Random Growth Processes on Graphs

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Contents

Zusammenfassung

In der vorliegenden Dissertation werden drei zufällige graphentheoretische Wachstumsmodelle betrachtet. Diese Modelle sind ballistisches Wachstum auf endlichen Graphen, Boolesche Perkolation auf gerichteten Graphen, sowie superkritische Galton-Watson-Verzweigungsprozesse mit Emigration.

Für das ballistische Wachstumsmodell auf endlichen Graphen erhalten wir verschiedene Resultate, die charakterisieren, wie die asymptotische Wachstumsrate von dem zugrundeliegenden Graphen abhängt. Außerdem beweisen wir, dass die Fluktuationen um diese Wachstumsrate stets durch einen zentralen Grenzwertsatz beschrieben werden.

Im Kontext von Boolescher Perkolation klären wir, für die Graphen $\mathbb{N}_0^n$ und $\mathbb{Z}^n$, $n \in \mathbb{N}$, wann alle bis auf endlich viele Punkte überdeckt werden. Wir zeigen auch, dass es für $n \geq 2$ unmöglich ist, den gerichteten n-ären Baum zu überdecken. Zudem präsentieren wir Zusammenhänge zwischen diesem Perkolationsmodell und dem sogenannten „Random Exchange Process“.

Schließlich untersuchen wir, wann superkritische Verzweigungsprozesse mit Emigration fast sicher aussterben und die erwartete Überlebenszeit endlich ist. Wir charakterisieren die Aussterbewahrscheinlichkeit in Abhängigkeit von der Populationsgröße sowie das asymptotische Wachstum der Population. Superkritische Verzweigungsprozesse mit Emigration verhalten sich gewissermaßen ähnlich wie subkritische Verzweigungsprozesse mit Immigration.

Summary

In the present thesis, we consider three different random graph-theoretic growth models. These models are called ballistic deposition on finite graphs, Boolean percolation on directed graphs, and supercritical Galton-Watson branching processes with emigration.

For our ballistic deposition model on finite graphs, we obtain various results, which characterize the relationship between the asymptotic growth rate and the underyling graph. Moreover, we prove that the fluctuations around this growth rate always satisfy a central limit theorem.

In the context of Boolean percolation, we clarify under which conditions all but finitely many points of the graphs $\mathbb{N}_0^n$ and $\mathbb{Z}^n$, $n \in \mathbb{N}$, are covered. We also prove, for $n \geq 2$, that it is impossible to cover the directed n-ary tree in this model. Besides, we present connections between this percolation model and the so-called random exchange process.

Finally, we study under which conditions supercritical branching processes with emigration become extinct almost surely, and whether the expected survival time is finite. We investigate the extinction probability in relation to the population size, and the asymptotic growth of the population. To some extent, supercritical branching processes with emigration behave similarly to subcritical branching processes with immigration.

List of Manuscripts

This thesis is based on the following three research articles.

1. **On the Growth of a Ballistic Deposition Model on Finite Graphs.** Braun, Georg. Preliminary version: arXiv:2001.09836.

   Published in *Markov Processes and Related Fields* **28**, 1-27, 2022.

2. **Boolean Percolation on Digraphs and Random Exchange Processes.** Braun, Georg. Preliminary version: arXiv:2111.04772.

   Submitted to *Journal of Applied Probability*, 16.04.2022.

3. **On Supercritical Branching Processes with Emigration.** Braun, Georg. Preliminary version: arXiv:2008.05178.

   Accepted for publication at *Journal of Applied Probability*, 04.10.2021.

Notation

Sets, Functions, Vectors

$\#M$	Cardinality of the set M
M^n	n-fold Cartesian product of the set M
$M \setminus N$	Set complement of N in M
M^c	Complement of the set M
$\mathbb{N}$	Set of all positive natural numbers
$\mathbb{N}_0$	Set of all non-negative natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{R}$	Set of all real numbers
$\mathbf{1}_M$	Indicator function of the set M
$\log(x)$	Natural logarithm of x
$\log_+(x)$	$\max(0, \log(x))$
$a_n \sim b_n$ as $n \to \infty$	Two sequences satisfy $\lim_{n\to\infty} a_n b_n^{-1} = 1$
$f(t) \sim g(t)$ as $t \to \infty$	Two functions satisfy $\lim_{t\to\infty} f(t)g(t)^{-1} = 1$
e_i	Vector with 1 in i-th entry and 0 elsewhere
δ_{ij}	Kronecker symbol
$\mathbf{1}$	Identity matrix

Probability Theory

$\mathbb{P}[A]$	Probability of the event A
$\mathbb{P}[A \mid B]$	Conditional probability of A given B
$\mathbb{E}[X]$	Expectation value of the random variable X
$\Longrightarrow$	Convergence in distribution
$N(\mu, \sigma^2)$	Normal distribution with parameters μ and σ^2
$\mathrm{Var}[X]$	Variance of the random variable X
$\mathrm{Cov}[X, Y]$	Covariance of the random variables X and Y
$X \stackrel{\mathrm{d}}{=} Y$	X and Y have the same distribution

Graph Theory

$\mathcal{G} = (V, E)$	Graph $\mathcal{G}$ with vertex set V and edge set E
$A(\mathcal{G})$	Adjacency matrix of the graph $\mathcal{G}$
$\emptyset$	Symbol for a root in a graph
$B_n(x)$	Open ball of radius n starting from the vertex x
$d(x, y)$	Distance from vertex x to vertex y in some graph
$[x]$	Closed neighborhood of the vertex x in an undirected graph
$\deg(x)$	Degree of the vertex x in an undirected graph
$\Delta\mathcal{G}$	Maximal degree of the undirected graph $\mathcal{G}$
$\mathrm{girth}(\mathcal{G})$	Length of the shortest cycle in the undirected graph $\mathcal{G}$
$\gamma(\mathcal{G})$	Asymptotic growth rate for ballistic deposition on $\mathcal{G}$
$\mathbb{N}_0^n$	Graph with vertex set $\mathbb{N}_0^n$ and all edges, which are of the form $(x, x + e_i)$, where $x \in \mathbb{N}_0^n$ and $i = 1, \ldots, n$
$\mathbb{Z}^n$	Graph with vertex set $\mathbb{Z}^n$ and all edges, which are of the form $(x, x + e_i)$, where $x \in \mathbb{N}_0^n$ and $i = 1, \ldots, n$
$\mathcal{D}_n$	Infinite directed n-ary tree, where we suppose $n \geq 2$
$\mathcal{B}$	Butterfly graph
$\mathcal{S}_n$	Undirected star graph with exactly n vertices
$\mathcal{C}_n$	Undirected circular graph with exactly n vertices
$\mathcal{K}_n$	Complete graph with exactly n vertices
$\mathcal{R}_n$	Undirected regular graph with exactly n vertices and $\Delta\mathcal{R}_n = n - 2$, where we suppose $n \geq 2$ is even
$V_\mu(\mathcal{G})$	Set of covered vertices in the graph $\mathcal{G}$ for the Boolean percolation model with the probability distribution μ

Introduction

1.1 Motivation

Graph theory is a branch of modern mathematics, which, roughly speaking, is concerned with aspects of social networks and similar systems.

For us, a graph $\mathcal{G}$ is a pair $\mathcal{G} = (V, E)$, where V is a non-empty set of vertices, and $E \subseteq V \times V$ is a binary relation, which represents the edges between vertices. If x, $y \in V$ and $(x, y) \in E$, we say that there is an edge from x to y in $\mathcal{G}$. Depending on the context, the vertices of a graph may be seen as the people in a social network, the places on a map, or possible locations for objects, for example in a game.

If the edge set $E \subseteq V \times V$ is symmetric, i.e., $(x, y) \in E$ implies $(y, x) \in E$ for all x, $y \in V$, we will say that $\mathcal{G}$ is an undirected graph. In this case, for specifying the edges in $\mathcal{G}$, instead of an ordered pair (x, y), we may also, equivalently, work with the set $\{x, y\}$. Then, instead of a subset of $V \times V$, the edge set E will be a subset of the powerset of V. Usually, in the context of undirected graphs, we will restrict our attention to graphs that do not contain any loops, i.e., $(x, x) \notin E$ respectively $\{x\} \notin E$ for all $x \in V$.

Although, in general, understanding the properties of a single graph is already a fascinating and challenging problem, in the following, we will discuss models in which a graph alters randomly over time. Note that, according to our definition, a graph $\mathcal{G} = (V, E)$ changes if and only if the vertex set V or the edge set E are modified in some way.

The literature dealing with random graph dynamics is abundant and diverse, not least because many models for a single random graph can simultaneously be seen as the dynamic evolution of a graph. Prominent examples include the Erdős-Rényi model or Galton-Watson trees.

The main part of this thesis separately studies three different random graph-theoretic growth models. While our choice is clearly subjective and the models differ a lot from each other, as we will explain in this chapter, they all share a fundamental property. The growth in these models is related to a Markov chain with time-homogeneous transition probabilities.

1.2 Time-Homogeneous Growth of Graphs

Markov processes form an important class of stochastic processes and are characterized by their limited memory. To put it simply, for a Markov process, the next state in the future depends on the current one but not on any further information from the past. In general, time may be discrete or continuous, and the state-space of the process may be a countable set or, more generally, a measurable space. If time is discrete and there are only countably many states, instead of Markov process, we may also use the term Markov chain. Famous examples of Markov processes are branching processes, random walks, and diffusion processes like Brownian motion.

A Markov process is called time-homogeneous if the probability that the process transitions from one state into a set of states within some time does not depend on the start time of this transition. The distribution of a time-homogeneous Markov process is uniquely determined by its initial condition and its transition probabilities. For time-homogeneous Markov chains, it suffices to specify, for all possible states, the transition probabilities between two consecutive moments in time. Typically, this is done by declaring a so-called transition matrix. While characterizing the long-term behavior of Markov processes is difficult in general, luckily, for time-homogeneous Markov chains, there is a relatively simple and widely accepted theory.

Under growth, we understand a time-dependent process, which is monotone increasing with respect to a partial order. In the context of graphs, a natural order is the subgraph relationship. Therefore, it is tempting to call a sequence of random graphs $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ a time-homogeneous growth process if, for all $n = 0, 1, \ldots$, the graph $\mathcal{G}_n$ is a subgraph of $\mathcal{G}_{n+1}$ almost surely, and the sequence $(\mathcal{G}_n)_{n\geq 0}$ is a time-homogeneous Markov chain.

However, from a conceptual point of view, this rather simplistic definition of time-homogeneous growth would have two fundamental disadvantages.

Firstly, growth is a process, which relies on a previously existing structure. In some sense, we, therefore, should rule out the possibility that a new component in the graph arises, which is not linked to the previous one.

Secondly, the direction, speed, and amount of growth may change over time. This is especially true for models in the real world. For example, the growth or decrease of a population may depend on the age of its individuals.

In fact, this second concern may even question whether the theory of time-homogeneous Markov chains is suitable for modeling real-world phenomena.

We propose the following notion of growth processes, which addresses both concerns and, in some sense, is a compromise regarding the second one.

Definition 1.1. *Let $\mathcal{G}_0 = (V_0, E_0)$, $\mathcal{G}_1 = (V_1, E_1)$, ... be a finite or infinite sequence of random finite graphs. Moreover, let $W_0 \subseteq V_0$, $W_1 \subseteq V_1$,... denote a sequence of random sets. Then, we call $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, ... a growth process on $\mathcal{G}_0$ if the following conditions hold.*

- *For all $n = 0, 1, \ldots$, the graph $\mathcal{G}_n$ is a subgraph of $\mathcal{G}_{n+1}$ almost surely.*
- *Almost surely, for all $n = 0, 1, \ldots$ and $e \in E_{n+1} \setminus E_n$, the initial or the final vertex of the edge e is contained in W_n.*
- *Almost surely, for all $n = 0, 1, \ldots$ and $x \in V_{n+1} \setminus V_n$, the vertex x is not isolated in $\mathcal{G}_{n+1}$.*

In this case, for all $n = 0, 1, \ldots$, we will say that the set W_n is an admissible growth zone for the graph $\mathcal{G}_n$.

If, furthermore, the sequence $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, ... is a Markov chain, we will call it a Markovian growth process. Moreover, if the transition probabilities of this Markov chain are time-homogeneous, we will refer to this sequence as a time-homogeneous growth process on $\mathcal{G}_0$.

In the following sections, we will explain how the three main models of this thesis relate to this notion of random growth processes on graphs. However, let us first give some comments regarding Definition 1.1 at this point.

First of all, in general, there may exist many different choices for the sequence of growth zones. Even if we impose that these sets are minimal, they are not unique in general, as we allow the growth of edges between two vertices, which both already existed before.

Moreover, in Definition 1.1, we let time evolve in discrete steps. But, only with minor changes, we can adapt our concept to continuous-time models.

Apart from that, we restrict ourselves to sequences of almost surely finite random graphs. This constraint has the advantage that it is relatively simple to define what is meant by a random graph and what it exactly means for the sequence $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, ... to be a Markov process.

Finally, by the first condition in Definition 1.1, for any growth process on a graph, we know that there exists the limit graph $\mathcal{G}_\infty := (V_\infty, E_\infty)$, where

$$V_\infty := \bigcup_{n \geq 0} V_n, \qquad E_\infty := \bigcup_{n \geq 0} E_n.$$

In general, $\mathcal{G}_\infty$ is an infinite random graph.

1.3 Ballistic Deposition on Finite Graphs

There are different models for ballistic deposition, but they typically allow the following description. Assume that on some structure consecutively, bricks are falling from above. The places at which the bricks arrive are supposed to be random, and two bricks are always glued together when they start to touch or share a surface. In this way, towers arise and grow randomly. Moreover, neighboring towers influence and accelerate each other's growth.

In the present thesis, we will study a version of ballistic deposition, in which the substrate is given by a deterministic undirected graph $\mathcal{G} = (V, E)$. Suppose that the vertex set V is finite and that the graph $\mathcal{G}$ is connected.

Recursively, we now define a time-homogeneous growth process $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1), \ldots$ as follows. Set $\mathcal{G}_0 := \mathcal{G}$, i.e., $V_0 := V$ and $E_0 := E$, and $W_0 := V$. Let $n \geq 0$ and assume the pair $(\mathcal{G}_n, W_n)$ is defined. Then, uniformly in the set W_n and independently from the past, we pick a random vertex y. Subsequently, set $V_{n+1} := V \cup \{z\}$, where $z \notin V_n$ denotes a new vertex, which is added to the graph, and $W_{n+1} := (W_n \cup \{z\}) \setminus \{y\}$.

Whenever we replace some $y \in W_n$ with a new vertex $z \in V_{n+1} \setminus V_n$ to form W_{n+1}, we call y and z equivalent. Then, for all $n \geq 0$ and $x \in W_n$, there is always a unique vertex in V, which is equivalent to x. We denote it by x_0.

We define E_{n+1} to arise from E_n by adding the edge (y, z), as well as all edges of the form (x, z), $x \in W_n$, for which x_0 and z_0 are neighbors in $\mathcal{G} = (V, E)$, and, moreover, there is no path from x to y in $\mathcal{G}_n$.

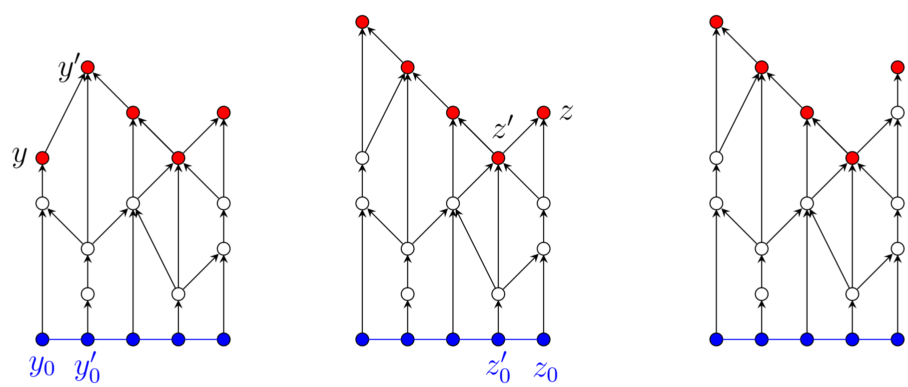


Figure 1.1: Illustration of the deposition model. The underlying graph $\mathcal{G}$ is depicted in blue. The vertices, which form the growth zone, are red. In the first step, the vertex y is selected, and in the next step, z gets chosen.

Let us briefly explain how we can formally define the height in this ballistic deposition model. Let $x \in V$, $n \in \mathbb{N}_0$, and denote by x' the unique element of the growth zone W_n, which is equivalent to x. Then, if $x = x'$, we define the height of x at time $n \geq 0$ as zero. If $x \neq x'$, we consider all paths in $\mathcal{G}_n$, which start in V and end in x', and do not use any other vertex from V. Then, we say a path of maximal length in this set is a backbone for x' at time $n \geq 0$, and the height of x is the number of edges in a backbone.

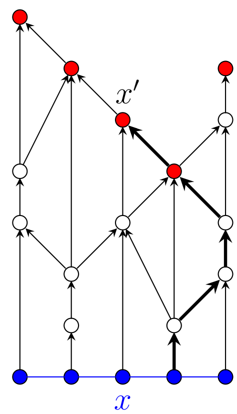


Figure 1.2: As before, the underlying graph is depicted in blue, and the vertices of the growth zone are red. We assume that x and x' are equivalent. A backbone for x' is indicated by thick edges. The height of x is equal to 5.

Suppose that the height configuration of our deposition model at time $n \geq 0$ is given by $(h_y)_{y \in V} \in \mathbb{N}_0^V$. Moreover, let $x \in V$ and assume that in the next step, the random element of the growth zone W_n, which gets chosen, is equivalent to the vertex x. Then, only the height h_x is increased, and the new height of the vertex x is given by

$$\tilde{h}_x = 1 + \max_{y \in [x]} h_y, \tag{1.1}$$

where $[x] := \{x\} \cup \{y \in V \mid \{x, y\} \in E\} \subseteq V$ denotes the closed neighborhood of the vertex x in the undirected graph $\mathcal{G}$.

The recurrence relation (1.1) and the resulting time-homogeneous Markov chain $H = (H_n)_{n \geq 0}$ for the height in our deposition process are the starting point of Chapter 2. Formally, for all $x \in V$, we will write $H_{x,n}$ for the height of the vertex x after n deposition or growth events, and for all $n \geq 0$, we set $H_n := (H_{x,n})_{x \in V}$.

However, we will work not only with H but also with a corresponding continuous-time Markov process $\tilde{H}$, which is defined as follows. Assume, for all $x \in V$, that ξ_x denotes a Poisson process with unit intensity, and that the family $(\xi_x)_{x\in V}$ is independent. Then, for all $x \in V$ and $t \in [0, \infty)$, in the process $\tilde{H}$, we will update the height of the vertex x at time t according to equation (1.1) if and only if the process ξ_x jumps at this moment.

Previously, for the family of circular graphs $\mathcal{C}_n$, $n \geq 1$, Atar, Athreya, and Kang observed in [4], that Kingman's subadditive ergodic theorem can be used to deduce a law of large numbers for the height in our continuous-time deposition process $\tilde{H}$. The same arguments are valid, more generally, for an undirected graph $\mathcal{G}$, which is finite and connected. In this context, the law of large numbers holds both for the minimal and maximal height, and in both cases, we arrive at the same limiting constant $\gamma(\mathcal{G}) \in (0, \infty)$. We will refer to $\gamma(\mathcal{G})$ as the growth rate or growth parameter of the graph $\mathcal{G}$.

In our study, we present many results, which describe how the growth parameter $\gamma(\mathcal{G})$ is related to the properties of the graph $\mathcal{G}$. Often, it will be useful to switch between discrete and continuous time.

The main result of Chapter 2 is the following central limit theorem, which is contained in Section 2.5.

Theorem 2.8. *Let $\mathcal{G} = (V, E)$ be an undirected graph, which is both finite and connected. Then, there exists $\sigma^2 = \sigma^2(\mathcal{G}) \in [0, \infty)$ with*

$$\frac{\max_{x\in V} H_{x,n} - n\ \frac{\gamma(\mathcal{G})}{\#V}}{n^{1/2}} \Longrightarrow N(0, \sigma^2) \quad \text{for } n \to \infty.$$

The same central limit theorem with the same constant σ^2 also holds if we replace $\max_{x\in V} H_{x,n}$ with $\min_{x\in V} H_{x,n}$. Moreover,

$$\sigma^2(\mathcal{G}) = 0 \quad \text{if and only if} \quad \mathcal{G} \text{ is isomorphic to } \mathcal{K}_{\#V}.$$

In the context of this theorem, a normal distribution with variance $\sigma^2 = 0$ will be identified with a Dirac law. Furthermore, for all $n \in \mathbb{N}$, $\mathcal{K}_n$ denotes a complete graph with exactly n vertices.

Our proof of Theorem 2.8 relies on applying renewal-theoretic arguments to the surface process $(\delta_n)_{n\geq 0}$, which is given by

$$\delta_n := (\delta_{x,n})_{x\in V}\,, \qquad \delta_{x,n} := H_{x,n} - \min_{y\in V} H_{y,n}, \quad x \in V.$$

1.4 Boolean Percolation on Directed Graphs

We propose a model for the spread of rumors through a network in which each individual invents one rumor. The ranges of these rumors are random, independent of each other, and obey a common probability law. How many individuals of the network are influenced by at least one rumor?

Let $\mathcal{G} = (V, E)$ be a graph and denote by $d : V \times V \to \mathbb{N}_0 \cup \{\infty\}$ the metric induced by $\mathcal{G}$, i.e., $d(x, y)$ is the distance from x to y for all x, $y \in V$. Note that, as $\mathcal{G}$ is a directed graph, in general, $d(x, y) \neq d(y, x)$ for x, $y \in V$. For all $x \in V$ and $n \in \mathbb{N}_0$, we will write $B_n(x)$ for the open ball of radius n starting in x. For example, $B_0(x) = \emptyset$ and $B_1(x) = \{x\}$ for all $x \in V$.

Let $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ be a probability vector and denote by $(Y_x)_{x\in V}$ a family of i.i.d. random variables, which satisfies $\mathbb{P}[Y_x = n] = \mu_n$ for all $n \geq 0$. In our interpretation, the random variable Y_x will represent the range of the rumor, which starts to spread from the vertex $x \in V$. More precisely, for all $x \in V$, the rumor invented in x spreads to all vertices in $B_{Y_x}(x)$. Overall, the set of all vertices, which are influenced by at least one rumor, is given by

$$V_\mu := \bigcup_{x\in V} B_{Y_x}(x).$$

In Chapter 3, we will study the properties of the random set V_μ for a few specific choices of $\mathcal{G}$, under the assumption $\mu_0 \in (0, 1)$. Mainly, we will investigate whether the set $V \setminus V_\mu$ is finite almost surely. Interestingly, in most of our cases, this question is related to the limit graph $\mathcal{G}_\infty$ of a time-homogeneous growth model. Let us explain this connection.

To simplify our arguments, let $n \geq 2$ be a natural number and assume that the graph $\mathcal{G} = (V, E)$ is the infinite directed n-ary tree $\mathcal{D}_n$. Let $\emptyset$ denote the unique root in $\mathcal{G}$, which satisfies $d(\emptyset, y) < \infty$ for all $y \in V$.

Now, we recursively define a time-homogeneous growth process $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, ... as follows. Initially, we set $\mathcal{G}_0 := (V_0, E_0)$, where $V_0 := \{\emptyset\}$ and $E_0 := \emptyset$, and $W_0 := \{\emptyset\}$. Let $n \geq 0$ and assume that the pair $(\mathcal{G}_n, W_n)$ is specified. Then, we set

$$\begin{aligned} V_{n+1} &:= V_n \cup \{y \in V \setminus V_n \mid \exists x \in W_n : Y_x > d(x, y)\}, \\ E_{n+1} &:= E_n \cup \{(x, y) \in E \mid x,\ y \in V_{n+1}\}, \\ W_{n+1} &:= V_{n+1} \setminus V_n. \end{aligned}$$

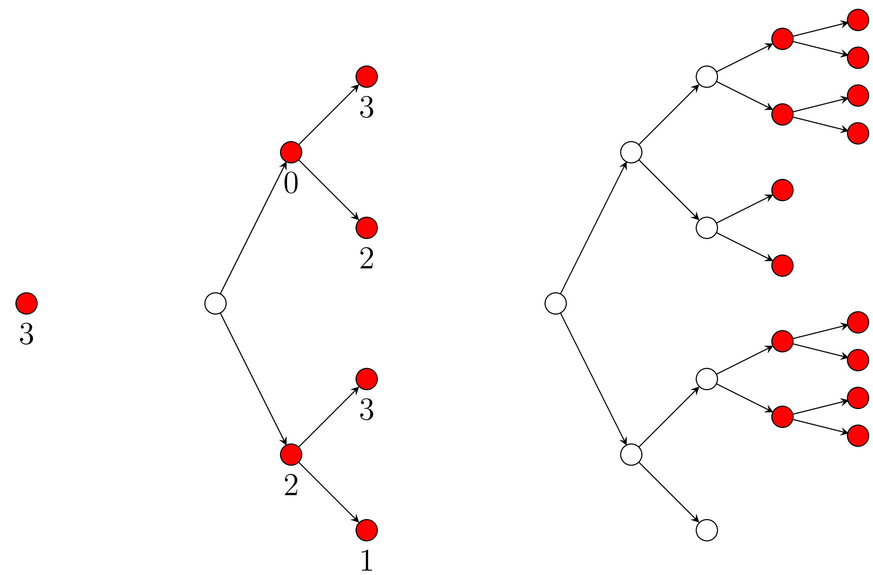


Figure 1.3: Illustration of the growth process if the underlying graph $\mathcal{G}$ is the infinite directed binary tree. The vertices, which belong to the growth zone, are depicted in red. The number next to a vertex x denotes the value of Y_x.

Let $\mathcal{G}_\infty = (V_\infty, E_\infty)$ denote the limit graph of the growth process $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, ..., and observe that $V_\infty \subseteq V_\mu \cup \{\emptyset\}$ almost surely since we have $V_n \subseteq V_\mu \cup \{\emptyset\}$ almost surely for all $n \geq 0$. Consequently, if the set $V \setminus V_\infty$ is finite, then so is the set $V \setminus V_\mu$. Furthermore, by Kolmogorov's 0-1 law, if the sequence of graphs $(\mathcal{G}_n)_{n\geq 0}$ is strictly growing forever with a positive probability, then V_μ contains an infinite path almost surely.

The main difference between our percolation model V_μ and the limit graph V_∞ of the growth process $(\mathcal{G}_n, W_n)_{n\geq 0}$ can be described as follows. In the percolation model V_μ, every vertex $y \in V$ covers itself for $Y_y \geq 1$. However, in the limit graph V_∞, the occurrence of every $y \in V$ is solely determined by all random variables Y_x, for which $0 < d(x, y) < \infty$.

Observe that, for all $n \geq 2$ and $\mathcal{G} = \mathcal{D}_n$, we can associate a multitype branching process $(Z_n)_{n\geq 0}$ to our percolation model as follows. For every $y \in V$ with $d(\emptyset, y) = n \in \mathbb{N}_0$, we identify the vertex y with an individual in the n-th generation of $(Z_n)_{n\geq 0}$ if $x \in V_\mu$ for all $x \in V$ with $0 \leq d(x, y) < n$. In this case, the type of y will be defined by

$$z_y := \max \{Y_x - d(x, y) \mid x \in V,\ d(x, y) < \infty\} \in \mathbb{N}.$$

So, we arrive at branching process $(Z_n)_{n\geq 0}$, which starts with a single individual of random type $Y_\emptyset$, provided $Y_\emptyset \geq 1$. In general, the type space is infinite. Hence, we distinguish between global and local extinction of $(Z_n)_{n\geq 0}$. This process dies out globally if the entire population vanishes at some moment and becomes extinct locally if every type only occurs finitely often.

In Chapter 3, we will also discuss connections between our Boolean percolation model and a rather classical Markov chain, which is sometimes called random exchange process. Let $(Y_n)_{n\geq 0}$ be a sequence of i.i.d. random variables, which take values in $\mathbb{N}_0$ and satisfy $\mathbb{P}[Y_0 = n] = \mu_n$ for all $n \geq 0$. Then, we recursively define a random exchange process $(R_n)_{n\geq 0}$ by

$$R_0 := Y_0, \qquad R_{n+1} := \max\{R_n - 1,\ Y_{n+1}\}, \quad n \geq 0. \tag{1.2}$$

By construction, the Markov chain $(R_n)_{n\geq 0}$ has time-homogeneous transition probabilities and is irreducible with respect to its state space $\mathcal{X} \subseteq \mathbb{N}_0$. We will write $P = (P_{m,k})_{m,k\in\mathcal{X}}$ for the transition matrix and $\rho(P)$ for the spectral radius of $(R_n)_{n\geq 0}$. More generally, if $A = (A_{m,k})_{m,k\geq 1}$ is a finite quadratic matrix, or a matrix with both countably infinite many columns and rows, and A is nonnegative and irreducible, we define

$$\rho(A) := \limsup_{n\to\infty} \left(A^n_{m,k}\right)^{1/n} \in [0,\infty],$$

and the value of $\rho(A)$ does not depend on the choice of m, $k \geq 1$.

Our main result of Chapter 3 reads as follows.

Theorem 3.3. *Let $n \geq 2$. Then, for any distribution μ, $\#V^c_\mu(\mathcal{D}_n) = \infty$ almost surely. Moreover, the following statements are equivalent.*

(A) *Almost surely, $V_\mu(\mathcal{D}_n)$ contains a path of infinite length.*

(B) *With a positive probability, $(Z_m)_{m\geq 0}$ will not die out globally.*

(C) *With a positive probability, $(Z_m)_{m\geq 0}$ will not die out locally.*

(D) *$\rho(M) > n^{-1}$, where $M := (P_{m,k})_{m,k\in\mathcal{X}\setminus\{0\}}$.*

Finally, we want to point out that random exchange processes are closely related to the class of autoregressive processes, which we will encounter in our study of supercritical branching processes with emigration in Chapter 4.

This was observed by Zerner in [98, Section 1], and for the convenience of the reader, we want to explain this connection in the following.

Let $(U_n)_{n\geq 0}$ denote a sequence of nonnegative random variables and $(T_n)_{n\geq 1}$ a sequence of real-valued variables. Assume that $(U_n, T_n)_{n\geq 1}$ is i.i.d. and independent of U_0. Then, in generalizing equation (1.2), we define a (random decrement) exchange process $(R_n)_{n\geq 0}$ by

$$R_0 := U_0, \qquad R_{n+1} := \max\{R_n - T_{n+1},\ U_{n+1}\}, \quad n \geq 0. \tag{1.3}$$

Again, the process $(R_n)_{n\geq 0}$ is a Markov chain with time-homogeneous transition probabilities. Formally, to arrive at equation (1.2), we may set $T_n := 1$ for all $n \geq 1$ and $U_n := Y_n$ for all $n \geq 0$.

Now, we consider the process $(M_n)_{n\geq 0}$ defined by $M_n := e^{R_n}$, $n \geq 0$. Moreover, set $A_n := e^{-T_n}$ and $B_n := e^{U_n}$ for all $n \geq 1$. Then, from equation (1.3), we can deduce the recurrence relation

$$M_{n+1} = \max\{e^{R_n - T_{n+1}},\ e^{U_{n+1}}\} = \max\{A_{n+1}M_n,\ B_{n+1}\}, \quad n \geq 0. \tag{1.4}$$

In particular, $(M_n)_{n\geq 0}$ is a time-homogeneous Markov process, and we refer to it as a max-autoregressive process. For example, by induction, we can verify the formula

$$M_n = \max_{m=0,\ldots,n} A_n \cdots A_{m+1} B_m, \quad n \geq 0.$$

By replacing the maximum in (1.4) with a sum, we arrive at the Markov process $(X_n)_{n\geq 0}$, which is given by $X_0 := M_0$ and

$$X_{n+1} := A_{n+1}X_n + B_{n+1}, \quad n \geq 0. \tag{1.5}$$

This Markov chain $(X_n)_{n\geq 0}$ is called a (random coefficient) first-order autoregressive process, and the recurrence relation (1.5) is the so-called random difference equation. Similar as for $(M_n)_{n\geq 0}$, we obtain the representation

$$X_n = \sum_{m=0}^{n} A_n \cdots A_{m+1} B_m, \quad n \geq 0.$$

Observe that, due to the definition of the sequence $(A_n, B_n)_{n\geq 1}$, these random variables and the process $(X_n)_{n\geq 0}$ only attain nonnegative values. However, in general, one might also consider autoregressive processes, for which this condition is not satisfied. While autoregressive and max-autoregressive processes differ from each other through their definition, as mentioned by Zerner in [98, Comments after Proposition 1.1], in the nonnegative case, one might expect that they behave similarly if the involved distributions are heavy-tailed.

1.5 Galton-Watson Branching Processes

Branching processes are a class of stochastic processes, which describe how the size of populations varies over time. The main assumption is that individuals independently of each other give birth to a random number of children. While there are many different models, partly in continuous time or with a continuous state-space, we will concentrate on a specific version of the classical Galton-Watson branching process. More precisely, we let time evolve in discrete steps $n = 0, 1, 2, \ldots$, and assume that there is one underlying probability distribution on $\mathbb{N}_0$, which characterizes the random number of children of each individual. Between two consecutive generations, each individual reproduces and dies, and we allow the occurrence of an emigration event. The sizes of these emigration events, sometimes called catastrophes, are assumed to be i.i.d. with respect to time. If at some moment, the number of migrants exceeds the population size, the population will become extinct.

If the emigration component is absent, we recover the classical Galton-Watson process. In this case, it is well-known that, unless each individual almost surely gives birth to exactly one child, the population survives forever with a positive probability if and only if the process is supercritical, i.e., the expected number of children of each individual is greater than one.

In Chapter 4, we will study the long-term behavior of supercritical branching process with emigration if the mean offspring of each individual is finite. Due to the presence of the emigration component, the population may become extinct almost surely. We will present various results on the extinction probabilities of these branching processes and also characterize how fast these processes grow if they do not die out eventually.

In our study, we allow that, between two consecutive generations, the reproduction behavior of the individuals and the number of subsequent emigrants depend on each other. However, we restrict ourselves to processes with time-homogeneous transition probabilities. Furthermore, we suppose that neither reproduction nor emigration dominates each other, i.e., the population may become arbitrarily large, but, at the same time, irrespective of the population size, the process may still become extinct in the future.

As for many versions of the Galton-Watson branching process, we arrive at a representation in the form of a time-homogeneous growth model $(\mathcal{G}_0, W_0)$, $(\mathcal{G}_1, W_1)$, $\ldots$, by considering the genealogical tree of the population. The initial graph $\mathcal{G}_0 = (V_0, E_0)$ consists of finitely many, deterministic, isolated vertices, and $W_0 := V_0$. Let $n \geq 0$ and assume the pair $(\mathcal{G}_n, W_n)$ is defined. To form $\mathcal{G}_{n+1} = (V_{n+1}, E_{n+1})$, according to the reproduction distribution, independently, we add to each vertex in W_n a random number of children. Then, for example, uniformly in the set of all new vertices, we delete as many vertices as specified by the emigration event. This gives us V_{n+1}. The edges, which we subsequently add to get E_{n+1}, start in W_n and connect each parent with all of its children, which have survived the emigration event. Finally, the new growth zone is $W_{n+1} := V_{n+1} \setminus V_n$.

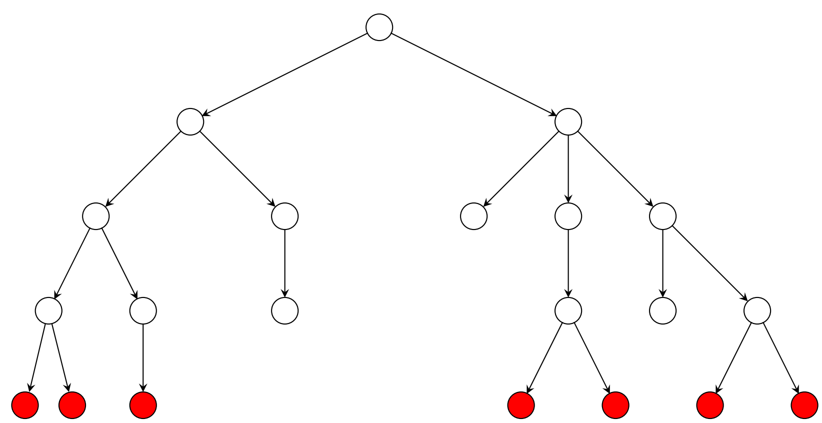


Figure 1.4: Illustration of a Galton-Watson tree, which starts from a single individual. The vertices of the growth zone are depicted in red.

Formally, for all $n \geq 0$, we may define the size of the n-th generation in our branching process by $Z_n := \#W_n$. Throughout Chapter 4, we will assume that $\lambda \in (1, \infty)$ denotes the mean number of children of an individual. Besides, we assume that ξ and Y are random variables, which take values in $\mathbb{N}_0$ and represent the offspring distribution of a single individual respectively the strength of an emigration event between two consecutive generations. The extinction time of $(Z_n)_{n\geq 0}$ will be denoted by τ and set equal to $+\infty$ if the process survives forever.

Arguably the most interesting result of Chapter 4 is the following limit theorem, which relates the extinction probabilities of $(Z_n)_{n\geq 0}$ to the strength of the emigration component Y as the initial population size $Z_0 = k$ tends to infinity.

Theorem 4.4. *Assume $\mathbb{P}[Y > n]$ is regularly varying for $n \to \infty$ with index $\alpha \in (0, \infty)$. Then, for all $N \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$,*

$$\limsup_{k\to\infty} \mathbb{P}[\tau < N \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} \leq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Furthermore, if all exponential moments of ξ are finite, then

$$\lim_{k\to\infty} \mathbb{P}[\tau < N \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} = \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Our study of supercritical branching processes with emigration, as well as many of our results, are strongly motivated by an observation, which links this branching process model to subcritical autoregressive processes. More precisely, in Section 4.3, we deduce a duality relation between these processes in the sense of Siegmund, compare [89], under the assumption that the number of children of each individual is almost surely constant. For the convenience of the reader, we, therefore, want to give a brief description of this duality concept at this point, but without aiming at full generality.

Assume $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ are Markov processes with time-homogeneous transition probabilities and suppose that the state-space of both processes is $[0, \infty)$. Then, with a slight abuse of notation, we say that X and Y are Siegmund dual if, for all $0 \leq a, b < \infty$ and $n \in \mathbb{N}_0$,

$$\mathbb{P}[X_n \geq a \mid X_0 = b] = \mathbb{P}[Y_n \leq b \mid Y_0 = a]. \tag{1.6}$$

Observe that, by inserting $a = 0$, in order for this condition to be satisfied, it is necessary that the process Y is almost surely absorbed at 0. Obviously, this assertion is satisfied for our branching process $(Z_n)_{n\geq 0}$. As we only allow emigration, but no immigration into the population, the process $(Z_n)_{n\geq 0}$ cannot be revived upon extinction.

Furthermore, if the duality condition (1.6) is satisfied, by inserting $b = 0$, we can relate the probability of eventual absorption of Y in 0 given $Y_0 = a$ to the behavior of the transition probability $\mathbb{P}[X_n \geq a \mid X_0 = 0]$ for $n \to \infty$. If the process X is positive recurrent, under suitable conditions, we may expect that these probabilities converge towards a stationary solution of X, and thus have a positive limit, provided a is large enough.

Article 1

On the Growth of a Ballistic Deposition Model on Finite Graphs

Georg Braun

Abstract. We revisit a ballistic deposition process introduced by Atar et al. in [4]. Let $\mathcal{G} = (V, E)$ be a finite connected graph and choose independently and uniformly vertices in $\mathcal{G}$. If a vertex $x \in V$ gets chosen and the previous height configuration is given by $h = (h_y)_{y \in V} \in \mathbb{N}_0^V$, the height h_x is replaced with

$$\tilde{h}_x := 1 + \max_{y \in [x]} h_y.$$

For different underlying graphs $\mathcal{G}$, we determine the asymptotic growth parameter $\gamma(\mathcal{G})$ of this model. We also present a central limit theorem for the height fluctuations around $\gamma(\mathcal{G})$ and a graph-theoretic reinterpretation of an inequality obtained in [4].

Keywords. ballistic deposition process; random surface; stochastic growth; random sequential adsorption.

2020 Mathematics Subject Classification. 60C05, 60J10.

2.1 Introduction

Let us start with an informal description of our random growth model.

In a city, there is an exclusive group of skyscraper owners. Once in a while, an owner decides to heighten his building until it is strictly higher than the skyscrapers of the group members he disrespects. If his building already achieves this, it will be raised by only one floor. How fast will the skyscrapers grow?

Ballistic growth models are typically studied on infinite graphs, when they are believed to belong to the KPZ universality class and, in two dimensions, exhibit fluctuations of the order $t^{1/3}$ as the time t goes to infinity (compare, for example, [80], [22], and [7]). However, exact results of this kind have been proven only for a few specific models (see, for example, [14] and [13]).

In this article, we restrict our attention to the case of finite underlying graphs and study the asymptotic growth of a specific deposition model. We obtain formulas for the asymptotic growth rate in some explicitly solvable cases and prove a classical central limit theorem for the fluctuations around this growth rate, which holds for arbitrary graphs. We also present an upper bound for the asymptotic growth parameter in terms of the spectral radius of the underlying graph. The proof of this inequality relies on the methods used by Atar, Athreya, and Kang in [4].

Let $\mathcal{G} = (V, E)$ be a connected undirected graph with a finite non-empty vertex set V and a non-empty edge set $E \subseteq \{\{x, y\} \mid x, y \in V,\ x \neq y\}$. The (closed) neighborhood of a vertex $x \in V$ is defined by

$$[x] := \{x\} \cup \{y \in V \mid \{x, y\} \in E\}.$$

As time goes by, we successively choose independently and uniformly vertices in the graph $\mathcal{G}$. If a vertex $x \in V$ gets chosen and the previous height of the process is given by $(h_y)_{y \in V} \in \mathbb{N}_0^V$, the height h_x of x will be replaced by

$$\tilde{h}_x := 1 + \max_{y \in [x]} h_y. \tag{2.1}$$

This rule defines the so-called next nearest neighbor ballistic deposition process. In our study of this model, it will be helpful to distinguish between the following two closely related versions of this process.

On the one hand, we can let the time evolve in discrete steps $n = 1, 2, \ldots$ and always choose exactly one vertex at these time points. Then, the height of a vertex $x \in V$ after n steps will be denoted by $H_{x,n}$, and the ballistic deposition process is $H := (H_n)_{n \in \mathbb{N}_0}$, where $H_n := (H_{x,n})_{x \in V}$, $n \geq 0$.

On the other hand, we may choose a family $(\xi_x)_{x \in V}$ of independent Poisson processes and change the height in a vertex $x \in V$ at time $t \in (0, \infty)$ if and only if the corresponding Poisson process ξ_x jumps at time t. Unless explicitly stated otherwise, we will assume that all Poisson processes have unit intensity. We will write $\tilde{H}_{x,t}$ for the height of $x \in V$ at time $t \in [0, \infty)$ and define $\tilde{H}_t := (\tilde{H}_{x,t})_{x \in V}$ for $t \in [0, \infty)$, as well as $\tilde{H} := (\tilde{H}_t)_{t \geq 0}$.

By construction, both H and $\tilde{H}$ are time-homogeneous Markov processes. Usually, we will assume the initial condition $H_{x,0} = \tilde{H}_{x,0} = 0$ for all $x \in V$, which ensures that both processes have the same countable state-space.

In [4], Atar, Athreya, and Kang considered the specific case of a circular graph $\mathcal{G} = \mathcal{C}_n$, which can be defined to have the vertex set $\{1, \dots, n\}$ and the edge set $\{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{n,1\}\}$. Then, as explained in [4, Section 1], Kingman's subadditive ergodic theorem yields the existence of the almost sure limit

$$\gamma(\mathcal{C}_n) := \lim_{t\to\infty} t^{-1} \max_{x\in V} \tilde{H}_{x,t} = \lim_{t\to\infty} t^{-1} \min_{x\in V} \tilde{H}_{x,t} \in (0,\infty). \tag{2.2}$$

In fact, these arguments apply in the same way to a general graph $\mathcal{G}$, and hence we always define the growth parameter $\gamma(\mathcal{G})$ by the right-hand side of equation (2.2). The asymptotic growth of our discrete-time deposition process is related to $\gamma(\mathcal{G})$ via

$$\gamma(\mathcal{G}) = \#V \lim_{n\to\infty} n^{-1} \max_{x\in V} H_{x,n} = \#V \lim_{n\to\infty} n^{-1} \min_{x\in V} H_{x,n}, \tag{2.3}$$

and these equations again hold almost surely. So, for studying the parameter $\gamma(\mathcal{G})$, we can switch from continuous-time to discrete-time or vice versa.

Let us now briefly summarize previous literature results on our model and mention related works. The main result of [4] is the inequality

$$3.21 < \gamma(\mathcal{C}_n) < 5.35 \quad \text{for all } n \geq 5. \tag{2.4}$$

The authors of [4] also claimed that this inequality is satisfied for $n = 4$. However, as we will verify in Section 2.4,

$$\gamma(\mathcal{C}_4) = 2 + \frac{2}{\sqrt{3}} \approx 3.1547.$$

This reveals that a minor calculation error has occurred in [4] for $n = 4$. Nevertheless, the inequality (2.4) and its proof given in [4] are correct.

In [28], Fleurke, Formentin, and Külske assumed that the vertices of the graph $\mathcal{G}$ are not chosen uniformly, but according to a fixed Markov chain with state-space V. They proved the existence of the limit $\gamma(\mathcal{G})$ in this more general setting and also established a sub-Gaussian concentration inequality for the maximal height.

In [70], Mountford and Sudbury studied homogeneous isotropic infinite graphs and related the growth parameter $\gamma(\mathcal{G})$ to the roughness of the surface.

In [66], Mansour, Rastegar, and Roitershtein discussed combinatorial problems related to our model in the case of $\mathcal{G} = \mathcal{C}_n$ and conjectured that

$$\lim_{n\to\infty} \gamma(\mathcal{C}_n) = 4.$$

In [79], Penrose and Yukich verified a law of large numbers and a central limit theorem for the total height of all accepted particles in a ballistic deposition process on $\mathbb{R}^d$. This result is based on insights on marked Poisson processes, which also yields information for other random sequential adsorption models.

In [86], Seppäläinen studied a ballistic deposition model on $\mathbb{Z}^n$ and proved the existence of a limiting shape, which is related to Eden's growth model respectively first passage percolation.

For convenience, let us now briefly explain how this article is organized.

In Section 2.2, we introduce various relevant graph theoretic concepts and notations. Then, in Section 2.3, we mainly concentrate on the class of star graphs. We also present an example of non-isomorphic graphs $\mathcal{G}$ and $\mathcal{H}$ with $\gamma(\mathcal{G}) = \gamma(\mathcal{H})$. In Section 2.4, a rather simple probabilistic approach is used to compute the growth parameter $\gamma(\mathcal{G})$ in a specific setting. Subsequently, in Section 2.5, we deduce a central limit theorem for the height fluctuations around $\gamma(\mathcal{G})$. Our proof, similar to [28], relies on a suitable renewal structure in the surface process of our deposition process. In Section 2.6, we give an upper bound for $\gamma(\mathcal{G})$ by using spectral graph theory. This result is based on some modifications of the arguments used in [4]. Finally, in Section 2.7, we briefly study a ballistic growth model, which arises through alteration of the deposition rule (2.1).

2.2 Graph-theoretic Preliminaries

The degree of a vertex $x \in V$ is $\deg(x) := \#[x] - 1$, and the maximal degree in $\mathcal{G}$ is $\Delta\mathcal{G} := \max_{x\in V} \deg(x)$. The graph $\mathcal{G}$ is regular if $\deg(x) = \deg(y)$ for all x, $y \in V$. A vertex $x \in V$ is called dominant in the graph $\mathcal{G}$ if $[x] = V$.

A path of length n in $\mathcal{G}$ is a tuple $(x_1, \ldots, x_{n+1}) \in V^{n+1}$ with $\{x_i, x_{i+1}\} \in E$ for all $i = 1, \ldots, n$. If, in addition, $x_1 = x_{n+1}$, $n \geq 1$, and $x_i \neq x_j$ for all $i, j = 1, \ldots, n$ with $i \neq j$, we will call $(x_1, \ldots, x_{n+1})$ a cycle in $\mathcal{G}$. The length of the shortest cycle of a graph $\mathcal{G}$ will be denoted by $\operatorname{girth}(\mathcal{G})$. If there is no cycle in $\mathcal{G}$, we set $\operatorname{girth}(\mathcal{G}) := \infty$. We define $d(x, x) := 0$ for all $x \in V$, and, for vertices $x \neq y$, we define $d(x, y)$ to be the length of the shortest path from x to y. Note that d is a metric on V. A permutation $(x_1, \dots, x_{\#V})$ of V will be called non-decreasing if the function $k \mapsto d(x_1, x_k)$ is non-decreasing.

The adjacency matrix of $\mathcal{G}$ is denoted by $A(\mathcal{G})$. For all $(x, y) \in V^2$, the corresponding entry of the matrix $A(\mathcal{G})$ is one, if $\{x, y\} \in E$, and zero otherwise. In particular, $A(\mathcal{G})$ is a quadratic matrix of dimension $\#V$. Moreover, for all $n \geq 1$, the entries of $A(\mathcal{G})^n$ count the number of paths of length n between two vertices.

Given two graphs $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$, we say that $\mathcal{G}$ is a subgraph of $\mathcal{G}'$ if both $V \subseteq V'$ and $E \subseteq E'$.

Let $n \geq 1$. We will write $\mathcal{S}_n$ to denote a star graph with n vertices. Formally, we choose $\{1, \dots, n\}$ and $\{\{1, 2\}, \dots, \{1, n\}\}$ as vertex, respectively edge set. Furthermore, we denote by $\mathcal{K}_n$ a complete graph with n vertices, in which there is an edge between any two different vertices. Observe that, for all $n \geq 1$, $\gamma(\mathcal{K}_n) = n$.

If $n \geq 2$ is even, we denote by $\mathcal{R}_n$ a regular graph, which consists of n vertices and satisfies $\Delta\mathcal{R}_n = n - 2$. Note that, by these conditions, the graph $\mathcal{R}_n$ is determined uniquely up to an isomorphism. For another graph $\mathcal{R}'_n$ with these properties, choose a vertex x in $\mathcal{R}_n$ and a vertex x' in $\mathcal{R}'_n$ and set $\varphi(x) := x'$. Then, by assumption, there are unique vertices y in $\mathcal{R}_n$ and y' in $\mathcal{R}'_n$, which are not connected to x respectively x'. Set $\varphi(y) := y'$. Then, again choose new vertices in $\mathcal{R}_n$ and $\mathcal{R}'_n$ and continue the described procedure. After finitely many steps, we arrive at a graph isomorphism φ between $\mathcal{R}_n$ and $\mathcal{R}'_n$.

Formally, we can assume that the vertex and edge set of $\mathcal{R}_n$ is given by $\{1, \dots, n\}$ and $\{\{1, 2\}, \dots, \{1, n-1\}, \{2, 3\}, \dots, \{2, n-2\}, \{2, n\}, \{3, 4\}, \dots\}$.

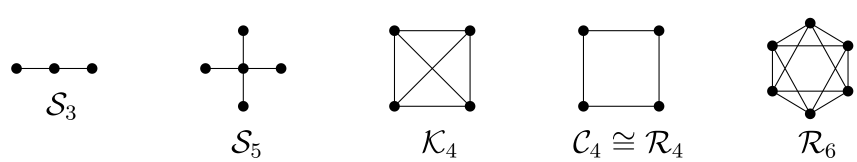


Figure 2.1: Some of the graphs we will study in this article.

By definition of our deposition model, each vertex x interacts with the growth of the height process only via its (closed) neighborhood $[x]$. Therefore, we call vertices x, $y \in V$ equivalent in $\mathcal{G}$ if $[x] = [y]$. The graph, which arises from $\mathcal{G}$ by identifying all equivalent vertices, will be denoted by $\hat{\mathcal{G}}$ and called an irreducible graph. Observe that the asymptotic growth in our model does not change if we replace $\mathcal{G}$ by $\hat{\mathcal{G}}$ and modify the intensity of the underlying Poisson processes accordingly. More precisely, the intensity of the Poisson process associated with a vertex $\hat{x}$ of $\hat{\mathcal{G}}$ has to equal the number of vertices in $\mathcal{G}$ that have been contracted into $\hat{x}$.

Also, note that we can reverse this transformation. Assume we are given a graph with positive integer intensities for all vertices. Then, we can stepwise choose the vertices with no unit intensity, define a new adjacent vertex with unit intensity and the same closed neighborhood, and reduce the intensity of the previously chosen vertex by one. We will use the term vertex cloning for this procedure. Again note that the order in which the vertices are chosen does not affect the resulting graph up to an isomorphism.

Example 2.1. For the butterfly graph $\mathcal{B}$, we obtain the following.

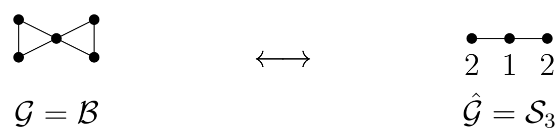


Figure 2.2: The height in our ballistic deposition process on the butterfly graph $\mathcal{B}$ is equivalent to the height of a modified deposition model on $\mathcal{S}_3$.

All in all, for studying the asymptotic growth in our ballistic deposition model, the following three settings are essentially the same.

(i) Arbitrary graphs $\mathcal{G}$ with unit intensities.
(ii) Arbitrary graphs $\mathcal{G}$ with positive integer intensities.
(iii) Irreducible graphs $\hat{\mathcal{G}}$ with positive integer intensities.

Observe that, when working in the setting (ii), the asymptotic growth rate changes linearly if we multiply all intensities by a fixed constant. Translating this into our original setting (i) therefore yields the following construction.

Construction 2.2. *For any graph $\mathcal{G}$ and $n \in \mathbb{N}$, there is a graph $\mathcal{H}$ with*

$$\gamma(\mathcal{H}) = n\ \gamma(\mathcal{G}).$$

Moreover, we can construct such a graph $\mathcal{H}$ by starting with $\mathcal{G}$ and then cloning each vertex exactly $n-1$ times.

Example 2.3. Let us illustrate Construction 2.2 for $n = 2$ and $\mathcal{G} = \mathcal{S}_3$.

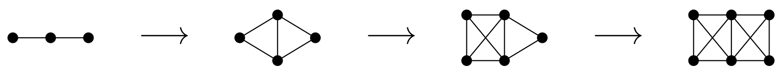


Figure 2.3: Successively, the vertices of $\mathcal{S}_3$ are cloned. Initially, we start with the dominant vertex, but the order in this procedure does not influence the resulting graph $\mathcal{H}$ up to an isomorphism. We know $\gamma(\mathcal{H}) = 2\gamma(\mathcal{S}_3)$.

2.3 On the Sequence of Star Graphs

Let $n \geq 3$ and consider the graph $\mathcal{S}_n$. Then, both in discrete and in continuous time, by stopping our deposition process at the moments at which the height of the dominant vertex is increased, we arrive at a process with i.i.d. increments. Hence, we can express $\gamma(\mathcal{S}_n)$ by identifying both the expectation of these increments and the mean waiting time between consecutive stops. In discrete-time, the height of the dominant vertex increases after a geometrically distributed time waiting time of mean n. From the deposition rule (2.1) and equation (2.3), we therefore obtain

$$\gamma(\mathcal{S}_n) = \sum_{k=0}^{\infty} \frac{1}{n}\left(1 - \frac{1}{n}\right)^k \sum_{\substack{r_1,\dots,r_{n-1} \geq 0 \\ r_1+\dots+r_{n-1}=k}} \frac{1}{(n-1)^k} \binom{k}{r_1,\dots,r_{n-1}} \left[1 + \max_j r_j\right],$$

which can be simplified into

$$\gamma(\mathcal{S}_n) = 1 + \frac{1}{n}\sum_{k=1}^{\infty} \frac{a_{n-1,k}}{n^k}, \quad a_{n,k} := \sum_{\substack{r_1,\dots,r_n \geq 0 \\ r_1+\dots+r_n=k}} \binom{k}{r_1,\dots,r_n} \max_{j=1,\dots,n} r_j. \quad (2.5)$$

In continuous time, the height of the dominant vertex increases after an exponentially distributed time W of mean one. By using (2.2), we get

$$\gamma(\mathcal{S}_n) = 1 + \mathbb{E}\left[\max_{j=1,\dots,n-1} U_{j,W}\right], \quad (2.6)$$

where we assume, for all $j \in \mathbb{N}$ and $\lambda \in (0, \infty)$, that the random variable $U_{j,\lambda}$ is Poisson distributed with mean λ. Moreover, here we suppose that the family $(U_{j,\lambda})_{j\in\mathbb{N},\lambda\in(0,\infty)}$ is independent itself and independent of W.

Interestingly, we can determine the exact value of $\gamma(\mathcal{S}_3)$ by working directly with equation (2.5). For all $k \geq 0$, we have

$$\begin{aligned} a_{2,2k} &= \sum_{l=0}^{2k} \binom{2k}{l} \max\{l, 2k-l\} = 2 \sum_{l=k+1}^{2k} \binom{2k}{l} l + k \binom{2k}{k} \\ &= 4k \sum_{l=k}^{2k-1} \binom{2k-1}{l} + k \binom{2k}{k} = k 2^{2k} + k \binom{2k}{k}. \end{aligned}$$

In the same way, we obtain

$$a_{2,2k+1} = 2(2k+1) \sum_{l=k}^{2k} \binom{2k}{k} = (2k+1)\, 2^{2k} + (2k+1) \binom{2k}{k}.$$

These two formulas allow us to directly verify the recurrence relation

$$a_{2,k} = 2 \, \frac{k}{k-1} \, a_{2,k-1} + 4 \, \frac{k-3}{k-2} \, a_{2,k-2} - 8 \, a_{2,k-3} \quad \text{for all } k \geq 3.$$

By neglecting the last term in this recursion, or by Stirling's approximation, we can verify that the generating function $g(s) := \sum_{k=3}^{\infty} \frac{a_{2,k}}{k} s^k$ is finite for all $s \in (0,1)$ small enough. Moreover, by using the recurrence relation and inserting $a_{2,0} = 0$, $a_{2,1} = 2$ and $a_{2,3} = 6$, it follows that the generating function $g = g(s)$ satisfies the differential equation

$$s \cdot g'(s) = 2s \left\{ (s \cdot g(s))' + 9s^2 \right\} + 4s^4 \left\{ (s^{-1} \cdot g(s))' + 3 \right\} - 8s^4 \left\{ g'(s) + 2 + 6s \right\}.$$

Using the initial condition $g(0) = 0$, we obtain, for all $s \in (0,1)$ small enough,

$$g(s) = \frac{2s^2(1+6s) - 1 + \sqrt{1-4s^2}}{2-4s}.$$

By monotone convergence, this formula for $g(s)$ is true for all $s \in [0, 1/2)$, and by equation (2.5), we can deduce

$$\gamma(\mathcal{S}_3) = 1 + \frac{1}{3}\left(\frac{2}{3} + \frac{6}{9}\right) + \frac{1}{9} \cdot g'\left(\frac{1}{3}\right) = 2 + \frac{1}{\sqrt{5}}.$$

Remarks. 1. The sequence $(a_{2,k})_{k\geq 1}$ is mentioned in the OEIS under A230137.

2. The series representation (2.5) is hard to work with, in general, but at least allows rather precise calculations. We obtain, for example,

$$\gamma(\mathcal{S}_4) = 2.72446357391224888\dots\ .$$

We could not find an integer coefficient polynomial, which might have this value as a root. Hence, we conjecture that $\gamma(\mathcal{S}_4)$ is transcendental.

Proposition 2.4. *There are non-isomorphic graphs $\mathcal{G}$, $\mathcal{H}$ with $\gamma(\mathcal{G}) = \gamma(\mathcal{H})$.*

Proof. For the butterfly graph $\mathcal{B}$, we find by a similar calculation

$$\gamma(\mathcal{B}) = 1 + \frac{1}{5}\sum_{k=1}^{\infty} a_{2,k}\left(\frac{2}{5}\right)^k = 1 + \frac{1}{5}\left(\frac{4}{5} + \frac{24}{25} + \frac{2}{5}\cdot g'\left(\frac{2}{5}\right)\right) = \frac{11}{3}.$$

Consequently, by applying Construction 2.2 with $n = 3$ to $\mathcal{B}$, we obtain a graph $\mathcal{H}$ with $\gamma(\mathcal{H}) = 11 = \gamma(\mathcal{K}_{11})$, which is not isomorphic to $\mathcal{K}_{11}$. □

We have the following combinatorial interpretation for equation (2.5). Assume there are n bins and m balls. Then we throw the balls independently of each other in one uniformly chosen bin. Denote by $Z_{n,m}$ the number of balls in the maximally loaded bin and let Y_n be a random variable, which is independent of $(Z_{n,m})_{m\geq 1}$ and geometrically distributed with mean n. Then, equation (2.5) reads as

$$\gamma(\mathcal{S}_n) = 1 + \mathbb{E}\left[Z_{n-1,Y_{n-1}}\right].$$

Properties of the random variable $Z_{n,m}$ for n, $m \to \infty$ and similar models have been studied by various authors, see, for example, [25], [82], and [69]. If the ratio $\lambda := m/n$ is assumed to be constant for $n \to \infty$, many properties of $Z_{n,m}$ can be deduced from the Poisson approximation. In [3, Theorem 1], Anderson proved that, for any $\lambda \in (0,\infty)$, there exists an integer sequence $(I_n)_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} \mathbb{P}\Big[\max_{j=1,\dots,n} U_{j,\lambda} \in \{I_n, I_n+1\}\Big] = 1.$$

For two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we write $a_n \sim b_n$ for $n \to \infty$ to denote $\lim_{n\to\infty} a_n b_n^{-1} = 1$. In [59], Kimber proved that, for all $\lambda \in (0,\infty)$, one can choose the sequence $(I_n)_{n\geq 0}$ so that $I_n \sim \log(n)\{\log(\log(n))\}^{-1}$ for $n \to \infty$, and furthermore

$$1 - \mathbb{P}\Big[\max_{j=1,\dots,n} U_{j,\lambda} \in \{I_n, I_n+1\}\Big] \sim \left(\frac{\lambda \log(\log(n))}{\log(n)}\right)^{1+B_n} \quad \text{for } n \to \infty,$$

where $(B_n)_{n\geq 1}$ is a sequence, which is dense in $[-1/2, 1/2]$ and, in general, may depend on λ. In [34], Gonnet used the Poisson approximation to study the asymptotic expectation value of $Z_{n,m}$ for a constant ratio $\lambda = m/n$ as n, $m \to \infty$. In [34, Section 4], he verified that, for fixed $\lambda \in (0, \infty)$,

$$f_n(\lambda) := \mathbb{E}\left[\max_{j=1,\dots,n} U_{j,\lambda}\right] \sim \frac{\log(n)}{\log(\log(n))} \quad \text{for } n \to \infty. \tag{2.7}$$

Arguably, this result is counterintuitive, since due to the convolution property of the Poisson distribution, $f_n(\lambda)$ is strictly increasing in λ and one could expect a linear dependency with respect to λ in (2.7). But, for $\lambda_1, \lambda_2 \in (0, \infty)$ with $\lambda_1 < \lambda_2$, we can verify $f_n(\lambda_1) \sim f_n(\lambda_2)$ for $n \to \infty$ as follows.

For example, by [60, Proposition 1], for any $\beta > 1$, there is $N \in \mathbb{N}$ with

$$\mathbb{P}[U_{1,\lambda_1} > n] > e^{-\lambda_1} \frac{\lambda_1^{n+1}}{(n+1)!} > \beta\, e^{-\lambda_2} \frac{\lambda_2^{\lfloor \beta n+1 \rfloor}}{\lfloor \beta n + 1 \rfloor !} \geq \mathbb{P}[U_{1,\lambda_2} > \beta n] \text{ for all } n \geq N.$$

Consequently, by stochastic dominance, $f_n(\lambda_2) \leq N + \beta f_n(\lambda_1)$ for all $n \geq N$, provided $N \in \mathbb{N}$ is large enough. As $f_n(\lambda_1) \to \infty$ for $n \to \infty$, by letting $\beta \to 1$, we can indeed deduce $f_n(\lambda_1) \sim f_n(\lambda_2)$ as $n \to \infty$.

Proposition 2.5.

$$\gamma(\mathcal{S}_n) \sim \frac{\log(n)}{\log(\log(n))} \quad \textit{as } n \to \infty.$$

In particular, $\gamma(\mathcal{S}_n) \to \infty$ *for* $n \to \infty$.

Proof. The convolution property of the Poisson distribution implies that the functions $f_n = f_n(\lambda)$, $n \geq 1$, are both monotone increasing and subadditive.

Let $r \in (0, \infty)$ and recall equation (2.6). Then, due to monotonicity,

$$\gamma(\mathcal{S}_n) = 1 + \int_0^\infty e^{-\lambda} f_{n-1}(\lambda)\ \mathrm{d}\lambda \geq \int_r^\infty e^{-\lambda} f_{n-1}(r)\ \mathrm{d}\lambda = e^{-r} f_{n-1}(r).$$

Now, apply (2.7) and let $r \to 0$ to conclude

$$\liminf_{n\to\infty} \gamma(\mathcal{S}_n) \frac{\log(\log(n))}{\log(n)} \geq 1.$$

On the other hand, for all $r \in (0, \infty)$, we have the estimate

$$\begin{aligned}\gamma(\mathcal{S}_n) &\leq 1 + \int_0^r e^{-\lambda} f_{n-1}(r) \, \mathrm{d}\lambda + \int_r^\infty e^{-\lambda} f_{n-1}(\lambda) \, \mathrm{d}\lambda \\ &= 1 + \left(1 - e^{-r}\right) f_{n-1}(r) + e^{-r} \int_0^\infty e^{-\lambda} f_{n-1}(\lambda + r) \, \mathrm{d}\lambda.\end{aligned}$$

Hence, by using the subadditivity of f_{n-1} and equation (2.6), we obtain

$$\gamma(\mathcal{S}_n) \leq 1 + f_{n-1}(r) + e^{-r} \int_0^\infty e^{-\lambda} f_{n-1}(\lambda) \, \mathrm{d}\lambda \leq 1 + f_{n-1}(r) + e^{-r}\gamma(\mathcal{S}_n).$$

By applying (2.7) and letting $r \to \infty$, we therefore conclude

$$\limsup_{n\to\infty} \gamma(\mathcal{S}_n) \frac{\log(\log(n))}{\log(n)} \leq 1. \qquad \square$$

2.4 Calculation of $\gamma(\mathcal{G})$ for Specific Graphs

Theorem 2.6. *Let $N \in \mathbb{N}_0$, n, $m \in \mathbb{N}$. Assume m is even, and $N \geq 1$ in the case of $m = 2$. Let $\mathcal{G} = (V, E)$ denote the graph, which arises from $\mathcal{R}_m$ by executing the following procedure.*

(i) *Clone each vertex of $\mathcal{R}_m$ exactly $n - 1$ times.*
(ii) *Add exactly N new vertices $x_1, \ldots, x_N$ to V.*
(iii) *Add to E all edges of the form $\{x_i, y\}$, where $y \in V \setminus \{x_i\}$.*

Note that $\#V = N + nm$. Set $\kappa := \frac{\#V}{2n}$ and $\tau := (\sqrt{\kappa^2 - 1} - \kappa + 1)^{-1}$. Then,

$$\gamma(\mathcal{G}) = \#V - \frac{n^2 m}{\#V} \tau \left\{ \tau + \frac{1}{2\kappa} \right\}^{-1}.$$

Proof. We define a stochastic process $(\Delta_n)_{n\geq 0}$ as follows. If at time $n \geq 0$, the height of one dominant vertex x_j, $j = 1, \ldots, n$, is maximal, then set $\Delta_n := 0$. Otherwise, there are at most two different vertices that share the maximal height. If there are two different vertices whose height is maximal, again $\Delta_n := 0$. Finally, assume $x \in V$ is the unique vertex of maximal height and not dominant in $\mathcal{G}$. Then, choose a vertex $y \in V$, whose height is maximal under all vertices, which are not equivalent to x. Then, we define $\Delta_n := H_{x,n} - H_{y,n}$.

It is not hard to see that the process $(\Delta_n)_{n\geq 0}$ is a time-homogeneous Markov chain. Assume, for example, that $\Delta_n = m$ for some $n \in \mathbb{N}$ and $m \geq 3$. Denote by x be the unique vertex of maximal height and by y the vertex, whose height is increased in the next step. Then, we know $\Delta_{n+1} = 0$ if y is dominant. If y is equivalent to x, we can conclude $\Delta_{n+1} = m + 1$. If x and y do not share an edge, we know $\Delta_{n+1} = m - 1$ instead. Finally, if x and y are connected by an edge, not equivalent, and y is not dominant in $\mathcal{G}$, then $\Delta_{n+1} = 1$.

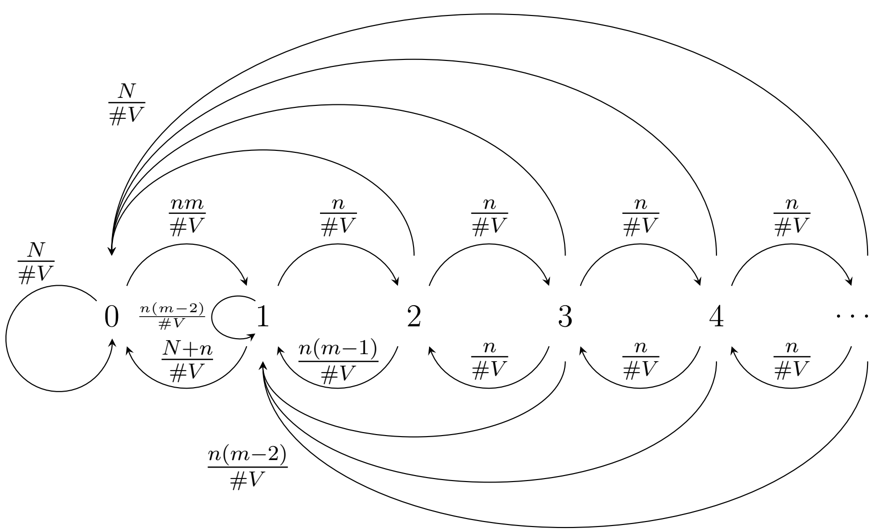


Figure 2.4: Transition probabilities of the Markov chain $(\Delta_n)_{n\geq 0}$.

The Markov chain $(\Delta_n)_{n\geq 0}$ is positive recurrent and we can calculate its stationary solution $\Pi = (\Pi(n))_{n\geq 0}$. From the recurrence relation

$$\Pi(n) = \frac{1}{2\kappa}\left(\Pi(n-1) + \Pi(n+1)\right) \quad \text{for all } \; n \geq 2,$$

we deduce the representation

$$\Pi(n) = c_1 \left(\kappa - \sqrt{\kappa^2 - 1}\right)^{n-1} \quad \text{for all } n \geq 1,$$

where $c_1 \in (0,\infty)$ is a fixed constant. Using the equation

$$\Pi(0) = \frac{N}{\#V} + \frac{\Pi(1)}{2\kappa} = \frac{N}{\#V} + \frac{c_1}{2\kappa},$$

as well as

$$\Pi(0) = 1 - \sum_{n\geq 1} \Pi(n) = 1 - c_1 \sum_{n=0}^{\infty} \left(\kappa - \sqrt{\kappa^2 - 1}\right)^n = 1 - c_1\tau, \tag{2.8}$$

we can identify

$$c_1 = \left(1 - \frac{N}{\#V}\right)\left\{\tau + \frac{1}{2\kappa}\right\}^{-1} = \frac{nm}{\#V}\left\{\tau + \frac{1}{2\kappa}\right\}^{-1}. \tag{2.9}$$

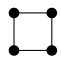
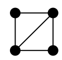
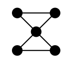
Observe that the transitions of $(\Delta_n)_{n\geq 0}$ yield information on the growth of the maximal height in $\mathcal{G}$. Each transition from a state $k \neq 1$ to 0 implies that the maximal height increases by one. The same also holds for all transitions from a state k to $k+1$ and all transitions from $k \neq 2$ to 1. On the other hand, we know that a transition from $k \geq 3$ to $k-1$ will not increase the maximal height. For the transitions from 1 to 0 and the transition from 2 to 1, we do not know for sure whether the maximal height increases. However, independent of the past, the conditional probability for this is $N/(N+n)$ respectively $(m-2)/(m-1)$.

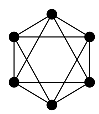
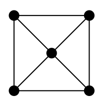
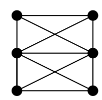
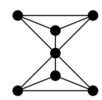
By applying Birkhoff's ergodic theorem to the snake chain $(\Delta_n, \Delta_{n+1})_{n\geq 0}$ and using equation (2.3), we obtain the following expression for $\gamma(\mathcal{G})$.

$$\begin{aligned}\gamma(\mathcal{G}) = \#V \Bigg(&\Pi(0) + \sum_{n\geq 1}\Pi(n)\frac{n}{\#V} + \sum_{n\geq 2}\Pi(n)\frac{N}{\#V} + \sum_{1\leq n\neq 2}\Pi(n)\frac{n(m-2)}{\#V} \\ &+ \Pi(1)\frac{N+n}{\#V}\frac{N}{N+n} + \Pi(2)\frac{n(m-1)}{\#V}\frac{m-2}{m-1}\Bigg) \\ = \#V &\left(\Pi(0) + \sum_{n\geq 1}\Pi(n)\frac{N+n(m-1)}{\#V}\right) = \#V - n\left(1 - \Pi(0)\right).\end{aligned}$$

The claim now follows by inserting (2.8) and (2.9). $\square$

Example 2.7.

$\mathcal{G}$				
$\gamma(\mathcal{G})$	$2+\frac{1}{\sqrt{5}}$	$2+\frac{2}{\sqrt{3}}$	$3+\frac{1}{\sqrt{3}}$	$\frac{11}{3}$
(N,n,m)	$(1,1,2)$	$(0,1,4)$	$(2,1,2)$	$(1,2,2)$

$\mathcal{G}$				
$\gamma(\mathcal{G})$	$3+\frac{3}{\sqrt{2}}$	$3+\frac{2\sqrt{21}}{7}$	$4+\frac{2}{\sqrt{5}}$	$4+\frac{3}{\sqrt{13}}$
(N,n,m)	$(0,1,6)$	$(1,1,4)$	$(2,2,2)$	$(1,3,2)$

2.5 A Central Limit Theorem around $\gamma(\mathcal{G})$

In order to state the main result of this section, let us introduce some notation. We write $Z_n \Longrightarrow Z$ for $n \to \infty$ to denote convergence in distribution. For $\sigma^2 \in [0,\infty)$ we write $N(0,\sigma^2)$ for the centered normal distribution with variance σ^2. In the case of $\sigma^2 = 0$, we identify this law with a Dirac measure.

Theorem 2.8. *For any graph $\mathcal{G} = (V,E)$, there is $\sigma^2 = \sigma^2(\mathcal{G}) \in [0,\infty)$ with*

$$\frac{\max_{x\in V} H_{x,n} - n\,\frac{\gamma(\mathcal{G})}{\#V}}{n^{1/2}} \Longrightarrow N(0,\sigma^2) \quad \textit{for } n \to \infty.$$

The same central limit theorem with the same constant σ^2 also holds if we replace $\max_{x\in V} H_{x,n}$ *with* $\min_{x\in V} H_{x,n}$. *Moreover,*

$$\sigma^2(\mathcal{G}) = 0 \quad \textit{if and only if} \quad \mathcal{G} \textit{ is isomorphic to } \mathcal{K}_{\#V}.$$

The results of this section are based on the study of the surface process $\delta = (\delta_n)_{n\geq 0}$, which is defined by

$$\delta_n := (\delta_{x,n})_{x\in V}\,, \qquad \delta_{x,n} := H_{x,n} - \min_{y\in V} H_{y,n}. \tag{2.10}$$

The process $(\delta_n)_{n\geq 0}$ is a time-homogeneous Markov chain, which can be seen as follows. Define an equivalence relation on the state-space of $(H_n)_{n\geq 0}$ by identifying two different height vectors if and only if the height difference is the same for all vertices. Then, the transition probabilities of $(H_n)_{n\geq 0}$ from one equivalence class to another do not depend on the representative of the former one. So, we obtain a Markov chain on the set of equivalence classes, which can be identified with $(\delta_n)_{n\geq 0}$ after describing each equivalence class by its unique normalized representative.

By modifying the start height H_0 respectively the initial state δ_0, we can ensure that the Markov chain $(\delta_n)_{n\geq 0}$ is irreducible and has the state-space

$$S := \left\{ (h_x)_{x\in V} \in \mathbb{N}_0^V \;\middle|\; h_x = 0 \text{ for a } x \in V \text{ and } h_y \neq h_z \text{ for all } \{y,z\} \in E \right\}.$$

We can describe the transition probabilities of $(\delta_n)_{n\geq 0}$ in the following way. Let $h = (h_x)_{x\in V} \in S$, $y \in V$, and set $m_y := \min\{h_x \mid x \neq y\}$. Then, we know

$$\mathbb{P}\big[\delta_{n+1} = \tilde{h} \mid \delta_n = h\big] = \frac{1}{\#V}, \quad \tilde{h}_x := \begin{cases} h_x - m_y, & x \neq y \\ 1 + \max_{z\in[y]} h_z - m_y, & x = y \end{cases}. \tag{2.11}$$

Observe that, for the transition from h to $\tilde{h}$, there is a unique vertex $x \in V$ with $\tilde{h}_x > h_x$, and this vertex is given by $x = y$. In particular, given an h, the state $\tilde{h}$ in (2.11) is uniquely determined by the choice of $y \in V$ and vice versa, and all non-zero transition probabilities of $(\delta_n)_{n\geq 0}$ are given by (2.11).

We will prove Theorem 2.8 by applying renewal arguments to $(\delta_n)_{n\geq 0}$ and making use of a random index central limit theorem. For fixed $h \in S$, we define the sequence of hitting times of h by

$$\tau_1^h := \inf\{n \geq 0 \mid \delta_n = h\}, \qquad \tau_{k+1}^h := \inf\left\{n > \tau_k^h \mid \delta_n = h\right\}, \quad k \in \mathbb{N}.$$

Before starting to analyze the Markov chain $(\delta_n)_{n\geq 0}$ formally, let us briefly mention a simple but rather important observation. Denote by $S_0 \subseteq S$ the set of all $h' \in S$ with $\max_{x\in V} h'_x \leq \#V$. Let $h \in S$ and $(x_1, \dots, x_{\#V})$ be a non-decreasing permutation of V. Moreover, assume that $h_{x_1} = \max_{y\in V} h_y$. Then, if $\delta_n = h$ for some $n \geq 0$, and in the following steps, the height of the vertices $x_1, \dots, x_{\#V}$ are increased one after the other exactly one time per vertex, it follows that $\delta_{n+\#V} \in S_0$. So, we say that $(x_1, \dots, x_{\#V})$ resets h.

Lemma 2.9. *Fix $\mathcal{G} = (V, E)$. Then, for all $h \in S$, there is an $n \in \mathbb{N}$ with*

$$\mathbb{P}\Big[\delta_n = h \mid \delta_0 = \tilde{h}\Big] \geq (\#V)^{-n} \quad \textit{for all } \tilde{h} \in S. \tag{2.12}$$

Consequently, each random variable τ_1^h, $h \in S$, has an exponential moment, which is finite for any initial distribution on S. In particular, the Markov chain $(\delta_n)_{n\geq 0}$ is positive recurrent and has a stationary solution π.

Proof. Let $h \in S$. Since S_0 is a finite subset of S, there is an $n_0 \in \mathbb{N}$ such that, for all $h' \in S_0$, the chain $(\delta_n)_{n\geq 0}$ can go from h' to h in $n(h') \leq n_0$ steps. Set $n := n_0 + \#V$. Let $\tilde{h} \in S$ be given. Choose $x_1 \in V$ with $h_{x_1} = \max_{y\in V} h_y$ and a non-decreasing permutation $(x_1, \dots, x_{\#V})$ of V. Denote by $h' \in S_0$ the unique element, which arises when $\tilde{h}$ is reset according to $(x_1, \dots, x_{\#V})$ and $m := n_0 - n(h') \in \mathbb{N}_0$. Now assume that $\delta_0 = \tilde{h}$ and, in the first m steps, only the height of x_1 increases. Then we arrive at a state, which again can be reset according to $(x_1, \dots, x_{\#V})$. Hence, we can go from $\tilde{h}$ to h' in $m + \#V$ steps, and as $(\delta_n)_{n\geq 0}$ may go from h' to h in $n(h')$ steps, the claim follows. $\square$

The following lemma ensures that the Markov chain $(\delta_n)_{n\geq 0}$ contains enough information about the growth of the process $(H_n)_{n\geq 0}$.

Lemma 2.10. *There are function $g_1 : S \times S \to \mathbb{N}_0^V$, $g_2 : S \times S \to \{0, 1\}$ with*

$$
\begin{aligned}
g_1(\delta_n, \delta_{n+1}) &= H_{n+1} - H_n \quad \textit{and} \\
g_2(\delta_n, \delta_{n+1}) &= \max_{x\in V} H_{x,n+1} - \max_{y\in V} H_{y,n} \quad \textit{almost surely.}
\end{aligned}
$$

Proof. The main step is to define $g_1(h, \tilde{h})$, as, given g_1, we can construct g_2, for example, by the formula

$$
g_2(h, \tilde{h}) := \sum_{x\in V} \mathbf{1}_{(g_1(h,\tilde{h}))_x > 0} \; \mathbf{1}_{\max\limits_{y\in[x]} h_y = \max\limits_{z\in V} h_z}.
$$

For the definition of g_1, recall the description of the transition probabilities of $(\delta_n)_{n\geq 0}$ given above in (2.11). Each transition of $(\delta_n)_{n\geq 0}$ corresponds to an increase of the value of one uniquely vertex, and clearly, this also holds for all transitions of our deposition process $(H_n)_{n\geq 0}$. By recalling our definition of $(\delta_n)_{n\geq 0}$ in (2.10), it is clear that these two vertices are always the same. Now, let h, $\tilde{h} \in S$, and $y \in V$ be as in (2.11). Then, we can define g_1 by

$$
\big(g_1(h, \tilde{h})\big)_x := \begin{cases} 1 + \max\limits_{z\in[x]} h_z - h_x, & x = y \\ \\ 0, & x \neq y. \end{cases}
$$

$\square$

Our last ingredient for the proof of Theorem 2.8 is a rather simple inequality. For its proof, we use the counterpart of the process $(\delta_n)_{n\geq 0}$ in our continuous-time deposition process. For all $t \in [0, \infty)$, we define

$$\tilde{\delta}_t := (\tilde{\delta}_{x,t})_{x\in V}, \qquad \text{where} \quad \tilde{\delta}_{x,t} := \tilde{H}_{x,t} - \min_{y\in V} \tilde{H}_{y,t}, \quad x \in V.$$

Roughly speaking, all previously mentioned arguments and results for $(\delta_n)_{n\geq 0}$ also hold for $(\tilde{\delta}_t)_{t\geq 0}$ with only minor changes.

Proposition 2.11. *Let $\mathcal{G} = (V, E)$, $\mathcal{G}' = (V', E')$ be two given graphs and assume that $\mathcal{G}$ is a subgraph of $\mathcal{G}'$. Then $\gamma(\mathcal{G}) \leq \gamma(\mathcal{G}')$, and*

$$\gamma(\mathcal{G}) = \gamma(\mathcal{G}') \quad \textit{if and only if} \quad \mathcal{G} = \mathcal{G}'.$$

Proof. We couple our continuous-time ballistic deposition processes on $\mathcal{G}$ and $\mathcal{G}'$ by assuming that they share the same underlying Poisson processes $(\xi_x)_{x\in V}$. This directly yields the inequality $\gamma(\mathcal{G}) \leq \gamma(\mathcal{G}')$.

Now assume $\mathcal{G} \neq \mathcal{G}'$, and let us prove $\gamma(\mathcal{G}) < \gamma(\mathcal{G}')$. For this purpose, note that, by induction over $\#V$, it suffices to consider the following two cases.

(i) $V' = V$ and $E' = E \cup \{\{x, y\}\}$ for a suitable choice of x, $y \in V$.

(ii) $V' = V \cup \{x'\}$ for a $x' \notin V$ and $E' = E \cup \{\{x, x'\}\}$ for a $x \in V$.

It turns out that both cases can be treated roughly in the same way, and we, therefore, start and mainly concentrate on the case (ii).

To verify (ii), we construct a new growth model on $\mathcal{G}'$, which evolves asymptotically faster than our deposition model on $\mathcal{G}$ and at most as fast as the deposition process on $\mathcal{G}'$. For this, let $\xi_{x'}$ be the Poisson process related to the vertex x' in the latter one. Moreover, let $(\tilde{\delta}_t)_{t\geq 0}$ be the time-continuous surface process of the ballistic deposition on $\mathcal{G}$. Fix $h \in S$ with $h_x = \max_{y\in V} h_y$ and a non-decreasing permutation $(x_1, \ldots, x_{\#V})$ with $x_1 = x$.

Our new growth process on $\mathcal{G}'$ arises by modifying the deposition rule (2.1). We will take the possible growth events of the vertex x' as well as the influence of x' on its neighbor x only into account if the current height fluctuations behave in a specific way. More precisely, the influence of x' at a point in time is only taken into account if both the Markov chain $(\tilde{\delta}_t)_{t\geq 0}$ is in the state h, and then, in the following, the first Poisson process, who jumps, is $\xi_{x'}$, followed by a jump of ξ_{x_1}, ξ_{x_2}, and so on until $\xi_{x_{\#V}}$ has jumped. After such an event has occurred, we again neglect the possible growth of x' or its influence on the growth of $x = x_1$ until the next time both $(\tilde{\delta}_t)_{t\geq 0}$ is in state h and, subsequently, the Poisson processes behave accordingly.

By definition, it is clear that the height of our new growth process is always smaller than in our original ballistic deposition process on $\mathcal{G}'$ since, in the latter, the influence of the vertex x' is always taken into account.

On the other hand, the maximal height in our new process always exceeds the maximal height of our ballistic deposition on $\mathcal{G}$, since, in this process, the vertex x' is always neglected. However, by construction, we know that the maximal height of our new process at a time $t \in (0, \infty)$ is always at least as big as the maximal height in our ballistic deposition on $\mathcal{G}$ plus the number of visits of $(\tilde{\delta}_t)_{t\geq 0}$ in h up to time t, which have been followed by the above-mentioned behavior of the underlying Poisson processes. Since $(\tilde{\delta}_t)_{t\geq 0}$ is a positive recurrent Markov chain and irreducible on S, Birkhoff's ergodic theorem yields that this second contribution strictly increases the asymptotic growth rate. Consequently, $\gamma(\mathcal{G}) < \gamma(\mathcal{G}')$.

Case (i) can be treated roughly in the same way as (ii). Instead of taking the growth of the vertex x' and its influence on x into account only sometimes, one now has to handle the influence of the edge $\{x, y\}$ in a similar way. $\square$

Proof of Theorem 2.8. By Lemma 2.10, we have the representation

$$R_n := \max_{x\in V} H_{x,n} - n\,\frac{\gamma(\mathcal{G})}{\#V} = \max_{x\in V} H_{x,0} + \sum_{k=1}^{n} f(\delta_{k-1}, \delta_k), \quad n \geq 0, \tag{2.13}$$

where $f(h, h') := g_2(h, h') - \frac{\gamma(\mathcal{G})}{\#V}$. Fix $h \in S$ and assume $H_0 := \delta_0 := h$. Consider the sequence $(W_n)_{n\geq 1}$ defined by

$$W_n := f\left(\delta_{\tau_n^h}, \delta_{\tau_n^h+1}\right) + \cdots + f\left(\delta_{\tau_{n+1}^h-1}, \delta_{\tau_{n+1}^h}\right).$$

The random variables $(W_n)_{n\geq 1}$ are i.i.d. by construction. Besides, since clearly $-1 \leq f \leq 1$, Lemma 2.9 yields

$$0 \leq \tilde{\sigma}^2 := \mathbb{E}\left[W_1^2\right] \leq \mathbb{E}\left[(\tau_2^h - \tau_1^h)^2\right] < \infty.$$

Set $K_n := \sup\{k \in \mathbb{N} \mid \tau_k^h \leq n\}$. We will now verify the following statements.

A) $n^{-1/2}\, R_{\tau_{K_n}^h} \Longrightarrow N(0, \sigma^2)$ for $n \to \infty$, where $\sigma^2 := \pi(h)\, \tilde{\sigma}^2$.

B) $n^{-1/2} \left| R_n - R_{\tau_{K_n}^h} \right| \Longrightarrow 0$ for $n \to \infty$.

Once we have established A) and B), Slutsky's theorem immediately gives

$$n^{-1/2}R_n = n^{-1/2}R_{\tau^h_{K_n}} + n^{-1/2}\Big(R_n - R_{\tau^h_{K_n}}\Big) \Longrightarrow N(0,\sigma^2) \quad \text{for } n \to \infty$$

and hence verifies the first claim of Theorem 2.8.

To prove A), note that by Kac's theorem we know $\tau^h_n \sim \pi(h)^{-1}n$ for $n \to \infty$ almost surely and $K_n \sim \pi(h)n$ for $n \to \infty$. Moreover, we have

$$n^{-1/2}R_{\tau^h_{K_n}} = n^{-1/2}\sum_{k=1}^{K_n} W_k.$$

Hence, the claim follows from Anscombe's theorem, see [41, Theorem 2.3].

To prove B), consider the estimate

$$\Big|R_n - R_{\tau^h_{K_n}}\Big| \le \sup\Big\{\Big|R_k - R_{\tau^h_{K_n}}\Big|;\ k = \tau^h_{K_n}, \tau^h_{K_n}+1, \dots, \tau^h_{K_n+1}\Big\}.$$

Now, we apply the inequality

$$|R_k - R_l| \le \max\left\{\frac{\gamma(\mathcal{G})}{\#V}, 1 - \frac{\gamma(\mathcal{G})}{\#V}\right\}|k-l| \le |k-l|, \quad k,l \ge 1,$$

which allows us to deduce

$$n^{-1/2}\left|R_n - R_{\tau^h_{K_n}}\right| \le n^{-1/2}\left(\tau^h_{K_n+1} - \tau^h_{K_n}\right).$$

By the Markov property of $(\delta_n)_{n\ge 0}$, $\left(\tau^h_{K_n+1} - \tau^h_{K_n}\right)_{n\ge 1}$ is i.i.d. and B) follows.

Until now, we have verified that, for deterministically chosen initial state h of H_0 respectively δ_0, a central limit theorem holds. The following argument shows that σ^2 does not depend on the choice of h and also, that we may choose a random initial condition.

Consider two ballistic deposition processes on $\mathcal{G}$ with different deterministic initial values and couple them by assuming that, with each step, the height of the same vertex is increased. Then, by our deposition rule (2.1), the maximal height difference cannot increase over time. So, if a central limit theorem holds for one process, it also holds for the other one. The same arguments also allow us to extend the central limit theorem to an arbitrary initial distribution on S.

Let us now continue with the second claim of Theorem 2.8. We start by proving it under the assumption $\delta_0 \sim \pi$. Note that

$$n^{-1/2}\left(\min_{x\in V} H_{x,n} - n\frac{\gamma(\mathcal{G})}{\#V}\right) = n^{-1/2}\left(\max_{x\in V} H_{x,n} - n\frac{\gamma(\mathcal{G})}{\#V}\right) - \frac{\max_{x\in V}\delta_{x,n}}{n^{1/2}}. \quad (2.14)$$

As $\delta_0 \sim \pi$, we know that the distribution of $\max_{x\in V}\delta_{x,n}$ does not depend on n. Therefore, the claim follows by applying Slutsky's theorem to (2.14) and using the central limit theorem for the maximal height.

So far, we have verified the central limit theorem for the minimal height under the assumption $\delta_0 \sim \pi$. Since $\pi(h) > 0$ for all $h \in S$, we conclude as above that the central limit theorem holds for arbitrary initial distributions.

Let us now prove the last claim of Theorem 2.8. If $\mathcal{G}$ is isomorphic to a complete graph, then clearly $\sigma^2 = 0$. Assume that $\mathcal{G}$ is not isomorphic to $\mathcal{K}_{\#V}$. Recall that we know $\sigma^2 = \tilde{\sigma}^2\pi(h)$ from A). So, we will show $\tilde{\sigma}^2 = \mathbb{E}[W_1^2] > 0$.

Choose $x \in V$ with $h_x = \max_{y\in V} h_y$ and a non-decreasing permutation $(x_1, \ldots, x_{\#V})$ of V with $x_1 = x$. Denote by $\tilde{h} \in S$ the unique state, at which the chain $(\delta_n)_{n\geq 0}$ arrives after starting in h and being reset according to $(x_1, \ldots, x_{\#V})$. As $(\delta_n)_{n\geq 0}$ is irreducible, there is a finite path along which $(\delta_n)_{n\geq 0}$ may go from $\tilde{h}$ and h. Let $h_1, h_2, \ldots, h_N$ denote the path with $h_1 = h_N = h$, which arises by concatenation. Let $M \in \mathbb{N}$ be the number of returns of $(\delta_n)_{n\geq 0}$ to h along this path. Then, we know

$$\mathbb{P}\left[W_1 + \ldots + W_M = c\right] > 0, \quad \text{where } c := \sum_{j=1}^{N-1} f(h_j, h_{j+1}).$$

If $c \neq 0$, then clearly $\tilde{\sigma}^2 > 0$ and the claim holds. Therefore, suppose $c = 0$. Then, we continue with the construction of another path $h'_1, \ldots, h'_{N+1}$ along which $(\delta_n)_{n\geq 0}$ can go from $h'_1 := h$ to $h'_{N+1} := h$. For this purpose, let $h'_1 := h_1$, and note that the permutation $(x_1, \ldots, x_{\#V})$ still resets h'_1. We define $h'_2, \ldots, h'_{\#V+2}$ by using the resetting event and note that $h'_{\#V+2} = \tilde{h}$. Then, we consider the same path from $\tilde{h}$ to h as before and, therefore, we can set $h'_k := h_{k-1}$ for all $1 \leq k \leq N+1$. Let $M' \in \mathbb{N}$ be the number of returns of $(\delta_n)_{n\geq 0}$ to h along $h'_1, \ldots, h'_{N+1}$. Then, by construction,

$$\mathbb{P}\left[W_1 + \cdots + W_{M'} = c'\right] > 0, \quad \text{where } \; c' := \sum_{j=1}^{N} f(h'_j, h'_{j+1}).$$

By construction of our route $h'_1, \dots, h'_n$ and our assumption $c = 0$, we have

$$c' = \sum_{j=1}^{N} f(h'_j, h'_{j+1}) = \sum_{j=1}^{N-1} f(h_j, h_{j+1}) + 1 - \frac{\gamma(\mathcal{G})}{\#V} = 1 - \frac{\gamma(\mathcal{G})}{\#V}.$$

By Proposition 2.11, $\gamma(\mathcal{G}) < \gamma(\mathcal{K}_{\#V}) = \#V$, and thus we deduce $c' > 0$. Clearly, this implies $\tilde{\sigma}^2 = \mathbb{E}[W_1^2] > 0$, and consequently $\sigma^2 > 0$. □

Remark. In our proof of Theorem 2.8, we have identified some kind of renewal structure in form of the Markov chain $(\delta_n)_{n\geq 0}$ and the resetting of its states. In fact, one can also try to prove Theorem 2.8 by imposing the condition $\delta_0 \sim \pi$, which guarantees that the process $(R_n)_{n\geq 0}$ defined by (2.13) is stationary. For this approach, one needs to ensure both adequate moment and mixing conditions, compare [12, Theorem 27.4]. Let $Y_n := R_{n+1} - R_n$, $n \geq 0$. Then, $-1 \leq Y_n \leq 1$ almost surely, and, by Birkhoff's ergodic theorem, $\mathbb{E}_\pi[Y_n] = 0$. Moreover, equation (2.12) verifies a so-called Doeblin condition for the Markov chain $(\delta_n)_{n\geq 0}$, which implies geometric ergodicity, compare [64, Chapter 2]. This, in return, yields exponentially fast mixing, see [16, Theorem 3.7]. Alternatively, one can also use the Dobrushin coefficient of the Markov chain $(\delta_n)_{n\geq 0}$, compare [27, Chapter 3.4].

As a consequence of the central limit theorem for stationary processes, we can also deduce the representation

$$\sigma^2 = \mathrm{Var}_\pi[Y_1^2] + 2\sum_{k=2}^{\infty} \mathrm{Cov}_\pi[Y_1, Y_k] \in [0, \infty).$$

However, from this formula, it is not clear when $\sigma^2 > 0$. We were able to answer this question only by working directly with the Markov chain $(\delta_n)_{n\geq 0}$.

Still, it is worth mentioning that above mentioned moment and mixing conditions of $(Y_n)_{n\geq 0}$ do not only imply the classical central limit theorem but also its functional version, compare [48, Corollary 1], as well as a law of the iterated logarithm, see [83, Theorem 2 and further comment].

2.6 A General Upper Bound for $\gamma(\mathcal{G})$

The following estimate is based on the arguments used by Atar, Athreya, and Kang in [4] to derive the upper bound for $\gamma(\mathcal{C}_n)$ given in (2.4). Our result holds for arbitrary graphs but is somewhat less sharp, as we have simplified some of the rather technical arguments from [4] in our more general setting.

Theorem 2.12. *Let $\mathcal{G} = (V, E)$ be a given graph and ρ the spectral radius of $A(\mathcal{G}) + \mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix with index set V. Then,*

$$\gamma(\mathcal{G}) \leq e \cdot \rho.$$

Proof. For all $m \in \mathbb{N}$, we define

$$T_m := \inf \Big\{ t > 0 \mid \max_{x \in V} \tilde{H}_{x,t} = m \Big\}.$$

We modify our time-continuous deposition process in the following way. At time T_m, the height in each vertex is set equal to m. Then the process evolves as usual until the maximal height again hits a multiple of m. At this particular time, the height of each vertex of the graph increases until it is again equal to the maximal height. By continuing this procedure, we arrive at a model, which grows at least as fast as our original process. By the law of large numbers, we, therefore, deduce, for all $m \in \mathbb{N}$,

$$\gamma(\mathcal{G}) \leq \frac{m}{\mathbb{E}[T_m]}.$$

Applying Markov's inequality, for all $m \in \mathbb{N}$ and $a \in (0, \infty)$, we find

$$\mathbb{E}[T_m] \geq am\,(1 - \mathbb{P}[T_m \leq am])$$

and, consequently,

$$\gamma(\mathcal{G}) \leq \frac{1}{a\,(1 - \mathbb{P}[T_m \leq am])}. \tag{2.15}$$

Note that $T_m \leq am$ if and only if there exist vertices $x_1, \ldots, x_m \in V$ and $0 < t_1 < \ldots < t_m \leq am$ such that $x_{i+1} \in [x_i]$ for all $i = 0, \ldots, m-1$ and in each time interval (t_i, t_{i+1}) the height of the vertex x_i increases strictly. The number of tuples $(x_1, \ldots, x_{m+1})$ satisfying $x_{i+1} \in [x_i]$ for all $i = 0, \ldots, m$ is $\|(A(\mathcal{G}) + \mathbf{1})^m\|$, where the norm $\|\cdot\|$ is defined as the sum of the absolute value of all entries. Since $A(\mathcal{G}) + \mathbf{1}$ is a nonnegative irreducible matrix, by the Perron-Frobenius theorem,

$$\rho := \lim_{n \to \infty} \sqrt[n]{\|(A(\mathcal{G}) + \mathbf{1})^n\|} \in (0, \infty).$$

Hence, for all $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$, such that, for all $m \geq m_0$,

$$\mathbb{P}[T_m \leq am] \leq (\rho + \varepsilon)^m \mathbb{P}\left[S_m \leq am\right],$$

where $S_m = \sum_{k=1}^{m} W_k$ for all $m \geq m_0$ and $(W_n)_{n\geq 1}$ denotes a sequence of i.i.d. exponentially distributed random variables with mean one. For all $\lambda \in (0, \infty)$, by Markov's inequality,

$$\begin{aligned}\mathbb{P}[S_m \leq am] &= \mathbb{P}\left[\exp(-\lambda S_m) > \exp(-a\lambda m)\right] \leq \exp(a\lambda m)\mathbb{E}\left[\exp(-\lambda W_1)\right]^m \\ &= \exp(a\lambda m)\left(\lambda+1\right)^{-m} = \exp\left(\left(a\lambda - \log(1+\lambda)\right)m\right).\end{aligned}$$

Minimizing over λ, we deduce that the optimal bound is $\lambda = (1-a)/a$ and

$$\mathbb{P}[T_m \leq am] \leq \#V \exp\left(m\left(1 - a + \log(a) + \log(\rho+\varepsilon)\right)\right).$$

Choose $a := \{e(\rho+\varepsilon)\}^{-1} \in (0,\infty)$. Then, $\log(a) = -\log(\rho+\varepsilon) - 1$, and thus

$$\mathbb{P}[T_m \leq am] \leq \#V \exp\left(-am\right) \to 0 \quad \text{for } m \to \infty.$$

Inserting this in equation (2.15) and letting $m \to \infty$, we deduce

$$\gamma(\mathcal{G}) \leq \frac{1}{a} = e\left(\rho+\varepsilon\right).$$

The claim now follows by letting $\varepsilon \to 0$. □

Remark. Note that $\rho \leq \Delta\mathcal{G}+1$ and equality holds if and only if $\mathcal{G}$ is a regular graph. By considering the case of a complete graph, we immediately see that the upper bound in Theorem 2.12 is optimal up to a constant. Hence, one might ask whether there exists a sequence of regular graphs $(\mathcal{G}_n)_{n\geq 1}$ satisfying $\gamma(\mathcal{G}_n) \sim e{\cdot}\Delta\mathcal{G}_n$ as $n \to \infty$. We want to include a minor result to this question.

Proposition 2.13. *Let $(\mathcal{G}_n)_{n\geq 0}$ be a sequence of regular graphs with $\Delta\mathcal{G}_n \to \infty$ for $n \to \infty$ and $\operatorname{girth}(\mathcal{G}_n) \geq 5$ for all $n \in \mathbb{N}$. Then,*

$$\liminf_{n\to\infty} \frac{\gamma(\mathcal{G}_n)}{\Delta\mathcal{G}_n} \geq \frac{2e-1}{(e-1)^2} \approx 1.506.$$

Proof. Fix $M \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, such that $m := \Delta\mathcal{G}_n + 1 > M$ for all $n \geq n_0$.

We construct a random growth model, which evolves slower than our original one. For simplicity, assume that the first three vertices x_1, x_2, and x_3, which grow, form a path (x_1, x_2, x_3) in $\mathcal{G}$. Now only take into account the neighbors of x_2 and x_3 and, for the time being, neglect the possible growth of any other vertex. We also neglect the possible growth of x_1, x_2, and x_3.

Whenever a neighbor x_4 of x_3 grows, we consider the path (x_2, x_3, x_4) instead of (x_1, x_2, x_3), we forget about the height of any vertex $y \notin \{x_2, x_3, x_4\}$, and our procedure starts from the beginning. In this case, the maximal height among the vertices, which we take into account, will increase by one unit.

If a neighbor x_3' of x_2 grows, we memorize its height. Then we will neglect further growth of x_3', but, in the future, we will take into account its neighbors, which differ from x_2. Then, if a neighbor $x_4' \neq x_2$ of x_3' grows, we will replace our path (x_1, x_2, x_3) by (x_2, x_3', x_4') and forget about the height of all vertices $y \notin \{x_2, x_3', x_4'\}$. In this case, we again arrive at our initial situation, and the maximal height among the vertices taken into account increases by one unit.

All in all, we will memorize the height of up to M neighbors of x_2. Once we have reached this limit, we will not take into account the potential growth of a neighbor of x_2 anymore. As the graph $\mathcal{G}_n$, $n \geq n_0$, is regular, and $\text{girth}(\mathcal{G}_n) \geq 5$, by counting the number of neighbors of x_2 in our continuous-time setting, whose heights are higher than x_2 and memorized, we obtain a time-homogeneous Markov process with the following transition rates.

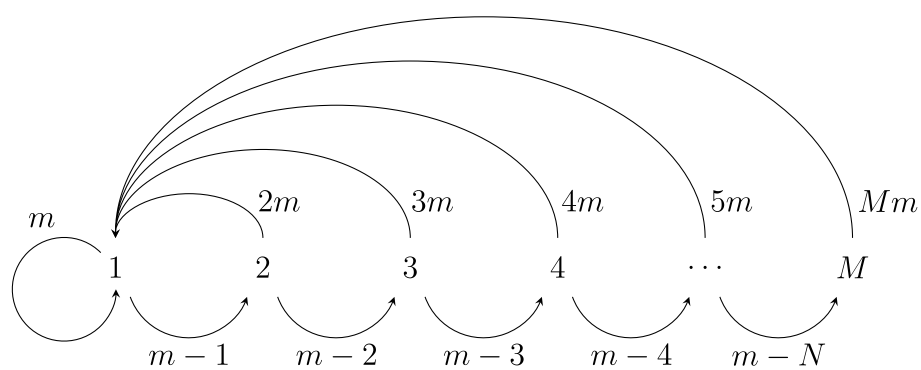


Figure 2.5: The number of memorized heights forms a time-homogeneous Markov process. We have indicated all transition rates.

Let $p_{m,M}(k)$, $k = 1, \ldots, M$, denote the invariant probability distribution. Then, by including the time-scaling induced by the transition rates, we know

$$\gamma(\mathcal{G}_n) \geq \left(\sum_{k=1}^{M-1} p_{m,M}(k) \left\{ (k+1)m - k \right\} + p_{m,M}(M) M m \right) \cdot \left(\sum_{k=1}^{M-1} p_{m,M}(k) \frac{km}{km + (m-k)} + p_{m,M}(M) \right).$$

As $n \to \infty$, we know $m = \Delta\mathcal{G}_n \to \infty$ and consequently $p_{m,M}(k) \to p_M(k)$, where $p_M(k)$, $k = 1, \dots, M$, denotes the invariant probability of time discrete Markov chain, whose transition probabilities are as follows.

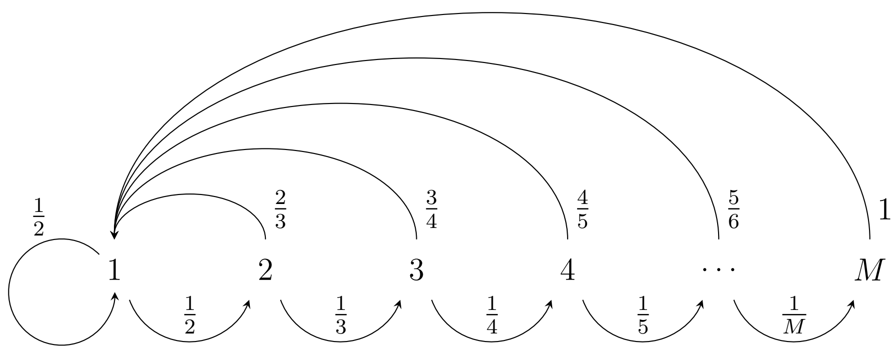


Figure 2.6: We arrive at this time-homogeneous Markov chain.

Now, by a simple calculation, we deduce

$$p_M(k) = \frac{1}{k!\,(1 + 1/2 + 1/6 + \ldots + 1/M!)}, \quad k = 1, \dots, M.$$

For all $M \in \mathbb{N}$, as $m = \Delta\mathcal{G}_n \to \infty$, we have

$$\frac{1}{m+1} \sum_{k=1}^{M-1} p_{m,M}(k) \left\{ (k+1)m - k \right\} \to \sum_{k=1}^{M-1} p_M(k)\ (k+1) \in (0, \infty),$$

$$\sum_{k=1}^{M-1} p_{m,M}(k) \frac{km}{km + (m-k)} \to \sum_{k=1}^{M-1} p_M(k) \frac{k}{k+1}.$$

Therefore, for all $M \in \mathbb{N}$, we conclude

$$\liminf_{n \to \infty} \frac{\gamma(\mathcal{G}_n)}{\Delta\mathcal{G}_n} \geq \left(\sum_{k=1}^{M-1} p_M(k)\ (k+1) \right) \left(\sum_{k=1}^{M-1} p_M(k) \frac{k}{k+1} \right).$$

Finally, we let $M \to \infty$ and note that

$$\sum_{k=1}^{M-1} p_M(k)\ (k+1) \to \frac{1}{e-1} \sum_{k=1}^{\infty} \frac{k+1}{k!} = \frac{2e-1}{e-1},$$
$$\sum_{k=1}^{M-1} p_M(k) \frac{k}{k+1} \to \frac{1}{e-1} \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = \frac{1}{e-1}. \qquad \square$$

Let us give a simple conclusion of both Proposition 2.5 and Theorem 2.12.

Corollary 2.14. *Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of graphs. Then,*

$$\lim_{n\to\infty} \gamma(\mathcal{G}_n) = \infty \qquad \textit{if and only if} \qquad \lim_{n\to\infty} \Delta\mathcal{G}_n = \infty.$$

2.7 A Modified Ballistic Deposition Model

For a better understanding of our growth model, it is natural to study ballistic deposition processes, which arise by modification of the recursion (2.1). The so-called nearest-neighbor ballistic deposition model is specified by the rule

$$\tilde{h}_x := \max_{y\in[x]} \{h_y + \delta_{xy}\},$$

where δ_{xy} is the Kronecker symbol. For any graph $\mathcal{G}$, we can define the asymptotic growth parameter $\tilde{\gamma}(\mathcal{G})$ in this new model in the same way as in the introduction. A simple coupling argument gives $\tilde{\gamma}(\mathcal{G}) \le \gamma(\mathcal{G})$ and

$$\gamma(\mathcal{S}_n) \le \tilde{\gamma}(\mathcal{S}_n) + 2 \quad \text{for all } n \in \mathbb{N}.$$

In particular, we see that Proposition 2.5, Theorem 2.12, and Corollary 2.14 also hold if we replace $\gamma(\mathcal{G})$ with $\tilde{\gamma}(\mathcal{G})$. However, direct calculations reveal differences between the growth models. Consider the complete graph $\mathcal{K}_n$ with $n \ge 1$ fixed. Then, by counting the number of vertices of maximal height in the discrete-time deposition model, we obtain the following Markov chain. Let Π denote the unique invariant distribution. Then, for all $k = 2, \dots, n$,

$$\Pi(k) = \frac{n-(k-1)}{n}\ \Pi(k-1) = \Pi(1) \prod_{l=1}^{k-1} \frac{n-l}{n},$$

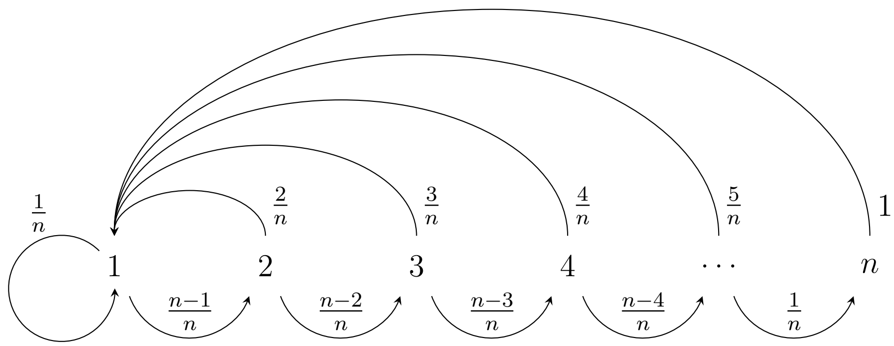


Figure 2.7: Counting the number of vertices of maximal height in $\mathcal{K}_n$ results in a time-homogeneous Markov chain with these transition probabilities.

and, using $\Pi(1) + \cdots + \Pi(n) = 1$, we find

$$\Pi(1) = \left(\sum_{k=1}^{n} \prod_{l=1}^{k-1} \frac{n-l}{n} \right)^{-1}.$$

By applying Birkhoff's ergodic theorem, we deduce

$$\tilde{\gamma}(\mathcal{K}_n) = n \sum_{k=1}^{n} \Pi(k) \frac{k}{n} = \sum_{k=1}^{n} \Pi(k) k.$$

For small values of $n \geq 1$, we can calculate Π and its expectation $\tilde{\gamma}(\mathcal{K}_n)$.

n	1	2	3	4	5
$\tilde{\gamma}(\mathcal{K}_n)$	1	$\frac{4}{3}$	$\frac{27}{17}$	$\frac{128}{71}$	$\frac{3125}{1569}$

For all $n \geq 1$, we find

$$\tilde{\gamma}(\mathcal{K}_n) = n \cdot \Pi(1) \cdot \sum_{k=1}^{n} k n^{-k} \frac{(n-1)!}{(n-k)!} = n \cdot \Pi(1) = \frac{n}{e^n \; n^{-n} \; \Gamma(n+1, n) - 1},$$

where we have identified Naor's urn distribution, compare [72, Appendix] and [50, Section 11.2.12], and inserted [65, Equation 8.8.10]. By a series expansion of the incomplete gamma function, see, for example, [32, Section 2.3],

$$\Gamma(n+1, n) \sim \frac{1}{2} \; \Gamma(n+1) \quad \text{for } n \to \infty.$$

This allows us to verify

$$e^n \, n^{-n} \, \Gamma(n+1,n) \sim e^n \, n^{-n} \, \frac{1}{2} \, \Gamma(n+1) \sim \sqrt{\frac{\pi}{2}} \, n^{1/2} \quad \text{as } n \to \infty,$$

and further deduce $\tilde{\gamma}(\mathcal{K}_n) \sim (2/\pi)^{1/2} \, n^{1/2}$ as $n \to \infty$.

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Article 2

Boolean Percolation on Digraphs and Random Exchange Processes

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Abstract. We study, in a general graph-theoretic formulation, a long-range percolation model introduced by Lamperti in [61]. For various underlying digraphs, we discuss connections between this model and random exchange processes. We clarify, for all $n \in \mathbb{N}$, under which conditions the lattices $\mathbb{N}_0^n$ and $\mathbb{Z}^n$ are essentially covered in this model. Moreover, for all $n \geq 2$, we establish that it is impossible to cover the directed n-ary tree in our model.

Keywords. boolean percolation; rumor spread and firework process; infinite paths in random graphs; long-range percolation; random exchange process; infinite type branching process; recurrence and transience; spectral radius.

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3.1 Introduction

Percolation theory is a fascinating area of modern probability, which tries to understand under which conditions infinite components arise in random structures. In the present article, we study the properties of a Boolean percolation model on directed graphs and relate this model to a classical Markov chain known as the random exchange process.

Let $\mathcal{G} = (V, E)$ be a directed graph with an infinite, countable vertex set V. For all vertices $x,\ y \in V$, we denote by $d(x, y) \in \mathbb{N}_0 \cup \{\infty\}$ the distance from x to y in $\mathcal{G}$. Note that $d : V \times V \to \mathbb{N}_0 \cup \{\infty\}$ is an extended quasimetric on V, which is symmetric if and only if the graph $\mathcal{G}$ is undirected, i.e., $(x, y) \in E$ implies $(y, x) \in E$ for all $x,\ y \in V$. Moreover, for all $x \in V$

and $n \in \mathbb{N}_0$, we denote by $B_n(x)$ the open ball of radius n starting from x, which is the set of all vertices $y \in V$ with $d(x,y) < n$.

Let $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ be a probability vector and $(Y_x)_{x\in V}$ a family of i.i.d. random variables satisfying $\mathbb{P}[Y_x = n] = \mu_n$ for all $n \geq 0$. In our percolation model, every vertex $x \in V$ will cover any vertex of $B_{Y_x}(x)$. Hence, the set of covered respectively uncovered vertices are

$$V_\mu := V_\mu(\mathcal{G}) := \bigcup_{x\in V} B_{Y_x}(x) \subseteq V, \qquad V_\mu^c := V_\mu^c(\mathcal{G}) := V \setminus V_\mu(\mathcal{G}).$$

As we are interested in the properties of the random sets V_μ and V_μ^c, we will always assume $\mu_0 \in (0,1)$, since $V_\mu = V$ almost surely in the case of $\mu_0 = 0$, and $V_\mu = \emptyset$ almost surely for $\mu_0 = 1$.

Let $x, y \in V$ and $V' = V_\mu$ or $V' = V_\mu^c$. Then, if both x and y are contained in V' and connected by a path in $\mathcal{G}$, which uses only vertices from V', we will say that x and y are in the same cluster.

To state our results, we introduce the following notation. Let $n \in \mathbb{N}$, $V = \mathbb{N}_0^n$ or $V = \mathbb{Z}^n$, and E be the set of all pairs $(x, x+e_j)$, where $x \in V$, $j = 1,\ldots,n$, and $e_j = (\delta_{ij})_{i=1,\ldots,n}$. Then, we denote the resulting graph $\mathcal{G} = (V,E)$ by $\mathbb{N}_0^n$ respectively $\mathbb{Z}^n$. Furthermore, for all $n \geq 2$, we define the infinite directed n-ary tree $\mathcal{D}_n := (V_n, E_n)$ by

$$V_n := \bigcup_{m\geq 0} \{1,\ldots,n\}^m, \qquad \text{where} \quad \{1,\ldots,n\}^0 := \emptyset,$$
$$E_n := \{(\emptyset,1),\ldots,(\emptyset,n)\} \cup \{(x,(x,j)) \mid x \in V_n \setminus \{\emptyset\},\ j = 1,\ldots,n\}.$$

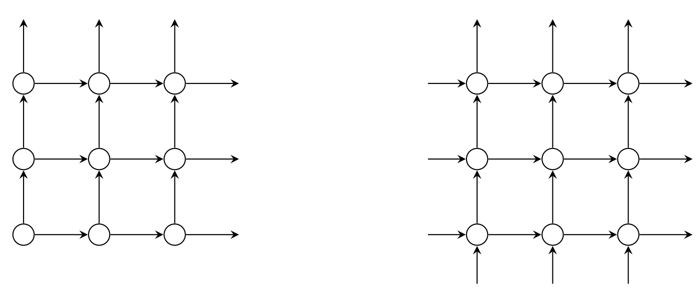

Figure 3.1: Illustration of the lattices $\mathbb{N}_0^2$ (left) and $\mathbb{Z}^2$ (right).

In this article, we will clarify under which conditions the graphs $\mathbb{N}_0^n$ and $\mathbb{Z}^n$ are (essentially) covered by a distribution μ, compare Theorem 3.1 and

Theorem 3.2 below. On the other hand, in Theorem 3.3, we will see that, for any distribution μ and $n \geq 2$, $\#V_\mu^c(\mathcal{D}_n) = \infty$ almost surely.

To the best of our knowledge, the present percolation model was first studied by Lamperti in [61] for $\mathcal{G} = \mathbb{N}_0$. This research was motivated by statistical physics and included the following description. At each location $n \in \mathbb{N}_0$, there is a fountain, which sprays water to the right and is wetting the segment from $n+1$ to $n+Y_n$. As $\mu_0 > 0$, with some positive probability, a fountain fails to operate at all.

Our percolation model and variants of it were studied by various authors, compare [63], [52], [53], [10], [30], and [51]. For a recent survey, see [51]. At this point, however, we want to postpone the discussion of how our new insights and results are related to these articles.

We can interpret our percolation model as the spread of a rumor through a network, a firework process, or a discrete version of Boolean percolation. This model was introduced by Gilbert in [33]. First points are chosen randomly in $\mathbb{R}^n$ according to a Poisson point process. Then, in the simplest case, around these points, the unit sphere is covered. For monographs, which are concerned with Boolean percolation, see [90], [43], [67], and [84].

In the present article, we also investigate a connection between the above described graph-theoretic percolation model and a rather classical Markov chain, which is sometimes called random exchange process. As far as we know, it was first observed by Zerner in [98, Section 1] that these two stochastic models are related to each other.

Let $(Y_n)_{n\geq 0}$ denote a sequence of i.i.d. random variables, which, as before, are distributed according to μ. Then, we set $X_0 := Y_0$ and recursively define

$$X_{n+1} := \max\{X_n - 1, Y_{n+1}\}, \quad n \in \mathbb{N}_0.$$

To the best of our knowledge, this process $(X_n)_{n\geq 0}$ first occurred in a statistical research article on deepwater exchange of a fjord, see [29]. Later, it was studied, in more general form, in [46] and [47]. In the following, we call the Markov chain $(X_n)_{n\geq 0}$ a (constant decrement) random exchange process. By construction, it has time-homogeneous transition probabilities and is irreducible on its state space $\mathcal{X}$, which is equal to $\mathbb{N}_0$ if μ is unbounded, and otherwise takes the form $\{0, 1, \ldots, n_0\}$, where $n_0 := \sup\{n \in \mathbb{N} \mid \mu_n \neq 0\}$. The transition matrix P associated with $(X_n)_{n\geq 0}$ is

$$P := P^\mu := \left(P^\mu_{x,y}\right)_{x,y\in\mathcal{X}}, \qquad \text{where} \quad P^\mu_{x,y} := \begin{cases} \mu_y, & y \geq x, \\ \sum_{z=0}^{x-1} \mu_z, & y = x-1, \\ 0, & y \leq x-2. \end{cases}$$

As $(X_n)_{n\geq 0}$ respectively P is irreducible, for all $z > 0$, Green's function

$$G(x,y|z) := \sum_{n=0}^{\infty} P^n_{x,y}\ z^n$$

either converges or diverges simultaneously for all x, $y \in \mathcal{X}$, compare [94, Chapter 1.1]. Therefore, independent of the choice of x, $y \in \mathcal{X}$, we can define the spectral radius of $(X_n)_{n\geq 0}$ respectively P by

$$\rho(P) := \limsup_{n\to\infty} \left(P^n_{x,y}\right)^{1/n} \in (0,1].$$

More generally, if A is an arbitrary irreducible matrix with nonnegative entries, we can define $\rho(A) \in [0,\infty]$ exactly in the same way.

Let us now state connections between the set V_μ of covered vertices in our percolation model and the Markov chain $(X_n)_{n\geq 0}$. We start by reformulating previous results in the following way.

Theorem 3.1. *For any law μ, the following statements are equivalent.*

(i) *Almost surely, $\#V^c_\mu(\mathbb{Z}) < \infty$.*

(ii) *Almost surely, $V_\mu(\mathbb{Z}) = \mathbb{Z}$.*

(iii) *The Markov chain $(X_n)_{n\geq 0}$ is not positive recurrent.*

(iv) *The expectation of μ is infinite, i.e., $\sum_{n\geq 0} n\mu_n = \infty$.*

It is not difficult to verify, more generally, that (i) and (ii) are equivalent if we replace $\mathbb{Z}$ by an arbitrary vertex-transitive graph.

By applying the Borel-Cantelli lemma, we can directly verify that (ii) and (iv) are equivalent statements. This equivalence was also observed, in a more general form, in [52, Section 2.2] and [10, Section 4]. For any graph $\mathcal{G} = (V,E)$, we have $V_\mu = V$ almost surely if and only if

$$\sum_{x\in V} \sum_{k\geq d(x,y)} \mu_k = \infty \quad \text{for all } y \in V.$$

For example, for all $n \geq 1$, $V_\mu(\mathbb{Z}^n) = \mathbb{Z}^n$ almost surely if and only if the n-th moment of μ diverges. This kind of phenomenon is well-known in the context of Boolean percolation models, and, thus, it seems convenient to include some previous literature results at this point.

In [43], Hall studied Boolean percolation on $\mathbb{R}^n$ with spheres of random i.i.d. radii and proved in [43, Theorem 3.1] that the entire space is almost surely covered if and only if the n-th moment of the radius distribution diverges. In [36], Gour established that if the n-th moment of the radius distribution is finite, there exists a critical value for the intensity of the underlying Poisson process. Recently, more results on phase transitions were deduced in [2] and [26]. However, there are also results on other aspects of Boolean percolation. For example, in [1], it was shown that this model is noise sensitive, and in [62], the capacity functional was studied.

In [6] and [11], Boolean percolation was studied on $[0,\infty)^n$ when, instead of the sphere around a point x, the set $x+[0,R_x)^d$ is occupied, where R_x is the radius associated which x. The results in [6] characterize under which conditions the entire space is essentially covered, and interestingly depend on whether $n=1$ or $n\geq 2$. In [11], Bezborodov observed, for $n=1$ and some radius distributions, that the covered volume fraction is one, but all clusters are bounded almost surely.

In [20], Coletti and Grynberg studied a model on $\mathbb{Z}^n$, in which first Bernoulli percolation with parameter $p\in(0,1)$ is performed, and then, independently, around the present points, random i.i.d. balls are covered. Again, the occupied region is almost surely $\mathbb{Z}^n$ if and only if the n-th moment of the radius distribution diverges. For a study of this percolation model on doubling graphs, also see [21].

Let us return to Theorem 3.1. The equivalence of (iii) and (iv) was first observed by Helland in [46, Section 3] and also mentioned by Kellerer in [56, comments after Theorem 2.6]. We can deduce it as follows. Due to the form of the transition probabilities of the Markov chain $(X_n)_{n\geq 0}$, any invariant measure $\tau=(\tau_x)_{x\in\mathcal{X}}$ has to satisfy

$$\tau_x=\sum_{z=0}^{x}\tau_z\mu_x+\tau_{x+1}\sum_{z=0}^{x}\mu_z,\quad\text{provided that } x,\, x+1\in\mathcal{X}.$$

Solving this recurrence relation yields the representation

$$\tau_x=\tau_0\left(\sum_{z\geq x}\mu_z\right)\left(\prod_{y=0}^{x-1}\sum_{z=0}^{y}\mu_z\right)^{-1},\quad x\in\mathcal{X}. \tag{3.1}$$

By a careful look at this formula indeed, it follows that (iii) and (iv) are equivalent. Moreover, if the distribution μ has a finite expectation, we can determine the stationary solution of $(X_n)_{n\geq 0}$ from (3.1) via normalization. In Section 3.3, we will present some concrete examples.

For results on positive recurrence of more general exchange processes with random decrements, the reader may consult [47, Section 2].

Theorem 3.2. *For any law μ, the following statements are equivalent.*

(a) *There exists $n \in \mathbb{N}$ with $\#V_\mu^c(\mathbb{N}_0^n) < \infty$ almost surely.*

(b) *For all $n \in \mathbb{N}$, $\#V_\mu^c(\mathbb{N}_0^n) < \infty$ almost surely.*

(c) *The Markov chain $(X_n)_{n\geq 0}$ is transient.*

(d) $\sum_{m\geq 0} \prod_{k=1}^{m} \sum_{l=0}^{k-1} \mu_l < \infty$.

Moreover, if one of these conditions is satisfied, then $\mathbb{E}[\#V_\mu^c(\mathbb{N}_0^n)] < \infty$ for all $n \in \mathbb{N}$ and there exists $\alpha \in (0, \infty)$ with $\mathbb{E}[\exp(\alpha \#V_\mu^c(\mathbb{N}_0))] < \infty$.

This theorem improves on previous works by revealing that, rather surprisingly, the value of $n \in \mathbb{N}$ does not influence whether all but finitely many points of the graph $\mathbb{N}_0^n$ are covered by a distribution μ. In the appendix of [61], Kesten proved that $\#V_\mu^c(\mathbb{N}_0) < \infty$ almost surely if and only if condition (d) in Theorem 3.2 is satisfied. This result was later rediscovered by various authors, partly in a different and more general form, compare [56, comments to Proposition 6.6], [52, Theorem 2.1], [30, Theorem 1], and [10, Section 3].

As observed by Zerner in [98, Proposition 1.1] and suggested by our notation, we can couple the set of covered points $V_\mu(\mathbb{N}_0)$ and the random exchange process $(X_n)_{n\geq 0}$ by using the same sequence of random variables $(Y_n)_{n\geq 0}$ in both definitions. Then, by construction,

$$\begin{aligned} V_\mu(\mathbb{N}_0) &= \{n \in \mathbb{N}_0 \mid \exists k \in \{0, \dots, n\} : Y_k > n - k\} \\ &= \left\{n \in \mathbb{N}_0 \;\middle|\; \max_{0\leq k\leq n} (Y_k - (n-k)) > 0\right\} \\ &= \{n \in \mathbb{N}_0 \mid X_n > 0\}. \end{aligned}$$

Consequently, we know that the Markov chain $(X_n)_{n\geq 0}$ is transient if and only if $\#V_\mu^c(\mathbb{N}_0) < \infty$ almost surely.

Let $n \geq 2$ and consider the infinite directed n-ary tree $\mathcal{D}_n = (V_n, E_n)$. Then, interestingly, we can associate a multitype branching process $(Z_m)_{m\geq 0}$ to our percolation model on $\mathcal{D}_n$ in the following way.

Let $k \in \mathbb{N}_0$ and $y \in V_n$ with $d(\emptyset, y) = k$. Then, we identify the vertex y with an individual of the k-th generation of $(Z_m)_{m\geq 0}$ if and only if $y \in V_\mu$ and y is contained in the same cluster as $\emptyset$. In other words, we demand that all vertices, which form the path from $\emptyset$ to y in $\mathcal{D}_n$, are contained in $V_\mu(\mathcal{D}_n)$. If this condition is satisfied, we define the type of y by

$$z_y := \max \{Y_x - d(x, y) \mid x \in V_n,\ d(x, y) < \infty\} .$$

By construction, if $Y_\emptyset \geq 1$, the branching process $(Z_n)_{n\geq 0}$ starts with one individual of type $Y_\emptyset$. However, on the event $Y_\emptyset = 0$, there are no individuals at all. For a vertex $y \in V_n$ to be identified with an individual in $(Z_m)_{n\geq 0}$, necessarily $y \in V_\mu$, i.e., there exists $x \in V_n$ with $Y_x > d(x, y)$. Hence, as $\mu_0 \in (0, 1)$, the type space of the branching process $(Z_m)_{n\geq 0}$ is $\mathcal{Z} = \mathcal{X} \setminus \{0\}$.

We can describe the reproduction in this branching process as follows. Every individual of type $x \geq 2$ has exactly n children, whose types are independent of each other. For each of them, the probability of type $y \in \mathcal{Z}$ is $M_{x,y} := P_{x,y}$. On the other hand, an individual of type 1 has n potential children, which are again independent of each other. For all $z \in \mathcal{Z}$, the probability that a given potential child is born and of type z is $M_{1,z} := P_{1,z}$. However, with probability μ_0, a potential child is not born.

As the type space $\mathcal{Z}$ is infinite in general, we distinguish between the local and global extinction of $(Z_m)_{m\geq 0}$. This process dies out globally if, at some moment, the total number of individuals vanishes. It dies out locally if, for all $z \in \mathcal{Z}$, only finitely many individuals of type z are born. While global extinction always implies local extinction, the reverse is not true in general for branching processes with infinitely many types.

Theorem 3.3. *Let $n \geq 2$. Then, for any distribution μ, $\#V_\mu^c(\mathcal{D}_n) = \infty$ almost surely. Moreover, the following statements are equivalent.*

(A) *Almost surely, $V_\mu(\mathcal{D}_n)$ contains a path of infinite length.*

(B) *With a positive probability, $(Z_m)_{m\geq 0}$ will not die out globally.*

(C) *With a positive probability, $(Z_m)_{m\geq 0}$ will not die out locally.*

(D) $\rho(M) > n^{-1}$, *where $M := (M_{x,y})_{x,y\in\mathcal{Z}}$.*

Note that, up to multiplication with $n \geq 2$, M is the mean matrix of the branching process $(Z_m)_{m\geq 0}$. To some degree, this explains why condition (D) is related to (B) and (C). Also, observe that M arises from the transition matrix P of the random exchange process $(X_n)_{n\geq 0}$ simply by deleting both the first row and column.

For an introduction to infinite type branching processes, we recommend Braunsteins' exposition in [17, Chapter 2] and the references mentioned therein. This presentation also explains the rather well-understood results on the extinction of finite type branching processes.

We also want to note that the branching processes $(Z_n)_{n\geq 0}$, which we consider in the present article, have a mean matrix of upper Hessenberg form. Recently, Braunsteins and Haupthenne studied the extinction of branching processes with a lower Hessenberg mean matrix in [18].

3.2 Proof of Theorem 3.2 and Theorem 3.3

Proof of Theorem 3.2. (b)$\Longrightarrow$(a). This implication is clear.

(b)$\Longrightarrow$(c)$\Longrightarrow$(a). From our coupling between $V_\mu(\mathbb{N}_0)$ and $(X_n)_{n\geq 0}$, we know that $(X_n)_{n\geq 0}$ is transient if and only if $V_\mu^c(\mathbb{N}_0) < \infty$ almost surely. In particular, the implications (b)$\Longrightarrow$(c) and (c)$\Longrightarrow$(a) follow.

(d)$\Longrightarrow$(a). By Kolmogorov's 0-1 law, for any probability distribution μ, either $\#V_\mu^c(\mathbb{N}_0) < \infty$ almost surely or $\#V_\mu^c(\mathbb{N}_0) = \infty$ almost surely. By identifying the expression in (d) with $\mathbb{E}[\#V_\mu^c(\mathbb{N}_0)]$, the implication follows.

Finally, let us assume that condition (a) holds for a distribution μ.

In the first step, we verify that we can restrict ourselves to the case $n = 1$. For this, suppose $\#V_\mu^c(\mathbb{N}_0^n) < \infty$ almost surely for some $n \geq 2$. Then, consider the subgraph $\mathcal{G}' = (V', E')$ of $\mathbb{N}_0^n$, which is induced by the vertex set V' of all $(x_1, \ldots, x_n) \in \mathbb{N}_0^n$ with $x_j = 0$ for all $j = 2, \ldots, n$. By construction, $\mathcal{G}'$ is isomorphic to $\mathbb{N}_0$, and we know that $V_\mu^c(\mathcal{G}') = V_\mu^c(\mathbb{N}_0^n) \cap V'$ is finite almost surely. Consequently, $\#V_\mu^c(\mathbb{N}_0) < \infty$ almost surely.

In the second step, we prove all remaining claims. As $0 < \mu_0 < 1$,

$$p := \mathbb{P}\left[V_\mu(\mathbb{N}_0) = \mathbb{N}\right] > 0.$$

Therefore, by the strong Markov property, we know that $\#V_\mu^c(\mathbb{N}_0)$ is geometrically distributed with parameter p. In particular, this random variable has a finite exponential moment, and we also deduce (d), i.e.,

$$\mathbb{E}\left[\#V_\mu^c(\mathbb{N}_0)\right] = \sum_{m\geq 0} q_m < \infty, \quad \text{where} \quad q_m := \mathbb{P}\left[m \in V_\mu^c(\mathbb{N}_0)\right]. \tag{3.2}$$

Let $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in \mathbb{N}_0^n$. For all $j = 1, \ldots, n$, let π_j denote the unique path from $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$ to x. Then, for all $j = 1, \ldots, n$, the path π_j consists of x_j edges in direction e_j, and, for all $i \neq j$, the paths π_i and π_j only share one vertex, which is their endpoint $x = (x_1, \ldots, x_n)$. So, for the event $\{x \in V_\mu^c(\mathbb{N}_0^n)\}$ to occur, it is necessary that for each $j = 1, \ldots, n$ there exists no vertex y contained in the path π_j such that $Z_y > d(y, x)$. This defines n independent events, whose probabilities can be described with the sequence $(q_m)_{m\geq 0}$ defined in (3.2). All in all,

$$\begin{aligned}
\mathbb{E}\left[\#V_\mu^c(\mathbb{N}_0^n)\right] &= \sum_{x\in\mathbb{N}_0^n} \mathbb{P}\left[x \in V_\mu^c(\mathbb{N}_0^n)\right] \leq \sum_{(x_1,\ldots,x_n)\in\mathbb{N}_0^n} \prod_{j=1}^n q_{x_j} \\
&= \sum_{x_1\geq 0} q_{x_1} \sum_{x_2\geq 0} q_{x_2} \cdots \sum_{x_{n-1}\geq 0} q_{x_{n-1}} \sum_{x_n\geq 0} q_{x_n} \\
&= \mathbb{E}\left[\#V_\mu^c(\mathbb{N}_0)\right]^n < \infty.
\end{aligned}$$

In particular, for all $n \geq 2$, $\#V_\mu^c(\mathbb{N}_0^n) < \infty$ almost surely, and condition (b) holds. Since we have already verified (b)$\Longrightarrow$(c), this finishes the proof. $\square$

Proof of Theorem 3.3. First, we verify that for all $n \geq 2$ and any law μ, $\#V_\mu^c(\mathcal{D}_n) = \infty$ almost surely. For this, for all $m \in \mathbb{N}$, we set

$$r_m := \mathbb{P}\left[\exists y \in V_\mu^c(\mathcal{D}_n) : \ d(\emptyset, y) = m\right].$$

As $0 < \mu_0 < 1$, we know $r_m \in (0,1)$ for all $m \in \mathbb{N}$. Moreover, for all $j = 1, \ldots, N$, we denote by $\mathcal{G}_j$ the induced subgraph obtained from $\mathcal{D}_n$ by restricting to all vertices, which can be reached from $j \in V_n$.

Let $m \geq 2$. Then, we know that there exists a $y \in V_\mu^c(\mathcal{D}_n)$ with $d(\emptyset, y) = m$ if and only if $Y_\emptyset \leq m$ and, for some $j = 1, \ldots, n$, there exists a vertex z in the graph $\mathcal{G}_j$ with $d(j, z) = m - 1$, which is not covered by any other vertex of $\mathcal{G}_j$. Note that the latter event is independent of $Y_\emptyset$ and that the graphs $\mathcal{G}_1, \ldots, \mathcal{G}_n$ are isomorphic to $\mathcal{D}_n$. Consequently, for all $m \geq 1$, we obtain the recurrence relation

$$r_{m+1} = (1 - (1 - r_m)^n) \; F(m+1), \qquad \text{where} \quad F(k) := \sum_{l=0}^{k} \mu_l. \tag{3.3}$$

Let $N \in \mathbb{N}$ with $F(N) > 1/2$. Then, by (3.3), for all $m \geq N$,

$$r_{m+1} \geq \left(1 - (1 - r_m)^2\right) F(N) = r_m(2 - r_m)F(N). \tag{3.4}$$

The map $f_N : [0,1] \to [0,1]$, $x \mapsto x(2-x)F(N)$, is monotone increasing. Hence, due to the estimate (3.4), iteration of the function f_N yields

$$r_m \geq f_N^{m-N}(r_N) \quad \text{for all } m \geq N+1.$$

The map f_N has the two fixpoints 0 and $x_N := 2 - F(N)^{-1} \in (0,1]$. Hence, by monotonicity, if $r_N \geq x_N$, then also $r_m \geq x_N$ for all $m \geq N$. On the other hand, if $r_N < x_N$, then, since f_N is concave and $f_N'(0) > 1$, $f_N^k(r_N) \to x_N$ for $k \to \infty$. In both cases, we can deduce

$$\liminf_{m \to \infty} r_m \geq x_N = 2 - F(N)^{-1}.$$

As $N \in \mathbb{N}$ can be chosen arbitrarily large in this argument, it follows that $r_m \to 1$ for $m \to \infty$. In particular, $V_\mu^c(\mathcal{D}_n)$ is almost surely non-empty.

By Kolmogorov's 0-1 law, we know that either $\#V_\mu^c(\mathcal{D}_n)$ is finite almost surely, or this random variable is infinite almost surely. In the first case, due to our assumption $\mu_0 \in (0,1)$, it would follow that $V_\mu^c(\mathcal{D}_n)$ is empty with a positive probability. Hence, we can conclude $\#V_\mu^c(\mathcal{D}_n) = \infty$ almost surely.

In the second step of this proof, we now verify that indeed the statements (A), (B), (C), and (D) are equivalent to each other.

(C)$\Longrightarrow$(B). This implication is clear.

(A)$\Longleftrightarrow$(B). If (A) holds, then, with a positive probability, $V_\mu(\mathcal{D}_n)$ contains an infinite path starting from the root $\emptyset$. On this event, $(Z_n)_{n\geq 0}$ does not die out globally, i.e., condition (B) holds. Conversely, if (B) holds, then, with a positive probability, $V_\mu(\mathcal{D}_n)$ contains an infinite path. By Kolmogorov's 0-1 law, (A) follows.

(C)$\Longleftrightarrow$(D). Since the mean matrix M of the branching process $(Z_n)_{n\geq 0}$ is irreducible, this equivalence follows from the theory of multitype branching processes, compare [17, Theorem 9], [31], or [9].

(B)$\Longrightarrow$(C). Suppose, for some distribution μ, that $(Z_n)_{n\geq 0}$ dies out locally almost surely but survives forever with a positive probability. Then, as $(Z_n)_{n\geq 0}$ starts with a single individual of random type $Y_\emptyset$, with some positive probability, $(Z_n)_{n\geq 0}$ survives forever, and no individuals of type 1 are born. On this event, we would know that $\#V^c_\mu(\mathcal{D}_n) < \infty$, and this is a contradiction to our first claim. The implication follows. □

3.3 Examples

Example 3.4. Let $m \in \mathbb{N}$, $m \geq 2$, and μ be the uniform distribution on $\{0, 1, \dots, m-1\}$. Then, by (3.1), the stationary solution τ of $(X_n)_{n\geq 0}$ is

$$\tau_n = \frac{m!}{m^m}(m-n)\frac{m^{n-1}}{n!}, \quad n \in \{0, \dots, m-1\}.$$

This law τ is a terminating member of the Kemp family of generalized hypergeometric probability distributions, compare [50, Section 2.4.1]. However, it also naturally arises from Naor's urn model [72, Appendix], also see [50, Section 11.2.12]. Assume that there are m balls in an urn, of which one is red, and the rest are white. In each step, pick one ball, and if it is white, replace it with a red ball. Continue until the first time T, at which a red ball gets chosen. Then, the distribution of T is

$$\mathbb{P}[T = n] = (m-1)!\ m^{-n}\ \frac{n}{(m-n)!}, \quad n \in \{1, \dots, m\},$$

and $m - T$, i.e. the number of tries not needed, has distribution τ.

Example 3.5. Let $p \in (0, 1)$ and μ be the geometric distribution with parameter $1 - p$. Then, by (3.1), the stationary solution τ of $(X_n)_{n\geq 0}$ is

$$\tau_n = \tau_0\ p^n \left(\prod_{k=1}^{n} \left(1 - p^k\right) \right)^{-1} = \tau_0\ \frac{p^n}{(p;p)_n}, \quad n \in \mathbb{N}_0,$$

where $(a;q)_n$ is the q-Pochhammer symbol. By normalisation,

$$\tau_0 = \left(\sum_{n\geq 0} \frac{p^n}{(p;p)_n} \right)^{-1} = \frac{1}{(p;p)_\infty} = \phi(p)^{-1},$$

where we have applied the q-binomial theorem, and ϕ denotes Euler's function. In [8, Section 4], Benkherouf and Bather discussed, in more general form, this distribution τ and referred to it as an Euler distribution. For more information, also see [50, Section 10.8.2].

Example 3.6. Assume that there exists $c \in (0, \infty)$ and $n_0 \in \mathbb{N}$ with

$$\sum_{k>n} \mu_k = \frac{c}{n} \quad \text{for all } n \geq n_0.$$

Then, as the expectation of μ is infinite, we know that the statements (i)-(iv) from Theorem 3.1 do hold. Moreover, it follows from the Gaussian ratio test that the condition (d) in Theorem 3.2 is satisfied if and only if $c > 1$. According to [47, Theorem 3.2], for any value of $c \in (0, \infty)$,

$$\lim_{n\to\infty} \mathbb{P}[X_n\, n^{-1} \leq y] = y^c (y+1)^{-c} \quad \text{for all } y \in (0, \infty).$$

This limit is an inverse Beta distribution with $\alpha = c$ and $\beta = 1$.

Example 3.7. Assume that $\mu = (\mu_n)_{n\geq 0}$ has finite support. Then, $(X_n)_{n\geq 0}$ has a stationary solution and $\rho(P) = 1$. Moreover, $\rho(M)$ is the spectral radius and Perron Frobenius eigenvalue of M. As in the proof of Theorem 3.3, we define, for the case $n = 2$,

$$r_m := \mathbb{P}\left[\exists y \in V_\mu^c(\mathcal{D}_2) : d(\emptyset, y) = m\right], \quad m \in \mathbb{N}.$$

Then, observe that, for all $m \geq n_0 := \sup\{n \in \mathbb{N} \mid \mu_n \neq 0\}$, the recurrence relation (3.3) simplifies into

$$r_{m+1} = (1 - (1 - r_m)^2) = r_m(2 - r_m).$$

This recursion is a modified version of the logistic equation. It follows that

$$r_m = 1 - \exp\left(-c\, 2^m\right) \quad \text{for all } m \geq n_0,$$

where $c \in (0, \infty)$ is a fixed parameter.

Example 3.8. Let $n \in \mathbb{N}$, $p \in (0, 1)$, $\mu_n := p$ and $\mu_0 := 1 - p$. Then, the matrix $M = M_{n,p}$ has dimension n and is of the form

$$M_{n,p} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & p \\ 1-p & 0 & \cdots & \cdots & 0 & p \\ 0 & 1-p & 0 & \cdots & 0 & p \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1-p & p \end{pmatrix}$$

The characteristic polynomial $\chi_{n,p} = \chi_{n,p}(z)$ of $M_{n,p}$ satisfies

$$\chi_{n,p}(z) = \det(z\mathbf{1}_n - M_{n,p}) = z\chi_{n-1,p}(z) - p(1-p)^{n-1},$$

and this recurrence relation can be deduced from a Laplace expansion of the first row of $M_{n,p}$. It follows from $\xi_{1,p}(z) = z - p$, that

$$\chi_{n,p}(z) = \frac{p(1-p)^n + (z-1)z^n}{p+z-1}.$$

We know that $\rho(M_{n,p})$ is the largest zero of this polynomial in $(0,1)$.

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Article 3

On Supercritical Branching Processes with Emigration

Georg Braun

Abstract. We study supercritical branching processes under the influence of an i.i.d. emigration component. We provide conditions under which the lifetime of the process is finite, respectively, has a finite expectation. A theorem of Kesten-Stigum type is obtained, and the extinction probability for a large initial population size is related to the tail behavior of the emigration.

Keywords. extinction probabilities; random difference equation; autoregressive process; recurrence and transience; Kesten-Stigum theorem.

2020 Mathematics Subject Classification. 60J80, 60J10.

4.1 Introduction

Branching processes are a fascinating class of stochastic processes, which model the evolution of a population under the assumption that different individuals give independently of each other birth to a random number of children. Timeless monographs, which present the theory of classical branching processes, include [44], [5], and [87].

In the present article, we study the consequences of including an i.i.d. emigration component between consecutive generations in the supercritical regime. Intuitively speaking, the properties of this model are determined by the interplay of two opposite effects, namely the explosive nature of supercritical branching processes and the decrease in the population size caused by emigration.

Formally, let $((\xi_{n,j})_{j\geq 1}, Y_n)_{n\geq 1}$ denote a sequence of i.i.d. random variables. Assume that $(\xi_{1,j})_{j\geq 1}$ is i.i.d. and let ξ be an independent copy of the family $(\xi_{n,j})_{n,j\geq 1}$. Moreover, suppose Y is an independent copy of $(Y_n)_{n\geq 1}$ and that both ξ and Y only take values in $\mathbb{N}_0$. Then, we define a branching process with emigration $(Z_n)_{n\geq 0}$ by setting $Z_0 := k \in \mathbb{N}$ and recursively

$$Z_{n+1} := \left(\sum_{j=1}^{Z_n} \xi_{n+1,j} - Y_{n+1} \right)_+ , \qquad n \geq 0. \tag{4.1}$$

Throughout this article, we will focus on the supercritical case and, more precisely, assume that

$$\lambda := \mathbb{E}[\xi] \in (1, \infty).$$

Naturally, our study will concentrate on the extinction time

$$\tau := \inf\{n \geq 1 \mid Z_n = 0\}, \quad \text{where } \inf \emptyset := \infty.$$

In Theorem 4.2, we will prove that τ is almost surely finite if and only if $\mathbb{E}[\log_+(Y)] = \infty$. Moreover, in Theorem 4.3, we show that $\mathbb{E}[\tau] < \infty$ if

$$0 < \sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}\left[Y \leq r(\lambda + \varepsilon)^m\right] < \infty \text{ for a } \varepsilon > 0 \text{ and an } r \in (0, \infty), \tag{4.2}$$

and, under some additional assumptions, $\mathbb{E}[\tau] = \infty$ provided that

$$\sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}\left[Y \leq r\lambda^m m^{-\theta}\right] = \infty \text{ for some } \theta \in (1, \infty),\ r \in (0, \infty). \tag{4.3}$$

The precise statements of all of our results will be given in Section 4.2. In Theorem 4.4, we relate the behavior of the extinction probabilities

$$q_k := \mathbb{P}[\tau < \infty \mid Z_0 = k], \quad k \geq 1,$$

to the tail behavior of Y as $k \to \infty$. We also present a strong limit theorem for the population size Z_n as $n \to \infty$, compare Theorem 4.5.

Our study is motivated by a simple observation, which links our model to subcritical autoregressive processes. We will explain it in Section 4.3. To the best of our knowledge, this connection has not been investigated in the literature so far.

While our criteria ensuring $\mathbb{E}[\tau] < \infty$ respectively $\mathbb{E}[\tau] = \infty$ are not exact, as illustrated by the following example, the gap is quite narrow.

Example 4.1. Assume that there are $c \in (0, \infty)$ and $n_0 \in \mathbb{N}$ with

$$\mathbb{P}[Y > n] = \frac{c}{\log n} \quad \text{for all } n \geq n_0.$$

Then $\mathbb{E}[\log_+(Y)] = \infty$ and hence $\tau < \infty$ almost surely. If $c > \log \lambda$, then, by Raabe's test, (4.2) holds. If $c \leq \log \lambda$, then, by the Gauss test, (4.3) holds.

Let us end this introduction by trying to give an overview of the literature dealing with branching processes with emigration.

In [91], Vatutin explored the critical case $\lambda = 1$ for $Y \equiv 1$ and $\sigma^2 := \mathrm{Var}[\xi] \in (0, \infty)$. He showed that τ has a regularly varying tail with exponent $-1 - 2/\sigma^2$ and, if all moments of ξ are finite, proved that $2Z_n/n\sigma^2$, conditioned on being positive, converges weakly to an exponential distribution. These results were improved by Vinokurov respectively Kaverin in [93] and [54], and more recently by Denisov, Korshunov, and Wachtel in [24]. The approach of [24] allows, more generally, a size-dependent offspring distribution and immigration.

More or less specific models of critical branching processes involving both immigration and emigration were studied in [71], [97] and [95].

The present article and most of the above literature deals with specific cases of controlled branching. This model was introduced by Sevastyanov and Zubkov in [88], who classified eventual extinction for a control function of linear and polynomial growth. We also refer to the related work by Zubkov in [99] and [100]. In [96], Yanev generalized this model by assuming random i.i.d. control functions $(\varphi_n)_{n\geq 1}$. Roughly speaking, the present article assumes $\varphi_n(m) := (m - Y_n)_+$, $m \in \mathbb{N}$. For a recent monograph on controlled branching, see [35].

We also want to mention some results for models in continuous time. In this scenario, the population size changes if exactly one individual gives birth to a random number of children or if an emigration event, sometimes called catastrophe, occurs. The case that each individual has either 0 or 2 children would correspond to a birth-and-death process. In [74] and [76], Pakes gave results for a catastrophe rate proportional to the population size. Moreover, Pakes studied this model with a size-independent emigration rate in [75], [77], and [78]. In the supercritical regime, he related the almost sure eventual extinction to the condition $\mathbb{E}[\log_+(Y)] = \infty$, see [75, Theorem 2.1 and Corollary 3.1]. This was also verified by Grey in [37], who proved the same result also for the time-discrete model of our present article under the assumption that $(\xi_{1,j})_{j\geq 1}$ and Y_1 are independent. However, this independence condition can be avoided, compare Theorem 4.2 of the present article.

4.2 Preliminaries and Statement of Results

In the following, we usually assume $Z_0 = k \in \mathbb{N}$, $\lambda \in (1, \infty)$, as well as

(H1) There exists a strictly increasing sequence $(k_n)_{n\geq 0}$ in $\mathbb{N}$, which satisfies $k_0 = k$ and $\mathbb{P}\Big[\sum_{j=1}^{k_n} \xi_{1,k} - Y_1 = k_{n+1}\Big] > 0$ for all $n \in \mathbb{N}_0$,

and

(H2) $\quad \mathbb{P}\Big[\sum_{j=1}^{n} \xi_{1,j} - Y_1 \leq n-1\Big] > 0$ for all $n \geq 1$.

In some of our results, we will also need Grey's restriction

(IND) The random variables $(\xi_{1,j})_{j\geq 1}$ and Y_1 are independent.

The conditions (H1) and (H2) ensure that neither emigration nor branching will dominate each other fully for the possibly rather small initial state respectively for a large population size.

If $\lambda > 1$, then condition (H1) is always satisfied provided that the initial population $Z_0 = k \in \mathbb{N}$ is chosen large enough. As similar arguments will occur in many of our proofs, let us briefly explain how this can be verified. By truncation of the offspring distribution ξ, also compare Observation 4.13 in Appendix A, we may restrict us to the case $\lambda < \infty$. Choose $\varepsilon > 0$ with $\lambda - \varepsilon > 1$. Then, by the law of large numbers,

$$\mathbb{P}\Big[\sum_{j=1}^{k} \xi_{1,j} > (\lambda - \varepsilon)k\Big] \to 0 \quad \text{for } k \to \infty.$$

In particular, we know that there exists $K \in \mathbb{N}$ such that this probability is smaller than $1/2$ for all $k \geq K$. Moreover, we can choose $N \in \mathbb{N}$ with $\mathbb{P}[Y_1 \geq N] < 1/2$. Then, for all $k \geq K$ with $N < \varepsilon k$, we deduce

$$\mathbb{P}[Z_1 > k \mid Z_0 = k] > 0.$$

So, if $\lambda > 1$, then condition (H1) holds if $Z_0 = k$ is chosen large enough. On the other hand, we know that condition (H2) holds, for example, in the case of $\mathbb{P}[\xi = 0] > 0$, or if the emigration distribution Y is unbounded and (IND).

Note that (H1) and (H2) are preserved if the value of $Z_0 = k \geq 1$ is increased. The state-space of $(Z_n)_{n\geq 0}$ may be affected by such a modification. However, this will not cause any problems in our study.

Let us give a final thought behind (H1) and (H2). Consider a modification of the branching process $(Z_n)_{n\geq 0}$, in which the population gets revived with exactly k individuals upon extinction. Then this renewal version of $(Z_n)_{n\geq 0}$ is a Markov chain, which is irreducible and exhibits an infinite state space containing 0 if and only if both (H1) and (H2) are satisfied. In [73], Pakes investigated this revived branching process when the emigration component is absent but with a more general resetting mechanism upon extinction. Moreover, both the subcritical and critical cases are studied in [73].

Theorem 4.2. *$\tau < \infty$ almost surely if and only if $\mathbb{E}[\log_+(Y)] = \infty$.*

If the process $(Z_n)_{n\geq 0}$ dies out almost surely, it is natural to ask whether its expected lifetime is finite or infinite.

Theorem 4.3. *Assume $\mathbb{E}[\log_+(Y)] = \infty$.*

(I) $\mathbb{E}[\tau] < \infty$, *provided that there are $\varepsilon > 0$ and $r \in (0, \infty)$ with*

$$0 < \sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}\left[Y \leq r(\lambda + \varepsilon)^m\right] < \infty.$$

(II) $\mathbb{E}[\tau] = \infty$, *provided that* $\mathbb{E}\left[\xi^{1+\delta}\right] < \infty$ *for a* $\delta > 0$, (IND), *and*

$$\sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}\left[Y \leq r\lambda^m m^{-\theta}\right] = \infty \quad \text{for a } \theta \in (1, \infty),\ r \in (0, \infty).$$

If the process $(Z_n)_{n\geq 0}$ does not become extinct almost surely, one might try to understand the distribution of τ and the extinction probabilities $(q_k)_{k\geq 1}$ in case of a large initial population size $k \geq 1$. For this, we use Karamata's concept of slow and regular variation and assume

(REG) $\mathbb{P}[Y > t]$ varies regularly for $t \to \infty$ with index $\alpha \in (0, \infty)$.

For a gentle introduction to slow and regular variation, we refer the reader to [68]. A measurable function $L : [0, \infty) \to (0, \infty)$ is called slowly varying for $t \to \infty$, if for all $c \in (0, \infty)$ one has $L(ct)/L(t) \to 1$ as $t \to \infty$. Moreover, a measurable function $f : [0,\infty) \to (0,\infty)$ is regularly varying for $t \to \infty$, if there exists $\alpha \in \mathbb{R}$, $t_0 \in [0,\infty)$ and a slowly varying function L satisfying $f(t) = t^\alpha L(t)$ for all $t \geq t_0$. In this case, the constant $\alpha \in \mathbb{R}$ is unique, and $-\alpha$ is called the index of f.

Theorem 4.4. *Assume* (REG) *and let* $N \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. *Then,*

$$\limsup_{k\to\infty} \mathbb{P}[\tau < N \mid Z_0 = k]\ \mathbb{P}[Y > k]^{-1} \leq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Furthermore, if all exponential moments of ξ *are finite, then*

$$\lim_{k\to\infty} \mathbb{P}[\tau < N \mid Z_0 = k]\ \mathbb{P}[Y > k]^{-1} = \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

By choosing $N = \infty$ in Theorem 4.4, we, in particular, obtain results on the extinction probabilities $(q_k)_{k\geq 1}$ for $k \to \infty$.

Besides τ and $(q_k)_{k\geq 1}$, one can also investigate the asymptotic behavior of the process $(Z_n)_{n\geq 0}$ conditioned on its non-extinction. Observe that the sequence $W_n := \lambda^{-n} Z_n$, $n \geq 0$, is a nonnegative supermartingale. Hence, as in the case without any migration, Doob's martingale convergence theorem yields the existence of the almost sure limit

$$W := \lim_{n\to\infty} W_n, \quad \text{and moreover } \ 0 \leq \mathbb{E}[W] \leq k.$$

Theorem 4.5.

(a) $\mathbb{P}[W > 0] > 0$ *if and only if*

$$\mathbb{E}[\xi \log_+(\xi)] < \infty \qquad \text{and} \qquad \mathbb{E}[\log_+(Y)] < \infty.$$

Furthermore, in this case, $\mathbb{P}[W > 0] = \mathbb{P}[\tau = \infty]$.

(b) *Assume* $\mathbb{P}[W > 0] > 0$, $\mathbb{P}[\xi = \lambda] < 1$ *and* (IND). *Then,*

$$\mathbb{P}[a < W < b] > 0 \quad \text{for all } 0 \leq a < b \leq \infty.$$

The proofs of Theorem 4.2 and Theorem 4.3 are somewhat similar and, therefore, together contained in Section 4.4. The arguments needed for Theorem 4.4 and Theorem 4.5 are different and slightly more technical, and thus, are carried out separately in Section 4.5 and Section 4.6.

4.3 Relation to the Random Difference Equation

In this section, we always assume $\xi \equiv \lambda$. Then, (4.1) simplifies into

$$Z_{n+1} = (\lambda Z_n - Y_{n+1})_+, \quad n \geq 0.$$

Consider the process $\big(\hat{Z}_n\big)_{n\geq 0}$ defined by $\hat{Z}_0 := Z_0 = k$ and

$$\hat{Z}_{n+1} := \lambda \hat{Z}_n - Y_{n+1}, \quad n \geq 0.$$

Then, by induction over $n \geq 0$, we can verify $Z_n = \big(\hat{Z}_n\big)_+$ and

$$\hat{Z}_n = \lambda^n k - \sum_{j=1}^{n} \lambda^{n-j} Y_j.$$

Let $m \in \mathbb{N}_0$. Then, for all $n \geq 0$, we know

$$\mathbb{P}[Z_n > m] = \mathbb{P}\big[\hat{Z}_n > m\big] = \mathbb{P}\Big[\lambda^{-n} m + \textstyle\sum_{j=1}^{n} \lambda^{-j} Y_j < k\Big] \tag{4.4}$$

$$= \mathbb{P}\Big[\lambda^{-n} m + \textstyle\sum_{j=1}^{n} \lambda^{-(n-j)} \lambda^{-1} Y_j < k\Big] = \mathbb{P}\big[\hat{X}_n < k\big], \tag{4.5}$$

where $(\hat{X}_n)_{n\geq 0}$ is the autoregressive process defined by $\hat{X}_0 := m$ and

$$\hat{X}_{n+1} := \lambda^{-1} \hat{X}_n + \lambda^{-1} Y_{n+1}, \quad n \geq 0.$$

The study of this random difference equation was initiated by Kesten in [57] in the more general random-coefficient version

$$X_{n+1} := A_{n+1} X_n + Y_{n+1}, \quad n \geq 0,$$

where the sequence $(A_n, Y_n)_{n\geq 1}$ is typically assumed to be i.i.d. and independent of X_0. The Markov chain $(X_n)_{n\geq 0}$ is sometimes called a perpetuity. For more information on this process, we also refer to the monograph [19] and the exposition in [49, Chapter 2].

Formally, the equations (4.4) and (4.5) establish that $(Z_n)_{n\geq 0}$ and $(\hat{X}_n)_{n\geq 0}$ are dual Markov chains in the sense of Siegmund, compare [89].

In the following, suppose $A_1 \geq 0$ and $Y_1 \geq 0$ almost surely. In the contractive case $\mathbb{E}[\log A_1] < 0$, it is well-known, that the condition

$$\mathbb{E}[\log_+(Y_1)] < \infty$$

is related to the existence of a stationary solution for $(X_n)_{n\geq 0}$, see, for example, [92, Theorem 1.6] or [19, Theorem 2.1.3]. This can be explained in the following way. Assume that $Y_0 := X_0$ has the same distribution as Y_1. Then, for fixed $n \geq 1$, by exchangeability

$$X_n = \sum_{j=0}^{n} A_n \cdots A_{j+1} Y_j \overset{\mathrm{d}}{=} \sum_{j=0}^{n} A_1 \cdots A_j Y_{j+1} =: X_n',$$

and $X_n' \to X_\infty$ almost surely for $n \to \infty$, provided the existence of the limit

$$X_\infty := \sum_{n\geq 0} A_1 \cdots A_n Y_{n+1}.$$

If $A_1 \equiv \lambda^{-1}$ and $Y_1 \geq 0$, then the limit X_∞ is almost surely finite if and only if $\mathbb{E}[\log_+(Y)] < \infty$, see, for example, Lemma 4.7 in Section 4.4.

Inserting $m = 0$ into (4.4) and (4.5) gives

$$\mathbb{P}[\tau = \infty] = \lim_{n\to\infty} \mathbb{P}[Z_n > 0] = \lim_{n\to\infty} \mathbb{P}\big[\hat{X}_n < k\big] = \mathbb{P}[\hat{X}_\infty < k], \tag{4.6}$$

where

$$\hat{X}_\infty := \lambda^{-1} \sum_{n\geq 0} \lambda^{-n} Y_{n+1}.$$

Consequently, in this way, we can recover the statement of Theorem 4.2. Furthermore, consider equation (4.6) and the following result, which was obtained by Grincevičius in [39].

Theorem 4.6 (Grincevičius)**.** *Assume $\mathbb{P}[Y_1 > t]$ varies regularly for $t \to \infty$ with index $\alpha \in (0, \infty)$, $\mathbb{E}[A_1^\alpha] < 1$, and $\mathbb{E}[A_1^\beta] < \infty$ for a $\beta > \alpha$. Then,*

$$\lim_{k\to\infty} \mathbb{P}\left[X_\infty > k\right] \ \mathbb{P}[Y_1 > k]^{-1} = \sum_{j=0}^{\infty} \mathbb{E}[A_1^\alpha]^j.$$

In the specific case $\xi \equiv \lambda$ and $A_1 \equiv \lambda^{-1}$, we can apply this theorem to recover the asymptotic formula for $(q_k)_{k\geq 1}$ as $k \to \infty$ given in Theorem 4.4. Note that X_∞ and $\hat{X}_\infty$ differ by the constant λ^{-1}, which explains why the limit in Theorem 4.4 is $\lambda^{-\alpha}/(1 - \lambda^{-\alpha})$ rather than $1/(1 - \lambda^{-\alpha})$.

It is worth mentioning that Grey questioned some parts of the original proof of Theorem 4.6 and gave a new, improved version in [38].

Finally, note that, in our case, all random variables involved in the definition of $(\hat{X}_n)_{n\geq 0}$ are nonnegative and, hence, Kellerer's theory of recurrence and transience of order-preserving chains is available, compare [55] and [56]. By again inserting $m = 0$ into (4.4) and (4.5), we deduce

$$\mathbb{E}[\tau] = \sum_{n\geq 0} \mathbb{P}[\tau > n] = \sum_{n\geq 0} \mathbb{P}[Z_n > 0] = \sum_{n\geq 0} \mathbb{P}[\hat{X}_n < k],$$

and hence conclude that $\mathbb{E}[\tau] = \infty$ if and only if the process $(\hat{X}_n)_{n\geq 0}$ is recurrent. A recent result by Zerner, see [98], states that this rather generally is the case if and only if there exists $b \in (0, \infty)$ with

$$\sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}[Y_1 \leq b\lambda^m] = \infty.$$

Clearly, in the specific case $\xi \equiv \lambda$, this result characterizes the finiteness of $\mathbb{E}[\tau]$ exactly and hence more precisely than Theorem 4.3.

Interestingly, Zerner's criterion does not only apply to general random-coefficient autoregressive processes but also rather general subcritical branching processes with immigration. For many of these models, the existence of a stationary solution is related to a finite logarithmic moment of the immigration, compare [45] and [81].

4.4 Proofs of Theorem 4.2 and Theorem 4.3

As a preparation, we start with two simple lemmas.

Lemma 4.7. *Let $(U_n)_{n\geq 1}$ denote a sequence of i.i.d. nonnegative random variables. Then, almost surely,*

$$\limsup_{n\to\infty} \frac{U_n}{n} = \begin{cases} 0, & \text{if} \quad \mathbb{E}[U_1] < \infty, \\ \infty, & \text{if} \quad \mathbb{E}[U_1] = \infty. \end{cases}$$

The proof of Lemma 4.7 follows directly from the Borel-Cantelli lemma, also see [40, Chapter 6, Proposition 1.1]. Lemma 4.7 is known in the context of supercritical branching processes with immigration and can be used to obtain some of Seneta's classical results on whether immigration increases the speed of divergence, also compare [85] and [23, Section 3.1.1].

We will also apply the following concentration estimate, which can be seen as a weaker but more general form of Chebyshev's inequality.

Lemma 4.8. *Let* $(V_n)_{n\geq 0}$ *denote a sequence of i.i.d. random variables and* $S_n := \sum_{j=1}^n V_j$ *for all* $n \geq 1$*. Assume* $\mathbb{E}[V_1] = 0$ *and that there is* $\delta \in (0,1]$ *with* $c := \mathbb{E}\big[|V_1|^{1+\delta}\big] < \infty$*. Then,*

$$\mathbb{P}[|S_n| > t] \leq 2cnt^{-1-\delta} \quad \text{for all } n \geq 1 \text{ and } t \in (0,\infty).$$

Proof. By the Marcinkiewicz-Zygmund inequality, see [40, Chapter 3, Corollary 8.2], we have

$$\mathbb{E}\big[|S_n|^{1+\delta}\big] \leq 2cn \quad \text{for all } n \geq 1.$$

Thus, the claim follows by applying Markov's inequality. □

The branching process, which is obtained from $(Z_n)_{n\geq 0}$ by neglecting any emigration, will be denoted by $(Z'_n)_{n\geq 0}$. Formally, we set $Z'_0 := k \geq 1$ and

$$Z'_{n+1} := \sum_{j=1}^{Z'_n} \xi_{n+1,j}, \quad n \geq 0.$$

We will also work with the stopping time

$$\tau' := \inf\left\{n \geq 1 \mid Z'_{n+1} \leq Y_{n+1}\right\}.$$

Observe that, by definition, $Z_n \leq Z'_n$ and $\tau \leq \tau'$ almost surely.

Proof of Theorem 4.2. First, let $\mathbb{E}[\log_+(Y)] = \infty$. Choose $\varepsilon > 0$ and set

$$T := \inf\left\{n \geq 1 \mid Z'_m \leq (\lambda+\varepsilon)^m \text{ for all } m \geq n\right\}.$$

Then, for all $n \geq 1$, Markov's inequality gives

$$\mathbb{P}\left[Z'_n > (\lambda+\varepsilon)^n\right] \leq k\Big(1 + \frac{\varepsilon}{\lambda}\Big)^{-n}.$$

Hence, by the Borel-Cantelli lemma, $T < \infty$ almost surely. Furthermore, applying Lemma 4.7 with $U_n := \log_+(Y_n)$ gives $Y_n \geq (\lambda+\varepsilon)^n$ for infinitely many $n \geq 1$ almost surely. This yields $\tau \leq \tau' < \infty$ almost surely.

Secondly, let $\mathbb{E}[\log_+(Y)] < \infty$. By truncation of the offspring distribution, compare Observation 4.13 from Appendix A, we can assume that the distribution ξ is bounded and $\sigma^2 := \mathrm{Var}[\xi] \in [0, \infty)$. Moreover, as (H1) and (H2) do hold, we can choose $Z_0 = k$ large enough to ensure that these conditions remain true after truncation.

Fix $\varepsilon > 0$ with $\lambda_1 := \lambda - 2\varepsilon > 1$ and let $\lambda_0 := \lambda - \varepsilon$. Then, for all $n \geq 1$, consider the following events

$$A_n := \left\{ \sum_{j=1}^{\lfloor \lambda_0^n \rfloor} \xi_{n+1,j} \geq \left(\lambda - \frac{\varepsilon}{2}\right) \lfloor \lambda_0^n \rfloor \right\}, \qquad B_n := \left\{ Y_n \leq \lambda_1^n \right\}.$$

For all $n \geq 1$, we find by Chebyshev's inequality

$$\mathbb{P}[A_n^c] \leq \mathbb{P}\left[\left| \sum_{j=1}^{\lfloor \lambda_0^n \rfloor} \xi_{1,j} - \lambda \lfloor \lambda_0^n \rfloor \right| > \frac{\varepsilon}{2} \lfloor \lambda_0^n \rfloor \right] \leq \left(\frac{\varepsilon}{2}\right)^{-2} \frac{\sigma^2}{\lfloor \lambda_0^n \rfloor}.$$

Since $\lambda_0 > 1$, due to the Borel-Cantelli lemma, almost surely only finitely many events A_n^c, $n \geq 1$, do occur. On the other hand, by Lemma 4.7, we know that almost surely all but finitely many events B_n, $n \geq 1$, do occur. Also, note that $(A_n)_{n\geq 1}$ is a sequence of independent events, and so is $(B_n)_{n\geq 1}$. All in all, we can fix $n_0 \in \mathbb{N}$ such that

$$\left(\lambda - \frac{\varepsilon}{2}\right) \lfloor \lambda_0^n \rfloor - \lambda_1^n \geq \left\lfloor \lambda_0^{n+1} \right\rfloor \quad \text{for all } n \geq n_0 \tag{4.7}$$

and

$$\min\left(\mathbb{P}[A], \mathbb{P}[B]\right) > \frac{1}{2}, \quad \text{where } A := \bigcap_{n \geq n_0} A_n, \quad B := \bigcap_{n \geq n_0} B_n. \tag{4.8}$$

Note that the value of n_0 does not depend on k. By using (4.8), we find

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] \geq \mathbb{P}[A] + \mathbb{P}[B] - 1 > 0.$$

Finally, by recalling (H1) and (H2), we can increase $Z_0 = k$ to ensure

$$\mathbb{P}[C] > 0, \qquad \text{where} \quad C := \left\{ Z_{n_0} \geq \lfloor \lambda_0^{n_0} \rfloor \right\}.$$

By inserting our construction of the events A and B and using (4.7), an inductive argument yields $Z_n \geq \lfloor \lambda_0^n \rfloor$ for all $n \geq n_0$ on the event $A \cap B \cap C$. Since the events $A \cap B$ and C are independent by definition,

$$\mathbb{P}[\tau = \infty] \geq \mathbb{P}[A \cap B \cap C] = \mathbb{P}[A \cap B]\, \mathbb{P}[C] > 0.$$

□

In fact, a careful look at the second part of this proof reveals the following result, which we need in the proof of Theorem 4.5.

Proposition 4.9. *Assume* $\mathbb{E}[\log_+(Y)] < \infty$. *Then,* $q_k \to 0$ *for* $k \to \infty$.

The proof of Proposition 4.9 is left to the reader.

Proof of Theorem 4.3. (I). Since $\tau \leq \tau'$, it suffices to verify $\mathbb{E}[\tau'] < \infty$. Fix $\varepsilon > 0$ and $r \in (0, \infty)$ according to the assumption and set

$$T := \inf\left\{n \geq 1 \mid Z'_m \leq r(\lambda+\varepsilon)^{m-1} \text{ for all } m \geq n\right\},$$
$$\hat{T} := \inf\left\{n > T \mid Y_n > r(\lambda+\varepsilon)^n\right\}.$$

Then, $\tau' \leq \hat{T}$ almost surely by construction, and hence it suffices to prove $\mathbb{E}[\hat{T}] < \infty$. As in the proof of Theorem 4.2, we know $T < \infty$ almost surely. For all $n \geq 1$, by applying Markov's inequality, we deduce

$$\mathbb{P}[T = n] \leq \mathbb{P}\left[Z'_{n-1} > r(\lambda+\varepsilon)^{n-2}\right] \leq \frac{\lambda k}{r}\left(1 + \frac{\varepsilon}{\lambda}\right)^{-n+2}. \tag{4.9}$$

Moreover, since $((\xi_{n,j})_{j\geq 1}), Y_n)_{n\geq 1}$ is i.i.d., we know

$$\mathbb{E}\big[\hat{T}\big] = \sum_{n\geq 1} \mathbb{E}\big[\hat{T} \mid T = n\big]\ \mathbb{P}[T = n] = \sum_{n\geq 1} \left(\mathbb{E}[T_n] + n\right)\mathbb{P}[T = n], \tag{4.10}$$

where

$$T_n := \inf\left\{m \geq 1 \mid Y_{n+m} > r(\lambda+\varepsilon)^{n+m}\right\}, \quad n \geq 1.$$

For all $n \geq 1$, we have

$$\begin{aligned}\mathbb{E}\left[T_n\right] &= 1 + \sum_{m\geq 1} \mathbb{P}\left[T_n > m\right] = 1 + \sum_{m\geq 1}\prod_{l=1}^{m} \mathbb{P}\left[Y \leq r(\lambda+\varepsilon)^{n+l}\right] \\ &= 1 + \left(\sum_{m\geq n+1}\prod_{l=1}^{m} \mathbb{P}\left[Y \leq r(\lambda+\varepsilon)^{l}\right]\right)\left(\prod_{l=1}^{n} \mathbb{P}\left[Y \leq r(\lambda+\varepsilon)^{l}\right]\right)^{-1}.\end{aligned}$$

Due to our choice of $r \in (0, \infty)$, we conclude that $1 \leq \mathbb{E}[T_n] < \infty$ for all $n \geq 1$. Note that we can insert this formula for $\mathbb{E}[T_n]$ into equation (4.10). Then, by using the estimate (4.9), we know $\mathbb{E}[\hat{T}] < \infty$, and the claim follows.

(II). First, note that by possibly increasing $r \in (0, \infty)$, we can guarantee that there exists $n_0 \in \mathbb{N}$ satisfying both $rn_0^{-\theta} < 1$ and

$$\mathbb{P}\left[Y \leq \kappa r\right] > 0, \quad \text{where } \kappa := \prod_{n \geq n_0} \left(1 - rn^{-\theta}\right) \in (0, 1).$$

Fix $r \in (0, \infty)$ accordingly. By possible increasing n_0, which results in an increase of κ, and by using our assumption, we can ensure

$$\sum_{n \geq n_0} \prod_{l=n_0}^{n} \mathbb{P}\left[Y \leq \kappa r \lambda^l l^{-\theta}\right] = \infty. \tag{4.11}$$

Fix η with $(1+\delta)^{-1} < \eta < 1$. Then, for all $n \geq n_0$, let

$$N_n := \lambda^n \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right), \quad f_n := \kappa \lambda^{\eta n}, \quad g_n := \kappa r n^{-\theta} \lambda^n.$$

By possibly increasing $n_0 \in \mathbb{N}$ and recalling $\eta < 1$, for all $n \geq n_0$,

$$\begin{aligned}
\lambda \lfloor N_n \rfloor - \lceil f_n \rceil &\geq \lambda^{n+1} \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right) - \kappa \lambda^{\eta n} n - 2 \\
&\geq \lambda^{n+1} \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right) - \lambda^{\eta n} n^2 \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right) \\
&= \left(\lambda^{n+1} \left(1 + \frac{1}{n}\right) - \lambda^{\eta n} n^2\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right) \\
&= \left(\lambda^{n+1} + \frac{\lambda^{n+1}}{n+1} + \frac{\lambda^{n+1}}{n(n+1)} - \lambda^{\eta n} n^2\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right) \\
&\geq \lambda^{n+1} \left(1 + \frac{1}{n+1}\right) \prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right).
\end{aligned}$$

Moreover, by possibly increasing $n_0 \in \mathbb{N}$, we can guarantee $g_n \geq n$ for all $n \geq n_0$. So, by inserting the definition of κ, for all $n \geq n_0$, we get

$$\begin{aligned}
\lceil g_{n+1} \rceil \leq g_{n+1} + 1 &\leq \left(1 + \frac{1}{n+1}\right) g_{n+1} \\
&< \lambda^{n+1} \left(1 + \frac{1}{n+1}\right) \left(\prod_{l=n_0}^{n} \left(1 - rl^{-\theta}\right)\right) r \ (n+1)^{-\theta}.
\end{aligned}$$

By combining the previous two estimates, for all $n \geq n_0$, we directly find

$$\lambda\lfloor N_n \rfloor - \lceil f_n \rceil - \lceil g_{n+1} \rceil \geq N_{n+1}. \tag{4.12}$$

For all $n \geq n_0$, we consider the event

$$D_n := \left\{ \sum_{j=1}^{\lfloor N_n \rfloor} \xi_{n,j} \geq \lambda\lfloor N_n \rfloor - \lceil f_n \rceil \right\}.$$

Since we assume $\mathbb{E}\big[\xi^{1+\delta}\big] < \infty$, by Lemma 4.8, there is $c \in (0, \infty)$ with

$$\mathbb{P}\left[D_n^c\right] \leq \mathbb{P}\left[\left| \sum_{j=1}^{\lfloor N_n \rfloor} \xi_{1,j} - \lambda\lfloor N_n \rfloor \right| > \lceil f_n \rceil \right] \leq \frac{2c\lfloor N_n \rfloor}{\lfloor f_n \rfloor^{1+\delta}}, \quad \text{for all } n \geq n_0.$$

Recalling our definition of N_n, f_n, and η, and noticing that the events D_n, $n \geq n_0$, are independent, we obtain

$$\mathbb{P}[D] > 0, \quad \text{where} \quad D := \bigcap_{n \geq n_0} D_n.$$

Consider the stopping time

$$T := \inf\left\{ n > n_0 \mid Y_n > g_n \right\}.$$

Then, by definition of T and $(g_n)_{n\geq 0}$, we deduce from (4.11)

$$\mathbb{E}[T] = \sum_{n\geq 0} \mathbb{P}[T > n] = n_0 + 1 + \sum_{n\geq n_0+1} \prod_{l=n_0+1}^{n} \mathbb{P}\left[Y \leq \kappa r l^{-\theta} \lambda^l\right] = \infty.$$

Besides, by (H1) and (H2), we may assume that the value of $Z_0 = k \geq 1$ is chosen large enough to ensure that

$$\mathbb{P}[C] > 0, \qquad \text{where} \quad C := \left\{ Z_{n_0} \geq N_{n_0} \right\}.$$

Note that, by construction, C and D are independent events. All in all, by (4.12), we can deduce that $\tau \geq T$ on the event $B := C \cap D$, which occurs with a positive probability. Finally, by (IND),

$$\mathbb{E}[\tau] \geq \mathbb{E}[\tau 1_B] \geq \mathbb{E}[T 1_B] = \mathbb{E}[T \mid B]\, \mathbb{P}[B] = \mathbb{E}[T]\, \mathbb{P}[B] = \infty. \qquad \square$$

4.5 Proof of Theorem 4.4

For convenience, we split the proof of Theorem 4.4 into smaller parts by formulating and separately proving the following two lemmas.

Lemma 4.10. *Assume* (REG)*. Then,*

$$C := \limsup_{k\to\infty} \; q_k \; \mathbb{P}[Y > k]^{-1} \leq \frac{\lambda^{-\alpha}}{1-\lambda^{-\alpha}}.$$

Lemma 4.11. *Assume* (REG) *and that all exponential moments of ξ are finite. Moreover, let $N \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Then,*

$$\liminf_{k\to\infty} \; \mathbb{P}[\tau < N \mid Z_0 = k] \; \mathbb{P}[Y > k]^{-1} \geq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Proof of Lemma 4.10. By truncation of the offspring distribution, compare Observation 4.13 from Appendix A, we may assume that ξ is almost surely bounded and particularly has finite exponential moments.

Let us verify $C < \infty$ as a first step. For this, choose $\varepsilon > 0$ with $\lambda_0 := \lambda - 2\varepsilon > 1$ and set $\lambda_1 := \lambda - \varepsilon$. Then, by Lemma 4.14 from Appendix B, there are $c_1, \ldots, c_N \in (0,\infty)$ such that the sequence

$$x_0 := 1, \qquad x_{n+1} := \begin{cases} \lambda_1 x_n - \lambda_0^{n+1}, & n \geq N \\ \\ \lambda_1 x_n - c_{n+1}, & n \leq N-1, \end{cases}$$

is strictly positive and satisfies $x_n \geq c^n$ for a $c > 1$ and all $n \geq 1$. Furthermore, for all $k \geq 1$, we consider the events

$$A_{k,n} := \left\{ \sum_{j=1}^{\lfloor k x_n \rfloor} \xi_{n+1,j} \geq \lambda_1 k x_n \right\}, \quad n \geq 0, \qquad A_k := \bigcap_{n\geq 0} A_{k,n}.$$

For all $k \geq 1$, we have

$$q_k = \mathbb{P}[\tau < \infty,\ A_k \mid Z_0 = k] + \mathbb{P}\big[\tau < \infty,\ A_k^c \mid Z_0 = k\big], \tag{4.13}$$

as well as

$$\mathbb{P}\big[\tau < \infty,\ A_k^c \mid Z_0 = k\big] \leq \sum_{n\geq 0} \mathbb{P}[A_{k,n}^c].$$

By the Cramér-Chernoff method or a sub-Gaussian concentration estimate, compare [15, Section 2.1 resp. Section 2.2], and by our knowledge concerning the sequence $(x_n)_{n\geq 0}$, $\mathbb{P}[\tau < \infty,\ A_k^c \mid Z_0 = k] \to 0$ for $k \to \infty$ exponentially fast. Therefore, by applying condition (REG), we deduce

$$\lim_{k\to\infty} \mathbb{P}\big[\tau < \infty,\ A_k^c \mid Z_0 = k\big]\ \mathbb{P}[Y > k]^{-1} = 0,$$

and, by recalling (4.13), we further conclude

$$C = \limsup_{k\to\infty}\ \mathbb{P}[\tau < \infty,\ A_k \mid Z_0 = k]\ \mathbb{P}[Y > k]^{-1}.$$

Fix $k \geq 1$ and let $Z_0 = k$. Then, by construction of A_k and $(x_n)_{n\geq 0}$,

$$\{\tau < \infty\} \cap A_k \subseteq \bigcup_{n=1}^{N} \{Y_n > kc_n\}\ \cup\ \bigcup_{n>N} \{Y_n > k\lambda_0^n\}.$$

Consequently,

$$C \leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1} \left(\sum_{n=1}^{N} \mathbb{P}\left[Y_n > c_n k\right] + \mathbb{P}\left[\sum_{n>N} Y_n \lambda_0^{-n} > k \right] \right). \quad (4.14)$$

Note that, on the one hand, due to (REG),

$$\lim_{k\to\infty} \sum_{n=1}^{N} \mathbb{P}\left[Y > c_n k\right]\ \mathbb{P}[Y > k]^{-1} = \sum_{n=1}^{N} c_n^{-\alpha} < \infty,$$

and on the other hand, by applying Theorem 4.6 with $A_1 \equiv \lambda_0^{-1}$,

$$\limsup_{k\to\infty}\ \mathbb{P}\left[\sum_{n>N} Y_n \lambda_0^{-n} > k \right] \mathbb{P}[Y > k]^{-1} < \infty.$$

All in all, by equation (4.14), we conclude that $C < \infty$.

In the second step, fix $0 < \delta < \lambda_1 = \lambda - \varepsilon$ and set $\lambda_2 := \lambda_1 - \delta$. Later, we will let $\varepsilon \searrow 0$ and $\delta \searrow 0$, when $\lambda_1 \nearrow \lambda$ and $\lambda_2 \nearrow \lambda$. For all $k \geq 1$, we define the events

$$D_{1,k} := \{Y_1 > \lambda_1 k\},\quad D_{2,k} := \{\delta k \leq Y_1 \leq \lambda_1 k\}\quad \text{and}\quad D_{3,k} := \{Y_1 < \delta k\}.$$

For every $k \geq 1$, given $Z_0 = k$, we can decompose $\{\tau < \infty\}$ by using the three events $D_{1,k}$, $D_{2,k}$, and $D_{3,k}$. From this decomposition, we obtain

$$q_k \leq \mathbb{P}[D_{1,k}] \ + \ \mathbb{P}[\tau < \infty,\ D_{2,k} \mid Z_0 = k] \ + \ \mathbb{P}[\tau < \infty,\ D_{3,k} \mid Z_0 = k].$$

Let us introduce the notation $q_r := q_{\lfloor r \rfloor}$ for $r \in (0, \infty)$. Note that the function $r \mapsto q_r$ is monotone non-increasing with respect to r, and

$$\mathbb{P}\left[\sum_{j=1}^{n} \xi_{1,j} \leq \Big(\lambda - \frac{\varepsilon}{2}\Big) n\right] \to 0 \quad \text{exponentially fast as } n \to \infty,$$

which can be verified, as in the first step, by the Cramér-Chernoff method or a sub-Gaussian concentration estimate. By combining these two remarks with our assumption (REG), for all $\varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned}
&\limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[\tau < \infty,\ D_{2,k} \mid Z_0 = k] \\
= &\limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[D_{2,k}]\ \mathbb{P}[\tau < \infty \mid D_{2,k},\ Z_0 = k] \\
\leq &\limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[Y \geq \delta k]\ q_{(\varepsilon/2)k} = 0,
\end{aligned}$$

where, for the last step, we have also inserted our knowledge $C < \infty$. Similarly, we can verify

$$\limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[\tau < \infty,\ D_{3,k} \mid Z_0 = k] \leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ q_{\lambda_2 k}.$$

All in all, and again by invoking on (REG),

$$\begin{aligned}
C &= \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ q_k \\
&\leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[Y > \lambda_1 k] \ + \ \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ q_{\lambda_2 k} \\
&= \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\ \mathbb{P}[Y > \lambda_1 k] \ + \ \limsup_{k\to\infty}\ \frac{\mathbb{P}[Y > \lambda_2 k]\ q_{\lambda_2 k}}{\mathbb{P}[Y > k]\ \mathbb{P}[Y > \lambda_2 k]} \\
&\leq \lambda_1^{-\alpha} + C\lambda_2^{-\alpha}.
\end{aligned}$$

Letting $\delta \searrow 0$ and $\varepsilon \searrow 0$, $C \leq \lambda^{-\alpha} + C\lambda^{-\alpha}$, and the claim follows. □

Proof of Lemma 4.11. Due to monotonicity, it suffices to prove the claim for $N < \infty$. Fix $\varepsilon > 0$. For all $k \geq 1$ and $l = 1, \ldots, N-1$, define

$$A_{k,l} := \left\{ \sum_{j=1}^{\lceil k(\lambda+\varepsilon)^l \rceil} \xi_{l,j} \leq k(\lambda + \varepsilon)^{l+1} \right\}, \quad A_k := \bigcap_{l=1}^{N-1} A_{k,l}.$$

Then, for all $l = 1, \dots, N-1$, $\mathbb{P}[A_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast due to the Cramér-Chernoff method. Hence, by (REG),

$$\begin{aligned} L_N^- :=& \liminf_{k\to\infty} \ \mathbb{P}[\tau < N \mid Z_0 = k] \ \mathbb{P}[Y > k]^{-1} \\ =& \liminf_{k\to\infty} \ \mathbb{P}[\tau < N, \ A_k \mid Z_0 = k] \ \mathbb{P}[Y > k]^{-1}. \end{aligned}$$

By inserting the definition of both A_k and $(Z_n)_{n\geq 0}$, we further obtain

$$L_N^- \geq \liminf_{k\to\infty} \ \mathbb{P}\left[\exists l \in \{1, \dots, N-1\} : \ Y_l \geq k(\lambda+\varepsilon)^l, \ A_k\right] \ \mathbb{P}[Y > k]^{-1}.$$

Again, we combine (REG) with the fact that for all $l = 0, \dots, N-1$, $\mathbb{P}[A_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast. This gives us

$$L_N^- \geq \liminf_{k\to\infty} \ \mathbb{P}\left[\exists l \in \{1, \dots, N-1\} : \ Y_l \geq k(\lambda+\varepsilon)^l\right] \ \mathbb{P}[Y > k]^{-1}.$$

Now, by applying the inclusion-exclusion principle, recalling that the sequence $(Y_m)_{m\geq 1}$ is i.i.d. and working with (REG), we obtain

$$L_N^- \geq \sum_{l=1}^{N-1} \lim_{k\to\infty} \mathbb{P}\left[Y_1 \geq k(\lambda+\varepsilon)^l\right] \ \mathbb{P}[Y > k]^{-1} = \sum_{l=1}^{N-1} (\lambda+\varepsilon)^{-\alpha l}.$$

The claim now follows by letting $\varepsilon \searrow 0$. □

Proof of Theorem 4.4. Because of both Lemma 4.10, which covers the case $N = \infty$, and Lemma 4.11, it suffices to prove that, for fixed $2 \leq N < \infty$,

$$L_N^+ := \limsup_{k\to\infty} \ \mathbb{P}[\tau < N \mid Z_0 = k] \leq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

As in the proof of Lemma 4.10, we can assume that the offspring distribution is bounded and, in particular, all exponential moments of ξ are finite. Let us verify, for arbitrary $\varepsilon_1 \in (0,1)$ with $\lambda_0 := \lambda - 2\varepsilon_1 > 1$,

$$L_N^+ \leq \sum_{l=1}^{N-1} \lambda_0^{-\alpha l}. \tag{4.15}$$

Let $\lambda_1 := \lambda - \varepsilon_1$ and define, for all $k \geq 1$ and $l = 1, \dots, N-1$, the events

$$B_{k,l} := \left\{ \sum_{j=1}^{\lfloor k\lambda_1^l \rfloor} \xi_{l,j} \geq k\left(\lambda - \frac{\varepsilon_1}{2}\right)\lambda_1^l \right\}, \quad B_k := \bigcap_{l=1}^{N-1} B_{k,l}.$$

For all $l = 1, \dots, N-1$, the Cramér-Chernoff method or a sub-Gaussian concentration estimate implies that $\mathbb{P}[B_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast, and hence

$$L_N^+ = \limsup_{k\to\infty} \mathbb{P}[\tau < N,\ B_k \mid Z_0 = k]\ \mathbb{P}[Y > k]^{-1}.$$

Consider the events

$$C_k := \left\{\exists l \in \{1, \dots, N-1\} :\ Y_l > k\lambda_0^l\right\}, \quad k \geq 1.$$

Then, by (REG),

$$\begin{aligned}\limsup_{k\to\infty}\ \mathbb{P}[C_k]\ \mathbb{P}[Y > k]^{-1} &\leq \limsup_{k\to\infty} \sum_{l=1}^{N-1} \mathbb{P}\left[Y_l > k\lambda_0^l\right]\ \mathbb{P}[Y > k]^{-1} \\ = \sum_{l=1}^{N-1} \lim_{k\to\infty} \mathbb{P}\left[Y > k\lambda_0^l\right]\ \mathbb{P}[Y > k]^{-1} &= \sum_{l=1}^{N-1} \lambda_0^{-\alpha l},\end{aligned}$$

and hence, in order to obtain the inequality (4.15), it suffices to show

$$\lim_{k\to\infty}\ \mathbb{P}\left[\tau < N,\ B_k,\ C_k^c \mid Z_0 = k\right]\ \mathbb{P}[Y > k]^{-1} = 0. \tag{4.16}$$

Let $\varepsilon_2 \in (0,1)$ and introduce, for all $k \geq 1$, the random variable

$$R_k := \#\left\{l = 1, \dots, N-1 \mid Y_l \geq \varepsilon_2 k\lambda_0^l\right\}.$$

Then, since (REG) holds and $(Y_m)_{m\geq 1}$ is i.i.d., we easily obtain

$$\limsup_{k\to\infty}\ \mathbb{P}\left[\tau < N,\ B_k,\ C_k^c,\ R_k \geq 2 \mid Z_0 = k\right]\ \mathbb{P}[Y > k]^{-1} = 0. \tag{4.17}$$

For a given $\varepsilon_1 > 0$, choose $0 < \varepsilon_2 < \varepsilon_1/2$. Then, for every $k \geq 1$, by inserting the definition of B_k, R_k, and $(Z_n)_{n\geq 0}$, we can deduce

$$\mathbb{P}[\tau < N,\ B_k,\ R_k = 0 \mid Z_0 = k] = 0.$$

In particular, we obtain

$$\limsup_{k\to\infty}\ \mathbb{P}\left[\tau < N,\ B_k,\ C_k^c,\ R_k = 0 \mid Z_0 = k\right]\ \mathbb{P}[Y > k]^{-1} = 0. \tag{4.18}$$

Combining (4.17) and (4.18), in order to verify (4.16), we only need to show

$$\limsup_{k\to\infty} \mathbb{P}\left[\tau < N,\ C_k^c,\ R_k = 1 \mid Z_0 = k\right] \mathbb{P}[Y > k]^{-1} = 0. \tag{4.19}$$

Let $k \geq 1$ and $l = 1, \ldots, N-1$. Then, we introduce events $B'_{k,l,1}, \ldots, B_{k,l,N-1}$ by

$$B'_{k,l,r} := \left\{ \sum_{j=1}^{\lfloor kx_{r-1} \rfloor} \xi_{r,j} \geq \lambda_1 k x_{r-1} \right\}, \quad r = 1, \ldots, N-1,$$

where $x_0 := 1$ and

$$x_{r+1} := \lambda_1 x_r - b_r, \qquad b_r := \begin{cases} \lambda_0^r, & r = l \\ \varepsilon_2 \lambda_0^r, & r \neq l. \end{cases}$$

For all $k \geq 1$, denote by B'_k the event that all events $B'_{k,l,r}$, $l, r = 1, \ldots, N-1$, occur. Then, again by the Cramér-Chernoff method or the sub-Gaussian concentration inequality, we know

$$\begin{aligned} &\limsup_{k\to\infty} \mathbb{P}\left[\tau < N,\ C_k^c,\ R_k = 1 \mid Z_0 = k\right] \mathbb{P}[Y > k]^{-1} \\ =&\limsup_{k\to\infty} \mathbb{P}\left[\tau < N,\ B'_k,\ C_k^c,\ R_k = 1 \mid Z_0 = k\right] \mathbb{P}[Y > k]^{-1}. \end{aligned}$$

According to Lemma 4.15 from Appendix B, for every $\varepsilon_1 > 0$, it is possible to choose $\varepsilon_2 > 0$ small enough to ensure

$$\mathbb{P}\left[\tau < N,\ B'_k,\ C_k^c,\ R_k = 1 \mid Z_0 = k\right] = 0 \quad \text{for all } k \geq 1,$$

where we have inserted the definition of the events B'_k and C_k^c, as well as the definition of the random variable R_k and the process $(Z_n)_{n\geq 0}$. In particular, by choosing $\varepsilon_2 > 0$ small enough, (4.19) follows. □

4.6 Proof of Theorem 4.5

In the following, we will again work with the branching process $(Z'_n)_{n\geq 0}$, but more generally assume $Z'_0 = k' \geq 1$ and possibly $k \neq k'$. Let $q' \in [0, 1)$ denote the extinction probability of $(Z'_n)_{n\geq 0}$ given $k' = 1$ and recall the existence of the almost sure martingale limit

$$W' := \lim_{n\to\infty} \lambda^{-n} Z'_n \in [0, \infty).$$

By the Kesten-Stigum theorem [58], it is known that $W' = 0$ almost surely if and only if $\mathbb{E}[\xi \log_+(\xi)] = \infty$. Moreover, if $\mathbb{E}[\xi \log_+(\xi)] < \infty$, then, given $(Z'_n)_{n\geq 0}$ survives forever, $W' > 0$ almost surely.

Our main idea is to divide the population into two groups. Then, if the emigration is weak, it may only affect one of these groups.

Lemma 4.12 (Decomposition)**.** *Fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$ and $k' := k_0 - k$. Let $Z_1^{(1)} := k$, $Z_1^{(2)} := k'$, and define, for all $n \geq 1$, recursively*

$$Z_{n+1}^{(1)} := \left(\sum_{j=1}^{Z_n^{(1)}} \xi_{n+1,j} - Y_{n+1} \right)_+, \qquad Z_{n+1}^{(2)} := \sum_{j=Z_n^{(1)}+1}^{Z_n^{(1)}+Z_n^{(2)}} \xi_{n+1,j},$$

Then,

$$\left(Z_n^{(1)}\right)_{n\geq 1} \overset{\mathrm{d}}{=} (Z_n)_{n\geq 0}, \qquad \left(Z_n^{(2)}\right)_{n\geq 1} \overset{\mathrm{d}}{=} (Z'_n)_{n\geq 0}, \tag{4.20}$$

and if (IND) *holds, then the processes $\left(Z_n^{(1)}\right)_{n\geq 0}$ and $\left(Z_n^{(2)}\right)_{n\geq 0}$ are independent. Moreover, for all $n \geq 1$,*

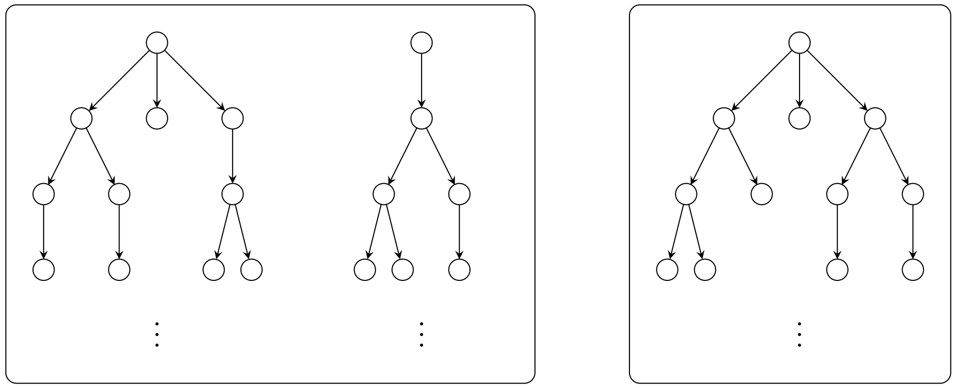
$$Z_n = Z_n^{(1)} + Z_n^{(2)} \quad \text{on the event} \quad \{Z_1 = k_0\} \cap \left\{Z_n^{(1)} > 0\right\}, \tag{4.21}$$
$$Z_n \geq Z_n^{(1)} + Z_n^{(2)} \quad \text{on the event} \quad \{Z_1 \geq k_0\} \cap \left\{Z_n^{(1)} > 0\right\}. \tag{4.22}$$

Proof of Lemma 4.12. By construction, $\left(Z_n^{(1)}\right)_{n\geq 0}$ and $\left(Z_n^{(2)}\right)_{n\geq 0}$ are time-homogeneous Markov chains with the same initial state and transition probabilities as $(Z_n)_{n\geq 0}$ respectively $(Z'_n)_{n\geq 0}$. Hence, (4.20) holds.

By inserting the definitions of $Z_n^{(1)}$ and $Z_n^{(2)}$, one can straightforwardly verify both (4.21) and (4.22). The details are therefore omitted.
Finally, assume (IND) and let $a_1, a_2, b_1, b_2 \in \mathbb{N}$. Then, for all $n, m \geq 1$,

$$\begin{aligned}
&\mathbb{P}\left[Z_{n+1}^{(1)} = a_2,\ Z_{m+1}^{(2)} = b_2 \mid Z_n^{(1)} = a_1,\ Z_m^{(2)} = b_1\right] \\
&= \mathbb{P}\left[\left(\textstyle\sum_{j=1}^{a_1} \xi_{n+1,j} - Y_{n+1}\right)_+ = a_2,\ \textstyle\sum_{j=a_1+1}^{a_1+b_1} \xi_{m+1,j} = b_2\right] \\
&= \mathbb{P}\left[\left(\textstyle\sum_{j=1}^{a_1} \xi_{n+1,j} - Y_{n+1}\right)_+ = a_2\right] \mathbb{P}\left[\textstyle\sum_{j=a_1+1}^{a_1+b_1} \xi_{m+1,j} = b_2\right] \\
&= \mathbb{P}\left[Z_{n+1}^{(1)} = a_2 \mid Z_n^{(1)} = a_1\right] \mathbb{P}\left[Z_{m+1}^{(2)} = b_2 \mid Z_m^{(2)} = b_1\right].
\end{aligned}$$

Hence transitions of $\left(Z_n^{(1)}\right)_{n\geq 1}$ and $\left(Z_n^{(2)}\right)_{n\geq 0}$ are independent, and the claim follows by recalling that the initial states are chosen constant. □



$\left(Z_n^{(1)}\right)_{n\geq 0}$ $\left(Z_n^{(2)}\right)_{n\geq 0}$

Figure 4.1: Illustration of the decomposition in Lemma 4.12. We divide the Galton-Watson tree in two groups. Under suitable conditions, the first group may survive forever even if the whole emigration component is always subtracted from it. On this event, the second group is shielded from emigration, and the entire population is the superposition of a branching process with emigration and branching process without migration.

Proof of Theorem 4.5. (a). Let $\mathbb{P}[W > 0] > 0$. Then $\mathbb{P}[\tau = \infty] > 0$ and hence, by Theorem 4.2, we immediately obtain $\mathbb{E}[\log_+(Y)] < \infty$. Besides, using $Z_n \leq Z'_n$ for $Z_0 = Z'_0 = k$ and applying the classical Kesten-Stigum theorem, see [58], we directly conclude $\mathbb{E}[\xi \log_+(\xi)] < \infty$.

On the contrary, let us assume $\mathbb{E}[\log_+(Y)] < \infty$ and $\mathbb{E}[\xi \log_+(\xi)] < \infty$. Then, due to Theorem 4.2, $\mathbb{P}[\tau = \infty] > 0$, and hence it suffices to show

$$\mathbb{P}[W > 0] = \mathbb{P}[\tau = \infty].$$

In order to obtain this claim, we note that $\{W > 0\} \subseteq \{\tau = \infty\}$ and verify

$$\mathbb{P}[W = 0,\ \tau = \infty] = 0. \tag{4.23}$$

By (H1) and (H2), we know $\{\tau = \infty\} = \{Z_n \to \infty\}$ almost surely. Moreover, W is monotone with respect to $Z_0 = k$. Consequently,

$$\begin{aligned} \mathbb{P}[W = 0,\ \tau = \infty] &\leq \liminf_{k\to\infty}\ \mathbb{P}[W = 0 \mid Z_0 = k] &(4.24)\\ &= 1 - \limsup_{k\to\infty}\ \mathbb{P}[W > 0 \mid Z_0 = k]. &(4.25)\end{aligned}$$

Choose $\varepsilon > 0$ with $\lambda - \varepsilon > 1$ and set $k_0(k) := \lfloor(\lambda - \varepsilon)k\rfloor$ for all $k \geq 1$. Then, by the strong law of large numbers,

$$\limsup_{k\to\infty} \mathbb{P}[Z_1 \geq k_0(k) \mid Z_0 = k] = 1 \tag{4.26}$$

and $k_0(k) - k \to \infty$ for $k \to \infty$. Fix $k \geq 1$, $k_0 = k_0(k)$, and assume $Z_0 = k$. Then, by making use of the notation introduced in Lemma 4.12 and (4.22),

$$\mathbb{P}[W > 0] \geq \mathbb{P}[Z_1 \geq k_0]\ \mathbb{P}\big[\forall n \geq 0:\ Z_n^{(1)} > 0,\ \lim_{n\to\infty} \lambda^{-n} Z_n^{(2)} > 0\big]. \tag{4.27}$$

Recalling (4.20) and Proposition 4.9, we know

$$\mathbb{P}\big[\forall n \geq 0:\ Z_n^{(1)} > 0\big] = 1 - q_k \to 1 \quad \text{for } k \to \infty. \tag{4.28}$$

On the other hand, by (4.20) and the Kesten-Stigum theorem [58],

$$\mathbb{P}\left[\lim_{n\to\infty} \lambda^{-n} Z_n^{(2)} > 0\right] = \mathbb{P}\left[W' > 0 \mid Z_0' = k_0(k) - k\right] = 1 - (q')^{k_0(k)-k},$$

and, since $k_0(k) - k \to \infty$ for $k \to \infty$, we further obtain

$$\lim_{k\to\infty} \mathbb{P}\left[\lim_{n\to\infty} \lambda^{-n} Z_n^{(2)} > 0\right] = 1. \tag{4.29}$$

By combining (4.26), (4.28), and (4.29) with (4.27), we conclude

$$\limsup_{k\to\infty} \mathbb{P}[W > 0 \mid Z_0 = k] = 1,$$

and hence (4.23) and the claim follows by recalling (4.24) and (4.25).

(b). First, let $a = 0$. Assume that there exists $b \in (0, \infty)$ satisfying $\mathbb{P}[0 < W < b] = 0$ and $\mathbb{P}[0 < W < b + \varepsilon] > 0$ for all $\varepsilon > 0$. Then, we choose $\varepsilon > 0$ and $\delta > 0$ with

$$\tilde{b} := \lambda^{-1}(b + \varepsilon) + \delta < b. \tag{4.30}$$

Also, fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$, and again recall the notation from Lemma 4.12. Then, by the decomposition (4.21) and (4.30),

$$\begin{aligned} &\mathbb{P}[0 < W < \tilde{b}] \\ &\geq \mathbb{P}\left[Z_1 = k_0\right]\ \mathbb{P}\Big[\lim_{n\to\infty} \lambda^{-n} Z_n^{(1)} \in \big(0, \lambda^{-1}(b + \varepsilon)\big)\,,\ \lim_{n\to\infty} \lambda^{-n} Z_n^{(2)} < \delta\Big]. \end{aligned}$$

By using (IND) and Lemma 4.12, we further deduce

$$\mathbb{P}[0 < W < \tilde{b}] \geq \mathbb{P}[Z_1 = k_0]\ \mathbb{P}[0 < W < b + \varepsilon]\ \mathbb{P}[0 < W' < \lambda\delta],$$

where we assume $Z_0 = k$ and $Z_0' = k_0 - k$. The first two probabilities on the right-hand side of this inequality are positive by construction. The third factor is also positive. This follows, for example, from the fact that W' has a strictly positive Lebesgue density on $(0, \infty)$, see, for example, [5, Chapter 1, Part C]. Consequently, $\mathbb{P}[0 < W < \tilde{b}] > 0$, which is a contradiction to our assumptions on $b \in (0, \infty)$. Hence, the claim is true if $a = 0$.

For arbitrary $a > 0$, again fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$ and also $\varepsilon > 0$ with $\varepsilon < b - a$. Then, by the same arguments as for $a = 0$, and again assuming $Z_0 = k$ and $Z_0' = k_0 - k$, we obtain

$$\mathbb{P}[a < W < b] \geq \mathbb{P}[Z_1 = k_0]\ \mathbb{P}[0 < W < \lambda\varepsilon]\ \mathbb{P}[\lambda a < W' < \lambda(b - \varepsilon)].$$

Since we have verified the claim for $a = 0$, we know that the second factor on the right-hand side of this inequality is positive. Our choice of ε implies that also the third factor is positive. Hence, as for $a = 0$, we can indeed deduce $\mathbb{P}[a < W < b] > 0$. □

4.7 Appendix A. Truncation of the reproduction law

In some of our proofs, we make use of the following observation.

Observation 4.13 (Truncation of the offspring distribution)**.** Let $(Z_n)_{n \geq 0}$ denote a branching process with emigration, which is defined recursively by equation (4.1) under the same assumptions as in the introduction. Moreover, let $\lambda := \mathbb{E}[\xi_{1,1}] \in (0, \infty]$ and assume that the distribution of $\xi_{1,1}$ is unbounded. Then, for every $s \in (0, \infty)$, there exists $N \in \mathbb{N}$ with the following property. If we define, for all $n, j \geq 1$,

$$\tilde{\xi}_{n,j} := \begin{cases} \xi_{n,j}, & \xi_{n,j} \leq N \\ N, & \xi_{n,j} > N, \end{cases}$$

as well as $\tilde{Z}_0 := Z_0 := k$, and, recursively

$$\tilde{Z}_{n+1} := \left(\sum_{j=1}^{\tilde{Z}_n} \tilde{\xi}_{n+1,j} - Y_{n+1} \right)_+, \quad n \geq 0,$$

then we arrive at a branching process with emigration $(\tilde{Z}_n)_{n\geq 0}$, which satisfies $\tilde{Z}_n \leq Z_n$ almost surely for all $n \geq 0$, has a bounded offspring distribution and, if $\lambda = \infty$, then $s < \mathbb{E}[\tilde{\xi}_{1,1}] < \infty$, respectively $0 < \lambda - \mathbb{E}[\tilde{\xi}_{1,1}] < s$ in case of $\lambda < \infty$. In particular, if $\tilde{\tau}$ denotes the extinction time of the process $(\tilde{Z}_n)_{n\geq 0}$, then $\tau \geq \tilde{\tau}$ almost surely.

Moreover, if the condition (IND) respectively (H2) is satisfied for the process $(Z_n)_{n\geq 0}$, then, by construction, the corresponding condition also holds for $(\tilde{Z}_n)_{n\geq 0}$. Finally, if $\lambda \in (1, \infty)$ and $s < \lambda - 1$, then, as for the process $(Z_n)_{n\geq 0}$, we know that the condition (H1) holds for $(\tilde{Z}_n)_{n\geq 0}$ provided $Z_0 = \tilde{Z}_0 = k$ is chosen large enough.

4.8 Appendix B. Notes on the recursion $x_{n+1} = ax_n - b_n$

The following two claims can be proved by elementary arguments. We omit the details.

Lemma 4.14. *Let $x_0 := 1$, $a \in (1, \infty)$; and $\varepsilon > 0$ with $a - \varepsilon_1 > 1$. Then there exists $N \in \mathbb{N}$ and $c_1, \ldots, c_N \in (0, \infty)$ such that for the sequence $(x_n)_{n\geq 0}$ defined by*

$$x_{n+1} := ax_n - b_n, \quad \text{where} \quad b_n := \begin{cases} (a-\varepsilon)^{n+1}, & n \geq N \\ c_{n+1}, & n \leq N-1, \end{cases}$$

there exists $c > 1$ with $x_n \geq c^n > 0$ for all $n \geq 1$.

Lemma 4.15. *Let $N \in \mathbb{N}$, $x_0 := 1$, $a \in (1, \infty)$, and $\varepsilon_1 > 0$ with $a - \varepsilon_1 > 1$. Then, there exists $\varepsilon_2 > 0$ with the following property. For arbitrary $l \in \{1, \ldots, N-1\}$, the recursion*

$$x_{n+1} := ax_n - b_n, \quad \text{where} \quad b_n := \begin{cases} (a-\varepsilon_1)^l, & n = l, \\ \varepsilon_2(a-\varepsilon_1)^n, & n \neq l, \end{cases}$$

defines reals $x_1, \ldots, x_N$ with $x_j \geq \varepsilon_1 > 0$ for all $j = 1, \ldots, N$.

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List of Figures

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