Algebraic Methods in Reconstruction of Varieties and Game Theory

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Javier Sendra Arranz
Madrid, Spanien

Tübingen
2024

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	18.07.2024
Dekan:	Prof. Dr. Thilo Stehle
1. Berichterstatter/-in:	Jun. Prof. Dr. Daniele Agostini
2. Berichterstatter/-in:	Prof. Dr. Hannah Markwig
3. Berichterstatter/-in:	Prof. Dr. Josef Schicho

To my father.

Abstract

In this thesis we study the synergies among theoretical, computational and applied algebraic geometry in three different settings, that correspond to each of the chapters: plane Hurwitz numbers, Hessian correspondence and algebraic game theory.

First, we study plane Hurwitz numbers from a theoretical and computational point of view. We explore how to reconstruct $\mathfrak{h}_d$ plane curves from the branch locus of the projection from a fix point, where $\mathfrak{h}_d$ is the plane Hurwitz number of degree d. We approach this recovery problem for the case of plane cubics and plane quartics. From a theoretical point of view, we compute the real plane Hurwitz numbers $\mathfrak{h}_3^{\text{real}}$ and $\mathfrak{h}_4^{\text{real}}$. Moreover, we introduce and study the notion of (real) Segre-Hurwitz numbers $\mathfrak{sh}_{d_1,d_2}$ and $\mathfrak{sh}_{d_1,d_2}^{\text{real}}$.

Secondly, we investigate the Hessian correspondence of hypersurfaces $H_{d,n}$. This is the map that associate to a degree d hypersurface in $\mathbb{P}^n$ its Hessian variety or second polar variety. We analyse the fibers of $H_{d,n}$ and its birationality for hypersurfaces of Waring rank at most $n+1$ and for hypersurfaces of degree 3 and 4. From a computational perspective, we explore how to recover a hypersurface from its Hessian variety. We also investigate the geometry of the catalecticant enveloping variety.

Thirdly, we explore the application of algebro-geometric tools to the study of the conditional independence (CI) equilibria of a collection of CI statements $\mathcal{C}$. We focus on the case where $\mathcal{C}$ is the global Markov property of an undirected graph. We investigate this notion of equilibrium through the algebro-geometric examination of the Spohn CI variety. We restrict our study to binary games and we analyse the dimension of these varieties. In the case where the graph is a disjoint union of complete graphs, we explore further algebro-geometric features of Spohn CI varieties.

Authorship

This thesis is based on the papers [1, 103, 104, 112]. The papers [1, 103, 104] are joint works with other researchers, whereas in [112] I am the only author. The article [1] is published in the journal *Discrete & Computational Geometry*. The article [104] is published in the journal *Journal of Symbolic Computation*. The papers [103, 112] have been submitted to journals for publication, and they can be found in *arXiv*.

Each chapter of this thesis starts with a preliminary section. These are Section 1.1, Section 2.1 and Section 3.1. They have been developed by myself and they provide the required background for each chapter.

Chapter 1 is based mostly on the article [1] coauthored with Daniele Agostini, Hannah Markwig, Clemens Nollau, Victoria Schleis and Bernd Sturmfels. All the authors contributed equally to the publication. I made major contributions to the development of the results appearing in Sections 1.2 and 1.5. The construction of examples in Section 1.5.4 is extra material to [1] that does not appear in the paper. The contribution to this section was equally distributed among Clemens Nollau and myself. Section 1.6 is based on an ongoing project with Clemens Nollau. Both authors contributed equally to the development of the results in Section 1.6.

Chapter 2 is mostly based on the paper [112]. This chapter is written by myself and I am the only author of this paper. Section 2.6 is based on an ongoing project with Leonie Kayser. I developed the results appearing in this section.

Chapter 3 is based on the papers [103, 104] with Irem Portakal. All authors have contributed equally to the publication. I made major contributions to the theoretical developments while the coauthor's contribution focused more in the game theory aspects.

Acknowledgements

First and foremost, I want to thank my advisor, Daniele Agostini. Thanks a lot for your knowledge, support, patient, and guidance. You have inspired me as a mathematician and as a person. You are my favourite and only advisor, and I am grateful to be your student.

Next, I would like to thank all my coauthors: Ángel David Ríos Ortiz, Bernd Sturmfels, Hannah Markwig, Clemens Nollau, Victoria Schleis, Daniele Agostini, Irem Portakal and Leonie Kayser. Thanks for all the fruitful mathematical discussions and all the knowledge I have learned from you. I also want to thank Fulvio Gesmundo and Pierpaola Santarsiero, for patiently explaining tensors to me. I am thankful to "la Caixa" Foundation and to the MPI MiS for the support during the first two years of my PhD.

I could not continue without expressing my gratitude to all the friends that I have shared these last three years with. You have made my life in Leipzig and in Tübingen much happier. Anna, it is incredibly difficult to fit in one sentence how grateful I am to you. Thank you for your love and support. Fabio, mio fratello napolitano, thank you for being there in the good and bad moments. Chiara, thank you for all the laughs together, even if sometimes they came from my Espanish accent. Claudia, you are my older hermanita matemática, and I thank you for all the help and support you have given me as so. Marie, thank you for teaching me to take a break and have a relaxing glass of wine. Claudia, Chiara and Marie, your strength has inspired me during these years. Simon, thank you for all the fun moments we have shared together, whether it's skiing, dancing or sharing dog memes. Giulio, here we go, thank you for beating me all the time at kickers. Stefano and Ángel, thank you for all the nights and all the blokus games. I also want to thank Giacomo, Víctor, Mirke, Pierpaola, Fulvio, Mima, Felix, Sachi, Clemens, Victoria, and Yelena, for being there in these last years.

I could not be more grateful to all the friends from home that were supporting me from the distance. Thank you for always welcomed me in Madrid as if I have always been there. Your warmth has kept me going through this adventure in Germany. In particular, I want to thank Celia, María, Alex, Séfora, Guille, Lidia, Pol, Álvaro, Antonio, and Sara.

Last, but not least, I want to thank my family. Mamá, Papá, Rafa, Bea and Heiko, without your love and support, I could not have reached this far. Thank you for being always there.

Contents

Introduction

In the past, the field of algebraic geometry was often perceived as abstract and theoretical branch of mathematics. However, Buchberger's work [19] on Gröbner bases in the 1960s ignited a breakthrough in the field. His algorithmic treatment of polynomial ideals gave rise to computational algebraic geometry, introducing a new perspective emphasized on computational methods for manipulating and solving polynomial systems of equations. This innovative point of view motivated and inspired many algebraic geometers to face the challenge of applying these techniques to other branches of mathematics as statistics, optimization, convex geometry; and even to other natural sciences as physics, biology, engineering, etc. In the last decades, the interplay among theoretical, computational and applied algebraic geometry have resulted into remarkable advances, which have been beneficial to the three perspectives.

The framework in which this thesis is located in the interconnection among the theoretical, applied and computational aspects of algebraic geometry, commented above. More precisely, in this thesis we explore these synergies, in the sense described above, on three distinct scenarios: plane Hurwitz numbers, Hessian varieties of hypersurfaces and algebraic game theory. From a computational point of view, the first two settings explore the reconstruction of algebraic varieties from distinct related varieties. In the frame of plane Hurwitz numbers, we are interested in recovering plane curves from the branch locus of a linear projection. In the case of Hessian varieties, we aim to reconstruct a hypersurface from its Hessian variety. From an applied point of view, we explore the use of algebro-geometric tools in the field of game theory.

This thesis is based on the papers [1, 103, 104, 112] and it is structured in three chapters each of them devoted to one of these scenarios. In the following, we briefly introduce the topic of each chapter and we comment its structure and main results.

Chapter 1

Chapter 1 is mostly based on the paper [1] with Daniele Agostini, Hannah Markwig, Clemens Nollau, Victoria Schleis and Bernd Sturmfels. This chapter is focused on the recovery of plane curves from their branch locus and plane Hurwitz numbers. Hurwitz theory is an important branch of enumerative geometry focused on the study of Hurwitz numbers. These numbers were introduced by Hurwitz in [72] and they count the branched coverings among curves of fixed genus and satisfying certain fixed ramification. Historically, Hurwitz theory has been approached from a theoretical point of view aiming to find the values of these numbers. In Chapter 1 we provide a new computational perspective to this enumerative problem: we seek to numerically and effectively reconstruct branched coverings from their branch loci. The main obstacle in this recovery problem emerges from the fact that the source curves of different branched coverings, with same branched locus, might lie in distinct ambient spaces.

We restrict our study to plane curves and plane Hurwitz numbers. The plane Hurwitz numbers $\mathfrak{h}_d$ were introduced by Ongaro and Shapiro in [99, 101] and they count the number of degree d simple branched covering of $\mathbb{P}^1$ that are linear projection of a degree d smooth plane curve. Our computational objective is to recover $\mathfrak{h}_d$ degree d plane curves from the branch locus of the projection from a fixed point. The only known nontrivial values of $\mathfrak{h}_d$ are $\mathfrak{h}_3 = 40$ and $\mathfrak{h}_4 = 120$, which were determined by Clebsch [30, 31] and Vakil [122] respectively. In Chapter 1, we analyse this recovery problem for plane cubics and quartics from a numerical point of view. Moreover, this computational approach inspired the theoretical analysis of real plane Hurwitz numbers $\mathfrak{h}_d^{\mathrm{real}}$, which we carry out for plane cubics and quartics.

The notion of plane Hurwitz numbers leads to the consideration of analogous enumerative problems on other surfaces. In Chapter 1 we introduce and study the concept of (real) Segre-Hurwitz numbers $\mathfrak{sh}_{d_1,d_2}$ which count the number of simple branched covering of $\mathbb{P}^1$ that are obtained by projecting a bidegree $(d_1 d_2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$ to the second factor. The investigation of Segre-Hurwitz numbers is the topic of an ongoing project with Clemens Nollau and it does not form not part of the contents of [1].

Chapter 1 is structured as follows. In Section 1.1 we briefly introduce the required background for the development of the chapter. Section 1.2 is devoted to the numeric approach of our recovery problem. In Section 1.3 we explore the combinatorics of our problem. The case of cubic plane curves is carried out in Section 1.4, whereas Section 1.5 is focused on the analysis of plane quartics. In Section 1.6 we introduce and analyse the notion of (real) Segre-Hurwitz numbers. The chapter concludes with a list of open questions in Section 1.7.

The main contribution of Chapter 1 are:

- The development of numerical and effective algorithms for reconstructing 40 plane cubics and 120 plane quartics from their branch locus respectively.
- The computation of the real plane Hurwitz numbers for plane cubics and quartics.
- The analysis of the relation between the real plane Hurwitz number and the number of real roots in the branch locus.
- The construction of the (real) Segre-Hurwitz numbers $\mathfrak{sh}_{d_1,d_2}$ and the computation of $\mathfrak{sh}_{3,d_2}$, $\mathfrak{sh}_{4,2}$ and $\mathfrak{sh}_{4,2}^{\mathrm{real}}$.

Chapter 2

Chapter 2 is based on the paper [112] and it focuses on the theoretical and computational study of the Hessian correspondence. This problem is motivated by a classical object in algebraic geometry: dual varieties. The Gaussian map or first polar map of a hypersurface X in $\mathbb{P}^n$ is the rational map from X to $(\mathbb{P}^n)^*$ defined by the first order derivatives of the defining polynomial of X. The dual variety X^* of a hypersurface X is the subvariety of $(\mathbb{P}^n)^*$ defined as the Zariski closure of the image of the Gaussian map. One of the most celebrated results concerning these varieties is the Biduality theorem (see [40, Theorem 1.2.2]), which states that $(X^*)^* = X$. As a consequence, we can recover effectively a hypersurface from its dual variety, and the map associating to a hypersurface its dual variety is birational onto its image. In Chapter 2 we investigate the analogous construction for second order derivatives. We carry out this study for hypersurfaces over an algebraically closed field of characteristic zero.

Let X be a degree d hypersurface in $\mathbb{P}^n$ defined by a polynomial F. The Hessian map or second polar map of X is the rational map that sends a point $p \in X$ to the evaluation of the Hessian matrix of F at p. Note that the target space of the Hessian map is the projective space of symmetric matrices of size $n+1$. The Hessian variety or second polar variety of X is the closure of the image of the Hessian map. In [27] it was shown that the Hessian map of a smooth hypersurface is finite onto its image. Moreover, this paper shows the strong relation between Hessian varieties and the classical Hesse problem, which study the locus of polynomials whose Hessian variety has zero determinant. We refer to [29] for further details on the Hesse problem. In [87], more features of Hessian varieties as their degree, are studied.

In Chapter 2 we do a further exploration of Hessian varieties from a theoretical and computational point of view. The Hessian correspondence $H_{d,n}$ is the map that associates to a degree d hypersurface in $\mathbb{P}^n$ its Hessian variety. From a theoretical point of view, we are interested in the understanding of the fibers of $H_{d,n}$. In the case of first order derivatives and dual varieties, the analogous map associating to a hypersurface its dual variety is birational onto its image. From this, the next question is posted: For which values of d and n is the Hessian correspondence $H_{d,n}$ also birational onto its image? We do not only want to analyse this question from a theoretical perspective, but we also add a computational perspective to the problem. In case where $H_{d,n}$ is birational onto its image, can we effectively recover a hypersurface from its Hessian variety? Recall that again, this question has a positive answer for dual varieties by the Biduality theorem.

Chapter 2 answers these questions for hypersurfaces with Waring rank at most $n + 1$ and hypersurfaces of degree 3 and 4. For the case of cubic hypersurfaces, we derive the birationality of $H_{3,n}$ for $n \geq 2$ from the birationality of the gradient map, which is a classical result by Bertini [6, 7]. For a modern and more general version we refer to [46, Theorem 3.2]. In Section 2.4 we provide an alternative approach to the birationality of the gradient map that we later use to recover a cubic hypersurface from its Hessian variety. In the case of quartic hypersurfaces, we use the syzygies of Veronese varieties to derive the birationality of $H_{4,n}$.

In the investigation of the cubic case, the catalecticant enveloping variety Φ_k^e emerges.

This variety is defined as the locus of $(k+1)$–dimensional subspaces of the space of degree $d-e$ forms in $n+1$ variables containing all the e–th partial derivatives of a degree d form. The algebro-geometric study of this variety is of particular interest in the research community dealing with tensors, due to its relation with the notion of e–gradient rank. This concept of rank is used in [56] to approach Comon's conjecture (see [98, Problem 15]), a relevant standing conjecture in tensor decomposition. In Section 2.6 we analyse the geometry of the catalecticant enveloping variety Φ_k^1 for $d \geq 3$. The study of Φ_k^1 for $d=3$ appears in [112], whereas the case $d \geq 4$ is the topic of an ongoing project with Leonie Kayser.

The structure of Chapter 2 is as follows. In Section 2.1 we introduce the preliminaries of the chapter. The definitions of Hessian varieties and Hessian correspondence are given in Section 2.2. Section 2.3 focuses on the Hessian correspondence of hypersurfaces with rank at most $n+1$. The Hessian correspondence for cubic hypersurfaces is analysed in Section 2.4, whereas in Section 2.5 we study the case of quartic hypersurfaces. Section 2.6 is devoted to the study of the catalecticant enveloping variety. Moreover, in this section we introduce the k–th polar correspondence $\mathcal{G}_{d,n}^k$ as the generalization of the Hessian correspondence to k–th order derivatives, and we analyse the case $k=d-1$. Finally, in Section 2.7 we present a list of open questions on this chapter.

The main contribution of this Chapter are:

- We prove that the Hessian correspondence for degree d hypersurfaces, with Waring rank at most $n+1$, is birational onto its image for d even and generically finite of degree 2^{k+1} for d odd. We provide a numerical algorithm for reconstructing a hypersurface with Waring rank at most $n+1$ from its Hessian variety.
- We show that $H_{d,n}$ is birational onto its image for $d=3,4$ and $(d,n) \neq (3,1)$. For $(d,n)=(3,1)$, we show that $H_{3,1}$ has degree 2.
- We provide an effective method for recovering a generic hypersurface from its Hessian variety for $d=3$ and $n \geq 2$ and for $d=4$ and n even or $n=1$.
- We compute the irreducible components of catalecticant enveloping variety Φ_k^1.
- We show that the $(d-1)$–th polar correspondence $\mathcal{G}_{d,n}^{d-1}$ is birational onto its image

Chapter 3

The last chapter of this thesis is in the frame of algebraic game theory and it is based on the papers [103, 104] with Irem Portakal. This emerging field focuses on approaching game theoretic problems from an algebraic point of view. This synergy between algebraic geometry and game theory is similar to that of the first area and statistics. In statistics, certain implicit statistical models can be described by polynomial equations. The field of algebraic statistics studies these models through the analysis of the algebraic varieties defined by the associated polynomials (see [42, 120]). This interplay has been beneficial to both areas, with remarkable developments in the last decades.

Similarly, in game theory, distinct notions of equilibria can be described using polynomial equations. Following the above strategy, algebraic game theory investigates these notions of equilibria through the algebro-geometric study of the variety defined by the corresponding polynomials. An example of such an interaction is that of the classical Nash equilibria which can be determined by studying a system of multilinear equations. For instance, one can use the Bernstein-Khovanskii-Kushnirenko (BKK) theorem to find upper bounds for the number of totally mixed Nash equilibria (see [92, 119]).

In [116, 117, 118], Spohn introduced and studied (from a game theoretic perspective) the notion of dependency equilibrium. This equilibrium arises from the distinction between collective and individual behavior of the players of a game. The situation where the players behave independently is modeled by the Nash CI equilibria, whereas the collective behavior corresponds to the dependency equilibria. As in the Nash case, the dependency equilibria can also be described by polynomial constrains. The variety defined by these equations is called the Spohn variety, and it was introduced in [105] where its algebro-geometric examination was carried out.

The Nash and dependency equilibria fall into opposite extremes in the spectrum of dependencies among the players. The Nash equilibrium corresponds to the case where all the players behave independently, whereas the situation where the players are dependent, and they behave collectively, is represented by the dependency equilibrium. However, there are many possible scenarios in between these two extremal cases. This gap between the Nash and dependency equilibria is filled by the concept of conditional independence (CI) equilibrium.

CI equilibrium was introduced in [105, Section 6] as the intersection of the dependency equilibrium and a statistical model given by a collection of conditional independence (CI) statements. This definition can be again translated into the algebraic setting leading to the notion of Spohn CI variety. In Chapter 3 we emphasise the case of undirected graphical models, where the dependencies among players are modeled through the edges of a graph whose vertices represent the players. We focus on the case of binary games where we study the CI equilibria associated to such graphs through the algebro-geometric analysis of the Spohn CI variety. For instance, we approach [105, Conjecture 24] which asks the open question of computing the dimension of these Spohn CI varieties.

We pay special attention to the case where the graph is the disjoint union of complete graphs. The graph with no edges and the complete graph are the two extremal examples of such graphs, and their corresponding CI equilibria coincide with the Nash and dependency equilibrium respectively. In general, this type of graphs models the scenario where the players are divided into independent groups, one for each connected component of the graph, and each group behaves collectively. In this case, the corresponding Spohn CI variety is called a Nash CI variety. The case of one edge graph is the next natural step to the Nash equilibria. In this case, the Nash CI variety is a curve for generic games, called Nash CI curve. In Chapter 3, we do a further examination of the properties of Nash CI varieties.

Chapter 3 is structured as follows. In Section 3.1 we give a brief introduction to algebraic statistics and algebraic game theory. Section 3.2 focuses on the algebro-geometric study of Nash CI curves. In Section 3.3 we analyse the dimension of Spohn CI varieties arising from undirected graphical models. The investigation of Nash CI varieties is carried out in Section 3.4. In Section 3.5 we investigate the notion of affine universality for certain families of Nash CI varieties. Finally, the open questions and new research lines are listed in Section 3.6.

The main contribution of Chapter 3 are:

- We show that generic Spohn CI varieties of one edge undirected graphical models are one dimensional. We compute the degree and the genus of Nash CI curves and we prove that generic Nash CI curves are smooth, irreducible and connected.
- We solve [105, Conjecture 24] by computing the dimension of Spohn CI varieties of undirected graphical models.
- We compute the equations of Nash CI varieties and their degrees. We show that Nash CI varieties are connected, and that smooth Nash CI surface are of general type.
- We prove that the set of totally mixed CI equilibria of a generic Nash CI variety is a real smooth manifold, which is either empty or it has the same dimension as the Nash CI variety.
- We analyse the affine universality, in the sense of [34], for Nash CI varieties where connected components of the associated graph have at most one edge. We show that the families of these varieties satisfy the Murphy's law.

Chapter 1

Recovery of plane curves from branch points

Hurwitz numbers count the branched coverings, up to isomorphism, between curves of fixed genus and satisfying certain fixed ramification. The theory of Hurwitz numbers was initiated by Hurwitz in [72] and, since then, it has become a fundamental branch of enumerative geometry. Classically, this theory has been studied from an abstract point of view with the aim of finding the value of Hurwitz numbers. In this chapter we provide a computational perspective to this problem: we aim to numerically and effectively reconstruct branched coverings from their branch locus.

The difficulty of this recovery problem arises from the fact that the source curves of different branched coverings with same branch locus might not lie in the same ambient space. To deal with this difficulty, we mainly restrict our study to plane curves and to plane Hurwitz numbers. The plane Hurwitz number $\mathfrak{h}_d$ counts the number of degree d simple branched coverings of $\mathbb{P}^1$ that are linear projections of a degree d smooth plane curve. Plane Hurwitz numbers were introduced and studied in Ongaro's 2014 PhD thesis and in his work with Shapiro [99, 101]. Our computational goal is to recover $\mathfrak{h}_d$ degree d plane curves from the branch locus of the projection from a fixed point. In [52, 70], the similar problem for surfaces is considered. In these papers, the authors study how to recover a surface in $\mathbb{P}^3$ from the branch locus of a linear projection to $\mathbb{P}^2$. Note that here the branch locus is a plane curve.

Presently, the only known nontrivial values of $\mathfrak{h}_d$ are $\mathfrak{h}_3 = 40$ and $\mathfrak{h}_4 = 120$. The former value was computed by Clebsch [30, 31] in a different setting. A modern view of this computation can be found in [99, Proposition 5.5.2]. The value $\mathfrak{h}_4 = 120$ was computed by Vakil in [122]. In this chapter we analyse our recovery problem for plane cubics and quartics from a numerical and symbolic point of view.

We consider also the situation over real numbers. The real Hurwitz numbers count branched coverings between real algebraic curves whose real involutions are compatible via the branched covering. For further literature on real Hurwitz numbers we refer to

[21, 62, 75]. Similarly, the real plane Hurwitz number $\mathfrak{h}_d^{\text{real}}$ counts the number of simple branched coverings coming from projecting a degree d real plane curve, with the same branch locus. The second aim of this chapter is to compute the real plane Hurwitz numbers for plane cubics and quartics.

From the investigation of plane Hurwitz numbers from a theoretical, computational and real point of view, the following question arises: what happens if we consider curves in other surfaces? At the end of this chapter we also consider curves in the Segre variety $S = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_2 : S \to \mathbb{P}^1$ be the projection to the second factor of S. Given a generic curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d_1, d_2), the projection $\pi_2 : C \to \mathbb{P}^1$ is a degree d_1 simple branch covering. We will define the Segre-Hurwitz number $\mathfrak{sh}_{d_1,d_2}$ as the number of coverings, up to isomorphism, of the form $\pi_2 : C \to \mathbb{P}^1$ with the same branch locus, where C has bidegree (d_1, d_2). For instance, the Segre- Hurwitz number $\mathfrak{sh}_4$ equals 135. This computation was done in [122] in slightly different setting. The last task of this chapter is to introduce this notion and investigate some of these numbers from a complex and real point of view.

This chapter is structured as follows. In Section 1.1 we introduce the background required for the rest of the chapter. A brief introduction to the theory of Hurwitz numbers is given in Section 1.1.1. Section 1.1.2 is devoted to plane Hurwitz numbers. In Sections 1.1.3 and 1.1.4 we present some real algebro-geometric tools needed for the study of real plane Hurwitz numbers. In Section 1.1.3 we recall some results on the real structure of space sextics. Section 1.1.4 serves as an introduction to del Pezzo surfaces of degree 1 and their real geometry.

In Section 1.2 we study our recovery problem from a numerical point of view. We do this by finding normal forms for the action of the group defining the notion of isomorphism of plane branched coverings. In Section 1.3, we analyze the combinatorial aspects of our problem. We use combinatoric tools to compute the real plane Hurwitz number for cubics and the real Hurwitz number for plane quadrics. In Section 1.4 we relate the information on Table 1.3 with the Galois group of the branching morphism, and study the recovery problem for cubics from a symbolic perspective following Clebsch ideas in [30, 31].

Section 1.5 is focused on the plane Hurwitz number for plane quartics. In Section 1.5.1 we summarize Vakil's ideas used to determine $\mathfrak{h}_4$ in [122]. Section 1.5.2 is devoted to the computation of the possible values of $\mathfrak{h}_4^{\text{real}}$. In Section 1.5.3, we provide an effective algorithm for determining the 120 plane quartics from their branch locus. The analysis of the relation between the real plane Hurwitz numbers $\mathfrak{h}_4^{\text{real}}$ and the number of real roots t in the branch locus is carried out in Section 1.5.4. We give a complete description of which values of the pair $(\mathfrak{h}_4^{\text{real}}, t)$ can not happen, and we construct examples for each of the possible values.

The development of an analogous theory to plane Hurwitz numbers for curves in the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1$ is carried out in Section 1.6. In this Section we introduce the notion of Segre-Hurwitz numbers $\mathfrak{sh}_{d_1,d_2}$ and their real version $\mathfrak{sh}^{\text{real}}_{d_1,d_2}$. We investigate the relation of these numbers with the classical Hurwitz numbers. In Section 1.6.1 we investigate the case of curves of bidegree $(4, 2)$.

Finally, in Section 1.7 we list some of the open problems and new research lines concerning this chapter.

The main contributions of this chapter are:

- The development of numerical algorithms for reconstructing the 40 and 120 plane cubics and plane quartics from their branch locus.
- The analysis of the combinatorics associated to the 40 cubics, as well as an algorithm for computing 40 cubics from their branch locus together with their combinatoric information exposed in Table 1.3.
- The computation of $\mathfrak{h}^{\text{real}}_3$ and H^{real}_4.
- The presentation of an algorithm for computing by radicals the 40 cubics from one of them.
- The computation of $\mathfrak{h}^{\text{real}}_4$ and the complete analysis of the relation between $\mathfrak{h}^{\text{real}}_4$ and the real roots in the base locus.
- The development of an effective algorithm for computing the 120 plane quartics from their branch locus.
- The construction of Segre-Hurwitz numbers and the computation of $\mathfrak{sh}_{3,d_2}$ for $d_2 \geq 2$, $\mathfrak{sh}_{4,2}$ and $\mathfrak{sh}^{\text{real}}_{4,2}$.

1.1 Preliminaries on plane Hurwitz numbers

One of the main goals of this chapter is to provide a computational approach to a classical topic in enumerative geometry, Hurwitz numbers. We aim to find effective methods for reconstructing branched coverings from their base loci. To do so, in Sections 1.1.1 and 1.1.2 we present the required background on Hurwitz numbers and plane Hurwitz numbers respectively. A second goal of this chapter is to analyse the real algebraic geometry behind plane Hurwitz numbers. For this purpose, we devote part of this section to introduce the tools from real algebraic geometry that we will use in the chapter. In Section 1.1.3 we revise the notion of space sextics and their real geometry, and in Section 1.1.4 we focus on del Pezzo surfaces of degree 1 and their real structure.

1.1.1 Hurwitz numbers

In this section we give a brief introduction to the theory of Hurwitz numbers. The study of these numbers goes back to Hurwitz in [72]. For further literature in this topic we refer to [22, 23, 62, 75, 99].

During this section, an algebraic curve will denote a smooth connected projective complex curve. A branched covering between two curves C_1 and C_2 is a nonconstant map

$$\pi : C_1 \to C_2.$$

Such a map is surjective and finite. We denote its degree by d. In particular, for generic $p \in C_2$, the fiber of p through f consists of d distinct points. A point $p \in C_2$ is a branch point if the fiber $\pi^{-1}(p)$ has a point q with multiplicity at least 2. In this case, we say that the point q is a ramification point. The set of branching points and the set of ramification points are called the branch locus and the ramification locus respectively. We say that two branched coverings $\pi : C_1 \to C_2$ and $\pi' : C_1' \to C_2$ are isomorphic if there exists an isomorphism $f : C_1 \to C_1'$ such that $\pi = \pi' \circ f$.

In general, given a branch locus B in C_2, the Hurwitz numbers enumerate the number of branched coverings up to isomorphism with fixed ramification and from curves with a fixed genus. For our purpose, we focus only on a certain type of ramifications. We say that a branched covering is simple if each branch point has only one ramification point in the fiber and the multiplicity of the ramification point is two. For simple branched coverings, the number w of branch points equals the number of ramification points counted with multiplicity. We will restrict our study to degree d simple branched coverings over the Riemann sphere $\mathbb{P}^1$. In other words, we assume that C_2 is isomorphic to the projective line. Under these assumptions, the Riemann-Hurwitz formula ([67, Section IV Corollary 2.4]) implies that

$$w = 2(g - 1 + d).$$

The Hurwitz space $\mathcal{H}_{g,d}$ is the space of degree d simple branched covering over $\mathbb{P}^1$ from a curve of genus g up to isomorphism. In [72], it was proven that $\mathcal{H}_{g,d}$ is a manifold. A functorial perspective is given in [51] where it is shown that the Hurwitz space is a scheme. We define the branching morphism as the map

$$\mathrm{br} : \mathcal{H}_{g,d} \longrightarrow \mathbb{P}^w$$

sending a branched covering to its branch locus. Here, we identify $\mathbb{P}^w$ with the w–th symmetric power of $\mathbb{P}^1$. The branching morphism is étale and finite (see [51, Corollary 6.4]). We define the Hurwitz number $H_{g,d}$ as the degree of the branching morphism.

In this chapter we mainly focus on plane curves. Therefore, we restrict our study to the Hurwitz space $\mathcal{H}_{g,d}$ where g is given by the genus–degree formula. In other words, we fix the genus g to be $g = (d-1)(d-2)/2$, which implies that the number of branch points is $w = d(d-1)$. Under this assumption, we denote the Hurwitz space

$$\mathcal{H}_{\frac{(d-1)(d-2)}{2},d}$$

and the Hurwitz number

$$H_{\frac{(d-1)(d-2)}{2},d}$$

by $\mathcal{H}_d$ and H_d respectively. Following [23], the number H_d can be found by counting monodromy representations, i.e. homomorphisms from the fundamental group of the target minus the branch points to the symmetric group over the fiber of the base point.

Lemma 1.1.1 (Hurwitz [72]). *The Hurwitz number H_d equals $1/d!$ times the number of tuples of transpositions $\tau = (\tau_1, \tau_2, \ldots, \tau_{d\cdot(d-1)})$ in the symmetric group $\mathbb{S}_d$ satisfying*

$$\tau_{d\cdot(d-1)} \circ \cdots \circ \tau_2 \circ \tau_1 = \mathrm{id},$$

where the subgroup generated by the τ_i acts transitively on the set $\{1, 2, \ldots, d\}$.

We now turn to branched covers that are real. This has been studied in [21, 62, 75]. A cover $f : C \to \mathbb{P}^1$ is called real if the Riemann surface C has an involution which is compatible with complex conjugation on the Riemann sphere $\mathbb{P}^1$. In this situation, the branch points in $\mathbb{P}^1$ can be real or pairs of complex conjugate points. We define the real Hurwitz number $H_d^{\mathrm{real}}(t)$ as the weighted count of degree d real covers f of $\mathbb{P}^1$ by a genus $\binom{d-1}{2}$ curve C having $d(d-1)$ fixed simple branch points, t of which are real. As before, each cover $f : C \to \mathbb{P}^1$ is weighted by $\frac{1}{|\mathrm{Aut}(f)|}$. The following result appears in [21, Section 3.3].

Lemma 1.1.2. *The real Hurwitz number $H_d^{\mathrm{real}}(t)$ equals $1/d!$ times the number of tuples τ as in Lemma 1.1.1 for which there exists an involution $\sigma \in \mathbb{S}_3$ such that*

$$\sigma \circ \tau_i \circ \cdots \circ \tau_1 \circ \sigma = (\tau_1 \circ \cdots \circ \tau_i)^{-1}$$

for $i = 1, \ldots, t-1$ and $\sigma \circ \tau_{t+i} \circ \sigma = \tau_{t'+1-i}$ for $i = 1, \ldots, t'$, where t is the number of real branch points and t' the number of pairs of complex conjugate branch points.

1.1.2 Plane Hurwitz numbers

In this section we introduce the notion of plane Hurwitz numbers. Broadly speaking, plane Hurwitz numbers count the branched coverings arising from the linear projection of a plane curve to a line. We consider the linear projection $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that takes a point $[x, y, z]$ in the projective plane to the point $[x, y]$ on the projective line. Geometrically, this is the projection with center $p = [0, 0, 1]$. Let C_A be a generic degree d plane curve defined by the homogeneous polynomial

$$A(x, y, z) \;=\; \sum_{i+j+k=d} \alpha_{ijk}\, x^i y^j z^k. \tag{1.1}$$

Here generic means smooth with no flex lines or bitangents passing through p. Thus, the restriction $\pi_A : C_A \to \mathbb{P}^1$ of π to C_A is a degree d simple branched cover with $d(d-1)$ branch points. We represent the branch points by a binary form

$$B(x,y) \;=\; \sum_{i+j=d(d-1)} \beta_{ij}\, x^i y^j. \tag{1.2}$$

Geometrically, the $d(d-1)$ branch points are the $d(d-1)$ tangent lines of $\mathbb{V}(A)$ passing through the point of projection p.

Example 1.1.3. *Let us Consider the smooth conic C defined by the polynomial $F = x^2 - y^2 + z^2$. The projection $\pi : C \to \mathbb{P}^1$ defines a double cover of $\mathbb{P}^1$. Given $q = [q_0, q_1] \in \mathbb{P}^1$, the fiber $\pi^{-1}(q)$ is the intersection of C with the lines defined by the linear form $l = q_1 x - q_0 y$. This intersection consists of the points $[q_0, q_1, z]$ such that $q_0^2 - q_1^2 + z^2 = 0$. Thus, q is a branch point if and only if $q_0^2 = q_1^2$. We deduce that the branch points are $[1,1]$ and $[1,-1]$ and the ramification points are $[1,1,0]$ and $[1,-1,0]$.*

We identify the space of degree d ternary forms with the projective space $\mathbb{P}^{\binom{d+2}{2}-1}$ with coordinates α_{ijk} as in (1.1). Similarly, we identify the space of degree $d(d-1)$ binary forms with $\mathbb{P}^{d(d-1)}$ with coordinates β_{ij} as in (1.2). Passing from the curve to its branch points defines a rational map from $\mathbb{P}^{\binom{d+2}{2}-1}$ to $\mathbb{P}^{d(d-1)}$. A point $q = [q_0, q_1]$ lies in the branch locus of π_A if and only if $\pi_A^{-1}(q)$ has a multiple point. In other words, if and only if the polynomial $A(q_0, q_1, z)$ has a multiple root. Therefore, the map sending A to the branch locus of π_A is given by

$$\mathrm{br} : \begin{array}{ccc} \mathbb{P}^{\binom{d+2}{2}-1} & \dashrightarrow & \mathbb{P}^{d(d-1)} \\ A & \mapsto & \mathrm{discr}_z(A) \end{array}, \tag{1.3}$$

where $\mathrm{discr}_z(A)$ denotes the discriminant of A with respect to the variable z. This discriminant is a binary form B of degree $d(d-1)$ in x, y whose coefficients are polynomials of degree $2d-2$ in α.

The goal of plane Hurwitz numbers is to, given a branch locus B, enumerate the branched covers of the form π_A for $A \in \mathbb{P}^{\binom{d+2}{2}-1}$ whose branch locus is B, up to isomorphism. Algebraically, we want to count the polynomials A such that $\mathrm{discr}_z(A) = B$ up to scalar. To define this enumerative problem properly, we need to establish the right notion of isomorphism. Let $\mathcal{G} \subset \mathrm{PGL}(3)$ be the subgroup of automorphisms of $\mathbb{P}^2$ that commute with the projection π. Geometrically, $\mathcal{G}$ is the group of automorphisms of $\mathbb{P}^2$ that fix p and the pencil of lines passing through p. We can explicitly describe $\mathcal{G}$ as the subgroup of $\mathrm{PGL}(3)$ of matrices of the form

$$\begin{pmatrix} g_0 & 0 & 0 \\ 0 & g_0 & 0 \\ g_1 & g_2 & g_3 \end{pmatrix} \tag{1.4}$$

with $g_0g_3 \neq 0$. We consider the action of $\mathcal{G}$ on $\mathbb{P}^{\binom{d+2}{2}-1}$ given by

$$g \ : \ x \mapsto g_0 x \, , \ y \mapsto g_0 y \, , \ z \mapsto g_1 x + g_2 y + g_3 z \quad \text{with } \ g_0 g_3 \neq 0. \tag{1.5}$$

Then, the discriminant $\mathrm{discr}_z(A)$ is invariant under the action (1.5).

Definition 1.1.4. *By [99, Corollary 5.2.1], the fiber of* br *over B is a finite union of $\mathcal{G}$-orbits. The number $\mathfrak{h}_d$ of these $\mathcal{G}$-orbits is called the* **plane Hurwitz number** *of degree d.*

We denote the quotient of $\mathbb{P}^{\binom{d+2}{2}-1}/\mathcal{G}$ by $\mathcal{PH}_d$. Note that the group $\mathcal{G}$ is not a reductive group. Therefore the quotient $\mathcal{PH}_d$ is not, a priory, a good quotient. In Section 1.2 we will see that, despite $\mathcal{G}$ not being reductive, $\mathcal{PH}_d$ is a good quotient which is an open subset of the weighted projective space $\mathbb{P}(2^3, 3^4, \ldots, d^{d+1})$. A point in this weighted projective space is identified with a vector $(U_2, \ldots, U_d)$ where U_i is a degree i form in x and y. Then, the point $(U_2, \ldots, U_d)$ corresponds to the $\mathcal{G}$–orbit of the polynomial

$$z^d + U_2(x,y)z^{d-1} + \cdots + U_{d-1}(x,y)z + U_d(x,y).$$

We refer to [68, 94] for the background on geometric invariant theory. Using this notation, we consider the branching morphism

$$\begin{array}{rccc} \mathrm{br}: & \mathcal{PH}_d & \dashrightarrow & \mathbb{P}^{d(d-1)} \\ & [A] & \mapsto & \mathrm{discr}_z(A) \end{array} . \tag{1.6}$$

The branching morphism (1.6) is generically finite and its degree is the plane Hurwitz number $\mathfrak{h}_d$.

Example 1.1.5. *For $d = 2$, we have $\mathfrak{h}_2 = 1$. Our polynomials are*

$$\begin{aligned} A \ &= \alpha_{200}x^2 + \alpha_{110}xy + \alpha_{101}xz + \alpha_{020}y^2 + \alpha_{011}yz + \alpha_{002}z^2, \\ \mathrm{discr}_z(A) \ &= (4\alpha_{002}\alpha_{200} - \alpha_{101}^2)x^2 + (4\alpha_{002}\alpha_{110} - 2\alpha_{011}\alpha_{101})xy + (4\alpha_{002}\alpha_{020} - \alpha_{011}^2)y^2, \\ B \ &= \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2. \end{aligned}$$

The equations $\mathrm{discr}_z(A) = B$ describe precisely one $\mathcal{G}$-orbit in $\mathbb{P}^5$. Indeed, up to change of coordinates, we can assume that $B = xy$. Then, our equations are

$$4\alpha_{002}\alpha_{200} = \alpha_{101}^2 \ \textit{and} \ 4\alpha_{002}\alpha_{020} = \alpha_{011}^2.$$

Since $A(0,0,1) \neq 0$, we can assume that $\alpha_{002} = 1$. Then, if $\mathrm{discr}A = B$, we get that

$$A = 4z^2 + \alpha_{101}^2x^2 + 4\alpha_{110}xy + 4\alpha_{101}xz + \alpha_{011}^2y^2 + 4\alpha_{011}yz.$$

One can check that such a polynomial lies in the $\mathcal{G}$–orbit of $z^2 + xy$. Therefore, the plane Hurwitz number $\mathfrak{h}_2$ is one. In general, the only point in the fiber $\mathrm{br}^{-1}(B)$ is the $\mathcal{G}$–orbit of $z^2 + B$.

Note that the dimension of $\mathcal{PH}_d$ is $d(d+3)/2-3$, whereas the dimension of the target space is $d(d-1)$. In particular, $\dim \mathcal{PH}_d < d(d-1)$ for $d \geq 4$. We denote the closure of the image of the branching morphism by $\mathcal{V}_d$. For $d = 2, 3$ we have that $\dim \mathcal{PH}_d = \dim \mathbb{P}^{d(d-1)}$, and hence, $\mathcal{V}_d = \mathbb{P}^{d(d-1)}$. On the other hand, for $d \geq 4$, $\mathcal{V}_d$ has positive codimension. For instance, for $d = 4$, we have that $\dim \mathcal{PH}_4 = \binom{6}{2} - 1 - 3 = 11$. Therefore, $\mathcal{V}_4 \subset \mathbb{P}^{12}$ is a hypersurface in $\mathbb{P}^{12}$.

Plane Hurwitz numbers $\mathfrak{h}_d$ were studied in Ongaro's 2014 PhD thesis and in his work with Shapiro [99, 101]. For instance, in [99, Proposition 5.2.1], it is shown that, given two degree d plane curves A and B, the two linear projections π_A and π_B are isomorphic as abstract branched coverings if and only if they are in the same $\mathcal{G}$–orbit. Therefore, we get an injective morphism $\mathcal{PH}_d \to \mathcal{H}_d$ that commutes with the branching morphisms of both $\mathcal{PH}_d$ and $\mathcal{H}_d$. We conclude that the plane Hurwitz number $\mathfrak{h}_d$ is compatible with the count performed by the Hurwitz number H_d and $\mathfrak{h}_d \leq H_d$. Now that we know that the notion of isomorphism of plane covering is compatible with the one of abstract covering, the next natural question is whether any abstract covering from a plane curve can be obtained as a linear projection. The following theorem answers this question.

Theorem 1.1.6. *[99, Theorem 5.1.1] Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d > 4$. Then, any degree d branched covering $C \to \mathbb{P}^1$ is isomorphic as a branched covering to a linear projection.*

Theorem 1.1.6 motivates the study of plane Hurwitz numbers. We deduce from Theorem 1.1.6 that plane Hurwitz numbers not only are a subcount of the usual Hurwitz numbers but, for $d > 4$, they count all the branched coverings from plane curves.

In Example 1.1.5 we showed that $\mathfrak{h}_2$ is one. Presently, the only known nontrivial values of plane Hurwitz numbers are $\mathfrak{h}_3 = 40$ and $\mathfrak{h}_4 = 120$. The former value is due to Clebsch [30, 31]. We first learned it from [99, Proposition 5.5.2], where a modern interpretation of the number is given. Mainly, it is shown that $\mathfrak{h}_3$ equals the Hurwitz number H_3 which was classically computed. The value $\mathfrak{h}_4 = 120$ was determined by Vakil in [122]. In Section 1.5.1, we revise Vakil's reasoning in this computation.

1.1.3 Real space sextics

In this section we provide a brief introduction to some real algebro-geometric features of certain class of curves: spaces sextics. These curves play a fundamental role in Vakil's computation of $\mathfrak{h}_4$ in [122]. In Section 1.2, we will introduce the notion of real plane Hurwitz numbers and in Section 1.5 we will compute the values of these numbers for plane quartics. In this computation, the understanding of the real geometry of these curves will be crucial. For literature on real algebraic geometry we refer to [10].

Definition 1.1.7. *A space sextic is a smooth genus 4 curve whose canonical model lies in a quadric cone in $\mathbb{P}^3$.*

In other words, a space sextic is a smooth space curve $C \subset \mathbb{P}^3$ that is the complete intersection of a quadric cone and a cubic surface. By the adjunction formula (see [44, Section 1.4.2]), one can check that the genus of such curve is indeed 4. Now, let C be a real space sextic. We denote the real points of C by $C(\mathbb{R})$. The connected components of $C(\mathbb{R})$ are called ovals. By Harnack's curve theorem (see [10, Section 11.6]), a real curve C of genus g has at most $(g+1)$ ovals. In our case, a real space sextic has at most $4+1=5$ ovals. In Figure 1.1 we see examples of real space sextics for each possible number of ovals. We say that $C(\mathbb{R})$ separates C if $C(\mathbb{C}) \setminus C(\mathbb{R})$ is not connected as a two dimensional real manifold.

Next, we analyze the possible distribution of the ovals of a real space sextic C. Let Q be the real quadric cone where C lies. Then, C does not pass through the singularity of Q. Note that the smooth locus of $Q(\mathbb{R})$ is the real cylinder $\mathbb{R}^1 \times \mathbb{P}^1_{\mathbb{R}}$, where $\mathbb{P}^1_{\mathbb{R}}$ is identified with the real circle. Therefore, $C(\mathbb{R})$ is contained in the cylinder $\mathbb{R}^1 \times \mathbb{P}^1_{\mathbb{R}}$. Recall that the fundamental group of $\mathbb{R}^1 \times \mathbb{P}^1_{\mathbb{R}}$ is $\mathbb{Z}$ and it is generated by the class of the curve $\{0\} \times \mathbb{P}^1_{\mathbb{R}}$.

Definition 1.1.8. *We say that an oval of C is small if it vanishes in the fundamental group of $\mathbb{R}^1 \times \mathbb{P}^1_{\mathbb{R}}$. Analogously, we say that an oval of C is big if it does not vanish in the fundamental group of the cylinder.*

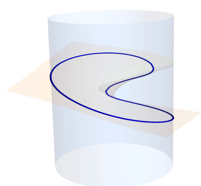
For instance, in Figure 1.1b we have a real space sextic with two ovals: one big and one small. In [109, Section 5], the author uses algebraic geometry to describe all the possible distributions of small and big ovals in a real space sextic. Consider the projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ whose center is the singularity q of Q. Recall that the singularity of Q is a real point. The image of Q is a smooth conic Q' and the restriction of π to the space sextic is a degree 3 simple covering of $Q' \simeq \mathbb{P}^1$. The fiber of a real point in $\mathbb{P}^1$ is the intersection of C with a real line L contained in Q passing through q. An equivalent definition of big and small ovals can be given in terms of this projection. An oval O of C is big if any real fiber of π cuts O in an odd number of points. Analogously, an oval O of C is small if any real fiber of π cuts O in an even number of points. Using this description, we can list the number of oval distributions of a real space sextic. For instance, a real space sextic C can not have four big ovals. Indeed, the fiber of a real point through $\pi|_C$ contains at least 1 point for each big oval. Using the degree of the branched covering $\pi|_C$, we deduce that C has at most 3 big ovals. The following result lists all the possible distributions of small and big ovals of a space sextic (see [109, Section 5]); Figure 1.1 shows pictures of all the possible cases provided in Theorem 1.1.9.

Theorem 1.1.9. *Let C be a real space sextic inside a quadric cone Q with vertex q. Then, C satisfies one of the following statements*

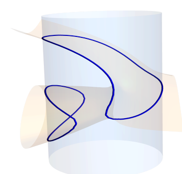
1. *C has one big oval.*
2. *C has one big oval and one small oval.*
3. *C has one big oval and two small ovals which lie in the same connected component of the complement of the big oval inside $Q(\mathbb{R}) \setminus \{q\}$.*

4. *C has one big oval and three small ovals.*
5. *C has one big oval and four small ovals.*
6. *C has three big ovals.*
7. *C has one big oval and two small ovals which lie in different connected components of the complement of the big oval inside $Q(\mathbb{R}) \setminus \{q\}$.*

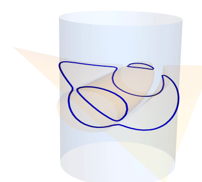
Moreover, $C(\mathbb{R})$ separates $C(\mathbb{C})$ if and only if one of the last two cases is satisfied.



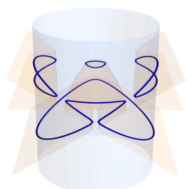
(a) Theorem 1.1.9 Case 1.



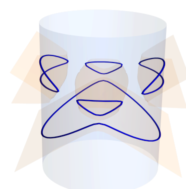
(b) Theorem 1.1.9 Case 2.



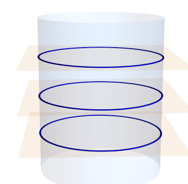
(c) Theorem 1.1.9 Case 3.



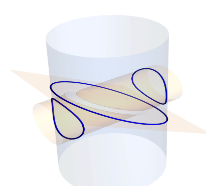
(d) Theorem 1.1.9 Case 4.



(e) Theorem 1.1.9 Case 5.



(f) Theorem 1.1.9 Case 6.



(g) Theorem 1.1.9 Case 7.

Figure 1.1: The seven pictures represents the seven case of Theorem 1.1.9. The space sextic is the intersection of the affine cone x^2+y^2-1 in light blue, with a cubic surface in light orange.

Let us now highlight the connection between real theta characteristics and the ovals of a real curve.

Definition 1.1.10. *A **theta characteristic** of a non singular curve C is a divisor Θ such that 2Θ is linearly equivalent to the canonical divisor of C. We say that a theta characteristic Θ is **even** or **odd** if the dimension of the space of global sections of Θ is even or odd, respectively.*

We refer to [40, Chapter 5] for further literature on theta characteristics. An example of a geometric application of theta characteristics is the case of space sextics, where odd theta characteristics coincide with tritangent planes to the curve (see [64, Theorem 2.2]). The number of even and odd theta characteristics is uniquely determined by the genus of the curve (see [93])

Theorem 1.1.11. *Let C be a genus g smooth curve. Then, the number of even and odd theta characteristics is $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$, respectively.*

Next we introduce the real variant of theta characteristics, which was defined in [81].

Definition 1.1.12. *A real theta characteristic of a real curve C is a theta characteristic Θ of C that is linearly equivalent to $\iota^*\Theta$, where ι is the involution defining the real structure on C.*

The number of real odd and even theta characteristics of a genus g real curve was derived in the main theorem of [81].

Theorem 1.1.13. *Let C be a genus g real smooth curve with $s > 0$ ovals.*

1. *If $C(\mathbb{R})$ separates C, the number of even and odd real theta characteristics of C is $2^{g-1}(2^{s-1}+1)$ and $2^{g-1}(2^{s-1}-1)$, respectively.*
2. *If $C(\mathbb{R})$ does not separate C, the number of even and odd real theta characteristics of C are both 2^{g+s-2}.*

We can apply Theorem 1.1.13 to compute the number of real theta characteristics by means of the distinct cases of Theorem 1.1.9. In Table 1.2, the number of even and odd theta characteristics is given for each of these cases. We highlighted before that the odd theta characteristics of a space sextic are in bijection with the tritangent planes of the curve. Similarly, real odd theta characteristics of a real space sextic are in bijection with real tritangent planes. Here, a real tritangent plane is a tritangent plane defined by a real equation, which differs from the notion of totally real tritangent plane. The latter is a real tritangent plane intersecting the real space sextic at 3 real points with multiplicity 2. In the same way that real tritangents are related to real theta characteristic, totally real tritangents are closely related to the notion of totally real theta characteristics (see [83]). For more literature on (totally) real tritangent planes we refer to [26, 64, 82].

1.1.4 Real del Pezzo surfaces of degree one

In [122], another key ingredient in the study of $\mathfrak{h}_4$ are del Pezzo surfaces of degree one. We give a brief introduction to this type of surfaces following [40, Chapter 8] and [91, Chapter IV]. With the purpose of investigating real plane Hurwitz numbers, we devote this section not only to del Pezzo surfaces of degree 1, but also to their real version. For the real algebraic geometry of del Pezzo surface we mainly follow [109].

Definition 1.1.14. *A del Pezzo surface is a smooth surface with ample anticanonical bundle. The degree of a del Pezzo surface is defined as the self intersection of the canonical class.*

For instance, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are del Pezzo surfaces of degree 9 and 8 respectively. The classification of del Pezzo surfaces is a well-known result in algebraic geometry (see [40, Theorem 8.1.15]). This classification states that a del Pezzo surface is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of k points in $\mathbb{P}^2$ in general position for $1 \leq k \leq 8$. The degree of the latest type of del Pezzo surfaces is $9-k$. For our purposes, we focus on del Pezzo surfaces of degree 1. By the above classification of del Pezzo surface, such a surface is isomorphic to the blow up of $\mathbb{P}^2$ at 8 points in general position. Here general position means that the following conditions are satisfied:

- All points are distinct.
- No three points are in a line.
- No six points are on a conic.
- No cubic passes through the points with one of them being singular.

An important concept related to del Pezzo surfaces is the notion of (-1)–curves.

Definition 1.1.15. *A (-1)–curve in a del Pezzo surface is a rational curve with self intersection -1.*

In [91, Section 26], one can find the complete description of the (-1)–curves of del Pezzo surfaces. Here we will only present the (-1)–curves for del Pezzo surfaces of degree 1. In total, there are 240 (-1)–curves in a del Pezzo surface of degree 1. These (-1)–curves can be described through the blow up description of the del Pezzo surface. Consider the blow up $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ where $\mathcal{P} = \{p_1, \ldots, p_8\}$ is a set of 8 points in $\mathbb{P}^2$ in general position. We now list all the (-1)–curves of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$.

Theorem 1.1.16. *[91, Theorem 26.2] The 240 (-1)–curves of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ are*

1. *For every $i \in [8]$, the exceptional divisor E_i of p_i is a (-1)–curve.*
2. *For every $i, j \in [8]$ distinct, the strict transform $\tilde{L}_{i,j}$ of the line $L_{i,j}$ passing through p_i and p_j is a (-1)–curve.*
3. *For every $i_1, i_2, i_3 \in [8]$ distinct, let Q_{i_1,i_2,i_3} be the strict transform of the quadric passing through $\mathcal{P} \setminus \{p_{i_1}, p_{i_2}, p_{i_3}\}$. Then, Q_{i_1,i_2,i_3} is a (-1)–curve.*
4. *For every $i, j \in [8]$ distinct, the strict transform $C_{i,j}$ of a cubic passing through $\mathcal{P} \setminus \{p_i, p_j\}$ with multiplicity 1, through p_i with multiplicity 2 and not passing through p_j, is a (-1)–curve.*
5. *For every $i_1, i_2, i_3 \in [8]$ distinct, let Q'_{i_1,i_2,i_3} be the strict transform of the quartic passing through $\mathcal{P} \setminus \{p_{i_1}, p_{i_2}, p_{i_3}\}$ with multiplicity 1 and with multiplicity 2 at p_{i_1}, p_{i_2} and p_{i_3}. Then, Q_{i_1,i_2,i_3} is a (-1)–curve.*

6. *For every $i, j \in [8]$ distinct, the strict transform $R_{i,j}$ of the plane quintic passing doubly through $\mathcal{P} \setminus \{i, j\}$ and with multiplicity 1 at p_i and p_j is a (-1)–curve.*

7. *For every $i \in [8]$, let S_i be the strict transform of the plane sextic passing doubly through $\mathcal{P} \setminus \{p_i\}$ and with multiplicity 3 at p_i. Then, S_i is a (-1)–curve.*

We are interested on counting the (-1)–curves of a del Pezzo surface of degree 1 up to the action of an involution of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ called the Bertini involution, which is defined as follows. The linear system of cubics passing through $\mathcal{P}$ has dimension 2 and it is generated by two cubics u, v. Similarly, the linear system of sextics passing doubly through $\mathcal{P}$ has dimension 4. This linear system is generated by u^2, uv, v^2 and a sextic w. Given a generic point $p \in \mathbb{P}^2$, there is a unique cubic u_p passing through $\mathcal{P} \cup \{p\}$. Consider the linear system E_p of sextics passing through p and vanishing double at $\mathcal{P}$. Then, E_p has dimension 3 and it is generated by $u_p u, u_p v$ and a sextic w_p. The intersection of w_p and u_p at 18 points counted with multiplicity. Among them, 16 are the 8 points of $\mathcal{P}$ with multiplicity 2. The two remaining points are p and an extra point denoted by q. In paricular, we get that the base locus of E_p is $\{2p_1, \ldots, 2p_8, p, q\}$. The Bertini involution of the plane is the rational involution that sends p to q. The Bertini involution of the plane lifts to an involution ι of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ called the Bertini involution.

Definition 1.1.17. *A Bertini pair $\{C_1, C_2\}$ is an unordered pair of (-1)–curves in $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ invariant under the Bertini involution.*

In [109, Secion 5], the list of Bertini pairs of a degree 1 del Pezzo surface is given. This list can also be found in [82, Section 2]. We now describe each of these pairs using the notation in Theorem 1.1.16. For every $i \in [8]$, $\{E_i, S_i\}$ is a Bertini pair. For every $i, j \in [8]$ distinct, $\{\tilde{L}_{i,j}, R_{i,j}\}$ form a Bertini pair. Similarly, there are 56 Bertini pairs of the form $\{Q_{i_1,i_2,i_3}, Q'_{i_1,i_2,i_3}\}$ for $i_1 < i_2 < i_3$. Finally, $\{C_{i,j}, C_{j,i}\}$ is a Bertini pair for $i \neq j$. In particular there are 120 Bertini pairs in a del Pezzo surface of degree 1.

The fact that the number 120 appears both in the setting of del Pezzo surfaces and space sextics is not a coincidence. We saw in Section 1.1.3 that a space sextic lies in a quadric cone. Such cone is isomorphic to the weighted projective space $\mathbb{P}(1, 1, 2)$. Now, let u and v be two linearly independent cubics passing through $\mathcal{P}$. Let w be a plane sextics such that $\{u^2, uv, v^2, w\}$ is a basis of the linear system of sextics vanishing doubly at $\mathcal{P}$. The map

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2 & \longrightarrow & \mathbb{P}(1,1,2) \\ p & \longmapsto & [u(p), v(p), w(p)] \end{array}$$

is a double cover of $\mathbb{P}(1, 1, 2)$. The Bertini involution is exactly the involution defined by this double cover. Moreover, the branch locus is a space sextic and the preimage of a tritangent plane consists of two (-1)–curves forming a Bertini pair. We conclude that the 120 tritangent planes of a space sextic are in bijection with the 120 (-1)–curves of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ up to the Bertini involution.

$X_{\mathbb{R}}$	# real (-1)–curves
$\mathbb{P}^2_{\mathbb{R}}(8,0)$	240
$\mathbb{P}^2_{\mathbb{R}}(6,2)$	126
$\mathbb{P}^2_{\mathbb{R}}(4,4)$	60
$\mathbb{P}^2_{\mathbb{R}}(2,6)$	26
$\mathbb{P}^2_{\mathbb{R}}(0,8)$	8
$\mathbb{D}_4(3,0)^1_2$	24
$\mathbb{D}_4(3,0)^0_3$	24
$\mathbb{D}_4(1,2)$	6
$\mathbb{D}_2(1,0)$	4
$\mathbb{G}_2(1,0)$	2
$\mathbb{B}_1$	0

Table 1.1: On the left column we list the possible real del Pezzo surfaces of degree 1. The right column shows the number of real (-1)–curves each type of real del Pezzo surface has.

Next we focus on the real structure of del Pezzo surfaces. The main reference we follow for the real geometry of these surfaces is [109]. Given a real algebraic variety X, we denote the blow up of X at $a+2b$ general points, a real points and b conjugate pairs of points, by $X(a,2b)$. Let $\mathbb{D}_d$ be the real algebraic surface associated to a real de Jonquiéres involution described in [109, Section 3]. Similarly, let $\mathbb{G}_2$ and $\mathbb{B}_1$ be the real surfaces associated to a real Geiser and Bertini involution respectively. We refer to [109, Sections 4 and 5] for further details on these surfaces.

Let X be a real del Pezzo surface whose real structure is given by an involution σ.

Definition 1.1.18. *A real (-1)–curve on X is a (-1)–curve C such that $\sigma(C)=C$. A real Bertini pair is a Bertini pair $\{C_1,C_2\}$ that is invariant under the real involution σ. In other words, $\sigma(C_1)=C_2$.*

Note that in the definition of a real Bertini pair we do not require the (-1)–curves in a real Bertini pair to be real. In [109, Corollary 5.3], a list of the real del Pezzo surfaces of degree 1 together with their number of real (-1)–curves is given (See Table 1.1).

Given a real del Pezzo surface X of degree 1, we define its associated surface as follows. Recall that we have a double cover $X \to Q \subset \mathbb{P}^3$, where Q is a real quadric cone. Locally, X is given by the equation $z^2 = f(x,y)$, where f is the local equation of Q. The associated surface of X is the surface locally defined by $z^2 = -f(x,y)$. Note that after a complex base change X is isomorphic to its associate surface. Let C be the space sextic given by the branch locus of the double cover $X \to Q \subset \mathbb{P}^3$. Then, the double cover establishes a bijection between the set of real tritangent planes of C and the set of real Bertini pairs of X. In [109, Corollary 5.3] a list of the real del Pezzo surfaces of degree 1 with their associated surfaces and the corresponding number of real Bertini pairs is given (see Table 1.2).

$X_{\mathbb{R}}$	$X'_{\mathbb{R}}$	Ovals	# real odd theta	# real even theta
$\mathbb{P}^2_{\mathbb{R}}(8,0)$	$\mathbb{B}_1$	1 big + 4 small	120	136
$\mathbb{P}^2_{\mathbb{R}}(6,2)$	$\mathbb{G}_2(1,0)$	1 big + 3 small	64	64
$\mathbb{P}^2_{\mathbb{R}}(4,4)$	$\mathbb{D}_2(1,0)$	1 big + 2 small	32	32
$\mathbb{P}^2_{\mathbb{R}}(2,6)$	$\mathbb{D}_4(1,2)$	1 big + 1 small	16	16
$\mathbb{P}^2_{\mathbb{R}}(0,8)$	$\mathbb{P}^2_{\mathbb{R}}(0,8)$	1 big	8	8
$\mathbb{D}_4(3,0)^1_2$	$\mathbb{D}_4(3,0)^1_2$	3 big	24	40
$\mathbb{D}_4(3,0)^0_3$	$\mathbb{D}_4(3,0)^0_3$	1 big + 1 + 1 small	24	40

Table 1.2: The first two columns list the pairs of del Pezzo surface together with its associated surface. The third column represents the distribution of the ovals of the corresponding space sextic. Here 1 big + 1 + 1 small refers to case 7 in Theorem 1.1.9. The fourth column represents the number of real odd theta characteristics of the space sextic, which equals the number of real tritangent planes to the space sextics and the number of real Bertini pairs. The last column indicates the number of real even theta characteristics.

1.2 Normal forms and numerical computations

In this section we approach our recovery problem from a numerical perspective. The main idea is to find useful normal forms for the action of $\mathcal{G}$ on $\mathbb{P}^{\binom{d+2}{2}-1}$. These normal forms allow us to translate the recovery problem into solving a polynomial system of equations that we can solve numerically using `HomotopyContinuation.jl` [17].

We use the same notation as in Section 1.1.2. We identify $\mathbb{P}^{\binom{d+2}{2}-1}$ with the space of plane curves (1.1) of degree d and use as homogeneous coordinates the α_{ijk}. In other words, we write a degree d ternary form as

$$A(x,y,z) \;=\; \sum_{i+j+k=d} \alpha_{ijk}\, x^i y^j z^k.$$

Similarly, we represent the branch points by a binary form

$$B(x,y) \;=\; \sum_{i+j=d(d-1)} \beta_{ij}\, x^i y^j.$$

We identify $\mathbb{P}^{d(d-1)}$ with the space of degree $d(d-1)$ binary forms and we use the coordinates β_{ij} as above. We consider the following projective space L_d of codimension

3 given by the equations

$$L_d \;=\; \mathbb{V}(\,\alpha_{10\,d-1}\,,\,\alpha_{d-1\,10}\,,\,\alpha_{00d}-\alpha_{01\,d-1}\,). \tag{1.7}$$

We now show that this linear space serves as normal form with respect to the group action on fibers of (1.3). Recall that the group that acts is the three-dimensional group $\mathcal{G} \subset \mathrm{PGL}(3)$ given in (1.5).

Theorem 1.2.1. *Let A be a ternary form of degree $d \geq 3$ such that*

$$\alpha_{00d}\left(\sum_{k=0}^{d-1}\frac{(k+1)(-1)^k}{d^k}\alpha_{10\,d-1}^{k}\alpha_{00d}^{d-k-1}\alpha_{d-k-1\,0\,k+1}\right)\;\neq\;0. \tag{1.8}$$

The orbit of A under the action of $\mathcal{G}$ on $\mathbb{P}^{\binom{d+2}{2}-1}$ intersects the linear space L_d in one point.

Proof. We find the unique point in $L_d \cap \mathcal{G}A$. Let $g \in \mathcal{G}$ be as in (1.4). Without loss of generality, we set $g_0 = 1$. Note that the coefficient of xz^{d-1} in gA equals $(d\alpha_{00d}g_1 + \alpha_{10\,d-1})g_3^{d-1}$. By setting $g_1 = -\frac{1}{d}\alpha_{10\,d-1}/\alpha_{00d}$, we obtain that the coefficient of xz^{d-1} of gA vanishes. The polynomial gA arises from A by the coordinate change $z \mapsto g_1x+g_2y+g_3z$. Thus, a monomial $x^iy^jz^{d-i-j}$ contributes the expression $x^iy^j(g_1x+g_2y+g_3z)^{d-i-j}$ to gA. This contributes to the monomials $x^{i'}y^{j'}z^{d-i'-j'}$ with $i' \geq i$ and $j' \geq j$. The coefficient of $x^{d-1}y$ in gA arises from applying the coordinate change to the subsum of A

$$\sum_{i=0}^{d-1}\alpha_{i0\,d-i}\,x^iz^{d-i} \;+\; \sum_{i=0}^{d-1}\alpha_{i1\,d-i-1}\,x^iyz^{d-i-1}.$$

Thus, the coefficient of $x^{d-1}y$ in gA is equal to

$$\sum_{i=0}^{d-1}\alpha_{i0\,d-i}(d-i)\,g_1^{d-i-1}g_2 \;+\; \sum_{i=0}^{d-1}\alpha_{i1\,d-i-1}\,g_1^{d-i-1}.$$

Inserting the above result for g_1, and setting the coefficient of $x^{d-1}y$ to zero, this affine-linear equation for g_2 can be solved. We obtain g_2 as a rational function in the α_{ijk}.

Next, we equate the coefficients of yz^{d-1} and z^d. The first can be computed from the sum $\alpha_{00d}z^d + \alpha_{01\,d-1}yz^{d-1}$ and equals $\alpha_{00d}\,d\,g_2g_3^{d-1} + \alpha_{01\,d-1}\,g_3^{d-1}$. The second is computed from the z^d coefficient of A only, and we find it to be $\alpha_{00d}\cdot g_3^d$. Setting these two to be equal, and solving for g_3, we obtain $g_3 = \frac{1}{\alpha_{00d}}(\alpha_{00d}\,d\,g_2 + \alpha_{01\,d-1})$. Evaluating our result for g_2, we obtain g_3 as a rational function in the α_{ijk}. Note that the expressions obtained for g_1, g_2, g_3 as rational functions in the α_{ijk} are unique. Therefore, the $\mathcal{G}$–orbit of A cuts L_d in a unique point. $\square$

Remark 1.2.2. *One of the assumption of Theorem 1.2.1 is that $d \geq 3$. Observe that the result does not hold for $d = 2$. Indeed, in this case, the $\mathcal{G}$-orbit of A consists of the conics of the form*

$$\begin{aligned} gA = {} & (\alpha_{002}g_1^2 + \alpha_{101}g_0g_1 + \alpha_{200}g_0^2)x^2 + (2\alpha_{002}g_1g_2 + \alpha_{011}g_0g_1 + \alpha_{101}g_0g_2 + \alpha_{110}g_0^2)xy \\ & + (2\alpha_{002}g_1g_3 + \alpha_{101}g_0g_3)xz + (\alpha_{002}g_2^2 + \alpha_{011}g_0g_2 + \alpha_{020}g_0^2)y^2 \\ & + (2\alpha_{002}g_2g_3 + \alpha_{011}g_0g_3)yz + \alpha_{002}g_3^2z^2. \end{aligned}$$

For generic α_{ijk}, there is no choice of $g \in \mathcal{G}$ making the coefficients of the monomials xy and xz both zero. Note that the parenthesized sum in Equation (1.8) is the zero polynomial for $d = 2$, but not for $d \geq 3$.

Example 1.2.3. *We show the solution in the two cases of primary interest: $d = 3$ and $d = 4$. For cubics ($d = 3$), the unique point gA in $L_3 \cap \mathcal{G}A$ is given by the group element g with*

$$\begin{aligned} g_0 &= 1, \\ g_1 &= -\frac{\alpha_{102}}{3\alpha_{003}}, \\ g_2 &= \frac{9\alpha_{003}^2\alpha_{210} - 3\alpha_{003}\alpha_{102}\alpha_{111} + \alpha_{012}\alpha_{102}^2}{3\alpha_{003}(3\alpha_{003}\alpha_{201} - \alpha_{102}^2)}, \\ g_3 &= \frac{9\alpha_{003}^3\alpha_{210} + 3\alpha_{003}\alpha_{012}\alpha_{201} - 3\alpha_{003}^2\alpha_{102}\alpha_{111} + \alpha_{003}\alpha_{012}\alpha_{102}^2 - \alpha_{102}^2\alpha_{012}}{\alpha_{003}(3\alpha_{003}\alpha_{201} - \alpha_{102}^2)}. \end{aligned}$$

For quartics ($d = 4$), the unique point gA in $L_4 \cap \mathcal{G}A$ is given by $g \in \mathcal{G}$, where

$$\begin{aligned} g_0 &= 1, \\ g_1 &= -\frac{\alpha_{103}}{4\alpha_{004}}, \\ g_2 &= \frac{64\alpha_{004}^3\alpha_{310} - 16\alpha_{004}^2\alpha_{103}\alpha_{211} + 4\alpha_{004}\alpha_{103}^2\alpha_{112} - \alpha_{013}\alpha_{103}^3)}{8\alpha_{004}(8\alpha_{004}^2\alpha_{301} - 4\alpha_{004}\alpha_{103}\alpha_{202} + \alpha_{103}^3)}, \end{aligned}$$

and g_3 is the quotient of

$$\begin{gathered} 64\alpha_{004}^4\alpha_{310} + 16\alpha_{004}^2\alpha_{013}\alpha_{301} - 16\alpha_{004}^3\alpha_{103}\alpha_{211} - 8\alpha_{004}\alpha_{013}\alpha_{103}\alpha_{202} + \\ 4\alpha_{004}^2\alpha_{103}^2\alpha_{112} + 2\alpha_{103}^3\alpha_{013} - \alpha_{004}\alpha_{013}\alpha_{103}^3 \end{gathered}$$

and $2\alpha_{004} \cdot (8\alpha_{004}^2\alpha_{301} - 4\alpha_{004}\alpha_{103}\alpha_{202} + \alpha_{103}^3)$.
One can derive similar formulas for the transformation to normal form when $d \geq 5$. The denominator in the expressions for g is the polynomial of degree d in α shown in (1.8).

Our task is to solve the equation $\mathrm{discr}_z(\hat{A}) = B$, for a fixed binary form B. This equation is understood projectively, meaning that we seek $\hat{A}$ in $\mathbb{P}^{\binom{d+2}{2}-1}$ such that

$\mathrm{discr}_z(\hat{A})$ vanishes at all zeros of B in $\mathbb{P}^1$. By Theorem 1.2.1, we may assume that $\hat{A}$ lies in the subspace L_d. Our system has extraneous solutions, namely ternary forms $\hat{A}$ whose discriminant vanishes identically. They must be removed when solving our recovery problem. We now identify them geometrically.

Proposition 1.2.4. *The base locus of the discriminant map (1.3) consists of two varieties. These have codimension* 3 *and* $2d-1$ *respectively in* $\mathbb{P}^{\binom{d+2}{2}-1}$. *The former consists of all curves that are singular at* $p = [0,0,1]$*, and the latter is the locus of non-reduced curves.*

Proof. The binary form $\mathrm{discr}_z(A)$ vanishes identically if and only if the univariate polynomial function $z \mapsto A(u,v,z)$ has a double zero $\hat{z}$ for all $u,v \in \mathbb{C}$. If p is a singular point of the curve $\mathbb{V}(A)$ then $\hat{z} = 0$ is always such a double zero. If A has a factor of multiplicity ≥ 2 then so does the univariate polynomial $z \mapsto A(u,v,z)$, and the discriminant vanishes. Up to closure, we may assume that this factor is a linear form, so there are $\binom{d}{2} - 1 + 2$ degrees of freedom. This shows that the family of nonreduced curves A has codimension $2d-1 = (\binom{d+2}{2} - 1) - (\binom{d}{2} + 1)$. The two scenarios define two distinct irreducible subvarieties of $\mathbb{P}^{\binom{d+2}{2}-1}$. For A outside their union, the binary form $\mathrm{discr}_z(A)$ is not identically zero. $\square$

We now present our solution to the recovery problem for cubic curves. Let B be a binary sextic with six distinct zeros in $\mathbb{P}^1$ and let A be a ternary cubic in L_3. Thus, A is a polynomial of the form

$$A \;=\; \alpha_{300}x^3 + \alpha_{201}x^2z + \alpha_{111}xyz + \alpha_{102}xz^2 + \alpha_{030}y^3 + \alpha_{021}y^2z + yz^2 + z^3.$$

Here we assume $p = [0,0,1] \notin \mathbb{V}(A)$, so that $\alpha_{003} = 1$. Since $A \in L_3$, we also get that $\alpha_{012} = 1$. The remaining six coefficients α_{ijk} are unknowns. The discriminant $\mathrm{discr}_z(A)$ has degree three in these variables and equals

$$(4\alpha_{201}^3 + 27\alpha_{300}^2)x^6 + (12\alpha_{111}\alpha_{201}^2 - 18\alpha_{201}\alpha_{300})x^5y + \cdots + (4\alpha_{021}^3 - \alpha_{021}^2 - \cdots + 4\alpha_{030})y^6.$$

We are looking for the values of α_{ijk} such that this expression vanishes at each of the six zeros of B. This gives a system of six inhomogeneous cubic equations in the six unknowns α_{ijk}. In order to remove the extraneous solutions described in Proposition 1.2.4, we further require that the leading coefficient of the discriminant is nonzero. We can write our system of cubic constraints in the α_{ijk} as follows:

$$\left.\begin{array}{r} \mathrm{rank}\begin{pmatrix} 4\alpha_{201}^3{+}27\alpha_{300}^2 & \cdots & 4\alpha_{021}^3{-}\alpha_{021}^2 - \cdots + 4\alpha_{030} \\ \beta_{60} & \cdots & \beta_{06} \end{pmatrix} \leq 1 \\ \text{and} \quad 4\alpha_{201}^3 + 27\alpha_{300}^2 \neq 0 \end{array}\right\}. \tag{1.9}$$

This polynomial system encodes exactly the recovery of plane cubics from six branch points.

The study of cubic curves tangent, in particular Corollary 1.2.5 below, to a pencil of six lines goes back to Cayley [25]. The formula $\mathfrak{h}_3 = 40$ was found by Clebsch [30, 31]. We shall discuss his remarkable work in Section 1.4. A modern proof for $\mathfrak{h}_3 = 40$ was given by Kleiman and Speiser in [80, Corollary 8.5]. We here present the argument given in Ongaro's thesis [99].

Corollary 1.2.5. *For general β_{ij}, the system (1.9) has $\mathfrak{h}_3 = 40$ distinct solutions $\alpha \in \mathbb{C}^6$.*

Proof. By [99, Proposition 5.5.2], every covering of $\mathbb{P}^1$ by a plane cubic curve is a shift in the group law of that elliptic curve followed by a linear projection from a point in $\mathbb{P}^2$. This implies that the classical Hurwitz number, which counts such coverings, coincides with the plane Hurwitz number $\mathfrak{h}_3$. The former is the number of six-tuples $\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)$ of permutations of $\{1, 2, 3\}$, not all equal, whose product is the identity, up to conjugation. We can choose $\tau_1, \ldots, \tau_5$ in $3^5 = 243$ distinct ways. Three of these are disallowed, so there are 240 choices. The symmetric group $\mathbb{S}_3$ acts by conjugation on the tuples τ, and all orbits have size six. The number of classes of allowed six-tuples is thus $240/6 = 40$. This is our Hurwitz number $\mathfrak{h}_3$. Now, the assertion follows from Theorem 1.2.1, which ensures that the solutions of (1.9) are representatives. □

Now, we focus on another normal form, shown in (1.10), which has desirable geometric properties. Let A be a ternary form (1.1) with $a_{00\,d} \neq 0$. We define a group element $g \in \mathcal{G}$ by

$$g_0 = 1\,,\ g_1 = -\frac{a_{10\,d-1}}{d \cdot a_{00d}}\,,\ g_2 = -\frac{a_{01\,d-1}}{d \cdot a_{00d}}\,,\ g_3 = 1.$$

Then, the coefficients of xz^{d-1} and yz^{d-1} in gA are zero. Thus, after this transformation, we have that

$$A \;=\; z^d \,+\, A_2(x,y)\cdot z^{d-2} \,+\, A_3(x,y)\cdot z^{d-3} \,+\, \cdots \,+\, A_{d-1}(x,y)\cdot z \,+\, A_d(x,y). \tag{1.10}$$

Here $A_i(x, y)$ is an arbitrary binary form of degree i. Its $i+1$ coefficients are unknowns. The group $\mathcal{G}$ still acts by rescaling x, y simultaneously with arbitrary non-zero scalars $\lambda \in \mathbb{C}^*$. Up to this action, we can identify the normal form (1.10) with the weighted projective space $\mathbb{P}(2^3, 3^4, \ldots, d^{d+1})$. In [97, Section 4.2] it is proven that the quotient $\mathcal{PH}_d$ is a good quotient and it can be seen as an open subset in the weighted projective space $\mathbb{P}(2^3, 3^4, \ldots, d^{d+1})$. Moreover, in [97] a more functorial perspective is given to $\mathcal{PH}_d$ and the plane Hurwitz numbers.

We next illustrate the utility of (1.10) by computing the planar Hurwitz number for d=4. Consider a general ternary quartic A. We record its 12 branch points by fixing the discriminant $B = \mathrm{discr}_z(A)$. Let $\hat{A} \in L_4$ be an unknown quartic in the normal form specified in Theorem 1.2.1. In particular, $\hat{A}$ has 13 terms, 11 of the form $\alpha_{ijk}x^i y^j z^k$ plus yz^3 and z^4. Our task is to solve the following system of 12 polynomial equations of degree five in the 11 unknowns α_{ijk}:

$$\mathrm{discr}_z(\hat{A}) = B \text{ up to scalar multiplication.} \tag{1.11}$$

The number of solutions of this system was found by Vakil [122] with geometric methods.

Theorem 1.2.6. *Let $B = \sum_{i+j=12} \beta_{ij} x^i y^j$ be the discriminant with respect to z of a general ternary quartic A. Then the polynomial system (1.11) has $\mathfrak{h}_4 = 120$ distinct solutions $\alpha \in \mathbb{C}^{11}$.*

Proof. We work with the normal form (1.10). Up to the action of $\mathcal{G}$, the 11–dimensional weighted projective space $\mathbb{P}(2^3, 3^4, 4^5)$ parametrizes the triples (A_2, A_3, A_4). Following Vakil [122], we consider a second weighted projective space of dimension 11, namely $\mathbb{P}(3^5, 2^7)$. This weighted projective space parametrizes pairs (U_4, U_6) where $U_i = U_i(x, y)$ is a binary form of degree i, up to a common rescaling of x, y by some $\lambda \in \mathbb{C}^*$. We define a rational map among our two weighted projective spaces as follows:

$$\begin{array}{rccc} \nu: & \mathbb{P}(2^3, 3^4, 4^5) & \dashrightarrow & \mathbb{P}(3^5, 2^7) \\ & (A_2, A_3, A_4) & \mapsto & (-4A_4 - \frac{1}{3}A_2^2,\ A_3^2 - \frac{8}{3}A_2A_4 + \frac{2}{27}A_2^3) \end{array}. \tag{1.12}$$

We compose ν with the following map into the space $\mathbb{P}^{12}$ of binary forms of degree 12:

$$\begin{array}{rccc} \mu: & \mathbb{P}(3^5, 2^7) & \dashrightarrow & \mathbb{P}^{12} \\ & (U_2, U_3) & \mapsto & 4 \cdot U_2^3 + 27 \cdot U_3^2 \end{array}. \tag{1.13}$$

The reasoning for the maps (1.12) and (1.13) is that they represent the formula of the discriminant $\mathrm{discr}_z(A)$ of the special quartic in (1.10). Thus, modulo the action of $\mathcal{G}$, we have

$$\pi \ = \ \mu \circ \nu,$$

where $\pi : \mathbb{P}^{14} \dashrightarrow \mathbb{P}^{12}$ is the branch locus map in (1.3). One checks this by a direct computation.

Vakil proves in [122, Proposition 3.1] that the map ν is dominant and its degree equals 120. We also verified this statement independently via a numerical calculation in affine coordinates using `HomotopyContinuation.jl` [17], and we certified its correctness using the method in [16]. This implies that the image of the map μ equals the hypersurface $\mathcal{V}_4$. In particular, $\mathcal{V}_4$ is the locus of all binary forms of degree 12 that are sums of the cube of a quartic and the square of a sextic. Vakil proves in [122, Theorem 6.1] that the map μ is birational onto its image $\mathcal{V}_4$. We verified this statement by a Gröbner basis calculation. This result implies that both ν and π are maps of degree 120, as desired. □

The hypothesis in Theorem 1.2.6 that ensures that B is the discriminant with respect to z of a generic ternary quartic is necessary. Indeed, for $d = 4$, the variety $\mathcal{V}_4$ is a hypersurface of $\mathbb{P}^{12}$. Hence, B is not a generic degree 12 binary form but it must be contained in $\mathcal{V}_4$. The degree of this hypersurface was determined by Vakil in the following result.

Theorem 1.2.7. *[122, Theorem 6.2] The degree of the hypersurface $\mathcal{V}_4$ equals 3762.*

From Theorem 1.2.7 we derive the following corollary.

Corollary 1.2.8. *If we prescribe 11 general branch points on the line $\mathbb{P}^1$ then the number of complex quartics A such that $\mathrm{discr}_z(A)$ vanishes at these points is equal to $120 \cdot 3762 = 451440$.*

Proof. Consider the space $\mathbb{P}^{12}$ of binary forms of degree 12. Vanishing at 11 general points defines a line in $\mathbb{P}^{12}$. That line meets the hypersurface $\mathcal{V}_4$ in 3762 points. By Theorem 1.2.6, each of these points in $\mathcal{V}_4 \subset \mathbb{P}^{12}$ has precisely 120 preimages A in $\mathbb{P}^{14}$ under the map (1.3). $\square$

Remark 1.2.9. *It was claimed in [99, equation (5.14)] and [101, page 608] that $\mathfrak{h}_3$ is equal to $120 \cdot (3^{10} - 1)/2 = 3542880$. That claim is not correct. The factor $(3^{10} - 1)/2$ is not needed.*

Remark 1.2.10. *We also verified that $\mathcal{V}_4$ has degree 3762, namely by solving 12 random affine-linear equations on the parametrization (1.13). The common Newton polytope of the resulting polynomials has normalized volume 31104. This is the number of paths tracked by the polyhedral homotopy in `HomotopyContinuation.jl`. We found $22572 = 3762 \times 6$ complex solutions. The factor 6 arises because U_2 and U_3 can be multiplied by roots of unity.*

Algorithm 1 describes the numerical methods of reconstructing 120 plane quartics from their branch locus.

Algorithm 1

Input: pair (U_4, U_6) where U_i is a binary form of degree i.

Output: 120 ternary quartics that approximate the 120 quartics in L_4 whose discriminant equals $U_4^3 + U_6^2$.

1. Construct the system of equations given by

$$\begin{array}{rcl} A_1A_3 - 4A_0A_4 - \frac{1}{3}A_2^2 & = & U_2 \\ A_1^2A_4 + A_0A_3^2 - \frac{8}{3}A_0A_2A_4 - \frac{1}{3}A_1A_2A_3 + \frac{2}{27}A_2^3 & = & U_3 \end{array} . \tag{1.14}$$

2. The system (1.14) has $12 = 5 + 7$ equations in the 12 unknown coefficients of $A \in L_4$. Compute the 120 complex solutions, which can be found easily with `HomotopyContinuation.jl` [17].

We finish this section introducing the notion of real plane Hurwitz numbers.

Definition 1.2.11. *We say that an orbit of the action of $\mathcal{G}$ on $\mathbb{P}^{\binom{d+2}{2}-1}$ is* **real** *if it contains a degree d form A with real coefficients. Analogously, a point in $\mathcal{PH}_d$ is real if the corresponding $\mathcal{G}$–orbit is real. We denote the real points of $\mathcal{V}_d$ by $\mathcal{V}_d(\mathbb{R})$. Given $B \in \mathcal{V}_d(\mathbb{R})$, we define the* **real plane Hurwitz number***, denoted by $\mathfrak{h}_d^{\mathrm{real}}(B)$, as the number of real points of $\mathcal{PH}_d$ in the fiber of B via the branching morphism.*

For instance, in the recovery method shown in Example 1.1.5 for $d = 2$, we see that if B is a real branch locus, then the unique $\mathcal{G}$–orbit in the fiber of B is also real. Therefore, $\mathfrak{h}_2^{\mathrm{real}}(B) = 1$ for every real branch locus B.

Along this section we have shown the usefulness of the normal forms in (1.7) and (1.10) for performing numerical computation. Another advantage of both normal forms is that the they read when a $\mathcal{G}$–orbit is real or not.

Proposition 1.2.12. *For a generic $A \in \mathbb{P}^{\binom{d+2}{2}}$, the $\mathcal{G}$–orbit of A is real if and only if its intersection with the projective subspaces (1.7) ((1.10) respectively) is a real point.*

Proof. Let $A \in \mathbb{P}^{\binom{d+2}{2}}$ such that the $\mathcal{G}$–orbit of A is real. We can assume that A is a real polynomial. By Theorem 1.2.1, there exists a unique $g \in \mathcal{G}$ such that gA is in L_d. Moreover, in the proof of Theorem 1.2.1, g is written by means of the coefficients of A. Since these coefficients are real, gA is a real polynomial. Therefore, the intersection of L_d and the $\mathcal{G}$–orbit of A is real point. The converse follows from the fact that the intersection of L_d and the $\mathcal{G}$–orbit of A is an element of the orbit. The proof for the normal form (1.10) follows from the same reasoning. □

We use Proposition 1.2.12 to compute numerically the possible real plane Hurwitz numbers for cubics and quartics. The numerical computations that we ran using `HomotopyContinuation.jl` suggested that $\mathfrak{h}_3^{\mathrm{real}}(B) = 8$ for any real branch locus B. In the case of plane quartics, the numerical experiments suggested that for $B \in \mathcal{V}_4(\mathbb{R})$, the real plane Hurwitz number $\mathfrak{h}_4^{\mathrm{real}}(B)$ is 8, 16, 24, 32, 64 or 120. These computations serve as an inspiration for the theoretical proofs of these values carried out in Sections 1.3 and 1.5.2.

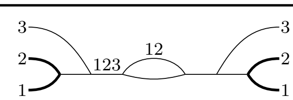
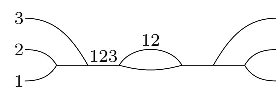
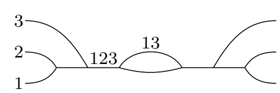
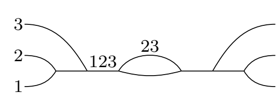
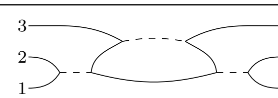
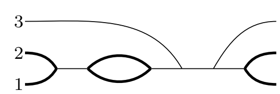
1.3 Hurwitz combinatorics

The theory of Hurwitz numbers is strongly related to combinatorics via the monodromy group. This is exhibited in Lemma 1.1.1, where the computation the Hurwitz numbers H_d is transform into a combinatorial problem. Analogously, Lemma 1.1.2 allows to compute real Hurwitz numbers using combinatorial tools. In this section we focus on the combinatorial features of our problem.

The two cases, we are primary interested in, are $d = 3$ and $d = 4$. From the proofs of Corollary 1.2.5 and Theorem 1.2.6, we infer that these two cases have different behaviors. This is also reflected in the fact that $\mathfrak{h}_3 = H_3 = 40$, but $\mathfrak{h}_4 = 120 < H_4 =$

7528620. The count in Lemma 1.1.1 can be realized by combinatorial objects known as monodromy graphs. These occur in different guises in the literature. Here we use the version that is defined formally in [62, Definition 3.1]. These represent abstract covers in the tropical setting of balanced metric graphs. In the following example we list the 40 monodromy graphs for $d = 3$.

Example 1.3.1 (Forty monodromy graphs)**.** *For $d = 3$, Lemma 1.1.1 yields $H_3 = 40$ six-tuples $\tau = (\tau_1, \tau_2, \ldots, \tau_6)$ of permutations of $\{1, 2, 3\}$, up to the conjugation action by $\mathbb{S}_3$. In Table 1.3 we list representatives for these 40 orbits (see also [100, Table 1]). Each tuple τ determines a monodromy graph as in [22, Lemma 4.2] and [62, Section 3.3]. Reading from the left to right, the diagram represents the cycle decompositions of the permutations $\tau_i \circ \cdots \circ \tau_1$ for $i = 1, \ldots, 6$. For instance, for the first type $\mathcal{A}_1$, we start at* $\mathrm{id} = (1)(2)(3)$*, then pass to* $(12)(3)$*, next to* (123)*, then to* $(12)(3)$*, etc. On the right end, we are back at* $\mathrm{id} = (1)(2)(3)$.

Type	***Real?***	***Six-Tuple*** τ	***Monodromy Graph***	***Clebsch***	$\mathbb{P}^3(\mathbb{F}_3)$
$\mathcal{A}_1$	✓(12)	(12)(13)(13)(13)(13)(12)		123	0010
$\mathcal{A}_2$	✓(12)	(12)(13)(13)(23)(23)(12)		1*a*	0100
$\mathcal{A}_3$	✗	(12)(13)(13)(13)(23)(13)		348	1022
$\mathcal{A}_4$	✗	(12)(13)(13)(13)(12)(23)		357	1012
$\mathcal{A}_{11}$	✗	(12)(13)(13)(23)(12)(13)		7*b*	1102
$\mathcal{A}_{12}$	✗	(12)(13)(13)(23)(13)(23)		4*c*	1201
$\mathcal{A}_5$	✗	(12)(13)(23)(23)(13)(12)		456	1020
$\mathcal{A}_6$	✗	(12)(13)(23)(23)(23)(13)		267	1011
$\mathcal{A}_7$	✗	(12)(13)(23)(23)(12)(23)		168	0012
$\mathcal{A}_{13}$	✗	(12)(13)(23)(12)(23)(12)		1*b*	1100
$\mathcal{A}_{14}$	✗	(12)(13)(23)(12)(12)(13)		7*c*	1201
$\mathcal{A}_{15}$	✗	(12)(13)(23)(12)(13)(23)		4*a*	0101
$\mathcal{A}_8$	✗	(12)(13)(12)(12)(13)(12)		789	1010
$\mathcal{A}_9$	✗	(12)(13)(12)(12)(23)(13)		159	0010
$\mathcal{A}_{10}$	✗	(12)(13)(12)(12)(12)(23)		249	1021
$\mathcal{A}_{16}$	✗	(12)(13)(12)(13)(23)(12)		1*c*	1200
$\mathcal{A}_{17}$	✗	(12)(13)(12)(13)(12)(13)		7*a*	0102
$\mathcal{A}_{18}$	✗	(12)(13)(12)(13)(13)(23)		4*b*	1101
$\mathcal{B}_1$	✓*(id)*	(12)(12)(13)(13)(12)(12)		*base*	1000
$\mathcal{B}_2$	✓*(id)*	(12)(12)(13)(13)(23)(23)		147	0001
$\mathcal{C}_1^\ell$	✓(12)	(12)(12)(12)(13)(13)(12)		2*a*	0110
$\mathcal{C}_2^\ell$	✗	(12)(12)(12)(13)(23)(13)		8*b*	1112
$\mathcal{C}_3^\ell$	✗	(12)(12)(12)(13)(12)(23)		5*c*	1222

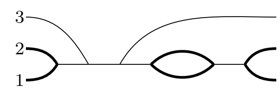
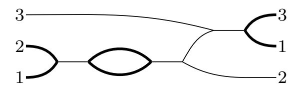
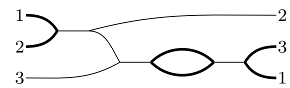
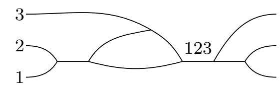
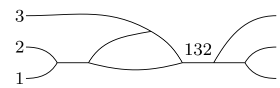
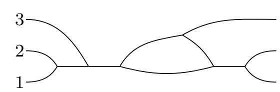
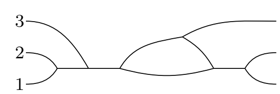
$\mathcal{C}_1^r$	✓ (12)	(12)(13)(13)(12)(12)(12)		$3a$	0120
$\mathcal{C}_2^r$	✗	(12)(13)(23)(13)(13)(13)		$6b$	1121
$\mathcal{C}_3^r$	✗	(12)(13)(12)(23)(23)(23)		$9c$	1211
$\mathcal{D}_1^\ell$	✓ *(id)*	(12)(12)(12)(12)(13)(13)		369	1002
$\mathcal{D}_1^r$	✓ *(id)*	(12)(12)(13)(13)(13)(13)		258	1001
$\mathcal{E}_1^\ell$	✗	(12)(12)(13)(23)(13)(12)		$2b$	1110
$\mathcal{E}_3^\ell$	✗	(12)(12)(13)(23)(23)(13)		$8c$	1221
$\mathcal{E}_5^\ell$	✗	(12)(12)(13)(23)(12)(23)		$5a$	0111
$\mathcal{E}_2^\ell$	✗	(12)(12)(13)(12)(23)(12)		$2c$	1220
$\mathcal{E}_4^\ell$	✗	(12)(12)(13)(12)(12)(13)		$5b$	1111
$\mathcal{E}_6^\ell$	✗	(12)(12)(13)(12)(13)(23)		$8a$	0112
$\mathcal{E}_1^r$	✗	(12)(13)(23)(13)(12)(12)		$3c$	1210
$\mathcal{E}_3^r$	✗	(12)(13)(13)(12)(13)(13)		$6c$	1212
$\mathcal{E}_5^r$	✗	(12)(13)(13)(12)(23)(23)		$9b$	1122
$\mathcal{E}_2^r$	✗	(12)(13)(12)(23)(12)(12)		$3b$	1120
$\mathcal{E}_4^r$	✗	(12)(13)(12)(23)(13)(13)		$6a$	0121
$\mathcal{E}_6^r$	✗	(12)(13)(23)(13)(23)(23)		$9a$	0122

Table 1.3: The first column lists the label given in Example 1.3.1 of the 40 six-tuples of permutations representing $H_3 = 40$. The second column indicates whether the corresponding six-tuple is real or not. The third column corresponds to the sex-tuples of permutations, and in the fourth column the monodromy graphs are indicated. In the case of a real monodromy graph, the coloring is also indicated. The two rightmost columns, labeled **Clebsch** and $\mathbb{P}^3(\mathbb{F}_3)$, will be explained in Section 1.4.

To identify real monodromy representations (see Lemma 1.1.2), we give a coloring as in [62, Definition 3.5]. Using [62, Lemma 3.5] we find eight real covers among the 40 *complex covers. We use [62, Lemma 2.3] to associate the real covers to their monodromy representations.*

We divide the 40 *classes into five types, $\mathcal{A}$ to $\mathcal{E}$, depending on the combinatorial type of the graph. Types $\mathcal{A}$ and $\mathcal{B}$ are symmetric under reflection of the ends, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are not. An upper index ℓ indicates that the cycle of the graph is on the left side of the graph, while r indicates that it is on the right side. The number of classes of each type is the multiplicity in [22, Lemma 4.2] and [100, Table 1]. Each class starts with the real types, if there are any, and proceeds lexicographically in τ. In the table, the edges of the monodromy graphs are labeled by the cycle they represent. If the edge is unlabeled, then the corresponding cycle is either clear from context or varies through all possible cycles in $\mathbb{S}_3$ of appropriate length.*

The combinatorics exposed in Table 1.3 allow us to compute the real Hurwitz number $H_3^{\mathrm{real}}(t)$.

Proposition 1.3.2. $H_3^{\mathrm{real}}(t) = 8$ *for* $t = 6, 4, 2, 0$.

Proof. We prove the proposition by investigating all monodromy representations in Table 1.3. Using explicit computations in `Oscar` [35], we identify all six-tuples τ that satisfy the conditions in Lemma 1.1.2. For a cover with 6 real branch points, we obtain 8 real monodromy representations, of types $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1^l, \mathcal{C}_1^r, \mathcal{D}_1^l$ and $\mathcal{D}_1^r$, listed in Table 1.3 with coloring. For a cover with 4 real branch points and a pair of complex conjugate branch points, we again obtain 8 real monodromy representations. These are the types $\mathcal{A}_3, \mathcal{A}_{12}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_2^l, \mathcal{C}_1^r, \mathcal{D}_1^l$ and $\mathcal{D}_1^r$. For two real branch points and two complex conjugate pairs, we again obtain 8 real monodromy representations, namely of types $\mathcal{A}_9, \mathcal{A}_{12}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{D}_1^l, \mathcal{D}_1^r, \mathcal{E}_3^\ell$ and $\mathcal{E}_1^r$. Finally, for three pairs of complex conjugate branch points, we find the 8 types $\mathcal{A}_5, \mathcal{A}_{17}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{D}_1^l, \mathcal{D}_1^r, \mathcal{E}_3^\ell$ and $\mathcal{E}_5^r$. □

Using Proposition 1.3.2, the real plane Hurwitz number for $d = 3$ are determined.

Corollary 1.3.3. *The real plane Hurwitz number for cubics is equal to eight. To be precise, the system (1.9) always has* 8 *real solutions, provided that the given parameters* β_{ij} *are real and generic.*

Proof. It derives from Corollary 1.2.5 and Proposition 1.3.2. Namely, we use the fact that plane covers are in bijection with abstract covers. Let $C \to \mathbb{P}^1$ be a real cover by an elliptic curve C. The shift of C that is referred to in the proof of Corollary 1.2.5 is real as well.

□

Remark 1.3.4. *Another proof of Corollary 1.3.3, following Clebsch [30, 31], appears in Section 1.4.*

In Algorithm 2 we present a method for: given a real binary sextic B, it computes numerically 40 plane cubics in the normal form of Theorem 1.2.1 with their corresponding labelling according to Table 1.3.

The situation is more interesting for $d = 4$, where we obtained the following result:

Theorem 1.3.5. *The real Hurwitz numbers for degree* 4 *coverings of* $\mathbb{P}^1$ *by genus* 3 *curves are*

$$H_4^{\mathrm{real}}(12) = 20590, \quad H_4^{\mathrm{real}}(10) = 15630, \quad H_4^{\mathrm{real}}(8) = 11110, \quad H_4^{\mathrm{real}}(6) = 7814,$$
$$H_4^{\mathrm{real}}(4) = 5654, \quad H_4^{\mathrm{real}}(2) = 4070, \quad H_4^{\mathrm{real}}(0) = 4350.$$

Proof. These numbers have been found by a direct computation using `Oscar` [35]. The code for this computation is available at the repository website MathRepo [47] of MPI-MiS via the link `https://mathrepo.mis.mpg.de/BranchPoints`. We start by constructing a list of all monodromy representations of degree 4 and genus 3. As monodromy representations occur in equivalence classes, we construct only one canonical representative for each class. This is the element of the equivalence class that is minimal with respect to the lexicographic ordering. We get a list of $H_4 = 7528620$ monodromy representations that was computed in about 6.5 hours. In other words, we embarked on a table just like Table 1.3, but its number of rows is now 7528620 instead of 40.

We next applied Cadoret's criterion in [21, Section 3.3, formula $(\star)$], which we stated in Lemma 1.1.2, to our big table. We start with our 7528620 tuples τ, computed as just described, and mentioned in Lemma 1.1.1. According to Cadoret's criterion, we must check for each 12-tuple τ whether there exists an involution σ that satisfies certain equations in the symmetric group $\mathbb{S}_4$. This depends on the number r of real branch points. Recall that $t = \{0, 2, 4, \ldots, 12\}$. For $t = 2, 4, \ldots, 12$, the only possible involutions σ are id, (12), (34) and (12)(34), by the structure of the canonical representative computed for the list. For $t = 0$, all involutions in $\mathbb{S}_4$ can appear. For each involution σ and each value of t, it took between 5 and 30 minutes to scan our big table, and to determine how many 12-tuples τ satisfy Cadoret's criterion for the pair (t, σ). For each t, we collected the number of tuples τ for which the answer was affirmative. This gave the numbers stated in Theorem 1.3.5. □

Algorithm 2

Input: a binary sextic B with real coefficients.

Output: 40 cubics A in L_3 (normal form of Theorem 1.2.1) along with their labeling by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{E}_6^r$ (See Table 1.3).

1. Compute numerically the 40 solutions of the system (1.9) using `HomotopyContinuation.jl`.

2. Fix 6 loops $\gamma_1, \ldots, \gamma_6$ in the Riemann sphere around the six roots of B that are compatible with complex conjugation. If all six roots of B are real then we use [62, Construction 2.4].

3. For each cubic A computed in the first step, we track numerically the three roots z of $A(x, y, z) = 0$ as $(x : y)$ cycles along γ_i. The resulting permutation of the three roots is the transposition $\tau_{A,i}$.

4. The label of the cubic A is the transposition $\tau_A = \tau_{A,1} \cdot \tau_{A,6}$. This transposition is unique up to conjugacy by $\mathbb{S}_3$. The 8 real cubics A are mapped to the 8 real monodromy representations, in the proof of Proposition 1.3.2.

5. Return the list of tuples (A, τ_A) where A is a cubic computed in the first step.

1.4 Plane cubics

In this section we study in further detail the case of plane cubics. In other words, we focus on the branching morphism $\mathrm{br} : \mathcal{PH}_3 \dashrightarrow \mathbb{P}^6$. This morphism is dominant and a generic fiber consists of 40 distinct points. In this section we study the Galois group of this map as well as the recovery algorithm for cubics.

We start by focusing in the the Galois group Gal_3 associated with the branching morphism for cubics. Here the Galois group of br is defined as in [65]. Informally, Gal_3 is the subgroup of geometry-preserving permutations of the $\mathfrak{h}_3$ solutions.

Theorem 1.4.1. *The Galois group* Gal_3 *for cubics is the simple group of order* 25920*, namely*

$$\mathrm{Gal}_3 \;=\; \mathrm{SU}_4(\mathbb{F}_2) \;=\; \mathrm{PSp}_4(\mathbb{F}_3) \;=\; W(E_6)/\pm\,. \tag{1.15}$$

This is the Weyl group of type E_6 modulo its center, here realized as 4×4 matrix groups over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$. The action of Gal_3 *on the* 40 *monodromy graphs in Table 1.3 agrees with the action of the symplectic group on the* 40 *points in the projective space $\mathbb{P}^3$ over $\mathbb{F}_3$.*

Theorem 1.4.1 is a modern interpretation of Clebsch's work [30, 31] on binary sextics. We first learned about the role of the Weyl group in (1.15) through Elkies' unpublished manuscript [45].

Remark 1.4.2. *The last two columns of Table 1.3 identify the* 40 *monodromy representations with $\mathbb{P}^3(\mathbb{F}_3)$ and with Clebsch's* 40 *cubics in [31]. The bijection we give respects the maps $\mathrm{Gal}_3 \simeq \mathrm{PSp}_4(\mathbb{F}_3) \hookrightarrow \mathbb{S}_{40}$. But it is far from unique. The same holds for Cayley's table in [24].*

Proof of Theorem 1.4.1. We consider cubics written in the normal form in (1.10). In other words, we consider cubics of the form $A = z^3 + A_2(x,y)z + A_3(x,y)$. The discriminant of A with respect to z is $\mathrm{discr}_z(A) = 4A_2^3 + 27A_3^2$. Given a binary sextic B, our task is to compute all pairs of binary forms (A_2, A_3) such that $4A_2^3 + 27A_3^2 = B$. This system of equations in the coefficients of A has 40 solutions, modulo the scaling of A_2 and A_3 by roots of unity. Let $U = \sqrt[3]{4} \cdot A_2$ and $V = \sqrt{-27} \cdot A_3$. We must solve the following problem. Given a binary sextic B, compute all decompositions into a binary quadric U and a binary cubic V:

$$B \;=\; U^3 - V^2. \tag{1.16}$$

This is precisely the problem addressed by Clebsch in [30, 31]. By considering the change of his labeling upon altering the base solution, he implicitly determined the Galois group as a subgroup of $\mathbb{S}_{40}$. The identification of this group with $W(E_6)$ modulo its center appears in a number of sources, including [71, 121]. These sources show that Gal_3 is also the Galois group of the 27 lines on the cubic surface. Todd [121] refers to permutations of the 40 Jacobian planes, and Hunt [71, Table 4.1] points to the 40

triples of trihedral pairs. The connection to cubic surfaces goes back to Jordan in 1870, and it was known to Clebsch.
As a subgroup of the symmetric group $\mathbb{S}_{40}$, our Galois group is generated by five permutations $\Gamma_1, \ldots, \Gamma_5$. These permutations correspond to consecutive transpositions $(\gamma_i \gamma_{i+1})$ of the six loops $\gamma_1, \gamma_2, \ldots, \gamma_6$ in Algorithm 2. Each generator is a product of nine 3-cycles in $\mathbb{S}_{40}$. Here are the formulas for $\Gamma_1, \ldots, \Gamma_5$ as permutations of the 40 rows in Table 1.3:

$$\Gamma_1 = (\mathcal{A}_{10}\mathcal{A}_6\mathcal{A}_1)(\mathcal{A}_8\mathcal{A}_7\mathcal{A}_3)(\mathcal{A}_9\mathcal{A}_5\mathcal{A}_4)(\mathcal{A}_{17}\mathcal{A}_{13}\mathcal{A}_{12})(\mathcal{A}_{18}\mathcal{A}_{14}\mathcal{A}_2)(\mathcal{A}_{16}\mathcal{A}_{15}\mathcal{A}_{11})(\mathcal{E}_2^r\mathcal{E}_6^r\mathcal{E}_3^r)(\mathcal{E}_4^r\mathcal{E}_1^r\mathcal{E}_5^r)(\mathcal{C}_3^r\mathcal{C}_2^r\mathcal{C}_1^r)$$

$$\Gamma_2 = (\mathcal{E}_4^\ell\mathcal{A}_{14}\mathcal{A}_{10})(\mathcal{E}_6^\ell\mathcal{A}_{15}\mathcal{A}_9)(\mathcal{E}_2^\ell\mathcal{A}_{13}\mathcal{A}_8)(\mathcal{B}_1\,\mathcal{E}_1^r\mathcal{E}_2^r)(\mathcal{D}_1^r\,\mathcal{C}_2^r\mathcal{C}_3^r)(\mathcal{B}_2\,\mathcal{E}_6^r\mathcal{E}_4^r)(\mathcal{E}_5^\ell\mathcal{A}_7\mathcal{A}_{17})(\mathcal{E}_1^\ell\mathcal{A}_5\mathcal{A}_{16})(\mathcal{E}_3^\ell\mathcal{A}_6\mathcal{A}_{18})$$

$$\Gamma_3 = (\mathcal{C}_3^\ell\,\mathcal{E}_5^\ell\,\mathcal{E}_4^\ell)(\mathcal{C}_1^\ell\,\mathcal{E}_1^\ell\,\mathcal{E}_2^\ell)(\mathcal{C}_2^\ell\,\mathcal{E}_3^\ell\,\mathcal{E}_6^\ell)(\mathcal{A}_{17}\mathcal{A}_{11}\mathcal{A}_{14})(\mathcal{A}_{18}\mathcal{A}_{12}\mathcal{A}_{15})(\mathcal{A}_{16}\mathcal{A}_2\mathcal{A}_{13})(\mathcal{E}_2^r\mathcal{E}_1^r\mathcal{C}_1^r)(\mathcal{E}_4^r\mathcal{C}_2^r\mathcal{E}_3^r)(\mathcal{C}_3^r\mathcal{E}_6^r\mathcal{E}_5^r)$$

$$\Gamma_4 = (\mathcal{D}_1^\ell\,\mathcal{C}_2^\ell\,\mathcal{C}_3^\ell)(\mathcal{E}_6^\ell\,\mathcal{B}_2\,\mathcal{E}_5^\ell)(\mathcal{E}_2^\ell\,\mathcal{E}_1^\ell\,\mathcal{B}_1)(\mathcal{A}_8\mathcal{A}_{16}\mathcal{E}_2^r)(\mathcal{A}_9\mathcal{E}_4^r\mathcal{A}_{17})(\mathcal{E}_3^r\mathcal{A}_3\mathcal{A}_{11})(\mathcal{E}_5^r\mathcal{A}_{12}\mathcal{A}_4)(\mathcal{A}_{15}\mathcal{E}_6^r\mathcal{A}_7)(\mathcal{A}_{13}\mathcal{A}_5\mathcal{E}_1^r)$$

$$\Gamma_5 = (\mathcal{C}_3^\ell\mathcal{C}_2^\ell\mathcal{C}_1^\ell)(\mathcal{E}_4^\ell\mathcal{E}_6^\ell\mathcal{E}_2^\ell)(\mathcal{E}_5^\ell\mathcal{E}_3^\ell\mathcal{E}_1^\ell)(\mathcal{A}_{10}\mathcal{A}_9\mathcal{A}_8)(\mathcal{A}_{17}\mathcal{A}_{18}\mathcal{A}_{16})(\mathcal{A}_4\mathcal{A}_3\mathcal{A}_1)(\mathcal{A}_{11}\mathcal{A}_{12}\mathcal{A}_2)(\mathcal{A}_{14}\mathcal{A}_{15}\mathcal{A}_{13})(\mathcal{A}_7\mathcal{A}_6\mathcal{A}_5)$$

A compatible bijection with the labels of Clebsch [31, Section 9] is given in the second-to-last column in Table 1.3. The last column indices a bijection with the 40 points in the projective space $\mathbb{P}^3$ over the three-element field $\mathbb{F}_3$. This bijection is compatible with the action of the matrix group $\mathrm{PSp}_4(\mathbb{F}_3)$. Under this bijection, the five generators of our Galois group are mapped to matrices of order 3:

$$\Gamma_1 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix}.$$

These are symplectic matrices with entries in $\mathbb{F}_3$, modulo scaling by $(\mathbb{F}_3)^* = \{\pm 1\} = \{1, 2\}$. A computation using GAP [86] verifies that these groups are indeed isomorphic. In the notation of the atlas of simple groups, our Galois group (1.15) is the group $O_5(3)$. $\square$

Remark 1.4.3. *The fundamental group of the configuration space of six points in the Riemann sphere* $\mathbb{P}^1$ *is the braid group* B_6*, which therefore maps onto the finite group* Gal_3*. One can check that the permutations and matrices listed above satisfy the braid relations that define* B_6*:*

$$\begin{cases} \Gamma_i\,\Gamma_{i+1}\,\Gamma_i \;=\; \Gamma_{i+1}\,\Gamma_i\,\Gamma_{i+1} & \text{for } i = 1,2,3,4 \text{ and} \\ \Gamma_i\,\Gamma_j = \Gamma_j\,\Gamma_i & \text{for } |i-j| \geq 2. \end{cases}$$

A first consequence of Theorem 1.4.1 is that our recovery problem for cubics is solvable by radicals.

Corollary 1.4.4. *Given a ternary cubic A, the 39 other solutions (U, V) to the equation (1.16) can be written in radicals in the coefficients of the binary forms A_2 and A_3.*

Proof. We view Gal_3 as a group of permutations on the 40 solutions and we consider the stabilizer subgroup of the given solution. That stabilizer has order $25920/40 = 3 \cdot 216$, and this is the Galois group of the other 39 solutions. It contains the Hesse group $\mathrm{ASL}_2(\mathbb{F}_3)$ as a normal subgroup of index 3. The group $\mathrm{ASL}_2(\mathbb{F}_3)$ is solvable, and has order 216. It is the Galois group of the nine inflection points of a plane cubic [65, Section II.2]. Therefore, the stabilizer is solvable, and hence (U, V) is expressible by radicals over (A_2, A_3). □

Clebsch explains how to write the 39 solutions in radicals in the coefficients of (A_2, A_3). We now give a brief description of his algorithm, which reveals the inflection points of a cubic. The input of the algorithm is a pair (A_2, A_3) where A_i is a binary form of degree i. It represents a cubic A as in Equation (1.10). The output is the other 39 cubics in the normal form (1.10) with same discriminant as A. The algorithm we present works for complex binary forms (A_1, A_2). Nevertheless, we assume that A_1 and A_2 are real. The reason for this assumption is to exhibit where the 8 real plane cubics appear in the algorithm. We set $\tilde{U} = \sqrt[3]{4} \cdot A_2$ and $\tilde{V} = \sqrt{-27} \cdot A_3$. We aim to compute a list of $39 + 1$ all pairs (U, V) that satisfy

$$U^3 - \tilde{U}^3 \;=\; V^2 - \tilde{V}^2. \tag{1.17}$$

Real solutions of (1.9) correspond to pairs (U, V) such that U and iV are real, where $i = \sqrt{-1}$. To solve Equation (1.17), we consider the cubic

$$\mathcal{C} = \left\{ (x : y : z) \in \mathbb{P}^2 : z^3 - 3\tilde{U}z + 2\tilde{V} = 0 \right\}.$$

Then, the nine inflection lines of $\mathcal{C}$ are given by the equation $z = \xi$, where $\xi = \alpha x + \beta y$ is a linear form such that $(\xi^3 - 3\tilde{U}\xi + 2\tilde{V})$ is the cube of a linear form $\eta = \gamma x + \delta y$. The coefficients α and β can be computed from $\tilde{U}$ and $\tilde{V}$ by radicals. Here, the Galois group is $\mathrm{ASL}_2(\mathbb{F}_3)$. Next, we compute (γ, δ) from the equation

$$\eta^3 = \xi^3 - 3\tilde{U}\xi + 2\tilde{V}.$$

For each of the pairs (α, β) above, this system has three solutions $(\gamma, \delta) \in \mathbb{C}^2$. This leads to 27 vectors $(\alpha, \beta, \gamma, \delta)$, all expressed by means of radicals of the coefficients of A_2 and A_3. For each of these tuples we fix $U = \tilde{U} - \xi^2 - \xi\eta - \eta^2$. Then,

$$U^3 - (\tilde{U}^3 - \tilde{V}^2) \;=\; -\frac{3}{4}\left(\eta^3 + 2\eta^2\xi + 2\eta\xi^2 + \xi^3 - 2\eta\tilde{U} - \xi\tilde{U}\right)^2.$$

The square root of the right hand side is a binary cubic V. In particular, each tuple $(\alpha, \beta, \gamma, \delta)$ leads to a pair (U, V) satisfying (1.17). Using this method, we construct

27 solutions to (1.9), three for each inflection point of the curve $\mathcal{C}$. Three of the 9 inflection points are real, and each of them yields one real solution to (1.9).
Finally, we compute the remaining $12 = 39 - 27$ solutions. Among these 12 solutions, 4 are reals. We label the inflection points of $\mathcal{C}$ by $1, 2, \ldots, 9$ such that the following triples are collinear:

$$\underline{123}, 456, 789, \quad \underline{147}, \underline{258}, \underline{369}, \quad 159, 267, 348, \quad 168, 249, 357. \tag{1.18}$$

If $1, 2, 3$ are real and $\{4, 7\}$, $\{5, 8\}$, $\{6, 9\}$ are complex conjugates, then precisely the four underlined lines are real. This labeling agrees with that used by Clebsch in [31, Section 9, page 50].
We now execute the formulas in [31, Section 11, Section 12]. For each of the 12 lines in (1.18), we compute two solutions (U, V) of (1.17). These solutions are expressed rationally by means of the data (α, β) computed above. Each new solution (U, V) arises twice from each triple of lines in (1.18), so we get 12 in total.

1.5 Plane quartics

We now focus on the case of plane quartics. In this setting, the quotient $\mathcal{PH}_4$ has dimension 11 and the branching morphism $\mathrm{br} : \mathcal{PH}_4 \dashrightarrow \mathbb{P}^{12}$ is not dominant as in the cubic case. The closure $\mathcal{V}_4$ of the image of br is a hypersurface of degree 3762 (see [122, Theorem 6.2]). As mentioned in Section 1.1.2, the degree of this map was computed in [122] by Vakil, concluding that the plane Hurwitz number for quartics is $\mathfrak{h}_4 = 120$. The goal of this section is to use Vakil's ideas on [122] to compute the real plane Hurwitz numbers for quartics and to develop an algorithm for recovering 120 degree 4 curves from their branch locus. Moreover, we explore the relation between the real plane Hurwitz number and the number of real roots in the base locus.

1.5.1 Vakil's construction

In this section we summarize Vakil's ideas used to determine $\mathfrak{h}_4$ in [122]. In [122, Section 2], in order to compute $\mathfrak{h}_4$, the author introduces distinct spaces and he studies the connection among them. We now briefly recall these objects.

As in Section 1.2, we consider the weighted projective space $\mathbb{P}(3^5, 2^7)$. We identify the points of $\mathbb{P}(3^5, 2^7)$ with the pair $[U_4(x, y), U_6(x, y)]$, where U_i is a degree i binary form. Let $\mathcal{B}$ be the open subset of $\mathbb{P}(3^5, 2^7)$ consisting of all points $[U_4, U_6]$ such that the degree 12 homogeneous polynomial $U_4(x, y)^3 + U_6(x, y)^2$ has distinct roots. In [122, Theorem 6.1] it is proven that the map

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{V}_4 \subset \mathbb{P}^{12} \\ [U_4, U_6] & \longmapsto & U_4^3 + U_6^2 \end{array}$$

is birational, and we deduce that $\mathcal{B}$ and $\mathcal{V}_4$ are birational. Therefore, we identify a point $[U_4, U_6] \in \mathcal{B}$ with the polynomial $U_4(x,y)^3 + U_6(x,y)^2$.

Recall that a space sextic C is a genus 4 curve in $\mathbb{P}^3$ that is the complete intersection of a quadric cone Q and a cubic surface. Let q be the singular point of Q. The restriction to C of the projection with center q is a degree 3 branched covering of $\mathbb{P}^1$. This covering is given by a line bundle $\mathcal{L}$ with two sections. A frame of C is an isomorphism between $\mathbb{P}(H^0(C,\mathcal{L}))$ and $\mathbb{P}^1$. In other words, a frame of C is map $Q \setminus \{q\} \to \mathbb{P}^1$ which contracts the ruling of Q. Such a map is the composition of the projection with center q and an automorphism of $\mathbb{P}^1$. We consider the space $\mathcal{E}$ of space sextics with a frame. Using that a quadric cone is isomorphic to the weighted projective space $\mathbb{P}(1,1,2)$, we can identify $\mathcal{E}$ with the space of degree 6 curves in $\mathbb{P}(1,1,2)$ defined by $X_2^3 + U_4(X_0,X_1)X_2 + U_6(X_0,X_1)$, where X_0, X_1, X_2 are the coordinates of $\mathbb{P}(1,1,2)$ and $U_i(X_0,X_1)$ are binary forms of degree i. Here the frame is given by the projection $\mathbb{P}(1,1,2) \dashrightarrow \mathbb{P}^1$ sending $[X_0,X_1,X_2]$ to $[X_0,X_1]$. By [122, Theorem 2.12], we have that the map

$$\begin{array}{ccl} \mathcal{B} & \longrightarrow & \mathcal{E} \\ [U_4, U_6] & \longmapsto & \mathbb{V}(X_2^3 + U_4(X_0,X_1)X_2 + U_6(X_0,X_1)) \subset \mathbb{P}(1,1,2) \end{array}$$

is an isomorphism. In particular, $\mathcal{E}$ and $\mathcal{V}_4$ are also birational. The birational morphism between both spaces sends a space sextic C to the branch locus of the degree 3 covering from C to $\mathbb{P}^1$ given by the frame.

At this point, we want to underline the relation between del Pezzo surfaces of degree 1 and elliptic fibrations. For us, a rational elliptic fibration is a flat family of genus 1 reduced curves over $\mathbb{P}^1$ with a section, with smooth total space and, at worst, multiplicative reduction. We refer to [114, Section VII.5] for the definition of multiplicative reduction. Let X be a del Pezzo surface of degree 1. We may construct from X an elliptic fibration as follows. We write X as the blow up of $\mathbb{P}^2$ at 8 points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in general position. Let p_0 be the ninth point in the base locus of the pencil of cubics passing through $\mathcal{P}$, and let X' be the blow up of X at p_0. Then, the map $X' \to \mathbb{P}^1$ defined by the pencil of cubics is a rational elliptic fibration. In this case, the section $\mathbb{P}^1 \to X'$ is given by the exceptional divisor of p_0. Moreover, for generic $\mathcal{P}$, the elliptic fibration has 12 singular fibers and they are all nodal curves. Let $\mathcal{A}$ be the space of such rational elliptic fibrations together with a frame. Here a frame is an isomorphism between $\mathbb{P}^1$ and the pencil of cubic passing through $\mathcal{P}$.

From the above construction we deduce that an elliptic fibration in $\mathcal{A}$ is determined by the data $(\mathcal{P}, u, v, [\lambda,\mu])$ where $\mathcal{P}$ is the set of 8 generic points in $\mathbb{P}^2$, u, v are two cubics vanishing at $\mathcal{P}$ and $[\lambda,\mu]$ is a point in $\mathbb{P}^1$. Here, the frame is the map sending $[1,0]$ to u, $[0,1]$ to v and $[1,1]$ to $\lambda u + \mu v$.

In [122, Theorem 2.12] it is shown that $\mathcal{A}$ is isomorphic to $\mathcal{B}$ and $\mathcal{E}$. The map from $\mathcal{B}$ to $\mathcal{A}$ is given by the Weierstrass model of rational elliptic fibrations (see [95, Section 3]). The map from $\mathcal{A}$ to $\mathcal{E}$ is given as follows. Let $(\mathcal{P}, u, v, [\lambda,\mu])$ be a point in $\mathcal{A}$ and let w be a plane sextic such that $\{u^2, uv, v^2, w\}$ is a basis of the plane sextics vanishing doubly at $\mathcal{P}$. Then, the map

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2 & \longrightarrow & \mathbb{P}(1,1,2) \\ p & \longmapsto & [\lambda u(p), \mu v(p), w(p)] \end{array}$$

is a double cover ramified over the space sextic. The isomorphism between $\mathcal{A}$ and $\mathcal{E}$ sends the elliptic fibration to this space sextic. Note that in this construction, the choice of w does not affect the frame $\mathbb{P}(1,1,2) \to \mathbb{P}^1$. Therefore, different choices of w give isomorphic space sextics with the same frame. Finally, the birational morphism from $\mathcal{A}$ to $\mathcal{V}_4$ sends a rational elliptic fibration to the image of the 12 singular fibers.

We now describe the map from $\mathcal{PH}_4$ to $\mathcal{A}$. Let C be a plane quartic and let $\widetilde{C}$ be the strict transform of Q in $\mathrm{Bl}_p\mathbb{P}^2$. Recall that $p = [0,0,1]$ is the center of the projection. We obtain the elliptic fibration X' as the double cover of $\mathrm{Bl}_p\mathbb{P}^2$ ramified over $\widetilde{C}$. The elliptic fibration $X' \to \mathbb{P}^1$ is given by the composition of the double cover $X' \to \mathrm{Bl}_p\mathbb{P}^2$ with the map $\mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^1$ given by the exceptional divisor. Note that the preimage of a point $q \in \mathbb{P}^1$ is a curve Y in X'. The restriction of the double cover $X' \to \mathrm{Bl}_p\mathbb{P}^2$ to Y is a double cover of $\mathbb{P}^1$ ramified at 4 points. By the Riemann–Hurwitz formula one deduces that the smooth fibers of the fibration must have genus 1.

We now revise Vakil's computation of $\mathfrak{h}_4$.

Theorem 1.5.1. *[122, Proposition 3.1] The map $\mathcal{PH}_4 \to \mathcal{A}$ has degree 120. In particular, $\mathfrak{h}_4 = 120$.*

Proof. Let $(\mathcal{P} = \{p_1, \ldots, p_8\}, u, v, [\lambda, \mu])$ be a point in $\mathcal{A}$ as above. Let p_0 be the ninth point of intersection of u and v and let X' be the blow up of $\mathbb{P}^2$ and $\mathcal{P} \cup \{p_0\}$. The elliptic fibration $X' \to \mathbb{P}^1$ sends a point q to $[\lambda u(q), \mu v(q)]$. Let $E_0, \ldots, E_8$ be the exceptional divisors of $p_0, \ldots, p_8$. Let Q' be the locus of points q in X' that in each fiber of the fibration satisfy $2q = E_0 + E_1$. This equality is considered in the group law of the fiber. The intersection of Q' with each fiber consists of 4 points. By [122, Lemma 2.2], Q' is a smooth curve and the restriction of the fibration to Q' yields to a degree 4 branched covering of $\mathbb{P}^1$ ramified at the 12 singular fibers. In [122, Proposition 3.1], it is shown that the map $Q' \to \mathbb{P}^1$ is canonical. In particular, Q' is not hyperelliptic, and by Riemann-Hurwitz formula we deduce that Q' has genus 3. The canonical model of Q' is a plane quartic in the fiber of the branching morphism.

The only choice involved in the construction of Q' was the selection of the (-1)–curve E_1, which is disjoint from E_0. Recall that E_0 is fixed since it is the section of the elliptic fibration. The number of such (-1)–curves disjoint from E_0 is 240 and they correspond to the (-1)–curves of the del Pezzo surface of degree 1 obtained by blowing down E_0 from X'. In [122, Proposition 3.1] it is shown that two such (-1)–curves give rise to the same plane quartic in $\mathcal{PH}_4$ if and only if they form a Bertini pair. In particular, the points in the fiber of $\mathcal{PH}_4 \to \mathcal{A}$ are in bijection with the set of Bertini pairs in $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$. In Section 1.1.4 we saw that the number of Bertini pairs is 120. $\square$

Let B be a generic branch locus in $\mathcal{V}_4$. Using the birational morphism between $\mathcal{V}_4$ and $\mathcal{E} \simeq \mathcal{A}$, the branch locus B is equivalent to a space sextic C and a rational elliptic fibration $X' \to \mathbb{P}^1$. As above, such elliptic fibration is encoded in the data $(\mathcal{P}, u, v, [\lambda, \mu])$. Then, the proof of Theorem 1.5.1 establishes a bijection between the 120 plane quartics with branch locus B, and the 120 Bertini pairs of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$. In Section 1.1.3, we saw that these are also in bijection with the 120 tritangent planes to C.

1.5.2 Real plane Hurwitz number for quartics

The first step in the study of the quartic case is the computation of the real plane Hurwitz number $\mathfrak{h}_4^{\mathrm{real}}(B)$ for $B \in \mathcal{V}_4(\mathbb{R})$. To determine the value of these numbers, we investigate where the base field plays a role in the proof of Theorem 1.5.1.

Theorem 1.5.2. *Let $B \in \mathcal{V}_4(\mathbb{R})$, then the real plane Hurwitz number $\mathfrak{h}_4^{\mathrm{real}}(B)$ equals* 8*,* 16*,* 24*,* 32*,* 64*, or* 120*.*

Proof. Let B be a generic real branch locus in $\mathcal{V}_4$, and let C be the real space sextic associated to B via the birational morphism between $\mathcal{E}$ and $\mathcal{V}_4$. In the complex setting taking a double cover of $\mathbb{P}(1,1,2)$ branched over C gives a del Pezzo surface of degree one, together with the Bertini involution ι. Over the real numbers we obtain two different del Pezzo surfaces, X and its associated surface X'. As showed in Section 1.1.4, upon base changing to the complex numbers, $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ become isomorphic. The possible topological types of real del Pezzo surfaces have been classified by several authors, we use the work of Russo [109]. Such classification is shown in Table 1.1. Similarly, Table 1.2 records the possible pairs of $\{X, X'\}$ together with the number of real tritangents of C.

Let L be a (-1)–curve in $X_{\mathbb{C}}$. Vakil's construction in Theorem 1.5.1 shows that the plane quartic associated to L in the fiber of B is obtained as follows. Blowing down L on $X_{\mathbb{C}}$ gives a del Pezzo surface of degree 2 which is a double cover of $\mathbb{P}^2$ ramified over the plane quartic we are looking for. The Bertini involution ι of X maps L to a (-1)-curve giving the same quartic, which leads to the bijection between the fiber $\mathrm{br}^{-1}(B)$ and the set of Bertini pairs of X. In the real setting, the pair $\{L, \iota(L)\}$ produces a real quartic curve if and only if is invariant under the complex conjugation on $X_{\mathbb{C}}$. In Section 1.1.4 we called such pair a real Bertini pair. Therefore, the real plane Hurwitz number equals the number of real Bertini pairs on $X_{\mathbb{C}}$. Mapping L to a line in the cone $\mathbb{P}(1,1,2)$ gives a tritangent to C [109, 5, example 4]. In this way real tritangents of C are the same as real Bertini pairs. The number of real tritangent is computed using Theorems 1.1.9 and 1.1.13. Table 1.2 relates these numbers (and hence the real plane Hurwitz numbers) with the real structure of X and X'. □

Remark 1.5.3. *Algorithm 1 is consistent with Theorem 1.5.2. Given binary forms U_2, U_3 with real coefficients, it correctly outputs* 8*,* 16*,* 24*,* 32*,* 64 *or* 120 *real quartics in the subspace L_4.*

Among the values computed in Theorem 1.5.2, we highlight the number 120. There exists $B \in \mathcal{V}_4(\mathbb{R})$ such that $\mathfrak{h}_4^{\mathrm{real}}(B) = \mathfrak{h}_4 = 120$. In particular, the fiber $\mathrm{br}^{-1}(B)$ consists of 120 real plane quartics. For our computational purpose, we are interested in understanding what happens when we replace $\mathbb{R}$ by $\mathbb{Q}$. To do so, we give an effective proof of Theorem 1.5.2.

In Section 1.5.1, we saw that a branch locus $B \in \mathcal{V}_4$ is equivalent to giving a rational elliptic fibration $X \to \mathbb{P}^1$ with a section. Such elliptic fibration is given as the data $(\mathcal{P}, u, v, [\lambda, \mu])$. For simplicity, in the rest of the section we will assume that $[\lambda, \mu] = [1, 1]$, and we represent the corresponding elliptic fibration through the tuple $(\mathcal{P}, u, v)$.

Corollary 1.5.4. *From any general configuration $\mathcal{P} = \{p_1, p_2, \ldots, p_8\} \subset \mathbb{P}^2$ with coordinates in $\mathbb{Q}$ and two cubics u, v with coefficients in $\mathbb{Q}$ vanishing at $\mathcal{P}$, we obtain an instance of (1.11) whose* 120 *complex solutions A all have coefficients in $\mathbb{Q}$.*

Proof. We construct the 120 quartics A with rational arithmetic from the coordinates in $\mathcal{P}$. Fix j and let w be a cubic that vanishes on $\mathcal{P}\backslash\{p_j\}$ but does not vanish at P_j. Consider the rational map

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ q & \longmapsto & [u(p), v(q), w(q)] \end{array}. \tag{1.19}$$

A fiber of (1.19) is described by the intersection two cubics vanishing in $\mathcal{P}\backslash\{p_j\}$. These two cubics intersect at 9 points: $\mathcal{P} \setminus \{p_j\}$ and two extra points which are the fiber of (1.19). Thus, the map (1.19) has degree 2 and its base locus is $\mathcal{P}\backslash\{p_j\}$. Blowing up this base locus we obtained the degree 2 cover of $\mathbb{P}^2$ from a del Pezzo surface of degree 2. The branch locus of this cover map is a quartic curve. Indeed, let L be a generic line in $\mathbb{P}^2$. The fiber of L through (1.19) is a plane smooth cubic E. The restriction of (1.19) to E is a degree 2 simple branched covering of $\mathbb{P}^1$. By the Riemann-Hurwitz formula we deduce that this branched covering has 4 branch points. We conclude that L intersect the branch locus of (1.19) at four points, and hence, this branch locus is a plane quartic.
We claim that this is the quartic Q that corresponds to the exceptional fiber of the blow-up at P_j in Vakil's construction. Indeed, let X' be the blowup of $\mathbb{P}^2$ at all the 9 base points of the pencil $[u, v]$ and let $C \subseteq X'$ be a general curve in the pencil. As shown in the proof of Theorem 1.5.1, the intersection of the Q and C is the set of points $p \in C$ such that $\mathcal{O}_C(2p) \cong \mathcal{O}_C(E_0 + E_1)$. Alternatively, these are exactly the branch points of the map $C \to \mathbb{P}^1$ induced by the linear system of $\mathcal{O}_C(E_0 + E_1)$. Thus, it is enough to prove that the restriction of the map $[u, v, w]\colon X' \to \mathbb{P}^2$ is given exactly by the linear system above. However, the map of $[u, v, w]$ on X' corresponds to the complete linear system $L = 3H - E_2 - \cdots - E_8 = -K_X + E_0 + E_1$ so that $\mathcal{O}_C(L) \cong \mathcal{O}_C(-K_X + E_0 + E_1)$ and we need to show that $\mathcal{O}_C(-K_X) \cong \mathcal{O}_C$. This follows from the adjunction formula, using the fact that C is an elliptic curve moving in a pencil.

Thus, we can recover the quartic as the branch locus of our degree 2 rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The ternary form A that defines this branch locus can be computed from the ideals of $\mathcal{P}$ and P_j using Gröbner bases, and this uses rational arithmetic over the ground field. This yields eight of the 120 desired quartics corresponding to the eight exceptional divisors. The remaining 112 quartics are found by repeating this process after some Cremona transformations have been applied to the pair $(\mathbb{P}^2, \mathcal{P})$. These transformations use only rational arithmetic and they preserve the del Pezzo surface, but they change the collection of eight (-1)-curves that are being blown down. We provide more details on the application of Cremona transformations in Section 1.5.3. $\square$

1.5.3 Algorithm for plane quartics

The proofs presented in the previous section provide an algorithm for our recovery problem. Recall that, given a branch locus $B \in \mathcal{V}_4$, we want to compute 120 plane quartics representing the fibers of the branching morphism. As explained in Section 1.5.2, a branch locus $B \in \mathcal{V}_4$ is equivalent to give 8 plane points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in general position and a basis of the pencil of cubics vanishing at $\mathcal{P}$. The input of our algorithms will consist on the data given by the last. In [26], it is explored how to move from $B \in \mathcal{V}_4$ to our input and vice-versa.

We start by fixing $\mathcal{P} = \{p_1, \ldots, p_8\}$ to be the set of 8 plane points in general position and two u, v cubic forms vanishing at $\mathcal{P}$. These two cubics form a basis of the pencil of cubics passing through $\mathcal{P}$. Let p_0 be the ninth point in the base locus of this pencil, and let X be the blow up of $\mathbb{P}^2$ at $\mathcal{P}$. In the poof of [122, Prposition 3.1], it is shown that the 120 quartics we aim to compute are in bijection with the 120 Bertini pairs of X. In Theorem 1.1.16, we listed all the Bertini pairs of X. From this list, the 120 (-1)–curves of the form E_i for $i \in [8]$, $L_{i,j}$ for $i < j \in [8]$, $Q_{i,j,k}$ for $i < j < k \in [8]$ and $C_{i,j}$ for $i < j \in [8]$, are representative of the 120 Bertini pairs. The aim of this section is to provide effective methods to compute the plane quartics associated to each of the above 120 (-1)–curves. This is done in Algorithms 3, 4, 5 and 6.

First, we focus on the plane quartics associated to the exceptional divisors of X. The computation of such plane quartics was exhibited in Corollary 1.5.4. For $i \in [8]$, we consider the linear system Λ_i of cubics vanishing at $\mathcal{P} \setminus \{p_i\}$. This linear system has (affine) dimension 3. Let w be a cubic vanishing at $\mathcal{P} \setminus \{p_i\}$ and not at p_i. Then, u, v, w generate Λ_i, and we get a rational map

$$\begin{array}{cccc} \varphi_i : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & [x, y, z] & \mapsto & [u(x, y, z), v(x, y, z), w(x, y, z)] \end{array}.$$

The rational map φ_i has degree 2 and it is ramified over the plane quartic C, which is the curve we aim to compute. Let $[r, s, t]$ be the plane coordinates of the target projective space of φ. A point $q = [q_0, q_1, q_2] \in \mathbb{P}^2$ is a ramification point of φ if and only if the Jacobian matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix} \tag{1.20}$$

vanishes at q. This allow us to compute the plane quartic C by computing the image through φ of the degree 9 plane curve defined by the determinant of the matrix (1.20). In particular, we can determine C using Gröbner bases. We now summarize the steps of such computation.

Algorithm 3

Input: The set of 8 plane points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in general position, an index $i \in [8]$ and two cubic forms u, v vanishing at $\mathcal{P}$.

Output: A plane quartic corresponding to the exceptional divisor of p_i.

1. Compute a cubic form vanishing at $\mathcal{P} \setminus \{p_i\}$ and not at p_i.
2. Compute the saturation J of the ideal
$$I \;=\; \left\langle 2 \times 2 \text{ minors of } \begin{pmatrix} u & v & w \\ r & s & t \end{pmatrix} \right\rangle \;+\; \left\langle \det \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix} \right\rangle$$
by the ideal of $\mathcal{P} \setminus \{p_i\}$, which is generated by u, v, w.
3. Determine the ideal K obtained by eliminating the variables x, y, z from J.
4. Return the quartic form in the variables r, s, t generating the ideal K.

The following example illustrate the steps of Algorithm 3.

Example 1.5.5. *Consider the* 8 *plane points*

$$\begin{array}{llll} p_1 = [1,0,0], & p_2 = [0,1,0], & p_3 = [0,0,1], & p_4 = [1,1,1], \\ p_5 = [1,2,3], & p_6 = [1,3,2], & p_7 = [1,-1,-2], & p_8 = [1,-2,-1], \end{array}$$

together with the two cubics

$$u = 24xy^2 + 7x^2z - 56xyz + y^2z + 25xz^2 - yz^2,$$
$$v = 24x^2y - 25x^2z + 8xyz - 7y^2z - 7xz^2 + 7yz^2.$$

One can check that u and v vanish at $\mathcal{P} = \{p_1, \ldots, p_8\}$. The ideal of $\mathcal{P} \setminus \{p_1\}$ is generated by the cubics u, v and

$$w := 168x^3 - 119x^2z - 80xyz + 7y^2z - 17xz^2 + 41yz^2.$$

Note that the cubic w does not vanish at p_1. Let I be the ideal as in Step 2 of Algorithm 3, and let J be the ideal obtained saturating I by the ideal generated by u, v, w. This saturation is needed because eliminating the variables x, y, z from the ideal I leads to the zero ideal. A computation in `Macaulay2` *shows that the elimination of the variables x, y, z from the ideal J is generated by the quartic form*

$$49r^4 + 252r^3s + 2676r^2s^2 + 22848rs^3 + 32256s^4 - 784r^3t - 4500r^2st - 2808rs^2t + 1344s^3t + 750r^2t^2 + 228rst^2 - 300s^2t^2 - 16rt^3 - 12st^3 + t^4.$$

The plane quartic defined by this polynomial corresponds to the exceptional divisor of p_1.

As before, we fix a branch locus B in the form of a blow up X of 8 plane points $\mathcal{P} = \{p_1, \ldots, p_8\}$ together with two cubics forms u, v vanishing at $\mathcal{P}$. Algorithm 3 provides a method for computing 8 plane quartics corresponding to the 8 exceptional divisors. We now provide a method for computing the 132 remaining plane quartics. The strategy is to use Cremona transformations to find automorphisms of the del Pezzo surface X exchanging the (-1)–curves. Recall that a Cremona transformation is a birational automorphism of a projective space $\mathbb{P}^k$. We are interested in plane Cremona transformations, i.e. birational maps from $\mathbb{P}^2$ to $\mathbb{P}^2$. For instance, in Section 1.1.4 we introduced the Bertini involution of the plane, which is a Cremona transformation of $\mathbb{P}^2$ that lifts to an automorphism of X.

After a change of coordinates, we can assume that the points p_1, p_2, p_3, p_4 are equal to $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$, respectively. Now we consider the rational map

$$\phi: \begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ [x, y, z] & \longmapsto & [yz, xz, xy] \end{array}. \tag{1.21}$$

The base locus of ϕ consists of the points p_1, p_2, p_3. Moreover, ϕ^2 sends $[x, y, z]$ to $[x^2yz, xy^2z, xyz^2]$. In particular, ϕ^2 is the identity away from the lines L_{12}, L_{13} and L_{23}; here L_{ij} is the line passing through p_i and p_j. Hence, ϕ is a rational involution and in particular it is a Cremona transformation. Note that we can build such a Cremona transformation for any 3 points in $\mathbb{P}^2$ not lying in a line. Given 3 points q_1, q_2, q_3 not collinear, we denote the corresponding Cremona transformation by ϕ_{q_1,q_2,q_3}. Using this notation, we have that $\phi = \phi_{p_1,p_2,p_3}$.

Now we consider the set of 8 plane points $\mathcal{P}' = \{p_1, p_2, p_3, \phi(p_4), \ldots, \phi(p_8)\}$. The pullback ϕ^*u and ϕ^*v of u and v through ϕ are degree 6 forms. Since u, v vanishes at p_1, p_2, p_3, we get that ϕ^*u and ϕ^*v are the multiplication of xyz and a cubic form u' and v' respectively.

Proposition 1.5.6. *The elliptic fibrations given by the data $(\mathcal{P}, u, v)$ and $(\mathcal{P}', u', v')$ are isomorphic.*

Proof. By construction, ϕ commutes with the maps from $\mathbb{P}^2$ to $\mathbb{P}^1$ given by u, v and u', v' respectively. Hence, it is enough to check that the ϕ lifts to an isomorphism, between $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ and $\mathrm{Bl}_{\mathcal{P}'}\mathbb{P}^2$. Let $\pi : Y \to \mathbb{P}^2$ be the blow up map of $\mathbb{P}^2$ at p_1, p_2, p_3 and let ϕ' be the composition $\phi \circ \pi$. Then, ϕ' is the resolution of the birational map ϕ. Moreover, the preimage of p_1, p_2 and p_3 are the strict transformations of L_{23}, L_{13} and L_{12} respectively. Therefore, $\phi' : Y \to \mathbb{P}^2$ is also the blow up map of $\mathbb{P}^2$ at p_1, p_2, p_3. The proof follows by blowing up the remaining 5 points of $\mathcal{P}$ and $\mathcal{P}'$ □

The difference between the two elliptic fibrations in Proposition 1.5.6 is that the exceptional divisor p_1, p_2, p_3 in $\mathrm{Bl}_{\mathcal{P}'}\mathbb{P}^2$ correspond to the (-1)–curves $\tilde{L}_{23}, \tilde{L}_{13}, \tilde{L}_{12}$ in $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ respectively. Therefore, by applying Algorithm 3 to $(\mathcal{P}', u', v')$ and the points p_1, p_2, p_3 we obtain the plane quartics associated to the (-1)–curves $\tilde{L}_{23}, \tilde{L}_{13}, \tilde{L}_{12}$. We now summarize the steps of the algorithm for computing the plane quartics corresponding to a (-1)–curve of the form $\tilde{L}_{ij}$.

Algorithm 4

Input: The set of 8 plane points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in general position, two distinct indexes $i, j \in [8]$ and two cubic forms u, v vanishing at $\mathcal{P}$.

Output: A plane quartic corresponding to the (-1)–curve $\tilde{L}_{ij}$ of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$.

1. Fix $k \in [8]$ such that $k \neq i, j$. Compute an automorphism g of $\mathbb{P}^2$ sending p_i,p_j,p_k to $[1, 0, 0], [0, 1, 0], [0, 0, 1]$, respectively.
2. Compute the set $\mathcal{P}' = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \cup \phi \circ g(\mathcal{P} \setminus \{p_i, p_j, p_k\})$.
3. Compute the two cubics u' and v' resulting from dividing $u \circ g^{-1} \circ \phi$ and $v \circ g^{-1} \circ \phi$ by xyz, respectively.
4. Return the output of Algorithm 3 applied to the 8 points $\mathcal{P}'$, the index of the point $[0, 0, 1]$ and the two cubics u', v'.

Example 1.5.7. *Consider the set of 8 points $\mathcal{P}$ and the two cubics u and v as in Example 1.5.5. Our goal is to apply Algorithm 4 to compute the plane quartic associated to the (-1)–curve $\tilde{L}_{2,3}$. To do so, we consider the Cremona transformation ϕ defined in (1.21). Applying ϕ to $\mathcal{P}$ as in Step 2 of Algorithm 4 we obtained the set $\mathcal{P}'$ consisting of the following 8 points*

$$\begin{array}{llll} p'_1 = [1, 0, 0], & p'_2 = [0, 1, 0], & p'_3 = [0, 0, 1], & p'_4 = [1, 1, 1], \\ p'_5 = [6, 3, 2], & p'_6 = [6, 2, 3], & p'_7 = [2, -2, -1], & p'_8 = [2, -1, -2]. \end{array}$$

The two new cubics u' and v' obtained by diving $u(yz, xz, xy)$ and $v(yz, xz, xy)$ by xyz are

$$u' = x^2y - 25xy^2 - x^2z + 56xyz - 7y^2z - 24xz^2,$$
$$v' = 7x^2y - 7xy^2 - 7x^2z + 8xyz - 25y^2z + 24yz^2.$$

Applying Algorithm 3 to $\mathcal{P}'$, u', v' and the point p'_1 we get the quartic form

$$65513r^4 - 848876r^3s + 3598518r^2s^2 - 5075756rs^3 + 1945673s^4 + 423360r^3t - 2927232r^2st + 5164992rs^2t - 2101248s^3t + 883008r^2t^2 - 1040256rst^2 + 841536s^2t^2 + 193536rt^3 - 27648st^3.$$

The plane quartic defined by the above polynomial corresponds to the (-1)–curve $\tilde{L}_{23}$.

Using Algorithm 4, we can compute the plane quartics corresponding to the 28 (-1)–curves of the form $\tilde{L}_{ij}$ for $i,j \in [8]$ distinct. We now present how to compute the plane quartics corresponding to (-1)–curves of the form Q_{ijk}. Let $\tilde{\phi}$ be isomorphism between $X = \mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$ and $X' = \mathrm{Bl}_{\mathcal{P}'}\mathbb{P}^2$ arising from the Cremona transformation ϕ. We have seen that $\tilde{L}_{12}$ is sent via $\tilde{\phi}$ to the exceptional divisor of p_3 in X'. As Example 1.5.7 shows, this allows us to apply Algorithm 3 to obtain three more plane quartics. In order to achieve all the 120 quartics, we analyse how $\tilde{\phi}$ acts on the (-1)–curves up to the Bertini involution. In Section 1.1.4, we listed the 120 (-1)–curves up to the Bertini involution. The notation we use for the (-1)–curves of X is the same one used in Section 1.1.4, whereas the (-1)–curves of X' will be denoted by adding a tilde. For instance, the exceptional divisors of X' are $E'_1, \dots, E'_8$. The structure of the Picard group of X and X' is well know (see [91, Section 25]). It is generated by the hyperplane class H and the 8 exceptional divisors. The isomorphism $\tilde{\phi}$ leads to an isomorphism $\tilde{\phi}^*$ between Pic$X$ and PicX' given by

$$\begin{array}{rcl} H & \longmapsto & 2H - E'_1 - E'_2 - E'_3 \\ E_1 & \longmapsto & \tilde{L}'_{23} = H - E'_2 - E'_3 \\ E_2 & \longmapsto & \tilde{L}'_{13} = H - E'_1 - E'_3 \\ E_3 & \longmapsto & \tilde{L}'_{12} = H - E'_1 - E'_2 \\ E_i & \longmapsto & E'_i \text{ for } i \geq 4 \end{array} . \tag{1.22}$$

Using Equation (1.22), we determine how $\tilde{\phi}$ acts on the set of (-1)–curves. For instance, for $i \geq 4$ we get that

$$\tilde{\phi}^*(\tilde{L}_{1i}) = 2H - E'_1 - E'_2 - E'_3 - (H - E'_2 - E'_3) - E'_i = H - E'_1 - E'_i = \tilde{L}'_{1i}.$$

Similarly, for $i,j \geq 4$ distinct, we have that

$$\tilde{\phi}^*(\tilde{L}_{ij}) = 2H - E'_1 - E'_2 - E'_3 - E'_i - E'_j = Q'_{[8]\setminus\{1,2,3,i,j\}}.$$

Since ϕ is a birational involution, we obtain that $\tilde{\phi}^*(Q_{i,j,k}) = \tilde{L}'_{[8]\setminus\{1,2,3,i,j,k\}}$. Therefore, using two Cremona transformations and Algorithm 3 we can compute the plane quartics corresponding to the (-1)–curves of the form Q_{ijk}. We now present the steps of the algorithm for computing these plane quartics.

Algorithm 5

Input: The set of 8 plane points $\mathcal{P} = \{p_1, \dots, p_8\}$ in general position, three distinct indexes $i, j, k \in [8]$ and two cubic forms u, v vanishing at $\mathcal{P}$.

Output: A plane quartic corresponding to the (-1)–curve Q_{ijk} of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$.

1. Fix $\{a, b, c, d, e\} = [8] \setminus \{i, j, k\}$. Compute an automorphism g of $\mathbb{P}^2$ sending p_a,p_b,p_c to $[1, 0, 0], [0, 1, 0], [0, 0, 1]$, respectively.

2. Compute the set
$$\mathcal{P}' = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \cup \phi \circ g(\mathcal{P} \setminus \{p_a, p_b, p_c\}).$$

3. Compute the two cubics u' and v' resulting from dividing $u \circ g^{-1} \circ \phi$ and $v \circ g^{-1} \circ \phi$ by xyz, respectively.

4. Return the plane quartics obtained by applying Algorithm 4 to the 8 points $\mathcal{P}'$, the two indexes d, e and the two cubics u', v'.

Example 1.5.8. *Consider the 8 points $\mathcal{P}$ and the cubics u and v as in Example 1.5.5. We aim to compute the plane quartic corresponding to the (-1)–curve $Q_{4,5,6}$. The first three steps of Algorithm 5 corresponds to the computation of $\mathcal{P}'$, u' and v' as in Example 1.5.7. The remaining step is to apply Algorithm 4 to the 8 points $\mathcal{P}'$, the cubics u' and v' and the indexes 7 and 8. To do so, we consider the change of coordinates g given by the matrix*
$$g = \begin{pmatrix} 3 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$
Note that
$$g(p_1') = [1, 0, 0], g(p_7') = [0, 1, 0] \text{ and } g(p_8') = [0, 0, 1].$$
We compute the 8 points $\tilde{\mathcal{P}}$ obtained from applying $\phi \circ g$ to $\mathcal{P}'$ as in Step 2 of Algorithm 4. These eight points are
$$\tilde{p}_1 = [1, 0, 0], \quad \tilde{p}_2 = [-2, 2, -4], \quad \tilde{p}_3 = [-2, -4, 2], \quad \tilde{p}_4 = [1, -7, -7],$$
$$\tilde{p}_5 = [4, -28, -112], \quad \tilde{p}_6 = [4, -112, -28], \quad \tilde{p}_7 = [0, 1, 0], \quad \tilde{p}_8 = [0, 0, 1].$$
The two cubics $\tilde{u}$ and $\tilde{v}$ obtained from applying the third step of Algorithm 4 are
$$\tilde{u} = 217x^2y - 5xy^2 + 224x^2z + 84xyz + y^2z - 16xz^2 - yz^2,$$
$$\tilde{v} = 7x^2y + 37xy^2 + 56x^2z + 12xyz + 7y^2z - 40xz^2 - 7yz^2.$$

Applying Algorithm 3 to the 8 points $\tilde{\mathcal{P}}$, the cubics $\tilde{u}$ and $\tilde{v}$, and the index $i = 1$ we obtain the quartic form

$$8222942755r^4 + 10151459084r^3s + 3099430482r^2s^2 + 4136946188rs^3+ \\ 2381975203s^4 - 1367877024r^3t - 17339917536r^2st - 22632159456rs^2t- \\ 4083767712s^3t + 1907639424r^2t^2 + 9072601344rst^2+ \\ 4457669760s^2t^2 - 795764736rt^3 - 1130194944st^3 + 80621568t^4.$$

defining the plane quartic corresponding to the (-1)–curve $Q_{4,5,6}$.

Algorithm 5 allows us to compute the 56 plane quartics corresponding to the 56 (-1)–curves of the form $Q_{i,j,k}$ for $i, j, k \in [8]$ distinct. Together with Algorithms 3 and 4, we can compute 92 of the 120 plane curves. The remaining 28 plane quartics can be similarly obtained using the fact that $\tilde{\phi}^*(C_{1,i}) = Q'_{23i}$ for $i \geq 4$. In particular, we can compute the plane quartic associated to a (-1)–curve of the form $C_{i,j}$ by applying a Cremona transformation and Algorithm 5. We present now the steps of this recovery algorithm.

Algorithm 6

Input: The set of 8 plane points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in general position, two distinct indexes $i, j \in [8]$ and two cubic forms u, v vanishing at $\mathcal{P}$.

Output: A plane quartic corresponding to the (-1)–curve Q_{ijk} of $\mathrm{Bl}_{\mathcal{P}}\mathbb{P}^2$.

1. Fix $k, m \in [8] \setminus \{i\}$. Compute an automorphism g of $\mathbb{P}^2$ sending p_i,p_k,p_m to $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ respectively.

2. Compute the set
$$\mathcal{P}' = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \cup \phi \circ g(\mathcal{P} \setminus \{p_a, p_b, p_c\}).$$

3. Compute the two cubics u' and v' resulting from dividing $u \circ g^{-1} \circ \phi$ and $v \circ g^{-1} \circ \phi$ by xyz respectively.

4. Return the plane quartics obtained by applying Algorithm 5 to the 8 points $\mathcal{P}'$, the two indexes j, k, m and the two cubics u', v'.

Using Algorithms 3, 4, 5 and 6 we can recover from a base locus B, the 120 plane quartics corresponding to the 120 fibers of the branching morphism. We present a summary of the steps of the recovery algorithm.

Algorithm 7

Input: The set of 8 plane points $\mathcal{P} = \{p_1, \dots, p_8\}$ in general position and two cubic forms u, v vanishing at $\mathcal{P}$.

Output: A list of 120 plane quartic corresponding to the 120 points in the fibers of the branching morphism.

1. For every $i \in [8]$, compute the plane quartic obtained by applying Algorithm 3 to $\mathcal{P}$, i and u, v.
2. For every $i, j \in [8]$ distinct, compute the plane quartic obtained by applying Algorithm 4 to $\mathcal{P}$, i, j and u, v.
3. For every $i, j, k \in [8]$ distinct, compute the plane quartic obtained by applying Algorithm 5 to $\mathcal{P}$, i, j, k and u, v.
4. For every $i, j \in [8]$ distinct, compute the plane quartic obtained by applying Algorithm 6 to $\mathcal{P}$, i, j and u, v.
5. Return the list of the 120 plane quartics computed in the previous steps.

1.5.4 Real branch locus and construction of examples

In our recovery problem for plane quartics, two different integer numbers related with real geometry arise. One of them is the real plane Hurwitz number $\mathfrak{h}_d^{\text{real}}(B)$ for a given real branch locus B, whose possible values were computed in Theorem 1.5.2. The second value is the number of real points in B, which will be denoted by $t(B)$. Note that $t(B) \in \{0, 2, 4, 6, 8, 10, 12\}$ since B is a degree 12 form.

In this section we study the relation between this two numbers. For instance, we deal with questions as: can it happen that $\mathfrak{h}_4^{\text{real}}(B) = 64$ and $t = 0$? What about $\mathfrak{h}_4^{\text{real}}(B) = 64$ and $t = 12$?. The following result gives a complete description of which pair $(\mathfrak{h}_4^{\text{real}}(B), t(B))$ can happen for $B \in \mathcal{V}_4(\mathbb{R})$.

Theorem 1.5.9. *There is no $B \in \mathcal{V}_4(\mathbb{R})$ such that the tuple $(\mathfrak{h}_4^{\text{real}}(B), t(B))$ corresponds to red entry in Table 1.4. For each green entry (h, t) of Table 1.4, there exists $B \in \mathcal{V}_4(\mathbb{R})$ such that the tuple $(\mathfrak{h}_4^{\text{real}}(B), t(B)) = (h, t)$.*

Proof. Let B be a real branch locus in $\mathcal{V}_4$, and assume that $\mathfrak{h}_4^{\text{real}}(B) \neq 24$. By Theorem 1.5.2 and Table 1.2, we can assume that the real structure of the corresponding del Pezzo surface $X_{\mathbb{R}}$ comes from the blow up of 8 points $\mathcal{P}$, t real and $4 - t/2$ complex conjugates pairs. The 12 points in $\mathbb{P}^1$ which are zeros of B correspond to the singular fibers under the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 : [x, y, z] \mapsto [u, v]$. This is the map given by two cubics u,v through $\mathcal{P}$. Algebraically, we seek points $(\alpha : \beta) \in \mathbb{P}^1$ such that the cubic $\beta \cdot u - \alpha \cdot v$

is singular. There are 12 such rational cubics, and the t real roots of B correspond to those rational cubics that are real.

The *Welschinger invariant* $W_r(\mathbb{P}^2, 3)$ is defined as the signed count of real rational cubics passing through t real points and $4-r$ pairs of complex conjugate points. By definition, $W_r(\mathbb{P}^2, 3) \leq t$. Example 4.3 in [113] provides a tropical computation of the Welschinger invariants for cubics and yields $W_r(\mathbb{P}^2, 3) = 2r$. We conclude that $2r \leq t$.

Assume now that $\mathfrak{h}_4^{\mathrm{real}}(B) = 24$ and let C be the space sextic associated to B. We distinguish two cases: C has 3 big ovals or 1 big oval and 2 small. Let Q be the real cone containing C and let q be its unique singular point, which is real. The restriction of the projection $\pi_q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center q to C gives a degree 3 simple covering of $\mathbb{P}^1$ ramified over B. Each root of B corresponds to a tangent line of C passing through q. Assume now that C has two small ovals. Then, each small oval has two real tangents passing through q. Therefore, $t \geq 4$. Now assume that C has 3 ovals. If t is positive, then there exists a real line L passing through q that it is tangent to C. Since L is real, it intersects each big oval of C with multiplicity at most 1. In particular, L intersects C in at most 4 points counted with multiplicity. This is a contradiction since $\pi_q|_C$ is a degree 3 cover. We conclude that $t = 0$.

It remains to prove that the rest of the possible cases happen. The rest of this section will be devoted to find examples for each of these cases. $\square$

			t						
$X_{\mathbb{R}}$	Ovals	$\mathfrak{h}_4^{\mathrm{real}}$	0	2	4	6	8	10	12
$\mathbb{P}^2_{\mathbb{R}}(0,8)$	1 big	8							
$\mathbb{P}^2_{\mathbb{R}}(2,6)$	1 big + 1 small	16							
$\mathbb{P}^2_{\mathbb{R}}(4,4)$	1 big + 2 small	32							
$\mathbb{P}^2_{\mathbb{R}}(6,2)$	1 big + 3 small	64							
$\mathbb{P}^2_{\mathbb{R}}(8,0)$	1 big + 4 small	120							
$\mathbb{D}_4(3,0)^1_2$	3 big	24							
$\mathbb{D}_4(3,0)^0_3$	1 big + 1 + 1 small	24							

Table 1.4: The first column of the table represents the real del Pezzo surface associated to a real branch locus B. The second column is the type of ovals of the space sextic associated to B, and the third column corresponds to the real plane Hurwitz number. Finally, t denotes the number of real roots of B.

The goal for the rest of this section is to find examples for each green entry of the Table 1.4. Our first approach for finding these examples was to randomly sample point in $\mathcal{V}_4$ and use the numerical methods described in Section 1.2 to compute the corresponding values of $\mathfrak{h}_4^{\mathrm{real}}$ and t. However, this method failed since most of the cases with $\mathfrak{h}_4^{\mathrm{real}} \geq 24$ were not obtained. A better result was achieved after sampling the points of $\mathcal{V}_4$ as points $(U_4, U_6) \in \mathbb{P}(3^5, 2^7)$ such that U_4 and U_6 have only real roots.

Unfortunately, using this sampling method there were still green cases of Table 1.4 that were not achieved. We now present two methods for constructing examples using the real algebraic geometry of plane quartics and space sextics, respectively.

The first approach for finding examples relies on the real structure of plane quartics and their relation with elliptic fibrations. Let B be a real branch locus in $\mathcal{V}_4$ with t real points and let X be the corresponding del Pezzo surface. Assume that the real structure of X is $X_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}}(2a, 2b)$ for $2a + 2b = 8$. In other words, X is the blow up of 8 generic points $\mathcal{P} = \{p_1, \ldots, p_8\}$ in the plane, where $2a$ points are real and there are b conjugate pairs of points. This places us in the first four rows of Table 1.4. Let p_0 be the ninth point in the base locus of the pencil of cubic vanishing at $\mathcal{P}$. Since $\mathcal{P}$ is real, p_0 is real. The blow up of $\mathbb{P}^2$ at $\mathcal{P} \cup \{p_0\}$ is an elliptic fibration denoted by X'. Note that X' inherits the real structure from the blow up. Therefore, the topological Euler characteristic of $X'_{\mathbb{R}}(\mathbb{R})$, as a real manifold, is

$$\mathcal{X}^{\text{top}}(X'_{\mathbb{R}}(\mathbb{R})) = \mathcal{X}^{\text{top}}(\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})) - 2a = -2a. \tag{1.23}$$

Among the 12 singular fibers of the elliptic fibration, t of them are real nodal plane cubics.

Definition 1.5.10. *We say that a real nodal fiber is hyperbolic if around the singularity, the real points of the curve look like the real points of the plane curve $x^3 - y^2 + x^2 = 0$ around the origin. In other words, locally the singularity looks like the left curve in Figure 1.2. Analogously, we say that a real nodal fiber is elliptic if around the singularity, the real points of the curve look like the real points of the plane curve $x^3 - y^2 - x^2 = 0$ around the origin. This local picture is illustrated in the right curve in Figure 1.2.*

We denote the number of hyperbolic and elliptic fibers $X'_{\mathbb{R}}$ by h and e. Since every real singular fiber of $X'_{\mathbb{R}}$ is either elliptic or hyperbolic, we conclude that

$$t = e + h. \tag{1.24}$$

The hyperbolic and elliptic fibers have topological Euler characteristic -1 and 1 respectively, and we get that

$$-2a = e - h. \tag{1.25}$$

From Equations (1.24) and (1.25) we deduce that we can compute t and a from e and h. By Theorem 1.5.2 and Table 1.2, we can recover $\mathfrak{h}_4^{\text{real}}$ from a. Therefore, we can compute the tuple $(t, \mathfrak{h}_4^{\text{real}})$ from e and h.

Figure 1.2: The left hand side illustrates the local picture of a real hyperbolic nodal singularity. The figure on the right hand side illustrates the local picture of a real elliptic nodal singularity.

We now explore the relation between e and h and the real geometry of plane quartics. Let Q be a real plane quartic in the fiber of B through the branching morphism. Recall that the twelve points of B correspond with the 12 tangent lines of Q passing through the point of projection $p = [0, 0, 1]$. These tangent lines can be of two types.

Definition 1.5.11. *Let F be the real quartic form defining Q and assume that $F(p) > 0$. Given $q \in Q(\mathbb{R})$, we say that the tangent line T_qQ is* hyperbolic *if the restriction of F to $T_qQ(\mathbb{R})$ is nonnegative around q. Similarly, we say that the tangent line T_qQ is* elliptic *if the restriction of F to $T_qQ(\mathbb{R})$ is nonpositive around q. In Figure 1.3 we illustrate in green and blue a hyperbolic and elliptic tangent, respectively.*

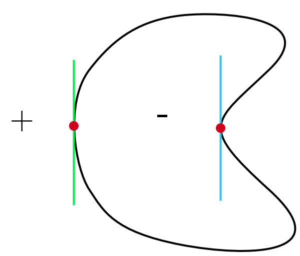


Figure 1.3: Picture of the real points of a real plane curve whose defining polynomial is negative in the bounded region and positive in the unbounded region. The green tangent line is a hyperbolic tangent. The blue tangent line is elliptic.

Lemma 1.5.12. *Let $C = \mathbb{V}(F)$ be a generic real plane quartic such that $F(p) > 0$. Let $X' \to \mathbb{P}^1$ be the elliptic fibration corresponding to the branch locus of C. Then, the number h of hyperbolic singular fibers of X' equals the number of hyperbolic tangent lines of C through p. Analogously, the number e of elliptic singular fibers of X' equals the number of elliptic tangent lines of C through p.*

Proof. We start recalling how to obtain the elliptic fibration X' from C. We consider the blow up X of $\mathbb{P}^2$ at the projection point $p = [0, 0, 1]$. We denote the strict transformation of C again by $\tilde{C}$. We obtain the elliptic fibration X' as the double cover of X ramified over $\tilde{C}$. The elliptic fibration $X' \to \mathbb{P}^1$ is given by the composition of the double cover $X' \to X$ with the map $X \to \mathbb{P}^1$ given by the exceptional divisor of X.

Now, let L be a real tangent line of C at $q \in C$ passing through p. Then, L corresponds to a real singular fiber E of the elliptic fibration and we get a double cover $\varphi : E \to L$. Let q' be the singular point of E, i.e. $q' = \varphi^{-1}(q)$. Now assume that L is elliptic. Then, there exists an open neighbourhood U of q in L such that q is the only ramification point in U and $F|_{U(\mathbb{R})}$ is nonpositive. Then, $\varphi^{-1}(U)$ is an open neighbourhood of q' whose only real point is q'. We deduce that the singularity at q' is elliptic. One may apply the same argument to show that hyperbolic tangent lines to C passing through p correspond to hyperbolic singularities. □

Our strategy is to use Lemma 1.5.12 to construct examples corresponding to green entries of the Table 1.4. In [63], Harnack degenerates singular plane quartics to constructs plane curves of given genus with a prescribed number of ovals (see [10, 125] for a modern reference). For instance, the same strategy is followed in [9] for the construction of the Trott curve. We use this classical real algebro-geometric technique to construct our examples. We exhibit this method in the following example.

Example 1.5.13. *Consider the two polynomials*

$$\begin{aligned}\tilde{F}_1 &= \frac{1}{4}(0.707x - 0.707z)^2 + (0.707z + 0.707x)^2 - y^2,\\ \tilde{F}_2 &= \frac{1}{7}(0.866z - 0.5x)^2 + (0.5x + 0.866z)^2 - y^2,\end{aligned} \tag{1.26}$$

and let $\tilde{F}$ be the product of $\tilde{F}_1$ and $\tilde{F}_2$. The plane quartic $\tilde{C}$ defined by $\tilde{F}$ has two irreducible components, each of them has degree 2 (see Figure 1.4). As in the construction of the Trott curve, we slightly modify our curve adding a degree 4 non-negative polynomial:

$$F = \tilde{F} + \frac{5}{10^4}\, y^2(x^2 + z^2). \tag{1.27}$$

Let C be the plane curve defined by F. Then, C is a smooth curve with 12 real tangent lines passing through $[0, 0, 1]$. Ten of them are hyperbolic and two of them are elliptic (see Figure 1.4). From equations (1.24) and (1.25) we deduce that $\mathfrak{h}_4^{\mathrm{real}} = 120$ and $t = 12$.

Similarly, in Table 1.5 we provide pictures of the plane quartics corresponding to the examples of the first five rows of Table 1.4. These examples have been constructed using the same method exhibited in Example 1.5.13: we obtain a smooth plane quartic by deforming a reducible plane quartic. The first two columns of Table 1.5 represent the real plane Hurwitz number and the number of real roots of the branch locus. The

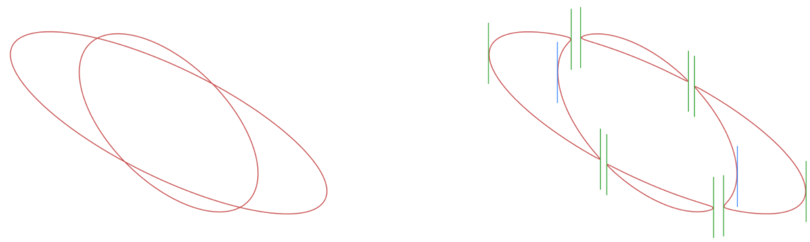
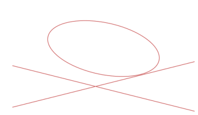
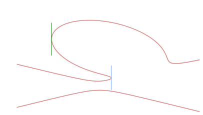
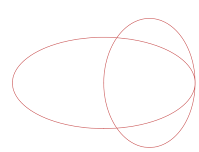
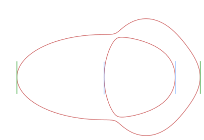
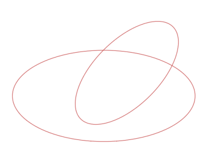
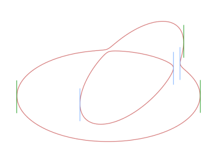
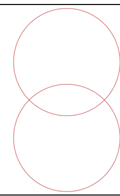
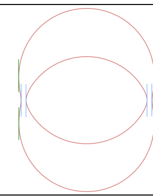
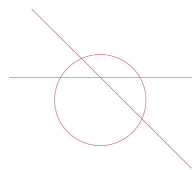
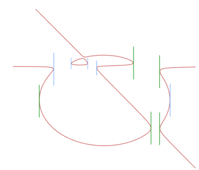
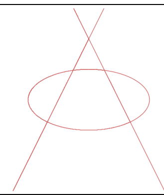
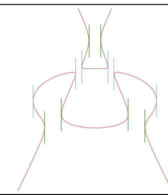
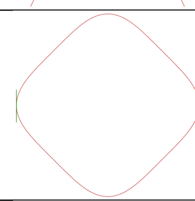
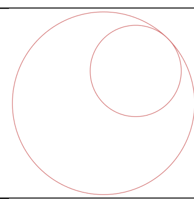
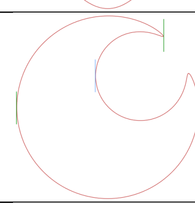
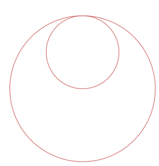
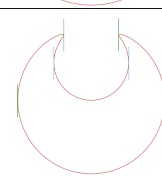
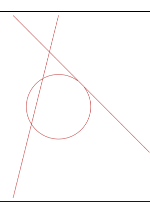
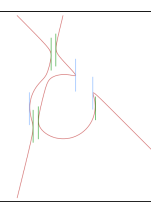
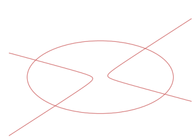
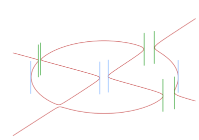
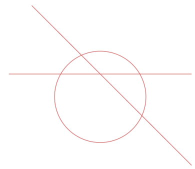
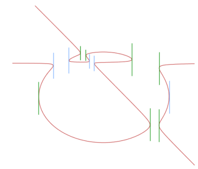
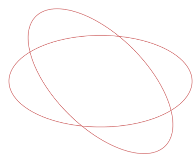
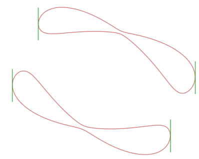
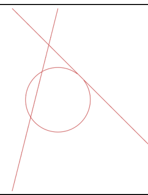
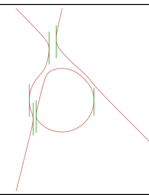
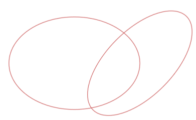
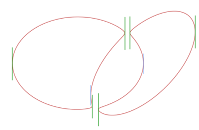
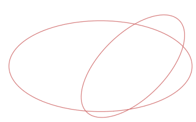
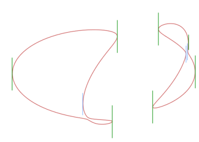
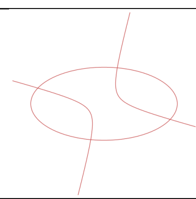
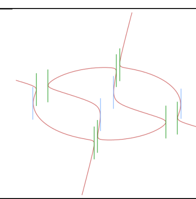
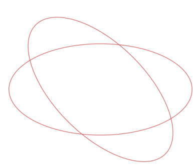
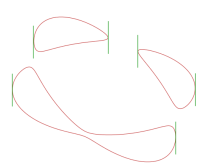
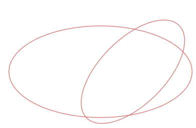
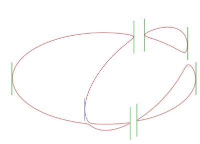


Figure 1.4: The curve on the left is the real picture of the plane quartic defined by the product of the polynomials in (1.26). The curve on the right is the real picture of the plane quartic defined by the polynomial (1.27). The tangent lines in green represent the hyperbolic tangent lines passing through the point of projection. The tangent lines in blue are elliptic. It has ten hyperbolic tangents and two elliptic.

third and fourth columns represent the number of hyperbolic and elliptic real tangents to the plane quartic. The fifth column of Table 1.5 shows the picture of the reducible real curve we start with. The last column corresponds to the smooth real plane quartic obtained by deforming the reducible curve on the fifth column. The values $\mathfrak{h}_4^{\mathrm{real}}, t, h, e$ of these plane quartics are given by the corresponding entries of the row. In the figures in the last row, the tangent lines in green are the hyperbolic tangents passing through the projection point, and the ones in blue correspond to the elliptic tangents.

$\mathfrak{h}_4^{\mathrm{real}}$	t	h	e	Singular quartic	Plane quartic
8	2	1	1		
8	4	2	2		
8	6	3	3		

8	8	4	4		
8	10	5	5		
8	12	6	6		
16	2	2	0		
16	4	3	1		
16	6	4	2		
16	8	5	3		
16	10	6	4		

16	12	7	5		
32	4	4	0		
32	6	5	1		
32	8	6	2		
32	10	7	3		
32	12	8	4		
64	6	6	0		
64	8	7	1		

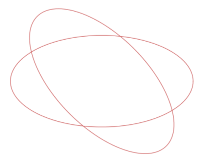
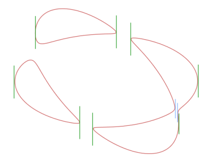
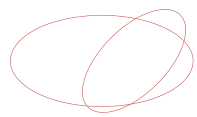
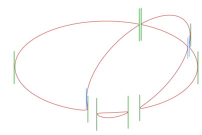
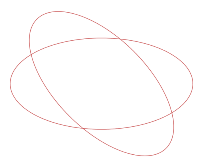
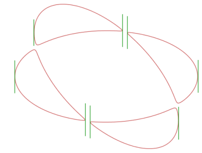
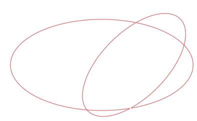
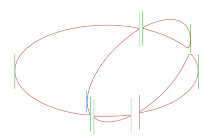
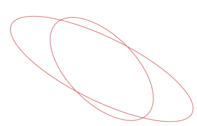
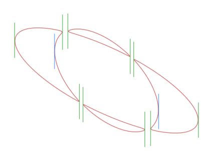
64	10	8	2		
64	12	9	3		
120	8	8	0		
120	10	9	1		
120	12	10	2		

Table 1.5: List of examples corresponding to the first 5 rows of Table 1.4. The first and second columns correspond to the real plane Hurwitz number and number of real points in the branch locus. The third and fourth columns are the number of hyperbolic and elliptic tangents. The fifth column corresponds to the reducible plane quartic we degenerate to obtain the smooth plane quartic. The last column provides the picture of the real plane quartic whose values of $\mathfrak{h}_4^{\mathrm{real}}, t, h, e$ are given by the first fourth columns.

Note that the previous method for constructing examples uses that the real structure of the del Pezzo surface coming from the blow up of real points and complex conjugate points. Therefore, this method can not provide examples where the real plane Hurwitz number is 24. This corresponds to the last two rows of Table 1.4. We use the real geometry of space sextics to construct those examples.

Recall that the real Hurwitz number of a real branch locus can be obtained through the oval distribution of corresponding real space sextic (see Table 1.4). Let $B \in \mathcal{V}_4(\mathbb{R})$ and let C be a real space sextic lying on the real quadric cone Q. Let q be the unique singular point of Q. Recall that the restriction to C of the linear projection

$\pi_q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from q is a degree 3 simple branched cover of $\mathbb{P}^1$ ramified over B. The real points of B correspond to real tangent lines to C passing through C. In particular, we can compute $\mathfrak{h}_4^{\text{real}}$ and t from the real picture of the space sextic. In the following example we illustrate how to do so.

Example 1.5.14. *Let x_0, x_1, x_2, x_3 be the coordinates of $\mathbb{P}^3$. We fix Q to be the quadric cone defined by the polynomial $G = x_0^2 + x_1^2 - 4x_2^2$. The singular point of Q is $q = [0,0,0,1]$ and the linear projection $\pi_q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ sends $[x_0, x_1, x_2, x_3]$ to $[x_0, x_1, x_2]$. Now, $Q' = \pi_q(Q)$ is the smooth plane quadric defined by G which is isomorphic to $\mathbb{P}^1$. The set of real points of $Q \setminus \{q\}$ equals the cylinder $\mathbb{R} \times \mathbb{P}^1_{\mathbb{R}}$. The blue surface in Figure 1.1a is a picture of the real points of $Q \setminus \{q\}$. The fibers of the projection π_q corresponds to vertical lines in the cylinder in Figure 1.5a. Consider the space sextic C given by the equations*

$$x_2^3 - x_0x_2x_3 - x_1x_3^2 = x_0^2 + x_1^2 - 4x_3^2 = 0.$$

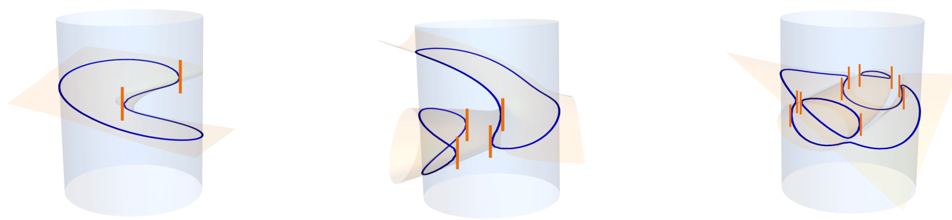
The real picture of this space sextic is illustrated in Figure 1.5a. Moreover, from the picture one can observe that C has two tangent lines passing through the singular point of the quadric surface. Hence, it corresponds to the case $\mathfrak{h}_4^{\text{real}} = 8$ and $t = 2$ in Table 1.4. Similarly, Figure 1.1b corresponds to the case $\mathfrak{h}_4^{\text{real}} = 16$ and $t = 4$, Figure 1.5c corresponds to the case $\mathfrak{h}_4^{\text{real}} = 32$ and $t = 10$, Figure 1.5d corresponds to the case $\mathfrak{h}_4^{\text{real}} = 64$ and $t = 10$, Figure 1.5e corresponds to the case $\mathfrak{h}_4^{\text{real}} = 120$ and $t = 8$, and Figure 1.5b corresponds to the case $\mathfrak{h}_4^{\text{real}} = 24$ and $t = 4$. Some of these examples were constructed using the same method used for constructing the examples in Table 1.5, but for space sextics. We refer to [48, Figure 1] for an application of such methods to space sextics.

We now focus on the case $\mathfrak{h}_4^{\text{real}} = 24$ and $t = 12$. The idea to find such an example is to interpret the cubic surface that intersects the quadric cone as a family of real plane cubics such that two of the fibers of the family are real plane cubics with 6 tangent lines through $[0,0,1]$. We consider the cubic surface Y given by the affine equation

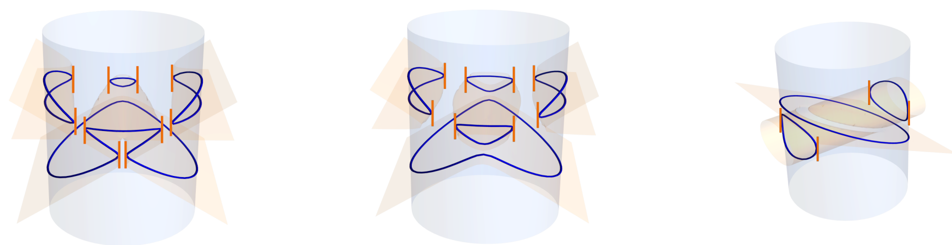
$$(x_1^2 + x_2^2 - 1)\left(x_2 - \frac{1}{3}x_0\right) + \frac{x_0}{1000}(x_1^2 + x_2^2) = 0. \tag{1.28}$$

We see Y a family of plane cubics in the parameter x_0. The fiber Y_{x_0} at $x_0 = -1.5$ and $x_0 = 1.5$ has 6 tangent lines through $[0,0,1]$ each (see Figure 1.6). Note that, for each value of $x_0 \neq 0$, the plane cubic is constructed applying the same technique used in Example 1.5.13. We start with a reducible cubic that is the union of a conic and a line, and we add a extra factor to correct the singularities.

Let C be the space sextic given by the intersection of Y with $\mathbb{V}(x_0^2 + x_1^2 - 4)$. The real picture of C is shown in Figure 1.7. The number tangent lines through $[0,0,0,1]$ is 12. These 12 vertical tangent lines arise from the six vertical tangent lines to $Y_{1.5}$ and $Y_{-1.5}$ (see Figure 1.6). Hence, in this case, $\mathfrak{h}_4^{\text{real}} = 24$ and $t = 12$.



(a) Case $(\mathfrak{h}_4^{\mathrm{real}}, t) = (8, 2)$. (b) Case $(\mathfrak{h}_4^{\mathrm{real}}, t) = (16, 4)$. (c) Case $(\mathfrak{h}_4^{\mathrm{real}}, t) = (32, 10)$.



(d) Case $(\mathfrak{h}_4^{\mathrm{real}}, t) = (64, 10)$. (e) Case $(\mathfrak{h}_4^{\mathrm{real}}, t)$=$(120, 10)$. (f) Case $(\mathfrak{h}_4^{\mathrm{real}}, t)$=$(24, 4)$.

Figure 1.5: Real pictures of a space sextic obtained by intersecting the affine cone x^2+y^2-1, in light blue, with a cubic surface in light orange. The real vertical tangent lines of the space sextic are illustrated in orange.

(a) Plane cubic Y_{x_0} for $x_0 = 1.5$. (b) Plane cubic Y_{x_0} for $x_0 = -1.5$.

Figure 1.6: Plane cubics obtained by fixing the value of x_0 in (1.28).

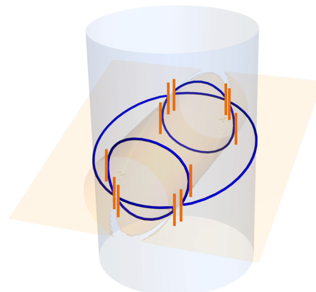


Figure 1.7: Real picture of the space sextic given by the Equations (1.28). It corresponds to the case $\mathfrak{h}_4^{\text{real}} = 24$ and $t = 6$.

Using the same strategy one can construct the remainder cases of the last row of Table 1.4. In Figure 1.8, we present the real pictures of space sextics corresponding to these cases.

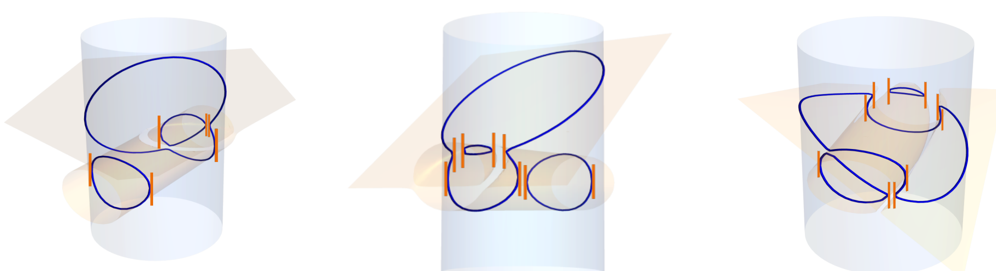


Figure 1.8: Real pictures of space sextics corresponding to the cases $t = 6, 8, 10$ (sorted from left to right) of the last row of Table 1.4.

1.6 Segre-Hurwitz numbers

The restriction of the Hurwitz theory to plane curves gives rise to the notion of plane Hurwitz numbers. As shown in [99, Proposition 5.2.1], the plane Hurwitz numbers is compatible with the classical Hurwitz numbers. This restriction allowed us to analyze the recovery problem associated to these enumerative problems. A natural question arises from the construction of plane Hurwitz numbers: what does it happen if, instead of the plane curves, we consider curves contained on other algebraic varieties? The theme of this section is to develop an analogous theory to plane Hurwitz numbers for curves lying on the Segre variety $S := \mathbb{P}^1 \times \mathbb{P}^1$.

Let $[x_0, x_1]$ and $[y_0, y_1]$ be the coordinates of the first and second factor of S, respectively. In addition, we consider the two projections $\pi_1, \pi_2 : S \to \mathbb{P}^1$ sending $([x_0, x_1], [y_0, y_1])$ to $[x_0, x_1]$ and $[y_0, y_1]$, respectively. For $d_1, d_2 \geq 0$, we denote the vector space of bi-homogeneous polynomials of bi-degree (d_1, d_2) by V_{d_1,d_2}. In other words,

$$V_{d_1,d_2} := H^0(S, \mathcal{O}_S(d_1, d_2)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_2)).$$

Thus, the dimension of V_{d_1,d_2} is $(d_1 + 1)(d_2 + 1)$. Similarly, $|\mathcal{O}_S(d_1, d_2)|$ denotes the projectivitation of V_{d_1,d_2}. Let C be a curve in S defined by a polynomial in $|\mathcal{O}_S(d_1, d_2)|$. A computation on the Chow ring of S together with the adjunction formula shows that the genus g of C is $g = (d_1 - 1)(d_2 - 1)$.

Given $F \in |\mathcal{O}_S(d_1, d_2)|$ generic, the restriction of π_2 to $\mathbb{V}(F)$ is a degree d_1 branched covering of $\mathbb{P}^1$. Indeed, given a point $p = [p_0, p_1] \in \mathbb{P}^1$, the fiber through $\pi_2|_{\mathbb{V}(F)}$ is given by the solutions to the equation $F(x_0, x_1, p_0, p_1) = 0$, which has d_1 solutions. Since F is chosen generic, we assume that this branched covering is simple. By the Riemann–Hurwitz formula, the number of branch points of such covering is $2d_2(d_1 - 1)$. Similar to the construction presented in Section 1.1.2, we consider the branching morphism

$$\mathrm{br} : |\mathcal{O}_S(d_1, d_2)| \dashrightarrow \mathbb{P}^{2d_2(d_1-1)} \tag{1.29}$$

sending $F \in |\mathcal{O}_S(d_1, d_2)|$ to the branch locus of $\pi_2 : \mathbb{V}(F) \to \mathbb{P}^1$. In order to give an algebraic expression of the map (1.29), we write a polynomial $F \in |\mathcal{O}_S(d_1, d_2)|$ as

$$F = \sum_{i=0}^{d_1} F_i(y_0, y_1) x_0^{d_1-i} x_1^i,$$

where F_i are homogeneous polynomials of degree d_2 in y_0, y_1. A point $q \in \mathbb{P}^1$ is a branch point if and only if the polynomial $F(x_0, x_1, q_0, q_1)$ has a double root. We can translate this condition algebraically using discriminants. The discriminant $\mathrm{disc}_{x_0}F$ of $F(x_0, 1, y_0, y_1)$ w.r.t. x_0 is a homogeneous polynomial of degree $2d_2d_1 - d_2$ in the variables y_0 and y_1. Thus, if $q \in \mathbb{P}^1$ is a branch point, then $\mathrm{disc}_{x_0}F(q)$ vanishes. Note that the number of branch points is $2d_2d_1 - 2d_2$ while $\mathrm{disc}_{x_0}F$ has degree $2d_2d_1 - d_2$. This means that $\mathbb{V}(\mathrm{disc}_{x_0}F)$ is the union of the branch locus of π and d_2 extra points. Note that $\mathrm{disc}_{x_0}F$ is the multiplication of F_0 with a polynomial $2d_2(d_1-1)$. We denote this polynomial by ΔF. In particular, the d_2 extra points in $\mathbb{V}(\mathrm{disc}_{x_0}F)$ are given by the roots of F_0, and the branch locus of π_2 is given by ΔF. Therefore, algebraically, the branching morphism is the rational map sending F to ΔF.

Example 1.6.1. *Fix $d_1 = d_2 = 2$. We write $F \in |\mathcal{O}_S(2, 2)|$ as $F = F_0x_0^2 + F_1x_0x_1 + F_2x_1^2$, where F_i are binary quadrics in the variables y_0, y_1. The restriction of π_2 to $\mathbb{V}(F)$ is a degree 2 simple branched covering of $\mathbb{P}^1$ ramified over 4 points. The discriminant of $F(x_0, 1, y_0, y_1)$ w.r.t. x_0 equals $F_0(F_1^2 - 4F_0F_2)$ and the branch locus of π_2 is given by the roots of $F_1^2 - 4F_0F_2$. Thus, the branching morphism sends F to $F_1 + 4F_0F_2$. A computation in `Macaulay 2` shows that this map is dominant.*

In the definition of plane Hurwitz numbers, we considered plane curves up to the action of the group $\mathcal{G}$, defined as the group of automorphisms $\mathbb{P}^2$ commuting with the linear projection. In this section, we are interested in the group of automorphisms of S commuting with the projection π_2. Such group is PGL(2) acting on S through the first factor. In other words, $g \cdot (p, q) = (g(p), q)$ for $g \in \mathrm{PGL}(2)$ and $(p, q) \in S$. This induces an action of PGL(2) on $|\mathcal{O}_S(d_1, d_2)|$ as follows:

$$g \cdot F = F(ax_0 + bx_1, cx_0 + dx_1, y_0, y_1) \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2) \text{ and } F \in |\mathcal{O}_S(d_1, d_2)|.$$

Since PGL(2) is reductive, we get a good quotient $\mathcal{SH}_{d_1,d_2} := |\mathcal{O}_S(d_1, d_2)|//\mathrm{PGL}(2)$ (see [94, 68]). Now, note that the map (1.29) is invariant under the action of PGL(2). Therefore, we obtain a map

$$\begin{array}{rccc} \mathrm{br}: & \mathcal{SH}_{d_1,d_2} & \dashrightarrow & \mathbb{P}^{2d_2(d_1-1)} \\ & [F] & \longmapsto & \Delta F, \end{array} \tag{1.30}$$

called the **branching morphism**. We denote the closure of the image of br by $\mathcal{V}_{d_1,d_2}$. For instance, from Example 1.6.1 we deduce that $\mathcal{V}_{2,2} = \mathbb{P}^4$. Next, we consider the morphism

$$\begin{array}{rccc} \iota: & \mathcal{SH}_{d_1,d_2} & \dashrightarrow & \mathcal{H}_{d_1,(d_1-1)(d_2-1)} \\ & [C] & \longmapsto & [\pi : C \to \mathbb{P}^1], \end{array}$$

and let $\widetilde{\mathcal{SH}}_{d_1,d_2}$ be the closure of the image of ι. It follows that br is the composition of ι with the branching morphism from the Hurwitz scheme $\mathcal{H}_{d_1,(d_1-1)(d_2-1)}$ to $\mathbb{P}^{2d_2(d_1-1)}$. In order to have an analogous plane Hurwitz number in this setting, we need br and ι to be generically finite into their image.

Definition 1.6.2. *We define the* **Segre-Hurwitz number** $\mathfrak{sh}_{d_1,d_2}$ *as the degree of the branching morphism* (1.30)*, whenever* br *is generically finite. If* br *is not generically finite, we fix* $\mathfrak{h}_{d_1,d_2} = \infty$.

Note that the branching morphism br is generically finite if and only if ι is generically finite. We are particularly interested in the case where ι is injective. In such situation, the counting done through the Segre-Hurwitz numbers is compatible with the classical Hurwitz number. In the case of plane Hurwitz numbers, this compatibility is a consequence of [99, Proposition 5.2.1], where it is shown that the map $\mathcal{PH}_d \to \mathcal{H}_d$, sending a plane curve to the branching cover given by the linear projection, is injective. We aim to understand for which values of d_1 and d_2, the map ι is injective.

Lemma 1.6.3. *Let* $C \in |\mathcal{O}_S(d_1, d_2)|$ *with* $d_2 \geq 2$*, then* $h^0(C, \mathcal{O}_C(0, 1)) = 2$.

Proof. We consider the following short exact sequence of sheaves

$$0 \to \mathcal{O}_S(-d_1, -d_2) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

Tensoring by $\mathcal{O}_S(0,1)$, we obtain that the sequence

$$0 \to \mathcal{O}_S(-d_1, 1-d_2) \to \mathcal{O}_S(0,1) \to \mathcal{O}_C(0,1) \to 0$$

is exact. By taking global sections we get the following long exact sequence

$$\begin{gathered} 0 \to H^0(S, \mathcal{O}_S(-d_1, 1-d_2)) \to H^0(S, \mathcal{O}_S(0,1)) \to H^0(C, \mathcal{O}_C(0,1)) \to \\ \to H^1(S, \mathcal{O}_S(-d_1, 1-d_2)) \to \cdots \end{gathered}.$$

By the Künneth formula (see [44, Theorem 2.10]), for $d_2 \geq 2$ we have that $H^0(S, \mathcal{O}_S(-d_1, 1-d_2))$ and $H^1(S, \mathcal{O}_S(-d_1, 1-d_2))$ are zero dimensional. We conclude that

$$h^0(C, \mathcal{O}_C(0,1)) = h^0(S, \mathcal{O}_S(0,1)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2.$$

□

Using Lemma 1.6.3, we derive the following proposition.

Proposition 1.6.4. *Let $d_1 = 3$ and $d_2 \geq 2$. Then, for $C_1, C_2 \in |\mathcal{O}_S(d_1, d_2)|$, $\pi_2 : C_1 \to \mathbb{P}^1$ and $\pi_2 : C_2 \to \mathbb{P}^1$ are isomorphic as branched covering if and only if C_1 and C_2 are equal as points in $\mathcal{SH}_{3,d_2}$. In particular, for $d_1 = 3$ and $d_2 \geq 2$, the map ι is injective.*

Proof. If C_1 and C_2 are equal as points in $\mathcal{SH}_{d_1,d_2}$, then there exists $g \in \mathrm{PGL}(2)$ such that $g \times \mathrm{Id}(C_1) = C_2$. In particular, $g \times \mathrm{Id}$ is an isomorphism between the two branched coverings.
Conversely, let $\phi : C_1 \to C_2$ be an isomorphism such that $\pi_2 = \pi_2 \circ \phi$. We need to check that ϕ comes from an automorphism of S. Since π_2 is the projection to the second factor of $\mathbb{P}^1$ we deduce that $\pi_2^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{C_i}(0,1)$ for $i \in \{1,2\}$. Since ϕ commutes with π we get that

$$\mathcal{O}_{C_1}(0,1) \simeq \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \phi^*\pi_2^*\mathcal{O}_{\mathbb{P}^1}(1) = \phi^*\mathcal{O}_{C_2}(0,1). \tag{1.31}$$

By Lemma 1.6.3, $|\mathcal{O}_{C_i}(0,1)| = \mathbb{P}^1$. In particular, we get an automorphism ϕ_2 of $\mathbb{P}^1$ such that $\phi_2 \circ \pi = \phi \circ \pi$. Since $\pi = \pi \circ \phi$ we conclude that $\phi_2 = \mathrm{Id}$. Now, $\phi^* K_{C_2}$ and K_{C_1} are isomorphic. By the adjunction formula, we deduce that

$$\begin{gathered}(d_1-2)\mathcal{O}_C(1,0) + (d_2-2)\mathcal{O}_C(0,1) = \mathcal{O}_C(d_1-2, d_2-2) = \phi^*\mathcal{O}_C(d_1-2, d_2-2) = \\ (d_1-2)\phi^*\mathcal{O}_C(1,0) + (d_2-2)\mathcal{O}_C(0,1).\end{gathered}$$

Together with equation (1.31), we deduce that

$$(d_1-2)\mathcal{O}_C(1,0) = (d_1-2)\phi^*\mathcal{O}_C(1,0).$$

Thus, for $d_1 = 3$ we get that $\mathcal{O}_C(1,0) = \phi^*\mathcal{O}_C(1,0)$. Again, by Lemma 1.6.3, we get an automorphism ϕ_1 of $\mathbb{P}^1$. In particular, we get an automorphisms $\phi_1 \times \phi_2 = \phi_1 \times \mathrm{Id}$ of S that maps C_1 to C_2, and C_1 and C_2 are equal as points in $\mathcal{SH}_{3,d_2}$. □

The condition $d_2 \geq 2$ in Proposition 1.6.4 is not very restrictive since for $d_2 = 1$, C has genus zero. Now, the dimension of $\mathcal{H}_{d_1,(d_1-1)}$ is $2d_2(d_1-1)$, while the dimension of $\mathcal{SH}_{d_1,d_2}$ is $(d_1+1)(d_2+1)-4$. In particular, we get that

$$\dim \mathcal{SH}_{d_1,d_2} - \dim \mathcal{H}_{d_1,(d_1-1)(d_2-1)} = (d_1-3)(d_2-1). \tag{1.32}$$

We deduce that for $d_3 = 3$ and $d_2 \geq 2$, $\widetilde{\mathcal{SH}}_{3,d_2} = \mathcal{H}_{3,2(d_2-1)}$. Moreover, from Proposition 1.6.4, we derive the following result.

Corollary 1.6.5. *For $d_1 = 3$ and $d_2 \geq 2$, ι is a birational morphism between $\mathcal{SH}_{3,d_2}$ and $\mathcal{H}_{3,2d_2-2}$, and $\mathcal{V}_{3,d_2} = \mathbb{P}^{4d_2}$. In particular, the Segre-Hurwitz number $\mathfrak{sh}_{3,d_2}$ coincides with the Hurwitz number $H_{3,2d_2-2}$*

We finish this section introducing the real version of the Segre-Hurwitz numbers. We say that a point in $\mathcal{SH}_{d_1,d_2}$ is real if the corresponding PGL(2)-orbit contains a real algebraic curve. Given a real branch locus B in $\mathcal{Z}_{d_1,d_2}(\mathbb{R})$, we define the real Segre-Hurwitz number $\mathfrak{sh}^{\mathrm{real}}_{d_1,d_2}(B)$ as the number of real point in the fiber of B through the branching morphism, whenever the branching morphism is finite. As a consequence of Corollary 1.6.5, we get that for $d_1 = 3$ and $d_2 \geq 2$, the real Segre-Hurwitz number coincides with the real Hurwitz number $H^{\mathrm{real}}_{3,2(d_2-1)}$.

1.6.1 Segre-Hurwitz numbers for $(4,2)$-curves

From Equation (1.32) we deduce that, for $d_1 \geq 4$ and $d_2 \geq 2$, the dimension of $\mathcal{SH}_{d_1,d_2}$ is lower than the dimension of $\mathcal{H}_{d_1,(d_1-1)(d_2-1)}$. In particular, for $d_1 \geq 4$ and $d_2 \geq 2$, the map ι is not dominant and $\widetilde{\mathcal{SH}}_{d_1,d_2}$ has positive codimension in $\mathcal{H}_{d_1,(d_1-1)(d_2-1)}$. The topic of this section is the first case where this behaviour happens: $(d_1,d_2) = (4,2)$. In this case $\widetilde{\mathcal{SH}}_{4,2}$ is a hypersurface in $\mathcal{H}_{4,3}$ We start by deriving the following result for $d_2 = 2$.

Proposition 1.6.6. *For $d_1 \geq 3$ and $d_2 = 2$, the map ι is injective.*

Proof. Let C_1 and C_2 be two curves in $\mathcal{SH}_{d_1,d_2}$ such that $\iota(C_1) = \iota(C_2)$. We say that a line bundle is g^1_d if its global sections define a degree d branched cover of $\mathbb{P}^1$ (see [2]). Then, both $\mathcal{O}_{C_1}(1,0)$ and $\phi^*\mathcal{O}_{C_2}(1,0)$ are $g^1_{d_2}$. Indeed, the map associated to both line bundles are π_1 and $\pi_1 \circ \phi$, and both maps are degree d_2 covers of $\mathbb{P}^1$. For $d_2 = 2$, $\mathcal{O}_{C_1}(1,0)$ and $\phi^*\mathcal{O}_{C_2}(1,0)$ are $g^1_{d_2}$. From the uniqueness of g^1_2 line bundles on curves with genus at least 2 (see [124, Proposition 20.5.7]), we deduce that $\mathcal{O}_{C_1}(1,0) \simeq \phi^*\mathcal{O}_{C_2}(1,0)$. Now, the proof follows using the same argument as in Proposition 1.6.4. □

A direct consequence of Proposition 1.6.6 is that the Segre-Hurwitz number $\mathfrak{sh}_{d_1,2}$ is compatible with the counting performed through the Hurwitz number H_{d_1,d_1-1}. In particular, $\mathfrak{sh}_{d_1,2} \leq H_{d_1,d_1-1}$ for $d_1 \geq 4$.

For the rest of the section we fix $(d_1,d_2) = (4,2)$. A generic curve C in $|\mathcal{O}_S(4,2)|$ has genus 3, and the projection $\pi_2 : C \to \mathbb{P}^1$ is a simple branched cover of degree 4 with 12 branch points. These branched coverings are of the same type as the ones considered while studying the plane Hurwitz number for quartic curves. This connection was observed in [122]. There, the author introduce the subscheme $\mathcal{D}$ of $\mathcal{H}_{4,3}$ of degree 4 simple branched covers of $\mathbb{P}^1$ ramified at 12 points whose source curve is hyperelliptic and of genus 3. Vakil proves that the map from $\mathcal{D}$ to $\mathbb{P}^{12}$ sending a branched cover to its branch locus has degree 135 onto its image, whose closure is the hypersurface $\mathcal{V}_4$. Recall that for $d_2 = 2$ the curves in $|\mathcal{O}(d_1, 2)|$ are hyperelliptic. By Proposition 1.6.6, we deduce that ι is a birational morphism between $\mathcal{SH}_{4,2}$ and $\mathcal{D}$. Therefore, we conclude that $\mathcal{V}_{4,2} = \mathcal{V}_4$ and the Segre Hurwitz number $\mathfrak{sh}_{4,2}$ is 135. We now compute the real Segre-Hurwitz number $\mathfrak{h}^{\mathrm{real}}_{4,2}$.

Theorem 1.6.7. *Given B real branch locus, $\mathfrak{sh}^{\mathrm{real}}(B)$ is either $7, 15, 31, 39, 63$ or 135.*

Proof. Let B be a generic branch locus in $\mathcal{V}_4$ and let C be the corresponding space sextic. We denote the cone containing C by Q. Using Theorem 1.1.11, we get that the number of even theta characteristics of C is 136. Note that this number equals $\mathfrak{sh}_{4,2} + 1$. This is because among these 136 even theta characteristic, one of them is obtained from Q as follows. Let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from the singular point of Q. The restriction of π to C is a degree 3 branched cover of $\mathbb{P}^1$ ramified over B. A fiber of this branched cover is a degree 3 divisor. Let Θ be the line bundle associated to this divisor. Then, Θ is an even theta characteristic. Indeed, $h^0(C, \Theta) = 2$ since the global sections of Θ define the degree 3 covering of $\mathbb{P}^1$. Now, by the adjunction formula, the canonical bundle of C is $\mathcal{O}_C(1)$. Since the fiber of two points via π is a divisor in $|\mathcal{O}_C(1)|$, we conclude that $2\Theta = \mathcal{O}_C(1)$.
In [122, Section 4], Vakil uses Recillas' trigonal construction to build a bijection among the 135 fibers of B through the branching morphism of $\mathcal{SH}_{4,2}$ and the set of even theta characteristics of C distinct than Θ. For further literature of Recillas' construction we refer to Recillas' original paper [108]. Now, assume that B is real. Then, Θ is a real theta characteristic. Moreover, using the fact that Recillas' construction preserves the real involution, we deduce that the above bijection sends real fibers of B to real even theta characteristics. The proof follows from Theorem 1.1.13. $\square$

The same study carry out in Section 1.5.4, allow us to analyse the relation between $\mathfrak{sh}^{\mathrm{real}}(B)$ and the number of real roots of B for $B \in \mathcal{V}_4(\mathbb{R})$. In particular, as a consequence of the proof of Theorem 1.5.9 and the examples constructed in Section 1.5.4, we derive the following result.

Corollary 1.6.8. *There is no $B \in \mathcal{V}_4(\mathbb{R})$ such that the tuple $(\mathfrak{sh}^{\mathrm{real}}_{4,2}(B), t(B))$ corresponds to a red entry in Table 1.6. For each green entry (sh, t) of Table 1.6, there exists $B \in \mathcal{V}_4(\mathbb{R})$ such that the tuple $(\mathfrak{sh}^{\mathrm{real}}_{4,2}, t(B)) = (sh, t)$.*

			t						
$X_{\mathbb{R}}$	Ovals	$\mathfrak{sh}_{4,2}^{\text{real}}$	0	2	4	6	8	10	12
$\mathbb{P}^2_{\mathbb{R}}(0,8)$	1 big	7							
$\mathbb{P}^2_{\mathbb{R}}(2,6)$	1 big + 1 small	15							
$\mathbb{P}^2_{\mathbb{R}}(4,4)$	1 big + 2 small	31							
$\mathbb{P}^2_{\mathbb{R}}(6,2)$	1 big + 3 small	63							
$\mathbb{P}^2_{\mathbb{R}}(8,0)$	1 big + 4 small	135							
$\mathbb{D}_4(3,0)^1_2$	3 big	39							
$\mathbb{D}_4(3,0)^0_3$	1 big + 1 + 1 small	39							

Table 1.6: The first column indicates the real del Pezzo surface associated to a real branch locus. The second column describe the oval distribution of the corresponding space sextic, and the third column indicates the value of the real Segre-Hurwitz number. The rest of the columns indicate the number t of real points in the branch locus.

1.7 Open problems

We finish this chapter with a section devoted to list, and comment, some open questions and future research lines.

One of the main open problems of this chapter is related to the computation of new (real) plane Hurwitz numbers. Only for $d \leq 4$ the plane Hurwitz numbers, and its real version, are known. The value of these numbers remains open for $d \geq 5$.

Question 1.7.1. *What are the values of $\mathfrak{h}_d$ and $\mathfrak{h}_d^{\text{real}}$ for $d \geq 5$?*

Of particular interest is the case of plane sextics. In [122], for the computation of $\mathfrak{h}_4$, Vakil exploits the fact that the double cover of the blow up of $\mathbb{P}^2$, at a point ramified over a plane quartic, is an elliptic fibration. For the case $d = 6$, the double cover of $\mathbb{P}^2$, ramified over a plane sextic, is a K3 surface. Based on this observation, one may try to determine $\mathfrak{h}_6$ and $\mathfrak{h}_6^{\text{real}}$ using the geometry of K3 surfaces.

In the study of plane Hurwitz numbers, we focused on simple branched covering. We can state the analogous enumerative question for nonsimple plane branched coverings giving rise to (real) nonsimple plane Hurwitz numbers.

Question 1.7.2. *What are the values of (real) nonsimple plane Hurwitz numbers?*

For instance, a plane cubic has 9 flex points. If a flex line passes through the center of the projection the corresponding ramification point (the flex point) has multiplicity 3. For plane quartics, we also could have bitangent lines passing through the center of projection. This will lead to branch points whose fibers are two ramification points, each with multiplicity 2. A future research line is the computation of the corresponding (real) plane Hurwitz numbers for not simple branched coverings. For instance, the Trott curve has its 28 bitangent real. Among these bitangents, we can find 4 of them passing through a point. The projection centered at this point leads to a degree 4 non simple branched covering of $\mathbb{P}^1$. Its branch locus consists of 6 points, four of them with multiplicity 2. The fiber of each of these four points consists of two ramification points with multiplicity 2. What is the real plane Hurwitz number in this case? We are aiming to answer these questions in an ongoing project together with Clemens Nollau. Analogously, a recovery problem can be stated in this situation from a numerical and symbolic point of view.

Concerning the (real) Segre-Hurwitz numbers introduced in Section 1.6, the computation of most of these numbers remains open. In general, it is not known for which values of d_1 and d_2, the Segre-Hurwitz number $\mathfrak{sh}_{d_1,d_2}$ is finite. In this chapter, we have seen that for $(d_1, d_2) = (d_1, 2)$ and $d_1 \geq 3$, the Segre-Hurwitz number is finite. Moreover, we have computed $\mathfrak{sh}_{3,2}$ and $\mathfrak{sh}_{4,2}$. The following question remains open.

Question 1.7.3. *What are the values of $\mathfrak{sh}_{d_1,2}$ and $\mathfrak{sh}^{\mathrm{real}}_{d_1,2}$ for $d_1 \geq 5$?*

From a computational point of view, in an ongoing on project with Clemens Nollau, we are studying the recovery algorithm for the 135 curves of bidegree $(4, 2)$ studied in Section 1.6.1.

The introduction of Segre-Hurwitz numbers in Section 1.6 creates a new frontier in the field of plane Hurwitz numbers. In this section we moved from plane curves to curves in $\mathbb{P}^1 \times \mathbb{P}^1$. What happens if we consider curves on other varieties? For instance, we can consider curves in the blow up of a point in $\mathbb{P}^2$. We have two natural projections from $\mathrm{Bl}_p\mathbb{P}^2$ to $\mathbb{P}^1$ given by the hyperplane class and the exceptional divisor, respectively. Each type of projection leads to a branched covering $C \to \mathbb{P}^1$, where C is a curve in $\mathrm{Bl}_p\mathbb{P}^2$. Up to the correct notion of isomorphism, what is the number of such branched coverings sharing the same fixed branch locus? Can we recover these curves in $\mathrm{Bl}_p\mathbb{P}^2$ from their branch locus? We can consider a more general setting. Let H_n be the Hirzebruch surface associated to the bundle $\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}$ of $\mathbb{P}^1$ and consider the bundle projection $\pi : H_n \to \mathbb{P}^1$. Given a generic smooth curve C in H_n, the restriction of π to C is a simple branched covering of $\mathbb{P}^1$. Up to isomorphism, can we count the branched coverings of this form with fixed branch locus? Similarly to the definition of plane Hurwitz number and Segre-Hurwitz numbers, such value can be properly defined. We call these numbers the n–Hirzebruch Hurwitz numbers. For instance, for $n = 0$ $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and the 0–Hirzebruch Hurwitz number coincide with the Segre-Hurwitz numbers. Moreover, the real version of these numbers can also be defined.

Question 1.7.4. *What are the values of the n–Hirzebruch Hurwitz numbers and their real version?*

Another way to generalize plane Hurwitz numbers is, instead of considering linear projection from $\mathbb{P}^2$ to $\mathbb{P}^1$, we can consider linear projections from $\mathbb{P}^n$ to $\mathbb{P}^1$. These projections have center a codimension 2 linear subspace. An analogous enumerative problem can be stated in this setting. An interesting case is when canonical curves are considered. Let C be a generic canonical genus g curve embedded in $\mathbb{P}^{g-1}$. A projection $\pi : \mathbb{P}^{g-1} \dashrightarrow \mathbb{P}^1$ leads to a degree $2g-2$ simple branched covering of $\mathbb{P}^1$ such that $\pi^*\mathcal{O}_{\mathbb{P}^1}$ is isomorphic to the canonical bundle of C. Up to isomorphism, what is the number of such branched coverings with the same fixed branch locus? For instance, a genus 4 canonical curve in $\mathbb{P}^3$ is a space sextic. A line in $\mathbb{P}^3$ defines a linear projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ whose restriction to a generic space sextic is a degree 6 simple branched covering of $\mathbb{P}^1$. How many of such coverings share the same branch locus?

We can slightly change the above problem as follows. Instead of considering arbitrary curves in $\mathbb{P}^3$, we can fix a normal surface S and count branched coverings obtained from projecting linearly curves in S to $\mathbb{P}^1$. By requiring S to be normal, we can assume that the curves we are counting do not intersect the singular locus of S. For instance, we can fix S to be $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^3$. Note that this is not the same situation analysed in Section 1.6. There, we studied maps to $\mathbb{P}^1$ defined by $|\mathcal{O}(0,1)|$. In this case, the projection is defined by $|\mathcal{O}(1,1)|$. We could also consider curves in a quartic cone Q, In this situation, we can also consider the projection with center the singular point of Q. Given a curve in C in Q, this projection defines a branched covering of $\mathbb{P}^1$. Similar enumerative problems can be considered in this setting. For instance, for space sextics in Q, the birationality between $\mathcal{E}$ and $\mathcal{V}_4$ shows that there is only one class of space sextics in Q whose above projection has a fixed branch locus (see [122, Theorem 2.12]).

Chapter 2

Hessian correspondence and catalecticant enveloping varieties

Throughout this chapter, we will work with an algebraically closed field $\mathbb{K}$ of characteristic 0. Moreover, let $V := \mathbb{K}^{n+1}$, for $n \geq 1$, with coordinates $x_0, \ldots, x_n$, and let $S^d V$ denote the d–th symmetric power of V.

Given a polynomial $F \in \mathbb{P}(S^d V)$, the Gaussian map is the rational map that sends a smooth point of $\mathbb{V}(F) \subseteq \mathbb{P}^n$ to its tangent hyperplane. Here we see this hyperplane as a point in the dual projective space $(\mathbb{P}^n)^*$. In other words, the Gaussian map is the rational map defined by the first order derivatives of F. The dual variety of F, denoted by $\mathbb{V}(F)^*$, is the closure of the image of the Gaussian map. Dual varieties are classical objects in both differential and algebraic geometry. We refer to [40, 55, 110] for further details on this topic. One of the most celebrated results concerning these varieties is the Biduality theorem (see [40, Theorem 1.2.2]), which states that $(\mathbb{V}(F)^*)^* = \mathbb{V}(F)$. A first consequence of the Biduality theorem is that the map associating to a hypersurface its dual variety is birational onto its image. Moreover, given a dual variety X, we can effectively compute the fiber of X through this map by computing X^*. In this chapter we we analyse the analogous construction for the second order derivatives.

Given $F \in \mathbb{P}(S^d V)$, the Hessian map or second polar map is the rational map

$$h_F : \mathbb{V}(F) \dashrightarrow \mathbb{P}(S^2 V)$$

sending a point p to the evaluation of the Hessian matrix of F at p. Here we use the identification of $S^2 V$ and the space of symmetric matrices. The Hessian variety or second polar variety is defined as the closure of the image of the Hessian map. In other words, the Hessian variety consists of all Hessian matrices of the hypersurface. By abuse of notation, we will often consider h_F as a rational map from $\mathbb{P}^n$ instead of from $\mathbb{V}(F)$. By Euler's formula, both rational maps have the same base locus. The Hessian map was studied in [27], where it is proven that h_F is finite when $\mathbb{V}(F)$ is smooth (see [27, Proposition 6]). In particular, the Hessian variety of a generic polynomial has dimension $n-1$. In [87] it is shown that if h_F has no base locus, then the Hessian variety has degree $d(d-2)^{n-1}$. In this chapter we carry out a further investigation of Hessian varieties by analysing the relation between a hypersurface and its Hessian variety.

The Hessian map and the Hessian variety is closely related to the classical Hesse problem. The Hessian polynomial of $F \in \mathbb{P}(S^dV)$ is the determinant of the Hessian matrix of F. The (d,n)–Gordan-Noether locus is the locus of polynomials in $\mathbb{P}(S^dV)$ whose Hessian polynomial vanishes. The Hesse problem deals with the description of the (d,n)–Gordan-Noether locus. For $n \leq 3$, a polynomial lies in this locus if and only if it is a cone (see [58]). Some remarkable contributions about Hessian polynomials and the Hesse problem are [18, 49, 53, 57, 88, 126]. We refer to [29, 110] for further information on the Gordan-Noether locus. In our setting, a polynomial F lies in the (d,n)–Gordan-Noether if and only if $h_F(\mathbb{P}^n)$ is contained in the hypersurfaces of symmetric matrices with vanishing determinant. In [27] the authors use the Hessian map to study the subvariety $\overline{S}_r(F)$ of $\mathbb{V}(F)$ defined as the Zariski closure of the points of $\mathbb{V}(F)$ where the Hessian matrix has rank r. This variety is strongly connected with the following generalization of the Hesse problem: can we describe the locus of the polynomials whose Hessian matrix has rank at most rank r? Note that for $r = n$ this locus is the Gordan-Noether locus previously defined.

Motivated by the Hesse problem, the relation between a hypersurface and its Hessian polynomial has recently gained the attention of algebraic geometers. We highlight the works [28], [38] and [39], where the birationality of the map associating a polynomial its Hessian polynomial is studied for certain d and n. In these papers, this map is called the Hessian map, and the image of this map is the Hessian variety of type (d,n). It should be noticed that our notion of Hessian map and Hessian variety differs from the concept introduced in these papers. We expect that our study of Hessian varieties could enlighten new ideas for approaching the Hesse problem and its related questions.

The Hessian correspondence $H_{d,n}$ is the rational map associating to a degree d hypersurface in $\mathbb{P}^n$ its Hessian variety. This is the main object of study of this chapter, and our goal is to study the fibers of this map. In the case of dual varieties, the analogous map is birational onto its image by the Biduality theorem. Therefore, the following questions appears naturally. Is it birational onto its image as its analogous for dual varieties? Is the Hessian correspondence generically finite? Is the fiber a unique polynomial up to change of coordinates in $\mathbb{P}^n$? Or up to isomorphism or birationality of the corresponding hypersurfaces? We are also interested in a computational approach to these questions. In the case of first order derivatives, we can effectively recover a hypersurface from its dual variety. Can we reconstruct a hypersurface from its Hessian variety? This is equivalent to find an effective algorithm for computing the fibers of the Hessian correspondence. In this chapter we answer the above questions for low Waring rank hypersurfaces, and for hypersurfaces of degree 3 and 4.

In Section 2.4, we will consider the rational map

$$\begin{array}{rccc}\alpha_n: & \mathbb{P}(S^3V) & \dashrightarrow & \mathrm{Gr}(n+1,S^2V) \\ & F & \longmapsto & \left\langle \dfrac{\partial F}{\partial x_0},\dots,\dfrac{\partial F}{\partial x_n}\right\rangle\end{array}.$$

This map is called the **gradient map** and in [6, 7] it was shown that its is birational onto its image. We refer to [20, 46] for a modern reference. In Section 2.4, we derive the birationality of $H_{3,n}$ for $n \geq 2$, which follows in a rather straightforward way from the birationality of α_n. However, here we present an alternative approach to the birationality of α_n, that in Section 2.4.3 can be turned into an effective algorithm for recovering a cubic from its Hessian. On the other hand, for the case of quartics, we need to use different ideas, coming from syzygies of Veronese varieties.

Of particular interest is also the closure of the image of gradient map. More generally, we define the variety $\mathcal{Z}_{e,k}$ as the Zariski closure of the set

$$\mathcal{Z}^{\circ}_{e,k} := \left\{\Gamma \in \mathrm{Gr}(k+1,S^{d-e}V) : \langle \nabla^e F\rangle = \Gamma \text{ for some } F \in S^dV\right\},$$

where $\nabla^e F$ denotes the set of all e–th partial derivatives of F. The motivation behind the study of this variety comes from tensor decomposition and apolarity theory. Among the distinct notions of ranks, partially symmetric ranks have recently attracted the attention of the tensor community. One of the main questions about partially symmetric ranks was posted by Comon (see [98, Problem 15]): if we regard a symmetric tensor as a partially symmetric tensor, are the symmetric rank and the partially symmetric rank equal? In the last years, affirmative answers have been given to this questions under certain assumptions in [4, 32, 50, 111, 127].

In [56], Comon's question is approach using simultaneous Waring ranks. More concretely, through the notion of e-th gradient ranks. Given a polynomial $F \in S^dV$, the **e–th gradient rank** is defined as the minimum r such that there exists linear forms $l_1,\dots,l_r$ such that any e–th partial derivative of F can be written as a linear combination of $l_1^{d-e},\dots,l_r^{d-e}$. It turns out that the study of the e–th gradient rank is closely related to the geometry of the variety $\mathcal{Z}_{e,k}$. Our goal is to analyse this variety through the **catalecticant enveloping variety**, defined as

$$\Phi^e_k := \left\{E \in \mathrm{Gr}(k+1,S^{d-e})V) : \langle \nabla^e F\rangle \subseteq E \text{ for some } F \in S^dV\right\}.$$

This chapter is structured as follows. In Section 2.1 we introduce the background required in the rest of the chapter. This preliminary section is divided in two subsections. Dual varieties, polar varieties and polar maps are presented in Section 2.1.1, and a brief introduction to Waring rank, secant varieties and apolarity theory is given in Section 2.1.2.

In Section 2.2 we introduce the main objects of study: Hessian maps, Hessian varieties and the Hessian correspondence $H_{d,n}$. We also introduce the rational map $H_{d,n,k}$, defined as the restriction of the Hessian correspondence to polynomials with Waring rank k.

Section 2.3 is devoted to the study of the Hessian correspondence for polynomials with Waring rank at most $n+1$. In Section 2.3.1, we compute the degree of $H_{d,n,k}$ for $k \leq n+1$ and, in Section 2.3.2, we provide a numerical algorithm for recovering the fibers of $H_{d,n,k}$ for $k \leq n+1$.

Section 2.4 studies the Hessian correspondence for cubic hypersurfaces. More precisely, in Section 2.4.1, we study $H_{3,1}$, its image and the computation of its fibers in detail. Afterwards, in Section 2.4.2, we analyse $H_{d,n}$ for $n \geq 2$. We use the gradient map α_n to study the fibers of $H_{3,n}$. Finally, in Section 2.4.3, an algorithm for recovering a cubic hypersurface, from its Hessian variety, is provided.

Section 2.5 is devoted to the Hessian correspondence for $d = 4$. In Section 2.5.1 we focus on $H_{4,1}$. In Section 2.5.2 we study $H_{4,n}$ for $n \geq 2$ and in Section 2.5.3 an effective algorithm for recovering F from $H_{4,n}(F)$, when n even, is derived.

Section 2.6 focuses on the generalization of the map α_n and on the varieties $\mathcal{Z}_{e,k}$ and Φ_k^e. We consider the variety $\mathrm{Sub}_{e,k}$ as the closure of the set of polynomials $F \in \mathbb{P}(S^dV)$ such that $\dim \langle \nabla^e F \rangle = k+1$ and we introduce the rational map $\alpha_{e,k} : \mathrm{Sub}_{e,k} \dashrightarrow \mathrm{Gr}(k+1, S^{d-e}V)$ sending F to $\langle \nabla^e F \rangle$. The map $\alpha_{1,n}$ is the gradient map, whose birationality was studied in [6, 7, 46]. In Section 2.6.1 we provide a new proof for the birationality of $\alpha_{1,k}$, which inspired the techniques use in Section 2.6.2 to analyse the irreducible components of the catalecticant enveloping variety for $d = 1$. Finally, in Section 2.6.3 we generalize the Hessian correspondence to higher order derivatives. We define the l–th polar correspondence $\mathcal{G}^l_{d,n}$ as the rational map sending a polynomial $F \in \mathbb{P}(S^dV)$ to its l–th polar variety. We study the $(d-1)$–th polar correspondence and the corresponding recovery problem.

The main contributions of this chapter are:

- For $k \leq n+1$, we prove that the $H_{d,n,k}$ is birational onto its image for d even, and generically finite of degree 2^{k+1} for d odd. Moreover, we provide a numerical algorithm for reconstructing a hypersurface of Waring rank at most $n+1$ from its Hessian variety.

- We show that $H_{3,1}$ has generically degree 2, we compute its image and the rational involution preserving the fibers. We present an effective algorithm for reconstructing the two cubic binary forms with same Hessian variety from their Hessian variety.

- We prove that $H_{d,n}$ is birational onto its image for $d = 3, 4$ and $(d, n) \neq (3, 1)$. Moreover, we compute the image of $H_{4,1}$.

- We provide an effective method for recovering a generic hypersurface from its Hessian variety for $d = 3$ and $n \geq 2$, and for $d = 4$ and n even or $n = 1$.

- We provide an effective algorithm for computing the parametrization of a Veronese variety of order 2 and of even dimension.
- We compute the irreducible components of Φ_k^1 and their dimension. We determine the dimension of $\mathcal{Z}_{1,k}$ and a irreducible component of its boundary.
- We prove that the $(d-1)$–th polar correspondence is birational onto its image.

2.1 Preliminaries on polar varieties and tensors

The background require for the development of this chapter is presented in this section. We give a brief introduction to some classical objects as dual varieties, polar varieties, Waring rank, catalecticant matrices and apolar ideals.

2.1.1 Polar varieties

This section is devoted to polar varieties. The main example of these varieties are dual varieties. The dual variety is a classical object in the fields of algebraic and differential geometry. This notion can be defined for varieties of arbitrary dimension, although in this section we only focus on dual varieties of hypersurfaces in $\mathbb{P}^n$. We refer to [40, 55, 110] for further details on this topic.

Definition 2.1.1. *Given $F \in \mathbb{P}(S^dV)$, the* Gaussian map *or* first polar map *is defined as the rational map*

$$\begin{array}{cccc} g_F^1: & \mathbb{V}(F) & \dashrightarrow & (\mathbb{P}^n)^* \\ & p & \longmapsto & \nabla F(p) = [\frac{\partial F}{\partial x_0}(p), \dots, \frac{\partial F}{\partial x_n}(p)]. \end{array} \tag{2.1}$$

In other words, the Gaussian map associates to each smooth point of $\mathbb{V}(F)$ its tangent hyperplane. The dual variety *of $\mathbb{V}(F)$, denoted by $\mathbb{V}(F)^*$, is the Zariski closure of the image of the Gaussian map, namely, the closure of $g_F^1(\mathbb{V}(F))$.*

The first natural question on dual varieties, that is answered in the next theorem, is about its dimension.

Theorem 2.1.2. *[55, Corollary 1.2] The dual variety of $\mathbb{V}(F)$ is a hypersurface if and only if $\mathbb{V}(F)$ is not a cone.*

Moreover, from [55, Proposition 3.2] one can deduce that, for generic $F \in \mathbb{P}(S^dV)$, the dual variety $\mathbb{V}(F)^*$ is a hypersurface of degree $d(d-1)^{n-1}$. Let $F \in \mathbb{P}(S^dV)$ be such that $\mathbb{V}(F)$ is not a cone. The F–discriminant Δ_F is the defining polynomial of the dual variety of $\mathbb{V}(F)$ up to scalar. Then, for generic F, Δ_F is a degree $d(d-1)^{n-1}$ polynomial. Note that, even when the dual variety of F is a hypersurface, it might happen that its degree is not $d(d-1)^{n-1}$. This phenomenon is exhibited, for instance, in the Plücker formulas (see [40, Section 1.2.3]), which relate the degree, genus and singularities of a plane curve and its dual curve. For instance, let C be a degree d plane curve whose singularities are δ nodes and κ cusps. Then, the degree of the dual curve is $d(d-1)-2\delta-3\kappa$, which differs from the expected degree.

Example 2.1.3. *Fix $n=2$ and $d=3$ and consider the polynomial $F=x_0^3+x_1^3+x_2^3$. Let C be the smooth plane curve defined by F. The Gaussian map $g_F^1 : C \to (\mathbb{P}^2)^*$ sends a point $p=[p_0,p_1,p_2]\in C$ to $[p_0^2,p_1^2,p_2^2]$. One can check that the dual variety C^* is the plane sextic defined by the polynomial*

$$z_0^6+z_1^6+z_2^6-2(z_0^3z_2^3+z_1^3z_2^3).$$

Here z_0,z_1,z_2 are the coordinates of the dual projective plane. Now let $G=x_0^3+x_2(x_1^2-x_0^2)$. The dual curve is given by the equation

$$4z_0^4+4z_1^4-8z_0^2z_1^2+4z_0^4z_2-36z_0z_1^2z_2-27z_1^2z_2^2.$$

Note that the plane curve defined by G has a nodal singularity and the degree of $\mathbb{V}(F)^*$ agrees with the Plücker formulas.

Suggested by many dual phenomena in mathematics, the name dual variety suggests that the dual variety of the dual variety is the initial hypersurface. Indeed, this phenomenon is known as the Biduality Theorem

Theorem 2.1.4. *[55, Theorem 1.1] Let X be a hypersurface which is not a cone. Then, $X=(X^*)^*$.*

A direct consequence of the Biduality Theorem is that we can recover effectively a generic hypersurface X from its dual variety by computing the dual variety of X^*.

Example 2.1.5. *Consider the polynomial*

$$G=z_0^6+z_1^6+z_2^6-2(z_0^3z_2^3+z_1^3z_2^3)$$

defining the a plane sextic in the dual projective space $(\mathbb{P}^2)^*$. The Gaussian map for G sends a point $q=[q_0,q_1,q_2]$ to

$$[q_0^5-q_0^2q_1^3-q_0^2q_2^3,q_1^5-q_1^2q_0^3-q_1^2q_2^3,q_2^5-q_2^2q_0^3-q_2^2q_0^3].$$

One checks that the image of $\mathbb{V}(G)$ through this map is the plane curve defined by $x_0^3+x_1^3+x_2^3$. Note that this computation is compatible with Example 2.1.3.

Next we consider the rational map

$$\begin{array}{rccc} \mathcal{G}_{d,n}^1: & \mathbb{P}(S^dV) & \dashrightarrow & \mathbb{P}(S^{d(d-1)^{n-1}}) \\ & F & \longmapsto & \Delta_F \end{array}, \tag{2.2}$$

which associates to a hypersurface its dual variety. By the Biduality theorem, $\mathcal{G}_{d,n}^1$ is injective, and hence, it is birational onto its image. Moreover, the fibers of $\mathcal{G}_{d,n}^1$ can be recovered effectively. Given a dual variety X, the fiber $\mathcal{G}_{d,n}^{-1}(X)$ is X^*.

Note that the construction of the dual variety of $\mathbb{V}(F)$ is based on the first order derivatives of F, since the tangent space at a smooth point of $\mathbb{V}(F)$ is given by the gradient of F. The generalization of the Gaussian map to higher order derivatives is called the k–polar map.

Definition 2.1.6. *For $k \geq 1$ and $F \in \mathbb{P}(S^dV)$, the k–polar map of F is the rational map*

$$\begin{array}{rccc} g_F^k : & \mathbb{V}(F) & \dashrightarrow & \mathbb{P}(S^kV) \\ & p & \longmapsto & \sum \dfrac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}(p) x_{i_1} \cdots x_{i_k}. \end{array} \tag{2.3}$$

The k–th polar variety is defined as the closure of the image of g_F^k.

Note that, $\mathbb{P}(S^1V)$ can be identified with $(\mathbb{P}^n)^*$ and thus (2.3), with $k = 1$, yields the usual Gaussian map (2.1). By abuse of notation, we will often consider g_F^k as a rational map from $\mathbb{P}^n$ instead of from $\mathbb{V}(F)$. Note that by Euler's formula, both rational maps have the same base locus. Moreover, this base locus is contained in the singular locus of $\mathbb{V}(F)$. In particular, for generic $F \in \mathbb{P}(S^dV)$, the k–th polar map has no base locus. In [87], the author proved that if g_F^k has no base locus, the k–th polar variety has dimension $n-1$ and degree $d(d-k)^{n-1}$.

Example 2.1.7. *Let $n = 1$ and consider the binary form $F = x_0^4 + x_0^3x_1 + x_0^2x_1^2 + x_0x_1^3 + x_1^4$. The 3–th polar map $g_F^3 : \mathbb{V}(F) \to \mathbb{P}(S^3V) \simeq \mathbb{P}^3$ is a linear map that sends a point $p = [p_0, p_1]$ to*

$$[12p_0 + 3p_1, 3p_0 + p_1, p_0 + 3p_1, 3p_0 + 12p_1].$$

In this case, the 4–polar variety is defined by the ideal

$$\langle 3z_1 - 6z_2 + z_3, z_0 - 9z_2 + 2z_3, 3321z_2^4 - 3051z_2^3z_3 + 1071z_2^2z_3^2 - 171z_2z_3^3 + 11z_3^4 \rangle.$$

One of the main objects we will study in Chapter 2 are Hessian maps and Hessian varieties. They will be defined in Section 2.2 as the second polar map and second polar variety respectively.

2.1.2 Waring rank, Secant varieties and apolar ideals

In this section we introduce some tensor related objects that will make their appearance in Chapter 2. For further details in these topics we refer to [5, 74, 84]. We start with a classical notion in tensor decomposition.

Definition 2.1.8. *The Waring rank of a polynomial $F \in \mathbb{P}(S^dV)$ is defined as the minimum integer $r \geq 0$ such that $F = l_1^d + \cdots + l_r^d$ for some linear forms $l_1, \ldots, l_r \in S^1V$. In this situation, the expression of F as sum of r powers of linear forms is called a Waring decomposition. The border rank of a polynomial $F \in \mathbb{P}(S^dV)$ is the minimum integer r such that F is contained in the Zariski closure of the set of degree d polynomials with Waring rank r.*

From these definitions it follows that the border rank of a polynomial is lower or equal than its Waring rank. However, the equality does not always hold. For instance, the Waring rank of $x_0x_1^2$ is 3 but its border rank is 2 (see [78, Example 1.4 and Example 2.10]).

We denote the Veronese variety of all polynomials of the form l^d, for $l \in \mathbb{P}(S^1V)$, by $V^{d,n}$. Then, the closure of the set of degree d polynomials of symmetric rank k is the k–th secant variety of $V^{d,n}$, denoted by $\sigma_k(V^{d,n})$. Therefore, the border rank of $F \in \mathbb{P}(S^dV)$ is the minimum integer $k \geq 1$ such $F \in \sigma_k(V^{d,n})$. The expected dimension of the secant variety $\sigma_k(V^{d,n})$ is

$$\text{expdim}\sigma_k(V^{d,n}) = \min\left\{sn+s-1, \binom{n+d}{n}-1\right\}.$$

In general, the expected dimension of $\sigma_k(V^{d,n})$ bigger or equal to its dimension. The k–defect, denoted by δ_k, is defined as the difference $\text{expdim}\sigma_k(V^{d,n}) - \dim\sigma_k(V^{d,n})$. We say the secant variety $\sigma_k(V^{d,n})$ is k–defective if $\delta_k > 0$. The Alexander–Hirschowitz Theorem shows when $\sigma_k(V^{d,n})$ is k–defective.

Theorem 2.1.9. *[12, Theorem 1.2] The dimension of the higher secant varieties $\sigma_k(V^{d,n})$ equals the expected dimension except for the cases shown in Table 2.1.*

d	n	k	δ_k	$\dim\sigma_k(V^{d,n})$
2	≥ 2	$2 \leq k \leq n$	$\binom{k}{2}$	$kn+k-1-\binom{k}{2}$
3	4	7	1	33
4	2	5	1	14
4	3	9	1	33
4	4	14	1	68

Table 2.1: List of7 secant varieties of $V^{d,n}$ that are defective.

For further references on decomposition of tensors and their relation to secant varieties we refer to [5].

Now we shift our point of view on tensor decompositions from secant varieties to apolar ideals. Consider the graded rings $S = \mathbb{K}[x_0,\dots,x_n]$ and $T = \mathbb{K}[y_0,\dots,y_n]$. We denote the d–graded piece of S and T by S_d and T_d respectively. In terms of the notation used in Section 2.1.1, S_d is identified with S^dV. The apolarity action is the action of T on S where y_i acts on S as the partial derivative $\frac{\partial}{\partial x_i}$. On the monomials, this action is defined on monomials as follows. Consider the monomials $x^\alpha = x_0^{\alpha_0}\cdots x_n^{\alpha_n} \in S$ and $y^\beta = y_0^{\beta_0}\cdots y_n^{\beta_n} \in T$ for $\alpha = (\alpha_0,\dots,\alpha_n), \beta = (\beta_0,\dots,\beta_n) \in \mathbb{N}^{n+1}$. Then y^β acts on x^α by

$$y^\beta \circ x^\alpha = \begin{cases} \displaystyle\prod_{i=0}^{n} \frac{(\alpha_i)!}{(\alpha_i-\beta_i)!} x^{\alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else} \end{cases},$$

where $\alpha \geq \beta$ if and only if $\alpha_i \geq \beta_i$ for every i. The apolaritity action defines for every $j \in \mathbb{N}$ a perfect pairing

$$S_i \times T_i \to S_0 \simeq \mathbb{K}$$

sending the tuple (F, G) to $G \circ F$.

Definition 2.1.10. *Given a polynomial $F \in S_d$, the $(i, d-i)$–catalecticant map of F is the linear map*

$$\begin{array}{rccc} F_{i,d-i}: & T_i & \longrightarrow & S_{d-i} \\ & G & \longmapsto & G \circ F \end{array}. \tag{2.4}$$

A first consequence of this definition is that the image of $F_{i,d-i}$ equals $\langle \nabla^i F \rangle$. Recall that $\nabla^i F$ is the set of all i–th partial derivatives of F. For instance, the catalecticant map $F_{1,d-1}$ is the map from T_1 to S_{d-1} sending y_i to the first order derivative of F with respect to x_i.

Example 2.1.11. *Fix $n = 2$ and $d = 4$, and consider the polynomial $F = x_0^4 + x_1^4 + x_2^4 + x_0^2 x_1^2$. The catalecticant map $F_{1,3} : T_1 \to S_3$ acts on the monomials of T_1 as follows*

$$y_0 \mapsto 4x_0^3 + 2x_0 x_1^2, \quad y_1 \mapsto 4x_1^3 + 2x_0^2 x_1, \quad y_2 \mapsto 4x_2^3.$$

In particular, $F_{1,3}$ is injective, i.e. it is a linear embedding. Similarly, the catalecticant map $F_{2,2} : T_2 \to S_2$ acts on the basis of monomials of T_2 as

$$y_0^2 \mapsto 12x_0^2 + 2x_1^2 \quad y_0 y_1 \mapsto 4x_0 x_1 \quad y_0 y_2 \mapsto 0 \quad y_1^2 \mapsto 12x_1^2 + 2x_0^2 \quad y_1 y_2 \mapsto 0 \quad y_2^2 \mapsto 12x_2^2.$$

In the basis of T_2 and S_2 given by the degree 2 monomials, the matrix of $F_{2,2}$ is

$$\begin{pmatrix} 12 & 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}.$$

In particular, the rank of $F_{2,2}$ is 4 and its kernel is generated by $y_0 y_2$ and $y_1 y_2$.

In Example 2.1.11, we show that the under certain basis, the matrix of the catalecticant map $F_{2,2}$ is symmetric. This is consequence of a more general fact. We write a polynomial F in S_d as

$$F = \sum_{|u|=d} a_u x^u.$$

In T_i we fix the standard basis given by the monomials, whereas in S_{d-i} we fix the basis given by the elements of the form $\frac{1}{u_0! \cdots u_n!} x^u$ for $u \in \mathbb{N}^{n+1}$ with $|u| = d - i$. Under these bases, the catalecticant map $F_{i,d-i}$ is given by the matrix

$$\mathrm{Cat}_F(d-i, i) := (a_{u+v})_{|u|=i, |v|=d-i}. \tag{2.5}$$

The matrix (2.5) is called the i–th catalecticant matrix. On of the first properties of catalecticant matrices is that for any $F \in S_d$, $\mathrm{Cat}_F(d-i,i)^{\mathrm{t}} = \mathrm{Cat}_F(i,d-i)$ (See [77, Section 1]). In particular, when d is even, $\mathrm{Cat}_F(d/2,d/2)$ is symmetric.

Example 2.1.12. *For degree d binary forms, the catalecticant matrix $\mathrm{Cat}_F(d-i,i)$ is the $(d-i+1,i+1)$–matrix*

$$\begin{pmatrix} a_{d,0} & a_{d-1,1} & \cdots & a_{d-i,i} \\ a_{d-1,1} & a_{d-2,2} & \cdots & a_{d-i-1,i+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,d-i} & a_{i-1,d-i+1} & \cdots & a_{0,d} \end{pmatrix}.$$

Definition 2.1.13. *Given $F \in S_d$, the* apolar ideal *of F, denoted by $F^{\perp}$, is the ideal of T defined as*

$$F^{\perp} := \{G \in T : \; G \circ F = 0\}.$$

The apolar ideal $F^{\perp}$ is a graded ideal of T and for $i \le d$, its i–th graded component is equal to the kernel of the catalecticant $F_{i,d-i}$. For $i \ge d+1$, the i–th graded component of $F^{\perp}$ is T_i.

Example 2.1.14. *Fix $n = 2$ and $d = 4$, and consider the polynomial $F = x_0^4 + x_1^4 + x_2^4 + x_0^2x_1^2 \in S_4$. In Example 2.1.11 we saw that $F_{1,3}$ is injective and the kernel of $F_{2,2}$ is generated by y_0y_2 and y_1y_2. Therefore, $(F^{\perp})_1 = 0$ and $(F^{\perp})_2 = \langle y_0y_2, y_1y_2\rangle$. One can check that all the generators of $(F^{\perp})_3$ are in the ideal generated by $(F^{\perp})_2$. In degree 4, the elements of $(F^{\perp})_4$ that are not obtained from $(F^{\perp})_2$ are linear combinations of $y_0^3y_1$ and $y_0y_1^3$. We conclude that the apolar ideal of F is*

$$F^{\perp} = \langle y_0y_2, y_1y_2, y_0^3y_1, y_0y_1^3\rangle + \langle T_5\rangle.$$

A celebrated result concerning apolar ideals is the Apolarity Lemma which relates apolar ideals with Waring decompositions (see [5, Lemma 2.80]). Using the apolar ideal, we consider the quotient $A_F := T/F^{\perp}$. Then, A_F is an Artinian Gorenstein ring (see [74, Section 2.3]). We call A_F the apolar Artinian Gorenstein ring of F. For further literature on Gorenstein rings we refer to [74]. One of the main results concerning apolar Artinian Gorenstein rings is due to Macaulay (see [89, Section 4])

Theorem 2.1.15. *[74, Lemma 2.12] There is a bijection between S_d and graded Artinian Gorenstein ring $A = T/I$ with socle in degree d.*

Note that the i–th graded component of A_F vanishes for $i \ge d+1$. A consequence of Theorem 2.1.15 is that the Hilbert function $h_{A_F}(t)$ of A_F is symmetric. In other words, $h_{A_F}(t) = h_{A_F}(d-t)$ for $0 \le t \le d$.

2.2 Hessian varieties and Hessian correspondence

The aim of this section is to present the main objects of study of this chapter: Hessian maps, Hessian varieties and the Hessian correspondence. These are the analogous to Gaussian maps, the dual varieties and the map $\mathcal{G}^1_{d,n}$ to second order varieties. Given a polynomial $F \in S^dV$, we denote the Hessian matrix of F by H_F.

Definition 2.2.1. *The Hessian map is defined as the rational map*

$$\begin{array}{rccc} h_F: & \mathbb{V}(F) & \dashrightarrow & \mathbb{P}(S^2V) \\ & p & \longmapsto & \mathrm{H}_F(p). \end{array} \tag{2.6}$$

The Hessian variety of F is the closure of $h_F(\mathbb{V}(F))$.

Clearly the Hessian map generalizes the Gaussian map and, taking into account that $\mathbb{P}(S^2V)$ can be identified with the set of the symmetric matrices, the Hessian map can be seen as the degree 2 polar map. In this case, the Hessian variety is the second polar variety. Let $z_{i,j}$ be the coordinate of $\mathbb{P}(S^2V)$ representing the monomial x_ix_j. By abuse of notation, we will often consider h_F, in (2.6), as a rational map from $\mathbb{P}^n$ instead of from $\mathbb{V}(F)$.

Example 2.2.2. *For $d = 4$ and $n = 2$ we consider the polynomial $F = x_0^4 + x_1^4 + x_2^4$. The Hessian matrix of F is*

$$\mathrm{H}_F = \begin{pmatrix} 12x_0^2 & 0 & 0 \\ 0 & 12x_1^2 & 0 \\ 0 & 0 & 12x_2^4 \end{pmatrix}.$$

In particular, the Hessian map of F is

$$\begin{array}{rccc} h_F: & \mathbb{V}(F) \subset \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ & [x_0, x_1, x_2] & \longmapsto & [12x_0^2, 0, 0, 12x_1^2, 0, 12x_2^4] \end{array},$$

and the Hessian variety of F is given by the ideal

$$\langle z_{0,1}, z_{0,2}, z_{1,2}, z_{0,0}^2 + z_{1,1}^2 + z_{2,2}^2 \rangle.$$

Similarly as in the case of the map $\mathcal{G}^1_{d,n}$, our goal is to define a rational map that associates a polynomial $F \in \mathbb{P}(S^dV)$ to its Hessian variety. The first difficulty we encounter is that the Hessian variety is not a hypersurface in $\mathbb{P}(S^2V)$. Therefore, the target space of such a map can not be a space of polynomials of certain degree. We have to consider a Hilbert scheme as the target space. Let $\Xi_{d,n}$ be the closure of the set $\{(x, F) \in \mathbb{P}(S^2V) \times \mathbb{P}(S^dV) : x \in \overline{h_F(\mathbb{V}(F))}\}$. The projection $\Xi_{d,n} \dashrightarrow \mathbb{P}(S^dV)$ defines a family of algebraic varieties parametrized by an open subset of $\mathbb{P}(S^dV)$. By [27, Proposition 6], the Hessian map of a smooth hypersurface is finite. Hence, the Hessian variety of smooth hypersurfaces is irreducible of dimension $n-1$. Therefore, there exists a dense open subset of $\mathbb{P}(S^dV)$ where the fibers of the family $\Xi_{d,n}$ are irreducible and they all have the same dimension. Let $p_{d,n}(t)$ be the Hilbert polynomial of the Hessian variety of a generic hypersurface in this open subset. In particular, $p_{d,n}(t)$ is the Hilbert polynomial of the Hessian variety of a generic element in $\mathbb{P}(S^dV)$.

Definition 2.2.3. *We define the* Hessian correspondence *as the rational map*

$$\begin{array}{rccc} H_{d,n}: & \mathbb{P}(S^dV) & \dashrightarrow & \mathrm{Hilb}^{p_{d,n}(t)}(\mathbb{P}(S^2V)) \\ & F & \longmapsto & \overline{h_F(\mathbb{V}(F))}. \end{array} \tag{2.7}$$

In other words, $H_{d,n}$ sends a hypersurface to its Hessian variety.

Therefore, h_F and $H_{d,n}$ can be seen as the generalization, to the second order of derivation, of the Gaussian map and $\mathcal{G}^1_{d,n}$, respectively.

Example 2.2.4. *For $d=3$ and $n=2$, we consider the polynomials $F=x_0x_1x_2$ and $G=x_0^3+x_1^3$. Their Hessian matrices are*

$$\mathrm{H}_F = \begin{pmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & x_0 \\ x_1 & x_0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathrm{H}_G = \begin{pmatrix} 6x_0 & 0 & 0 \\ 0 & 6x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. One can check that the Hessian varieties of F and G are

$$\mathbb{V}(z_{0,0}, z_{1,1}, z_{2,2}, z_{0,1}z_{0,2}z_{1,2}) \quad \textit{and} \quad \mathbb{V}(z_{0,1}, z_{0,2}, z_{1,2}, z_{2,2}, z_{0,0}^3 + z_{1,1}^3)$$

respectively. We observe that the Hessian variety of F is a singular cubic curve in a plane in $\mathbb{P}^5$, whereas the Hessian variety of G consists of three distinct points in a line in $\mathbb{P}^5$. In particular, the Hilbert polynomials of these two Hessian varieties are distinct. In Section 2.4, we will see that $p_{3,n}(t) = \binom{t+n}{n} - \binom{t+n-3}{n}$. We conclude that G lies in the base locus of $H_{3,2}$.

Example 2.2.4 shows that the family of Hessian varieties $\Xi^1_{d,n} \dashrightarrow \mathbb{P}(S^dV)$ is not flat since different fibers may have different Hilbert polynomial (see [67, Theorem 9.9]). As a consequence, we get that the Hessian correspondence has a base locus consisting of the polynomials F whose Hessian variety does not have Hilbert polynomial $p_{d,n}(t)$. Analogously, we could define the Hessian correspondence for varieties of polynomials contained in this base locus. The main example we are interested in is the case of the variety of polynomials with Waring rank k.

Definition 2.2.5. *We define the* Hessian correspondence of polynomials with Waring rank *k as the rational map*

$$\begin{array}{rccc} H_{d,n,k}: & \sigma_k(V^{d,n}) & \dashrightarrow & \mathrm{Hilb}^{p_{d,n,k}(t)}(\mathbb{P}(S^2V)) \\ & F & \longmapsto & \overline{h_F(\mathbb{V}(F))}. \end{array} \tag{2.8}$$

where $p_{d,n,k}(t)$ is the Hilbert polynomial of the Hessian variety of a generic hypersurface in $\sigma_k(V^{d,n})$.

Example 2.2.4 shows that $p_{3,2}(t)$ and $p_{3,2,2}(t)$ are distinct. In general, it does not hold that $p_{d,n} = p_{d,n,k}$ and $H_{d,n}$ and $H_{d,n,k}$ are distinct maps. However, for any d and n there exists a k such that $\sigma_k(V^{d,n}) = \mathbb{P}(S^dV)$, and $H_{d,n} = H_{d,n,k}$; for instance, if k is the Waring rank of a generic polynomial. This Waring rank is achieved when the dimension of $\sigma_k(V^{d,n})$ (see Theorem 2.1.9) equals the dimension of $\mathbb{P}(S^dV)$.

The natural question, and the central aim of this chapter, is the extension of the results concerning the maps g^1 and $\mathcal{G}^1_{d,n}$ to h_F, $H_{d,n}$ and $H_{d,n,k}$. More concretely, we analyse whether $H_{d,n}$ and $H_{d,n,k}$ are birational onto its image. From a computational approach, given a generic variety X in the image of $H_{d,n}$ ($H_{d,n,k}$, respectively), our goal is to recover, when possible, the polynomials whose Hessian variety is X. That is, to determine $F \in \mathbb{P}(S^dV)$ such that $H_{d,n}(F) = X$.

We observe that the cases $d = 1$ and $d = 2$ have a direct answer. For $d = 1$, the problem is trivial since the Hessian is the zero matrix. Let now $d = 2$. For $F \in \mathbb{P}(S^2V)$, the Hessian matrix H_F is a symmetric matrix with constant entries. Therefore, for $F \in \mathbb{P}(S^2V)$, the Hessian map is a constant map sending $\mathbb{V}(F)$ to the point $H_F \in \mathbb{P}(S^2V)$. We get that $p_{2,n}(t) = 1$ and $\mathrm{Hilb}^{p_{2,n}(t)}(\mathbb{P}(S^2V)) = \mathbb{P}(S^2V)$. Therefore, the Hessian correspondence for quadrics is the map $H_{2,n} : \mathbb{P}(S^2V) \to \mathbb{P}(S^2V)$ that sends F to H_F. Applying Euler formula twice, we get that $2F = (x_0, \dots, x_n)\mathrm{H}_F(x_0, \dots, x_n)^{\mathrm{t}}$. In particular, we get that $H_{2,n}$ is the isomorphism between degree two polynomials and symmetric matrices.

2.3 Hessian correspondence and Waring rank

Let us start this section with a motivating example. Let $n = 3$ and $d = 3$, and consider the polynomials $F = x_0^3 + x_1^3 + x_2^3$ and $G = x_0^3 + x_1^3 - x_2^3$. Both F and G have Waring rank 3 and their Hessian matrices are the diagonal matrices

$$\begin{pmatrix} 6x_0 & 0 & 0 \\ 0 & 6x_1 & 0 \\ 0 & 0 & 6x_2 \end{pmatrix} \text{ and } \begin{pmatrix} 6x_0 & 0 & 0 \\ 0 & 6x_1 & 0 \\ 0 & 0 & -6x_2 \end{pmatrix}$$

respectively. In particular, the corresponding Hessian maps h_F and h_G factors through the closed embedding of $\mathbb{V}(z_{01}, z_{02}, z_{12})$ in $\mathbb{P}(S^2V)$. Moreover, one can check that the Hessian varieties of F and G are both equal to

$$\mathbb{V}(z_{01}, z_{02}, z_{12}, z_{00}^3 + z_{11}^3 + z_{22}^3) \subset \mathbb{P}(S^2V) \simeq \mathbb{P}^5.$$

Similarly, the Hessian varieties of $x_0^3 - x_1^3 + x_2^3$ and $-x_0^3 + x_1^3 + x_2^3$ equal $H_{3,2}(F)$ too. In particular, we deduce that there exists a locus in $\mathbb{P}(S^3V)$ where $H_{3,2}$ is not injective but has degree at least 4. In this section we study how the Hessian correspondence acts on the locus of these type of polynomials.

In (2.7), we defined the Hessian correspondence as a rational map from $\mathbb{P}(S^dV)$ to certain Hilbert scheme. Similarly, in (2.8), we defined $H_{d,n,k}$ as the Hessian correspondence for polynomials in $\sigma_k(V^{d,n})$. In this section we analyse the Hessian correspondence $H_{d,n,k}$ for polynomials with Waring rank $k \leq n+1$. In this case, under this assumption, the secant variety $\sigma_k(V^{d,n})$ has a dense $\mathrm{PGL}(n+1)$–orbit. We analyse the interaction of the action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}(S^dV)$ and the Hessian map. This study allows us to conclude Section 2.3.1 with the computation of the degree of $H_{d,n,k}$ for $k \leq n+1$. In Section 2.3.2, we provide a numerical algorithm for computing the fibers of $H_{d,n,k}$.

2.3.1 Degree of $H_{d,n,k}$ for $k \leq n+1$

For any d we consider the group action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}(\mathrm{S}^dV)$ given by $g \cdot F = F \circ g^{\mathrm{t}}$ for $g \in \mathrm{PGL}(n+1)$ and $F \in \mathbb{P}(\mathrm{S}^dV)$. This defines a representation $\rho : \mathrm{PGL}(n+1) \to \mathrm{PGL}(S^dV)$. Since the target space of the Hessian map is $\mathbb{P}(S^2V)$, the representation ρ for $d = 2$ will play a fundamental role.

Example 2.3.1. *Fix $n = 1$ and $d = 1$. Consider the basis of S^2V given by the monomials x_0^2, x_0x_1, x_1^2. Then,*

$$\begin{array}{l} g \cdot x_0^2 = a^2x_0^2 + 2acx_0x_1 + c^2x_1^2 \\ g \cdot x_0x_1 = abx_0^2 + (ad+bc)x_0x_1 + cdx_1^2 \\ g \cdot x_1^2 = b^2x_0^2 + 2bdx_0x_1 + d^2x_1^2 \end{array}, \ \textit{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2).$$

Therefore, the representation ρ is the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

The next lemma analyzes how this action interacts with the Hessian map.

Lemma 2.3.2. *Let $n \geq 1$ and $d \geq 3$. Then, for $F \in \mathbb{P}(\mathrm{S}^dV)$ and $g \in \mathrm{PGL}(n+1)$, we have that $h_{g \cdot F} = \rho(g^{\mathrm{t}})^{\mathrm{t}} \circ h_F \circ g^{\mathrm{t}}$. In other words, the following diagram commutes:*

$$\begin{tikzcd} \mathbb{V}(F) \arrow[r, "h_F"] & \mathbb{P}(S^2V) \arrow[d, "\rho(g^{\mathrm{t}})^{\mathrm{t}} \cdot"] \\ \mathbb{V}(g \cdot F) \arrow[u, "g^{\mathrm{t}}"] \arrow[r, "h_{g \cdot F}"'] & \mathbb{P}(S^2V) \end{tikzcd}$$

In particular, we have that $H_{d,n}(g \cdot F) = \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{d,n}(F))$.

Proof. By the Leibniz Rule, for $i, j \in [n]$ we have that

$$\frac{\partial^2 g \cdot F}{\partial x_i \partial x_j} = \sum_{k,l} g^{\mathrm{t}}_{k,i} g^{\mathrm{t}}_{l,j} \frac{\partial^2 F}{\partial x_k \partial x_l} \circ g^{\mathrm{t}}.$$

Hence, the coordinate $z_{i,j}$ of the composition $h_{g^{\mathrm{t}} \cdot F} \circ (g^{\mathrm{t}})^{-1}$ is

$$\sum_{k,l} g_{i,k} g_{j,l} \frac{\partial^2 F}{\partial x_k \partial x_l} = \sum_{k<l} (g_{i,k} g_{j,l} + g_{i,l} g_{j,k}) \frac{\partial^2 F}{\partial x_k \partial x_l} + \sum_k g_{i,k} g_{j,k} \frac{\partial^2 F}{\partial x_k^2}. \tag{2.9}$$

On the other hand, the coordinate $z_{i,j}$ of $\rho(g^{\mathrm{t}})^{\mathrm{t}}$ is

$$\sum_{k<l} (g_{i,k} g_{j,l} + g_{i,l} g_{j,k}) z_{k,l} + \sum_k g_{i,k} g_{j,k} z_{k,k}.$$

By (2.9), the coordinate $z_{i,j}$ of $\rho(g^{\mathrm{t}})^{\mathrm{t}} \circ h_F$ is equal to the coordinate $z_{i,j}$ of $h_{g \cdot F} \circ (g^{\mathrm{t}})^{-1}$. The equality $H_{d,n}(g \cdot F) = \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{d,n}(F))$ follows from the first part of the lemma and the fact that $\mathbb{V}(g \cdot F) = (g^{\mathrm{t}})^{-1}(\mathbb{V}(F))$. □

As a consequence of Lemma 2.3.2, we derive the following result.

Proposition 2.3.3. *Let $F \in \mathbb{P}(S^d V)$ and $g \in \mathrm{PGL}(n+1)$. Then, $g \cdot H_{d,n}^{-1}(H_{d,n}(F)) = H_{d,n}^{-1}(H_{d,n}(g \cdot F))$. In particular the fibers of $H_{d,n}(F)$ and $H_{d,n}(g \cdot F)$ via $H_{d,n}$ are isomorphic.*

Proof. Let $\tilde{F} \in \mathbb{P}(S^d V)$ with same Hessian variety as F. By Lemma 2.3.2, we get that

$$H_{d,n}(g \cdot \tilde{F}) = \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{d,n}(\tilde{F})) = \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{d,n}(F)) = H_{d,n}(g \cdot F).$$

In particular, $g \cdot \tilde{F}$ has same Hessian variety as $g \cdot F$. Hence, $g \cdot H_{d,n}^{-1}(H_{d,n}(F))$ is contained in $H_{d,n}^{-1}(H_{d,n}(g \cdot F))$. Similarly, since $g^{-1} \cdot g \cdot F = F$, we deduce that $g^{-1} \cdot H_{d,n}^{-1}(H_{d,n}(g \cdot F))$ is contained in $H_{d,n}^{-1}(H_{d,n}(F))$. Now, the proof follows applying g to the previous inclusion. □

Note that Lemma 2.3.2 and Proposition 2.3.3 also hold if $H_{d,n}$ is replaced by $H_{d,n,k}$, for any k. Now, assume that $k \leq n+1$ and let $\mathcal{U}_k \subseteq \sigma_k(V^{d,n})$ be the subset of polynomials of the form $l_1^d + \cdots + l_k^d$ such that $l_1, \ldots, l_k \in S^1 V$ are linearly independent. Note that $\mathcal{U}_k$ contains a nonempty dense open subset of $\sigma_k(V^{d,n})$. Moreover, since $k \leq n+1$, it is the $\mathrm{PGL}(n+1)$–orbit of the polynomial $F_k := x_0^d + \cdots + x_{k-1}^d$. Therefore, by Proposition 2.3.3, in order to understand the fibers of $H_{d,n,k}$ restricted to $\mathcal{U}_k$, it is enough to study the fiber of the Hessian variety of the polynomial $F_k := x_0^d + \cdots + x_{k-1}^d$.

Note that up to scalar multiplication, the Hessian matrix of F_k is the diagonal matrix

$$\begin{pmatrix} x_0^{d-2} & & & & & \\ & \ddots & & & & \\ & & x_{k-1}^{d-2} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

In particular, the Hessian variety of F_k is a hypersurface in the $(k-1)$–th dimensional linear subspace

$$\mathbb{P}(U) := \mathbb{V}(z_{i,j} : (i,j) \notin \{(0,0), \ldots, (k-1, k-1)\}) \subset \mathbb{P}(S^2V). \tag{2.10}$$

Note that $\mathbb{P}(U)$ equals $\overline{h_{F_k}(\mathbb{P}^n)}$. Now, let $G \in \mathbb{P}(S^dV)$ with the same Hessian variety as F_k. This implies that $\frac{\partial^2 G}{\partial x_i x_j} = 0$ for $(i,j) \notin \{(0,0), \ldots, (k-1, k-1)\}$. From the Euler's formula we deduce that the first order derivatives of G, with respect to $x_k, \ldots, x_n$, vanish. Hence, G is a polynomial in the variables $x_0, \ldots, x_{k-1}$. Therefore, in order to study $H_{d,n,k}$, we assume that $k = n+1$.

Lemma 2.3.4. *Let $G \in \mathbb{P}(S^dV)$ with the same Hessian variety as F_{n+1}. Then, $G = \lambda_0 x_0^d + \cdots + \lambda_n x_n^d$ for $\lambda_0, \ldots, \lambda_n \neq 0$.*

Proof. Let a_w be the coefficient of $x^w := x_0^{w_0} \cdots x_n^{w_n}$ for $w = (w_0, \ldots, w_n)$ and $|w| := w_0 + \cdots + w_n = d$. Now, fix w such that $|w| = d$ and $w \neq d\, e_i$ for some i. Here, e_i is the i–th standard vector $(0, \ldots, 0, 1, 0, \ldots, 0)$. In particular, there exists $i \neq j$ with $w_i, w_j \geq 1$. We deduce that a_w is the coefficient of the monomial $x^{w-e_i-e_j}$ of $\frac{\partial^2 G}{\partial x_i x_j}$. Since G and F_{n+1} have the same Hessian variety, from equation (2.10) we get that $\frac{\partial^2 G}{\partial x_i x_j} = 0$. Therefore, $a_w = 0$ for $w \notin \{d\, e_0, \ldots, d\, e_n\}$, and $G = \lambda_0 x_0^d + \cdots + \lambda_n x_n^d$ for $\lambda_i \in \mathbb{K}$. Now assume that $\lambda_n = 0$. Then, the Hessian variety of G would be a hypersurface in a projective space of dimension $n-1$. This is a contradiction since the Hessian variety of F_{n+1} has dimension $n-1$. We conclude that $\lambda_i \neq 0$ for every i. □

Let Γ be the linear subspace spanned by $x_0^d, \ldots, x_n^d$ in S^dV. We identify a point $\lambda := [\lambda_0, \ldots, \lambda_n] \in \mathbb{P}(\Gamma)$ with the polynomial $F_\lambda := \lambda_0 x_0^d + \cdots + \lambda_n x_n^d$. Using this notation F_{n+1} equals $F_{\mathbb{1}}$, where $\mathbb{1} := [1, \ldots, 1]$. We denote the n–dimensional torus of $\mathbb{P}(\Gamma)$ by $\mathbb{T}$. From Lemma 2.3.4, we deduce that if G has the same Hessian variety as $F_{\mathbb{1}}$, then G must lie in the torus $\mathbb{T}$. Given $F_\lambda \in \mathbb{P}(\Gamma)$, we consider the rational map

$$\begin{array}{cccc} h_\lambda : & \mathbb{P}^n & \dashrightarrow & \mathbb{P}(U) \\ & [x_0, \ldots, x_n] & \longmapsto & [\lambda_0 x_0^{d-2}, \ldots, \lambda_n x_n^{d-2}] \end{array},$$

Note that for $F_\lambda \in \mathbb{T}$, h_λ is well-defined everywhere and it has degree $(d-2)^n$. The image of $\mathbb{V}(F_\lambda)$ is a hypersurface defined by a homogeneous polynomial $\tilde{F}_\lambda$. We denote the degree of $\tilde{F}_\lambda$ by $\tilde{d}$. Note that the Hessian variety of F_λ is the intersection of the hypersurface defined by $\tilde{F}_\lambda$ and $\mathbb{P}(U)$.

Lemma 2.3.5. *For $F_\lambda \in \mathbb{T}$, the degree $\tilde{d}$ of the polynomial $\tilde{F}_\lambda$ equals $d(d-2)^{n-1}$ if d is odd, and $d(d-2)^{n-1}/2^n$ if d is even.*

Proof. Since h_λ is finite,the product of the degree of $\tilde{F}_\lambda$ and the degree of the restriction of h_λ to $\mathbb{V}(F_\lambda)$ equals $d(d-2)^{n-1}$. By Lemma 2.3.2, it is enough to prove the statement for $\tilde{F}_{\mathbb{1}}$. Hence, it is sufficient to check that

$$\deg h_{\mathbb{1}}|_{\mathbb{V}(F_{\mathbb{1}})} = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ 2^n & \text{if } d \text{ is even.} \end{cases}$$

Let p be a generic point in $\mathbb{V}(F_{\mathbb{1}})$ and let $q \in \mathbb{V}(F_{\mathbb{1}})$ be such that $h_{\mathbb{1}}(p) = h_{\mathbb{1}}(q)$. Then, there exist $\mu \neq 0$ and $\xi_0, \ldots, \xi_n$ $(d-2)$–roots of the unit, such that $q_i = \mu\xi_i p_i$ for every $i \in \{0, \ldots, n\}$. Since $\sum q_i^d = 0$, we deduce that

$$\xi_0^2 p_0^d + \cdots + \xi_n^2 p_n^d = p_0^d + \cdots + p_n^d = 0.$$

In particular, we have that $(\xi_0^2 - 1)p_0^d + \cdots + (\xi_n^2 - 1)p_n^d = 0$. Assume that $n \geq 2$. Since this equality holds for $p \in \mathbb{V}(F_{\mathbb{1}})$ generic and $\dim \mathbb{V}(F_{\mathbb{1}}) \geq 1$, we get that $\xi_i^2 = 1$ for every i. Since $\xi_i^{d-2} = 1$, we deduce that $\xi_i = \pm 1$ if d is even or $\xi_i = 1$ if d is odd. For $n = 1$, we have that $p_0^d = -p_1^d$, $p_0, p_1 \neq 0$ and $\xi_0^2 p_0^d = -\xi_1^2 p_1^d$. This implies that $\xi_0^2 = \xi_1^2$. Since we can assume that $\xi_0 = 1$, we deduce that $\xi_0^2 = \xi_1^2 = 1$. Hence, we conclude that the fiber $h_{\mathbb{1}}^{-1}(h_{\mathbb{1}}(p))$ is the point p if d is odd or the 2^n points of the form $[\pm p_0, \ldots, \pm p_n]$ if d is even. $\square$

A first consequence of Lemma 2.3.5 is the computation of the Hilbert polynomial $p_{d,n,k}(t)$ for $k \leq n+1$. By Lemma 2.3.2, $p_{d,n,k}(t)$ is the Hilbert polynomial of $H_{d,n,k}(F_{\mathbb{1}})$. The Hessian variety of $F_{\mathbb{1}}$ is the intersection of $\mathbb{P}(U) \simeq \mathbb{P}^{k-1}$ and a hypersurface of degree $\tilde{d}$. From Lemma 2.3.5 we conclude that

$$p_{d,n,k} = \begin{cases} \binom{t+n}{n} - \binom{t+n-d(d-2)^{n-1}/2^n}{n} & \text{for } d \text{ even} \\ \binom{t+n}{n} - \binom{t+n-d(d-2)^{n-1}}{n} & \text{for } d \text{ odd} \end{cases}$$

for $k \leq n+1$.

Lemma 2.3.5 allows us to consider the rational map

$$\tilde{H}_{d,n} : \mathbb{P}(\Gamma) \dashrightarrow \mathbb{P}(S^{\tilde{d}} U)$$

sending F_λ to $\tilde{F}_\lambda$. Note that $\tilde{H}_{d,n}$ is well defined in the torus $\mathbb{T}$. In general, using Lemma 2.3.2, and the fact that the $\mathrm{PGL}(n+1)$–orbit of F_k is dense in $\sigma_k(V^{d,n})$ for $k \leq n+1$, we get that the Hessian variety of a generic element in $\sigma_k(V^{d,n})$ is a hypersurface of degree $\tilde{d}$ in a k–dimensional subspace of $\mathbb{P}(S^2V)$. In particular, the target space of $H_{d,n,k}$ is the projectivization of the $\tilde{d}$–th symmetric power of the universal bundle of $\mathrm{Gr}(k+1, S^2V)$ and $\tilde{H}_{d,k}$ is the restriction of $H_{d,n,k}$ to $\mathbb{P}(\Gamma)$. Hence, to study the fibers of the map $H_{d,n,k}$ it is enough to focus on $\tilde{H}_{d,n}$.

Proposition 2.3.6. *For $d \geq 3$, the restriction of $\tilde{H}_{d,n}$ to the torus $\mathbb{T}$ is finite and unramified. For d odd it has degree 2^n and for d even it is an isomorphism.*

Proof. We first compute the set theoretical fiber of $\tilde{H}_{d,n}(F_{\mathbb{1}})$. Let $F_\lambda \in \mathbb{P}(\Gamma)$ such that $\tilde{F}_\lambda = \tilde{F}_{\mathbb{1}}$. Consider a point $p = [p_0, p_1, 0, \ldots, 0]$, with $p_0 p_1 \neq 0$, in the hypersurface defined by F_λ. In other words, $\lambda_0 p_0^d + \lambda_1 p_1^d = 0$. Since $\tilde{F}_{\mathbb{1}} = \tilde{F}_\lambda$, there exists $q = [q_0, \ldots, q_n] \in \mathbb{V}(F_{\mathbb{1}})$ such that $h_\lambda(p) = h_{\mathbb{1}}(q)$. In other words, there exists a solution to the system of equations

$$\begin{array}{lcll} \lambda_0 p_0^d + \lambda_1 p_1^d & = & 0, & \\ q_0^d + q_1^d & = & 0, & \\ \lambda_0 p_0^{d-2} q_1^{d-2} - \lambda_1 p_1^{d-2} q_0^{d-2} & = & 0, & \\ q_i^{d-2} & = & 0 & \text{for } i = 2, \ldots, n. \end{array} \tag{2.11}$$

Since $\lambda_i, p_i, q_i \neq 0$, for $i = 0, 1$, we have that

$$\lambda_1 \left(\frac{p_1 q_0}{p_0 q_1} \right)^{d-2} = \lambda_0 = -\lambda_1 \left(\frac{p_1}{p_0} \right)^d .$$

Multiplying by $(q_0/q_1)^2$ the previous equation we get that

$$p_0^2 q_1^2 = p_1^2 q_0^2. \tag{2.12}$$

Assume first that d is even, i.e., $d = 2a$ for $a \in \mathbb{N}$. Using equations (2.11) and (2.12) we get that

$$\lambda_0 (p_0 q_1)^{d-2} = (p_1 q_0)^{d-2} \lambda_1 = (p_1^2 q_0^2)^{a-1} \lambda_1 = (p_0^2 q_1^2)^{a-1} \lambda_1 = (p_0 q_1)^{d-2} \lambda_1.$$

In particular, we deduce that $\lambda_0 = \lambda_1$. Applying this argument to points in $\mathbb{V}(F_\lambda)$ of the form $[p_0, 0, \ldots, 0, p_i, 0, \ldots, 0]$, we deduce that $\lambda_0 = \lambda_i$, and we conclude that $F_\lambda = F_n$.
Assume now that d is odd. Applying the same reasoning as in the even case to the equation

$$(\lambda_0 p_0^{d-2} q_1^{d-2})^2 = (\lambda_1 p_1^{d-2} q_0^{d-2})^2,$$

we deduce that $\lambda_0^2 = \lambda_1^2 = \cdots = \lambda_n^2$. Now assume that $\lambda_0^2 = \lambda_1^2 = \cdots = \lambda_n^2$, i.e. $F_\lambda = \sum \pm x_i^d$. Thus, h_λ is the map sending $[x_0, \ldots, x_n]$ to $[\pm x_0^{d-2}, \ldots, \pm x_n^{d-2}]$. Let p be a point in $\mathbb{V}(\tilde{F}_{\mathbb{1}})$. Then, there exists $q \in \mathbb{V}(F_{\mathbb{1}})$ such that $h_{\mathbb{1}}(q) = [q_0^{d-2}, \ldots, q_n^{d-2}] = p$. Consider the point $q' = [\pm q_0, \ldots, \pm q_n]$. Then, we have that

$$F_\lambda(q') = \sum \pm(\pm q_i)^d = \sum (\pm)^2 q_i^d = F_{\mathbb{1}}(q) = 0,$$

and we derive that $q' \in \mathbb{V}(F_\lambda)$. Now, we get that

$$h_\lambda(q') = [\pm(\pm q_0)^{d-2}, \dots, \pm(\pm q_n)^{d-2}] = [(\pm)^2 q_0^{d-2}, \dots, (\pm)^2 q_n^{d-2}] = h_{\mathbb{1}}(q) = p.$$

We conclude that $h_\lambda(\mathbb{V}(F_\lambda)) = h_{\mathbb{1}}(\mathbb{V}(F_{\mathbb{1}}))$, and hence, $\tilde{F}_\lambda$ and $\tilde{F}_{\mathbb{1}}$ are equal. In particular, we deduce that the reduced structure of the fiber $\tilde{H}_{d,n}^{-1}(\tilde{F}_{\mathbb{1}})$ is the point $F_{\mathbb{1}}$ if d is even or the 2^n points of the form $\sum \pm x_i^d$ if d is odd. By the Generic Smoothness Theorem (see [67] Corollary 10.7), we get that a generic fiber of $\tilde{H}_{d,n}$ is reduced. By Proposition 2.3.3, all the fibers of the restriction of $\tilde{H}_{d,n}$ to $\mathbb{T}$ are isomorphic via the torus action and we conclude that the schematic structure of the fibers is reduced. Moreover, since the fibers of $\tilde{H}_{d,n}|_{\mathbb{T}}$ are all reduced of the same degree, we deduce that $\tilde{H}_{d,n}|_{\mathbb{T}}$ is étale. In particular, for d even it is an isomorphism. $\square$

Recall that for $k \leq n+1$, $\mathcal{U}_k \subseteq \sigma_k(V^{d,n})$ is the nonempty open subset of polynomials of the form $l_1^d + \cdots + l_k^d$ such that $l_1, \dots, l_k \in S^1 V$ are linearly independent. As a consequence of Proposition 2.3.6, we derive the following theorem.

Theorem 2.3.7. *For $d \geq 3$ and $k \leq n+1$, $H_{d,n,k}$ is generically finite. Furthermore, for d odd, it has degree 2^{k-1} and, for d even, $H_{d,n,k}$ is birational onto its image.*

Proof. By Proposition 2.3.3, the fibers of the restriction of $H_{d,n,k}$ to $\mathcal{U}_k$ are all isomorphic to the fiber of $H_{d,n,k}(F_k)$ via the action of $\mathrm{PGL}(n+1)$. In particular, by Lemma 2.3.4, these fibers are isomorphic to the fibers of the restriction of $\tilde{H}_{d,k}$ to the torus $\mathbb{T}$. Then, the proof follows from Proposition 2.3.6. $\square$

A first consequence of Theorem 2.3.7 is the case of the Hessian correspondence for $n = 1$ and $d = 3$. In this setting, a generic polynomial in $\mathbb{P}(S^3\mathbb{C}^2)$ has rank 2 (see Theorem 2.1.9), and hence, $H_{3,1,2}$ and $H_{3,1}$ coincide. In particular, we derive the following result.

Corollary 2.3.8. *The restriction of $H_{3,1}$ to the open subset of cubic binary forms with distinct roots is étale of degree 2.*

2.3.2 Recovery algorithm

Now we present an algorithm for computing the fibers of $H_{d,n,k}$ for $k \leq n+1$. In other words, given a generic variety X in the image of $H_{d,n,k}$, we aim to compute the polynomials F such that the Hessian variety of F is X. By Theorem 2.3.7 we know that, for d even, such polynomial is unique whereas, for d odd, the fiber of X consists in 2^{k-1} polynomials. As before, the main idea is to exploit Lemma 2.3.2. We recall that in $\mathbb{P}(S^2V)$, the Veronese variety $V^{2,n}$ coincides with the variety of rank one symmetric matrices.

Lemma 2.3.9. *Let $F \in \mathcal{U}_k$ and let H be the smallest linear subspace containing the Hessian variety of F. Then, H intersects $V^{2,n}$ in k points.*

Proof. By Lemma 2.3.2, we can assume that $F = x_0^d + \cdots + x_{k-1}^d$. Under this assumption, the equations of H are $z_{ij} = 0$ for $i \neq j$ and $z_{ii} = 0$ for $i \geq k$. One can check that the only rank one matrices in H are the diagonal matrices whose diagonal is the standard vector e_i for $i \leq k-1$. Moreover, the multiplicity in the intersection is one. □

Now, assume that we are given X in $H_{d,n,k}(\mathcal{U}_k)$, and let H be the smallest linear subspace containing X. We know that H has dimension $k-1$ and, by Lemma 2.3.9, we has that H intersects $V^{2,n}$ in k points, which correspond to the squares of certain linear forms $l_1, \ldots, l_k$. Then, by Lemma 2.3.2, any $F \in H_{d,n,k}^{-1}(X)$ can be written as $F = \lambda_1 l_1^d + \cdots + \lambda_k l_k^d$. In general, determining $l_1, \ldots, l_k$ requires solving a system of quadratic equations, which in some cases needs to be solved by numerical methods as `HomotopyContinuation.jl` [17].

Once we have determined $l_1, \ldots, l_k$ from X, we construct the linear change of coordinates $g \in \mathrm{PGL}(n+1)$ such that $g \cdot (l_1^d + \cdots + l_k^d) = x_0^d + \cdots + x_{k-1}^d$. By Proposition 2.3.3, we reduce the computation of the fibers of $H_{d,n,k}$ to the computation of the fibers of $\tilde{H}_{d,k}$.

Lemma 2.3.10. *Let F_λ be an element of the torus $\mathbb{T} \subset \mathbb{P}(\Gamma)$ and let $\mu = (\mu_0, \ldots, \mu_n)$ be such that $\mu_i^d = \lambda_i$ for $i = 0, \ldots, n$. Then,*

$$\tilde{F}_\lambda = \tilde{F}_{\mathbb{1}}\left(\frac{1}{\mu_0^2} z_0, \ldots, \frac{1}{\mu_n^2} z_n\right).$$

Proof. Let $\lambda \in \mathbb{T}$ and $\mu = (\mu_0, \ldots, \mu_n)$ such that $\mu_i^d = \lambda_i$ for $i = 0, \ldots, n$. We consider the diagonal matrix $g_\mu \in \mathrm{PGL}(n+1)$ whose diagonal is $(\mu_0, \ldots, \mu_n)$. Similarly, consider the linear automorphisms g_{μ^2} of $\mathbb{P}^n$ given by the diagonal matrix whose diagonal is $(\mu_0^2, \ldots, \mu_n^2)$. Analogously to Lemma 2.3.2, the composition $g_{\mu^2} \circ h_{\mathbb{1}}$ equals $h_\lambda \circ g_\mu^{-1}$ Since $g_\mu \cdot F_{\mathbb{1}}$ equals F_λ, we deduce that $g_{\mu^2}(\mathbb{V}(\tilde{F}_{\mathbb{1}})) = \mathbb{V}(\tilde{F}_\lambda)$. This implies that $\tilde{F}_\lambda = \tilde{F}_{\mathbb{1}}\left(\frac{1}{\mu_0^2} z_0, \ldots, \frac{1}{\mu_n^2} z_n\right)$. □

Now, let X be an element in the image of $\tilde{H}_{d,n}$. Recall X is a hypersurface of $\mathbb{P}(U)$ of degree $\tilde{d}$ as in Lemma 2.3.5. Let $\tilde{F}$ be the polynomial defining X. By Lemma 2.3.10, we get that $\tilde{F} = \tilde{F}_{\mathbb{1}}(\frac{1}{\mu_0^2} z_0, \ldots, \frac{1}{\mu_n^2} z_n)$ for some $\mu \in \mathbb{T}$. This identity defines a system of polynomial equations in the coordinates of μ. A solution μ of this system of equations gives a polynomial F_{μ^d} such that $\tilde{H}_{d,h}(F_{\mu^d}) = \tilde{F}$, where $\mu^d = (\mu_0^d, \ldots, \mu_n^d)$. Moreover, this system of equations can be solved by radicals.

In Algorithm 8 we summarize the steps of the recovery algorithm for reconstructing a generic hypersurface of rank $k \leq n+1$ from its Hessian variety.

Algorithm 8

Input: the ideal I defining a variety X in the image of $H_{d,n,k}$ for $k \leq n+1$.

Output: The polynomials in the fiber of X via $H_{d,n,k}$.

1. Compute the smallest linear subspace H containing X by taking the linear generators of the saturation of I.
2. Determine (numerically) the k points in the intersection $H \cap V^{2,n}$. Let $l_1, \dots, l_k$ the linear forms obtained by pulling back these k points thought the Veronese embedding of $V^{2,n}$.
3. Compute $g \in \mathrm{PGL}(n+1)$ such that $g \cdot (l_1^d + \cdots + l_k^d) = x_1^d + \cdots + x_k^d$.
4. Determine $\tilde{X} := \rho(g^{\mathrm{t}})^{\mathrm{t}}(X)$, which is a hypersurface in $\mathbb{P}(U)$. Let $\tilde{F}$ be the defining polynomial of $\tilde{X}$.
5. Determine a solution $\mu = (\mu_1, \dots, \mu_k) \in (\mathbb{K}^*)^k$ to the system of equations given by $\tilde{F} = \tilde{F}_{\mathbb{1}}\left(\frac{1}{\mu_1^2} z_0, \dots, \frac{1}{\mu_k^2} z_{k-1}\right)$.
6. If d is even return
$$(\mu_1 l_1)^d + \cdots + (\mu_k l_k)^d$$
else return the 2^{k-1} polynomials
$$\pm(\mu_1 l_1)^d \pm \cdots \pm (\mu_k l_k)^d.$$

Example 2.3.11. *Consider the variety $X \in \operatorname{Im} H_{5,1,2}$ given by the ideal*

$$\langle z_{0,0} - z_{1,1}, 3z_{0,1}^5 + 25z_{0,1}^4 z_{1,1} + 30z_{0,1}^3 z_{1,1}^2 + 50z_{0,1}^2 z_{1,1}^3 + 15z_{0,1} z_{1,1}^4 + 5z_{1,1}^5\rangle.$$

We apply Algorithm 8 to compute the two polynomials $F_1, F_2 \in \mathbb{P}(S^5V)$ whose Hessian variety is X. The smallest linear subspace H containing X is defined by the equation $z_{0,0} = z_{1,1}$. The equation of the Veronese variety $V^{2,1} \subset \mathbb{P}^2$ is defined by the polynomial $z_{0,1}^2 - 4z_{0,0}z_{1,1}$. Then, the intersection $H \cap V^{2,1}$ consists of the points $[1,2,1]$ and $[1,-2,1]$. The pullback of these points through the Veronese embedding are the linear forms $l_1 = x_0 + x_1$ and $l_2 = x_0 - x_1$. We consider the matrix

$$g = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then, $g \cdot (l_1^5 + l_2^5) = x_0^5 + x_1^5$. Now, using Example 2.3.1 we get that

$$\rho(g^{\mathrm{t}})^t = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

One can check that $\rho(g^{\mathrm{t}})^{\mathrm{t}}(X)$ is defined by the ideal $\langle z_{0,1}, 4z_{0,0}^5 + z_{1,1}^5\rangle$. On the other hand, the Hessian variety of $F_{\mathbb{1}} = x_0^5 + x_1^5$ is defined by the ideal $\langle z_{0,1}, z_{0,0}^5 + z_{1,1}^5\rangle$. In particular, $\tilde{F}_{\mathbb{1}} = z_{0,0}^5 + z_{1,1}^5$. Let $(\mu_0, \mu_1) \in (\mathbb{K}^*)^2$ be such that

$$4z_{0,0}^5 + z_{1,1}^5 \text{ and } \frac{1}{\mu_0^{10}} z_{0,0}^5 + \frac{1}{\mu_1^{10}} z_{1,1}^5$$

are equal up to scalar. This implies that $\mu_1^{10} = 4\mu_0^{10}$, and hence, $\mu_1^5 = \pm 2\mu_0^5$. Then, the polynomials F_1 and F_2 are of the form $\mu_0^5 l_0 + \mu_1^5 l_1^5$. We conclude that $F_1 = (x_0+x_1)^5 + 2(x_0 - x_1)^5$ and $F_1 = (x_0+x_1)^5 - 2(x_0-x_1)^5$. One can check that the Hessian variety of these two polynomials is indeed X.

2.4 Hessian correspondence of degree 3 hypersurfaces

In this section we carry out the study of the Hessian correspondence for cubic hypersurfaces. For $F \in \mathbb{P}(S^3V)$, its second derivatives are linear forms and the Hessian map is a linear map. In [87, Theorem 3.1], it is shown that for $F \in \mathbb{P}(S^3V)$, h_F is not a linear embedding if and only if F is a cone. Therefore, for generic $F \in \mathbb{P}(S^3V)$, the Hessian map h_F is a linear embedding. Moreover, we deduce that, for $F, G \in \mathbb{P}(S^3V)$ generic such that $H_{3,n}(F) = H_{3,n}(G)$, it holds that $\mathbb{V}(F)$ and $\mathbb{V}(G)$ are isomorphic as varieties. The next proposition analyzes the relation between two generic hypersurfaces of degree three with same Hessian variety.

Proposition 2.4.1. *Let $F \in \mathbb{P}(S^3V)$ be generic and $G \in \mathbb{P}(S^3V)$ be such that $H_{3,n}(F) = H_{3,n}(G)$. Then, there exists $g \in \mathrm{PGL}(n+1)$ such that $F \circ g = G$.*

Proof. $H_{3,n}(F)$ is contained in $h_F(\mathbb{P}^n) \cap h_G(\mathbb{P}^n)$. Assume that $h_F(\mathbb{P}^n) \neq h_G(\mathbb{P}^n)$. Since F is generic, the Hessian map of F is a linear embedding and $\dim H_{3,n}(F) = \dim h_F(\mathbb{P}^n) \cap h_G(\mathbb{P}^n) = n-1$. Then, $\mathbb{V}(F)$ is contained in $h_F^{-1}(h_G(\mathbb{P}^n))$, which is a hyperplane. This implies that the linear form defining this hyperplane is contained in the ideal defined by F. We reach a contradiction since F has degree 3, and we get that $h_F(\mathbb{P}^n) = h_G(\mathbb{P}^n)$. Let $g = h_F^{-1} \circ h_G$. Now, the proof follows from the fact that g is an automorphism of $\mathbb{P}^n$ that maps $\mathbb{V}(G)$ to $\mathbb{V}(F)$. $\square$

Remark 2.4.2. *As a consequence of the proof of Proposition 2.4.1 we deduce that, for generic $F \in \mathbb{P}(S^3V)$, $h_F(\mathbb{P}^n)$ is the unique n dimensional linear subspace containing the Hessian variety. In particular, $h_F(\mathbb{P}^n)$ is the smallest linear subspace containing $h_{3,n}(F)$.*

We are interested in giving a better description of the fibers of $H_{3,n}$. For instance, we saw in Corollary 2.3.8 that for $d = 3$ and $n = 1$ the Hessian correspondence has generically degree 2. However, to derive this result we used that a generic cubic binary form can be expressed as a sum of two cubes. In general, it is not true that a cubic form in $n+1$ variables has Waring rank $n+1$. As a consequence we distinguish two cases in our study, $n=1$ and $n \geq 2$.

2.4.1 Case $n = 1$

In this section we set $d=3$ and $n=1$. In Corollary 2.3.8 we showed that the restriction of $H_{3,1}$ to the open subset of polynomials with distinct roots is a degree 2 étale map. The aim of this section is to find the involution preserving the fibers of $H_{3,1}$, to provide a recovery algorithm for $H_{3,1}$ and to compute the image of $H_{3,1}$.

For $F \in \mathbb{P}(S^3V)$, $\mathbb{V}(F)$ consists of three points and h_F is a linear map. The locus where h_F is not a linear embedding coincides with the twisted cubic $V^{3,1}$ in $\mathbb{P}(S^3V)$ of cubes of linear forms. Indeed, let $F \in V^{3,1}$. After a change of coordinates we may assume $F = x_0^3$ for which h_F is not a linear embedding. Now, assume that F is not in $V^{3,1}$. We can assume that F is $x_0^2x_1$ or $x_0x_1(x_0+x_1)$. One can check that h_F is a linear embedding in both cases. Hence, for $F \notin V^{3,1}$, h_F is an embedding and $H_{3,1}(F)$ consists of 3 points in $\mathbb{P}^2$ counted with multiplicity. This implies that the Hilbert polynomial $p_{3,1}(t)$ is constantly 3. For our computational purposes, we will replace the Hilbert scheme $\mathrm{Hilb}^3\mathbb{P}^2$ by the third symmetric power of $\mathbb{P}^2$, denoted by $\mathrm{Sym}^3\mathbb{P}^2$. Based on this, we consider $H_{3,1}$ as the rational map $H_{3,1} : \mathbb{P}(S^3V) \dashrightarrow \mathrm{Sym}^3\mathbb{P}^2$ sending F to its Hessian variety. In Corollary 2.3.8 we showed that the restriction of $H_{3,1}$ to the open subset of polynomials with distinct roots is a degree 2 étale map. Moreover, in Section 2.3.1 we saw that $x_0^3+x_1^3$ and $x_0^3-x_1^3$ have the same Hessian variety. However, for our purposes, it will be useful to focus on a different example.

Example 2.4.3. *Consider the polynomials $F = x_0x_1(x_0-x_1)$ and $G = (x_0-2x_1)(2x_0-x_1)(x_0+x_1)$. Then, one can check that the Hessian varieties of F and G are both equal to $\{[1,-1,0],[0,-1,1],[1,0,-1]\}$. In particular, there is no other polynomial in $\mathbb{P}(S^3V)$ with the same Hessian variety as F and G.*

Let $F, G \in \mathbb{P}(S^3V)$ with distinct roots. Then, since $\mathrm{PGL}(2)$ is 3–transitive in $\mathbb{P}^1$, the set of elements $g \in \mathrm{PGL}(2)$ such that $g \cdot F = G$ consists of 6 elements corresponding to the permutation of the roots of F. If F and G have 2 distinct roots, then the above set is isomorphic to 6 copies of $\mathbb{P}^1$. In this situation, for $F \in \mathbb{P}(S^3V) \setminus V^{3,1}$, we introduce the subgroup

$$\Sigma_F := \{g \in \mathrm{PGL}(2) : \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{3,1}(F)) = H_{3,1}(F)\}.$$

As a consequence of Lemma 2.3.2 we deduce the following proposition.

Proposition 2.4.4. *Let $F \in \mathbb{P}(S^3V) \setminus C$. Then, there is a surjection*

$$\begin{array}{ccc} \Sigma_F & \to & H_{3,1}^{-1}(H_{3,1}(F)) \\ g & \mapsto & g \cdot F. \end{array} \tag{2.13}$$

Moreover, the preimage of a polynomial G consists of the elements of $\mathrm{PGL}(2)$ *that map the three roots of G to the three roots of F.*

Proof. Let $g \in \mathrm{PGL}(2)$ be such that $\rho(g^{\mathrm{t}})^{\mathrm{t}}$ preserves $H_{3,1}(F)$ and let $G = g \cdot F$. By Lemma 2.3.2, we deduce that $H_{3,1}(G) = \rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{3,1}(F)) = H_{3,1}(F)$, and we conclude that F and G have the same Hessian variety. Hence, the map (2.13) is well defined. Now, let $G \in \mathbb{P}(S^3V)$ with same Hessian variety as F. Since the action of PGL(2) on $\mathbb{P}^1$ is 3–transitive, there exists $g \in \mathrm{PGL}(2)$ such that $g \cdot F = G$ and by Lemma 2.3.2, $\rho(g^{\mathrm{t}})^{\mathrm{t}}(H_{3,1}(F)) = H_{3,1}(G) = H_{3,1}(F)$. Hence, we conclude that g lies in Σ_F and the map (2.13) is surjective. □

Example 2.4.5. *Let F and G be the polynomials in Example 2.4.3. Then, Σ_F consists in the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}.$$

The first six matrices are the automorphisms of $\mathbb{P}^1$ that preserve $\mathbb{V}(F)$. The last six matrices are the automorphisms g of $\mathbb{P}^1$ such that $g \cdot F = G$. In particular, the surjection in (2.13) sends the first six matrices to F and the last six matrices to G.

In Corollary 2.3.8, we restrict the study of $H_{3,1}$ to the open subset of polynomials with Waring rank 2, which coincide with the polynomials with distinct roots. Now, we focus on the case where the polynomial has a double root. For $F = x_0^2 x_1$, $H_{3,1}(F) = \{2[1,0,0],[0,1,0]\}$ where $2[1,0,0]$ denotes the point $[1,0,0]$ with multiplicity 2. Then, Σ_F consists of $g \in \mathrm{PGL}(2)$ such that $\rho(g^{\mathrm{t}})^{\mathrm{t}}([1,0,0]) = [1,0,0]$ and $\rho(g^{\mathrm{t}})^{\mathrm{t}}([0,1,0]) = [0,1,0]$. One can check that

$$\Sigma_F = \left\{ \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} : [a_0, a_1] \in \mathbb{P}^1 \right\} \simeq \mathbb{P}^1.$$

Since for every $g \in \Sigma_F$, $g \cdot F = F$, we get that $H_{3,1}^{-1}(H_{3,1}(F)) = \{F\}$. By a similar argument as in the proof of Proposition 2.3.8, we get that for $F \in \mathbb{P}(S^3V)$, with two distinct roots, $H_{3,1}^{-1}(H_{3,1}(F)) = \{F\}$ set-theoretically.

Remark 2.4.6. (Recovery of the fibers of $H_{3,1}$) *The above study provides an effective method for computing preimages through $H_{3,1}$. Assume that we are given $\{p_1, p_2, p_3\}$ in the image of $H_{3,1}$ where p_1, p_2, p_3 are distinct. Consider the ideal I defined by the 2×2 minors of the matrices*

$$\begin{pmatrix} a(a-2b) & ac-ad-bc & c(c-2d) \\ p_{1,0} & p_{1,1} & p_{1,2} \end{pmatrix}, \begin{pmatrix} b(b-2a) & bd-ad-bc & d(d-2c) \\ p_{2,0} & p_{2,1} & p_{2,2} \end{pmatrix},$$
$$\text{and } \begin{pmatrix} a^2-b^2 & ac-bd & c^2-d^2 \\ p_{3,0} & p_{3,1} & p_{3,2} \end{pmatrix},$$

where $p_i = [p_{i,0}, p_{i,1}, p_{i,2}]$. *Then,* $\mathbb{V}(I)$ *is the subvariety of* $\mathrm{PGL}(2)$ *consisting of the elements* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2)$ *such that*

$$\rho(g^{\mathrm{t}})^{\mathrm{t}}([1,-1,0]) = p_1, \ \ \rho(g^{\mathrm{t}})^{\mathrm{t}}([0,-1,1]) = p_2, \ \text{ and } \rho(g^{\mathrm{t}})^{\mathrm{t}}([1,0,-1]) = p_3.$$

Then, $\mathbb{V}(I)$ *consists of two involutions* g_1 *and* g_2. *In particular, these two involutions can be explicitly computed, since the polynomials*

$$\begin{gathered} p_{2,0}(p_{1,1}p_{3,0} - p_{1,0}p_{3,1})a^2 + 2p_{3,0}(p_{1,0}p_{2,1} - p_{1,1}p_{2,0})ab + p_{1,0}(p_{2,0}p_{3,1} - p_{2,1}p_{3,0})b^2, \\ p_{2,2}(p_{1,2}p_{3,1} - p_{1,1}p_{3,2})c^2 + 2p_{3,2}(p_{1,1}p_{2,2} - p_{1,2}p_{2,1})cd + p_{1,2}(p_{2,1}p_{3,2} - p_{2,2}p_{3,1})d^2 \end{gathered} \tag{2.14}$$

lie in I. *For instance, the first polynomial in Equation 2.14 is given by the sum*

$$p_{2,0}p_{3,0}\begin{vmatrix} a(a-2b) & ac-ad-bc \\ p_{1,0} & p_{1,1} \end{vmatrix} - p_{1,0}p_{3,0}\begin{vmatrix} b(b-2a) & bd-ad-bc \\ p_{2,0} & p_{2,1} \end{vmatrix}$$
$$-p_{1,0}p_{2,0}\begin{vmatrix} a^2-b^2 & ac-bd \\ p_{3,0} & p_{3,1} \end{vmatrix}.$$

Solving these equations lead to 4 possible solutions. Discarding the two solutions that do not vanish at I, *we get* g_1 *and* g_2. *Then, the two polynomials in* $H_{3,1}^{-1}(\{p_1, p_2, p_3\})$ *are* $g_1 \cdot x_0x_1(x_0 - x_1)$ *and* $g_2 \cdot x_0x_1(x_0 - x_1)$.

Example 2.4.7. *Consider the variety* X *in* $\mathbb{P}^2$ *consising on the points* $[0,0,0]$, $[3,-1,-1]$ *and* $[3,1,-1]$. *Following the algorithm presented in Remark 2.4.6, we consider the polynomials in Equation (2.14) which, in this case, are*

$$f_1 = -a(a-2b) \quad \text{and} \quad f_2 = c(c-2d)\ .$$

One can check that the solutions of $\{f_1 = f_2 = 0\}$ *of the form* $a = c = 0$ *or* $a - 2b = c - 2d = 0$ *together with the* 2×2 *minors of the matrices in Remark 2.4.6 lead to the solution* $a = b = c = d = 0$, *which is discarded. From the solutions of* $\{f_1 = f_2 = 0\}$ *of the form* $a = c - 2d = 0$ *and* $c = a - 2b = 0$ *lead to the matrices*

$$g_1 = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

We obtain the polynomials

$$\begin{aligned} F_1 &= g_1 \cdot x_0x_1(x_0 - x_1) = 2x_1(3x_0 + x_1)(-3x_0 + x_1), \\ F_2 &= g_2 \cdot x_0x_1(x_0 - x_1) = x_0(x_0 + x_1)(x_0 - x_1). \end{aligned}$$

One can check that the Hessian varieties of F_1 *and* F_2 *consist of the three initial points* $[0,0,0]$, $[3,-1,-1]$ *and* $[3,1,-1]$.

As a consequence of Corollary 2.3.8, there exists a rational involution in φ in $\mathbb{P}(S^3V)$ that preserves the fibers of $H_{3,1}$. Let $F, G \in \mathbb{P}(S^3V)$ with distinct roots such that $H_{3,1}(F) = H_{3,1}(G)$. Let $h_1 \in \mathrm{PGL}(2)$ be such that $h_1 \cdot F = x_0x_1(x_0 - x_1)$ and consider the automorphism

$$h_2 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

appearing in Example 2.4.5. Then, for $g = h_1^{-1}h_2h_1$ we get that $g \cdot F = G$. This defines a rational map $\psi : \mathbb{P}(S^3V) \dashrightarrow \mathbb{P}(\mathrm{Mat}(2,2))$ sending F to $h_1^{-1}h_2h_1$ given by

$$[a_0, a_1, a_2, a_3] \mapsto \begin{pmatrix} 9a_3a_0 - a_1a_2 & 2a_1^2 - 6a_2a_0 \\ 6a_3a_1 - 2a_2^2 & a_2a_1 - 9a_3a_0 \end{pmatrix}, \tag{2.15}$$

where $a_0, \ldots, a_3$ are the coordinates of $\mathbb{P}(S^3V)$. Note that the determinant of the matrix in (2.15) is the equation defining the secant variety of $V^{3,1}$. Then, the desired rational involution $\iota : \mathbb{P}(S^3V) \dashrightarrow \mathbb{P}(S^3V)$ sends F to $\psi(F) \cdot F$. In coordinates, ι is given by four homogeneous polynomial of degree 7. One can check that the determinant of the matrix in (2.15) divides these four polynomials. We conclude that ι is given by three polynomials of degree four. A computation using the software Macaulay2 [60] shows that the four polynomials defining the involution ι are

$$-2a_1^3 + 9a_0a_1a_2 - 27a_0^2a_3,\ 3(-a_1^2a_2 + 6a_0a_2^2 - 9a_0a_1a_3),\ 3(a_1a_2^2 - 6a_1^2a_3 + 9a_0a_2a_3)$$
$$2a_2^3 - 9a_1a_2a_3 + 27a_0a_3^2.$$

Moreover, the radical of the ideal generated by these polynomials is the ideal of the twisted cubic C.

We finish this section with the computation of the image of $H_{3,1}$. To do so, we embed $\mathrm{Sym}^3\mathbb{P}^2$ in $\mathbb{P}^9$ through the global sections of $\mathcal{O}_{(\mathbb{P}^2)^3}(1,1,1)^{S_3}$.

Proposition 2.4.8. *The image of $H_{3,1}$ is a subvariety of $\mathbb{P}^9$ of dimension 3, degree 10 and its ideal is generated by 3 linear forms and 10 cubic forms.*

Proof. The result is obtained by a direct computation using the software Macaulay2 [60]. □

2.4.2 Case $n \geq 2$

In Section 2.4.1 we gave a precise description of the fibers $H_{3,1}$, proving that $H_{3,1}$ is generically finite of degree 2. For $n \geq 2$ we prove that $H_{3,n}$ is birational onto its image.

To achieve this goal, first we introduce the required algebraic objects for studying the map $H_{3,n}$ for $n \geq 2$. For $F \in \mathbb{P}(S^3V)$, we consider the image of the catalecticant map $F_{1,2}$ (see Equation (2.4)). This is the linear subspace $\langle \nabla^1 F \rangle$ of S^2V generated by the first order derivatives of F. The strategy for studying the map $H_{3,n}$ for $n \geq 2$ is to look at $h_F(\mathbb{P}^n)$ instead of $H_{3,n}(F)$. The following result connects $\langle \nabla^1 F \rangle$ and $h_F(\mathbb{P}^n)$.

Lemma 2.4.9. *For $F \in \mathbb{P}(S^3V)$, $\mathbb{P}(\langle \nabla^1 F \rangle) = h_F(\mathbb{P}^n)$.*

Proof. By the Euler formula, the (a,b)-th entry of $2\frac{\partial F}{\partial x_i}$, as a symmetric matrix in $\mathbb{P}(S^2V)$, is the third derivative $\frac{\partial^3 F}{\partial x_i \partial x_a \partial x_b}$. On the other hand, let $\{e_0, \dots, e_n\}$ be the canonical basis of $\mathbb{P}^n$. Then, $h_F(\mathbb{P}^n)$ is the span of $\{h_F(e_0), \dots, h_F(e_n)\}$. The (j,k)-th entry of $h_F(e_i)$ is the coefficient of x_i in $\frac{\partial^2 F}{\partial x_i \partial x_j}$, which is $\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$. Hence, the symmetric matrices $2\frac{\partial F}{\partial x_i}$ and $h_F(e_i)$ coincide, and $\mathbb{P}(\langle \nabla^1 F \rangle) = h_F(\mathbb{P}^n)$. $\square$

Inspired by Lemma 2.4.9 and Remark 2.4.2, we consider the gradient map, which is defined as the rational map

$$\begin{array}{cccc} \alpha_n : & \mathbb{P}(S^dV) & \dashrightarrow & \mathrm{Gr}(n+1, S^{d-1}V) \\ & F & \longmapsto & \langle \nabla^1 F \rangle \end{array} . \tag{2.16}$$

The birationality of the gradient map onto its image was studied by Bertini in [6, 7], where it is shown that for cubic forms, α_n is birational for $d \geq 3$ and $(d,n) \neq (3,1)$. Moreover, in [90] the locus where α_n is not injective is described. We refer to [46, Theorem 3.2] for a modern version of these results. A further reference on this topic is [20]. Our strategy is to use by Lemma 2.4.9 and Remark 2.4.2 to derive the birationality of $H_{3,n}$ from the birationality of α_n.

Example 2.4.10. *For $n = 1$, the Grassmannian $\mathrm{Gr}(2, S^2V)$ is isomorphism to $\mathbb{P}(S^2V)^* \simeq \mathbb{P}^2$. Then, α_1 is a rational map from $\mathbb{P}^3$ to $\mathbb{P}^2$. Let $[a_0, a_1, a_2, a_3]$ be a point in $\mathbb{P}^3$ corresponding to the polynomial $F = a_0x_0^3 + a_1x_0^2x_1 + a_2x_0x_1^2 + a_3x_1^3$. Then, $\alpha_1(F)$ is the line in $\mathbb{P}^2$ generated by the points $[6a_0, 2a_1, a_2]$ and $[a_1, 2a_2, 6a_3]$. Using the expression of this line as a point in the dual projective plane, we get that in coordinates α_1 corresponds to the map*

$$[a_0, a_1, a_2, a_3] \longmapsto [12a_1a_3 - 2a_2^2, a_1a_2 - 36a_0a_3, 12a_0a_2 - 2a_1^2].$$

Note that in these coordinates, the base locus of α_1 coincide with the twisted cubic $V^{3,1}$. Moreover, one can check that α_1 is dominant and the generic fiber is one dimensional.

We now present a proof for the birationality of the gradient map distinct from the ones appearing in the literature. As we will see in Section 2.4.3, one advantage of our proof is that it can be used to reconstruct a cubic hypersurface from its Hessian variety. Another advantage of our method is that, as we will see in Section 2.6, it can be used to compute the irreducible components of the catalecticant enveloping variety.

For $0 \leq k \leq n$, let Sub_k be the variety of polynomials $F \in \mathbb{P}(S^3V)$ such that $\operatorname{rank} F_{1,2} \leq k+1$. In other words, Sub_k consists of all polynomials such that the dimension of $\langle \nabla^1 F \rangle$ is at most $k+1$. Therefore, the ideal of Sub_k is described by the $(k+2) \times (k+2)$ minors of the corresponding catalecticant matrix (see [77]). We denote the open subset of Sub_k consisting of the polynomials F with $\operatorname{rank} F_{1,2} = k+1$ by Sub_k°.

Example 2.4.11. *For $k = 0$, Sub_0 is the variety defined by the 2×2 minors of the Catalecticant matrix $\mathrm{Cat}(2, 1)$. In [74, Remark 1.24] it is shown that these minors coincide with the equations of the Veronese variety $V^{3,n}$, and Sub_0 equals $V^{3,n}$.*

Our interest on the varieties Sub_k lies in the fact that Sub_{n-1} is the base locus of α_n. In [74, Lemma 1.22], the following description of Sub_k is given:

$$\mathrm{Sub}_k = \{F \in \mathbb{P}(S^3V) : \exists U \in \mathrm{Gr}(k+1, V) \text{ s.t. } F \in \mathbb{P}(S^3U)\}, \tag{2.17}$$

i.e. Sub_k consists of all the polynomials that, after a change of coordinates, can be written by means of at most $k+1$ variables. Moreover, by [74, Proposition 1.23], we get that Sub_k is an irreducible variety of dimension

$$(k+1)(n-k) + \binom{3+k}{3} - 1. \tag{2.18}$$

Example 2.4.12. *For $k = 1$, the dimension of Sub_1 is $2n + 1$. Note that $\sigma_2(V^{3,n})$ is contained in Sub_1. Indeed, a generic element F of $\sigma_2(V^{3,n})$ is of the form $l_1^3 + l_2^3$ for l_1, l_2 linear forms. Up to a change of coordinates, we can assume that $F = x_0^3 + x_1^3$. Then, $\langle \nabla^1 F \rangle$ is generated by x_0^2 and x_1^2, and F lies in Sub_1. Therefore, we conclude that $\sigma_2(V^{3,n})$ is contained in Sub_1. By Theorem 2.1.9, we get that $\dim \sigma_2(V^{3,n}) = \dim \mathrm{Sub}_1 = 2n+1$. Since Sub_1 is irreducible, we conclude that the reduced structure of Sub_1 is the second secant variety of $V^{3,n}$. This statement is improved in [77, Theorem 3.3], where the author shows that the scheme structure of Sub_1 and $V^{3,n}$ is the same.*

Similarly to the definition of α_n, for $0 \leq k \leq n+1$, we consider the rational map

$$\begin{array}{rccc} \alpha_k : & \mathrm{Sub}_k & \dashrightarrow & \mathrm{Gr}(k+1, S^2V) \\ & F & \longmapsto & \langle \nabla^1 F \rangle \end{array}. \tag{2.19}$$

The domain of definition of α_k is Sub_k° and its base locus is Sub_{k-1}. For instance, α_{n-1} is the analogous to the gradient map (2.16) but for the base locus of α_n. We denote the image of α_k and its closure by $\mathcal{Z}_k^\circ$ and $\mathcal{Z}_k$, respectively.

Example 2.4.13. *For $k = 0$, the target space of α_0 is $\mathbb{P}(S^2V)$. On the other hand, by Example 2.4.11, the domain of α_0 is the Veronese variety $V^{3,n}$. In particular, any $F \in \mathrm{Sub}_0$ can be written as $F = l^3$ for $l \in \mathbb{P}(S^1V)$, and $\langle \nabla^1 F \rangle$ is the line of S^2V generated by l^2. We deduce that α_0 sends l^3 to l^2. Hence, α_0 is the composition of the inverse of the Veronese embedding of $V^{3,n}$ and the Veronese embedding of $V^{2,n}$. We conclude that $\mathcal{Z}_0 = V^{2,n}$.*

Example 2.4.14. *For $k = 1$, we saw in Example 2.4.12 that $\mathrm{Sub}_1 = \sigma_2(V^{3,n})$. Let $F \in \mathrm{Sub}_1 \setminus \mathrm{Sub}_0$. Then, either $F = l_1^3 + l_2^3$ or $F = l_1^2 l_2$ for distinct $l_1, l_2 \in S^1V$. In the first case, $\langle \nabla^1 F \rangle = \langle l_1^2, l_2^2 \rangle$. Therefore, $\alpha_1(F)$ is the secant line of $V^{2,n}$ passing through l_1^2 and l_2^2. In the second case, $\langle \nabla^1 F \rangle = \langle l_1^2, l_1 l_2 \rangle$, and hence, $\alpha_1(F)$ is a tangent line of $V^{2,n}$ at l_1^2. We deduce that $\mathcal{Z}_1$ is the variety of secant and tangent lines to $V^{2,n}$.*

From the irreducibility of Sub_k we deduce that $\mathcal{Z}_k$ is irreducible for every k. A challenging question concerning the variety $\mathcal{Z}_k$ is the study of the boundary

$$\partial\mathcal{Z}_k := \overline{\mathcal{Z}_k \setminus \mathcal{Z}_k^\circ}.$$

With the purpose of investigated this boundary, in Section 2.6 we will introduce the catalecticant enveloping variety. Once we have introduced the map α_k, we are interested in its birationality. Using the birationality of th gradient map (see [46, Theorem 3.2]) and (2.17), one can check that α_k is birational onto its image for $k \neq 1$. The case $k = 1$ is exhibit in the following example.

Example 2.4.15. *We fix $k = 1$ and let $F \in \mathrm{Sub}_1$. Up to linear change of coordinates we can assume that F is a polynomial in the variables x_0 and x_1. In other words, $F \in \mathbb{P}(S^3U)$ where U is the linear subspace of V generated by the first two coordinates. Let $G \in \mathrm{Sub}_1$ such that $\alpha_1(F) = \alpha_1(G)$. Then, G is also contained in $\mathbb{P}(S^3U)$. Indeed, assume on the contrary that for instance the variables x_2 appears in G. Then, there exists a derivative of G where x_2 appears. This is a contradiction since $\langle\nabla^1 G\rangle = \langle\nabla^1 F\rangle$ is included in S^2U. Therefore the fiber $\alpha_1^{-1}(\langle\nabla^1 F\rangle)$ is included in $\mathbb{P}(S^3U)$. The restriction of α_1 to $\mathbb{P}(S^3U)$ is exactly the map presented in Example 2.4.10. We conclude that the generic fiber of α_1 is one dimensional.*

Now we provide a different proof for the birationality of α_k motivated by our computational approach to the recovery problem. Consider the vector space $(S^2V)^{\oplus n+1}$ with coordinates $y^l_{i,j}$, where $y^l_{i,j}$ corresponds to the monomial $2x_ix_j$ if $i \neq j$ or to the monomial x_i^2 if $i = j$ in the l–th direct summand. Let W be the image of the linear map

$$\begin{array}{rccc} \iota: & \mathbb{P}(S^3V) & \to & \mathbb{P}\left((S^2V)^{\oplus n+1}\right) \\ & F & \mapsto & \left(\dfrac{\partial F}{\partial x_0}, \dots, \dfrac{\partial F}{\partial x_n}\right). \end{array} \tag{2.20}$$

By the Euler's formula, ι is a linear embedding and $W \simeq \mathbb{P}(S^3V)$. The following lemma exhibit the connection between α_k and the projective subspace W.

Lemma 2.4.16. *Let $\Gamma \in \mathrm{Im}\,\alpha_k$. Then $F \in \mathrm{Sub}_k^\circ$ is in the fiber $\alpha_k^{-1}(\Gamma)$ if and only if $\iota(F) \in \mathbb{P}(\Gamma^{\oplus n+1})$. In particular, the closure of the fibers of $\alpha_k^{-1}(\Gamma)$ is the linear subspace $\iota^{-1}(W \cap \mathbb{P}(\Gamma^{\oplus n+1}))$.*

Proof. Let $\Gamma \in \mathrm{Im}\,\alpha_k$ and consider $F \in \mathrm{Sub}_k^\circ$ such that $\alpha_k(F) = \Gamma$. This implies that $\langle\nabla^1 F\rangle = \Gamma$ and all the first partial derivatives of F lie in Γ. Therefore, $\iota(F) \in \mathbb{P}(\Gamma^{\oplus n+1})$. Conversely, assume that $\iota(F) \in \mathbb{P}(\Gamma^{\oplus n+1})$, then all the first partial derivatives lie in Γ. Hence, $\langle\nabla^1 F\rangle$ is contained in Γ. Since $F \in \mathrm{Sub}_k^\circ$, we get that $\dim\langle\nabla^1 F\rangle = \dim\Gamma$ and we conclude that $\Gamma = \alpha_k(F)$.
From the above, we deduce that $\alpha_k^{-1}(\Gamma)$ is equal to the intersection of Sub_k^0 and $\iota^{-1}(W \cap \mathbb{P}(\Gamma^{\oplus n+1}))$. Now, the proof follows by taking closures. $\square$

Note that Lemma 2.4.16 provides a method for computing the fibers of α_k by means of linear algebra. Let $\Gamma \in \operatorname{Im} \alpha_k$. Then, the closure of $\alpha_k^{-1}(\Gamma)$ equals $\iota^{-1}(W \cap \mathbb{P}(\Gamma^{\oplus n+1}))$. Since ι is an embedding, the computation is reduced to determine the intersection $W \cap \mathbb{P}(\Gamma^{\oplus n+1})$. In order to carry out explicit computations, we first compute the equations of the linear subspace W.

Lemma 2.4.17. *The equations of W are*

$$\begin{cases} y^i_{i,j} = y^j_{i,i} & \text{for } i < j, \\ y^i_{j,i} = y^j_{i,i} & \text{for } j < i, \\ y^l_{i,j} = y^j_{i,l} = y^i_{j,l} & \text{for } i < j < l. \end{cases} \tag{2.21}$$

Proof. We write $G \in \mathbb{P}(S^3V)$ as

$$G = \sum_{i=0}^{n} a_i x_i^3 + \sum_{i \neq j}^{n} b_{i,j} x_i^2 x_j + \sum_{i<j<l}^{n} 2c_{i,j,l} x_i x_j x_l \quad \text{for } a_i, b_{i,j}, c_{i,j,k} \in \mathbb{K}. \tag{2.22}$$

Then, its first order derivatives are:

$$\frac{\partial G}{\partial x_i} = 3a_i x_i^2 + \sum_{j \neq i}^{n} (2b_{i,j} x_i x_j + b_{j,i} x_j^2) + \sum_{j<l: j,l \neq i}^{n} 2c_{i,j,l} x_j x_l.$$

Now, the proof follows from the following equalities

$$y^l_{i,j} = \begin{cases} 3a_l & \text{if } i = j = l, \\ b_{l,j} & \text{if } i = l \text{ and } j \neq i, \\ b_{l,i} & \text{if } j = l \text{ and } j \neq i, \\ b_{j,l} & \text{if } i = j \neq l, \\ c_{i,j,l} & \text{if } i, j, l \text{ are distinct.} \end{cases}$$

$\square$

By Lemma 2.4.16, the strategy to study the birationality of α_k is to find an $F \in \mathrm{Sub}_k^\circ$ such that $W \cap \mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1})$ has dimension zero. In the following example we analyse the case $k = 2$.

Example 2.4.18. *Let $k = 2$ and consider the polynomial $F = x_0x_1x_2$. Then $\langle \nabla^1 F \rangle$ is generated by the monomials x_1x_2, x_0x_2 and x_0x_1. Now, the equations of $\mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1})$ are*

$$y^0_{00} = y^0_{11} = y^0_{22} = y^1_{00} = y^1_{11} = y^1_{22} = y^2_{00} = y^2_{11} = y^2_{22} = 0.$$

On the other hand, the equations of W are

$$\begin{array}{c} y^0_{01} - y^1_{00} = y^0_{02} - y^2_{00} = y^1_{12} - y^2_{11} = y^1_{01} - y^0_{11} = y^2_{02} - y^0_{22} = y^2_{12} - y^1_{22} = 0 \\ y^2_{01} - y^1_{02} = y^2_{01} - y^0_{12} = 0 \end{array}.$$

From these equations, we deduce that the intersection $W \cap \mathbb{P}(\langle \nabla^1 F\rangle^{\oplus n+1})$ is the point $[x_1x_2, x_0x_2, x_0x_1]$. By Lemma 2.4.16, we conclude that

$$\alpha_{1,3}^{-1}(\Gamma) = \iota_1^{-1}([x_1x_2, x_0x_2, x_0x_1]) = [x_1x_2x_0 + x_0x_2x_1 + x_0x_1x_2] = F,$$

and hence, α_2 is birational onto its image.

The next result generalizes this example for $k \geq 3$.

Lemma 2.4.19. *Let $k \geq 3$ and let $F = \sum_{i=0}^{k-1} x_i^2 x_{i+1} + x_k^2 x_0$. Then, $\mathbb{P}(W) \cap \mathbb{P}(\langle \nabla^1 F\rangle^{\oplus n+1})$ has dimension zero.*

Proof. From the first order derivatives of F one can check that the equations of $\Gamma := \mathbb{P}(W) \cap \mathbb{P}(\langle \nabla^1 F\rangle^{\oplus n+1})$ are

$$\begin{cases} y^l_{i,i} = y^l_{i+1,i+2} & \text{for } i \leq k-2, \\ y^l_{k-1,k-1} = y^l_{0,k}\,, & \\ y^l_{k,k} = y^l_{0,1}\,, & \\ y^l_{i,j} = 0 & \text{for } (i,j) \notin \{(0,0),\dots,(k,k),(0,1),\dots,(k-1,k),(0,k)\}. \end{cases} \tag{2.23}$$

together with the equations (2.21). We claim that the non vanishing coordinates of Γ are $y^1_{0,0},\dots,y^k_{k-1,k-1},y^0_{k,k}$ and $y^0_{0,1},y^1_{1,2},\dots y^{k-1}_{k-1,k},y^k_{0,k}$. First of all, note that for $i,j,l \leq n$, $y^l_{i,j} = 0$ is an equation of Γ if i, j or l is greater than k. Indeed, it holds for i or j greater than k. Assume that $i,j \leq k$ and $j > k$. By equation (2.23), if $i \neq j$ we get $y^l_{i,j} = y^j_{i,l} = 0$. If $i = j$, $y^l_{i,i} = y^i_{i,l} = 0$. Similarly, we claim that for $i < j < l$, $y^l_{i,j} = y^j_{i,l} = y^l_{i,j} = 0$. The first three equalities appear in (2.21). Since $i+1 < l$ we get that, by equation (2.23) $y^j_{i,l} = 0$ for $(i,l) \neq (0,k)$. Assume that $(i,l) = (0,k)$. For $j > 1$, $y^k_{0,j} = 0$. So we assume that $j = 1$. Then, since $k \geq 3$, $y^0_{1,k} = 0$. Using that for $i < j < l$, $y^l_{i,j} = 0$, one can check that $y^i_{i,i} = 0$ for $i \leq n$. For example, for $i \leq k-2$, we get that by equation (2.21), $y^i_{i,i} = y^i_{i+1,i+2} = 0$. Now, for $i \neq k$ and $l \neq i+1$, $y^l_{i,i} = 0$. Indeed, it is enough to check this for $i,l \leq k$ and $l \neq i$. Assume first that $l > i+1$. Then, $y^l_{i,i} = y^l_{i,l}$, which, by equation (2.21) equals to zero for $(i,l) \neq (0,k)$. In this case, $y^k_{0,0} = y^k_{0,1} = 0$. A similar argument shows that $y^k_{i,i} = 0$ for $l < i-1$. For $l = i-1, i$ the claim follows from equation (2.21) and the fact that for $i < j < l$, $y^l_{i,j} = 0$. Similarly, one can check that $y^l_{k,k} = 0$ for $l \neq k$.
Finally, the proof follows from the fact that the coordinates $y^1_{0,0},\dots,y^k_{k-1,k-1},y^0_{k,k}$ and $y^0_{0,1},y^1_{1,2},\dots y^{k-1}_{k-1,k},y^k_{0,k}$ are all equal by equations (2.21) and (2.23). □

Remark 2.4.20. *Euler's formula allows us to write an homogeneous polynomial by means of its first order derivatives. In this sense, Lemma 2.4.19 is a generalization of Euler's formula for generic polynomials since it allows us to recover a homogeneous polynomial F of degree 3 from $\langle \nabla^1 F\rangle$.*

As consequence of Lemma 2.4.19 we derive the birationality of α_k.

Proposition 2.4.21. *For $2 \leq k \leq n$, α_k is birational onto its image and $\dim \mathcal{Z}_k = \dim \mathrm{Sub}_k$.*

Proof. Let F be as in Lemma 2.4.19, and $G \in \mathbb{P}(S^3V)$ such that $\alpha_k(F) = \alpha_k(G)$. Then, $\iota(F)$ and $\iota(G)$ lie in $W \cap \mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1})$. By Lemma 2.4.19, $\iota(F) = \iota(G)$. Since ι is injective, we deduce that $F = G$. So, since $F \in \mathrm{Sub}_k$, we deduce that $\{F\} = \alpha_k^{-1}(\alpha_k(F))$. In particular, we conclude that α_k is birational onto its image and that $\dim \mathrm{Sub}_k = \dim \mathcal{Z}_k$. □

Now, using Proposition 2.4.21 we can prove the main result of this section.

Theorem 2.4.22. *For $n \geq 2$, $H_{3,n}$ is birational onto its image.*

Proof. Let F be a generic polynomial in $\mathbb{P}(S^3V)$. As mentioned in Remark 2.4.2, $h_F(\mathbb{P}^n)$ is the unique n–dimensional projective subspace containing $H_{3,n}(F)$. We consider the rational map $\beta_n : \mathrm{Im}H_{3,n} \dashrightarrow \mathrm{Gr}(n+1, S^2V)$ that sends X to the smallest projective subspace containing X. By Lemma 2.4.9, we get that $\alpha_n = \beta_n \circ H_{3,n}$. The proof follows now from Proposition 2.4.21. □

2.4.3 Recovery algorithm

The goal of this section is to present a recovery algorithm for the Hessian correspondence for $d = 3$ and $n \geq 2$. In other words, given X in the image of $H_{3,n}$, we give an effective method for computing the unique polynomial F (up to scalar) whose Hessian variety is X. We exploit the relation between α_n and $H_{3,n}$ exhibited in Section 2.4.2.

We fix $n \geq 2$. Assume that we are given the ideal I of a generic variety X in the image of $H_{3,n}$. By Theorem 2.4.22, we deduce that there exists a unique $F \in \mathbb{P}(S^3V)$ such that $H_{3,n}(F) = X$. Let $\mathbb{P}(\Gamma)$ be the smallest linear subspace containing X. Then, by Lemma 2.4.9, we have that $\mathbb{P}(\Gamma) = h_F(\mathbb{P}^n) = \alpha_n(F)$, In particular, F is the unique element in the fiber $\alpha_n^{-1}(\Gamma)$. Therefore, we can determine F by computing $\alpha_n^{-1}(\Gamma)$ as shown in Lemma 2.4.16. In Algorithm , we outline the main steps for this purpose.

Let us illustrate the algorithm in the following examples.

Example 2.4.23. *For $n = 2$ the Hessian variety of a polynomial in $\mathbb{P}(S^3V)$ lies in $\mathbb{P}^5$. Consider the variety $X = \mathbb{V}(z_{0,0}, z_{1,1}, z_{2,2}, z_{0,1}z_{0,2}z_{1,2}) \subset \mathbb{P}^5$. The smallest plane containing X is $\mathbb{V}(z_{0,0}, z_{1,1}, z_{2,2})$. In Example 2.4.18 we computed the intersection of W with $\mathbb{V}(z_{0,0}, z_{1,1}, z_{2,2})^{\oplus n+1}$, which consists in the point $[x_1x_2, x_0x_2, x_0x_1]$. Using Euler's formula we get that $F = x_0x_1x_2 + x_1x_0x_2 + x_2x_0x_1 = 3x_0x_1x_2$ is the unique polynomial such that $H_{3,2}(F) = \mathbb{V}(z_{0,0}, z_{1,1}, z_{2,2}, z_{0,1}z_{0,2}z_{1,2})$.*

Example 2.4.24. *For $n = 3$, the Hessian variety of a polynomial in $\mathbb{P}(S^3V)$ lies in $\mathbb{P}^9$. Consider the variety*

$$X = \mathbb{V}(z_{0,2}, z_{1,3}, z_{0,0}-z_{1,2}, z_{1,1}-z_{2,3}, z_{2,2}-z_{0,3}, z_{3,3}-z_{0,1}, z_{1,2}^2z_{2,3}+z_{2,2}z_{2,3}^2+z_{2,2}^2z_{3,3}+z_{1,2}z_{3,3}^2)$$

Algorithm 9

Input: the ideal I of $X \in \mathrm{Im} H_{3,n}$.

Output: the unique polynomial $F \in \mathbb{P}(S^3V)$ such that $H_{3,n}(F) = \mathbb{V}(I)$.

1. Compute the smallest projective subspace $\mathbb{P}(\Gamma)$ containing $\mathbb{V}(I)$ by taking the degree one part of the saturation of I.
2. Determine $W \cap \mathbb{P}(\Gamma^{\oplus n+1}) \subseteq \mathbb{P}\left((S^2V)^{\oplus n+1}\right)$. By Lemma 2.4.19, this intersection is a point $[F_0, \cdots, F_n] \in \mathbb{P}\left((S^2V)^{\oplus n+1}\right)$.
3. Compute F via Euler's formula: $F = \sum x_i F_i$.

---

in $\mathbb{P}^9$. The smallest linear subspace containing X is

$$\mathbb{P}(E) = \mathbb{V}(z_{0,2}, z_{1,3}, z_{0,0} - z_{1,2}, z_{1,1} - z_{2,3}, z_{2,2} - z_{0,3}, z_{3,3} - z_{0,1}).$$

One can check that the equations of $\mathbb{P}(E^4)$ are exactly the ones described in (2.23). In particular, the intersection $\mathbb{P}(E^4) \cap W$ was computed in Lemma 2.4.19. We deduce that the unique form in $\mathbb{P}(S^3V)$ whose Hessian variety is X is

$$F = x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_0.$$

Remark 2.4.25. *In [90, 46], the locus where α_n is not injective is described as the Zariski closure of the set of forms F in $\mathbb{P}(S^dV)$ that, after a change of coordinates F can be written as the sum*

$$F(x_0, \ldots, x_n) = F_1(x_0, \ldots, x_a) + F_2(x_{a+1}, \ldots, x_n),$$

for $0 \leq a \leq n-1$ and $F_1, F_2 \neq 0$. This is exactly the locus where Algorithm 9 fails.

We conclude this section with some comments on how to apply Algorithm 9 to the restriction of $H_{3,n}$ to Sub_k. The proof of Theorem 2.4.22 uses the birationality of α_n onto its image. As a consequence, the input of Algorithm 9 must be the Hessian variety of a polynomial in Sub_n. Nevertheless, in Proposition 2.4.21 we prove the birationality of α_k for $k \geq 2$. Therefore, for $k \geq 2$, the restriction of $H_{3,n}$ to Sub_k is birational onto its image and Algorithm 9 provides a method for recovering its fibers.

2.5 Hessian correspondence of degree 4 hypersurfaces

In this section we study the Hessian correspondence for $d = 4$. In Section 2.4, the crucial idea was to, focus on $h_F(\mathbb{P}^n)$ instead of on the Hessian variety of F. For cubic hypersurfaces we saw that h_F is linear and $h_F(\mathbb{P}^n)$ is the smallest linear subspace containing the Hessian variety. In the case of $d=4$, the Hessian map is not linear, but it is defined by degree 2 polynomials. We will see that for $d=4$, the Hessian map of a generic quartic hypersurface is a Veronese embedding of order two. Along this section we analyze the Hessian correspondence $H_{4,n}$ through the Veronese variety $h_F(\mathbb{P}^n)$.

Let $U \subset \mathbb{P}(S^4V)$ be the open subset of polynomials F whose second order derivatives are linearly independent as elements in S^2V. In particular, for $F \in U$, its second derivatives form a basis of S^2V.

Example 2.5.1. *Let $F=\sum_{i\neq j} x_i^2x_j^2 \in S^4V$. Its second order derivatives are*

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} 2\sum_{k\neq i} x_k^2 & \text{if } i=j \\ 4x_ix_j & \text{if } i\neq j \end{cases}.$$

One can check that the set of all second order derivatives form a basis of S^2V if and only if the $(n+1)\times(n+1)$–matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{2.24}$$

has maximal rank. Now, the determinant of the matrix (2.24) is $(-1)^n n$, and we conclude that the set of second order derivatives of F is a basis of S^2V.

Example 2.5.1 shows that the open subset U is nonempty. Therefore we get that, for $F \in U$, h_F is a Veronese embedding and $h_F(\mathbb{P}^n)$ is a Veronese variety. In particular, we deduce that the Hessian variety of $F \in U$ is the intersection of the Veronese variety $h_F(\mathbb{P}^n)$ and a quadric hypersurface. This is because a quartic hypersurface in $\mathbb{P}^n$ corresponds to a quartic under the Veronese embedding.

Example 2.5.2. *Fix $n=2$ and consider the polynomial $F=x_0^2x_1^2+x_0^2x_2^2+x_1^2x_2^2$ as in Example 2.5.1. Then, the Hessian map h_F extends to a Veronese embedding of $\mathbb{P}^n$ to $\mathbb{P}(S^2V)$:*

$$\begin{array}{cccc} h_F: & \mathbb{P}^n & \longrightarrow & \mathbb{P}(S^2V)\simeq\mathbb{P}^5 \\ & [x_0,x_1,x_2] & \longmapsto & [x_1^2+x_2^2, 2x_0x_1, 2x_0x_2, , x_0^2+x_2^2, 2x_1x_2, x_0^2+x_1^2] \end{array}.$$

The image of h_F is a Veronese variety defined by the ideal

$$\langle z_{0,2}^2+z_{1,2}^2-2z_{0,0}z_{2,2}-2z_{1,1}z_{2,2}+2z_{2,2}^2, z_{0,1}z_{0,2}+z_{0,0}z1,2-z_{1,1}z_{1,2}-z_{1,2}z_{2,2},$$
$$z_{0,0}z_{0,2}-z_{0,2}z_{1,1}-z_{0,1}z_{1,2}+z_{0,2}z_{2,2}, z_{0,1}^2-2z_{0,0}z_{1,1}+2z_{1,1}^2+z_{1,2}^2-2z_{1,1}z_{2,2},$$
$$z_{0,0}z_{0,1}+z_{0,1}z_{1,1}-z_{0,2}z_{1,2}-z_{0,1}z_{2,2}, z_{0,0}^2-z_{1,1}^2-z_{1,2}^2+2z_{1,1}z_{2,2}-z_{2,2}^2\rangle$$

Moreover, one can check that the pullback of the polynomial $G = -z_{0,0}^2 + 2z_{0,0}z_{1,1} - z_{1,1}^2 + 2z_{0,0}z_{2,2} + 2z_{1,1}z_{2,2} - z_{2,2}^2$ *via* h_F *is* F. *We conclude that the Hessian variety of* F *is the intersection of* $h_F(\mathbb{P}^n)$ *and the quadric hypersurface defined by* G.

A first consequence of h_F being a Veronese embedding is the computation of the Hilbert polynomial $p_{4,n}$.

Proposition 2.5.3. *The Hilbert polynomial* $p_{4,n}(t)$ *is*

$$p_{4,n}(t) = \binom{2t+n}{n} - \binom{2t-4+n}{n}.$$

Proof. Let $F \in \mathbb{P}(S^4V)$ be generic and denote its Hessian variety by X. Then, h_F is a Veronese embedding and X is the intersection of a Veronese variety $\mathcal{V}$ and a quadric hypersurface. A method for computing Hilbert polynomials is to use the equality $p_{4,n}(t) = \chi(\mathcal{O}_X(t))$ (see [67, Section 3 Exercise 5.2]). Using the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{V}}(-2) \to \mathcal{O}_{\mathcal{V}} \to \mathcal{O}_X \to 0$$

we get that $\chi(\mathcal{O}_X(t)) = \chi(\mathcal{O}_{\mathcal{V}}(t)) - \chi(\mathcal{O}_{\mathcal{V}}(t-2))$. The proof follows from the fact that $\mathcal{O}_{\mathcal{V}}(t) = \mathcal{O}_{\mathbb{P}^n}(2t)$. □

Another consequence of h_F being an embedding is that, for generic $F, G \in U$ with same Hessian variety, one has that $\mathbb{V}(F)$ and $\mathbb{V}(G)$ are isomorphic. However, a priori this isomorphism might not extend to an automorphism of $\mathbb{P}^n$. We improve this statement in two steps. First we show that if $H_{4,n}(F) = H_{4,n}(G)$, then there exists $g \in \mathrm{PGL}(n+1)$ such that $g \cdot F = G$. In Theorem 2.5.10 we prove that the only possible g such that $g \cdot F = G$ is the identity. Moreover, for n even we provide an effective algorithm for computing F from its Hessian variety.

For proving the main results of the section, we show that for $n \geq 2$ $h_F(\mathbb{P}^n)$ is the unique Veronese variety containing the Hessian variety of F. Note that this is not true for $n = 1$. Indeed, for $n = 1$, the Hessian variety of a binary quartic consists of 4 points in the plane that are the intersection of 2 smooth conics. In particular, the Hessian variety is contained in a pencil of Veronese varieties. Therefore, we need to carry out separately the case $n = 1$.

2.5.1 Case $n = 1$

In this case, for $F \in \mathbb{P}(S^4V) \simeq \mathbb{P}^4$ generic, $H_{4,1}(F)$ consists of 4 points in $\mathbb{P}^2$. These four points define a pencil of quadrics denoted by Q_F. We consider $H_{4,1}$ as the map $H_{4,1} : \mathbb{P}^4 \dashrightarrow \mathrm{Gr}(2, H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ sending F to Q_F. In the next proposition we analyze the birationality of $H_{4,1}$.

Proposition 2.5.4. *For a generic* $F \in \mathbb{P}(S^4V)$, $H_{4,1}$ *is birational onto its image.*

Proof. Let $a_0, \dots, a_4$ be the coordinates of $\mathbb{P}^4$ and let $F = \sum a_i x_0^{4-i} x_1^i$. Let $b_0, \dots, b_5$ be the coordinates of $\mathbb{P}^5 \simeq \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ corresponding to the monomials $z_{0,0}^2, z_{0,0}z_{0,1}, z_{0,1}^2, z_{0,0}z_{1,1}, z_{0,1}z_{1,1}, z_{1,1}^2$. Using the software MACAULAY2 [60], one can check that the ideal of $h_F(\mathbb{V}(F))$ is generated by the quadrics

$$\begin{aligned} Q_1 &:= 3a_4 z_{0,0}^2 - 3a_3 z_{0,0}z_{0,1} + 2a_2 z_{0,1}^2 + a_2 z_{0,0}z_{1,1} - 3a_1 z_{0,1}z_{1,1} + 3a_0 z_{1,1}^2 \\ Q_2 &:= 9a_3^2 z_{0,0}^2 + (-36a_2a_3 + 72a_1a_4)z_{0,0}z_{0,1} + (20a_2^2 - 144a_0a_4)z_{0,1}^2 + \\ &\quad (16a_2^2 - 18a_1a_3)z_{0,0}z_{1,1} + (-36a_1a_2 + 72a_0a_3)z_{0,1}z_{1,1} + 9a_1^2 z_{1,1}^2. \end{aligned} \tag{2.25}$$

In particular, we see that for generic F, the pencil of quadrics $H_{4,1}(F)$ is not contained in the hyperplane $\{b_2 - 2b_3 = 0\}$. Hence, we deduce that $H_{4,1}(F)$ and this hyperplane intersect in the quadric Q_1. Since F is uniquely determined by Q_1 and viceversa, we conclude that $H_{4,1}$ is generically injective. $\square$

In Remark 2.4.6, we showed how to determine the fiber of $H_{3,1}$. The proof of Proposition 2.5.4 provides a method for computing the fibers of $H_{4,1}$.

Remark 2.5.5. *(Recovery of the fiber of $H_{4,1}$) Let $X \in \operatorname{Im} H_{4,1}$, and let Q be the unique quadric in the intersection of X with the hyperplane defined by the equation $b_2 - 2b_3 = 0$, where $b_0, \dots, b_5$ are the coordinates of $\mathbb{P}^5$; as in the proof of Proposition 2.5.4. Then, using equation (2.25) we can compute the unique element in $H_{4,1}^{-1}(X)$.*

Example 2.5.6. *Consider the variety $X \in \operatorname{Im} H_{4,1}$ whose ideal I is generated by the quadrics*

$$z_{0,1}^2 - z_{0,0}z_{1,1} \text{ and } z_{0,0}^2 + 2z_{0,0}z_{1,1} + z_{1,1}^2.$$

Therefore, the degree 2 component I_2 of I is a line of $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(2)))$. The intersection of this line with the hyperplane given by the equation $b_2 - 3b_3 = 0$ is the quadric

$$3z_{0,0}^2 + 4z_{0,1}^2 + 2z_{0,0}z_{1,1} + 3z_{1,1}^2.$$

Using the method exhibited in Remark 2.5.5, we get a quartic binary form

$$F = x_0^4 + 2x_0^2x_1^2 + x_1^4.$$

One can check that the Hessian variety of F is X.

We finish this section with the computation of the image of $H_{4,1}$. Using the Plücker coordinates, we compute this image in $\mathbb{P}^{14}$.

Proposition 2.5.7. *The image of $H_{4,1}$ in $\mathbb{P}^{14}$ has dimension 4, degree 29, and is generated by 3 linear forms, the quadrics generating the ideal of $\operatorname{Gr}(2,6)$ and 7 cubic forms.*

Proof. This result is obtained by a direct computation using the software MACAULAY2 [60]. $\square$

2.5.2 Case $n \geq 2$

As mentioned at the beginning of this section, the idea for studying $H_{4,n}$ is to look at the Veronese variety $h_F(\mathbb{P}^n)$. For $n \geq 2$ we will prove that for $F \in U$, $h_F(\mathbb{P}^n)$ is the unique Veronese variety containing the Hessian variety. In the rest of the subsection we fix $n \geq 2$. The strategy to study the Veronese varieties containing the Hessian variety is to look at its first order syzygies. Let R be the homogeneous coordinate ring of $\mathbb{P}(S^2V)$ and let M be a finitely generated graded R–module. Let

$$M \longleftarrow F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2 \longleftarrow \cdots$$

be the minimal free resolution of M where $F_i = \oplus_j R(-j)^{\beta_{i,j}}$. The integers $\beta_{i,j}$ are called the graded Betti numbers and the image of δ_1 is called the module of first order syzygies of M. We say that the first order syzygies are linear if the image of δ_1 is generated by elements of degree 1, or equivalently, the entries of the map δ_1 are linear forms (see [43] for further details).

Lemma 2.5.8. *Let X_1 and X_2 be two projective subvarieties of $\mathbb{P}^N$ of codimension greater than 1, with ideals generated by the same numbers of quadrics and with linear first order syzygies. Moreover, assume that X_1 is irreducible, nondegenerate and $X_1 \cap X_2 = X_1 \cap Q$ where Q is a quadric hypersurface. Then, $X_1 = X_2$.*

Proof. Let I_1 and I_2 be the ideals of X_1 and X_2 respectively, and let q be the quadratic form defining Q. Then, $I_1 + I_2 = \langle \alpha_1, \ldots, \alpha_m, q\rangle$, where $\alpha_1, \ldots, \alpha_m$ are the quadrics generating I_1. We can assume that I_2 is generated by $q, \alpha_2, \ldots, \alpha_m$. Since the first order syzygies of X_1 and X_2 are linear, there exist l_1 and l_2 linear forms such that $D(l_1) \cap X_1 = \mathbb{V}(\alpha_2, \ldots, \alpha_m)$ and $D(l_2) \cap X_2 = \mathbb{V}(\alpha_2, \ldots, \alpha_m)$. Since X_1 is generated by quadrics, it is nondegenerate. Therefore, $D(l_1) \cap X_1$ and $D(l_2) \cap X_1$ are non-empty. Since the ideal of X_1 is irreducible, $D(l_1) \cap D(l_2) \cap X_1 \neq \emptyset$. Similarly, $D(l_2) \cap X_2 \neq \emptyset$. Hence, $D(l_1) \cap D(l_2) \cap X_1 \subseteq D(l_2) \cap X_2 \neq \emptyset$. Since X_1 is irreducible, we get that $X_1 \subseteq X_2$, and therefore, $I_2 \subseteq I_1$. Since I_1 and I_2 are generated by the same number of linearly independent quadrics, we conclude that $X_1 = X_2$. $\square$

Using this lemma, we derive the following proposition.

Proposition 2.5.9. *For $F \in U$, $h_F(\mathbb{P}^n)$ is the unique Veronese variety containing $H_{4,n}(F)$.*

Proof. Assume that the Hessian variety is contained in two Veronese varieties $V_1 = h_F(\mathbb{P}^n)$ and V_2. Then, there exists a quadratic form q in the ideal of V_2 such that $H_{4,n}(F) \subseteq \mathbb{V}(q) \cap V_1 \subsetneq V_1$. Since both $\mathbb{V}(F)$ and $h_F^{-1}(\mathbb{V}(q))$ have degree 4, we deduce that $H_{4,n}(F) = \mathbb{V}(q) \cap V_1$ and we get that $V_1 \cap V_2 = \mathbb{V}(q) \cap V_1$. By Lemma 2.5.8 we conclude that $V_1 = V_2$. $\square$

Let F and G be two polynomials in U with the same Hessian variety. By the previous proposition, $H_{4,n}(F)$ is contained in a unique Veronese variety. Hence, we deduce that $h_F(\mathbb{P}^n) = h_G(\mathbb{P}^n)$. Therefore, $h_G^{-1} \circ h_F$ is an automorphism of $\mathbb{P}^n$ that maps $\mathbb{V}(F)$ to $\mathbb{V}(G)$. Thus, we deduce that a generic fiber of $H_{4,n}$ is contained in an orbit of the action $\mathrm{PGL}(n+1)$ on $\mathbb{P}(S^4V)$. The following result provides a better description of a generic fiber of $H_{4,n}$, namely as a point.

Theorem 2.5.10. *For $n \geq 2$, $H_{4,n}$ is birational onto its image.*

Proof. To prove that $H_{4,n}$ is birational it is enough to check that on U it is injective. Let $F, G \in U$ such that $H_{4,n}(F) = H_{4,n}(G)$. By Proposition 2.5.9, $h_F(\mathbb{P}^n) = h_G(\mathbb{P}^n)$. Fixing $g = h_F^{-1} \circ h_G$ we get that $g(\mathbb{V}(G)) = \mathbb{V}(F)$. Therefore, $g^{\mathrm{t}} \cdot F = G$ and $h_G = h_F \circ g$. As in the proof of Lemma 2.3.2, we get that

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = \sum_k g_{k,i} g_{k,j} \frac{\partial^2 F}{\partial x_k^2} \circ g + \sum_{k<l} (g_{k,i} g_{l,j} + g_{l,i} g_{k,j}) \frac{\partial^2 F}{\partial x_k \partial x_l} \circ g.$$

Let M be the automorphism of $\mathbb{P}(S^2V)$ given by the linear forms

$$l_{i,j} = \sum_k g_{k,i} g_{k,j} z_{k,k} + \sum_{k<l} (g_{k,i} g_{l,j} + g_{l,i} g_{k,j}) z_{k,l}.$$

Then, $h_G = M \circ h_F \circ g$. Since $g = h_F^{-1} \circ h_G$, we deduce that M restricted to $h_F(\mathbb{P}^n)$ is the identity. Using that $h_F(\mathbb{P}^n)$ is nondegenerate, we get that $M = \mathrm{Id}$. Therefore, for every i, j we get that $l_{i,j} = \lambda z_{i,j}$ for $\lambda \in \mathbb{K}^*$. For $l_{i,i} = \lambda z_{i,i}$ we deduce that $g_{k,i} = 0$ and $g_{i,i} = g_{k,k}$ for $i \neq k$. We conclude that $g = \mathrm{Id_n}$ and $F = G$. □

2.5.3 Recovery algorithm

Once we have proven that $H_{4,n}$ is birational, our next goal is to find an effective algorithm for recovering F from $H_{4,n}(F)$. We present an algorithm for the case where n is even.

First step: computation of the Veronese variety $\mathcal{V}_X$

Given X, a generic element in the image of $H_{4,n}$, the first step of the algorithm is to compute the unique Veronese variety $\mathcal{V}_X$ containing X. Let $m+1$ be the number of quadratic forms generating the ideal of X. Since $n \geq 2$, we have that $m \geq 2$. Among these quadrics, m of them generate the ideal of $\mathcal{V}_X$ containing X.

Lemma 2.5.11. *Let $X \in H_{4,n}(U)$ and let $I_2(X)$ be the vector space of quadrics containing X. Let W be a vector subspace of $I_2(X)$ of codimension 1. Then, $\mathbb{V}(W)$ is a Veronese variety if and only if $\mathbb{V}(W)$ has linear syzygies.*

Proof. By [61, Theorem 2.2], the Veronese variety has linear syzygies. Conversely, let $W = \langle \alpha_1, \ldots, \alpha_m \rangle \subseteq I_2(X)$ be a subspace of codimension 1 with linear syzygies. By Proposition 2.5.9 there exists a unique Veronese variety $\mathcal{V}_X$ containing X. Let $W' \subset I_2(X)$ be the subspace generated by the quadrics containing $\mathcal{V}_X$, which has codimension 1 in $I_2(X)$. Assume that $W \neq W'$. Then, there exists $q \in W$ such that $q + W' = I_2(X)$. Therefore, $\mathcal{V}_X \cap \mathbb{V}(W) = \mathcal{V}_X \cap \mathbb{V}(q)$. By Lemma 2.5.8, we conclude that $\mathcal{V}_X = \mathbb{V}(W)$. □

From the above result, we deduce that, in order to find $\mathcal{V}_X$, it is enough to find $W \subset I_2(X)$ as in Lemma 2.5.11. Let J be the ideal of $\mathcal{V}_X$. Then, the minimal free resolution of J looks like

$$0 \longleftarrow J \longleftarrow R(-2)^a \longleftarrow R(-3)^b \longleftarrow \cdots,$$

where R is the coordinate ring of $\mathbb{P}(S^2V)$.

Lemma 2.5.12. *Let $X \in H_{2,n}(U)$ and let I be its ideal. Then $\beta_{0,j} = 0$ for $j \neq 2$, $\beta_{0,2} = a + 1$, and $\beta_{1,3} = b$, where $\beta_{i,j}$ are the graded Betti numbers of I.*

Proof. Since X has codimension 1 in $\mathcal{V}_X$, and it is the image through the Hessian map of a degree 4 hypersurface, we deduce that I is generated by $a + 1$ quadrics and hence, $\beta_{0,j} = 0$ for $j \neq 2$ and $\beta_{0,2} = a + 1$. Let $q_0, \ldots, q_a$ be generators of I such that $q_1, \ldots, q_a$ generate the ideal of $\mathcal{V}_X$. Now, since X is contained in $\mathcal{V}_X$, we have that $\beta_{1,3} \geq b$. Assume that $\beta_{1,3} > b$. This implies that there exists a linear syzygy of the form $\sum_{i=0}^{a} l_i q_i$, where l_i are linear forms and l_0 is non-zero. In particular, we get that

$$X \cap D(l_0) = \mathbb{V}(q_1, \ldots, q_a) \cap D(l_0) = \mathcal{V}_X \cap D(l_0).$$

Since $\mathcal{V}_X$ is nondegenerate, this intersection is non empty and we get that $X = \mathcal{V}_X$. Since X has lower dimension that $\mathcal{V}_X$ we conclude that $\beta_{1,3} = b$. □

Let I be the ideal of X. From the previous lemma we deduce that I has a minimal free resolution of the form

$$0 \longleftarrow I \xleftarrow{\pi_1} R(-2)^{a+1} \xleftarrow{M} R(-3)^b \oplus F \longleftarrow \cdots, \tag{2.26}$$

where F is a free module, and M is a block matrix of the form

$$M = \left(\begin{array}{c|c} \frac{M_1}{0} & M_2 \end{array} \right). \tag{2.27}$$

Here M_1 is an $a \times b$ matrix whose entries are linear forms. By Lemma 2.5.11, the vector subspace W of $I_2(X)$ generated by the first a coordinates of π_1 generates the ideal J. In order to find generators of J, it is enough to find a free resolution of the form (2.26).

We present a method for obtaining such a resolution using linear algebra. Assume we are given a minimal free resolution of I

$$0 \longleftarrow I \xleftarrow{\pi_2} R(-2)^{a+1} \xleftarrow{N} R(-3)^b \oplus F \longleftarrow \cdots .$$

We obtain the following isomorphisms of sequences

$$\begin{tikzcd}
0 & I \arrow[l] \arrow[d, equal] & R(-2)^{a+1} \arrow[l, "\pi_2"'] \arrow[d, "g"'] & R(-3)^b \oplus F \arrow[l, "N"'] \arrow[d, "h_1 \oplus h_2"] \\
0 & I \arrow[l] \arrow[d, equal] & R(-2)^{a+1} \arrow[l, "\pi_1"] \arrow[d, equal] & R(-3)^b \oplus F \arrow[l, "M"] \arrow[d, "h_1^{-1} \oplus h_2^{-1}"] \\
0 & I \arrow[l] & R(-2)^{a+1} \arrow[l, "\pi_1"] & R(-3)^b \oplus F \arrow[l, "M \circ (h_1 \oplus h_2)"]
\end{tikzcd}$$

where the first isomorphisms g and $h_1 \oplus h_2$ exist by the uniqueness of minimal free resolutions and M is as in (2.27). In particular, $M \circ (h_1 \oplus h_2)$ is a matrix of the same form as (2.27). Hence, there exists a $g \in \mathrm{GL}(a+1)$ such that

$$\begin{tikzcd}
0 & I \arrow[l] \arrow[d, equal] & R(-2)^{a+1} \arrow[l, "\pi_2"'] \arrow[d, "g"'] & R(-3)^b \oplus F \arrow[l, "N"'] \arrow[d, equal] \\
0 & I \arrow[l] & R(-2)^{a+1} \arrow[l, "\pi"] & R(-3)^b \oplus F \arrow[l, "M'"]
\end{tikzcd}$$

is an isomorphism to a complex of the form (2.26). In particular, the first a entries of the map $\pi_2 \circ g^{-1}$ generate the ideal of $\mathcal{V}_X$. The computation of g is achieved by solving the linear system in the entries of g given by requiring the first b entries of the last row of $g \circ N$ to vanish.

Let us illustrate the previous procedure by an example.

Example 2.5.13. *Consider the hypersurface in $\mathbb{P}^2$ given by the polynomial*

$$F = x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 + 3x_0^2 x_1 x_2.$$

The ideal of $X := H_{4,2}(F) \subset \mathbb{P}^5$ is

$$\begin{aligned} I = \langle & z_0^2 - z_0z_3 + z_2z_3 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + 4z_1z_5 + \tfrac{1}{2}z_2z_5 - 4z_4z_5 - 3z_5^2, \\ & z_0z_1 + z_2z_4 - \tfrac{3}{2}z_0z_5 + \tfrac{1}{2}z_2z_5 + z_3z_5, z_0z_2 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + 2z_1z_5 - 2z_4z_5 - z_5^2, \\ & z_1^2 + \tfrac{1}{4}z_0z_3 - \tfrac{1}{4}z_2z_3 - z_1z_4 - 2z_1z_5 - \tfrac{1}{8}z_2z_5 + z_4z_5 + z_5^2, z_1z_2 - \tfrac{1}{2}z_0z_5 - \tfrac{1}{2}z_2z_5, \\ & z_2^2 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + z_5^2, z_1z_3 + z_2z_4 - \tfrac{1}{4}z_5^2\rangle. \end{aligned} \tag{2.28}$$

The first part of the free resolution of I is

$$0 \longleftarrow R \xleftarrow{A_0} R(-2)^7 \xleftarrow{A_1} R(-3)^8 \oplus R(-4)^6,$$

where $R = \mathbb{K}[z_0, z_1, z_2, z_3, z_4, z_5]$. The submatrix N of A_1 with linear entries is

$$\begin{pmatrix} -z_2-4z_4 & \frac{1}{2}z_5 & 0 & -4z_4 & 0 & z_1-\frac{3}{2}z_5 & -\frac{1}{4}z_3 & 0 \\ -2z_5 & 4z_4 & \frac{1}{2}z_5 & 0 & -4z_4 & -z_0-4z_4 & -z_1+z_4+\frac{1}{2}z_5 & -z_2-4z_4 \\ 0 & 2z_5 & -z_2 & 0 & 0 & -4z_5 & z_0-z_3 & -2z_5 \\ z_0-z_3+8z_4 & -z_1+\frac{1}{2}z_5 & \frac{1}{4}z_3 & -z_2+8z_4 & -\frac{1}{2}z_5 & \frac{1}{2}z_5 & \frac{1}{4}z_3+\frac{1}{8}z_5 & z_1-\frac{3}{2}z_5 \\ 4z_5 & z_0-4z_4 & z_1-z_4-\frac{3}{2}z_5 & 2z_5 & -z_2+4z_4 & -z_3+4z_4-\frac{1}{2}z_5 & \frac{1}{2}z_5 & -z_3+4z_4 \\ z_3-4z_4+\frac{1}{2}z_5 & 0 & -\frac{1}{4}z_3-\frac{1}{8}z_5 & z_0-4z_4 & z_1-\frac{1}{2}z_5 & 0 & 0 & \frac{1}{2}z_5 \\ 2z_5 & -4z_4 & -\frac{1}{2}z_5 & 0 & 4z_4 & z_0+4z_4 & z_1-z_4-\frac{1}{2}z_5 & z_2+4z_4 \end{pmatrix}$$

Choosing

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we get that the last row of $g \cdot N$ is zero. Therefore, the six generators of the Veronese variety correspond to the first six entries of the matrix $A_0 \circ g^{-1}$. We conclude that the ideal of the unique Veronese variety containing X is the zero locus of the ideal J given by

$$\begin{aligned} J = \langle z_0^2 - z_0z_3 + z_2z_3 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + 4z_1z_5 + \tfrac{1}{2}z_2z_5 - 4z_4z_5 - 3z_5^2&, \\ z_0z_1 - z_1z_3 - \tfrac{3}{2}z_0z_5 + \tfrac{1}{2}z_2z_5 + z_3z_5 + \tfrac{1}{4}z_5^2&, \\ z_1^2 + \tfrac{1}{4}z_0z_3 - \tfrac{1}{4}z_2z_3 - z_1z_4 - 2z_1z_5 - \tfrac{1}{8}z_2z_5 + z_4z_5 + z_5^2&, \\ z_0z_2 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + 2z_1z_5 - 2z_4z_5 - z_5^2&, \\ z_1z_2 - \tfrac{1}{2}z_0z_5 - \tfrac{1}{2}z_2z_5&, \\ z_2^2 + 4z_0z_4 - 4z_2z_4 - 4z_3z_4 + z_5^2\rangle&. \end{aligned} \tag{2.29}$$

Second step: computation of an isomorphism $\varphi : \mathcal{V}_X \to \mathbb{P}^n$.

The next step of our algorithm is to find an isomorphism $\varphi : \mathcal{V}_X \to \mathbb{P}^n$, which is the inverse of a Veronese embedding $v_2 : \mathbb{P}^n \to \mathcal{V}_X$. We set $L \simeq \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$.

Lemma 2.5.14. *For n even, $L \simeq \omega_{\mathcal{V}_X} \otimes \mathcal{O}_{\mathbb{P}(S^2V)}(k+1)|_{\mathcal{V}_X}$ where $2k = n$.*

Proof. Since φ is the inverse of a Veronese embedding, we get that $L^{\otimes 2} = \mathcal{O}_{\mathbb{P}(S^2V)}(1)|_{\mathcal{V}_X}$. Since φ is an isomorphism, we have that $\varphi^*\omega_{\mathbb{P}^n} = \omega_{\mathcal{V}_X}$ and hence, $L^{\otimes -n-1} \simeq \omega_{\mathcal{V}_X}$. Let $k \in \mathbb{N}$ such that $n = 2k$. Then,

$$L = L^{\otimes n+2} \otimes L^{\otimes -n-1} = \mathcal{O}_{\mathbb{P}(S^2V)}(1)|_{\mathcal{V}_X}^{\otimes k+1} \otimes \omega_{\mathcal{V}_X}.$$

□

In particular, φ is given by a basis of the space of global sections of

$$\omega_{\mathcal{V}_X} \otimes \mathcal{O}_{\mathbb{P}(S^2V)}(k+1)|_{\mathcal{V}_X}.$$

Let ι be the inclusion of $\mathcal{V}_X$ in $\mathbb{P}^N = \mathbb{P}(S^2V)$. Let $R = \mathbb{K}[z_0, \dots, z_N]$ be the homogeneous coordinate ring of $\mathbb{P}^N$ and let J be the ideal of $\mathcal{V}_X$. In [76], the Betti numbers of the Veronese embbeding given by $|\mathcal{O}_{\mathbb{P}^n}(2)|$ are computed. In particular, we deduce that the minimal free resolution of R/J is of the form

$$0 \longleftarrow R \xleftarrow{A_1} P_1 \xleftarrow{A_2} \cdots \xleftarrow{A_r} R(k-N)^{n+1} \longleftarrow 0. \tag{2.30}$$

where r is the codimension of $\mathcal{V}_X$ in $\mathbb{P}^N$.

Proposition 2.5.15. *For $n = 2k$, the line bundle L is the sheafification of the graded R/J–module*

$$(R/J)^{n+1}/\mathrm{Im}\, A_r^t \tag{2.31}$$

Proof. By [67, Corollary 7.12], we have that $\omega_{\mathcal{V}_X} \simeq \iota^*\mathcal{E}\mathrm{xt}^r_{\mathbb{P}^N}(\iota_*\mathcal{O}_{\mathcal{V}_X}, \omega_{\mathbb{P}^N})$. Using [59, Proposition 7.7], we get that

$$\omega_{\mathcal{V}_X} \otimes \mathcal{O}_{\mathbb{P}(S^2V)}(k+1)|_{\mathcal{V}_X} \simeq \iota^*\mathcal{E}\mathrm{xt}^r_{\mathbb{P}^N}(\iota_*\mathcal{O}_{\mathcal{V}_X}, \omega_{\mathbb{P}^N}(k+1)) \simeq \iota^*\mathcal{E}\mathrm{xt}^r(\iota_*\mathcal{O}_{\mathcal{V}_X}, \mathcal{O}_{\mathbb{P}^N}(k-N)).$$

Applying $\mathcal{H}\mathrm{om}(-, R[k-N])$ to (2.30) we get the complex

$$0 \longleftarrow R^{n+1} \xleftarrow{A_r^t} \cdots \xleftarrow{A_2^t} R[k-N] \longleftarrow 0.$$

We deduce that $\mathcal{E}\mathrm{xt}^r(\iota_*\mathcal{O}_{\mathcal{V}_X}, \mathcal{O}_{\mathbb{P}^N}(k-N))$ is the sheafification of the r–th cohomology group of this complex which is $R^{n+1}/\mathrm{Im} A_r^t$. The proposition follows by tensoring this module by R/J. □

Let M denote the graded R/J–module in equation (2.31). From Proposition 2.5.15 we deduce that the space of global sections of L is the zero graded piece of M, i.e. $M_0 = \mathbb{K}^{n+1}$. Let $e_0, \dots, e_n$ be a basis of M_0 and let $\mathcal{U} = D(z_0) \cap \mathcal{V}_X$. We get an inclusion

$$M_0 = H^0(\mathcal{V}_X, \omega_{\mathcal{V}_X}(k+1)) \hookrightarrow H^0(U, \omega_{\mathcal{V}_X}(k+1)) = (M_{z_0})_0,$$

where M_{z_0} denotes the localization of M by z_0. On the other hand, since L is trivial on $\mathcal{U}$, we get an isomorphism $\phi : (M_{z_0})_0 \to ((R/J)_{z_0})_0$. This allows us to define the map

$$\begin{array}{rccc} \varphi: & U & \longrightarrow & \mathbb{P}^n \\ & z & \longmapsto & [\phi(e_0)(z), \cdots, \phi(e_n)(z)]. \end{array}$$

We illustrate the previous reasoning by an example.

Example 2.5.16. *Consider the Veronese surface $\mathcal{V}$ in $\mathbb{P}^5$ given by the ideal J of Equation (2.29). The free resolution of R/J is*

$$0 \to R \to R(-2)^6 \to R(-3)^8 \xrightarrow{A} R(-4)^3 \to 0,$$

where

$$A = \begin{pmatrix} z_2 + 4z_4 & -\frac{1}{2}z_5 & 4z_4 \\ 2z_5 & z_2 & 0 \\ -z_1 + \frac{3}{2}z_5 & -\frac{1}{4}z_3 & \frac{1}{2}z_5 \\ z_0 + 4z_4 & -z_1 + z_4 + \frac{1}{2}z_5 & 4z_4 \\ z_3 - 4z_4 & -z_1 + z_4 + \frac{3}{2}z_5 & z_2 - 4z_4 \\ 4z_5 & z_0 - z_3 & 2z_5 \\ -\frac{1}{2}z_5 & \frac{1}{4}z_3 + \frac{1}{8}z_5 & -z_1 + \frac{1}{2}z_5 \\ z_3 - 4z_4 + \frac{1}{2}z_5 & \frac{1}{2}z_5 & z_0 - 4z_4 \end{pmatrix}.$$

By Proposition 2.5.15, $L \simeq \omega_{\mathcal{V}} \otimes \mathcal{O}_{\mathbb{P}^5}(2)|_{\mathcal{V}}$ is the sheafification of the graded module

$$M := (R/J)^3/\langle \textit{rows of } A\rangle.$$

The space of global sections of L is isomorphic to the zero graded piece of M, which is $\mathbb{K}^3$. Let $\{e_0, e_1, e_2\}$ be the usual basis of this space. Now consider the principal affine open subset $D(z_1)$. Then,

$$H^0(D(z_1), L) = ((M)_{z_1})_0 = ((R/J)_{z_1})_0^{\oplus 3}/\langle \textit{rows of } A_{z_1}\rangle \overset{\phi}{\simeq} ((R/J)_{z_1})_0,$$

where $A_{z_1} = \dfrac{1}{z_1}A$. One can check that in $\langle$rows of $A_{z_1}\rangle$ we have the elements

$$e_0 + \frac{1}{2}\left(\frac{z_0}{z_1} + \frac{z_2}{z_1}\right) e_1 \textit{ and } - e_2 + \frac{1}{8}\left(-2\frac{z_0}{z_1} + 6\frac{z_2}{z_1} + 4\frac{z_3}{z_1} + \frac{z_5}{z_1}\right) e_1.$$

Therefore, we get that

$$\phi(e_0) = -\frac{1}{2}\left(\frac{z_0}{z_1} + \frac{z_2}{z_1}\right)\phi(e_1) \textit{ and } \phi(e_2) = \frac{1}{8}\left(-2\frac{z_0}{z_1} + 6\frac{z_2}{z_1} + 4\frac{z_3}{z_1} + \frac{z_5}{z_1}\right)\phi(e_1).$$

Thus, we get an isomorphism

$$\begin{array}{ccc} D(z_1) \cap \mathcal{V} & \xrightarrow{\varphi} & \mathbb{P}^2 \\ \left[\dfrac{z_0}{z_1}, \cdots, \dfrac{z_5}{z_1}\right] & \longmapsto & \left[-4\left(\dfrac{z_0}{z_1} + \dfrac{z_2}{z_1}\right), 8, -2\dfrac{z_0}{z_1} + 6\dfrac{z_2}{z_1} + 4\dfrac{z_3}{z_1} + \dfrac{z_5}{z_1}\right]. \end{array}$$

Note that once we have the map φ, we can compute a parametrization of $\mathcal{V}$. This parametrization is the inverse of φ and is given by a basis $\{F_0, \dots, F_5\}$ of $S^2\mathbb{K}^3$. In particular, we get that for generic $[x, y, z] \in \mathbb{P}^2$ we must have that

$$[-4(F_0 + F_2), 8F_1, -2F_0 + 6F_2 + 4F_3 + F_5] = [x, y, z].$$

This defines some linear equations in the coefficients of $F_0, \dots, F_5$. A generic solution of such linear system of equations will give a parametrization of V. In our example, one can check that the polynomials

$$x_0^2, x_1^2, -x_0^2 - 2x_0x_1, x_2^2, x_0x_2, 8x_0^2 + 12x_0x_1 + 8x_1x_2 - 4x_2^2$$

provide a parametrization of the Veronese variety.

Remark 2.5.17. *The computation of the isomorphism $\varphi : \mathcal{V}_X \to \mathbb{P}^n$ is the only step of the algorithm where we use that n is even. For n odd, the canonical bundle of $\mathbb{P}^n$ has even degree. As a consequence, we can not write L by means of $\omega_{\mathcal{V}_X}$ and $\mathcal{O}_{\mathbb{P}}^n(2)$. One needs to find a $\mathcal{O}_{\mathbb{P}^N}$–module whose restriction to $\mathcal{V}_X$ is a line bundle of odd degree.*

Third step: computation of $\varphi(X)$

Once we have computed an isomorphism $\varphi : \mathcal{V}_X \to \mathbb{P}^n$, we compute the image of X through φ. This image is a hypersurface defined by a degree 4 polynomial G. Let $F \in \mathbb{P}(S^4V)$ be such that $H_{4,n}(F) = X$. In other words, F is the polynomial we want to compute. Then, the composition $\varphi \circ h_F$ is an automorphisms of $\mathbb{P}^n$ that sends $\mathbb{V}(F)$ to $\mathbb{V}(G)$. In particular, we get that there exists $g \in \mathrm{PGL}(n+1)$ such that $G = g \cdot F$. Moreover, by Lemma 2.3.2 we get the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "h_F"] \arrow[d, "(g^{\mathrm{t}})^{-1}"'] & \mathcal{V}_X \arrow[dl, "\varphi"'] \arrow[d, "\rho(g^{\mathrm{t}})^{\mathrm{t}}"] \\
\mathbb{P}^n \arrow[r, "h_G"'] & h_G(\mathbb{P}^n)
\end{tikzcd}.$$

Therefore, we get that $\rho(g^{\mathrm{t}})^{\mathrm{t}} = h_G \circ \varphi$. Note that the representation ρ is injective and the preimages through ρ can be computed by a solving linear system of equations. In particular, we can recover g from the composition $h_G \circ \varphi$. Finally, the polynomial F is computed by applying g^{-1} to G.

The algorithm

Summarizing the previous reasoning, we outline the algorithm

Algorithm 10

Input: the ideal I defining a variety inside the image of $H_{4,n}$ for $n = 2k$

Output: the unique $F \in \mathbb{P}(S^4V)$ such that $H_{4,n}(F) = \mathbb{V}(I)$.

1. Compute the unique Veronese variety $\mathcal{V}$ containing $\mathbb{V}(I)$.
2. Determine an isomorphism $\varphi : \mathcal{V} \to \mathbb{P}^n$.
3. Compute the quartic form G defining $\varphi(\mathbb{V}(I))$.
4. Determine $g^{\mathrm{t}} = \rho^{-1}((h_G \circ \varphi)^{\mathrm{t}})$.
5. Return, $F = g^{-1} \cdot G$.

Example 2.5.18. *Consider the variety X given by the ideal I in Equation (2.28). By Example 2.5.13, the ideal J of the unique Veronese variety $\mathcal{V}_X$ is given by (2.29). By Example 2.5.16, we have an isomorphism $\varphi : \mathcal{V}_X \to \mathbb{P}^2$ sending $[z_0, \cdots, z_5]$ to*

$$[-4(z_0 + z_2), 8z_1, -2z_0 + 6z_2 + 4z_3 + z_5].$$

The image of X through this map is given by the polynomial

$$G = 2x_0^4 - 8x_0^3x_1 + 6x_0^2x_1^2 + x_0x_1^3 + 4x_0^3x_2 - 48x_0^2x_1x_2 + 12x_0x_1^2x_2 - 96x_0x_1x_2^2 - 64x_1x_2^3.$$

One can check that, modulo J, the composition $h_G \circ \varphi$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -4 & 4 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 8 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}. \tag{2.32}$$

In this case, the representation ρ is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix} \longmapsto \begin{pmatrix} a_0^2 & a_0b_0 & b_0^2 & a_0c_0 & b_0c_0 & c_0^2 \\ 2a_0a_1 & a_1b_0 + a_0b_1 & 2b_0b_1 & a_1c_0 + a_0c_1 & b_1c_0 + b_0c_1 & 2c_0c_1 \\ a_1^2 & a_1b_1 & b_1^2 & a_1c_1 & b_1c_1 & c_1^2 \\ 2a_0a_2 & a_2b_0 + a_0b_2 & 2b_0b_2 & a_2c_0 + a_0c_2 & b_2c_0 + b_0c_2 & 2c_0c_2 \\ 2a_1a_2 & a_2b_1 + a_1b_2 & 2b_1b_2 & a_2c_1 + a_1c_2 & b_2c_1 + b_1c_2 & 2c_1c_2 \\ a_2^2 & a_2b_2 & b_2^2 & a_2c_2 & b_2c_2 & c_2^2 \end{pmatrix}.$$

One can check that the unique $g \in \mathrm{PGL}(3)$ such that $\rho(g^{\mathrm{t}})^{\mathrm{t}}$ equals the matrix (2.32) is

$$g = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

We conclude that $H_{4,2}^{-1}(X)$ is the polynomial

$$g^{-1} \cdot G = -256x_0^3x_1 - 768x_0^2x_1x_2 - 256x_1^3x_2 - 256x_0x_2^3.$$

Note that this polynomial, up to multiplication by constants, coincides with the polynomial we started with in Example 2.5.13.

2.6 Catalecticant enveloping variety

In the study of the Hessian correspondence for cubic hypersurface the rational map α_n, associating to a polynomial F the linear subspace $\langle \nabla^1 F \rangle$, played a crucial role. Motivated by this rational map, in Section 2.4.2 we introduced the variety $\mathcal{Z}_n$. This variety is strongly related to Comon's question (see [98, Problem 15]) through the notion of e–th gradient rank (see [56]). The goal of this section is to generalize these objects from cubic forms to forms of arbitrary degree. To investigate these generalizations, we introduce and study the catalecticant enveloping variety.

Fix $d, n \in \mathbb{N}$. We recall that $V = \mathbb{K}^{n+1}$ and that S^dV is the space of homogeneous polynomials of degree d in $x_0, \ldots, x_n$. For $0 < e < d$ and $0 \leq k$ we consider the closed subvariety of $\mathbb{P}(S^dV)$

$$\mathrm{Sub}_{e,k} := \left\{ F \in \mathbb{P}(S^dV) \mid \operatorname{rank} F_{e,d-e} \leq k+1 \right\}.$$

In other words, $\mathrm{Sub}_{e,k}$ consists of all $F \in \mathbb{P}(S^dV)$ such that $\dim \langle \nabla^k F \rangle \leq k+1$. Let $\mathrm{Sub}^{\circ}_{e,k}$ be the open subset of $\mathrm{Sub}_{e,k}$ consisting of the polynomials $F \in \mathbb{P}(S^dV)$ such that $\operatorname{rank} F_{e,d-e} = k+1$. By [74, Lemma 3.5], $\mathrm{Sub}^{\circ}_{e,k}$ is a dense open subset of $\mathrm{Sub}_{e,k}$. Following this notation, $\mathrm{Sub}_{1,k}$ corresponds to the variety Sub_k introduced in Section 2.4.2. Recall that T is the polynomial ring $\mathbb{K}[y_0, \ldots, y_n]$ acting on $S^\bullet V = \mathbb{K}[x_0, \ldots, x_n]$ by the apolarity action introduced in Section 2.1.2. Note that for any $F \in \mathbb{P}(S^dV)$,

$$\mathrm{rank} F_{e,d-e} \leq \min\{\dim T_e, \dim S^{d-e}V\}.$$

Therefore, we can assume that $k < \min\left\{\binom{e+n}{n}, \binom{d-e+n}{n}\right\}$. Note that for $e \leq d/2$ this minimum is $\binom{n+e}{n}$.

Example 2.6.1.

- *In Example 2.4.12 we showed that, for $d = 3$, $\mathrm{Sub}_{1,0}$ is the Veronese variety $V^{3,n}$. For $d \geq 3$, $\mathrm{Sub}_{1,0}$ is also the Veronese variety $V^{d,n}$ (See [74, Remark 1.24]). More generally, [106, Corollary 5.5] states that $\mathrm{Sub}_{e,0}$ is the Veronese variety $V^{d,n}$ for all $0 < e < d$.*

- *For* $k = \min\left\{\binom{e+n}{n}, \binom{d-e+n}{n}\right\} - 1$*, we have that* $\mathrm{Sub}_{e,k} = \mathbb{P}(S^d V)$*.*

In Sections 2.4.2, we used that $\mathrm{Sub}_{1,k}$ is irreducible for $d = 3$. However, $\mathrm{Sub}_{e,k}$ might not be irreducible for $e \geq 2$. For instance, in [37] it is proved that, for $d = 3$, $\mathrm{Sub}_{4,4}$ has two components of dimension 12 (see also [74, Example 3.6]). In [74, Chapter 7.2] the irreducibility of $\mathrm{Sub}_{e,k}$ is studied in more detail. For instance, [74, Corollary 7.10] gives sufficient conditions for $\mathrm{Sub}_{e,k}$ to be reducible. Nevertheless, even if $\mathrm{Sub}_{e,k}$ might be reducible, it is always connected (see [74, Theorem 7.6]). Moreover, $\mathrm{Sub}_{1,k}$ is irreducible for $d \geq 3$. This can be deduced from the following description of $\mathrm{Sub}_{1,k}$ (see [74, Proposition 1.23])

$$\mathrm{Sub}_{1,k} = \{F \in \mathbb{P}(S^d V) \,:\, \exists U \in \mathrm{Gr}(k+1, V) \text{ s.t. } F \in \mathbb{P}(S^d U)\}. \tag{2.33}$$

In other words, $\mathrm{Sub}_{1,k}$ consists of all polynomials such that, after a change of coordinates, they can be written by means of $k+1$ variables. Using this description, in [74, Proposition 1.23], the dimension of $\mathrm{Sub}_{1,k}$ is determined:

$$\dim \mathrm{Sub}_{1,k} = (k+1)(n-k) + \binom{d+k}{d} - 1. \tag{2.34}$$

In Section 2.4.2, we introduced the rational map $\alpha_{1,k} \subset \mathbb{P}(S^3 V)$ sending a cubic form F to $\langle \nabla^1 F \rangle$. Now we introduce the generalization of this map to any d and e. On the variety $\mathrm{Sub}_{e,k}$ we define the rational map

$$\begin{array}{cccc} \alpha_{e,k} : & \mathrm{Sub}_{e,k} & \dashrightarrow & \mathrm{Gr}(k+1, S^{d-e} V) \\ & F & \mapsto & \langle \nabla^e F \rangle \end{array}. \tag{2.35}$$

Note that the domain of definition of $\alpha_{e,k}$ is $\mathrm{Sub}^{\circ}_{e,k}$. Using this notation, the rational map α_k defined in (2.19) coincides with $\alpha_{1,k}$ for $d = 3$, and for $d \geq 3$ the map $\alpha_{1,k}$ is the generalization of α_k to higher degrees. We denote the image of $\alpha_{e,k}$ and its closure by $\mathcal{Z}^{\circ}_{e,k}$ and $\mathcal{Z}_{e,k}$ respectively.

Example 2.6.2. *Fix* $k = 0$*. In Example 2.6.1 we saw that* $\mathrm{Sub}_{1,0}$ *equals the Veronese variety* $V^{d,n}$*. Moreover, the target space of* $\alpha_{d,0}$ *is* $\mathbb{P}(S^{d-e} V)$*, and one can check that for a linear form* $l \in S^1 V$*,* $\alpha_{d,0}(l^d)$ *equals* l^{d-e}*. Therefore,* $\alpha_{d,0}$ *is the composition of the inverse of the Veronese embedding of* $V^{d,n}$ *with the Veronese embedding of* $V^{d-e,n}$*. In particular,* $\alpha_{d,0}$ *is an isomorphism onto its image and* $\mathcal{Z}_{d,0} = V^{d-e,n}$*.*

Example 2.6.3. *Assume that* $d = 3$ *and* $n = k = 2$*. By Proposition 2.4.21, we have that* $\alpha_{1,2}$ *is a birational morphism between* $\mathbb{P}(S^3 V) = \mathbb{P}^9$ *and* $\mathrm{Gr}(3, S^2 V)$*, which is also 9–dimensional. We get that* $\alpha_{1,2}$ *is dominant and* $\mathcal{Z}_{1,2} = \mathrm{Gr}(3, S^2 V)$*. More generally, assume that* $d = 3$ *and* $k = 2 \leq n$*. As above, for* $U \in \mathrm{Gr}(3, V)$ *we have that* $\alpha_{1,2}(\mathbb{P}(S^3 U))$ *equals* $\mathrm{Gr}(3, S^2 U) \subset \mathrm{Gr}(3, S^2 V)$*. In particular, we get that*

$$\mathcal{Z}_{1,2} = \bigcup_{U \in \mathrm{Gr}(3,V)} \mathrm{Gr}(3, S^2 U).$$

Example 2.6.4. *Let $d = 2$ and $e = 1$. Then, for $F \in \mathrm{Sub}_{1,k}$, $\langle \nabla^1 F\rangle$ equals the support of F. Recall that the support of $F \in \mathbb{P}(S^dV)$ is the smallest linear subspace U of V such that $F \in \mathbb{P}(S^dU)$. In particular, we deduce that, given $\mathbb{P}(U) \in \mathrm{Im}\,\alpha_{1,k}$, the closure of the fiber $\alpha_{1,k}^{-1}(\mathbb{P}(U))$ is $\mathbb{P}(S^2U)$. Moreover, for $d = 2$ and $k \geq 0$, $\alpha_{1,k}$ is dominant and the dimension of a generic fiber of $\alpha_{1,k}$ is $\binom{d+k}{k} - 1$.*

For $e = 1$ and $k = n$, $\alpha_{1,n}$ coincides with the gradient map, which is birational onto its image for $d \geq 3$ and $(d, n) \neq (3, 1)$ (see [46, Theorem 3.2]). Similarly, Example 2.6.2 shows that $\alpha_{d,0}$ is an isomorphism onto its image for every d. On the contrary, Example 2.6.4 shows that for $d = 2$, the generic fiber of $\alpha_{1,k}$ has positive dimension. With the aim of computing the dimension of $\mathcal{Z}_{e,k}$, we post the following question: For which values of d, n, e, k, is the map $\alpha_{e,k}$ birational onto its image?

To approach this question, we introduce the dual version of the map $\alpha_{e,k}$. We define the rational map $\beta_{e,k}$ assigning to a form the kernel of its catalecticant map

$$\begin{array}{rccc} \beta_{e,k}: & \mathrm{Sub}_{e,k} & \dashrightarrow & \mathrm{Gr}(\binom{n+e}{n} - k, T_e) \\ & F & \mapsto & \mathrm{Ker}\, F_{e,d-e} \end{array}. \tag{2.36}$$

Recall that $F_{e,d-e}$ is the catalecticant map of F from T_e to $S^{d-e}V$. Since $\mathrm{Ker}\, F_{e,d-e}$ is the degree e component of the apolar ideal of F, $\beta_{e,k}$ associates to $F \in \mathrm{Sub}^\circ_{e,k}$ the vector space $F_e^\perp \subset F^\perp$. The apolarity action gives an isomorphism between the dual space S^eV^* and T_e. In particular, $\mathrm{Gr}(k+1, S^eV)$ and $\mathrm{Gr}(\binom{n+e}{n} - k, T_e)$ are isomorphic and we get the following commutative diagram relating $\alpha_{d-e,k}$ and $\beta_{e,k}$

$$\begin{tikzcd} \mathrm{Sub}_{d-e,k} \arrow[r, dashed, "\alpha_{d-e,k}"] \arrow[d, equal] & \mathrm{Gr}(k, S^eV) \arrow[d, "(-)^\perp"', "\cong"] \\ \mathrm{Sub}_{e,k} \arrow[r, dashed, "\beta_{e,k}"] & \mathrm{Gr}(\binom{n+e}{n} - k, T_e) \end{tikzcd}. \tag{2.37}$$

The commutativity of Diagram (2.37) implies that any property satisfied by $\alpha_{d-e,k}$ is satisfied by $\beta_{e,k}$ and vice-versa. The relation between $\beta_{e,k}$ and apolar ideals allows us to use apolarity theory to approach the birationality of $\alpha_{e,k}$. Note that by Theorem 2.1.15, the apolar ideals is deeply related to Artinian Gorenstein rings.

In the setting of apolar ideals, the birationality of $\beta_{e,k}$ is equivalent to ask whether we can recover a polynomial $F \in \mathrm{Sub}_{e,k}$ from the e–th graded components of its apolar ideal. This question was partially answered in the setting of Artinian Gorenstein rings in [11] for $k = \tilde{k}$, where

$$\tilde{k} = \min\left\{\binom{e+n}{n}, \binom{d-e+n}{n}\right\} - 1.$$

This is the case where $F_{e,k}$ has maximal rank and $\mathrm{Sub}_{e,k} = \mathbb{P}(S^dV)$. Let F be a form in S^dV and let I be its apolar ideal. Since the Hilbert function of the apolar Artinian

Gorenstein ring T/I is symmetric, we get that the degree d component I_d of I is a hyperplane of T_d. In particular, I_d can be seen as a point in the dual vector space T_d^*. Through the isomorphism between the T_d^* and S^dV given by the apolarity action, I_d corresponds to the polynomial F. Therefore, we have an effective method for recovering F from the degree d part of its apolar ideal. Now, assume that all the generators of $F^\perp$ have degree at most $\tilde{d}$. Then, we can recover $(F^\perp)_d$ from $(F^\perp)_e$ for $\tilde{d} \leq e \leq d$. The even case of the following theorem can be found in [73, Example 4.7], whereas the odd case corresponds to [11, Theorem B].

Theorem 2.6.5. *[11, Theorem A] Let $d \geq 3$ and $n \geq 3$, let $F \in \mathbb{P}(S^dV)$ be a generic, and let $m(F)$ be the maximum degree of the minimal homogeneous generators of $F^\perp$. Then,*

1. *For $d = 2t-2$, $m(F) \leq t$.*
2. *For $d = 2t-1$ $m(F) \leq t$.*

From Theorem 2.6.5, we deduce the following result.

Corollary 2.6.6. *For $d \geq 3$ and $n \geq 3$, we have that*

1. *for $d = 2t-2$ and $e \leq t-2$, $\alpha_{e,\tilde{k}}$ is birational onto its image.*
2. *for $d = 2t-1$ and $e \leq t-1$, $\alpha_{e,\tilde{k}}$ is birational onto its image.*

In particular, in the above cases $\mathcal{Z}_{e,\tilde{k}}$ is rational and $\dim \mathcal{Z}_{e,\tilde{k}} = \binom{d+n}{n} - 1$.

Proof. Assume that $d = 2t-2$ and $e \leq t-2$. By Diagram 2.37, $\alpha_{e,\tilde{k}}$ is birational if and only if $\beta_{d-e,\tilde{k}}$ is birational. Let $F \in \mathrm{Sub}_{e,\tilde{k}} = \mathbb{P}(S^dV)$ be generic and let $G \in \mathbb{P}(S^dV)$ be such that $\beta_{d-e,\tilde{k}}(F) = \beta_{d-e,\tilde{k}}(G)$. In other words, $(F^\perp)_{d-e} = (G^\perp)_{d-e}$. By Theorem 2.6.5, the minimal generators of $F^\perp$ have degree at most $t \leq d-e$. In particular, we get that $(F^\perp)_d = (G^\perp)_d$, and hence, $F = G$. The same reasoning shows that for $d = 2t-1$ and $e \leq t-1$, $\alpha_{e,\tilde{k}}$ is birational onto its image. □

Due to the complicated structure of the variety $\mathrm{Sub}_{e,k}$, the study of the birationality of $\alpha_{e,k}$ for $k < \tilde{k}$ is much complicated. In particular, the study $\mathcal{Z}_{e,k}$ becomes more challenging. A natural, but difficult, question we can consider is the description of the boundary

$$\partial \mathcal{Z}_{e,k} = \overline{\mathcal{Z}_{e,k} \setminus \mathcal{Z}_{e,k}^\circ}$$

of $\mathcal{Z}_{e,k}$. We introduce the catalecticant enveloping variety as a first approach to this question.

Definition 2.6.7. *For $e \leq d$ and $l < \binom{d-e+n}{n}$, we define the catalecticant enveloping variety of degree d as*

$$\Phi_k^e := \left\{ \Gamma \in \mathrm{Gr}(k+1, S^{d-e}V) : \langle \nabla^e F \rangle \subseteq \Gamma \ \textit{for some } F \in \mathrm{Sub}_{e,k} \right\}.$$

In Section 2.6.2, we will show that Φ_k^e is a projective variety. Since $\mathcal{Z}_{e,k}^\circ$ is contained in Φ_k^e, we deduce that $\partial\mathcal{Z}_{d,n}$ is contained in Φ_k^e. We aim to study this boundary from the irreducible components of Φ_k^e.

Example 2.6.8. *For $k = 0$, we have that $\Phi_0^d = \mathcal{Z}_{d,0}$. In particular, by Example 2.6.2, we conclude that $\Phi_0^d = V^{d-e,n}$.*

2.6.1 Birationality of $\alpha_{1,k}$

As commented above, the map $\alpha_{1,n}$ coincides with the gradient map, which is birational onto its image for $d \geq 3$ and $(d, n) \neq (3, 1)$ (see [46, Theorem 3.2]). In this section we present a new proof of this fact, that in Section 2.6.2 will help us to compute the irreducible components of the catalecticant enveloping variety Φ_k^1. Moreover, the new reasoning presented in this section will provide in Section 2.6.3 a method for recovering a degree d hypersurface from its $(d-1)$–th polar variety.

Before restricting to the case $e = 1$, we first study the geometry of the fibers of $\alpha_{e,k}$. To do so, we consider the linear embedding

$$\begin{array}{cccc} \iota_e : & \mathbb{P}(S^d V) & \longrightarrow & \mathbb{P}\left(\mathrm{Hom}_{\mathbb{K}}\left(T_e, S^{d-e}V\right)\right) \\ & F & \longmapsto & F_{e,d-e} \end{array}.$$

Note that ι_e is a linear embedding by Euler's formula. For instance, for $e = 1$, the columns of $F_{1,d-1}$ are the first order derivatives of F. Therefore, the map ι_1 sends a degree d polynomial to its gradient. For $d = 3$, ι_1 coincides with the map (2.20). In general, the map ι_e sends a polynomial to the tensor of its e–th partial derivatives. We denote the image of ι_e by W_e.

Proposition 2.6.9. *Given $\Gamma \in \mathrm{Im}\,\mathcal{Z}_{e,k}$, the closure of $\alpha_{e,k}^{-1}(\Gamma)$ consists of all the degree d polynomials F such that $\langle \nabla^e F\rangle$ is contained in Γ. In particular, the closure of a fiber of $\alpha_{e,k}$ is a linear subspace.*

Proof. Consider the variety

$$\Sigma = \{(\Gamma, \varphi) \in \mathrm{Gr}(k+1, S^{d-e}V) \times W_e : \mathrm{Im}\,\varphi \subseteq \Gamma\}$$

together with the projection $\pi : \Sigma \to \mathrm{Gr}(k+1, S^{d-e}V)$. Note that the image of this projection is $\mathcal{Z}_{e,k}$. Given $\Gamma \in \mathcal{Z}_{e,k}^\circ$, the fiber $\Sigma_\Gamma := \pi^{-1}(\Gamma)$ is the linear subspace of maps $\varphi \in W_e$ whose image is contained in Γ. We now consider the map

$$\begin{array}{cccc} \alpha_{e,k} \times \iota_e : & \mathrm{Sub}_{e,k}^\circ & \longrightarrow & \Sigma \\ & F & \longmapsto & (\alpha_{e,k}(F), \iota_e(F)) \end{array}.$$

This map is well-defined since the image of $\iota_e(F)$ equals $\langle \nabla^e F\rangle = \alpha_{e,k}(F)$ for $F \in \mathrm{Sub}_{e,k}^\circ$. Moreover, $\alpha_{e,k} \times \iota_e$ is an isomorphism onto its image since ι_e is a linear embedding. The inverse map is given by the composition of the projection to W_e

and ι_e^{-1}. Now, the composition $\pi \circ (\alpha_{e,k} \times \iota_e)$ equals $\alpha_{e,k}$. Thus, for $\Gamma \in \mathcal{Z}^{\circ}_{e,k}$, we get that

$$\overline{\alpha_{e,k}^{-1}(\Gamma)} = \overline{(\alpha_{e,k} \times \iota_e)^{-1}(\Sigma_\Gamma)} = \iota_e^{-1}(\Sigma_\Gamma).$$

The proof follows from the fact that ι_e is a linear map and that Σ_Γ is a linear subspace of W_e. □

Remark 2.6.10. *Note that Proposition 2.6.9 provides an effective method for computing the closure of the fibers of $\alpha_{e,k}$ using linear algebra. Recall that W_e is a linear subspace. Given $\Gamma \in \operatorname{Im} \alpha_{e,k}$, we can compute its fiber by computing the intersection of W_e with the linear subspace of maps whose images are contained in Γ.*

For the rest of this section we fix $d = 1$. In order to derive the birationality of $\alpha_{1,k}$, we follow the same strategy used in Section 2.4.2: find a polynomial $F \in \mathrm{Sub}^{\circ}_{1,k}$ such that the intersection of W_1 and the linear subspace of maps whose images are contained in $\operatorname{Im} F_{1,d-1}$ has dimension zero. For this purpose, we first compute the equations of W_1.

As mentioned above, we identify the target space of ι_1 with $\mathbb{P}((S^{d-1}V)^{\oplus n+1})$. Through this identification, we can rewrite the map $\alpha_{1,k}$ as the rational map sending a polynomial F to its gradient ∇F. For $0 \le i \le n$ and $w = (w_0, \dots, w_n)$ such that $|w| := w_0 + \cdots + w_n = d-1$, let y^i_w be the coordinate of $\mathbb{P}((S^{d-1}V)^{\oplus n+1})$ representing the term $(w_i+1)x^w$ in the i–th direct summand.

Lemma 2.6.11. *The equations of W_1 in the coordinates y^i_w are*

$$y^{i_1}_{w-e_{i_1}} - y^{i_2}_{w-e_{i_2}} = 0, \tag{2.38}$$

for every $w \in \mathbb{N}^{n+1}$ with $|w| = d$ and $0 \le i_1 < i_2 \le n$ such that $w_{i_1}, w_{i_2} \ge 1$.

Proof. Let z_w be the coordinate of $\mathbb{P}(S^dV)$ representing the monomial x^w. In other words, a point in $\mathbb{P}(S^dV)$ is a degree d homogeneous polynomial of the form

$$G = \sum_{|w|=d} z_w x^w.$$

In particular, the first order derivatives of G are

$$\frac{\partial G}{\partial x_i} = \sum_{|w|=d, w_i \ge 1} (w_i+1) a_w x^{w-e_i}.$$

We deduce that for $|v| = d-1$, the coordinate $\iota_1(G)^i_v$ of $\iota_1(G)$ equals z_{v+e_i}. In particular, ι_e is a monomial map and the equations of W_e are linear binomials. The proof follows from the fact that the only linear binomial relations among the coordinates of the map ι_1 are of the form

$$\iota_1(G)^{i_1}_{w-e_{i_1}} = \iota_1(G)^{i_2}_{w-e_{i_2}}$$

for $|w| = d$ and $w_{i_1}, w_{i_2} \ge 1$. □

Using Lemma 2.6.11, we address the birationality of $\alpha_{1,k}$.

Theorem 2.6.12. *The rational map $\alpha_{1,k}$ is birational onto its image except in the following cases:*

1. *If $d = 2$ and $1 \leq k \leq n$, then the fibers are $\alpha_{1,k}^{-1}(U) = \mathbb{P}(S^2U) \subseteq \mathbb{P}(S^2V)$.*

2. *If $d = 3$ and $k = 1$, the fibers are one-dimensional.*

Proof. The case $d = 2$ was presented in Example 2.6.4. The case $d = 3$ and $k = 1$ was explored in Section 2.4.2. So we can assume that $d \geq 3$ and $(d,k) \neq (3,1)$. In Proposition 2.4.21 the result was proved for $d = 3$. Assume now that $d \geq 4$. To prove that $\alpha_{1,k}$ is birational it is enough to find $F \in \mathrm{Sub}^{\circ}_{1,k}$ such that $\alpha_{1,k}^{-1}(\alpha_{1,k}(F))$ is the point F. By Proposition 2.6.9, we get that a fiber of $\alpha_{1,k}$ is a point if and only if it is zero dimensional. Moreover, given $F \in \mathrm{Sub}^{\circ}_{1,k}$, the closure of the fiber $\alpha_{1,k}^{-1}(\langle \nabla^1 F\rangle)$ is isomorphic to $W_1 \cap \mathbb{P}((\langle \nabla^1 F\rangle)^{\oplus n+1})$ via ι_1. Therefore, it is enough to prove that there exists $F \in \mathrm{Sub}^{\circ}_{1,k}$ such that the dimension of $W_1 \cap \mathbb{P}((\langle \nabla^1 F\rangle)^{\oplus n+1})$ is zero.

Assume that $k = n$. As in the proof of Proposition 2.4.21, we consider the polynomial

$$F = \sum_{i \in \mathbb{Z}_{n+1}} x_i^{d-1} x_{i+1},$$

where $\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$. Note that using this notation $x_{n+1} = x_0$. For every $0 \leq i \leq n$, the i–th first order derivative of F is

$$\frac{\partial F}{\partial x_i} = (d-1)x_i^{d-2}x_{i+1} + x_{i-1}^{d-1}.$$

Note that the two monomials appearing in a first order derivative do not appear in the other first order derivatives. Therefore, the first order derivatives of F are linearly independent as elements in $S^{d-1}V$ and $F \in \mathrm{Sub}^{\circ}_{1,n}$.
Now, we compute the equations of the image of $F_{1,d-1}$ in $\mathbb{P}(S^{d-1}V)$. Let z_w be the coordinate of $\mathbb{P}(S^{d-1}V)$ corresponding to the monomial x^w for $|w| = d-1$. For $i \in \mathbb{Z}_{n+1}$, we consider the vectors

$$a_i = (d-2)e_i + e_{i+1} \text{ and } b_i = (d-1)e_{i-1}, \text{ where } e_i = (0,\dots,0,\underset{(i)}{1},0,\dots,0).$$

Note that $a_n = e_0 + (d-2)e_n$ and $b_0 = (d-1)e_n$. Using this notation, we get that $\frac{\partial F}{\partial x_i} = (d-1)x^{a_i} + x^{b_i}$. Then, the equations of $\langle \nabla^1 F\rangle$ are

$$\begin{aligned} &z_w = 0 && \text{for } |w| = d-1,\ w \neq a_i, b_i\, \forall i \in \mathbb{Z}_{n+1}, \\ &z_{a_i} - (d-1)z_{b_i} = 0 && \text{for } i \in \mathbb{Z}_{n+1}. \end{aligned} \tag{2.39}$$

In particular, the equations of $\mathbb{P}(\langle\nabla^1 F\rangle^{\oplus n+1})$ in the coordinates y^i_w used in Lemma 2.6.11 are

$$\begin{aligned}
&y^i_w = 0 && \text{for } w \neq a_j, b_j\, \forall j \in \mathbb{Z}_{n+1},\\
&y^i_{a_j} - \frac{d-1}{d} y^i_{b_j} = 0 && \text{for } j = i-1,\\
&y^i_{a_j} - y^i_{b_j} = 0 && \text{for } j = i,\\
&y^i_{a_j} - \frac{d-1}{2} y^i_{b_j} = 0 && \text{for } j = i+1,\\
&y^i_{a_j} - (d-1) y^i_{b_j} = 0 && \text{otherwise.}
\end{aligned} \tag{2.40}$$

Let W_F be the intersection of $\mathbb{P}((\langle\nabla^1 F\rangle)^{\oplus n+1})$ and W_1. We claim that W_F is zero dimensional. The equations of W_F are given by the Equations (2.38) together with the Equations in (2.40). Let $i_1, i_2 \in \mathbb{Z}_{n+1}$ with $i_1 \neq i_2$. By Equation (2.38) we get that $y^{i_1}_{a_{i_2}} = y^{i_2}_{a_{i_2}+e_{i_1}-e_{i_2}}$. Since $d \geq 4$, $a_{i_2} + e_{i_1} - e_{i_2} \neq a_j, b_j$ for every $j \in \mathbb{Z}_{n+1}$. Therefore, by Equation (2.40) we deduce that for $i_1 \neq i_2$, $y^{i_1}_{a_{i_2}} = 0$ is among the equations defining W_F. Moreover, using Equation (2.40) we get that $y^{i_1}_{b_{i_2}} = 0$ for $i_1 \neq i_2$ are also equations of W_F. Now, by Equations (2.40) and (2.38) we get that for every i

$$y^i_{a_i} = y^i_{b_i} = y^{i-1}_{b_i+e_i-e_{i-1}} = y^{i-1}_{a_{i-1}}.$$

Hence, we deduce that the equations of W_F are

$$\begin{aligned}
&y^i_w = 0 \text{ for } (i,w) \neq (i,\alpha_i), (i,\beta_i),\\
&y^0_{a_0} = y^0_{b_0} = y^1_{a_1} = y^1_{b_1} = \cdots = y^n_{a_n} = y^n_{b_n}.
\end{aligned}$$

From this system of equations we derive that W_F is zero dimensional. In particular, $\alpha^{-1}_{1,n}(\langle\nabla^1 F\rangle)$ is equal to the reduced point $\{F\}$, and $\alpha_{1,n}$ is birational onto its image for $d \geq 4$.

It remains to check the case $k \leq n-1$. Let F be a generic element in $\mathrm{Sub}_{1,k}$. By Equation (2.33), there exists $U \in \mathrm{Gr}(k+1, S^1 V)$ such that $F \in \mathbb{P}(S^d U)$. Without loss of generality, we can assume that U is generated by the linear forms $x_0, \ldots, x_k$. Now let $G \in \mathrm{Sub}_{e,k}$ be such that $\alpha_{e,k}(G) = \alpha_{e,k}(F)$. Then, the partial derivatives of G with respect to $x_{k+1}, \ldots, x_n$ vanish. This implies that $G \in \mathbb{P}(S^d U)$ and $\alpha^{-1}_{e,k}(\langle\nabla^1 F\rangle)$ equal $(\alpha_{e,k}|_{\mathbb{P}(S^d U)})^{-1}(\langle\nabla^1 F\rangle)$. The birationality of $\alpha_{e,k}|_{\mathbb{P}(S^d U)}$ follows from the case $n = k$. Therefore, we conclude that $\alpha^{-1}_{e,k}(\langle\nabla^1 F\rangle) = \{F\}$ and $\alpha_{1,k}$ is birational for $d \geq 4$. □

Corollary 2.6.13. *For $d \geq 3$ and $(d,k) \neq (3,1)$, the dimension of $\mathcal{Z}_{1,k}$ is*

$$\dim \mathcal{Z}_{1,k} = (k+1)(n-k) + \binom{d+k}{d} - 1.$$

Proof. Follows from Theorem 2.6.12 and Equation (2.34). □

A direct consequence of Theorem 2.6.12 is that $\beta_{e-1,k}$ is also birational onto its image for $d \neq 2$ and $(d,k) \neq (3,1)$. In particular, this means that in these cases we can recover a generic polynomial $F \in \mathrm{Sub}_{1,k}$ from the degree $d-1$ component of its apolar ideal.

2.6.2 Irreducible components of catalecticant enveloping variety for $d = 1$

In this section we study the catalecticant enveloping variety Φ_k^e, which is defined as

$$\Phi_k^e := \left\{\Gamma \in \mathrm{Gr}(k+1, S^{d-e}V) : \langle \nabla^e F\rangle \subseteq \Gamma \text{ for some } F \in \mathrm{Sub}_{e,k}\right\}$$

Our goal is to analyse $\mathcal{Z}_{e,k}$ and $\partial\mathcal{Z}_{e,k}$ through the investigation of Φ_k^e. We start this study by checking that it is a projective algebraic variety.

Proposition 2.6.14. *The catalecticant enveloping variety Φ_k^e is a projective subvariety of $\mathrm{Gr}(k+1, S^{d-e}V)$.*

Proof. Consider the incidence variety

$$\Sigma := \{(\Gamma, \varphi) \in \mathrm{Gr}(k+1, S^{d-e}V) \times W_e : \mathrm{Im}\,\varphi \subseteq \Gamma\}$$

together with the projection $\pi : \Sigma \to \mathrm{Gr}(k+1, S^{d-e}V)$. We claim that $\pi(\Sigma) = \Phi_k^e$. Indeed, $\Gamma \in \mathrm{Gr}(k+1, S^{d-e}V)$ is contained in $\pi(\Sigma)$ if and only if $\mathrm{Im}\, F_{e,d-e} \subseteq \Gamma$. Since $\mathrm{Im}\, F_{e,d-e} = \langle \nabla^e F\rangle$, this is equivalent to Γ being contained in Φ_k^e. The proof follows from the fact that Σ is projective. $\square$

Example 2.6.15. *Fix $d = 3$ and $k = 1$, and let $\Gamma \in \Phi_1^1$. Then, there exists $F \in \mathrm{Sub}_1$ such that $\langle \nabla^1 F\rangle \subseteq \Gamma$. If F is contained in Sub_1°, we get that $\Gamma \in \mathcal{Z}_1$. By Example 2.4.14, Γ is a tangent or secant line to $V^{2,n}$. If $F \in \mathrm{Sub}_0$, then the only requirement imposed to Γ is that $\mathbb{P}(\Gamma)$ and $V^{2,n}$ have nonempty intersection. We conclude that Φ_1^1 is the variety of projective lines in $\mathbb{P}(S^2V)$ intersecting the Veronese variety $V^{2,n}$.*

For $0 \leq l \leq k$, we consider the variety

$$\Phi_{l,k}^e := \{\Gamma \in \mathrm{Gr}(k+1, S^{d-e}V) : \Lambda \subseteq \Gamma \text{ for some } \Lambda \in \mathcal{Z}_{e,l}\}.$$

Using this notation, $\Phi_{k,k}^e = \mathcal{Z}_{e,k}$. Since $\mathcal{Z}_{e,l}$ is closed, $\Phi_{l,k}^e$ is also closed because it is the image of a bundle of Grassmannians. Moreover, $\Phi_{l,k}^e$ is contained in Φ_k^e. Indeed, for a generic element Γ in $\Phi_{l,k}^e$ there exists $\Lambda \in \mathcal{Z}_{e,k}^\circ$ such that $\Lambda \subseteq \Gamma$. In particular, there exists $F \in \mathrm{Sub}_{e,k}^\circ$ such that $\Lambda = \langle \nabla^e F\rangle \subseteq \Gamma$. Therefore, Γ is contained in Φ_k^e. By Proposition 2.6.14, we deduce that $\Phi_{l,k}^e$ is contained in Φ_k^e, and we obtain that

$$\Phi_k^e = \bigcup_{0 \leq l \leq k} \Phi_{l,k}^e. \tag{2.41}$$

Example 2.6.16. *By Example 2.6.8, we have that* $\Phi^1_0 = \mathcal{Z}_{1,0} = V^{d-1,n}$. *Then,*

$$\Phi^1_{0,k} = \{\Gamma \in \mathrm{Gr}(k+1, S^{d-1}V) : \mathbb{P}(\Gamma) \cap V^{d-1,n} \neq \emptyset\}.$$

In other words, $\Phi^1_{0,k}$ *is the variety of* k*–planes in* $\mathbb{P}(S^{d-1}V)$ *that intersect the Veronese variety* $V^{d-1,n}$. *Now, we consider the incidence variety*

$$\Sigma := \{(p, \Gamma) \in V^{d-1,n} \times \mathrm{Gr}(k+1, S^{d-1}V) : p \in \mathbb{P}(\Gamma)\}$$

together with the projections $\pi_1 : \Sigma \to V^{d-1,n}$ *and* $\pi_2 : \Sigma \to \mathrm{Gr}(k+1, S^{d-1}V)$. *Note that* π_2 *is generically finite onto its image, which is* $\Phi^1_{0,k}$. *Thus,* $\dim \Phi^1_{0,k} = \dim \Sigma$. *On the other hand, the fibers of* π_2 *are all Grassmannians of the form* $\mathrm{Gr}(k, \binom{d-1+n}{n} - 1)$. *We conclude that* $\Phi^1_{0,k}$ *is irreducible and its dimension is*

$$n + k\left(\binom{n+d-1}{n} - k - 1\right). \tag{2.42}$$

In Example 2.6.16 we showed the irreducibility og $\Phi^1_{0,k}$. Here we used that $\mathcal{Z}_{1,k}$ is irreducible. In general, $\mathcal{Z}_{e,l}$ might not be irreducible and, hence, $\Phi^e_{l,k}$ might be reducible. This obstacle arises from the fact that $\mathrm{Sub}_{e,k}$ is possibly reducible for $e \geq 2$. This is the main reason why we focus on the case $e = 1$. For the rest of this section, we set $e = 1$. By abuse of notation, during this section we simplify the notation and we denote Φ^1_k and $\Phi^1_{l,k}$ by Φ_k and $\Phi_{l,k}$ respectively. Similarly, we denote $\alpha_{1,k}$ and $\mathcal{Z}_{1,k}$ by α_k and $\mathcal{Z}_k$.

Proposition 2.6.17. *For* $l \leq k \leq n$, $\Phi_{l,k}$ *is irreducible. Moreover, the irreducible components of* Φ_k *are of the form* $\Phi_{l,k}$ *for some* $l \leq k$.

Proof. Consider the incidence variety

$$\Sigma := \{(E, \Gamma) \in \mathcal{Z}_l \times \mathrm{Gr}(k+1, S^{d-1}V) : E \subseteq \Gamma\}$$

together with the projection $\pi_1 : \Sigma \to \mathcal{Z}_l$ and $\pi_2 : \Sigma \to \mathrm{Gr}(k+1, S^{d-1}V)$. All the fibers of π_1 are irreducible and of the same dimension. Since $\mathcal{Z}_l$ is irreducible, by [66, Theorem 11.14] we deduce that Σ is irreducible too. Since the image of π_2 is $\Phi_{l,k}$, we conclude that $\Phi_{l,k}$ is irreducible. Hence, from (2.41), we conclude that the irreducible components of Φ_k are of the form $\Phi_{l,k}$ for some $l \leq k$. □

From Proposition 2.6.17, the following question arises: for which values of l and k, is $\Phi_{l,k}$ an irreducible component of Φ_k? For instance, Example 2.6.15 shows that for $d = 3$, $\Phi_1 = \Phi_{0,1}$ and $\Phi_{1,1} = \mathcal{Z}_1$ is contained in $\Phi_{0,1}$. In particular, we get that $\Phi_{1,k} \subset \Phi_{0,k}$ for $d = 3$.

Example 2.6.18. *Fix* $d = 3$. *For* $k = 2$, Φ_2 *is the union of* $\Phi_{0,2}$, $\Phi_{1,2}$ *and* $\Phi_{2,2} = \mathcal{Z}_2$. *Since* $\mathcal{Z}_1$ *is contained in* $\Phi_{0,1}$ *we deduce that* $\Phi_{1,2}$ *is contained in* $\Phi_{0,2}$. *For* $n = 2$, *we showed in Example 2.6.3 that* $\Phi_2 = \Phi_{2,2} = \mathrm{Gr}(3, S^2V)$. *Assume now that* $n \geq 3$.

Then, the catalecticant enveloping variety Φ_2 is the union of $\Phi_{0,2}$ and $\mathcal{Z}_2$. Now, by Proposition 2.4.21, the dimension of $\mathcal{Z}_2$ is equal to $3n+3$. On the other hand, using Equation (2.42), and that $n \geq 3$, we deduce that $\dim \Phi_{0,2} > \dim \mathcal{Z}_2$. Hence, $\Phi_{0,2}$ is an irreducible component of Φ_2. Now we show that $\mathcal{Z}_2$ is also an irreducible component of Φ_2. As in Example 2.4.18, we consider the polynomial $F = x_0x_1x_2 \in \mathrm{Sub}_2^\circ$. We claim that $\langle \nabla^1 F \rangle \in \mathcal{Z}_2$ does not lie in $\Phi_{0,2}$. On the contrary, assume that $\langle \nabla^1 F \rangle \in \Phi_{0,2}$. Then, there exists $G \in \mathrm{Sub}_0$ such that $\langle \nabla^1 G \rangle \subseteq \langle \nabla^1 F \rangle$. In particular, $\iota(G)$ lies in $W \cap \mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1})$. By Example 2.4.18, we derive that $\iota(G) = \iota(F)$, and hence, $F = G$. This is a contradiction since $F \notin \mathrm{Sub}_0$. Therefore, we conclude that $\mathcal{Z}_2$ is not contained in $\Phi_{0,2}$ and that the irreducible components of Φ_2 are $\mathcal{Z}_2$ and $\Phi_{0,2}$.

In Example 2.6.18, the two main ingredients for computing the irreducible components of Φ_2 were the dimension of $\phi_{l,2}$, for $l = 0, 2$, and Example 2.4.18. This example is generalized in the proof of Theorem 2.6.12. In order to extend these arguments to any d and k, we start by generalizing the computation performed in Example 2.4.18 as follows. Consider the polynomial $F = x_0x_1x_2 \in \mathbb{P}(S^3V)$. Then, $\langle \nabla^1 F \rangle = \langle x_1x_2, x_0x_2, x_0x_1 \rangle$. Now, we consider the vector space $\Gamma_{2,k} = \langle \nabla^1 F \rangle + \langle x_0x_3, x_0x_4, \ldots x_0x_k \rangle$. One can check that $W \cap \mathbb{P}(\Gamma_{2,k}^{\oplus n+1})$ has dimension zero. This implies that, for every $G \in \mathbb{P}(S^3V)$ such that $\langle \nabla^1 G \rangle \subseteq \Gamma_{2,k}$, one has $G = F$. We now generalize this example to any d and k.

Fix $1 \leq l \leq k \leq n$. As in the proof of Theorem 2.6.12, we consider the polynomial

$$F = \sum_{i \in \mathbb{Z}_{l+1}} x_i^{d-1} x_{i+1}, \tag{2.43}$$

where $\mathbb{Z}_{l+1} = \mathbb{Z}/(l+1)\mathbb{Z}$ is identified with the set $\{0, 1, \ldots, l\}$. Following the notation of Theorem 2.6.12, let z_u be the coordinate of $S^{d-1}V$ corresponding to the monomial x^u. For $i \in \mathbb{Z}_{l+1}$, we consider the vectors

$$a_i = (d-2)e_i + e_{i+1} \text{ and } b_i = (d-1)e_{i-1}.$$

Here, $a_l = e_0 + (d-2)e_l$ and $b_0 = (d-1)e_l$. Similarly to Equation (2.39), the equations of $\langle \nabla^1 F \rangle$ in $S^{d-1}V$ are

$$\begin{array}{ll} z_w = 0 & \text{for } |w| = d-1, \ w \neq a_i, b_i \forall i \in \mathbb{Z}_{l+1}, \\ z_{a_i} - (d-1) z_{b_i} = 0 & \text{for } i \in \mathbb{Z}_{l+1}. \end{array}$$

Note that $\dim \langle \nabla^1 F \rangle = l + 1$, and hence, $F \in \mathrm{Sub}_{1,l}^\circ$. We now consider a slightly different linear subspace of $S^{d-1}V$. Let Γ be the linear subspace

$$\Gamma := \langle \nabla^1 F \rangle + \langle x_0^{d-2} x_{l+1}, \ldots, x_0^{d-2} x_k \rangle. \tag{2.44}$$

For $l = k$, we have that $\Gamma = \langle \nabla^1 F \rangle$. Moreover, $\dim \Gamma = l + 1 + (k - l) = k + 1$, and hence, Γ is contained in $\Phi_{l,k}$. For $l + 1 \leq i \leq k$ we consider the integer vector $c_i = (d-2)e_0 + e_i$, i.e. $x^{c_i} = x_0^{d-2} x_i$. The equations of Γ in $\mathbb{P}(S^{d-1}V)$ are obtained by removing the equations $z_{c_{l+1}} = \cdots = z_{c_k} = 0$ from the equations of $\langle \nabla^1 F \rangle$. In other words, the equations of Γ are

$$\begin{aligned} & z_w = 0 && \text{for } w \neq a_i, b_i \text{ for } i \in \mathbb{Z}_{l+1} \text{ and } w \neq c_i \text{ for } l+1 \leq i \leq k, \\ & z_{a_i} - (d-1) z_{b_i} = 0 && \text{for } i \in \mathbb{Z}_{l+1}. \end{aligned}$$

As in Section 2.6.1, we identify the space $\mathbb{P}(\mathrm{Hom}_{\mathbb{K}}(T_1, S^{d-1}V))$ with $\mathbb{P}((S^{d-1}V)^{\oplus n+1})$. We use the coordinates y^i_w of $\mathbb{P}((S^{d-1}V)^{\oplus n+1})$ introduced in Lemma 2.6.11.

Lemma 2.6.19. *For $1 \leq l \leq k$ and $d \geq 3$ and $(l, d) \neq (1, 3), (2, 3)$, the intersection of W_1 and $\mathbb{P}(\Gamma^{\oplus n+1})$ is zero dimensional. In particular, F is the unique polynomial such that $\langle \nabla^1 F \rangle \subseteq \Gamma$.*

Proof. For $l = k$, $\Gamma = \langle \nabla^1 F \rangle$ and the proof follows from the proof of Theorem 2.6.12. Assume that $l < k$. As in Theorem 2.6.12, the equations of $\mathbb{P}(\Gamma^{\oplus n+1})$ are

$$\begin{aligned} & y^i_w = 0 && \text{for } w \neq a_j, b_j \forall j \in \mathbb{Z}_{l+1} \text{ and } w \neq c_i \text{ for } l+1 \leq i \leq k, \\ & y^i_{a_j} - \frac{d-1}{d} y^i_{b_j} = 0 && \text{for } j = i-1, \\ & y^i_{a_j} - y^i_{b_j} = 0 && \text{for } j = i, \\ & y^i_{a_j} - \frac{d-1}{2} y^i_{b_j} = 0 && \text{for } j = i+1, \\ & y^i_{a_j} - (d-1) y^i_{b_j} = 0 && \text{otherwise.} \end{aligned} \tag{2.45}$$

In the proof of Theorem 2.6.12, $W_1 \cap \mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1})$ is zero dimensional. Therefore, it is enough to check that

$$W_1 \cap \mathbb{P}(\langle \nabla^1 F \rangle^{\oplus n+1}) = W_1 \cap \mathbb{P}(\Gamma^{\oplus n+1}).$$

To derive this equality, we show that $y^i_{c_j}$ vanishes at $W_1 \cap \mathbb{P}(\Gamma^{\oplus n+1})$ for $i = 0, \dots, n$ and $l+1 \leq j \leq k$. Assume that $i \neq 0$. By Equation (2.38) we obtain that

$$y^i_{c_j} = y^0_{(d-3)e_0 + e_i + e_j}.$$

Since $j \geq l+1$ and $i \neq 0$, we get that $(d-3)e_0 + e_i + e_j$ is distinct from $a_m, b_m, c_{m'}$ for every $0 \leq m \leq l$ and $l+1 \leq m' \leq k$. Therefore, by Equation (2.45), we deduce that $y^i_{c_j} = 0$ for $i \neq 0$.
Assume now that $i = 0$. Since $1 \leq l$ and $l+1 \leq j$, we deduce that $j \geq 2$. From the equations of W_1, we get that

$$y^0_{c_j} = y^j_{b_1}.$$

By Equation (2.45), $y^j_{b_1}$ vanishes if and only if $y^j_{a_1}$ vanishes. Now, from the equations of W_1 we have that $y^j_{a_1} = y^2_{(d-2)e_1 + e_j}$. Since $j > l$, we deduce that $(d-2)e_1 + e_j$ is distinct from $a_m, b_m, c_{m'}$ for every $0 \leq m \leq l$ and $l+1 \leq m' \leq k$. Therefore, $y^j_{a_1} = 0$. By Equation (2.45) we deduce that $y^0_{c_j} = y^j_{b_1} = y^0_{c_j} = 0$. We conclude that for every $i = 0, \dots n$ and $l+1 \leq j \leq k$, $y^i_{c_j}$ vanishes at $W_1 \cap \mathbb{P}(\Gamma^{\oplus n+1})$. $\square$

In the next proposition, we use Lemma 2.6.19 to compute the dimension of $\Phi_{l,k}$.

Proposition 2.6.20. *For $d \geq 3$, the dimension of $\Phi_{l,k}$ is*

$$\dim \Phi_{l,k} = (l+1)(n-l) + \binom{d+l}{d} - 1 + (k-l)\left(\binom{d-1+n}{n} - k - 1\right). \tag{2.46}$$

Proof. Note that for $l = 0$, Equations (2.42) and (2.46) agree. A similar computation as in Example 2.6.16 shows that Equation (2.46) holds for $(d,l) = (3,1)$. Assume now that $(d,l) \neq (3,1)$ and $l \geq 1$. Consider the variety

$$\Sigma := \{(\Lambda, \Gamma) \in \mathcal{Z}_l \times \Phi_k : \Lambda \subseteq \Gamma\}$$

together with the projections $\pi_1 : \Sigma \to \mathcal{Z}_l$ and $\pi_2 : \Sigma \to \Phi_k$. For $\Lambda \in \mathcal{Z}_l$, we have that

$$\pi_1^{-1}(\Lambda) \simeq \mathrm{Gr}(k-l, S^{d-1}V/\Lambda).$$

Therefore, the fibers of π_1 are irreducible and they all have dimension

$$(k-l)\left(\binom{d-1+n}{n} - (l+1) - (k-l)\right) = (k-l)\left(\binom{d-1+n}{n} - k - 1\right).$$

Since $\mathcal{Z}_l$ is irreducible, by [66, Theorem 11.14] we deduce that Σ is irreducible too. By [67, Proposition 9.5], we obtain that

$$\dim \Sigma = \dim \mathcal{Z}_{1,l} + (k-l)\left(\binom{d-1+n}{n} - k - 1\right).$$

Now, note that the image of π_2 is $\Phi_{l,k}$. We claim that π_2 is birational onto its image. Indeed, take Γ as in (2.44), and let (Λ, Γ) such that $\Lambda \subseteq \Gamma$. In other words, $(\Lambda, \Gamma) \in \pi_2^{-1}(\Gamma)$. Since $\Lambda \in \mathcal{Z}_l$, there exists $G \in \mathrm{Sub}_{1,l}$ such that $\langle \nabla^1 G \rangle \subseteq \Lambda$. In particular, $\langle \nabla^1 G \rangle \subseteq \Gamma$, and by Lemma 2.6.19 we deduce that $G = F$, where F is as in (2.43). In particular, $\Lambda = \langle \nabla^1 F \rangle$ and the fiber $\pi_2^{-1}(\Gamma)$ is a point. We conclude that for $(l,d) \neq (1,3)$ and $l \geq 1$, $\Phi_{l,k}$ and Σ are birational, and hence,

$$\dim \Phi_{l,k} = \dim \mathcal{Z}_{1,l} + (k-l)\left(\binom{d-1+n}{n} - k - 1\right).$$

Now, the proof follows from Corollary 2.6.13. $\square$

We use the formula for the dimension of $\Phi_{l,k}$, provided in Proposition 2.6.20, to analyze the irreducible components of Φ_k. For this purpose, we study the relation between the dimensions of $\Phi_{l,k}$ and $\Phi_{l',k}$ for distinct l and l'.

Lemma 2.6.21. *For $d \geq 3$, and $0 \leq l \leq k-1$ we have that*

1. $\dim \Phi_{l,k} > \dim \Phi_{l+1,k}$ *for* $(l,k) \neq (n-1,n)$.

2. $\dim \Phi_{n-1,n} = \dim \Phi_{n,n} - 1$.

3. $\dim \Phi_{n-2,n} > \dim \Phi_{n,n}$ *for* $d \geq 4$. *For* $d = 3$, $\dim \Phi_{n-2,n} = \dim \Phi_{n,n} + n - 3$.

Proof. Using Proposition 2.6.20, we have that

$$\dim \Phi_{l,k} - \dim \Phi_{l+1,k} = 2l + 1 - n - k + \binom{d+n-1}{d-1} - \binom{d+l}{d-1}. \tag{2.47}$$

In particular, for $k = n$ and $l = n - 1$, we get that

$$\dim \Phi_{n-1,n} - \dim \Phi_{n,n} = 2(n-1) + 1 - n - n = -1.$$

Therefore, $\dim \Phi_{n-1,n} = \dim \Phi_{n,n} - 1$. Similarly, for $n \geq 2$ we have that

$$\begin{aligned} \dim \Phi_{n-2,n} - \dim \Phi_{n,n} &= 2(n-1) - 2 - 2n + \binom{d+n-1}{d-1} - \binom{d+n-2}{d-1} \\ &= \binom{d+n-1}{d-1} - \binom{d+n-2}{d-1} - 4 \\ &= \binom{d+n-2}{d-2} - 4. \end{aligned}$$

For $n \geq 2$ and $d \geq 4$, $\binom{d+n-2}{d-2} \geq \binom{4}{2} > 4$. Hence, we conclude that $\dim \Phi_{n-2,n} > \dim \Phi_{n,n}$ for $d \geq 4$. For $d = 3$, we get that

$$\dim \Phi_{n-2,n} - \dim \Phi_{n,n} = \binom{n+1}{1} - 4 = n - 3.$$

Assume now $(l,k) \neq (n-1,n)$. In order to check that $\dim \Phi_{l,k} > \dim \Phi_{l+1,k}$, it is enough to prove that $\dim \Phi_{l,l+1} > \dim \Phi_{l+1,l+1}$. Using Equation (2.47), we get

$$\dim \Phi_{l,l+1} - \dim \Phi_{l+1,l+1} = l - n + \binom{d+n-1}{d-1} - \binom{d+l}{d-1}.$$

Since $l \leq n-2$, we obtain that

$$\dim \Phi_{l,l+1} - \dim \Phi_{l+1,l+1} \geq l - n + \binom{d+n-1}{d-1} - \binom{d+n-2}{d-1} = l - n + \binom{d+n-2}{n}.$$

For $d \geq 3$,

$$\binom{d+n-2}{n} \geq \binom{n+1}{n} > n.$$

Therefore, $\dim \Phi_{l,l+1} > \dim \Phi_{l+1,l+1} \geq$ for $l \leq n-2$. We conclude that $\dim \Phi_{l,k} > \dim \Phi_{l+1,k}$ for $(l,k) \neq (n-1,n)$. $\square$

Lemma 2.6.21 together with Lemma 2.6.19 allow us to compute the irreducible components of $\Phi_{l,k}$.

Theorem 2.6.22. *The only values of $0 \leq l \leq k$ such that $\Phi_{l,k}$ is not an irreducible component of Φ_k are*

1. *$l = 1$ for $k \leq n-1$ and $d = 3$.*

2. *$l = 1, n-1$ for $k = n$ and $d = 3$.*

3. *$l = n-1$ for $k = n$ and $d \geq 4$.*

For $k \leq n-1$ and $d \geq 4$, $\Phi_{l,k}$ is an irreducible component of Φ_k for any $0 \leq l \leq k$. Moreover, $\Phi_{n-1,n}$ is a divisor of $\Phi_{n,n}$.

Proof. First, we fix $k \leq n-1$ and $d \geq 4$. Assume that $\Phi_{l,k}$ is not an irreducible component of Φ_k for $0 \leq l \leq k$. Then, there exists $m \neq l$ such that $\Phi_{l,k} \subseteq \Phi_{m,k}$. By Lemma 2.6.21, if $l < m$, then $\dim \Phi_{l,k} > \dim \Phi_{m,k}$. Hence, $m < l$ and $l \geq 1$. Let F and Γ be as in (2.43) and (2.44) respectively. Then, $F \in \mathrm{Sub}^{\circ}_{1,l}$ and Γ lies in $\Phi_{l,k}$. Since $\Phi_{l,k} \subset \Phi_{m,k}$, $\Gamma \in \Phi_{m,k}$ and, hence, there exists $G \in \mathrm{Sub}_{1,m}$ such that $\operatorname{Im} G_{1,D-1} \subset \Gamma$. By Lemma 2.6.19, $G = F$. This is a contradiction since $G \notin \mathrm{Sub}^{\circ}_{1,l}$ as $m < l$. We conclude that $\Phi_{l,k}$ is an irreducible component of Φ_k for $0 \leq l \leq k \leq n-1$.
Now assume that $k \leq n-1$ and $d = 3$. We recall that in this case $\Phi_{1,k}$ is contained in $\Phi_{0,k}$. The same argument as above shows that, for $l \neq 1$, $\phi_{l,k}$ is an irreducible component of Φ_k
Assume now that $k = n$. The same argument used above shows that $\Phi_{l,n}$ is an irreducible component of Φ_n for $l \neq n-1$ and $d \geq 4$, and for $l \neq 1$ and $d = 3$. It remains to check that $\Phi_{n-1,n}$ is included in $\Phi_{n,n}$. Let $\Gamma \in \Phi_{n-1,n}$ be generic. There exists $F \in \mathrm{Sub}^{\circ}_{1,n-1}$ such that $\langle \nabla^1 F \rangle \subset \Gamma$. We can assume that F is a polynomial in the variables $x_0, \ldots, x_{n-1}$. Therefore, $\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n-1}}$ are linearly independent and they generate $\langle \nabla^1 F \rangle$. In particular, there exists $g \in S^{d-1}V$ such that Γ is generated by $\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n-1}}$ and g. We write g as

$$g = \sum_{i=0}^{d-1} g_i(x_0, \ldots, x_{n-1}) x_n^i,$$

where g_i is a degree $d-i-1$ homogeneous polynomial in the variables $x_0, \ldots, x_{n-1}$. For $\mu \in \mathbb{K}$ we consider the polynomial

$$G_\mu = F + \mu \left(\sum_{i=0}^{d-1} \frac{1}{i+1} g_i \cdot x_n^{i+1} \right).$$

The first order derivatives of G_μ are

$$\frac{\partial G_\mu}{\partial x_j} = \begin{cases} \dfrac{\partial F}{\partial x_j} + \mu \left(\displaystyle\sum_{i=0}^{d-1} \frac{1}{i+1} \frac{\partial g_i}{\partial x_j} x_n^{i+1} \right) & \text{for } 0 \leq j \leq n-1 \\ \mu \displaystyle\sum_{i=0}^{d-1} g_i x_n^i = \mu g & \text{for } j = n \end{cases}.$$

Since $\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n-1}}, g$ are linearly independent, for $\mu \neq 0$ we have that $G_\mu \in \mathrm{Sub}^\circ_{1,n}$ and $\mathrm{Im}\,(G_\mu)_{1,d-1}$ lies in $\Phi_{n,n}$. However, for $\mu = 0$ we have that $G_0 = F$ and $\mathrm{Im}\,(G_0)_{1,d-1} = \langle \nabla^1 F \rangle$ is not a point in $\Phi_{n,n}$. We consider the map

$$\begin{array}{rccc} \delta: & \mathbb{K} \setminus \{0\} & \longrightarrow & \Phi_{n,n} \\ & \mu & \longmapsto & \langle \nabla^1 G_\mu \rangle \end{array}.$$

By the Curve-to-Projective Extension Theorem (see [124, Theorem 17.5.1]), we can extend the map δ to $\mu = 0$. For $i = 0, \ldots, n-1$, we consider the map

$$\begin{array}{rccc} \delta_i: & \mathbb{K} & \longrightarrow & \mathbb{P}(S^{d-1}V) \\ & \mu & \longmapsto & \dfrac{\partial G_\nu}{\partial x_i} \end{array}.$$

Then, for $\mu \neq 0$, $\delta_i(\mu)$ is contained in $\delta(\mu)$. Since this is a closed condition, we obtain that $\delta_i(0) = \frac{\partial F}{\partial x_i}$ is contained in $\delta(0)$. Thus, $\delta(0)$ contains $\langle \nabla^1 G_0 \rangle = \langle \nabla^1 F \rangle$. Moreover, since $g \in \mathrm{Im}\,(G_\mu)_{1,d-1}$ for $\mu \neq 0$, we deduce that $g \in \delta(0)$. This implies that $\delta(0) = \langle \nabla^1 F \rangle + \langle g \rangle = \Gamma$. Hence, $\Gamma \in \Phi_{n,n}$, and we conclude that $\Phi_{n-1,n}$ is contained in $\Phi_{n,n}$. By Lemma 2.6.21, we deduce that $\Phi_{n-1,n}$ is a divisor of $\Phi_{n,n}$. $\square$

Corollary 2.6.23. *For $d \geq 3$, $\Phi_{n-1,n}$ is an irreducible component of $\partial \mathcal{Z}_{1,n}$.*

Proof. Recall that $\mathcal{Z}_n = \Phi_{n,n}$. By Theorem 2.6.22 we get that $\Phi_{n-1,n}$ is a divisor of $\mathcal{Z}_n$. By Lemma 2.6.19, the intersection of W_1 and $\mathbb{P}(\Gamma^{\oplus n+1})$ is zero dimensional for a generic $\Gamma \in \Phi_{n-1,n}$. In particular, for a generic $\Gamma \in \Phi_{n-1,n}$, there exists a unique polynomial $F_\Gamma \in \mathbb{P}(S^d V)$ such that $\langle \nabla^1 F_\Gamma \rangle$ is contained in Γ. Moreover, F_Γ is contained in $\mathrm{Sub}_{1,n-1}$. Assume that $\Phi_{n-1,n}$ is not contained in $\partial \mathcal{Z}_{1,n}$. Then, a generic $\Gamma \in \Phi_{n-1,n}$ is contained in $\mathcal{Z}_n^\circ$. In particular, there exists $F \in \mathrm{Sub}^\circ_{1,n}$ such that $\langle \nabla^1 F \rangle$ is contained in Γ. Since Γ is generic, we get that $F = F_\Gamma$. This is a contradiction since F_Γ is not contained in $\mathrm{Sub}^\circ_{1,n}$. Therefore, $\Phi_{n-1,n}$ is contained in $\partial \mathcal{Z}_n$. Since $\dim \partial \mathcal{Z}_n \leq \dim \mathcal{Z}_n - 1$, we deduce that $\Phi^1_{n-1,n}$ is an irreducible component of $\partial \mathcal{Z}_n$. $\square$

Example 2.6.24. *For $n = 1$, we have that $\Phi_1 = \Phi_{0,1} \cup \mathcal{Z}_1$. By Theorem 2.6.22 and Corollary 2.6.23, we deduce that $\Phi_1 = \mathcal{Z}_1$ and $\partial \mathcal{Z}_1 = \Phi^1_{0,1}$.*

2.6.3 k–polar correspondence

In Section 2.1.1 we introduced the k–polar maps and the k–polar varieties, and in Section 2.2 we defined the Hessian map and the Hessian variety as the second polar map and polar variety, respectively. Using these objects, the Hessian correspondence was introduced. In this Section we generalize the notion of Hessian correspondence to higher order derivatives.

Recall that, for $F \in \mathbb{P}(S^dV)$ and for $1 \le k \le d$, the k–polar map is the rational map

$$\begin{array}{rccc} g_F^k: & \mathbb{V}(F) & \dashrightarrow & \mathbb{P}(S^kV) \\ & p & \longmapsto & \displaystyle\sum_{u_0+\cdots+u_n=k} \frac{\partial^k F}{\partial x_0^{u_0}\cdots\partial x_n^{u_n}}(p)x_0^{u_0}\cdots x_n^{u_n}, \end{array}$$

and the k–polar variety is the closure of the image of the k–polar map. Let z_u be the coordinate of $\mathbb{P}(S^kV)$ corresponding with the monomial $x^u = x_0^{u_0}\cdots x_n^{u_n}$, where $u \in \mathbb{N}^{n+1}$ such that $|u| = u_0 + \cdots + u_n = k$. Analogously to the definition of the Hessian correspondence, we aim to define the k–polar correspondence as the map that associate to a degree d homogeneous polynomial its k–polar variety. Let $p_{d,n}^k(t)$ be the Hilbert polynomial of the k–polar variety of a generic polynomial in $\mathbb{P}(S^dV)$. For instance, as presented in Section 2.1.1, $p_{d,n}^1$ is the Hilbert polynomial of a hypersurface of degree $d(d-1)^{n-1}$ in $(\mathbb{P}^n)^*$.

Example 2.6.25. *For d even, we fix k to be $d/2$. In this setting, the catalecticant matrix of $F_{k,k}$ of a polynomial $F \in \mathbb{P}(S^{2k}V)$ is a square matrix of dimension $\binom{k+n}{n}$. Since the determinant of the square catalecticant matrix $\mathrm{Cat}(k,k)$ is nonzero, we deduce that for generic F, $\langle \nabla^k F\rangle$ has dimension $\binom{k+n}{n}$. In particular, for F generic the set of all k–th derivatives of F form a basis of S^kV. We conclude that for generic $F \in \mathbb{P}(S^{2k}V)$, g_F^k is a Veronese embedding. In this case, the pullback of a quadric hypersurface through g_F^k is a hypersurface of degree $2k$. This implies that for $F \in \mathbb{P}(S^{2k}V)$ generic, the k–polar variety of F is the intersection of a Veronese variety and a quadric hypersurface. The same argument as in Proposition 2.5.3 shows that*

$$p_{2k,n}^k = \binom{kt+n}{n} - \binom{k(t-2)+n}{n}.$$

Definition 2.6.26. *We define the k–**polar correspondence** as the rational map*

$$\begin{array}{cccc} \mathcal{G}_{d,n}^k & \mathbb{P}(S^dV) & \dashrightarrow & \mathrm{Hilb}^{p_{d,n}^k(t)}(\mathbb{P}(S^kV)) \\ & F & \longmapsto & \overline{g_F^k(\mathbb{V}(F))} \end{array}. \tag{2.48}$$

As with the Hessian correspondence, we aim to understand the generic fibers of the k–th polar correspondence. We can extend the questions we proposed for $H_{d,n}$ to $\mathcal{G}_{d,n}^k$: for which value of k,d,n is $\mathcal{G}_{d,n}^k$ birational onto its image? And for which values is generically finite? If so, what is its degree? When has the generic fiber positive dimension? Moreover, we are interested in finding effective methods for recovering a hypersurface from its k–polar variety.

For $k = 1$, $\mathcal{G}^1_{d,n}$ is the rational map (2.2), which is birational onto its image by the Biduality Theorem. For $k = 2$, $\mathcal{G}^2_{d,n}$ coincides with the Hessian correspondence. In particular, in Sections 2.4 and 2.5 we answered the above questions for $k = 2$ and $d = 3, 4$. In this section we show that $\mathcal{G}^{d-1}_{d,n}$ is birational onto its image for $(d, n) \neq (3, 1)$. Moreover, we provide an algorithm to recover a degree d hypersurface from its $(d-1)$–polar variety. The strategy we follow is the same used in Section 2.4.2. We use the birationality of the map $\alpha_{1,n}$, proved in Theorem 2.6.12, to derive the birationality of $\mathcal{G}^{d-1}_{d,n}$.

Let $F \in \mathbb{P}(S^dV)$ and $k = d-1$. The $(d-1)$–partial derivatives of F are linear forms. Therefore, the $(d-1)$–polar map of F is a linear map from $\mathbb{P}^n$ to $\mathbb{P}(S^{d-1}V)$. Moreover, for $F \in \mathbb{P}(S^dV)$, the $(d-1)$–polar map g^{d-1}_F is not a linear embedding if and only if F is a cone (see [87, Theorem 3.1]). We deduce that for a generic $F \in \mathbb{P}(S^dV)$, g^{d-1}_F is a linear embedding.

Example 2.6.27. *Let $F = x_0^d + \cdots + x_n^d$. The only non zero $(d-1)$–partial derivatives of F are*

$$\frac{\partial^{d-1}F}{\partial x_i^{d-1}} = (d-1)!x_i \text{ for } i = 0, \ldots, n.$$

The $(d-1)$–polar map is the linear map defined in coordinates by

$$z_{e_i} = (d-1)!\, x_i \text{ for } i = 0, \ldots, n \text{ and } z_u = 0 \text{ else},$$

where $e_i = (0, \ldots, 0, \underset{(i)}{1}, 0, \ldots, 0)$. We observe that g^k_F is a linear embedding of $\mathbb{P}^n$ and

$$g^{d-1}_F(\mathbb{P}^n) = \mathbb{V}(z_u : u \neq e_0, \ldots, e_n).$$

The $(d-1)$–polar variety of F is the intersection of the hypersurface $z^d_{e_0} + \cdots + z^d_{e_n}$ with $g^{d-1}_F(\mathbb{P}^n)$.

Now, given $F \in \mathbb{P}(S^dV)$ generic and $G \in \mathbb{P}(S^dV)$ with the same $(d-1)$–polar variety, then g^{d-1}_F and g^{d-1}_G are linear embeddings. We deduce that there exists $g \in \mathrm{PGL}(n+1)$ such that

$$g \cdot F = G.$$

This means that the generic fibers of the $(d-1)$–polar correspondence are contained in a $\mathrm{PGL}(n+1)$–orbit. We aim to give a better description of these fibers. As done for $H_{3,n}$, the strategy is to investigate $g^{d-1}_F(\mathbb{P}^n)$ instead of the $(d-1)$–polar variety and then use the study of $\alpha_{1,k}$ performed in Section 2.6.1. We consider the linear automorphism h of $\mathbb{P}(S^{d-1}V)$ that sends the monomial x^u to $u!x^u$, where $u! = (u_0)! \cdots (u_n)!$

Lemma 2.6.28. *For $F \in \mathbb{P}(S^d V)$, it holds that*

$$g_F^k(\mathbb{P}^n) = h(\mathbb{P}(\langle \nabla^1 F \rangle)).$$

Proof. We write F as $F = \sum_{|w|=d} a_w x^w$. Then, the $(d-1)$–partial derivatives of F are of the form

$$\frac{\partial^u F}{\partial x^u} = \sum_{i=0}^{n} (u+e_i)! a_{u+e_i} x_i.$$

Therefore,

$$g_F^k(e_i) = \sum_{|u|=d-1} (u+e_i)! a_{u+e_i} x^u.$$

On the other hand, we have that

$$\frac{\partial F}{\partial x_i} = \sum_{|u|=d-1} (u_i+e_i) a_{u+e_i} x^u.$$

Now, the proof follows from the equality $g_F^k(e_i) = h(\frac{\partial F}{\partial e_i})$. □

Using Lemma 2.6.28, we can derive the birationality of $\mathcal{G}_{d,n}^{d-1}$ from the birationality of the gradient map $\alpha_{1,n}$ (see [46, Theorem 3.2]).

Theorem 2.6.29. *The map $\mathcal{G}_{d,n}^{d-1}$ is birational onto its image for $(d,n) \neq (3,1)$.*

Proof. We follow the same strategy as in the proof of Theorem 2.4.22. Let $F \in \mathbb{P}(S^d V)$ generic and let X be its $(d-1)$–polar variety, then $g_F^{d-1}(\mathbb{P}^n)$ is the unique n–dimensional linear subspace containing X. Indeed, assume that X is contained in the two distinct n–dimensional linear subspaces H_1 and H_2. Since X has dimension $n-1$, X must be equal to the linear subspace $H_1 \cap H_2$, which is a contradiction. As a consequence, we can define the rational map

$$\delta_n : \operatorname{Im} \mathcal{G}_{d,n}^{d-1} \dashrightarrow \operatorname{Gr}(n+1, S^{d-1}V)$$

sending a k–polar variety X to the smallest linear subspace containing it. By Lemma 2.6.28, we get

$$\alpha_{1,n} = \delta_n \circ \mathcal{G}_{d,n}^{d-1}.$$

Therefore, the birationality of $\mathcal{G}_{d,n}^{d-1}$ follows from the birationality of $\alpha_{1,n}$ proven in Theorem 2.6.12 (see also [46, Theorem 3.2]). □

The proof of Theorem 2.6.29, and the study of the map $\alpha_{1,n}$ carried out in Section 2.6.1, allow us to develop a recovery algorithm for $\mathcal{G}_{d,n}^{d-1}$. In other words, we provide an effective method to, given a k–polar variety X, recover the unique polynomial F (up to scalar) whose k–polar variety is X. Now, the steps of the algorithm are presented.

Algorithm 11

Input: the ideal I of $X \in \operatorname{Im} \mathcal{G}_{d,n}^{d-1}$.

Output: the unique polynomial $F \in \mathbb{P}(S^d V)$ such that $\mathcal{G}_{d,n}^{d-1}(F) = \mathbb{V}(I)$.

1. Compute the smallest projective subspace $\mathbb{P}(\Gamma)$ containing $\mathbb{V}(I)$ by taking the degree one part of the saturation of I.
2. Determine $W_1 \cap \mathbb{P}(\Gamma^{\oplus n+1}) \subseteq \mathbb{P}\left((S^2V)^{\oplus n+1}\right)$. By Theorem 2.6.12, this intersection is a point $[F_0 : \cdots : F_n] \in \mathbb{P}\left((S^{d-1}V)^{\oplus n+1}\right)$.
3. Compute F via the Euler's formula: $F = \sum x_i F_i$.

Example 2.6.30. *Fix $d = 5$ and $n = 2$, and consider the subvariety X of $\mathbb{P}(S^4V) \simeq \mathbb{P}^{14}$ given by the ideal*

$$\langle z_{3,1,0} z_{3,0,1} z_{2,1,1}^3 \rangle + \langle z_v : v \in \mathbb{N}^{n+1} \ s.t.\ |v| = 4 \text{ and } v \neq (3,1,0), (3,0,1), (2,1,1) \rangle.$$

Let Γ be the smallest projective subspace containing X. In other words, Γ is given by the equation

$$z_v = 0 \text{ for } v \neq (3,1,0), (3,0,1), (2,1,1).$$

The intersection $W \cap \mathbb{P}(\Gamma^{\oplus n+1})$ is given by Equations (2.38) and $y_u^i = 0$ for $u \neq (3,1,0), (3,0,1), (2,1,1)$. From the equations of W we deduce that

$$y_{3,1,0}^0 = y_{4,0,0}^1$$

which, by the equations of Γ, vanishes. Similarly, we have the following equalities

$$y_{3,1,0}^1 = y_{2,2,0}^0, \ \ y_{3,0,1}^0 = y_{4,0,0}^0, \ \ y_{3,0,1}^2 = y_{2,0,2}^0, \ \ y_{2,1,1}^1 = y_{1,2,1}^0, \ \ y_{2,1,1}^2 = y_{1,1,2}^0,$$

and all the variables at the right hand side of the above equalities vanish due to the equations of Γ. Now, using the equations of W we deduce that

$$y_{3,1,0}^2 = y_{3,0,1}^1 = y_{2,1,1}^0.$$

We conclude that the intersection of W with $\mathbb{P}(\Gamma^{\oplus n+1})$ is the point

$$[x_0^2 x_1 x_2, x_0^3 x_2, x_0^3 x_1].$$

Using the third step of Algorithm 11, we conclude that $x_0^3 x_1 x_2$ is the unique polynomial in $\mathbb{P}(S^5V)$ whose 4–th polar variety is X.

2.7 Open problems

We finish Chapter 2 by listing some of the open problems and future research lines related to the Hessian correspondence. The main open question concerning the Hessian correspondence is:

Question 2.7.1. *Is the Hessian correspondence $H_{d,n}$ birational onto its image for $(d,n) \neq (3,1)$?*

In this chapter we have analysed this question for cubic and quartic hypersurfaces and for hypersurfaces of Waring rank at most $n+1$. Our results answer affirmatively Question 2.7.1 for $d \leq 4$. A similar question could be asked in more generality for the k–th polar correspondence.

Question 2.7.2. *For which values d, n, k the map $\mathcal{G}^k_{d,n}$ is birational onto its image?*

For $k = 1$ this question is answered by the Biduality Theorem (see Theorem 2.1.4). Furthermore, the outcome of Chapter 2 gives an affirmative answer to this question for $k = 2$ and $d = 3, 4$ and for $k = d - 1$. From a computational perspective, another intriguing question is the development of recovery algorithms. In the case where $\mathcal{G}^k_{d,n}$ is birational, can we recover effectively a hypersurface from its k–th polar variety?. For instance, for the case of the Hessian correspondence for quartic hypersurfaces, $H^4_{d,n}$ is birational, but the recovery question remains open for n odd.

Concerning the catalecticant enveloping variety, lot of interesting questions arise. The most challenging and ambitious question would be the computation of the irreducible components of Φ^e_k. The difficulty of this problem arises from the complexity of the irreducible components of $\mathrm{Sub}_{e,k}$.

Question 2.7.3. *What is the relation between the irreducible components of* $\mathrm{Sub}_{e,k}$*, $\mathcal{Z}_{e,k}$ and Φ^e_k?*

A more approachable question would be the computation of the irreducible components of Φ^e_k in the cases where $\mathrm{Sub}_{e,k}$ is irreducible or in the cases where its irreducible components are well understood. All these questions are deeply related to the birationality of the map $\alpha_{e,k}$. Some other interesting problems would be the computation of the Chow classes of the irreducible components of Φ^1_k in the Grassmannian.

The main motivation for studying the catalecticant enveloping variety is the study of the variety $\mathcal{Z}_{d,n}$. Some of the main questions concerning this variety are:

Question 2.7.4. *What are the equations of $\mathcal{Z}_{d,n}$? Can we describe the boundary $\partial\mathcal{Z}_{d,n}$?*

In Section 2.6.2 we computed an irreducible component of $\partial\mathcal{Z}_{d,n}$. Does it has more irreducible components? In an ongoing project with Leonie Kayser we are studying these questions and their relation to tensor decomposition.

The final comment concerns the application of the analysis carried out in Chapter 2 to the classical Hesse problem. Recall that the (d,n)–Gordan-Noether locus is the locus of polynomials in $\mathbb{P}(S^dV)$ whose Hessian polynomial vanishes. The Hesse problem concerns the description of the (d, n)–Gordan-Noether locus. For $n \leq 3$, a polynomial lies in this locus if and only if it is a cone (see [58]). Consider the hypersurface $\mathcal{S}$ of $\mathbb{P}(S^2V)$ consisting of all singular symmetric matrices. The relation between the Hessian map and the (d, n)–Gordan-Noether locus relies on the fact that a polynomial F lies in the (d, n)–Gordan-Noether if and only if $h_F(\mathbb{P}^n)$ is contained in $\mathcal{S}$. For instance, for $d = 3$ and for $F \in \mathbb{P}(S^3V)$ generic, $h_F(\mathbb{P}^n)$ is an n–dimensional projective subspace. Therefore, F lies in the $(3, n)$-Gordan-Noether locus if and only $h_F(\mathbb{P}^n)$ lies in the Fano scheme $F_n(\mathcal{S})$ of n–planes contained in $\mathcal{S}$. In particular, by Lemma 2.4.9, the Hesse problem for cubics can be translated to the study of the intersection of $\mathcal{Z}_{1,n}$ and $F_n(\mathcal{S})$. We expect that our study of $h_F(\mathbb{P}^n)$ and the catalecticant enveloping variety could enlighten new ideas for approaching the Hesse problem.

Chapter 3

Algebraic game theory

Game theory is an area that has historically benefited greatly from external ideas. One of the most known examples of this assertion is the application of the Kakutani fixed-point theorem from topology to show the existence of Nash equilibria [96]. Beyond topology, algebraic geometry has also played an important role in advancing game theory, giving rise to the field of algebraic game theory. For instance, one can compute Nash equilibria by studying systems of multilinear equations. This leads to finding upper bounds for the number of totally mixed Nash equilibria of generic games which uses mixed volumes of polytopes and the BKK theorem [92, 119].

From a game theoretic perspective, we work in the setting of normal form games. Such a game consists on the data

$$(n, (d_1, \ldots, d_n), (X^{(1)}, \ldots, X^{(n)})),$$

where n is the number of players and d_i is the number of pure strategies the i–th player can select. Furthermore, for $i \in [n]$, $X^{(i)}$ is a tensor of the format $d_1 \times \cdots \times d_n$ called the payoff table of the i–th player. We denote by

$$V = \mathbb{R}^{d_1 \cdots d_n - 1}$$

the vector space of tensors of such a format. The entry $X^{(i)}_{j_1 \cdots j_n}$ represents the profit the player i receives when player 1 chooses the pure strategy j_1, the player 2 chooses the pure strategy j_2, etc.

A mixed strategy is a tensor p in the closed probability simplex $\overline{\Delta}$ of V. The entry $p_{j_1,\ldots,j_n}$ is the probability that the player 1 chooses the pure strategy j_1, the player 2 chooses the pure strategy j_2, etc. Once the players have chosen a mixed strategy p, the expected payoff of the i–th player is defined as the following scalar product:

$$PX^{(i)} = \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots j_n} p_{j_1 \cdots j_n}.$$

A Nash equilibrium is a rank one mixed strategy where no player can increase their expected payoff by changing their mixed strategy while assuming the other players have fixed mixed strategies. A totally mixed Nash equilibrium is a Nash equilibrium in the open probability simplex Δ of V.

In [3], Aumann introduced the concept of correlated equilibria, which is a generalization of Nash equilibria. Recently, correlated equilibria was studied via the use of oriented matroids and convex geometry [13]. These two classical notions of equilibria model the situation where the players behave independently and they decide individually. In [117], Spohn introduced yet another notion of equilibria, known as dependency equilibria. The dependency equilibria arises from the distinction among decisions made under individual rationality and decisions made collectively. In contrast with the individual decisions of the players treated by the classical Nash and correlated equilibria, the dependency equilibria models the collective behaviour of the players.

This distinction between the individual and collective behavior of the players is illustrated in the classical example of the prisoners' dilemma. The prisoners' dilemma describes the situation where two criminal partners (the two players) are arrested and imprisoned. Each criminal is interrogated and they have two options (two pure strategies): either they testify or they remain silent. If both criminals testify, both are sentenced to two years of jail, whereas if both remain silent, they get one year of jail each. However, if only one testifies, that criminal goes free and the other goes to jail for three years. In this game, the Nash and correlated equilibrium yield to the mutual betrayal of the players, which is not an optimal mixed strategy. In [117], it is shown that the mutual non-betrayal is a dependency equilibrium, leading to a more optimal mixed strategy for the game. This example highlights the significance of the collective behaviour in certain games, showing the importance of dependency equilibria in the field of game theory.

The dependency equilibria and Nash equilibria lie in opposite extremes of the spectrum of dependencies among the players. The collective behaviour of the players is modeled by the dependency equilibrium, whereas the Nash equilibrium corresponds to the case where the players behave independently. In between these two extremes, we may encounter more complex dependencies among the players. For instance, which notion of equilibria models the scenario where three players behave collectively while the rest behave independently? The gap between these extremes is filled by the concept of conditional independence (CI) equilibria. CI equilibria were introduced in [105, Section 6] as the intersection of the dependency equilibria and a statistical model given by a collection of conditional independence (CI) statements. In this chapter we focus on the case of undirected graphical models, where the dependencies among players are model through the edges of a graph whose vertices represent the players.

Graphical models are widely used to build complicated dependency structures among random variables. One of the early developers of the axioms for conditional independence statements was Spohn [116], who, quite coincidentally (or not), introduced dependency equilibria. We model a $(d_1 \times \cdots \times d_n)$-player game X in normal form as an undirected graphical model of a graph $G = ([n], E)$. The vertices of the graph correspond to n discrete random variables $\mathcal{X}_1, \ldots, \mathcal{X}_n$ which represent the players of the game X. Their state spaces $[d_1], \ldots, [d_n]$ represent the set of pure strategies of each player. An edge between two random variables represents the dependency among those players. The transition between edges and dependencies is done through the global Markov properties, which give a collection of CI statements $\mathcal{C} := \mathrm{global}(G)$ to be satisfied by the players. This collection of CI statements imposes quadratic constrains on the probability simplex Δ. The variety defined by these equations is the independence variety $\mathcal{M}_{\mathcal{C}}$ and its intersection with Δ is the independence model. The set of totally mixed CI equilibria is defined as the intersection of the set of totally mixed dependency equilibria and $\mathcal{M}_{\mathcal{C}}$. Using this language, the dependency equilibria and the Nash equilibria fall in the two extreme graphs: the complete graph and the graph with no edge respectively.

By construction, the set of totally mixed CI equilibria is a semialgebraic set. For the graph with no edges, i.e. the Nash case, this semialgebraic set generically consists on a finite number of points. The algebro-geometric study of the set of Nash equilibria can be found in [119, Chapter 6]. For more complicated graphs, the geometry of this semialgebraic set is no longer so simple. For instance, its expected dimension is positive for graphs with at least one edge (see [105, Section 6]). One of the main tools for studying sophisticated semialgebraic sets is through the algebro-geometric properties of its algebraic closure. Such techniques have been proven to be extremely beneficial in fields such as optimization, convex geometry, and algebraic statistics (see e.g. [8, 14]).

In the case of the complete graph, i.e. the dependency equilibrium case, the algebro-geometric examination of the set of dependency equilibria was carried out in [105] through the analysis of the Spohn variety which, broadly speaking, is the algebraic closure of this semialgebraic set. In other words, the algebraic version of the dependency equilibria is the Spohn variety. For the CI equilibria, its corresponding associated algebraic variety is the Spohn CI variety. In this chapter we study the geometric features of these semialgebraic sets i.e. CI equilibria, through the algebro-geometric analysis the Spohn CI varieties. We focus on binary games, i.e. $d_1 = \cdots = d_n = 2$ and on Spohn CI varieties arising from the global Markov property of undirected graph.

This chapter is structured as follows. In Section 3.1, we give a brief introduction to graphical models and algebraic game theory through two subsections. The background on graphical models required in this chapter is introduced in Section 3.1.1. In Section 3.1.2 we define the Nash and dependency equilibria, we highlight their relation with algebraic geometry and we present the main results concerning Spohn varieties. Finally, Spohn CI varieties and CI equilibria are also introduced in Section 3.1.2.

In the spectrum of dependencies among the players, the next case to Nash equilibria is the CI equilibria of one edge graphs. Section 3.2 is devoted to Spohn CI varieties of one edge graphical models. The section is structured in three subsections. In Section 3.2.1 we show that for generic games these varieties are curves called Nash CI curves. Section 3.2.2 investigates the genus and degree of these curves, and in Section 3.2.3 their smoothness is analysed.

The dimension of Spohn CI varieties is studied in Section 3.3. In Section 3.4 we analyse the Spohn CI varieties of graphs whose connected components are complete. We call these varieties Nash conditional independence (CI) varieties. The properties of these varieties are studied in Section 3.4.1. In Section 3.5 we analyse the affine universality of certain families of Nash CI varieties.

Finally, a list of open problems and new research lines is discussed in Section 3.6.

The main contributions of this chapter are:

- We show that generic Spohn CI varieties of one edge undirected graphical models are one dimensional. We compute the degree and the genus of Nash CI curves. We prove that generic Nash CI curves are smooth, irreducible and connected.
- We solve [105, Conjecture 24] by computing the dimension of Spohn CI varieties of undirected graphical models.
- We compute the equations of Nash CI varieties and their degree. We show that Nash CI varieties are connected, and that smooth Nash CI surface are of general type.
- We analyse the possible singularities of Nash CI varieties. We prove that the set of totally mixed CI equilibria of a generic Nash CI variety is a real smooth manifold, which is either empty or it has the same dimension as the Nash CI variety.
- We analyse the affine universality, in the sense of [34], for Nash CI varieties where connected components of the associated graph have at most one edge. We show that the families of these varieties satisfy the Murphy's law.

3.1 Preliminaries on algebraic game theory

The field of algebraic statistics explores the synergies between statistics and algebraic geometry, providing algebraic methods for addressing statistical problems. In Section 3.1.1 we present the necessary background on algebraic statistics and graphical models and we refer to [42, 120] for further details. Analogously, the goal of algebraic game theory is to approach game theoretic problems from an algebro-geometric perspective. In Section 3.1.2 we give a brief introduction to game theory and its relation with algebraic geometry. In this section, we define the main object of study of this chapter: Spohn conditional independence varieties. For further details in algebraic game theory we refer to [105, 119].

3.1.1 Algebraic statistics and graphical models

A brief introduction to algebraic statistics and graphical models is presented in this section. One of the main synergies between algebraic geometry and statistics arises from the concept of independence variety. These varieties allow us to approach the geometry of independence statistical models from an algebraic perspective. We will mainly focus on independence models arising from graphs, which are called graphical models. For further details on these topics we refer to [42, 120].

Let $\mathcal{X} = (\mathcal{X}_i \mid i \in [n])$ be an n–dimensional discrete random vector and let $[d_i] := \{1, \ldots, d_i\}$ be the set of values taken by the discrete random variable $\mathcal{X}_i$. Then, $\mathcal{X}$ takes values in

$$\mathcal{R} := \prod_{i\in[n]} [d_i].$$

For $A \subset [n]$, we consider the subvector $\mathcal{X}_A = (\mathcal{X}_i)_{i\in A}$, which takes values in

$$\mathcal{R}_{\mathcal{A}} := \prod_{i\in A} [d_i].$$

We say that the random vector $\mathcal{X}$ satisfies the conditional independence (CI) statement $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C$ for $A, B, C \subset [n]$ disjoint subset if $\mathcal{X}_A$ is conditionally independent to $\mathcal{X}_B$ given $\mathcal{X}_C$. If C is the empty set, $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_\emptyset$ is denoted by $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B$. We refer to [120, Section 4.1] for the definition of conditional independence. We say that a CI statement $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C$ is saturated if $A \cup B \cup C = [n]$. For $j_1 \in [d_1], \ldots, j_n \in [d_n]$, we denote the probability $P(\mathcal{X}_i = j_i : \forall\, i \in [n])$ by $p_{j_1\cdots j_n}$. Similarly, for $A, B, C \subset [n]$ disjoint and for $j_A \in \mathcal{R}_A$, $j_B \in \mathcal{R}_B$, $j_C \in \mathcal{R}_C$, we denote the probability $P(\mathcal{X}_A = j_A, \mathcal{X}_B = j_B, \mathcal{X}_C = j_C)$ by $p_{j_A j_B j_C +}$. In other words, we can write $p_{j_A j_B j_C +}$ as

$$p_{j_A j_B j_C +} = \sum_{j_D \in \mathcal{R}_D} p_{j_A j_B j_C j_D}$$

where $D = [n] \setminus (A \cup B \cup C)$. The probability $p_{j_1\cdots j_n}$ can be seen as a tensor of the format $d_1 \times \cdots \times d_n$. Let $V = \mathbb{R}^{d_1\cdots d_n}$ be the space of all tensors of this format. By abuse of notation, we also denote the coordinates of V by $p_{j_1\cdots j_n}$ for $j_i \in [d_i]$. We denote the closed probability simplex of V by $\overline{\Delta}$. In other words,

$$\overline{\Delta} := \left\{ p \in V : \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} p_{j_1\cdots j_n} = 1 \text{ and } p_{j_1\cdots j_n} \geq 0\ \forall j_1 \in [d_1], \ldots, j_n \in [d_n] \right\}.$$

In this chapter we are interested in positive probabilities. That is, we focus on the open probability simplex

$$\Delta := \Delta^\circ_{d_1\ldots d_n - 1} = \{p \in \overline{\Delta} : p_{j_1,\ldots,j_n} > 0\ \forall j_1 \in [d_1], \ldots, j_n \in [d_n]\}.$$

The following result is one of the central foundations of algebraic statistics since it allows to translate CI statements in the discrete setting into quadratic constrains on the variables $p_{j_1\cdots j_n}$.

Proposition 3.1.1. *[120, Proposition 4.1.6] The conditional independence statement $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C$ holds if and only if*

$$p_{i_A i_B i_C +} p_{j_A j_B i_C +} - p_{j_A i_B i_C +} p_{i_A j_B i_C +} \tag{3.1}$$

for all $i_A, j_A \in \mathcal{R}_A$, $i_B, j_B \in \mathcal{R}_B$, and $i_C \in \mathcal{R}_C$.

The quadratic polynomials in (3.1) lie in the coordinate ring of V. In other words, they are polynomials in the ring $\mathbb{R}[p_{j_1\cdots j_n} : j_1 \in [d_1], \ldots, j_n \in [d_n]]$.

Definition 3.1.2. *Given a CI statement $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B | \mathcal{X}_C$, the conditional independence (CI) ideal, denoted by*

$$I_{\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C},$$

is the ideal generated by all the quadratic polynomials as in equation (3.1). Given a collection $\mathcal{C}$ of CI statements, we define the CI ideal $I_{\mathcal{C}}$ of $\mathcal{C}$ as the ideal

$$I_{\mathcal{C}} := \sum_{(\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C) \in \mathcal{C}} I_{\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C}.$$

Example 3.1.3. *Fix $n = 2$ and consider the CI statement $\mathcal{X}_1 \perp\!\!\!\perp \mathcal{X}_2$. Here C is empty. Then, the independence ideal of $\mathcal{X}_1 \perp\!\!\!\perp \mathcal{X}_2$ is*

$$I_{\mathcal{X}_1 \perp\!\!\!\perp \mathcal{X}_2} = \langle p_{i_1 i_2} p_{j_1 j_2} - p_{i_1 j_2} p_{j_1 i_2} : i_1, j_1 \in [d_1] \text{ and } i_2, j_2 \in [d_2] \rangle .$$

Due to our algebro-geometric approach, we work in the setting of projective geometry. Let $\mathbb{P}(V)$ be the projectivization of the vector space V with coordinates $p_{j_1\cdots j_n}$. In particular, there is the quotient map

$$\pi : V \setminus \{0\} \to \mathbb{P}(V).$$

The image of the probability simplex is the set of points $p \in \mathbb{P}(V)$ whose entries all have the same sign. Moreover, the restriction of π to $\overline{\Delta}$ is an isomorphism (of semialgebraic sets) onto its image. The inverse map sends $p \in \pi(\overline{\Delta})$ to

$$(\cdots : \frac{p_{j_1\cdots j_n}}{p_+} : \cdots), \text{ where } p_+ := \sum_{j_1,\ldots,j_n} p_{j_1\cdots j_n}.$$

This isomorphism allows us to work in the projective setting. Similarly, $\pi(\Delta)$, which is isomorphic to Δ, consists of all the points $p \in \mathbb{P}(V)$ whose entries are nonzero and they all have the same sign. By abuse of notation, we denote $\pi(\overline{\Delta})$ and $\pi(\Delta)$ also by $\overline{\Delta}$ and Δ respectively. Using this notation, the CI ideal $I_{\mathcal{C}}$ is an homogeneous ideal in the graded coordinate ring of the projective space $\mathbb{P}(V)$.

Definition 3.1.4. *The* independence variety $\mathcal{M}_{\mathcal{C}}$ *of a collection of CI statements* $\mathcal{C}$ *is the subvariety of* $\mathbb{P}(V)$ *obtained by removing all the irreducible components of* $\mathbb{V}(I_{\mathcal{C}})$ *contained in the hyperplanes of the form* $\{p_{j_1\cdots j_n} = 0\}$*. Analogously, we define the* independence model $\mathcal{M}_{\Delta,\mathcal{C}}$ *as the intersection of* $\mathcal{M}_{\mathcal{C}}$ *with the open probability simplex of* $\mathbb{P}(V)$*.*

From Proposition 3.1.1, the independence model $\mathcal{M}_{\Delta,\mathcal{C}}$ is the statistical model of all positive probabilities distributions satisfying all the CI statements of $\mathcal{C}$. From an algebraic perspective, the ideal defining the independence variety $\mathcal{M}_{\mathcal{C}}$ is obtained by saturating the ideal $I_{\mathcal{C}}$ by the ideals of the hyperplanes $\{p_{j_1\cdots j_n} = 0\}$.

Remark 3.1.5. *In [120, Chapter 4] the independence variety* $\mathcal{M}_{\mathcal{C}}$ *is defined as the variety of* $\mathbb{P}(V)$ *defined by the ideal* $I_{\mathcal{C}}$*. Note that this definition differs from Definition 3.1.4. The motivation behind this distinction comes from the game theoretic setting, where we will mainly consider positive probabilities.*

Example 3.1.6. *The independence variety given by the ideal computed in Example 3.1.3 is the Segre variety* $\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1}$ *in* $\mathbb{P}(V) = \mathbb{P}^{d_1 d_2 - 1}$*. More general, for* $n \in \mathbb{N}$ *let* $\mathcal{C}$ *the set of all CI statements of* $\mathcal{X}$*. One can check that the independence variety* $\mathcal{M}_{\mathcal{C}} \subset \mathbb{P}(V)$ *equals the Segre variety* (3.6)*. In Example 3.1.11 we will provide a geometric argument for this equality.*

In the following we focus on a special class of independence models whose collection of CI statements $\mathcal{C}$ arises from undirected graphs. These models are called undirected graphical models. Let $G = ([n], E)$ be an undirected graph. The vertex $i \in [n]$ represents the random variable $\mathcal{X}_i$ and each edge $(i, j) \in E$ denotes the dependence between the random variables $\mathcal{X}_i$ and $\mathcal{X}_j$. We consider Markov properties associated to the graph G, that is certain conditional independence statements that must be satisfied by all random vectors $\mathcal{X}$ consistent with the graph G.

Definition 3.1.7. *A pair of vertices* $(a, b) \in [n]$ *is said to be* separated *by a subset of vertices* $C \subset [n]\backslash\{a, b\}$*, if every path from* a *to* b *contains a vertex* $c \in C$*. Let* $A, B, C \subseteq [n]$ *be disjoint subsets of* $[n]$*. We say that* C *separates* A *and* B *if* a *and* b *are separated by* C *for all* $a \in A$ *and* $b \in B$*. The* global Markov property $\mathrm{global}(G)$ *associated to* G *consists of all conditional independence statements* $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C$ *for all disjoint sets* A*,* B*, and* C *such that* C *separates* A *and* B *in* G*.*

Remark 3.1.8. *There are also pairwise and local Markov properties where*

$$\mathrm{pairwise}(G) \subseteq \mathrm{local}(G) \subseteq \mathrm{global}(G).$$

We refer to [120, Chapter 13] for the definition of these other Markov properties. The definition of independence variety in [120, Chapter 5] gives rise to three possibly distinct independence varieties associated of an undirected graph:

$$\mathcal{M}_{\mathrm{pairwise}(G)}, \mathcal{M}_{\mathrm{local}(G)}, \text{ and } \mathcal{M}_{\mathrm{global}(G)}.$$

Nevertheless, for positive probability distributions, the pairwise, local and global Markov property are all equivalent by Pearl and Paz in [102]. In particular, using our definition of independence variety (see Definition 3.1.4) we get that $\mathcal{M}_{\mathrm{pairwise}(G)}$, $\mathcal{M}_{\mathrm{local}(G)}$, and $\mathcal{M}_{\mathrm{global}(G)}$ are all equal.

In this thesis we focus on independence varieties $\mathcal{M}_{\mathcal{C}}$ where $\mathcal{C}$ is the global Markov property of an undirected graph with n vertex. In general, graphical models refer to independence model arising from a Markov property of a directed or undirected graphs. During this chapter we use the notion of graphical models to refer to independence models of undirected global Markov properties. By Remark 3.1.8, the distinction among pairwise, local and global Markov properties is not required. In the rest of this chapter, unless it is specified, the collection of CI statements $\mathcal{C}$ denotes the global Markov property of an undirected graph G with n vertices.

In [85], it is shown that all CI statements of global Markov properties can be derived from the saturated global Markov property. Note that for saturated CI statements, the quadrics in (3.1) are binomials and, hence, the independence ideal is a binomial ideal. However, these independence ideals might not be toric in general. Recall that an ideal is toric if it is binomial and prime. We refer to [33] for further details on toric ideals and toric varieties.

We say that a graph is chordal or decomposable if there are no induced cycles of length greater or equal to four. In [54, Theorem 4.4] it is shown that the independence ideal of a graph G with respect to the global Markov property is toric if and only if G is chordal (see also [120, Theorem 13.3.1]). Therefore, for decomposable graphs $\mathcal{M}_{\mathrm{global}(G)}$ equals $\mathbb{V}(I_{\mathrm{global}(G)})$ and the independence variety is toric. During this chapter we exploit the monomial map of these toric varieties in our game theoretic setting. However, we would like to use this toric structure for any graph, not only for decomposable graphs. The saturation used in Definition 3.1.4 allows us to endow the independence variety of a nondecomposable graph with the structure of a toric variety. A clique of an undirected graph G is a complete subgraph of G. Let $\mathcal{D}$ be the set of all maximal cliques of G. For a clique $C \in \mathcal{D}$, we consider the torus

$$\mathbb{T}_C := (\mathbb{C}^*)^{2^{|C|}} \text{ with coordinates } \sigma^{(C)}_{j_C} \text{ for } j_C = (j_i)_{i\in[C]} \in [2]^{|C|}, \tag{3.2}$$

where $[C]$ denotes the set of vertices of C and $|C|$ denotes the number of vertices.

Proposition 3.1.9. *[120, Proposition 13.2.5] The independence variety $\mathcal{M}_{\mathrm{global}(G)}$ of an undirected graph G is the toric variety in $\mathbb{P}(V)$ defined by the monomial map*

$$\phi: \quad \mathbb{T} := \prod_{C\in\mathcal{D}} \mathbb{T}_C \quad \longrightarrow \quad \mathbb{P}(V), \tag{3.3}$$

given by

$$p_{j_1\cdots j_n} = \prod_{C\in\mathcal{D}} \sigma^{(C)}_{j_C}. \tag{3.4}$$

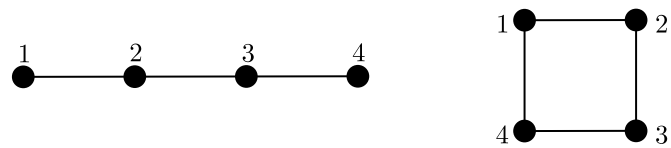


Figure 3.1: Line graph and cycle on four vertices

Remark 3.1.10. *Note that Proposition 3.1.9 does not hold for the definition of independence variety used in [120, Chapter 4]. Using this definition, the toric variety defined by (3.3) is contained in the independence variety. By Hammersley–Clifford Theorem ([120, Theorem 13.2.3]) and [120, Proposition 13.2.5], this toric variety is the unique irreducible component of the independence variety not contained in the hyperplanes $\{p_{j_1\cdots j_n} = 0\}$. In our notation, this irreducible component is what we defined as independence variety in Definition 3.1.4.*

Example 3.1.11. *Consider the graph G with n vertices and no edges. The set of maximal cliques $\mathcal{D}(G)$ coincides with the set of vertices of G. Then, the parametrization (3.4) can be seen as the monomial map*

$$\begin{array}{ccc} \mathbb{P}^{d_1-1}\times\cdots\times\mathbb{P}^{d_n-1} & \longrightarrow & \mathcal{M}_{\mathcal{C}} \\ ([\sigma_1^{(1)},\ldots,\sigma_{d_1}^{(1)}],\cdots,[\sigma_1^{(n)},\ldots,\sigma_{d_n}^{(n)}]) & \longmapsto & [\cdots,\underbrace{\sigma_{j_1}^{(1)}\cdots\sigma_{j_n}^{(n)}}_{(p_{j_1\cdots j_n})},\cdots]\end{array} .$$

This map coincides with the Segre embedding and we conclude that $\mathcal{M}_{\mathcal{C}}$ is the Segre variety (3.6).

The simplicial complex of cliques or clique complex of the undirected graph G is the simplicial complex whose d–dimensional faces are the cliques of G with d vertices. A formula for the dimension of $\mathcal{M}_{\mathrm{global}(G)}$ is derived in [69, Corollary 2.7]. The following result specifies this dimension formula for the case $d_1 = \cdots = d_n = 2$.

Proposition 3.1.12. *Let $G = ([n], E)$ be an undirected graph and assume that $d_1 = \cdots = d_n = 2$. Then, $\dim \mathcal{M}_{\mathrm{global}(G)}$ is the number of non-empty faces of the associated simplicial complex of cliques. In other words, the dimension of $\mathcal{M}_{\mathrm{global}(G)}$ equals the number of cliques of G.*

Example 3.1.13. *Consider a 4-player game modeled with two different graphical models as in Figure 3.1. The parametrization of the independence variety $\mathcal{M}_{\mathcal{C}}$ for the line graph and the cycle are*

$$p_{j_1j_2j_3j_4} = \sigma^{(12)}_{j_1j_2}\sigma^{(23)}_{j_2j_3}\sigma^{(34)}_{j_3j_4} \quad \textit{and} \quad p_{j_1j_2j_3j_4} = \sigma^{(12)}_{j_1j_2}\sigma^{(23)}_{j_2j_3}\sigma^{(34)}_{j_3j_4}\sigma^{(14)}_{j_1j_4} \tag{3.5}$$

respectively. For the line graph, the associated simplicial complex of cliques consists of 4 cliques of size 1 and 3 cliques of size 2. Thus, the independence variety $\mathcal{M}_{\mathcal{C}}$ of the line graph is 7-dimensional. For the cycle graph, the associated simplicial complex of cliques consists of 4 cliques of size 1 and 4 cliques of size 2. Therefore, the independence variety $\mathcal{M}_{\mathcal{C}}$ is 8-dimensional.

Example 3.1.14. *Consider a 7-player game with binary choices modeled by the decomposable graph G in Figure 3.2. The monomial map of the independence variety $\mathcal{M}_{\mathcal{C}}$ of G is*

$$p_{j_1\cdots j_7} = \sigma^{(123)}_{j_1j_2j_3}\sigma^{(2345)}_{j_2j_3j_4j_5}\sigma^{(2356)}_{j_2j_3j_5j_6}\sigma^{(567)}_{j_5j_6j_7}.$$

The dimension of $\mathcal{M}_{\mathcal{C}}$ is $7+13+9+2=31$ which is the number of non-empty faces of the associated simplicial complex of cliques.

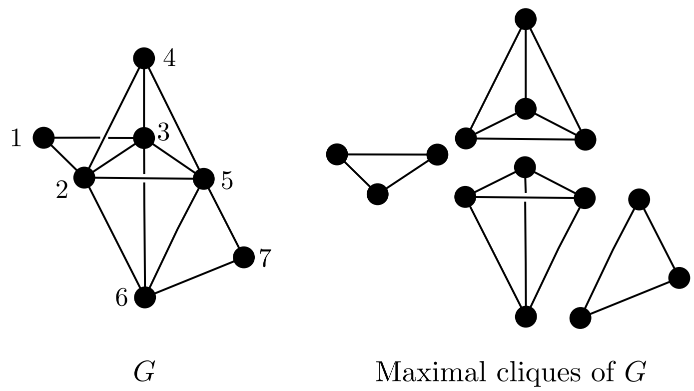


Figure 3.2: The chordal decomposable graph G has 4 maximal cliques. Two of them have 3 vertices and the other two have 4 vertices.

3.1.2 Game theory meets algebraic geometry

We work in the setting of normal form games X with n players, labelled by $1,\ldots,n$. For $i\in[n]$, the i–th player can select among d_i pure strategies. The set of pure strategies of the i–th player is denoted by $[d_i]$. The game is defined by fixing n payoff tables $X^{(1)},\ldots,X^{(n)}$. Each payoff table $X^{(i)}$ is a tensor of format $d_1\times\cdots\times d_n$ with real entries. For $j_1\in[d_1],\ldots,j_n\in[d_n]$, the entry $X^{(i)}_{j_1\ldots j_n}$ represents the payoff that player i receives when player 1 chooses pure strategy j_1, player 2 chooses pure strategy j_2, etc. As in Section 3.1.1, $V=\mathbb{R}^{d_1\cdots d_n}$ denotes the real vector space of all tensors of format $d_1\times\cdots\times d_n$ and let $p_{j_1\ldots j_n}$ be the coordinates of V. The entry $p_{j_1\cdots j_n}$ represents the probability that the first player chooses the pure strategy j_1, the second player chooses the pure strategy j_2, etc. We denote the closed and open probability simplex of V by $\overline{\Delta}$ and Δ respectively. A point $p\in\overline{\Delta}$ is called a mixed strategy. The expected payoff of the i–th player is defined as the following scalar product:

$$PX^{(i)} = \sum_{j_1=1}^{d_1}\cdots\sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots j_n}p_{j_1\cdots j_n}.$$

Example 3.1.15. *[105, Example 1] Two friends want to go to a concert and they have to decide between two artists: Kaos Urbano or Gata Cattana. In the language of* *normal form games, our game X has two players, i.e. $n = 2$. Each player can decide between two pure strategies: Kaos Urbano $= 1$ and Gata Cattana $= 2$, and $d_1 = d_2 = 2$. The payoff tables of this game are the matrices*

$$X^{(1)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad X^{(2)} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} .$$

The highest entry of $X^{(1)}$ is $X^{(1)}_{11} = 3$, which means that player 1 is very happy if both players go to the first concert. On the contrary, she will be unhappy if both friends attend to distinct concerts. This is reflected by the entries $X^{(1)}_{12} = X^{(1)}_{21} = 0$. The entry p_{11} of a mixed strategy represents the probability of the both friends choosing to go to the concert of Kaos Urbano. The two expected payoffs are given by

$$PX^{(1)} = 3p_{11} + 2p_{22} \textit{ and } PX^{(2)} = 2p_{11} + 3p_{22}.$$

The reason why we consider the positive real points is primarily rooted in the technical aspects inherent to the definition of dependency equilibrium. In particular, as we consider the conditional expected payoffs, it is essential to avoid the cases where the denominator in the conditional probabilities becomes zero. An additional reason is our investigation of universality for Spohn CI varieties in Section 3.5, similar to Datta's universality result for totally mixed Nash equilibria [34]. The extension of the definition of dependency equilibrium to include the boundary of the probability simplex is explored in the ongoing work [107].

The classical frame where game theory is developed is that of vector spaces and affine geometry. As in Section 3.1.1, and motivated by our algebro-geometric perspective, we work in the setting of projective geometry. Let $\mathbb{P}(V)$ be the projectivization of the vector space V with coordinates $p_{j_1 \cdots j_n}$. Recall that the open simplex of $\mathbb{P}(V)$ is isomorphic to Δ via the projection $V \setminus \{0\} \to \mathbb{P}(V)$. By abuse of notation, we denote the open simplex of $\mathbb{P}(V)$ also by Δ.

In the projective space $\mathbb{P}(V)$ we consider the Segre variety

$$\mathbb{P}^{d_1 - 1} \times \cdots \times \mathbb{P}^{d_n - 1}, \tag{3.6}$$

which coincides with the space of rank one tensors. This Segre variety is embedded in $\mathbb{P}(V)$ via the Segre parametrization

$$p_{j_1 \cdots j_n} = \sigma^{(1)}_{j_1} \cdots \sigma^{(n)}_{j_n}, \tag{3.7}$$

where $\sigma^{(i)}_j$ for $j \in [d_i]$ are the coordinates of $\mathbb{P}^{d_i - 1}$.

Definition 3.1.16. *A* Nash equilibrium *is a rank one mixed strategy where no player can increase their expected payoff by changing their mixed strategy while assuming the other players have fixed mixed strategies. A* totally mixed Nash equilibrium *is a Nash equilibrium in the open simplex Δ.*

In particular, a (totally mixed) Nash equilibrium lies in the intersection of the Segre variety of rank one tensor and $\overline{\Delta}$ or Δ respectively. This intersection is the image through the Segre parametrization of the product $\Delta_1 \times \cdots \times \Delta_n$, where Δ_i is the open simplex of $\mathbb{P}^{d_i-1}$.

Remark 3.1.17. *The classical theory of Nash equilibria analyses the situation where the players do not communicate between each other and they behave independently. Geometrically, one can exhibit this fact via the Segre parametrization (3.7). Note that the coordinates $p_{j_1\cdots j_n}$ of a mixed strategy are indexed by $j_1 \in [d_1], \ldots, j_n \in [d_n]$. Therefore, all players are jointly needed for fixing a mixed strategy. However, in the Nash setting, mixed strategies are rank one tensors. In particular, the Segre parametrization (3.7) splits the coordinate $p_{j_1\cdots j_n}$ in the product of the coordinates corresponding to each factor of the Segre variety. Each of these coordinates represents the probability of an isolated player choosing a pure strategy. Thus, the process of choosing a mixed strategy is split into each player choosing independently their mixed strategies.*

The relation between the Nash equilibria and algebraic geometry is explained in [119, Chapter 6], where the set of totally mixed equilibria is written as the intersection of the open simplex $\Delta_1 \times \cdots \times \Delta_n$ with an algebraic subvariety of the Segre variety (3.6). To define this variety, we work in the affine setting. From Definition 3.1.16, we get that a mixed strategy

$$p = (p^{(1)}, \ldots, p^{(n)}) \in \overline{\Delta}_1 \times \cdots \times \overline{\Delta}_n$$

is a Nash equilibrium if for every player i, we have that

$$PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} q^{(i)}_{j_i} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \geq 0 \tag{3.8}$$

for any $q^{(i)} \in \overline{\Delta}_i$. Choosing $q^{(i)}$ to be the vertices of $\overline{\Delta}_i$ we obtain that p is a Nash equilibrium if and only if

$$PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k\cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \geq 0 \tag{3.9}$$

for every $i \in [n]$ and every $k \in [d_i]$. Now, using Equation (3.9) and that $p^{(i)}_1 + \cdots + p^{(i)}_{d_i} =$

1, we get that for $i \in [n]$ and $k \in [d_i]$ the following holds

$$PX^{(i)} = \sum_{m=1}^{d_i} p_m^{(i)} \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_m=1}^{d_m}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots m \cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \leq$$

$$\sum_{m \neq k}^{d_i} p_m^{(i)} PX^{(i)} + p_k^{(i)} \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} =$$

$$PX^{(i)} - p_k^{(i)} \left(PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \right).$$

This implies that

$$p_k^{(i)} \left(PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \right) \leq 0.$$

Since $p_k^{(i)} \geq 0$, by Equation (3.9) we deduce that

$$p_k^{(i)} \left(PX^{(i)} - \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1 \cdots k \cdots j_n} p^{(1)}_{j_1} \cdots p^{(i-1)}_{j_{i-1}} p^{(i+1)}_{j_{i+1}} \cdots p^{(n)}_{j_n} \right) = 0 \tag{3.10}$$

for every k. For each player i and any two pure strategies $k, l \in [d_i]$, we consider the multihomogeneous polynomial

$$\sum_{j_1 \in [d_1]} \cdots \widehat{\sum_{j_i \in [d_i]}} \cdots \sum_{j_n \in [d_n]} \left(X^{(i)}_{j_1 \cdots k \cdots j_n} - X^{(i)}_{j_1 \cdots l \cdots j_n} \right) \sigma^{(1)}_{j_1} \cdots \widehat{\sigma^{(i)}_{j_i}} \cdots \sigma^{(n)}_{j_n} \tag{3.11}$$

From Equation (3.10), we get that a totally mixed Nash equilibrium lies in the variety defined by all the polynomials in (3.11). The following result shows that the converse also holds.

Theorem 3.1.18. *[119, Theorem 6.4] The set of totally mixed Nash equilibria of a game X consists of all rank one tensors p in the intersection of $\Delta_1 \times \cdots \times \Delta_n$ and the variety defined by all the polynomials in (3.11).*

Example 3.1.19. *[105, Example 5] We compute the Nash equilibria of the game in Example 3.1.15. In this case, the Segre variety (3.6) is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, and the polynomials in (3.11) are*

$$2\sigma_2^{(2)} - 3\sigma_1^{(2)} \text{ and } 3\sigma_2^{(1)} - 2\sigma_1^{(1)}.$$

By Theorem 3.1.18, the only totally mixed Nash equilibrium is the point

$$\left(\frac{6}{25}, \frac{9}{25}, \frac{4}{25}, \frac{6}{25} \right).$$

One can check that the only pure Nash equilibria are $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$.

The notion of dependency equilibria arises when the Segre variety (3.6) is not considered, and it describes the situation where the players behave collectively. The concept of dependency equilibrium was introduced by the philosopher Wolfgang Spohn in [117, 118]. To incorporate causal dependencies among the players, one considers conditional probabilities.

Definition 3.1.20. *The conditional expected payoff of the i–th player conditionated on her having fixed pure strategy $k \in [d_i]$ is defined as the expected payoff conditioned on player i having fixed pure strategy $k \in [d_i]$, i.e. is given by the expression*

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k\cdots j_n} \frac{p_{j_1\cdots k\cdots j_n}}{p_{+\cdots+k+\cdots+}},$$

where

$$p_{+\cdots+k+\cdots+} = \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} p_{j_1\cdots k\cdots j_n}.$$

Note that the fraction

$$\frac{p_{j_1\cdots k\cdots j_n}}{p_{+\cdots+k+\cdots+}}$$

is the conditional probability of players choosing the pure strategies $j_1, \ldots, k, \ldots, j_n$ given that the i–th player chooses the pure strategy $k \in [d_i]$.

Definition 3.1.21. *We say that a tensor p in $\overline{\Delta}$, or in Δ respectively, is a (totally mixed) dependency equilibrium if the conditional expected payoffs of each player do not depend on the pure strategy. In other words, p is a dependency equilibrium if for every $i \in [n]$ and for every $k, k' \in [d_i]$ the following equality holds:*

$$\sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k\cdots j_n} \frac{p_{j_1\cdots k\cdots j_n}}{p_{+\cdots+k+\cdots+}} = \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k'\cdots j_n} \frac{p_{j_1\cdots k'\cdots j_n}}{p_{+\cdots+k'+\cdots+}}. \tag{3.12}$$

In [105], the authors established the algebro-geometric foundations for the theory of dependency equilibrium by studying an algebraic variety in complex projective space obtained by relaxing the reality and positivity constraints. The Spohn CI variety arises from this relaxation of the game theoretic constraints.

Definition 3.1.22. *The Spohn variety $\mathcal{V}_X$ of the game X is the variety defined by the 2×2 minors of the following $d_i \times 2$ matrices of linear forms $M_1, \ldots, M_n$:*

$$M_i = M_i(P) := \begin{bmatrix} \vdots & \vdots \\ p_{+\cdots+k+\cdots+} & \sum_{j_1=1}^{d_1} \cdots \widehat{\sum_{j_i=1}^{d_i}} \cdots \sum_{j_n=1}^{d_n} X^{(i)}_{j_1\cdots k\cdots j_n} p_{j_1\cdots k\cdots j_n} \\ \vdots & \vdots \end{bmatrix}. \tag{3.13}$$

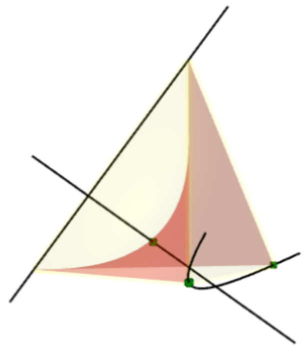


Figure 3.3: [105, Figure 1] We illustrate the Spohn variety of Example 3.1.23. The three irreducible components of $\mathcal{V}_X$ are illustrated in black. The open simplex is illustrated in light yellow. The red surface corresponds to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1$. The three green points represent the three Nash equilibria computed in Example 3.1.19.

Note that the 2×2 minors of the matrices $M_1, \dots, M_n$ provide the same expressions as those in (3.12), after getting rid of the denominators. Thus, the set of totally mixed dependency equilibria of the game X is the intersection $\mathcal{V}_X \cap \Delta$.

Example 3.1.23. *[105, Example 5] We continue with Examples 3.1.15 and 3.1.19. We now compute the Spohn variety and the dependency equilibria. For the game in Example 3.1.15, the matrices M_1 and M_2 are the following 2×2 matrices:*

$$M_1 = \begin{pmatrix} p_{11} + p_{12} & 3p_{11} \\ p_{21} + p_{22} & 2p_{22} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} p_{11} + p_{21} & 2p_{11} \\ p_{12} + p_{22} & 3p_{22} \end{pmatrix}.$$

The Spohn variety $\mathcal{V}_X$ is the subvariety of $\mathbb{P}^3$ defined by the ideal generated by the determinant of M_1 and M_2. A computation in `Macaulay2` *shows that the ideal of $\mathcal{V}_X$ is the intersection*

$$\langle p_{11}, p_{22} \rangle \cap \langle 2p_{12} - 3p_{21} - p_{22}, p_{11} - p_{22} \rangle \cap \langle 2p_{12} + 3p_{21}, 3p_{11}p_{21} + p_{11}p_{22} + 3p_{21}p_{22} \rangle.$$

We observe that the Spohn variety has three irreducible components: two of them are lines and the third one is a conic. The only component of $\mathcal{V}_X$ that intersects the open simplex Δ is the line defined by

$$2p_{12} - 3p_{21} - p_{22} = p_{11} - p_{22} = 0.$$

The intersection of this line with Δ and $\mathbb{P}^1 \times \mathbb{P}^1$ is the unique totally mixed Nash equilibria computed in Example 3.1.19. In Figure 3.3 we illustrate the three components of $\mathcal{V}_X$ and its intersection with Δ and $\mathbb{P}^1 \times \mathbb{P}^1$.

The dimension and degree of the Spohn variety is computed in the following theorem.

Theorem 3.1.24. *[105, Theorem 6] For generic payoff tables $X^{(1)}, \dots, X^{(n)}$, the Spohn variety is irreducible of codimension $d_1 + \cdots + d_n - n$ and degree $d_1 \cdots d_n$. Moreover, the set of totally mixed Nash equilibria is the intersection*

$$\mathcal{V}_X \cap \left(\mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_n-1}\right) \cap \Delta. \tag{3.14}$$

Theorem 3.1.24 shows the relation between the dependency equilibria and the totally mixed Nash equilibria. The intersection of the set of totally mixed dependency equilibria with the Segre variety (3.6) coincides with the set of totally mixed Nash equilibria. This was shown in Example 3.1.23.

Recall that the dependency equilibria model the situation where the n players of the game behaves collectively. On the contrary, the Nash equilibrium deals with the case where all the players behave independently. From a game theoretic perspective there is a big gap between both models. For instance, what happens when the players are divided in two independent groups and each group acts collectively? The concepts of Spohn conditional independence variety and conditional independence equilibria were introduced in [105, Section 6] with the aim of filling this gap between the Nash equilibria and the dependency equilibria.

From a geometric point of view, this gap can also be perceived from the dimension of the Spohn variety. By Theorem 3.1.24, the Spohn variety is high dimensional and it contains the set of totally mixed equilibria. On the other hand, the variety defined by the polynomials (3.11) is generically zero dimensional (see [119, Section 6.4]). The Spohn conditional independence variety fills this dimensional gap between both varieties.

We think of the n players of the game as n random variables whose state space is the set of pure strategies of the players. Using this probabilistic perspective, we may see the dependencies among the players as conditional independent statements among the random variables. This shows the importance of the independence varieties in our construction. Let $\mathcal{C}$ be a collection of CI statements and let $\mathcal{M}_{\mathcal{C}}$ be the corresponding independence variety. Recall from Definition 3.1.4 that $\mathcal{M}_{\mathcal{C}}$ has no irreducible component contained in the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$.

Definition 3.1.25. *Let $\mathcal{C}$ be a collection of CI statements. The* Spohn conditional independence (CI) variety *$\mathcal{V}_{X,\mathcal{C}}$ of a game X is variety obtained after removing the irreducible components of the intersection $\mathcal{V}_C \cap \mathcal{M}_{\mathcal{C}}$ lying in the hyperplanes*

$$\{p_{j_1 j_2 \cdots j_n} = 0\} \text{ and } \{p_{++\cdots+} = 0\},$$

where $\{p_{++\cdots+} = 0\}$ denotes the hyperplanes defined by the linear forms in the first column of the matrices $M_1, \dots, M_n$ in (3.13).

Geometrically, the Spohn CI variety is the Zariski closure of the points in $\mathcal{V}_X$ satisfying the CI statements in $\mathcal{C}$ and not lying in the above hyperplanes. Algebraically, the ideal of $\mathcal{V}_{X,\mathcal{C}}$ is obtained by saturating the ideal of $\mathcal{V}_X \cap \mathcal{M}_{\mathcal{C}}$ by the ideals of the hyperplanes.

Definition 3.1.26. *The set of totally mixed conditional independence (CI) equilibria is defined as the intersection of the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ with the open simplex Δ. In other words, the set of totally mixed CI equilibria is the set of all totally mixed dependency equilibria satisfying the CI statements in $\mathcal{C}$.*

The goal of this chapter is to analyse the set of totally mixed CI equilibria arising from the global Markov property of undirected graphs. We address this study through the algebro-geometric analysis of Spohn CI varieties of undirected graphs. By Remark 3.1.8, the Spohn CI varieties of global, local and pairwise undirected Markov properties agree.

In the rest of this chapter, we assume that $\mathcal{C}$ is a collection of CI statements coming from the global Markov property of an undirected graph with n vertices. We identify the players of the game with the vertices of the graph. The edges of the graph represent the dependencies among the players.

Example 3.1.27.

- *Let G be the complete graphs with n vertices. The collection of CI statements is empty and $\mathcal{M}_{\mathcal{C}} = \mathbb{P}(V)$. Therefore, the Spohn CI variety coincides with the Spohn variety and the CI equilibria with the dependency equilibria.*

- *Let G be the graph with n vertices and no edges. In this case, $\mathcal{C}$ is the collection of all CI statements and by Example 3.1.11, $\mathcal{M}_{\mathcal{C}}$ is the Segre variety (3.6). One can check that the Spohn CI variety in $\mathcal{M}_{\mathcal{C}}$ is given by the equations (3.11). By Theorem 3.1.24, the set of totally mixed CI equilibria, for no edge graphical models, coincides with the set of totally mixed Nash equilibria.*

The two graphs in Example 3.1.27 correspond to opposite extremes in the spectrum of graphs with n vertices. From a game theoretic perspective, these graphs model the two extremes cases in the spectrum of dependencies: the collective (dependency equilibria) and the independent (Nash equilibria) behaviour of the players. More complex graphs enable us to model complicated dependencies among the players, thereby filling the gap between the the Nash and dependency equilibria in the scope of dependencies. For instance, consider the situation where the players are divided into two disjoint groups $S_1, S_2 \subseteq [n]$ and each group acts collectively. This scenario is modeled by the graph that is the disjoint union of the complete graphs whose set of vertices are S_1 and S_2 respectively.

The dependencies among the players described by a graph can also be illustrated geometrically through the monomial map (3.4). In the case of the graph with no edges this was depicted in Remark 3.1.17. The following example describes this geometric perspective for the graph in Figure 3.4.

Example 3.1.28. *Consider now the undirected graph G depicted in Figure 3.4. The CI equilibria associated to this graph models the situation where the second and the third* *players behave collectively and the first player behaves independently. The cooperation between the second and third player is represented by the edge among the corresponding vertices of G. This dependency among the players can also be illustrated through the monomial map* (3.4)*. In this setup, this map can be seen as the Segre embedding*

$$\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2d_3-1} \longrightarrow \mathbb{P}(V)$$

given by

$$p_{j_1j_2j_3} = \sigma_{j_1}\tau_{j_2j_3} \text{ for } j_i \in [d_i],$$

where σ_{j_1} and $\tau_{j_2j_3}$ are the coordinates of $\mathbb{P}^{d_1-1}$ and $\mathbb{P}^{d_2d_3-1}$ respectively. In particular, $\mathcal{M}_\mathcal{C}$ is the Segre variety $\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2d_3-1}$. As in Remark 3.1.17, the Segre embedding of $\mathcal{M}_\mathcal{C}$ splits the coordinate $p_{j_1j_2j_3}$ in the product of two factors. The factor σ_{j_1} represents the probability of the first players choosing independently the pure strategy j_1. The factor $\tau_{j_2j_3}$ is the probability of the second and third player choosing the pure strategies j_2 and j_3 respectively. Thus, the process of choosing a mixed strategy is split into the first player choosing independently her mixed strategy and the second and third player choosing collectively their mixed strategy.

In the Nash case, i.e. for graphs with no edges, the Spohn CI variety is defined by the polynomials (3.11) and it is generically zero dimensional (see [119, Section 6.4]), whereas the Spohn CI variety of more complicated graphs is expected to have higher dimension (see [105, Section 6]). Therefore, the set of totally mixed CI equilibria is expected to have positive dimension. As a consequence, the geometry of this semialgebraic set is more complicated than in the Nash setting. Our goal is to understand this challenging geometry through the algebro-geometric study of the Spohn CI variety. In this chapter we approach this for binary games. In other words, in the rest of the chapter we assume that $d_1 = \cdots = d_n = 2$. Thus, the ambient space where all of our varieties lie is $\mathbb{P}(V) = \mathbb{P}^{2^n-1}$. For binary games, the matrices (3.13) are square matrices and the Spohn variety $\mathcal{V}_X$ is defined by the n determinants of these matrices. Therefore, the codimension of the Spohn CI variety in $\mathcal{M}_\mathcal{C}$ is at most n. In [105, Section 6] the following conjecture is proposed.

Conjecture 3.1.29. *[105, Conjecture 24] Let G be the undirected graph with n vertices that models a generic n-player game X with binary choices in normal form. Let $\mathcal{C} =$ global(G) and $\mathcal{M}_\mathcal{C}$ be the discrete conditional independence variety of G. Then, the corresponding Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ has codimension n in $\mathcal{M}_\mathcal{C}$.*

In [105], Conjecture 3.1.29 is proven for $n \leq 3$ using a direct computation in `Macaulay2` [60].

Proposition 3.1.30. *[105, Proposition 25] For generic payoff tables, the Spohn CI variety of a graph with $n \leq 3$ vertices is irreducible and has codimension n in $\mathcal{M}_\mathcal{C}$.*

In Section 3.2 we prove Conjecture 3.1.29 for one edge graphs. In Section 3.3 we generalize this result to any undirected graphical model proving Conjecture 3.1.29.

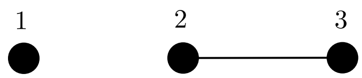


Figure 3.4: Graph with 3 vertices and one edge.

Example 3.1.31. *We consider 3–player games with binary choices, i.e.* $n = 3$ *and* $d_1 = d_2 = d_3 = 2$*. By Theorem 3.1.24, for generic payoff tables, the Spohn variety for a generic 3-player game with binary choices is a fourfold of degree 8 in* $\mathbb{P}^7$*. Moreover, by intersecting* $\mathcal{V}_X$ *with the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *we obtain the totally mixed Nash equilibria as in [105, Example 3]. Now consider the* 3 *vertices graph whose only edge connects the second and the third vertices. This graph is illustrated in Figure 3.4. One can check that the independence variety associated to this graph is the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^3$ *embedded in* $\mathbb{P}^7$*. In Proposition 3.1.30 it is shown that for generic games, the Spohn CI variety associated to this graph is an irreducible curve. This curve is contained in the Spohn variety and it contains the set of totally mixed Nash equilibria.*

Example 3.1.32. *Now, let us consider a non-generic game: "El Farol bar" problem for three players. Three friends would like to meet in a bar which has limited seats. Each player has two choices:*

Go=1 or Stay=2.

The people who go to the bar are rewarded when there are few of them (in our case two) and forfeited if the bar is overcrowded, i.e. if all three go to the bar. Those who choose to stay at home are rewarded when the bar is overcrowded and forfeited if they are the only one who showed up. We represent this in terms of $2 \times 2 \times 2$ *payoff tensor for each player.*

$$\begin{array}{rc} & \begin{array}{cccccccc} 111 & 121 & 112 & 122 & 211 & 221 & 212 & 222 \end{array} \\ \begin{array}{r} X^{(1)} = \\ X^{(2)} = \\ X^{(3)} = \end{array} & \begin{pmatrix} -1 & 2 & 2 & 0 & 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 1 & 2 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 \end{pmatrix}. \end{array}$$

A computation in `Macaulay 2` *shows that for this game, the Spohn variety* $\mathcal{V}_X$ *is a 5-dimensional reducible variety of degree 8 in* $\mathbb{P}^7$*. The Spohn CI variety associated to the graph in Figure 3.4 is a reducible surface of degree 8 in the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^3$*. In particular, this specific game does not satisfy Theorem 3.1.24 nor Proposition 3.1.30.*

Now we introduce the notion of payoff region. To do so, we momentarily leave behind the projective setting and we work on the vector space V.

Definition 3.1.33. *For a game* X*, we define the* **payoff map** *as the linear map*

$$\begin{array}{cccc} \pi_X : & V & \longrightarrow & \mathbb{R}^n \\ & p & \longmapsto & (PX^{(1)}, \dots, PX^{(n)}) \end{array}. \tag{3.15}$$

The image of the probability simplex $\overline{\Delta}$ via the payoff map is called the cooperative payoff region. *The image of all rank one tensors in the probability simplex through π_X is called the* noncooperative payoff region. *In [105, Section 5], the authors introduce the concept of* dependency payoff region, *which is defined as*

$$\mathcal{P}_X := \pi_X(\overline{\Delta} \cap \mathcal{V}_X),$$

where, $\mathcal{V}_X$ denotes the cone of the Spohn variety in V.

In [105, Section 5] it is shown that the dependency payoff region is a semialgebraic set and the authors analyze its boundary.

Definition 3.1.34. *The* conditional independence (CI) payoff region *is the image of the set of totally mixed CI equilibria through the payoff map (3.15).*

3.2 Nash CI curves

In the spectrum of dependencies, the next case to Nash equilibria is the situation where two players cooperate and the rest behave independently. In terms of graphical models, this situation is described by one edge graphs. In this section we analyse the Spohn CI variety and the CI equilibria of one edge undirected graphical models.

As set previously, we focus on games with binary choices of pure strategies, i.e. $d_1 = \cdots = d_n = 2$. Let G be a one edge graph with n vertices, representing the n players of the game. Up to labeling, we can assume that the $n-2$ isolated vertices of G correspond to the first $n-2$ players of the game, whereas the two vertices connected by the only edge are associated to the last two players. During this section, $\mathcal{M}_\mathcal{C}$ and $\mathcal{V}_{X,\mathcal{C}}$ will respectively denote the independence variety and the Spohn CI variety associated to the one edge graph G.

Remark 3.2.1. *For a one edge graph G, the monomial map in (3.4) corresponds to the Segre embedding*

$$\left(\mathbb{P}^1\right)^{n-2} \times \mathbb{P}^3 \longrightarrow \mathbb{P}^{2^n-1}$$

given by

$$p_{j_1\dots j_n} = \sigma^{(1)}_{j_1} \cdots \sigma^{(n-2)}_{j_{n-2}} \tau_{j_{n-1}j_n} \tag{3.16}$$

for $j_1, \dots, j_n \in [2]$. Here $\sigma^{(i)}_1, \sigma^{(i)}_2$ are the coordinates of the i–th $\mathbb{P}^1$ factor of the Segre variety for $i \in [n-2]$. Analogously, $\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}$ are the coordinates of the $\mathbb{P}^3$ factor of the Segre variety. Note that this notation differs from the one used in Proposition 3.1.9. We conclude that the independence variety $\mathcal{M}_\mathcal{C}$ of a one edge graph G is the Segre variety $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^3$ in $\mathbb{P}^{2^n-1}$.

By Remark 3.2.1, we have that for one edges graphs $\mathcal{M}_\mathcal{C}$ has dimension $n+1$. Thus, in this case, Conjecture 3.1.29 asks whether for generic payoff tables the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ of one edge graphical models is a curve or not. The goal of this section is to give a positive answer to this question and study some of the geometric properties of these curves as their degree, genus, connectedness, irreducibility and smoothness.

Example 3.2.2. *For $n = 2$, the graph G is the complete graph with two vertices. In this case, $\mathcal{M}_{\mathcal{C}} = \mathbb{P}^3$ and the Spohn CI variety coincides with the Spohn variety. By [105, Theorem 8], for generic payoff tables, $\mathcal{V}_{X,\mathcal{C}}$ is an elliptic curve in $\mathbb{P}^3$. In particular, $\mathcal{V}_{X,\mathcal{C}}$ is a smooth irreducible curve of genus 1, and Conjecture 3.1.29 is satisfied.*

Example 3.2.2 shows that Conjecture 3.1.29 holds for two players games. Moreover, it shows that for generic payoff tables, the Spohn CI variety of the complete graph with two edges is an smooth irreducible curve of genus 1. This is no longer true is we drop the generic condition from the assumptions. For instance, Example 3.1.23 shows a 2 players game where $\mathcal{V}_{X,\mathcal{C}} = \mathcal{V}_X$ is reducible and singular. These examples lead to the question of whether a generic Spohn CI variety of one edge graphical models is smooth and irreducible. We answer this question in Section 3.2.3. To do so, in the next section we first compute the equations of these varieties.

3.2.1 Equations of the Nash CI curve

The first step for understanding the geometry of the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ of one edge graphical models is the description of its ideal. Our approach to determine generators of the ideal of $\mathcal{V}_{X,\mathcal{C}}$ is to exploit the properties of graphical models presented in Section 3.1.1, mainly Proposition 3.1.9. In this section we illustrate this strategy for the case of one edge graphical models. Through the analysis of these generators we prove Conjecture 3.1.29 for one edge graphs.

To compute the ideal of $\mathcal{V}_{X,\mathcal{C}}$, we evaluate the determinants of the matrices M_i in (3.13) at the parametrization (3.16) of $\mathcal{M}_{\mathcal{C}}$. The evaluation of the determinant of M_i, for $i \leq n-2$, at this parametrization is

$$\begin{vmatrix} \sigma_1^{(i)} \sum\limits_{j_1,\dots,\widehat{j_i},\dots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1}j_n} & \sigma_1^{(i)} \sum\limits_{j_1,\dots,\widehat{j_i},\dots,j_n} X^{(i)}_{j_1\cdots \underset{(i)}{1} \cdots j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1}j_n} \\ \sigma_2^{(i)} \sum\limits_{j_1,\dots,\widehat{j_i},\dots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1}j_n} & \sigma_2^{(i)} \sum\limits_{j_1,\dots,\widehat{j_i},\dots,j_n} X^{(i)}_{j_1\cdots \underset{(i)}{2} \cdots j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1}j_n} \end{vmatrix}.$$

From the first and second row we can extract the factors $\sigma_1^{(i)}$ and $\sigma_2^{(i)}$ respectively. Analogously, from the first column we can extract the factor

$$\sum_{j_1,\dots,\widehat{j_i},\dots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1}j_n}.$$

Therefore, we obtain that, for $i \leq n-2$, the evaluation of the determinant of M_i at the Segre parametrization (3.16) is the product of

$$\sigma_1^{(i)}\sigma_2^{(i)}\left(\sum_{j_1,\dots,\widehat{j_i},\dots,j_n}\sigma_{j_1}^{(1)}\cdots\widehat{\sigma_{j_i}^{(i)}}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}j_n}\right) \tag{3.17}$$

with the polynomial

$$F_i := \begin{vmatrix} 1 & \sum_{j_1,\dots,\widehat{j_i},\dots,j_n} X^{(i)}_{j_1\cdots \underset{(i)}{1}\cdots j_n}\sigma_{j_1}^{(1)}\cdots\widehat{\sigma_{j_i}^{(i)}}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}j_n} \\ 1 & \sum_{j_1,\dots,\widehat{j_i},\dots,j_n} X^{(i)}_{j_1\cdots \underset{(i)}{2}\cdots j_n}\sigma_{j_1}^{(1)}\cdots\widehat{\sigma_{j_i}^{(i)}}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}j_n} \end{vmatrix} = \tag{3.18}$$

$$\sum_{j_1,\dots,\widehat{j_i},\dots,j_n}\left(X^{(i)}_{j_1\cdots 2\cdots j_n} - X^{(i)}_{j_1\cdots 1\cdots j_n}\right)\sigma_{j_1}^{(1)}\cdots\widehat{\sigma_{j_i}^{(i)}}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}j_n}.$$

Similarly, evaluating the determinant of M_{n-1} at the parametrization, we obtain that the determinants of M_{n-1} and M_n are, respectively, the product of $\sum_{j_1,\dots,j_{n-2}}\sigma_{j_1}^{(1)}\cdots\sigma_{j_{n-2}}^{(n-2)}$ with

$$F_{n-1} := \det\begin{pmatrix} \tau_{11}+\tau_{12} & \sum_{j_1,\dots,j_{n-2},j_n} X^{(n-1)}_{j_1\cdots j_{n-2}1j_n}\sigma_{j_1}^{(1)}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{1j_n} \\ \tau_{21}+\tau_{22} & \sum_{j_1,\dots j_{n-2},j_n} X^{(n-1)}_{j_1\cdots j_{n-2}2j_n}\sigma_{j_1}^{(1)}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{2j_n} \end{pmatrix},$$

$$F_n := \det\begin{pmatrix} \tau_{11}+\tau_{21} & \sum_{j_1,\dots,j_{n-1}} X^{(n)}_{j_1\cdots j_{n-1}1}\sigma_{j_1}^{(1)}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}1} \\ \tau_{12}+\tau_{22} & \sum_{j_1,\dots,j_{n-1}} X^{(n)}_{j_1\cdots j_{n-1}2}\sigma_{j_1}^{(1)}\cdots\sigma_{j_{n-2}}^{(n-2)}\tau_{j_{n-1}2} \end{pmatrix}.$$

Once we have computed the ideal of $\mathcal{M}_{\mathcal{C}}\cap\mathcal{V}_X$ in the coordinate ring of $\mathcal{M}_{\mathcal{C}}$, we proceed to saturate it by the family of hyperplanes.

Definition 3.2.3. *We denote the variety defined by $F_1,\dots,F_n$ in $\mathcal{M}_{\mathcal{C}}$ by C_X.*

By construction, the variety C_X is contained in $\mathcal{M}_{\mathcal{C}}\cap\mathcal{V}_X$ and C_X is union of irreducible components of $\mathcal{M}_{\mathcal{C}}\cap\mathcal{V}_X$.

Lemma 3.2.4. *For any game X, the variety C_X contains the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$. Moreover, $\mathcal{V}_{X,\mathcal{C}}$ is the union of irreducible components of C_X.*

Proof. Since C_X is contained in $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$, it is enough to check that C_X contains all the irreducible components of $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$ not contained in the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$. Recall that these hyperplanes are the ones we remove from $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$ for constructing $\mathcal{V}_{X,\mathcal{C}}$ in Definition 3.1.25. Note that the equations of C_X are obtained by removing the factors (3.17) from the equations of $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$. Hence, it is enough to check that these factors lead to components of $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$ contained in the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$.
Assume first that $\sigma_1^{(i)}$ is zero, then $p_{j_1 \cdots j_n} = 0$ for $j_i = 1$. Thus, the factor $\sigma_1^{(i)}$ leads to components of $\mathcal{M}_\mathcal{C} \cap \mathcal{V}_X$ lying in the family of hyperplanes. Similarly, if the factor

$$\sum_{j_1,\ldots,\widehat{j_i},\ldots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1} j_n}$$

vanishes, we obtain that

$$p_{+\cdots+1+\cdots+} = \sigma_1^{(i)} \sum_{j_1,\ldots,\widehat{j_i},\ldots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1} j_n} = 0.$$

Therefore, if the factor

$$\sum_{j_1,\ldots,\widehat{j_i},\ldots,j_n} \sigma_{j_1}^{(1)} \cdots \widehat{\sigma_{j_i}^{(i)}} \cdots \sigma_{j_{n-2}}^{(n-2)} \tau_{j_{n-1} j_n}$$

gets removed in the process of saturation. □

In Proposition 3.4.8 we will see in a wider setting that for generic n payoff tables, the Spohn CI variety of one edge graphical models coincides with C_X. This will also follow from Theorem 3.2.23, where we will see that C_X is irreducible for generic payoff tables. With a small abuse of the notation, we will also refer to the variety C_X as the Spohn CI variety.

Remark 3.2.5. *Note that $\mathcal{V}_{X,\mathcal{C}} \subseteq C_X$ for any payoff tables. The converse inclusion only holds for generic payoff tables and we can not drop the generic condition in this statement. For instance, one can fix the payoff table $X^{(1)}$ such that the determinant in* (3.18) *is*

$$F_1 = \begin{vmatrix} 1 & 0 \\ 1 & \sigma_1^{(2)} \cdots \sigma_1^{(n-2)} \tau_{11} \end{vmatrix} = \sigma_1^{(2)} \cdots \sigma_1^{(n-2)} \tau_{11}.$$

Then, the variety C_X is contained in some of the hyperplanes that we removed. Therefore, in this case the Spohn CI variety is empty but C_X is not.

This section is devoted to the study of the variety C_X. By definition, C_X is the intersection of n divisors in $\mathcal{M}_\mathcal{C}$ with multi-degree

$$(0,1,\ldots,1),\ldots,(1,\ldots,1,\underset{(i)}{0},1,\ldots,1),\ldots,(1,\ldots,1,0,1,1),(1,\ldots,1,2),(1,\ldots,1,2).$$

Lemma 3.2.6. *For any game X, the variety C_X is nonempty.*

Proof. Let D_i be the divisor of $\mathcal{M}_\mathcal{C}$ defined by F_i. Then, C_X is the intersection of $D_1, \dots, D_n$ in $\mathcal{M}_\mathcal{C}$. Let $[D_i]$ be the class of D_i in the Chow ring of $\mathcal{M}_\mathcal{C}$. To prove that C_X is nonempty, it is enough to check that $[D_1]\cdots[D_n]$ is nonzero in the Chow ring of $\mathcal{M}_\mathcal{C}$. In the proof of Proposition 3.2.12 we will see that this product is nonzero. □

The first interesting example of such a variety is for two players games. This situation was depicted in Example 3.2.2. Now we illustrate an example where we compute the equations of C_X for a particular 3–player game.

Example 3.2.7. *We consider the 3-player game from [119, Section 6.2] with the following payoff table*

$$\begin{array}{r} \\ X^{(1)} = \\ X^{(2)} = \\ X^{(3)} = \end{array} \begin{array}{c} \begin{array}{cccccccc} 111 & 121 & 112 & 122 & 211 & 221 & 212 & 222 \end{array} \\ \begin{pmatrix} 0 & 6 & 11 & 1 & 6 & 4 & 6 & 8 \\ 12 & 7 & 6 & 8 & 10 & 12 & 8 & 1 \\ 11 & 11 & 3 & 3 & 0 & 14 & 2 & 7 \end{pmatrix}. \end{array} \tag{3.19}$$

Following the above formulas, the ideal of C_X is generated by the polynomials

$$F_1 = 6\tau_{11} - 5\tau_{12} - 2\tau_{21} + 7\tau_{22},$$
$$F_2 = (5\sigma_1^{(1)} - 2\sigma_2^{(1)})\tau_{11}\tau_{21} - (\sigma_1^{(1)} + 4\sigma_2^{(1)})\tau_{12}\tau_{21} + (4\sigma_1^{(1)} + 9\sigma_2^{(1)})\tau_{11}\tau_{22} + (7\sigma_2^{(1)} - 2\sigma_1^{(1)})\tau_{12}\tau_{22},$$
$$F_3 = (8\sigma_1^{(1)} - 2\sigma_2^{(1)})\tau_{11}\tau_{12} + (8\sigma_1^{(1)} + 12\sigma_2^{(1)})\tau_{12}\tau_{21} + (8\sigma_1^{(1)} - 7\sigma_2^{(1)})\tau_{11}\tau_{22} + (8\sigma_1^{(1)} + 7\sigma_2^{(1)})\tau_{21}\tau_{22}.$$

A `Macaulay2` *computation shows that the Spohn CI variety C_X is an irreducible curve of genus 3 and degree 8. It is contained in the Spohn variety which is four-dimensional irreducible variety of degree 4 whose ideal is generated by the determinants of three 2×2 matrices M_i depicted in (3.13). Moreover, C_X contains the two totally mixed Nash equilibria of X computed in [119, Section 6.2].*

One of the main invariants of an algebraic variety is the dimension. Next we use the equations of the Spohn CI variety for one edge graphical models to determine its dimension. Note that C_X is the intersection of n equations in $\mathcal{M}_\mathcal{C}$, which has dimension $n+1$. By Lemma 3.2.6, $\dim C_X \geq 1$ and we expect C_X to be a curve inside $\mathcal{M}_\mathcal{C}$.

Definition 3.2.8. *We say that C_X is a* Nash conditional independence (CI) curve *if it has dimension 1.*

For instance, Examples 3.1.23, 3.2.2 and 3.2.7 are examples of Nash CI curves. The following result shows that Conjecture 3.1.29 holds for the case of one edge graphs.

Theorem 3.2.9. *For a generic game X, C_X is a curve.*

Proof. Seeing the payoff tables $X^{(1)}, \dots, X^{(n)}$ as variables in n copies of $\mathbb{P}^{2^n-1}$, we consider the variety C defined as the projective subvariety

$$\mathbb{V}(F_1, \dots, F_n)$$

of $\left(\mathbb{P}^{2^n-1}\right)^n \times \mathcal{M}_{\mathcal{C}}$. Here $F_1, \dots, F_n$ are the n polynomials defining C_X introduced above. Also, we consider the projection

$$\begin{array}{rccc} \pi : & C & \longrightarrow & \left(\mathbb{P}^{2^n-1}\right)^n \\ & (X^{(1)}, \dots, X^{(n)}, p) & \longmapsto & (X^{(1)}, \dots, X^{(n)}). \end{array}$$

Note that, given a point $X \in \left(\mathbb{P}^{2^n-1}\right)^n$, the fiber of X through π is C_X. Using that the dimension of the fibers of π is upper semicontinuous, we obtain that there exists an open subset $U \subseteq \left(\mathbb{P}^{2^n-1}\right)^n$ such that, for every $X \in U$, $\dim C_X \leq 1$. Moreover, since for every X, C_X is the zero locus of n equations, C_X has codimension at most n in $\mathcal{M}_{\mathcal{C}}$. Hence $\dim C_X \geq 1$ for every $X \in \left(\mathbb{P}^{2^n-1}\right)^n$. Thus, we conclude that for every $X \in U$, $\dim(C_X) = 1$. In order to check that U is dense, we prove that it is non-empty. For $k \leq n-2$ we fix $X^{(k)}$ such that the following equation holds

$$F_k = \prod_{i \neq k}^{n-2} (k\sigma_1^{(i)} - \sigma_2^{(i)})(\tau_{11} + k\tau_{12} + k^2\tau_{21} + k^3\tau_{22}). \tag{3.20}$$

Note that this is possible because if we expand the right-hand side of the above equation, we get a polynomial of the same format as F_k. We denote the $n-1$ linear forms in (3.20) by $L_{k,1}, \dots, L_{k,n-1}$ respectively. In particular,

$$F_k = L_{k,1} \cdots L_{k,n-1}$$

for $k \leq n-2$ and $L_{k,i}$ is a linear form in the i–th factor of $\mathcal{M}_{\mathcal{C}}$. Then, the irreducible components of $\mathbb{V}(F_1, \dots, F_{n-2})$ are of the form $\mathbb{V}(L_{1,i_1}, \dots, L_{n-2,i_{n-2}})$ for $i_1, \dots, i_{n-2} \in [n-1]$. Note that for any $i \leq n-2$, the linear forms $L_{1,i}, \dots, L_{n-2,i}$ define distinct points in the i–th $\mathbb{P}^1$ factor of $\mathcal{M}_{\mathcal{C}}$. On the other hand, consider the matrix M given by the coefficients of the linear forms $L_{1,n-1}, \dots, L_{n-2,n-1}$. Any submatrix of any dimension of M is a Vandermonde matrix. Therefore, all the minors of M are nonzero. Hence, the linear subspace defined by any number of the linear forms $L_{1,i}, \dots, L_{n-2,i}$ has the expected dimension. In particular, we deduce that $\mathbb{V}(L_{1,i_1}, \dots, L_{n-2,i_{n-2}})$ has the expected dimension for any $i_1, \dots, i_{n-2} \in [n-1]$. This implies that $\mathbb{V}(F_1, \dots, F_{n-2})$ has dimension 3 and its irreducible components are isomorphic to $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^2$, or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Following the same idea, one can fix $X^{(n-1)}$ and $X^{(n)}$ such that F_{n-1} and F_n are respectively as follows

$$F_{n-1} = \prod_{i=1}^{n-2} ((n-1)\sigma_1^{(i)} - \sigma_2^{(i)})(A_{n-1}\tau_{11}\tau_{21} + B_{n-1}\tau_{12}\tau_{21} + C_{n-1}\tau_{11}\tau_{22} + D_{n-1}\tau_{12}\tau_{22}),$$
$$F_n = \prod_{i=1}^{n-2} (n\sigma_1^{(i)} - \sigma_2^{(i)})(A_n\tau_{11}\tau_{12} + B_n\tau_{22}\tau_{11} + C_n\tau_{12}\tau_{21} + D_n\tau_{21}\tau_{22}),$$

where

$$D_{n-1} = B_{n-1} + C_{n-1} - A_{n-1}, \;\; D_n = B_n + C_n - A_n,$$

and the coefficients A_k, B_k, C_k depend linearly on $X^{(k)}$ for $k \geq n-1$. As before, we denote the linear factors of F_{n-1} and F_n by $L_{n-1,1}, \ldots, L_{n-1,n-2}$ and $L_{n,1}, \ldots, L_{n,n-2}$ respectively. Similarly, we denote the quadratic factors of F_{n-1} and F_n by Q_{n-1} and Q_n. Then, the irreducible components of C_X are of the form

$$\mathbb{V}(L_{1,i_1}, \ldots, L_{n,i_n}),\ \mathbb{V}(L_{1,i_1}, \ldots, L_{n-1,i_{n-1}}, Q_n),\ \mathbb{V}(L_{1,i_1}, \ldots, L_{n-2,i_{n-2}}, L_{n,i_n}, Q_{n-1}),$$

or

$$\mathbb{V}(L_{1,i_1}, \ldots, L_{n-2,i_{n-2}}, Q_{n-1}, Q_n).$$

As before, the components of the form $\mathbb{V}(L_{1,i_1}, \ldots, L_{n,i_n})$ have the expected dimension. On the other hand, a computation in `Macaulay2` [60] shows that the intersection of the quadric $\mathbb{V}(Q_n)$ and the variety defined by any number of the linear forms $L_{1,n-1}, \ldots, L_{n-2,n-1}$ has the expected dimension for generic A_n, B_n, C_n. Therefore, the irreducible components of the form $\mathbb{V}(L_{1,i_1}, \ldots, L_{n-1,i_{n-1}}, Q_n)$ have the expected dimension. A similar argument shows the the remaining components also have the expected dimension. Hence, the intersection of the threefold $\mathbb{V}(F_1, \ldots, F_{n-2})$ with $\mathbb{V}(F_{n-1}, F_n)$ has dimension 1, and we conclude that U is non-empty. □

Remark 3.2.10. *In the proof of Theorem 3.2.9, we constructed a game where C_X is one dimensional. Moreover, one can check that for this game, none of the irreducible components of C_X are contained in the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$. This implies that the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ of one edge graphs equals C_X for generic games X. We will see this reasoning in detail in Proposition 3.4.8.*

Note that again the generic condition in Theorem 3.2.9 can not be dropped. As shown in Example 3.1.32, there exist payoff tables for which the variety C_X is not a curve. This fact will be explored in Section 3.5.

Corollary 3.2.11. *For generic payoff tables, the Nash CI curve is a complete intersection of n divisors of $\mathcal{M}_{\mathcal{C}}$.*

One may show that the specific Nash CI curve constructed in the proof of Proposition 3.2.9 is connected. By the semicontinuity theorem, we then obtain that for generic games, the Nash CI curve is connected. In Section 3.2.2, we yet use a different argument to prove in Lemma 3.2.15 that the Nash CI curve is connected.

3.2.2 Degree and genus of Nash CI curves

In Theorem 3.2.9, we have seen that C_X is a curve for generic payoff tables. This section is devoted to the computation of the degree and the genus of the Nash CI curve C_X. Moreover, we prove that the Nash CI curve C_X is connected. The main strategy for analysing these properties of Nash CI curves is to use Corollary 3.2.11.

Proposition 3.2.12. *For generic payoff tables, the degree of C_X is the coefficient of the monomial $x_1 \cdots x_{n-2}x_{n-1}^3$ in the polynomial*

$$\prod_{i=1}^{n-2}\left(\sum_{k\neq i}^{n-1} x_k\right)\left(2x_{n-1}+\sum_{k=1}^{n-2} x_k\right)^2\left(\sum_{k=1}^{n-1} x_k\right). \tag{3.21}$$

Proof. By Corollary 3.2.11, C_X is the complete intersection of n divisors $D_1, \ldots, D_n$ of $\mathcal{M}_\mathcal{C}$, where $D_i := \mathbb{V}(F_i)$. We compute the degree by multiplying the classes of these divisors with the class of $H \cap \mathcal{M}_\mathcal{C}$ in the Chow ring of $\mathcal{M}_\mathcal{C}$ where H is a generic hyperplane of $\mathbb{P}^{2^n-1}$. Using Künneth's formula (see [44, Theorem 2.10]), we obtain the Chow rings of $\mathcal{M}_\mathcal{C}$ as follows:

$$A_\bullet(\mathcal{M}_\mathcal{C}) \simeq \left(\bigotimes_{i=1}^{n-2} A_\bullet(\mathbb{P}^1)\right) \otimes A_\bullet(\mathbb{P}^3) \simeq \mathbb{Z}[x_1, \ldots, x_{n-1}]/\langle x_1^2, \ldots, x_{n-2}^2, x_{n-1}^4\rangle.$$

The classes $[D_i]$ in $A_\bullet(\mathcal{M}_\mathcal{C})$ correspond to the first Chern classes of the line bundles $\mathcal{O}_{\mathcal{M}_\mathcal{C}}(D_i)$. Let F be a multi-homogeneous polynomial in $\mathbb{C}[\sigma_1^{(1)}, \ldots, \tau_{22}]$, and let D be the divisor $D = \mathbb{V}(F)$ in $\mathcal{M}_\mathcal{C}$. For $i \leq n-2$, we denote the degree of F with respect to the variables $\sigma_1^{(i)}, \sigma_2^{(i)}$ by d_i (resp. for d_{n-1} and $\tau_{j_1j_2}$). Then, the line bundle associated to D is

$$\mathcal{O}_{\mathcal{M}_\mathcal{C}}(D) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(d_1)) \otimes \cdots \otimes \pi_{n-2}^*(\mathcal{O}_{\mathbb{P}^1}(d_{n-2})) \otimes \pi_{n-1}^*(\mathcal{O}_{\mathbb{P}^3}(d_{n-1})),$$

where π_i is the projection from $\mathcal{M}_\mathcal{C}$ to the corresponding factor of the product. We denote this line bundle by $\mathcal{O}_{\mathcal{M}_\mathcal{C}}(d_1, \ldots, d_{n-1})$. In particular, we obtain that

$$\mathcal{O}_{\mathcal{M}_\mathcal{C}}(D_i) = \begin{cases} \mathcal{O}_{\mathcal{M}_\mathcal{C}}(1, \ldots, 1, \underset{(i)}{0}, 1, \ldots, 1) & \text{for } i \leq n-2, \\ \mathcal{O}_{\mathcal{M}_\mathcal{C}}(1, \ldots, 1, 2) & \text{for } i = n-1, n. \end{cases}$$

Furthermore, since the first Chern class of $\mathcal{O}_{\mathcal{M}_\mathcal{C}}(d_1, \ldots, d_{n-1})$ is $d_1x_1 + \cdots + d_{n-1}x_{n-1}$, the following holds:

$$[D_i] = \mathcal{O}_{\mathcal{M}_\mathcal{C}}(D_i) = \begin{cases} \displaystyle\sum_{k\neq i}^{n-1} x_k & \text{for } i \leq n-2, \\ \displaystyle 2x_{n-1} + \sum_{k=1}^{n-2} x_k & \text{for } i = n-1, n. \end{cases}$$

See [44, Chapters 1 and 2] for more details on these computations. Therefore, we deduce that

$$[D_1]\cdots[D_n] = \prod_{i=1}^{n-2} x_k \left(\sum_{k\neq i}^{n-1} x_k\right)\left(2x_{n-1}+\sum_{k=1}^{n-2} x_k\right)^2.$$

Finally, using that for a generic hyperplane H it holds that $[H \cap \mathcal{M}_{\mathcal{C}}] = x_1 + \cdots + x_{n-1}$, we get that $[D_1] \cdots [D_n] [H \cap \mathcal{M}_{\mathcal{C}}]$ corresponds to the class of the polynomial (3.21) in $A_\bullet(\mathcal{M}_{\mathcal{C}})$ and hence the statement follows. $\square$

Next, we compute the arithmetic genus of C_X. For this purpose, we first determine the Euler characteristic of $\mathcal{O}_{C_X}$. By Corollary 3.2.11, we get the following exact sequence (Koszul complex):

$$\begin{aligned} 0 \to \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-\sum_{k=1}^{n} D_i \right) \to \bigoplus_{i_1 < \cdots < i_{n-1}}^{n} \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-\sum_{k=1}^{n-1} D_{i_k} \right) \to \cdots \to \\ \to \bigoplus_{i_1 < i_2}^{n} \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-D_{i_1} - D_{i_2} \right) \to \bigoplus_{i}^{n} \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-D_i \right) \to \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \to \mathcal{O}_{C_X} \to 0. \end{aligned} \tag{3.22}$$

We define χ_k as the Euler characteristic of the $(k+2)$–th term from the right of the previous exact sequence, i.e.

$$\chi_k := \chi \left(\bigoplus_{i_1 < \cdots < i_k}^{n} \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-D_{i_1} - \cdots - D_{i_k} \right) \right) = \sum_{i_1 < \cdots < i_k}^{n} \chi \left(\mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \left(-D_{i_1} - \cdots - D_{i_k} \right) \right). \tag{3.23}$$

Then, using (3.22) we obtain that

$$\chi(\mathcal{O}_{C_x}) = \chi(\mathcal{O}_{\mathcal{M}_{\mathcal{C}}}) + \sum_{k=1}^{n} (-1)^k \chi_k. \tag{3.24}$$

By Künneth formula, and since $\chi(\mathcal{O}_{\mathbb{P}^n}) = 1$, we deduce that $\chi(\mathcal{O}_{\mathcal{M}_{\mathcal{C}}}) = 1$. Thus, it only remains to compute χ_k. For the following lemma, our convention is that $\binom{a}{b} = 0$, if $a < b$.

Lemma 3.2.13. *For* $3 \leq k \leq n$,

$$\chi_k = \binom{n-2}{k} \chi(\mathcal{O}_{k,1}) + 2\binom{n-2}{k-1} \chi(\mathcal{O}_{k,2}) + \binom{n-2}{k-2} \chi(\mathcal{O}_{k,3}),$$

where

1. $\chi(\mathcal{O}_{k,1}) = (-1)^{n+1} \binom{k-1}{3} (k-2)^k (k)^{n-2-k}$.
2. $\chi(\mathcal{O}_{k,2}) = (-1)^{n+1} \binom{k}{3} (k-2)^k (k-1)^{n-1-k}$.
3. $\chi(\mathcal{O}_{k,3}) = (-1)^{n+1} \binom{k+1}{3} (k-2)^k (k-2)^{n-k}$.

Moreover, we have that $\chi_1 = 0$ *and* $\chi_2 = (-1)^{n+1}$.

Proof. The idea is to use Künneth formula as above. First of all, we observe that only the following three types of line bundles can appear in the sum of equation (3.23):

- If $i_1, \ldots, i_k \leq n-2$, the line bundle of the corresponding term of (3.23) is of the form
$$\mathcal{O}_{\mathcal{M}_\mathcal{C}}(-k, \ldots, \underset{(i_1)}{-k+1}, \ldots, \underset{(i_k)}{-k+1}, \ldots, -k).$$
Moreover, in (3.23) these terms appear $\binom{n-2}{k}$ times.

- If $i_1, \ldots, i_{k-1} \leq n-2$ and $i_k > n-2$, the line bundle of the corresponding term of (3.23) is of the form
$$\mathcal{O}_{\mathcal{M}_\mathcal{C}}(-k, \ldots, \underset{(i_1)}{-k+1}, \ldots, \underset{(i_{k-1})}{-k+1}, \ldots, -k-1).$$
Moreover, in (3.23) these terms appear $2\binom{n-2}{k-1}$ times.

- If $i_1, \ldots, i_{k-2} \leq n-2$ and $i_{k-1} > n-2$, the line bundle of the corresponding term of (3.23) is of the form
$$\mathcal{O}_{\mathcal{M}_\mathcal{C}}(-k, \ldots, \underset{(i_1)}{-k+1}, \ldots, \underset{(i_{k-2})}{-k+1}, \ldots, -k-2).$$
Moreover, in (3.23) these terms appear $\binom{n-2}{k-2}$ times.

To deal with each of these three cases, we again use Künneth formula. In our setting, this formula states that for $d_1, \ldots, d_{n-1} \in \mathbb{Z}$,

$$h^k\left(\mathcal{M}_\mathcal{C}, \mathcal{O}_{\mathcal{M}_\mathcal{C}}(d_1, \ldots, d_{n-1})\right) = \sum_{k_1+\cdots+k_{n-1}=k} h^{k_{n-1}}\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d_{n-1})\right) \prod_{i=1}^{n-2} h^{k_i}\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i)\right).$$

In particular, using that all the first $n-2$ factors of $\mathcal{M}_\mathcal{C}$ are isomorphic, we get that the Euler characteristic of each of the line bundles of the above cases does not depend on the choice of the i_l that is lower or equal than $n-2$. Denoting the line bundles of the three cases by $\mathcal{O}_{k,1}$, $\mathcal{O}_{k,2}$, and $\mathcal{O}_{k,3}$ respectively, we get that

$$\chi_k = \binom{n-2}{k}\chi(\mathcal{O}_{k,1}) + 2\binom{n-2}{k-1}\chi(\mathcal{O}_{k,2}) + \binom{n-2}{k-2}\chi(\mathcal{O}_{k,3}). \tag{3.25}$$

To compute each of the terms of the above expression we use that for $d > 0$,

$$h^k\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)\right) = \begin{cases} 0 & \text{if } k \neq n \text{ or } d \leq n, \\ \binom{d-1}{n} & \text{if } k = n \text{ and } d > n. \end{cases}$$

In particular, for $d_1, \ldots, d_{n-1} > 0$ such that either $d_i \leq 1$ for some $i \leq n-2$ or $d_i \leq 3$ for $i \geq n-1$, we obtain that

$$h^k\left(\mathcal{M}_\mathcal{C}, \mathcal{O}_{\mathcal{M}_\mathcal{C}}(-d_1, \ldots, -d_{n-1})\right) = 0 \ \ \forall k.$$

On the contrary, if $d_i > 1$ for every $i \leq n-2$ and $d_{n-1}, d_n > 3$, we get that

$$h^k\left(\mathcal{M}_{\mathcal{C}}, \mathcal{O}_{\mathcal{M}_{\mathcal{C}}}(-d_1, \ldots, -d_{n-1})\right) = \begin{cases} 0 & \text{if } k \neq n+1, \\ \binom{d_{n-1}-1}{3} \prod_{i=1}^{n-2} (d_i - 1) & \text{if } k = n+1. \end{cases} \tag{3.26}$$

Now, note that for all the line bundles appearing in χ_1 it holds that $d_{n-1} \leq 3$. Thus, we can conclude that $\chi_1 = 0$. Analogously, the only line bundle appearing in the expression of χ_2 with a non-zero cohomology group is $\mathcal{O}(-D_{n-1} - D_n) = \mathcal{O}(-2, \ldots, -2, -4)$. Using equation (3.26) one deduces that the only non-zero cohomology group of this line bundle is the $(n+1)$–th cohomology group and its dimension is 1. Hence, we conclude that $\chi_2 = (-1)^{n+1}$. A similar argument shows that, for $k \geq 3$, the only non-zero cohomology group of $\mathcal{O}_{k,1}$, $\mathcal{O}_{k,2}$, and $\mathcal{O}_{k,3}$ is the $(n+1)$–th cohomology group and we deduce the following formulas:

- $\chi(\mathcal{O}_{k,1}) = (-1)^{n+1}\binom{k-1}{3}(k-2)^k k^{n-2-k}$.
- $\chi(\mathcal{O}_{k,2}) = (-1)^{n+1}\binom{k}{3}(k-2)^k (k-1)^{n-1-k}$.
- $\chi(\mathcal{O}_{k,3}) = (-1)^{n+1}\binom{k+1}{3}(k-2)^k (k-2)^{n-k}$.

Now, the proof of the lemma follows from these expressions and equation (3.25). □

Finally, as a consequence of Lemma 3.2.13 and Equation (3.24), the following corollary can be derived:

Corollary 3.2.14. *For generic payoff games, the Euler characteristic of C_X is*

$$\chi(\mathcal{O}_{C_X}) = 1 + (-1)^{n+1} + \sum_{k=3}^{n} (-1)^k \chi_k$$

where χ_k is as in Lemma 3.2.13.

Now that the Euler characteristic has been computed, we deal with the arithmetic genus. For this purpose, we first need to ensure that the Nash CI curve is connected.

Lemma 3.2.15. *For generic payoff tables, $h^0(C_X, \mathcal{O}_{C_X}) = 1$. In particular, C_X is connected.*

Proof. The idea is to split the exact sequence (3.22) in short exact sequences and apply the long exact sequence of cohomology on each of them. To do so, let $\mathcal{F}_n$ be the first sheaf (starting from the left) appearing in the exact sequence (3.22). Similarly, let $\mathcal{F}_{n-1}$ be the second sheaf of the exact sequence, and so on. We denote the morphisms from $\mathcal{F}_i$ to $\mathcal{F}_{i-1}$ by ϕ_i. Then, (3.22) can be written as

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{\phi_n} \mathcal{F}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{O}_{\mathcal{M}_{\mathcal{C}}} \xrightarrow{\phi_0} \mathcal{O}_{C_X} \longrightarrow 0. \tag{3.27}$$

Let K_i be the kernel of ϕ_i. Then, the above exact sequence splits in the following n short exact sequences:

$$\begin{array}{ll}
(E_0): & 0 \longrightarrow K_0 \longrightarrow \mathcal{O}_{\mathcal{M}_\mathcal{C}} \longrightarrow \mathcal{O}_{C_X} \longrightarrow 0, \\
(E_1): & 0 \longrightarrow K_1 \longrightarrow \mathcal{F}_1 \longrightarrow K_0 \longrightarrow 0, \\
\vdots & \qquad\qquad\qquad \vdots \\
(E_i): & 0 \longrightarrow K_i \longrightarrow \mathcal{F}_i \longrightarrow K_{i-1} \longrightarrow 0, \\
\vdots & \qquad\qquad\qquad \vdots \\
(E_{n-1}): & 0 \longrightarrow K_{n-1} \longrightarrow \mathcal{F}_{n-1} \longrightarrow K_{n-2} \longrightarrow 0.
\end{array}$$

Now, we consider the long exact sequence in the cohomology of (E_0):

$$0 \longrightarrow H^0(\mathcal{M}_\mathcal{C}, K_0) \longrightarrow H^0(\mathcal{M}_\mathcal{C}, \mathcal{O}_{\mathcal{M}_\mathcal{C}}) \longrightarrow H^0(C_X, \mathcal{O}_{C_X}) \longrightarrow H^1(\mathcal{M}_\mathcal{C}, K_0) \longrightarrow \cdots .$$

By the exactness of this sequence, if $h^1(\mathcal{M}_\mathcal{C}, K_0) = 0$, we would get a surjection

$$H^0(\mathcal{M}_\mathcal{C}, \mathcal{O}_{\mathcal{M}_\mathcal{C}}) \longrightarrow H^0(C_X, K_0) \longrightarrow 0.$$

Since $h^0(\mathcal{M}_\mathcal{C}, \mathcal{O}_{\mathcal{M}_\mathcal{C}}) = 1$, this would imply that $h^0(C_X, \mathcal{O}_{C_X}) = 1$. Thus, it is enough to check that $h^1(\mathcal{M}_\mathcal{C}, K_0) = 0$. To do so, we focus on the long exact sequence in cohomology arising from (E_1):

$$\cdots \longrightarrow H^1(\mathcal{M}_\mathcal{C}, \mathcal{F}_1) \longrightarrow H^1(\mathcal{M}_\mathcal{C}, K_0) \longrightarrow H^2(\mathcal{M}_\mathcal{C}, K_1) \longrightarrow H^2(\mathcal{M}_\mathcal{C}, \mathcal{F}_1) \longrightarrow \cdots .$$

By the computations performed in the proof of Lemma 3.2.13, we know that for $j \geq 1$ and $k \leq n$, $h^k(\mathcal{M}_\mathcal{C}, \mathcal{F}_j) = 0$. Thus, from the above sequence, we conclude that

$$H^1(\mathcal{M}_\mathcal{C}, K_0) \simeq H^2(\mathcal{M}_\mathcal{C}, K_1).$$

Recursively, we get that

$$H^i(\mathcal{M}_\mathcal{C}, K_{i-1}) \simeq H^{i+1}(\mathcal{M}_\mathcal{C}, K_i)$$

for every $i \leq n-1$. In particular, by the exactness of (3.27), we have that $K_{n-1} = \mathcal{F}_n$. Hence, we obtain that

$$H^1(\mathcal{M}_\mathcal{C}, K_0) \simeq H^2(\mathcal{M}_\mathcal{C}, K_1) \simeq \cdots \simeq H^n(\mathcal{M}_\mathcal{C}, K_{n-1}) = H^n(\mathcal{M}_\mathcal{C}, \mathcal{F}_n).$$

Now, the proof follows from the vanishing of the cohomology group $H^n(\mathcal{M}_\mathcal{C}, \mathcal{F}_n)$. □

Remark 3.2.16. *Lemma 3.2.15 analyses the connectedness of Nash CI curves as complex algebraic varieties. This notion differs from the connectedness of the set of real points of a Nash CI curve. Recall that the connected components of the set of real points of a real algebraic curve are called ovals. By Harnack's Theorem (see [10, Section 11.6]), the number of ovals of a real algebraic curve is at most $g+1$, where g is the genus of the curve. In particular, the computation of the genus of a Nash CI curve can be used to understand the number of components of the corresponding set of totally mixed CI equilibria.*

Using that C_X is connected, we may derive the arithmetic genus of the Nash CI curve.

Corollary 3.2.17. *For generic payoff tables, C_X is a connected curve of arithmetic genus*

$$p_a(C_X) = (-1)^n + \sum_{k=3}^{n} (-1)^{k+1} \chi_k$$

where χ_k is as in Lemma 3.2.13.

Example 3.2.18. *Consider the 3-player game from Example 3.2.7. By Lemma 3.2.12, we obtain the degree of C_X is 8 which is the coefficient of the monomial $x_1 x_2^3$ in the polynomial*

$$x_2 \left(2x_2 + x_1\right)^2 \left(x_1 + x_2\right).$$

By Corollary 3.2.17 and Lemma 3.2.13, the genus of C_X is

$$p_a(C_X) = (-1)^3 + \chi_3 = -1 + 4 = 3.$$

Note that this computation matches with the `Macaulay2` *computation in Example 3.2.7.*

Using Corollary 3.2.17 in Table 3.1, we can determine effectively the genus of the Nash CI curve for different values of n.

n	genus	degree
3	3	8
4	23	30
5	175	146
6	1469	880
7	13491	6276
8	135859	51562
9	1494879	478670

Table 3.1: Genus and degree of the Nash CI curve C_X.

Remark 3.2.19. *The genus of C_X can be also computed combinatorially via [79, Theorem 1], and more generally by the motivic arithmetic genus formula in [36]. The main idea is to determine the discrete mixed volume of Newton polytopes* $\mathrm{Newt(F_i)}$ *for $i \in [n]$.*
Another method for computing the genus of the Nash CI curve is using the adjunction formula. Let $i : C_X \hookrightarrow \mathcal{M}_\mathcal{C}$ be the closed immersion of C_X in $\mathcal{M}_\mathcal{C}$. Applying the adjunction formula (see [44, Section 1.4.3]) we get that

$$\omega_{C_X} \simeq i^*(\omega_{\mathcal{M}_\mathcal{C}} \otimes \mathcal{O}_{\mathcal{M}_\mathcal{C}}(D_1) \otimes \cdots \otimes \omega_{\mathcal{M}_\mathcal{C}}(D_n)) \simeq i^*(\mathcal{O}_{\mathcal{M}_\mathcal{C}}(n-3, \ldots, n-3, n-2)) . \quad (3.28)$$

Since C_X is a complete intersection, it is Gorenstein and hence,

$$\deg(\omega_{C_X}) = 2p_a(C_X) - 2.$$

As a consequence of Equation (3.28), one can compute the degree of ω_{C_X} using the Chow ring of $\mathcal{M}_{\mathcal{C}}$ as in Lemma 3.2.12. Using this method we obtain that $2p_a(C_X) - 2$ is equal to the coefficient of the monomial $x_1 \cdots x_{n-2}x_{n-1}^3$ in the polynomial

$$\prod_{i=1}^{n-2}\left(\sum_{k\neq i}^{n-1} x_k\right)\left(2x_{n-1} + \sum_{k=1}^{n-2} x_k\right)^2 \left((n-3)\sum_{k=1}^{n-2} x_k + (n-2)x_{n-1}\right).$$

However, our computations in `Macaulay2` *show that this method is less effective than the formula in Corollary 3.2.17.*

3.2.3 Smoothness of the Nash CI curve

One of the main properties of an algebraic variety is the smoothness. In this section we prove that for generic payoff tables, the Nash CI curve C_X is smooth. Together with Lemma 3.2.15, we will deduce that a generic Nash CI curve is irreducible. Our strategy is to study the linear system of the divisors defining C_X in order to apply Bertini's Theorem (see [67, Theorem 8.18]).

As a consequence of the study of these linear systems, we give an answer to the universality of divisors for C_X. The universality of divisors asks whether any divisor of the same multidegree of the defining polynomials of C_X can be obtained from a game. The study of the universality of divisors for C_X is motivated by the Nash case. We denote the n polynomials in (3.11) by $G_1, \ldots, G_n$ and let N_X be the variety defined by $\{G_1, \ldots, G_n\}$ in the Segre variety $(\mathbb{P}^1)^n \subset \mathcal{M}_{\mathcal{C}}$. The variety N_X is the Spohn CI variety of the graphical model of the graph with no edges, i.e. N_X is the Spohn CI variety of the collection of all CI statements. By Theorem 3.1.18 the intersection of N_X with the open simplex is the set of totally mixed Nash equilibria. For $1 \leq i \leq n$, let $\tilde{V}_i$ be the vector space of multi-homogeneous polynomials with multi-degree $(1, \ldots, 1, \underset{(i)}{0}, 1, \ldots, 1)$, i.e.

$$\tilde{V}_i := H^0\left(\left(\mathbb{P}^1\right)^n, \mathcal{O}_{(\mathbb{P}^1)^n}(1, \ldots, 0, \ldots, 1)\right).$$

Note that $\tilde{V}_i$ has dimension 2^{n-1}. The universality of divisors asks whether for any $i \in [n]$ and for any polynomial $G \in \tilde{V}_i$, there exists a payoff table $X^{(i)}$ such that $G_i = G$. In other words, the universality of divisors aims to determine whether for any i the linear map

$$\begin{array}{ccc} \mathbb{R}^{2^n} & \longrightarrow & \tilde{V}_i \\ \tilde{X}^{(i)} & \longmapsto & G_i \end{array}$$

is surjective or not. The surjectivity of this map is equivalent to the linear map

$$\begin{array}{ccc} \left(\mathbb{R}^{2^n}\right)^n & \longrightarrow & \tilde{V}_1 \times \cdots \times \tilde{V}_n \\ (X^{(1)}, \ldots, X^{(n)}) & \longmapsto & (G_1, \ldots, G_n). \end{array} \tag{3.29}$$

In [119, Corollary 6.7] a positive answer is given to the universality of divisors for N_X, and hence, the map (3.29) is surjective. The next layer to N_X in our spectrum of Spohn CI varieties is the Nash CI curve. We carry out a similar analysis for the case of the Nash CI curve.

For $i \leq n-2$, let V_i be the vector space of multi-homogeneous polynomials of multi-degree $(1,\ldots,\underset{(i)}{0},\ldots,1)$, i.e.

$$V_i := H^0\left(\mathcal{M}_{\mathcal{C}}, \mathcal{O}_{\mathcal{M}_{\mathcal{C}}}(1,\ldots,\underset{(i)}{0},\ldots,1)\right).$$

Similarly, we define the vector spaces V_{n-1} and V_n as

$$V_{n-1} = V_n = H^0\left(\mathcal{M}_{\mathcal{C}}, \mathcal{O}_{\mathcal{M}_{\mathcal{C}}}(1,\ldots,1,2)\right).$$

The dimension of V_{n-1} and V_n is $10 \cdot 2^{n-1}$. For every i, we consider the linear map

$$\begin{array}{rccc} \phi_i : & \mathbb{R}^{2^n} & \longrightarrow & V_i \\ & X^{(i)} & \longmapsto & F_i. \end{array}$$

We denote the image of ϕ_i by Λ_i. In other words, ϕ_i is the linear map sending the i–th payoff table to the divisor D_i and Λ_i is the linear system of divisors in V_i arising from a game. Assembling together these maps, we consider the linear map

$$\begin{array}{ccc} \left(\mathbb{R}^{2^n}\right)^n & \longrightarrow & V_1 \times \cdots \times V_n \\ (X^{(1)},\ldots,X^{(n)}) & \longmapsto & (F_1,\ldots,F_n). \end{array} \tag{3.30}$$

In this case, the universality of divisors asks whether the map (3.30) is surjective or not. In Proposition 3.2.20 we compute the image of the map (3.30).

Proposition 3.2.20. *Let Λ_i be the linear system defined by the image of ϕ_i. The image $\Lambda_1 \times \ldots \times \Lambda_n$ of the linear map (3.30) has dimension $2^{n-1}(n+1)$. In particular, the map (3.30) is not surjective.*

Proof. First of all, we note that, as in the Nash case, ϕ_i is surjective for $i \leq n-2$. Thus, for $i \leq n-2$, $\Lambda_i = V_i$ and $\dim \Lambda_i = 2^{n-1}$. For $i = n-1$, we can rewrite V_{n-1} as the tensor product

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\right)^{\otimes(n-2)}.$$

Expanding F_{n-1}, we obtain

$$\begin{aligned} F_{n-1} \;=\; & \sum_{j_1,\ldots,j_{n-2}} \sigma^{(1)}_{j_1}\cdots\sigma^{(n-2)}_{j_{n-2}} \Big(A^{(n-1)}_{j_1\cdots j_{n-2}}\tau_{11}\tau_{21} + B^{(n-1)}_{j_1\cdots j_{n-2}}\tau_{12}\tau_{21} + \\ & C^{(n-1)}_{j_1\cdots j_{n-2}}\tau_{11}\tau_{22} + D^{(n-1)}_{j_1\cdots j_{n-2}}\tau_{12}\tau_{22}\Big) \end{aligned} \tag{3.31}$$

where the coefficients are defined as

$$\begin{aligned}
A^{(n-1)}_{j_1\cdots j_{n-2}} &= X^{(n-1)}_{j_1\cdots j_{n-2}21} - X^{(n-1)}_{j_1\cdots j_{n-2}11}, \quad C^{(n-1)}_{j_1\cdots j_{n-2}} = X^{(n-1)}_{j_1\cdots j_{n-2}22} - X^{(n-1)}_{j_1\cdots j_{n-2}11},\\
B^{(n-1)}_{j_1\cdots j_{n-2}} &= X^{(n-1)}_{j_1\cdots j_{n-2}21} - X^{(n-1)}_{j_1\cdots j_{n-2}12}, \ D^{(n-1)}_{j_1\cdots j_{n-2}} = B^{(n-1)}_{j_1\cdots j_{n-2}} + C^{(n-1)}_{j_1\cdots j_{n-2}} - A^{(n-1)}_{j_1\cdots j_{n-2}}.
\end{aligned}$$

From this expression we deduce that $\Lambda_{n-1} = W_{n-1} \otimes \left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\right)^{\otimes(n-2)}$, where

$$W_{n-1} = \{A\tau_{11}\tau_{21} + B\tau_{12}\tau_{21} + C\tau_{11}\tau_{22} + (B + C - A)\tau_{12}\tau_{22} : \text{ for } A, B, C \in \mathbb{R}\}.$$

Thus, W_{n-1} is a linear subspace of dimension 3 and hence, $\dim \Lambda_{n-1} = 3 \cdot 2^{n-2}$. Analogously, for F_n we get the expression

$$\begin{aligned}
F_n \quad = \quad & \sum_{j_1,\ldots,j_{n-2}} \sigma^{(1)}_{j_1} \cdots \sigma^{(n-2)}_{j_{n-2}} \Big(A^{(n)}_{j_1\cdots j_{n-2}}\tau_{11}\tau_{12} + B^{(n)}_{j_1\cdots j_{n-2}}\tau_{12}\tau_{21} + \\
& C^{(n)}_{j_1,\ldots,j_n}\tau_{11}\tau_{22} + D^{(n)}_{j_1,\ldots,j_n}\tau_{21}\tau_{22}\Big)
\end{aligned} \tag{3.32}$$

where the coefficients are defined as

$$\begin{aligned}
A^{(n)}_{j_1\cdots j_{n-2}} &= X^{(n)}_{j_1\cdots j_{n-2}12} - X^{(n)}_{j_1\cdots j_{n-2}11}, \quad C^{(n)}_{j_1\cdots j_{n-2}} = X^{(n)}_{j_1\cdots j_{n-2}22} - X^{(n)}_{j_1\cdots j_{n-2}11},\\
B^{(n)}_{j_1\cdots j_{n-2}} &= X^{(n)}_{j_1\cdots j_{n-2}12} - X^{(n)}_{j_1\cdots j_{n-2}21}, \ D^{(n)}_{j_1\cdots j_{n-2}} = B^{(n)}_{j_1\cdots j_{n-2}} + C^{(n)}_{j_1\cdots j_{n-2}} - A^{(n)}_{j_1\cdots j_{n-2}}.
\end{aligned}$$

The dimension of Λ_n follows from this expression as before. Then, the image of the map (3.30) is $\Lambda_1 \times \cdots \times \Lambda_n$ and it has dimension $2^{n-1}(n+1)$. It follows that the map (3.30) is not surjective since the dimension of $V_1 \times \cdots \times V_n$ is $2^{n-1}(n+8)$. □

In contrast with the Nash case, from Proposition 3.2.20, we conclude that the universality for divisors does not hold. In particular, the linear systems $\Lambda_1, \ldots, \Lambda_n$ might have base locus. From Bertini's theorem (see [67, Theorem 8.18]), we deduce that for generic payoff tables, C_X is smooth away from these base loci. Hence, in order to study the smoothness of C_X we need to compute the base locus of each of these linear systems. Since ϕ_i is surjective for $i \in [n-2]$, we get that Λ_i is complete for $i \in [n-2]$ and, hence, base point free. The next lemma computes the base locus of Λ_{n-1} and Λ_n.

Lemma 3.2.21.

1. *The base locus of Λ_{n-1} is*

$$\left(\mathbb{P}^1\right)^{n-2} \times \left(L^{(n-1)}_1 \cup L^{(n-1)}_2 \cup L^{(n-1)}_3\right)$$

where $L^{(n-1)}_1, L^{(n-1)}_2, L^{(n-1)}_3$ are the lines of $\mathbb{P}^3$ defined by the equations

$$\{\tau_{11} = \tau_{12} = 0\}, \quad \{\tau_{21} = \tau_{22} = 0\}, \quad \textit{and} \quad \{\tau_{11} + \tau_{12} = \tau_{21} + \tau_{22} = 0\},$$

respectively.

2. *The base locus of* Λ_n *is*

$$\left(\mathbb{P}^1\right)^{n-2} \times \left(L_1^{(n)} \cup L_2^{(n)} \cup L_3^{(n)}\right)$$

where $L_1^{(n)}, L_2^{(n)}, L_3^{(n)}$ *are the lines of* $\mathbb{P}^3$ *defined by the equations*

$$\{\tau_{11} = \tau_{21} = 0\}, \quad \{\tau_{12} = \tau_{22} = 0\}, \quad \textit{and} \quad \{\tau_{11} + \tau_{21} = \tau_{12} + \tau_{22} = 0\},$$

respectively.

Proof. As in the proof of Proposition 3.2.20, we write Λ_{n-1} as the tensor product

$$\Lambda_{n-1} = W_{n-1} \otimes \left(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\right)^{\otimes(n-2)}.$$

Thus, to compute the base locus of Λ_{n-1} it is enough to compute the base locus of W_{n-1} in $\mathbb{P}^3$. The elements of this linear system are polynomials of the form

$$A\tau_{11}\tau_{21} + B\tau_{12}\tau_{21} + C\tau_{11}\tau_{22} + (B + C - A)\tau_{12}\tau_{22}$$

for $[A : B : C] \in \mathbb{P}^2$. So, for $[A : B : C] \in \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$ we obtain the polynomials $\tau_{11}\tau_{21} - \tau_{12}\tau_{22}, \tau_{12}(\tau_{21} + \tau_{22}), \tau_{22}(\tau_{11} + \tau_{12})$, which generate W_i. Then, one can check that the base locus of these three polynomials is exactly $L_1^{(n-1)} \cup L_2^{(n-1)} \cup L_3^{(n-1)}$. Thus, we conclude that the base locus of Λ_{n-1} is $\left(\mathbb{P}^1\right)^{n-2} \times \left(L_1^{(n-1)} \cup L_2^{(n-1)} \cup L_3^{(n-1)}\right)$. A similar computation derives the statement for Λ_n. □

For generic payoff tables, $\mathbb{V}(F_1, \ldots, F_{n-2}, F_{n-1})$ and $\mathbb{V}(F_1, \ldots, F_{n-2}, F_n)$ are surfaces and their intersection is the Nash CI curve C_X. By Bertini's theorem (see [67, Theorem 8.18]), we obtain that for generic payoff tables $\mathbb{V}(F_1, \ldots, F_{n-2}, F_{n-1})$ (respectively $\mathbb{V}(F_1, \ldots, F_{n-2}, F_n)$) is smooth away from the base loci Λ_{n-1} (respectively Λ_n). The next proposition states that these surfaces are indeed smooth.

Proposition 3.2.22. *For generic payoff tables,* $\mathbb{V}(F_1, \ldots, F_{n-2}, F_{n-1})$ *and* $\mathbb{V}(F_1, \ldots, F_{n-2}, F_n)$ *are smooth surfaces.*

Proof. We denote $\mathbb{V}(F_1, \ldots, F_{n-2}, F_{n-1})$ and $\mathbb{V}(F_1, \ldots, F_{n-2}, F_n)$ by $S_{X,1}$ and $S_{X,2}$ respectively. We prove the result for $S_{X,2}$ (the analogous proof works for $S_{X,1}$). By Bertini's theorem, it is enough to check smoothness at the points of the base locus of Λ_n. Let

$$\Omega_i = S_{X,2} \cap \left(\left(\mathbb{P}^1\right)^{n-2} \times L_i^{(n)}\right).$$

Then, $\Omega_1 \cup \Omega_2 \cup \Omega_3$ is the intersection of $S_{X,2}$ with the base locus of Λ_n. We check the Jacobian criterion at the points of $\Omega_1 \cup \Omega_2 \cup \Omega_3$.

We will focus on Ω_1 (similar reasoning works for Ω_2 and Ω_3). First, we prove that Ω_1 is smooth. Let $\tilde{F}_i$ be the polynomial resulting from restricting F_i to Ω_1, i.e., substituting

τ_{11} and τ_{21} by 0 in F_i. Then, $\Omega_1 = \mathbb{V}(\tilde{F}_1, \ldots, \tilde{F}_{n-2})$. Moreover, $\tilde{F}_1, \ldots, \tilde{F}_{n-2}$ are generic elements of complete linear systems of $(\mathbb{P}^1)^{n-2} \times L_1^{(n)}$. By Bertini's Theorem (see [67, Theorem 8.18]), Ω_1 is smooth.

Now we check the Jacobian criterion for $S_{X,2}$ at Ω_1. Let $J_{S_{X,2}}(x)$ be the Jacobian matrix of $F_1, \ldots, F_{n-2}, F_n$ at x, written as:

$$J_{S_{X,2}}(x) := \begin{array}{c} \\ \begin{matrix} \vdots \\ \frac{\partial}{\partial \sigma^{(i)}_{j_i}} \\ \vdots \\ \frac{\partial}{\partial \tau_{12}} \\ \frac{\partial}{\partial \tau_{22}} \\ \hline \frac{\partial}{\partial \tau_{11}} \\ \frac{\partial}{\partial \tau_{21}} \end{matrix} \end{array} \begin{array}{c} \begin{array}{ccc|c} F_1 & \cdots & F_{n-2} & F_n \end{array} \\ \left(\begin{array}{c|c} A(x) & B(x) \\ \hline C(x) & D(x) \end{array}\right) \end{array}.$$

Note that either τ_{11} or τ_{21} appear in any monomial of F_n. Therefore, for $x \in \Omega_1$, $B(x) = 0$. Now, $A(x)$ is the Jacobian of $\tilde{F}_1, \ldots, \tilde{F}_{n-2}$ w.r.t. $\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots, \sigma_1^{(n-2)}, \sigma_2^{(n-2)}, \tau_{12}, \tau_{22}$. Since, Ω_1 is smooth, $A(x)$ has maximal rank. Hence, $J_{S_{X,2}}(x)$ has maximal rank if and only if $D(x)$ has maximal rank. Thus, a point $x \in \Omega_1$ is a singular point of $S_{X,2}$ if and only if it lies in $\mathbb{V}(\frac{\partial F_n}{\partial \tau_{11}}, \frac{\partial F_n}{\partial \tau_{21}})$. These two derivatives are

$$\frac{\partial F_n}{\partial \tau_{11}} = \sum_{j_1,\ldots,j_{n-2}} \sigma_{j_1}^{(1)} \cdots \sigma_{j_{n-2}}^{(n-2)} (A^{(n)}_{j_1\cdots j_{n-2}} \tau_{12} + C^{(n)}_{j_1\cdots j_{n-2}} \tau_{22}),$$

$$\frac{\partial F_n}{\partial \tau_{21}} = \sum_{j_1,\ldots,j_{n-2}} \sigma_{j_1}^{(1)} \cdots \sigma_{j_{n-2}}^{(n-2)} (B^{(n)}_{j_1\cdots j_{n-2}} \tau_{12} + D^{(n)}_{j_1\cdots j_{n-2}} \tau_{22}).$$

As before, $\frac{\partial F_n}{\partial \tau_{11}}$ is a generic element of a complete linear system of $(\mathbb{P}^1)^{n-2} \times L_1^{(n)}$. In particular, we deduce that for generic payoff tables, $\mathbb{V}(\tilde{F}_1, \ldots, \tilde{F}_{n-2}, \frac{\partial F_n}{\partial \tau_{11}})$ is the intersection of $n-1$ divisors, and each of these divisors is generic in a complete linear system of $(\mathbb{P}^1)^{n-2} \times L_1^{(n)}$. By Bertini's Theorem (see [67, Theorem 8.18]), for generic payoff tables, $\mathbb{V}(\tilde{F}_1, \ldots, \tilde{F}_{n-2}, \frac{\partial F_n}{\partial \tau_{11}})$ has dimension 0. Now, we write $\frac{\partial F_n}{\partial \tau_{21}}$ as

$$\frac{\partial F_n}{\partial \tau_{21}} = \sum_{j_1,\ldots,j_{n-2}} \sigma_{j_1}^{(1)} \cdots \sigma_{j_{n-2}}^{(n-2)} (B^{(n)}_{j_1\cdots j_{n-2}} (\tau_{12} + \tau_{22}) + (C^{(n)}_{j_1\cdots j_{n-2}} - A^{(n)}_{j_1\cdots j_{n-2}}) \tau_{22}).$$

Note that for generic payoff tables, $\mathbb{V}(\frac{\partial F_n}{\partial \tau_{21}})$ defines a smooth hypersurface in $(\mathbb{P}^1)^{n-2} \times L_1^{(n)}$. By Bertini's Theorem (see [67, Theorem 8.18]), for generic payoff tables

$$\mathbb{V}\left(\tilde{F}_1, \ldots, \tilde{F}_{n-2}, \frac{\partial F_n}{\partial \tau_{11}}, \frac{\partial F_n}{\partial \tau_{21}}\right)$$

has the expected dimension, and hence, it is empty. We conclude that $S_{X,2}$ is smooth at Ω_1. The same argument follows for Ω_2 and Ω_3. □

In particular, applying Bertini's theorem to $\mathbb{V}(F_1,\dots,F_{n-2},F_{n-1})$ and the linear system Λ_n, we obtain that for generic payoff tables, the singular locus of C_X lies in $\mathbb{V}(F_1,\dots,F_{n-2},F_{n-1})$ with the base locus of Λ_n. Similarly, applying the same argument to $\mathbb{V}(F_1,\dots,F_{n-2},F_n)$ and Λ_{n-1}, we deduce that the singular locus lies in the intersection of $\mathbb{V}(F_1,\dots,F_{n-2},F_n)$ with the base locus of Λ_{n-1}. This implies that for generic payoff tables, the singular locus of C_X lies in the intersection of the base loci of Λ_{n-1} and Λ_n. This intersection is

$$\left(\mathbb{P}^1\right)^{n-2} \times \{q_1,q_2,q_3,q_4,q_5\},$$

where $q_1,\dots,q_5$ are $[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1],[1:-1:-1:1]$ respectively. In the next theorem, we deduce the smoothness from studying locally the smoothness at the points of this intersection.

Theorem 3.2.23. *For generic payoff tables, the Nash CI curve C_X is smooth and irreducible.*

Proof. By Lemma 3.2.15, C_X is connected. Therefore, it is enough to prove the smoothness of C_X in order to conclude the irreducibility. For $i \in [5]$, let S_i be the intersection of C_X with $(\mathbb{P}^1)^{n-2} \times \{q_i\}$. By Bertini's theorem, the singular locus of C_X lies in the intersection of C_X with $S_1 \cup \cdots \cup S_5$. The strategy we follow is to apply locally the Jacobian Criterion on the points of this intersection. The reasoning is similar to the proof of Proposition 3.2.22. We analyze the smoothness at the points in S_1 (similarly for $S_2,\dots,S_5$). First, one can check using Bertini's Theorem that S_1 is smooth. For $x \in S_1$, the Jacobian matrix of $F_1\dots,F_n$ with respect to $\sigma_1^{(1)},\sigma_2^{(1)},\dots,\sigma_1^{(n-2)},\sigma_2^{(n-2)},\tau_{12},\tau_{21},\tau_{22}$ at x is of the form

$$J_{C_X}(x) := \begin{array}{c} \\ \vdots \\ \frac{\partial}{\partial \sigma_{j_i}^{(i)}} \\ \vdots \\ \frac{\partial}{\partial \tau_{12}} \\ \frac{\partial}{\partial \tau_{21}} \\ \frac{\partial}{\partial \tau_{22}} \end{array} \begin{array}{c} \begin{array}{ccc|cc} F_1 & \cdots & F_{n-2} & F_{n-1} & F_n \end{array} \\ \left(\begin{array}{c|c} A(x) & 0 \\ \hline C(x) & D(x) \end{array}\right) \end{array}.$$

Then, for $x \in S_1$, the matrix $A(x)$ coincide with the Jacobian of S_1 at x. Thus, we conclude that $x \in S_1$ is a singular point of C_X if and only if the rank of $D(x)$ is not maximal. A similar argument as in Proposition 3.2.22 shows that the intersection of S_1 with the variety defined by the 2×2 minors of $D(x)$ is empty. Hence, we conclude that C_X is smooth at S_1. □

Remark 3.2.24. *By [105, Remark 3.3], the maximum number of totally mixed Nash equilibria for a generic n-player game with binary choices is not zero. It is in particular the number of derangements of the set $[n]$ [119, Corollary 6.9]. Let X be a generic game for which there exists a totally mixed Nash equilibrium. Since the Nash CI curve C_X is smooth and contains totally mixed Nash equilibria of X, the real points of C_X are Zariski dense (e.g. [115, Theorem 5.1]). As a consequence of Theorem 3.2.23, we deduce that for such a game, the set of totally mixed Spohn CI equilibria of the Nash CI is a smooth manifold of dimension 1.*

3.3 Dimension of Spohn CI varieties

Theorem 3.2.9 shows that Conjecture 3.1.29 holds for one edge graphical models. In this section, we generalize this result proving Conjecture 3.1.29 for any undirected graphical model. In the case of one edge graphs, the independence variety is a Segre variety. The parametrization of this Segre variety played a fundamental role in the computation of the dimension of this Spohn CI variety. For general graphs, our strategy is to use the monomial map (3.4).

Let $G = ([n], E)$ be an undirected graph with n vertices, and let $\mathcal{D}$ be the set of the maximal cliques of G. Recall that the monomial map (3.4)

$$\phi : \quad \mathbb{T} := \prod_{C \in \mathcal{D}} \mathbb{T}_C \quad \longrightarrow \quad \mathcal{M}_{\mathcal{C}},$$

in Proposition 3.1.9 given by

$$p_{j_1 \cdots j_n} = \prod_{C \in \mathcal{D}} \sigma^{(C)}_{j_C}. \tag{3.33}$$

The torus $\mathbb{T}_C$ is defined in (3.2) and its coordinates are $\sigma^{(C)}_{j_C}$, where

$$j_C = (j_i)_{i \in [C]} \in [2]^{|c|}.$$

We want to evaluate the determinants of the matrices $M_1, \ldots, M_n$ in (3.13) by (3.33). This is the same strategy used in Section 3.2.1 for computing the equations of the Nash CI curve. As in the Nash CI curve case, we distinguish two cases depending on whether the graph has isolated vertices or not. For $i \in [n]$, we denote the set of maximal cliques of G containing the vertex i by $\mathcal{D}_i$, and we consider the set

$$[\mathcal{D}_i] := \left(\bigcup_{C \in \mathcal{D}_i} [C] \right) \setminus \{i\}.$$

In other words, $[\mathcal{D}_i]$ is the set of vertices of G distinct than i that are contained in a clique in $\mathcal{D}_i$. The cardinal of $[\mathcal{D}_i]$ is denoted by c_i. Note that if i is an isolated vertex, $[\mathcal{D}_i]$ is empty. Now, for $j = (j_k)_{k \in [\mathcal{D}_i]} \in [2]^{c_i}$ and $a \in [2]$, we consider the index

$j(a) = (j_k)_{k\in[\mathcal{D}_i]\cup\{i\}} \in [2]^{c_i+1}$ where $j_i = a$. Given such index and a clique $C \in \mathcal{D}_i$, we also consider the index $j_C(a) = (j_k)_{k\in[C]} \in [2]^{|C|}$. Using this notation, we define the monomial

$$\mathfrak{S}^{(i)}_{j,a} := \prod_{C\in\mathcal{D}_i} \sigma^{(C)}_{j_C(a)}$$

for $a \in [2]$ and $j = (j_k)_{k\in[\mathcal{D}_i]} \in [2]^{c_i}$. Then, the evaluation of the determinant of M_i at (3.33) is the determinant of the matrix

$$\begin{pmatrix} \displaystyle\sum_{j\in[2]^{n-1}} \mathfrak{S}^{(i)}_{j,1} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 1\cdots j_n} \mathfrak{S}^{(i)}_{j,1} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} \\ \displaystyle\sum_{j\in[2]^{n-1}} \mathfrak{S}^{(i)}_{j,2} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 2\cdots j_n} \mathfrak{S}^{(i)}_{j,2} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} \end{pmatrix}. \tag{3.34}$$

From the first column of (3.34), we deduce that the determinant of (3.34) is the product of

$$\sum_{j\in[2]^{n-c_i-1}} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} \tag{3.35}$$

and the polynomial

$$\det \begin{pmatrix} \displaystyle\sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,1} & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 1\cdots j_n} \mathfrak{S}^{(i)}_{j,1} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} \\ \displaystyle\sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,2} & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 2\cdots j_n} \mathfrak{S}^{(i)}_{j,2} \prod_{C\notin\mathcal{D}_i} \sigma^{(C)}_{j_C} \end{pmatrix} \tag{3.36}$$

If i is not an isolated vertex of G, we define the polynomial F_i as the determinant (3.36). Assume now that i is an isolated vertex of the graph. By abuse of notation, we also denote the maximal clique defined by this isolated vertex by i. In this case, the determinant (3.36) is

$$\det \begin{pmatrix} \sigma^{(i)}_1 & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 1\cdots j_n} \sigma^{(i)}_1 \prod_{C\in\mathcal{D}\setminus\{i\}} \sigma^{(C)}_{j_C} \\ \sigma^{(i)}_2 & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 2\cdots j_n} \sigma^{(i)}_2 \prod_{C\in\mathcal{D}\setminus\{i\}} \sigma^{(C)}_{j_C} \end{pmatrix}.$$

We obtain that the above determinant is the product of $\sigma^{(i)}_1\sigma^{(2)}_2$ and the determinant

$$\det \begin{pmatrix} 1 & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 1\cdots j_n} \prod_{C\in\mathcal{D}\setminus\{i\}} \sigma^{(C)}_{j_C} \\ 1 & \displaystyle\sum_{j\in[2]^{n-1}} X^{(i)}_{j_1\cdots 2\cdots j_n} \prod_{C\in\mathcal{D}\setminus\{i\}} \sigma^{(C)}_{j_C} \end{pmatrix}. \tag{3.37}$$

For an isolated vertex i of the graph, we define the polynomial F_i as the determinant (3.37). We denote the variety defined by $F_1, \dots, F_n$ in $\mathbb{T}$ by Y_X. By construction Y_X is contained in $\phi^{-1}(\mathcal{M}_{\mathcal{C}} \cap \mathcal{V}_X)$.

Lemma 3.3.1. *For any X, $\phi^{-1}(\mathcal{V}_{X,\mathcal{C}})$ is contained in Y_X.*

Proof. We proceed as in Lemma 3.2.4. Note that to construct the polynomials $F_1, \dots, F_n$ we have removed some factors of the determinant (3.34). The image via ϕ of the varieties defined by each of these factors are contained in some of the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$. For instance, assume that the factor (3.35) vanishes. By (3.33) we get that

$$p_{+\cdots+1+\cdots+} = \left(\sum_{j \in [2]^{n-c_i-1}} \prod_{C \notin \mathcal{D}_i} \sigma_{j_C}^{(C)} \right) \left(\sum_{j \in [2]^{c_i}} \prod_{C \in \mathcal{D}_i} \sigma_{j_C(1)}^{(C)} \right) = 0.$$

Therefore, Y_X is obtained by removing some components from $\phi^{-1}(\mathcal{M}_{\mathcal{C}} \cap \mathcal{V}_X)$ contained the preimage via ϕ of the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$. We deduce that the preimage of the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ through ϕ is contained in Y_X. $\square$

Our strategy is to analyse the dimension of Y_X to compute the dimension of $\mathcal{V}_{X,\mathcal{C}}$. To do so, we analyse the base loci of the linear systems defined by $F_1, \dots, F_n$.

Note that F_i is a multihomogeneous polynomial in the coordinates of $\mathbb{T}$. Its multidegree depends on whether i is an isolated vertex or not. Assume that $i \in [n]$ is an isolated vertex. Then, for $C \in \mathcal{D}$, the degree of F_i in the coordinates of $\mathbb{T}_C$ is 0 if $C = i$ and 1 otherwise. In other words, the multidegree of F_i is given by the integer vector

$$\sum_{C \in \mathcal{D} \setminus \{i\}} e_C.$$

Assume now that i is not an isolated vertex. The multidegree of F_i is given by the integer vector

$$\sum_{C \notin \mathcal{D}} e_C + \sum_{C \in \mathcal{D}} 2e_C.$$

We denote the space of multihomogeneous polynomials in the coordinates of $\mathbb{T}$, of the same multidegree as F_i, by V_i. In particular, F_i is contained in V_i for any game X. For $i \in [n]$ we consider the linear map

$$\begin{array}{ccc} \mathbb{R}^{2^n} & \longrightarrow & V_i \\ X^{(i)} & \longmapsto & F_i. \end{array}$$

We denote the image of this map by Λ_i. We use Bertini's Theorem (see [67, Theorem 8.18]) to compute the dimension of Y_X. To apply this strategy, we analyse the base locus of Λ_i. First, if i is an isolated vertex, F_i looks like the polynomials (3.11).

Similarly to the universality of divisors in the Nash case, one can check that $\Lambda_i = V_i$ for isolated vertices. Now, assume that i is not an isolated vertex. Then, F_i can be written as a linear combination of polynomials that are the product of a determinant of the form

$$\det \begin{pmatrix} \sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,1} & \sum_{j\in[2]^{c_i}} Y^{(i)}_{j(1)}\mathfrak{S}^{(i)}_{j,1} \\ \sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,2} & \sum_{j\in[2]^{c_i}} Y^{(i)}_{j(2)}\mathfrak{S}^{(i)}_{j,2} \end{pmatrix}, \tag{3.38}$$

for some $Y^{(i)}_{j(1)}, Y^{(i)}_{j(2)} \in \mathbb{C}$, and a multihomogeneous polynomial L of multidegree

$$\sum_{C\notin\mathcal{D}_i} e_C. \tag{3.39}$$

Moreover, for any polynomial that is the product of L and (3.38), there exists a game X such that F_i equals this product. We denote the vector space of all multihomogeneous polynomials of the form (3.38) by W_i. Then, Λ_i is the tensor product of W_i and the complete linear system of multihomogeneous polynomials with multidegree (3.39). In particular, Λ_i and W_i have the same base locus.

Lemma 3.3.2. *For $i \in [n]$ not being an isolated vertex, the linear system W_i is generated by the polynomials*

1. *For $a \in [2]^{c_i}$, $\mathfrak{S}^{(i)}_{a,1}\left(\sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,2}\right)$.*
2. *For $a \in [2]^{c_i}$, $\mathfrak{S}^{(i)}_{a,2}\left(\sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,1}\right)$.*
3. $\mathfrak{S}^{(i)}_{\mathbb{1},1}\mathfrak{S}^{(i)}_{\mathbb{1},2} - \sum_{j,k\in[2]^{c_i}\setminus\{\mathbb{1}\}} \mathfrak{S}^{(i)}_{j,1}\mathfrak{S}^{(i)}_{k,2}$*, where $\mathbb{1} = (1,\ldots,1) \in [2]^{c_i}$.*

Proof. We write the determinant (3.38) as

$$\sum_{j,k\in[2]^{c_i}} A^{(i)}_{j,k}\mathfrak{S}^{(i)}_{k,1}\mathfrak{S}^{(i)}_{j,2}, \tag{3.40}$$

where

$$A^{(i)}_{j,k} = Y^{(i)}_{j(2)} - Y^{(i)}_{k(1)}.$$

Note that for $j,k \in [2]^{c_i}$, we have that

$$A^{(i)}_{j,k} - A^{(i)}_{j,\mathbb{1}} - A^{(i)}_{\mathbb{1},k} + A^{(i)}_{\mathbb{1},\mathbb{1}} = Y^{(i)}_{j(2)} - Y^{(i)}_{k(1)} - Y^{(i)}_{j(2)} + Y^{(i)}_{\mathbb{1}(1)} - Y^{(i)}_{\mathbb{1}(2)} + Y^{(i)}_{k(1)} + Y^{(i)}_{\mathbb{1}(2)} - Y^{(i)}_{\mathbb{1}(1)} = 0,$$

and we deduce that

$$A^{(i)}_{j,k} = A^{(i)}_{j,\mathbb{1}} + A^{(i)}_{\mathbb{1},k} - A^{(i)}_{\mathbb{1},\mathbb{1}} \text{ for } j,k \neq \mathbb{1}.$$

Therefore, we can write the polynomial (3.40) as

$$\sum_{j\in[2]^{c_i}} A^{(i)}_{j,\mathbb{1}}\mathfrak{S}^{(i)}_{\mathbb{1},1}\mathfrak{S}^{(i)}_{j,2} + \sum_{j\in[2]^{c_i}\setminus\{\mathbb{1}\}} A^{(i)}_{\mathbb{1},j}\mathfrak{S}^{(i)}_{j,1}\mathfrak{S}^{(i)}_{\mathbb{1},2} + \sum_{j,k\in[2]^{c_i}\setminus\{\mathbb{1}\}} (A^{(i)}_{j,\mathbb{1}} + A^{(i)}_{\mathbb{1},k} - A^{(i)}_{\mathbb{1},\mathbb{1}})\mathfrak{S}^{(i)}_{k,1}\mathfrak{S}^{(i)}_{j,2}.$$

The proof follows by fixing in the above expression all the coefficients $A^{(i)}_{j,\mathbb{1}}, A^{(i)}_{\mathbb{1},j}$ except one to be zero. □

Once we have computed the generators of W_i, we deal with the computation of their base loci.

Lemma 3.3.3. *For $i \in [n]$ not being an isolated vertex, the base locus of W_i is*

$$\mathbb{V}(G_1, G_2) \cup \mathbb{V}(\mathfrak{S}^{(i)}_{a,1} : a \in [2]^{c_i}) \cup \mathbb{V}(\mathfrak{S}^{(i)}_{a,2} : a \in [2]^{c_i}) \tag{3.41}$$

where

$$G_1 = \sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,2} \text{ and } G_2 = \sum_{j\in[2]^{c_i}} \mathfrak{S}^{(i)}_{j,1}$$

Proof. Let Z_1, Z_2, Z_3 be the three varieties in the union (3.41) respectively, and let Z be the variety defined by the ideal generated by all the polynomials listed in Lemma 3.3.2. We have to show that $Z_1 \cup Z_2 \cup Z_3 = Z$. First, note that the first row of the matrix in (3.38) vanishes at Z_2. In particular, the determinant (3.38) vanishes at Z_2, and hence, Z_2 is contained in Z. Similarly, Z_3 is contained in Z. Now, the first column of the matrix (3.38) vanishes at Z_1. Therefore Z_1 is also contained in Z.

Next, we assume that p is a point in Z not contained in Z_1. Then, either $G_1(p) \neq 0$ or $G_2(p) \neq 0$. Assume that $G_1(p)$ does not vanish. Note that the first type of polynomials in Lemma 3.3.2 are of the form $\mathfrak{S}^{(i)}_{a,1}G_1$ for $a \in [2]^{c_i}$. Since $G_1(p) \neq 0$, we deduce that $\mathfrak{S}^{(i)}_{a,1}$ vanishes at p for $a \in [2]^{c_i}$. Therefore, p is contained in Z_2. Similarly, if $G_2(p) \neq 0$, then $p \in Z_3$. We conclude that $Z = Z_1 \cup Z_2 \cup Z_3$. □

Lemma 3.3.3 allows us to prove Conjecture 3.1.29.

Theorem 3.3.4. *Conjecture 3.1.29 holds for any undirected graphical model.*

Proof. Let $\mathcal{V}_{X,\mathcal{C}}$ be the Spohn CI variety of a generic game X and let $\widetilde{\mathcal{V}}_{X,\mathcal{C}}$ be the preimage of $\mathcal{V}_{X,\mathcal{C}}$ through the monomial map ϕ in (3.33). By Lemma 3.3.1, $\widetilde{\mathcal{V}}_{X,\mathcal{C}}$ is contained in Y_X. Let B_X be the intersection of Y_X and the union of the base loci of $\Lambda_1, \ldots, \Lambda_n$, and let $\widetilde{Y}_X$ be the Zariski closure of $Y_X \setminus B_X$ in $\mathbb{T}$. Recall that the base locus of Λ_i is either empty if i is an isolated vertex, or it is given by Lemma 3.3.3. By Bertini's Theorem (see [67, Theorem 8.18]), we get that $Y_X \setminus B_X$ and $\widetilde{Y}_X$ have codimension n in $\mathbb{T}$ for a generic game X. Now, note that for $i \in [n]$, the image of the base locus of Λ_i via ϕ is contained in the union of the hyperplanes $\{p_{j_1 j_2 \cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$. This implies that $\widetilde{\mathcal{V}}_{X,\mathcal{C}}$ is contained in $\widetilde{Y}_X$, and we deduce that

$$n = \mathrm{codim}_{\mathbb{T}} \tilde{Y} \leq \mathrm{codim}_{\mathbb{T}} \widetilde{\mathcal{V}}_{X,\mathcal{C}}.$$

Now, the proof follows from the fact that $\mathrm{codim}_{\mathbb{T}} \widetilde{\mathcal{V}}_{X,\mathcal{C}} \leq \mathrm{codim}_{\mathcal{M}_{\mathcal{C}}} \mathcal{V}_{X,\mathcal{C}}$ and that $\mathrm{codim}_{\mathcal{M}_{\mathcal{C}}} \mathcal{V}_{X,\mathcal{C}} \leq n$.

□

We deduce the following result as a consequence of Proposition 3.1.12 and Theorem 3.3.4.

Corollary 3.3.5. *Let $G = ([n], E)$ be an undirected graphical model and $\mathcal{C} = \mathrm{global}(G)$. Then, for generic payoff tables, the dimension of the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ is the number of positive dimensional faces of the associated simplicial complex of the cliques. In other words, for generic payoff tables, the dimension of $\mathcal{V}_{X,\mathcal{C}}$ is the number of cliques of G with at least two vertices.*

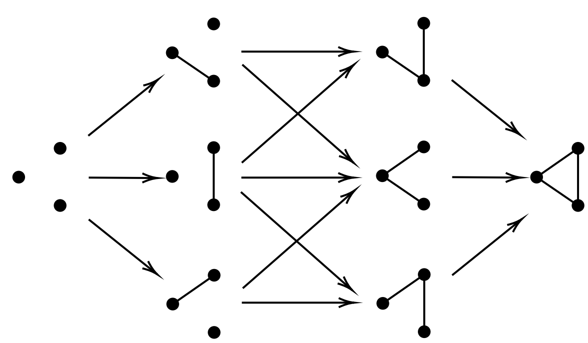


Figure 3.5: The poset structure of the set of subgraphs of the complete graph on 3 vertices with respect to the inclusion.

Example 3.3.6. *For generic payoff tables, if $G = ([n], E)$ is a line graph or a cycle (see Figure 3.1), the dimension of the Spohn CI variety is determined by counting the number of edges. This is because the clique complex consists exclusively of one-dimensional simplices. On the other hand in the case of the decomposable graph from Figure 3.2, the Spohn CI variety is $31 - 7 = 24$ dimensional.*

Next, we explore how Spohn CI varieties of different undirected graphs are related. Let

$$G = ([n], E(G)) \text{ and } G' = ([n], E(G'))$$

be two undirected graphs. We say that G is a subgraph of G', denoted by $G \subseteq G'$, if $E(G) \subseteq E(G')$.

Lemma 3.3.7. *Let $G \subseteq G'$ and let $\mathcal{V}_{X,\mathcal{C}}$ and $\mathcal{V}'_{X,\mathcal{C}}$ be the Spohn CI variety of G and G' respectively. Then, $\mathcal{V}_{X,\mathcal{C}}$ is a subvariety of $\mathcal{V}'_{X,\mathcal{C}}$. The analogous inclusion holds for the corresponding sets of totally mixed CI equilibria.*

Proof. Let $G \subseteq G'$, then, we have that global(G') is contained in global(G). This implies that $I_{\text{global}(G')} \subseteq I_{\text{global}(G)}$, and hence, $\mathcal{M}_{\text{global}(G)}$ is a subvariety of $\mathcal{M}_{\text{global}(G')}$. In particular, we deduce that the Spohn CI variety, corresponding to G, is contained in the Spohn CI variety corresponding to G'. □

Let G be the complete graph with n vertices. The inclusion of graphs gives to the set of subgraphs of G, with n vertices, a structure of poset. By Lemma 3.3.7, we get a poset structure on the set of Spohn CI varieties (similarly with totally mixed CI equilibria). In this poset the initial and terminal objects are the set of totally mixed Nash equilibria and the set of totally mixed dependency equilibria. In other words, the set of totally mixed CI equilibria always contains the set of totally mixed Nash equilibria and it is always contained in the set of totally mixed dependency equilibria.

Example 3.3.8. *For $n = 3$ we have 8 subgraphs of the complete graph on 3 vertices: one with no edges, 3 with one edge, 3 with two edges, and the complete graph. The poset structure of the set, formed by these 8 graphs, is shown in Figure 3.5. In particular, we get a similar picture for the corresponding independence varieties and Spohn CI varieties. A* `Macaulay2` *computation shows that the dimension of the independence varieties for a graph with 0, 1, 2 or 3 edges is 3, 4, 5 and 7 respectively. Therefore, by Theorem 3.3.4, the dimension of the corresponding Spohn CI varieties are 0, 1, 2, and 4 respectively. This shows that, in the poset of Spohn CI varieties, there might be dimensional gaps.*

3.4 Nash CI varieties

The goal of this section is to analyze the algebro-geometric properties of the Spohn CI variety of undirected graphical models whose connected components are all cliques. An example of such graphs is one edge undirected graphs, where the connected components are isolated vertex (which are complete) or the complete graph of two vertices. In this section, we generalize some of the results of Section 3.2 to this type of Spohn CI varieties.

Let $(s_1, \ldots, s_k)$ be a partition of the set $[n]$ with $\emptyset \neq s_i \subseteq [n]$ and $|s_i| = n_i$. In other words, $[n]$ is the disjoint union of $s_1, \ldots, s_k$ and $n_1 + \cdots + n_k = n$. Given such a partition, we consider the complete graphs

$$G_1, \ldots, G_k$$

on the set of vertices $s_1, \ldots, s_k$, respectively. Note that up to the labeling of the vertices, the integers $n_1, \ldots, n_k$ carry all the information of the partition. Thus, throughout the section, we denote the partition as $\mathbf{n} := (n_1, \ldots, n_k)$ where $1 \leq n_1 \leq \cdots \leq n_k \leq n$. From a game theoretic point of view, this modeling can be seen as players forming k groups, where each group's members act dependently within the group but independently from all other players. We define the graph

$$G_{\mathbf{n}} := G_1 \sqcup \cdots \sqcup G_k$$

and denote the discrete conditional independence variety of $G_{\mathbf{n}}$ by $\mathcal{M}_{\mathbf{n}}$. We first compute the independence variety $\mathcal{M}_{\mathbf{n}}$. In the following proposition, we see how connected components of any undirected graphical model $G = ([n], E)$ get translated to products in the (not necessarily the positive part) discrete conditional independence variety.

Proposition 3.4.1. *Let $G = ([n], E)$ be an undirected graphical model with k connected components G_i. Then*

$$\mathcal{M}_{\mathrm{global}(G)} = \mathcal{M}_{\mathrm{global}(G_1)} \times \cdots \times \mathcal{M}_{\mathrm{global}(G_k)}.$$

Proof. For simplicity, we will assume that G has two connected components, $G_1 = ([n_1], E_1)$ and $G_2 = ([n_2], E_2)$. Then, we have that $[n_1] \perp\!\!\!\perp [n_2] \,|\, \emptyset \in \mathrm{global}(G)$. By Proposition 3.1.1, the corresponding independence ideal equals the ideal defining the Segre variety $\mathbb{P}^{d_{i_1} \cdots \, d_{i_{n_1}} - 1} \times \mathbb{P}^{d_{j_1} \cdots \, d_{j_{n_2}} - 1}$. In particular,

$$\mathcal{M}_{\mathrm{global}(G)} \subseteq \mathbb{P}^{d_{i_1} \cdots \, d_{i_{n_1}} - 1} \times \mathbb{P}^{d_{j_1} \cdots \, d_{j_{n_2}} - 1}.$$

Consider the following parametrization of the Segre variety

$$p_{i_1 \cdots i_{n_1} j_1 \cdots j_{n_2}} = \sigma^{(1)}_{i_1 \cdots i_{n_1}} \sigma^{(2)}_{j_1 \cdots j_{n_2}}. \tag{3.42}$$

If $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C \in \mathrm{global}(G_1)$, then $\mathcal{X}_A \perp\!\!\!\perp \mathcal{X}_B \mid \mathcal{X}_C \sqcup [n_2] \in \mathrm{global}(G)$. In particular, evaluating the Segre parametrization (3.42) in the independence ideal of the latter CI statement we deduce that

$$\mathcal{M}_{\mathrm{global}(G)} \subseteq \mathcal{M}_{\mathrm{global}(G_1)} \times \mathbb{P}^{d_{j_1} \cdots \, d_{j_{n_2}} - 1}.$$

Similarly, we get that

$$\mathcal{M}_{\mathrm{global}(G)} \subseteq \mathbb{P}^{d_{i_1} \cdots \, d_{i_{n_1}} - 1} \times \mathcal{M}_{\mathrm{global}(G_2)}$$

and, hence,

$$\mathcal{M}_{\mathrm{global}(G)} \subseteq \mathcal{M}_{\mathrm{global}(G_1)} \times \mathcal{M}_{\mathrm{global}(G_2)}.$$

On the other hand, every CI statement in $\mathrm{global}(G)$ is of the form

$$\mathcal{X}_{A_1 \sqcup B_1} \perp\!\!\!\perp \mathcal{X}_{A_2 \sqcup B_2} | \mathcal{X}_{A_3 \sqcup B_3},$$

where $\mathcal{X}_{A_1} \perp\!\!\!\perp \mathcal{X}_{A_2}|\mathcal{X}_{A_3}$ and $\mathcal{X}_{B_1} \perp\!\!\!\perp \mathcal{X}_{B_2}|\mathcal{X}_{B_3}$ are in global(G_1) and global(G_2), respectively. By axioms $C1$ and $C2$ and the definition of separation on undirected graphical models [85, page 29], it is enough to consider CI statements where

$$\bigsqcup_{i\in[3]} A_i \sqcup \bigsqcup_{j\in[3]} B_i = [n].$$

By Proposition 3.1.1, the quadrics generating the corresponding independence ideal of $\mathcal{X}_{A_1\sqcup B_1} \perp\!\!\!\perp \mathcal{X}_{A_2\sqcup B_2}|\mathcal{X}_{A_3\sqcup B_3}$ are of the form

$$p_{abc\,\alpha\beta\gamma}\cdot p_{a'b'c\,\alpha'\beta'\gamma} - p_{a'bc\,\alpha'\beta\gamma}\cdot p_{ab'c\,\alpha\beta'\gamma} \tag{3.43}$$

for $a, a' \in \mathcal{R}_{A_1}$, $b, b' \in \mathcal{R}_{A_2}$, $c \in \mathcal{R}_{A_3}$, $\alpha, \alpha' \in \mathcal{R}_{B_1}$, $\beta, \beta' \in \mathcal{R}_{B_2}$, and $\gamma \in \mathcal{R}_{B_3}$. We claim that such quadric lies inside the ideal of $\mathcal{M}_{\mathcal{C}_1} \times \mathcal{M}_{\mathcal{C}_2}$. Indeed, the quadrics

$$q_1 = \sigma^{(1)}_{abc}\sigma^{(1)}_{a'b'c} - \sigma^{(1)}_{a'bc}\sigma^{(1)}_{ab'c},\ \ q_2 = \sigma^{(2)}_{\alpha\beta\gamma}\sigma^{(2)}_{\alpha'\beta'\gamma} - \sigma^{(2)}_{\alpha'\beta\gamma}\sigma^{(2)}_{\alpha\beta'\gamma}.$$

lie in the ideal of $\mathcal{M}_{\mathcal{C}_1} \times \mathcal{M}_{\mathcal{C}_2}$. Then, the expression

$$\sigma^{(2)}_{\alpha\beta\gamma}\sigma^{(2)}_{\alpha'\beta'\gamma}q_1 + \sigma^{(1)}_{a'bc}\sigma^{(1)}_{ab'c}q_2$$

coincides with the evaluation of the Segre parametrization (3.42) in the quadric (3.43). Hence, we conclude that $\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}_1} \times \mathcal{M}_{\mathcal{C}_2}$. □

Proposition 3.4.1 allows to compute the discrete conditional independence variety $\mathcal{M}_{\mathbf{n}}$ of the graphical model $G_{\mathbf{n}}$ with binary choices for a partition $\mathbf{n}$ of the set $[n]$. Alternatively, since $G_{\mathbf{n}}$ is decomposable, by Proposition 3.1.9 and [54, Theorem 4.2] one can conclude the following corollary.

Corollary 3.4.2. *Let $\mathbf{n} = (n_1, \dots, n_k)$ be a partition of the set $[n]$, and let $G_{\mathbf{n}}$ be a binary undirected graphical model whose connected components are $G_1, \dots, G_k$, where each G_i is a complete graph on n_i vertices. Then,*

$$\mathcal{M}_{\mathbf{n}} = \mathbb{P}^{2^{n_1}-1} \times \cdots \times \mathbb{P}^{2^{n_k}-1}$$

is the Segre variety.

Example 3.4.3. *Consider the partition $(\{1,2\},\{3,4\})$ of the set $[4]$, i.e. $\mathbf{n} = (2,2)$. The corresponding graph G_4 is the graph appearing in Figure 3.7 which is the disjoint union of two cliques on 2 vertices. The corresponding independence variety is $\mathcal{M}_{\mathbf{n}} = \mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15}$.*

Definition 3.4.4. *We define the* **n–Nash conditional independence (CI) variety***, denoted by $N_{X,\mathbf{n}}$, as the Spohn CI variety of $G_{\mathbf{n}}$. We define the set of* **totally mixed n–Nash conditional independence (CI) equilibria** *as the intersection of*

$$N_{X,\mathbf{n}} \cap (\Delta_1 \times \cdots \times \Delta_k)\,,$$

where Δ_i is the open simplex of the corresponding factor of the Segre variety $\mathcal{M}_{\mathbf{n}}$.

In other words, the set of totally mixed $\mathbf{n}$–Nash CI equilibria is the set of totally mixed CI equilibria for the independence variety $\mathcal{M}_{\mathbf{n}}$.

Example 3.4.5. *Certain totally mixed* $\mathbf{n}$*-Nash CI equilibria have already appeared in this chapter.*

- *For* $\mathbf{n} = (1, \ldots, 1)$*, the ideal of the variety* $N_{X,\mathbf{n}}$ *is generated by the polynomials* (3.11)*. By Theorem 3.1.18, the intersection of* $N_{X,\mathbf{n}}$ *with the open simplex is the set of totally mixed Nash equilibria. Hence, the set of totally mixed Nash equilibria and the set of totally mixed Nash CI equilibria coincide.*
- *For* $\mathbf{n} = (1, \ldots, 1, 2)$*,* $N_{X,\mathbf{n}}$ *is the Nash CI curve.*
- *For* $\mathbf{n} = (n)$*,* $N_{X,\mathbf{n}}$ *is the Spohn variety and the set of totally mixed Nash CI equilibria is the set of totally mixed dependency equilibria.*

As a consequence of Theorem 3.3.4 and Corollary 3.4.2, we deduce the following result.

Proposition 3.4.6. *Let* $\mathbf{n} = (n_1, \ldots, n_k)$ *a partition of* $[n]$*. Then, for generic payoff tables, the dimension of* $N_{X,\mathbf{n}}$ *is*

$$\dim N_{X,\mathbf{n}} = \dim \mathcal{M}_{\mathbf{n}} - n = 2^{n_1} + \cdots + 2^{n_k} - k - n.$$

Note that Theorem 3.2.9 agrees with Proposition 3.4.6 for the case of Nash CI curves.

3.4.1 Equations and Properties of Nash CI varieties

We now compute the equations of a generic Nash CI variety. From this computation, we will deduce that generic Nash CI varieties are complete intersections in Segre varieties. This allows to determine some properties Nash CI varieties such as their degree. We follow the same strategy as in Section 3.2.1. We evaluate the equations of the Spohn variety at the parametrization of the Segre variety and we remove the factors that lead to components in the hyperplanes we are saturating by. We carried out this computation in Section 3.3 in a more general setting. There, we computed the polynomials $F_1, \ldots, F_n$ and we considered the variety Y_X defined by them. The restriction of these polynomials to our particular case provides the equations of N_X.

Given a partition $\mathbf{n} = (n_1, \ldots, n_k)$ of $[n]$, we label the n players of the game by

$$(1, 1), \ldots, (1, n_1), \ldots, (k, 1), \ldots, (k, n_k).$$

We denote the payoff tables of the game by

$$X^{(1,1)}, \ldots, X^{(1,n_1)}, \ldots, X^{(k,1)}, \ldots, X^{(k,n_k)}.$$

By Proposition 3.1.9, the parametrization of $\mathcal{M}_{\mathbf{n}}$ is given by the Segre embedding

$$p_{j_{11}\cdots j_{1n_1}\cdots j_{k1}\cdots j_{kn_k}} := \sigma^{(1)}_{j_{11}\cdots j_{1n_1}} \cdots \sigma^{(k)}_{j_{k1}\cdots j_{kn_k}} \tag{3.44}$$

where $j_{il_i} \in [2]$ for $i \in [k]$ and $l_i \in [n_i]$. Here $\sigma^{(i)}_{j_{i1}\cdots j_{in_i}}$ are the coordinates of the i–th factor of $\mathcal{M}_{\mathbf{n}}$. Now, let (i, l_i) be a player such that $n_i \geq 2$. In other words, the corresponding vertex is not isolated. Then, the polynomial $F_{(i,l_i)}$ computed in Section 3.3 is the determinant

$$\det\begin{pmatrix} \sum\limits_{j_{i1}\cdots\widehat{j_{il_i}}\cdots j_{in_i}} \sigma^{(i)}_{j_{i1}\cdots 1\cdots j_{in_i}} & \sum\limits_{j_{11}\cdots\widehat{j_{il_i}}\cdots j_{kn_k}} X^{(i,l_i)}_{j_{11}\cdots 1\cdots j_{kn_k}} \sigma^{(1)}_{j_{11}\cdots j_{1n_1}} \cdots \sigma^{(i)}_{j_{i1}\cdots 1\cdots j_{in_i}} \cdots \sigma^{(k)}_{j_{k1}\cdots j_{kn_k}} \\ \sum\limits_{j_{i1}\cdots\widehat{j_{il_i}}\cdots j_{in_i}} \sigma^{(i)}_{j_{i1}\cdots 2\cdots j_{in_i}} & \sum\limits_{j_{11}\cdots\widehat{j_{il_i}}\cdots j_{kn_k}} X^{(i,l_i)}_{j_{11}\cdots 2\cdots j_{kn_k}} \sigma^{(1)}_{j_{11}\cdots j_{1n_1}} \cdots \sigma^{(i)}_{j_{i1}\cdots 2\cdots j_{in_i}} \cdots \sigma^{(k)}_{j_{k1}\cdots j_{kn_k}} \end{pmatrix}. \tag{3.45}$$

Now, assume that $n_i = 1$, i.e. the player $(i, l_i) = (i, 1)$ corresponds to an isolated vertex of the graph. In this case, we have that

$$F_{(i,1)} = \sum_{j_{11}\cdots\widehat{j_{i1}}\cdots j_{kn_k}} \left(X^{(i,1)}_{j_{11}\cdots 2\cdots j_{kn_k}} - X^{(i,1)}_{j_{11}\cdots 1\cdots j_{kn_k}}\right) \sigma^{(1)}_{j_{11}\cdots j_{1n_1}} \cdots \widehat{\sigma^{(i)}_{j_{i1}}} \cdots \sigma^{(k)}_{j_{k1}\cdots j_{kn_k}}. \tag{3.46}$$

As in Lemma 3.3.2, we get that

$$N_{X,\mathbf{n}} \subseteq \mathbb{V}(F_{(1,1)}, \ldots, F_{(k,n_k)}) \subseteq \mathcal{M}_{\mathbf{n}}. \tag{3.47}$$

where $F_{(i,l_i)}$ is as in (3.45) if $n_i \geq 2$ or as in (3.46) if $n_i = 1$. Note that for $\mathbf{n} = (1, \ldots, 1, 2)$, the polynomials $F_{(1,1)}, \ldots, F_{(k,n_k)}$ coincide with the polynomials defining the variety C_X in Definition 3.2.3. We want to show that for generic payoff tables we get an equality in (3.47). To do so, we analyse the linear systems of $\mathcal{M}_{\mathbf{n}}$ defined by these polynomials.

Let $D_{(i,j_i)}$ be the divisor in $\mathcal{M}_{\mathbf{n}}$ defined by the polynomial $\mathbb{V}(F_{(i,j_i)})$. The divisor $D_{(i,j_i)}$ lies in the linear system defined by the line bundle

$$\mathcal{O}(1, \ldots, 1, \underset{(i)}{(1 - \delta_{1,n_i})2}, 1, \ldots, 1)) \tag{3.48}$$

of $\mathcal{M}_{\mathbf{n}}$, where $\delta_{i,j}$ is 1 if $i = j$, and 0 if $i \neq j$. In other words,

$$\mathcal{O}(1, \ldots, 1, \underset{(i)}{(1 - \delta_{1,n_i})2}, 1, \ldots, 1)) = \begin{cases} \mathcal{O}(1, \ldots, 1, \underset{(i)}{0}, 1, \ldots, 1)) & \text{if } n_i = 1, \\ \mathcal{O}(1, \ldots, 1, \underset{(i)}{2}, 1, \ldots, 1)) & \text{if } n_i > 1. \end{cases}$$

Now, we consider the map that sends a payoff table $X^{(i,j_i)}$ to the divisor $D_{(i,j_i)}$. More precisely, for (i, j_i), we consider the map

$$\begin{array}{rccl} \phi_{(i,j_i)}: & \mathbb{R}^{2^n} & \longrightarrow & H^0(\mathcal{M}_{\mathbf{n}}, \mathcal{O}(1, \ldots, 1, \underset{(i)}{(1 - \delta_{1,n_1})2}, 1, \ldots, 1)) \\ & X^{(i,j_i)} & \longmapsto & F_{(i,j_i)} \end{array}. \tag{3.49}$$

We denote the image of ϕ_{i,j_i} by $\Lambda_{(i,j_i)}$. In Section 3.3 we studied these linear systems. In particular, in Lemma 3.3.2 the generators of $\Lambda_{(i,j_i)}$ were computed. The next result is the translation of Lemma 3.3.2 to the setting of Nash CI varieties.

Lemma 3.4.7. *For $n_i = 1$, the linear system $\Lambda_{(i,1)}$ is complete. For $n_i \geq 2$, we have that*

$$\Lambda_{(i,l_i)} \simeq W_{(i,l_i)} \otimes \bigotimes_{j\neq i} H^0(\mathbb{P}^{2^{n_j}-1}, \mathcal{O}(1)),$$

where $W_{(i,l_i)}$ is the linear system of $\mathbb{P}^{2^{n_i}-1}$ generated by the polynomials

1. *for* $(j_1,\dots,\widehat{j_{l_i}},\dots,j_{n_i}) \in [2]^{n_i-1}$, $\sigma^{(i)}_{j_1\cdots 1\cdots j_{n_i}}\left(\sum_{m_1,\dots,\widehat{m_{l_i}},\dots,m_{n_i}} \sigma^{(i)}_{m_1\cdots 2\cdots m_{n_i}}\right)$,
2. *for* $(m_1,\dots,\widehat{m_{l_i}},\dots,m_{n_i}) \in [2]^{n_i-1}$, $\sigma^{(i)}_{m_1\cdots 2\cdots m_{n_i}}\left(\sum_{j_1,\dots,\widehat{j_{l_i}},\dots,j_{n_i}} \sigma^{(i)}_{j_1\cdots 1\cdots j_{n_i}}\right)$,
3. $\sigma^{(i)}_{1\cdots 1\cdots 1}\sigma^{(i)}_{1\cdots 2\cdots 1} - \sum_{\substack{(j_1\cdots\widehat{j_{l_i}}\cdots j_{n_i})\neq(1,\dots,1)\\(m_1\cdots\widehat{m_{l_i}}\cdots m_{n_i})\neq(1,\dots,1)}} \sigma^{(i)}_{j_1\cdots 1\cdots j_{n_i}}\sigma^{(i)}_{m_1\cdots 2\cdots m_{n_i}}$.

Using Lemma 3.4.7, we derive the following result

Proposition 3.4.8. *For generic payoff tables,*

$$N_{X,\mathbf{n}} = \mathbb{V}(F_{(1,1)},\cdots,F_{(k,n_k)}).$$

In particular, for generic payoff tables, $N_{X,\mathbf{n}}$ is a complete intersection in $\mathcal{M}_{\mathbf{n}}$.

Proof. We consider the variety

$$\mathcal{X} := \{(X,p) \in V^n \times \mathbb{P}^{2^n-1} : p \in \mathbb{V}(F_{(1,1)},\cdots,F_{(k,n_k)})\}$$

together with the projection $\pi : \mathcal{X} \to V^n$. Here, we identify $X \in V^n$ with the game $X = (X^{(1,1)},\dots,X^{(k,n_k)})$. We denote the fiber of X via π by $\mathcal{X}_X$. Note that π is surjective and, for any X, $\dim \mathcal{X}_X \geq \dim \mathcal{M}_{\mathbf{n}} - n$. Let H be a hyperplane of $\mathbb{P}^{2^n-1}$ of the form $\{p_{j_1j_2\cdots j_n} = 0\}$ or $\{p_{++\cdots+} = 0\}$. We consider the variety

$$\Sigma_H := \overline{\mathcal{X} \setminus (V^n \times H)}.$$

For $X \in V^n$, we denote the intersection of $\mathcal{X}_X$ with Σ_H by $\Sigma_{H,X}$. Note that for $X \in V^n$, $\Sigma_{H,X}$ contains the Nash CI variety $N_{X,\mathbf{n}}$. By Theorem 3.3.4, the restriction of π to Σ_H is dominant. We want to show that $\mathcal{X}_X$ equals $N_{X,\mathbf{n}}$ for generic $X \in V^n$. This is equivalent to show that for any hyperplane H of the form $\{p_{j_1j_2\cdots j_n} = 0\}$ or $\{p_{++\cdots+} = 0\}$, we have that $\Sigma_{H,X} = \mathcal{X}_X$ for generic $X \in V^n$.

If $\mathcal{X}$ has no irreducible component in $V^n \times H$, then Σ_H is dense in $\mathcal{X}$. Hence, $\Sigma_{H,X} = \mathcal{X}_X$ for generic X. Assume now that $\mathcal{X}$ has an irreducible component contained in $V^n \times H$. Let $\mathcal{X}_1$ be the union of these irreducible components. If the restriction of π to $\mathcal{X}_1$ is not dominant, then $\Sigma_{H,X} = \mathcal{X}_X$ for generic X. Assume that $\pi|_{\mathcal{X}_1}$ is dominant. Since $\pi|_{\mathcal{X}_1}$ is closed, it is surjective. Assume that there exists $X \in V^n$ such that $\mathcal{X}_X$ has dimension $\dim \mathcal{M}_{\mathbf{n}} - n$ and $\mathcal{X}_X$ has no irreducible component contained in H. Then, the intersection of $\mathcal{X}_X$ and $\mathcal{X}_1$ has dimension at most $\dim \mathcal{M}_{\mathbf{n}} - n - 1$. Using that the dimension of the fibers of $\pi|_{\mathcal{X}_1}$ is upper semicontinuous, we get that the generic fiber of $\pi|_{\mathcal{X}_1}$ has dimension at most $\dim \mathcal{M}_{\mathbf{n}} - n - 1$. This is a contradiction since the dimension of the fibers of $\pi|_{\mathcal{X}_1}$ is at least $\dim \mathcal{M}_{\mathbf{n}} - n$.

Therefore, it is enough to show that there exists $X \in V^n$ such that $\mathcal{X}_X$ has dimension $\dim \mathcal{M}_{\mathbf{n}} - n$ and $\mathcal{X}_X$ has no irreducible component contained in H. Assume that H is defined by $p_{m_{(1,1)}\cdots m_{(k,n_k)}} = 0$ for fixed $m_{(1,1)}, \ldots, m_{(k,n_k)} \in [2]$. By Lemma 3.4.7, we can choose X such that

$$F_{(i,l_i)} = q_{(i,l_i)} \prod_{j \neq i}^{k} l_j^{(i,l_i)},$$

where $q_{(i,l_i)}$ is an element in the linear system $W_{(i,l_i)}$ and $l_j^{(i,l_i)}$ is a generic element of the complete linear system $H^0(\mathbb{P}^{2^{n_j}-1}, \mathcal{O}_{\mathbb{P}^{2^{n_j}-1}}(1))$. Then, the irreducible components of $\mathcal{X}_X$ are defined by n polynomials of the form $q_{(i,l_i)}$ or $l_j^{(i,l_i)}$. Since the linear forms $l_j^{(i,l_i)}$ are generic in a complete linear system, by Bertini's Theorem (see [67, Theorem 8.18]), it is enough to check that the intersection of any number of quadrics of the form $q_{(i,l_i)}$ has the expected dimension and none of its irreducible components is contained in H. This can be checked on each of the factors of the Segre variety $\mathcal{M}_{\mathbf{n}}$. In other words, given a player i, we need to check that there exist $q_{(i,1)} \in W_{(i,1)}, \ldots, q_{(i,n_i)} \in W_{(i,n_i)}$ such that for any subset $S \subseteq [n_i]$, the variety

$$\mathbb{V}(q_{(i,l)} : l \in S) \subset \mathbb{P}^{2^{n_i}-1}$$

is a complete intersection and it has no irreducible component in the hyperplane $H_i := \{\sigma^{(i)}_{m_{(i,1)}\cdots m_{(i,n_i)}} = 0\}$. For simplicity, we will assume that $S = [n_i]$. The same arguments can be apply to any subset of $[n_i]$

For a player (i,l), we fix the index

$$\widetilde{m}_{(i,l)} = \begin{cases} 1 & \text{if } m_{(i,l)} = 2 \\ 2 & \text{if } m_{(i,l)} = 1 \end{cases}.$$

By Lemma 3.4.7, we can set the quadric $q_{(i,l)}$ to be the product

$$q_{(i,l)} = \sigma^{(i)}_{\widetilde{m}_{(i,1)}\cdots \underset{(l)}{m_{(i,l)}} \cdots \widetilde{m}_{(i,n_i)}} \left(\sum_{a_1,\ldots,\widehat{a_l},\ldots,a_{n_i}} \sigma^{(i)}_{a_1 \cdots \underset{(l)}{\widetilde{m}_{(i,l)}} \cdots a_{n_i}} \right) \tag{3.50}$$

We denote the linear forms in (3.50) by $\mathfrak{S}_{(i,l)}$ and $g_{(i,l)}$ respectively. Up to labelling of the players, the irreducible components of $\mathbb{V}(q_{(i,1)}, \dots, q_{(i,n_i)})$ are linear subspaces of the form $\mathbb{V}(\mathfrak{S}_{(i,1)}, \dots, \mathfrak{S}_{(i,j)}) \cap \mathbb{V}(g_{(i,j+1)}, \dots, g_{(i,n_i)})$ for $j \leq n_i$. First of all, note that $\mathbb{V}(\mathfrak{S}_{(i,1)}, \dots, \mathfrak{S}_{(i,j)})$ has the expected dimension since its the zero locus of j distinct monomials. Now, for $l > j$, the monomial $\sigma^{(i)}_{m_{(i,1)} \cdots \widetilde{m}(i,l) \cdots m_{(i,n_i)}}$ appears in $g_{(i,l)}$ and it does not appear in any of the other linear forms $g_{(i,j+1)}, \dots, g_{(i,n_i)}$. This implies that $\mathbb{V}(g_{(i,j+1)}, \dots, g_{(i,n_i)})$ has also the expected dimension. Moreover, the monomial $\sigma^{(i)}_{m_{(i,1)} \cdots \widetilde{m}(i,l) \cdots m_{(i,n_i)}}$ does not appear neither in the linear forms $\mathfrak{S}_{(i,1)}, \dots, \mathfrak{S}_{(i,j)}$. Thus, the intersection

$$\mathbb{V}(\mathfrak{S}_{(i,1)}, \dots, \mathfrak{S}_{(i,j)}) \cap \mathbb{V}(g_{(i,j+1)}, \dots, g_{(i,n_i)})$$

has the expected dimension. It remains to show that this intersection is not contained in $H_i = \{\sigma^{(i)}_{m_{(i,1)} \cdots m_{(i,n_i)}} = 0\}$. This follows from the fact that the variable $\sigma^{(i)}_{m_{(i,1)} \cdots m_{(i,n_i)}}$ does not appear in the linear forms $\mathfrak{S}_{(i,1)}, \dots, \mathfrak{S}_{(i,j)}, g_{(i,j+1)}, \dots, g_{(i,n_i)}$. We conclude that for $q_{(i,l)}$ as in (3.50) and for generic linear forms $l_j^{(i,l)}$, $\dim \mathcal{X}_X = \dim \mathcal{M}_{\mathbf{n}} - n$ and $\mathcal{X}_X$ has no irreducible components contained in the hyperplane H. Therefore, for generic $X \in V^n$, we have that $\mathcal{X}_X = \Sigma_{H,X}$.

A similar argument shows that the same holds for hyperplanes of the form $\{p_{++\cdots+} = 0\}$. Since there are only a finite number of hyperplanes of this form, we deduce that for generic $X \in V^n$, $\mathcal{X}_X$ has no irreducible component included in these hyperplanes. We conclude that $\mathcal{X}_X$ equals the Spohn CI variety $\mathcal{V}_{X,\mathcal{C}}$ for generic payoff tables. □

By Proposition 3.4.8, $N_{X,\mathbf{n}}$ is the complete intersection of the divisors $D_{(1,1)}, \dots, D_{(k,n_k)}$. Recall that $D_{(i,l)}$ is the divisor defined by $F_{(i,l)}$ and it lies in the linear system given by the line bundle (3.48). Now we compute the degree of generic Nash CI varieties.

Proposition 3.4.9. *For generic payoff tables, the degree of $N_{X,\mathbf{n}}$ is the coefficient of the monomial*

$$x_1^{2^{n_1}-1} \cdots x_k^{2^{n_k}-1}$$

in the polynomial

$$\left(\sum_{i=1}^{k} x_i\right)^{\sum_i 2^{n_i} - n - k} \prod_{\beta=1}^{k} \left(\sum_{i=0}^{k} x_i + (-1)^{\delta_{1,n_\beta}} x_\beta\right)^{n_\beta}, \tag{3.51}$$

where

$$\delta_{i,\beta} = \begin{cases} 0 & \text{if } i \neq \beta \\ 1 & \text{if } i = \beta \end{cases}.$$

Proof. We compute the degree of $N_{X,\mathbf{n}}$ using the Chow ring of $\mathcal{M}_{\mathbf{n}}$, which is given by

$$\mathcal{A}_\bullet(\mathcal{M}_{\mathbf{n}}) \simeq \mathbb{Z}[x_1, \dots, x_k]/\langle x_1^{2^{n_1}}, \dots, x_k^{2^{n_k}}\rangle. \tag{3.52}$$

The line bundle of the divisor $D_{(\beta,j_\beta)}$ is

$$\mathcal{O}_{\mathcal{M}_{\mathbf{n}}}(D_{(\beta,j_\beta)}) = \mathcal{O}(1,\ldots,1,\underset{(i)}{(1-\delta_{1,n_\beta})2},1,\ldots,1)).$$

Thus, we get that the class of $D_{(\beta,j_\beta)}$ in $\mathcal{A}_\bullet(\mathcal{M}_{\mathbf{n}})$ is given by

$$\left[D_{(\beta,j_\beta)}\right] = \sum_{i=0}^{k} x_i + (-1)^{\delta_{1,n_\beta}} x_\beta.$$

Let H be a generic hyperplane. We deduce that

$$[D_{(1,1)}]\cdots[D_{(k,n_n)}][H\cap\mathcal{M}_{\mathbf{n}}]$$

is equal to the class of the polynomial (3.51) in $\mathcal{A}_\bullet(\mathcal{M}_{\mathbf{n}})$. Thus, we conclude that the degree of $N_{X,\mathbf{n}}$ is the coefficient of the monomial $x_1^{2^{n_1}-1}\cdots x_k^{2^{n_k}-1}$ of the polynomial (3.51). We refer to [44, Chapters 1 and 2] for more details on this computation. $\square$

Note that Proposition 3.4.9 for Nash CI curves coincides with Proposition 3.2.12. Now we analyze the connectedness of generic Nash CI varieties.

Proposition 3.4.10. *Let $\mathbf{n} \neq (1,\ldots,1)$. Then, for generic payoff tables, $N_{X,\mathbf{n}}$ is connected.*

Proof. It follows similarly as in the proof of Lemma 3.2.15. By Proposition 3.4.8, we get the following exact sequence (Koszul complex):

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{\phi_n} \mathcal{F}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{O}_{\mathcal{M}_{\mathbf{n}}} \xrightarrow{\phi_0} \mathcal{O}_{N_{X,\mathbf{n}}} \longrightarrow 0, \tag{3.53}$$

where

$$\mathcal{F}_l := \bigoplus_{(i_1,j_1)<\cdots<(i_l,j_l)}^{n} \mathcal{O}_{\mathcal{M}_{\mathbf{n}}}\left(-\sum_{l=1}^{k} D_{(i_l,j_l)}\right).$$

Let K_i be the kernel of ϕ_i. Then, the above exact sequence splits into the following n short exact sequences:

$$\begin{array}{ll}
(E_0): & 0 \longrightarrow K_0 \longrightarrow \mathcal{O}_{\mathcal{M}_{\mathbf{n}}} \longrightarrow \mathcal{O}_{N_{X,\mathbf{n}}} \longrightarrow 0 \\
(E_1): & 0 \longrightarrow K_1 \longrightarrow \mathcal{F}_1 \longrightarrow K_0 \longrightarrow 0 \\
\vdots & \qquad\qquad\quad \vdots \\
(E_i): & 0 \longrightarrow K_i \longrightarrow \mathcal{F}_i \longrightarrow K_{i-1} \longrightarrow 0 \\
\vdots & \qquad\qquad\quad \vdots \\
(E_{n-1}): & 0 \longrightarrow K_{n-1} \longrightarrow \mathcal{F}_{n-1} \longrightarrow K_{n-2} \longrightarrow 0.
\end{array}$$

In order to check that $N_{X,\mathbf{n}}$ is connected, we show that $h^0(N_{X,\mathbf{n}},\mathcal{O}_{N_{X,\mathbf{n}}}) = 1$. From the long exact sequence of (E_0), this is equivalent to show that $h^1(\mathcal{M}_{\mathbf{n}},K_0) = 0$.

Now, by Künneth formula, we obtain that

$$H^{\alpha}(\mathcal{M}_{\mathbf{n}}, \mathcal{O}(-d_1, \ldots, -d_k)) = 0$$

for $\alpha \neq \sum_i 2^{n_i} - k$ or if some $d_i < 2^{n_i}$. Since $\mathbf{n} \neq (1, \ldots, 1)$, $\sum_i 2^{n_i} - k \geq n + 1$ and the equality only holds for $\mathbf{n} = (1, \ldots, 1, 2)$. Thus, we conclude that $H^l(\mathcal{M}_{\mathbf{n}}, \mathcal{F}_i) = 0$ for $l \leq n$. From the long exact sequence of (E_i) we deduce that we deduce that

$$H^i(\mathcal{M}_{\mathbf{n}}, K_{i-1}) \simeq H^{i+1}(\mathcal{M}_{\mathbf{n}}, K_i)$$

for every $i \in [n-1]$. In particular, we get that

$$H^1(\mathcal{M}_{\mathbf{n}}, K_0) \simeq H^n(\mathcal{M}_{\mathbf{n}}, K_{n-1}) = H^n(\mathcal{M}_{\mathbf{n}}, \mathcal{F}_n) = 0.$$

□

Nash CI curves were studied in Section 3.2, and they correspond to Nash CI varieties whose partition is $\mathbf{n} = (1, \ldots, 1, 2)$. From Proposition 3.4.6, one can check that this is the only partition $\mathbf{n}$ such that $N_{X,\mathbf{n}}$ is a curve for generic payoff tables. Similarly, we could ask for which partitions $\mathbf{n}$ the dimension formula in Proposition 3.4.6 equals 2. In other words, when is $N_{X,\mathbf{n}}$ generically a surface? It turns out that the only partitions for which $N_{X,\mathbf{n}}$ is generically a surfaces are $\mathbf{n} = (1, \ldots, 1, 2, 2)$ or $\mathbf{n} = (2, 2)$.

Definition 3.4.11. *We say that the Nash CI variety $N_{X,\mathbf{n}}$ is a* Nash CI surface *if it has dimension 2 and $\mathbf{n} = (1, \ldots, 1, 2, 2)$ or $\mathbf{n} = (2, 2)$.*

Example 3.4.12. *For the partition $(\{1, 2\}, \{3, 4\})$ in Example 3.4.3, we have that for generic payoff tables, the Nash CI variety is a Nash CI surface. In Example 3.4.16 we will illustrate a particular example of a Nash CI surface associated to the partition $(\{1, 2\}, \{3, 4\})$ and we compute the $(2, 2)$–Nash CI equilibria. On the other hand, in Example 3.3.8 for generic 3–players games, the Spohn CI variety of a graph with three vertices and two edges is a surface. However, since the graph is connected but not complete, it is not a Nash CI surface.*

In Section 3.3 we described the poset structure on the set of Spohn CI varieties (and CI equilibria) of a game. We saw that the next step to the Nash case is the Nash CI curve, whose associated graph has only one edge. Similarly, Nash CI surface lies in the second layer of this filtration. The partition $\mathbf{n}$ of a Nash CI surface equals $(1, \ldots, 1, 2, 2)$ or $\mathbf{n} = (2, 2)$. The only possible subpartitions of these partitions are of the form $(1, \ldots, 1)$, $(1, \ldots, 1, 2)$ or $(1, \ldots, 1, 2, 1, 1)$ (see Figure 3.7). The first subpartition corresponds to the Nash case, whereas the last two subpartitions correspond to Nash CI curves. In particular, any Nash CI surface contains only two Nash CI curves and their intersection is contained in the set of totally mixed Nash equilibria. In Example 3.4.16 we will explore this filtration in a particular example for $\mathbf{n} = (2, 2)$.

Figure 3.6: The two only graphs with n vertices and two edges up to labelling.

Remark 3.4.13. *Note that the study of Spohn CI varieties associated to one edge graphical models is completely covered by the Nash CI curve. However, for two edges graphical models, there are two possible graphs. These graphs are illustrated in Figure 3.6. The first graph of Figure 3.6 is the graph associated to Nash CI surfaces, whereas the second graph is not associated to a Nash CI variety. By Proposition 3.4.1 and Example 3.3.8, the associated independence variety of this second graph has dimension* $n+2$*. By Theorem 3.3.4, the associated Spohn CI variety of this graph is a surface (which is not a Nash CI surface) for generic payoff tables. Moreover, the graphs in Figure 3.6 are the only graphs for which a generic Spohn CI variety is a surface.*

The canonical bundle encodes important properties of an algebraic variety. We now determine the canonical bundle of generic Nash CI varieties. From Proposition 3.4.8, we deduce that $N_{X,\mathbf{n}}$ is Gorenstein. Thus, we can use the adjunction formula (see [44, Chapter 1.4.2]) to compute the canonical bundle of $N_{X,\mathbf{n}}$.

Lemma 3.4.14. *For generic payoff tables, we have that*

$$\omega_{N_{X,\mathbf{n}}} = \iota^*\mathcal{O}\left(n + n_1(1-2\delta_{1,n_1}) - 2^{n_1}, \ldots, n + n_k(1-2\delta_{1,n_k}) - 2^{n_k}\right),$$

where ι is the inclusion of $N_{X,\mathbf{n}}$ in $\mathcal{M}_{\mathbf{n}}$. In particular, if $n + n_i(1-2\delta_{1,n_i}) - 2^{n_i} > 0$ for every $i \in [k]$, then $\omega_{N_{X,\mathbf{n}}}$ is ample.

Proof. Using the adjunction formula we have that

$$\begin{aligned}\omega_{N_{X,\mathbf{n}}} = \iota^*\left(\omega_{\mathcal{M}_{\mathbf{n}}} \otimes \mathcal{O}\left(2(1-\delta_{1,n_1}), 1, \ldots, 1\right) \otimes \cdots \otimes \mathcal{O}\left(1, \ldots, 1, 2(1-\delta_{1,n_k})\right)\right) = \\ \iota^*\left(\omega_{\mathcal{M}_{\mathbf{n}}} \otimes \mathcal{O}\left(n + n_1(1-2\delta_{1,n_1}), \ldots, n + n_k(1-2\delta_{1,n_k})\right)\right).\end{aligned}$$

Now, the result follows from the fact that $\omega_{\mathcal{M}_{\mathbf{n}}} = \mathcal{O}\left(-2^{n_1}, \ldots, -2^{n_k}\right)$. □

For smooth algebraic surfaces, one important invariant is the Kodaira dimension, which plays a fundamental role in the classification of smooth algebraic surfaces. For instance, a smooth surface X is rational or ruled if and only of its Kodaira dimension is -1 (see [67, Theorem 6.1]). We now use Lemma 3.4.14 to compute the Kodaira dimension for smooth Nash CI surfaces. The value of the Kodaira dimension of a surface is one of the integers -1, 0, 1 or 2. We say that a smooth surface is of general type if its Kodaira dimension equals 2. We refer to [67, Section V.6] for further details on Kodaira dimensions.

Corollary 3.4.15. *The canonical bundle of Nash CI surfaces is ample. In particular, any smooth Nash CI surface has Kodaira dimension 2 and is of general type.*

Proof. In order to prove that a Nash CI surface is of general type, we need to check that the Kodaira dimension is 2. Thus, it is enough to show that the canonical bundle is ample. In the case of a Nash CI surface, we have that $(n_1, \dots, n_k) = (1, \dots, 1, 2, 2)$ or $(n_1, n_2) = (2, 2)$ where $n \geq 4$. From Lemma 3.4.14, we deduce that

$$\omega_{N_{X,\mathbf{n}}} = \iota^* \mathcal{O}\left(n-3, \dots, n-3, n-2, n-2\right) \quad \text{or} \quad \omega_{N_{X,\mathbf{n}}} = \iota^* \mathcal{O}\left(1,1\right),$$

which are ample. □

Note that Corollary 3.4.15 refers to smooth Nash CI surfaces. In Example 3.4.16 we will present a particular Nash CI surface that is not smooth nor irreducible. However, we expect this behavior to be a special case and not a generic situation. In Section 3.2.3, the smoothness and irreducibility of a generic Nash CI curve is derived. In the case of surfaces, we conjecture that a generic Nash CI surface is smooth and irreducible. This question is a more challenging problem than in the curve situation and it remains open. At the end of this section we investigate in more detail the smoothness of Nash CI varieties using Bertini's Theorem (see [67, Theorem 8.18]).

We now focus on a concrete example of a Nash CI surface corresponding to the partition $\mathbf{n} = (2, 2)$. The aim of this example is to show a complete description of a Nash CI surface and its CI equilibria from the perspective of both algebraic geometry and game theory. We compute the algebro-geometric properties of the Nash CI surface and the two Nash CI curves that it contains. We determine the Nash CI equilibria of the Nash CI surface and the Nash CI curve and we compute the set of totally mixed Nash equilibria. We compute the payoff regions associated to each of these equilibrium sets. We conclude that the Nash CI curves and the Nash CI surfaces provide better expected payoffs than the classical totally mixed Nash equilibria.

Example 3.4.16. *In the following, we construct a $(2 \times 2 \times 2 \times 2)$-game X to present the detailed computations of Nash CI equilibria. The study of the CI payoff region of this game shows that there are totally mixed Nash CI equilibria that give better expected payoffs than the totally mixed Nash equilibria. We set the payoff tables whose nonzero entries are as follows:*

$$\begin{array}{c} X^{(1)}_{1111} = X^{(2)}_{1112} = X^{(3)}_{1111} = X^{(4)}_{1211} = 1,\ \ X^{(2)}_{1121} = X^{(4)}_{2111} = -10,\ \ X^{(2)}_{2221} = X^{(4)}_{2122} = -16, \\ X^{(1)}_{2111} = X^{(2)}_{1212} = X^{(3)}_{1121} = X^{(4)}_{1212} = 3,\ \ X^{(2)}_{1221} = X^{(4)}_{2112} = -14,\ \ X^{(2)}_{2121} = X^{(4)}_{2121} = -12, \\ X^{(1)}_{1211} = X^{(2)}_{2112} = X^{(3)}_{1112} = X^{(4)}_{1221} = X^{(1)}_{2122} = X^{(3)}_{2221} = 2, \\ X^{(1)}_{2211} = X^{(2)}_{2212} = X^{(3)}_{1122} = X^{(4)}_{1222} = X^{(2)}_{1222} = X^{(4)}_{2212} = 4. \end{array}$$

Let G_i be an undirected graphical model from Figure 3.7 and $\mathcal{C}_i = \mathrm{global}(G_i)$ for $i \in [4]$. We denote the Nash CI variety of the graph G_i by N_{X,G_i}. The graphical model G_4 is the disjoint union of two cliques and thus the independence variety $\mathcal{M}_{\mathcal{C}_4}$ is the Segre variety $\mathbb{P}^3 \times \mathbb{P}^3$. The Nash CI variety N_{X,G_4} is a subvariety of $\mathbb{P}^{15}$ lying in the intersection of

$\mathcal{M}_{\mathcal{C}_4}$ and the Spohn variety $\mathcal{V}_X$. Let $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ be the coordinates of the first and second $\mathbb{P}^3$ factor of $\mathcal{M}_{\mathcal{C}_4}$. As a subvariety of $\mathcal{M}_{\mathcal{C}_4}$, the Nash CI variety N_{X,G_4} is defined by the following four polynomials that are products of a linear form l_i and a quadratic form q_i for $i \in [4]$:

$$\begin{aligned}
l_1q_1 &:= (\sigma_{11}^{(2)} - 2\sigma_{22}^{(2)})(2\sigma_{11}^{(1)}\sigma_{21}^{(1)} + \sigma_{21}^{(1)}\sigma_{12}^{(1)} + 3\sigma_{11}^{(1)}\sigma_{22}^{(1)} + 2\sigma_{12}^{(1)}\sigma_{22}^{(1)}),\\
l_2q_2 &:= (\sigma_{21}^{(2)} - 2\sigma_{12}^{(2)})(2\sigma_{11}^{(1)}\sigma_{12}^{(1)} + \sigma_{21}^{(1)}\sigma_{12}^{(1)} + 3\sigma_{11}^{(1)}\sigma_{22}^{(1)} + 2\sigma_{21}^{(1)}\sigma_{22}^{(1)}),\\
l_3q_3 &:= (\sigma_{11}^{(1)} - 2\sigma_{22}^{(1)})(2\sigma_{11}^{(2)}\sigma_{21}^{(2)} + \sigma_{21}^{(2)}\sigma_{12}^{(2)} + 3\sigma_{11}^{(2)}\sigma_{22}^{(2)} + 2\sigma_{12}^{(2)}\sigma_{22}^{(2)}),\\
l_4q_4 &:= (\sigma_{21}^{(1)} - 2\sigma_{12}^{(1)})(2\sigma_{11}^{(2)}\sigma_{12}^{(2)} + \sigma_{21}^{(2)}\sigma_{12}^{(2)} + 3\sigma_{11}^{(2)}\sigma_{22}^{(2)} + 2\sigma_{21}^{(2)}\sigma_{22}^{(2)}).
\end{aligned} \tag{3.54}$$

These polynomials are computed using Formula (3.45). One can check that N_{X,G_4} is a Nash CI surface and it is the union of 14 complete intersection surfaces in $\mathbb{P}^3 \times \mathbb{P}^3$. One of them is the zero locus of the ideal $\langle l_1, l_2, l_3, l_4\rangle$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Four of these surfaces are the zero loci of ideals of the form $\langle l_{i_1}, l_{i_2}, l_{i_3}, q_{i_4}\rangle$ for $\{i_1, i_2, i_3, i_4\} = [4]$ distinct. Each of these surfaces are isomorphic to the disjoint union of two planes. Similarly, we get six varieties that are the zero loci of ideals of the form $\langle l_{i_1}, l_{i_2}, q_{i_3}, q_{i_4}\rangle$ for $\{i_1, i_2, i_3, i_4\} = [4]$ distinct. The varieties defined by the ideals $\langle l_1, l_2, q_3, q_4\rangle$ and $\langle q_1, q_2, l_3, l_4\rangle$ are empty, whereas the other 4 ideals define surfaces that are the product of two smooth conics. We get four surfaces defined by ideals of the form $\langle l_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}\rangle$ for $\{i_1, i_2, i_3, i_4\} = [4]$ distinct. Each of these surfaces is the disjoint union of 4 quadric surfaces in $\mathbb{P}^3$. Finally, the ideal $\langle q_1, q_2, q_3, q_4\rangle$ leads to a surface which is the product of two degree 4 curves in $\mathbb{P}^3$. We conclude that N_{X,G_4} is the union of 14 complete intersection surfaces but it has 30 irreducible components.

The quadratic surface defined by q_i in the corresponding $\mathbb{P}^3$ does not intersect the open simplex $\Delta_3^\circ \subset \mathbb{P}^3$ for $i \in [4]$. In particular, we deduce that the only component of N_{X,G_4} intersecting $\Delta \subset \mathbb{P}^{15}$ is $\mathcal{L} := \mathbb{V}(l_1, l_2, l_3, l_4)$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, the set of totally mixed Nash CI equilibria of G_4 is $\mathcal{L} \cap (\Delta_3^\circ \times \Delta_3^\circ)$. Equivalently, as a subset of $\mathbb{P}^{15}$, the set of totally mixed CI equilibria is

$$\left\{ p \in \mathbb{P}^{15} : \begin{array}{cc} p_{1111} = 2p_{1122} = 2p_{2211} = 4p_{2222}, & p_{1212} = 2p_{1221} = 2p_{2112} = 4p_{2121} \\ p_{1112} = 2p_{1121} = 2p_{2212} = 4p_{2221}, & p_{1211} = 2p_{2111} = 2p_{1222} = 4p_{2122} \\ p_{2222}p_{2121} = p_{2221}p_{2122}, & p_{2222}, p_{2121}, p_{2221}, p_{2122} > 0 \end{array} \right\}.$$

Note that the set of totally mixed Nash CI equilibria is contained in a 3-dimensional projective space defined by the 12 linear equations in the previous expression. We identify this projective space with $\mathbb{P}^3$. Let z_0, z_1, z_2, z_3 be the coordinates of $\mathbb{P}^3$ corresponding to $p_{2222}, p_{2221}, p_{2122}, p_{2121}$. We may view $\mathcal{L}$ as the Segre surface $\mathbb{V}(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$ and the set of totally mixed Nach CI equilibria as the intersection of $\mathbb{V}(z_0z_3 - z_1z_2)$ with the open simplex Δ_3°. In Figure 3.7, we illustrate the poset of subgraphs of G_4 and similarly a poset of inclusions of Spohn CI varieties. The Segre surface[1] $\mathcal{L}$ contains

[1] we use in this paragraph the colors appearing in Figure 3.7

two components of Nash CI curves N_{X,G_2} and N_{X,G_3}, and in their intersection lies the ***set of totally mixed Nash equilibria****. The only components of the Nash CI curves N_{X,G_2} and N_{X,G_3} intersecting the open simplex are the line $L_1 = \mathbb{V}(z_0 - z_1, z_2 - z_3)$ and $L_2 = \mathbb{V}(z_0 - z_2, z_1 - z_3)$. The intersection of L_1 and L_2 is the unique totally mixed Nash equilibrium which is the point $p = [1, 1, 1, 1]$. Note that $\mathcal{L}$ is a ruled surface and through each point q in $\mathcal{L}$ there are exactly two lines contained in $\mathcal{L}$ passing through q. In our case, the totally mixed Nash equilibrium is the point p in $\mathcal{L}$ and the set of totally mixed Nash CI equilibria of the graphs G_1 and G_2 correspond to the two lines in S passing through p respectively. This is illustrated in Figure 3.7.*

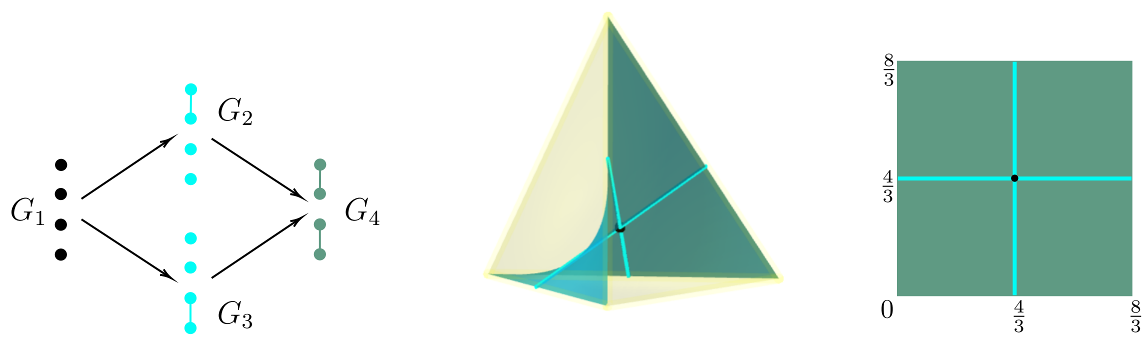


Figure 3.7: Poset of subgraphs of the 4 vertex graph G_4, their CI equilibria and payoff regions.

Now, we compute the CI payoff region associated to the Nash CI surface and the two Nash CI curves. In the coordinates z_0, z_1, z_2, z_3, the sum of all the coordinates $p_{i_1 i_2 i_3 i_4}$ equals $\frac{1}{9}(z_0 + z_1 + z_2 + z_3)$. We denote the cone of $\mathcal{L}$ in $\mathbb{A}^4$ by $\tilde{\mathcal{L}}$. Then, we identify the set of totally mixed Nash CI equilibria $\mathcal{L} \cap \Delta_3^\circ$ with the intersection of $\tilde{\mathcal{L}}$ and the open simplex

$$\tilde{\Delta}^\circ = \left\{ (z_0, z_1, z_2, z_3) \in \mathbb{A}^4 : z_0 + z_1 + z_2 + z_3 = \frac{1}{9} \;\; \textit{and} \;\; z_0, z_1, z_2, z_3 > 0 \right\}.$$

In the coordinates z_0, z_1, z_2, z_3, the restriction of the payoff map to the set of totally mixed Nash CI equilibria is

$$\begin{array}{rccc} \pi_X : & \tilde{\Delta} \cap \tilde{\mathcal{L}} & \longrightarrow & \mathbb{R}^4 \\ & (z_0, z_1, z_2, z_3) & \longmapsto & \left(PX^{(1)}, PX^{(2)}, PX^{(3)}, PX^{(4)}\right), \end{array}$$

where the expected payoffs are

$$\begin{array}{ll} PX^{(1)} = 24(z_0 + z_2), & PX^{(2)} = -24(z_1 + z_3) \\ PX^{(3)} = 24(z_0 + z_1), & PX^{(4)} = -24(z_2 + z_3) \end{array}.$$

Note that

$$PX^{(1)} + PX^{(2)} = PX^{(3)} - PX^{(4)} = \frac{8}{3}.$$

Therefore, we can consider the payoff map π_X as the map from $\tilde{\Delta} \cap \tilde{\mathcal{L}}$ to $\mathbb{R}^2$ sending (z_0, z_1, z_2, z_3) to $(PX^{(1)}, PX^{(3)})$. Restricting the Segre parametrization to Δ, we obtain the following parametrization of $\tilde{\Delta}^\circ \cap \tilde{\mathcal{L}}$:

$$\begin{array}{cccc} \varphi : & \mathbb{R}^2_{>0} & \longrightarrow & \tilde{\Delta} \cap \tilde{\mathcal{L}} \\ & (a,b) & \longmapsto & \left(\dfrac{ab}{9(a+1)(b+1)}, \dfrac{a}{9(a+1)(b+1)}, \dfrac{b}{9(a+1)(b+1)}, \dfrac{1}{9(a+1)(b+1)} \right). \end{array}$$

In particular, we get that the composition $\pi_X \circ \varphi$ sends a point $(a,b) \in \mathbb{R}^2_{>0}$ to

$$\frac{8}{3}\left(\frac{b}{b+1}, \frac{a}{a+1}\right).$$

Therefore, the CI payoff region is the open square

$$\left(0, \frac{8}{3}\right) \times \left(0, \frac{8}{3}\right)$$

in $\mathbb{R}^2$.

Similarly, for the two Nash CI curves, we get that the payoff regions are the open segments

$$\left\{\frac{4}{3}\right\} \times \left(0, \frac{4}{3}\right) \text{ and } \left(0, \frac{4}{3}\right) \times \left\{\frac{4}{3}\right\}$$

respectively as illustrated in Figure 3.7. In these coordinates, the totally mixed Nash point is

$$\left(\frac{1}{36}, \frac{1}{36}, \frac{1}{36}, \frac{1}{36}\right).$$

The corresponding expected payoff in $\mathbb{R}^2$ is the point $(\frac{4}{3}, \frac{4}{3})$. In particular, we see that in the image of Nash CI curves there are totally mixed Nash CI equilibria that give better expected payoffs than the totally mixed Nash equilibria. For instance, the points

$$\left(\frac{3}{72}, \frac{3}{72}, \frac{1}{72}, \frac{1}{72}\right) \text{ and } \left(\frac{3}{72}, \frac{1}{72}, \frac{3}{72}, \frac{1}{72}\right)$$

lie in the two Nash CI curves respectively and they give better expected payoff than the totally mixed Nash equilibria. Similarly, for any totally mixed Nash CI equilibria on a Nash CI curve, there exists a totally mixed Nash CI equilibrium in $\mathcal{L}$ which gives better expected payoffs.

The next property of Nash CI varieties to be studied is the smoothness. We use Bertini's Theorem (see [67, Theorem 8.18]) to analyze the singularities of a generic $\mathbf{n}$–Nash CI variety. To do so, we apply the study of the base locus of the linear systems $\Lambda_{(i,l)}$ done in Section 3.3. Recall that linear system $\Lambda_{(i,l)}$ is defined as the image of the map $\phi_{(i,l)}$ in (3.49). As in Section 3.3, if $n_i = 1$ we get that $\phi_{(i,1)}$ is surjective and $\Lambda_{(i,j_i)}$

is complete. By Lemma 3.4.7, we deduce that the base locus of $\Lambda_{(i,l)}$ is the base locus of $W_{(i,l)}$. Moreover, by Lemma 3.3.3, if $n_i \geq 2$, $W_{(i,j_i)}$ has a base locus given by

$$\mathbb{V}(G_1, G_2) \cup \mathbb{V}(\sigma^{(i)}_{j_1\cdots 1\cdots j_{n_i}} : j_1, \ldots, j_{n_i} \in [2]) \cup \mathbb{V}(\sigma^{(i)}_{m_1\cdots 2\cdots m_{n_i}} : m_1, \ldots, m_{n_i} \in [2]),$$

where

$$G_1 := \sum_{m_1,\ldots,\widehat{m_{l_i}},\ldots,m_{n_i}} \sigma^{(i)}_{m_1\cdots 2\cdots m_{n_i}} \quad \text{and} \quad G_2 := \textstyle\sum_{j_1,\ldots,\widehat{j_{l_i}},\ldots,j_{n_i}} \sigma^{(i)}_{j_1\cdots 1\cdots j_{n_i}}.$$

Since this base locus is included in the intersection of $\mathcal{M}_{\mathcal{C}}$ and the hyperplanes $\{p_{j_1 j_2\cdots j_n} = 0\}$ and $\{p_{++\cdots+} = 0\}$, we derive the following result.

Corollary 3.4.17. *For generic payoff tables, the Nash CI variety is smooth away from the hyperplanes $\{p_{j_{11}\cdots j_{kn_k}} = 0\}$ and $\{p_{++\cdots+} = 0\}$. In particular, for generic payoff tables the set of totally mixed $\mathbf{n}$–Nash equilibria is a smooth semialgebraic manifold.*

Remark 3.4.18. *Let X be a generic game for which there exists a totally mixed Nash equilibrium. Since the Nash CI variety $N_{X,\mathbf{n}}$ is smooth in Δ and contains totally mixed Nash equilibria of X, the real points of $N_{X,\mathbf{n}}$ are Zariski dense (see [115, Theorem 5.1]). As a consequence of Theorem 3.3.4, we deduce that for such a game, the set of totally mixed Nash CI equilibria is a smooth manifold of dimension $2^{n_1} + \cdots + 2^{n_k} - k - n$.*

3.5 Universality theorems

In Section 3.2.3 we introduced the notion of universality of divisors. This notion was motivated by the Nash equilibria case, i.e. the case of Nash CI varieties for $\mathbf{n} = (1, \ldots, 1)$, where this concept of universality holds. In the case of Nash CI curves, we saw in Section 3.2.3 that the universality of divisors does not hold anymore. From Lemma 3.3.3 applied to Nash CI varieties, we deduce that the universality of divisors does not hold for Nash CI varieties with $\mathbf{n} \neq (1, \ldots, 1)$. In this section we focus on a different kind of universality: the affine universality. Let $\mathcal{X}$ be a set or family of projective algebraic varieties. We say that $\mathcal{X}$ satisfies the affine universality if for any affine algebraic variety S, there exists $X \in \mathcal{X}$ such that S is isomorphic to an affine open subset of X. The motivation behind the study of the affine universality lies in the fact that it measures the difficulty of a family of varieties. For instance, if a family $\mathcal{X}$ satisfies the affine universality means that the most complicated singularity appears in one of the varieties of the family. Our goal is to study this notion of universality for certain families of Nash CI varieties. More precisely, we aim to understand how complicated the set of Nash CI equilibria can be. For this purpose we will fix the base field to be $\mathbb{R}$. Nevertheless, most of our results hold for arbitrary fields.

The study of universality results in the game theoretic context was initiated by Datta in [34]. There, Datta explores the universality for two families: the set of all sets of totally mixed Nash equilibria for binary games with arbitrary players, and the set all sets of totally mixed Nash equilibria for 3–player games with arbitrary strategies. We focus on the first family. The notion of universality studied in [34] is slightly different to our notion of affine universality. In [34], the notion of isomorphism used is the notion of stable isomorphism in the category of semialgebraic sets, whereas in this section we use the notion of isomorphism of affine varieties. In other words, two real affine algebraic varieties are isomorphic if their coordinate rings are isomorphic. In the category of semialgebraic sets, Datta proves the following universality result:

Theorem 3.5.1. *[34, Theorem 1] Any real affine algebraic variety is isomorphic, in the category of semialgebraic sets, to the set of totally mixed Nash equilibria of a game with binary choices.*

We now translate Datta's universalities results to our notion of isomorphism. Recall that the Nash CI variety $N_{X,(1,\ldots,1)}$ is given by the polynomials (3.11) in the Segre variety $\mathcal{M}_{(1,\ldots,1)} = (\mathbb{P}^1)^n$. Let $\sigma_1^{(i)}, \sigma_2^{(i)}$ be the coordinates of the i–th factor of $\mathcal{M}_{(1,\ldots,1)}$. We consider the open subset $U_{X,0}$ of $N_{X,(1,\ldots,1)}$ defined by

$$\sigma_2^{(1)}, \ldots, \sigma_2^{(n)} \neq 0.$$

Note that Theorem 3.5.1 is a consequence of [34, Theorem 6]. In the proof of [34, Theorem 6], given any real algebraic variety S, the author constructs a game X such that the affine open subset $U_{X,0}$ of $N_{X,(1,\ldots,1)}$ and S have isomorphic coordinate rings. Therefore, [34, Theorem 1,Theorem 6] can be rephrased as follows using the notion of isomorphism of algebraic varieties.

Theorem 3.5.2. *Let $S \subset \mathbb{R}^m$ be a real affine algebraic variety. Then, there exists an n-player game with binary choices with payoff tables $X^{(1)}, \ldots, X^{(n)}$ such that $U_{X,0} \simeq S$.*

The goal of this section is to derive analogous results to Theorem 3.5.2 for $\mathbf{n}$–Nash CI varieties whose partition $\mathbf{n}$ is of the form $(n_1, \ldots, n_k) = (1, \ldots, 1, 2, \ldots, 2)$. Given such partition, we denote by l the number of times that 2 appears in the partition. In particular, $k = n - l$ and these partitions only depend on the value of l. Moreover, the independence variety of these partitions is

$$\mathcal{M}_{\mathbf{n}} = \left(\mathbb{P}^1\right)^{n-2l} \times \left(\mathbb{P}^3\right)^l .$$

for $2l \leq n$. We denote the corresponding $\mathbf{n}$–Nash CI variety by $Y_{X,l}$. For instance, $Y_{X,0}$ equals $N_{X,(1,\ldots,1)}$. For generic payoff tables, $Y_{X,1}$ is the Nash CI curve, and $Y_{X,2}$ is the Nash CI surface. Let $U_{X,l} \subset Y_{X,l}$ be the affine open subset defined by

$$\sigma_2^{(1)}, \ldots, \sigma_2^{(n-2l)}, \sigma_{22}^{(n-2l+1)}, \ldots, \sigma_{22}^{(n-l)} \neq 0.$$

In this setting, the affine universality asks whether any real affine algebraic variety is isomorphic to the affine open subset $U_{X,l}$ of $N_{X,\mathbf{n}}$ for some game. Theorem 3.5.2 gives a positive answer to this question for $l = 0$. However, for $l \geq 1$, the affine universality does not hold anymore. Indeed, $Y_{X,l}$ is the intersection of n divisors in a variety of dimension $n+l$. Thus, the dimension of $Y_{X,l}$ is greater or equal to l. We deduce that for $l \geq 1$, the affine universality fails for affine real varieties of dimension at most $l-1$. We overcome this dimension problem in two different ways, providing two partial answer to the affine universality for $l \geq 1$. First we add artificially extra dimensions to the given affine varieties by taking the product with an affine space. In the second approach, we artificially require the affine variety to have dimension at least l.

Lemma 3.5.3. *For every n-player game with binary choices with payoff tables $\tilde{X}^{(1)}, \ldots, \tilde{X}^{(n)}$, there exists an $(n+2l)$-player game with binary choices with payoff tables $X^{(1)}, \ldots, X^{(n+2l)}$ such that*

$$U_{X,l} \simeq U_{\tilde{X},0} \times \mathbb{R}^l.$$

Proof. Let $G_1, \ldots, G_n$ be the polynomials defining $U_{\tilde{X},0}$ in $\mathbb{A}^n$. We consider an $(n+2l)$–player game with payoff tables $X^{(1)}, \ldots, X^{(n+2l)}$. Let $\tilde{\mathbf{n}}$ be the partition of $n+2l$ where 1 and 2 appear n and l times respectively. Let $\sigma_j^{(i)}$ for $j \in [2]$ and $i \in [n]$ be the coordinates of the n factors of $\mathcal{M}_{\tilde{\mathbf{n}}}$ corresponding to $\mathbb{P}^1$, and let $\sigma_{j_1,j_2}^{(n+i)}$ for $j_1, j_2 \in [2]$ and $i \in [l]$ be the coordinates of the $\mathbb{P}^3$ factors of $\mathcal{M}_{\tilde{\mathbf{n}}}$. We denote the $n+2l$ equations of $Y_{X,l}$ by $F_1, \ldots, F_n, F_{n+1,1}, F_{n+1,2}, \ldots, F_{n+l,1}, F_{n+l,2}$. Similarly, we denote the polynomials defining the open subset $U_{X,l}$ by $F'_1, \ldots, F'_n, F'_{n+1,1}, F'_{n+1,2}, \ldots, F'_{n+l,1}, F'_{n+l,2}$. In other words, F'_i is obtained by substituting in F_i the variables $\sigma_2^{(1)}, \ldots, \sigma_2^{(n)}, \sigma_{2,2}^{(n+1)}, \ldots, \sigma_{2,2}^{(n+2l)}$ by 1. For the player $(n+i, 1)$, we can fix the coefficients $Y^{(n+i,1)}$ such that the determinant (3.38) equal

$$\det \begin{pmatrix} \sigma_{11}^{(n+i)} + \sigma_{12}^{(n+i)} & 0 \\ \sigma_{21}^{(n+i)} + \sigma_{22}^{(n+i)} & \sigma_{22}^{(n+i)} \end{pmatrix} = \sigma_{22}^{(n+i)}(\sigma_{11}^{(n+i)} + \sigma_{12}^{(n+i)}).$$

In particular, for $i \in [l]$ we can choose the payoff table $X^{(n+i,1)}$ such that

$$F_{n+i,1} = \sigma_2^{(1)} \cdots \sigma_2^{(n)} (\sigma_{1,1}^{(n+i)} + \sigma_{1,2}^{(n+i)}) \prod_{j \neq i}^{l} \sigma_{2,2}^{(n+j)}.$$

Similarly, we fix $X^{(n+i,2)}$ such that

$$F_{n+i,2} = \sigma_2^{(1)} \cdots \sigma_2^{(n)} (\sigma_{1,1}^{(n+i)} + \sigma_{1,2}^{(n+i)}) \prod_{j \neq i}^{l} \sigma_{2,2}^{(n+j)},$$

for every $i \in [l]$. In particular, $F'_{n+i,1} = \sigma_{1,1}^{(n+i)} + \sigma_{2,1}^{(n+i)}$ and $F'_{n+i,2} = \sigma_{1,1}^{(n+i)} + \sigma_{1,2}^{(n+i)}$ and we get that

$$U_{X,l} = \mathbb{V}(F'_1, \ldots, F'_n) \times \mathbb{R}^l.$$

Now, we fix the payoff tables of the first n players to be

$$X^{(i)}_{j_1,\dots,j_{n+2l}} = \begin{cases} \tilde{X}^{(i)}_{j_1,\dots,j_n} & \text{if } j_{n+1} = \dots = j_{n+2l} = 2 \\ 0 & \text{else} \end{cases}.$$

Then, the polynomials $F'_1, \dots, F'_n$ are equal to the polynomials $G_1, \dots, G_n$ and we conclude that

$$U_{X,l} = \mathbb{V}(F'_1, \dots, F'_n) \times \mathbb{R}^l \simeq U_{\tilde{X},l} \times \mathbb{R}^l.$$

□

From Theorem 3.5.2 and Lemma 3.5.3 we deduce the following first universality theorem for Nash CI varieties.

Theorem 3.5.4. *Let $l \in \mathbb{N}$ and let $S \subseteq \mathbb{R}^m$ be an affine real algebraic variety. Then, there exists $n \geq l$ and an n–players game X with binary choices such that $U_{X,l} \simeq S \times \mathbb{R}^l$.*

A consequence of Theorem 3.5.4 is that, for any l, the space of all varieties $Y_{X,l}$ for any binary game X with any number of players satisfies Murphy's law. We say that the set or family of algebraic varieties $\mathcal{X}$ satisfies Murphy's law if, for any singularity type, there exists a variety X in $\mathcal{X}$ with this singularity type. For further details on the Murphy's law in algebraic geometry we refer to [123].

Corollary 3.5.5. *For any $l \in \mathbb{N}$, the space of all varieties $Y_{X,l}$ for any binary game X with any number of players satisfies the Murphy's law.*

Proof. The proof follows from Theorem 3.5.4 and the fact that $S \times \mathbb{A}^k$ has the same singularity type as S. □

One of the main consequences of the Murphy's law in our setting is that any singularity type can appear in the set of totally mixed Nash CI equilibria associated to the Spohn CI varieties $Y_{X,l}$.

In Theorem 3.5.4 we solved the dimension problem by artificially adding extra dimensions. In our second approach, we force the dimension to be at least l.

Theorem 3.5.6. *Let $l \in \mathbb{N}$ and let $S \subseteq \mathbb{R}^n$ be a real affine algebraic variety defined by $G_1, \dots, G_m \in \mathbb{R}[x_1, \dots, x_n]$ with $m \leq n - l$. For every $i \in \{1, \dots, n\}$, let δ_i be the maximum of the degrees of x_i in $G_1, \dots, G_m$. Then, there exists a $(\delta + n + l)$–player game with binary choices such that the affine open subset W_X of C_X is isomorphic to S, where $\delta = \delta_1 + \dots + \delta_n$.*

Proof. We follow the same notation as in Lemma 3.5.3. Consider a $(\delta + n + l)$–players game X. We label the last $2l$ players by $(1,1), (1,2), \dots, (l,1), (l,2)$. The variety $Y_{X,l}$ lies in the Segre variety $(\mathbb{P}^1)^{\delta+n-l} \times (\mathbb{P}^3)^l$. We denote the coordinates of the $\mathbb{P}^1$ factors by $\sigma^{(i)}_j$ for $j \in [2]$ and $i \in [\delta + n - l]$. Similarly, the coordinates of the $\mathbb{P}^3$ factors are denoted by $\sigma_{j_1,j_2}^{(\delta+n-l+i)}$ for $j_1, j_2 \in [2]$ and $i \in [l]$. Moreover, we denote the polynomials defining $U_{X,l}$ by

$$F'_1, \ldots, F'_{\delta+n-l}, F'_{1,1}, F'_{1,2}, \ldots, F'_{l,1}, F'_{l,2}.$$

As in the proof of Lemma 3.5.3, we fix the payoff tables of the players $(1,1), \ldots, (l,2)$ such that

$$F'_{i,1} = \sigma_{1,1}^{(\delta+n-l+i)} + \sigma_{2,1}^{(\delta+n-l+i)} \quad \text{and} \quad F'_{\delta+n-l+i,2} = \sigma_{1,1}^{(\delta+n-l+i)} + \sigma_{1,2}^{(\delta+n-l+i)}$$

for every $i \in [l]$. In particular, we deduce that

$$\mathbb{V}(F'_{1,1}, \ldots, F'_{l,2}) = \left(\mathbb{P}^1\right)^{\delta+n}.$$

Following the proof of [34, Theorem 6], there exists a $(\delta + n)$–players game $\tilde{X}$ such that $U_{\tilde{X},0} = S$. Moreover, we can assume that the last $n - m$ payoff tables of the game vanish. Now, we fix the payoff tables of the first $\delta + n - l$ of the game X as follows:

$$X^{(i)}_{j_1,\ldots,j_{\delta+n-l},j_{1,1},\ldots,j_{l,2}} = \begin{cases} \tilde{X}^{(i)}_{j_1,\ldots,j_{\delta+n-l}} & \text{if } j_{1,1} = \cdots = j_{l,2} = 2 \\ 0 & \text{else} \end{cases}$$

for $i \in [\delta + n - l]$. One can check that the polynomials $F'_1, \ldots, F'_{\delta+n-l}$ are equal to (but with different variables) the $\delta + n - l$ polynomials defining $U_{\tilde{X},0}$. Using that $n - m \geq l$, we deduce that

$$U_{X,l} = \mathbb{V}(F'_1, \ldots, F'_{\delta+n-l}) \cap \left(\mathbb{P}^1\right)^{\delta+n} \simeq U_{\tilde{X},0} \simeq S.$$

□

Remark 3.5.7. *In [34], Datta's universality theorem refers to the set of totally mixed Nash equilibria. An analogous statement for the set of totally mixed CI equilibria can be obtained in our setting. Namely, given l and a real affine algebraic variety S, there exists a game with binary choices such that $U_{X,l} \cap \Delta$ is isomorphic to $S \times \mathbb{R}^l$ (Corollary 3.5.4). As in [34], here we use the notion of stable isomorphism in the category of semialgebraic sets. To derive these results one should argue as in [34]: the set of real points of a real affine algebraic variety is isomorphic to the set of real points of a real affine algebraic variety whose real points are contained in the probability simplex. Now, assuming the latter, the statement follows from Proposition 3.5.4. An analogous statement also holds for Theorem 3.5.6.*

Note that the proof of Theorem 3.5.2 given in [34] provides a method for, given the real affine algebraic variety, finding a game satisfying the statements of the theorem. Together with the proofs of Theorems 3.5.4 and 3.5.6, we obtain a method for, given a feasible real affine algebraic variety, constructing a game satisfying the assertions of the theorems.

Example 3.5.8. *Consider the real plane curve defined by*

$$G_1 = x_1^2 + x_2^2 - 1.$$

By Theorem 3.5.6, this plane curve can be described by a 7-player game with binary choices. Consider the polynomials

$$\begin{aligned}
F_1 &= \sigma_2^{(2)}\sigma_2^{(4)}\sigma_2^{(5)}\left(\sigma_1^{(3)}\tau_{22} - \tau_{11}\sigma_2^{(3)}\right),\\
F_2 &= \sigma_2^{(1)}\sigma_2^{(5)}\left(\sigma_2^{(3)}\sigma_1^{(4)}\tau_{22} - \sigma_1^{(3)}\sigma_2^{(4)}\tau_{11}\right),\\
F_3 &= \sigma_2^{(2)}\sigma_2^{(4)}\tau_{22}\left(\sigma_1^{(1)}\sigma_2^{(5)} - \sigma_1^{(5)}\sigma_2^{(1)}\right),\\
F_4 &= \sigma_2^{(3)}\tau_{22}\left(\sigma_2^{(1)}\sigma_1^{(2)}\sigma_2^{(5)} - \sigma_1^{(1)}\sigma_2^{(2)}\sigma_1^{(5)}\right),\\
F_5 &= \sigma_2^{(1)}\sigma_2^{(3)}\tau_{22}\left(\sigma_2^{(4)}\sigma_1^{(2)} + \sigma_2^{(2)}\sigma_1^{(4)} - \sigma_2^{(2)}\sigma_2^{(4)}\right),\\
F_6 &= \sigma_2^{(1)}\sigma_2^{(2)}\sigma_2^{(3)}\sigma_2^{(4)}\sigma_2^{(5)}\tau_{22}(\tau_{11} + \tau_{12}),\\
F_7 &= \sigma_2^{(1)}\sigma_2^{(2)}\sigma_2^{(3)}\sigma_2^{(4)}\sigma_2^{(5)}\tau_{22}(\tau_{11} + \tau_{21}).
\end{aligned}$$

One can check that the open subset of $\mathbb{V}(F_1, \ldots, F_7)$ defined by $\sigma_2^{(1)} \cdots \sigma_2^{(5)}\tau_{22} \neq 0$ is isomorphic to $\mathbb{V}(G_1)$. Moreover, the tuple $(F_1, \ldots, F_7)$ lies in the image of the linear map (3.30). Computing its preimage, one can determine a linear subspace of games X such that C_X is defined by the equations $F_1, \ldots, F_7$ and the open subset U_X is isomorphic to $\mathbb{V}(G_1)$. For example, the possible payoff tables corresponding to the players 1, 5, 6, and 7 are

$$X^{(1)}_{i_1\cdots i_7} = \begin{cases} 1 & \textit{if } (i_1, \ldots, i_7) \in \{(2,2,1,2,2,2,2), (1,2,2,2,1,1,1)\},\\ 0 & \textit{else.}\end{cases}$$

$$X^{(5)}_{i_1\cdots i_7} = \begin{cases} 1 & \textit{if } (i_1, \ldots, i_7) \in \{(2,1,2,2,2,2,2), (2,2,2,1,2,2,2), (2,2,2,2,1,2,2)\},\\ 0 & \textit{else.}\end{cases}$$

$$X^{(6)}_{i_1\cdots i_7} = X^{(7)}_{i_1\cdots i_7} = \begin{cases} 1 & \textit{if } (i_1, \ldots, i_7) \in \{(2,2,2,2,2,2,2)\},\\ 0 & \textit{else.}\end{cases}$$

3.6 Open problems

We now list the open questions and new research lines concerning this chapter.

During this chapter we have focused on binary games and we have analysed Spohn CI varieties arising from undirected graphical models. However, there are still many properties of these varieties that remain unknown. For instance, in Section 3.4.1 we used Bertini's theorem to study the possible singularities of generic Nash CI varieties. For one edge graphs, we proved that a generic Nash CI curves is smooth and irreducible. The smoothness and irreducibility of generic Nash CI varieties remains open.

Question 3.6.1. *For any partition* $\mathbf{n}$ *of* $[n]$*, is the Nash CI variety* $N_{X,\mathbf{n}}$ *smooth and irreducible for generic payoff tables?*

A particularly interesting case of Question 3.6.1 is the case of Nash CI surfaces.

In Section 3.4, we have computed some invariants of Nash CI varieties as their connectedness and their degrees. However, for general Spohn CI varieties of undirected graphical models, their dimension is the only invariant we have computed. The degree, smoothness, irreducibility, connectedness, etc of these varieties remain open.

Another feature to be studied is the real algebraic geometry of Nash CI varieties, or more general Spohn CI varieties. For instance, we have check that generic Nash CI varieties are connected as complex varieties. However, it might happen that the set of real points of these varieties is disconnected.

Question 3.6.2. *How many components can the set of real points of generic Nash CI varieties have?*

For instance, in the case of the Nash CI curve, the number of components of the real points is bounded by $g+1$, where g is the genus. For $s \leq g+1$, does it exist a game such that its Nash CI curve has s ovals?

Recall that the set of totally mixed CI equilibria is a semialgebraic set. In the study of the geometry of the seminalgebraic sets it is fundamental the understanding of its boundary. Again this boundary can be study from its algebraic closure.

Question 3.6.3. *What can we say about the algebraic closure of the boundary of the set totally mixed CI equilibria?*

In the case of Nash CI surface, we expect the boundary to be one dimensional. What type of curve is its algebraic closure? Another semialgebraic set appearing in the game theoretic context is the payoff region of the set of totally mixed CI equilibria. In Example 3.4.16 we presented an example of such semialgebraic set (see Figure 3.7). The study of this semialgebraic set remains open.

Assume that we are given a mixed strategy p of a game X and a graph G. From a game theoretic point of view, a very interested problem is finding the distance from a fixed mixed strategy to the set of totally mixed Spohn CI equilibria of G. The difficulty of this question is measured through the Euclidean distant (ED) degree. We refer to [15, 41] for further details on ED degrees and metric algebraic geometry. Therefore, another open question is

Question 3.6.4. *What is the ED degree of Nash CI varieties, and more generally Spohn CI varieties of undirected graphical models?*

Other opens problem concerns the study of the boundary equilibria. During our study we have focused on totally mixed equilibria. This could be seen in the construction of the Spohn CI variety, where we removed all components lying in the boundary of the open simplex. What happen if we consider these boundary components? Is still Conjecture 3.1.29 true?

Finally, all along this chapter, we have assumed that all of our games have binary strategies. In other words, we have assumed that $d_1 = \cdots = d_n = 2$. The algebro-geometric study of the Spohn CI variety and the CI equilibria for nonbinary games remains open.

Bibliography

[1] Daniele Agostini, Hannah Markwig, Clemens Nollau, Victoria Schleis, Javier Sendra-Arranz, and Bernd Sturmfels. Recovery of plane curves from branch points. *Discrete & Computational Geometry*, pages 1–25, 2023.

[2] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joe Harris. Geometry of algebraic curves: Volume i, 1994.

[3] Robert J Aumann. Subjectivity and correlation in randomized strategies. *Journal of mathematical Economics*, 1(1):67–96, 1974.

[4] Edoardo Ballico and Alessandra Bernardi. Tensor ranks on tangent developable of Segre varieties. *Linear and Multilinear Algebra*, 61(7):881–894, 2013.

[5] Alessandra Bernardi, Enrico Carlini, Maria Virginia Catalisano, Alessandro Gimigliano, and Alessandro Oneto. The hitchhiker guide to: secant varieties and tensor decomposition. *Mathematics*, 6(12):314, 2018.

[6] Eugenio Bertini. Sui sistemi di ipersuperficie di s_r aventi le stesse prime polari: Nota i. *Rend. Accad. Lincei*, 7:217–227, 1898.

[7] Eugenio Bertini. Sui sistemi di ipersuperficie di s_r aventi le stesse prime polari: Nota ii. *Rend. Accad. Lincei*, 7:275–281, 1898.

[8] Grigoriy Blekherman, Pablo A Parrilo, and Rekha R Thomas. *Semidefinite optimization and convex algebraic geometry.* SIAM, 2012.

[9] Richard Blum and AP Guinand. A quartic with 28 real bitangents. *Canadian Mathematical Bulletin*, 7(3):399–404, 1964.

[10] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36. Springer Science & Business Media, 2013.

[11] Mats Boij, Emanuela De Negri, Alessandro De Stefani, and Maria Evelina Rossi. On the rate of generic gorenstein k-algebras. *arXiv preprint arXiv:2304.14842*, 2023.

[12] Maria Chiara Brambilla and Giorgio Ottaviani. On the Alexander–Hirschowitz theorem. *Journal of Pure and Applied Algebra*, 212(5):1229–1251, 2008.

[13] Marie-Charlotte Brandenburg, Benjamin Hollering, and Irem Portakal. Combinatorics of correlated equilibria. *arXiv preprint arXiv:2209.13938*, 2022.

[14] Paul Breiding, Türkü Özlüm Çelik, Timothy Duff, Alexander Heaton, Aida Maraj, Anna-Laura Sattelberger, Lorenzo Venturello, and Oğuzhan Yürük. Nonlinear algebra and applications. *Numerical Algebra, Control and Optimization*, 13(1):81–116, 2023.

[15] Paul Breiding, Kathlén Kohn, and Bernd Sturmfels. Metric algebraic geometry, 2024.

[16] Paul Breiding, Kemal Rose, and Sascha Timme. Certifying zeros of polynomial systems using interval arithmetic. *ACM Transactions on Mathematical Software*, 49(1):1–14, 2023.

[17] Paul Breiding and Sascha Timme. HomotopyContinuation.jl: A package for homotopy continuation in Julia. In *Mathematical Software – ICMS 2018*, pages 458–465. Springer, Springer International Publishing, 2018.

[18] Davide Bricalli, Filippo Francesco Favale, and Gian Pietro Pirola. On the Hessian of cubic hypersurfaces. *arXiv preprint arXiv:2305.10871*, 2023.

[19] Bruno Buchberger. Ein algorithmus zum auffinden der basiselemente des restklassenringes nach einem nulldimensionalen polynomideal. *Ph. D. Thesis, Math. Inst., University of Innsbruck*, 1965.

[20] Weronika Buczyńska, Jarosław Buczyński, Johannes Kleppe, and Zach Teitler. Apolarity and direct sum decomposability of polynomials. *Michigan Mathematical Journal*, 64(4):675–719, 2015.

[21] Anna Cadoret. Counting real Galois covers of the projective line. *Pacific journal of mathematics*, 219(1):53–81, 2005.

[22] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. Tropical Hurwitz numbers. *Journal of Algebraic Combinatorics*, 32(2):241–265, 2010.

[23] Renzo Cavalieri and Eric Miles. *Riemann surfaces and algebraic curves: a first course in Hurwitz theory*. London Mathematical Society Student Texts. Cambridge University Press, 2016.

[24] A Cayley. Algorithm for the characteristics of the triple-functions. *Journal für die reine und angewandte Mathematik*, 1879.

[25] Arthur Cayley. On the cubic curves inscribed in a given pencil of six lines. *Quarterly J. of Pure and Appl. Maths*, 9:210–221, 1868.

[26] Turku Ozlum Celik, Avinash Kulkarni, Yue Ren, and Mahsa Sayyary Namin. Tritangents and their space sextics. *Journal of Algebra*, 538:290–311, 2019.

[27] A.D.R. Choudary and Alexandru Dimca. On the dual and Hessian mappings of projective hypersurfaces. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 101, pages 461–468. Cambridge University Press, 1987.

[28] Ciro Ciliberto and Giorgio Ottaviani. The Hessian map. *International Mathematics Research Notices*, 2022(8):5781–5817, 2022.

[29] Ciro Ciliberto, Francesco Russo, and Aron Simis. Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian. *Advances in Mathematics*, 218(6):1759–1805, 2008.

[30] A Clebsch. Zur theorie der binären formen sechster ordnung und zur dreitheilung der hyperelliptischen functionen. *Mathematische Annalen*, 2(2):193–197, 1870.

[31] Alfred Clebsch. *Zur Theorie der binären Formen sechster Ordnung und zur Dreitheilung der hyperelliptischen Functionen*, volume 14. Dieterichschen Buchhandlung, 1869.

[32] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1254–1279, 2008.

[33] David A Cox, John B Little, and Henry K Schenck. *Toric varieties*, volume 124. American Mathematical Soc., 2011.

[34] Ruchira S Datta. Universality of Nash equilibria. *Mathematics of Operations Research*, 28(3):424–432, 2003.

[35] Wolfram Decker, Christian Eder, Claus Fieker, Max Horn, and Michael Joswig, editors. *The Computer Algebra System OSCAR: Algorithms and Examples*. Algorithms and Computation in Mathematics. Springer, 2024.

[36] Sandra Di Rocco, Christian Haase, and Benjamin Nill. A note on discrete mixed volume and Hodge–Deligne numbers. *Advances in Applied Mathematics*, 104:1–13, 2019.

[37] Susan J Diesel. Irreducibility and dimension theorems for families of height 3 Gorenstein algebras. *Pacific Journal of Mathematics*, 172:365–397, 1996.

[38] Alexandru Dimca and Gabriel Sticlaru. On the birationality of the Hessian maps of quartic curves and cubic surfaces. *Rendiconti Lincei*, 33(4):885–891, 2023.

[39] Igor Dolgachev and Vassil Kanev. Polar covariants of plane cubics and quartics. *Advances in Mathematics*, 98:216–301, 1993.

[40] Igor V Dolgachev. *Classical algebraic geometry: a modern view.* Cambridge University Press, 2012.

[41] Jan Draisma, Emil Horobeţ, Giorgio Ottaviani, Bernd Sturmfels, and Rekha R Thomas. The Euclidean distance degree of an algebraic variety. *Foundations of computational mathematics*, 16:99–149, 2016.

[42] Mathias Drton, Bernd Sturmfels, and Seth Sullivant. *Lectures on algebraic statistics*, volume 39. Springer Science & Business Media, 2008.

[43] David Eisenbud. *The geometry of syzygies: a second course in algebraic geometry and commutative algebra.* Springer, 2006.

[44] David Eisenbud and Joe Harris. *3264 and all that: A second course in algebraic geometry.* Cambridge University Press, 2016.

[45] Noam D Elkies. The identification of three moduli spaces. *arXiv preprint math/9905195*, 1999.

[46] Maksym Fedorchuk. Direct sum decomposability of polynomials and factorization of associated forms. *Proceedings of the London Mathematical Society*, 120(3):305–327, 2020.

[47] Claudia Fevola and Christiane Görgen. The mathematical research-data repository MathRepo. *arXiv preprint arXiv:2202.04022*, 2022.

[48] Sergey Finashin and Viatcheslav Kharlamov. Two kinds of real lines on real del Pezzo surfaces of degree 1. *Selecta Mathematica*, 27:1–23, 2021.

[49] Alfredo Franchetta. Sulle forme algebriche di s4 aventi l'hessiana indeterminata. *Rend. Mat*, 13:1–6, 1954.

[50] Shmuel Friedland. Remarks on the symmetric rank of symmetric tensors. *SIAM Journal on Matrix Analysis and Applications*, 37(1):320–337, 2016.

[51] William Fulton. Hurwitz schemes and irreducibility of moduli of algebraic curves. *Annals of Mathematics*, pages 542–575, 1969.

[52] Matteo Gallet, Niels Lubbes, Josef Schicho, and Jan Vršek. Reconstruction of rational ruled surfaces from their silhouettes. *Journal of Symbolic Computation*, 104:366–380, 2021.

[53] Alice Garbagnati and Flavia Repetto. A geometrical approach to Gordan-Noether's and Franchetta's contributions to a question posed by Hesse. *Collectanea mathematica*, 60:27–41, 2009.

[54] Dan Geiger, Christopher Meek, and Bernd Sturmfels. On the toric algebra of graphical models. *The Annals of Statistics*, 34(3):1463–1492, 2006.

[55] Israel M Gelfand, Mikhail M Kapranov, Andrei V Zelevinsky, Israel M Gelfand, Mikhail M Kapranov, and Andrei V Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Springer, 1994.

[56] Fulvio Gesmundo, Alessandro Oneto, and Emanuele Ventura. Partially symmetric variants of Comon's problem via simultaneous rank. *SIAM Journal on Matrix Analysis and Applications*, 40(4):1453–1477, 2019.

[57] Rodrigo Gondim and Francesco Russo. On cubic hypersurfaces with vanishing Hessian. *Journal of Pure and Applied Algebra*, 219(4):779–806, 2015.

[58] Paul Gordan and Max Noether. Ueber die algebraischen formen, deren hesse'sche determinante identisch verschwindet. *Mathematische Annalen*, 10:547–568, 1876.

[59] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. Springer, 2010.

[60] Daniel R Grayson and Michael E Stillman. Macaulay2, a software system for research in algebraic geometry, 2002.

[61] Mark L Green. Koszul cohomology and the geometry of projective varieties. *Journal of Differential Geometry*, 19(1):125–171, 1984.

[62] Mathieu Guay-Paquet, Hannah Markwig, and Johannes Rau. The combinatorics of real double Hurwitz numbers with real positive branch points. *International Mathematics Research Notices*, 2016(1):258–293, 2016.

[63] Axel Harnack. Ueber die vieltheiligkeit der ebenen algebraischen curven. *Mathematische Annalen*, 10(2):189–198, 1876.

[64] Corey Harris and Yoav Len. Tritangent planes to space sextics: the algebraic and tropical stories, 2017.

[65] Joe Harris. Galois groups of enumerative problems. *Duke Mathematical Journal*, 46:685–724, 1979.

[66] Joe Harris. *Algebraic geometry: a first course*, volume 133. Springer Science & Business Media, 2013.

[67] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[68] Victoria Hoskins. Moduli problems and geometric invariant theory. *Lecture notes 2016*, 7(28):12, 2015.

[69] Serkan Hoşten and Seth Sullivant. Gröbner bases and polyhedral geometry of reducible and cyclic models. *Journal of Combinatorial Theory, Series A*, 100(2):277–301, 2002.

[70] Eriola Hoxhaj, Jean Michel Menjanahary, and Josef Schicho. Using algebraic geometry to reconstruct a Darboux cyclide from a calibrated camera picture. *Applicable Algebra in Engineering, Communication and Computing*, pages 1–17, 2023.

[71] Bruce Hunt. *The geometry of some special arithmetic quotients*. Springer, 2006.

[72] Adolf Hurwitz. Über Riemann'sche flächen mit gegebenen verzweigungspunkten. *Mathematische Annalen*, 39:1–60, 1891.

[73] Anthony Iarrobino. Compressed algebras: Artin algebras having given socle degrees and maximal length. *Transactions of the American Mathematical Society*, 285(1):337–378, 1984.

[74] Anthony Iarrobino and Vassil Kanev. *Power sums, Gorenstein algebras, and determinantal loci*. Springer Science & Business Media, 1999.

[75] Ilia Itenberg and Dimitri Zvonkine. Hurwitz numbers for real polynomials. *Commentarii Mathematici Helvetici*, 93(3):441–474, 2018.

[76] Tadeusz Józefiak, Piotr Pragacz, and Jerzy Weyman. Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices. In *Tableaux de Young et foncteurs de Schur en algèbre et géométrie - TORUN, Pologne, 1980*, number 87-88 in Astérisque, pages 109–189. Société mathématique de France, 1981.

[77] Vassil Kanev. Chordal varieties of Veronese varieties and catalecticant matrices. *Journal of Mathematical Sciences*, 94:1114–1125, 1999.

[78] Leonie Kayser. The Waring problem for polynomials, January 2022. Available at https://leokayser.github.io/.

[79] Askold G. Khovanskii. Newton polyhedra and the genus of complete intersections. *Functional Analysis and its applications*, 12(1):38–46, 1978.

[80] S Kleiman and Robert Speiser. Enumerative geometry of nonsingular plane cubics. *Algebraic geometry: Sundance 1988*, 116:85–113, 1991.

[81] Vyacheslav Alekseevich Krasnov. On theta characteristics of real algebraic curves. *Mathematical Notes*, 64:347–350, 1996.

[82] Avinash Kulkarni, Yue Ren, Mahsa Sayyary Namin, and Bernd Sturmfels. Real space sextics and their tritangents. In *Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*, pages 247–254, 2018.

[83] Mario Kummer. Totally real theta characteristics. *Annali di Matematica Pura ed Applicata (1923-)*, 198(6):2141–2150, 2019.

[84] Joseph M Landsberg. *Tensors: geometry and applications: geometry and applications*, volume 128. American Mathematical Soc., 2011.

[85] Steffen L Lauritzen. *Graphical models*, volume 17. Clarendon Press, 1996.

[86] Steve Linton. GAP: groups, algorithms, programming. *ACM Communications in Computer Algebra*, 41(3):108–109, 2007.

[87] Luis E Lopez. Images of the polar maps for hypersurfaces. *arXiv preprint arXiv:0811.0754*, 2008.

[88] Christoph Lossen. When does the Hessian determinant vanish identically? (on Gordan and Noether's proof of Hesse's claim). *Bulletin of the Brazilian Mathematical Society, New Series*, 35:71–82, 2004.

[89] Francis Sowerby Macaulay. *The algebraic theory of modular systems*, volume 19. Cambridge University Press, 1994.

[90] Carmelo Mammana. Una caratterizzazione delle ipersuperficie che non sono individuate dal loro sistema primo polare. *Le Matematiche*, 12:125–134, 1957.

[91] Yu I Manin. *Cubic forms: algebra, geometry, arithmetic*. Elsevier, 1986.

[92] Richard D McKelvey and Andrew McLennan. The maximal number of regular totally mixed Nash equilibria. *Journal of Economic Theory*, 72(2):411–425, 1997.

[93] David Mumford. Theta characteristics of an algebraic curve. In *Annales Scientifiques de l'École Normale Supérieure*, volume 4, pages 181–192, 1971.

[94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[95] David Mumford and Kalevi Suominen. Introduction to the theory of moduli. In *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.)*, pages 171–222. Wolters-Noordhoff Publishing, Groningen, 1972.

[96] John F Nash Jr. Equilibrium points in n-person games. *Proceedings of the national academy of sciences*, 36(1):48–49, 1950.

[97] Clemens Nollau. Branched coverings from plane curves, October 2022. Available at https://clemensnollau.wixsite.com/infinityisok/post/my-diploma-thesis.

[98] Luke Oeding. Report on" geometry and representation theory of tensors for computer science, statistics and other areas". *arXiv preprint arXiv:0810.3940*, 2008.

[99] Jared Ongaro. *Plane Hurwitz Number*. PhD thesis, University of Nairobi, 2014.

[100] Jared ONGARO. Formulae for calculating Hurwitz numbers. *Journal of Advanced Studies in Topology*, 10(2), 2019.

[101] Jared Ongaro and Boris Shapiro. A note on planarity stratification of Hurwitz spaces. *Canadian mathematical bulletin*, 58(3):596–609, 2015.

[102] Judea Pearl and Azaria Paz. Graphoids: graph-based logic for reasoning about relevance relations or when would x tell you more about y if you already know z? In *Proceedings of the 7th European Conference on Artificial Intelligence - Volume 2*, page 357–363. North-Holland, 1986.

[103] Irem Portakal and Javier Sendra-Arranz. Game theory of undirected graphical models. *arXiv preprint arXiv:2402.13246*, 2024.

[104] Irem Portakal and Javier Sendra-Arranz. Nash conditional independence curve. *Journal of Symbolic Computation*, 122:102255, 2024.

[105] Irem Portakal and Bernd Sturmfels. Geometry of dependency equilibria. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste: an International Journal of Mathematics*, 54, 2022.

[106] Mario Pucci. The Veronese variety and catalecticant matrices. *Journal of Algebra*, 202(1):72–95, 1998.

[107] Mantas Radzvilas, Gerard J Rothfus, and Wolfgang Spohn. Dependency equilibria: Extending Nash equilibria to entangled belief systems. *work in progress*.

[108] Sevin Recillas. Jacobians of curves with g_4^1's are the Prym's of trigonal curves. *Bol. Soc. at. exicana (2)*, 19(1):9–13, 1974.

[109] Francesco Russo. The antibirational involutions of the plane and the classification of real del Pezzo surfaces. *Algebraic geometry*, pages 289–312, 2002.

[110] Francesco Russo. *On the geometry of some special projective varieties*. Springer, 2016.

[111] Anna Seigal. Ranks and symmetric ranks of cubic surfaces. *Journal of Symbolic Computation*, 101:304–317, 2020.

[112] Javier Sendra-Arranz. The hessian correspondence of hypersurfaces of degree 3 and 4. *arXiv preprint arXiv:2307.10415*, 2023.

[113] Eugenii Shustin. A tropical calculation of the welschinger invariants of real toric del Pezzo surfaces. *arXiv preprint math/0406099*, 2004.

[114] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer, 2009.

[115] Frank Sottile. Real algebraic geometry for geometric constraints. *arXiv preprint arXiv:1606.03127*, 2016.

[116] Wolfgang Spohn. Stochastic independence, causal independence, and shieldability. *Journal of Philosophical logic*, 9:73–99, 1980.

[117] Wolfgang Spohn. Dependency Equilibria and the Casual Structure of Decision and Game Situations. *Homo Oeconomicus*, 20:195–255, 2003.

[118] Wolfgang Spohn. Dependency equilibria. *Philosophy of Science*, 74(5):775–789, 2007.

[119] Bernd Sturmfels. *Solving systems of polynomial equations*. Number 97. American Mathematical Soc., 2002.

[120] Seth Sullivant. *Algebraic statistics*, volume 194. American Mathematical Soc., 2018.

[121] John Arthur Todd. On the simple group of order 25920. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 189(1018):326–358, 1947.

[122] Ravi Vakil. Twelve points on the projective line, branched covers, and rational elliptic fibrations. *Mathematische Annalen*, 320(1):33–54, 2001.

[123] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Inventiones mathematicae*, 164(3):569–590, 2006.

[124] Ravi Vakil. The rising sea: Foundations of algebraic geometry. *preprint*, 2017.

[125] Oleg Viro. From the sixteenth Hilbert problem to tropical geometry. *Japanese Journal of Mathematics*, 3:185–214, 2008.

[126] Junzo Watanabe and Michiel de Bondt. On the theory of Gordan-Noether on homogeneous forms with zero Hessian (improved version). In *Polynomial Rings and Affine Algebraic Geometry: PRAAG 2018, Tokyo, Japan, February 12- 16 1*, pages 73–107. Springer, 2020.

[127] Xinzhen Zhang, Zheng-Hai Huang, and Liqun Qi. Comon's conjecture, rank decomposition, and symmetric rank decomposition of symmetric tensors. *SIAM Journal on Matrix Analysis and Applications*, 37(4):1719–1728, 2016.