Hodge Theory of Nondegenerate Minimal Toric Hypersurfaces

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CHAPTER 1

Introduction

1.1 Abstract

In this thesis we study toric hypersurfaces in the context of higher-dimensional algebraic geometry. The topics are quite complicated but restricting to generic situations and almost smooth birational models (minimal models), we are able to get good results. We ask how to calculate invariants like the Plurigenera or the Hodge numbers of toric hypersurfaces. Deforming such hypersurfaces within the surrounding toric variety we study a Kodaira-Spencer map, parameterizing infinitesimal deformations one-to-one and the infinitesimal Torelli theorem, bridging deformation theory and Hodge theory, both by very explicit formulas, though for this part we restrict to surfaces in toric 3-folds.

People familiar to toric geometry and toric hypersurfaces should get a stronger insight of how much interest this topic is in higher-dimensional algebraic geometry. People working on complex algebraic geometry should find these results helpful both for checking open problems within this region and getting a large set of examples. Based on knowledge in complex algebraic geometry we use several preworks like famous works on toric hypersurfaces and many written books on complex geometry, algebraic geometry and toric geometry. We list some particular important sources as well as the notation at the end

of the introduction referencing to the sources within the introduction by the letters a) to g).

1.2 Historical Motivation

Let us broach just one issue of this thesis by giving some historical motivation: Given a nondegenerate Laurent polynomial

$$f = \sum_{m \in \Delta \cap M} a_m \cdot x^m \tag{1.1}$$

with n-dimensional Newton polytope Δ let

$$Z_f := \{f = 0\} \subset T.$$

We are looking for a good compactification of Z_f realized as closure in a suitable projective toric variety. The Newton polytope Δ defines a projective toric variety $\mathbb{P}_\Delta$ via its normal fan and there is a diagram

$$\begin{array}{ccccc} & & \overbrace{\tilde{Z}_f}^{=\text{ smooth compactification of } Z_f} & \subseteq & \tilde{\mathbb{P}} \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Z_f & \subseteq & \underbrace{Z_{\Delta,f}}_{=\text{ closure of } Z_f \text{ in } \mathbb{P}_\Delta} & \subseteq & \mathbb{P}_\Delta \end{array}$$

where p denotes a toric resolution of singularities and $\tilde{Z}_f$ the preimage of $Z_{\Delta,f}$ under p. In 1986 Vladimir I. Danilov and Askold G. Khovanskii invented ideas how the *Hodge-Deligne numbers* of Z_f and the *Hodge numbers* of $\tilde{Z}_f$ could be calculated. In examples this program works nice in lower dimensions or if the normal fan Σ_Δ of Δ is simplicial.

Criticism and open questions:

- The mixed Hodge components

$$H^{n-k,k-1} H_c^{n-1}(Z_f, \mathbb{C}) \tag{1.2}$$

  of weight $n-1$ and their dimensions are of particular interest for us. There is neither a natural basis for these components nor a uniform equation or formula for the *Hodge-Deligne* numbers depending just on $n \in \mathbb{N}$ (the formulas of Danilov and Khovanskii get very complicated already for $n = 4$ or $n = 5$).

- The authors stay somehow „loose“ in specifying a nice compactification of Z_f: They do not construct or deal with special compactifications, that are of great importance in higher-dimensional algebraic geometry, though their methods are quite general and if Σ_Δ is simplicial the compactification $Z_{\Delta,f}$ is already useful enough for the calculations of the Hodge numbers.

First improvement (due to V. Batyrev):

One breakthrough was made in ([Bat22]). There it is constructed a projective toric variety $\mathbb{P}$ to a fan Σ, such that the closure

$$Z_f \subset Y_f \subset \mathbb{P}$$

of Z_f gets a **minimal model** of Z_f, that is

- Y_f has terminal singularities and
- The canonical divisor K_{Y_f} is nef.

In the first part of this thesis we heavily exploit these results: Concerning the birational model $Y = Y_f$ we extend results of ([DK86]) by giving an explicit formula for the *plurigenera* $P_m(Y)$ of Y, that specialize to the *geometric genus* for $m = 1$. The *Kodaira dimension* and the *canonical volume* of Y are then gotten from the asymptotic behavior of the plurigenera.

Second improvement:

We find a vector space representation of the mixed Hodge components (1.2) as quotient of a vector space $L^*(k \cdot \Delta)$ by a subspace $U_{f,k}$ of which we specify generators. This is necessary for dealing with the concept of an *infinitesimal variation of Hodge structure* (due to Philipp Griffiths) for toric hypersurfaces in the last chapter.

A deficit is that these generators of $U_{f,k}$ are not linearly independent for $k \geqslant 3$, that is the goal to obtain the Hodge-Deligne numbers directly is much to optimistic. But still the results we get are sufficient for our purposes.

Further common methods with Danilov, Khovanskii and Batyrev:

- We reduce the calculation of the Hodge component $H^{n-k,k-1}(Y_f, \mathbb{C})$ to the calculation of 1.2 (similarly to (b)) but restricting to $n = 3$.

- We follow (e) to define the *jacobian ring* R_f and the *(interior) module* $R_{Int,f}$ over R_f which identifies the components (1.2) with vector spaces defined by the lattice geometry of integral multiples of Δ.

1.3 The combinatorial construction of minimal models

In the dissertation ([Fine83]) Jonathan Fine, a student of Miles Reid, came up with the idea of associating a polytope $F(\Delta)$, the so called *Fine interior of* Δ, to a lattice polytope $\Delta \subset M_{\mathbb{R}}$ (in connection with the resolution of singularities). This polytope has been resumed by Miles Reid and many years later by V. V. Batyrev, with most success in (a). This polytope plays a decisive role in the construction of $\mathbb{P}$: At the very basic $\mathbb{P}$ is shown to exists if and only if the Fine interior $F(\Delta)$ is nonempty.

In case Δ is reflexive $F(\Delta)$ equals the origin and the adjoint divisor $K_{\mathbb{P}} + Y$ is trivial. In general given a lattice polytope Δ with $F(\Delta) \neq \varnothing$ the Fine interior equals the polytope associated to this adjoint divisor, thereby generalizing the classical case of Calabi-Yau minimal toric hypersurfaces. Besides to construct $\mathbb{P}$ another polytope $C(\Delta)$, the *canonical closure of* Δ has to be recalled from (a) (definition 3.1.5). The following inclusions set some superficial understanding on the relationship between these polytopes

$$\operatorname{convhull}\big(\operatorname{Int}(\Delta) \cap M\big) \subseteq F(\Delta) \subsetneq \Delta \subseteq C(\Delta),$$

Quite often the last inclusion turns out to be an equality, for example if Δ is reflexive, though at least in higher dimensions there are examples where $C(\Delta)$ is just a rational polytope.

1.4 Birational invariants of Y

In (a) it is shown that the **Kodaira dimension** of $Y = Y_f$ is given by

$$\kappa(Y) = \min\Big(\dim\, F(\Delta),\, n-1\Big).$$

We get a higher result

Theorem 1.4.1. *Let Δ be an n-dimensional lattice polytope with $k := \dim\, F(\Delta) \geqslant 0$. The plurigenera $P_m(Y) := h^0(Y, mK_Y)$ are given by*

$$P_m(Y) = \begin{cases} l(m \cdot F(\Delta)) - l^*((m-1) \cdot F(\Delta)), & k = n \\ l(m \cdot F(\Delta)) + l^*((m-1) \cdot F(\Delta)), & k = n-1 \\ l(m \cdot F(\Delta)) & k < n-1, \end{cases}$$

with exception of the special case $n = 0$ and $m = 1$.

This answers a problem stated by M. Reid in ([Rei87, (4.12),(4.13)]). Restricting to birational models of Y with at most terminal singularities $P_m(Y)$, $m \geqslant 1$ are birational invariants.

Thereafter we deduce an explicit formula for the **Canonical volume** of Y (see Corollary 4.3.1), that is the maximal self-intersection number K_Y^{n-1} of the canonical class K_Y of Y. We are very granted V.V. Batyrev, who already knew and proved this formula earlier, for the hint that it follows pretty easily from Theorem 1.4.1 and the characterization of K_Y^{n-1} as leading coefficient of $P_m(Y)$, considered as a polynomial in m of degree n.

1.5 Infinitesimal deformations of Y_f

Beginning with chapter 5 we get results which are partly oriented on known results and proofs, though basically we work with skillful and own ideas and much more general methods, approximately comparable to the relationship between projective and toric hypersurfaces. Throughout the chapters 5 and 6 we set 3 restrictions on Δ (conditions (+))

- $n = 3$
- $l^*(\Delta) > 0$

- $C(\Delta)$ is a lattice polytope.

Concerning more general assumptions we guess the conditions $n \geqslant 2$ and $F(\Delta) \neq \emptyset$ are sufficient for our results (with the exception $n = 2$, dim $F(\Delta) \in \{0, 1\}$), though this requires some efforts in proving (see [Gie22a], still missing the generalization $F(\Delta) \neq \emptyset$, unpublished).

Let $U_{reg}(\Delta)$ denote the set of nondegenerate Laurent polynomials with Newton polytope Δ and let $B = B(\Delta)$ denote the projectivization of $U_{reg}(\Delta)$. The second projection

$$\mathcal{X} := \{(x, f) \in \mathbb{P} \times B \,|\, x \in Y_f\} \overset{pr_2}{\to} B$$

defines a natural **deformation** of Y_f over B. Given a tangent vector $v : C \to B$ at $f \in B$, where C denote the *dual numbers* we build the pullback diagram

$$\begin{array}{ccc} \mathcal{X} \times C & \xrightarrow{pr_1} & \mathcal{X} \\ {\scriptstyle pr_2 \times id}\downarrow & & \downarrow{\scriptstyle pr_2} \\ C & \xrightarrow{v} & B \end{array} \tag{1.3}$$

to construct an *infinitesimal deformation* (abbreviate inf.def.) $\mathcal{X}_v : \mathcal{X} \times C \overset{pr_2 \times id}{\to} C$ of Y_f in $\mathcal{X}$ (that is also one in $\mathbb{P}$). Switching from $\mathcal{X}$ to $\mathcal{X}_v$ some information get losted, though restricting to $\mathcal{X}_v$ is still enough to obtain usefull results.

In ([KoSp58]) Kunihiko Kodaira and Donald C. Spencer introduced a linear map, the **Kodaira-Spencer map**, that allows to parameterize (infinitesimal) deformations of (algebraic) varieties. Their thoughts were influenced by methods from complex analysis and differential equations, but later the ideas were extended mainly by people working on deformation theory, Hodge theory and (complex) algebraic geometry.

We study this situation under the assumptions (+) : For this we identify

$$\begin{array}{ccc} L(C(\Delta))/\mathbb{C} \cdot f \cong \{\text{ Inf. def. of } Y \text{ in } \mathbb{P} \}/\text{iso.} & \xrightarrow{\kappa_{\mathbb{P},f}} & H^1(Y, T_Y) \\ \cup\!\mid & \nearrow{\scriptstyle \kappa_f} & \wr\!\shortparallel \\ L(\Delta)/\mathbb{C} \cdot f \cong \{\text{ Inf. def. of } Y \text{ in } \mathcal{X} \}/\text{iso.} & & \{\text{ Inf. def. of } Y \}/\text{iso.} \end{array} \tag{1.4}$$

$\kappa_{\mathbb{P},f}$ is a connecting homomorphism (called the **Kodaira-Spencer map**) and κ_f its restriction. These maps parameterize the *infinitesimal deformations* of Y in $\mathbb{P}$ (in $\mathcal{X}$ respectively) one to one.

The kernel of $\kappa_{\mathbb{P},f}$:

The family $\mathcal{X} \to B$ is *isotrivial*, that is all fibres are isomorphic, if and only if $\kappa_f \equiv 0$ for all $f \in B$. The kernel of $\kappa_{\mathbb{P},f}$ has been calculated for curves in toric surfaces ($n = 2$) by Jan Koelman in (c) and for (quasi-)smooth hypersurfaces in projective or weighted projective spaces $\mathbb{P}$ by Phillip Griffiths, Joseph Steenbrink and others, see (f). Other questions on $\kappa_{\mathbb{P},f}$ include the study of isotrivial deformations, an iterated Kodaira-Spencer map, the *Shafarevich conjecture* (see [Kov05]) and extensions of κ_f say to a logarithmic context. The main result of chapter 5 is

Theorem 1.5.1. *Given the conditions* (+)

$$\ker(\kappa_{\mathbb{P},f}) \cong \text{Lie Aut}(\mathbb{P}), \tag{1.5}$$

where $\text{Aut}(\mathbb{P})$ *denotes the automorphism group of* $\mathbb{P}$ *and* $\text{Lie Aut}(\mathbb{P})$ *the Lie algebra of* $\text{Aut}(\mathbb{P})$.

The cokernel of $\kappa_{\mathbb{P},f}$:

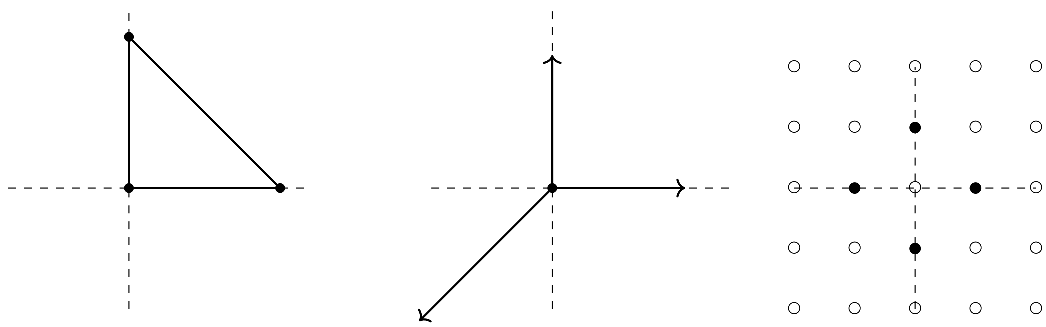
Concerning all infinitesimal deformations there might be some additional, not in $Im(\kappa_{\mathbb{P},f})$, namely

- deformations of Y_f induced by deformations of $\mathbb{P}$ in $H^1(\mathbb{P}, T_{\mathbb{P}})$.
- In some (more exceptional) cases there are other (non-projective) deformations (Example: K3-surfaces, see section 1.9, Example 5.6.2).

These questions have been studied in an explicit way for Calabi-Yau toric hypersurfaces in ([Mav03]): If Δ is (quite general n-dimensional and) reflexive then $H^1(Y, T_Y)$ splits into infinitesimal deformations of Y inside $\mathbb{P}$ and „non-polynomial“ infinitesimal deformations of Y. The later are induced by deformations of $\mathbb{P}$ inside a larger toric variety $\mathbb{P}_{\Sigma(\Gamma^*)}$, where Γ^* varies over the 2-dimensional faces of the dual polytope of Δ.

1.6 Explicit calculations of the kernels of the Kodaira-Spencer maps

Classically to the fan Σ (or similarly any other complete fan) there are integral vectors $R(N, \Sigma)$, so called roots (due to *M. Demazure*, [Dem70]).



Example: The 2-dimensional standard simplex (left), its normal fan (middle) and the 4 roots of the normal fan (thick, right)

The point is that each root α_i gives an element $z(\alpha_i)$ in the Lie algebra of the automorphism group of $\mathbb{P}$ and

$$\text{Lie Aut}(\mathbb{P}) \cong \text{Lie}(T) \oplus \bigoplus_{i=1}^{r} z(\alpha_i) \cdot \mathbb{C} \qquad r := |R(N, \Sigma)|, \tag{1.6}$$

see ([Cox95]). In view of the last section we specify Laurent polynomials supported on $C(\Delta)$ and Δ giving bases for $\ker(\kappa_{\mathbb{P},f})$ and $\ker(\kappa_f)$: Here we heavily exploit the results from Bruns and Gubeladze ([BG99]) and thereby define certain new Laurent polynomials $w_{-\alpha}(f)$ for every root α of Σ:

$$w_{-\alpha}(f) := \sum_{m \in \Delta \cap M} ht_{-\alpha}(m) \cdot a_m \cdot x^{m-\alpha}.$$

Our results:

Corollary and Theorem 1.6.1. *Given the conditions* (+)

$$\ker(\kappa_{\mathbb{P},f}) \cong \left\langle x_i \cdot \frac{\partial f}{\partial x_i} \mid i = 1, 2, 3, \quad w_{-\alpha}(f), \quad \alpha \in R(N, \Sigma) \right\rangle.$$

The roots of Σ_Δ form a subset of the roots of Σ and

$$\ker(\kappa_f) \cong \Big\langle\, x_i \cdot \frac{\partial f}{\partial x_i} \mid i = 1,2,3, \quad w_{-\alpha}(f) \quad \alpha \in R(N, \Sigma_\Delta)\, \Big\rangle.$$

This result is much explicit both in specifying a basis of Laurent polynomials for the kernel and in dealing with a concrete situation (f varying in B), being useful in examples.

1.7 Mixed Hodge components of Z_f

In chapter 7 we deal with the Hodge components of $H^{n-1}(Y_f, \mathbb{C})$. Apart from one result, that should generalize with minor changes, we do not restrict to $n = 3$. The cohomological data of Y_f that is of interest is already contained in the affine part Z_f, that is to say given the inclusions $j : Z_f \to Y_f$ and $\iota : Y_f \to \mathbb{P}$ the pullback homomorphism

$$j^* H^{n-1}(Y_f, \mathbb{C}) = \operatorname{Gr}_W^{n-1} H^{n-1}(Z_f, \mathbb{C}) \quad (\text{modulo } \iota^* H^2(\mathbb{P}, \mathbb{Q})),$$

defines an isomorphism onto a graded component $\operatorname{Gr}_W^{n-1} := W^{n-1}/W^{n-2}$, modulo cohomology classes that arise as restrictions of cohomology classes from $\mathbb{P}$.

We recall the definition of the (graded) *jacobian ring of Batyrev* R_f and the (graded) *interior module* $R_{Int,f}$ over R_f, that settles an isomorphism

$$H^{n-k,k-1} H^{n-1}(Z_f, \mathbb{C}) = \operatorname{Gr}_F^{n-k} \operatorname{Gr}_W^{n-1} H^{n-1}(Z_f, \mathbb{C}) \cong R^k_{Int,f},$$

as is shown in (e), thereby reducing the calculation of the mixed Hodge components of weight $n-1$ to a lattice geometric problem.

Illustrating the module $R_{Int,f}$ at some 3-dimensional polytopes gave us intuition that the construction of $R_{Int,f}$ given in (e) might be improved: The original construction defines $R_{Int,f}$ as a *graded* R_f-module, leaving a picture for the homogeneous components $R^k_{Int,f}$ widely open. Concerning the dimensions the situation is not much better: $\dim R^k_{Int,f}$ stays mysterious except for $k = 1, 2, n, n+1$ and that the dimensions are symmetric around the middle index. We introduce slightly different polynomials $g_\Gamma(f)$, where $\Gamma \leqslant \Delta$,

$$g_\Gamma(f) := \sum_{m \in M \cap \Delta} a_m \cdot (\langle n_\Gamma, m\rangle - \operatorname{Min}_\Delta(n_\Gamma)) \cdot x^m$$

to give a more precise presentation of $R^k_{Int,f}$:

Proposition 1.7.1. *Given an n-dimensional lattice polytope Δ with $l^*(\Delta) > 0$ and $f \in U_{reg}(\Delta)$ take facets $\Gamma_1, ..., \Gamma_{n+1}$ of Δ with $n_{\Gamma_1}, ..., n_{\Gamma_{n+1}}$ affine linearly independent. Then*

$$R^k_{Int,f} \cong L^*(k \cdot \Delta)/U_{f,k} \qquad \text{for } k = 1, ..., n+1,$$

where $U_{f,k}$ denotes the $\mathbb{C}$-vector space spanned by

$$g_{\Gamma_i}(f) \cdot x^v \quad i = 1, ..., n+1, \quad v \in \operatorname{Int}((k-1) \cdot \Delta) \cap M \tag{1.7}$$
$$g_{\Gamma}(f) \cdot x^v \quad \Gamma \leqslant \Delta, \quad v \in \operatorname{Int}((k-1) \cdot \Gamma) \cap M. \tag{1.8}$$

This seems promising but still for $k \geqslant 3$ the generators in (1.7) and (1.8) fail to be linear independent. Nevertheless this result is sufficient for us.

1.8 Infinitesimal Variation of Hodge structures

Given the conditions (+) in chapter 8 we define a **period map**

$$B \ni f \mapsto [H^0(Y_f, \Omega^2_{Y_f})].$$

This map is holomorphic and by results of Griffiths the differential of $\mathcal{P}_{B,f}$ factors through κ_f (working with κ_f and not $\kappa_{\mathbb{P},f}$ is no restriction here)

$$\begin{tikzcd} L(\Delta)/\mathbb{C} \cdot f \arrow[r, "\kappa_f"] \arrow[dr, "d\mathcal{P}_{B,f}"'] & H^1(Y_f, T_{Y_f}) \arrow[d, "\Phi_f"] \\ & \operatorname{Hom}(R^1_{Int,f}, R^2_{Int,f}) \end{tikzcd} \tag{1.9}$$

The classical **infinitesimal Torelli Theorem (ITT)** for Y_f asks Φ_f to be injective. The period map arises from complex geometry and the dimension of its kernel might very well depend on the (less generically) chosen f. The ITT is of interest as roughly speaking it gives information in as much the classical Hodge numbers serve for the classfication of (smooth) algebraic varieties. Of course the period map $\mathcal{P}_{B,f}$ itself and questions concerning the injectivity of this map (the *global Torelli theorem*) would give more direct and global geometric information.

After dealing with some preparations on a quotient $\mathcal{M}(\Delta)$ of B by a canonical action of the torus T on B in a rather technical part, we turn to the kernel of $d\mathcal{P}_{B,f}$: In (e) the differential $d\mathcal{P}_{B,f}$ is shown to be simply induced by the addition map

$$\begin{aligned} L(\Delta) &\to \operatorname{Hom}(L^*(\Delta), L^*(2\cdot\Delta)) \\ m &\mapsto (m' \mapsto m + m'). \end{aligned} \tag{1.10}$$

Here we work with the representation of Proposition (1.7.1) for $R^k_{Int,f}$.

Our idea is straightforward: Working with diagram 1.9 calculate the kernel of $d\mathcal{P}_{B,f}$ and compare it with the kernel of κ_f to deduce on the kernel of $\Phi_{f|Im\,\kappa_f}$. Despite of the simpleness of (1.10) we did not tackled the problem of calculating $\ker(d\mathcal{P}_{B,f})$ but instead end up with a conjecture, where only the inclusion $\supseteq$ is clear.

Conjecture 1.8.1. *Let Δ be a 3-dimensional lattice polytope with $(0,0,0) \in \operatorname{Int}(\Delta) \cap M$ and $\operatorname{Int}(\Delta) \cap M \nsubseteq E$ (plane). Then*

$$\begin{aligned} \ker(d\mathcal{P}_{B,f}) \overset{\text{mod Lie}(T)}{=} &\langle g_\Gamma(f) \cdot x^w \in L(\Delta)/\mathbb{C}\cdot f \mid \Gamma \leqslant \Delta \text{ a facet}, \\ &w + v \in \big(\operatorname{Int}(\Delta) \cup \operatorname{Int}(\Gamma)\big) \cap M, \quad \forall\, v \in \operatorname{Int}(\Delta) \cap M\rangle. \end{aligned}$$

1.9 Some set of illustrating examples

We mention/work with some Examples.

- **Smooth projective curves in toric surfaces**. Given a lattice polygon Δ with $F(\Delta) \neq \varnothing$ we have $l^*(\Delta) > 0$ and $C(\Delta) = \Delta$ is automatic. Computing the main invariants of Y reduces to the genus formula $g(Y) = |\operatorname{Int}(\Delta) \cap M|$. The kernel $\ker(\kappa_f)$ has been computed in the dissertation of J. Koelman (c). The ITT is known to be true except if

$$\dim F(\Delta) = 1, \quad |\operatorname{Int}(\Delta) \cap M| \geqslant 3,$$

  that is if Y_f is hyperelliptic of genus $\geqslant 3$. In this case the ITT fails, though this failure could not be seen from the infinitesimal defomrations of Y_f in $Im(\kappa_f)$.

- **Nondegenerate surfaces** $Y \subset \mathbb{P}^3$ **of degree** $d \geqslant 4$ (the case of hypersurfaces $Y \subset \mathbb{P}^n$, $n \geqslant 4$ is almost the same). Here $\Delta = d \cdot \Delta_3$ with Δ_3 the standard 3-simplex). The invariants $P_m(Y)$ and K_Y^2 are known (see [BHPV04]). Given $d \geqslant 5$ the Kodaira-Spencer map κ_f is surjective. In case $d = 4$ then Y is a K3-surface and there are non-algebraic deformations of Y (in this case $\kappa_{\mathbb{P},f}$ is not surjective).

$$\ker(\kappa_f) = J_f^d, \qquad J_f := \left(\frac{\partial f_{hom}}{\partial x_0}, ..., \frac{\partial f_{hom}}{\partial x_3}\right) \trianglelefteq \mathbb{C}[x_0, ..., x_3],$$

  where f_{hom} denotes the homogenization of f. This result is due to *Griffiths* and J_f is called the *jacobian ring of Griffiths*. The ITT is known to be true, see (f).

- **Kanev/Todorov surfaces**. Beginning with certain 3-dimensional lattice polytopes Δ we study minimal surfaces Y with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = \underbrace{1,}_{\text{Kanev surfaces}} \quad \underbrace{2.}_{\text{Todorov surfaces}}$$

  in section 4.4 and section 8.4. Depending on the coefficients $(a_m)_{m \in \Delta \cap M}$ we obtain examples with

$$\dim \ker(\Phi_{f|Im\,\kappa_f}) = \begin{cases} 2, & Y_f = \text{ Kanev surfaces} \\ 3, & Y_f = \text{ Todorov surfaces} \\ 0, & \quad\text{in both cases} \end{cases} \tag{1.11}$$

  that is the ITT may fail though all of our counterexamples seem to be known see ([Cat78], [SSU85]).

1.10 Improvements, open problems and subsequent issues

Deformation theory, Hodge theory and moduli spaces of (higher dimensional) algebraic varieties or toric hypersurfaces:

- Generalize the results beginning with chapter 5 by replacing the conditions (+) by $n \geqslant 2$ and $F(\Delta) \neq \varnothing$ (with the exception $n = 2$, dim $F(\Delta) \leqslant 1$). This seems to get a bit complicated but should be done since it puts everything into a nice framework, see ([Gie22b]) for some attempts. We omit such attempts here due to the larger amount of work and the non-triviality of finding (counter-)examples in this context, that would be necessary for a better understanding.

- Given say $\kappa(Y) = 2$ work with the invariants ($\chi := \chi(Y)$, $K := K_Y^2$) of Y and (similarly to [Cat11] or other articles) the (Giesecker-)moduli space $\mathcal{M}_{\chi,K}$ of surfaces of general type with these invariants. Determine the dimension of those $[X] \in \mathcal{M}_{\chi,K^2}$ „isomorphic" to a nondegenerate toric hypersurface, or say „deformation equivalent", or „up to smoothing"(similarly to what is done in [Mav03] for Calabi-Yau varieties). Generalize this to higher dimensions $n \geqslant 4$. This enters a vast region.

- The work we have done in this thesis includes other topics from Hodge-theory concerning toric hypersurfaces: The *generic Picard number*, the *Noether-Lefschetz locus*, the *Hodge numbers* or even *the Hodge conjecture* just to mention very important ones (see [BrGr10], [BrGr17], [BrMo22]).

- Study coker(κ_f) and the remaining part $\ker(\Phi_{f|coker\,\kappa_f})$ of the ITT. Generalize this to $n \geqslant 4$ (This might get very complicated).

Higher-dimensional algebraic geometry:

- In higher-dimensional algebraic geometry the plurigenera of a sufficiently smooth (projective) variety X are of much interest: There are current (open) questions concerning $P_m(X)$ for the case $\kappa(X) = n$, on the pluricanonical embeddings of X or connections between the Kodaira dimension and the Iitaka fibration of X for $0 \leqslant \kappa(X) \leqslant n$ (see [BiZh15], [ViZh08]). Work with the deformation equivalence of $P_m(X)$ on projective birational models of X with say terminal singularities (see [Tsu02]) to see in as much our formulas for $P_m(Y)$ and the subsequent results in ([Gie22a]) apply.

Generalizing the framework:

- Work with *quasismooth* toric hypersurfaces (defined by the quotient construction of simplicial projective toric varieties), and similar questions for this setting, compare the two situations.

- Replace the assumption $f \in U_{reg}(\Delta)$ on f by other generic conditions on $f \in L(\Delta)$ introduced by I. Moissejewitsch, M. Michailowitsch Kapranov and Andrei Zelevinsky (see [GKZ94]) to get a more complete picture.

- Generalize the theory/ideas/results of this thesis to complete intersections in toric varieties.

- Working with $f \in L(\Delta)$ having Newton polytope slightly smaller than Δ might get interesting for examples and subfamilies of $\mathcal{X} \to B$ with monomial bases.

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1.12 Main references to our topics:

(a) *Minimal models of toric hypersurfaces:* The article ([Bat22]) of V.V. Batyrev.

(b) *The Hodge-Deligne numbers:* The article ([DK86]) of V.I. Danilov and A.G. Khovanskiĭ.

(c) *Infinitesimal deformations and the Kodaira Spencer map for curves in toric surfaces:* The PhD thesis ([Koe91]) of J. Koelman.

(d) *The automorphism group of a (normal projective) toric variety and its Lie algebra:* The article ([BG99]) of W. Bruns and J. Gubeladze.

(e) *The jacobian ring and the variations of (mixed) Hodge structures for non-degenerate toric hypersurfaces:* The article ([Bat93]) of V.V. Batyrev.

(f) *The infinitesimal Torelli theorem for smooth hypersurfaces in $\mathbb{P}^n$:* The chapter 6 of the second volume of the classical book of C. Voisin on Hodge theory and complex algebraic geometry ([Voi03]).

(g) *General results on toric varieties, including toric vanishing theorems and toric Serre duality theorems:* The classical book ([CLS11]) of D.A. Cox, J.B. Little and H.K. Schenck.

1.13 Notation

We recapitulate some *standard notation* (though most of our notions are defined within this thesis).

w.r.t.: „with respect to“
$|A|$: The cardinality of a (finite) set A.
gcd: The greatest common divisor.

convhull(S): The convex hull of a subset S of some vector space.
$\langle v_1, ..., v_k\rangle$: The span of $v_1, ..., v_k$ as a ($\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$)-vector space (in case $k = 2$ also the scalar product).

M: The standard lattice $\mathbb{Z}^n$. $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.
N: The dual lattice of M. $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.
T: The torus $N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

Given a rational polytope $P \subset M_{\mathbb{R}}$ let

Cone($P \times \{1\}$): The cone over the polytope P.
Int(P): The *relative* interior of P, that is the interior of P in an affine subspace of $M_{\mathbb{R}}$ of the same dimension as P.
Bound(P): The set $P \setminus \mathrm{Int}(P)$.

$L(P)$: The $\mathbb{C}$-vector space with basis the characters $\{\chi^m \mid m \in P \cap M\}$.
$L^*(P)$: The $\mathbb{C}$-vector space with basis the characters $\{\chi^m \mid m \in \mathrm{Int}(P) \cap M\}$.
$l(P)$: $\dim_{\mathbb{C}} L(P) = |P \cap M|$.

$l^*(P)$: $\dim_{\mathbb{C}} L^*(P) = |\operatorname{Int}(P) \cap M|$.
Σ_Δ: The normal fan of Δ.
$\mathbb{P}_\Delta$: The projective toric variety to the normal fan Σ_Δ of Δ.
$\mathbb{P}_\Sigma$: The toric variety to the fan Σ.

Given an n-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ and $f \in L(\Delta)$ let
$U_{reg}(\Delta)$: The set of nondegenerate Laurent polynomials with Newton polytope Δ.

We assume the following standard notation:

$$\Delta = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geqslant -b_i, \quad i = 1, ..., r\}, \tag{1.12}$$

and setting $\Gamma \leqslant \Delta$ to denote a facet of Δ we use the notation

$$\Gamma_i := \{x \in M_{\mathbb{R}} \mid \langle x, \nu_i \rangle \geqslant -b_i\} \cap \Delta, \qquad i = 1, ..., r.$$

Z_f: $\{f = 0\} \subset T$.

$Z_{\Delta,f}$: The Zariski closure of Z_f in $\mathbb{P}_\Delta$.

$Z_{\Sigma,f}$: The Zariski closure of Z_f in $\mathbb{P}_\Sigma$.

A variety X over a field k is understood in the sense of Hartshorne (see [Hart77, Ch.I, Def. after Remark 3.1.1, Ch.II, Prop.2.6]). We always assume $k = \mathbb{C}$. Given a smooth projective variety X, $\dim X = n$, let

Ω_X^p: The sheaf of differential p-forms on X
K_X: The canonical line bundle Ω_X^n (in case X is singular K_X is defined in section 5.1).
T_X: The tangent sheaf $(\Omega_X^1)^*$ on X

Given another normal projective algebraic variety Y, we denote a morphism between X and Y by $X \xrightarrow{p} Y$ and a rational map by $X \overset{p}{\dashrightarrow} Y$.

Given a sheaf $\mathcal{F}$ on X ($\mathcal{F}$ is always either a coherent sheaf of $\mathcal{O}_X$-modules or one of the constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) and Cartier divisors D, D' and $D_1, ..., D_n$ on X let

$\mathcal{O}_X(D)$: The invertible sheaf (of $\mathcal{O}_X$-modules) associated to D (In case D is a Weil divisor $\mathcal{O}_X(D)$ is just a rank 1 reflexive sheaf)

$p_*(\mathcal{F})$: The pushforward sheaf of $\mathcal{F}$ under p (given an open subset $\varnothing \neq U \subset Y$: $p_*(\mathcal{F})(U) := \mathcal{F}(p^{-1}(U))$).

$\mathcal{F}^*$: The dual sheaf of $\mathcal{F}$.

$H^k(X, \mathcal{F})$: The k-th sheaf cohomology group of $\mathcal{F}$.

$H^{p,q}(X, \mathbb{C})$: The Hodge component of $H^{p+q}(X, \mathbb{C})$ of type (p, q).

$H_c^{p,q} H^k(Z_f, \mathbb{C})$: The mixed Hodge component of coh. with support of Z_f of weigth k and type (p, q).

$h^{p,q}(X, \mathbb{C})$: The dimension of $H^{p,q}(X, \mathbb{C})$.

$h_c^{p,q} H^k(Z_f, \mathbb{C})$: The dimension of $H_c^{p,q} H^k(Z_f, \mathbb{C})$.

$D \sim_{lin} D'$: D is linear equivalent to D'.

$D_1 .... D_n$: The (topological) intersection number of $D_1, ..., D_n$ (compare [Laz00, Ch. 1.1.C]).

D^n : Abbreviation for $\underbrace{D....D}_{\text{n-times}}$.

CHAPTER 2

Toric varieties and nondegenerate toric hypersurfaces

In this first chapter we recall some basic definitions from algebraic geometry, the definition of projective toric varieties, toric morphisms, divisors on toric varieties and nondegenerate toric hypersurfaces. By an (algebraic) variety we always mean an irreducible variety over the complex numbers $\mathbb{C}$. A curve (surface) is an algebraic variety of dimension 1 (2).

Let X be a normal projective variety: A divisor D on X is a *Weil divisor*, that is a $\mathbb{Z}$-linear combination of subvarieties of X of codimension 1. D is *Cartier* if there is a covering $\{U_i\}_{i \in I}$ of X such that

$$D_{|U_i} = div(f_i)$$

is the divisor associated to a rational function f_i. D is $\mathbb{Q}$-Cartier if $m \cdot D$ is Cartier for some $m \in \mathbb{N}_{\geqslant 1}$. Whether D is ($\mathbb{Q}$-)Cartier or not just depends on the linear equivalence class of D. The sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are isomorphic as $\mathcal{O}_X$-modules if and only if $D \sim_{lin} D'$ (see [Rei79, App. to §1]).

X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier. Given normal algebraic varieties X, Y, a surjective morphism $\phi : X \to Y$ let D be a Cartier divisor on X with $D_{|U} = div(f)$ on some Zariski open subset U. Then on $V := \phi^{-1}(U)$ the pullback $\phi^*(D)$ of D is defined as

$$\phi^*(D)_{|V} = div(f \circ \phi_{|V}).$$

$\phi^*(D)$ is again Cartier and the functor ϕ^* respects linear equivalence.

Definition 2.0.1. *A divisor D on an n-dimensional complete normal variety X is called nef if*

$$D.C \geqslant 0$$

for all irreducible curves $C \subset X$. A nef divisor D is called big if $D^n > 0$.

For X, Y normal algebraic varieties,

$$\phi : X \to Y$$

a surjective morphism and $D \subset Y$ a Cartier divisor, then D is nef if and only if $\phi^*(D)$ is nef ([Laz00, Ex.1.4.4]).

2.1 Toric Vaieties

In this section we recall some basic facts about toric varieties from ([CLS11]): Let M denote an n-dimensional lattice $\mathbb{Z}^n$ with dual lattice N. We write $M_{\mathbb{R}}$ for $M \otimes \mathbb{R}$ and let

$$T := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$$

be the n-dimensional torus. By a rational polytope $F \subset M_{\mathbb{R}}$ we mean a polytope, whose vertices have coordinates in $\mathbb{Q}$. We may represent a rational polytope F as intersection of finitely many half-planes

$$F = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geqslant -b_i,\, i = 1, ..., r\},$$

where $\nu_i \in N$ are primitive and $b_i \in \mathbb{Q}$. To F is associated a normal fan Σ_F. The normal fan consists of a collection of cones in $N_{\mathbb{R}}$, such that the 1-dimensional cones in Σ_F have cone generators n_i and different n_i's span a cone in Σ_F if and only if the facets the n_i are normal to intersect.

We denote the i-dimensional cones of a fan Σ by $\Sigma[i]$. For $i = 1$ we identify $\Sigma[1]$ with the primitive lattice points n_i generating the rays. For Δ an n-dimensional lattice polytope or more generally a rational polytope let

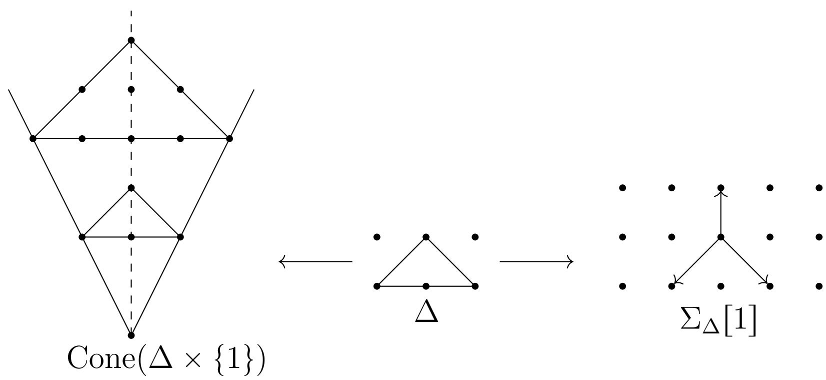
$$\mathrm{Min}_{\Delta}(\nu) := \min_{m \in \Delta} \langle m, \nu \rangle, \quad \nu \in N,$$

such that $\mathrm{Min}_{\Delta}(\nu_i) = -b_i$. Given a complete fan Σ, that is a fan whose support equals $N_{\mathbb{R}}$, we associate a complete normal toric variety, which we denote by $\mathbb{P}_\Sigma$. If $\Sigma = \Sigma_F$ is the normal fan to F, $\mathbb{P}_{\Sigma_F}$ is projective and we denote it by $\mathbb{P}_F$. Since we construct toric varieties via fans all toric varieties we consider are normal as algebraic varieties. If $\Delta \subset M_\mathbb{R}$ is a lattice polytope there is a different way to construct $\mathbb{P}_\Delta$: Take the cone $\mathrm{Cone}(\Delta \times \{1\})$ over Δ and the semigroup algebra

$$S_\Delta := \mathbb{C}[\mathrm{Cone}(\Delta \times \{1\}) \cap (M \times \mathbb{Z})].$$

By ([CLS11, Thm.7.A.1])

$$\mathbb{P}_\Delta \cong \mathrm{Proj}(S_\Delta).$$



The polytope Δ in the middle and two possible constructions of $\mathbb{P}_\Delta$: Via the cone over Δ (on the left) and via the normal fan (on the right).

To $\sigma \in \Sigma[n-k]$ is associated a k-dimensional torus orbit $\mathcal{O}(\sigma)$ of $\mathbb{P}_\Sigma$. The closure $V(\tau)$ of $\mathcal{O}(\tau)$ in $\mathbb{P}_\Sigma$ equals

$$V(\tau) = \bigcup_{\tau \leqslant \sigma} \mathcal{O}(\sigma).$$

By definition the **canonical divisor** $K_{\mathbb{P}_\Sigma}$ is the divisor on $\mathbb{P}_\Sigma$ associated to a *rational differential form*, for example the form

$$\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}$$

which is regular on T. We get

$$K_{\mathbb{P}_\Sigma} = -\sum_{\nu_i \in \Sigma[1]} D_i. \tag{2.1}$$

Proposition 2.1.1. *([CLS11, Prop.4.2.7])*
$\mathbb{P}_\Sigma$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Sigma$ is simplicial, that is the generators ν_i of σ are linearly independent over $\mathbb{R}$.

$\square$

Construction 2.1.2. Given lattices N and N', fans Σ and Σ' in N and N' and a homomorphism of lattices

$$\overline{\phi} : N \to N'$$

with $\mathbb{R}$-linear extension $\overline{\phi}_\mathbb{R} : N'_\mathbb{R} \to N_\mathbb{R}$ assume that for every $\sigma' \in \Sigma'$ there is $\sigma \in \Sigma$ with

$$\overline{\phi}_\mathbb{R}(\sigma') \subset \sigma.$$

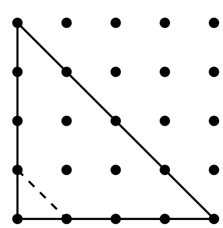
Using $T \cong N \otimes_\mathbb{Z} \mathbb{C}^*$, $T' \cong N' \otimes_\mathbb{Z} \mathbb{C}^*$ the homomorphism

$$n \otimes z \mapsto \overline{\phi}(n') \otimes z$$

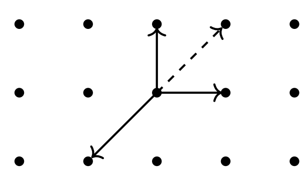
between tori continues to a morphism $\phi : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma$ between toric varieties ([CLS11, Ch.3.3, Thm.3.3.4]). By definition ϕ is a **toric morphism**. By ([HLY02, Prop.2.1.4]) any irreducible fiber of ϕ admits again the structure of a toric variety.

Proposition 2.1.3. *([CLS11, Ch.3.3])*
Given two complete fans Σ and Σ' such that $\Sigma[1]$ and $\Sigma'[1]$ belong to the same lattice N and Σ' refines Σ there is an induced birational morphism $p : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma$.

$\square$



$\Delta = 4 \cdot \Delta_2$



$\Sigma_\Delta[1]$

Illustration of the blow up of $\mathbb{P}^2$ at a torus fixed point: The normal fan Σ_Δ gets refined by inserting the dashed ray $(1,1)$. By cutting off the vertex $(0,0)$ in the picture for Δ on the left we get a new polytope Δ' whose normal fan $\Sigma_{\Delta'}$ equals the refinement on the right.

Proposition 2.1.4. *([CLS11, Prop.3.3.7])*
Let $N' \subset N$ be a sub-lattice of finite index $(N : N')$ and let $\Sigma' \subset N'_{\mathbb{R}}$ be a fan. With respect to the lattice N, the fan Σ is denoted by Σ'. Then the inclusion $N' \subset N$ induces a finite toric morphism $\mathbb{P}_{\Sigma'} \to \mathbb{P}_{\Sigma}$ of degree $(N : N')$.

□

Example 2.1.5. For the following figure let

$$N_2 := \{ (2n_1, 2n_2, 2n_3) \mid \quad n_1, n_2, n_3 \in N \} \subset N.$$

and $\mathbb{P}_{\Sigma} = \mathbb{P}^2$. Write $(t_0 : t_1 : t_2)$ for the homogeneous coordinates on $\mathbb{P}^2$ and

$$T = \left\{ \left(1, \frac{t_1}{t_0}, \frac{t_2}{t_0}\right) \in \mathbb{P}^2 \,|\, t_0, t_1, t_2 \neq 0 \right\}$$

for the torus in $\mathbb{P}^2$. The inclusion $\overline{\phi}_2 : N' \to N$ induces the following homomorphism $(\phi_2)_T$ between tori

$$(\phi_2)_T\left(1, \frac{t_1}{t_0}, \frac{t_2}{t_0}\right) := \left(1, \left(\frac{t_1}{t_0}\right)^2, \left(\frac{t_2}{t_0}\right)^2\right).$$

$(\phi_2)_T$ continues to the toric morphism

$$\begin{aligned} &\phi_2 : \mathbb{P}^2 \to \mathbb{P}^2 \\ &(t_0 : t_1 : t_2) \mapsto (t_0^2 : t_1^2 : t_2^2). \end{aligned}$$

Given a toric variety $\mathbb{P}$ and $l \in \mathbb{N}_{\geqslant 1}$ in Construction 5.3.2 we recall a *multiplication map* ϕ_l due to *Fujino* generalizing the map ϕ_2 from above.

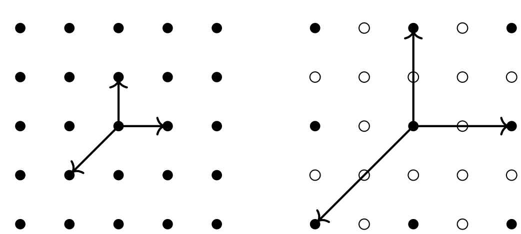


Illustration of the fan Σ_{Δ_2} of $\mathbb{P}^2$ with respect to the standard lattice N (on the left) and with respect to the sub-lattice N_2 (on the right).

Construction 2.1.6. To a divisor

$$D = \sum_{i=1}^{r} a_i D_i, \quad a_i \in \mathbb{Z}$$

on a complete toric variety $\mathbb{P}_\Sigma$ we associate a polytope

$$P_D := \{x \in M_\mathbb{R} | \langle x, \nu_i \rangle \geqslant -a_i, \quad \nu_i \in \Sigma[1]\}, \tag{2.2}$$

which is at least rational, and which computes the global sections of D, that is (compare [CLS11, Prop.4.3.3])

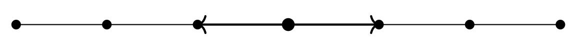
$$H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m,$$

where for $m = (m_1, ..., m_n)$ the function χ^m denotes the character

$$t = (t_1, ..., t_n) \mapsto t^m := t_1^{m_1} \cdot ... \cdot t_n^{m_n}$$

of T. Note that given $k \geqslant 1$ the polytope P_{kD} associated to $k \cdot D$ equals $k \cdot P_D$ ([CLS11, Exc. 4.3.2]).

Example 2.1.7. Up to isomorphism there is only one complete 1-dimensional fan Σ, namely



with toric variety $\mathbb{P}_\Sigma \cong \mathbb{P}^1$. It follows that given complete fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$, where

$$\dim\, N'_\mathbb{R} = \big(\dim\, N_\mathbb{R}\big) + 1$$

and a toric morphism $p : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma$ of relative dimension 1, the fiber F of p is isomorphic to $\mathbb{P}^1$ since p is proper.

Example 2.1.8. Let Δ be an n-dimensional lattice simplex, that is $|\Sigma_\Delta[1]| = n + 1$, say

$$\Sigma_\Delta[1] = \{\nu_0, ..., \nu_n\},$$

then $\mathbb{P}_\Delta$ is denoted a **fake weighted projective space**. In this case there are unique $q_0, ..., q_n \in \mathbb{N}_{\geqslant 1}$ with $\gcd(q_0, ..., q_n) = 1$ and

$$\sum_{i=0}^{n} q_i \nu_i = 0.$$

The numbers $q_0, ..., q_n$ are the weights of $\mathbb{P}_\Delta$. If $\nu_0, ..., \nu_n$ generate the lattice N then $\mathbb{P}_\Delta$ is a **weighted projective space**. The weights $q_0, ..., q_n$ determine $\mathbb{P}_\Delta$ up to isomorphism and we write

$$\mathbb{P}_\Delta = \mathbb{P}(q_0, ..., q_n).$$

If

$$[N : \mathbb{Z} \cdot q_0 + ... + \mathbb{Z} \cdot q_n] > 1,$$

then $\mathbb{P}_\Delta$ depends on $q_0, ..., q_n$ and the torsion in $\mathrm{Cl}(\mathbb{P}_\Delta)$ (see [Kas09]).

2.2 Nondegenerate hypersurfaces in toric varieties

Definition 2.2.1. *Let f be a Laurent polynomial with presentation*

$$f = \sum_{m \in A} a_m z^m, \quad a_m \in \mathbb{C} \tag{2.3}$$

for some finite nonempty subset $A \subset M$. The support of f is defined as

$$\mathrm{Supp}(f) := \Big\{ m \in A \,|\, a_m \neq 0 \Big\}.$$

The Newton polytope is the convex hull of $\mathrm{Supp}(f)$.

The presentation in (2.3) we be our standard notation for f. We always assume that the affine span of A over $\mathbb{R}$ equals $M_\mathbb{R}$, though A is not required to generate M affinely over $\mathbb{Z}$.

Definition 2.2.2. *Given a Laurent polynomial f with Newton polytope Δ we call f nondegenerate w.r.t. Δ (or Δ-regular) if Z_f is smooth and for every face σ of Δ with associated torus orbit $\mathcal{O}(\sigma)$ of $\mathbb{P}_\Delta$ the intersection $Z_{\Delta,f} \cap \mathcal{O}(\sigma)$ is either empty or smooth of codimension one in $\mathcal{O}(\sigma)$.*

Remark 2.2.3. This condition may also be expressed by saying that Z_f is smooth and for every face Γ of Δ

$$f_{|\Gamma},\, x_1 \cdot \frac{\partial f_{|\Gamma}}{\partial x_1}, ..., x_n \cdot \frac{\partial f_{|\Gamma}}{\partial x_n} \tag{2.4}$$

have no common zero in $(\mathbb{C}^*)^n$, where

$$f_{|\Gamma} := \sum_{m \in A \cap \Gamma} a_m x^m.$$

We denote the set of nondegenerate Laurent polynomials $f \in L(\Delta)$ by $U_{reg}(\Delta)$. Throughout this thesis f is always assumed to be nondegenerate. We take abbreviations like: For a lattice polytope Δ and a given f to mean that f is a nondegenerate Laurent polynomial with Newton polytope Δ.

Remark 2.2.4. Given an n-dimensional lattice polytope Δ and $A = \Delta \cap M$ the condition for f to be Δ-regular is a Zariski open nonempty condition on the coefficients $(a_m)_{m \in A}$, given by the non-vanishing locus of the *principal A-determinant* E_A (see [GKZ94, Ch.10]) for the case that A affinely generates M and [Bat03, Prop.2.16] else).

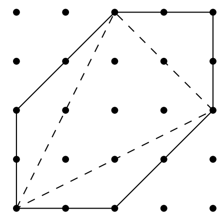
Let us check at hand that the condition in Remark 2.2.3 is true on a nonempty Zariski open subset of $\mathbb{C}^{\#A}$ if A consists just of the vertices of an n-dimensional lattice polytope Δ:

Let Γ be a k-dimensional face of Δ. Choose $k+1$ affine linear independent vertices of Γ and take the k-dimensional simplex Γ' which is the convex hull of these vertices. Assume that the assertion holds for simplices then it is fulfilled for Γ'.

Varying the coefficients of the remaining vertices of Γ in a Zariski open subset does not violate the condition in Remark 2.2.3. Thus iterating over all faces Γ of Δ we have to intersect finitely many nonempty Zariski open subsets to get our subset. We are left to deal with the case that Δ is a simplex with vertices $v_0, ..., v_n$. We apply an unimodular transformation to Δ (this is allowed by [GKZ94, Ch.9 Prop.1.4, Ch.10 Thm.1.2]) such that $v_0, ..., v_{n-1} \in \{x_n = 0\}$. Setting

$$x_n \cdot \frac{\partial f}{\partial x_n} = 0$$

in (2.4) we must have $x^{v_n} = 0$ and there is no solution in the torus $(\mathbb{C}^*)^n$.



A lattice polygon Δ and a subsimplex whose vertices are also vertices of Δ.

Construction 2.2.5. Given $f \in L(\Delta)$ the closure Y_f in the toric variety $\mathbb{P}_\Sigma$ to an n-dimensional complete fan Σ is a Weil divisor linear equivalent to

$$Y_f \sim_{lin} -\sum_{\nu_i \in \Sigma[1]} \mathrm{Min}_\Delta(\nu_i) \cdot D_i. \tag{2.5}$$

(see [Bat22, Prop.7.1]). Therefore we are just interested in the linear equivalence class of Y_f. Since Δ is always a lattice polytope Y is an integral divisor. The divisor Y is Cartier if and only if Min_Δ is a *support function* for Y, that is

$$\mathrm{Min}_\Delta : N_\mathbb{R} \to \mathbb{R}$$

is linear on each cone of Σ and $\mathrm{Min}_\Delta(N) \subset \mathbb{Z}$. Similarly Y is $\mathbb{Q}$-Cartier if just $\mathrm{Min}_\Delta(N) \subset \mathbb{Q}$.

CHAPTER 3

Minimal models of nondegenerate toric hypersurfaces

In this chapter we introduce the necessary definitions and methods both from the combinatorial point of view and from the background in algebraic geometry to construct minimal models of nondegenerate toric hypersurfaces. Largely we follow the article ([Bat22]).

3.1 Modifications of the Newton polytope Δ

Definition 3.1.1. *An n-dimensional rational polytope $\Delta \subset M_{\mathbb{R}}$ has a presentation*

$$\Delta = \{x \in M_{\mathbb{R}} \,|\, \langle x, \nu_i \rangle \geqslant \mathrm{Min}_{\Delta}(\nu_i),\, \nu_i \in \Sigma_{\Delta}[1]\}$$

and we define the Fine interior $F(\Delta)$ of Δ as

$$F(\Delta) := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geqslant \mathrm{Min}_{\Delta}(\nu) + 1,\, \nu \in N \backslash \{0\}\}.$$

Remark 3.1.2. The Fine interior was introduced by J. Fine in ([Fine83]). In general it is only a rational polytope though if $\dim \Delta = 2$ and Δ is a lattice polytope then $F(\Delta)$ turns out to be a lattice polytope as well, namely it equals the convex span of the interior lattice points of Δ ([Bat17, Prop.2.9]).

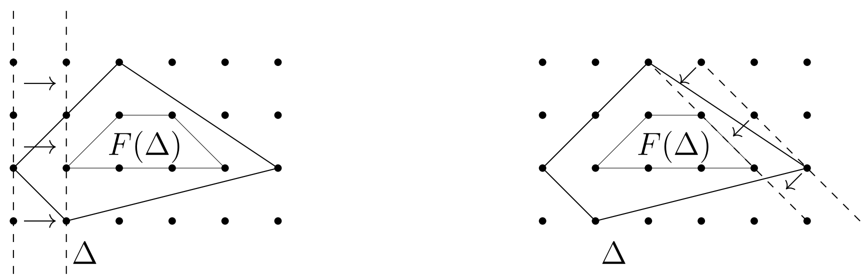


Illustration of the construction of the Fine interior $F(\Delta)$ from Δ.

Remark 3.1.3. In order to construct the Fine interior $F(\Delta)$ of Δ we have to move every hyperplane which touches some face of Δ „one step“ into the interior of Δ. In general it is not enough to move just the hyperplanes defining facets one step into the interior (see Figure 3.1). Even worse it might happen that an hyperplane cuts out a facet of Δ but does not touch $F(\Delta)$ after replacing one step into the interior of Δ. This happens if and only if $\Delta \neq C(\Delta)$ (see Lemma 3.1.7 below).

Obviously we have $L(F(\Delta)) = L^*(\Delta)$. If $\Delta := d \cdot \Delta_n$ with Δ_n the n-dimensional standard simplex we have

$$F(\Delta) = (d - n - 1) \cdot \Delta_n.$$

This is because if we just move the facets of Δ one step into the interior we already get the lattice polytope $(d - n - 1) \cdot \Delta_n = \text{convhull}(\text{Int}(\Delta) \cap M)$.

Definition 3.1.4. Let Δ be a rational polytope with $F(\Delta) \neq \varnothing$. The set of lattice points $\nu \in N \backslash \{0\}$ with

$$\text{Min}_{F(\Delta)}(\nu) = \text{Min}_{\Delta}(\nu) + 1$$

is called the support $S_F(\Delta)$ of $F(\Delta)$ to Δ.

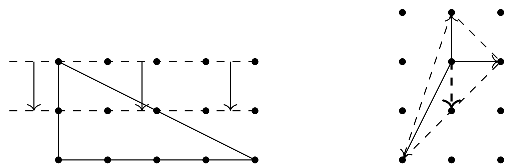


Figure on the support vectors: On the left Δ and on the right $\Sigma_\Delta[1]$. $F(\Delta)$ equals the unique interior lattice point of Δ. By ([Bat22, Prop.3.11]) the support vectors $S_F(\Delta)$ are always contained in the convex span of the rays $\Sigma_\Delta[1]$. In particular $S_F(\Delta)$ is a finite set. The above pictures show that $(0,-1) \in S_F(\Delta)$.

Definition 3.1.5. *For Δ a rational polytope with $F(\Delta) \neq \emptyset$*

$$C(\Delta) := \{x \in M_{\mathbb{R}} \,|\, \langle x, \nu \rangle \geqslant \mathrm{Min}_\Delta(\nu) \quad \forall\, \nu \in S_F(\Delta)\}$$

is called the canonical closure of Δ. Δ is canonically closed if $C(\Delta) = \Delta$.

Remark 3.1.6. *([Bat22, Prop.3.17(b), Cor.3.19, Prop.4.4])*

Given a lattice polytope Δ with $F(\Delta) \neq \emptyset$

$$\Delta \subset C(\Delta), \quad C(C(\Delta)) = C(\Delta), \quad F(C(\Delta)) = F(\Delta).$$

Besides

- $\dim(\Delta) = 2:$ $C(\Delta) = \Delta$.
- $\dim(\Delta) = 3$: $C(\Delta) = \Delta$ in all known examples.
- $\dim(\Delta) \geqslant 4$: $C(\Delta)$ is just a rational polytope in general.

Let us summarize useful properties of $C(\Delta)$ in a technical lemma:

Lemma 3.1.7. *([Bat22, Prop.3.17a), Cor.3.18, Prop.4.3]*
An n-dimensional lattice polytope Δ with $F(\Delta) \neq \emptyset$ is canonically closed if and only if $\Sigma_\Delta[1] \subset S_F(\Delta)$. Besides $S_F(C(\Delta)) = S_F(\Delta)$ and for $\nu \in S_F(\Delta)$

$$\mathrm{Min}_{C(\Delta)}(\nu) = \mathrm{Min}_\Delta(\nu).$$

Construction 3.1.8. *([Bat22, Thm.6.3])*
Let $\Delta \subset M_{\mathbb{R}}$ be an n-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Define $\tilde{\Delta}$ as Minkowski sum

$$\tilde{\Delta} := C(\Delta) + F(\Delta)$$

The normal fan $\Sigma_{\tilde{\Delta}}$ is the coarsest refinement of $\Sigma_{C(\Delta)}$ and $\Sigma_{F(\Delta)}$. Besides

$$\Sigma_{\tilde{\Delta}}[1] \subset S_F(\Delta).$$

Let Σ be a simplicial fan with $\Sigma[1] = S_F(\Delta)$, which refines $\Sigma_{\tilde{\Delta}}$.

3.2 The construction of minimal models

For convenience we write $\mathbb{P}$ instead of $\mathbb{P}_\Sigma$ and denote the closure of Z_f in $\mathbb{P}$ by Y_f or Y. There is a diagram

$$\begin{tikzcd} & \mathbb{P} \arrow[d, "\pi"] & \\ & \mathbb{P}_{\tilde{\Delta}} \arrow[dl, "\rho"'] \arrow[dr, "\theta"] & \\ \mathbb{P}_{C(\Delta)} & & \mathbb{P}_{F(\Delta)} \end{tikzcd} \tag{3.1}$$

where π and ρ are birational since both $\Sigma_{\tilde{\Delta}}$ refines $\Sigma_{C(\Delta)}$ and Σ refines $\Sigma_{\tilde{\Delta}}$ in N. The morphism θ is birational if and only if

$$\dim(F(\Delta)) = \dim(\Delta).$$

Summarizing results: We explain the notions „terminal “and „canonical“ singularities in the Appendix of this chapter (section 3.5).

Theorem 3.2.1. *([Bat22, Thm.7.5])*
Let Δ be an n-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then the closures $Z_{\tilde{\Delta},f}$ and Y_f of Z_f define normal algebraic varieties, since they do not contain any $(n-2)$-dimensional torus orbit of $\mathbb{P}_{\tilde{\Delta}}$ and $\mathbb{P}$.

Proposition 3.2.2. *([Bat22, Prop.7.4])*
The divisor $Y \subset \mathbb{P}$ is nef, big and $\mathbb{Q}$-Cartier.

Definition 3.2.3. *Given a normal projective variety Y birational to Z_f with at most terminal singularities and K_Y nef Y is called a minimal model of Z_f.*

Theorem 3.2.4. *([Bat22, Cor.6.6])*
Given an n-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ with $F(\Delta) \neq \emptyset$, the toric variety $\mathbb{P}$ has at most terminal singularities. The adjoint divisor $K_{\mathbb{P}} + Y$ is nef.

Remark 3.2.5. The adjunction formula

$$K_Y = (K_{\mathbb{P}} + Y)_{|Y}$$

applies since Y does not contain any $(n-2)$-dimensional torus orbit of $\mathbb{P}$ (see [Bat22, Thm.7.5]).

Corollary 3.2.6. *[Bat22, Thm.8.2]*
Let Δ be an n-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then $Y = Y_f$ is a minimal model of Z_f.

Remark 3.2.7. $\mathbb{P}_{\tilde{\Delta}}$ has at most canonical singularities and the morphism $\pi : \mathbb{P} \to \mathbb{P}_{\tilde{\Delta}}$ is crepant, that is $\pi^*(K_{\mathbb{P}_{\tilde{\Delta}}}) = K_{\mathbb{P}}$ and similarly for $Z_{\tilde{\Delta},f}$ and the morphism $Y_f \to Z_{\tilde{\Delta},f}$ (see [Bat22, Cor.6.5, Thm.8.1]).

3.3 Further properties of Y

Lemma 3.3.1. *$Y \subset \mathbb{P}$ is Cartier if and only if $C(\Delta)$ is a lattice polytope.*

Proof. We work with the representation of the linear equivalence class of Y_f from Construction 2.2.5. By Lemma 3.1.7 given $\nu \in \Sigma[1] = S_F(\Delta)$

$$\mathrm{Min}_{\Delta}(\nu) = \mathrm{Min}_{C(\Delta)}(\nu).$$

The function $\mathrm{Min}_{C(\Delta)} : N_{\mathbb{R}} \to \mathbb{R}$ is linear on the cones of Σ since Σ refines the normal fan of $C(\Delta)$. Thus

$$Y \subset \mathbb{P} \text{ Cartier } \Leftrightarrow \mathrm{Min}_{C(\Delta)}(N) \subset \mathbb{Z} \Leftrightarrow C(\Delta) \text{ is a lattice polytope.}$$

□

Remark 3.3.2. *(Compatibility of nondegeneracy of f with the fan Σ)*

Given an n-dimensional lattice polytope Δ and a Laurent polynomial f the singular locus $\mathbb{P}_{sing}$ of $\mathbb{P}$ always equals the union of torus orbits of $\mathbb{P}$ ([CLS11, Prop.11.1.2]). We mention two easier cases

- $C(\Delta) = \Delta$: Here Σ refines Σ_Δ and Y intersects the toric strata of $\mathbb{P}$ transversely in a subset of codimension 1 ([Bat94], see also [Tre10, Prop.5.1.3]) and Y behaves nondegenerate w.r.t. $\mathbb{P}$ just as Z_Δ does w.r.t. $\mathbb{P}_\Delta$. As a consequence
$$Y_{sing} = Y \cap \mathbb{P}_{sing}.$$
- $C(\Delta)$ is a lattice polytope: Here $Y \subset \mathbb{P}$ is Cartier by Lemma 3.3.1 and $Y_{sing} \supset Y \cap \mathbb{P}_{sing}$ by the jacobian criterion for smoothness (see [Hart77, Ch.I.5 Definition]).

General situation: In general Y should define a quasismooth hypersurface in $\mathbb{P}$. This would imply
$$Y_{sing} \subset Y \cap \mathbb{P}_{sing}$$
by ([BaCo94, Def.3.2, Rem.3.3]).

Remark 3.3.3. For an n-dimensional lattice polytope Δ with $F(\Delta) \neq \varnothing$ and a given f, the polytope associated to $Y = Y_f$ equals $C(\Delta)$ by Construction 2.1.6, that is
$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(Y)) \cong L(C(\Delta)).$$
Besides the polytope associated to $Y + K_{\mathbb{P}}$ equals $F(\Delta)$ by Construction 2.2.5 and formula (2.2.5).

Definition 3.3.4. *Given a normal projective surface Y, we write $\kappa(Y)$ for the Kodaira dimension of Y, which is defined to be the Kodaira dimension of a resolution of singularities Y' of Y. The latter could be defined as the number $\kappa := \kappa(Y')$ measuring the growth of the plurigenera of Y', that is*
$$a \cdot m^\kappa \leqslant h^0(Y', mK_{Y'}) \leqslant A \cdot m^\kappa$$
for some constants $a, A > 0$ and all sufficiently large and divisible $m \in \mathbb{N}$ (see [Laz00, Cor.2.1.38]).

Theorem 3.3.5. *([Bat22, Thm.9.2])*
Let $\Delta \subset M_{\mathbb{R}}$ be an n-dimensional lattice polytope with $F(\Delta) \neq \varnothing$ and $k := \dim\, F(\Delta)$. Then Y has Kodaira dimension

$$\kappa(Y) = \begin{cases} n-1, & k = n \\ n-1 & k = n-1 \\ k & k < n-1. \end{cases}.$$

Example 3.3.6. Given $\Delta = d \cdot \Delta_n$ with $d \geqslant n+1$ then

$$F(\Delta) = (d-n-1) \cdot \Delta_n \neq \varnothing, \quad C(\Delta) = \Delta$$

and $\mathbb{P} = \mathbb{P}_\Delta = \mathbb{P}^n$. Given an f the closure $Y = Z_{\Delta,f} \subset \mathbb{P}^n$ is smooth (even slightly more special) and K_Y is nef.

Definition 3.3.7. A *reflexive* polytope Δ is a lattice polytope with facet presentation

$$\Delta = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geqslant -1\}.$$

Example 3.3.8. Given such a polytope

$$F(\Delta) = \{0\}, \quad C(\Delta) = \Delta \qquad \Rightarrow \qquad \tilde{\Delta} = \Delta$$

and $S_F(\Delta)$ equals the lattice points on the boundary of the dual polytope of Δ by ([Bat22, Prop.4.9]).

3.4 Three-dimensional lattice polytopes Δ with $l^*(\Delta) = 1$ and $\dim\, F(\Delta) = 3$

Example 3.4.1. There are 49 three-dimensional lattice polytopes Δ with

$$\mathrm{Int}(\Delta) \cap M = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \dim\, F(\Delta) = 3.$$

(see [Sch18, Appendix A.3]). We list them in Tables 9.1 and 9.2 and picture those with $C(\Delta) = \Delta$ in Figures 9.1 and 9.2.

Remark 3.4.2.

- The 49 polytopes are similar to reflexive polytopes: There is exactly one $\Gamma \leqslant \Delta$ with integral distance 2 to $(0,0,0)$ (see the pictures). All other facets have distance 1 to $(0,0,0)$.
- Up to unimodular equivalence there are just 5 rational polytopes P occurring as Fine interior $F(\Delta) = P$.
- For P one of these 5 polytopes there is exactly one maximal polytope Δ w.r.t. „$\subset$“with $F(\Delta) = P$.
- Given a maximal polytope Δ up to replacing Δ the Fine interior $F(\Delta)$ is proportional to Δ.

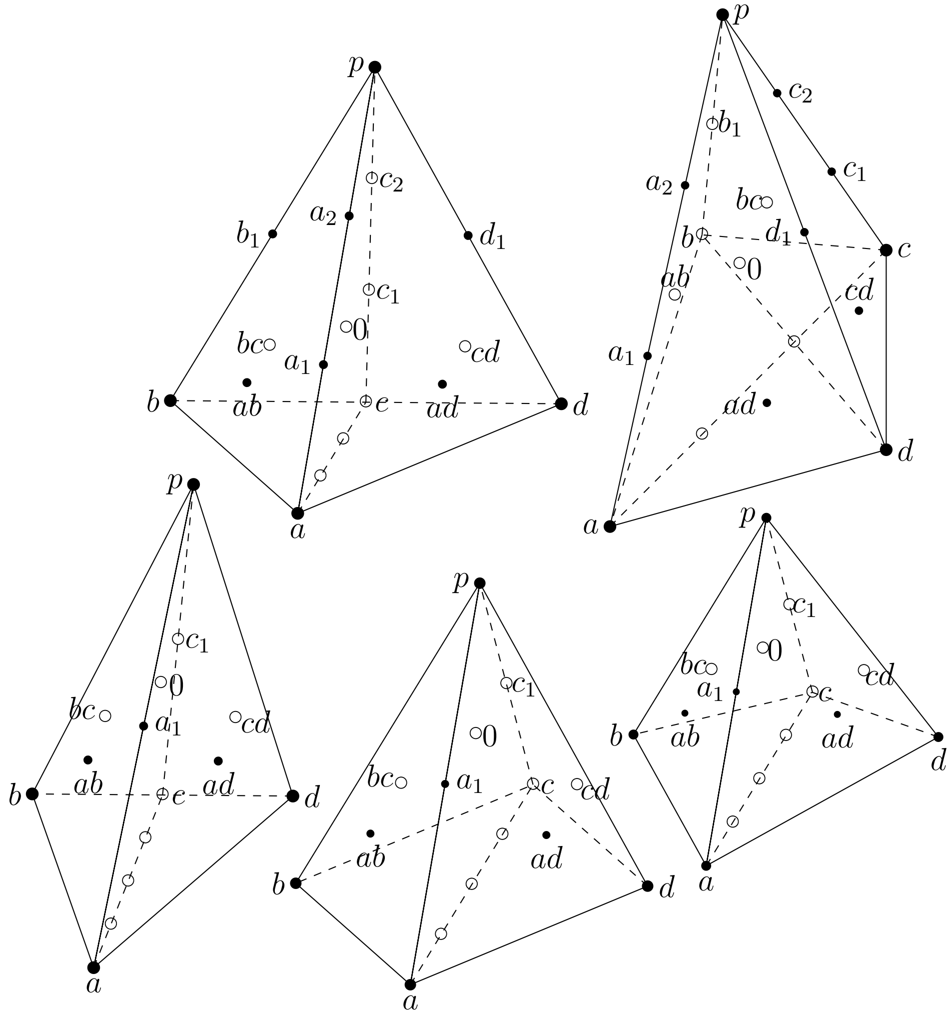
Dividing the 49 polytopes into classes $a), b), c), d)$ and $e)$ according to their Fine interior, we picture the maximal polytopes in Figure 3.3. In $a)$ and $c)$ the maximal polytopes are simplices, $\mathbb{P}_\Delta$ is a fake weighted projective space to the weights

$$a) : (2,1,1,1), \quad c) : (2,2,1,1).$$

In $c)$, $d)$ and $e)$ the Fine interior $F(\Delta)$ equals

$$\begin{aligned}
&c) : F(\Delta) = \langle (0,0,0), (1,1/2,2), (1,1/4,1), (1,3/4,1) \rangle \\
&d) : F(\Delta) = \langle (0,0,0), (1,-1/2,1), (1,-1/2,0), (1,-3/4,1/2), (1,-1/4,1/2) \rangle \\
&e) : F(\Delta) = \langle (0,0,0), (1,3/2,-1), (1,3/4,0), (1,1/2,0), (1,3/4,-1/2) \rangle
\end{aligned}$$

Concerning the other 46 polytopes we list the Fine interior in the Tables 9.1 and 9.2 and Figure 3.3.



The maximal polytopes in the classes a), b), c), d) and e).

$a)$:$\Delta = \langle a = (2, 1, -2), b = (2, 0, 1), d = (2, 2, 1), p = (-4, -2, 1)\rangle$

$b)$:$\Delta = \langle a = (2, -3, 1), b = (2, -1, 2), c = (2, 0, 1), d = (2, -1, 0), p = (-4, 3, -2)\rangle$

$c)$:$\Delta = \langle a = (2, 1, 5), p = (-2, -1, -3), b = (2, 0, 1), d = (2, 2, 1)\rangle$

$d)$:$\Delta = \langle a = (2, -1, 3), b = (2, 0, 1), c = (2, -1, -1), d = (2, -2, 1),$
$p = (-2, 1, -1)\rangle$

$e)$:$\Delta = \langle a = (2, 0, 1), b = (2, 1, -1), c = (2, 4, -3), d = (2, 1, 1), p = (-2, -2, 1)\rangle$

3.5 Appendix: Singularities of the minimal model program

In order to complete the definition of this chapter we mention the following results:

Definition 3.5.1. *[Rei83, (1.11)]*
Consider a fan Σ in $N_{\mathbb{R}}$ and a cone σ of Σ. Then σ is called canonical (of index $j \in \mathbb{N}$), if there exists a primitive vector $m \in M$ such that

$$\langle m, \nu_i \rangle = j \quad \text{for } \nu_i \in \sigma[1]$$

and

$$\langle m, n \rangle \geqslant j \quad \text{for } n \in \sigma \cap N,\ n \notin \{(0, ..., 0)\} \cup \sigma[1]. \tag{3.2}$$

σ is called terminal if we have strict inequality in (3.2). The fan Σ is called canonical (terminal) if all its cones are canonical (terminal).

Definition 3.5.2. *([Rei87])*
A normal algebraic variety Y (over $\mathbb{C}$) is said to have at most terminal (canonical) singularities if K_Y is $\mathbb{Q}$-Cartier and writing m for the smallest natural number with mK_Y Cartier, then for every resolution of singularities $\sigma : Y' \to Y$ we have

$$mK_{Y'} = \sigma^*(mK_Y) + \sum_{i=1}^{r} a_i E_i, \tag{3.3}$$

where a_i are integers with $a_i > 0$ ($a_i \geqslant 0$). Here $E_1, ..., E_r$ are the exceptional divisors of σ.

According to [Rei83, (1.12)] we have the following result:

Theorem 3.5.3. *The toric variety $\mathbb{P}$ has at most canonical (terminal) singularities if and only if the fan Σ is canonical (terminal).*

Remark 3.5.4. *([Rei87])*
A normal algebraic surface X has canonical (terminal) singularities if and only if it has at most rational double points (is smooth). A variety with terminal singularities has a locus of singularities of codimension $\geqslant 3$.

CHAPTER 4

The plurigenera of minimal models

In this section we compute the plurigenera $h^0(Y, mK_Y)$ of a minimal toric hypersurface and derive from this a formula for the maximal self intersection number K_Y^{n-1}. Together with the vanishing $q(Y) = 0$ this allows us to compute the main invariants of the algebraic surfaces from section 3.4.

4.1 Two toric vanishing Theorems

There are two vanishing Theorems which we will use several times:

Theorem 4.1.1. *(Demazure's vanishing Theorem), ([CLS11, Thm.9.2.3]) Let $\mathbb{P}$ be a complete normal toric variety and D a $\mathbb{Q}$-Cartier nef divisor on $\mathbb{P}$. Then*

$$H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D)) = 0, \quad p > 0.$$

Theorem 4.1.2. *(Batyrev-Borisov vanishing Theorem) , ([CLS11, Thm.9.2.7]) For D a $\mathbb{Q}$-Cartier nef divisor D on a complete normal toric variety $\mathbb{P}$*

$$H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-D)) = \begin{cases} 0 & p \neq \dim P_D \\ \bigoplus\limits_{m \in L^*(-P_D)} \mathbb{C} \cdot \chi^m & p = \dim P_D \end{cases},$$

Remark 4.1.3. *(Toric Serre duality)*
Let $\mathbb{P}$ be an n-dimensional complete normal toric variety, then for D a $\mathbb{Q}$-Cartier divisor we have

$$H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D))^* \cong H^{n-p}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-D + K_{\mathbb{P}}))$$

by ([CLS11, Thm.9.2.10a)]). There is an action of the character group M on the cohomology groups and in the situation of Theorem 4.1.2 we have splittings

$$H^{n-p}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + D)) \cong \bigoplus_{m \in M} H^{n-p}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + D))_m \cdot \chi^m$$
$$H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-D)) \cong \bigoplus_{m \in M} H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-D))_m \cdot \chi^m.$$

By ([CLS11, Ex.9.12 formula (9.2.9)]) Serre duality restricts to a duality

$$H^{n-p}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + D))_m \cong \left(H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-D))_{-m}\right)^*.$$

Due to the change of sign at m we get

$$H^{n-p}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + D)) \cong \begin{cases} 0 & p \neq \dim P_D \\ \bigoplus\limits_{m \in L^*(P_D)} \mathbb{C} \cdot \chi^m & p = \dim P_D \end{cases}$$

In chapter 5 we deal with a complicated vanishing Theorem where the sheaves of differential p-forms $\Omega^p_{\mathbb{P}}$ appear for $1 \leqslant p \leqslant n$.

4.2 The Plurigenera

Theorem 4.2.1. *Let Δ be an n-dimensional lattice polytope with $k := \dim F(\Delta) \geqslant 0$. Then the plurigenera $P_m(Y) := h^0(Y, mK_Y)$ are given by*

$$P_m(Y) = \begin{cases} l(m \cdot F(\Delta)) - l^*((m-1) \cdot F(\Delta)), & k = n \\ l(m \cdot F(\Delta)) + l^*((m-1) \cdot F(\Delta)), & k = n-1 \\ l(m \cdot F(\Delta)) & k < n-1. \end{cases}$$

with exception of the special case $n = 0$ and $m = 1$.

Proof.

$$H^0(\mathbb{P}, m(K_{\mathbb{P}} + Y)) \cong L(m \cdot F(\Delta)) \quad m \in \mathbb{N}_{>0},$$

since the polytope associated to $m \cdot (K_{\mathbb{P}} + Y)$ equals $m \cdot F(\Delta)$. We use an ideal sheaf sequence for Y and apply the adjunction formula

$$K_Y = (Y + K_{\mathbb{P}})_{|Y}$$

to get an exact sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}, (m-1)(K_{\mathbb{P}} + Y) + K_{\mathbb{P}}) \to H^0(\mathbb{P}, m(K_{\mathbb{P}} + Y)) \\ &\to H^0(Y, mK_Y) \to H^1(\mathbb{P}, (m-1)(K_{\mathbb{P}} + Y) + K_{\mathbb{P}}) \to 0 \end{aligned} \tag{4.1}$$

Here we have applied Theorem 4.1.2 to the divisor $K_{\mathbb{P}} + Y$ which is $\mathbb{Q}$-Cartier and nef by (Theorem 3.2.4).

$$H^1(\mathbb{P}, m(K_{\mathbb{P}} + Y)) = 0$$

by Theorem 4.1.1. If $m = 1$ then

$$h^0(\mathbb{P}, K_{\mathbb{P}}) = 0, \quad h^1(\mathbb{P}, K_{\mathbb{P}}) = 0$$

by Remark 4.1.3 and $P_1(Y)$ is given by

$$P_1(Y) = l(F(\Delta)) = l^*(\Delta).$$

Given $m \geqslant 2$ we apply Serre duality to the $\mathbb{Q}$-Cartier divisor $D := (m-1) \cdot (K_{\mathbb{P}} + Y)$ and use Remark 4.1.3

$$H^0(\mathbb{P}, D + K_{\mathbb{P}}) \cong \begin{cases} 0, & \dim F(\Delta) \leqslant n-1 \\ L^*((m-1) \cdot F(\Delta)), & \dim F(\Delta) = n \end{cases}$$

and

$$H^1(\mathbb{P}, D + K_{\mathbb{P}}) \cong \begin{cases} L^*((m-1) \cdot F(\Delta)), & \dim F(\Delta) = n-1 \\ 0, & \dim F(\Delta) \neq n-1 \end{cases}$$

The result follows by adding the dimensions in the exact sequence 4.1.

□

Example 4.2.2. If $\dim(\Delta) = 2$ and $F(\Delta) \neq \varnothing$ then $Y := Z_\Delta$ is a smooth curve and we distinguish

$$Y = \begin{cases} \text{elliptic curve} & \text{if } \dim F(\Delta) = 0 \\ \text{hyperelliptic curve, } g(Y) \geqslant 2 & \text{if } \dim F(\Delta) = 1 \\ \text{non-hyperelliptic curve, } g(Y) \geqslant 3 & \text{if } \dim F(\Delta) = 2 \end{cases}$$

This assertion follows from

$$g(Y) = P_1(Y) = l^*(\Delta).$$

Given $\dim F(\Delta) = 1$ then by Example 2.1.7

$$\mathbb{P}_{F(\Delta)} \cong \mathbb{P}^1$$

and $\theta \circ \pi : \mathbb{P} \to \mathbb{P}^1$ induces a morphism $Y \to \mathbb{P}^1$ of degree 2. It remains to check that this morphism coincides with the morphism induced by the linear system $|K_Y|$ to see that Y is hyperelliptic (see [Gie22a, section 6]).

4.3 The invariants K_Y^{n-1} and $q(Y)$

The lattice normalized volume $\mathrm{Vol}_\mathbb{Z}(F)$ of a rational polytope F may be defined by

$$\mathrm{Vol}_\mathbb{Z}(F) = \lim_{m\to\infty} \frac{l(m \cdot F) \cdot (\dim F)!}{m^{\dim F}} \tag{4.2}$$

([BR09, Lemma 3.19]). Here normalized means that the standard n-simplex Δ_n has $\mathrm{Vol}_\mathbb{Z}(\Delta_n) = 1$. See ([BR09]) for details.

Corollary 4.3.1. *Let Δ be an n-dimensional lattice polytope, where $n \geqslant 2$, with $k := \dim F(\Delta) \geqslant 0$. Then*

$$K_Y^{n-1} = \begin{cases} \mathrm{Vol}_\mathbb{Z}(F(\Delta)) + \sum\limits_{Q \leqslant F(\Delta)} \mathrm{Vol}_\mathbb{Z}(Q) & k = n \\ 2 \cdot \mathrm{Vol}_\mathbb{Z}(F(\Delta)) & k = n-1 \\ 0 & k < n-1 \end{cases}$$

Proof. By ([Laz00, Remark after Def. 2.2.31]) we have

$$K_Y^{n-1} = \lim_{m\to\infty} \frac{(n-1)! \cdot h^0(Y, mK_Y)}{m^{n-1}}.$$

By the formula of Theorem (4.2.1) given dim $F(\Delta) < n-1$ then $K_Y^{n-1} = 0$. Given dim $F(\Delta) = n-1$ then

$$\lim_{m\to\infty} \frac{(n-1)! \cdot l^*((m-1)\cdot F(\Delta)}{m^{n-1}} = \lim_{m\to\infty} \frac{(n-1)! \cdot l(m \cdot F(\Delta))}{m^{n-1}} = \mathrm{Vol}_{\mathbb{Z}}(F(\Delta))$$

by ([BR09, Thm.4.1]). If dim $F(\Delta) = n$ then write

$$P_m(Y) = l(m\cdot F(\Delta)) - l((m-1)\cdot F(\Delta)) + l((m-1)\cdot F(\Delta)) - l^*((m-1)\cdot F(\Delta)).$$

By formula 4.2 we have

$$\lim_{m\to\infty} \frac{l(mF(\Delta)) - l((m-1)F(\Delta))}{m^{n-1}/(n-1)!} = \mathrm{Vol}_{\mathbb{Z}}(F(\Delta)).$$

Finally by ([BR09, Thm.5.6]) we have

$$\lim_{m\to\infty} \frac{l((m-1)F(\Delta)) - l^*((m-1)F(\Delta))}{m^{n-1}/(n-1)!} = \sum_{Q \leqslant F(\Delta)} \mathrm{Vol}_{\mathbb{Z}}(Q).$$

□

We deduce from the article of Danilov and Khovanskii:

Proposition 4.3.2. *Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then*

$$q(Y) := h^0(Y, \Omega_Y^1) = 0.$$

Proof. By the *Hodge decomposition* it is enough to show that $H^1(Y, \mathbb{C}) = 0$ (see chapter 7). By ([DK86, Prop.3.4]) we have

$$H^1(Z_\Delta, \mathbb{C}) = 0.$$

Choose a partial toric resolution of singularities $\sigma : \mathbb{P}' \to \mathbb{P}_\Delta$ modifying just the 1-dimensional torus orbits of $\mathbb{P}_\Delta$ such that $\sigma : Z' \to Z_\Delta$ is a resolution of singularities. This works since by nondegeneracy Z_Δ does not pass through the torus fixed points of $\mathbb{P}_\Delta$. For E a σ-exceptional curve on Z' we have $E \cong \mathbb{P}^1$. An argument using the Mayer-Vietoris sequence shows that

$$h^1(Z', \mathbb{C}) = 0.$$

Since Z' is gotten by blowing up Y at several points ([BHPV04, Ch. III Cor.4.4]), $H^1(Y, \mathbb{C}) = 0$. □

4.4 Kanev and Todorov surfaces

The main invariants of (smooth) algebraic surfaces include the *geometric genus* $p_g(Y) := P_1(Y)$, the irregularity $q(Y)$ and K_Y^2.

Example 4.4.1. In case $n = 3$ and Δ is reflexive then

$$p_g(Y) = 1, \quad q(Y) = 0,$$

and by the adjunction formula $K_Y = \mathcal{O}_Y$. Y is called a *K3 surface*.

For the 49 examples from section 3.4 we get

$$K_Y^2 \in \{1, 2\}, \quad p_g(Y) = 1, \quad q(Y) = 0.$$

Definition 4.4.2. *(compare ([Cat78a]))*
A Kanev (or Kunev/Kynev) surface is a minimal complex projective surface Y with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 1.$$

Remark 4.4.3. By ([Cat78, Thm.2.2]) these surfaces are simply connected. Surfaces Y with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 2$$

are deal with in ([CD89]). Given such a surface Y the linear system $|2 \cdot K_Y|$ defines a finite morphism $\phi_{2 \cdot K_Y} : Y \to \mathbb{P}^3$. If the image of Y is a quadric cone then Y has fundamental group

$$\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}. \tag{4.3}$$

For our Examples the spaces $H^0(Y, m \cdot K_Y)$ have monomial bases by Theorem 4.2.1 and we may compute such bases for $m = 2$ in $c), d)$ and $e)$:

$$\begin{aligned}
c): \quad & w_0 := (0,0,0), w_1 := (2,1,4), w_2 := (2,1,3), w_3 := (2,1,2) \\
d): \quad & w_0 := (0,0,0), w_1 := (2,-1,2), w_2 := (2,-1,1), w_3 := (2,-1,0) \\
e): \quad & w_0 := (0,0,0), w_1 := (2,1,0), w_2 := (2,2,-1), w_3 := (2,3,-2).
\end{aligned}$$

This notation means that the characters $t \mapsto t^{w_i}$ build a basis of $H^0(Y, 2 \cdot K_Y)$.

In $c), d)$ and $e)$ we have $2 \cdot w_2 = w_1 + w_3$. Since the morphism ϕ_{2K_Y} associated to $|2 \cdot K_Y|$ restricted to the torus T is given by

$$t = (t_1, t_2, t_3) \mapsto (t^{w_1}, t^{w_2}, t^{w_3})$$

$\phi_{2K_Y}(Y)$ is a quadric cone in $c), d)$ and $e)$ and (by [CD89]) $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 4.4.4. Interestingly some Kanev surfaces and surfaces of the second type where $K_Y^2 = 2$ are closely related to K3 surfaces: For Y a Kanev surface $2K_Y$ defines a finite morphism $\phi_{2K_Y} : Y \to \mathbb{P}^2$ of degree 4. If this morphism factors through a K3 surface with R.D.P. then in the literature Y is called *special*. These special Kanev surfaces are of particular interest (see chapter 8).
For Y a surface with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 2$$

the morphism $\phi_{2K_Y} : Y \to \mathbb{P}^3$ is of degree $1, 2, 4$ or 8. If ϕ_{2K_Y} factors through a K3 surface with R.D.P. then Y is called a *Todorov surface*. In fact for the definition of a Todorov surface there condition $K_Y^2 = 2$ is weakened to the condition

$$K_Y^2 \in \{1, ..., 8\}$$

([Mor87]). Some of these surfaces are known to fail the infinitesimal Torelli Theorem.

Example 4.4.5. Consider the sub-lattice

$$M' := \{(m_1, m_2, m_3) \in M | \, m_1 \in 2 \cdot \mathbb{Z}\} \subset M$$

of index 2 in M with dual lattice $N' \supset N$. Let Δ' be the polytope Δ with respect to the lattice M'. Then Δ' is a lattice polytope since all vertices of Δ have even first coordinate and Δ' turns out to be reflexive. The inclusion $N \to N'$ induces a degree 2 toric morphism

$$\phi_1 : \mathbb{P}_\Delta \to \mathbb{P}_{\Delta'}.$$

We compute bases of $H^0(Y, 2K_Y)$:

$$\begin{aligned} a) : &\quad (0,0,0), (2,1,0), (2,1,-1) \\ b) : &\quad (0,0,0), (2,-1,1), (2,-2,1). \end{aligned}$$

It follows that given an f as in (2.3) with $a_m = 0$ for

$$m \in (M \backslash M') \cap \Delta$$

then ϕ_{2K_Y} factors through the restriction of ϕ_1 to Y. Since Δ' is reflexive $Z_{\Delta'}$ gets a K3 surface with R.D.P (Example 3.3.8 and Remark 4.4.1) and Y is special.

Thus we find subfamilies of special Kanev surfaces by setting the following monomials to zero (only in the maximal polytopes)

$$a), b) : b_1, d_1, ab, bc, ad, cd.$$

Similarly in $c), d)$ and $e)$ we get Todorov surfaces if we set the coefficients to the following monomials to zero:

$$c), d), e) : ab, ad, bc, cd.$$

CHAPTER 5

The Kodaira-Spencer maps $\kappa_{\mathbb{P},f}$ and κ_f and their kernels

Given an n-dimensional Newton polytope Δ with $F(\Delta) \neq \varnothing$ the toric variety $\mathbb{P}$ does not depend on the Laurent polynomial f. In fact even more: $\mathbb{P}$ is the same for all lattice polytopes Δ with $F(\Delta) \neq \varnothing$ and fixed $C(\Delta)$. In this way we get explicit deformations of Y_f by varying $f \in U_{reg}(\Delta)$.

Throughout this and the next chapter we restrict to (conditions (+))

- $n = 3$
- $l^*(\Delta) > 0$
- $C(\Delta)$ is a lattice polytope

The first point ensures that Y is smooth. The second point is exploited in order to prove $h^0(Y, T_Y) = 0$ (see Proposition 5.4.3) and the third point guarantees that Y defines a Cartier divisor on $\mathbb{P}$ by Lemma 3.3.1.

In this rather technical chapter we introduce a *Kodaira-Spencer* map κ_f parameterizing one-to-one the infinitesimal deformations of Y_f arising when varying f. We extend κ_f by introducing a second *Kodaira-Spencer* map $\kappa_{\mathbb{P},f}$ parameterizing the infinitesimal deformations of Y_f in $\mathbb{P}$.

We abstractly identify $\ker(\kappa_{\mathbb{P},f})$ and $\ker(\kappa_f)$ with some vector spaces, postponing the explicit calculations of

$$\begin{aligned} \ker(\kappa_{\mathbb{P},f}) &\subset L(C(\Delta))/\mathbb{C}\cdot f \\ \ker(\kappa_f) \ &\subset L(\Delta)/\mathbb{C}\cdot f \end{aligned}$$

to the next chapter.

5.1 Tangent sheaf, Normal sheaf and Sheaves of differential p-forms

Since $\mathbb{P}$ is not necessarily smooth we have to recall the notion of reflexive sheaves which weakens the definition of locally free sheaves.

Definition 5.1.1. *A coherent sheaf $\mathcal{F}$ on a normal variety X is called reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, where $\mathcal{F}^{**}$ denotes the double dual (reflexive hull) of the sheaf $\mathcal{F}$.*

Remark 5.1.2. *([CLS11, Prop.8.0.1, Thm.8.0.4])*
If X is normal and $j : U \subset X$ an open subset with $\mathrm{Codim}(X\backslash U) \geqslant 2$, a reflexive sheaf is uniquely determined by its restriction to U, that is

$$\mathcal{F} \cong j_*(\mathcal{F}_{|U}). \tag{5.1}$$

Conversely if $\mathcal{F}$ is a coherent sheaf with $\mathcal{F}_{|U}$ locally free and $\mathrm{codim}(X\backslash U) \geqslant 2$ then $j_*(\mathcal{F}_{|U})$ is reflexive ([Sch08, Prop.2.12]). The dual of a coherent sheaf on a normal variety is always reflexive, in particular the reflexive hull of a coherent sheaf is reflexive.

Remark 5.1.3. With this definition the map

$$\begin{aligned} \{\text{Weil divisors on } X\} &\to \{\text{rank one reflexive sheaves on } X\} \\ D &\mapsto \mathcal{O}_X(D) \end{aligned}$$

gets linear, that is $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes_r \mathcal{O}_X(D')$.

If $\mathcal{F}$ is reflexive and $\mathcal{L}$ a line bundle on X then $\mathcal{F} \otimes \mathcal{L}$ is easily seen to be reflexive by checking the condition with the double dual stalk-wise using ([Hart77, Ch.3, Prop.6.8]).

Definition 5.1.4. *Given an n-dimensional normal algebraic variety X we define the reflexive sheaves*

$$\Omega^p_X := \iota_* \Omega^p_U \quad 1 \leqslant p \leqslant n,$$
$$T_X := (\Omega^1_X)^*,$$

where $\iota : U \to X$ denotes the inclusion of the smooth locus of X.

Remark 5.1.5. If X is smooth then

$$T_X \cong \Omega^{n-1}_X \otimes \mathcal{O}_X(-K_X)$$

by ([Hart77, Ch. II Ex.5.16b)]). Applying this to the inclusion of the smooth locus $\iota : U \to \mathbb{P}$ and taking the push-forward under the inclusion ι gives

$$T_{\mathbb{P}} \cong \Omega^2_{\mathbb{P}} \otimes_r \mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}}),$$

which we will use later.

Definition 5.1.6. *Assume that $n := \dim \Delta = 3$ and that $Y \subset \mathbb{P}$ is Cartier. Then we define the normal sheaf of Y in $\mathbb{P}$ as $N_{Y/\mathbb{P}} := \mathcal{O}_{\mathbb{P}}(Y)_{|Y} = (\mathcal{I}/\mathcal{I}^2_Y)^*$.*

Remark 5.1.7. Given a normal variety X there is still a different method for the construction of the tangent sheaf T_X: Let $\Omega^p_{X,\text{Kähl}}$ denote the sheaf of Kähler p-differentials on X and

$$T_{X,\text{Kähl}} := (\Omega^1_{X,\text{Kähl}})^*$$

ist dual (compare [Hart77, Ch.II.8]). $T_{X,\text{Kähl}}$ is reflexive since $\Omega^1_{X,\text{Kähl}}$ is coherent, and coincides with T_X on the smooth locus U of X. As a consequence

$$T_{X,\text{Kähl}} \cong T_X.$$

Note that

$$H^0(X, T_{X,\text{Kähl}}) \cong \text{Lie}(\text{Aut}(X)), \tag{5.2}$$

by ([MuOd15, Ch.VI.1]), where $\text{Lie}(\text{Aut}(X))$ denotes the Lie algebra of the automorphism group of X. In particular $h^0(X, T_X) = 0$ if $\text{Aut}(X)$ is a finite group.

For V a normal projective *toric* variety $\text{Aut}(V)$ is an algebraic group ([Cox95, Prop.4.3]) of finite type. If V is not toric it is possible that $h^0(Y, T_Y) = 0$ though $\text{Aut}(Y)$ is a discrete space with infinitely many components.

Example 5.1.8. Take the *Fermat quartic* $Y \subset \mathbb{P}^3$

$$0 = x_0^4 + ... + x_3^4,$$

which defines a K3 surface. Then $h^0(Y, T_Y) = 0$ but Y has infinite automorphism group (see [ShIn10, Thm.5]).

5.2 Kodaira-Spencer maps

Let $B := \pi(U_{reg}(\Delta))$, where $\pi : L(\Delta) \to \mathbb{P}L(\Delta)$ denotes the natural projection. Take

$$\mathcal{X} := \{(x, f) \in \mathbb{P} \times B |\, x \in Y_f\}. \overset{pr_2}{\to} B. \tag{5.3}$$

and the normal sheaf

$$N_{Y/\mathcal{X}} := \left(\mathcal{I}_Y/\mathcal{I}_Y^2\right)^* = \mathcal{O}_{\mathcal{X}}(Y)_{|Y},$$

where $\mathcal{I}_Y$ denotes the ideal sheaf of Y in $\mathcal{X}$. Under the conditions (+) there are two tangent sheaf sequences

$$0 \to T_Y \to T_{\mathbb{P}|Y} \to N_{Y/\mathbb{P}} \to 0. \tag{5.4}$$

$$0 \to T_Y \to T_{\mathcal{X}|Y} \to N_{Y/\mathcal{X}} \to 0 \tag{5.5}$$

Definition 5.2.1. *The two coboundary maps*

$$\begin{aligned} \kappa_{\mathbb{P}} = \kappa_{\mathbb{P},f} &: H^0(Y, N_{Y/\mathbb{P}}) \to H^1(Y, T_Y) \\ \kappa = \kappa_f &: \quad H^0(Y_f, N_{Y/\mathcal{X}}) \to H^1(Y, T_Y) \end{aligned}$$

are called Kodaira-Spencer maps for $Y \subset \mathbb{P}$ and $Y \subset \mathcal{X}$.

To motivate these Kodaira-Spencer maps geometrically let us recall some facts from deformation theory: For this let $D := \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ denote the *dual numbers*, then the underlying topological space of D is just a point, but obviously D is a non-reduced scheme.

Definition 5.2.2. *A deformation of Y_f over D (also called a first order infinitesimal deformation) is a flat surjective morphism $\mathcal{Y} \to D$ such that* *the fiber over the underlying point $\{f\}$ of D equals Y_f. If $\mathcal{Y} \subset \mathcal{X} \times D$ (or $\mathcal{Y} \subset \mathbb{P} \times D$) and*

$$\begin{tikzcd} \mathcal{Y} \arrow[rr] \arrow[dr] & & \mathcal{X} \times D \arrow[dl, "pr_2"] \\ & D & \end{tikzcd}$$

commutes (similarly for $\mathcal{Y} \subset \mathbb{P} \times D$) then $\mathcal{Y} \to D$ is called an infinitesimal deformation of Y_f in $\mathcal{X}$ (in $\mathbb{P}$).

A tangent vector in $T_{B,f}$ is the same as a morphism

$$D \to B$$

which maps the underlying point of D to f ([Hart77, Ch. II Ex.2.8]). Given such a tangent vector we get an induced infinitesimal deformation

$$\mathcal{X} \times_B D \to D$$

of Y_f in $\mathcal{X}$ by using the fiber product.

Remark 5.2.3. We recall ([Ser06])

$$\begin{aligned} H^0(Y, N_{Y/\mathcal{X}}) &\cong \{\text{inf. def. of } Y \text{ in } \mathcal{X}\}/\text{iso.} \\ H^0(Y, N_{Y/\mathbb{P}}) &\cong \{\text{inf. def. of } Y \text{ in } \mathbb{P}\}/\text{iso.} \\ H^1(Y, T_Y) &\cong \{\text{inf. def. of } Y/\text{iso.} \end{aligned}$$

Then κ_f (and $\kappa_{\mathbb{P},f}$) map an infinitesimal deformation of Y_f in $\mathcal{X}$ (in $\mathbb{P}$) onto its equivalence class in $H^1(Y_f, T_{Y_f})$. In Proposition 5.5.1 we prove that $\kappa_{\mathbb{P},f}$ restricts to κ_f. The dimension $\dim Im(\kappa_f)$ is called the *number of moduli* of $\mathcal{X} \to B$.

Remark 5.2.4. Taking the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(Y) \to N_{Y/\mathbb{P}} \to 0 \tag{5.6}$$

and the vanishing $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = 0$ due to Demazure we get

$$H^0(Y, N_{Y/\mathbb{P}}) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(Y))/H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \cong L(C(\Delta))/\mathbb{C} \cdot f.$$

where we have used Remark 3.3.3.

Remark 5.2.5. The normal sheaf $N_{Y_f/\mathcal{X}}$ is trivial, that is

$$N_{Y_f/\mathcal{X}} = \bigoplus_{i=1}^{l(\Delta)-1} \mathcal{O}_{Y_f} \cong T_{B,f} \otimes_{\mathbb{C}} \mathcal{O}_{Y_f}.$$

One argument works as follows: Write $\{f\} = H_1 \cap ... \cap H_{l(\Delta)-1}$ as intersection of projective hyperplanes intersecting transversely. Switching to hyperplanes $H_i' \sim_{lin} H_i$ with $f \notin H_1' \cap ... \cap H_{l(\Delta)-1}'$ and setting $G_i = pr_2^*(H_i)$, $G_i' = pr_2^*(H_i')$ we get

$$\mathcal{O}_{Y_f}(G_{i|Y_f}) \cong \mathcal{O}_{Y_f}(G'_{i|Y_f}) \cong \mathcal{O}_{Y_f}$$

and the first result follows. B is a nonempty open subset of $\mathbb{P}^{l(\Delta)-1}$ and $T_{B,f} \cong T_{\mathbb{C}^{l(\Delta)},f}/\mathbb{C} \cdot f$. In particular

$$H^0(Y_f, N_{Y_f/\mathcal{X}}) \cong L(\Delta)/\mathbb{C} \cdot f.$$

5.3 Mavlyutov's Vanishing Theorem

Theorem 5.3.1. *([CLS11, Thm.9.3.3])*
Let V be an n-dimensional complete toric variety to a simplicial fan. If D is a nef Cartier divisor on V, then

$$H^p(V, \Omega_V^q \otimes \mathcal{O}(D)) = 0$$

for $p > q$.

Construction 5.3.2. *(Multiplication morphism)*
([Fuj06, 2.5, Prop.3.2], [CLS11, Lemma 9.2.6, Proof of Thm. 9.3.1])
Let V be a normal toric variety, D a divisor on V and $m \in \mathbb{N}_{\geqslant 1}$ such that $m \cdot D$ is Cartier. There is a construction due to *Fujita*:

Namely given $l \in \mathbb{N}_{\geqslant 1}$ the map $\overline{\phi}_l : N \to N$ given by

$$n \mapsto l \cdot n$$

induces a toric morphism $\phi_l : V \to V$. There results an injection (Remark 5.1.3)

$$\begin{aligned} H^p(V, \Omega_V^q \otimes_r \mathcal{O}(D)) &\to H^p(V, \Omega_V^q \otimes_r \mathcal{O}(lD)) \\ &\cong H^p(V, \Omega_V^q \otimes \mathcal{O}(lD)), \quad p, q \geqslant 0. \end{aligned}$$

This result becomes powerful for us especially when combined with Theorem 5.3.1 above.

5.4 The Computation of $\ker(\kappa_{\mathbb{P},f})$

The following three sections deal with cohomological applications of what we have introduced before.

Theorem 5.4.1. *Under the conditions* (+)

$$\ker(\kappa_{\mathbb{P},f}) \cong \text{Lie Aut}(\mathbb{P}). \tag{5.7}$$

Proof. Given $Y = Y_f$ by the tangent sheaf sequence (5.4), Remark 5.1.7 and Proposition 5.4.3 below we have to show that

$$H^0(Y, T_{\mathbb{P}|Y}) \cong H^0(\mathbb{P}, T_{\mathbb{P}}).$$

The ideal sheaf sequence

$$0 \to T_{\mathbb{P}} \otimes \mathcal{O}(Y) \to T_{\mathbb{P}} \to T_{\mathbb{P}|Y} \to 0$$

produces the cohomology sequence

$$\begin{aligned} 0 &\to H^0(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)) \to H^0(\mathbb{P}, T_{\mathbb{P}}) \to H^0(Y, T_{\mathbb{P}|Y}) \\ &\to H^1(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)). \end{aligned}$$

We conclude with the following Lemma.

□

Lemma 5.4.2.

$$h^0(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)) = h^1(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)) = 0.$$

Proof. By Remark 5.1.3 $T_{\mathbb{P}} \otimes \mathcal{O}(-Y)$ is reflexive and with Remark 5.1.5 we get

$$T_{\mathbb{P}} \otimes \mathcal{O}(-Y) \cong \Omega^2_{\mathbb{P}} \otimes_r \mathcal{O}(-Y - K_{\mathbb{P}}).$$

By Construction 5.3.2 replacing $\mathcal{O}(-Y - K_{\mathbb{P}})$ by a multiple

$$\mathcal{O}(-mY - mK_{\mathbb{P}})$$

which is a line bundle and $\otimes_r$ by $\otimes$ does not affect the cohomology groups. Besides

$$H^k(\mathbb{P}, \Omega^2_{\mathbb{P}} \otimes \mathcal{O}(-mY - mK_{\mathbb{P}})) \cong H^{3-k}(\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathcal{O}(mY + mK_{\mathbb{P}})) \quad k = 0, 1.$$

by Serre duality ([CLS11, Thm. 9.2.10b)]). The right hand side is 0 by Theorem 5.3.1. □

Proposition 5.4.3. *Given a 3-dimensional lattice polytope Δ with $l^*(\Delta) > 0$ and $Y = Y_f$ then*
$$H^0(Y, T_Y) = 0.$$

Proof. $h^0(Y, \mathcal{O}(K_Y)) = l^*(\Delta) > 0$ by assumption and Theorem 4.2.1. The vanishing

$$h^0(Y, T_Y) \leqslant h^0(Y, T_Y \otimes \mathcal{O}(K_Y)) = h^0(Y, \Omega^1_Y) = 0$$

follows from Proposition 4.3.2 and Remark 5.1.5. □

It seems hard (or at least much harder than we have done) to generalize this Proposition to the case $F(\Delta) \neq \emptyset$.

5.5 The computation of $\ker(\kappa_f)$

Proposition 5.5.1. *Under the conditions* (+) $\kappa_{\mathbb{P},f}$ *restricts to* κ_f*.*

The following reduction step has essentially been carried out in ([Koe91, Ch.2.1] and [Voi03, Lemma 6.15]).

Proof. The two tangent sheaf sequences

$$0 \to T_Y \to T_{\mathcal{X}|Y} \to N_{Y/\mathcal{X}} \to 0$$

$$0 \to T_Y \to T_{\mathbb{P}|Y} \to N_{Y/\mathbb{P}} \to 0$$

are related via the differential

$$(pr_1)_* : T_{\mathcal{X}|Y} \to T_{\mathbb{P}|Y},$$

of the first projection $pr_1 : \mathcal{X} \to \mathbb{P}$. pr_1 induces an isomorphism

$$Y_f \times \{f\} \to Y_f$$

$(pr_1)_*$ restricts to the identity on T_Y. The map

$$(pr_1)_* : N_{Y/\mathcal{X}} \cong H^0(Y, N_{Y/\mathcal{X}}) \otimes \mathcal{O}_Y \subset H^0(Y, N_{Y/\mathbb{P}}) \otimes \mathcal{O}_Y \to N_{Y/\mathbb{P}}$$

is just given by multiplication of sections. The result of the Proposition follows from the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(Y, T_{\mathcal{X}|Y}) & \longrightarrow & H^0(Y, N_{Y/\mathcal{X}}) & \xrightarrow{\kappa_f} & H^1(Y, T_Y) \\
 & & \Big\downarrow{\scriptstyle (pr_1)_*} & & \Big\downarrow & & \Big\downarrow{\scriptstyle id} \\
0 & \longrightarrow & H^0(Y, T_{\mathbb{P}|Y}) & \longrightarrow & H^0(Y, N_{Y/\mathbb{P}}) & \xrightarrow{\kappa_{\mathbb{P},f}} & H^1(Y, T_Y)
\end{array}$$

□

Remark 5.5.2. Given the conditions (+) ($C(\Delta)$ should be a lattice polytope) replacing $\Delta \mapsto C(\Delta)$ gives $\kappa_f = \kappa_{\mathbb{P},f}$, that is not much is lost in restricting to κ_f.

5.6 Appendix: Additional infinitesimal deformations of Y_f

Remark 5.6.1. Apparently

$$H^1(Y, N_{Y/\mathbb{P}}) = 0$$

by the exact sequence (5.6) and Theorem 4.1.1. Thus the cokernel of $\kappa_{\mathbb{P},f}$ equals

$$H^1(Y, T_{\mathbb{P}|Y})$$

and the infinitesimal deformations of $\mathbb{P}$ in $H^1(\mathbb{P}, T_{\mathbb{P}})$ induce infinitesimal deformations of Y (see [Ser06, Prop.3.4.23]).

We see this by an ideal sheaf sequence for $Y \subset \mathbb{P}$ and Lemma 5.4.2

$$0 \to H^1(\mathbb{P}, T_{\mathbb{P}}) \to H^1(Y, T_{\mathbb{P}|Y}) \to H^2(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)).$$

Very remarkably

$$H^1(\mathbb{P}, T_{\mathbb{P}})$$

parameterizes all infinitesimal deformations of $\mathbb{P}$ by a Theorem of Schlessinger ([Sch71]) since dim $\mathbb{P} = 4$ and $\mathbb{P}$ has just isolated quotient singularities, though for general singular toric varieties Y the situation is much more complicated (see [IlTu18]).

Example 5.6.2. Let $\mathbb{P} = \mathbb{P}^3$, $\Delta = 4 \cdot \Delta_3$ then $\dim Im(\kappa_{\mathbb{P}}) = 19$ and $H^1(\mathbb{P}, T_{\mathbb{P}}) = 0$. But $h^1(Y, T_Y) = 20$ since Y is a K3 surface ([Huy16]). In ([Gie22b]) we show that if $n := \dim \Delta \geqslant 4$ then such a phenomenon does not occur.

CHAPTER 6

Explicit bases of the kernels of $\kappa_{\mathbb{P},f}$ and κ_f

In this chapter we compute explicit bases of $\ker(\kappa_{\mathbb{P},f}) \subset L(C(\Delta))/\mathbb{C} \cdot f$ and $\ker(\kappa_f) \subset L(\Delta)/\mathbb{C} \cdot f$.

6.1 A basis for $\ker(\kappa_{\mathbb{P},f})$

This part is rather technical and summarizing. The statments, most essential for the first reading, are formula 6.1, formula 6.2 and Corollary 6.1.5.

Definition 6.1.1.

$$
\begin{aligned}
R(N, \Sigma) := \{\alpha \in M \mid \langle \alpha, n(\alpha) \rangle = 1 \textit{ for some } n(\alpha) \in \Sigma[1] \\
\textit{and } \langle \alpha, n_j \rangle \leqslant 0 \textit{ for } n_j \in \Sigma[1] \backslash \{n(\alpha)\}\}
\end{aligned}
$$

denote the Demazure roots of the fan Σ (see [Cox95]). Likewise we define $R(N, \Sigma_{C(\Delta)})$ and $R(N, \Sigma_\Delta)$ by replacing Σ by $\Sigma_{C(\Delta)}$ and Σ_Δ.

There are inclusions

$$
\Sigma_{C(\Delta)}[1] \underbrace{\subset}_{\text{(since } \Sigma \text{ refines } \Sigma_{C(\Delta)})} \Sigma[1] \underbrace{\subset}_{\text{(Figure 3.2)}} \operatorname{convhull}(\Sigma_\Delta[1]). \tag{6.1}
$$

Lemma 6.1.2. *Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then*

$$R(N, \Sigma_\Delta) \subset R(N, \Sigma) = R(N, \Sigma_{C(\Delta)}).$$

Proof. To the second equality: Let $\alpha \in R(N, \Sigma_{C(\Delta)})$, that is

$$\langle \alpha, n(\alpha) \rangle = 1, \quad \langle \alpha, n_j \rangle \leqslant 0 \quad \text{ for } n_j \in \Sigma_{C(\Delta)}[1] \backslash \{n(\alpha)\}.$$

$\Rightarrow \langle \alpha, n_j \rangle \leqslant 0$ for $n_j \in \Sigma[1] \backslash \{n(\alpha)\}$, that is $\alpha \in R(N, \Sigma)$. Conversely assume $\alpha \in R(N, \Sigma)$. If $n_i \notin \Sigma_{C(\Delta)}[1]$ then α would have scalar product $\leqslant 0$ with all vectors in $\Sigma_{C(\Delta)}[1]$ and thus would be zero since Σ refines $\Sigma_{C(\Delta)}$, a contradiction. The first inclusion follows similarly by using (6.1). □

We ask for a basis of Laurent polynomials for

$$\text{Lie Aut}(\mathbb{P}) \subset L(C(\Delta))/\mathbb{C} \cdot f.$$

Remember the results from ([BG99]): Given $f \in B$ there is a map

$$\begin{aligned} &\phi_f : T \to B \\ &(t_1, t_2, t_3) \mapsto \Big((x_1, x_2, x_3) \mapsto f(t_1 x_1, t_2 x_2, t_3 x_3) \Big). \end{aligned}$$

By differentiating ϕ_f we get an injective homomorphism $(d\phi_f)_e : \text{Lie}(T) \to T_{B,f}$ where $e = (1, 1, 1)$ with

$$Im(d(\phi_f)_e) = \Big\langle x_1 \cdot \frac{\partial f}{\partial x_1}, ..., x_3 \cdot \frac{\partial f}{\partial x_3} \Big\rangle.$$

For $m \in M \cap C(\Delta)$ and $\alpha \in R(N, \Sigma_{C(\Delta)})$ define

$$ht_{-\alpha}(m) := \max\{k \in \mathbb{N}_{\geqslant 0} |\, m - k \cdot \alpha \in C(\Delta)\}. \tag{6.2}$$

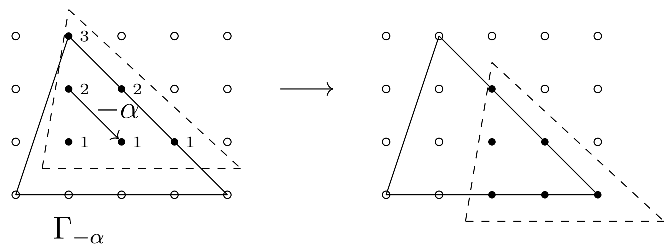
Given $\alpha \in R(N, \Sigma_{C(\Delta)})$ we denote by $\Gamma_{-\alpha} \leqslant C(\Delta)$ the facet to which $n(\alpha)$ is normal.

Remark 6.1.3. Assuming

$$\Gamma_{-\alpha} = \{x \in M_{\mathbb{R}} |\, \langle x, n_\Gamma \rangle = b_\Gamma\} \cap C(\Delta)$$

and $m \in M \cap C(\Delta)$ then

$$ht_{-\alpha}(m) = \langle m, n_\Gamma \rangle - b_\Gamma. \tag{6.3}$$



On the left: The vector $-\alpha$ and all lattice points $m \in C(\Delta)$ with $ht_{-\alpha}(m) > 0$. On the right: A replacement and the support vectors of $w_{-\alpha}(f)$ (thick).

The function $ht_{-\alpha}$ continues linearly to a map $S_{C(\Delta)} \to S_{C(\Delta)}$, which respects the grading on $S_{C(\Delta)}$ (see section 2.1 for the definition of $S_{C(\Delta)}$). Define a graded automorphism $e^{\lambda}_{-\alpha} : S_{C(\Delta)} \to S_{C(\Delta)}$ by

$$e^{\lambda}_{-\alpha}(x^m) := x^m \cdot (1 + \lambda \cdot x^{-\alpha})^{ht_{-\alpha}(m)} \qquad \lambda \in \mathbb{C}$$

(see [BG99, section 3]). $e^{\lambda}_{-\alpha}$ induces an automorphism of $\mathbb{P}_{C(\Delta)}$ by the description $\mathbb{P}_{C(\Delta)} \cong \mathrm{Proj}(S_{C(\Delta)})$ and by functoriality of taking „Proj “.

Corollary 6.1.4. *([BG99, Lemma 3.1, Thm.3.2b), Thm.5.4])*
Lie Aut$(\mathbb{P})$ *has a basis of derivations acting on* $L(C(\Delta))$ *as follows*

$$\begin{aligned} x_i \frac{\partial}{\partial x_i} : &\quad x^m \mapsto m_i \cdot x^m, \quad i = 1, 2, 3, \\ z(\alpha) := \frac{\partial e^{\lambda}_{-\alpha}}{\partial \lambda}_{|\lambda=0} : &\quad x^m \mapsto ht_{-\alpha}(m) \cdot x^{m-\alpha}, \quad \alpha \in R(N, \Sigma_{C(\Delta)}). \end{aligned}$$

□

By definition of the tangent sheaf sequence the homomorphism

$$j : H^0(\mathbb{P}, T_{\mathbb{P}}) \cong H^0(Y, T_{\mathbb{P}|Y}) \to H^0(Y, N_{Y/\mathbb{P}}) \cong L(C(\Delta))/\mathbb{C} \cdot f$$

is given by applying the derivations from Lie Aut$(\mathbb{P})$ to f and restricting to $Y = Y_f$. We get

Corollary 6.1.5. *Under the conditions* (+)

$$\ker(\kappa_{\mathbb{P},f}) \subset L(C(\Delta))/\mathbb{C} \cdot f$$

equals the span of the Laurent polynomials

$$x_1 \cdot \frac{\partial f}{\partial x_1}, ..., x_3 \cdot \frac{\partial f}{\partial x_3}, \quad w_{-\alpha}(f), \quad \alpha \in R(N, \Sigma_{C(\Delta)}),$$

where

$$w_{-\alpha}(f) := \sum_{m \in \Delta \cap M} ht_{-\alpha}(m) \cdot a_m \cdot x^{m-\alpha}.$$

□

6.2 Examples

Example 6.2.1. Given

$$\Delta = d \cdot \Delta_3,$$

and f then Y_f is a smooth degree d surface in $\mathbb{P}^3$. By ([Voi03, Lemma 6.15])

$$\ker(\kappa_{\mathbb{P},f}) = \ker(\kappa_f) \cong J^d_{f,griff}, \tag{6.4}$$

if we work with the family $\mathcal{X} \to U_{reg}(\Delta)$ (if we projectivize then we have to mod out f from the kernel). Here $J^d_{f,griff}$ denotes the d-th homogeneous component of *Griffiths Jacobian ideal*

$$J_{f,griff} := \left(\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_3}\right) \trianglelefteq \mathbb{C}[x_0, ..., x_3],$$

since

$$\dim \operatorname{Aut}(\mathbb{P}^3) = \dim PGL(3, \mathbb{C}) = 15$$

where $PGL(3, \mathbb{C})$ denotes the *projective linear group*. There are 12 roots in $R(N, \Sigma)$. These roots are given by

$$\pm e_i, \quad i = 1, 2, 3, \quad \pm e_i \mp e_j, \quad i, j = 1, 2, 3, \quad i \neq j$$

and if $d \geqslant 4$ Proposition 6.1.5 restricts to (6.4) up to homogenization.

Example 6.2.2. For the maximal polytope in a) we have

$$R(N, \Sigma) = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

and for the maximal polytope in $c)$ we have

$$R(N,\Sigma) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In $b), d)$ and $e)$ we have $R(N,\Sigma) = \varnothing$. Starting with a maximal polytope Δ we get for the number of moduli:

$$\begin{aligned} a) &: \dim Im(\kappa_f) = 12, \quad & b) &: \quad \dim Im(\kappa_f) = 14, \\ c) &: \dim Im(\kappa_f) = 10, \quad & d), e) &: \dim Im(\kappa_f) = 11. \end{aligned}$$

By ([Tod80, Thm.2]) all Kanev surfaces vary in a single family with number of moduli $h^1(Y, T_Y) = 18$. Similarly by ([SSU85, (1.3.2)]) all minimal surfaces Y with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 2, \quad \pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$$

vary in a single family with number of moduli $h^1(Y, T_Y) = 16$.

6.3 A basis for $\ker(\kappa_f)$

$x_1 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_3 \cdot \frac{\partial f}{\partial x_3}$ belong to $L(\Delta)$ but given $\Delta \neq C(\Delta)$ the Laurent polynomials $w_{-\alpha}(f)$ need not have support on Δ as the following example shows. We found this example with a computer search.

Example 6.3.1. Consider the following polytope

$$\Delta = \left\langle \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 10 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\rangle,$$

with $l^*(\Delta) = 3$, $\dim F(\Delta) = 1$ and $C(\Delta)$ has the additional vertex $(1, -1, 1)$. We obtain a family of elliptic surfaces $\mathcal{X} \to B$.

$$R(N,\Sigma) = \left\{ \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

and

$$R(N,\Sigma_\Delta) = R(N,\Sigma) \cup \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

The column vector $-\alpha = (1, 0, 1)$ belongs to the facet

$$\Gamma_{-\alpha} = \left\langle \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 10 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

of $C(\Delta)$.

$$\underbrace{\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}}_{\text{vertex of } \Delta} \notin \Gamma_{-\alpha}$$

and

$$-\alpha + (-1, -1, -1) = (0, -1, 0) \notin \Delta.$$

Thus only 6 of the roots in $R(N, \Sigma)$ reduce the number of moduli.

Theorem 6.3.2. *Under the conditions* $(+)$

$$\ker(\kappa_f) \cong \Big\langle x_i \frac{\partial f}{\partial x_i}, \quad i = 1, ..., 3 \\ w_{-\alpha}(f), \quad \alpha \in R(N, \Sigma_\Delta) \Big\rangle.$$

Proof. The proof is rather technical. By Proposition 5.5.1

$$\ker(\kappa_f) = \ker(\kappa_{\mathbb{P},f}) \cap L(\Delta)$$

and $R(N, \Sigma_\Delta) \subset R(N, \Sigma_{C(\Delta)})$ by Lemma 6.1.2. Let $R := R(N, \Sigma_{C(\Delta)}) \backslash R(N, \Sigma_\Delta)$. The Theorem is a consequence of the three points below.

- $\alpha \in R(N, \Sigma_\Delta) \Rightarrow w_{-\alpha}(f) \in L(\Delta)$.
- $\alpha \in R \Rightarrow w_{-\alpha}(f) \notin L(\Delta)$.
- Varying $\alpha \in R$ the $w_{-\alpha}(f)$ are linearly independent in $L(C(\Delta))/L(\Delta)$.

The necessity of the first two points is obvious and the last point assures that no linear combination of the $w_{-\alpha}(f)$, where $\alpha \in R$, lies in $\ker(\kappa_f)$.

First point: To $\alpha \in R(N, \Sigma_\Delta)$ is associated both $\Gamma_{-\alpha} \leqslant \Delta$ and $\Gamma'_{-\alpha} \leqslant C(\Delta)$. We show

$$\Gamma_{-\alpha} \subset \Gamma'_{-\alpha}, \tag{6.5}$$

since then for $m \in M \cap \Delta$, $m \notin \Gamma_{-\alpha}$ we get $m - \alpha \in \Delta$, that is $w_{-\alpha}(f) \in L(\Delta)$. Concerning (6.5): Given $n_i \in \Sigma_\Delta[1]$ with $\langle \alpha, n_i \rangle = 1$ and $n_j \in \Sigma_{C(\Delta)}[1]$ with $\langle \alpha, n_j \rangle = 1$ then $n_i = n_j$ by (6.1). It follows $\Gamma_{-\alpha} \subset \Gamma'_{-\alpha}$ since

$$\mathrm{Min}_{C(\Delta)}(n_i) = \mathrm{Min}_\Delta(n_i).$$

Thus $\Gamma_{-\alpha} \subset \Gamma'_{-\alpha}$.

Second point: There is a facet $\Gamma_{-\alpha}$ of $C(\Delta)$ such that

$$m - \alpha \in C(\Delta) \quad \text{for } m \in C(\Delta) \cap M, \quad m \notin \Gamma_{-\alpha}.$$

First assume that $\Gamma_{-\alpha} \cap \Delta$ is also a facet of Δ. There is $n_j \in \Sigma_\Delta[1] \backslash \{n_{\Gamma_{-\alpha}}\}$ with $\langle \alpha, n_j \rangle > 0$ since $\alpha \notin R(N, \Sigma_\Delta)$. Given $m \in Vert(\Gamma_j)$, then $m \in \mathrm{Supp}(f)$ and $m - \alpha \notin \Delta$ since

$$\langle m - \alpha, n_j \rangle < \mathrm{Min}_\Delta(n_j).$$

$\Rightarrow w_{-\alpha}(f) \notin L(\Delta)$. Assume that $\Gamma_{-\alpha} \cap \Delta$ is a face of Δ of dimension $< n-1$. The convex span

$$\langle m \in Vert(\Delta) \,|\, m - \alpha \notin \Delta \rangle$$

is of dimension $\geqslant n-1$. $\Rightarrow$ there is $m \in Vert(\Delta)$ with

$$m \notin \Gamma_{-\alpha}, \quad m - \alpha \notin \Delta,$$

that is $w_{-\alpha}(f) \notin L(\Delta)$.

Third point: Given a fixed facet $\Gamma = \Gamma_{-\alpha}$ of $C(\Delta)$ all

$$\alpha \in R(N, \Sigma_{C(\Delta)}) \backslash R(N, \Sigma_\Delta)$$

with $\Gamma_{-\alpha} = \Gamma$ build the lattice points on a lattice polytope $P \subset M_\mathbb{R}$. Given $\alpha \in Vert(P)$ there is $m \in \mathrm{Supp}(f)$ such that $x^{m-\alpha}$ does not appear in the support of any other $w_{-\alpha'}(f)$. Thus $w_{-\alpha}(f)$ does not appear with nonzero coefficient in any relation between the $w_{-\alpha'}(f)$. We then break down

P vertex by vertex.

Let Γ_1, Γ_2 be two different facets of $C(\Delta)$ and $\alpha_1, \alpha_2 \in R(N, \Sigma_{C(\Delta)}) \backslash R(N, \Sigma_\Delta)$ roots to these facets. Given a relation in

$$L(C(\Delta))/L(\Delta)$$

in which both $w_{-\alpha_1}(f)$ and $w_{-\alpha_2}(f)$ appear with nonzero coefficients there is $v \in \mathrm{Supp}(f)$ with

$$\langle v - \alpha_1, n_1 \rangle < \mathrm{Min}_\Delta(n_1), \quad v - \alpha_1 + \alpha_2 \in M \cap \Delta.$$

Then

$$\langle v - \alpha_1 + \alpha_2, n_1 \rangle \geqslant \mathrm{Min}_\Delta(n_1),$$

but $\langle \alpha_2, n_1 \rangle \leqslant 0$ since α_2 is a root for $n_2 \neq n_1$, a contradiction.

$\square$

CHAPTER 7

Hodge components and Jacobian rings

In this chapter we recall the purely combinatorial construction of the *jacobian ring* R_f of Batyrev and the *interior module* $R_{Int,f}$ over R_f. Both R_f and $R_{Int,f}$ are $\mathbb{N}_{\geqslant 0}$-graded. $R_{Int,f}$ serves to calculate the cohomology classes in $H^{n-1}(Z_f, \mathbb{C})$ of minimal weight W^{n-1}. We represent

$$R^k_{Int,f} = L^*(k \cdot \Delta)/U_{f,k}$$

for some subspace $U_{f,k}$ we specify generators of (see Proposition 7.2.3). The big problem remaining open is to specify a basis for $U_{f,k}$. This seems to be out of reach for us, though in some cases like $\Delta = d \cdot \Delta_n$ this problem might be written of from the combinatorics of the multiples $k \cdot \Delta$.

7.1 The Jacobian ring of Batyrev

In the sections 7.1 and 7.2 we do not restrict to the dimension $n = 3$.

Definition 7.1.1. *Let Δ be an n-dimensional lattice polytope and S_Δ denote as in section 2.1 the subalgebra of $\mathbb{C}[x_0, x_1^{\pm}, ..., x_n^{\pm}]$ spanned as $\mathbb{C}$-vector space by 1 and all monomials*

$$x_0^k x_1^{m_1} ... x_n^{m_n}, \quad \textit{where } k \in \mathbb{N}_{\geqslant 1}, \quad m_1, ..., m_n \in \mathbb{Z},$$

such that the rational point

$$\frac{m}{k} := \left(\frac{m_1}{k}, ..., \frac{m_n}{k}\right)$$

belongs to Δ*. Denote the* k*-th graded piece of* S_Δ *by* S_Δ^k*. The subalgebra* S_Δ^* *is defined in the same way except that we require that* m/k *belongs to the interior of* Δ*.*

We identify lattice points $m \in M$ with monomials $x^m \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

Construction 7.1.2. *[Bat93]*
Given a Laurent polynomial f in the variables $x_1, ..., x_n$ with n-dimensional Newton polytope Δ let

$$F(x_0, x_1, ..., x_n) := x_0 f(x_1, ..., x_n) - 1$$

which is an equation for the complement $T \backslash Z_f \subset \overline{T} := (\mathbb{C}^*)^{n+1}$.
Consider the logarithmic derivatives

$$F_i(x_0, x) := x_i \frac{\partial}{\partial x_i} F(x_0, x) = x_i \frac{\partial (x_0 f)}{\partial x_i} \quad 0 \leqslant i \leqslant n.$$

Definition 7.1.3. *[Bat93]*
The graded ideal $J_{\Delta,f}$ within S_Δ generated by $F_0, ..., F_n$ is called the *jacobian ideal* and the quotient ring

$$R_f := S_\Delta / J_{\Delta,f}$$

is called the *Jacobian ring of Batyrev* (associated to the Laurent polynomial f). We denote the k-th homogeneous component of R_f by R_f^k.

Theorem 7.1.4. *[Bat93, Thm.4.8]*
The jacobian ring R_f *is a graded ring. It is finite dimensional as* $\mathbb{C}$*-vector space if and only if* f *is nondegenerate. In this case the dimensions of* R_f^k *are independent of the polynomial* f*.* f *is nondegenerate if and only if* $F_0, ..., F_n$ *are algebraically independent over* $\mathbb{C}$*.*

Definition 7.1.5. *We denote the homogeneous ideal* $R_f \cap S_\Delta^*$ *of* R_f *by* $R_{Int,f}$ *and its* k*-th homogeneous component by* $R_{Int,f}^k$*. We call* $R_{Int,f}$ *the interior* R_f*-module.*

Remark 7.1.6. Poincaré duality on $H^2(Y, \mathbb{C})$ restricts to an isomorphism

$$(R_{Int,f}^k)^* \cong R_{Int,f}^{4-k}$$

by ([Bat93, Remark 9.5, Prop.9.7]). In ([Bat93, Ch.9]) $R_{Int,f}$ is denoted by H_f and should not be confused with the dualizing R_f-module D_f.

7.2 Construction of the components $R^k_{Int,f}$

Remark 7.2.1. By definition

$$R^1_{Int,f} \cong L^*(\Delta)$$

(independently of f). More interestingly

$$R^k_{Int,f} \cong L^*(k\Delta)/\Big(J^k_{\Delta,f} \cap L^*(k\Delta)\Big), \tag{7.1}$$

Since $J^k_{\Delta,f}$ is a graded ideal of S_Δ inductively

$$J^k_{\Delta,f} = L((k-1)\Delta) \cdot J^1_{\Delta,f}$$

Aim: Switching to different generators of $J^1_{\Delta,f}$ allows us to describe the relations in $R^k_{Int,f}$ more explicitly.

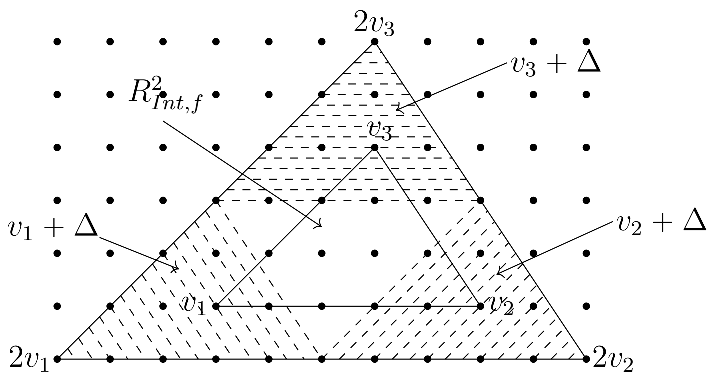


Illustration of the construction of $R^2_{Int,f}$ for a 2-simplex $\Delta = \langle v_1, v_2, v_3\rangle$ and f having support on the vertices of Δ. The shaded regions do not belong to $R^2_{Int,f}$, there are 4 points left in $R^2_{Int,f}$.

Construction 7.2.2. Let Δ be an n-dimensional lattice polytope with given f as in (2.3). Given a facet $\Gamma = \Gamma_i \leqslant \Delta$, where $i \in \{1, ..., r\}$, define

$$g_\Gamma(f) := \sum_{m\in M\cap\Delta} a_m \cdot (\langle n_i, m\rangle + b_i) \cdot x_0 x^m \underbrace{=}_{\text{(transforming)}} +b_i F_0 + \sum_{j=1}^{n} (n_i)_j F_j \in J^1_{\Delta,f},$$

where $n_i = ((n_\Gamma)_1, ..., (n_\Gamma)_n)$ and $b_i = -\operatorname{Min}_\Delta(n_i)$. The first representation implies that

$$\operatorname{Supp}(g_\Gamma(f)) \subset \big(\Delta \cap M\big) \backslash \big(\Gamma \cap M\big).$$

In case Δ is an n-simplex and

$$\operatorname{Supp}(f) = \operatorname{Vert}(\Delta)$$

then $g_{\Gamma_i}(f) \approx x^{v_i}$ where v_i is the vertex opposite to Γ_i ($\approx$ means up multiplication with a nonzero scalar). Conversely the matrix

$$\begin{pmatrix} b_1 & (n_{\Gamma_1})_1 & ... & (n_{\Gamma_1})_n \\ \vdots & \vdots & & \\ b_r & (n_{\Gamma_r})_1 & ... & (n_{\Gamma_r})_n \end{pmatrix} \tag{7.2}$$

has rank $n+1$ since $n_{\Gamma_1}, ..., n_{\Gamma_r}$ span $N_\mathbb{R}$ and $(b_1, ..., b_r) \neq (0, ..., 0)$. Thus $F_0, ..., F_n$ are linear combinations of $g_{\Gamma_1}(f), ..., g_{\Gamma_r}(f)$ and we get new generators

$$J_{\Delta,f} = (g_{\Gamma_1}(f), ..., g_{\Gamma_r}(f)). \tag{7.3}$$

Proposition 7.2.3. *Let Δ be an n-dimensional lattice polytope with $l^*(\Delta) > 0$ and a given f. Given $\Gamma_1, ..., \Gamma_{n+1} \leqslant \Delta$ with $n_{\Gamma_1}, ..., n_{\Gamma_{n+1}}$ affine linear independent. Then*

$$R^k_{Int,f} = L^*(k \cdot \Delta)/U_{f,k} \quad k = 1, ..., n+1$$

where $U_{f,k}$ denotes the vector space over $\mathbb{C}$ spanned by

$$g_{\Gamma_i}(f) \cdot x^v \quad i = 1, ..., n+1, \quad v \in \operatorname{Int}((k-1) \cdot \Delta) \cap M \tag{7.4}$$

$$g_\Gamma(f) \cdot x^v \quad \Gamma \leqslant \Delta, \qquad v \in \operatorname{Int}((k-1) \cdot \Gamma) \cap M \tag{7.5}$$

If $k = 2$ these polynomials are linearly independent.

Proof. The inclusion

$$U_{f,k} \subseteq \Big(J^k_{\Delta,f} \cap L^*(k\Delta)\Big).$$

is a consequence of the definition of $J_{\Delta,f}$.

To show: All relations

$$h \cdot g \in L^*(k \cdot \Delta), \qquad h \in J^1_{\Delta,f}, \quad g \in L((k-1) \cdot \Delta) \tag{7.6}$$

are of type (7.4) or (7.5). First if $\mathrm{Supp}(g) \subseteq \mathrm{Int}((k-1)\cdot\Delta) \cap M$ then the equation 7.6 is a linear combination of the relations (7.4) and without restriction

$$\mathrm{Supp}(g) \subset \mathrm{Bound}((k-1)\cdot\Delta) \cap M.$$

Write

$$h = \sum_{\Gamma} c_\Gamma \cdot g_\Gamma(f)$$

and take $F \leqslant \Delta$ with

$$\mathrm{Supp}(g) \cap \Big((k-1)\cdot F \cap M\Big) \neq \varnothing,$$

(this assumption makes sense since $\mathrm{Supp}(g) \neq \varnothing$). Then

$$\mathrm{Supp}(h) \cap (F \cap M) = \varnothing$$

by (7.6). In case $c_\Gamma = 0$ except for $\Gamma = F$, then

$$\mathrm{Supp}(g) \subset ((k-1)\cdot F)$$

and we get a relation of type (7.5). Restricting h to $F \cap M$ we get

$$h_{|F\cap M} = \sum_{\Gamma} c_\Gamma \cdot \big(-b_\Gamma F_{0|\Gamma_1} + \sum_{i=1}^{n} (n_\Gamma)_i F_{i|\Gamma_1}\big) \in \langle F_{0|\Gamma_1}, ..., F_{n|\Gamma_1}\rangle.$$

Expanding and restricting to F this means

$$\sum_{\Gamma} c_\Gamma \sum_{m\in F\cap M} a_m\big(\langle n_\Gamma, m\rangle - b_\Gamma\big)x_0 x^m = 0. \tag{7.7}$$

The left hand side in (7.7) equals

$$\sum_{\Gamma} c_\Gamma \cdot \big(-b_\Gamma F_{0|\Gamma_1} + \sum_{i=1}^{n} (n_\Gamma)_i F_{i|\Gamma_1}\big) \in \langle F_{0|\Gamma_1}, ..., F_{n|\Gamma_1}\rangle.$$

The F_i are algebraically independent over $\mathbb{C}$ (Theorem 7.1.4), since f is nondegenerate with respect to Δ. Besides $f_{|\Gamma_1}$ remains nondegenerate with respect to Γ_1 (by the definition of nondegeneracy) and it follows that there is (up to scaling) only one relation between $F_{0|\Gamma_1}, ..., F_{n|\Gamma_1}$, the one of the second type.

For $k = 2$ we know the dimension ([Bat93, Thm.9.8])

$$\dim R^2_{Int,f} = l^*(2 \cdot \Delta) - (n+1) \cdot l^*(\Delta) - \sum_{\Gamma} l^*(\Gamma),$$

where sum ranges over all facets Γ of Δ. Thus the last statement follows by comparing dimensions. □

Example 7.2.4. Let Δ be a simplex with vertices $v_0, ..., v_n$ and assume that f has support on the vertices of Δ. Then as already noted

$$g_{\Gamma_i}(f) = x^{v_i},$$

where Γ_i denotes the facet opposite to v_i. In this case we get a *monomial* basis for example of $R^2_{Int,f}$ by taking the quotient of $L^*(2 \cdot \Delta)$ by the span of

$$x^{v_i + v} \quad i = 0, ..., n, \quad v \in \big(\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma_i)\big) \cap M.$$

Remark 7.2.5. For $k > 2$ the polynomials in the Proposition will not be linear independent over $\mathbb{C}$ since we have the trivial relations

$$g_{\Gamma_i}(f) \cdot g_{\Gamma_j}(f) \cdot x^v - g_{\Gamma_j}(f) \cdot g_{\Gamma_i}(f) \cdot x^v$$

for $v \in L^*((k-2)\Delta)$. It would be very interesting to find a minimal set of relations between the g_{Γ_i} in $R^k_{Int,f}$ (see Remark 7.4.4 below).

7.3 Hodge and mixed Hodge structures

In this section we recall some *general* facts on (mixed) Hodge structure, thereby introducing the necessary notation. Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \varnothing$ and a given f. The „interesting“ cohomology classes of $Y = Y_f$ lie in $H^2(Y, \mathbb{C})$ by Proposition 4.3.2, Poincaré duality and since Y is compact.

There is a *Hodge decomposition* ([Del75])

$$H^2(Y, \mathbb{C}) \cong H^{2,0}(Y) \oplus H^{1,1}(Y) \oplus H^{0,2}(Y),$$

with $H^{p,2-p}(Y) = \overline{H^{2-p,p}(Y)}$. Equivalently there is a descending filtration

$$H^2(Y, \mathbb{C}) = F^0 \supset ... \supset F^3 = 0, \quad \text{where } F^i = \bigoplus_{p \geqslant i} H^{p,2-p}(Y).$$

with $H^k(Y, \mathbb{C}) = F^p \cap \overline{F^{3-p}}$. The filtration F^i is said to put a *Hodge structure of weight* 2 on the complex vector space $H := H^2(Y, \mathbb{C})$. The inclusion $\iota : Z_f \to T$ gives a pullback homomorphism

$$\iota^* : H^i(T, \mathbb{C}) \to H^i(Z_f, \mathbb{C}), \tag{7.8}$$

which is an isomorphism for $i < 2$ and injective for $i = 2$ by the Lefschetz theorem for hypersurfaces in tori ([DK86, Remark 3.10]) and the cohomology groups of T are well known since T is homotopy equivalent to $S^1 \times S^1$. We define the *primitive cohomology* of Z_f

$$PH^2(Z_f, \mathbb{C}) = \operatorname{coker}\Big(H^2(T, \mathbb{C}) \to H^2(Z_f, \mathbb{C})\Big).$$

There are two filtrations:

$$\begin{aligned} H^2(Z_f, \mathbb{C}) &= F^0 \supset ... \supset F^3 = 0 && \text{(Hodge filtration)} \\ 0 = W_1 \subset ... \subset W_4 &= H^2(Z_f, \mathbb{Q}) && \text{(Weight filtration)} \end{aligned}$$

Set

$$\mathrm{Gr}_F^i := F^i/F^{i+1}, \quad \mathrm{Gr}_W^j := W_j/W_{j-1}.$$

The filtration F^i induces a Hodge structure of weight r on

$$\mathrm{Gr}_W^r := (W_r/W_{r-1}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Definition 7.3.1. *Given $j \in \{0, 1, 2\}$ and $i \in \{0, ..., j + 2\}$ the vector spaces*

$$H^{p,2-p}(Y), \quad H^{i,j+2-i} H^2(Z_f, \mathbb{C}) := \mathrm{Gr}_F^i \, \mathrm{Gr}_W^{j+2} H^2(Z_f, \mathbb{C})$$

are called the Hodge components of $H^2(Y, \mathbb{C})$ and $H^2(Z_f, \mathbb{C})$ and their dimensions the Hodge numbers of Y and the Hodge-Deligne numbers of Z_f.

Remark 7.3.2. The inclusion $j : Z_f \to Y = Y_f$ induces a pullback homomorphism

$$j^* : H^k(Y, \mathbb{C}) \to H^k(Z_f, \mathbb{C})$$

with

$$j^*(F^i H^2(Y, \mathbb{C})) \subset F^i H^2(Z_f, \mathbb{C}), \quad j^* H^2(Y, \mathbb{C}) \subseteq W_2 H^2(Z_f, \mathbb{C})$$

(see [Voi02, Ch.7]).

Remark 7.3.3. The natural (intersection) pairing

$$H^2(Y,\mathbb{C}) \otimes H^2(Y,\mathbb{C}) \to \mathbb{C}$$

implies that

$$H^{1,1}(Y) \cong \left(H^{2,0}(Y) \oplus H^{0,2}(Y)\right)^{\perp}$$

(see [Voi02, Lemma 7.30]), where $\perp$ means orthogonal w.r.t. this pairing.

7.4 The Hodge components of $H^2(Z_f,\mathbb{C})$ and $H^2(Y_f,\mathbb{C})$

Remark 7.4.1. By ([Bat93, Cor.3.10]) the image of

$$H^2(T,\mathbb{C}) \to H^2(Z_f,\mathbb{C})$$

is contained in $\operatorname{Gr}_W^4 H^2(Z_f,\mathbb{C})$. The filtrations F^i and W_j respect the primitive cohomology.

Remark 7.4.2. We have an isomorphism of $\mathbb{C}$ vector spaces ([Bat93, Thm.6.9, Cor.6.10])

$$PH^2(Z_f,\mathbb{C}) \cong R_f.$$

This allows us to transport the Hodge and the weight filtration onto R_f (see [Bat93, Thm.6.9, Thm.8.2]): The Hogde filtration on R_f is given by the reverse integral grading on R_f and the weight filtration on R_f^k is induced by the subdivision of $k\cdot\Delta$ into j-dimensional faces where $j=0,...,3$. In this thesis we just need the following result:

Theorem 7.4.3. *([Bat93, Prop.9.2])*
There is an isomorphism

$$\operatorname{Gr}_W^{n-1} H^{n-1}(Z_f,\mathbb{C}) \cong R_{Int,f}.$$

which respects the Hodge filtration, that is

$$Gr_F^p Gr_W^{n-1} H^{n-1}(Z_f,\mathbb{C}) \cong R_{Int,f}^{n-p}.$$

Remark 7.4.4. Knowing a minimal set of relations between the g_Γ's would allow us to compute the dimensions of the Hodge components of $H^{n-1}(Z_f,\mathbb{C})$ of minimal weight.

Remark 7.4.5. Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \varnothing$. Write

$$Y \backslash Z_f = \bigcup_{i=1}^{h} G_i \cup S$$

where $G_i := V(\tau_i)$ are irreducible curves and S equals the union of the fixed points of $\mathbb{P}$, that are isolated on Y.

Construction 7.4.6. Let $Z := Y \backslash D$, where

$$D := G_1 + ... + G_h.$$

There is the following *Gysin exact sequence* (compare [DK86, Proof of Thm.3.7]):

$$\begin{aligned} 0 \to H^1(Z_f, \mathbb{C}) &\xrightarrow{r} \bigoplus_{i=1}^{h} H^0(G_i, \mathbb{C}) \xrightarrow{k_*} H^2(Y, \mathbb{C}) \\ &\xrightarrow{j^*} H^2(Z_f, \mathbb{C}) \xrightarrow{r} \bigoplus_{i=1}^{h} H^1(G_i, \mathbb{C}) \to 0, \end{aligned}$$

since $h^1(Y, \mathbb{C}) = h^3(Y, \mathbb{C}) = 0$. The inclusion $j : Z_f \to Y$ yields the pullback homomorphism j^*, r is called the *residue map* and k_* the so called *Gysin map*. To be more precise k_* is the homomorphism Poincaré dual to

$$k^* := \bigoplus_{i=1}^{h} k_i^* : H^2(Y, \mathbb{C}) \to \bigoplus_{i=1}^{h} H^2(G_i, \mathbb{C}),$$

where $k_i : G_i \to Y$ denotes the inclusion (compare [Voi02, 7.3.2]). r is in fact a topological map (see [CMSP17, Ch.3.2]).
Setting

$$Z := Z_f \cup S$$

we get

$$H^k(Z_f, \mathbb{C}) \cong H^k(Z, \mathbb{C}) \quad k \geqslant 1,$$

that is we may replace Z_f by Z in the above sequence. The cohomology groups $H^k(G_i, \mathbb{C})$ for $k = 0, 1$ carry Hodge structures of weight 0 and 1. Then

$$r(\operatorname{Gr}_W^j H^k(Z_f, \mathbb{C})) \subset \bigoplus_{i=1}^{h} \operatorname{Gr}_W^{j-2} H^{k-1}(G_i, \mathbb{C}) \quad k = 1, 2.$$

and the image of $H^0(G_i, \mathbb{C})$ under k_* equals the image of $\mathcal{O}_Y(G_i)$ under the first chern map

$$c_1 : \mathrm{Pic}(Y) \to H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{Z})$$

(compare [Voi02, Thm.11.33, Ch.11 Ex.1]). Thus we obtain the exact sequence

$$\begin{aligned} 0 \to Gr_W^2 H^1(Z_f, \mathbb{C}) &\xrightarrow{r} \bigoplus_{i=1}^{h} H^0(G_i, \mathbb{C}) \xrightarrow{k_*} H^2(Y, \mathbb{C}) \\ &\xrightarrow{j^*} Gr_W^2 H^2(Z_f, \mathbb{C}) \to 0. \end{aligned} \tag{7.9}$$

Theorem 7.4.7. *Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \varnothing$ and let $Y = Y_f$ be a minimal model of Z_f. Then*

$$H^{p,2-p}(Y) \approx R_{Int,f}^{p+1} \tag{7.10}$$

where $\approx$ means up to cohomology classes in $\mathbb{P}$. These cohomology classes are integral (that is they lie in $H^2(Y, \mathbb{Z})$, constant on the whole family and lie in $H^{1,1}(Y)$).

□

7.5 Appendix: The algorithm of Danilov and Khovanskii

In 1986 Vladimir I. Danilov and Askold G. Khovanskii invented ideas how the Hodge-Deligne numbers of Z_f and the Hodge numbers of a smooth compactification of Z_f could be calculated (see [DK86]). We shortly sketch the inductive character of their algorithmic work: The authors work in arbitrary dimension n but with a smooth (or at least quasismooth) birational model of Z_f. For this take a toric resolution of singularities $p : \tilde{\mathbb{P}} \to \mathbb{P}_\Delta$, in which case the preimage

$$\tilde{Z}_f := p^{-1}(Z_{\Delta,f})$$

is smooth. *Poincaré duality* yields a perfect pairing

$$H_c^k(\tilde{Z}_f, \mathbb{C}) \times H^{2n-2-k}(\tilde{Z}_f, \mathbb{C}) \to \mathbb{C}$$

respecting the Hodge filtrations, where $H_c^k(\tilde{Z}_f, \mathbb{C})$ denotes the cohomology with compact support. For the calculations of main interest is the following invariant of euler type

$$e^{p,q}(X) := \sum_{k=0}^{n} (-1)^k \cdot h_c^{p,q} H_c^k(X, \mathbb{C}) \tag{7.11}$$

for X a variety. Writing $\tilde{Z}_f = Z_f \cup \bigcup_{\Gamma' \leqslant \Delta'} Z_{\Gamma',f}$ for some polytope Δ' majorizing Δ, then

$$e^{p,q}(\tilde{Z}_f) = e^{p,q}(Z_f) + \sum_{\Gamma' \leqslant \Delta'} e^{p,q}(Z_{\Gamma',f}) \tag{7.12}$$

(see [DK86, 5.2] for details). For $p + q > n - 1$

$$e^{p,q}(Z_f) = \begin{cases} e^{p,q}(T) & p = q \\ 0 & p \neq q \end{cases}$$

by a Lefschetz theorem. Thus (7.12) serves to get an inductive calculation of $e^{p,q}(\tilde{Z}_f)$ for $p + q > n - 1$. By *Poincaré duality*

$$e^{p,q}(\tilde{Z}_f) = e^{n-1-p,n-1-q}(\tilde{Z}_f)$$

is also known for $p + q < n - 1$ and the last remaining number $e^{p,n-1-p}(Z_f)$ is gotten from the others and the relation

$$\sum_q e^{p,q}(Z_f) = (-1)^{p+n-1} \cdot \binom{n}{p+1} + \phi_{n-p}(\Delta)$$

The term $\phi_{n-p}(\Delta)$ depends on the dimensions $l^*(j \cdot \Delta)$ for $j \geqslant 1$ (see [DK86, 4.4] for the last term). Last but not least

$$e^{p,q}(Z_f) = \pm h_c^{p,q} H_c^{n-1}(Z_f, \mathbb{C}), \quad e^{p,q}(\tilde{Z}_f) = \pm h^{p+q}(\tilde{Z}_f, \mathbb{C}) \tag{7.13}$$

due to Theorems of *Grothendieck* and *Lefschetz*, giving the Hodge-Deligne numbers (at least in theory).

Examples from: We list some examples from ([DK86, 5.11]):

$n = 1 \Rightarrow Z_f$ equals $(l(\Delta) + 1)$ different points in $\mathbb{C}^*$

$n = 2$:

$h^{0,0}H_c^1(Z_f)$	$h^{0,1}H_c^1(Z_f)$
$h^{1,0}H_c^1(Z_f)$	$h^{1,1}H_c^1(Z_f)$

=

$\Pi - 1$	$l^*(\Delta)$
$l^*(\Delta)$	0

$$\begin{aligned} \text{where } \Pi &:= |\{\text{points in the 1-skeleton of } \Delta\}| \\ &= l(\Delta) - l^*(\Delta). \end{aligned}$$

$n = 3$:

$h^{2,0}H_c^2(Z_f)$	$h^{2,1}H_c^2(Z_f)$	$h^{2,2}H_c^2(Z_f)$
$h^{1,0}H_c^2(Z_f)$	$h^{1,1}H_c^2(Z_f)$	$h^{1,2}H_c^2(Z_f)$
$h^{0,0}H_c^2(Z_f)$	$H^{0,1}H_c^2(Z_f)$	$h^{0,2}H_c^2(Z_f)$

=

$l^*(\Delta)$	0	0
$\sum_\Gamma l^*(\Gamma)$	$h^{1,1}$	0
$\Pi - 1$	$\sum_\Gamma l^*(\Gamma)$	$l^*(\Delta)$

$$\text{where } h^{1,1} := l^*(2 \cdot \Delta) - 4 \cdot l^*(\Delta) - 3 - \sum_\Gamma l^*(\Gamma).$$

CHAPTER 8

The infinitesimal Torelli Theorem (ITT)

Varying f we are keen to know whether the Hodge components of $H^2(Y_f, \mathbb{C})$ determine Y_f in the family $\mathcal{X} \to B$ up to isomorphism. One could make this problem precise by introducing and studying properties of the *period map* $\mathcal{P}_B$. This leads to so called *Torelli type Theorems*. These are not really „uniform Theorems“ as there are algebraic varieties failing them, but such varieties are often very exceptional.
In this chapter we restrict to the *infinitesimal Torelli Theorem* for $\mathcal{X} \to B$. We ask if the kernel of the differential $d\mathcal{P}_{B,f}$ at f, which factors through the Kodaira-Spencer map for $\mathcal{X} \to B$, is strictly larger than the kernel of κ_f.
For this we establish a conjecture on the kernel of $d\mathcal{P}_{B,f}$ which though we do not prove. We conclude by showing that some of the surfaces of general type to our 49 polytopes Δ with

$$\mathrm{Int}(\Delta) \cap M = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dim\, F(\Delta) = 3$$

fail this infinitesimal Torelli Theorem.

8.1 Properties of the period map and its differential

Definition 8.1.1. *([Voi02, Thm.9.3, Ch.10.1.2, Ch.10.1.3])*
Let Δ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and $f \in B$. The period map $\mathcal{P}_{B,f}$ for the 2-th cohomology is defined by

$$\begin{aligned}\mathcal{P}_{B,f} : &B \to \Gamma \backslash D \\ &f' \mapsto [H^{2,0}(Y_{f'})],\end{aligned}$$

where

$$H^{2,0}(Y_{f'}) \subset H^2(Y_{f'}, \mathbb{C}) \cong H^2(Y_f, \mathbb{C})$$

Here

$$\begin{aligned}&D: \text{ period domain (a quasiprojective variety)} \\ &\Gamma: \text{ The monodromy group } \pi_1(B, f)\end{aligned}$$

We refer to ([Voi02, Ch.10], [Voi03, Ch.3]) for details.

Remark 8.1.2. $H^{2,0}(Y_{f'})$ determines the Hodge structure on $H^2(Y_{f'}, \mathbb{C})$ by Remark 7.3.3. The period map $\mathcal{P}_{B,f}$ is holomorphic (see [Voi02, Ch.10]). The *Torelli Theorem* asks if the Hodge structure of a fiber of pr_2 determines this fibre (up to isomorphism). We study the situation infinitesimally.

Construction 8.1.3. *(Result of Griffiths) ([Voi02, Thm.10.21])*
The differential $d\mathcal{P}_{B,f}$ of $\mathcal{P}_{B,f}$ fits into a diagram

$$\begin{tikzcd} T_{B,f} \arrow[r, "\kappa_f"] \arrow[dr, "d\mathcal{P}_{B,f}"'] & H^1(Y_f, T_{Y_f}) \arrow[d, "\Phi_f"] \\ & \mathrm{Hom}(H^0(Y_f, \Omega^2_{Y_f}), H^1(Y_f, \Omega^1_{Y_f})) \end{tikzcd} \tag{8.1}$$

Φ_f is the homomorphism between cohomology groups induced by cup product and the contraction

$$T_{Y_f} \times \Omega^2_{Y_f} \to \Omega^1_{Y_f}.$$

This diagram is important since it connects the Hodge-theoretic homomorphism $d\mathcal{P}_{B,f}$ with the Kodaira-Spencer map κ_f. Given the conditions (+) we may replace κ_f by $\kappa_{\mathbb{P},f}$ by replacing Δ by $C(\Delta)$ (see Remark 5.5.2).

Remark 8.1.4. Starting with a smooth proper deformation $\mathcal{Y} \to S$ of Y_f with S smooth, we define a Kodaira Spencer map $\kappa_{S,f}$ and a period map $\mathcal{P}_{S,f}$ just as in the Definitions 5.2.1 and 8.1.1. The result of Griffiths remains valid: $d\mathcal{P}_{S,f}$ factors through $\kappa_{S,f}$ and Φ_f, that is Φ_f is universal.

Definition 8.1.5. *The infinitesimal Torelli Theorem (short: ITT) for Y_f asks if Φ_f is injective. The infinitesimal Torelli Theorem for Y_f in $\mathcal{X} \overset{pr_2}{\to} B$ asks if $\Phi_{f|Im\,\kappa_f}$ is injective. The infinitesimal Torelli Theorem for $\mathcal{X} \overset{pr_2}{\to} B$ asks if $\Phi_{f|Im\,\kappa_f}$ is injective for $f \in B$.*

If κ_f is surjective of course the first and the third definition coincide. Choose a reference point $f \in B$ and define a map from B into a *mixed period domain* D_{mix} (which has to be defined) by

$$f' \overset{\phi_f}{\mapsto} R^1_{Int,f'} \subset R_{Int,f'} \cong R_{Int,f}.$$

ϕ_f is the entry of a *mixed period map*, which maps f' onto the mixed Hodge components of $H^2(Z_{f'}, \mathbb{C})$. Apparently ϕ_f just depends on the affine part Z_f and not on the particular compactification Y_f.

Remark 8.1.6. *([Voi02, Ch. 9.2.1, Ch.10])*
Just as the Hodge decomposition fits into a general context by introducing Hodge structures, the differential $d\mathcal{P}_{B,f}$ fits into the context of an infinitesimal variation of Hodge structure by introducing a Gauß-Manin connection: We remember the construction without going into details:

Varying $f \in B$ the vector spaces $H^2(Y_f, \mathbb{C})$ build a holomorphic vector bundle $R^2(pr_2)_*(\mathbb{C})$ and similarly do $H^{p,2-p}(Y_f)$ build the vector bundle $\mathcal{H}^{p,2-p}$. The *Gauß-Manin connection*

$$\overline{\nabla}^{p,2-p} : \mathcal{H}^{p,2-p} \to \mathcal{H}^{p-1,3-p} \otimes \Omega^1_B$$

is a connection, which could be defined fibrewise

$$\overline{\nabla}^{p,2-p}_f : H^{p,2-p}(Y_f) \to H^{p-1,3-p}(Y_f) \otimes \Omega^1_{B,f}$$

as the map with

$$d\mathcal{P}_{B,f}(v)(z) = \overline{\nabla}^{2,0}_f(z)(v),$$

for $v \in T_{B,f}$ and $z \in H^{2,0}(Y_f)$, where the second bracket stand on the right hand side means the contraction between $T_{B,f}$ and $\Omega^1_{B,f}$.

The cohomology groups with rational coefficients $H^2(Y_f, \mathbb{Q})$ form a *local subsystem* $R^2(pr_2)_*(\mathbb{Q})$ of $\mathcal{H}^{1,1}$ with

$$\overline{\nabla}^{1,1}_f(z) = 0 \quad \text{for } z \in H^2(Y_f, \mathbb{Q}).$$

We justify the fact that the above map ϕ_f is enough for us (here we are a bit pedantic, but this is more clear to us than in ([Bat93, Prop.11.8])) by the following lemma:

Lemma 8.1.7. *$d\phi_f$ has image in $\mathrm{Hom}(R^1_{Int,f}, R^2_{Int,f})$ and $d\mathcal{P}_{B,f}$ factors as follows*

$$\begin{tikzcd} T_{B,f} \arrow[r, "d\phi_f"] \arrow[dr, "d\mathcal{P}_{B,f}"'] & \mathrm{Hom}(R^1_{Int,f}, R^2_{Int,f}) \arrow[d] \\ & \mathrm{Hom}(H^0(Y_f, \Omega^2_{Y_f}), H^1(Y_f, \Omega^1_{Y_f})) \end{tikzcd} \tag{8.2}$$

where the vertical map denotes the inclusion.

Proof.

$$H^0(Y_f, \Omega^2_{Y_f}) \cong R^1_{Int,f}, \quad H^1(Y_f, \Omega^1_{Y_f}) \approx R^2_{Int,f},$$

by Theorem 7.4.7, where $\approx$ means up to cohomology classes that come from classes of $\mathbb{P}$. The image of $d\phi_f$ lies in

$$\mathrm{Hom}(R^1_{Int,f}, R^2_{Int,f})$$

by the result of Griffiths (Construction 8.1.3).
<u>To show:</u> Given $v \in T_{B,f}$ and $z \in R^1_{Int,f}$ then

$$d\mathcal{P}_{B,f}(v)(z) \in R^2_{Int,f}.$$

We prove

$$R_{Int,f} = \Big(\bigoplus_{i=1}^{h} H^2(G_i, \mathbb{C}) \Big)^{\perp}, \tag{8.3}$$

w.r.t the pairing

$$\langle , \rangle : H^2(Y_f, \mathbb{C}) \otimes H^2(Y_f, \mathbb{C}) \to \mathbb{C}$$

Given this assertion is valid, take $v \in T_{B,f}$, $x \in \bigoplus_{i=1}^{h} H^2(G_i, \mathbb{C})$ and $z \in R^1_{Int,f}$ arbitrary.

$$\Rightarrow \langle \overline{\nabla}_f^{2,0}(z)(v), x \rangle = -\langle z, \overline{\nabla}_f^{1,1}(x)(v) \rangle$$

by ([Voi03, Prop.5.19 formula (5.14)]). Besides

$$\overline{\nabla}_f^{1,1}(x) = 0$$

by Remark 8.1.6, that is

$$\overline{\nabla}_f^{2,0}(z)(v) \in \left(\bigoplus_{i=1}^{h} H^2(G_i, \mathbb{C}) \right)^{\perp} = R_{Int,f}$$

and $\overline{\nabla}_f^{2,0}(z)(v) \in R^2_{Int,f}$ by Construction 8.1.3.

Concerning (8.3): Since

$$G_1 + ... + G_r$$

is an snc-divisor we get the inclusion „$\supset$“ in (8.3). The Gysin exact sequence 7.9 is Poincaré dual to an exact sequence

$$0 \to \left(\mathrm{Gr}_W^2 \, H^2(Z_f, \mathbb{C}) \right)^* \to H^2(Y, \mathbb{C}) \xrightarrow{k^*} \bigoplus_{i=1}^{r} H^2(G_i, \mathbb{C}) \tag{8.4}$$

Poincaré duality on $H^2(Y, \mathbb{C})$ restricts to an isomorphism

$$(R^k_{Int,f})^* \cong R^{4-k}_{Int,f}$$

by Remark 7.1.6, that is dualizing the first term in (8.4) simply means to reverse the Hodge filtration. Let $y \in \bigoplus_{i=1}^{r} H^0(G_i, \mathbb{C})$, $x = k_*(y) \in \bigoplus_{i=1}^{r} H^2(G_i, \mathbb{C})$ and $z \in R_{Int,f}$. Then

$$\langle x, z \rangle = \langle k_*(y), z \rangle = \langle y, k^*(z) \rangle = 0$$

by the projection formula ([Voi02, 7.3.2]), proving (8.3).
By the definitions of $\mathcal{P}_B$ and ϕ as holomorphic maps mapping a Laurent polynomial f onto the same vector spaces it is clear that $d\mathcal{P}_{B,f}$ factors through $d\phi_f$. We are left to show that for $v \in T_{B,f}$ and $z \in R^1_{Int,f}$ we have

$$d\mathcal{P}_{B,f}(v)(z) \in R^2_{Int,f}.$$

We first show that

$$R_{Int,f} = \Big(\bigoplus_{i=1}^{r} H^2(G_i,\mathbb{C})\Big)^{\perp} \tag{8.5}$$

with respect to the intersection pairing

$$\langle\, ,\rangle : H^2(Y_f,\mathbb{C}) \otimes H^2(Y_f,\mathbb{C}) \to \mathbb{C}.$$

We call the cohomology classes on the right of (8.3) also the *primitive cohomology* classes of Y_f.
Let $v \in T_{B,f}$, x be a primitive cohomology class of Y_f and $z \in R^1_{Int,f}$. Then as in ([Voi03, Prop.5.19 formula (5.14)]) we have

$$\langle \overline{\nabla}_f^{2,0}(z)(v), x\rangle = -\langle z, \overline{\nabla}_f^{1,1}(x)(v)\rangle.$$

But as noted in Remark 8.1.6

$$\overline{\nabla}_f^{1,1}(x) = 0$$

and thus $\overline{\nabla}_f^{2,0}(z)(v) \in R_{Int,f}$ and by Griffiths transversality $\in R^2_{Int,f}$.

□

Remark 8.1.8. *(see [Bat93, Prop.11.8])*
$d\phi_f$ is simply induced by the addition of lattice points

$$L(\Delta) \to \mathrm{Hom}(L^*(\Delta), L^*(2\Delta)).$$

8.2 Smooth and stable points of $\mathcal{M}(\Delta)$

We do not restrict to $n = 3$ in this section. Given an n-dimensional lattice polytope Δ and some polynomial f, the torus T acts on $U_{reg}(\Delta)$

$$(t_1, ..., t_n).f(x_1, ..., x_n) = f(t_1x_1, ..., t_nx_n), \quad (t_1, ..., t_n) \in T.$$

$T \cdot f$ denotes the orbit of f under T.

Let $\mathcal{M}(\Delta) := B/T$ be the quotient of B by T. We omit equivalence classes and write $f \in \mathcal{M}(\Delta)$.

Definition 8.2.1. *([MuFo82, Def.1.7])*
Let $v \in \mathbb{C}^{l(\Delta)+1}$ and $x = [v]$. The point x is stable if the orbit $T \cdot v$ is closed and of dimension $\dim \Delta = n$. The second condition is equivalent to the condition

$$\#\operatorname{Stab}_T(v) < \infty,$$

where

$$\operatorname{Stab}_T(v) := \{t \in T \mid t.v = v\}$$

denotes the stabilizer of v w.r.t. the action of T.

Let $r := l(\Delta) - 1$. The set $(\mathbb{P}^r)^s$ of stable points of $\mathbb{P}^r$ is Zariski open in $\mathbb{P}^r$ ([MuFo82, §4]), but might be empty. Let

$$\Delta_r := \langle e_m \mid \quad m \in M \cap \Delta \rangle$$

denote the r-dimensional standard simplex embedded into an affine hyperplane in $\mathbb{R}^{l(\Delta)}$. Then there is a map $\pi : \Delta_r \to \Delta$ given by

$$\sum_{m \in M \cap \Delta} \lambda_m \cdot e_m \mapsto \sum_{m \in M \cap \Delta} \lambda_m \cdot m \quad \text{for} \sum_{m \in M \cap \Delta} \lambda_m = 1.$$

To Δ_r is associated the toric variety $\mathbb{P}^r$ and given

$$a := (a_m)_{m \in M \cap \Delta} \in \mathbb{P}^r$$

there is a natural $(\mathbb{C}^*)^r$ orbit through a of some dimension $k \in \{0, ..., r\}$. We denote the k-dimensional face of Δ_r corresponding to this orbit by $\Gamma(a)$.

Proposition 8.2.2. *([KSZ, Prop.3.5])*
$(\mathbb{P}^r)^s \neq \varnothing$ if and only if

$$(0, ..., 0) \in \operatorname{Int}(\Delta) \cap M.$$

More precisely $a \in (\mathbb{P}^r)^s$ is stable if and only if $\pi(\Gamma(a))$ has full dimension n and contains $(0, ..., 0)$ in its interior.

Corollary 8.2.3. *If $(0, ..., 0) \in \operatorname{Int}(\Delta) \cap M$ and $a := (a_m)_{m \in M \cap \Delta} \in \mathbb{R}^{l(\Delta)}$ is such that f as in (2.3) lies in $U_{reg}(\Delta)$, then a is stable.*

Proof. For m a vertex of Δ we have $a_m \neq 0$ and thus $\pi(\Gamma(a)) \supset \operatorname{Int}(\Delta)$, in particular $(0, ..., 0) \in \pi(\Gamma(a))$ and $\pi(\Gamma(a))$ is full-dimensional. $\square$

Corollary 8.2.4. *If* $(0,...,0) \in \mathrm{Int}(\Delta)$ *then the quotient* $\mathcal{M}(\Delta)$ *is smooth.*

Proof. By definition $L(\Delta)\backslash U_{reg}(\Delta) = \{E_A = 0\}$, where $A := \Delta \cap M$ by Remark 2.2.4. In effect $U_{reg}(\Delta)$ is affine.

The orbit $T \cdot f$ is closed and n-dimensional since all $f \in \mathcal{M}(\Delta)$ are stable w.r.t. the action of T on $\mathbb{P}^{l(\Delta)-1}$. Applying Luna's slice Theorem ([Dre12, Prop.5.7]) to the affine variety $U_{reg}(\Delta)$ and the projection

$$U_{reg}(\Delta) \to \mathcal{M}(\Delta)$$

gives us that $\mathcal{M}(\Delta)$ is smooth at f if $\mathrm{Stab}_{T\times\mathbb{C}^*}(f)$ contains just the neutral element $(1,...,1)$.

By definition

$$\mathrm{Stab}_T(f) = \{(t,1) = (t_1,...,t_n,1) \in T\times\mathbb{C}^* \mid \quad t^m a_m = a_m \quad \forall m \ \ \textit{with}\ a_m \neq 0\}.$$

Thus

$$\mathrm{Stab}_T(f) \subset \{t \in T \,|\, t^{v_i} = 1 \quad \forall \text{ vertices } v_1,...,v_k\} \times \{1\}^{\cdot}$$

Consider n vertices $v_1,...,v_n$ which span $M_{\mathbb{R}}$ and apply an unimodular transformation $U : M \to M$ such that

$$U(v_1),...,U(v_{n-1}) \in \{(m_1,...,m_n) \in M |\, m_n = 0\}.$$

Replace v_i by $U(v_i)$ and note that for

$$(t_1,...,t_n) \in \mathrm{Stab}_T(f)$$

the entry t_n is uniquely determined by $t_1,...,t_{n-1}$ and the relation $1 = t^{v_n}$. Further if $(t_1,...,t_{n-1}) = (1,...,1)$ then also $t_n = 1$. By this we have reduced the assertion $\mathrm{Stab}_T(f) = \{(1,...,1)\}$ to the lower-dimensional problem that the only solution of

$$1 = t^{v_1} = ... = t^{v_{n-1}}$$

is $t = (1,...,1)$. We continue inductively.

□

Remark 8.2.5. Given $(0,...,0) \in \mathrm{Int}(\Delta) \cap M$ and $f \in \mathcal{M}(\Delta)$ then f is smooth and stable. Let $\phi_f : T \to B$ denote the map

$$(t_1,...,t_n) \mapsto f(t_1x_1,...,t_nx_n).$$

Then as in section 6.1 the tangent space at the orbit $T \cdot f$ is given by

$$T_{T\cdot f,f} = Im((d\phi_f)_e) = \langle x_1\frac{\partial f}{\partial x_1},...,x_n\frac{\partial f}{\partial x_n}\rangle,$$

where $(d\phi_f)_e$ denotes the differential of ϕ_f at $e = (1,...,1)$. We get

$$\begin{aligned} T_{\mathcal{M}(\Delta),f} &\cong T_{B,f}/T_{T\cdot f,f} \\ &\cong L(\Delta)/\langle f, x_1\frac{\partial f}{\partial x_1},...,x_n\frac{\partial f}{\partial x_n}\rangle \cong R^1_f \end{aligned}$$

where the first isomorphism follows from ([Bat93, Cor.11.3]).

8.3 Explicit description of $d\mathcal{P}_{\mathcal{M}(\Delta),f}$

Construction 8.3.1. *(Computational description of $d\mathcal{P}_{\mathcal{M}(\Delta),f}$)*
Assume $(0,0,0) \in \mathrm{Int}(\Delta) \cap M$. By Corollary 8.2.4 all points $f \in \mathcal{M}(\Delta)$ are smooth. To the deformation $\mathcal{X} \to B \to \mathcal{M}(\Delta)$ there is a Kodaira-Spencer map $\kappa_{\mathcal{M}(\Delta),f}$ and a period map $\mathcal{P}_{\mathcal{M}(\Delta),f}$. We study the resulting map

$$R^1_f \overset{d\phi_f}{\to} \mathrm{Hom}(R^1_{Int,f}, R^2_{Int,f})$$

and its kernel.

The assertion of the following elementary Remark is also a consequence of diagram 8.1 and the results of Chapter 5.

Remark 8.3.2. *(Elementary proof that $\ker(\kappa_f) \subset \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$)*
Let $\alpha \in R(N, \Sigma_\Delta)$. Then there is $\Gamma_{-\alpha} \leqslant \Delta$ such that if $v \in \mathrm{Int}(\Delta) \cap M$ then by definition of the roots

$$v - \alpha \in \mathrm{Int}(\Delta) \cap M \quad \text{or} \quad v - \alpha \in \mathrm{Int}(\Gamma_{-\alpha}) \cap M.$$

By formula (6.3)

$$g_{\Gamma_{-\alpha}}(f) \cdot x^{-\alpha} = w_{-\alpha}(f).$$

We obtain

$$w_{-\alpha}(f) \cdot x^v = g_{\Gamma_{-\alpha}}(f) \cdot x^{v-\alpha} \in L^*(2\Delta).$$

Since $g_{\Gamma_{-\alpha}}(f) \cdot x^{v-\alpha} \in J^2_{\Delta,f}$ we verified explicitly that $w_{-\alpha}(f) \in \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$.

The kernel $\ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$:

Remark 8.3.3. Assume $(0,0,0) \in \mathrm{Int}(\Delta) \cap M$. First by definition

$$\ker(d\mathcal{P}_{\mathcal{M}(\Delta),f}) = \{h \in L(\Delta) \mid \forall v \in \mathrm{Int}(\Delta) \cap M : \ h \cdot x^v \in J^2_{\Delta,f}\}. \tag{8.6}$$

and by Proposition 7.2.3

$$L^*(2 \cdot \Delta) \cap J^2_{\Delta,f} = \langle g_\Gamma(f) \cdot x^w \mid \Gamma \leqslant \Delta, \\ w \in \mathrm{Int}(\Delta) \cap M \text{ or } w \in \mathrm{Int}(\Gamma) \cap M\rangle.$$

Obviously

$$\{g_\Gamma(f) \cdot x^w \mid \Gamma \leqslant \Delta \text{ a facet }, \\ w + v \in \big(\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma)\big) \cap M, \quad \forall\, v \in \mathrm{Int}(\Delta) \cap M\}.$$

is contained in $\ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$. In general given $h \in \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$

$$h \cdot x^v = \sum_\Gamma h_{\Gamma,v} \cdot g_\Gamma(f) \tag{8.7}$$

for $v \in \mathrm{Int}(\Delta) \cap M$ and $\mathrm{Supp}(h_{\Gamma,v}) \subset (\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma)) \cap M$.

Note: Restricting the condition $w \in \mathrm{Int}(\Delta) \cap M$ to 4 facets $\Gamma_1, ..., \Gamma_4$ with $n_{\Gamma_1}, ..., n_{\Gamma_4}$ affine linearly independent this representation is unique by Proposition 7.2.3.

Remark 8.3.4. $h_{\Gamma,v}$ is completely determined by $h_{\Gamma,(0,0,0)}$: Given $h \in \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$ and $v \in \mathrm{Int}(\Delta) \cap M$ there are

$$h_{\Gamma,v} \in \big(\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma)\big) \cap M$$

such that

$$h = \sum_\Gamma h_{\Gamma,v} \cdot g_\Gamma(f) \cdot x^{-v}$$

Setting $h_\Gamma := h_{\Gamma,(0,0,0)}$ the relation

$$\sum_\Gamma h_\Gamma \cdot g_\Gamma(f) \cdot x^v = h \cdot x^v = \sum_\Gamma h_{\Gamma,v} \cdot g_\Gamma(f)$$

implies

$$h_{\Gamma,v} = h_\Gamma \cdot x^v$$

by linear independence of the $g_\Gamma(f)$'s.

Conjecture 8.3.5. *Let Δ be a 3-dimensional lattice polytope with* $(0,0,0) \in \mathrm{Int}(\Delta) \cap M$ *and* $\mathrm{Int}(\Delta) \cap M \nsubseteq E$ *(plane). Then*

$$\begin{aligned}\ker(d\mathcal{P}_{\mathcal{M}(\Delta),f}) = &\langle g_\Gamma(f) \cdot x^w \in R^1_f \mid \Gamma \leqslant \Delta \text{ a facet},\\ & w + v \in \big(\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma)\big) \cap M, \quad \forall\, v \in \mathrm{Int}(\Delta) \cap M\rangle.\end{aligned}$$

The inclusion $\supseteq$ is clear. The problem with the opposite inclusion is that it might happen that

$$\sum_\Gamma h_\Gamma \cdot g_\Gamma(f) \in L(\Delta). \tag{8.8}$$

but $h_\Gamma \cdot g_\Gamma(f) \notin L(\Delta)$ for several Γ's. Maybe under the additional assumption

$$\mathrm{Int}(\Delta) \cap M \nsubseteq \text{ plane}$$

a proof of the conjecture gets more handable. But we could not finish a proof and just end up with two Lemmas in the direction of a possible proof: Suppose given an element of the kernel as in (8.8). Let

$$H_{n,l} := \{x \in M_\mathbb{R} \mid \langle x, n\rangle = l\} \quad n \in N, \quad l \in \mathbb{Z}.$$

Lemma 8.3.6. *Let* $R_\Gamma := (\mathrm{Int}(\Delta) \cup \mathrm{Int}(\Gamma)) \cap M$*.*

$$\mathrm{Supp}(h_\Gamma) \subset \mathrm{Cone}(\mathrm{Int}(\Gamma) \cap M) \cap H_{n_\Gamma, -l_\Gamma} \cap M, \tag{8.9}$$

where l_Γ denotes the smallest natural number $\geqslant 1$ with

$$\varnothing \neq H_{n_\Gamma, -l_\Gamma} \cap R_\Gamma$$

(if l_Γ does not exist, then $\mathrm{Supp}(h_\Gamma) = \varnothing$*).*

Proof. If

$$(0,0,0) \neq m \in \mathrm{Supp}(h_\Gamma)$$

then some multiple $r \cdot m$, $r \in \mathbb{N}_{\geqslant 1}$, lies in $\mathrm{Int}(\Gamma) \cap M$ by (8.7). Thus $m \in \mathrm{Cone}(\mathrm{Int}(\Gamma) \cap M)$. If there were $m' \in \mathrm{Int}(\Delta) \cap M$ with

$$0 > \langle m', n_\Gamma\rangle > \langle m, n_\Gamma\rangle$$

then

$$\begin{aligned}&(r-1)\cdot m+m' \in \mathrm{Int}(\Delta)\cap M\\ &\Rightarrow r\cdot m+m' = \Big(\underbrace{(r-1)\cdot m+m'}_{\in \mathrm{Int}(\Delta)\cap M}\Big)+\underbrace{m}_{\in \mathrm{Supp}(h_\Gamma)}\ \underbrace{\in}_{by (8.7)} R_\Gamma\cap M\end{aligned}$$

by (8.7), a contradiction. □

Lemma 8.3.7. *Assume that*

$$h_\Gamma \cdot g_\Gamma(f) \in L(\Delta)$$

for some facet Γ. *Then*

$$x^m \cdot g_\Gamma(f) \in L(\Delta) \quad \forall\, m \in \mathrm{Supp}(h_\Gamma) \tag{8.10}$$

Proof. This is simple: Assume to the contrary that $x^m \cdot g_\Gamma(f) \notin L(\Delta)$, say that this polynomial jumps out of a facet Γ_1. Then choose $n \in N$ suitable and find a vertex $v \in Vert(\Gamma_1)$ such that $m+v \notin \Delta \cap M$ and this vector remains left in $\mathrm{Supp}(h_\Gamma \cdot g_\Gamma(f))$. □

8.4 Examples and Counterexamples to the ITT

Remark 8.4.1. Assume

$$\sum_\Gamma h_\Gamma \cdot g_\Gamma(f) \in \ker(d\mathcal{P}_{B,f}).$$

Given $w \in \mathrm{Supp}(h_\Gamma)$ by Lemma (8.3.6)

$$\langle w, n_\Gamma\rangle = -l_\Gamma \leqslant 0, \quad \text{and } \langle w, n_\Gamma\rangle = 0 \Leftrightarrow w = (0,0,0).$$

In the latter case $h_\Gamma \cdot x^w = h_\Gamma \equiv 0 \in R^1_f$. Now given the conjecture 8.3.5 and given

$$x^w \cdot g_\Gamma(f) \in \ker(d\mathcal{P}_{B,f})$$

we get

$$\langle w, n_{\tilde\Gamma}\rangle \geqslant 0, \quad \forall\, \tilde\Gamma \neq \Gamma$$

for else $x^w \cdot g_\Gamma(f) \notin L(\Delta)$. Thus if $\langle w, n_\Gamma \rangle = -1$, then $x^w \cdot g_\Gamma(f) \in \ker(\kappa_f)$. The case

$$\langle w, n_\Gamma \rangle \leqslant -2, \quad \langle w, n_{\tilde\Gamma} \rangle \geqslant 0 \quad \forall\, \tilde\Gamma \neq \Gamma$$

remains, which seems to be very exceptional. In the following two examplex assume that Conjecture 8.3.5 is valid:

Example 8.4.2. By ([Fle86, Thm.3.1]) if

$$n \geqslant 2 \quad \text{and} \quad (d, n) \neq (3, 3)$$

then the ITT for smooth hypersurfaces $Y \subset \mathbb{P}^n$ of degree d is known to be true. Assume that $n = 3$ and note that in this case by [Voi03, Lemma 6.15] the Kodaira-Spencer map is surjective if $d \geqslant 5$.
Therefore we may prove the ITT for nondegenerate smooth surfaces in $\mathbb{P}^3$: We have $\Delta = d \cdot \Delta_3$. But obviously $l_\Gamma = 1$ for all facets $\Gamma \leqslant \Delta$ with the notation of Lemma 8.3.6 and we are done with the remark above.

Example 8.4.3. Given a 3-dimensional polytope Δ with

$$\operatorname{Int}(\Delta) \cap M = \{(0,0,0)\}$$

For Δ reflexive, that is $C(\Delta) = \Delta$ and $F(\Delta) = \{(0,0,0)\}$, we get a K3 surface and K3-surfaces fulfill the ITT (see [Huy16, Ch.6]). We may check $\ker(\Phi_{f|Im\,\kappa_f}) = \{0\}$ here: We have $l_\Gamma = 1$ for all facets Γ. Given $g_\Gamma(f) \cdot x^w \in \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f})$ if $\langle w, n_\Gamma \rangle < 0$ then $\langle w, n_\Gamma \rangle = -1$ and we are done.

Example 8.4.4. Switching to the (maximal) polytopes from section 3.4, in all examples there is exactly one facet Γ with distance 2 to the origin (all other facets have distance 1 to the origin, Remark 3.4.2). The facet Γ has 2 (or 3) interior lattice points, that we denote them by ac_1, ac_2 in a) and b) and by ac_1, ac_2, ac_3 in c), d) and e), and

$$p + ac_i \in \Delta.$$

p is the only vertex opposite to Γ and thus given $f \in B$

$$g_\Gamma(f) \cdot x^{ac_i} \in \ker(d\mathcal{P}_{\mathcal{M}(\Delta),f}), \quad g_\Gamma(f) \cdot x^{ac_i} \notin \ker(\kappa_f).$$

On the other hand side if all coefficients $(a_m)_{m \in M \cap \Delta}$ are nonzero in f, then

$$g_\Gamma(f) \cdot x^{ac_i} \notin L(\Delta) \quad \text{for }.$$

There are some lattice points on Δ lying between the facet Γ and the plane parallel to Γ through $(0,0,0)$, which prevent $g_\Gamma(f) \cdot x^v$ to have support on Δ.

Corollary 8.4.5. *For Δ one of the 5 maximal polytopes and a given f generic the ITT holds for $\mathcal{X} \to \mathcal{M}(\Delta)$ at Y_f. But if we set the coefficients to the following monomials (in all cases $a), b), c), d), e)$) to zero, then Y_f fails the ITT:*

$$ab, ad, bc, cd.$$

□

In $a)$ and $c)$ there is an additional facet Γ' with an $v \in Int(\Gamma') \cap M$ such that

$$g_\Gamma(f) \cdot x^v \in L(\Delta)$$

but in this case $v \in -R(N, \Sigma_\Delta)$, that is nothing changes (compare section 6.2). In fact it follows with results from ([Cat78]) that

$$a), b) : \dim \ker(\Phi_{f|Im\,\kappa_f}) = 2, \quad c), d), e) : \dim \ker(\Phi_{f|Im\,\kappa_f}) = 3.$$

Interestingly in $c), d)$ and $e)$ the surfaces Y_f failing the ITT are exactly those surfaces in the family $\mathcal{X} \to B$ which are Todorov surfaces.

Remark 8.4.6. By section 4.4 the minimal model Y gets a Kanev surface, in which case $h^{1,1}(Y) = 19$ or a surface with

$$p_g(Y) = 1, \quad q(Y) = 0, \quad K_Y^2 = 2, \quad \pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z},$$

in which case $h^{1,1}(Y) = 18$.

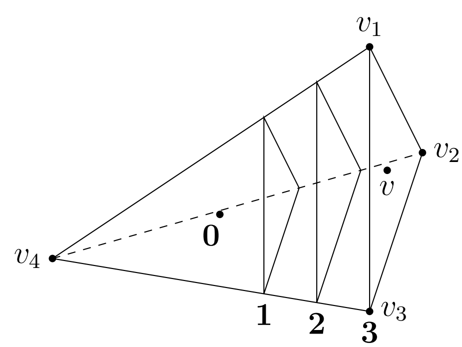
There are known results on Kanev and Todorov surfaces failing the ITT: There is a 14-dimensional family of Kanev surfaces, containing the 12-dimensional family of *special* Kanev surfaces, such that every member of this family fails the ITT (see [Cat78a]). Likewise a generic Todorov surfaces Y with $K_Y^2 = 2$ fails the ITT (see [SSU85, (1.4.2.1)]). We guess that our examples are not new (compare the above monomials with the monomials in Example 4.4.5) but of course we applied different methods and our computations are more explicit.

Concerning the other cases $C(\Delta) \neq \Delta$ or $\dim F(\Delta) \in \{1, 2\}$ we guess that there are other counterexamples of polytopes Δ with $\dim F(\Delta) = 1$ yielding proper elliptic surfaces in toric 3-folds (for example from the lists [Sch18, Appendix A.1,A.2]). For other „new“ counterexamples we are more skeptical. See the Example below for a naive approach.

Example 8.4.7. We sketch a possible simplex Δ with $l^*(\Delta) = 1$ ($(0,0,0)$ is the only interior lattice point) and one facet $\Gamma = \langle v_1, v_2, v_3 \rangle$ with distance 3 to $(0,0,0)$ and an interior lattice point $v \in \mathrm{Int}(\Gamma) \cap M$. In case $v_4 + v \in \Delta \cap M$ then we would get another counterexample to the ITT by choosing an f which has support on the vertices of Δ (In this case $g_\Gamma(f) \cdot x^v \equiv 0$ in $R^2_{Int,f}$ but v is not a root of Σ_Δ since the facet Γ has distance $3 > 1$ to $(0,0,0)$). But all lattice polytopes Δ with

$$\dim \Delta = \dim F(\Delta) = 3, \quad l^*(\Delta) = 1$$

have been classified in ([BKS19]) and there are just the 49 polytopes we have studied.

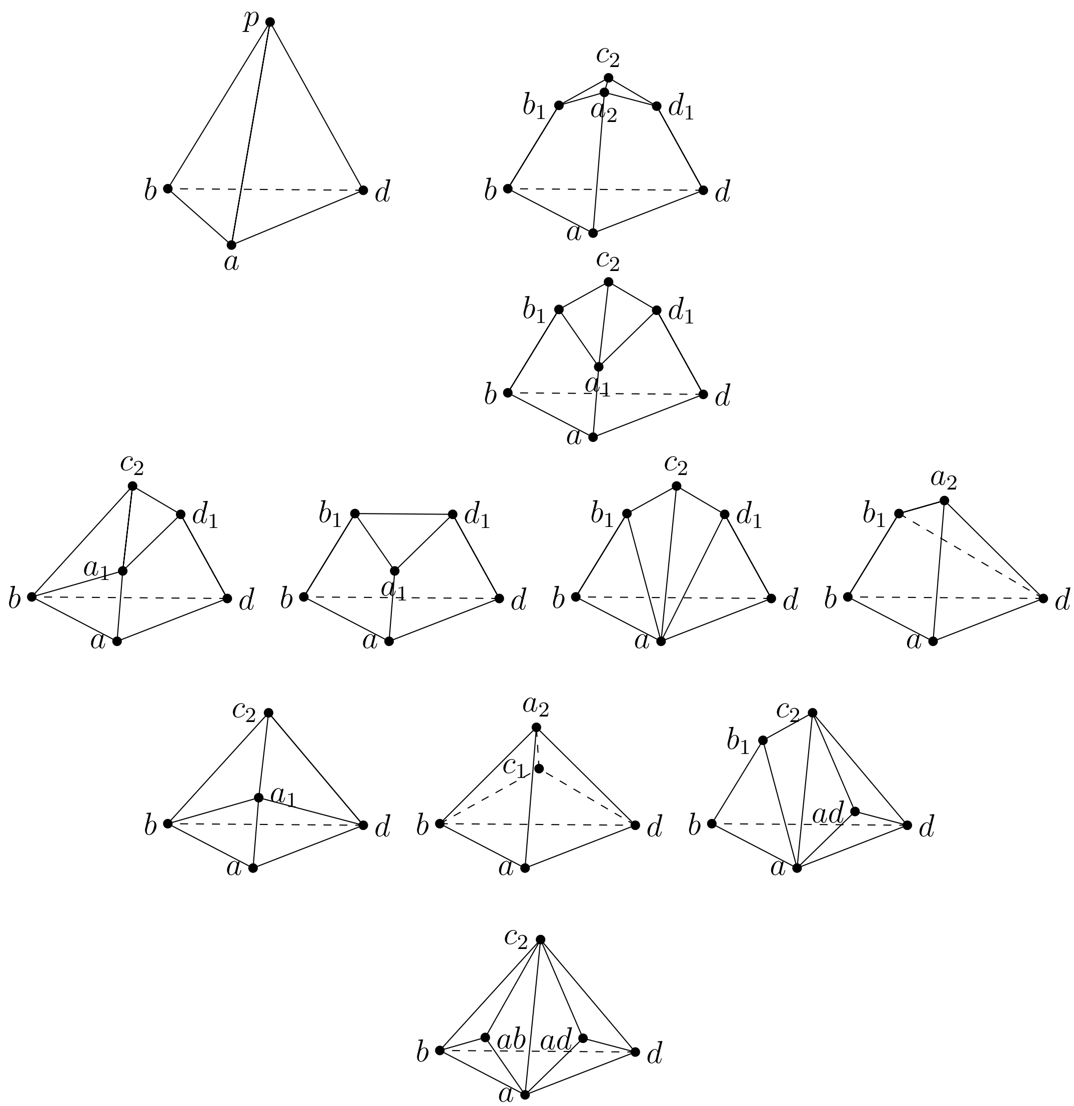


Such a lattice polytope does not exist.

CHAPTER 9

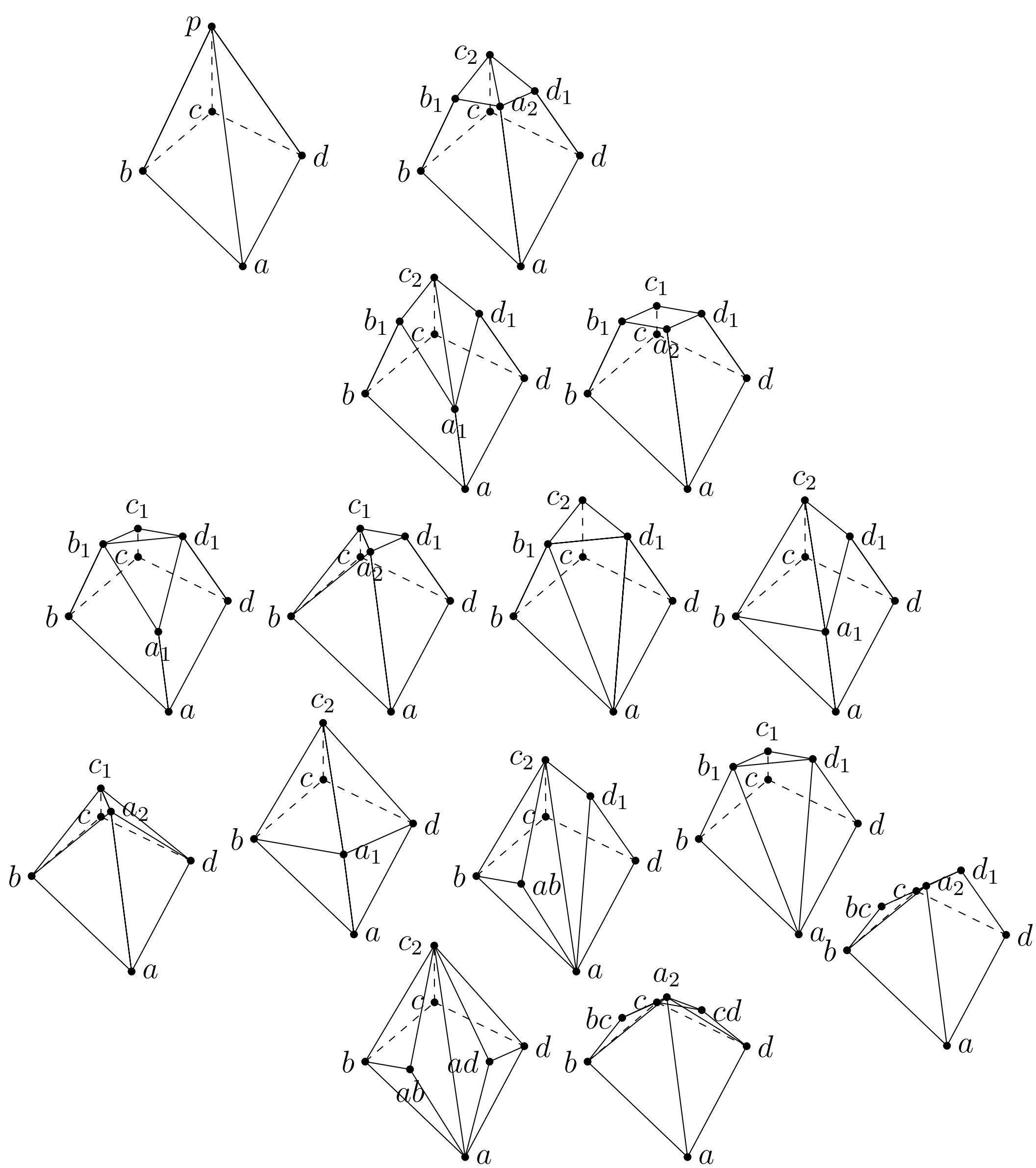
Appendix

9.1 The polytopes Δ with $C(\Delta) = \Delta$ in a)



The 11 canonically closed polytopes out of 20 polytopes in the first class a). The polytopes are ordered in rows descendingly by their number of lattice points (The maximal polytope is additionaly put in the first row on the left).

9.2 The polytopes Δ with $C(\Delta) = \Delta$ in b)



The 15 canonically closed polytopes out of 26 polytopes in the second class b) with the same convention on the rows as in case a) and also with the maximal polytope put in the first row on the left.

9.3 Data of all polytopes in $a)$ and $b)$

Polytopes in $a)$ such that $\Delta_{can} := \langle a, b, d\rangle$ sorted as in Figure 9.1 from the top to the bottom and from left to right. The arrows indicate that the polytopes are not canonically closed and the ID of the canonical closure is the polytope above the arrows (e.g. ID5389063 has canonical closure ID546219)

$F(\Delta) = \langle(0,0,0),(1,1/3,0),(1,2/3,0),(1,1/2,-1/2)\rangle$
$p := (-4,-2,1), a_2 := (-2,-1,0), c_2 := (-2,-1,1), b_1 := (-1,-1,1),$
$d_1 := (-1,0,1), a_1 := (0,0,-1), 0 := (0,0,0), c_1 := (0,0,1),$
$ab := (1,0,0), bc := (1,0,1), ad := (1,1,0), cd := (1,1,1),$
$b := (2,0,1), a := (2,1,-2), ac_1 := (2,1,-1), ac_2 := (2,1,0),$
$c := (2,1,1), d := (2,2,1)$

ID	spanning set for polytope Δ	number of points on Δ
547444	Δ_{can}, p	18
474457	$\Delta_{can}, a_2, c_2, d_1, b_1$	17
$\Rightarrow$ 545932	Δ_{can}, a_2, c_2	15
$\Rightarrow$ 532384	$\Delta_{can}, a_2, c_2, d_1$	16
$\Rightarrow$ 532606	$\Delta_{can}, a_2, d_1, b_1$	16
483109	$\Delta_{can}, d_1, b_1, c_2, a_1$	16
534669	$\Delta_{can}, c_2, d_1, a_1$	15
534866	$\Delta_{can}, b_1, a_1, d_1$	15
534667	$\Delta_{can}, c_2, d_1, b_1$	15
546062	Δ_{can}, b_1, a_2	15
546205	Δ_{can}, a_1, c_2	14
546219	Δ_{can}, c_1, a_2	14
$\Rightarrow$ 547524	Δ_{can}, a_2	11
$\Rightarrow$ 546863	Δ_{can}, a_2, bc	12
$\Rightarrow$ 539063	$\Delta_{can}, a_2, bc, cd$	13
536498	$\Delta_{can}, b_1, ad, c_2$	14
537834	$\Delta_{can}, ab, ad, c_2$	13
$\Rightarrow$ 547525	Δ_{can}, c_2	11
$\Rightarrow$ 546862	Δ_{can}, ab, c_2	12
$\Rightarrow$ 546663	Δ_{can}, ad, c_2	12

Polytopes in the class b) sorted as in Figure 9.2 from top to bottom, left to right. (with $\Delta_{can} := \langle a, b, c, d\rangle$ and the same convention as in Table 9.1)

$$F(\Delta) = \langle(0,0,0),(1,-1,1/2),(1,-2/3,1/3),(1,-1/2,1/2),(1,-2/3,2/3)\rangle$$
$$p := (-4,3,-2), c_2 := (-2,2,-1), a_2 := (-2,1,-1), b_1 := (-1,1,0)$$
$$d_1 := (-1,1,-1), 0 := (0,0,0), a_1 := (0,-1,0), c_1 := (0,1,0), cd := (1,0,0),$$
$$ad := (1,-1,0), ab := (1,-1,1), bc := (1,0,1), ac_2 := (2,-1,1),$$
$$ac_1 := (2,-2,1), d := (2,-1,0), c := (2,0,1), a := (2,-3,1), b := (2,-1,2)$$
.

ID	spanning set for polytope Δ	number of points on Δ
545317	Δ_{can}, p	18
354912	$\Delta_{can}, c_2, a_2, d_1, b_1$	17
$\Rightarrow$ 533513	Δ_{can}, c_2, a_2	15
$\Rightarrow$ 481575	$\Delta_{can}, c_2, a_2, d_1$	16
372528	$\Delta_{can}, d_1, b_1, c_2, a_1$	16
372973	$\Delta_{can}, b_1, d_1, a_2, c_1$	16
$\Rightarrow$ 490511	$\Delta_{can}, b_1, d_1, a_2$	15
388701	$\Delta_{can}, a_1, d_1, b_1, c_1$	15
$\Rightarrow$ 499287	$\Delta_{can}, a_1, d_1, b_1$	14
490485	$\Delta_{can}, c_1, a_2, d_1$	15
490481	$\Delta_{can}, c_2, b_1, d_1$	15
490478	$\Delta_{can}, d_1, c_2, a_1$	15
535952	Δ_{can}, a_2, c_1	14
536013	Δ_{can}, a_1, c_2	14
495687	$\Delta_{can}, d_1, c_2, ab$	14
$\Rightarrow$ 539313	Δ_{can}, d_1, c_2	13
499291	$\Delta_{can}, c_1, b_1, d_1$	14
$\Rightarrow$ 538356	Δ_{can}, b_1, d_1	13
499470	$\Delta_{can}, a_2, bc, d_1$	14
$\Rightarrow$ 539304	Δ_{can}, a_2, d_1	13
501298	$\Delta_{can}, c_2, ab, ad$	13
$\Rightarrow$ 547246	Δ_{can}, c_2	11
$\Rightarrow$ 540602	Δ_{can}, c_2, ab	12
501330	$\Delta_{can}, a_2, bc, cd$	13
$\Rightarrow$ 547240	Δ_{can}, a_2	11
$\Rightarrow$ 540663	Δ_{can}, a_2, bc	12

Bibliography

[BaCo94] V. V. Batyrev, D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Mathematical Journal, Vol.75, No.2, (1994).

[Bat93] V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Mathematical Journal, Volume 69, No.2, (1993).

[Bat94] V. V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, Journal of Algebraic Geometry **3** (1994), no. 3, 493–535.

[Bat03] V. V. Batyrev, *Cohomology, Periods and the Hodge structure of toric hypersurfaces*, (2003), https://www.math.arizona.edu/ swc/aws/2004/04BatyrevNotes.pdf.

[Bat17] V. V. Batyrev, *The stringy Euler number of Calabi-Yau hypersurface in toric varieties and the Mavlyutov duality*, Pure Appl. Math. Q. **13** (2017), no. 1, 1–47.

[Bat22] V. V. Batyrev, *Canonical models of toric hypersurfaces*, (2022), arXiv:2008.05814 [math.AG]

[BG99] W. Bruns, J. Gubeladze, *Polytopal Linear Groups*, Journal of Algebra 218, 715-737 (1999).

[BHPV04] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, (2004).

[BiZh15] C. Birkar, D. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, (2015), arXiv:1410.0938[math.AG].

[BKS19] V. Batyrev, A. Kasprzyk, K. Schaller, *On the Fine interior of three-dimensional canonical Fano polytopes*, (2019), arXiv:1911.12048 [math.AG].

[BR09] M. Beck, S. Robins, *Computing the Continuous Discretely. Integer-Point Enumeration in Polyhedra*, Springer (2009).

[BrGr10] U. Bruzzo, A. Grassi, *Picard group of hypersurfaces in toric 3-folds*, (2010), arXiv:1011.1003 [math.AG]

[BrGr17] U. Bruzzo, A. Grassi, *On the Hodge conjecture for hypersurfaces in toric varieties*, (2017), arXiv:1708.04944

[BrMo22] U. Bruzzo, W.D. Montoya, *Deformation of pairs and Noether-Lefschetz loci in toric varieties*, (2022), arXiv:2203.00664

[Cat78a] F. Catanese, *Surfaces with $K^2 = p_g = 1$ and their period mapping*, Algebraic Geometry Summer Meeting, Copenhagen, (1978), 1 – 29.

[Cat78] F. Catanese, *The moduli and the global period mapping of surfaces with $K^2 = p_g = 1$: a counterexample to the global Torelli problem* , Compositio Mathematica, Tome 41 (1980) no. 3, pp. 401-414.

[Cat11] F. Catanese, *A Superficial Working Guide to Deformations and Moduli*, (2011), arXiv:1106.1368.

[CD89] F. Catanese, O. Debarre, *Surfaces with $K^2 = 2, p_g = 1, q = 0$.*, Journal für die reine und angewandte Mathematik, Volume: 395, page 1-55, (1989).

[CLS11] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124, Amer. Math. Soc., Providence, RI, (2011).

[CMSP17] J. Carlson, S. Müller-Stach, C. Peters, *Period Mappings and period domains*, Cambridge studies in advanced Mathematics, (2017).

[Cox95] D. Cox, *The homogeneous coordinate ring of a toric variety*, Department of Mathematics and Computer Science, Amherst College, (1995).

[Del75] P. Deligne, *Théorie de Hodge III*, Publ. Math. IHES **44**, 5-77, (1975).

[Dem70] M. Demazure, *Sous-groupes alg ebriques de rang maximum du groupe de Cremona*, Ann. Sci. EcoleNorm. Sup. (4)3(1970). Effective Iitaka fibrations

[DK86] V. I. Danilov, A. G. Khovanskii, *Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers*, Izv. Akad Nauk SSSR Ser Mat **50** (1986), 925–945.

[Dre12] J. Drézet, *Luna's slice theorem and applications*, Jaroslaw A. Wisniewski. Algebraic group actions and quotients, Hindawi Publishing Corporation, pp.39-90, (2004).

[Fle86] H. Flenner, *The infinitesimal Torelli problem for zero sets of sections of vector bundles*, Math. Zeitschrift 193, 307-322, Springer Verlag (1986).

[Fine83] J. Fine, *Resolution and Completion of Algebraic Varieties*, Ph.D. University of Warwick 1983.

[Fuj06] O. Fujino, *Multiplication maps and vanishing theorems for toric varieties*, Math. Z. **257**, 631–641, (2007).

[Gie22a] J. Giesler, *The plurigenera of nondegenerate minimal toric hypersurfaces*, (2022), arXiv:2112.10523 [math.AG].

[Gie22b] J. Giesler, *The Kodaira-Spencer map for nondegenerate toric hypersurfaces*, (2022), arXiv:2203.01092 [math.AG].

[GKZ94] I. Gelfand, M. Kapranov, A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Springer-Birkhäuser, (1994).

[Hart77] R. Hartshorne, *Algebraic geometry*, Graduate texts in Mathematics, **52**, Springer-Verlag, Berlin (1977).

[HLY02] Y. Hu, C. Liu, S. Yau, *Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces*, Adv. Theor. Math. Phys. **6**, p.457-505, (2002).

[Huy16] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge studies in advanced mathematics, (2016).

[IlTu18] N. Ilten, C. Turo, *Deformations of Smooth Complete Toric Varieties: Obstructions and the Cup Product*, (2018), arXiv:1812.09254 [math.AG].

[JK20] , C. Jiang, Y. Kawamata, *Algebraic Varieties: Minimal Models and FiniteGeneration*, (2020), http://faculty.bicmr.pku.edu.cn/ tianzhiyu/BCHM/KawamataBook.pdf

[Kas09] A. Kasprzyk, *Bounds on Fake Weighted Projective Space*, (2009), arXiv:0805.1008 [math.AG].

[KoSp58] K. Kodaira, D. C. Spencer, *On Deformations of Complex Analytic Structures, I*, Annals of Mathematics, Second Series, Vol. 67, No. 2, pp. 328-401, (1958).

[Koe91] R. J. Koelman, *The number of moduli of families of curves on toric surfaces*, PhD Thesis, University of Nijmegen, (1991).

[Kov05] S. Kovács, *Strong non-isotriviality and rigidity*, Recent progress in arithmetic and algebraic geometry, Book chapter, (2005).

[KSZ] M. Kapranov, B. Sturmfels, V. Zelevinsky, *Quotients of toic varieties*, Math. Ann. **290**, 643-655, (1991).

[Laz00] R. Lazarsfeld, *Positivity in algebraic geometry 1. Classical setting: line bundles and linear series*, vol. 48 of Ergebnisse der Matihematik und ihrer Grenzgebiete **3**, (2000).

[Mav03] A. Mavlyutov, *Embedding of Calabi-Yau deformations into toric varieties*, Mathematische Annalen volume 333, pages 45–65 (2005).

[Mav08] A. Mavlyutov, *Cohomology of rational forms and a vanishing theorem on toric varieties*, Journal für die reine und angewandte Mathematik, (2008).

[Mor87] D. R. Morrison, *On the moduli of Todorov surfaces*, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, pp. 313-355 (1987).

[MuFo82] D. Mumford, *Geometric invariant theory*, Springer Verlag, second Enlarged Edition, (1982).

[MuOd15] D. Mumford, T. Oda, *Algebraic Geometry II*, Texts and Readings in Mathematics, (2015).

[Oda88] T. Oda, *Conves Bodies and Algebraic Geometry*, Springer-Verlag, Berlin Heidelberg New York, 1988. Effective Iitaka fibrations

[Ok87] M. Oka, *On the deformation of a certain type of algebraic varieties*, Preprint, (1987).

[Rei79] M. Reid, *Canonical* 3*-folds*, Algebraic Geometry Angers (1979).

[Rei83] M. Reid, *Decomposition of toric morphisms*, in Arithmetic and geometry, Progr. Math. **36**, Birkhäuser Boston, Boston, MA (1983), 395–418.

[Rei87] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., **46**, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.

[Sch71] M. Schlessinger, *Rigidity of Quotient Singularities*, Inventiones mathematicae, Volume: 14, page 17-26, (1971).

[Sch08] K. Schwede, *Generalized divisors and reflexive sheaves on normal varieties*, (2008), https://www.math.utah.edu/ schwede/Notes/GeneralizedDivisors.pdf

[Sch18] K. Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart-Karls-Universität Tübingen, (2018).

[ShIn10] T. Shioda, H. Inose, *On Singular K3 Surfaces*, Complex Analysis and Algebraic Geometry (2010).

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, A series of comprehensive studies in Mathematics, Springer, (2006).

[SSU85] M-H. Saito, Y. Shimizu, S. Usui, *Variation of Mixed Hodge Structure and the Torelli Problem*, Advanced Studies in Pure Mathematics 10, (1987).

[Tod80] A. Todorov, *Surfaces of general type with* $p_g = 1$ *and* $(K.K) = 1.I$, Annales scientifiques de l'École Normale Supérieure, Serie 4, Volume 13 no. 1, (1980), 1-21.

[Tre10] J. Treutlein, *Birationale Eigenschaften generischer Hyperflächen in algebraischen Tori*, Dissertation (2010).

[Tsu02] H. Tsuji, *Deformation Invariance of Plurigenera*, (2001), arXiv:math/0012225 [math.AG].

[ViZh08] E. Viehweg, D. Zhang, *Effective Iitaka fibrations*, (2008), arXiv:0707.4287 [math.AG].

[Voi02] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Cambridge University Press, (2002).

[Voi03] C. Voisin, *Hodge Theory and Complex Algebraic Geometry II*, Cambridge University Press, (2003).