Functors between categories of vector spaces; Introduction

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§ 1 Introduction

In the classification of natural vector bundles over differentiable manifolds – a problem posed by J.W. Milnor – D.B.A. Epstein (1969) came across the question of what the functors between categories of finite dimensional vector spaces look like. Together with M. Kneser, he was able to completely clarify the structure in [2] for the case that the vector space's base fields are of characteristic zero. When he first worked on the problem, Epstein needed additional assumptions about the functors, namely that they are "rational" or "continuous" in a certain sense [1]. With these premises, he succeeded in classifying natural vector bundles [3].

Although the results from [1] have been qualitatively improved in [2], I would prefer a discussion of the first paper, as it provides a deeper insight into the mathematics of these functors.

Let $\underline{V}(K)$ be the category of finite dimensional vector spaces over the field K, with K-linear mappings as morphisms. When bases are chosen in the vector spaces of a category, the morphisms are given by matrices of field elements. If K and L are two fields, then a functor

$$F: \underline{V}(K) \to \underline{V}(L)$$

induces, for each vector space $V \in \underline{V}(K)$, a homomorphism of the matrix semigroups

$$F: \mathrm{Hom}_K(V,V) \to \mathrm{Hom}_L(FV,FV).$$

In particular, we obtain a representation of the group GL(V,K) of regular endomorphisms of V. The classical results from representation theory on GL(V) and symmetric groups provide important information about these functors. The mentioned properties "rational" or "continuous" of the functors play a decisive role [4: Chap. III-V].

In order to be able to formulate the results later in the chapter, we need some definitions.

1.0 Definitions:

1.0.1: Let K, L be fields, and C a fixed vector space over L. We have the constant functor from V(K) to V(L), which assigns the fixed space C to every K-vector space and assigns to every morphism – even a 0-morphism

from $\underline{V}(K)$ – the identical mapping $1_C \in \underline{V}(L)$. For $\dim_L C=n$ we speak of the constant functor of dimension n.

1.0.2: Let K be $\subseteq$ L, let V and V' be vector spaces of dimension n or n' over K, and let W and W' be vector spaces of dimension m, m' over L. A function

$$\phi: \mathrm{Hom}_K(V,V') \rightarrow \mathrm{Hom}_L(W,W')$$

is called polynomial or rational if there are m·m' polynomial or rational functions q_{ij}, i= 1,–,m, j=1, –,m' over L, in n·n' variables, so that

$$((a_{kl})) = \begin{pmatrix} q_{11}(a_{11},-,a_{nn}') & \cdots & q_{1m}'(a_{11},-,a_{nn}') \\ \vdots & & \vdots \\ q_{m1}(a_{11},-,a_{nn}') & \cdots & q_{mm}'(a_{11},-,a_{nn}') \end{pmatrix}$$

applies to all matrices $(a_{kl}) \in \mathrm{Hom}_K(V,V')$.

1.0.3: If the base field has characteristic zero and a norm is defined on it, a norm ||—|| of a matrix A= (aij), with coefficients aij from this field, is defined as:

$$||A|| = +\sqrt{\sum_{i,j} |a_{ij}|^2}$$

If, in the following, we require normalised fields, we will restrict ourselves to the base fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers, in view of Pontrjagin's theorem [14: p.59].

1.0.4: If K and L are equal to $\mathbb{R}$ or $\mathbb{C}$, the continuity of a function

$$\phi: \mathrm{Hom}_K(V,V') \rightarrow \mathrm{Hom}_L(W,W')$$

is defined with regard to that norm ||—||.

1.0.5: This also explains what is meant by a continuous, rational, or polynomial representation over the field L

$$\phi: \mathrm{GL}(V) \rightarrow \mathrm{Hom}_L(W,W')$$

of the general linear group.

The structure of these three classes of representations is now precisely known, so it is useful to transfer the conceptualisation to functors:

1.0.6: If $K \subseteq L$, then a functor

$$F: \underline{V}(K) \rightarrow \underline{V}(L)$$

is called "polynomial" or "rational" if the mappings

$$F: \mathrm{Hom}_K(V,W) \to \mathrm{Hom}_L(FV,FW)$$

of the morphism sets induced by it are polynomial or rational for all pairs (V,W) of vector spaces from $\underline{V}(K)$.

If K and L are equal to $\mathbb{R}$ or $\mathbb{C}$, the functor is called continuous if the induced morphism mappings are continuous.

The representation of GL(V) induced by F, $V \in \underline{V}(K)$, then has the respective ones of the three functor properties.

1.0.7: A functor $F: \underline{V}(K) \to \underline{V}(L)$ is called a subfunctor of $G: \underline{V}(K) \to \underline{V}(L)$ if, for all $V \in \underline{V}(K)$, FV is a subspace of GV, and, for all morphisms f from $\underline{V}(K)$, Ff coincides with the restriction of Gf on FV.

A subfunctor of a rational, polynomial or continuous functor itself has the corresponding property.

1.0.8: A functor is called irreducible if 0 and F are the only subfunctors.

For example, F is irreducible if for all $V \in \underline{V}(K)$ the induced representation of GL(V) is irreducible, because if there were a real non-trivial subfunctor F' of F, there would also be a vector space $V \in \underline{V}(K)$, so that F'V would be a real non-trivial subspace of FV, which would remain invariant under all morphisms $F'g = Fg \mid_{F'V}$.

1.1 Classical representation theory of GL(V)

I would like to briefly outline the approach of classical representation theory on GL(V), because it sheds light on the approach of the present categorical investigations:

For two natural numbers n and f there is a $_{K}V$ vector space over K on which GL(n,K) and the symmetric group S(f) can be represented together so that the representations are centralised, namely the tensor product of f copies of the standard space V of dimension n over K, called the f-th tensor power $\otimes^f_K V$. GL is represented canonically: For an $A \in GL(n,K)$, the f-fold Kronecker product $\otimes^f_K A$ operates on $\otimes^f_K V$. S(f) is represented by permutation of the factors, i.e., if $\sigma \in S(f)$, then σ acts on a typical element as:

$$\sigma: v_1 \otimes \ldots \otimes v_f \to v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(f)}.$$

The representations of S(f) are known since two reports by Frobenius in 1900 and 1903. A. Young had developed the theory independently; the Young diagrams used are named after him (a good presentation can be found in v. d. Waerden [13]). The irreducible components of the

representations of GL can now be calculated, since they are centralised with the representations of the symmetric groups [4].

1.2 Examples

1.2.a: The fact that the described representations are centralised can be expressed categorically as follows:

Let $\otimes^f$: $\underline{V}(K) \to \underline{V}(L)$ be the functor defined by

$V \to \otimes^f_K V \otimes_K L$ for $V \in \underline{V}(K)$

$g \to \otimes^f_K g \otimes_K 1_L$ for $g: V \to W \in \underline{V}(K)$

(For f=0, let $\otimes^f$ be the constant functor of dimension 1.)

Let the group ring LS(f) be represented as above, i.e. for

$$\sum_{\sigma \in S(f)} \alpha_\sigma \cdot \sigma \quad \in LS(f)$$

apply

$$\alpha: v_1 \otimes \ldots \otimes v_f \longmapsto \sum_{\sigma \in S(f)} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(f)} \otimes \alpha_\sigma$$

Then α causes a natural transformation of the functor $\otimes^f$ into itself, since

$$\begin{array}{ccc} \otimes^f V \otimes_K L & \xrightarrow{\alpha_V} & \otimes^f V \otimes_K L \\ \downarrow \otimes^f g \otimes 1_L & & \downarrow \otimes^f g \otimes 1_L \\ \otimes^f W \otimes_K L & \xrightarrow{\alpha_W} & \otimes^f W \otimes_K L \end{array}$$

is commutative for all $g: V \to W \in \underline{V}(K)$.

If one restricts $\otimes^f g \otimes^1 L$ onto $\mathrm{Im}(\alpha V)$, we have

$$\otimes^f g \otimes 1 : \mathrm{Im}(\alpha_V) \longrightarrow \mathrm{Im}(\alpha_W),$$

i.e., the functor $\mathrm{Im}\alpha$: $\underline{V}(K) \to \underline{V}(L)$ with

$V \to \mathrm{Im}(\alpha V)$

$g \to \otimes^f g \otimes^1 | \mathrm{Im}(\alpha V)$

is a subfunctor of $\otimes^f$ with the epimorphic natural transformation

$\alpha: \otimes^f \to \mathrm{Im}\,\alpha$.

Functor $\otimes^f$ is a naturally polynomial functor (even "homogeneous of degree f", i.e. the coefficients of the matrices $\otimes^f A$ are monomials, simple power products of the coefficients of A of degree f, and indeed all possible power

products occur), as are Imα as a subfunctor, and direct sums of $\otimes^{fi}$ or Imα_i. The fact that these are already all possible polynomial and even all possible rational or continuous functors for fields of characteristic zero is proven in the main theorem, 1. 7.

1.2.b: Let θ: K $\rightarrow$ L_1 be a field embedding (i.e., a field monomorphism), where L_1 is a finite field extension of L. By means of θ, L_1 is a (K,L)-module. We have a functor F: $\underline{V}(K) \rightarrow \underline{V}(L)$, which maps V onto $V \otimes_K L_1$. In particular, we obtain a non-continuous functor $\underline{V}(\mathbb{R}) \rightarrow \underline{V}(\mathbb{R})$ if we use a non-continuous field embedding θ: $\mathbb{R} \rightarrow \mathbb{C}$.

If one adjoins all eigenvalues of the images of diagonalisable endomorphisms to L, then one obtains an extension field L_1 of L, such that the functor

$$\underline{V}(K) \xrightarrow{F} \underline{V}(L) \xrightarrow{\otimes^1} \underline{V}(L_1)$$

is splitting. We can define:

1.3.3 Definition: The smallest field L_1, such that the functor $\otimes^1 \circ F$ is splitting, is called the splitting-field (Zerfällungskörper) of F.

In the present work, work is mostly carried out on the splitting field.

1.4 Diagonal functors

1.4.1 Definition: A functor F is called a diagonal functor if the images Ff of all diagonalisable endomorphisms f are also diagonalisable. The well-known theorem [13] is of interest here:

1.4.2 Theorem: An endomorphism g is diagonalisable if all eigenvalues lie in the field and for each eigenvalue c with multiplicity k applies:

Rank (g–cE)= n–k.

Every diagonal functor is therefore in particular splitting. Example 1.2.b shows that not every functor is diagonal, even if the field is extended; Example 1.2.c shows that not every functor is splitting.

The main results of the work can now be formulated:

Let K be an infinite field, F: $\underline{V}(K) \rightarrow \underline{V}(L)$ a non-constant diagonal functor. Then there is a field embedding of K in L (i.e. a field monomorphism). In particular, K and L have the same characteristic.

1.5.1 Definitions: Let $\underline{V}(K)^n = V(K) \times \ldots \times \underline{V}(K)$ (n copies) and $\underline{V}(K)^0$ be the category with a single object (K as vector space over itself) and a morphism (1_K).

Δ: $\underline{V}(K) \to \underline{V}(K)^n$

denotes the diagonal, i.e.,

$V \xrightarrow{\Delta} (V, \dots, V)$

$f \xrightarrow{\Delta} (f, \dots, f)$.

A functor F is called a direct sum of subfunctors F_α if for all $V \in \underline{V}(K)$ we have:

$$FV \cong \bigoplus_\alpha F_\alpha V,$$

and for all $\underline{f} \in \underline{V}(K)$ we have

$$Ff \cong \bigoplus_\alpha F_\alpha f,$$

where

$$\begin{array}{ccc} FV & \xrightarrow{\cong} & \bigoplus_\alpha F_\alpha V \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle \bigoplus_\alpha F_\alpha f} \\ FW & \xrightarrow{\cong} & \bigoplus_\alpha F_\alpha W \end{array}$$

is commutative.

1.6 Theorem:

Let $F: \underline{V}(K) \to \underline{V}(L)$ be a diagonal functor, where charK = charL = 0. Then F decomposes into a direct sum of functors $F_\alpha: \underline{V}(K) \to \underline{V}(L)$, and for each F_α there is a functor $G_\alpha: \underline{V}(K)^{n(\alpha)} \to \underline{V}(L)$ that is additive in all variables, so that F is a direct summand of the composite functor

$$\underline{V}(K) \xrightarrow{\Delta} \underline{V}(K)^{n(\alpha)} \xrightarrow{G_\alpha} \underline{V}(L).$$

Explicitly, $F_\alpha V$ is a direct summand of $G_\alpha(V, \dots, V)$.

1.7 Theorem:

Let $K \subseteq L$ be fields of characteristic zero and $F: \underline{V}(K) \to \underline{V}(L)$ be a rational or continuous functor, where, for the latter case, $K = \mathbb{R}$. Then F is diagonal (and therefore splitting), and 1.6 can be applied to F. The following more general result even holds:

(i) F is the direct sum of irreducible subfunctors F_β.

(ii) For an irreducible functor F' there is a natural number n and a primitive idempotent $\alpha \in \mathbb{Q}S(n)$, so that $F' \cong \mathrm{Im}\alpha$.

(A primitive idempotent is an idempotent that is not the sum of two non-vanishing orthogonal idempotents [4: p.57]).

(iii) If $\alpha \in \mathbb{Q}S(n)$ and $\beta \in \mathbb{Q}S(m)$ are primitive idempotents, then $Im\alpha \cong Im\beta$ if n=m and the left-$\mathbb{Q}S(n)$-modules $\mathbb{Q}S(n)\alpha$ and $\mathbb{Q}S(n)\beta$ are isomorphic.

1.8 Remark:

The direct sums in 1.7 and 1.8 are not necessarily finite. Consider, for example, the functor that assigns to each vector space E its tensor algebra $\bigoplus_{i=0}^{\infty} E^i$[10: S. 422]. On the other hand, due to the finite dimensions of the vector spaces in question, every functor sum is locally finite, i.e. if $F \cong \bigoplus_{\alpha} F_\alpha$, then, for every $V \in V(K)$, $F_\alpha V=0$ for almost all α.

1.9 Remark:

In an abelian category, the usual theorems about irreducible or simple objects apply. For example, for non-isomorphic simple objects A and B it is Hom(A,B)=0 and Hom(A,A) is a division ring (Schur's lemma [4]); if an object is a direct sum of irreducible subobjects, then each subobject is a direct sum of a subset of the irreducible subobjects. In our case, the categories $\underline{V}(K)$ and the category of functors from $\underline{V}(K)$ to $\underline{V}(L)$ are abelian. If two decompositions $\bigoplus_{\alpha} F_\alpha$ and $\bigoplus_{\beta} F_\beta$ of F into irreducible subfunctors are given, then for every α there is therefore a β with $F_\alpha \cong F_\beta$, and the number of isomorphic F_α in a given isomorphism class is the same in every decomposition. In particular, such a number is finite, since for a vector space V whose image $F_\alpha V$ is non-zero, all isomorphic images $F_\beta V$ are also non-zero, yet the subspace $\bigoplus_{F_\beta \cong F_\alpha} F_\beta V$ of FV is finite-dimensional.

1. 10 Overview:

When, for a given splitting functor $F: \underline{V}(K) \to V(L)$ we consider for each vector space $V \in \underline{V}(K)$, the representation of the abelian group of scalar multiplications induced by F, the known decompositions of the spaces FV into weight spaces under this representation induce a decomposition of the functor into a direct sum of weight functors [10: Theorem II.7, p. 43]. A weight of a functor is understood to be a mapping $\alpha: K \to L$ which assigns, to each field element $\mu \in K$, an eigenvalue of the functor image $F(\mu_V)$ of the scalar multiplication of μ on V. (The definition – as it turns out – is independent of the choice of V.) This is explained in more detail in §3.

§§4 and 5 of the paper clarify the structure of these weights for rational, continuous and diagonal functors. It is this part of the work that was improved in the continuation work by Epstein and Kneser [2], namely it turned out that the structure of the weights is already determined when the

base fields have the characteristic zero, without the restrictive conditions on the functors.

The weight functors occurring in the decomposition of a functor are "homogeneous" (definition in 6.3). In §6 it is shown how they can be transformed into a composite functor

$$\underline{V}(K) \xrightarrow{\Delta} \underline{V}(K)^n \xrightarrow{G} \underline{V}(L)$$

can be embedded. Here, Δ is the diagonal, G is additive in each variable. The results of the paragraph are summarised in 1.6.

In §7, Theorem 1.6 is finally refined into the main theorem 1.7.

The mentioned functor G arises from "deviation functors" $F^{(n)}$, whereby a construction of Eilenberg and MacLane is transferred to functors [6]. This is the subject of §3.

In §2 of the article, Epstein gave pointers for further research. With regard to the achievements in [2] it becomes dispensable.

In my numbering, I follow the original source [1] very closely in order to make cross-referencing easier.

REFERENCES (original papers)

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[2] D.B.A. Epstein and M. Kneser: Functors between categories of vector spaces. Springer Lecture Notes 99 {Category Theory, Homology Theory, and their Applications III, Battelle Institute Conference 1968, 154-170.

[3] D.B.A. Epstein: Natural vector bundles. Ibid., 171-195.

[6] Eilenberg and MacLane: On the Groups H(π,n), II. Annals Math. 60 (1954), 75-83.