Reduction of the Classical Mechanics with respect to the Similarity Group Sim(3)

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Abstract

One of the most important goals of research in physics is to find the most basic and universal theories that describe our universe. Many theories assume the presence of absolute space and time in which physical objects are located, and physical processes occur. However, it is more fundamental to understand time as relative to the motion of another object, such as the number of swings of a pendulum and the position of an object primarily relative to other objects. The purpose of this thesis is to explain how classical mechanics can be formulated using the principle of relationalism (introduced below) on a most elementary space which is freed from absolute entities: shape space. In shape space, only the relative orientation and length of subsystems are considered. A sufficient requirement for the validity of the principle of relationalism is that when the scale variable of a system changes, all parameters of the theory that depend on the length change accordingly. In particular, the principle of relationalism requires an appropriate transformation of the coupling constants of the interaction potentials in classical physics. Consequently, this change leads to a transformation of Planck's measuring units, which allows us to derive a metric on shape space in a unique way. In particular, we explain in two different ways how to find the unique metric of shape space, taking into account the crucial role of rulers in determining the geometry of a space.

In order to find out the classical equations of motion on shape space, the method of "symplectic reduction of Hamiltonian systems" is extended to include scale transformations. In particular, we will give the derivation of the reduced Hamiltonian and symplectic form on shape space, and in this way, the reduction of a classical system with respect to the entire similarity group is achieved.

One can alternatively use the Lagrangian formalism of mechanics to derive the reduced equations of motion on shape space. It will be explained how the Principle of Relationalism makes the Lagrangian of the classical mechanics scale-invariant, which in turn ensures the existence of laws of motion on shape space. In order to find out these laws of motion, the Boltzman-Hammel equations of motion in an anholonomic frame on tangent space to system's absolute configuration space $T(Q)$, is adapted to the $Sim(3)$-fiber bundle structure of the configuration space Q. The derived equations of motion on shape space enable us, among others, to predict the evolution of the shape of a classical system without any reference to its absolute position, orientation, or size in absolute space. Under the action of the group of scale transformations Sc, the internal configuration space $Q_{int} := \frac{Q}{E(3)}$ becomes a fiber-bundle whose base space is shape space. It has been explicitly shown that the connection form of the Q_{int} considered as the Sc-fiber-bundle is flat.

After treating the general N-body system, shape equations of motion of a three body system are derived explicitly as an illustration of the general method, after which some cosmological implications of the scale-invariant classical mechanics are presented. In particular, we explain how the observed universe's accelerated expansion follows from the conservation of the dilational momentum in the modified Newtonian theory. Finally, we compare the present work with two other approaches to relational physics and discuss their essential differences.

This thesis is based on the preprints [1][1] and [2] .

[1] "Reproduced with permission from Springer Nature Journal"

Zusammenfassung

Eines der wichtigsten Ziele der physikalischen Forschung ist es, die grundlegendsten und universellsten Theorien zu finden, die unser Universum beschreiben und dessen Verhalten erklären. Viele Theorien gehen von der Existenz eines absoluten Raums und einer absoluten Zeit aus, in denen sich die physikalischen Objekte befinden und die physikalische Prozesse stattfinden. Es ist jedoch fundamentaler, die Zeit relativ zur Bewegung eines anderen Objekts zu verstehen, z. B. die Anzahl der Schwingungen eines Pendels und die Position eines Objekts von vornherein relativ zu anderen Objekten zu definieren. Diese Dissertation soll erklären, wie die klassische Mechanik unter Verwendung des Prinzips des Relationalismus (das unten eingeführt wird) auf einem elementarsten Raum formuliert werden kann, der von absoluten Grössen befreit ist: dem Shaperaum. Im Shaperaum werden nur die relative Orientierung und Länge von Subsystemen berücksichtigt. Eine hinreichende Voraussetzung für die Gültigkeit des Prinzips des Relationalismus ist, dass durch die Änderung der Skalenvariablen eines Systems (des Universums) alle von der Länge abhängigen Parameter der Theorie entsprechend geändert werden. Insbesondere das Prinzip des Relationalismus erfordert in der klassischen Physik eine bestimmte Transformation der Kopplungskonstanten der Wechselwirkungspotentiale. Diese Änderung führt folglich zu einer Transformation der Planckschen Maßeinheiten, die es uns ermöglicht, auf eindeutige Weise eine Metrik auf dem Shaperaum herzuleiten.

Um die klassischen Bewegungsgleichungen auf dem Shaperaum zu finden, wird die Methode der "Symplektischen Reduktion Hamiltonscher Systeme" um die Skalierungtransformationen erweitert. Insbesondere werden wir die Herleitung der reduzierten Hamiltonian und der symplektischen Form auf dem Shaperaum angeben. Damit wird die Reduktion eines klassischen Systems bezüglich der gesamten Ähnlichkeitsgruppe erreicht.

Wir können alternativ den Lagrange-Formalismus der Mechanik verwenden, um die reduzierten Bewegungsgleichungen auf dem Shaperaum herzuleiten. Es wird erklärt, wie das Prinzip des Relationalismus die Lagrange-Funktion der klassischen Mechanik skaleninvariant macht, was wiederum die Existenz den Bewegungsgesetzen im Shaperaum sicherstellt. Um diese Bewegungsgesetze herauszufinden, werden die Boltzman-Hammel-Bewegungsgleichungen in einem nichtholonomen System im Tangentialraum zum absoluten Konfigurationsraum $T(Q)$ des Systems an die $Sim(3)$-Faserbündelstruktur des Konfigurationsraums Q angepasst. Die hergeleiteten Bewegungsgleichungen im Shaperaum ermöglichen es uns unter anderem, die Entwicklung der Form eines klassischen Systems ohne Bezug auf seine absolute Position, Orientierung oder Größe im absoluten Raum vorherzusagen. Dazu reichen die Angabe eines Punktes und eines Geschwindigkeitsvektors auf dem Shaperaum als Anfangsbedinugen aus, wenn wir die zwei Erhaltungsgrössen D und L als Teil des Bewegungsgesetzes auf dem Shaperaum betrachten. Der Internalkonfigurationsraum $Q_{int} := \frac{Q}{E(3)}$ wird unter der Wirkung der Gruppe der Skalentransformationen Sc zu einem Faserbündel, dessen Basisraum der Shaperaum ist. Es wird explizit gezeigt, dass die Connectionform des als Sc-Faserbündel betrachteten Q_{int} flach ist.

Nach der Behandlung des N-Körpersystems werden zur Veranschaulichung der allgemeinen Methode explizit Shapegleichungen der Bewegung eines Dreikörpersystems hergeleitet, wonach einige kosmologische Implikationen der modifizierten(skaleninvarianten) klassischen Mechanik vorgestellt werden. Insbesondere, erklären wir, wie die beschleunigte Ausdehung des beobachteten Universums aus der Erhaltung des Dehnungsimpulses in der modifizierten Newtonschen Theorie folgt. Abschließend, werden die Inhalte der vorliegenden Arbeit mit zwei anderen Ansätzen der relationalen Physik verglichen und deren wesentliche Unterschiede diskutiert.

Diese Arbeit basiert auf den Aufsätzen [1][2] und [2] .

[2] "Reproduziert mit Genehmigung von Springer Nature Journal"

Preface

This manuscript can be divided to six parts.

In the *first part* (Chapter 1), we review the foundations of classical mechanics with an emphasis on the ideas of *Gottfried Wilhelm Leibniz*, and compare the Leibnizian relational worldview with absolute worldview of *Isaac Newton*. One of the most basic building blocks of Newtonian mechanics is the idea of an absolute time and space, the existence of which were assumed by Newton when formulating the laws of motion. We will take the absoluteness of space and time into question and will review how Newton's absolute time can be deduced from the change in the positions of particles. By this approach, time loses its status as a primitive notion in physics, and takes an emergent status instead. Our dynamics will be defined on shape space[3] S, for which, in contrast to the configuration space used in Newtonian Mechanics, time and space are not absolute entities. The central new concept in the discussion is the *principle of relationalism*, which leads us to consider particular constants of nature as homogeneous functions of proper degrees on the universe's configuration space. The theory developed along these lines in Sections (1.4.1) and (1.4.2) has the full similarity group $Sim(3)$ as its symmetry group.

In the *second part* (Chapters 2 and 3) of this dissertation we give a review of the literature on symplectic reduction of classical systems with respect to the Euclidean group $E(3)$. Here we follow [3] and [4] to a big extend.

In the *third part* of the dissertation (in Chapter 4), we explain how these methods can be expanded to include scale transformations, and consequently how the reduction of a classical system with respect to the full similarity group $sim(3)$ (considered throughout this manuscript to be the symmetry group), is achieved. Here we explain among others, how the kinetic metric of configuration space leads in a unique way to a metric on shape space, using the principle of relationalism.

[3]Quotient of the absolute configuration space $Q \cong \mathbb{R}^{3N}$ with respect to the similarity group $sim(3)$, which comprises all global spatial translations, rotations, and scalings.

Given the known procedures for deriving the reduced equations of motion with respect to the Euclidean group $E(3)$ in the Hamiltonain formalism using a symplectic structure on phase space, we derive the reduced symplectic form, and the Hamiltonian of a N-particle system on its reduced phase space with respect to the similarity group. This suffices to determine the evolution of a classical system on shape space, given its initial shape and shape velocities, without any reference to the system's orientation, position, or scale in absolute space.

At Chapter 5, we revisit the principle of relationalism and present a more concrete mathematical expression of it both in Newtonian and Leibnizian world-views. We also state other principles in relational mechanics and compare them with ours.

In the remaining parts of this manuscript we aim at deriving equations of motion of a classical system on shape space in the context of Lagrangian mechanics. In particular, in the *fourth part*, following [5], [6], and [7] to a big extend, we first review in Chapter 6 the geometric setting on the center of mass configuration space Q_{cm} as a $SO(3)$-fiber-bundle. We then explain in Chapter 7 how this setting can be expanded to scale transformations, and the construction of the $Sim(3)$-fiber bundle is discussed. Here we explain among others, how a metric $\mathbf{N}$ on shape space can be derived in a unique way. Under the action of the group of scale transformations Sc, the internal configuration space $Q_{int} := \frac{Q}{E(3)}$ becomes a fiber bundle, whose base space is shape space. It has been explicitly shown that the connection form of the Q_{int} considered as the Sc-fiber-bundle is flat. In Chapter 8, we first review the Lagrangian formulation of mechanics in anholonoimic frames, and their Boltzmann-Hamel equations of motion. Thereafter, we derive the equations of motion on shape space.

In the *fifth part* (in Chapters 9 and 10) we derive explicitly the shape equations of motion of a three-body system, and at last we discuss some cosmological consequences like accelerated expansion of the universe, and the total collision singularity in the classical mechanics.

Research in relational physics has a rich and long history, and there are many important attempts at implementing relational ideas in physics. See for instance [8],[9],[10] for more information. In the last part (in Chapter 11), we will give a quick comparison of our work with two of the other approaches in relational physics. The first alternative approach (denoted here by BKM) is based on the mechanical similarities in Newtonian mechanics, as is developed and elaborated in [11],[12],[13],[14]. The second approach(denoted here by BDGZ) is based on the geodesic dynamics on shape space, as is developed and expanded in [15], [16], [17]. In particular, we will explain how the BKM-approach and the BDGZ-approach differ from our work but still both satisfy the Principle of Relationalism.

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Symbols

x	A point on $Q_{cm} = \frac{Q}{\mathbb{R}^3}$
q	A point on $Q_{int} = \frac{Q_{cm}}{SO(3)}$
s	A point on shape space $S = \frac{Q}{Sim(3)}$
$\boldsymbol{r}_i$	i's Jacobi vector of an N-particle system
λ	Scale variable of a system
$\dot{\lambda}$	Scale velocity of a system
$\dot{\boldsymbol{\lambda}}$	Scale velocity of a system measured in internal units
A	Moment of inertia tensor of a N-particle system
$\mathbf{A}$	Gauge fields on Q_{int}
$\mathcal{A}_g$	Action of $g \in Sim(3)$ on $T(Q)$
$\mathbf{D}$	Dilational momentum operator
D	value of system's dilational momentum measured in internal units
$\mathbf{M}$	Mass metric on Q_{cm} or Q
$\mathbf{M}^{(m)}$	The measured mass metric on Q_{cm} or Q
B	Metric on Q_{int}
$\mathbf{N}$	Metric on shape space S
I_x	The canonical isomorphism from tangent space to cotangent space of Q
α, β, γ	Euler angles connecting a body frame and the space frame
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	Fixed laboratory frame, or space frame
$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$	Body frame
g	Rotation which brings the space frame to the body frame
$\mathbf{J} = \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times \frac{\partial}{\partial x_\alpha}$	Total angular momentum
Ω^a	Components of angular velocity in space frame
Ω'^a	Components of angular velocity in body frame
$\mathbf{J}$	$= \sum_{a=1}^{3} \mathbf{e}_a J_a = \sum_{a=1}^{3} \mathbf{e}'_a L_a$
J_a	$= (\mathbf{e}_a \mid \mathbf{J})$
L_a	$= (\mathbf{e}'_a \mid \mathbf{J})$ Left invariant vector fields on $SO(3)$
$J_a \mathbf{r}_i$	$= \mathbf{e}_a \times \mathbf{r}_i$
$L_a \mathbf{r}_i$	$= \mathbf{e}'_a \times \mathbf{r}_i = g\big(\mathbf{e}_a \times \sigma_i(q)\big)$
$\omega^a(J_b)$	$= \delta^a_b$
$\omega'^a(L_b) = \theta^a(L_b)$	$= \delta^a_b$
θ^a	Left invariant one forms on $SO(3)$
$g^{-1}dg$	$= \sum_{a=1}^{3} \theta^a R(\mathbf{e}_a)$
ψ^a	Right invariant 1-forms on $SO(3)$
dgg^{-1}	$= \sum_{a=1}^{3} \psi^a R(\mathbf{e}_a)$
$\boldsymbol{k}$	Curvature tensor of shape space
$\boldsymbol{c}$	Speed of light
c	Letter used to characterize scale transformations by a factor $c \in \mathbb{R}^+$
Sc	Group of spatial scale transformations(of matter)
G_{rs}	Group of spatial rotations and scale transformations

Contents

Chapter 1

Relationalism

1.1 Shape Space as the Physical Configuration Space

Physics aims to give a most accurate description, prediction, and understanding of nature and its phenomena. Imagine an experimental physicist in a laboratory, watching a specific physical phenomenon unfolding itself in front of his eyes. Imagine now an identical universe, which is translated by some amount, rotated by some angle, and dilated by some scale factor with respect to this universe. Would anything different from the first universe be observed by the physicist in the lab watching the phenomenon he was interested in? In other words, can the experimenter tell in which of these two possible universes he finds himself/herself? In fact, the physicist is fully blind to all of these global operations. By moving all the objects in the universe one meter to the left, the distances between the objects would not change at all. That is why the physicist would never see (measure) any difference concerning his state, the state of his environment, or even the universe. Intuitively[1], one expects neither any difference in how the phenomenon would unfold in front of him. From an internal point of view, the universe seems exactly

[1] for a relationalist

the same, whether it is located here or one meter to its left. It looks exactly the same after a total rotation of the universe by some degree or scaling of the universe (hence all the inter-particle distances) with some constant. One could object that by scaling the universe, the distances between the objects would also get scaled; hence, an internal observer would be able to observe this difference. However, as length measurements require rulers, and the same factor scales up the inter-particle distances of the rulers as it does all other distances, the internal observer can not notice any difference. Thus, two configurations of the universe that can be transformed into each other by a member of the similarity group $Sim(3)$ are kinematically indistinguishable from an internal point of view. There might be a difference for an external observer, but any discussion on what an external observer of the entire universe would see is purely academic; at best, it has a philosophical meaning but is irrelevant to any physical descriptions.

To illustrate the core concepts of the subject matter and review the definition of shape space, let us start with an example of a toy universe that consists of only three particles located in absolute space $\mathbb{R}^3$. Three coordinates specify each particle's position; hence nine numbers are needed to specify the configuration of this system. However, as we explained in the last paragraph, this is what an *external* observer watching these three particles in absolute space would say. From an *internal* point of view, for example, from the point of view of one of the three particles, not more than 2 degrees of freedom can be observed: the two angles of the triangle formed by these three particles. As explained in the last paragraph, this is because the absolute position (of, for instance, the center of mass), orientation, and scale of the system of three particles are unobservable from an internal point of view. One needs three numbers to specify the system's center of mass, three numbers to specify the system's orientation (e.g., Euler angles w.r.t. some frame of reference), and one number to specify the scale of the system. In other words, the similarity group $Sim(3)$ is seven-dimensional; hence its action on the system's configuration space would lead to a seven-dimensional orbit. As the configuration space of a three-particle system was $3 \times 3 = 9$ dimensional, two dimensions remain, which are called the shape degrees of freedom. In general, observations are always internal; thus, they always take place in shape space. Observations always give a quantity in terms of a pre-defined unit of that quantity. Hence, the numbers we register as the result of measurements are always comparative data, not absolute. Since from an internal point of view, just two angles are observable, one concludes that there are just two physical degrees of freedom for our toy universe. In other words, the physical configuration space(shape space) is two-dimensional, in contrast to the absolute configuration space, which is nine-dimensional.

This toy model can be generalized to N-particles, where N may be as big as the number of elementary particles in the whole universe. In that case, the absolute configuration space Q is a $3N$-dimensional, homogeneous space with $Sim(3)$ as its structure group. The fibers F are the orbits (generically seven-dimensional) generated by the action of $Sim(3)$ on absolute configuration space. The base space $S = \frac{Q}{Sim(3)}$ is then isomorphic to the shape space of the universe.
S can be understood as the *equivalence classes* of points on absolute configuration space where two points are being set equivalent if and only if they can be transformed into each other by a similarity transformation, i.e., for x,y being two points in Q

$$x \sim y \ if \ \exists g \in Sim(3) \mid x = gy$$

In the physics literature one frequently uses the terms "relational configuration space" or *shape space* instead of "physical configuration space". How the objects are located with respect to each other, or, equivalently, which shape they form is the only observable from an internal point of view. Thus formulating the dynamics in this base manifold, S is more fundamental than any description in absolute space. A curve in S corresponds to a unique evolution of the physical degrees of freedom of the system under investigation. These so-called physical degrees of freedom are the only quantities visible (or sensible) to internal observers. Adding the gauge degrees of freedom(the global $Sim(3)$ degrees of freedom), any curve in S represents an infinite number of trajectories in absolute space[2], all of which describe the same phenomena (see discussion above).

[2]Note that a description on phase space, however, often simplifies the equations of motion.

Finally, we want to mention that by considering the shape space of dimension $3N-7$, we tacitly have assumed the existence of a 3-dimensional absolute space. In a more general setting, one should rather start with a m-dimensional shape space where m is not necessarily $3N-7$, and argue how and under which circumstances an apparent three-dimensional absolute space would emerge for subsystems. Given that the effective three-dimensionality of absolute space (at least locally) is an empirical fact, we will consider the case $m = 3N-7$ in the present manuscript and postpone the more general setting to future works.

1.2 Time as an emergent concept

According to the worldview of Newton, there exists an absolute three-dimensional space[3], in which physical objects, e.g., particles, move. The positions of the particles then change as time passes, and Newton's laws tell how the positions change. *Time* is an ever-flowing external entity that exists independently of matter and space. In that sense, time generates the dynamics. Without it, there is no concept of motion. However, Newton acknowledged that only *relative* positions are experimentally observable. In the scholium[4] of his famous book *Principia* [18] he announces to explain how the existence of these absolute structures can be derived from the relative motions of the observable entities. He even claims that this was the central motivation for writing the book [19]; however, he does not return to this later in his book.

Leibniz, on the contrary, was unsatisfied with this way of describing nature. Accepting that the point-like particles of Newtonian Mechanics are the fundamental constituents of the universe, one expects a physical theory to explain, among others, the behavior (in this case, the motions) of these particles. In order to do this, Newton added two extra invisible entities to his description of nature: absolute space and absolute time. Those are essential entities of Newton's laws of motion, especially in his law of inertia. Leibniz, in contrast, thought of them as extra

[3]Whose existence is independent of matter.

[4]A scholium is an explanatory note in a book written by the author himself.

structures that are nonphysical and do not exist independently in nature. In his own words [20]:

"*As for my own opinion, I have said more than once, that I hold space to be something merely relative, as time is, that I hold it to be an order of coexistences, as time is an order of successions.*"

Downgrading space to the *order of coexistences* seems to us a clear denial of Newton's notion of absolute space reviewed above. Also, Ernst Mach expressed in the late 19th century his critique of absolute time [21]:

"*we must not forget that all things in the world are connected with one another and depend on one another, and that we ourselves and all our thoughts are also a part of nature. It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the change of things; made because we are not restricted to any one definite measure, all being interconnected. A motion is termed uniform in which equal increments of space described correspond to equal increments of space described by some motion with which we form a comparison, as the rotation of the earth. A motion may, with respect to another motion, be uniform. But the question whether a motion is in itself uniform, is senseless. With just as little justice, also, may we speak of an absolute time — of a time independent of change. This absolute time can be measured by comparison with no motion; it has therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception.*"

In a relational theory, time is an emergent concept, and it is most rational to define time in such a way that a change in time always relates to a change in the configuration of the system, for example, the change in positions of the atoms forming a pendulum. Time is defined along the trajectories. It is not a concept based on the configurations alone. Any monotonically increasing function f can be used to define the increment of time via

$$\delta t = f(\mid \delta \mathbf{x}_1 \mid, ..., \mid \delta \mathbf{x}_N \mid)$$

for infinitesimally small increments of the particle's positions $\mid \delta \mathbf{x}_i \mid$. Given an arbitrary trajectory on configuration space, including (among others) the two configurations A and B, any monotonous function of the arc length of that trajectory going from configuration A to configuration B can serve, for instance, as a definition of the duration of time passed between A and B. As we will review below, the duration can be indeed defined as a function of the changes in positions in a unique way such that the Newtonian equations of motion are valid. Of course, we are allowed to use any other suitable function as a definition of time. However, any different choice would make the form of the equations of motion different from the ones Newton wrote down. Among all possibilities, the Newtonian time has the advantage of bringing the equations of motion to their simplest form.

1.3 Emergence of time in Classical Mechanics

Originally the principle of least action was developed to justify the equations of motion in different theories. Different brilliant thinkers expressed the idea behind the principle in many different ways. *Pierre Louis Maupertuis* (1698-1759) is usually credited as the first who gave a concrete formulation of the least action principle (although it is suggested that Leibniz was even earlier). Maupertuis' motivation was to rationalize the (by then known) laws of ray optics and mechanics with theological arguments based on design or purpose to explain natural phenomena. Here is a quote from Maupertuis which sheds some light on his views [22]:

"*The laws of movement and of rest deduced from this principle being precisely the same as those observed in nature, we can admire the application of it to all phenomena. The movement of animals, the vegetative growth of plants ... are* *only its consequences; and the spectacle of the universe becomes so much the grander, so much more beautiful, the worthier of its Author, when one knows that a small number of laws, most wisely established, suffice for all movements.*"

Given an initial point in the system's configuration space under consideration q_A and a final point q_B, the path chosen by nature minimizes a functional depending on the trajectories connecting the two endpoints. This functional is usually called "action". In other words: the trajectory between q_A and q_B taken by the system minimizes the value of the action functional. More precisely, the chosen trajectory is a stationary point of the action, but in most cases, the only stationary point is a minimum.

Maupertuis proposed to define action as the integral of the so-called *Vis viva* (Latin for "living force"). The term *Vis viva* was introduced by Leibniz during the 1680s by his observation that the sum of the products of the constituting masses of a system (i.e., a multi-particle system) with the squares of their respective velocities is almost constant during (elastic) collisions, i.e., $\sum_{i=1}^{N} m_i v_i^2 = C$. This is, of course, what we now call the principle of energy conservation (in modern terms, Leibniz's *Vis viva* becomes 2 times the kinetic energy). It seemed to oppose the theory of conservation of momentum advocated by the rival camp (Sir Isaac Newton and Rene Descartes).
Maupertuis' suggestion, therefore, comes down to the following action functional

$$W = \int_{t_A}^{t_B} 2K dt = \int_{q_A}^{q_B} p dq \tag{1.1}$$

which is the right formula for systems where the kinetic energy is quadratic in the velocities. Here the letter t stands for the absolute time of Newton.

Maupertuis' principle states that for the true trajectories (the ones chosen by nature), Maupertius' action W is stationary on all trial trajectories with fixed initial and final positions q_A and q_B and fixed energy[5] $E = K + V$.

$$(\delta W)_E = 0 \tag{1.2}$$

[5]For the most general mechanical systems, energy is $E = \sum_i p_i \dot{q}_i - L$ which reduces to the well-known expression $T + V$ in cases where the Lagrangian can be written as $L = T - V$ with T being a quadratic form in the velocities, and V being independent of the velocities.

where in variational calculus, the constraint of fixed endpoints is usually left implicit, and every other constraint on the trial trajectories (thus, in this case, fixed energy) is written down explicitly. Note that in (1.1) no constraint is imposed on the value of t_B, as for different paths, a different amount of absolute time is needed to reach the endpoint q_B. In other words: t_b stands for the absolute time (moment) at which the configuration q_B is reached, and this varies as the path taken between q_A and q_B changes.

However, in the modern physics literature, the most common action principle is the minimization of Hamilton's action denoted by S. It is defined as an integral along the *spacetime* trajectory $q(t)$ connecting two configurations $q_A = q(t_A)$ and $q_B = q(t_B)$

$$S = \int_{t_A}^{t_B} L(q, \dot{q}) dt \tag{1.3}$$

The statement of **Hamilton's principle** then becomes: among all possible trajectories $q(t)$ that can connect the two configurations q_A and q_B during the exact given time interval $t_B - t_A = T$, the chosen trajectories are those making S minimal (respectively stationary). Thus, Hamilton's principle can be written as

$$(\delta S)_T = 0 \tag{1.4}$$

where, as before, the extra constraint of constant travel time is assumed and explicitly denoted as a subscript. Bear in mind that there may be more than one trajectory satisfying these constraints of fixed endpoints and travel time, see [23]. As mentioned, contrary to Maupertuis' principle, the allowable trial trajectories of Hamilton's principle do not need to satisfy the constant energy constraint *a priori*: the conservation of energy is here a consequence of Hamilton's principle for time-invariant systems (i.e., the Lagrangian does not have an explicit time dependence). Thus, Hamilton's principle (1.4) is applicable to both conservative (time-invariant) and non-conservative systems (i.e., systems that have an explicitly time-dependent Lagrangian due to, for instance, time-dependent potentials $V(q, t)$), while Maupertuis' principle (1.2) is restricted to conservative systems. For conservative systems, one can show that Hamilton and Maupertuis' principles are equivalent and related to each other through the famous Legendre transformations. The results from the action principles are curves that stand for the system's trajectory. It provides us with a manifestly covariant way of describing its evolution. For nonholonomic systems, however, non of these action principles are applicable.

So far, Hamilton's action principle seems more general and powerful than Maupertuis'. *Carl Gustav Jacob Jacobi* (1804 - 1851) thought so, too. However, he wanted to take it one step further by taking Newton's intuition of the existence of an absolute time more seriously by treating time as a variable in the variational calculations. In Newton's spirit, the value of time is as important as the value of the x-component of a particle's position, for example, or any generalized coordinate. Both of them are absolute and have physical reality. So, suppose one wants to apply Hamilton's principle properly. In that case, one should not use the absolute Newtonian time t as an independent variable, but in contrary, all $n+1$ variables q_1,...,q_n,t should be considered as functions of some arbitrary independent variable τ. It enables one to include the variation of t in the variational principle.

Thus, we aim to write Hamilton's action principle for a system containing $n+1$ degrees of freedom (see[24]). For consistency from now on in this chapter, we denote the derivative with respect to the Newtonian(absolute) time $\frac{\partial}{\partial t}$ by a dot and with respect to the independent variable (used to parametrize the $n+1$ physical degrees of freedom) $\frac{\partial}{\partial \tau}$ by prime. Starting with Hamilton's action functional (1.3) for the well-known Lagrangian $L(q, \dot{q})$ of classical mechanics, which is the difference between the kinetic and potential energy of the system, and rewriting it in terms of the independent variable τ we get

$$S = \int_{\tau_A}^{\tau_B} L(q, \frac{q'}{t'})t' d\tau \tag{1.5}$$

from which the new Lagrangian (for this new system which has $n+1$ degrees of freedom) can be read off, namely

$$L_{new} = Lt'.$$

Although we arrived at this by a simple mathematical step (change of integration variable in (1.3)), be aware of the important physical difference between (1.3) and

(1.5). In the latter we are varying the space-time curves connecting space-time events $A = (q_A, t_A)$ and $B = (q_B, t_B)$.
Now as no t appears in L_{new}, t is then by definition a cyclic variable. Hence its conjugate momentum

$$p_t = \frac{\partial L_{new}}{\partial t'} = \frac{\partial (Lt')}{\partial t'} = L + (\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial t'})t'$$

$$= L - (\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{q'_i}{t'^2})t' = L - \sum_{i=1}^{n} p_i \dot{q}_i$$

is a constant of motion. In the third equation, the chain rule is used. In the fourth equation, we used $\dot{q} = \frac{q'}{t'}$, hence $\frac{\partial \dot{q}_i}{\partial t'} = -\frac{q'_i}{t'^2}$.
But the expression derived for p_t coincides (up to a minus sign) with the first integral of the Lagrangian equations of motion (for scleronomic systems[6] see [25]) which is defined in the literature as the total energy E of the system.
In short, if t is a cyclic variable (which is the case when the corresponding Lagrangian L of the system we started with is conservative, i.e., L has no explicit time dependence), then

$$p_t = -E \tag{1.6}$$

is a constant of motion. This may also be considered an alternative derivation of the energy conservation theorem for conservative systems.

It is well known that n_c cyclic variables can be eliminated from the variational problem resulting in the reduction of the original variational problem by n_c degrees of freedom using the general reduction procedure (see, e.g., [24]). For this reason, cyclic variables are also called *ignorable variables* in Hamilton's formulation of mechanics. In the present case, the ignorable variable is t, and we are interested in reduction with respect to the variable t. The modified Lagrangian becomes

$$\bar{L}_{new} = L_{new} - p_t t' = Lt' - p_t t' = (L - p_t)t' = \sum_{i=1}^{n} p_i \dot{q}_i t'$$

[6]where the equations of constraints do not have explicit time dependence.

Hence, the modified action functional is as follows

$$\bar{S} = \int_{\tau_A}^{\tau_B} \bar{L}_{new} = \int_{\tau_A}^{\tau_B} \sum_{i=1}^{n} p_i \dot{q}_i t' d\tau = \int_{\tau_A}^{\tau_B} 2Kt' d\tau \tag{1.7}$$

where expression (B.4) is used for the kinetic energy K.

Note here that because $t'd\tau = dt$, the modified action (1.7) can simply be rewritten as Maupertuis' action (1.1) . However, Jacobi's dissatisfaction with Maupertuis' principle was of the same fundamental kind as his dissatisfaction with Hamilton's principle – with which by the way he started his considerations in the first place. In Maupertuis' action, the absolute time t is used as an independent variable for integration. However, in the Newtonian worldview, t itself must be the subject of the variational calculation, just like any of the q_i. It matters at which absolute time $t \in [t_A, t_B]$ a given configuration q_C (which locates on the true trajectory somewhere between q_A and q_B) is reached; as much as it matters at which value of the generalized coordinate q_i a specific value of some q_j is reached $(i \neq j)$. It is only to this end that one uses the variational principle. So Jacobi's concerns are quite justified for a convinced follower of Newtonian philosophy.

It is worth emphasizing that Jacobi's point (of putting time variable and position variables of a mechanical system on equal footing) is a novel formal difference from the works of his predecessors and truly finds its fundamental motivation in the Newtonian worldview. However, his point does not make a practical difference to Maupertuis' principle if the system subject of variational calculations is conservative and the initial value of the absolute time is additionally provided [7]. It is because, for conservative systems, the motion in time is constrained by (1.6). So the velocity by which the configuration point of the system moves on its trajectory can be calculated throughout the whole trajectory, and hence we can calculate exactly at which absolute time any intermediate configuration q_C has been reached. In this way, no need for variation of t remains. A simple calculation can elaborate on this point and leads to a formula for the exact value of

[7] which is indeed the case in Jacobi's principle, as you choose a spacetime point as your lower boundary of integration 1.7.

the absolute time at an intermediate configuration.

$$
\begin{aligned}
t_C = t_A + \int_{\tau_A}^{\tau_C} t' d\tau = t_A + \int_{q_A}^{q_C} \frac{dl}{\sqrt{2K}} \\
= t_A + \int_{q_A}^{q_C} \frac{dl}{\sqrt{2(E-V)}} \\
= t_A + \int_{\tau_A}^{\tau_C} \frac{\sqrt{< q' \mid q' >}}{\sqrt{2(E-V)}} d\tau
\end{aligned}
$$

Here $q(\tau)$ stands for the true trajectory of the conservative system in configuration space (in this case also a solution of Maupertuis' principle), $V = V\big(q(\tau)\big)$, and the norm $< . \mid . >$ on configuration space is defined with respect to the mass tensor (see appendix B). In the second equality, the expression of the kinetic energy in Newtonian theory (i.e., (1.8)) is used to substitute the increment of absolute time dt with the line element of configuration space dl. therefore, the use of absolute time as an independent variable in Maupertuis' principle is *a posteriori* satisfied. From Jacobi's analysis, it becomes clear that the circumstances under which Maupertuis' principle is applied (constancy of the total energy) leave no room for a variation of absolute time (its value is fixed for any intermediate configuration as we have just calculated).

Let us now move on with the last step of the reduction of a cyclic variable, which is the elimination of its velocity using the equation of motion of its conjugate momentum, in this case eliminating t' using (1.6). We equip the configuration space with a Riemannian metric and set this metric equal to the mass tensor $\mathbf{M}$. Then the kinetic energy can be expressed as

$$K = \frac{1}{2}\big(\frac{dl}{dt}\big)^2 \tag{1.8}$$

in which dl denotes the line element (with respect to $\mathbf{M}$ as metric)[8] of configuration space. So, in Newtonian theory, the velocity with which the configuration point of a system moves in configuration space is $\sqrt{2K}$. Again, in the present case, since we use τ as our independent variable, (1.8) needs to be rewritten as

$$K = \frac{1}{2}\big(\frac{dl}{d\tau}\big)^2 / t'^2. \tag{1.9}$$

[8] $dl^2 = \mathbf{M}_{ij} dx^i dx^j$

Now, using this and the momentum equation (which is equivalent to the energy theorem(1.6)), the cyclic variable (i.e., the remaining t' in the modified action (1.7)) can eventually be eliminated by inserting

$$t' = \frac{1}{\sqrt{2(E-V)}}\frac{dl}{d\tau} \tag{1.10}$$

into (1.7)

$$\begin{aligned}
\bar{S} &= \int_{\tau_A}^{\tau_B} 2Kt'd\tau = \int_{\tau_A}^{\tau_B} 2K\frac{1}{\sqrt{2(E-V)}}\frac{dl}{d\tau}d\tau \\
&= \int_{\tau_A}^{\tau_B} 2(E-V)\frac{1}{\sqrt{2(E-V)}}\frac{dl}{d\tau}d\tau \\
&= \int_{\tau_A}^{\tau_B} \sqrt{2(E-V)}\frac{dl}{d\tau}d\tau \\
&= \int_{A}^{B} \sqrt{2(E-V)}dl
\end{aligned}$$

It finally leads to the reduced action functional

$$\bar{S} = \int_{\tau_A}^{\tau_B} \sqrt{2(E-V)}\frac{dl}{d\tau}d\tau = \int_{A}^{B} \sqrt{2(E-V)}dl \tag{1.11}$$

where the last equation shows the invariance of this expression with respect to re-parametrizations. As usual, the minimizing paths of the action satisfy

$$(\delta\bar{S}) = 0 \tag{1.12}$$

and we have arrived at what is called **Jacobi's principle**. Note that constancy of the total energy E here is not a constraint imposed manually in the variational calculation (as in (1.2)) but a consequence of Hamilton's principle for time-invariant systems; hence we did not write a letter E explicitly in (1.12). As is evident from (1.11), absolute time t does not appear in its formulation. The solution of this principle is a path in configuration space without any reference to the motion in absolute time. However, the motion in absolute time (which was, of course, the question of Jacobi in the first place) can quickly be recovered from (1.10), the integration of which gives us the physical time t as a function of the independent parameter τ (which now parametrizes all $n+1$ degrees of freedom as Jacobi wanted).

More recently [15], Julian Barbour preferred to rewrite Jacobi's action (1.11) as

$$\bar{S} = 2\int_{\tau_A}^{\tau_B} \sqrt{E-V}\sqrt{\tilde{K}}d\tau \tag{1.13}$$

where $\tilde{K} := \frac{1}{2}\frac{dq}{d\tau}.\frac{dq}{d\tau}$ with the inner product defined with respect to the mass tensor, again. It can be rewritten as $\tilde{K} = \frac{1}{2}\sum_{i=1}^{N}\frac{d\mathbf{r}_i}{d\tau}.\frac{d\mathbf{r}_i}{d\tau}$ where dot denotes the standard Euclidean metric on $\mathbb{R}^3$, and the index i runs over the number of particles. Note that $\tilde{K}$ has nothing to do with the kinetic energy, which is a Newtonian term. When one uses the physical time to express the velocities, $\tilde{K}$ becomes, by definition, the kinetic energy we are all familiar with.

The Lagrangian L read off from (1.13) is used to define the canonical momenta

$$p_i := \frac{\partial L}{\partial(\frac{dq_i}{d\tau})} = m_i\sqrt{\frac{E-V}{\tilde{K}}}\frac{dq_i}{d\tau} \tag{1.14}$$

and the corresponding Euler-Lagrange equation becomes

$$\frac{dp^i}{d\tau} = \frac{\partial L}{\partial q_i} = -\sqrt{\frac{\tilde{K}}{E-V}}\frac{\partial V}{\partial q_i}. \tag{1.15}$$

Remember that we had the total freedom to choose any independent variable τ since the action (1.13) is reparametrization invariant. One possibility is to choose a parametrization that is such that $\tilde{K} = E - V$. We already know that this specific option for τ mimics the absolute time of Newton for two reasons. First, inside Newtonian Mechanics for a conservative system, the kinetic energy is, of course, $E - V$, and as τ here has been chosen such that $\tilde{K}$ becomes equal to $E - V$, we can conclude that this specific choice for τ marches in steps with absolute time. Another reason even more convincing is that for this specific τ, equation (1.15) takes its familiar form, namely Newton's second law.
Now from $\tilde{K} = \frac{1}{2}\sum_{i=1}^{N}\frac{d\mathbf{r}_i}{d\tau}.\frac{d\mathbf{r}_i}{d\tau} = E - V$ one easily deduces

$$dt = \frac{\sqrt{\sum_{i=1}^{N} m_i d\mathbf{r}_i.d\mathbf{r}_i}}{\sqrt{2(E-V)}} = \frac{dl}{\sqrt{2(E-V)}}, \tag{1.16}$$

where dl is again the line element of the configuration space with respect to the mass tensor. As Barbour said, from this, one can see how *change creates* *time*. See also [26] for a beautiful presentation of the notion of relational time from a physical and historical point of view. The sentence of Mach we quoted at the beginning of this chapter "*It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the change of things*", is fully reflected in (1.16). Equation (1.16) illustrates another fully holistic feature of the Newtonian theory, which was invisible to us in the way Newton was advocating it. A change in the position of the farthest objects causes the time we are experiencing to move forward! However, this idea is not far from how astronomers defined and used the so-called *ephemeris time* in practice. The motion of planet earth with respect to the farthest stars was used historically to keep track of the time passing, which is a similar concept. Time is a derived notion, not a primitive one, as Leibniz emphasized.

1.4 Principle of Relationalism

In the first part of this section, we have already mentioned the difference between the two alternative worldviews of Newtonian absolutism and Leibnizian relationalism. Given, that:

1. Most of our current physical theories (like Classical Mechanics, Quantum Mechanics, ...) are based on the Newtonian worldview;

2. Predictions of our current physical theories are compatible with the empirical data to an astonishingly high degree of accuracy and give a pretty clear explanation for the occurrence of numerous natural phenomena;

3. We think the relational worldview should be adopted in physics,

it is at first sight unclear whether a realistic relational theory can be formulated at all because it is not clear whether statements 2 and 3 are compatible with each other. In the following, we will explain how statements 2 and 3 can both be true. To this end, we introduce the **Principle of Relationalism** as follows:

Two possible universes, differing from each other just by the action of a global similarity transformation $Sim(3)$, are observationally[9] indistinguishable.

If a theory based on the Newtonian worldview satisfies the Principle of Relationalism, it can be recast into an empirically equivalent theory based on the Leibnizian worldview.

1.4.1 Scale invariant Classical Gravity

A natural question to ask now is whether Classical Mechanics satisfies the Principle of Relationalism or not. As the interaction potential functions like Newton's gravitational potential $V = \frac{Gm_1m_2}{r_{12}}$, or Coulomb potential $V = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}}$ defined on the absolute space, though being manifestly rotational and translational invariant, are clearly not scale-invariant, the answer of the above question seems to be *negative*, and as a result of it, the hope for a relational understanding of Newtonian Mechanics seems to be vanished[10]. However, prior to the above question, we should have asked another more primitive question. *Do we already know everything about Newtonian theory of classical Mechanics?* The answer to this question may be "Yes" if we had a derivation of the value of, for instance, the gravitational coupling constant G from Newtonian theory. In other words, we have the opinion that in a complete physical theory based on the Newtonian worldview, there exists a theoretical derivation of the value of the gravitational constant G, which must, of course, coincide with the value observed in our universe. So even if the Newtonian worldview is the correct view, the Newtonian

[9] With observationally indistinguishable we mean kinematically and dynamically indistinguishable.

[10] See however Chapter (11) for a way of understanding Newtonian mechanics as a relational theory, utilizing mechanical similarities. We call this alternative way the BKM-approach.

theory of gravitation may very well contain a foundational incompleteness (or gap) in it, as will be explained in more detail below in Section 1.4.2 [11]. In the following, we propose a way to partially fill this gap in a manner compatible with the Principle of Relationalism.

We can always render an arbitrary potential function defined on Q_{cm} or absolute space, scale-invariant, by postulating a special scale-dependent transformation law for its coupling. This law should be precisely the inverse of the transformation law of the potential function without its coupling. In this way, the scale-invariance of the total potential function we started with on the absolute space is established. Take any potential function

$$V(r_1, ..., r_N) = Y f(r_1, ..., r_N)$$

with Y being its coupling constant. Now apply a scale transformation

$$r_i \to c r_i$$

with $c \in \mathbb{R}^+$. Under this transformation, the function f and its coupling constant Y will transform as the following

$$\begin{cases} f(r_1, ..., r_N) \to f'(r_1, ..., r_N) := f(cr_1, ..., cr_N) \\ Y \to Y'. \end{cases}$$

Then clearly the potential V transforms as

$$V \to V' = Y' f'$$

Now by requiring V to be scale invariant i.e. $V' = V$, we can deduce the required transformation law of Y, namely

$$Y' = Y \frac{f}{f'}.$$

In other words, if f is a homogeneous function of degree k on absolute configuration space (as is the case for gravitational potential), Y must also be a homogeneous function of degree $-k$.

[11]Specifically a quotation there, from Albert Einstein would be illuminating in this matter.

This way the potential $V(x)$ on absolute space uniquely projects down to a potential function $V_s(q)$ on the reduced configuration space $\frac{\mathbb{R}^{3N}}{\mathbb{R}} \cong \mathbb{R}^{3N-1}$ w.r.t. the scale transformations. Denote this projection by $\pi : \mathbb{R}^{3N} \to \mathbb{R}^{3N-1}$. Then for each $q \in \frac{\mathbb{R}^{3N}}{\mathbb{R}} \cong \mathbb{R}^{3N-1}$ and $x \in \pi^{-1}(q)$ we have

$$V_s(q) = V(x) \tag{1.17}$$

This assignment is indeed independent of x (as long as it lies on that fiber above q) because $V(x)$ is a scale-invariant function on absolute configuration space. In other words, one unique value of the potential is given to each equivalence class of configurations under scale transformations [12]. But, of course, to find out the unique value for a given shape, one has to choose a representative [13] of the equivalence class, and this representative may as well be our good old representation[14] in which $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$. This way, we can make the classical gravity scale-invariant and compatible with the Principle of Relationalism. Equivalently one can say that in the gauge where the length of the international prototype meter bar is chosen to be the length unit (i.e., 1 meter), the measured value of G in our universe[15] in its current state becomes the above value.

1.4.2 Constants of Nature

In the last subsection, we have introduced a transformation law for the value of the gravitational coupling G. This transformation is obviously in conflict with the general belief that G is a constant. Hence we found it necessary to clarify this point and clear up possible confusions which may arise in this regard. To be more precise, we argue which constants of nature (numbers appearing in laws of nature of the respective theories) have to remain unchanged, given that the universe has to look and work the same way after a global scale transformation. We will see that G is not among those unchanging "*constants of nature*", neither

[12] Or similarity transformation
[13] By choosing a unit for length and time
[14] The SI units
[15] strictly speaking near Earth

is Planck's constant $\hbar$, nor the vacuum permittivity ϵ_0.

Observationally, we experience a vast number of regularities and fascinating patterns in nature, and our quest to understand the reason for their occurrence leads us to the discovery of the laws of nature in which a collection of dimensionless numbers appear whose exact values are not derived in any way inside the theory, but rather determined experimentally. In the words of *John D. Barrow* [27] these dimensionless numbers capture at the same time our greatest knowledge and our greatest ignorance about the universe.

To make it clear to the reader which numbers we are referring to, the nice correspondence of Albert Einstein with Ilse Rosenthal-Schneider [28] on this topic is very helpful. Einstein writes:

"*Now let there be a complete theory of physics in whose fundamental equations the "universal" constants* c_1,c_2,...,c_n *occur. The quantities may somehow be reduced g,cm,sec. The choice of these three units are obviously quite conventional. Each of these* c_1,...,c_n *has a dimension in these units. We now will choose conditions in such a way that* c_1,c_2,c_3 *have such dimensions that it is not possible to construct from them a dimensionless product* $c_1^\alpha c_2^\beta c_3^\gamma$*. Then one can multiply* c_4*,* c_5*, etc., in such a way by factors built from powers of* c_1,c_2,c_3 *that these new symbols* c_4^**,* c_5^**,* c_6^* *are pure numbers. These are the genuine universal constants of the theoretical system which have nothing to do with conventional units. My expectation now is that these constants* c_4^* *etc., must be basic numbers whose values are established through the logical foundation of the whole theory. Or could put it like this: In a reasonable theory there are no dimensionless numbers whose values are only empirically determinable. Dimensionless constants in the laws of nature, which from the purely logical point of view can just as well have different values, should not exist. To me, with my "trust in god" this appears to be evident, but there will be few who are of the same opinion.*"

To Max Planck, it seemed natural that the three dimensional constants G, $\hbar$,

$\boldsymbol{c}$ which appear in physical theories, determine the three basic measuring units. The units derived from them retain their natural significance as long as the law of gravitation and that of propagation of light in a vacuum remain valid. Therefore, they must always be the same when measured by the most widely differing intelligent beings according to the most widely differing methods. He defined the Planck mass and length and time units as

$$L_P = \sqrt{\frac{G\hbar}{\boldsymbol{c}^3}} = 1.616 \times 10^{-35} m$$

$$M_P = \sqrt{\frac{\hbar \boldsymbol{c}}{G}} = 2.177 \times 10^{-5} g$$

$$T_p = \sqrt{\frac{G\hbar}{\boldsymbol{c}^5}} = 5.390 \times 10^{-44} s$$

These can be considered as Einstein's dimensional constants c_1, c_2, c_3.

So far, we are aware of four distinct forces of nature, i.e., gravity, electromagnetism, and weak and strong forces. The strength of the former three of these forces (compared to the strong force) can be considered dimensionless (or Einstein's pure) numbers that define our world. The value of these dimensionless numbers are

$$\alpha_{EM} := \frac{e^2}{4\pi\epsilon_0 \hbar \boldsymbol{c}} \approx \frac{1}{137.036}$$

$$\alpha_G := \frac{G m_p^2}{\hbar \boldsymbol{c}} \approx 5 \times 10^{-39}$$

$$\alpha_W := \frac{G_F m_p^2 \boldsymbol{c}}{\hbar^3 \sim 1.03 \times 10^{-5}} \approx 10^{-15}$$

Universes for which the value of any of these three dimensionless numbers are different from the above values are observationally different (because the balances of the forces differ).

Coming back to the proposed transformation law

$$G \to cG$$

of the gravitational constant under a scale transformation by the factor $c \in \mathbb{R}^+$

$$r \to cr$$

one can immediately see from α_G that it changes the balance of Forces in Nature and leads to observable differences, which violates the Principle of Relationalism. However, if this transformation of G is accompanied by a transformation $\hbar \to c\hbar$ of the Planck's constant, the strength of gravity would remain unchanged. It is well known from Quantum mechanics that even the slightest change of the value of $\hbar$ would lead to a sudden release or absorption of an enormous amount of energy due to the dependence of the atomic orbital energy levels on the value of $\hbar$, i.e., for the hydrogen atom $E_n = -\frac{me^4}{8h^2\epsilon_0^2}\frac{1}{n^2}$, and this again violates the Principle of Relationalism. However, if a transformation $\epsilon_0 \to \frac{\epsilon_0}{c}$ of the vacuum permittivity is also taking place along the mentioned transformations of G and $\hbar$, the value of energy levels remain unchanged so that the mentioned principle is respected.

To summarize, after performing a scale transformation by a factor $c \in \mathbb{R}^+$

$$r \to cr$$

on the whole universe, the Principle of Relationalism requires the following transformation of (Einstein's dimensionful) constants

$$G \to cG \tag{1.18}$$

$$\hbar \to c\hbar \tag{1.19}$$

$$\epsilon_0 \to \frac{\epsilon_0}{c} \tag{1.20}$$

To appreciate the consistency of these transformation laws more, notice that they automatically induce the expected transformation of the Bohr radius $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$, classical electron's radius $r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e \boldsymbol{c}^2}$, and the fine structure constant α_{EM}, namely $a_0 \to ca_0$, $r_e \to cr_e$ and $\alpha_{EM} \to \alpha_{EM}$. So all atoms in the universe get bigger by exactly the same scaling factor for the universe itself, and the relative strength of the electromagnetic force remains unchanged. They also automatically result in a scale-invariant electrical force (Coulomb's potential).

Imagine an experiment by which one wants to figure out the velocity of an object. We argue that after performing a global scale transformation, the velocity of the same object during the same experiment remains unchanged. One can see this in the following way: From the formulas of Planck's system of units mentioned above, one can easily see that a scale transformation

$$r \to cr$$

reshuffles the values of these units expressed in SI[16] as

$$L_p \to cL_p$$

$$M_p \to M_p$$

$$T_p \to cT_p$$

As expected, the natural unit of length gets bigger by the same factor $c \in \mathbb{R}^+$. The time (measured in Planck's unit) also gets dilated and runs faster by the same factor.

Hence the measured speed v of an object transforms under a global scale transformation as follows

$$v = \frac{\Delta x}{\Delta t} \to v' = \frac{\Delta x'}{\Delta t'} = \frac{c\Delta x}{c\Delta t} = v$$

where Δx stands, for instance, for the distance between two other objects (which are needed to define the start and end point of any interval in space), and Δt for the time (measured in Plank unit) the object takes to travel between those two reference objects. The primed versions have the same quantities; however, after scale transforming the universe and measuring everything in new Planck units. The same can be said about the velocity of light[17] **c**, where one measures the time needed for light to path the distance between two objects. Note that in the relation $\Delta t' = c\Delta t$, the Principle of Relationalism is tacitly invoked in equating the number of ticks(or steps) of our new clock in the scaled universe for the duration of a physical phenomenon (in this example the passage of light of an object between the two reference objects), and the number of ticks of the old

[16]Here SI is being thought of as some measures of hypothetical absolute space and time, in the sense that they are exempted from the global transformation we perform on the universe. So we scale everything in the universe except the SI standard meter stick and the SI standard clock as resembling the absolute distance and absolute time duration, and a hypothetical mean to compare the new Plank's units after expansion to the old ones inside the absolute framework that classical theories and quantum mechanics are presented.

[17]To be more precise, the average two-way light's velocity is meant here. No experimental way exists to measure the light's direct one-way velocity due to the conventionality of simultaneity.

clock in the old (smaller) universe while the same phenomenon is taking place. So the measured speed of any object in universes before and after global scale transformations comes out the same. The Principle of Relationalism, and the characteristics of Plank units together are responsible for this result.
As a consequence of the constancy of the speed of light (measured in Planck units), $\mathbf{c} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ under scale transformations, one can deduce the corresponding transformation law of vacuum Permeability, namely

$$\mu_0 \to c\mu_0$$

The dilation of time under scale transformation in the presented way is also compatible with the operational SI definition of the time unit, i.e., the *second* is defined as the duration of 9192631770 cycles of the radiation corresponding to the transition between two energy levels of the ground state of the cesium-133 atom at rest at a temperature of absolute zero. By performing a scale transformation $r \to cr$, the wavelength of the emitted photon transforms correspondingly ($\lambda_{photon} \to c\lambda_{photon}$), and hence the time required for one cycle , i.e. $T = \frac{\lambda_{photon}}{\mathbf{c}}$, transforms as $T \to cT$. Therefore, the SI second will also get dilated by the same factor $c \in \mathbb{R}^+$. This shows the expected coherence between Planck and SI units of time under global scalings.
It is worth mentioning that the above feature of the modified Newtonian theory is not keen on using the Planck units. The true homogeneous function of the first degree on Q which must lead to the value $6.6743 \times 10^{-11} \frac{m^3}{kg.s^2}$ for the current state of the universe also allows the existence of Kepler pairs now[18]. The unit of length and time defined as the semi-major axis, and the orbital period of a Kepler pair, change under a global scale transformation Sc exactly as their Planck counterparts in the modified Newtonian theory. It is because of the mechanical similarities in the modified Newtonian theory. The homogeneity of the potential function of zero's degree in modified Newtonian theory guarantees that the Kepler pair's orbital period becomes longer by a factor $c \in \mathbb{R}^+$ after a mechanical similarity transformation of the universe (by the factor c) has been performed.

Another point worth emphasizing is that even though the value of G depends

[18]Whether or not these Kepler pairs form in a typical universe governed by the modified Newtonian theory is a separate question to be addressed in future work.

directly on the units chosen for the measurement of distance and duration, once a set of units is chosen (e.g., SI units), there is no justification in Newton's theory as why the value of G should be what it turns out to be (in the chosen set of units). We have called this issue a foundational gap in Newton's theory and proposed a way to partially fill this gap so that the principle of relationalism can be directly implemented in modern physics (which includes undoubtedly more than just the gravitational interaction alone). Completion of the absolute physical theories in line with the idea of G, $\hbar$, ϵ_0 being homogeneous functions of the mentioned degrees on the universe's configuration space not only partially fills[19] the foundational gap in Newtonian mechanics, but also directly address Einstein's justified concern about dimensionless constants of nature in his correspondence with Isle Rosenthal-Schneider. Namely, if in the ultimate (absolute) physical theory G, $\hbar$, ϵ_0 emerge as homogeneous functions on the configuration space (as anticipated by the direct implementation of the principle of relationalism), an immediate justification of the (otherwise surprisingly fine-tuned) value of the dimensionless constants of the ultimate theory characteristic of our universe, is provided, as the latter are specific ratios of the former.

Remember that Jacobi's principle stated that the path taken by a classical system minimizes the Jacobi action $\bar{S} = \int_{x_1}^{x_2} \sqrt{E - V} ds$ with x_1 and x_2 standing for the initial and final configuration of the system. For the path $\mathbf{x}(t)$ that minimizes this action one has the energy conservation equation $E = \frac{1}{2} M(\frac{d\mathbf{x}}{dt}, \frac{d\mathbf{x}}{dt}) + V$. Hence along this path (which is the only physical path in the sense that only this path is realized by nature) one has $K := \frac{1}{2} M(\frac{d\mathbf{x}}{dt}, \frac{d\mathbf{x}}{dt}) = E - V$. Now Jacobi action along this path can be rewritten as $\bar{S} = \int_{x_1}^{x_2} \sqrt{K} ds$. If one now performs a scale transformation $r \to cr$, with $c \in \mathbb{R}^+$, the system naturally gets bigger. However, the velocity of the constituting particles of the system measured in the new Planck units of time and length remain unchanged. Moreover, the length of the path between cx_1 and cx_2 measured in the new Planck length also remains unchanged. So in this way, one sees now that the action of classical mechanics is invariant under scale transformations.

[19] We use the word partially cause the exact expression of these functions are not determined and are yet to be discovered in a future complete physical theory. Only then is one allowed to drop the word "partially".

Chapter 2

Symplectic reduction of phase space with respect to a symmetry group

In this section, we will review how the collective motion of a multi-particle system (in particular, the rotations and translations of the system) is gotten rid of in the Hamiltonian formalism using the reduction procedure of Mardsen/Weinstein. We follow [3],[4] to a big extend. First, we review the expression of the laws of classical mechanics using a symplectic structure on phase space. This level of abstraction for formulating Classical Mechanics seems at first sight to be an unnecessary complication. However, its power lies in its generality and is beneficial, compared to less abstract formulations, when dealing with curved spaces. It is the case in many mechanical systems with constraints, as well as in reduced spaces like internal configuration space $Q_{int} = \frac{Q}{E(3)} = \frac{\mathbb{R}^{3N}}{E(3)}$ or shape space $S = \frac{Q}{sim(3)}$ of a N-particle system.

2.1 Definition of a Hamiltonian System in Symplectic Phase Space

Denote the configuration space of a N-particle system with $Q \cong \mathbb{R}^{3N}$. A symplectic form σ on Q is a closed non-degenerate differential two-form. Closed means that the exterior derivative of σ vanishes, i.e. $d\sigma = 0$, and non-degenerate means that if there exists some $u \in T_x(Q)$ such that $\sigma(u, v) = 0$ for all $v \in T_x(Q)$, then $u = 0$. The Hamiltonian H is a function on $T^*(Q)$ to which one can associate the respective Hamiltonian flow, which is a vector field X_H on Q defined by the equation $\sigma(X_H, Y) = dH$ for all $Y \in T(Q)$. Symplectic geometry is well suited for investigating mechanical systems. Starting with a system's configuration space Q, its phase space $T^*(Q)$ is canonically symplectic. Denoting the configuration coordinates by q_i, and the remaining coordinates needed on $T^*(Q)$ by p_i, the canonical symplectic form becomes $\sigma = \sum_{i=1}^{N} dq_i \wedge dp_i$. The Hamiltonian flow associated to a physical Hamiltonian $H = \sum_{i=1}^{N} \frac{p_i^2}{2} + V$ leads to an evolution of the system's initial state $(q_i(0), p_i(0)) \in T^*(Q)$, which is compatible with Newton's laws of motion. In the following, we explain this construction more precisely.

The cotangent space $T_x^*(Q)$ at $x \in Q$ is isomorphic to the tangent space $T_x(Q)$ by the induced isomorphism defined through the following equation

$$
\begin{aligned}
I_x : T_x(Q) &\to T_x^*(Q) \qquad (2.1)\\
I_x(v).u &= K_x(u, v)
\end{aligned}
$$

for $u, v \in T_x(Q)$. Here . stands for the pairing of vectors $T(Q)$ and covectors $T^*(Q)$.

Setting $p := I_x(v)$ and writing $p = (p_1, p_2, ..., p_N)$ as a tuple, we get from the definition of K_x

$$p_k = m_k v_k \qquad (2.2a)$$

$$p.u = \sum_{k=1}^{N} (p_k, u_k). \qquad (2.2b)$$

Thus, we have obtained the induced variables x and p, constituting a coordinate system of the cotangent space $T^*(Q) \cong Q \times \mathbb{R}^{3N}$. x and p are often called the

coordinate and momentum variables.
Now we define the canonical one-form θ on the cotangent bundle $T^*(Q)$. For

$$(u, w) \in T(T^*(Q))$$

being a tangent vector at

$$(x, p) \in T^*(Q) \cong Q \times \mathbb{R}^{dN}$$

we define

$$\theta_{(x,p)}(u, w) := p.u \; . \tag{2.3}$$

If u is a vector field on Q, then $dx_k^i(u) = u_k^i$ in Cartesian coordinates, so that the canonical one-form θ can be expressed in the following form

$$\theta = p.dx = \sum (p_k, dx_k). \tag{2.4}$$

The exterior derivative of θ reads

$$d\theta = dp \wedge dx = \sum (dp_k \wedge dx_k).$$

A scalar product K_x^* on the cotangent space $T_x^*(Q)$ can be defined as

$$K_x^*(q, p) := K_x(I_x^{-1}(q), I_x^1(p)) = \sum \frac{(q_k \mid p_k)}{m_k} \tag{2.5}$$

for $q, p \in T_x^*(Q)$.
The Hamiltonian of a system is a function on $T^*(Q)$ of the following form

$$H = K^*(p, p) + U, \tag{2.6}$$

where U is a potential function invariant under translations ($\mathbb{R}^n$) and rotations ($SO(3)$).
The triple

$$(T^*(Q), d\theta, H)$$

constitutes a Hamiltonian system.
Hamilton's equations of motion are given by the Hamiltonian vector field $\mathbf{X}_H$, which is defined through

$$d\theta(\mathbf{X}_H, \mathbf{Y}) = dH(\mathbf{Y})$$

$\forall \mathbf{Y} \in T(Q)$.

Before moving to the next section, we briefly discuss how a symplectic form can be used to express electromagnetic laws of motion for a charged particle [29] for the purpose of illustration. Using the isomorphism (2.1), the magnetic vector potential $\mathbf{A}$ can be considered as a 1-form. Then the magnetic field $\mathbf{B}$ becomes a 2-form $d\mathbf{A}$ on configuration space Q. Gauss's law for magnetism $\nabla.\mathbf{B} = 0$ is in this formalism expressed as $d\mathbf{B} = 0$. Then one defines a new phase space $(T^*(Q), \sigma_\mathbf{B})$ which differs from the previous phase space $(T^*(Q), \sigma)$ in that the canonical symplectic from σ on $T^*(Q)$ is replaced with

$$\sigma_\mathbf{B} = \sigma + \pi^*\mathbf{B},$$

where $\pi : T^*(Q) \to Q$. Denoting the electric potential function by ϕ, and denoting the velocity or momentum of the charged particle (which are identified to each other by the metric) by $\mathbf{v}$, the Hamiltonian becomes $H = \frac{1}{2} \mid\mid \mathbf{v} \mid\mid^2 + \phi$. The Hamiltonian vector field which defines the dynamics in the presence of Electromagnetic field, can then be derived from $\sigma_\mathbf{B}(\mathbf{X}_H, \mathbf{Y}) = dH(\mathbf{Y})$ for all $\mathbf{Y} \in T(Q)$.

2.2 Momentum mappings

The symplectic structure on $T^*(Q)$ enables us to express Noether's theorem more naturally. The action of a Lie-group G on $T^*(Q)$ can be generated by a vector field $\mathbf{a}_x$ on $T^*(Q)$, known as the infinitesimal generator of the action. Integral curves of $\mathbf{a}_x$ are the G-orbits on $T^*(Q)$. Noether's theorem then ensures the existence of a function μ on $T(Q)$, preserved by the action and conjugated to $\mathbf{a}_x$ by the symplectic form, i.e. $\sigma(\mathbf{a}_x, Y) = d\mu(Y)$ for all vector fields Y. This function is called the momentum map, and it is preserved by the Hamiltonian flow. Here we review the important concept of momentum mapping, which will be used frequently in the process of reduction.

If a group G acts on the manifold Q and $(.,.)$ is a G-invariant Riemannian metric, we define for

$$\mathbf{a} \in \mathbf{G}$$

$$v_x \in T_x Q$$

the momentum map μ as follows

$$\mu : T(Q) \equiv T^*(Q) \rightarrow \mathbf{G}^* \tag{2.7a}$$

$$\mu(v_x).\mathbf{a} := (\mathbf{a}_x, v_x) \tag{2.7b}$$

$$\mathbf{a}_x = \frac{d(e^{t\mathbf{a}}x)}{dt} \mid_{t=0} \in T_x^*(Q). \tag{2.7c}$$

$T(Q)$ and $T^*(Q)$ are identified with the metric.

There is an intrinsic formulation of the connection form in terms of the momentum map. Remember the definition of the inertia tensor(or operator) A. It was a linear operator in $\wedge^2(3)$, and there existed an isomorphism R between $\wedge^2(3)$ and $\mathbf{so}(3)$ (see appendix D), and since the tangent space and the cotangent space of Q are identified through the Riemannian metric on Q, we are able to redefine the inertia operator as follows

$$A : \mathbf{G} \rightarrow \mathbf{G}^* \tag{2.8a}$$

$$A_x(\mathbf{a}).\mathbf{b} = (\mathbf{a}_x, \mathbf{b}_x). \tag{2.8b}$$

The connection form is then

$$\omega(v_x) = A_x^{-1}(\mu(v_x)). \tag{2.9}$$

The horizontal distribution is the kernel of the momentum map μ.

Alternatively, one can also think of G as the group of symplectic transformations (preserving $d\theta$) and $\mathbf{G}$ as the Lie-algebra of G (which is identified with the tangent space to G at the identity). For every $\mathbf{a} \in \mathbf{G}$ we get a one-parameter subgroup of G by $exp(t\mathbf{a})$.

If for any $\mathbf{a} \in \mathbf{G}$ there exists a function $F_{\mathbf{a}}$ on $T^*(Q)$ satisfying $d\theta\mathbf{a}_x = -dF_{\mathbf{a}}$, then the action of G is called strongly symplectic. The function $F_{\mathbf{a}}$ depending linearly on $\mathbf{a}$, can be expressed in the form $F_{\mathbf{a}}(x,p) = \mu(x,p).\mathbf{a}$ which is the defining property of the momentum map μ of $T^*(Q)$ to G^*.

If the action of G is moreover exactly symplectic, G leaves θ invariant – then there is a simple equation that gives us the momentum map

$$\mu(x,p).\mathbf{a} = \theta(\mathbf{a}_x). \tag{2.10}$$

As we see below, momentum mappings cover linear and angular momentum.

It is well-known and intuitively clear that the expression of the connection form (2.9) for the $SO(3)$ fiber bundle, in Jacobi coordinates $\boldsymbol{r}_i$ becomes

$$\omega = R\Big(A_x^{-1}(\sum_{j=1}^{N-1} \boldsymbol{r}_j \times d\boldsymbol{r}_j)\Big) \tag{2.11}$$

where $d\boldsymbol{r}_j$ is the 3 dimensional vector valued one form, which, if applied to a vector on configuration space Q, gives the velocity vector of just the $j's$ particle.

2.3 Marsden-Weinstein method of Reduction of dynamical systems

Consider a symplectic manifold $P = T^*(Q)$, the symplectic form σ on this manifold, and a σ-preserving symplectic group G acting on P. The adjoint Ad_g and coadjoint Ad_g^* representations of G on the Lie-algebra space $\mathbf{G}$ and its dual space $\mathbf{G}^*$ respectively, are defined in appendix C (see C.1).
Let μ be the Ad^*-equivariant momentum mapping associated with the action of G. That is

$$\mu : P \to \mathbf{G}^* \tag{2.12a}$$

$$\mu(gx) = Ad^*_{g^{-1}}\mu(x), \forall x \in P. \tag{2.12b}$$

For $r \in \mathbf{G}^*$, $\mu^{-1}(r)$ is a submanifold of P. The isotropy subgroup G_r of G at $r \in \mathbf{G}^*$ is defined as the following

$$G_r = \{\forall g \in G \mid Ad^*_{g^{-1}}r = r\}. \tag{2.13}$$

Define then the manifold

$$P_r := \frac{\mu^{-1}(r)}{G_r} \tag{2.14}$$

with its canonical projection

$$\pi_r : \mu^{-1}(r) \to P_r. \tag{2.15}$$

P_r is called the **reduced phase space**.
With the help of the inclusion map

$$i_r : \mu^{-1}(r) \to P$$

we can get a unique symplectic form σ_r on P_r

$$\pi_r^* \sigma_r = i_r^* \sigma. \tag{2.16}$$

And at last, if the Hamiltonian H on P is invariant under the action of G, the Hamiltonian vector field $\mathbf{X}_H$ projects to a vector field $\mathbf{X}_{H_r}$ on P_r, namely

$$\pi_{r_*} \mathbf{X}_H = \mathbf{X}_{H_r}. \tag{2.17}$$

with

$$\pi_r^* H_r = i_r^* H. \tag{2.18}$$

Hence one obtains the **reduced system**

$$(P_r, \sigma_r, H_r).$$

Chapter 3

Example: Reduction with respect to the Euclidean group $E(3)$

As an illustration of the general framework of symplectic reduction discussed in the previous section, we review the reduction of phase space of a classical system with respect to the Euclidean group, following [3] to a considerable extent.

3.1 Reduction with respect to the translation group $G = \mathbb{R}^3$

The translation group $\mathbb{R}^3$ forms an exact symplectic group on $T^*(Q)$. Any member of this group $a \in \mathbb{R}^3$ acts on $T^*(Q)$ as follows, which leaves the one-form θ invariant

$$(x_1, ..., x_N, p_1, ..., p_N) \to (x_1 + a, ..., x_N + a, p_1, ..., p_N).$$

For $\mathbf{a} \in \mathbb{R}^3$, where $\mathbb{R}^3$ now stands for the Lie-algebra of the translation group, the infinitesimal generator of the subgroup $a(t) = exp(t\mathbf{a})$ has the form

$$\mathbf{a}_x(x,p) = (\mathbf{a}, ...\mathbf{a}, 0, ..., 0) \in T^*_x(Q)$$

so that the momentum map $\mu_t : T^*(Q) \to \mathbb{R}^3$ is given by:
$\mu_t(x,p).\mathbf{a} = \theta(\mathbf{a}_x) = \sum(p_k, \mathbf{a}) = (\sum p_k \mid \mathbf{a}) \Rightarrow$

$$\mu_t(x,p) = \sum p_k. \tag{3.1}$$

This way, we obtain the usual linear momentum.
Now in order to perform the reduction of the phase space $T^*(Q)$ with respect to the translation group $\mathbb{R}^3$ we apply the Marsden-Weinstein method. For $\lambda \in \mathbb{R}^3$, $\mu_t^{-1}(\lambda)$ is a submanifold of $T^*(Q)$ determined by $\sum p_k = \lambda$. This submanifold is isomorphic with $Q \times \mathbb{R}^{(N-1)3}$ for any λ. It is clear that the isotropy subgroup at λ, denoted by G_λ, is the whole group of translations $\mathbb{R}^3$. So the reduced phase space

$$P_\lambda = \frac{\mu_t^{-1}(\lambda)}{\mathbb{R}^3}$$

can be identified with $\frac{Q}{\mathbb{R}^3} \times \mathbb{R}^{3(N-1)}$, and therefore with

$$P_\lambda \cong Q_{cm} \times \mathbb{R}^{3(N-1)}.$$

This reduced space can, in turn, be thought of as a submanifold of $T^*(Q)$ determined by the following conditions

$$\sum m_k x_k = 0 \tag{3.2a}$$
$$\sum p_k = \lambda. \tag{3.2b}$$

What we are interested in is the case $\lambda = 0$. The submanifold $\frac{\mu_t^{-1}(0)}{\mathbb{R}^3}$ can then be identified with the cotangent bundle $T^*(Q_{cm})$

$$P_{\lambda=0} \cong T^*(Q_{cm}).$$

The reduced symplectic form on $T^*(Q_{cm})$ is then the restriction of $d\theta$ (which was the form on $T^*(Q)$) on $T^*(Q_{cm})$. For notational convenience, both of them are denoted by the same letter. So one arrives at the reduced Hamiltonian system with respect to the group of 3-dimensional spatial translations, i.e., $(T^*(Q_{cm}), d\theta, H)$.

Note that the identification between the reduced phase space with respect to the group of translations $P_\lambda = \frac{\mu_t^{-1}(\lambda)}{\mathbb{R}^3}$, and the cotangent bundle of the center of mass system $T^*(Q_{cm})$ holds only for $\lambda = 0$. In the remaining part of this subsection, we explain one way to see this point more clearly.

Theorem: $P_\lambda = \frac{\mu_t^{-1}(\lambda)}{\mathbb{R}^3}$, and $T^*(Q_{cm})$ cannot be identified to each other if $\lambda \neq 0$.

Proof: Consider a generic point $y \in T^*(Q)$. This point can symbolically be denoted as $y = \begin{pmatrix} \overrightarrow{p}_1 \\ ... \\ \overrightarrow{p}_N \end{pmatrix} (x)$ where $\overrightarrow{p}_i$ stands for the momentum of i'th particle (here the isomorphism between 1-forms and vectors is invoked too), and $x \in Q \cong \mathbb{R}^{3N}$ stands for a point in the configuration space of the multiparticle system. One can view $T^*(Q)$ simply as a $2N \cdot 3$ dimensional space, which is coordinatized by $\overrightarrow{x}_1, ..., \overrightarrow{x}_N, \overrightarrow{p}_1, ..., \overrightarrow{p}_N$. As each of these vectors consists of 3 numbers, they are indeed a collection of $6N$ numbers. As explained in the last paragraph, $\frac{\mu_t^{-1}(0)}{\mathbb{R}^3}$ can be considered as a $2(N-1).3$ dimensional submanifold of $T^*(Q)$ given by the constraints (3.2). So far, so good. One can alternatively view $T^*(Q)$ as follows: take the configuration space Q of the system, and attach to each point $x \in Q$ a $(3N)$-dimensional vector-space. This vector space is thought to be the collection of all possible elements $\begin{pmatrix} \overrightarrow{p}_1 \\ ... \\ \overrightarrow{p}_N \end{pmatrix}$ and denote this vector space by V_Q. Now comes the tricky point. The condition (3.2.a) gives us a fixed $3(N-1)$ dimensional surface in the absolute configuration space Q. By definition, this solid surface can be identified by Q_{cm} (one can even call this surface Q_{cm} no matter whether it is embedded into some bigger space or not). For the moment, we denote this surface by Q_{cm}^p, where p reminds us that this surface is part of a bigger configuration space Q. Now condition (3.2.b) selects a subspace of the vector-space which was attached to each point on Q, hence also to each point on the surface Q_{cm}^p. Clearly this subvector-space consists of special elements $\begin{pmatrix} \overrightarrow{p}_1 \\ ... \\ \overrightarrow{p}_N \end{pmatrix}$; namely the ones with $\sum \overrightarrow{p}_k = \lambda$. We denote this subvectorspace by $V_\lambda \subset V_Q$. In this way $\frac{\mu_t^{-1}(\lambda)}{\mathbb{R}^3}$ (which again was a submanifold of $T^*(Q)$ realized by constraints (3.2)) can be viewed as $Q_{cm}^p \times V(\lambda)$.

On the other hand, the cotangent space over the center of the mass system, i.e., $T^*(Q_{cm})$ is on its own an independently existing $2(N-1)d$ dimensional space, without any need of ambient space. Similarly this space $T^*(Q_{cm})$ can be viewed as $Q_{cm} \times V_{cm}$, where V_{cm} is just the $3(N-1)$ dimensional vector space attached to each point of the center of mass configuration space Q_{cm}.
Now if one tries to embed $T^*(Q_{cm})$ into $T^*(Q)$ one can indeed perfectly fit Q_{cm} on Q^p_{cm}, but one can never fit V_{cm} on V_λ unless $\lambda = 0$. The reason is that $V_\lambda \bigcap V^p_{cm} = \oslash$. Here V^p_{cm} denotes the embedding of V_{cm} in V_Q. Any element $v \in V_{cm}$ will assign a set of velocities to the particles. Pulled up to the absolute space in the center of mass system (so condition (3.2).a being valid), these velocities add up to zero (so they have to be elements of $V_{\lambda=0}$); otherwise we would immediately move out of the surface Q^p_{cm} and that results does not fit with $T^*(Q_{cm})$ on $Q^p_{cm} \times V_\lambda$ for $\lambda \neq 0$.

3.2 Reduction with respect to the Rotation group $G = SO(3)$

We now proceed to the angular momentum defined on $T^*(Q)$. The rotation group $SO(3)$ plays here the role of an exact symplectic group (preserving $d\theta$) whose action on $T^*(Q)$ is defined for (x,p) and $g \in SO(3)$ by

$$(x,p) \to (gx, gp). \tag{3.3}$$

For the case of vanishing linear momentum, i.e. $\lambda = 0$, we note that $SO(3)$ acts actually on $T^*(Q_{cm})$ as the conditions (3.2) are invariant under $SO(3)$. If λ is non-vanishing, only a subgroup of $SO(3)$ acts on $\frac{\mu_t^{-1}(\lambda)}{\mathbb{R}^3}$.

Consider some $\mathbf{a} = R_\xi \in \mathbf{so}(3)$, where $\xi \in \wedge^2(3)$ is the two-vector corresponding to the Lie-algebra element $\mathbf{a}$, and the correspondence is given by the isomorphism $R : \wedge^2(3) \to \mathbf{so}(d)$ defined in appendix D, see (D.4). The infinitesimal generator of the subgroup $exp(t\mathbf{a})$, is given by

$$\mathbf{a}_x(x,p) = \big(R_\xi(x), R_\xi(p)\big) = \big(R_\xi(x_1), ..., R_\xi(x_N), R_\xi(p_1), ..., R_\xi(p_N)\big)$$

where (D.6) is used. Therefor, the momentum mapping

$$\mu_r : T^*(Q_{cm}) \to \mathbf{so}^*(3)$$

can be calculated as follows:

$$\mu_r(x,p).\mathbf{a} = \theta_{(x,p)}(\mathbf{a}_x) = \sum_{k=1}^{N}(p_k \mid R_\xi(x_k)) = (\sum_{k=1}^{N} p_k \wedge x_k \mid \xi) = (R_{\sum_{k=1}^{N} p_k \wedge x_k} \mid R_\xi)$$

where (D.7).e has been used, and in the last equality, the fact that the mapping R is isometric is invoked. Hence one ends up with

$$\mu_r(x,p) = R_{-\sum_{k=1}^{N} x_k \wedge p_k} \tag{3.4}$$

Here we have identified $\mathbf{so}(d)$ and $\mathbf{so}^*(d)$ through the scalar product on $\mathbf{so}(d)$, namely $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} tr(\boldsymbol{\alpha}\boldsymbol{\beta}^T)$.

One can prove that for an exact symplectic group (transformations that leave the 1-form θ invariant), the associated momentum mapping is Ad^* -equivariant.

Now we use the Marsden-Weinstein reduction procedure for the rotation group $SO(3)$.

Let $\mathbf{a} \in \mathbf{so}(3) \cong \mathbf{so}^*(d)$. Then $\mu_r^{-1}(\mathbf{a})$ is a submanifold of $T^*(Q_{cm})$. Factoring out the orbits of the isotropy subgroup $G_\mathbf{a}$ of $SO(3)$ at $\mathbf{a}$, we obtain a reduced phase space $\frac{\mu_r^{-1}(\mathbf{a})}{G_\mathbf{a}}$. This process is merely an elimination of the angular momentum.

An important question now pops up: is the reduced phase space $\frac{\mu_r^{-1}(\mathbf{a})}{G_\mathbf{a}}$ diffeomorphic to the cotangent bundle $T^*(Q_{int})$ of the internal space $Q_{int} = \frac{Q}{\mathbb{R}^3 \circ SO(d)}$ as it was the case for the translations group?

The answer is NO, for $a \neq 0$.

For the N-body problem in $\mathbb{R}^3$ the dimension of the phase space reduces by 4 when eliminating the angular momentum. This is because the Lie-algebra $\mathbf{so}(3)$ is 3 dimensional, and the isotropy subgroup $G_\mathbf{a}$ for $a \neq 0$ turns out to be $SO(2)$. So the condition $\mu_r = \mathbf{a}$ in phase space diminishes the dimension by 3, and factoring out the $SO(2)$ orbits does by 1. Intuitively, once a single member of the 3-dimensional space $\mathbf{so}(3)$ has been chosen for the total angular momentum $\mathbf{a}$ of the Hamiltonian system, we end up on a sub-manifold with three dimensions

less. Now, in the original space, start rotating the whole system about an axis parallel to the total angular momentum vector and passes through the system's center of mass. It is indeed an $SO(2)$ rotation. It is clear that the value of the momentum map μ_r does not change at all by applying this $SO(2)$ rotation. So this constitutes the isotropy subgroup.
On the other hand $\dim\big(T^*(Q_{int})\big)$ is by 6 smaller than $\dim\big(T^*(Q_{cm})\big)$. Thus

$$dim\big(\frac{\mu_r^{-1}(a)}{G_a}\big) = dim\big(T^*(Q_{int})\big) + 2. \tag{3.5}$$

So, in general, the reduced phase space with respect to rotations $SO(3)$ is diffeomorphic to the cotangent bundle of the internal space $T^*(Q_{int})$. From the discussion above, it is clear that the total group $SO(3)$ becomes an isotropy subgroup $G_{\mathbf{a}}$ if and only if $\mathbf{a} = 0$, and in this case one has

$$\frac{\mu_r^{-1}(0)}{SO(3)} \cong T^*(Q_{int}). \tag{3.6}$$

Generally speaking, the reduced phase space is diffeomorphic to the fiber product $T^*(\frac{Q}{G}) \times_f (\frac{Q}{G_a})$ over the quotient $\frac{Q}{G}$, keeping in mind that $\frac{Q}{G_a}$ is naturally identified with the coadjoint orbit bundle $Q \times_G (\frac{G}{G_a})$ over $\frac{Q}{G}$ (see [30],[31]).

Now we want to study the symplectic form $\sigma_{\mathbf{a}}$ on the reduced phase space $P_{\mathbf{a}} = \frac{\mu_r^{-1}(\mathbf{a})}{G_{\mathbf{a}}}$. Since $\sigma_{\mathbf{a}}$ is defined by

$$\pi_\mu^* \sigma_\mu = i_\mu^* \sigma$$

and $\sigma = d\theta$. we have furthermore $i_\mu^* d\theta = d(i_\mu^* \theta)$ in our case. Remember that the maps used are $\pi_r : \mu^{-1}(r) \to P_r$ and $i_r : \mu^{-1}(r) \to T^*(Q_{cm})$.

For notational convenience, we work in the following on the tangent bundle over configuration space which is isomorphic to the cotangent bundle. Through this isomorphism, the tangent bundle can be endowed with a canonical symplectic form, which we denote by the same letter we used for the cotangent bundle, i.e.

$$\theta_{(x,v)} = \sum m_k < v_k, dx_k >= K(v, dx) \tag{3.7}$$

where each tangent space is equipped with a scalar product given by

$$K_x(u, v) = \sum m_k (u_k \mid v_k) \tag{3.8}$$

for $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_N)$ of $T_x(Q)$.
Define

$$\omega_x^D : \mathbf{so}(3) \to T_x(Q)$$

dual to $\omega_x : T_x \to \mathbf{so}(3)$ where the following isomorphisms has been taken into account $\mathbf{so}(3) \cong \mathbf{so}^*(3)$ and $T_x(Q) \cong T_x^*(Q)$. For $\mathbf{a} \in \mathbf{so}(3)$ and $v \in T_x(Q)$, ω_x^D is defined by

$$(\omega_x(v) \mid \mathbf{a}) =: K_x(v, \omega_x^D(\mathbf{a})). \tag{3.9}$$

One can prove that for any $v \in T_x(Q)$ the vector $v - \omega_x^D \mu_r(x,p)$ with $I_x^{-1}(p) = v$ is vibrational (horizontal). To this end, it suffices to show that $v - \omega_x^D \mu_r(x,p)$ and $R_\xi(x)$ are orthogonal for any $\xi \in \wedge^2\mathbb{R}^d$, in other words showing $K_x(R_\xi, v - \omega_x^D \mu_r(x,p)) = 0$, for $\forall \xi \in \wedge^2\mathbb{R}^3$ (see [3]).

Taking

$$(x, v)$$

as a coordinate system on $T(Q)$, the submanifold $\mu_r^{-1}(\mathbf{a})$ is determined in $T(Q)$ by the condition $R_{-\sum m_k x_k \wedge v_k} = \mathbf{a}$. Let now

$$w = v - \omega_x^D \mu_r(x,p)$$

with $I_x^{-1}(p) = v$, then the pair

$$(x, w)$$

meets the condition $R_{-\sum m_k x_k \wedge w_k} = 0$, so that it serves as coordinate system in $\mu_r^{-1}(0)$ under that condition. A coordinate system on $\mu_r^{-1}(\mathbf{a})$ can then be given by the pair $(x, w + \omega_x^D \mathbf{a})$. With this in mind we rewrite the canonical one-form θ at point $(x,v) \in T(Q_{cm})$, which obviously has $T_{(x,v)}(T(Q_{cm}))$ as its domain

$$\theta_{(x,v)} = K(v, dx) = K_x(w, dx) + K_x\big(\omega_x^D \mu_r(x,p), dx\big) = K_x(w, dx) + \big(\mu_r(x,p) \mid \omega_x \circ dx\big)$$

In the last step, we have used (3.9). Consequently on $\mu_r^{-1}(\mathbf{a})$ we have

$$i_{\mathbf{a}}^* \theta_{(x,v)} = K_x(w, dx) + (\mathbf{a} \mid \omega_x \circ dx) \tag{3.10}$$

where $v = w + \omega_x^D \mathbf{a}$. Thus the canonical two-form $d\theta$ restricts to $d(i_{\mathbf{a}}^* \theta)$ on $\mu_r^{-1}(\mathbf{a})$;

$$d(i_a^* \theta) = d\big(K_x(w, dx)\big) + d(\mathbf{a} \mid \omega \circ dx) \tag{3.11}$$

In all these equations dx should be viewed as a vector $[dx_1, dx_2, ..., dx_{N-1}]$. So one can formally act on it by the connection form ω and then take the exterior derivative and so on.
Since w is horizontal i.e. $w \in T_{x,hor} \cong T_{\pi(x)}(Q_{int})$ the first term on the right hand side of (3.11) is invariant under the action of $SO(3)$, and hence in one-to-one correspondence with the canonical two-form on $T(Q_{int}) \cong T^*(Q_{int})$. In contrast, the second term of the same side, depending on x, cannot project to a two-form on Q_{int}. In fact, $(\mathbf{a} \mid d\omega)$ is not horizontal (its value changes if we act with the group on it, or to be more precise, acting by any member of $G/G_{\mathbf{a}}$). We recall that the horizontal part of $d\omega$ is defined as the curvature form.

Last but not least, we discuss how an invariant metric[1] on the total space of fiber-bundles induces metrics on horizontal and vertical subspaces. In the context of molecular physics, this is known as *splitting of energy into vibrational and rotational parts* [3], as the total space $T(Q_{cm})$ is the tangent bundle over the center of mass configuration space of a molecule, and the group of 3-dimensonal rotations being the structure group.
Recall the decomposition

$$T_x(Q_{cm}) = T_{x,rot} \oplus T_{x,hor}$$

and the orthogonal projections

$$P_x : T_x \to T_{x,rot}$$

and

$$H_x := (1_x - P_x) : T_x \to T_{x,hor}$$

where 1_x denotes the identity element in $T_x(Q_{cm})$. With the help of the connection form $\omega_x : T_x(Q_{cm}) \to \mathbf{so}(d)$, one has the following orthogonal decomposition for any $v \in T_x(Q_{cm})$

$$\omega_x(v) = P_x(v) \Rightarrow v = P_x(v) + H_x(v)$$

Hence for any v and $u \in T_x(Q_{cm})$ one has

$$K_x(v, u) = K_x\big(P_x(v), P_x(u)\big) + K_x\big(H_x(v), H_x(u)\big) \tag{3.12}$$

[1] Invariant under the action of the structure group.

Now, if we set $v = u$, we obtain the kinetic energy expressed as the sum of rotational and vibrational energies. However, this does not mean there is no coupling between the rotational and vibrational motions. The coupling rather manifests itself into the dynamics through the connection form ω.

We now focus on the second term on the r.h.s of (3.12). Let π be the natural projection of Q_{cm} onto Q_{int}. Differentiation of that map $\pi_* : T(Q_{cm}) \to T(Q_{int})$ restricted on $T_{x,hor}$ gives an isomorphism of $T_{x,hor}$ with $T_{\pi(x)}(Q_{int})$. Let X,$Y \in T_m(Q_{int})$ for some $m \in Q_{cm}$. Then, at every point x with $\pi(x) = m$, one has unique horizontal vectors v and u satisfying $\pi_*(v) = X$ and $\pi_*(u) = Y$. If the metric K_x is $SO(3)$-invariant, i.e., $K_{gx}(gv, gu) = K_x(v, u)$ then the vibrational energy (second term of (3.12)) induces a Riemannian metric B on Q_{int} by

$$B_m(X, Y) := K_x(v, u) \tag{3.13}$$

One can easily verify that this definition is independent of the choice of x with $\pi(x) = m$.

We now look at the restriction of the vibrational energy to the submanifold $\mu_{\mathbf{a}}^{-1}$. Using the coordinate

$$w = v - \omega_x^D \mu_a(x, p)$$

(with $I_x^{-1}(p) = v$, paring the vectors and covectors (2.1)) the vibrational energy is written as $k_x(w, w)$ with $R_{-\sum m_k x_k \wedge w_k} = 0$, and is in one-to-one correspondence with the kinetic energy of the internal motion.

Now we turn to the first term on the r.h.s. of (3.12), the rotational energy. Considering the definition of the inertia operator A_x of the configuration x, as a linear operator in $\wedge^2\mathbb{R}^3$, one can calculate (see [3])

$$K_x(P_x(v), P_x(v)) = K_x\big(\omega_x(v), \omega_x(v)\big) = (RA_x^{-1}R^{-1}\mu_r(x, p) \mid \mu_r(x, p))$$

If the system's Hamiltonian is rotation invariant, the angular momentum is conserved, i.e., $\mu = \mathbf{a}$, and the last expression becomes a function of just the space variables x, namely $(RA_x^{-1}R^{-1}\mu \mid \mu)$. This function is, in fact, invariant under $G_{\mathbf{a}}$ (easily verifiable by using (D.7d)) and thus projects down to a function on the reduced phase space, which can be seen as centrifugal potential.

Now all the necessary ingredients are available for the reduction of the Hamiltonian system with respect to $SO(3)$. Remembering the inclusion map and the projection map

$$i_{\mathbf{a}} : \mu^{-1}(\mathbf{a}) \to P = T(Q_{cm}) \tag{3.14a}$$

$$\pi_{\mathbf{a}} : \mu^{-1}(\mathbf{a}) \to P_{\mathbf{a}} = \frac{\mu^{-1}(\mathbf{a})}{G_{\mathbf{a}}}. \tag{3.14b}$$

The reduced phase space $\frac{\mu^{-1}(\mathbf{a})}{G_{\mathbf{a}}}$ carries the symplectic form $\sigma_{\mathbf{a}}$ which, as discussed before, is related to the canonical form $d\theta$ through

$$i^*_{\mathbf{a}} d\theta = \pi^*_{\mathbf{a}} \sigma_{\mathbf{a}}.$$

On $\frac{\mu^{-1}(\mathbf{a})}{G_{\mathbf{a}}}$ the reduced Hamiltonian H_μ is defined by

$$H_{\mathbf{a}} \circ \pi_{\mathbf{a}} = H \circ i_{\mathbf{a}}.$$

Note that in the above equations defining the reduced symplectic form $\sigma_{\mathbf{a}}$ and the reduced Hamiltonian $H_{\mathbf{a}}$, the use of $\pi^{-1}_{\mathbf{a}}$ is avoided because the projection map $\pi_{\mathbf{a}}$ is not invertible (it sends an entire fiber to a point in the reduced space). This form and the Hamiltonian are expressed in coordinates (x, v) on $\mu_r^{-1}(\mathbf{a})$ with $v = w + \omega_x^D \mathbf{a}$ as follows

$$\pi^*_{\mathbf{a}} \sigma_{\mathbf{a}} = i^*_{\mathbf{a}} d\theta = d\big(K(w, dx)\big) + d(\mathbf{a} \mid \omega) \tag{3.15a}$$

$$H_{\mathbf{a}} \circ \pi_{\mathbf{a}} = H \circ i_{\mathbf{a}} = \frac{1}{2} K(w, w) + \frac{1}{2}\big(RA_x^{-1}R^{-1}\mu_r(x, p) \mid \mu_r(x, p)\big) + U \tag{3.15b}$$

where $R_{-\sum m_k x_k \wedge w_k} = 0$.

The r.h.s. of these equations are invariant under G_a, and hence can be thought of as quantities on the reduced phase space. The first expressions on the r.h.s. of (3.15a) and (3.15b) are in one-to-one correspondence with the canonical two-from, and the *kinetic energy* on $T^*(Q_{int}) \cong T(Q_{int})$ respectively. The second expressions on the r.h.s. of (3.15a) and (3.15b) can be seen as the source of the *Coriolis force* and the *centrifugal potential* of the system's projected motion on the internal space (so the internal motion) respectively.

Note, that if $a = 0$, the reduced phase space is diffeomorphic to the cotangent bundle $T^*(Q_{int})$ of the internal space Q_{int}, and the symplectic form $\sigma_{\mathbf{a}}$ becomes the canonical two form on $T^*(Q_{int})$. The reduced Hamiltonian H_a is then a sum of the kinetic energy of internal (horizontal) motions and the potential on Q_{int} (which is exactly the same potential as the one up on Q_{cm}). If the system's motion on absolute space is planar (so $Q \cong \mathbb{R}^{2N}$), and $\mathbf{a} \neq 0$ the reduced phase space is still diffeomorphic to $T^*(Q_{int})$, but the symplectic from $\sigma_{\mathbf{a}}$ is the canonical one plus a two-form which can be seen as a "magnetic field" on Q_{int}. The reduced Hamiltonian $H_{\mathbf{a}}$ also becomes the sum of kinetic and potential energies plus centrifugal potential. In both these cases, the system's motion is internal (horizontal). That means that it can be described on $T^*(Q_{int})$ or in terms of internal coordinates and their conjugate momenta.

Chapter 4

Reduction with respect to the similarity group $Sim(3)$

4.1 Metrics on the internal and shape space

Following [7] we next review how the kinetic metric on absolute configuration space induces a metric on the internal configuration space $Q_{int} = \frac{Q_{cm}}{SO(3)}$. After that, we explain a new way to derive a metric $\mathbf{N}$ on the $Sim(3)$-reduced tangent bundle ($\frac{T(Q)}{Sim(3)}$) from the mass metric $\mathbf{M}$ on absolute configuration space in a unique way. Moreover, we also explain how the unique metric N on shape space S can be derived.

Metric on the internal space:

Let us recall how the metric on the internal space $Q_{int} = \frac{Q_{cm}}{SO(3)}$ was derived from the $SO(3)$-invariant mass metric (3.8) on the center of mass system

$$\mathbf{M}_x(u, v) = \sum m_k < u_k \mid v_k > \tag{4.1a}$$

$$\mathbf{M}_x(u, v) = \mathbf{M}_{gx}(gu, gv) \tag{4.1b}$$

where $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_N) \in T_x(Q_{cm})$ being any two tangent vectors of Q_{cm} at the point $x \in Q_{cm}$.
Given two internal vectors $v', u' \in T_q(Q_{int})$, there are unique vectors $u, v \in T_x(Q_{cm})$ [1] so that

$$\begin{cases} \pi(x) = q \\ \pi_*(u) = u' \\ \pi_*(v) = v'. \end{cases}$$

Now the metric B on Q_{int} can be defined in the following way

$$B_q(v', u') := \mathbf{M}_x(v, u). \tag{4.2}$$

Since the metric M is $SO(3)$-invariant, it does not make any difference to which $x \in \pi^{-1}(q)$ the internal vectors v, u had been lifted for the value assigned by B_q. This is, in fact, crucial for the well-definedness of the reduced metric.

The kinetic energy of a N-particle system in the center of mass frame is $K = \frac{1}{2}\sum_{\alpha=1}^{N-1} \mid \dot{\boldsymbol{r}}_\alpha \mid^2$. Using $\dot{\boldsymbol{r}}_{b\alpha} = \boldsymbol{\omega} \times \boldsymbol{r}_{b\alpha} + \frac{\partial \boldsymbol{r}_{b\alpha}}{\partial q^\mu}\dot{q}^\mu$, and the expression

$$\boldsymbol{A}_\mu(q) = I^{-1}\boldsymbol{a}_\mu$$

for the gauge potentials, where

$$\boldsymbol{a}_\mu = \boldsymbol{a}_\mu(q) := \sum_{\alpha=1}^{N-1} \boldsymbol{r}_{b\alpha} \times \frac{\partial \boldsymbol{r}_{b\alpha}}{\partial q^\mu} \tag{4.3}$$

and A being the moment of inertia tensor with components

$$A_{ij} = A_{ij}(q) := \sum_{\alpha=1}^{N-1} (\mid \boldsymbol{r}_{b\alpha} \mid^2 \delta_{ij} - r_{b\alpha i} r_{b\alpha j})$$

one can write down the kinetic energy as

$$K = \frac{1}{2} < \boldsymbol{\omega} \mid A \mid \boldsymbol{\omega} > + < \boldsymbol{\omega} \mid A \mid \boldsymbol{A}_\mu > \dot{q}^\mu + \frac{1}{2} h_{\mu\nu}\dot{q}^\mu \dot{q}^v \tag{4.4}$$

with

$$h_{\mu\nu} = h_{\mu\nu}(q) = \sum_{\alpha=1}^{N-1} \frac{\partial \boldsymbol{\rho}_\alpha}{\partial q^\mu} . \frac{\partial \boldsymbol{\rho}_\alpha}{\partial q^\nu}. \tag{4.5}$$

[1] namely their horizontal lifts (4.7).

The velocity of a system's configuration in Jacobi coordinates is given by a vector

$$| v >= [\dot{\boldsymbol{r}}_{s1}, ..., \dot{\boldsymbol{r}}_{s,n-1}]$$

and in orientational and internal coordinates by the vector

$$| v >= [\dot{\theta}^i, \dot{q}^\mu]$$

where θ^i are the Euler angles which turn the space frame to the body frame of a configuration. If one decides to use the components of the body angular velocity $\boldsymbol{\omega}$ instead of the time derivatives of the Euler angles for denoting vectors in $T(SO(3))$ the configuration's velocity can alternatively be expressed as

$$| v >= [\boldsymbol{\omega}, \dot{q}^\mu]$$

in angular velocity and internal basis. This last combination forms an anholonomic frame or vielbein on $T(Q_{cm})$. Remember the relation between the body components of angular velocity and derivatives of Euler angles, i.e.,

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -sin\beta cos\gamma & sin\gamma & 0 \\ sin\beta sin\gamma & cos\gamma & 0 \\ cos\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}.$$

So, the (kinetic)metric tensor $m_{Q_{cm}}$ in angular and internal basis vectors $\{\omega^i, \dot{q}^\mu\}$, where $i = 1, 2, 3$, and $q = 1, ..., 3N - 6$ becomes as follows

$$< v \mid v >= \begin{bmatrix} \boldsymbol{\omega}^T & \dot{q}^\mu \end{bmatrix} \begin{bmatrix} A & A\boldsymbol{A}_\nu \\ \boldsymbol{A}_\mu^T A & h_{\mu\nu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \dot{q}^\nu \end{bmatrix} = (m_{Q_{cm}})_{ab} v^a v^b$$

So, the metric on Q_{cm} in angular and internal basis vectors $[\boldsymbol{\omega}, \dot{q}^\mu]$ is given by

$$\mathbf{M}_{ab} = \begin{bmatrix} A & A\boldsymbol{A}_\nu \\ \boldsymbol{A}_\mu^T A & h_{\mu\nu} \end{bmatrix}. \tag{4.6}$$

Decomposition of an arbitrary system's velocity in horizontal and vertical parts gives

$$| v >=| v_v > + | v_h >$$

$$[\boldsymbol{\omega}, \dot{q}^\mu] = [\boldsymbol{\omega} + \boldsymbol{A}_v \dot{q}^v, 0] + [-\boldsymbol{A}_\nu \dot{q}^\nu, \dot{q}^\mu]$$

Correspondingly, the kinetic energy of the system can also be thought of as the addition of two separate vertical and horizontal kinetic energies, i.e.

$$K = K_v + K_h = \frac{1}{2}(\boldsymbol{\omega} + \boldsymbol{A}_\mu \dot{q}^\mu) A (\boldsymbol{\omega} + \boldsymbol{A}_\nu \dot{q}^\nu) + \frac{1}{2} B_{\mu\nu} \dot{q}^\mu \dot{q}^v$$

where $B_{\mu\nu}$ is the metric on internal space

$$B_{\mu\nu} = h_{\mu\nu} - \boldsymbol{A}_\mu A \boldsymbol{A}_\nu.$$

So, in summary, to a vector

$$\mid v' >= \dot{q}^\mu$$

on internal space Q_{int} we associate a vector $\mid v_h >$ on Q_{cm} which is called its *horizontal lift*, connecting the two fibers. In the basis made of angular and shape velocities, the horizontal lift of v' takes the from

$$\mid v_h >= [-\boldsymbol{A}_\mu \dot{q}^\mu, \dot{q}^\mu] \tag{4.7}$$

Then, the metric $B_{\mu\nu}$ on the internal space can be found by the following defining equation

$$< v'_1 \mid v'_2 >= B_{\mu\nu} \dot{q}_1^\mu \dot{q}_2^\nu :=< v_{1h} \mid v_{2h} >= \mathbf{M}_{ab} v_{1h}^a v_{2h}^b$$

which leads to

$$B_{\mu\nu} = h_{\mu\nu} - \boldsymbol{A}_\mu A \boldsymbol{A}_\nu \tag{4.8}$$

For more information about the derivation of the metric on internal space $Q_{int} = \frac{Q}{E(3)}$, we highly recommend [7].

Metrics on shape space:

Now we are ready to derive a metric N on shape space $S = \frac{Q}{Sim(3)}$. Since the mass metric $\mathbf{M}$ is not scale invariant, i.e.,

$$\mathbf{M}_{cx}(Sc_* u, Sc_* v) = \mathbf{M}_{cx}(cu, cv) = c^2 \mathbf{M}_{cx}(v, u) = c^2 \mathbf{M}_x(v, u) \neq \mathbf{M}_x(v, u) \tag{4.9}$$

it is generally believed that, contrary to Q_{int}, it does not uniquely induce a metric on shape space S. However, once one uses measuring units built from matter instead of absolute measuring units, one sees that the mass metric uniquely induces a metric on Shape space. We first review how a metric on shape space is derived with the introduction of a conformal factor and then give our derivation of the unique metric on shape space, and explain the metric's uniqueness, and the relationship between realistic units of length and conformal factors.

As is explained in [16] one can introduce a new $Sim(3)$-invariant metric on Q, which subsequently induces a metric on shape space in a natural way. As the mass metric $\mathbf{M}$ is already rotation- and translation-invariant, the easiest way to arrive at a similarity-invariant metric is to multiply the mass metric by a function $f(x)$ (the so-called conformal factor) so that the whole expression

$$\mathbf{M}'_x := f(x)\mathbf{M}_x$$

becomes scale invariant, i.e.,

$$\forall c \in \mathbb{R}^+, \forall u, v \in T_x(Q) : f(cx)\mathbf{M}_{cx}(Sc_* u, Sc_* v) = f(x)\mathbf{M}_x(u, v).$$

Note that the function f must be translation- and rotation-invariant so that it does not spoil the Euclidean invariance of the mass metric. As $\mathbf{M}'_x = f(x)\mathbf{M}_x$ is now a metric invariant under the whole similarity group we are ready to write down the metric N on shape space:

$$N_s(v', u') := \mathbf{M}'_x(v, u) = f(x)\mathbf{M}_x(v, u), \tag{4.10}$$

where

$$\begin{cases} \pi(x) = s \\ \pi_*(u) = u' \\ \pi_*(v) = v' \end{cases}$$

with the projection map $\pi : Q_{cm} \to S = \frac{Q}{sim(3)}$.

When the action of scale transformation on $T(Q)$ is defined by the differential of the scale transformations, i.e., Sc_*, from the behavior of the mass metric $\mathbf{M}$ under scale transformation (4.9) one sees that any rotation- and translation-invariant homogeneous function[2] of degree -2 perfectly meets all the requirements of a conformal factor. For instance

$$f(x) = \sum_{i<j} \| \boldsymbol{x}_i - \boldsymbol{x}_j \|^{-2} \tag{4.11}$$

[2]A function of r variables $x_1, ..., x_r$ is being called homogeneous of degree n if $f(cx_1, ..., cx_r) = c^n f(x_1, ..., x_r), \forall c$

or

$$f(x) = I_{cm}^{-1} \tag{4.12}$$

where

$$I_{cm}(x) = \sum_j m_j \mid\mid \boldsymbol{x}_j - \boldsymbol{x}_{cm} \mid\mid^2 = \frac{1}{\sum_i m_i} \sum_{i<j} m_i m_j \mid\mid \boldsymbol{x}_j - \boldsymbol{x}_i \mid\mid^2$$

are two legitimate examples of conformal factors (as suggested in [16]). However, as the introduction of arbitrary conformal factors leads to arbitrary metrics on shape space and leads to the appearance of unphysical forces (see appendix A for clarification), this treatment seems unsatisfactory to us. By paying more attention to the important role of the measurement units in determination of the geometry of space, below we propose another way to derive the metric on shape space which does not have the problem(arbitrariness) just mentioned.

Bearing in mind that measurements of the velocity are, in essence, an experimental task, the transformation law of the velocities under scale transformations of the system (or any other transformation of the system) must also include experimental reasoning. Based on the principle of relationalism, we showed that the behavior of rods and clocks under scale transformations of the system is such that the measured velocities of objects (or parts of the system) are invariant. It is a natural consequence of the simultaneous expansion of the measuring rod and the corresponding dilation of the unit of time (See Section (1.4) for an explanation of this fact). Hence, a velocity vector

$$v_x = (v_1, ..., v_N) \in T_x(Q_{cm})$$

of an N-particle system transforms under scale transformations of the system as follows

$$x \to cx$$

$$v_x = (v_1, ..., v_N) \in T_x(Q_{cm}) \to v_{cx} = (v_1, ..., v_N) \in T_{cx}(Q_{cm})$$

Given the above action $\mathcal{A}_c$ of $c \in Sc \subset Sim(3)$ on velocities(or on $T(Q)$); the mass metric is a $\mathcal{A}_{Sc}$-invariant metric on Q, as can be seen by a short calculation:

$$\mathbf{M}_x(v_x, u_x) \to \mathbf{M}_{cx}(\mathcal{A}_c v_x, \mathcal{A}_c u_x) = \mathbf{M}_x(v_x, u_x) \tag{4.13}$$

where the equalities $\mathbf{M}_x = \mathbf{M}_{cx} = \mathbf{M}$ and $\mathcal{A}_c v_x = v_x$ has been used. Considering $T(Q_{int}) = T(\frac{Q}{E(3)})$ as a $\mathcal{A}_{Sc}$-fiber-bundle, the mass metric B on $Q_{int} = \frac{Q}{E(3)}$ (defined previously by expression (4.2)) induces a unique metric

$$\mathbf{N}_s : T(Q_{int})/\mathcal{A}_{Sc} \times T(Q_{int})/\mathcal{A}_{Sc} \to \mathbb{R}$$

We arrive in this way at the metric $\mathbf{N}$ as follows

$$\mathbf{N}_s(v', u') := B_q(v, u) \tag{4.14}$$

where

$$\begin{cases} \pi(q) = s \\ \pi'(u) = u' \\ \pi'(v) = v' \end{cases}$$

with the projection map defined as follows

$$\pi : Q_{int} \to S$$

$$\pi' : T(Q_{int}) \to T(Q_{int})/\mathcal{A}_{Sc}$$

Because the above construction is $\mathcal{A}_{Sc}$-invariant, to which $q \in \pi^{-1}(s)$ the pair of shape vectors $v', u' \subset T_q(Q_{int})/\mathcal{A}_{Sc}$ are lifted, does not make any difference for the value assigned by $\mathbf{N}_s$ to them. Hence, the metric $\mathbf{N}$ is also well defined. This method brings one uniquely to the shape (kinetic)metric $\frac{1}{2}\mathbf{N}$ on $T(Q_{int})/\mathcal{A}_{Sc}$ without the need of introducing a conformal factor and the mentioned ambiguity involved with it.

Alternatively, one can complement the DGZ-derivation of metric on shape space (4.10), and remove the involved arbitrariness in it as follows. As seen before, Mathematically, a metric $\mathbf{G}$ on a manifold Q is called scale-invariant if and only if

$$\forall v_1, v_2 \in T_q(Q) : \mathbf{G}_q(v_1, v_2) = \mathbf{G}_{cq}(Sc_* v_1, Sc_* v_2) \tag{4.15}$$

where $Sc_* : T(Q) \to T(Q)$ denotes the push forward of vectors along the scale transformations $Sc : q \to cq$ on Q. Since $Sc_* v = cv$, we saw that the mass metric $\mathbf{M}$ is not scale-invariant in this sense(4.9). However, what one physically measures and is relevant is not $\mathbf{M}$ but

$$\mathbf{M}_q^{(m)}(v_1, v_2) = \frac{\mathbf{M}_q(v_1, v_2)}{\mathbf{M}_q(\boldsymbol{q}_i - \boldsymbol{q}_j, \boldsymbol{q}_i - \boldsymbol{q}_j)} \tag{4.16}$$

where $1 < i, j < N$ are two particles that are used to define the unit of length. This is another way to realize how the arbitrariness of the metric on shape space criticised before disappears by the usage of real measuring units instead of "inaccessible absolute units". The **measured mass metric** is on its own scale invariant in the mathematical sense mentioned above. One could say that part of the arbitrariness of the conformal factor is now in fact shifted to the arbitrariness in the choice of a length unit, i.e., which particles i and j one chooses to define the length unit. However, one should realize that all reasonable choices of length unit will lead to the same metric N on shape space. A reasonable choice of length unit would be such that leads to no fictitious forces.

It is worth noting that N is a metric on $T(S) = T(Q_{int})/Sc_*$, while $\mathbf{N}$ is a metric on $T(Q)/\mathcal{A}_{Sc}$. Thus, these are metrics on two different vector bundles. Although we intuitively expect them to represent the same physical entity, their mathematical equivalence is not obvious to us. Throughout the rest of this text we will always work with $T(Q)/\mathcal{A}_{Sc}$ and use $\mathbf{N}$. With some abuse of notation, we denote both bundles by $T(S)$, but it is clear from the context which bundle is meant.

4.2 Reduction of the theory

For reduction of classical mechanics w.r.t. scale transformations, we now use the methods explained in chapters two and three. One of the reasons that so far nobody has gone after the extension of these formalisms to the similarity group is that the potential function defined on absolute space, though manifestly rotational and translational invariant, is clearly not scale invariant (take Newtonian gravity as an example). However, as explained in Chapter (2), scale transformation becomes an additional symmetry of the (modified) classical physics (see equation (1.17)), and this enables us to perform a symplectic reduction of the system's phase space with respect to the whole similarity group $G = sim(3)$.

Besides having a similarity invariant potential function on absolute configuration space (see equation (1.17)), in order to reduce the classical systems with respect to the similarity group (which was initially argued for and motivated in Chapter 1) with the help of symplectic reduction methods explained in Chapter (2), we have to change the connection form (2.11) to the following one

$$\omega = \omega_r + \omega_s = R\bigg(A_x^{-1}(\sum_{j=1}^{N-1} r_j \times dr_j)\bigg) + I_3 \mathbf{D}_x^{-1}\bigg(\sum_{j=1}^{N-1} r_j . dr_j\bigg) \tag{4.17}$$

in which I_3 is the 3×3 identity matrix, and we have defined the operator

$$\mathbf{D}_x : \mathbb{R} \to \mathbb{R}$$

$$\dot{\boldsymbol{\lambda}} \to D$$

expansion velocity $\to$ *dilatational momentum*

as

$$\mathbf{D}_x(\dot{\boldsymbol{\lambda}}) := \sum_{j=1}^{N-1} r_j^2 \dot{\boldsymbol{\lambda}} \tag{4.18}$$

and we call it the *dilatational tensor*. Here $\dot{\boldsymbol{\lambda}}$ stands for the rate of change of scale of the system (scale velocity so to speak)[3]

$$\dot{\boldsymbol{\lambda}} := \frac{\dot{\lambda}}{\lambda} \tag{4.19}$$

with

$$\lambda := max \mid \boldsymbol{x}_i - \boldsymbol{x}_j \mid \tag{4.20}$$

for $i, j = 1, ..., N$ being the system's scale variable.

We constructed this operator in direct analogy to the inertia tensor A_x. The inertia tensor sends an angular velocity (which can be represented as a vector in $\mathbb{R}^3$) to another vector in $\mathbb{R}^3$ which represents the total angular momentum of the whole system (object). In the same way, the dilatational tensor $\mathbf{D}_x$ takes an expansion velocity, which can be represented by just a number in $\mathbb{R}$ to a measure

[3]Here of course we assume that all measurements are conducted with the use of special Newtonian rods and clocks, which are isolated from the materialized universe and do not get affected by them in any way or by transformations we perform on the materialized universe. Practically of course such measuring instruments do not exist, but the existence of absolute space and absolute time in Newtonian worldview justifies their hypothetical existence.

of the total expansion of the system (dilatational momentum D) which again can be represented by another number in $\mathbb{R}$. As the Lie algebra of the group $\frac{sim(3)}{trans(3)}$ can be considered to be the addition of 3×3 skew-symmetric matrices of $\mathbf{so}(3)$, and real multiples of 3×3 identity matrix I_3, one recognizes the correct structure in this connection form. If one takes a random vector of $T_x(M)$ and acts on it by this connection form, the first term of (7.12) gives a member of $\mathbf{so}(3)$, and the second term, a number multiplied by the identity matrix I_3. So, it does what it is expected to do.

4.3 Symplectic reduction of phase space

Now that the metric (7.8) on shape space, and the connection form (7.12) is given, the way to get the reduced Hamiltonian equations of motion with respect to the similarity group is paved.

The first step is to find the momentum mapping corresponding to the following group

$$G = SO(3) \times \mathbb{R}^+$$

To this end consider first the Lie-algebra of G. It can be written as

$$\mathbf{G} = zI_3 + \mathbf{so}(3)$$

where $z \in \mathbb{R}$ and $\mathbf{so}(3)$ are as usual the skew-symmetric 3×3 matrices representing the Lie-algebra of the rotations group.

The action of G on $T^*(Q_{cm})$ is as follows

$$(x_1, ..., x_{N-1}; p_1, ..., p_{N-1}) \tag{4.21}$$

$$\downarrow$$

$$(cgx_1, ..., cgx_{N-1}; gp_1, ..., gp_{N-1})$$

where, as before, $c \in R^+$ and $g \in SO(3)$ (and all the lengths and time intervals are measured by the absolute Newtonian rods and clocks as seen before (??).). The momentum mapping corresponding to scale transformations is

$$\mu_{scale} = \sum_{j=1}^{N-1} r_j . dr_j$$

Hence, the momentum mapping for the group [4] G becomes as follows

$$\mu_{sim} = I_3 \sum_{j=1}^{N-1} r_j . dr_j + \sum_{j=1}^{N-1} R_{-r_j \wedge p_j} \tag{4.22}$$

We call μ_{sim} the similarity momentum [5]. Remember that mathematically the similarity momentum is the following mapping

$$\mu_{sim} : P = T^*(Q_{cm}) \to \mathbb{R}I_3 + \mathbf{so}(3) := \mathbf{G} \cong \mathbf{G}^* \tag{4.23}$$

In fact $\mathbf{G}$ is the summation of a skewsymmeric matrix with a real multiple of the identity matrix.
Let $a \in \mathbf{G} \cong \mathbf{G}^*$. It can be rewritten as

$$a = DI_3 + \mathbf{L}$$

for some $D \in \mathbb{R}$ standing for dilatational momentum (diagonal part of a), and some $\mathbf{L} \in \mathbf{so}(3)$ standing for the angular momentum (the non-diagonal part of a). As before, $\mu_G^{-1}(a)$ is a submanifold of $P = T^*(Q_{cm})$. The reduced phase space $\frac{\mu_G^{-1}(a)}{G_a}$ is then achieved by quotienting $\mu_G^{-1}(a)$ with respect to the isotropy group G_a.
The isotropy group corresponding to rotations was already discussed in Section (4). Now it is time to discuss the same question for the group of scale transformations. For simplicity, consider a system that is purely expanding. The dilatational momentum of this system is

$$D = \sum_{i=1} \mathbf{r}_i . \mathbf{v}_i$$

Perform a scale transformation $\mathbf{r}_i \to c\mathbf{r}_i$. Having the corresponding transformation law of the velocities (4.21) in mind, the dilatational momentum transforms

[4] of both rotations and scale transformations

[5] with slight abuse of notation, since starting with the center of mass system translations are not taken into account.

as follows

$$D \to D' = \sum_i (c\mathbf{r}_i).\mathbf{v}_i = cD \qquad (4.24)$$

This should be compared to the system's angular momentum, which is also in general[6] not invariant under rotations[7]. From (4.24), it becomes clear that any scale transformation changes the value of the dilatational momentum, except if $D = 0$. Hence the isotropy subgroup of the scale transformations is $\oslash$ when $D \neq 0$, and $\mathbb{R}^+$ when $D = 0$.

Now we are in a position to compare the reduced phase space $\frac{\mu_G^{-1}(a)}{G_a}$ with respect to $G = \mathbb{R}^+ \circ SO(3)$ with the cotangent bundle of shape space $T^*(S)$. Remember that $dim\big(T^*(Q_{cm})\big) = 6N - 6$ and $dim\big((T^*(S)\big) = 6N - 14$. Thus, when going from the cotangent bundle of the center of the mass system $T^*(Q_{cm})$ to the cotangent bundle of the shape space $T^*(S)$, eight dimensions get eliminated.

Choosing a specific value $a \in \mathbf{G}$ for the similarity momentum of the system reduces the dimension of $T^*(Q_{cm})$ by four. Consequently, taking the quotient with respect to the corresponding isotropy group G_a, leads to the elimination of one extra dimension when $\mathbf{L} \neq 0$ and $D \neq 0$ (because in this case $G_a = SO(2)$) and four extra dimensions when $\mathbf{L} = D = 0$. For this latter case, eight dimensions are eliminated in total, and hence the reduced phase space becomes isomorphic to the cotangent bundle of shape space, i.e.,

$$\frac{\mu_G^{-1}(0)}{G_0} \cong T^*(S).$$

For the generic case of $\mathbf{L}, D \neq 0$ we have

$$dim\big(\frac{\mu_G^{-1}(a)}{G_a}\big) = dim\big(T^*(S)\big) + 3 \qquad (4.25)$$

We wish now to discuss the canonical form and its reduction with respect to $G = SO(3) \times \mathbb{R}^+$. In particular, we seek the reduced symplectic form σ_a on the reduced phase space

$$P_a = \frac{\mu_r^{-1}(a)}{G_a}$$

[6]whenever the system's angular momentum is non-vanishing

[7]Note, however, using instead of absolute Newtonian rods and clocks internal rods and clocks, dilatational momentum becomes invariant under scale transformations.

starting as usual with the canonical one-form

$$\theta = \sum m_k < v_k, dx_k >= K(v, dx)$$

as we saw in (3.7). Remember that σ_a was defined by $\pi_\mu^* \sigma_\mu = i_\mu^* \sigma$ with $\sigma = d\theta$ and the inclusion map i_μ and the projection map π_a were defined as follows

$$i_a : \mu_{sim}^{-1}(a) \to P = T^*(Q_{cm}) \tag{4.26a}$$

$$\pi_a : \mu_{sim}^{-1}(a) \to P_a = \frac{\mu_{sim}^{-1}(a)}{G_a} \tag{4.26b}$$

Define an operator dual to the connection form (7.12) as

$$\omega_x^D : \mathbf{G} = \mathbb{R}I_3 + \mathbf{so}(3) \to T_x(Q_{cm})$$

For $a \in \mathbf{G}$ and $v \in T_x(Q_{cm})$, ω_x^D was defined in (3.9). We mentioned in Section (3.2) that for any $v \in T_x(Q)$ the vector $v - \omega_x^D \mu_{sim}(x, p)$ with $I_x^{-1}(p) = v$ is horizontal.
We choose (x, v) as coordinate system in $T(Q_{cm})$. The submanifold $\mu_{sim}^{-1}(a)$ is determined in $T(Q_{cm})$ by the condition

$$I_3 \sum_{j=1}^{N-1} x_j.v_j + \sum_{j=1}^{N-1} R_{-m_j x_j \wedge v_j} = a$$

Now, with the help of ω^D decompose a vector $v \in T(Q_{cm}) \cong T^*(Q_{cm})$ into horizontal and vertical parts, i.e., $v = w + \omega_x^D \mu_{sim}(x, p)$ where again $I_x^{-1}(p) = v$. Rewrite the canonical one-form θ as the following

$$\theta_{(x,v)} = \mathbf{M}(v, dx) = \mathbf{M}\big(w + \omega_x^D \mu_{sim}(x, p), dx\big) = \mathbf{M}(w, dx) + K\big(\omega_x^D \mu_{sim}(x, p), dx\big)$$

$$= \mathbf{M}(w, dx) + \big(\mu_{sim}(x, p) \mid \omega_x \circ dx\big)$$

Hence, on $\mu_{sim}^{-1}(a)$ we get the following one-form

$$i_a^* \theta_{(x,v)} = \mathbf{M}(w, dx) + (a \mid \omega_x \circ dx) \tag{4.27}$$

where we again used the horizontal-vertical decomposition of vectors $v = w + \omega_x^D a$. The canonical two-from on $\mu_{sim}^{-1}(a)$ then becomes

$$d(i_a^* \theta) = d\big(\mathbf{M}(w, dx)\big) + d(a \mid \omega \circ dx) \tag{4.28}$$

Since w is horizontal, i.e., $w \in W_{x,hor} \cong T_{\pi(x)}(S)$ the first term in the right hand side of (4.28) is invariant under $G = \mathbb{R}^+ \times SO(3)$, and hence in one-to-one correspondence with the canonical two-form on $T(S) \cong T^*(S)$. Contrary to this, the second term of the same side depends on x and hence does not project to any two-form on $T^*(S)$. As a matter of fact, $(a \mid d\omega)$ is not horizontal (its value changes if we act with the group on it, or to be more precise, acting by any member of G/G_a).

Having the following substitutions for the similarity momentum

$$a = DI_3 + \mathbf{L}$$

and for the connection form (7.12),

$$\omega = \omega_r + \omega_s$$

where as before

$$\omega_r : T(Q_{cm}) \to \mathbf{so}(3)$$

and

$$\omega_s : T(Q_{cm}) \to \mathbb{R}I_3$$

we rewrite the second term in (4.28) as follows

$$d(a \mid \omega \circ dx) = d\big(DI_3 + \mathbf{L} \mid (\omega_r + \omega_s) \circ dx\big) = d(\mathbf{L} \mid \omega_r \circ dx) + d(DI_3 \mid \omega_s \circ dx)$$

Here, we also used the fact that a diagonal and a skew-symmetric matrix are perpendicular to each other w.r.t. the matrix inner product $(A \mid B) = \frac{1}{2}tr(AB^T)$. Hence, the two-form on reduced phase space $\mu_{sim}^{-1}(a)$ can be rewritten:

$$d(i_a^*\theta) = d\big(\mathbf{M}(w, dx)\big) + d(\mathbf{L} \mid \omega_r \circ dx) + d(DI_3 \mid \omega_s \circ dx) \qquad (4.29)$$

As mentioned before, in general ($\mathbf{L} \neq 0$ and $D \neq 0$), neither the second term nor the third term is in one-to-one correspondence with two-forms on $T^*(S)$. It is because of the non-trivial transformation laws of $\mathbf{L}$ and D under the group $\frac{G}{G_a}$. However, as these terms are invariant under G_a, they can legitimately be considered as differential forms on reduced phase space P_a. The second term is the source of *Coriolis force* [3]. We call the last term **dilatational force**. Note that even when the overall angular momentum of the system under consideration vanishes, i.e., **L**$= 0$, its overall expansion (or contraction) causes extra complication for the reduced motion on S so that it fails to be describable in $T^*(S)$. So the dilatational force, like the Coriolis force, is a non-Hamiltonian force (cannot be derived from a potential function on S).

A similar procedure to the one seen in the previous section leads to the splitting of the kinetic energy. Having the decomposition

$$T_x(Q_{cm}) = T_{x,sim} \oplus T_{x,hor}$$

and the orthogonal projections

$$P_x = T_x \to T_{x,sim}$$

and

$$H_x = 1_x - P_x$$

in mind, for any v and $u \in T_x(Q_{cm})$ one has

$$\mathbf{M}_x(v,u) = \mathbf{M}_x\big(P_x(v), P_x(u)\big) + \mathbf{M}_x\big(H_x(v), H_x(u)\big)$$

Now, if we set $v = u$, we obtain the kinetic energy expressed as the sum of similarity and shape energies. In particular, the similarity energy can be written down as

$$\mathbf{M}_x\big(P_x(v), P_x(v)\big) = \big(RA_x^{-1}R^{-1}\mathbf{L} \mid \mathbf{L}\big) + \big(I_3\mathbf{D}^{-1}D \mid I_3 D\big) \tag{4.30}$$

where R and A_x were defined as follows

$$R : \mathbb{R}^3 \to \mathbf{so(3)}$$

$$R(a) = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix}$$

for any $a \in \mathbb{R}^3$, as is explained in more extend in appendix D, and

$$A_x : \mathbb{R}^3 \to \mathbb{R}^3$$

$$A_x(\boldsymbol{v}) = \sum_{j=1}^{N-1} \boldsymbol{r}_j \times (\boldsymbol{v} \times \boldsymbol{r}_j), \boldsymbol{v} \in \mathbb{R}^3, x \in Q_{cmns}$$

The symplectic form σ_a on the reduced phase space $P_a = \frac{\mu_{sim}^{-1}(a)}{G_a}$ can as before be derived from the canonical 2-from $d\theta$ through

$$i_a^* d\theta = \pi_a^* \sigma_a$$

and the reduced Hamiltonian H_a on P_a is defined through $H_a \circ \pi_a = H \circ i_a$.[8] Choosing the coordinate system (x, v) on $\mu_{sim}^{-1}(a)$, where $v = \omega^D(a)$, one gets the following expressions for the reduced two-form, and reduced Hamiltonian

$$\pi_a^* \sigma_a = i_a^* d\theta = d\big(\mathbf{M}(w, dx)\big) + d(\mathbf{L} \mid \omega_r) + d(DI_3 \mid \omega_s) \tag{4.31}$$

$$H_a \circ \pi_a = H \circ i_a = \frac{1}{2}\mathbf{M}(w, w) + \frac{1}{2}\big(RA_x^{-1}R^{-1}\mathbf{L} \mid \mathbf{L}\big) + \frac{1}{2}\big(I_3\mathbf{D}^{-1}D \mid I_3 D\big) + U \tag{4.32}$$

The last term at the right-hand side of (4.31) can be understood as a new kind of fictitious force, which we called *dilatational force*. It has a similar nature as the Coriolis force. The third term on the right-hand side of (4.32) can also be understood as a new kind of potential, which we call **scale potential**. The scale force, defined as the gradient of the scale potential, is in its nature similar to the centrifugal force.

[8] We have defined the inclusion map as follows

$$i_a : \mu_{sim}^{-1}(a) \to P = T^*(Q_{cm})$$

Chapter 5

Principle Of Relationalism Revisited

We have explained in Chapter (1), that the ideas of *Gottfried Wilhelm Leibniz*, and *Ernst Mach* on the foundation of mechanics, and in particular, about the absolute 3-dimensional space and the absolute time, as two of the building blocks of the classical physics ([20],[21]), can partly be expressed, and made more concise in terms of the **Principle of Relationalism**, which amounts to the following statement

Two possible universes, differing from each other just by the action of a global similarity transformation $Sim(3)$, are observationally[1] *indistinguishable.*

This statement includes, among others, the invariance of the laws of physics under any $Sim(3)$ transformation of the universe. The dynamical indistinguishably of the two alternative universes mentioned above requires the introduction of a particular action of the group of scale transformations Sc on particles' velocities(or on phase space), as will be explained in more detail in this paper. Different

[1] *with observationally indistinguishable we mean kinematically and dynamically indistinguishable.*

absolute theories may require different actions. For some classes of absolute theories, there may exist no action of Sc on the universe's phase space for which the considered theory could satisfy the mentioned principle. In this case, the nonexistence of a relational reformulation of the absolute theory under consideration follows immediately. We explained that a direct implementation of this principle requires the gravitational coupling G appearing in Newton's theory of gravitation to be a homogeneous function of degree one on the configuration space[2], and not a constant. Hence, in a given absolute frame of reference[3], after a global scale transformation of the universe by a factor $c \in \mathbb{R}^+$

$$(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) \to (c\boldsymbol{x}_1, ..., c\boldsymbol{x}_N)$$

the gravitational coupling must transform as

$$G \to G' = cG$$

This makes the Newtonian gravitational potential $V = \sum_{i,j=1}^{N} \frac{Gm_im_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}$ scale-invariant, and together, because of its rotation and translation invariance, $V : Q \to \mathbb{R}$ uniquely projects down to a function $V_s : S \to \mathbb{R}$ on shape space. The new theory, which promotes the gravitational coupling from a constant (as is the case in the Newtonian theory) to a function with the mentioned properties, is called the *modified Newtonian Theory*.

The above construction alone is, however, not sufficient for implementing the Principle of Relationalism, as it would make the strength of the gravitational force (measured, for instance, with respect to the strong nuclear force) scale-dependent. This would cause a clear violation of the mentioned principle. In order to avoid this and other violations, Planck's constant $\hbar$, and the vacuum permittivity ϵ_0, have also been promoted to homogeneous functions on Q of degrees 1 and -1 respectively. These, in particular, ensure the scale independence of the relative strength of the four known forces in nature [23].

It is also essential to take the transformations of the *measuring units* under the

[2]Whose exact form is yet unknown, but not needed for the aim of this paper.

[3]A hypothetical immaterial frame attached to the absolute space. One can call it god's frame of reference if one wishes so.

global scale transformations into account. Max Planck has derived units of duration, length, and mass from G, $\hbar$, ϵ_0, and light's average velocity $\boldsymbol{c}$ by dimensional analysis, i.e.

$$
\begin{aligned}
L_P &= \sqrt{\frac{G\hbar}{\boldsymbol{c}^3}} \\
M_P &= \sqrt{\frac{\hbar \boldsymbol{c}}{G}} \qquad (5.1) \\
T_p &= \sqrt{\frac{G\hbar}{\boldsymbol{c}^5}}
\end{aligned}
$$

One of the main advantages of this set of units is their accessibility in all regions of the universe in which the same laws of physics as ours (i.e., the laws of quantum mechanics, electromagnetism, and gravity) hold.

Seen from the absolute frame of reference and measured with some hypothetical fixed absolute units of duration and length[4], Planck's units will change under a global scale transformation of the universe as follows

$$
\begin{aligned}
L_p &\rightarrow L'_p = cL_p \\
M_p &\rightarrow M'_p = M_p \qquad (5.2) \\
T_p &\rightarrow T'_p = cT_p
\end{aligned}
$$

These transformations follow immediately from the expressions (5.1) by taking the mentioned degree of homogeneity of the functions G, ϵ_0, $\hbar$ into account. It shows the expected behavior for Planck's length unit. Additionally, it shows an increase in Planck's unit of time, mimicking a time dilation for internal observers (which keep measuring with the new Planck's units).

Given the above information, we want to discuss now how the velocity of different objects transform under a similarity transformation of the universe. In other words, given the initial state of a classical N-body system(universe) by the

[4]God's immaterial rods and clocks if you wish. They can also be called the absolute SI units, coinciding with the real (experimental) SI units, just for a specific scale of the universe. For instance, the absolute SI units and real SI units coincide for the universe now (at this moment of time), and before performing any scale transformation of the universe. The performance of such a transformation changes the internal units, but not the absolute ones. The existence of these absolute units in Newtonian Mechanics can be inferred from the existence of absolute space and time.

$3N$ position variables $(\boldsymbol{x}_1, ..., \boldsymbol{x}_N)$ and the $3N$ velocity variables $(\mathbf{v}_1, ..., \mathbf{v}_N)$, what will the new velocities $(\mathbf{v}'_1, ..., \mathbf{v}'_N)$ become, after we have changed (jumped) the system's configuration to $(\boldsymbol{x}'_1, ..., \boldsymbol{x}'_N) = (g\boldsymbol{x}_1, ..., g\boldsymbol{x}_N)$ for any $g \in Sim(3)$?
If the performed transformation belongs to the euclidean subgroup, i.e. $g \in E(3) \subset Sim(3)$, the transformation of the velocities is already well-known[5]. So we need just to discuss the transformation of the velocities under the group of scale transformations $Sc \subset Sim(3)$.
As the measurement of a velocity is basically an experimental task, the transformation law of the velocities under scale transformation of the system (or, in fact, any other transformation of the system) must also include some experimental reasoning. We will explain here that based on the Principle of Relationalism, the behavior of rods and clocks of the modified Newtonian theory under a universe's scale transformation is such that the measured velocities of objects (or parts of the system) remain invariant under such transformations. This invariance is a natural consequence of the simultaneous expansion of the measuring rod and the dilation of the unit of time we came across previously. To be more precise, we have already argued above that a global scale transformation of the universe by a factor $c \in \mathbb{R}^+$

$$x = (\boldsymbol{x}_1, ..., \boldsymbol{x}_N) \to x' = (c\boldsymbol{x}_1, ..., c\boldsymbol{x}_N)$$

causes the following transformation of the gravitational coupling G, Planck's $\hbar$, and the vacuum permittivity ϵ_0

$$G \to cG$$

$$\hbar \to c\hbar$$

$$\epsilon_0 \to \frac{\epsilon_0}{c}$$

These transformations, in turn, changed the behavior of rods and clocks through a change of their (Planck) units (5.1) as given in (5.2). The measured speed $\mathbf{v}$ of an object, e.g., a particle, gets transformed under a global scale transformation

[5] Velocities are invariant under the action of the group of spatial translations $T(3)$. Under the action of the group of spatial rotations $SO(3)$, the velocities transform to $(\mathbf{v}'_1, ..., \mathbf{v}'_N) = (g\mathbf{v}_1, ..., g\mathbf{v}_N)$, where g is the rotation connecting the old configuration to the new one.

as follows

$$\mathbf{v} = \frac{\Delta \boldsymbol{x}}{\Delta t} \rightarrow \mathbf{v}' = \frac{\Delta \boldsymbol{x}'}{\Delta t'} = \frac{c\Delta \boldsymbol{x}}{c\Delta t} = \mathbf{v} \tag{5.3}$$

where $\Delta\mathbf{x}$ stands for instance for the distance between two other objects (which are needed to define the start and end point of any interval in space), and Δt stands for the time duration (measured in Planck unit) that the object needs to travel between those two reference objects. The primed versions are the same quantities after scale transforming the universe and measuring everything in the new Planck units. So, the measured velocities of the objects[6] become invariant under the scale transformations. In the relation, $\Delta t' = c\Delta t$, which is used in (5.3), the Principle of Relationalism is silently invoked in equating the number of ticks (or steps) of our new clock in the scaled universe for the duration of a physical phenomenon (in this example the passage of an object between the two reference objects), and the number of ticks of the old clock in the old (smaller) universe while the same phenomenon is taking place. So the measured speed of any object in universes before and after global scale transformations remains the same. The Principle of Relationalism and the characteristics of the Plank units together are responsible for this result. Analogously, the system's configuration velocity, which is the collection of all the velocities of its constituent particles, is also invariant under a global scale transformation, i.e.

$$\begin{gathered} v_x = (\mathbf{v}_1, ..., \mathbf{v}_N) \in T_x Q \\ \downarrow \\ v_{cx} = (\mathbf{v}_1, ..., \mathbf{v}_N) \in T_{cx} Q \end{gathered} \tag{5.4}$$

This means that the group of scale transformations $Sc \subset Sim(3)$ acts on the

[6] With the new measuring instruments

phase space of a modified Newtonian N-body system as follows

$$\begin{pmatrix} \boldsymbol{x}_1 \\ . \\ . \\ . \\ \boldsymbol{x}_N \\ \boldsymbol{p}_1 \\ . \\ . \\ . \\ \boldsymbol{p}_N \end{pmatrix} \xrightarrow{Sc} \begin{pmatrix} c\boldsymbol{x}_1 \\ . \\ . \\ . \\ c\boldsymbol{x}_N \\ \boldsymbol{p}_1 \\ . \\ . \\ . \\ \boldsymbol{p}_N \end{pmatrix} \tag{5.5}$$

where $\boldsymbol{p}_i := m_i\mathbf{v}_i$. Note that the absolute velocities of the Newtonian world-view are not related in any way to the observable relative motions we were dealing with so far (e.g., in 5.3). Instead, they are derived from the unobservable motion in absolute space, i.e., the motion between two "space points" instead of two reference objects. As the distance between space points is not affected by a scale transformation of the matter in the universe, absolute velocities in modified Newtonian theory change under scale transformation $Sc : \boldsymbol{x}_i \to c\boldsymbol{x}_i$ by a factor of $\frac{1}{c}$, contrary to the relative velocities. In other words, the group of scale transformations $Sc \subset Sim(3)$ acts on the absolute phase space of a modified Newtonian N-body systems as follows

$$\begin{pmatrix} \boldsymbol{x}_1 \\ . \\ . \\ . \\ \boldsymbol{x}_N \\ \boldsymbol{p}_1 \\ . \\ . \\ . \\ \boldsymbol{p}_N \end{pmatrix} \xrightarrow{Sc} \begin{pmatrix} c\boldsymbol{x}_1 \\ . \\ . \\ . \\ c\boldsymbol{x}_N \\ c^{-1}\boldsymbol{p}_1 \\ . \\ . \\ . \\ c^{-1}\boldsymbol{p}_N \end{pmatrix} \tag{5.6}$$

where $\boldsymbol{p}_i := m_i\mathbf{v}_i$. In two of the other approaches to relational physics (BKM and BDGZ)[7], the action of the group Sc on absolute phase space is taken to be different from (5.6), i.e., being (11.12) and (11.18) respectively. We are led to (5.5) or (5.6) directly by the Principle of Relationalism. However, one can alternatively arrive at (5.6) by mechanical similarity transforamtions in the modified Newtonian theory. In Chapter (11) this is discussed in more detail. Using (5.5), we have shown (in Section 7.2 or Section 4.1) that the mass metric $\mathbf{M}$ on Q uniquely induces a metric $\mathbf{N}$ on shape space S in a straight forward way. In particular, this has been done without the need for any conformal factor contrary to the BDGZ-approach.

It is worth mentioning that the properties of the internal measuring units according to the modified Newtonian theory (which also lead us to (5.5)) is not keen on using the Planck units. The true homogeneous function of the first degree on Q which must lead to the value $6.6743 \times 10^{-11} \frac{m^3}{kg.s^2}$ for the current state of the universe allows the existence of Kepler pairs now[8]. The unit of length and time defined as the semi-major axis, and the orbital period of a Kepler pair, change under a global scale transformation Sc exactly in the same way as their Planck counterparts in the modified Newtonian theory do. It is because of (5.6). The homogeneity of the potential function of zeroth degree in modified Newtonian theory guarantees that the Kepler pair's orbital period becomes longer by a factor $c \in \mathbb{R}^+$ (11.1) after performing a mechanical similarity transformation of the universe (by the factor c).

For clarity, we give here a more mathematical formulation of the principle of relationalism for particle-based mechanical theories and compare it with two other principles (postulates) in relational mechanics.
Consider an absolute physical theory, built upon the notions of absolute space $\mathbb{R}^3$ and absolute time $\mathbb{R}$ of Newton. Denote the absolute time by the letter $t^{(\boldsymbol{n})}$. The state space η of a system according to a theory can be specified by the systems configuration[9] x, and its time derivatives of sufficient order $\dot{x}, \ddot{x}$,... depending on

[7]Considered in Chapter (11) for the purpose of comparison with our work

[8]Whether or not these Kepler pairs form in a typical universe governed by the modified Newtonian theory is a separate question to be addressed in future.

[9]Or by the theory's "beables".

the theory. The theory's law of motion is usually given by a system of coupled differential equations for the $3N$ configuration variables[10] with the absolute time t being the independent variable. A time-dependent isomorphism can express the solutions of this system of equations (law of motion)

$$O^{v,\dot{v},\dots}_{t^{(\boldsymbol{n})}} : Q \to Q$$

In other words, the (absolute time) parameterized curves $O^{v,\dot{v},\dots}_{t^{(\boldsymbol{n})}}(x_0)$ with $-\infty < t^{(n)} < \infty$, all satisfy the system of differential equations constituting the (absolute) theory's law of motion. As the law of motion of the (modified or original) Newtonian mechanics is a second-order differential equation, just the explicit specification of the velocities suffices for the determination of the solution emerging from an arbitrary configuration x, and hence for the construction of the time-dependent isomorphism $O^{v}_{t^{(\boldsymbol{n})}}$.

The mathematical expression of the Principle of Relationalism is as follows:

$$\forall x_0 \in Q, \forall g \in Sim(3) : O^{\mathcal{A}_g v}_{t^{(\boldsymbol{n})}}(gx_0) = gO^{v}_{t^{(\boldsymbol{n})}}x_0 \tag{5.7}$$

where $\mathcal{A}_g v$ is the transformed velocity under the action of $g \in Sim(3)$. As it is well known that velocities in classical mechanics are invariant under group T^3 of spatial translations[11], and covariant under the group $SO(3)$ of spatial rotations[12], it suffices to specify only the action of the group Sc of scale transformations.

As the Newtonian world-view is quite central to the above formulation of the principle, some comments on its relation to the Leibnizian world-view are in order. One can define the infinitesimal increment of a relational time variable as a monotonically increasing positive function of the infinitesimal increment of the system's actual configuration $\boldsymbol{x}$, i.e.,[13]

$$\delta t := f(\mid \delta \boldsymbol{x}_1 \mid, \mid \delta \boldsymbol{x}_2 \mid, ..., \mid \delta \boldsymbol{x}_N \mid)$$

where $\mid . \mid$ denotes the Euclidean norm on the absolute space $\mathbb{R}^3$. For instance, one may take f as the function for the arc length on the configuration space Q.

[10] For particle-based mechanical theories.

[11] $\forall g \in T^3 : \mathcal{A}_g v_x = v_x$ where $\mathcal{A}_g v \in T_{gx}(Q)$ and $v_x \in T_x(Q)$

[12] $\forall g \in SO(3) : \mathcal{A}_g v_x = g v_x$ where $\mathcal{A}_g v \in T_{gx}(Q)$ and $v_x \in T_x(Q)$

[13] See also [7], and [23] to better understand the concept of relational time.

It is explained in [7] that the ephemeris time (11.13), which corresponds to a specific choice f_e for the function f, replaces perfectly the absolute time $t^{(\boldsymbol{n})}$ in the Newtonian mechanics. As total speeding up or slowing down of all universe's particles by the same factor would not lead to any observable effect[14], without loss of generality one can set $E = 0$ in the denominator of f_e. In this way, all possible universes can be considered as having vanishing total energy. Identify Newton's absolute time $t^{(\boldsymbol{n})}$ with the ephemeris time of a universe with vanishing total energy and pathing through a configuration x_0. Call this universe the *reference universe*. Its role is to provide a relational representation of Newton's absolute time. After performing any transformation on the universe, a relational time variable may run differently compared to the absolute time $t^{(\boldsymbol{n})}$, i.e., compared to the ephemeris time of the reference universe. An equivalent expression of the Principle of Relationalism is as follows:

$$\forall x_0 \in Q, \forall g \in Sim(3) : O_{t'}^{u'}(gx_0) = gO_t^u x_0 \tag{5.8}$$

where t and t' denote the two alternative universe's internal times, corresponding to the same shape, which can be moreover assumed to be initially synchronized

$$t \mid_{x_0} = t' \mid_{x_0' := gx_0} = 0$$

The specification of the initial velocity of the transformed universe

$$u' := \frac{dx'}{dt'} \mid_{x' = gx_0}$$

from the previous universe's initial velocity

$$u := \frac{dx}{dt} \mid_{x_0}$$

depends partly on the choice of the new internal time variable t' (so the choice of f for the new universe), and partly on the action of the group Sc on the absolute phase space.

The mentioned principle moreover entails the nonexistence of any experiment carried out by the subsystems of the two alternative universes (emerging from x_0 and gx_0 respectively), which could signify a difference between the two alternative universes, and hence be conclusive on the question: "In which one of the two

[14]For any internal observer, which by definition must be a subsystem of the universe.

alternative universes am I finding myself?". This point, among others, specifies an action of the group of scale transformations Sc on the phase space as explained before (see (5.3) or (5.4)).

A theory satisfying **Barbour's fundamental postulate of relational mechanics** [32]:

> "*An initial point and an undirected line through it in shape space should uniquely determine a solution.*"

or satisfying the **Barbour-Bertotti's postulate of relational mechanics**[33]:

> "*A point and a direction in shape space should uniquely determine a solution.*"

known also as the *Mach-Poincare's criterion*[34], certainly satisfies the above *principle of relationalism*, but the other way around is not always true.
The somehow less appropriately chosen word "*initial*" in the expression of Barbour's fundamental postulate refers, in fact, to a **special shape** on the curve achieved by projecting the solution of a Newtonian N-body problem to shape space S. This special point is defined as the minimum of the complexity (11.7) along the solution curve. Complexity is a $Sim(3)$-invariant function on Q; hence it is also a function on S. At this special shape, the dilational momentum D [15] vanishes, and the system's moment of inertia is at its absolute minimum along the considered N-body's solution curve on Q. It is just at this special shape that the specification of a direction (or undirected line) on shape space would uniquely determine a whole solution. These data would not be sufficient for a generic shape, and the additional specification of D at a generic shape is needed to determine a solution. Equivalently one can say[16] that the resulting dynamics on shape space do not deal with plain geometrical paths on S, but centered paths (Paths with the extra structure of a "central point"); and a dynamics which depends appropriately on the "distance" from the central point.

[15] Which is a dynamical variable in the Newtonian theory of gravitation but a constant of motion of the modified Newtonian theory.
[16] As was pointed out to us by Sheldon Goldstein in private correspondence.

In contrast, Barbour-Bertotti's postulate is a statement about every point in shape space and is, therefore, a stronger[17] criterion than Barbour's fundamental postulate.

For the purpose of comparison, consider an absolute theory that contains some interaction potentials depending on the particles' relative velocities or relative accelerations or ... . Such a theory (under some circumstances) can still perfectly satisfy the principle of relationalism. However, the reformulation of its law of motion on shape space would certainly neither satisfy the fundamental postulate of Barbour nor Barbour-Bertotti's postulate. Hence, the principle of relationalism is a broader statement and a less stringent requirement than the other two postulates mentioned above.

The BKM-approach satisfies Barbour's fundamental postulate but does not satisfy Barbour-Bertotti's postulate. This last criterion is, however, satisfied by the BDGZ-approach. In fact, in the DGZ-approach [16], it has been shown that even a similarity invariant interaction potential can be explained by a geodesic motion on shape space; for a suitable choice of conformal factor[18], and hence still satisfies the Barbour-Bertotti's postulate.

[17] More stringent.

[18] Namely multiplying the conformal factor used for the non-interacting theory on shape space with the scale-invariant potential.

Chapter 6

Geometry of Q_{cm} as $SO(3)$-fiber bundle

In this section, we review the geometry of Q_{cm} as a principal fibre bundle, with $G = SO(3)$ being it's structure group. We introduce coordinate systems on Q_{cm} which are adopted to the fiber bundle structure, introduce the connection form, and discuss the decomposition of the mass metric $\mathbf{M}$ on Q_{cm}. We do these as preparation for describing a classical dynamical system in Lagrangian formalism, and deriving the reduced equations of motion in coordinates on the respective base space in the next section. As reference, one can find more information regarding these topics in [5],[6], [7].

6.1 Fiber bundle structure and definition of connection form

The absolute configuration space of a N-particle system is the set

$$Q = \{x = (\mathbf{x}_1, ..., \mathbf{x}_N) \mid \mathbf{x}_i \in \mathbb{R}^3\} \cong \mathbb{R}^{3N}$$

The center of mass system

$$Q_{cm} = \{x = (\mathbf{x}_1, ..., \mathbf{x}_N) \mid \sum_{j=1}^{N} m_j \mathbf{x}_j = 0\} \cong \mathbb{R}^{3(N-1)}$$

with $\mathbf{x}_j \in \mathbb{R}^3$, is stratified by the action of the symmetry group G (e.g. $SO(3)$). A new coordinate system on Q adapted to the projection $Q \to Q_{cm}$ is given by the following linear transformation

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ . \\ . \\ . \\ \mathbf{x}_N \end{pmatrix} \to \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ . \\ . \\ . \\ \boldsymbol{r}_{N-1} \\ \mathbf{R}_{cm} \end{pmatrix}$$

where $\boldsymbol{R}_{cm} = \frac{1}{\sum_{i=1}^{n} m_i} \sum_{\alpha=1}^{n} m_\alpha \boldsymbol{r}_\alpha$ is the center of mass of the system, and the $N-1$ vectors $\boldsymbol{r}_i$ are the mass weighted Jacobi vectors defined as

$$\mathbf{r}_j := (\frac{1}{\mu_j} + \frac{1}{m_{j+1}})^{-\frac{1}{2}} (\mathbf{x}_{j+1} - \frac{1}{\mu_j} \sum_{i=1}^{j} m_i \mathbf{x}_i) \tag{6.1}$$

With $\mu_j := \sum_{i=1}^{j} m_i$. Then, the center of mass configuration space Q_{cm} can be expressed in these coordinates by

$$Q_{cm} \cong \{x = (\mathbf{r}_1, ..., \mathbf{r}_{N-1}) \mid \mathbf{r}_j \in \mathbb{R}^3, j = 1, ..., N-1\} \tag{6.2}$$

By the introduction of the coordinate transformation $(\mathbf{x}_1, ..., \mathbf{x}_N) \to (\mathbf{r}_1, ..., \mathbf{r}_{N-1}, \mathbf{R}_{cm})$ on the absolute configuration space Q, the center of mass kinetic energy and the center of mass angular momentum naturally separates off from the system's total kinetic energy and the total angular momentum. Specifically system's total kinetic energy

$$T = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \mid \dot{\boldsymbol{x}}_\alpha \mid^2 = \sum_{\alpha,\beta=1}^{N} K_{\alpha\beta} (\dot{\boldsymbol{x}}_{s\alpha} . \dot{\boldsymbol{x}}_{s\beta})$$

with $K_{\alpha\beta} = m_\alpha \delta_{\alpha\beta}^{(3)}$ being the $3N \times 3N$ kinetic tensor (B.1), transforms to

$$T = \frac{1}{2} \sum_{\alpha=1}^{N-1} \mid \dot{\boldsymbol{r}}_\alpha \mid^2 + \frac{\sum_{i=1}^{n} m_i}{2} \mid \dot{\mathbf{R}}_{cm} \mid^2 \tag{6.3}$$

One can view each element of Q_{cm} as a $3 \times (N-1)$ matrix. The stratification of Q_{cm} can then be suitably described by the rank of this matrix. Namely

$$Q_{cm} = Q_{cm0} \cup Q_{cm1} \cup Q_{cm2} \cup Q_{cm3} \tag{6.4a}$$

$$Q_{cmk} := \{x \in Q_{cm} \mid rank(x) = k\} \tag{6.4b}$$

For instance, Q_{cm0} denotes the simultaneous total collision of all the particles, Q_{cm1} denotes the linear configurations, Q_{cm2} the planar configurations (all the particles are located on a single plane in $\mathbb{R}^3$). Members of Q_{cm0} and Q_{cm1} are called singular configurations.

Action of the rotation group $SO(3)$ on Q_{cm}, is as follows

$$x \to gx = (g\mathbf{r}_1, ..., g\mathbf{r}_{N-1})$$

for all $g \in SO(3)$. It is well known that the above $SO(3)$ action can be used to define an equivalence relation on Q_{cm}, with help of which one defines a quotient space $Q_{int} := \frac{Q_{cm}}{SO(3)}$, as the space of the equivalence classes. We denote the corresponding projection map by $\pi : Q_{cm} \to \frac{Q_{cm}}{SO(3)}$.

The isotropy group at $x \in Q_{cm}$ is defined as the following subgroup of the structure group $SO(3)$

$$G_x := \{g \in SO(3) \mid gx = x\} \tag{6.5}$$

and the $SO(3)$ orbit through x is defined as follows

$$O_x := \{gx \mid g \in SO(3)\} \tag{6.6}$$

One can easily verify the following equalities

$$G_x = \begin{cases} e & : x \in Q_{cm2} \cup Q_{cm3} \\ SO(2) & : x \in Q_{cm1} \\ SO(3) & : x \in Q_{cm0} \end{cases}$$

and

$$Ox \cong \frac{SO(3)}{G_x} \cong \begin{cases} SO(3) & : x \in Q_{cm2} \cup Q_{cm3} \\ S^2 & : x \in Q_{cm1} \\ 0 & : x \in Q_{cm0} \end{cases}$$

In this sense, Q_{cm} is stratified into a union of strata, i.e. $Q_{cm} = Q_{cmns} \cup Q_{cm1} \cup Q_{cm0}$, Where ns stands for non-singular. So all $x \in \partial Q_{cmns} = Q_{cm} \backslash Q_{cmns}$ are

singular configurations. Q_{cmns} contains all the two and three dimensional configurations in the absolute space.
Since each stratum is $SO(3)$ invariant, we can consider Q as a stratified fiber bundle where each stratum have it's own projection

$$\begin{cases} Q_{cmns} \to \frac{Q_{cmns}}{SO(3)} \\ Q_{cm1} \to \frac{Q_{cm1}}{SO(3)} \\ Q_{cm0} \to \frac{Q_{cm0}}{SO(3)} \end{cases}$$

whose fibers are $O_x \cong \frac{SO(3)}{G_x}$. In particular Q_{cmns} is a $SO(3)$ principal fiber bundle, as the structure group has a free action on Q_{cmns}.

Associated with the group action of $SO(3)$ on Q_{cm}, a *moving frame*

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$$

of the absolute space $\mathbb{R}^3$, given by

$$\mathbf{e}'_1 = g\mathbf{e}_1, \mathbf{e}'_2 = g\mathbf{e}_2, \mathbf{e}'_3 = g\mathbf{e}_3$$

can be attached to the mechanical system, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ stands for the (fixed) *space frame*. The Euler angles $\phi^1, \phi^2, \phi^3 = \alpha, \beta, \gamma$ are used to specify the orientation of the moving frame w.r.t. the space frame. Hence any point on Q_{cm} can be specified by $3N-6$ internal coordinates q^i relative to the body frame, and the three Euler angles ϕ^a. The components of a vector in $\mathbb{R}^3$ with respect to the body frame will be denoted from now on by a subscript "b". It is clear that the relation between components of a vector in the space frame and the body frame is as follows

$$\boldsymbol{v}_b = g^{-1}\boldsymbol{v}_s$$

with g, as already mentioned, is the rotation which takes the space frame to the body frame. For notational simplicity, from now on we drop the subscript s, whenever we mean the expression of vectors w.r.t. the space frame.

The moment of inertia tensor expressed in Jacobi coordinates, is the following map $A_x : \mathbb{R}^3 \to \mathbb{R}^3$

$$A_x(\boldsymbol{v}) = \sum_{j=1}^{N-1} \boldsymbol{r}_j \times (\boldsymbol{v} \times \boldsymbol{r}_j) \tag{6.7}$$

for any $\boldsymbol{v} \in \mathbb{R}^3, x \in Q_{cmns}$. You can parallel transport the lab frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in absolute space to the center of mass of the system under study. Denote the vectors connecting the center of mass to the particle α by $\boldsymbol{x}_\alpha$. In these coordinates, the moment of inertia tensor takes the more familiar form

$$A_x(\boldsymbol{v}) = \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{x}_\alpha \times (\boldsymbol{v} \times \boldsymbol{x}_\alpha)$$

Note the operator A_x^{-1} just exists for $x \in Q_{cmns}$. The reason is that A_x sends all vectors to zero if any x belongs to $Q_{cm1} \cup Q_{cm0}$, hence can not be inverted.

For a velocity vector $\dot{x} \in T_x(Q_{cm})$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{3N} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ . \\ . \\ . \\ \dot{\boldsymbol{x}}_N \end{bmatrix}$$

denote $d\boldsymbol{x}_i$ as the 1-form defined as follows

$$d\boldsymbol{x}_i(\dot{x}) := \dot{\boldsymbol{x}}_i$$

Similarly, the 1-forms $d\boldsymbol{r}_i$'s are defined for any $1 \leq i \leq N-1$.

For any $x \in Q_{cmns}$, a connection form

$$\omega_x : T_x(M_{ns}) \to \mathbf{so(3)}$$

is known to be defined with the help of A, as follows

$$\omega_x := R(A_x^{-1}(\sum_{j=1}^{N-1} \boldsymbol{r}_j \times d\boldsymbol{r}_j)) \tag{6.8}$$

Here $R : \mathbb{R}^3 \to \mathbf{so(3)}$ is defined as

$$R(a) = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix} \tag{6.9}$$

for any $a \in \mathbb{R}^3$, as explained in more extend in appendix D, and the expression of one form $d\boldsymbol{r}_j$ in orientational and internal coordinates is given by

$$d\boldsymbol{r}_j = \sum_{i=1}^{3N-6} \frac{\partial \boldsymbol{r}_j}{\partial q^i} dq^i + \sum_{a=1}^{3} \frac{\partial \boldsymbol{r}_j}{\partial \phi^a} d\phi^a$$

6.2 Local expression of the Connection form and the metric in orientational and internal coordinates

Now we want to review (based on [5], and [6]) the expression of the connection form 6.8, and the kinetic metric in terms of some new coordinates, which shall be introduced on the center of mass configuration space Q_{cmns}.
Consider an open subset $U \subset \frac{Q_{cmns}}{SO(3)}$, and define a local section

$$\sigma : U \to Q_{cmns}$$

Then, any point $x \in \pi^{-1}(U)$ can be expressed as

$$x = g\sigma(q) = \big(g\boldsymbol{\sigma}_1(q), ..., g\boldsymbol{\sigma}_{N-1}(q)\big) \tag{6.10}$$

with $g \in SO(3)$, and $q \in U \subset \frac{Q_{cmns}}{SO(3)}$ a point on the internal configuration space Q_{int}.
Note that

$$\sigma(q) = \begin{pmatrix} \boldsymbol{\sigma}_1(q) \\ ... \\ \boldsymbol{\sigma}_{N-1}(q) \end{pmatrix} = \begin{pmatrix} \sum_{a=1}^{3} C_1^a \mathbf{e}_a \\ ... \\ \sum_{a=1}^{3} C_{N-1}^a \mathbf{e}_a \end{pmatrix}$$

determines a way to put the multi-particle system with internal configuration (or shape in the next section) $q \in U$, in the absolute space $\mathbb{R}^3$. This is achieved by choosing the point $x \in Q_{cm}$ on the fiber above q, to which the internal configuration q is meant to be lifted as follows

$$\boldsymbol{r}_{3(i-1)+a} := C_i^a$$

So specifying a section σ, comes down to the specification of a set of $3(N-1)$ real valued functions on Q_{int}, i.e.

$$C_i^a : Q_{int} \to \mathbb{R}$$

For another subset $V \subset Q_{int}$ with $V \cap U \neq \emptyset$, one can define another local section

$$\tau : V \to Q_{cm}$$

such that $\sigma(q) = h\tau(q)$ for $(q,h) \in V \times SO(3)$. The two local sections σ and τ on $V \cap U$ are then related by $\tau = k\sigma$, where k is a $SO(3)$-valued function on $U \cap V$.

The vertical and horizontal[1] vectors (and one forms) can be expressed with respect to a local trivialization of the center of mass configuration space[2]

$$Q_{cm} \cong \mathbb{R}^{3N-3} \cong U \times SO(3)$$

To this end, rewrite the connection form

$$\begin{aligned} \omega &= R\big(A_x^{-1} \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{x}_\alpha \times d\boldsymbol{x}_\alpha\big) \\ &= R(A_x^{-1}(\sum_{j=1}^{N-1} \boldsymbol{r}_j \times d\boldsymbol{r}_j)) \end{aligned} \tag{6.11}$$

and the total angular momentum operator

$$\mathbf{J} = \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{x}_\alpha \times d\boldsymbol{x}_\alpha = \sum_{\alpha=1}^{N-1} \boldsymbol{r}_\alpha \times d\boldsymbol{r}_\alpha$$

with respect to the laboratory frame $\mathbf{e}_a$ as follows

$$\omega = \sum_{a=1}^{3} R(\mathbf{e}_a)\omega^a = \sum_{a=1}^{3} R(\mathbf{e}'_a)\omega'^a$$

where $\omega^a := \omega.R(\mathbf{e}_a)$, $\omega'^a := \omega'.R(\mathbf{e}'_a)$, and the total angular momentum expressed both in the fixed laboratory frame and the moving frame as

$$\mathbf{J} = \sum_{a=1}^{3} \mathbf{e}_a J_a = \sum_{a=1}^{3} \mathbf{e}'_a L_a$$

[1]in this case they are known as rotational and vibrational vectors in the literature.
[2]technically speaking we mean the non-singular stratum of configuration space.

where $J_a = (\mathbf{e}_a \mid \mathbf{J})$, and $L_a = (\mathbf{e}'_a \mid \mathbf{J})$.
Remember that

$$J_a \mathbf{r}_j = \mathbf{e}_a \times \mathbf{r}_j$$

$$L_a \mathbf{r}_j = \mathbf{e}'_a \times \mathbf{r}_j = g\big(\mathbf{e}_a \times \boldsymbol{\sigma}_j(q)\big)$$

The forms $\{\omega^a, dq^i\}$ with $a = 1, 2, 3$ and $i = 1, ..., 3N-6$ constitute[3] a local basis of the space of 1-forms on $T^*(Q_{cm})$. Moreover, the vector fields L_a and

$$\big(\frac{\partial}{\partial q^i}\big)^* := \frac{\partial}{\partial q^i} - \sum_a \wedge_i^a L_a \tag{6.12}$$

form a local basis of the space of vector fields $T_x(Q_{cm})$. Here $\big(\frac{\partial}{\partial q^i}\big)^*$ is called the *horizontal lift* of $\frac{\partial}{\partial q^i}$ from internal configuration space Q_{int} to Q_{cm}. It is defined by the conditions

$$\omega\big((\frac{\partial}{\partial q^i})^*\big) = 0$$

$$\pi_*(\frac{\partial}{\partial q^i})^* = \frac{\partial}{\partial q^i}$$

which explain its name. These conditions lead to the following expression for the functions $\wedge_i^a$ in (6.12)

$$\wedge_i^a = \big\langle A_x^{-1}(\sum_{\alpha=1}^{N-1} \boldsymbol{r}_\alpha \times \frac{\partial \boldsymbol{r}_\alpha}{\partial q^i}) \mid \mathbf{e}_a \big\rangle \tag{6.13}$$

Note also that $\frac{\partial \sigma}{\partial q^i}$ is a vector field (so a differential operator) on Q_{cm}. So, we have the following identities

$$\omega^a(J_b) = \delta_b^a, \omega^a\big((\frac{\partial}{\partial q^i})^*\big) = 0$$

$$dq^i(J_b) = 0, dq^i\big((\frac{\partial}{\partial q^i})^*\big) = \delta_j^i$$

$$\omega'^a(L_b) = \delta_b^a, \omega'^a\big((\frac{\partial}{\partial q^i})^*\big) = 0$$

$$dq^i(L_b) = 0, dq^i\big((\frac{\partial}{\partial q^i})^*\big) = \delta_j^i$$

[3]mathematically speaking one should write $\pi^* dq^i$ instead of dq^i there.

By writing out the connection form in the local coordinates (q, g), one can calculate the local expression of ω^a

$$\omega^a = \psi^a + \sum_{i=1}^{3N-6} \wedge_i^a dq^i \tag{6.14}$$

$$\omega'^a = \theta^a + \sum_{i=1}^{3N-6} \wedge_i'^a dq^i \tag{6.15}$$

where ψ^a and θ^a are respectively the three right and left invariant one-forms on $SO(3)$ defined through

$$dgg^{-1} =: \sum_{a=1}^{3} \psi^a R(\mathbf{e}_a) \tag{6.16}$$

$$g^{-1}dg =: \sum_{a=1}^{3} \theta^a R(\mathbf{e}_a) \tag{6.17}$$

The ψ^a and J_a can be expressed in terms of Euler angles, and are dual to each other i.e.

$$\psi^a(J_b) = \delta_b^a$$

$$\theta^a(L_b) = \delta_b^a$$

It can be shown that the rotational vector fields J_a, and the vibrational vector fields $(\frac{\partial}{\partial q_i})^*$ satisfy the following commutation relations [6]

$$[J_a, J_b] = -\sum_{c=1}^{3} \epsilon_{abc} J_c \tag{6.18a}$$

$$[(\frac{\partial}{\partial q_i})^*, (\frac{\partial}{\partial q_j})^*] = -\sum_{c=1}^{3} F_{ij}^c J_c \tag{6.18b}$$

$$[(\frac{\partial}{\partial q_i})^*, J_a] = 0 \tag{6.18c}$$

The second equation implies that two independent vibrational vectors are coupled in a way to lead to an infinitesimal rotation, and exactly because of this, vibrations and rotations are inseparable. Another fact is that the distribution spanned by the vectors $(\frac{\partial}{\partial q_i})^*$ is not completely integrable. If it was so, there existed a submanifold to which $(\frac{\partial}{\partial q_i})^*$ are tangential, and this surface could be identified with the internal space Q_{int}. The constraint of constancy of total angular momentum is realized by selecting the distribution spanned by $(\frac{\partial}{\partial q_i})^*$. This constraint is a non-holonomic one for the case of non-vanishing angular momentum, and a holonomic one in the case of vanishing angular momentum.

According to the orthogonal decomposition $T_x(M_{ns}) = V_x \oplus H_x$, it is known that the kinetic metric can be expressed in terms of the one-forms dq^α, and ω^a as follows

$$ds^2 = \sum_{\alpha,\beta=1}^{3N-6} a_{\alpha\beta} dq^\alpha dq^\beta + \sum_{a,b} A_{ab}\omega^a\omega^b \tag{6.19}$$

where

$$\begin{cases} a_{\alpha\beta} := ds^2((\frac{\partial}{\partial q^\alpha})^*, (\frac{\partial}{\partial q^\beta})^*) \\ A_{ab} := ds^2(J_a, J_b) = \mathbf{e}_a.A_{\sigma(q)}(\mathbf{e}_b) \end{cases}$$

Remember that $\omega^a(J_b) = \psi^a(J_b) = \delta^a_b$, and notice that $(a_{\alpha\beta})$ defines a Riemannian metric on the internal space $\frac{Q_{cmns}}{SO(3)}$, and A is the inertia tensor. One way to see why the coefficients A_{ab} appearing in (6.19) coincide with the coefficients of inertia tensor (6.7) is through the following calculation

$$ds^2(J_a, J_b) = \sum_{\alpha=1}^{N-1}(d\boldsymbol{r}_\alpha J_a).(d\boldsymbol{r}_\alpha J_b)$$

$$= \sum_{\alpha=1}^{N-1}(\mathbf{e}_a \times \boldsymbol{r}_\alpha).(\mathbf{e}_b \times \boldsymbol{r}_\alpha) = \mathbf{e}_a. \sum_{\alpha=1}^{N-1} \boldsymbol{r}_\alpha \times (\mathbf{e}_b \times \boldsymbol{r}_\alpha)$$

$$= A_{ab}$$

To clarify the notation, we add here that $d\boldsymbol{r}_\alpha$ is a 1-form which tells us how much (infinitesimally) the position of the α's particle, i.e. $\boldsymbol{r}_\alpha$, would change, when we let it act on an infinitesimal displacement (a tangent vector). In this particular case, we have chosen the infinitesimal generator of rotation about the a's axis, i.e. J_a. So, from this it should be clear how the substitution $d\boldsymbol{r}_\alpha J_a = \mathbf{e}_a \times \boldsymbol{r}_\alpha$ used above, is justified.

It is worth mentioning that we have so far seen two different kinds of vector fields defined on $SO(3)$. Namely L_a and J_a, for $a = 1, 2, 3$. The first set of vector fields L_a, are duals to the one-forms θ^a defined by (6.15). L_a coincides with the direction of movement along the fiber (which is itself part of $T(Q_{cm})$), if we rotate the system in $\mathbb{R}^3$ around the a'th *axis of body frame* ($\mathbf{e}'_a = g\mathbf{e}_a$), without changing it's shape. The second set of vector fields J_a, are however dual to the one-forms ψ^a defined by (6.14). They coincides with the direction of movement along the fiber, if we rotate the system in $\mathbb{R}^3$ around the a'th *axis of space frame* $\mathbf{e}_a$, without changing it's shape. Because changing the shape causes extra total rotation, and make the direction of the movement of the configuration point deviate from L_a or J_a, depending on how fast and in which way system's shape is changing, one needs the second terms in (6.14) and (6.15).

A Local expression of the connection form can be given in terms of the left invariant one forms on $SO(3)$, and local section σ of the fiber bundle. Let's take the Euler angles (α, β, γ) as coordinates on $SO(3)$, and q^α with $\alpha = 1, ..., 3N-6$, as local coordinates on $U \subset Q_{int}$. So, g and q can be expressed in terms of them respectively. A point x on the center of mass system, can hence be expressed by the coordinates $x = x(q^1, ..., q^{3N-6}, \phi, \theta, \psi)$. The connection form at the point $g\sigma(q)$, can then be written as follows

$$\omega_{g\sigma(q)} = dgg^{-1} + g\omega_{\sigma(q)}g^{-1} = g(g^{-1}dg + \omega_{\sigma(q)})g^{-1} \tag{6.20}$$

where

$$\omega_{\sigma(q)} := R\Big(A^{-1}_{\sigma(q)} \big(\sum_{j=1}^{N-1} \boldsymbol{\sigma}_j(q) \times d\boldsymbol{\sigma}_j(q) \big) \Big) \tag{6.21}$$

Now take the fixed laboratory basis $\mathbf{e}_a$ of $\mathbb{R}^3$, with $a = 1, 2, 3$. We introduced three left invariant one-forms θ^a on $SO(3)$, and $3 \times (3N-6)$ functions $\wedge^a_\alpha$ by the following equations

$$g^{-1}dg := \sum_{a=1}^{3} \theta^a R(e_a) \tag{6.22a}$$

$$\omega_{\sigma(q)=x} =: \sum_{a=1}^{3} \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha R(\mathbf{e}_a) \tag{6.22b}$$

$$= \sum_{\alpha=1}^{3N-6} \wedge_\alpha dq_\alpha = \sum_{\alpha=1}^{3N-6} R(\lambda_\alpha) dq_\alpha \tag{6.22c}$$

in which the following is used

$$R(\lambda_\alpha) = \wedge_\alpha(q) = \sum_{a=1}^{3} \wedge^a_\alpha(q) R(\mathbf{e}_a)$$

to express ω in a compacter form. The coefficients $\wedge^a_\alpha$ in the last equation were given by (6.13), which are also known as gauge potentials.

Now we rewrite the connection form (6.20) in terms of θ^a and $\wedge^a_\alpha$

$$\begin{aligned}\omega_{g\sigma(q)} &= g(g^{-1}dg + \omega_{\sigma(q)})g^{-1} \\ &= g(\sum_{a=1}^{3} \theta^a R(\mathbf{e}_a) + \sum_{a=1}^{3} \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha R(\mathbf{e}_a))g^{-1} \\ &= g(\sum_{a=1}^{3}(\theta^a + \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha) R(\mathbf{e}_a) g^{-1} \\ &= \sum_{a=1}^{3}(\theta^a + \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha) g R(\mathbf{e}_a) g^{-1} \\ &= \sum_{a=1}^{3}(\theta^a + \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha) R(g\mathbf{e}_a)\end{aligned}$$

in the third line of the above calculation, we could move g from the behind of the term $\sum_{a=1}^{3}(\theta^a + \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha)$ to the front of it, cause the forms dq^α by definition commute with g, and the forms θ^a are left invariant. Thus, the connection form (6.8) expressed in local coordinates $(q^1, ..., q^{3N-6}, \alpha, \beta, \gamma)$, takes the following form

$$\omega_{g\sigma(q)} = \sum_{a=1}^{3} \omega'^a R(\mathbf{e}'_a) = \sum_{a=1}^{3} \omega'^a R(g\mathbf{e}_a) \tag{6.23a}$$

$$\omega'^a := \theta^a + \sum_{\alpha=1}^{3N-6} \wedge^a_\alpha(x) dq^\alpha \tag{6.23b}$$

As mentioned before, from the fixed space frame $\mathbf{e}_a$, a moving frame $\mathbf{e}'_a$ can be defined as $\mathbf{e}'_a = g\mathbf{e}_a$. One can think of ω'^a as components of ω in the moving frame, cause $\omega.R(\mathbf{e}'_a) = \omega'^a$.

The **horizontal lift**[4] $(\frac{\partial}{\partial q^{\alpha_0}})^*$, of a local vector field $\frac{\partial}{\partial q^{\alpha_0}}$ on U to a point $x \in Q_{cm}$

[4] $\omega_{g\sigma(q)}\big((\frac{\partial}{\partial q^\alpha})^*\big) = 0$ and $\pi_*\big((\frac{\partial}{\partial q^\alpha})^*\big) = \frac{\partial}{\partial q^\alpha}$ are the defining criteria of horizontal lift of a local vector field $\frac{\partial}{\partial q^\alpha}$ on $U \subset Q_{int}$.

can be shown to be given by (see [16])

$$(\frac{\partial}{\partial q^{\alpha_0}})^* = \frac{\partial}{\partial q^{\alpha_0}} - \sum_{a=1}^{3} \wedge_{\alpha_0}^{a}(x) L_a \tag{6.24}$$

with

$$\wedge_{\alpha_0}^{a} =:= \langle A_{\sigma(s_0)}^{-1} \big(\sum_{i=1}^{N-1} \boldsymbol{r}_i \times \frac{\partial \boldsymbol{r}_i}{\partial q^{\alpha}} \big) \mid e'_a \rangle$$

In the above expression as usual L_a denote the left invariant vector fields on $SO(3)$, which are dual to θ^a, i.e. $\theta^a(L_b) = \delta_b^a$. So

$$dq^{\alpha}, \omega'^{a}$$

and

$$(\frac{\partial}{\partial q^{\alpha}})^*, L_a$$

form local bases of the 1-forms, and of the vector fields on $\pi^{-1}(U) \cong U \times SO(3)$, respectively. They are in accordance with the decomposition $T_x(Q_{cm}) = V_x \oplus H_x$. As mentioned before, L_a can be identified with the infinitesimal rotation with respect to axis $\mathbf{e}'_a$ of the body frame. Technically speaking, we have to use $\pi^* dq^{\alpha}$, the pull back of dq^{α} under the bundle's projection map, but for the sake of notational simplicity we still used dq^{α}.
Expression (6.24) can be derived by requiring $\omega_x((\frac{\partial}{\partial q^{\alpha_0}})^*) = 0$, which is of course how a horizontal lift w.r.t. a connection should be.

Chapter 7

Ingredients for the Lagrangian reduction with respect to the similarity group

Which equations of motion does the evolution of shape of a classical system satisfy, is the central question to be answered in this paper. To this end, we explain in this section how the necessary ingredients, i.e. the metric $\mathbf{N}$ on shape space, and the connection form ω for the $Sim(3)$ fiber bundle Q_{cm} can be derived.
The mass metric $\mathbf{M}$ of the absolute configuration space $Q \cong \mathbb{R}^{3N}$, induces metrics on the reduced configuration spaces, like the internal configuration space $Q_{int} = \frac{Q}{E(3)}$, or shape space $S = \frac{Q}{Sim(3)}$ in a natural fashion. We first review the derivation of the metric on Q_{int}, following [7]. Then, we explain how the principle of relationalism can be used to derive a metrical structure $\mathbf{N}$ on $T(Q)/\mathcal{A}_{Sim(3)}$ from the mass metric $\mathbf{M}$ on the absolute configuration space in a unique way. This principle at the same time enforces the interaction potentials being scale invariant, which is the key to the full decoupling of dynamics on shape space from the gauge degrees of freedom, i.e., $Sim(3)$ degrees of freedom.

7.1 Metric on the internal space

Let us review how the metric B on the internal space $Q_{int} = \frac{Q_{cm}}{SO(3)}$ can be derived from the $SO(3)$-invariant mass metric $\mathbf{M}$ on the the center of mass configuration space Q_{cm}, i.e.,

$$\mathbf{M}_x(u,v) = \sum m_k < \boldsymbol{u}_k \mid \boldsymbol{v}_k > \tag{7.1a}$$

$$\mathbf{M}_x(u,v) = \mathbf{M}_{gx}(gu,gv) \tag{7.1b}$$

where $u = (\boldsymbol{u}_1, ..., \boldsymbol{u}_N)$ and $v = (\boldsymbol{v}_1, ..., \boldsymbol{v}_N)$ are members of $T_x(Q_{cm})$, so being any two tangent vectors of Q_{cm} at the point $x \in Q_{cm}$.
Given two internal vectors

$$v', u' \in T_q(Q_{int})$$

there are unique vectors $u, v \in T_x(Q_{cm})$ [1] so that

$$\begin{cases} \pi(x) = q \\ \pi_*(u) = u' \\ \pi_*(v) = v' \end{cases}$$

where $\pi_* : T_q(Q_{cm}) \to T_{\pi(q)}(Q_{int})$ denotes the differential of the projection map $\pi : Q_{cm} \to Q_{int}$. Now, the metric B on Q_{int} can be defined by the following equation:

$$B_q(v', u') := \mathbf{M}_x(v, u) \tag{7.2}$$

As the metric $\mathbf{M}$ was $SO(3)$-invariant, to which $x \in \pi^{-1}(q)$ the internal vectors v, u had been lifted, would not make any difference for the value assigned by B_q, as it should be, to ensure the derived metric being well-defined.

The kinetic energy of a N-particle system in the center of mass frame, coordinatized by the $N-1$ Jacobi vectors $\boldsymbol{r}_\alpha$ is as follows

$$K = \frac{1}{2} \sum_{\alpha=1}^{N-1} \mid \dot{\boldsymbol{r}}_\alpha \mid^2$$

[1] namely their horizontal lifts (6.24)

Now consider the Jacobi vectors $\boldsymbol{r}_{b\alpha}$ in body frame, and denote the system's angular velocity with respect to the body frame by $\boldsymbol{\omega}$. Using

$$\dot{\boldsymbol{r}}_{b\alpha} = \boldsymbol{\omega} \times \boldsymbol{r}_{b\alpha} + \frac{\partial \boldsymbol{r}_{b\alpha}}{\partial q^{\mu}} \dot{q}^{\mu}$$

and the expression for the so called gauge potentials[2]

$$\boldsymbol{A}_{\mu}(q) := A^{-1} \boldsymbol{a}_{\mu}$$

with

$$\boldsymbol{a}_{\mu} = \boldsymbol{a}_{\mu}(q) := \sum_{\alpha=1}^{N-1} \boldsymbol{r}_{b\alpha} \times \frac{\partial \boldsymbol{r}_{b\alpha}}{\partial q^{\mu}} \tag{7.3}$$

and A being the moment of inertia tensor

$$A_{ij} = A_{ij}(q) := \sum_{\alpha=1}^{N-1} (\mid \boldsymbol{r}_{b\alpha} \mid^2 \delta_{ij} - r_{b\alpha i} r_{b\alpha j})$$

one can write down the kinetic energy as follows (see [7])

$$K = \frac{1}{2} < \boldsymbol{\omega} \mid A \mid \boldsymbol{\omega} > + < \boldsymbol{\omega} \mid A \mid \boldsymbol{A}_{\mu} > \dot{q}^{\mu} + \frac{1}{2} h_{\mu\nu} \dot{q}^{\mu} \dot{q}^{v} \tag{7.4}$$

with

$$h_{\mu\nu} = h_{\mu\nu}(q) = \sum_{\alpha=1}^{N-1} \frac{\partial \boldsymbol{r}_{\alpha}}{\partial q^{\mu}} . \frac{\partial \boldsymbol{r}_{\alpha}}{\partial q^{\nu}} \tag{7.5}$$

The velocity of a systems configuration in Jacobi coordinates is given by a vector

$$\mid v >= [\dot{\boldsymbol{r}}_1, ..., \dot{\boldsymbol{r}}_{N-1}]$$

and in orientational and internal coordinates by the vector

$$\mid v >= [\dot{\theta}^i, \dot{q}^{\mu}]$$

with $1 \leq i \leq 3$, and $1 \leq \theta \leq 3N - 6$. The θ^i's are the Euler angles, which turn the space frame to the body frame of a configuration. If one decides to use the components of body angular velocity $\boldsymbol{\omega}$ instead of the time derivatives of Euler angles for denoting vectors in $T(SO(3))$ the configuration's velocity can alternatively be expressed as

$$\mid v >= [\boldsymbol{\omega}, \dot{q}^{\mu}]$$

[2]In this context, the word gauge refers to the freedom which exists in the choice of a body frame for a given system.

in angular velocity and internal basis. This last combination form an anholonomic frame or vielbein on $T(Q_{cm})$.
Remember the relation between the body components of angular velocity and derivatives of Euler angles

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -sin\beta cos\gamma & sin\gamma & 0 \\ sin\beta sin\gamma & cos\gamma & 0 \\ cos\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

So, the expression for the mass metric $\mathbf{M}$ in angular and internal coordinates $\{\omega^i, \dot{q}^\mu\}$ becomes as follows

$$< v \mid v >= \begin{bmatrix} \boldsymbol{\omega}^T & \dot{q}^\mu \end{bmatrix} \begin{bmatrix} A & A\boldsymbol{A}_\nu \\ \boldsymbol{A}_\mu^T A & h_{\mu\nu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \dot{q}^\nu \end{bmatrix} = \mathbf{M}_{ab} v^a v^b$$

With $1 \leq a, b \leq 3N - 3$. In other words, the metric on Q_{cm} in angular and internal basis vectors $[\boldsymbol{\omega}, \dot{q}^\mu]$ is given as follows

$$\mathbf{M}_{ab} = \begin{bmatrix} A & A\boldsymbol{A}_\nu \\ \boldsymbol{A}_\mu^T A & h_{\mu\nu} \end{bmatrix} \tag{7.6}$$

$h_{\mu\nu}$ can be considered as the restriction of the mass metric metric $\mathbf{M}$ of Q_{cm}, on the section determined by the choice made for the body frame for each internal configuration $q = (q_1, ..., q_{3N-6})$.

Decomposition of an arbitrary configuration velocity, in horizontal and vertical parts, becomes as follows

$$\mid v >=\mid v_v > + \mid v_h >$$

$$[\boldsymbol{\omega}, \dot{q}^\mu] = [\boldsymbol{\omega} + \boldsymbol{A}_v \dot{q}^v, 0] + [-\boldsymbol{A}_\nu \dot{q}^\nu, \dot{q}^\mu]$$

Correspondingly, the kinetic energy of the system can be also thought of as the addition of two separate vertical and horizontal kinetic energies, i.e.

$$K = K_v + K_h = \frac{1}{2}(\boldsymbol{\omega} + \boldsymbol{A}_\mu \dot{q}^\mu) A (\boldsymbol{\omega} + \boldsymbol{A}_\nu \dot{q}^\nu) + \frac{1}{2} B_{\mu\nu} \dot{q}^\mu \dot{q}^v$$

where $B_{\mu\nu}$ is a new metric on internal space, which is in contrast to $h_{\mu\nu}$, invariant under changing the choice of the body frame [7]

$$B_{\mu\nu} = h_{\mu\nu} - \boldsymbol{A}_\mu A \boldsymbol{A}_\nu$$

So, in summary to a vector

$$| v' >= \dot{q}^\mu$$

on the internal space Q_{int}, we associate a vector $| v_h >$ on Q_{cm}, which is called it's *horizontal lift*, connecting the two $SO(3)$-fibers. In angular velocity and shape basis, horizontal lift of v' takes the from

$$| v_h >= [-\boldsymbol{A}_\mu \dot{q}^\mu, \dot{q}^\mu]$$

Then the metric $B_{\mu\nu}$ on the internal space Q_{int}, can be found by the following defining equation

$$< v_1' | v_2' >= B_{\mu\nu}\dot{q}_1^\mu \dot{q}_2^\nu :=< v_{1h} | v_{2h} >= \mathbf{M}_{ab} v_{1h}^a v_{2h}^b$$

and this leads directly to

$$B_{\mu\nu} = h_{\mu\nu} - \boldsymbol{A}_\mu A \boldsymbol{A}_\nu \tag{7.7}$$

For more detailed explanations we suggest the reader to look at [7], where a clear and complete presentation of this topic can be found. Later we will see in an explicit example of three particle system how the metric B looks like.

7.2 Metric on the Sim(3)-reduced tangent bundle

It is generally believed as the mass metric $\mathbf{M}$ is not scale invariant, it would not directly induce a metric on the $Sim(3)$-reduced tangent bundle $T(Q)/Sim(3)$, in contrary to the $E(3)$-reduced tangent bundle $T(Q)/E(3) \equiv T(Q_{int})$. However, as we will explain below, once one takes the behaviour of real measuring units[3] built out of the matter under spatial scale transformation of the universe into account, one sees how the metric $\mathbf{N}$ on $T(Q)/Sim(3)$ is derived in a unique way.

As the measurement of the velocity is essentially an experimental task, the transformation law of the velocities under scale transformation of the system (or any

[3]Contrary to the absolute measuring units which are unaffected by any transformation performed on the universe.

other transformation of the system), must also include experimental reasoning. We will explain here, that based on the Principle of Relationalism, the behavior of rods and clocks under system's scale transformations is such that the measured velocities of objects (or parts of the system), remain invariant. This is a natural consequence of simultaneous expansion of measuring rod and dilation of unit of time we came across in Chapter (1). To be more precise, we have argued in Chapter (1) that a global scale transformation of the universe by a factor $c \in \mathbb{R}^+$

$$x = (\mathbf{x}_1, ..., \mathbf{x}_N) \to x' = (c\mathbf{x}_1, ..., c\mathbf{x}_N)$$

causes the following transformation of gravitational constant G, Planck's constant $\hbar$, and vacuum permittivity ϵ_0

$$G \to cG$$
$$\hbar \to c\hbar$$
$$\epsilon_0 \to \frac{\epsilon_0}{c}$$

and this in turn changes the behavior of rods and clocks, through change of their (Planck) units[4]

$$L_p \to cL_p$$
$$M_p \to M_p$$
$$T_p \to cT_p$$

The measured speed $\mathbf{v}$ of an object, e.g. a particle, gets then transformed under a global scale transformation as follows

$$\mathbf{v} = \frac{\Delta \mathbf{x}}{\Delta t} \to \mathbf{v}' = \frac{\Delta \mathbf{x}'}{\Delta t'} = \frac{c\Delta \mathbf{x}}{c\Delta t} = \mathbf{v}$$

where $\Delta\mathbf{x}$ stands for instance for the distance between two other objects (which are needed to define the start and end point of any interval in space), and Δt for the time (measured in Plank unit) the object takes to travel between those two reference objects. The primed versions are the same quantities, but after scale transforming the universe and measuring everything in the new Plank units. So,

[4]Remember the following measuring units derived from constants of nature by Max Planck: $L_P = \sqrt{\frac{G\hbar}{\boldsymbol{c}^3}}$, $M_P = \sqrt{\frac{\hbar\boldsymbol{c}}{G}}$, $T_p = \sqrt{\frac{G\hbar}{\boldsymbol{c}^5}}$.

the measured velocities of objects [5] become invariant under scale transformations. Analogously, the system's configuration velocity, which is the collection of velocities of it's constituent particles, is also invariant under a global scale transformation, i.e.

$$v_x = (\mathbf{v}_1, ..., \mathbf{v}_N) \in T_x Q$$
$$\downarrow$$
$$v_{cx} = (\mathbf{v}_1, ..., \mathbf{v}_N) \in T_{cx} Q$$

Given the above action $\mathcal{A}_{Sc}$ of the group of scale transformations Sc on $T(Q)$, the mass metric $\mathbf{M}$ turns out to be scale invariant (or more precisely $\mathcal{A}_{Sc}$-invariant), in addition to being translation and rotation invariant.

In order to see more clearly how the metric on shape space can be derived, consider a centre of mass configuration velocity

$$v_x = (\mathbf{v}_1, ..., \mathbf{v}_N) \in T_x(Q_{cm})$$

of a N-particle system. This vector transforms under the dilatations of the system as

$$x \to cx$$

$$v_x = (\mathbf{v}_1, ..., \mathbf{v}_{3N}) \in T_x(Q_{cm}) \to \mathcal{A}_c v_x = (\mathbf{v}_1, ..., \mathbf{v}_{3N}) \in T_{cx}(Q_{cm})$$

With this action of scale transformations on $T(Q_{cm})$, the mass metric is indeed scale invariant as can be seen by a short calculation

$$\mathbf{M}_x(v_x, u_x) \to \mathbf{M}_{cx}(\mathcal{A}_c v_x, \mathcal{A}_c u_x) = \mathbf{M}_x(v_x, u_x)$$

In other words, the mass metric $\mathbf{M}$ expressed in special coordinates on $T(Q_{cm})$ built from internal units (e.g., Planck's units), is scale invariant.

The metric $\mathbf{N} : T(Q_{int})/\mathcal{A}_{Sc} \times T(Q_{int})/\mathcal{A}_{Sc} \to \mathbb{R}$ is subsequently given as follows

$$\mathbf{N}_s(v', u') := \mathbf{M}_x(v, u) \tag{7.8}$$

where

$$\begin{cases} \pi(x) = s \\ \pi(u) = u' \\ \pi'(v) = v' \end{cases}$$

[5] with the new measuring instruments

with the projection map defined as follows $\pi : Q_{int} \to S$, $\pi' : T(Q_{int}) \to T(Q_{int})/\mathcal{A}_{Sc}$. Because the above construction is $\mathcal{A}_{Sc}$-invariant, to which $q \in \pi^{-1}(s)$ the pair of shape vectors $v', u' \subset T_q(Q_{int})/\mathcal{A}_{Sc}$ are lifted, does not make any difference for the value assigned by $\mathbf{N}_s$ to them. Hence, the metric $\mathbf{N}$ is well defined.

7.3 Connection form for $Sim(3)$ fiber bundle

Beside having a similarity invariant potential function on absolute configuration space (see Chapter 1), and a metric $\mathbf{N}$ on shape space S, in order to be able to derive the Lagrangian equations of motion on S (which is the topic of the next section), we need to have the suitable connection form ω on the absolute configuration space Q, compatible with the $Sim(3)$ fiber bundle structure.
Here we first discuss some features of the similarity group, and we present two representations of this group, and it's Lie-algebra $\mathbf{sim}(3)$. These will be subsequently used in the construction of the connection form of the $Sim(3)$ fiber bundle. At the end of this section, we will show by an explicit calculation that shape space S has the same curvature as the internal configuration space $Q_{int} := \frac{Q}{E(3)}$.

Similarity group $Sim(3)$ acts on any point $\mathbf{x} \in \mathbb{R}^3$ of absolue space as follows

$$\mathbf{x} \to \mathbf{x}' = cR\mathbf{x} + \mathbf{t}$$

where

$$c \in \mathbb{R}^+$$

stands for the spatial scale transformations,

$$R \in SO(3)$$

for the 3×3 matrix representation of spatial rotations [6], and

$$\mathbf{t} = (t_1, t_2, t_3)^T \in \mathbb{R}^3$$

[6]which can be parametrized for instance by the three Euler angels.

for spatial translations.

As the group of rotations $SO(3)$ does not form a normal subgroup of $Sim(3)$, while groups of translations $T(3) \cong \mathbb{R}^3$ and scale transformations $Sc \cong \mathbb{R}^+$ both does, one recognises a semidirect product structure in $Sim(3)$,[7] i.e.

$$Sim(3) = Sc \times T(3) \rtimes SO(3)$$

If one thinks of absolute space as a section $\mathbb{R}^3 \times \{1\} \in \mathbb{R}^4$ one can give a representation of $Sim(3)$ in terms of the 4×4 matrices of the form

$$\begin{bmatrix} cR_{11} & cR_{12} & cR_{13} & t_1 \\ cR_{21} & cR_{22} & cR_{23} & t_2 \\ cR_{31} & cR_{32} & cR_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.9}$$

The Lie-algebra **sim**(3) of the similarity group is then given by the matrices

$$\begin{bmatrix} \dot{c} & \omega_3 & -\omega_2 & v_1 \\ -\omega_3 & \dot{c} & \omega_1 & v_2 \\ \omega_2 & -\omega_1 & \dot{c} & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.10}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ are possible angular and linear velocity vectors. The so called *similarity velocity* can be characterized by the triple

$$\boldsymbol{\delta} = (\boldsymbol{v}, \boldsymbol{\omega}, \dot{c})$$

Alternatively, one can give a representation of the similarity group by expressing the position of the particle in absolute space $\mathbb{R}^3$, on real projective space $\mathbb{R}P^4$ as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

[7] Applying a translation and then a rotation is equivalent to applying the rotation and then a translation by the rotated translation vector. Hence $E(3)$ is a semidirect product of $T(3)$ and $O(3)$, i.e. $E(3) = T(3) \rtimes O(3)$

Matrices representing $Sim(3)$ on $\mathbb{R}P^4$ are then of the following form

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & t_1 \\ R_{21} & R_{22} & R_{23} & t_2 \\ R_{31} & R_{32} & R_{33} & t_3 \\ 0 & 0 & 0 & c^{-1} \end{bmatrix}$$

A simple calculation shows indeed

$$\begin{bmatrix} R & \mathbf{t} \\ 0 & c^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{x} + \mathbf{t} \\ c^{-1} \end{pmatrix} \cong \begin{pmatrix} c(R\mathbf{x} + \mathbf{t}) \\ 1 \end{pmatrix}$$

In this representation, in contrary to the previous one, any $g \in Sim(3)$ is thought of as a translation and rotation followed by a dilatation. Correspondingly, the matrix representation of the Lie-algebra $\mathbf{sim}(3)$ is given by

$$\begin{bmatrix} 0 & \omega_3 & -\omega_2 & v_1 \\ -\omega_3 & 0 & \omega_1 & v_2 \\ \omega_2 & -\omega_1 & 0 & v_3 \\ 0 & 0 & 0 & -\dot{c} \end{bmatrix}$$

The action of $Sim(3)$ on the configuration space Q of a multiparticle system, can consequently be given from the previous actions in a straight forward way.

We use the upper left 3×3 block of (7.9) to construct a representation of the group

$$G_{rs} := \frac{Sim(3)}{trans(3)} = SO(3) \times \mathbb{R}^+$$

comprising all the rotations and scale transformations, on the centre of mass configuration space Q_{cm}. This group has a direct product structure and acts on $\mathbb{R}^3$ as follows

$$\begin{bmatrix} cR_{11} & cR_{12} & cR_{13} \\ cR_{21} & cR_{22} & cR_{23} \\ cR_{31} & cR_{32} & cR_{33} \end{bmatrix}$$

This action makes Q_{cm} into a $SO(3) \times \mathbb{R}^+$ fiber bundle. The Lie-algebra $\mathbf{g_{rs}}$ of G_{rs} consists of the matrices of the following form

$$\mathbf{g_{rs}} = \mathbf{so}(3) + I_3\dot{c} = \begin{bmatrix} \dot{c} & \omega_3 & -\omega_2 \\ -\omega_3 & \dot{c} & \omega_1 \\ \omega_2 & -\omega_1 & \dot{c} \end{bmatrix} \tag{7.11}$$

The letter I_3 stands for the 3×3 identity matrix, and $\dot{c} \in \mathbb{R}$ being the generator of scale transformations. One can arrive at the expression of the connection form

$$\omega = T(Q_{cm}) \rightarrow \mathbf{g_{rs}}$$

for the G_{rs} fiber bundle, by modifying (6.8) in the following way

$$\omega = \omega_r + \omega_s = R\Big(A_x^{-1}(\sum_{j=1}^{N-1} \boldsymbol{r}_j \times d\boldsymbol{r}_j)\Big) + I_3 D_x^{-1}\Big(\sum_{j=1}^{N-1} \boldsymbol{r}_j . d\boldsymbol{r}_j\Big) \tag{7.12}$$

Here, we have defined the operator

$$D_x : \mathbb{R} \rightarrow \mathbb{R}$$

as

$$D_x(\dot{\boldsymbol{\lambda}}) := \sum_{j=1}^{N-1} \boldsymbol{r}_j^2 \dot{\boldsymbol{\lambda}} \tag{7.13}$$

and it can be called the *dilational tensor*. The letter $\dot{\boldsymbol{\lambda}}$ stands for the rate of change of scale of the system (scale velocity so to speak)[8]

$$\dot{\boldsymbol{\lambda}} := \frac{\dot{\lambda}}{\lambda} \tag{7.14}$$

where

$$\lambda := max \mid \mathbf{x}_i - \mathbf{x}_j \mid \tag{7.15}$$

for i, j varying between $1, 2, .., N$; being one choice for the system's scale variable.

We have constructed this operator in direct analogy to the moment of inertia tensor A_x. The inertial tensor transfers an angular velocity (which can be represented as a vector in $\mathbb{R}^3$) to another vector in $\mathbb{R}^3$, which represents the total angular momentum of the whole system. Similarly, the dilatational tensor D_x takes an expansion velocity, which in turn can be represented by a number in $\mathbb{R}$, to a measure of the total expansion of the system (dilatational momentum D),

[8]Here of course we are assuming all measurements are being conducted with the use of special Newtonian rods and clocks, which are isolated from the material universe and do not get affected by them in any way, or by any transformations we perform on the material universe. Practically, such measuring instruments of course do not exist, but existence of absolute space and absolute time of Newtonian philosophy justifies hypothetical existence of them.

which again can be represented by another number in $\mathbb{R}$. Since the Lie-algebra of G_{rs} can be represented by the matrices (7.11), one recognizes the correct structure in the connection form (7.12), for the $SO(3) \times \mathbb{R}^+$ fiber bundle. Taking a random vector of $T_x(Q_{cm})$ and acting on it with this connection form, the first term yields a member of $\mathbf{so}(3)$, and the second term a number, which in turn is multiplied by the identity matrix, yielding in sum a matrix of the above form (hence a member of the Lie-algebra of the bundle's structure group). Thus, it does what it is expected to do.

Last but not least, let us investigate the curvature C of the connection form

$$\omega_s = D_x^{-1}\Big(\sum_{j=1}^{N-1} \boldsymbol{r}_j.d\boldsymbol{r}_j\Big)$$

Theorem: The connection form ω_s is flat.

ω_s can equally well be expressed in the Cartesian coordinates $x_1,...,x_{3N}$ on the absolute configuration space Q as follows

$$\omega_s = D_x^{-1}\big(\sum_{i=1}^{3N} m_{\lfloor\frac{i-1}{3}\rfloor+1} x_i dx_i\big) \tag{7.16}$$

where for every rational number p, the largest integer smaller than p is denoted by $\lfloor p \rfloor$. Given two arbitrary horizontal vectors $v, v' \in T_x(Q)$ as the input of the curvature 2-form, it is known [24] that

$$C(v, v') = -\frac{1}{2}\omega_s([v, v']) \tag{7.17}$$

where $[.,.]$ is the Lie-bracket of the extension of horizontal vectors v and v' to horizontal vector fields. Choosing a basis $\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_{3N}}$ of the tangent space $T_x(Q)$, the Lie-bracket of two vector fields $v = \sum_{i=1}^{3N} v_i \frac{\partial}{\partial x_i}$ and $v' = \sum_{i=1}^{3N} v'_i \frac{\partial}{\partial x_i}$ can be computed by

$$[v, v'] = \sum_i \sum_j (v_j \frac{\partial v'_i}{\partial x_j} - v'_j \frac{\partial v_i}{\partial x_j}) \frac{\partial}{\partial x_i}$$

As both vectors $v, v' \in T_x(Q)$ are horizontal, they satisfy the following conditions

$$\omega_s(v) = \omega_s(v') = 0$$

Using (7.16) the above conditions can be translated into relations (or constraint equations) in the variables $x_1,...,x_{3N},v_1,...,v_{3N}$, which after some rearrangement of terms become as follows

$$v_{3N} = \frac{1}{m_N x_{3N}} \sum_{i=1}^{3N-1} m_{\lfloor \frac{i-1}{3} \rfloor + 1} x_i v_i$$

$$v'_{3N} = \frac{1}{m_N x_{3N}} \sum_{i=1}^{3N-1} m_{\lfloor \frac{i-1}{3} \rfloor + 1} x_i v'_i$$

As all other $6N-1$ variables involved are independent, for all $j = 1, ..., 3N$, and $i = 1, ..., 3N-1$; all the following derivatives vanish

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v'_i}{\partial x_j} = 0$$

which simplify the above expression for $[v, v']$ greatly

$$[v, v'] = \sum_{j=1}^{3N} (v_j \frac{\partial v'_{3N}}{\partial x_j} - v'_j \frac{\partial v_{3N}}{\partial x_j}) \frac{\partial}{\partial x_{3N}} = 0$$

In the last equality $\frac{\partial v_{3N}}{\partial x_j} = \frac{m_{\lfloor \frac{i-1}{3} \rfloor + 1}}{m_N x_{3N}} v_j$ is used. Hence from (7.17) it follows that the connection form ω_s has a vanishing curvature. This means that shape space $S = \frac{Q}{sim(3)}$ is exactly as curved as the internal configuration space $Q_{int} = \frac{Q}{E(3)}$. That Q_{int} is curved, or in other words, that the curvature of the connection form ω_r given by (6.8) is non-vanishing, has been shown in [35]. So, in the process of quotienting out the flat configuration space Q by the action of the similarity group $Sim(3)$, the only stage which causes curvature in the final base space is the quiotienting with respect to the group of rotations $SO(3)$.

For the purpose of the reduction of the classical mechanics w.r.t. the scale transformations, we will use the geometric setting explained in this section, and Chapter (5). One of the reasons why reduction w.r.t. the similarity group has not been studied as extensively as the euclidean group is obviously that the potential function defined on the absolute space, though being manifestly rotational and transnational invariant, is clearly not scale invariant (take Newtonian gravity potential as an example). However, as explained in Chapter (1), as a consequence of the principle of relationalism, scale transformations become an additional symmetry of the classical physics. This enable the shape degrees of freedom to have an autonomous evolution, fully decoupled from the system's transnational, orientational, and scale degrees of freedom.

Chapter 8

Reduced Lagrangian equations of motion in shape coordinates

In this section, we seek the equations of motion of a N-particle system in shape, orientation, and scale coordinates, and their velocities. For this purpose, the Lagrangian of the system must first be expressed in terms of the new coordinates and velocities, and then the equations of motion can be derived. Since the angular velocities used to quantify the rate of rotation of a system are not derivatives of the three Euler angles (or other variables), the Lagrange, or Euler-Lagrange equations of motion cannot be used. For such cases, the Boltzmann-Hammel equations of motion must be used. In the first subsection, following [36], we give a brief review of the formulation of mechanics in quasi-coordinates and quasi-velocities, and then based on [5] and [6], we derive the equations of motion of classical systems in shape, orientation, and scale coordinates, and shape, angular, and scale velocities.

8.1 Equations of motion in quasi-coordinates

The generalized coordinates are the set of coordinates defining the degrees of freedom of a system. For instance for a rigid body moving in $\mathbb{R}^3$, there are 6 generalized coordinates (3 specifying the position of the body and 3 the orientation of it), i.e.

$$q = [q_1, ..., q_6] := [x, y, z, \alpha, \beta, \gamma]$$

The generalized speeds are obviously the derivatives of the generalized coordinates $\dot{q} = [\dot{x}, \dot{y}, \dot{z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}]$. These coordinates q_k can be called *true coordinates*, in the sense that if the velocities $\dot{q}_k$ are known functions of time, an integration with respect to time determines the respective coordinates, and hence state of the system.

On the other hand, one may define generalized speeds which cannot be written as time derivative of any coordinates, for instance, they are defined as linear combinations of the time derivatives of generalized coordinates. Such generalized speeds are called *quasi-velocities*, and the generalized coordinates corresponding to the these velocities are called *quasi-coordinates*. The word quasi in the last expression "quasi-coordinates" should be understood as nonexistent. As the most famous example of quasi-velocities, one can mention angular velocity components of a rigid body, which are linear combinations of derivatives of Euler angles, but they are themselves not time derivatives of any coordinates. Quasi-coordinates and quasi-velocities were first introduced to derive the so called Bolzmann-Hammel equations of motion, which will be shortly discussed below.

In the analysis of non-holonomic systems, quasi-velocity formulation cast the dynamical equations of motion in a form requiring fewer equations. For a system possessing n degrees of freedom with m nonholonomic constraints, the usage of fundamental nonholonomic lagrangian formalism leads to $2n + m$ equations of motion ($2n$ equations for the system state and m algebraic relations that must be solved for the multipliers). If quasi-velocity formalism is used, the same problem can be described by a system of $2n - m$ degrees of freedom (see [37],[38]).

Equations of motion for the classical mechanics in true coordinates are the known

Lagrange equations

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = F_i \tag{8.1}$$

with K being the system's kinetic energy, and F_i being the generalized force associated to generalized coordinate q_i, for $i \in [1, n]$.
Consider now the usage of the quasi-velocities Y_i, which are defined as N independent linear combinations of the $\dot{q}_k$'s, i.e.

$$Y_i := \alpha_{i1}\dot{q}_1 + \alpha_{i2}\dot{q}_2 + ... + \alpha_{in}\dot{q}_n = \sum_{r=1}^{N} \alpha_{ir}\dot{q}_r$$

with α_{ir} being known functions of the generalized coordinates q_k. Constructing a $N \times N$ matrix α from α_{ij}'s, one can write the the definition of quasi-velocities Y_k's more compactly

$$\begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_N \end{bmatrix} = \alpha \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ . \\ . \\ . \\ \dot{q}_N \end{bmatrix}$$

Having the above relations between the quasi-velocities Y_i's, and the generalized (true) velocities $\dot{q}_j$'s in mind, one can define a set of differential forms dy_k as

$$dy_k := \sum_{r=1}^{N} \alpha_{rk} dq_r$$

The above equations cannot always be integrated to obtain the variable y_k. In such cases, the differential forms dy_k are naturally called *nonintegrable*, and cannot be thought of as differential of some configuration variable y_k. The quantities dy_k are called *differentials of quasi-coordinates*, with some abuse of words, cause they are not really differentials, and the variables y_k are undefined.
If the quasi-velocities are known, the true velocities can be calculated using

$$\dot{q}_k = \beta_{kl} Y_l$$

where $\alpha_{sk}\beta_{kl} = \delta_{sl}$. Here δ_{sl} is the Kronecker Delta. It is easy to check that

$$\frac{\partial \dot{q}_k}{\partial Y_l} = \frac{\partial q_k}{\partial y_l} = b_{kl}$$

For a function $f(q_1, ..., q_N, t)$, with the partial derivative with respect to a quasi-coordinate y_l one means the following

$$\frac{\partial f}{\partial y_l} := \frac{\partial f}{\partial q_k}\frac{\partial q_k}{\partial y_l} = \frac{\partial f}{\partial q_k} b_{kl}$$

The kinetic energy $K(q_1, ..., q_N, \dot{q}_1, ..., \dot{q}_N)$ can be expressed in the new variables i.e. $K'(q_1, ..., q_N, Y_1, ..., Y_N)$, where the prime here indicates just the difference in the variables of the function. It can be shown (see [29]) that the equations of motion expressed in these new coordinates $q_1, ..., q_N, Y_1, ..., Y_N$ become of the following form

$$\frac{d}{dt}\frac{K'}{\partial Y_k} - \frac{\partial K'}{\partial y_k} + \gamma_{kij}\frac{\partial K'}{\partial Y_i} Y_j = F'_k \tag{8.2}$$

for $k = 1, 2, ..., N$, where

$$F'_k = \sum_{s=1}^{n} F_i b_{ik}$$

are the generalized forces corresponding to the virtual displacements δy_k, and

$$\gamma_{kij} := b_{sk} b_{lj} \big(\frac{a_{ij}}{q_l} - \frac{\partial a_{il}}{\partial q_s}\big)$$

These are known as the *Boltzmann-Hammel* equations of motion. As already mentioned, the partial derivative with respect to the quasi-coordinate appearing in second term of (8.2) should be understood as $\frac{\partial K'}{\partial y_k} := \frac{\partial f}{\partial q_l}\frac{\partial q_l}{\partial y_k} = \frac{\partial f}{\partial q_l} b_{lk}$. The N equations of (8.2) are equations of motion in quasi-coordinates. If y_1,...,y_N are true coordinates the coefficients γ_{kij} all vanish, and the Boltzmann-Hammel equations (8.2) take back the form of Lagrange equations (8.1).

Starting on the configuration space Q_{cm} with $n = \dim(Q_{cm}) = 3N - 3$, and a system with the lagrangian $L : T(Q) \to \mathbb{R}$, which usually is of the form

$$L(x, \dot{x}) = \frac{1}{2}\mathbf{M}_{ij}\dot{x}^i\dot{x}^j + V(x)$$

we impose m linear scleronomic (time independent) nonholonomic constraints, i.e. constraints of the form

$$a^{\sigma}_i(x)\dot{x}^i = 0 \tag{8.3}$$

where $1 \leq \sigma \leq m$. Define a vector space isomorphism Ψ^j_i on the tangent space $T(Q_{cm})$. The first m rows of Ψ^j_i is set to be identical with the constraint matrix, i.e.

$$\Psi^{\sigma}_i(q) = a^{\sigma}_i(q)$$

and the remaining rows can be chosen freely as long as the resulting matrix Ψ is invertible. This transformation Ψ can be viewed as change of basis of $T(Q_{cm})$

$$\Psi : \big(\frac{\partial}{\partial x^i}\big)_{i=1}^n \rightarrow \big(\frac{\partial}{\partial \theta^i}\big)_{i=1}^n \tag{8.4}$$

This new basis is called the *quasi-basis*. Hence a vector $v \in T(Q_{cm})$ can be expressed in either bases, i.e.

$$v = \dot{x}^i \frac{\partial}{\partial x^i} = u^j \frac{\partial}{\partial \theta^j}$$

where $u^j = \Psi_i^j \dot{x}^i$ are the components of the *quasi-velocities*.
As is well known, the basis vectors transform like $\frac{\partial}{\partial x^i} = \Psi_i^j \frac{\partial}{\partial \theta^j}$ and $\frac{\partial}{\partial \theta^i} = (\Psi^{-1})_i^j \frac{\partial}{\partial x^j}$ and the set of n one-forms dual to the quasi basis (i.e. the quasi coordinate forms) are $d\theta^i = \Psi_j^i dx^j$. Bear in mind that the one-forms $d\theta^j$ are not exact.

8.2 Lagrangian in quasi-coordinates

Take the local coordinates

$$(s, g, \lambda, \dot{s}, \dot{g}, \dot{\boldsymbol{\lambda}})$$

on

$$T\big(\pi^{-1}(U)\big) \subset T(Q_{cm})$$

with $U \subset S$, which are adopted coordinates to the bundles projection map

$$\pi : Q_{cm} \rightarrow S$$

So $(s, g, \lambda) \in \pi^{-1}(U)$, and $(\dot{s}, \dot{g}, \dot{\boldsymbol{\lambda}}) \in T_{\sigma(s)}(\pi^{-1}(U))$. Here $s = (s^\alpha)$ for $\alpha \in [1, ..., 3N-7]$, are for instance the $3N-7$ independent angles between the $N-1$ Jacobi vectors $\mathbf{r}_i$, and $\dot{\boldsymbol{\lambda}} := \frac{\dot{\lambda}}{\lambda}$ being the scale velocity.
Having the connection form (6.23) in mind, one can introduce a $\mathbf{so}(3)$-valued variable ([5],[6])

$$\Pi = \epsilon + \sum_{\alpha=1}^{3N-6} \wedge_\alpha(x)\dot{q}^\alpha \tag{8.5}$$

where

$$\epsilon = g^{-1}\dot{g}$$

and

$$\wedge_\alpha(x) = \sum_{a=1}^{3} \wedge_\alpha^a(x) R(\mathbf{e}_a)$$

The vectors associated with Π and ϵ will be denoted by Ω' and Ω respectively i.e.

$$R(\Omega') = \Pi$$

and

$$R(\Omega) = \epsilon$$

Thus

$$(s, g, \lambda, \dot{s}, \Omega', \dot{\boldsymbol{\lambda}})$$

constitutes a local (quasi)coordinate system on $T(\pi^{-1}(U))$. Bear in mind that Ω' denotes the angular velocity of the system in body frame, and hence the angular momentum of the system would become $\mathbf{L} = gA_{\sigma(s)}\Omega'$ in space frame. This angular momentum vector expressed in body frame becomes of course $A_{\sigma(s)}\Omega'$. As usual, g stands for the rotation which brings the space frame to the body frame.

As discussed in Section (5.2), the moment of inertial tensor can be expressed as follows

$$A_{ab} = ds^2(L_a, L_b)$$

with L_a being the left invariant vector fields on $SO(3)$ (which were dual to θ^a). The mass metric (6.19) on Q_{cm} expressed in coordinates $(s, \lambda, \alpha, \beta, \gamma)$ becomes subsequently as follows

$$ds^2 = \sum_{\alpha,\beta=1}^{3N-7} N_{\alpha\beta} ds^\alpha ds^\beta + \sum_{i=1}^{N-1} \mid \frac{\boldsymbol{r}_i}{\lambda} \mid^2 (d\lambda)^2 + \sum_{a,b=1}^{3} A_{ab}\omega'^a\omega'^b \tag{8.6}$$

where $\boldsymbol{r}_i$'s are as before the Jacobi vectors of system, and hence are unique functions of the s_α's and λ. The ω'^a's are components of the connection form of rotations (6.11) in body frame (6.15). By setting the instantaneous unit of length equal to the system's scale variable λ, one gets the following expression for the metric

$$ds^2 = \sum_{\alpha,\beta=1}^{3N-7} N_{\alpha\beta} ds^\alpha ds^\beta + \sum_{i=1}^{N-1} \mid \boldsymbol{r}_i \mid^2 (d\lambda)^2 + \sum_{a,b=1}^{3} A_{ab}\omega'^a\omega'^b \tag{8.7}$$

where now all the $\boldsymbol{r}_i$'s and A are expressed in the internal (expanding) length unit. It is worth mentioning that the independent angles s_i, and the scale coordinate are orthogonal coordinates on $Q_{int} = \frac{Q_{cm}}{SO(3)}$, as they bring the metric tensor in a diagonal form.

Consider a system with a similarity invariant Lagrangian $L(s, g, \lambda, \dot{s}, \Omega', \dot{\boldsymbol{\lambda}})$, i.e.

$$L(s, hg, c\lambda, \dot{s}, \Omega', \dot{\lambda}) = L(s, g, \lambda, \dot{s}, \Omega', \dot{\boldsymbol{\lambda}}) \tag{8.8}$$

$\forall h \in SO(3)$, and $\forall c \in \mathbb{R}^+$. Note that Ω' is left $SO(3)$-invariant. Such a function L on $T(Q_{cm})$ reduces naturally to a function $L^*(s, \dot{s}, \Pi, \dot{\lambda})$ on $\frac{T(Q)}{Sim(3)}$. The similarity invariance of the Lagrangian in classical mechanics, is a consequence of the similarity invariance of kinetic and potential energies, as already explained based on the principle of relationalism. It is important to remember that the units (of time and spatial distance) in which the Lagrangian has the property (8.8) are internal units.

We write the similarity invariant Lagrangian of classical mechanics[1] as follows

$$L = \frac{1}{2}\sum_{\alpha,\beta=1}^{3N-7} N_{\alpha\beta}\dot{s}^\alpha \dot{s}^\beta + \frac{1}{2}\sum_{i=1}^{N-1}(\boldsymbol{r}_i\dot{\boldsymbol{\lambda}})^2 + \frac{1}{2}\sum_{a,b} A_{ab}\Omega'^a\Omega'^b - V(s) \tag{8.9}$$

where V is a similarity invariant potential function, hence depending only on the $3N-7$ coordinates s_i. Notice that our previous expression for dilational momentum (7.13) is derivable from the above Lagrangian:

$$D = \frac{\partial L}{\partial \dot{\boldsymbol{\lambda}}} = \sum \mid \boldsymbol{r}_j \mid^2 \dot{\boldsymbol{\lambda}}$$

As the Lagrangian (8.9) is scale independent, D is a constant of motion[2].

[1]For the non-singular configurations

[2]Bear again in mind, that the units in which D is measured (and is constant), are all internal units.

8.3 Reduced Euler-Lagrange equations of motion

In this section finally we discuss the equations of motion in the anholonomic frame $(s, g, \lambda, \dot{s}, \Omega', \dot{\boldsymbol{\lambda}})$ on $T(Q_{cm})$ introduced in the last section.

Let x^λ, $\lambda = 1, ..., 3N-3$ be a local coordinate system on $W \subset Q_{cm}$. From this coordinate system, one can derive a basis for the vector fields, i.e. $\frac{\partial}{\partial x^\lambda}$, and a basis for the 1-forms, i.e. dx^λ on Q_{cm}. Let Z_λ and Z^λ be another local basis of the vector fields and 1-forms (dual to each other) on W. The later vector fields and one forms, are related to the former ones[3] by

$$Z_\lambda = \sum_{\mu=1}^{3N-3} B^\mu_\lambda \frac{\partial}{\partial x^\mu}$$

$$Z^\lambda = \sum_{\mu=1}^{3N-3} A^\lambda_\mu dx^\mu$$

where their duality requires $\sum_\lambda A^\mu_\lambda B^\lambda_\nu = \delta^\mu_\nu$. If the above relations for Z^λ's are integrable, there exists true coordinates z^λ on Q_{cm}, for which $Z_\lambda = \frac{\partial}{\partial z^\lambda}$, and $Z^\lambda = dz^\lambda$. Otherwise, the Z_λ's form an anholonomic basis of $T(Q_{cm})$, and the z^λ's become the corresponding quasi-coordinates on Q_{cm}.
Differentiation of Z^λ leads to [16]

$$dZ^\lambda = \sum_{\sigma<\kappa} \gamma^\lambda_{\sigma\kappa} Z^\kappa \wedge Z^\sigma$$

$$\gamma^\lambda_{\sigma\kappa} := \sum_{\mu\nu} \left(\frac{\partial A^\lambda_\mu}{\partial x^\nu} - \frac{\partial A^\lambda_\nu}{\partial x^\mu} \right) B^\mu_\sigma B^\nu_\kappa$$

In the expression of the Lagrangian function $L(x, \dot{x})$, one can replace the coordinate velocities $\dot{x}^\lambda$ with a new set of velocity coordinates[4]

$$\dot{z}^\lambda = \sum_\mu A^\lambda_\mu(x) \dot{x}^\mu$$

[3] Which were derived from the coordinate system x^λ

[4] Which typically are quasi-velocities

In these new variables the Lagrangian is denoted by L', i.e.

$$L'(x, \dot{z}) = L(x, \dot{x})$$

As explained in Section (7.1), The Euler-Lagrange equations in terms of $(x^\lambda, \dot{x}^\lambda)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\lambda}\right) - \frac{\partial L}{\partial x^\lambda} = 0$$

for $\lambda = 1, ..., 3N-3$; takes the following form of Boltzmann-Hammel equations of motion in terms of $(x^\lambda, \dot{z}^\lambda)$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{z}^\sigma}\right) - Z_\sigma L' + \sum_{\mu,\kappa} \gamma^\mu_{\sigma\kappa} \frac{\partial L'}{\partial \dot{z}^\mu} \dot{z}^\kappa = 0 \tag{8.10}$$

for $\sigma = 1, ..., 3N-3$.

In [6], a derivation of the reduced Lagrangian equation of motion on $Q_{int} = \frac{Q_{cm}}{SO(3)}$ is explained. In the rest of this section, we will present an extension of this work to find the reduced Lagrangian equations of motion on shape space $S = \frac{Q_{cm}}{SO(3)\times Sc}$. So we consider Q_{cm} as a $SO(3) \times Sc$ fiber bundle, and make a coordinate transformation on Q_{cm} from the Euler angles and Jacobi coordinates, to the Euler angles, scale, and the shape coordinates, i.e.

$$(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{N-1}, \alpha, \beta, \gamma) \rightarrow (s_1, ..., s_{3N-7}, \lambda, \alpha, \beta, \gamma)$$

A basis of 1-forms on Q_{cm} in the new coordinates is given as the following

$$Z^a := \omega^a$$

$$Z^4 := \omega_s$$

$$Z^{4+i} := ds^i$$

where $a = 1, 2, 3$ and $i = 1, 2, ..., 3N-7$. The ω^a's and ω_s can be read from the new connection form (7.12). In particular, the ω^a's are components of the rotations connection form (6.14) form (w.r.t. the fixed space frame). Their dual vector fields are as follows

$$Z_a = J_a$$

$$Z_4 = \frac{\partial}{\partial \lambda}$$

$$Z_{4+i} = \partial^*_{4+i} = (\frac{\partial}{\partial s^i})^*$$

Given a section (or lifting map) $\sigma : S \to Q_{cm}$, the horizontal lift of a vector $\frac{\partial}{\partial s^\alpha} \in T_{s_0}(S)$ at the point s_0 of shape space S, to the point $\sigma(s_0) \in Q_{cm}$ is given as follows

$$(\frac{\partial}{\partial s^\alpha})^* = \frac{\partial}{\partial s^\alpha} - D_r^{-1}\big(\sum_{j=1}^{N-1} \boldsymbol{r}_j . \frac{d\boldsymbol{r}_j}{ds^\alpha}\big)\frac{\partial}{\partial \lambda} - \sum_{a=1}^{3} \beta^a_\alpha(s_0) L_a \tag{8.11}$$

where analogous to (6.24) one has

$$\beta^a_\alpha := \big\langle A^{-1}_{\sigma(s_0)}\big(\sum_{i=1}^{N-1} \boldsymbol{r}_i \times \frac{\partial \boldsymbol{r}_i}{\partial s^\alpha}\big) \mid e'_a \big\rangle$$

Taking the exterior derivatives of Z^λ leads to the factors $\gamma^\lambda_{\sigma\kappa}$. They become

$$\gamma^a_{bc} = -\epsilon_{bca}$$

$$\gamma^a_{4+i,4+j} = -\boldsymbol{k}^a_{ij}$$

with $\boldsymbol{k}^a_{ij}$

$$\boldsymbol{k}^c_{ij} = \frac{\partial \beta^c_j}{\partial s^i} - \frac{\partial \beta^v_i}{\partial s^j} - \sum_{a,b=1}^{3} \epsilon_{abc} \beta^a_i \beta^b_j$$

and all other $\gamma^\lambda_{\sigma\kappa}$ vanishing. $\boldsymbol{k}^c_{ij}$ are the components of the curvature tensor of shape space

$$\boldsymbol{k}^c = d\omega^c - \sum_{a<b}^{3} \epsilon_{abc}\omega^a \wedge \omega^b = \sum_{i<j}^{3N-7} \boldsymbol{k}^c_{ij} ds^i \wedge ds^j$$

for the connection form ω of $SO(3) \times Sc$ fiber bundle given by (7.12). Note that besides the well-known interconnection of changes in Euler angels, which manifest themselves in the structure constants $\gamma^a_{bc} = -\epsilon_{abc}$, the only non-vanishing couplings are the coupling of shape variables s^i, to the orientational variables (Euler angles). Intuitively, as the scale of a mechanical system can easily be changed, without resulting any changes in either total orientation, or shape of the system, one expects vanishing of corresponding γ factors. In other words, the connection form ω_s seen as a connection form on $Q_{int} = \frac{Q_{cm}}{SO(3)}$ must have vanishing curvature. At the end of Section (6.3) we have already shown by an explicit calculation, that ω_s is indeed a flat connection.

Consider the coordinates

$$(\alpha, \beta, \gamma, \lambda, s^i; \Omega^1, \Omega^2, \Omega^3, \dot{\boldsymbol{\lambda}}, \dot{s}^i)$$

on TQ_{cm}, where the quasi-velocities Ω^a's are defined using (6.14), i.e.

$$\Omega^a := \omega^a(\frac{d}{dt}) = \psi^a_t + \sum_i \beta^a_i \dot{s}^i = \psi^a(\frac{d}{dt}) + \sum_i \beta^a_i \dot{s}^i$$

These are the components of angular velocity with respect to the fixed space frame.

As seen before (8.9), the Lagrangian describing a mechanical N-particle system expressed in the above coordinates becomes as follows

$$L = \frac{1}{2} \sum_{\alpha,\beta=1}^{3N-7} \mathbf{N}_{\alpha\beta} \dot{s}^\alpha \dot{s}^\beta + \frac{1}{2} \sum_{i=1}^{N-1} (\boldsymbol{r}_i \dot{\boldsymbol{\lambda}})^2 + \frac{1}{2} \sum_{a,b} A_{ab} \Omega^a \Omega^b - V(s)$$

The Boltzmann-Hammel equations of motion (8.10) in this new coordinate system then becomes

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{s}^i}) - (\frac{\partial}{\partial s^i})^* L - \sum_a \sum_j \boldsymbol{k}^a_{ij} \frac{\partial L}{\partial \Omega^a} \dot{s}^j = 0 \tag{8.12}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \Omega^a}) - J_a L - \sum_{b,c} \epsilon_{acb} \frac{\partial L}{\partial \Omega^b} \Omega^c = 0 \tag{8.13}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\boldsymbol{\lambda}}}) - \frac{\partial L}{\partial \lambda} = 0 \tag{8.14}$$

Chapter 9

3 body system

Take three particles located at the positions

$$\mathbf{x}_1 = (x_1, y_1, z_1)$$

$$\mathbf{x}_2 = (x_2, y_2, z_2)$$

$$\mathbf{x}_3 = (x_3, y_3, z_3)$$

The configuration of this system in centre of mass frame can be characterized by two Jacobi vectors

$$\boldsymbol{r}_1 = (\frac{1}{m_1} + \frac{1}{m_2})^{-1/2}(\boldsymbol{x}_2 - \boldsymbol{x}_1)$$

$$\boldsymbol{r}_2 = (\frac{1}{m_1 + m_2} + \frac{1}{m_3})^{-1/2}(\boldsymbol{x}_3 - \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1 + m_2})$$

As shape variables we introduce the two angles formed by the interparticle vectors, i.e.

$$s_1 := cos^{-1}(\frac{(\boldsymbol{x}_2 - \boldsymbol{x}_1).(\boldsymbol{x}_3 - \boldsymbol{x}_1)}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_1 \mid\mid \boldsymbol{x}_3 - \boldsymbol{x}_1 \mid})$$

$$s_2 := cos^{-1}(\frac{(\boldsymbol{x}_3 - \boldsymbol{x}_2).(\boldsymbol{x}_1 - \boldsymbol{x}_2)}{\mid \boldsymbol{x}_3 - \boldsymbol{x}_2 \mid\mid \boldsymbol{x}_1 - \boldsymbol{x}_2 \mid})$$

and the scale variable of the system is chosen like in (7.15) as follows

$$\lambda := max \mid \boldsymbol{x}_i - \boldsymbol{x}_j \mid$$

As explained before, system's rotational degrees of freedom can be taken care of by three Euler angles α, β, γ which connects the space frame and the body frame[1].

So we have the following coordinate transformation on absolute configuration space Q of the 3 particle system

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ x_3 \\ y_3 \\ z_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \\ \alpha \\ \beta \\ \gamma \\ \lambda \\ s_1 \\ s_2 \end{pmatrix}$$

From our previous discussions and some calculations, we can calculate the mass metric $\mathbf{M}$ on Q in new coordinates. It becomes as follows

$$\begin{aligned} dl^2 = & \frac{m_3(m_1+m_2)\lambda^2 sin^2(s_2)}{m_1 m_2 sin^2(s_1+s_2)} ds_1^2 + \frac{m_3(m_1+m_2)\lambda^2 sin^2(s_1)}{m_1 m_2 sin^2(s_1+s_2)} ds_2^2 \\ & + \big(1 + \frac{m_2}{m_1} + \frac{m_3(m_1+m_2) sin^2(s_2)}{m_1 m_2 sin^2(s_1+s_2)}\big) d\lambda^2 \\ & + \sum_{a,b} A_{ab}\omega^a\omega^b + (m_1+m_2+m_3)(dx_{cm}^2 + dy_{cm}^2 + dz_{cm}^2) \end{aligned} \tag{9.1}$$

Using the above line element, one can express the infinitesimal increment of Newton's absolute time dt in terms of system's motion (infinitesimal increments of particles spatial positions), i.e. (see [1],[26])

$$dt = \frac{dl}{\sqrt{E-V}} \tag{9.2}$$

This is the increment of the ephemeris time for our 3 particle universe. Now it becomes clear that for a spatially larger 3 particle system by a factor $c > 1$,

[1]Define the space frame simply equal to body frame at some specific time, for instance at the initial time.

the same amount of relational motion (ds_1, ds_2) leads [2] to a longer increment of ephemeris time[3] dt, if distances are measured with the fixed (non-scalable) absolute rod (unit of length) attached to Newton's absolute space (w.r.t. which by the way changes of λ can be measured and communicated). Considering ticks of clocks as specific amount of relational motion of the system (where clock itself is part of), the relation between the seconds of clocks after and before system's spatial scale transformation $\boldsymbol{x}_i \to c\boldsymbol{x}_i$ becomes $T \to T' = cT$. Of course again this difference in rate of clocks ticking can only have meaning if we use the absolute Newtonian clock which is unaffected by what happens with the matter in universe. This is also in complete agreement with our previous discussions (see Chapter 1) about relation between behaviour of Plank's time unit under systems global spatial scale transformations $\boldsymbol{x}_i \to c\boldsymbol{x}_i$, namely $T_p \to T'_p = cT_p$. This was there derived directly from the Principle of Relationalism.

As can be seen from (8.9), the Lagrangian of the 3 particle system in the center of mass frame is the following function on Q_{cm}

$$L = \frac{1}{2}\frac{m_3(m_1+m_2)\lambda^2 sin^2(s_2)}{m_1 m_2 sin^2(s_1+s_2)}\dot{s}_1^2 + \frac{1}{2}\frac{m_3(m_1+m_2)\lambda^2 sin^2(s_1)}{m_1 m_2 sin^2(s_1+s_2)}\dot{s}_2^2 \tag{9.3}$$
$$+\big(1+\frac{m_2}{m_1}+\frac{m_3(m_1+m_2)sin^2(s_2)}{m_1 m_2 sin^2(s_1+s_2)}\big)\dot{\boldsymbol{\lambda}}^2$$
$$+\frac{1}{2}\sum_{a,b} A_{ab}\Omega^a\Omega^b - V$$

where as before $\dot{\boldsymbol{\lambda}} = \frac{\dot{\lambda}}{\lambda}$, and

$$V = G\big(\frac{m_1 m_2}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_1 \mid} + \frac{m_1 m_3}{\mid \boldsymbol{x}_3 - \boldsymbol{x}_1 \mid} + \frac{m_2 m_3}{\mid \boldsymbol{x}_3 - \boldsymbol{x}_2 \mid}\big) \tag{9.4}$$

Even though the actual scale (λ) of the system appears explicitly in the above lagrangian, it is in fact a scale invariant lagrangian, if one takes the transformation of time unit and gravitational constant after performance of a global scale transformation [4] into account. Here we elaborate a bit more on this point. Regarding

[2]consider for simplicity the case where all collective momenta(linear, angular, and dilational) are vanishing.

[3]longer by the same factor with which the spatial size of the system had been multiplied.

[4]As disscused before, in a larger universe, clocks tick slower, and G becomes larger.

the units used in (9.3), one has measured the lengths w.r.t the absolute rod of Newton (w.r.t. which the diameter of our system happens to be λ), and one uses the ephemeris unit of time [5] for time measurements. Now one sees the quantity $\lambda\dot{s} = \lambda\frac{ds}{dt_e}$ where subscript e stands for ephemeris, is an invariant quantity under systems spatial scaling $\boldsymbol{x}_i \to \boldsymbol{x}'_i = c\boldsymbol{x}_i$, as such a transformation leads to

$$\lambda \to \lambda' = c\lambda$$

and

$$\delta t_e \to \delta t'_e = c\delta t_e$$

So one gets

$$\lambda\dot{s} = \lambda\frac{ds}{dt_e} \to \lambda'\dot{s}' = \lambda'\frac{ds}{dt'_e} = c\lambda\frac{ds}{c.dt_e} = \lambda\frac{ds}{dt_e} = \lambda\dot{s}$$

Now if one wants to be realistic about length measurements and include this reality in the theory, one has to use a length unit built out of the matter. So, an internal(or relational) length unit must be used, instead of the invisible absolute Newton's length unit which was coming from absolute space. Take for instance the diameter of system as the internal length unit. That means $\lambda = 1$. For any increment of system's shape (ds_1, ds_2), calculate the increment of time by using the formula for increment of ephemeris time (9.2) in which we also set $\lambda = 1$, i.e. $dt = dt_e \mid_{\lambda=1}$. Then the expression of Lagrangian (9.3) with usage of relational units of time and length becomes as follows

$$L = \frac{1}{2}\frac{m_3(m_1+m_2)sin^2(s_2)}{m_1m_2sin^2(s_1+s_2)}\dot{s}_1^2 + \frac{1}{2}\frac{m_3(m_1+m_2)sin^2(s_1)}{m_1m_2sin^2(s_1+s_2)}\dot{s}_2^2 \tag{9.5}$$
$$+\big(1+\frac{m_2}{m_1}+\frac{m_3(m_1+m_2)sin^2(s_2)}{m_1m_2sin^2(s_1+s_2)}\big)\dot{\boldsymbol{\lambda}}^2$$
$$+\frac{1}{2}\sum_{a,b}A_{ab}\Omega^a\Omega^b - V(s_1,s_2)$$

where now it's scale invariance has become explicitly apparent.
One has to be careful with the interpretation of the third term here containing the scale velocity $\dot{\boldsymbol{\lambda}}$. It has the same origin as the forth term in the lagrangian.

[5] which is a kind of internal clock for the system, w.r.t. which the second law of Newton remains valid.

The forth term causes the coriolis and centrifugal forces in a rotating frame of reference(like in a body frame of a rotating system), and similarly the third term causes dilational forces in a spatially expanding frame of reference (like a body frame of an expanding system with an internal unit of length). Note also that the scale velocity $\dot{\boldsymbol{\lambda}} = \frac{\dot{\lambda}}{\lambda}$ has just the dimension of inverse of time, as it is the ratio of change of system's scale $\delta\lambda$ during one second of internal time δt_e to the system's scale λ. At an instant of time t_0, when one is viewing the expanding system from some point of the absolute space, one can manually set the absolute unit of length equal to the instantaneous scale of the system at that time and express $\dot{\boldsymbol{\lambda}}$ in this new absolute length unit. It becomes simply $\dot{\boldsymbol{\lambda}} = \dot{\lambda} = \frac{\delta\lambda}{\delta t_e|_{\lambda=1}}$, where during the small observation time interval $[t_0, t_0 + \delta t_e \mid_{\lambda=1}]$ we have fixed the absolute length unit to $\lambda \mid_{t=t_0}$, and hence can measure the new scale of the system at the end of the time interval $\lambda \mid_{t=t_0+\delta t_e|_{\lambda=1}} = \lambda \mid_{t=t_0} + \delta\lambda$.

The equations of motion of the two shape degrees of freedom s^1, and s^2 for the case of vanishing total angular velocity $\omega_t = 0$ [6] can be given using (8.12)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}^i}\right) - \left(\frac{\partial}{\partial s^i}\right)^* L = 0$$

After some lengthy calculations we end up with the following coupled second order nonhomogeneous non-linear differential equations for the shape degrees of freedom of a nonrotating three body system

$$\begin{aligned} sin^2(s_2)sin(s_1+s_2)\ddot{s}_1 - 3sin^2(s_2)cos(s_1+s_2)\dot{s}_1^2 \\ +2sin(s_2\big(cos(s_2)sin(s_1+s_2) - sin(s_2)cos(s_1+s_2)\big)\dot{s}_2\dot{s}_1 \\ +2\dot{\boldsymbol{\lambda}}sin(s_1+s_2)sin^2(s_2)\dot{s}_1 \\ +sin(s_1)\big(cos(s_1)sin(s_1+s_2) - sin(s_1)cos(s_1+s_2)\big)\dot{s}_2^2 \\ +2\dot{\boldsymbol{\lambda}}^2 sin^2(s_2)cos(s_1+s_2) + \frac{m_1m_2}{m_3(m_1+m_2)}\frac{\partial V}{\partial s_1} = 0 \end{aligned} \tag{9.6}$$

and

$$sin^2(s_2)sin(s_1+s_2)\ddot{s}_2$$

[6]This is always the only relevant physical situation if the system under consideration is supposed to represent our universe. This follows from the fact that the frame build from the background stars and galaxies is an inertial reference frame.

$$+cos(s_1+s_2)\big(sin^2(s_1-2sin^2(s_2)\big)\dot{s}_2^2$$

$$+2sin(s_2)\Big(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Big)\dot{s}_1\dot{s}_2$$

$$+2\dot{\boldsymbol{\lambda}}sin^2(s_2)sin(s_1+s_2)\dot{s}_2$$

$$+sin(s_2)\Big(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Big)\dot{s}_1^2$$

$$+2\dot{\boldsymbol{\lambda}}^2sin(s_2)\Big(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Big)$$

$$-\frac{m_1m_2}{m_3(m_1+m_2)}\frac{\partial V}{\partial s_2}=0 \tag{9.7}$$

If additionally the system is non-expanding, i.e., $\dot{\boldsymbol{\lambda}}=0$, the reduced equations of motion on shape space become as follows

$$sin^2(s_2)sin(s_1+s_2)\ddot{s}_1-3sin^2(s_2)cos(s_1+s_2)\dot{s}_1^2$$

$$+2sin(s_2)\Bigg(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Bigg)\dot{s}_2\dot{s}_1$$

$$+sin(s_1)\Bigg(cos(s_1)sin(s_1+s_2)-sin(s_1)cos(s_1+s_2)\Bigg)\dot{s}_2^2$$

$$+\frac{m_1m_2}{m_3(m_1+m_2)}\frac{\partial V}{\partial s_1}=0 \tag{9.8}$$

and

$$sin^2(s_2)sin(s_1+s_2)\ddot{s}_2$$

$$+cos(s_1+s_2)\big(sin^2(s_1-2sin^2(s_2)\big)\dot{s}_2^2$$

$$+2sin(s_2)\Bigg(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Bigg)\dot{s}_1\dot{s}_2$$

$$+sin(s_2)\Bigg(cos(s_2)sin(s_1+s_2)-sin(s_2)cos(s_1+s_2)\Bigg)\dot{s}_1^2$$

$$-\frac{m_1m_2}{m_3(m_1+m_2)}\frac{\partial V}{\partial s_2}=0 \tag{9.9}$$

Now we have to discuss the structure of potential (9.4) in some more detail. As discussed before, the potential function can be considered as the product of two functions G and f on the absolute configuration space Q, i.e. $V=Gf$. The form of the function f is known from the time of Isaac Newton, and in the special case of our 3 body system, can be read off from (9.4). In contrary to f, the function G has remained unknown to this date. All we know about G is that, it has now a value of about $6.67384(80) \times 10^{-11} m^3.kg^{-1}.s^{-2}$ on and near earth, with a relative uncertainty of 2×10^{-5}, which makes it by far the least precisely known natural constant. As different contradictory results has been achieved so far by different methods of measurement of G at different times, there is no consensus on it's correct value, and the above mentioned value is just the average of results achieved by different methods[39].

Following our discussion on Natural constants in Chapter 1, one candidate function for G is

$$G := \sqrt{I_{cm}} = \sqrt{\sum_{i=1}^{N} m_i \mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid^2} \tag{9.10}$$

which obviously satisfies the requirement of being homogeneous of degree 1 under scale transformations[7]. For the region

$$0 \leq sin(s_1), sin(s_2) < sin(s_1 + s_2)$$

on the 3-body shape space, one has

$$\lambda = \mid \boldsymbol{x}_2 - \boldsymbol{x}_1 \mid$$

and a lift of the shape $s = (s_1, s_2)$ to the absolute configuration space can be given as follows

$$\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} \frac{sin(s_2)cos(s_1)}{sin(s_1+s_2)}\lambda \\ \frac{sin(s_2)sin(s_1)}{sin(s_1+s_2)}\lambda \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{x}_{cm} = \begin{bmatrix} \frac{m_2\lambda}{m_1+m_2+m_3} + \frac{m_3 sin(s_2)cos(s_1)}{(m_1+m_2+m_3)sin(s_1+s_2)}\lambda \\ \frac{m_3 sin(s_2)sin(s_1)}{(m_1+m_2+m_3)sin(s_1+s_2)}\lambda \\ 0 \end{bmatrix}$$

[7]This is however not a realistic candidate as a simple estimation of it's value for our universe would differ from the measured value of G by many orders of magnitude. One can ad hocly divide the proposed function by some appropriate number to make sure it's estimated value is compatible with the measured value, and it's degree of homogeneity remains unchanged.

Now by putting all these back in (9.10), and after some calculations, one gets the following expression for G

$$G = \lambda\Big(\frac{sin^2(s_2)}{sin^2(s_1+s_2)}\big(m_3+m_3^2(1-\frac{2}{M})\big)-2\frac{m_2m_3}{M}\frac{sin(s_2)cos(s_1)}{sin(s_1+s_2)}+m_2(1-\frac{m_2}{M})\Big)^{1/2}$$

and for f

$$f = \lambda^{-1}\Bigg(m_1m_2+m_1m_3\frac{sin(s_1+s_2)}{sin(s_2)}+\frac{m_2m_3}{\frac{sin^2(s_2)}{sin^2(s_1+s_2)}-2\frac{sin(s_2)cos(s_1)}{sin(s_1+s_2)}+1}\Bigg)$$

Finally one can write down the potential function $V = Gf$ of our three body system as follows

$$V = \Bigg(m_1m_2+m_1m_3\frac{sin(s_1+s_2)}{sin(s_2)}+\frac{m_2m_3}{\frac{sin^2(s_2)}{sin^2(s_1+s_2)}-2\frac{sin(s_2)cos(s_1)}{sin(s_1+s_2)}+1}\Bigg)$$

$$\times\Big(\frac{sin^2(s_2)}{sin^2(s_1+s_2)}\big(m_3+m_3^2(1-\frac{2}{M})\big)-2\frac{m_2m_3}{M}\frac{sin(s_2)cos(s_1)}{sin(s_1+s_2)}+m_2(1-\frac{m_2}{M})\Big)^{1/2}$$

which manifests it's scale invariance now explicitly.
By using this function in the two reduced Euler-Lagrange equations (9.8), (9.9) we finally obtain the reduced equations of motion on shape space of the 3 body system.

It is worth to mention in the general case, that even though the potential function V is scale independent [8], it turns out from (9.6),(9.9) that the rate of change of scale leaves it's trace on the shape dynamics, just as the rate of change of system's orientation (angular velocity) does, but contrary to the rate of change of system's position (linear tranlational velocity).

For the most simple case where the system under consideration is neither rotating nor expanding w.r.t. the absolute space, the Lagrangian function becomes

$$L = \frac{1}{2}\frac{m_3(m_1+m_2)sin^2(s_2)}{m_1m_2sin^2(s_1+s_2)}\dot{s}_1^2+\frac{1}{2}\frac{m_3(m_1+m_2)sin^2(s_1)}{m_1m_2sin^2(s_1+s_2)}\dot{s}_2^2+V(s_1s_2) \quad (9.11)$$

[8] So scale transformations are among the symmetry transformation of our mechanical system

Chapter 10

Cosmological consequence of the scale invariant mechanics

In this chapter, we consider two situations that highlight the empirical difference between the original and the modified Newtonian theory.

10.1 Accelerated expansion of the universe

In a N-particle system the following quantity

$$D = \sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2 \dot{\boldsymbol{\lambda}}$$

has been called the dilatational momentum of the system. We have already seen at the end of Section (7.3), that the dilatational momentum of the system in internal units is a constant of motion of the scale invariant theory. This was a consequence of the scale invariance of the kinetic energy of the MNT in internal units, and the scale invariance of the potential energy. Without loss of generality, we choose the absolute Newtonian units of time and length by setting them identical to the internal units at some instant of time, i.e. t_0, and set

$$t_0 = t_{internal} = t_{external}$$

so that both internal and external clocks are moreover synchronized at this instant. One can prove the following statement

$$D_{absolute} = cD_{relational} \tag{10.1}$$

where c is the scale factor of the system at each instant of time

$$c := \frac{\lambda\mid_{now}}{\lambda\mid_{t_0}} \tag{10.2}$$

With this notation $\dot{c} = \dot{\boldsymbol{\lambda}}$ in external units.
In order to prove (10.1), one should remember that the scale velocity of the system for an internal observer is understood as follows

$$\dot{\boldsymbol{\lambda}}_{internal} = \frac{\delta\lambda\mid_{during\ 1\ internal\ second\ in\ internal\ length\ unit(\lambda\mid_{now})}}{\lambda\mid_{now}\ (measured\ in\ internal\ length\ unit)}$$

As $\lambda\mid_{now}$ itself is the internal length unit, obviously the denominator of the last fraction is simply 1. How the numerator of this fraction is to be understood is explained in the discussion after equation (9.5). The dualism assumed(or sought) in this work gives two ways of understating or interpreting $\dot{\boldsymbol{\lambda}}_{internal}$. The Newtonian world view (as explained before), allows us to keep the length of the internal length unit seen from absolute space just during the time interval $[now, now + 1 internal\ second]$ constant, by setting it equal to $\lambda\mid_{now}$ during the whole mentioned time interval. With respect to this momentary length unit, the internal observer then measures $\delta\lambda$. Now as the internal observer cannot communicate with the external observer to perform this measurement [1], a new internal length unit built from a virialized subsystem (a stable non-expanding non-contracting subsystems seen from absolute space)can be used by the internal observer just for the purpose of measuring $\delta\lambda$. Both previous methods lead to the same numerical value for $\dot{\boldsymbol{\lambda}}_{internal}$. Now if one insists on the Leibnizian world view, one can view $\dot{\boldsymbol{\lambda}}_{internal}$ just as a variable appearing in the law of motion on shape space. There may (or may not) be some deeper reasoning behind it's value and dynamics from the shape space(Leibnizian) point of view, about which we don't speculate in this work.

[1] to have access to the momentary constant length unit

Analogously, the scale velocity of the system for the external observer is understood as follows

$$\dot{\boldsymbol{\lambda}}_{external} = \frac{\delta\lambda \mid_{during\ 1\ external\ second\ in\ external\ length\ unit(\lambda|_{t_0})}}{\lambda \mid_{now} (measured\ in\ external\ length\ unit)}$$

As at any moment of time[2]

$$external\ second = \frac{1}{c} \times internal\ second$$

and

$$external\ length\ unit(\lambda \mid_{t_0}) = \frac{1}{c} \times internal\ length\ unit(\lambda \mid_{now})$$

one immediately recognizes that the numerical value of the numerator of $\dot{\boldsymbol{\lambda}}$ for both observers is equal, but the numerical value of the denominator for external observer is c times bigger than the internal observer. Hence

$$\dot{\boldsymbol{\lambda}}_{external} = \frac{1}{c}\dot{\boldsymbol{\lambda}}_{internal} \tag{10.3}$$

which by using (10.2) can be rewritten as

$$\dot{\lambda} = \lambda \mid_{t_0} \dot{\boldsymbol{\lambda}}_{internal}$$

Equation (10.3) together with $\boldsymbol{r}_{external} = c\boldsymbol{r}_{internal}$ to calculate the dilatational momentum $D_{external}$ as follows

$$D_{external} = \sum_{j=1}^{N-1} \mid \boldsymbol{r}_{j,external} \mid^2 .\dot{\boldsymbol{\lambda}}_{external}$$

$$= \sum_{j=1}^{N-1} \mid c\boldsymbol{r}_{j,internal} \mid^2 .\frac{1}{c}\dot{\boldsymbol{\lambda}}_{internal}$$

$$= c\sum_{j=1}^{N-1} \mid \boldsymbol{r}_{j,internal} \mid^2 .\dot{\boldsymbol{\lambda}}_{internal}$$

$$= cD_{internal}$$

and this completes the proof of the equation (10.1).

[2]for $c > 1$ internal clock is running slower than internal clock because $T_p \to T_p' = cT_p$ as discussed in Chapter 1.

Knowing the value of dilatational momentum in internal units $D_{internal} = D$ is a constant of motion (as discussed at the end of Section 7.2), one can derive the time evolution of scale variable in external units. To this end, remember the following expression of dilatational momentum for external observer, which is written down purely "in terms of external units of time and length" (which are identical to the respective internal units at time t_0)

$$D_{external} = \sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2 \dot{\boldsymbol{\lambda}}_{external} = \sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2 \frac{\dot{\lambda}}{\lambda}$$

after solving for $\dot{\lambda}$ one has

$$\dot{\lambda} = \frac{\lambda D_{external}}{\sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2} = \frac{\lambda c D}{\sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2} = \frac{\lambda \frac{\lambda}{\lambda |_{t_0}} D}{\sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2} = \frac{\lambda^2 D}{\lambda \mid_{t_0} \sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2}$$

This expression would be greatly simplified if one defines the scale variable λ as

$$\lambda := \sqrt{\sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2}$$

which also has a clear intuitive justification. Putting this back in the previous equation one ends up with

$$\dot{\lambda} = \frac{D}{\lambda \mid_{t_0}} = D \tag{10.4}$$

as both D and $\lambda \mid_{t_0}$ are constants, this means that the external observer would see the scale variable (roughly size of the system with the above choice for λ) changing with a constant speed D (of course measured with respect to external units of time and length). One subsequently has

$$\dot{\boldsymbol{\lambda}}_{external} = \frac{\dot{\lambda}}{\lambda} = \frac{D}{\lambda\lambda \mid_{t_0}} = \frac{D}{\lambda} \tag{10.5}$$

From (10.4) and (10.2) one sees that the scale factor c is a linear function of time for the external observer

$$c = \frac{\lambda \mid_{now}}{\lambda \mid_{t_0}} = \frac{\dot{\lambda}(t_{external} - t_0) + \lambda \mid_{t_0}}{\lambda \mid_{t_0}} = \frac{\frac{D}{\lambda|_{t_0}}(t_{external} - t_0) + \lambda \mid_{t_0}}{\lambda \mid_{t_0}}$$

$$= \frac{D}{\lambda^2 \mid_{t_0}} t_{external} - \frac{D}{\lambda^2 \mid_{t_0}} t_0 + 1 = \frac{D}{\lambda^2 \mid_{t_0}} t_{external} + 1 = D t_{external} + 1$$

where in the last two equalities we have set t_0 being the initial time i.e. $t_0 = 0$, and have chosen $\lambda \mid_{t_0}$ as the absolute unit of length respectively.
From (10.4) it also becomes evident that in a contracting universe (where $D < 0$) the external observer would see after a finite amount of time $\frac{\lambda|_{now}}{D}$ measured by his external clock, the scale variable $\lambda = \sqrt{\sum_{j=1}^{N-1} \mid \boldsymbol{r}_j \mid^2}$ becoming zero. How would an internal observer experience this?
As *internal second*= $c\times$ *external second* for the case when $c < 1$ the internal clock speeds up compared to the external clock. The duration $\frac{\lambda|_{now}}{D}$ of external time, which was needed for the scale variable (or the scale factor) to reach the value zero from $\lambda \mid_{now}$ (or $c \mid_{now}$), would be measured by the internal clock to be

$$\int_{\lambda=\lambda|_{now}}^{\lambda=0} \frac{\frac{\lambda|_{now}}{D} s_{ex}}{s_{in}} d\lambda = \frac{\lambda \mid_{now}}{D} \int_{\lambda=\lambda|_{now}}^{\lambda=0} \frac{s_{ex}}{s_{in}} d\lambda$$

$$= \frac{\lambda \mid_{now}}{D} \int_{c=c|_{now}}^{c=0} \frac{s_{ex}}{s_{in}} dc = \frac{\lambda \mid_{now}}{D} \int_{c=c|_{now}}^{c=0} \frac{1}{c} dc = \infty$$

So for the internal observer it would take for ever to see what the external observer would see in just $\frac{\lambda|_{now}}{D}$ external seconds s_{ex}. This means, **in a contracting universe big crunch would never be seen in any finite time by the inhabitants of that universe.**

Now we wish to consider an expanding universe, i.e. $D > 0$. From (10.4) it becomes evident that the external observer sees the scale of the universe increases for ever with a constant rate $\dot{\lambda} = D$. How would an internal observer experience this?
Using (10.3) and (10.5) one can calculate the rate of change of scale variable for internal observer [3]

$$\dot{\boldsymbol{\lambda}}_{internal} = c\dot{\boldsymbol{\lambda}}_{external} = c\frac{D}{\lambda} = D$$

As already discussed once, $\dot{\boldsymbol{\lambda}}_{internal}$ means if the internal observer chooses momentarily a new unit of length, for instance let the size of a virialized subsystem be the new internal length unit (e.g. astronomical unit: au), then the amount of change of universe's scale variable (measured in au) during one internal second,

[3] note that the bold and normal version of scale velocity are identical for internal observer if the system's scale variable λ is chosen as internal unit of length.

divided by length of the universes scale variable (measured again in au) is D. So

$$\frac{d\lambda(in\ au)}{dt_{internal}}/\lambda(in\ au) = D \Rightarrow \frac{d\lambda(in\ au)}{dt_{internal}} = D \times \lambda(in\ au)$$

$$\Rightarrow \dot{\boldsymbol{\lambda}}_{internal}(in\frac{au}{s_{internal}}) = D \times \lambda(in\ au)$$

This means that **the internal observer sees an accelerated expanding universe**.

10.2 Increase in strength of gravity in regions far from matter concentration

Consider a three body system with two heavy masses $m_1 = m_2 = M$ located near each other at positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and the third body with a light mass $m_3 = m$ located at position $\boldsymbol{x}_3$. Let the origin of the spatial coordinate system coincide with the systems center of mass. As before (9.10) the so called gravitational constant for this three body universe can be calculated as follows

$$G = \sqrt{M(\mid \boldsymbol{x}_1 \mid^2 + \mid \boldsymbol{x}_2 \mid^2) + m \mid \boldsymbol{x}_3 \mid^2} \tag{10.6}$$

The gravitational potential of the third (light) body is given as usual by $V_3 = Gf_3$

$$V_3 = \sqrt{M(\mid \boldsymbol{x}_1 \mid^2 + \mid \boldsymbol{x}_2 \mid^2) + m \mid \boldsymbol{x}_3 \mid^2} \times \left(\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid} + \frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid} \right)$$

Hence, the total gravitational force exerted on the third body becomes

$$\boldsymbol{F}_3 = -\nabla_3 V_3 = -\nabla_3 (G f_3) = -G \nabla_3 f_3 - f_3 \nabla_3 G$$

$$= \boldsymbol{F}_{Ne} + \delta \boldsymbol{F}$$

where Ne stands for Newton, and ∇_3 is gradient w.r.t. the coordinates of the third particle. A short calculation then leads to the following results

$$\boldsymbol{F}_{Ne} = \sqrt{M(\mid \boldsymbol{x}_1 \mid^2 + \mid \boldsymbol{x}_2 \mid^2) + m \mid \boldsymbol{x}_3 \mid^2} \times \left(\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid^2} \hat{\boldsymbol{x}}_{31} + \frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid^2} \hat{\boldsymbol{x}}_{32} \right)$$

and

$$\delta \boldsymbol{F} = \frac{-m\boldsymbol{x}_3}{\sqrt{M(\mid \boldsymbol{x}_2 \mid^2 + \mid \boldsymbol{x}_1 \mid^2) + m \mid \boldsymbol{x}_3 \mid^2}} \times \left(\frac{mM}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid} + \frac{mM}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid} \right)$$

Now consider the case where $\frac{m}{M} << 1$, hence the systems center of mass is located in the middle of bodies 1 and 2. Furthermore choose the axis of spatial frame such that

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

with $y >> 1$. Then the exerted force on the third body approximately becomes the sum of the following two terms

$$\boldsymbol{F}_{Ne} \sim -\sqrt{2M + my^2} \frac{2mM}{y^2} \sim \frac{1}{y}$$

$$\delta \boldsymbol{F} \sim -\frac{my}{\sqrt{2M + my^2}} \left(\frac{2mM}{\sqrt{1 + y^2}} \right)$$

which points out an increase in the strength of the gravitational pull felt by the third particle compared to the Newtonian gravitational force[4]. This indicates that the gravitational force experienced by a test particle positioned in regions of space far from where most of the matter is concentrated, increases compared with the Newtonian gravitational force.

Moreover one sees that the $\frac{1}{y}$ decay of the gravitational force would make it possible that the third particle orbits the other two heavy masses with a constant velocity independent of y. This could at first sight mimic the flat rotation curves observed in the galaxies. So there seems to be some indications which could liberate us from invoking the existence of the (otherwise unobservable) *dark matter*, at least when it comes to the explanation of the flat rotation curves of the galaxies. There are however two caveats in the preceding argumentation, which seems to prevent us explaining the flat rotation curves by the modified theory presented in this paper.

First caveat is that in the above analysis we acted as there exists only one galaxy in the universe (whose behavior we approximated by a 3-body system). In reality

[4]Which decays as $\frac{1}{y^2}$

however, the considered 3-body system is located in a homogeneous and isotropic background distribution of galaxies. To account for this, we split G into a local and a global part, i.e.

$$G = G_{local} + G_{global}$$

where as before

$$G_{local} = \sqrt{M(\mid \boldsymbol{x}_1 \mid^2 + \mid \boldsymbol{x}_2 \mid^2) + m \mid \boldsymbol{x}_3 \mid^2}$$

and

$$G_{global} = \sqrt{\sum_{i=4}^{N} m_i \mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid^2}$$

Here the first three particles model, for instance, our milkyway galaxy, and the rest of the particles model all the other galaxies which are located at far larger distances compared to the interparticle distances of the first three particles. As before $\boldsymbol{x}_{cm}$ denotes the center of mass of the first three particles, which is located almost in the middle of the particles 1 and 2, as we assumed $M >> m$. In a more or less fixed homogeneous and isotropic background for this 3 body system, the gravitational forces acted on the third particle originating from the global (background) structure of the universe, averages out to zero, and as the value of G_{global} is a constant number, it doesn't change the functional form of the potential (or the force) experienced by the particle 3 due to the local structure around it (so due to particles 1 and 2). In other words, for the potential energy of the third particle one can write

$$V_3 = (G_{local} + G_{global}) \times (\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid} + \frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid} + \sum_{i=4}^{N} \frac{m_i m_3}{\mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid})$$

where as the location of all other galaxies are far greater than the inter-particle distances of our galaxy, we have used the approximation

$$\forall i > 3 : \mid \boldsymbol{x}_i - \boldsymbol{x}_3 \mid \approx \mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid$$

Hence for the force on the third particle one gets

$$\boldsymbol{F}_3 = -\nabla_3 V_3 = -G\nabla_3 (\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid} + \frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid} + \sum_{i=4}^{N} \frac{m_i m_3}{\mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid})$$

$$-(\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid} + \frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid} + \sum_{i=4}^{N} \frac{m_i m_3}{\mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid})\nabla_3(G)$$

$$= -G\nabla_3(\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid}+\frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid})-(\frac{m_1 m_3}{\mid \boldsymbol{x}_1 - \boldsymbol{x}_3 \mid}+\frac{m_2 m_3}{\mid \boldsymbol{x}_2 - \boldsymbol{x}_3 \mid}+\sum_{i=4}^{N} \frac{m_i m_3}{\mid \boldsymbol{x}_i - \boldsymbol{x}_{cm} \mid})\nabla_3(G_{local})$$

$$= \boldsymbol{F}_{Ne} + \delta \boldsymbol{F}$$

As the dominant part of G comes from G_{global} which is more or less a constant number for the scales of the 3-body system, one sees that $\boldsymbol{F}_{Ne}$ would not anymore behave as $\frac{1}{y}$. Because of this the possibility of finding an explanation of the flat rotation curves along these lines is less probable.

The second caveat lies behind the form of function (10.6), or (9.10) for G. In fact all we know from the true function G is that, because of the principle of relationalism it must be a homogeneous function of degree 1 in inter-particle distances. For instance $G = \sum_{i=1}^{N} \mid \boldsymbol{x}_i \mid$ is a simpler function that satisfies this requirement as well. However as the current value of G measured on Earth is of the order of 10^{-11} one should adhocly multiply the expression of G by a very small factor to make the G-function compatible with the experiments. The true function of G needs yet to be found and explained probably from some deeper theoretical reasoning and mechanism.

Chapter 11

Comparison with two other approaches in relational physics

Last but not least, to clarify the physical aspects of our work, we give a short comparison with some of the other main approaches in the literature. The notations in this section slightly differ from the rest of this paper and will be mentioned every time.

One of the established approaches in relational physics (see, for instance, [11] and the references in it) denoted here by the BKM-approach uses a property of the Newtonian mechanics known as mechanical similarity. This property says that if

$$x(t) = \big(\boldsymbol{x}_1(t), ..., \boldsymbol{x}_N(t)\big)$$

is a solution to the Newtonian N-body problem with a homogeneous potential function V of degree k, i.e. $x(t)$ satisfies

$$\frac{d^2x(t)}{dt^2} = \nabla V \mid_{x(t)}$$

then

$$x'(t') = \big(c\boldsymbol{x}_1(t'), ..., c\boldsymbol{x}_N(t')\big)$$

is also a solution of the theory with

$$t' = tc^{1-\frac{k}{2}} \tag{11.1}$$

i.e., $x'(t')$ satisfies

$$\frac{d^2x'(t')}{dt'^2} = \nabla V \mid_{x'(t')}$$

For a Newtonian N-body system, denote $\boldsymbol{x}_a$, $\boldsymbol{p}^a$, and m_a for the position, momentum, and the mass of the particle a. Systems with vanishing total energy are considered, where the energy is the well-known expression

$$E_{tot} = \sum_{a=1}^{N} \frac{\boldsymbol{p}^a.\boldsymbol{p}^a}{2m_a} + V_{New} \tag{11.2}$$

$$V_{New} = -\sum_{a<b} \frac{m_a m_b}{r_{ab}} \tag{11.3}$$

with $r_{ab} :=\parallel \boldsymbol{x}_a - \boldsymbol{x}_b \parallel$. So the Newton's gravitational coupling G is considered to be the constant 1 in the BKM-approach. It is then explained as the dilational momentum D of a Newtonian gravitational system is a monotonic function (along its solution curves); it can be used as the system's time variable (instead of the absolute Newtonian time). The transformed Hamiltonian of the Newtonian system in the new coordinates and the new time variable is shown to be

$$H(D) = ln\big(\sum_{a=1}^{N} \boldsymbol{\pi}^a.\boldsymbol{\pi}^a + D^2\big) - ln\big(I_{cm}^{\frac{1}{2}} \mid V_{New} \mid\big) \tag{11.4}$$

where $\boldsymbol{\pi}^a$ denote here the shape momenta, defined as

$$\boldsymbol{\pi}^a := \sqrt{\frac{I_{cm}}{m_a}} - D\boldsymbol{\sigma}_a \tag{11.5}$$

and $\boldsymbol{\sigma}_a$ is the following choice for the pre-shape coordinates

$$\boldsymbol{\sigma}_a := \sqrt{\frac{m_a}{I_{cm}}}\boldsymbol{r}_a^{cm}$$

coordinatizing pre-shape space PS, the quotient of configuration space by global translations and scale transformations. By restricting themselves to systems with vanishing angular momentum, the authors of [11] have left quotienting with respect to the group of rotations out of consideration.

After defining some new shape momenta as follows

$$\boldsymbol{\omega}^a := \frac{\boldsymbol{\pi}^a}{D}$$

from their previous expressions (11.5), and introducing a new time variable

$$\lambda := log(D)$$

the Hamiltonian (11.4) is cast into the following apparently time independent form

$$H_0 = log\big(\sum_{a=1}^{N} \boldsymbol{\omega}^a.\boldsymbol{\omega}^a + 1\big) - logC_s \tag{11.6}$$

with

$$C_s = \frac{\sqrt{I_{cm}}}{m_{tot}^3} \mid V_{New} \mid \tag{11.7}$$

called the complexity. The Hamiltonian H_0 leads to the following equations of motion for the (pre)shape coordinates and their momenta

$$\frac{d\boldsymbol{\sigma}}{d\lambda} = \frac{\partial H_0}{\partial \boldsymbol{\omega}^a} \tag{11.8}$$

$$\frac{d\boldsymbol{\omega}^a}{d\lambda} = -\frac{\partial H_0}{\partial \boldsymbol{\sigma}_a} - \boldsymbol{\omega}^a \tag{11.9}$$

the second of which shows dissipative dynamics for the (pre)shape momenta $\boldsymbol{\omega}^a$. Here are some comments on this approach for the purpose of comparison.
First of all, as the starting theory for the description of the N-body system in the absolute phase space $T^*(Q)$ given by the Hamiltonian (11.2),(11.3) is clearly not scale invariant, one should at first glance not even hope to find some law of motion on shape space of this system. In other words, two solutions, $O_t\,|1\rangle$, and $O_t\,|2\rangle$ emerging from the following two initial absolute states of the system

$$|1\rangle = (\boldsymbol{x}_1, ..., \boldsymbol{x}_N, \boldsymbol{p}_1, ..., \boldsymbol{p}_N)$$

and

$$|2\rangle = (c\boldsymbol{x}_1, ..., c\boldsymbol{x}_N, \boldsymbol{p}_1, ..., \boldsymbol{p}_N)$$

project down to two different curves on shape space. At this point, Barbour et al. (see [11],[12]) came up with a clever idea to transform the momenta $\boldsymbol{p}_a$ under the spatial scale transformations in such a way to force the second orbit to describe the same path on shape space as the first orbit, i.e.

$$\forall t : \pi(O_t\,|1\rangle) = \pi\big(O_{t'}\,|2\rangle\,\big)$$

where as usual $\pi : Q \to S = \frac{Q}{Sim(3)}$ is the fiber-bundle's projection map. To this end, dynamical similarities in Newtonian mechanics have been invoked to find the required transformation law for the momenta. As the potential function (11.3) considered here is homogeneous of degree $k = -1$, the new time variable after performance of a scale transformation by the factor c becomes

$$t' = c^{3/2} t \tag{11.10}$$

So for $c > 1$, the initial velocities[1] should slow down by a factor $c^{-3/2}$. Hence, the correct transformed state must be

$$|2\rangle' = (c\boldsymbol{x}_1, ..., c\boldsymbol{x}_N, c^{-3/2}\boldsymbol{p}_1, ..., c^{-3/2}\boldsymbol{p}_N) \tag{11.11}$$

Therefore, according to this approach, the group of scale transformations Sc acts on the phase space as follows

$$\begin{pmatrix} \boldsymbol{x}_1 \\ . \\ . \\ . \\ \boldsymbol{x}_N \\ \boldsymbol{p}_1 \\ . \\ . \\ . \\ \boldsymbol{p}_N \end{pmatrix} \xrightarrow{\text{Sc}} \begin{pmatrix} c\boldsymbol{x}_1 \\ . \\ . \\ . \\ c\boldsymbol{x}_N \\ c^{-3/2}\boldsymbol{p}_1 \\ . \\ . \\ . \\ c^{-3/2}\boldsymbol{p}_N \end{pmatrix} \tag{11.12}$$

This transformation means, by the way, that from the Newtonian absolute perspective[2] a larger universe by a factor c runs slower by a factor $c^{-3/2}$. Now the orbits of the theory (11.2), (11.3) emerging out of the initial states $|1\rangle$ and its scale transformed version $|2\rangle'$, indeed project down to the same curve on S as sought. However, does this mean that the inhabitants(observers) of the two alternative Newtonian N-body universes will not be able to tell whether they are located along the orbit $O_t\,|1\rangle$ or $O_t\,|2\rangle'$?

[1]expressed with respect to the absolute units of duration and length.
[2]with the usage of the absolute immaterial rods and clocks.

A first test that may lead us to the answer is the investigation of the observed velocities(so in relational length and ephemeris time units) of particles or subsystems under dynamical similarity transformations. The observers of these Newtonian universes, which are basically some subsystems, have only access to some internal rods and clocks. As the rods are built from matter, they will change their size by a factor of c after the system undergoes a dynamical similarity transformation. It is well-known [7] that the ephemeris time defined as

$$\delta t_e := \frac{\sqrt{\sum_{i=1}^{N} m_i \delta \boldsymbol{x}_i . \delta \boldsymbol{x}_i}}{\sqrt{2(E-V)}} \tag{11.13}$$

for a N-body universe with total energy E, mimics perfectly the flow of the absolute time (provides the most accurate internal clock[3]) in Newtonian mechanics. It makes the relation between the absolute motions in space and the increment of Newton's absolute time evident. In other words, it reveals how the seconds of Newton's absolute time is related to the absolute displacements(w.r.t. the immaterial absolute unit of length) of all the particles in the whole universe. However, this perfect matching between the absolute time, and the ephemeris time of the universe, gets destroyed (distorted) by a dynamical similarity transformation. It is because the system's(universe's) total energy is not an invariant of the mentioned transformation. So, the ephemeris time of the new universe achieved by a dynamical similarity transformation would no longer coincide with the absolute time of Newton(which is, according to Newton, unaffected by whatever transformation you are making on the material universe[4]). Because of this, one may get an impression that a violation of the principle of relationalism is facing the BKM-approach. In the following part, we explain why this is not the case.

If one wants to be realistic(relational) about spatial displacements $\delta \boldsymbol{x}_i$'s appearing in (11.13) one has to choose the distance between the i'th and j'th particle (for some $1 < i, j < N$) as unit of length. However, not all choices of i and j are good here. Careless choice of i and j would lead to a violation of energy

[3]The degree of accuracy of an internal time variable can be tested by the degree of accuracy of Newton's second law for the chosen internal time variable. In Newtonian mechanics, bad choices of internal time will lead to the appearance of fictitious forces not originating from the gradient of the interaction potential V considered in theory.

[4]"Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external"[40].

conservation because of the appearance of fictitious forces. One careless choice, for example, would be two particles that (in the absolute space) are moving accelerated towards or away from each other. On the other hand, if the mass m_i is much larger than the mass m_j, and the particle j is moving on a circular orbit (in absolute space) around the particle i, then these i and j particles constitute a good choice for the length unit. An example of such a good choice of length unit for our universe is the well-known astronomical unit au. Analogously, if one wants to be realistic about time durations, one has to choose also a good unit of time. In the previous example, the period of the lighter particle j around the much heavier particle i would constitute such a good time unit which we denote by su [5]. Note also, by changing the unit of length or time, the unit of energy will also get changed, and with that, the denominator of (11.13) which is the square root of the system's kinetic energy.

Imagine we have a Newtonian N-particle universe located in absolute space and changing its location with absolute time. Furthermore, imagine this universe finds itself in the following absolute state

$$|1\rangle = (\boldsymbol{x}_1, ..., \boldsymbol{x}_N, \boldsymbol{p}_1, ..., \boldsymbol{p}_N)$$

Also, imagine that the state is such that a good choice of length and time unit (e.g., au, su) can be made. Define the absolute length unit (Alu) of the absolute space and the absolute time unit (Atu) of the absolute time to be the au and su of this universe at state $| 1 >$, i.e.,

$$Alu := au \,|_{|1\rangle}$$

$$Atu := su \,|_{|1\rangle}$$

The formula (11.13) then tells you how the absolute time is marching forward. Performance of a dynamical similarity transformation brings the universe from state $|1\rangle$ to state $|2\rangle'$ given in (11.11). It is important to remember that the units in which $|2\rangle'$ is expressed are the absolute units. Under a dynamical similarity transformation, the ephemeris time (11.13) transforms as

$$\delta t'_e = c^{3/2}\delta t_e = c^{3/2}\delta t^{(\boldsymbol{n})} \tag{11.14}$$

[5] abbreviation for the solar year unit

It is because the kinetic energy

$$E - V = \sum_{a=1}^{N} \frac{\boldsymbol{p}_a^2}{2m_a}$$

changes by a factor of c^{-3} under the dynamical similarity transformations, as can be seen, form (11.12). So the transformation of the ephemeris time of the two systems related to each other by a dynamical similarity transformation is compatible with the prescription (11.10). How would the new ephemeris time behave if one uses the relational length and time units of the new universe, i.e., $au \mid_{|2\rangle'}$, $su \mid_{|2\rangle'}$ instead of the old universe/absolute units (which we do not have direct access to anymore)? As the new relational units are related to the old relational units/absolute units by the following qualities

$$au \mid_{|2\rangle'} = c.au \mid_{|1\rangle} = c.Alu$$

$$su \mid_{|2\rangle'} = c^{\frac{3}{2}}.su \mid_{|1\rangle} = c^{\frac{3}{2}}.Atu$$

the new kinetic energy in new relational units becomes

$$K' = c^{-3}K[\frac{kg.Alu^2}{Atu^2}] = c^{-3}K[\frac{kg.(c^{-1}au \mid_{|2\rangle'})^2}{(c^{-3/2}su \mid_{|2\rangle'})^2}] = c^{-2}K[\frac{kg.au^2 \mid_{|2\rangle'}}{su^2 \mid_{|2\rangle'}}]$$

So the denominator of (11.13) changes by a factor c^{-1}. As the numerical value for particle displacements $\delta\boldsymbol{x}_i$ in absolute space measured w.r.t. the new length unit, changes with the factor c^{-1}, the numerator of (11.13) changes also with a factor of c^{-1}. All these together mean that the new ephemeris time (after performing a dynamical similarity transformation of the universe) seen in new relational units would coincide with the absolute time (the old ephemeris time variable). So there is neither a kinematical nor a dynamical violation of the principle of relationalism to be expected in the BKM-approach, and the orbits (Newtonian universes) emanating from $|1\rangle$ and $|2\rangle'$ are for internal observers fully indistinguishable. This point is a very remarkable and unexpected feature of the original Newtonian Mechanics exploited by Barbour and his collaborators in [11],[12],[13],[14] and the references in them.

Another approach is based on the simplest kind of dynamics on shape space, i.e., the geodesic evolution on S, considered first in [15] and further expanded in [16]. In this approach, the gravitational potential function is replaced with a homogeneous function of degree -2, e.g.

$$V = I_{cm}^{-\frac{1}{2}} V_{New} \tag{11.15}$$

This approach partly relies on the following (presumable) transformation of velocities under scale transformations:

$$v' = cv \tag{11.16}$$

It leads in turn to the following behavior of the norm of a configuration's velocity v under the global scale transformations

$$\|v'\|^2 = \mathbf{M}_{cx}(v', v') = \mathbf{M}_{cx}(cv, cv) = c^2\mathbf{M}_{cx}(v, v) = c^2\mathbf{M}_x(v, v) = c^2\|v\|^2 \tag{11.17}$$

where $\mathbf{M}$ as usual denotes the mass metric on the absolute configuration space Q. Hence, the multiplication of $\|v'\|$ with a homogeneous function of degree -2 on Q , e.g. (11.15) makes the integrand of the (Jacobi's) action $\bar{S} = \int \sqrt{-V} dl$ [6] scale-invariant. Subsequently, reinterpreting the potential function (11.15) as a conformal factor, a new (similarity invariant) metric $\mathbf{M}' = V\mathbf{M}$ on Q can be defined, whose geodesics coincide with the evolution of Newtonian systems with the potential (11.15) on $(Q, \mathbf{M})$. It leads to a geodesic law of motion on shape space.

We emphasize here that in (11.17), a specific transformation law for the velocities is assumed. In other words, it is assumed that the group of scale transformations

[6] $dl^2 = \mathbf{M}_{ij} dx^i dx^j$

acts on the absolute phase space as follows

$$\begin{pmatrix} \boldsymbol{x}_1 \\ . \\ . \\ . \\ \boldsymbol{x}_N \\ \boldsymbol{p}_1 \\ . \\ . \\ . \\ \boldsymbol{p}_N \end{pmatrix} \xrightarrow{\text{Sc}} \begin{pmatrix} c\boldsymbol{x}_1 \\ . \\ . \\ . \\ c\boldsymbol{x}_N \\ c\boldsymbol{p}_1 \\ . \\ . \\ . \\ c\boldsymbol{p}_N \end{pmatrix} \tag{11.18}$$

In other words, the action of Sc on $T(Q)$ is considered to be by push forward Sc_*. This transformation is both different from the BKM-approach mentioned above, where the velocities(or momenta) w.r.t. the absolute Newtonian units of time and length, get scaled by a factor $c^{-\frac{3}{2}}$ (see (11.11)), and also from our work(see (5.6)). The fact that the DGZ velocity transformation (11.16), is mathematically compatible with (or follows from) the push forward of vectors under scale transformations in configuration space, i.e.,

$$Sc : Q \to Q$$

$$Sc_* : T(Q) \to T(Q)$$

$$v' = Sc_* v$$

does not give this transformation a physically natural or privileged place over other possible transformations of velocities previously considered. We believe that a lack of attention to the *physical origin of the notion of velocity* can lead to confusion here. In physics, velocity is a derived notion[7], and it immediately depends on the way we measure space and time intervals, without which one cannot talk about velocities. So, any physically serious statement about the transformation of velocities under some transformation of the universe's configuration must

[7]Derived from the more primitive notions of space and time. Considering time itself is a derived notion, as a relationalist would, velocity is merely a concept derived from space in a non-empty universe.

therefore include a discussion of the changes in the relevant measurement tools, such as that presented in this thesis for the modified Newtonian theory. The arbitrariness in the metric of the shape space in the DGZ-approach[16], caused by the arbitrariness in the choice of a conformal factor, is in our opinion due to the forgotten connection of length measures with real rulers. As explained at the end of the Section (IV.ii), the measured mass metric (4.16) is on its own scale-invariant. A conformal factor would be required if we had access to absolute rulers and could thus measure absolute lengths. Since all rules are themselves subsystems of the universe, they are also subject to the transformations applied to the universe. Taking this physical fact into account resolves the mentioned arbitrariness and provides us with the unique metric $\mathbf{N}$ in shape space.

Note also that (11.18) differs from the mechanical similarity transformations of an absolute theory with a homogeneous potential function of degree -2, according to which the momenta transform as $\boldsymbol{p}_i \to c^{-2}\boldsymbol{p}_i$. This discrepancy does not pose a problem in the DGZ- approach, since in the relational world-view the rate (with respect to the absolute time) at which the system's actual shape moves along a *given path* in shape space is a gauge freedom. This is because if the actual shape of the universe traverses a given curve (a geodesic in the BDGZ-approach) on shape space at different speeds (with respect to absolute time), no objective (relational) change can be observed by the inhabitants of that universe. On the other hand, if one takes an absolute physical theory (like the Newtonian or the modified Newtonian theory) as the starting point for finding the the relational laws of motion, a change in the absolute velocities (by some factor) generically has dynamical effects on the shape space, i.e., leads to alternative universes moving along *different paths* on shape space. As mentioned before, mechanical similarities provide a way to prevent this problem, by introducing a suitable action of $Sim(3)$ on the theory's absolute state space. It is worth emphasizing that the action of scale transformations on phase space, as used in our work, is a direct consequence of the way we implemented the Principle of Relationalism in (absolute) modified Newtonian theory, which turns out also to be compatible with the mechanical similarities of this theory. It is in particular, neither an additional postulate nor a gauge fixing condition.

Another difference worth mentioning is the potential function (11.15) used in the BDGZ-approach, being a homogeneous function of degree -2, which clearly differs both from the BKM-approach (11.3), and our work, where they are homogeneous functions of degree -1, and 0 (hence scale-invariant) respectively. Our scale invariant potential function can be incorporated into the geodesic DGZ-approach by choosing a new conformal factor $f' := Vf$, which is the multiplication of the conformal factors introduced in [16] by our scale-invariant potential. Nevertheless, the presence of a potential function of the type (11.15), which keeps the system's moment of inertia constant, is a characteristic of this approach and is absent in the other two approaches. Whether a specific gauge can be found in which the DGZ-approach coincides with the BKM-approach or with our work remains an open question of the DGZ-approach.

We think that the mathematical definition of the notion of scale-invariance in Riemannian geometry (see 4.16) is less relevant from the physical point of view. This is due to the use of the differential of the scale transformation Sc as the action of Sc on $T(Q)$ and its decoupling from the physical theory. Here we define a new notion that is more relevant to physics. A metric $\mathbf{G}$ on the configuration space Q is called *mechanical similarity invariant* if and only if

$$\forall v_1, v_2 \in T_q(Q), \mathbf{G}_q(v_1, v_2) = \mathbf{G}_{cq}(c^{\frac{k}{2}} v_1, c^{\frac{k}{2}} v_2) \tag{11.19}$$

where k is the degree of homogeneity of the potential function of the physical theory. The factor $c^{\frac{k}{2}}$ results from the combined effect of the time transformation required by the theory's mechanical similarity (11.1)[8], and ruler's expansion. Thus, instead of defining the action of Sc on $T(Q)$ by push forward Sc_*, we define the action of Sc by the mechanical similarity transformation on $T(Q)$. The mass metric $\mathbf{M}$ is mechanical similarity invariant for the modified Newtonian theory, but not for the original Newtonian theory. As explained in this section, a mechanical similarity invariant metric such as $\mathbf{M}$ for the modified Newtonian theory defines a unique metric on shape space. For the modified Newtonian theory, this unique metric is the same as that arising from the scale-invariant measured mass metric $\mathbf{M}^{(m)}$(see 4.16) at the end of Section (4.1).

[8]This type of time transformation was required for the dynamical equivalence of the two alternative universes governed by the absolute theory under consideration.

Chapter 12

Summary

In this thesis we have reviewed the method of symplectic reduction of a dynamical system with respect to a symmetry group and its application in the reduction of classical mechanics with respect to the Euclidean group $E(3)$.
We introduced a new physical principle, namely the *principle of relationalism*, as a guideline for the implementation of relational ideas in modern physics. This principle enabled us to make classical physics scale invariant, which in turn ensured the existence of laws of motion on shape space. Thus, it enabled us to achieve a complete relational reading of classical physics, which was hidden behind some foundational incompletenesses of classical theories, e.g., the lack of any derivation for the so-called *constants of nature* within the theory.
We then performed the symplectic reduction of the new scale-invariant theory on the absolute configuration space with respect to the similarity group $Sim(3)$ and derived the reduced symplectic form, the metric and the Hamiltonian on the system's shape space S. More precisely, we gave the dynamics of an N-particle system on its internal configuration space $Q_{int} = \frac{Q}{E(3)}$ (as derived in [3] or reviewed in Section 3) and derived its reduced dynamics with respect to the group of scale transformations using the principle of relationalism. The appearance of two new forces on the reduced phase space was remarkable.

In the remaining parts of the thesis we have worked with the Lagrangian formalism. We first reviewed the derivation of the Euler-Lagrange equations of motion in nonholonomic frames and the reduced equations of motion on the internal configuration space $Q_{int} = \frac{Q}{E(3)}$ of classical mechanics. Then, using the Principle of Relationalism, we extended the discussed methods to the whole similarity group $Sim(3)$. In particular, we constructed representations of the group $Sim(3)$ and its Lie-algebra $\mathbf{sim}(3)$ on Q, discussed how a vector on shape space can be lifted horizontally to the center of mass configuration space Q_{cm}, constructed the connection form ω_s for the action of the group of scale transformations Sc, and showed that this connection form is flat. As a consequence of the latter, quotienting out the configuration space w.r.t. the scale transformations Sc would not produce any additional curvature in the resulting base space. Thus, the curvature in the shape space is caused solely by the quotienting w.r.t. the rotation group $SO(3)$. Using these new ingredients, we have derived the reduced equations of motion for a N-particle system for its shape degrees of freedom.

As the simplest nontrivial example of the formalism, we have explicitly derived the equations of motion for the shape degrees of freedom of a three-particle system. We then discussed some cosmological consequences of the theory. In particular, we have shown that an expanding universe must necessarily be an accelerating expanding universe for internal observers, and that the total collision of all particles of a contracting system cannot occur in a finite amount of (internal) time.

At the end, we presented a comparison of our work with two other approaches to relational physics. In particular, we compared the used action of the group Sc on the absolute phase space in each approach. We explained how the principle of relationalism (as formulated in Section 1.4 and Chapter 5) itself defines an action of Sc on the absolute phase space of the modified Newtonian theory, which in turn enabled us to find the metric $\mathbf{N}$ of shape space. Alternatively, we discussed that by taking the role of rulers in determining the geometry of space into account, the measured mass metric is itself scale-invariant, so again no arbitrary conformal factor needs to be introduced to find the metric of the shape space. In particular, we explained the relationship between the choice of a length unit and the choice of a conformal factor, and elaborated that all reasonable choices of length units lead to the same metric on shape space. We have also seen that *Barbour's fundamental postulate of relational mechanics*, as expressed in [32], and *Barbour-Bertotti's postulate of relational mechanics* [33], are special cases of the (more general) *principle of relationalism* introduced in Chapter 1.

Appendix A

Geodesics as true evolution on shape space?

One can assume[1] that the true evolution on shape space is exactly along the geodesics corresponding to the metric (4.10) on shape space, as is the case in [15] and [16]. Of course, horizontal lifting of any geodesic from shape space S to absolute configuration space Q also results in a geodesic with respect to a conformal metric $\mathbf{M}'$ there. Consider now the Jacobi action on Q

$$\bar{S} = \int_{\tau_A}^{\tau_B} \sqrt{(E-V)}\frac{dl}{d\tau}d\tau = \int_{\tau_A}^{\tau_B} \sqrt{(E-V)}dl \tag{A.1}$$

and take $E = 0$. To ensure that the corresponding action principle $\bar{S} = \int_{\tau_A}^{\tau_B} \sqrt{-V}dl$ [2] always leads to a geodesics on absolute configuration space, V must indeed take the role of a conformal factor, i.e. $V = -f(x)$. Thus the expression $\sqrt{-V}dl$ becomes the line element with respect to the new metric $\mathbf{M}'$, as is explained in [16]. So, using (4.11) for the conformal factor (as one choice among many options), one gets $V = \sum_{i<j} \frac{1}{||\boldsymbol{x}_i - \boldsymbol{x}_j||^2}$, where the summation goes over all particles. The characteristic feature of this potential is its homogeneity of degree -2, which is a necessary condition for a conformal factor. As the

[1] Arguments for this assumption are usually based on simplicity and beauty which are being seen in reducing all of physics to geometry as strongly promoted by Descartes.

[2] $dl^2 = g_{ij}dx^i dx^j$

main classical interaction potentials between particles are known to be inversely proportional to their distance from each other, the previous result seems at first glance to be empirically inadequate. However, it may be the case that under certain circumstances an effective potential function of degree -1 could emerge for certain subsystems of the universe whose shape is evolving along a geodesic on shape space. Julian Barbour probed this possibility (in [15]) by starting with the following homogeneous function of degree -2

$$U = -\frac{W^2}{2}$$

as the potential on absolute space, where $W = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|}$. Then, in an inertial frame of reference with the spacial Newtonian time parameter t (see (1.16)), the equations of motion are

$$\frac{dp^i}{dt} = -W\frac{\partial W}{\partial x^i} \tag{A.2}$$

which is the Newton's law if the W behind the differentiation is replaced with G. He then goes on and argues that if the system is virialized,[3] the value of W remains effectively constant, and hence the motions of Newtonian type (hence described by homogenous potentials of degree -1) emerge effectively. Note that the vanishing of the dilatational momentum as a consequence of best matching with respect to scale transformations, enforces the constancy of the moment of inertia I_{cm}. So one may think that the experimentally verified Hubble expansion is clearly in contradiction with constancy of I_{cm}. But as Julian Barbour argued, constancy of I_{cm} would not prevent the matter from clumping, which would increase $-W$ and hence the gravitational constant. Take for instance, a planet orbiting a sun in a universe that in general is becoming clumpier. According to Barbour's theory, the gravitational force should then become stronger, and hence distance between the planet and the sun should adiabatically decrease. However if one insists that the strength of gravity in the distinguished inertial frame is

[3] Virialized means a system of gravitationally interacting particles that is stable. The smaller structures can still interact with each other, but the clusters as a whole doesn't expand or collapse. When a cluster is virialized the merging process and the collapse of matter have finished and the formation process of the galaxy is done. A system is virialized when the potential energy is twice the negative kinetic energy. From this one can find the condition $R_{vir} \cong \frac{R_{max}}{2}$ where R_{vir} is the radius when the cluster is virialized, and R_{max} is the radius (of the moment) at which the cluster starts to collapse. So by looking at the radius and the density of a cluster one can deduce if a cluster is virialized or not.

constant (as is the case in normal Newtonian theory) one is forced to have an adiabatic increase of all scales in this frame. This would then mimic a Hubble-type expansion in this frame. However, he does mention that he probably cannot go around the Hubble red shift in this way... .

Barbour then tried another option to achieve a scale invariant geodesic theory by using the unique conserved quantity I_{cm} of his 2003 shape geodesic theory. Newtonian potentials can be converted into homogeneous potentials of degree -2 if multiplied by an appropriate power of $\mu = \sqrt{I_{cm}} = \sqrt{\sum_{i<j} m_i m_j r_{ij}^2}$.
Thus, starting with the general Newtonian potential

$$V = \sum_{k=-\infty}^{\infty} a_k V_k$$

with V_k's being homogeneous functions of degree k, one goes over to the homogeneous potential of degree -2 (which is required for geodesic dynamics on shape space) as follows

$$\tilde{V} = \sum_{k=-\infty}^{\infty} b_k V_k \mu^{-(2+k)}$$

and according to Newton's law of motion, one ends up with the following equations of motion

$$\frac{dp^i}{dt} = -\sum_{k=-\infty}^{\infty} b_k \mu^{-(2+k)} \frac{\partial V_k}{\partial x^i} + \sum_{k=-\infty}^{\infty} (2+k) b_k \mu^{-(2+k)} V_k \frac{1}{\mu}\frac{\partial \mu}{\partial x^i} \tag{A.3}$$

In order to make the connection with the observations (hence also to V), one has to set $b_k \mu^{-(2+k)} = a_k$. Then the equations of motion for the modified (scale-invariant) potential (A.3) turns into the following

$$\frac{dp^i}{dt} = -\sum_{k=-\infty}^{\infty} a_k \frac{\partial V_k}{\partial x^i} + C(t) \sum_j m_i m_j \frac{\partial r_{ij}^2}{\partial x^i} \tag{A.4}$$

with $C(t) = \frac{\sum_{k=-\infty}^{\infty} (2+k) a_k V_k}{2 \sum_{i<j} m_i m_j r_{ij}^2}$.
So the real force acting on the i'th particle can be decomposed into the Newtonian-type forces and a residue which is a cosmological force. However in [41] it is claimed that also this attempt turns out to be empirically inadequate, since it

fails to show the formation of clusters for an expanding N-body solutions, and therefore fails to explain the emergence of stars and galaxies.
In a recent publication [16], these ideas of emergence of interacting theories on absolute space, from a free (non-interacting) theory on Shape space are being elaborated, and carefully expanded to the quantum mechanics.

In this thesis we did not insist on geodesity of dynamics on shape space. We rather sought the projected dynamical law of the modified Newtonian theory from the absolute configuration space down to the shape space. The principle of relationalism was used to make classical physics invariant under global scale transformations.

Appendix B

Mass tensor

Originally, the Kinetic energy K of a classical N-particle system is expressed(defined) in Cartesian coordinates as follows

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{x}}_i^2 = \frac{1}{2}[\dot{\mathbf{x}}_1, ..., \dot{\mathbf{x}}_N]\mathbf{M}\begin{bmatrix}\dot{\mathbf{x}}_1\\ ...\\ \dot{\mathbf{x}}_N\end{bmatrix} \tag{B.1}$$

where $\dot{\mathbf{x}}_i := \frac{d\mathbf{x}_i}{dt}$ with $\mathbf{x}_i = \begin{bmatrix} x_{3i-2}\\ x_{3i-1}\\ x_{3i}\end{bmatrix}$ and $\mathbf{M}$ is the so called mass matrix which is in this case just a block diagonal $3N \times 3N$ matrix with $\begin{bmatrix} m_j & 0 & 0\\ 0 & m_j & 0\\ 0 & 0 & m_j\end{bmatrix}$ as its j's block.

Here, as usual, configuration space is coordinatized by $x_1, x_2, ..., x_{3N}$ which are in turn the collection of Cartesian coordinates $x_{3i-2}, x_{3i-1}, x_{3i}$ used to denote the position(vector in $\mathbb{R}^3$) of the i'th particle $\mathbf{x}_i$.

Now, if the system suffers from a number of holonomic constraints, generalized coordinates $q_1, q_2, ..., q_f$ (with $f < 3N$ standing for total number of remaining degrees of freedom), can be used to coordinatize the new (generalized) configuration space. If one rewrites the kinetic energy K in terms of these new generalized coordinates q_j and their velocities $\dot{q}_j$ one ends up usually with a much more complicated expression than (B.1) and, in fact, the metric looses independence of the configuration, and its simple diagonal quadratic form in the velocities. In generalized coordinates, it is quadratic but not necessarily homogeneous in the velocities $\dot{q}_j$, and has in general a non-trivial dependence on the coordinates q_j (through M).

If the coordinate transformation between the set of Cartesian coordinates $x_1, ..., x_{3N}$ and the generalized coordinates $q_1, ..., q_f$ is time-independent (see [7]), the kinetic energy can be written as

$$K = \frac{1}{2} \sum_{k,l} M_{kl} \dot{q}_k \dot{q}_l = \frac{1}{2} [\dot{q}_1, ..., \dot{q}_f] \mathbf{M} \begin{bmatrix} \dot{q}_1 \\ ... \\ \dot{q}_f \end{bmatrix}, \tag{B.2}$$

where

$$\begin{aligned} M_{kl} &= \sum_{j=1}^{N} m_j \frac{d\mathbf{r}_j}{dq_k} . \frac{d\mathbf{r}_j}{dq_l} \\ &= \sum_{j=1}^{N} \sum_{i=0}^{2} m_j \frac{dx_{3j-i}}{dq_k} \frac{dx_{3j-i}}{dq_l} \end{aligned}$$

are elements of the $f \times f$ matrix $\mathbf{M}$.

The Lagrangian of classical mechanics is known to be $L = K - V$, where the potential V is usually independent of the generalized velocities $\dot{q}_i$. The conjugate momentum to q_i is defined as

$$p_i = \frac{\partial L}{\partial q_i} = \frac{\partial K}{\partial \dot{q}_i} = \sum_{j=1}^{f} M_{ij} \dot{q}_j \tag{B.3}$$

Thus, the expression (B.2) for the kinetic energy which involved just the velocities can be rewritten as

$$K = \frac{1}{2} \sum_{i=1}^{f} p_i \dot{q}_i \tag{B.4}$$

Appendix C

Adjoint and Coadjoint actions of a Lie-group

Let G be a Lie group, and $\mathbf{G}$ its Lie algebra, and $\mathbf{G}^*$ be the dual vector space of $\mathbf{G}$. The adjoint representation of $g \in G$ on $\mathbf{G}$ is defined by

$$Ad_g(Y) = \frac{d}{dt} \mid_{t=0} (ge^{tY}g^{-1}) \tag{C.1}$$

for $Y \in \mathbf{G}$.
The coadjoint action of $g \in G$ on $\mathbf{G}^*$ is characterized by

$$< Ad_g^*(\xi), Y >=< \xi, Ad_{g^{-1}}(Y) > \tag{C.2}$$

for $\xi \in \mathbf{G}^*$
Here, $<,>$: $\mathbf{g}^* \times \mathbf{g} \to \mathbf{R}$ is the dual pairing.
In summary: $ad_g(x) = gxg^{-1}$, $Ad_g = (ad_g)_* : \mathbf{G} \to \mathbf{G}$ being called the adjoint action , $Ad_g^* : \mathbf{G}^* \to \mathbf{G}^*$ being called the coadjoint action.

Appendix D

Isomorphism R

There exists an isoporphism R between the Lie-algebra $\mathbf{so}(3)$ of the rotation group, and the linear space $\wedge^2\mathbb{R}^3$ of all antisymmetric tensors of order 2, which we want to explain shortly.
Take $\boldsymbol{e}_1, ...\boldsymbol{e}_3$ as an orthonormal basis of $\mathbb{R}^3$. Then $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$ with $i < j$ constitutes an orthonormal basis of $\wedge^2\mathbb{R}^3$. The inner product in $\wedge^2\mathbb{R}^2$ is defined as the following

$$(u \wedge v \mid x \wedge y) = \begin{vmatrix} (u \mid x) & (u \mid y) \\ (v \mid x) & (v \mid y) \end{vmatrix}. \tag{D.1}$$

One can easily check that for two two-vectors (or tensors of order 2) $\xi = \sum_{i<j} \xi_{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$ and $\zeta = \sum_{k<l} \zeta_{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l$, definition (D.1) leads to the following

$$(\xi \mid \zeta) = \sum_{i<j} \xi_{ij} \zeta_{ij}. \tag{D.2}$$

Now we identify the Lie-algebra of the rotation group in 3 dimensions $\mathbf{so}(3)$ with the space of two forms (anti-symmeric tensors) $\wedge^2\mathbb{R}^3$ by the isomorphism R

$$R : \wedge^2\mathbb{R}^3 \xrightarrow{\sim} \mathbf{so}(3) \tag{D.3}$$

$$\xi \to R_\xi.$$

So for u,v,$x \in \mathbb{R}^3$ we define the following

$$R_{u\wedge v}(x) := (v \mid x)u - (u \mid x)v. \tag{D.4}$$

$R_{u\wedge v}$ is in fact a 3 dimensional square matrix and its multiplication by a 3 dimensional vector x is given by the last equation. For $\xi \in \wedge^2\mathbb{R}^3$ and $x = \sum x_j\boldsymbol{e}_j \in \mathbb{R}^3$ one can also write the above formula as

$$R_\xi(x) = \sum_i(\sum_j \xi_{ij}x_j)\boldsymbol{e}_i. \tag{D.5}$$

That is, R_ξ is an antisymmetric matrix with entries ξ_{ij}.
Given the natural scalar product of the Lie algebra; $(\alpha \mid \beta) = \frac{1}{2}tr(\alpha\beta^T)$ for $\alpha,\beta \in \mathbf{so}(3)$ one can show that the identification R is even an isometry from $\wedge^2\mathbb{R}^d$ to **so**(d).

As explained in [3], the space $\wedge^2\mathbb{R}^3$ can be identified with $\mathbb{R}^3$ by $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \rightarrow \boldsymbol{e}_3$ and its cyclic permutations. Hence if one sets

$$\xi_{12} = \phi^3, \xi_{23} = \phi^1, \xi_{31} = \phi^2$$

the two vector

$$\xi = \sum_{i<j} \xi_{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

is identified with

$$\phi = \sum \phi^i\boldsymbol{e}_i.$$

So in this case R becomes a linear isomorphism from $\mathbb{R}^3$ to $\mathbf{so}(3)$ i.e.

$$R : \mathbb{R}^3 \rightarrow \mathbf{so}(3)$$

$$R_\xi(x) = R_\phi(x) = -\phi \times x \tag{D.6}$$

for $x \in \mathbb{R}^3$.
Alternatively, R_{e_1} is the matrix (ξ_{ij}) with the only nonzero elements $\xi_{23} = -\xi_{32} = 1$.
One can also show([3]) that R is Ad-equivariant i.e. $R_{g\phi} = Ad_gR(\phi) = gR(\phi)g^{-1}$.

The map R, and the inertial tensor A_x have the following properties:

$$R_a b = a \times b \tag{D.7a}$$

$$R_{ga} = g R_a g^{-1} \tag{D.7b}$$

$$R_a . R_b =< a \mid b > \tag{D.7c}$$

$$A_{gx}(a) = g A_x(g^{-1}a) := Ad_g A_x(a) \tag{D.7d}$$

$$(x \mid R_\xi y) = (x \wedge y \mid \xi) \tag{D.7e}$$

$$(R_\xi x \mid R_\eta y) = (R_\xi x \wedge y \mid \eta) \tag{D.7f}$$

where $a, b \in \mathbb{R}^3$ and $g \in SO(3)$.

Bibliography

[1] S. Tokasi and P. Pickl, "Reduction of classical mechanics on shape space," *preprint arXiv:2203.07041(2022) (to appear in Found Phys Springer Nature journal).*

[2] S. Tokasi, "Lagrangian equations of motion of classical many body systems on shape space obeying the modified newtonian theory," *preprint arXiv:2208.00229(2022).*

[3] T. Iwai, "A geometric setting for classical molecular dynamics," vol. 47, no. 2, pp. 199–219, 1987.

[4] J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry," *Reports on mathematical physics*, vol. 5, no. 1, pp. 121–130, 1974.

[5] T. Iwai and H. Yamaoka, "Stratified reduction of classical many-body systems with symmetry," *Journal of Physics A: Mathematical and General*, vol. 38, no. 11, p. 2415, 2005.

[6] T. Iwai, "The mechanics and control for multi-particle systems," *Journal of Physics A: Mathematical and General*, vol. 31, no. 16, p. 3849, 1998.

[7] R. G. Littlejohn and M. Reinsch, "Gauge fields in the separation of rotations and internal motions in the n-body problem," *Reviews of modern physics*, vol. 69, no. 1, p. 213, 1997.

[8] J. Barbour and H. Pfister., "Mach's principle: from newton's bucket to quantum gravity (vol. 6).," *Springer Science Business Media.*, 1995.

[9] J. Barbour, “The end of time: The next revolution in physics.,” *Oxford university press*, 2001.

[10] A. Andre Koch Torres and A. K. T. Assis, “Relational mechancis,” *Montreal: Apeiron, 1999.*, 1995.

[11] J. Barbour, T. Koslowski, and F. Mercati, “A gravitational origin of the arrows of time.,” *arXiv preprint arXiv:1310.5167 (2013).*

[12] J. Barbour, T. Koslowski, and F. Mercati., “Identification of a gravitational arrow of time,” *Physical review letters 113.18 (2014): 181101.*

[13] D. Sloan, “Dynamical similarity,” *Physical Review D 97.12 (2018): 123541.*

[14] S. Gryb and D. Sloan, “When scale is surplus.,” *Synthese 199.5 (2021): 14769-14820*, 2021.

[15] J. Barbour, “Scale-invariant gravity: particle dynamics,” *Classical and quantum gravity*, vol. 20, no. 8, p. 1543, 2003.

[16] D. Dürr, S. Goldstein, and N. Zanghì, “Quantum motion on shape space and the gauge dependent emergence of dynamics and probability in absolute space and time,” *Journal of Statistical Physics*, vol. 180, no. 1, pp. 92–134, 2020.

[17] S. Goldstein and N. Zanghi, “Remarks about the relationship between relational physics and a large kantian component of the laws of nature.,” *arXiv preprint arXiv:2111.09609 (2021).*

[18] I. Newton, “The principia: mathematical principles of natural philosophy,” *Univ of California Press*, 1999.

[19] I. Newton, “Scholium to the definitions in philosophiae naturalis principia mathematica, bk. 1 (1689), trans. andrew motte (1729), rev. florian cajori, berkeley: University of california press, 1934. pp. 6-12, paragraph xiv,”

[20] H. G. Alexander, “The leibniz-clarke correspondence,” *Philosophy*, vol. 32, no. 123, 1956.

[21] E. Mach, “The science of mechanics: A critical and historical exposition of its principles,” *Open court publishing Company*, 1893.

[22] P.-L. M. de Maupertuis, “Les loix du mouvement et du repos deduite d’un principe metaphysique,” *Histoire de l’Academie Royale des Sciences et des Belles-Lettres da Berlin [. . .] pour l’anncee 1746*, vol. 286, 1748.

[23] C. G. Gray, “Principle of least action,” *Scholarpedia*, vol. 4, no. 12, p. 8291, 2009.

[24] C. Lanczos, “The variational principles of mechanics,” *University of Toronto press*, 2020.

[25] H. Goldstein, “Classical mechanics (3rd ed.),” *United States of America: Addison Wesley, 1980*.

[26] J. Barbour, “The nature of time,” *arXiv preprint arXiv:0903.3489*, 2009.

[27] J. D. Barrow, “The constants of nature: from alpha to omega-the numbers that encode the deepest secrets of the universe,” *PANTHEON BOOKS, NEW YORK, 2002*.

[28] I. Rosenthal-Schneider, “Reality and scientific truth: Discussions with einstein, von laue, and planck,” *Wayne State University Press, 1980*.

[29] B. Coquinot, P. M. Garcia, and E. M. Galcerán, “The b-geometry of magnetic fields,” 2020.

[30] A. Weinstein, “A universal phase space for particles in yang-mills fields,” *Letters in Mathematical Physics*, vol. 2, no. 5, pp. 417–420, 1978.

[31] V. Guillemin and S. Sternberg, “Symplectic techniques in physics,” *Cambridge university press*, 1990.

[32] J. Barbour, “Relationism in classical dynamics.,” *The Routledge Companion to Philosophy of Physics. Routledge, 2021. 46-57*.

[33] J. Barbour and B. Bertotti., “Mach’s principle and the structure of dynamical theories.,” *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 382(1783), 295-306*.

[34] J. Barbour, "The definition of mach's principle.," *Foundations of Physics 40.9 (2010): 1263-1284.*

[35] A. Guichardet, "On rotation and vibration motions of molecules," vol. 40, no. 3, pp. 329–342, 1984.

[36] J. I. Neimark and N. A. Fufaev, "Dynamics of nonholonomic systems, translations of mathematical monographs, vol. 33," *American Mathematical Society, Providence, Rhode Island*, vol. 518, pp. 65–70, 1972.

[37] E. Jarzebowska, "Quasi-coordinates based dynamics modeling and control design for nonholonomic systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e118–e131, 2009.

[38] J. M. Maruskin and A. M. Bloch, "The boltzmann-hamel equations for optimal control," pp. 554–559, 2007.

[39] C. Speake and T. Quinn, "The search for newton's constant," *Physics Today*, vol. 67, no. 7, p. 27, 2014.

[40] I. B. C. Isaac Newton and A. Whitman, "The principia: Mathematical principles of natural philosophy.," *Berkeley, Calif: University of California Press.*, 1999.

[41] J. Barbour, T. Koslowski, and F. Mercati, "A gravitational origin of the arrows of time," *arXiv preprint arXiv:1310.5167*, 2013.