequence of this is that the "true" structural properties/complexity of the proofs are not reflected in the structure of the definition of them making the proofs somewhat awkward as mathematical objects. Using higher-order rule definitions to introduce a certain *functional closure* it is possible to introduce proofs as mathematical objects on a higher level of abstraction, which results in a certain generalisation of the idea of general proof theory. From that perspective

general proof theory becomes a sort of structural theory of a certain class of functional closures. This related to theory of higher-order abstract syntax.

(v) *Paradoxes*
Paradoxes provide a natural example of definitions which are not well-founded and for which, therefore, principle (1) can be shown to fail. This gives an interesting view of paradoxical phenomena, making them something that is perfectly legitimate to construct, with the consequence that certain principles that we are used to from "well-behaved" definitions, fail to work. It is actually the paradoxes that shed critical light on a standard notion of consequence based on (1) and the closure properties expressed by it. It is one thing to introduce the Russell set, quite another thing to assume that the resulting definition has certain closure properties. It is one of the great achievements of Prawitz that in Appendix A of Natural Deduction he pointed at the positive significance of paradoxical reasoning, showing that it leads to non-normalisable proofs.

(vi) *Denial and negation*
To the duality between closure and reflection, that is, between the introduction of an atom on the right and left side of the consequence sign (turnstile), there corresponds the duality between assertion and negation. This is made explicit in a Schütte-Tait-style one sided sequent calculus, where introducing A on the left side can be viewed as claiming the negation of A. Relying on reflection, this negation is a sort of "negation by failure", as it is based on considering all possible ways of asserting A. If each of them fails, A fails as well. This theory of "assertion vs. negation-by-failure" can be complemented with a theory of "negation vs. position-by-failure" leading to an even more symmetric framework. This would mean to actually remove an asymmetry in the form of a definition. We would not only consider definitions in which an atom is equated with defining conditions, but also definitions, in which an atom is equated with defining consequences. In a way, this links the theory of definitional reflection to theories of dual intuitionistic and bi-intuitionistic logic.

(vii) *Duality*
Overall, the dualities exhibited by definitional reflection, correspond to the dualities in constructive approaches to logic as found in Girard's linear logic and the geometry of interaction, and also to ideas in game-theoretical and dialogical approaches to logic. The ideas that led Lorenzen to give up his admissibility-based operative logic in favour of a dialogue-based approach (way before Girard) rely on an explicit acknowledgement of the duality between "right" and "left" as a basic foundational principle of logic.

(viii) *Extensional vs. intensional harmony*
Normally, harmony principles are formulated in an extensional way, that is, extensionally equivalent elimination rules are equally appropriate. However an appropriate theory should take intensional notions into account such as Došen's notion of isomorphism, according to which entities are isomorphic when moving back and forth between them creates an identity proof. This

requires that with consequences terms are associated and corresponding equalities are defined. This leads to a certain type theory based on definitional reflection, in which, however, whole consequence statements are the types (see below).

(ix) *Categorial proof theory*
Categorial proof theory as initiated by Lambek and developed into a mature discipline by Došen is based on a fundamental assumptions it shares with definitional reflection: the idea that consequence is the most fundamental notion, and that terms should be associated with consequences rather than with propositions. Thus the right way of term association would be $f : A \vdash B$ rather than $x : A \vdash t(x) : B$ (the latter being the approach of type theories based on the Curry-Howard-correspondence). It needs to be investigated how far definitional reflection can be developed categorially, or conversely, how available categorial results can be made fruitful for the study of definitional reflection.

There are many other subjects that can and need to be considered in the context of definitional reflection. Our central claim is that the definitional approach has the significance to put general proof theory onto a new level of abstraction.