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Abstract

In this paper we generalize a chaos control method developed by Ott, Grebogi, and Yorke (1990) to control saddle points in $\mathbb{R}^2$ which are embedded in a strange attractor of a chaotic system. Our generalized method admits to control any unstable equilibrium in $\mathbb{R}^n$. We apply our findings to control the dynamics of the chaotic asset pricing model of Brock and Hommes (1998). In this model chaotic price movements are caused by heterogeneous market participants. We introduce a control authority which trades the risky asset like the other market participants. Using our control approach, it is possible for the authority to stabilize the market price with minimum effort.

1 Introduction

During the last 20 years chaos theory became an increasing field of interest in economics and finance, a field which originally arose from physics and mathematics [1].

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[1] Lorenz (1963), and May (1976).

Fundamental work was done by Day (1983), Benhabib and Day (1982), Grandmont (1985), and Scheinkman and LeBaron (1989).

Chaotic systems with its erratic fluctuations give us a broader perspective and an improved understanding of observed economic and financial time series. Instead of assuming linear relations and using linear stochastic variables to describe erratic behavior, chaos theory allows to model fluctuations endogenously. In a chaotic environment the existing attractor is a non periodic orbit. A trajectory stays in this attracting set without reaching any point twice. This attracting set is called a strange attractor. Trajectories on this strange attractor exhibit sensitive dependence on initial conditions. This is known as the "butterfly effect": A small intervention in the model has a high impact on the trajectory. After a short time the trajectory displays a completely different path than without this intervention. However, the global behavior of chaotic models is similar to stochastic models: Fluctuations are erratic and unpredictable in the long run.

A recent asset pricing model based on chaotic price fluctuations is given by Brock and Hommes (1998). In their model heterogeneous groups of investors with different expectations for the future market price generate demand and supply for the risky asset.[2] The resulting price is an equilibrium price which clears the market. Depending on the success of previous predictions market participants then adapt their future price beliefs. These adaptions of investors' beliefs give the model the necessary nonlinear structure to allow chaotic dynamics. Extensions of this model have been suggested by Chiarella and He (1999) and Gaunersdorfer (2000), recently.

A common interest of market participants is the reduction of price volatility. However, if market fluctuations are stochastic, there is no possibility to reduce volatility without interfering in the market process. On the other hand, if fluctuations are chaotic there exists a first control approach developed in the physics by Ott, Grebogi, and Yorke (1990) (OGY). They stabilize saddle points embedded in the strange attractor

[2] Heterogeneous modelling of asset markets are in line with the recent literature. See e. g. Brock and Hommes (1997a), Day and Huang (1990), De Fontnouvelle (1996), and LeBaron (1999).

of a chaotic 2-dimensional system only with small perturbations.[3] Small interventions are sufficient due to the sensitivity of a chaotic system. A successful application of this method was shown in a clinical experiment by Garfinkel, Spano, Ditto, and Weiss (1992) who controlled a chaotic heart beat with small electric pulses. In social sciences this method was first introduced by Holyst, Hagel, Haag, and Weidlich (1996) in a model of two competing firms with asymmetrical investment strategies.[4] However, this method is restricted to control saddle points in $\mathbb{R}^2$.

The plan of this paper is as follows: In Section 2 we show how the OGY approach can be extended to control any unstable equilibrium and generalize it for chaotic systems in $\mathbb{R}^n$. In Section 3 we apply our control approach to the chaotic asset pricing model of Brock and Hommes (1998). We assume the existence of a control authority which wants to reduce market volatility by stabilizing the asset price on its fundamental value. It acts like an ordinary market participant generating demand or supply for the risky asset. However, for these interventions the control authority has only limited resources available. Albeit, in a chaotic environment it is able to stabilize any unstable equilibrium with our approach. Section 4 concludes.

2 Control method

We use the sensitivity property to control chaotic systems with minimal perturbations. Only small interventions are necessary to cause huge variations along the system trajectory. In any strange attractor there are equilibria of different periods embedded. These

[3] A saddle point possesses a stable and an unstable manifold. Roughly speaking a stable manifold of a fixed point is a surface contained in the phase space of a dynamical system which has the property that trajectories starting on this surface will converge to the fixed point. Trajectories on the unstable manifold diverge from the fixed point, respectively. See Wiggins (1988), pp. 26.

[4] A different approach to control unstable equilibriums was suggested by Romeiras and Dayawansa (1992), and Shinbrot, Grebogi, Ott, and Yorke (1993). They use the pole placement method, an approach which is well known from applications in engineering sciences: See e. g. Ogata (1990), pp. 776. This method was used by Kaas (1998) to control a chaotic macroeconomic model.

equilibria are never reached by an uncontrolled trajectory because they are unstable and have a repelling neighborhood. However, by waiting till the trajectory comes close to the saddle point and then pushing it with a small intervention onto the stable manifold, the chaotic system can be stabilized and the trajectory converges to the saddle point. The intervention tends to zero the closer the trajectory gets to the saddle point. This procedure guarantees only small interventions in the model which do not destroy the overall chaotic behavior. We now show how this control idea can be extended to control any unstable equilibrium in $\mathbb{R}^n$.[5]

Because interventions take place only if the trajectory is close to the unstable fixed point, it is sufficient to consider its linearized neighborhood. Assume, the chaotic map

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t) = \begin{pmatrix} f_1(x_{1,t}, ..., x_{n,t}) \\ \vdots \\ f_n(x_{1,t}, ..., x_{n,t}) \end{pmatrix} \tag{1}$$

with $\mathbf{x}'_t = (x_{1,t}, ..., x_{n,t}) \in \mathbb{R}^n$ possesses an unstable fixed point $\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*)$ embedded in the strange attractor of (1). To linearize the neighborhood of the fixed point we use the Jacobian

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{\mathbf{x}^*}$$

to get

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{x}^* + \mathbf{J}(\mathbf{x}^* - \mathbf{x}_t) \\ \Delta\mathbf{x}_{t+1} &= \mathbf{J}\Delta\mathbf{x}_t \qquad\qquad \text{with } \Delta\mathbf{x}_t = (\mathbf{x}^* - \mathbf{x}_t). \end{aligned} \tag{2}$$

Using the eigenvectors $\mathbf{e}_i$ of the Jacobian as a new basis [6] we can write $\Delta\mathbf{x}_t = a_1\mathbf{e}_1 +$

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[5] We restrict our discussion on controlling fixed points, but the procedure to control equilibria of higher order $\mathbf{x}^* = \mathbf{f}^k(\mathbf{x}^*)$, $k = 2, 3, ...$ is similar. Instead of $\mathbf{f}(\mathbf{x})$ we consider the map $\mathbf{g}(\mathbf{x}) = \mathbf{f}^k(\mathbf{x})$ and treat the equilibria as fixed points $\mathbf{x}^* = \mathbf{g}(\mathbf{x}^*)$.

[6] For the moment we assume the Jacobian possesses n independent real eigenvectors. In appendix A.1, p. 20 we consider also the case when the fixed point possesses complex eigenvectors.

$... + a_n \mathbf{e}_n$ and get

$$\begin{aligned} \Delta \mathbf{x}_{t+1} &= \mathbf{J}\left(a_1 \mathbf{e}_1 + ... + a_n \mathbf{e}_n\right) \\ &= a_1 \lambda_1 \mathbf{e}_1 + ... + a_n \lambda_n \mathbf{e}_n. \end{aligned} \tag{3}$$

The control target is to direct the trajectory on the stable manifold of the fixed point. The stable manifold is a subspace of the state space $\mathbb{W}^s \subseteq \mathbb{R}^n$ on which all trajectories converge to the fixed point for $t \to \infty$. In contrast, on the unstable manifold the trajectories converge in negative time $t \to -\infty$ to the fixed point. In the linearized neighborhood of the fixed point the eigenvectors of the Jacobian $\mathbf{J}$ span the subspaces of these manifolds. The eigenvectors with eigenvalues $|\lambda_i| < 1$ span the stable manifold $\mathbb{W}^s = span\{\mathbf{e}_1, ..., \mathbf{e}_m\}$ and the eigenvectors with eigenvalues $|\lambda_i| > 1$ span the unstable manifold $\mathbb{W}^u = span\{\mathbf{e}_{m+1}, ..., \mathbf{e}_n\}$. The trajectory on the stable manifold has to be orthogonal to the unstable manifold. If in a n-dimensional state space the unstable manifold is $(n-m)$-dimensional $(n > m)$ we need $n-m$ control variables p_j to direct the trajectory during one period on the stable manifold. We assume the perturbations $\delta \mathbf{p}_t = \left(\delta p_{1,t} \quad \cdots \quad \delta p_{n-m,t}\right)'$ in each period are restricted to $|\delta p_{j,t}| \leq \delta p_{j,\max}$. This leads to the following control rule:

$$\delta \mathbf{p}_t = \begin{cases} \mathbf{W}^{-1}\mathbf{F}^{-1}\boldsymbol{\lambda}\mathbf{F}\Delta\mathbf{x}_t & if \quad |\mathbf{W}^{-1}\mathbf{F}^{-1}\boldsymbol{\lambda}\mathbf{F}\Delta\mathbf{x}_t| \leqq \delta \mathbf{p}_{\max} \\ \mathbf{0} & otherwise. \end{cases} \tag{4}$$

See appendix A.1 or the derivation of equation (4). $\mathbf{W}$ contains the derivatives of $\mathbf{f}$ with respect to the control variables p_j at the fixed point $\mathbf{x}^*$

$$\mathbf{W} = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_{n-m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial p_1} & \cdots & \frac{\partial f_n}{\partial p_{n-m}} \end{bmatrix}_{\mathbf{x}^*}.$$

$\mathbf{F}$ is a matrix consisting of the vectors $\mathbf{f}_i$. These are the contravariant basis vectors to the eigenvectors $\mathbf{e}_i$.[7] The control is started as soon as the trajectory is close to the fixed point and the computed perturbations do not exceed the maximum value $\delta \mathbf{p}_{\max}$.

[7] The contravariant basis $\mathbf{f}_i$ to the basis $\mathbf{e}_i$, $i = 1, ..., n$ is defined by the following properties: $\mathbf{f}_i' \mathbf{e}_j = 0$ for $i \neq j$ and $\mathbf{f}_i' \mathbf{e}_j = 1$ for $i = j$.

In case the fixed point has no stable manifold, however, there is no subspace on which a trajectory converges to the equilibrium. The condition to direct the trajectory into the fixed point then simply is $\Delta \mathbf{x}_{t+1} = \mathbf{0}$ which leads to the following control rule:[8]

$$\delta \mathbf{p}_t = \begin{cases} -\mathbf{W}^{-1}\mathbf{J}\Delta\mathbf{x}_t & if \quad |-\mathbf{W}^{-1}\mathbf{J}\Delta\mathbf{x}_t| \leqq \delta\mathbf{p}_{\max} \\ \mathbf{0} & otherwise. \end{cases} \tag{5}$$

In equation (5) the vector with the perturbations of the control variables $\delta \mathbf{p}_t$ is n-dimensional. n control variables are necessary to control the trajectory during one period straight into the fixed point.

With equations (4) and (5) at hand we are now able to control unstable equilibria in chaotic systems of arbitrary dimension. In the following section we use our findings to control the chaotic asset pricing model of Brock and Hommes (1998).

3 The Control of the Asset Pricing Model of Brock and Hommes (1998)

Brock and Hommes (1998) propose an asset pricing model with adaptive beliefs of heterogeneous investors. Chaotic behavior is inherent in their model for specific parameter values. In this section we demonstrate how the chaotic asset price fluctuations can be stabilized using our control method.

3.1 Asset Pricing Model

We review the model, first. Investors are assumed to invest in a risky and a risk free asset, only. They have the following wealth function

$$\widetilde{W}_{t+1} = RW_t + \underbrace{(\widetilde{p}_{t+1} + \widetilde{y}_{t+1} - Rp_t)}_{excess\ return\ per\ share} z_t, \tag{6}$$

[8] See appendix A.2.

where a tilde denotes a random variable. $1/R$ is the riskless discount factor. p_t denotes the price of the shares ex dividend at time t and $\{\widetilde{y}_t\}$ is the stochastic dividend process. z_t is the number of shares kept in stock at time t.

Brock and Hommes (1998) distinguish investor groups with different future beliefs. Each group computes the optimal number of assets in stock with a mean variance utility function

$$\arg\max_z \left\{ E_{h,t}\left(\widetilde{W}_{t+1}\right) - \frac{a}{2} V_{h,t}\left(\widetilde{W}_{t+1}\right)\right\}.$$

$E_{h,t}(\ )$ and $V_{h,t}(\ )$ are the beliefs of the conditional expectation and variance of investor type h. The parameter a determines the riskaversion which is assumed to be equal for all investors. Also the conditional variance of the *excess return per share* $V_{h,t}\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1} - Rp_t\right) \equiv \sigma^2$ is assumed to be constant. Hence, we get for the variance of the investor's portfolio $V_{h,t}\left(\widetilde{W}_{t+1}\right) = z_{h,t}^2 \sigma^2$. Each type h investor has to maximize

$$\arg\max_z \left\{ RW_t + \left(E_{h,t}\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1}\right) - Rp_t\right) z_{h,t} - \frac{a}{2} z_{h,t}^2 \sigma^2 \right\} \tag{7}$$

with the result

$$z_{h,t} = \frac{\left(E_{h,t}\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1}\right) - Rp_t\right)}{a\sigma^2}. \tag{8}$$

The optimal number of shares depends only on the price and the dividend process. The wealth W_t in equation (7) has no influence on the maximization.

Each investor group $h = 1, ..., j$ has a market share of $n_{h,t}$ at time t. The equilibrium of demand and supply implies

$$\sum_{h=1}^{j} n_{h,t} \frac{E_{h,t}\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1}\right) - Rp_t}{a\sigma^2} = z_{s,t}. \tag{9}$$

$z_{s,t}$ denotes the supply of shares per investor in t. For their further analysis Brock and Hommes (1998) assume zero supply of outside shares $z_{s,t} = 0$.

To get a first impact of the solution of the stochastic difference equation(9) consider the case of homogeneous investors

$$E_t\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1}\right) = Rp_t \tag{10}$$

with the information set $\mathcal{F}_t = \{p_t, p_{t-1}, ..., y_t, y_{t-1}, ...\}$, first. Furthermore, let us assume a growing stochastic dividend process [9]

$$\widetilde{y}_{t+1} = \alpha y_t + \widetilde{\varepsilon}_{t+1} \qquad \text{with } \alpha > 1, \widetilde{\varepsilon}_{t+1} \text{ is IID, and } E(\varepsilon) = 0. \tag{11}$$

Using the forward solution technique to solve equation (10) we get the fundamental solution

$$p_t^* = y_t \frac{\alpha}{R - \alpha} \tag{12}$$

which satisfies the "no bubble" condition $\underset{\tau\to\infty}{lim} E_t\left(\widetilde{p}_{t+\tau}\right)/R^\tau = 0$ for $R > \alpha$.

Denote the price deviation from the fundamental solution as

$$x_t = p_t - p_t^* \tag{13}$$

and assume the heterogeneous beliefs are of the following form

$$E_{h,t}\left(\widetilde{p}_{t+1} + \widetilde{y}_{t+1}\right) = E_t\left(p_{t+1}^* + \widetilde{y}_{t+1}\right) + f_h\left(x_{t-1}, ..., x_{t-L}\right),$$

where $E_t\left(p_{t+1}^* + \widetilde{y}_{t+1}\right)$ is the conditional expectation on the fundamental solution on the information set $\mathcal{F}_t$, and f_h is a deterministic function of each investor type h investor for the estimated deviation from the fundamental solution in period $t+1$. Investors predict the next deviation using past deviations up to a time lag L. We therefore can rewrite equation (9)

$$\begin{aligned} Rp_t &= E_t\left(p_{t+1}^* + \widetilde{y}_{t+1}\right) + \sum_{h=1}^{j} n_{h,t} f_h\left(x_{t-1}, ..., x_{t-L}\right) \\ Rx_t + Rp_t^* &= E_t\left(p_{t+1}^* + \widetilde{y}_{t+1}\right) + \sum_{h=1}^{j} n_{h,t} f_h\left(x_{t-1}, ..., x_{t-L}\right). \end{aligned}$$

Using (10) we get

$$Rx_t = \sum_{h=1}^{j} n_{h,t} f_h\left(x_{t-1}, ..., x_{t-L}\right). \tag{14}$$

[9] Brock and Hommes (1998) assume an IID dividend process $\{\widetilde{y}_t\}$ with constant expectation $E_t\left(\widetilde{y}_t\right) = \overline{y}$.

Now the adaptive dynamics of the asset pricing model are introduced. The investors adapt their beliefs of future price deviations every period by switching from an investor group with unsuccessful prediction in the past to a successful investor group. We get new market shares $n_{h,t}$ immediately after the price deviation x_t is determined. To incorporate this adaption of beliefs, the market share in equation (14) has to be lagged by one period

$$Rx_t = \sum_{h=1}^{j} n_{h,t-1} f_{h,t-1}\left(x_{t-1}, ..., x_{t-L}\right), \tag{15}$$

where $n_{h,t-1}$ are the relevant market fractions to determine the price deviation x_t. Before determining the new market fractions of each investor type, a performance measure has to be defined. Brock and Hommes (1998) use the realized excess profit:

$$\begin{aligned} \pi_{h,t-1} &= \left(p_{t+1} + y_{t+1} - Rp_t\right) z_{h,t-1} \\ &= \left(p_t^* + x_t + \alpha y_{t-1} + \varepsilon_t - Rp_{t-1}^* - Rx_{t-1}\right) z_{h,t-1}. \end{aligned}$$

Using equation (12) and (8) we get

$$\begin{aligned} \pi_{h,t-1} &= \left(x_t - Rx_{t-1} + \varepsilon_t\right) \frac{\left(E_{h,t-1}\left(\widetilde{p}_t + \widetilde{y}_t\right) - Rp_{t-1}\right)}{a\sigma^2} \\ &= \left(x_t - Rx_{t-1} + \varepsilon_t\right) \frac{\left(f_h\left(x_{t-2}, ..., x_{t-L-1}\right) - Rx_{t-1}\right)}{a\sigma^2}. \end{aligned}$$

We assume further that the investors have to gather costly information to make up their beliefs of the future price deviations. We subtract information cost C_h from the performance measure and get

$$\pi_{h,t-1} = \left(x_t - Rx_{t-1} + \varepsilon_t\right) \frac{\left(f_h\left(x_{t-2}, ..., x_{t-L-1}\right) - Rx_{t-1}\right)}{a\sigma^2} - C_h. \tag{16}$$

If investors of group h rely not only on last but also on previous performances of the prediction technique, we get the following adjusted performance measure

$$u_{h,t-1} = \pi_{h,t-1} + \eta u_{h,t-2}. \tag{17}$$

Using the discrete choice theory [10] Brock and Hommes (1998) model agents' choice of expectation strategy by extending equation (17) with a stochastic variable $\widetilde{\nu}_{h,t-1}$

$$U_{h,t-1} = u_{h,t-1} + \frac{\widetilde{\nu}_{h,t-1}}{\beta}. \tag{18}$$

The introduction of a stochastic component has the objective to make the model more realistic:[11] In a heterogeneous complex environment there exists more degrees of freedom. $\widetilde{\nu}_{h,t-1}$ represents the unmodelled heterogeneity and $u_{h,t-1}$ is only the observable part.[12] The *intensity of choice parameter* β determines the influence of the stochastic component. For a low β the stochastic component has a large influence on the performance measure and the previous profits of an expectation strategy (16) have almost no effect on choosing the expectation strategy for the following period and vice versa for large β. It is assumed, $\widetilde{\nu}_{h,t-1}$ to be double exponential distributed. The shape of the distribution function of a double exponential distributed random variable is close to the one of a normal distributed variable, but has the big advantage of the existence of a closed form solution.[13] With this distribution we get for the possibility that in the next period an investor joins the group h [14]

$$n_{h,t-1} = \frac{exp(\beta u_{h,t-1})}{Z_{t-1}} \tag{19}$$

where

$$u_{h,t-1} = \pi_{h,t-1} + \eta u_{h,t-2},$$

and

$$Z_t = \sum_{h=1}^{n} exp(\beta u_{h,t-1}).$$

[10] For the discrete choice theory see Anderson, de Palma, and Thisse (1992). Further implementation of the discrete choice theory in economic models are found in Brock and Hommes (1997b), Goeree and Hommes (2000), and Brock and de Fontnouvellel (2000).

[11] See De Fontnouvelle (1996), p. 11.

[12] For a broader discussion on the stochastic component, see Anderson, de Palma, and Thisse (1992), p. 33.

[13] See Anderson, de Palma, and Thisse (1992), p. 39.

[14] See De Fontnouvelle (1996), p. 12, and Anderson, de Palma, and Thisse (1992), p. 39.

Equation (**??**)can be interpreted as new market fraction, and together with (14) we have the model's difference equations to compute future price deviations x.

3.2 The Dynamic and Control of the Asset Pricing Model

To investigate the dynamics, Brock and Hommes (1998) assume a linear expectation

$$f_{h,t} = g_h x_{t-1} + b_h \tag{20}$$

where g_h is the trend factor and b_h the bias of investor type h. A *fundamentalist*, believing on a return of the price to its fundamental value, has a trend factor and a bias equal to zero: $g_h = b_h = 0$. A *pure trend chaser* has a trend factor $g > 0$, and the opposite ($g < 0$) is called a *contrarian*. In the following the decision of the investors do only depend on the performance of the last period and therefore $\eta = 0$. The dividend payment (11) is assumed to be deterministic $\widetilde{\varepsilon}_t = 0$. We get the following equation for the model

$$\begin{aligned} x_{t+1} &= F(x_t, x_{t-1}, x_{t-2}) \qquad (21) \\ &= \frac{1}{R}\sum_{h=1}^{j}\frac{1}{Z_t}exp\left(\beta\left[\frac{1}{a\sigma^2}(x_t - Rx_{t-1})\left((g_h x_{t-2} + b_h) - Rx_{t-1}\right) - C_h\right]\right)\cdot \\ &\quad \cdot (g_h x_t + b_h) \end{aligned}$$

with

$$Z_t = \sum_{h=1}^{j} exp\left(\beta\left[\frac{1}{a\sigma^2}(x_t - Rx_{t-1})\left((g_h x_{t-2} + b_h) - Rx_{t-1}\right) - C_h\right]\right).$$

Equation (21) is a third order difference equation with steady state $x^* = F(x^*, x^*, x^*) = 0$.

To control the steady state, we first transform (21) into a three dimensional first order equation system using time delay coordinates $\mathbf{x}_t = (x_{1,t}\ x_{2,t}\ x_{3,t})' = (x_t\ x_{t-1}\ x_{t-2})'$:

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \end{pmatrix} = \mathbf{F}(\mathbf{x}_t) = \begin{bmatrix} F(x_{1,t}, x_{2,t}, x_{3,t}) \\ x_{1,t} \\ x_{2,t} \end{bmatrix}. \tag{22}$$

In a second step we linearize (22) in the neighborhood of the fixed point using the Jacobian

$$\mathbf{x}_{t+1} = \mathbf{x}_t^* + \mathbf{J}\Delta\mathbf{x}$$

with

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \frac{\partial F}{\partial x_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{x^*,x^*,x^*} .$$

3.2.1 Fundamentalists versus Contrarians

We consider the simple case with two different types of investors, first. One group consists of fundamentalists who believe the price deviation from its fundamental value in the next period will be zero. The parameter of their prediction function are $g_1 = b_1 = 0$. A second group, the contrarians can be described by the parameter values $g_2 = -1.5$, $b_2 = 0$. Assume further $R = 1.1$ and $a\sigma^2 = 1$. The forecasts of future prices are costless for the contrarians, therefore $C_2 = 0$. The prediction of the fundamental value is more sophisticated and causes expenses of $C_1 = 1$. The intensity of choice is set to $\beta = 15$. With these parameter values the system possesses chaotic fluctuations as shown in Figure 1. When the contrarians dominate the market the trajectory diverge from its fundamental value. But if the price moves to far away from the fundamental value the fundamentalists will dominate the market, and strive the market price back to its fundamental value. The market domination of contrarians and fundamentalists alternate in a chaotic manner.[15]

[Figure 1]

We now introduce a control authority which wants to reduce the market volatility

[15] For a detailed explanation of the behavior of the trajectory see Brock and Hommes (1998).

by controlling the steady state $\mathbf{x}^* = \mathbf{0}$. The steady state possesses the Jacobian

$$\mathbf{J} = \begin{bmatrix} -1.364 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{x}^*=\mathbf{0}} .$$

By mere observation of the last column of $\mathbf{J}$, we recognize that in the neighborhood of the fixed point the last component of the state vector $\mathbf{x}_t$ in period t $x_{3,t}$ has no influence on the following state. For the Jacobian we get two distinct eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1.364$ with the corresponding eigenvectors $\mathbf{e}_1 = (0 \;\; 0 \;\; 1)^{'}$ and $\mathbf{e}_2 = (0.7398 \quad -0.5425 \;\; 0.3979)^{'}$. Due to $|\lambda_2| > 1$ the fixed point is unstable.

Because the trajectory is in a $\mathbb{R}^3$ space we cannot apply the original control approach of OGY, but we can use our more general control method. To do this we have to complete the eigenvectors to a basis of $\mathbb{R}^3$ by adding a vector $\mathbf{e}_3^*$ which is independent of $\mathbf{e}_1$ and $\mathbf{e}_2$.[16] Using the contravariant basis we get the following linear approximation for the system dynamics in the neighborhood of the fixed point

$$\Delta\mathbf{x}_{t+1} = (\mathbf{f}_1^{\prime}\Delta\mathbf{x}_t)\lambda_1\mathbf{e}_1 + (\mathbf{f}_2^{\prime}\Delta\mathbf{x}_t)\lambda_2\mathbf{e}_2 + (\mathbf{f}_3^{\prime}\Delta\mathbf{x}_t)\mathbf{J}\mathbf{e}_3^*.$$

Because of $\mathbf{x}^* = \mathbf{0}$ we can write

$$\mathbf{x}_{t+1} = (\mathbf{f}_1^{\prime}\mathbf{x}_t)\lambda_1\mathbf{e}_1 + (\mathbf{f}_2^{\prime}\mathbf{x}_t)\lambda_2\mathbf{e}_2 + (\mathbf{f}_3^{\prime}\mathbf{x}_t)\mathbf{J}\mathbf{e}_3^*. \tag{23}$$

In this example it is sufficient to direct the trajectory on the stable manifold $\mathbb{W}^s = span\,\{\mathbf{e}_1, \mathbf{e}_3^*\}$, because $\mathbf{e}_2$ gives the direction of the unstable manifold, only.

To control the equilibrium the control authority has to find a reasonable parameter to target the trajectory with small perturbations in this subspace. For this it can act like an ordinary market participant with the only concern to direct the market price to its fundamental value $\mathbf{x}^* = \mathbf{0}$. It does not maximize any profit function as other investors do. Therefore, the authority acts as a biased investor on the market with a market share of $n_{c,t}$ and biased parameter b_c. Conditioned on this intervention the

[16] For simplicity we use $\mathbf{e}_3^* = (0 \;\; 1 \;\; 0)^{'}$.

market shares of the other investor groups are reduced by the factor $(1 - n_{c,t})$. With an intervention of $n_{c,t}$ we get the following revision of $x_{1,t+1} = F(x_{1,t}, x_{2,t}, x_{3,t})$ in equation (22)

$$\begin{aligned} x_{1,t+1} &= \frac{1}{R}\left(\sum_{h=1}^{j} \frac{1}{Z_t} exp\left(\beta\left[\frac{1}{a\sigma^2}\left(x_{1,t} - Rx_{2,t}\right)\left(g_h x_{3,t} + b_h - Rx_{2,t}\right) - C_h\right]\right) \cdot \right. \\ &\quad \left. \cdot \left(1 - n_{c,t}\right)\left(g_h x_{1,t} + b_h\right) + n_{c,t} b_c\right). \end{aligned} \tag{24}$$

The vector $\mathbf{w}$, which determines the influence of the perturbation in the neighborhood of $\mathbf{x}^*$ is given by

$$\mathbf{w} = \begin{pmatrix} \frac{\partial F}{\partial n_{c,t}} \\ 0 \\ 0 \end{pmatrix} \tag{25}$$

with

$$\frac{\partial F}{\partial n_{c,t}} = \frac{1}{R}\left(\sum_{h=1}^{j} \frac{-exp\left(-\beta C_h\right)}{Z_t} b_h + \frac{b_c}{Z_t}\right) \qquad \text{and} \qquad Z_t = \sum_{h=1}^{j} exp\left(-\beta C_h\right).$$

We now add the influence of a perturbation $\delta n_{c,t}$ to equation (23)

$$\mathbf{x}_{t+1} = (\mathbf{f}_1'\mathbf{x}_t)\lambda_1\mathbf{e}_1 + (\mathbf{f}_2'\mathbf{x}_t)\lambda_2\mathbf{e}_2 + (\mathbf{f}_3'\mathbf{x}_t)\mathbf{J}\mathbf{e}_3^* + \mathbf{w}\delta n_{c,t} \tag{26}$$

and are able to quantify $\delta n_{c,t}$. The perturbation has to direct the trajectory into the stable subspace.[17] This leads to the condition

$$\mathbf{f}_2'\mathbf{x}_{t+1} = (\mathbf{f}_2'\mathbf{x}_t)\lambda_2 + (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^* + \mathbf{f}_2'\mathbf{w}\delta n_{c,t} = 0$$

and we get

$$\delta n_{c,t} = \frac{-(\mathbf{f}_2'\mathbf{x}_t)\lambda_2 - (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^*}{\mathbf{f}_2'\mathbf{w}}. \tag{27}$$

It is assumed that any fraction of the asset can be traded.

[17] The condition $\mathbf{f}_2'\mathbf{x}_{t+1} = 0$ has to be fulfilled.

In the first simulation we assume the control institution can have a maximum market share of 1 %. We get the following control law

$$\delta n_{c,t} = \begin{cases} \frac{-(\mathbf{f}_2'\mathbf{x}_t)\lambda_2 - (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^*}{\mathbf{f}_2'\mathbf{w}} & if \quad 0 \leq \frac{-(\mathbf{f}_2'\mathbf{x}_t)\lambda_2 - (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^*}{\mathbf{f}_2'\mathbf{w}} \leq 0.01 \\ 0 & otherwise. \end{cases} \tag{28}$$

We set the starting state of the trajectory to $\mathbf{x}_0 = (-0.1 \;\; 0.1 \;\; 0)'$ and $b_c = 0.2$.

[Figure 2]

As shown in Figure 2 a) the trajectory is controlled after 22 periods to its steady state $\mathbf{x}^*$. The dotted line shows the uncontrolled trajectory and in Figure 2 b) we see the necessary control perturbations. The biggest intervention of the control authority occurs in period 22: $\delta n_{c,22} = 0.0552\%$. Despite of the maximum market share of 1 % we get much smaller interventions. This is caused by the market behavior. If the asset price diverges too far from its fundamental value a growing number of fundamentalists push the price back in the neighborhood of $\mathbf{x}^*$. Only small interventions are then necessary to keep the trajectory in the steady state. Note, that successive control interventions are so small that they cannot be seen from Figure 2 b). However, without these tiny interventions the trajectory would leave the unstable steady state again.

The initial value $\mathbf{x}_0$ possesses a big influence on the control speed. Figure 3 visualizes the control speed for different initial values of the trajectory.[18] We varied $x_{1,0}$, $x_{2,0}$ in the range of $[-1,\ 1]$ and fixed $x_{3,0}$ to zero.

[Figure 3]

Surprisingly, there are starting points close to the fixed point, which have a longer control time than points further apart. As we can see in Figure 3, variation of $x_{1,0}$ is

[18] For computing the control surface we put a grid on the plane $[-1,\ 1] \times [-1,\ 1]$ with a distance of 0.005 to each neighboring point and computed the control success for each starting point. We computed the number of periods the trajectory needs to reach an ε-neighborhood ($\varepsilon = 0.01$), which it never will leave.

much more sensitive for the control success than $x_{2,0}$. There are two regions with low control speed for high $x_{1,0}$-values combined with low $x_{2,0}$-values and vice versa.

The closer the trajectory gets to its fundamental value the smaller are the perturbations. If we neglect small interventions we can implement a minimum boundary in the control law (28) $\delta n_{\min} = 0.1\%$

$$\delta n_{c,t} = \begin{cases} \frac{-(\mathbf{f}_2'\mathbf{x}_t)\lambda_2 - (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^*}{\mathbf{f}_2'\mathbf{w}} & if \qquad 0.001 \leq \frac{-(\mathbf{f}_2'\mathbf{x}_t)\lambda_2 - (\mathbf{f}_3'\mathbf{x}_t)\mathbf{f}_2'\mathbf{J}\mathbf{e}_3^*}{\mathbf{f}_2'\mathbf{w}} \leq 0.01 \\ 0 & otherwise. \end{cases} \tag{29}$$

The control authority waits with its stabilizing intervention until it exceeds $\delta n_{\min}$. Only if the trajectory diverges to far from $\mathbf{x}^*$ it will be dragged back. Applying the revised control law (29) to our example we get in Figure 4 a) and b) the same control success as before but in the time horizon of 100 periods only during 5 periods interventions are necessary compared to 77 without a minimum boundary. On the other hand in those few intervention periods higher perturbations are necessary to keep the trajectory in the neighborhood of $\mathbf{x}^*$.

[Figure 4]

3.2.2 Four Belief Types

In a second example we consider the case of four different belief types: One fundamentalist ($g_1 = b_1 = 0$), two biased trend chasers ($g_{2,3} = 0.9$, $b_{2,3} = \pm 0.2$), and one pure trend chaser ($g_4 = 1.01$, $b_4 = 0$). We further assume $R = 1.01$, $\beta = 95$, $C_{1,2,3,4} = 0$ and $a\sigma^2 = 1.0$. For these market situation Brock and Hommes (1997a) have found chaotic price behavior with the following Jacobian for the unstable steady state $\mathbf{x}^* = \mathbf{0}$

$$\mathbf{J} = \begin{bmatrix} 2.478 & -1.8 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{30}$$

The Jacobian has two conjugate complex eigenvalues $\lambda_{1,2} = 1.239 \pm 0.515i$ and one real eigenvalue $\lambda_3 = 0$.[19] Because of the eigenvalues $\bmod \lambda_{1,2} > 1$ and $|\lambda_3| < 1$, $\mathbf{x}^*$ has a 2-dimensional unstable manifold $\mathbb{W}^u \subset \mathbb{R}^2$ and a 1-dimensional stable manifold $\mathbb{W}^s \subset \mathbb{R}$. To control the system it is necessary to direct the trajectory onto the stable manifold of the steady state. As shown in Appendix A.1 the complex eigenvectors cannot be used as a basis. We replace them by $\mathbf{e}_1^* = (1 \;\; 0 \;\; 0)'$ and $\mathbf{e}_2^* = (0 \;\; 1 \;\; 0)'$ to get a basis for the $\mathbb{R}^3$ space.[20]

From the previous section we know two control variables are needed if the unstable manifold is 2-dimensional. But in our example $\mathbf{x}_t$ is a time delay vector. In period t only the first coordinate of the vector $\mathbf{x}_t$ can be directly influenced by a perturbation. The second and third coordinate are already determined in t. On this account we have to direct the trajectory during two periods on the stable manifold. In this case only one control variable is sufficient.

With a similar control authority as in the first example, there influence of two successive interventions on the trajectory is given by

$$\mathbf{x}_{t+2} = \mathbf{J}^2\mathbf{x}_t + \mathbf{w}\delta n_{c,t+1} + \mathbf{J}\mathbf{w}\delta n_{c,t} \tag{31}$$

or using the contravariant vectors

$$\mathbf{x}_{t+2} = (\mathbf{f}_1'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_1^* + (\mathbf{f}_2'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_2^* + (\mathbf{f}_3'\mathbf{x}_t)\,\lambda_3^2\mathbf{e}_3 + \mathbf{w}\delta n_{c,t+1} + \mathbf{J}\mathbf{w}\delta n_{c,t}. \tag{32}$$

To ensure that the trajectory is on the stable manifold in $t+2$ the conditions $\mathbf{f}_1'\mathbf{x}_{t+2} = 0$ and $\mathbf{f}_2'\mathbf{x}_{t+2} = 0$ have to hold. This leads to the following linear equation system which has to be solved for the perturbations $\delta n_{c,t}$ and $\delta n_{c,t+1}$

$$\begin{aligned}
\mathbf{f}_1'\,(\mathbf{f}_1'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_1^* + \mathbf{f}_1'\,(\mathbf{f}_2'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_2^* + \mathbf{f}_1'\mathbf{w}\delta n_{c,t+1} + \mathbf{f}_1'\mathbf{J}\mathbf{w}\delta n_{c,t} &= 0 \\
\mathbf{f}_2'\,(\mathbf{f}_1'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_1^* + \mathbf{f}_2'\,(\mathbf{f}_2'\mathbf{x}_t)\,\mathbf{J}^2\mathbf{e}_2^* + \mathbf{f}_2'\mathbf{w}\delta n_{c,t+1} + \mathbf{f}_2'\mathbf{J}\mathbf{w}\delta n_{c,t} &= 0.
\end{aligned} \tag{33}$$

[19] The corresponding eigenvectors are $\mathbf{e}_{1,2}^c = (0.001 \pm 0.732i \;\; 0.210 \pm 0.504i \;\; 0.289 \pm 0.286i)'$ and $\mathbf{e}_3 = (0 \;\; 0 \;\; 1)'$.

[20] $\mathbf{e}_1^*$, $\mathbf{e}_2^*$, and $\mathbf{e}_3$ are an orthogonal basis of $\mathbb{R}^3$. This basis corresponds with its contravariant basis.

Using equation (25) we get the vector $\mathbf{w}$

$$\mathbf{w} = \begin{pmatrix} 0.\overline{0495} \\ 0 \\ 0 \end{pmatrix}.$$

We find, that with the restriction of positive market interventions ($\delta n_{c,t} > 0$) we do not get any control success. Only if negative market shares (short selling) of the control authority are allowed, the chaos control is possible. Figure 5 a) shows the controlled trajectory with an initial value $\mathbf{x}_0 = (-0.2 \;\; 0.2 \;\; 0)^{'}$ and $\delta n_{\max} = 1\%$.

[Figure 5]

After 18 periods the fundamental value is stabilized.[21] A closer look on the perturbations in Figure 5 b) makes it obvious why negative market shares are necessary: The perturbations changes its sign every period. Therefore $\delta n_{c,t}$ and $\delta n_{c,t+1}$, computed with the equation system (33), will not fulfill the conditions $\delta n_{c,t} > 0$ and $\delta n_{c,t+1} > 0$.

As in the example before, introducing a minimum intervention of $\delta n_{\min} = 0.01\%$, to reduce the number of control interventions, leads to the same control result, and to an increase in the cumulated perturbations.

Finally we investigate again the control speed for different initial values. As it can be seen in Figure 6 more groups of different investor types lead to a higher sensitivity of the control speed on the starting values: Already small variations of the initial value can change the control success dramatically. Figure 6 reveals a symmetry of the control speed with respect to the fixed point $\mathbf{x}^*$.

[Figure 6]

4 Summary

In this paper we first have shown how the chaos control method of OGY can be expanded to stabilize not only saddle points in the $\mathbb{R}^2$ space but also any unstable equi-

[21] The dotted line shows again the uncontrolled trajectory.

librium in the $\mathbb{R}^n$ space, embedded in the strange attractor. With our method the trajectory is directed on the stable manifold of the equilibrium. If no stable manifold exists, we control the trajectory straight into the equilibrium. Depending on the dimension of the unstable manifold we get a linear equation system to compute the control interventions.

We applied our control method on the asset pricing model of Brock and Hommes (1998). This model explains chaotic price movements endogenous by heterogeneous market participants and their adaption policy. If such chaotic price behavior exists a control authority is able to stabilize the market price. In our first example with two investor groups the control authority was able to push any chaotic price trajectory, close to the equilibrium price, with their demand or supply for the risky asset during one period onto the stable manifold of the equilibrium. In a more complex market structure with four investor groups two successive market interventions were necessary for the control, due to the dimension of the unstable manifold of the equilibrium.

In a chaotic market environment no longer big interventions are necessary to control erratic price fluctuations. Using the properties of chaotic systems small interventions are sufficient for a control authority to stabilize any unstable equilibrium.

A Appendix: Derivation of the Control Rules

To derive the control rules we distinguish two cases:

Case 1: The unstable fixed point possesses a m-dimensional.

Case 2: The fixed point possesses an n-dimensional unstable manifold and no stable manifold.

A.1 The Equilibrium has a Stable Manifold

In equation (3) we use the eigenvectors as basis:

$$\Delta \mathbf{x}_{t+1} = a_1 \lambda_1 \mathbf{e}_1 + ... + a_n \lambda_n \mathbf{e}_n.$$

It is possible to substitute the coordinates a_1,..., a_n by the contravariant basis $\mathbf{f}_1, ..., \mathbf{f}_n$, and $\Delta\mathbf{x}_t$. For equation (2) we get [22]

$$\Delta\mathbf{x}_{t+1} = (\mathbf{f}_1'\Delta\mathbf{x}_t)\,\lambda_1\mathbf{e}_1 + ... + (\mathbf{f}_n'\Delta\mathbf{x}_t)\,\lambda_n\mathbf{e}_n. \tag{34}$$

Now the control target is to direct the trajectory on the stable manifold $\mathbb{W}^s$ using $n-m$ control variables p_j. The influence of the parameter shifts $\delta p_{j,t}$ can be incorporated in equation (34) via the vectors $\mathbf{w}_j$ which contain the derivatives of $\mathbf{f}$ with respect to p_j [23]

$$\mathbf{w}_j = \begin{pmatrix} \frac{\partial f_1}{\partial p_j} \\ \vdots \\ \frac{\partial f_n}{\partial p_j} \end{pmatrix}.$$

We get the following equation

$$\Delta\mathbf{x}_{t+1} = \sum_{i=1}^{n} (\mathbf{f}_i'\Delta\mathbf{x}_t)\mathbf{e}_1\lambda_i + \sum_{j=1}^{n-m} \mathbf{w}_j\delta p_{j,t} \tag{35}$$

To be on the stable manifold in period $t+1$, $\Delta\mathbf{x}_{t+1}$ has to be orthogonal to the contravariant basis vectors $\mathbf{f}_{m+1}...\mathbf{f}_n$

$$\mathbf{f}_{m+1}'\Delta\mathbf{x}_{t+1} = \mathbf{f}_{m+2}'\Delta\mathbf{x}_{t+1} = ... = \mathbf{f}_n'\Delta\mathbf{x}_{t+1} = 0.$$

We get the following linear equation system for $\delta\mathbf{p}_t' = (\delta p_{j,t}\ \ ...\ \ \delta p_{n-m,t})$

$$\underset{((n-m)\times(n-m))}{\begin{bmatrix} \lambda_{m+1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix}} \underset{((n-m)\times n)}{\begin{bmatrix} \mathbf{f}_{m+1}' \\ \vdots \\ \mathbf{f}_n' \end{bmatrix}} \underset{(n\times 1)}{\Delta\mathbf{x}_t} = \underset{((n-m)\times n)}{\begin{bmatrix} \mathbf{f}_{m+1}' \\ \vdots \\ \mathbf{f}_n' \end{bmatrix}} \underset{(n\times(n-m))}{\begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_{n-m} \end{bmatrix}} \underset{((n-m)\times 1)}{\delta\mathbf{p}_t} \tag{36}$$

or simply

$$\boldsymbol{\lambda}\mathbf{F}\Delta\mathbf{x}_t = \mathbf{F}\mathbf{W}\delta\mathbf{p}_t.$$

[22] From the definition of the contravariant basis (see footnote 7) it follows immediately: $a_i = \mathbf{f}_i'\Delta\mathbf{x}_t$, $i = 1, ..., n$.

[23] For a successful control the vectors $\mathbf{w}_j$ must be independent.

This leads to the following control rule:

$$\delta \mathbf{p}_t = \begin{cases} \mathbf{W}^{-1}\mathbf{F}^{-1}\boldsymbol{\lambda}\mathbf{F}\Delta\mathbf{x}_t & if \qquad |\mathbf{W}^{-1}\mathbf{F}^{-1}\boldsymbol{\lambda}\mathbf{F}\Delta\mathbf{x}_t| \leqq \delta\mathbf{p}_{\max} \\ \mathbf{0} & otherwise. \end{cases} \tag{37}$$

All components of the vector $\delta\mathbf{p}_t$ have to be smaller than the corresponding components of the vector $\delta\mathbf{p}_{\max}$ before the control is started.

Till now we assumed all eigenvalues to be real. If however the Jacobian possesses k complex eigenvalues, the corresponding complex eigenvectors cannot be used as basis vectors for the state space $\mathbb{R}^n$. To define a basis for $\mathbb{R}^n$ the complex eigenvectors have to be replaced by real vectors $\mathbf{e}_1^*, ..., \mathbf{e}_k^*$, which span the space $\mathbb{R}^n = span\,\{\mathbf{e}_1^*, ..., \mathbf{e}_k^*, \mathbf{e}_{k+1}, ..., \mathbf{e}_n\}$ together with the real eigenvectors.

To direct the trajectory on the m-dimensional stable manifold we use the corresponding contravariant basis and compute

$$\Delta\mathbf{x}_{t+1} = \sum_{i=1}^{k} (\mathbf{f}_i'\Delta\mathbf{x}_t)\mathbf{J}\mathbf{e}_i^* + \sum_{i=k+1}^{n} (\mathbf{f}_i'\Delta\mathbf{x}_t)\mathbf{e}_i\lambda_i + \sum_{j=1}^{n-m} \mathbf{w}_j\delta p_{j,t}. \tag{38}$$

In the first term of equation (38) $\mathbf{J}$ cannot be substituted by λ_i because the vectors $\mathbf{e}_i^*$ are no eigenvectors of $\mathbf{J}$. Similar to equation (36) we get to the following linear system to solve for the $(n-m)$-tuple $\delta\mathbf{p}_t$

$$\underset{((n-m)\times n)}{\begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_{n-m}' \end{bmatrix}} \underset{(n\times(n-m))}{\begin{bmatrix} \mathbf{J}\mathbf{e}_1^* & \cdots & \mathbf{J}\mathbf{e}_k^* & \lambda_{k+1}\mathbf{e}_{k+1} & \cdots & \lambda_{n-m}\mathbf{e}_{n-m} \end{bmatrix}} \underset{((n-m)\times n)}{\begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_{n-m}' \end{bmatrix}} \underset{(n\times 1)}{\Delta\mathbf{x}_t} =$$

$$= \underset{((n-m)\times n)}{\begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_{n-m}' \end{bmatrix}} \underset{(n\times(n-m))}{\begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_{n-m} \end{bmatrix}} \underset{((n-m)\times 1)}{\delta\mathbf{p}_t}. \tag{39}$$

Like equation (37) we start the control if $\delta\mathbf{p}_t \leq \delta\mathbf{p}_{\max}$.

A.2 The Equilibrium has no Stable Manifold

If there exists no stable manifold the trajectory has to be directed straight into the fixed point. To control the trajectory in the fixed point we need n parameters $p_1 ..., p_n$. With the following equation

$$\Delta \mathbf{x}_{t+1} = \mathbf{J}\Delta \mathbf{x}_t + \sum_{j=1}^{n} \mathbf{w}_j \delta p_{j,t} \tag{40}$$

we can compute the perturbation vector $\delta \mathbf{p}_t$. Because the condition for the trajectory in the fixed point is simply $\Delta \mathbf{x}_{t+1} = \mathbf{0}$, it is not necessary to change the basis. From

$$\mathbf{J}\Delta \mathbf{x}_t + \mathbf{W}\delta \mathbf{p}_t = \mathbf{0} \qquad \text{with } \underset{(n\times n)}{\mathbf{W}} = \left(\underset{(n\times 1)}{\mathbf{w}_1} \ldots \underset{(n\times 1)}{\mathbf{w}_n} \right)$$

we get

$$\delta \mathbf{p}_t = -\mathbf{W}^{-1}\mathbf{J}\Delta \mathbf{x}_t.$$

If we restrict each parameter shift again to a maximum absolute value of $\delta \mathbf{p}_{\max}$ the control law reads finally

$$\delta \mathbf{p}_t = \begin{cases} -\mathbf{W}^{-1}\mathbf{J}\Delta \mathbf{x}_t & if \quad |-\mathbf{W}^{-1}\mathbf{J}\Delta \mathbf{x}_t| \leqq \delta \mathbf{p}_{\max} \\ \mathbf{0} & otherwise. \end{cases} \tag{41}$$

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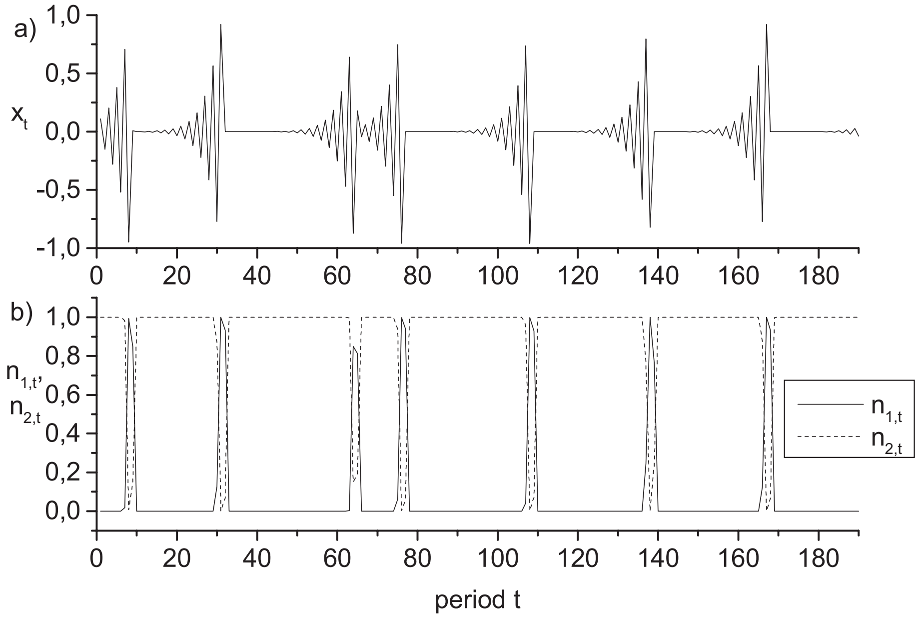

Figure 1: a) Chaotic movement of the price trajectory b) Market share of the fundamentalists $n_{1,t}$ and the contrarians $n_{2,t}$

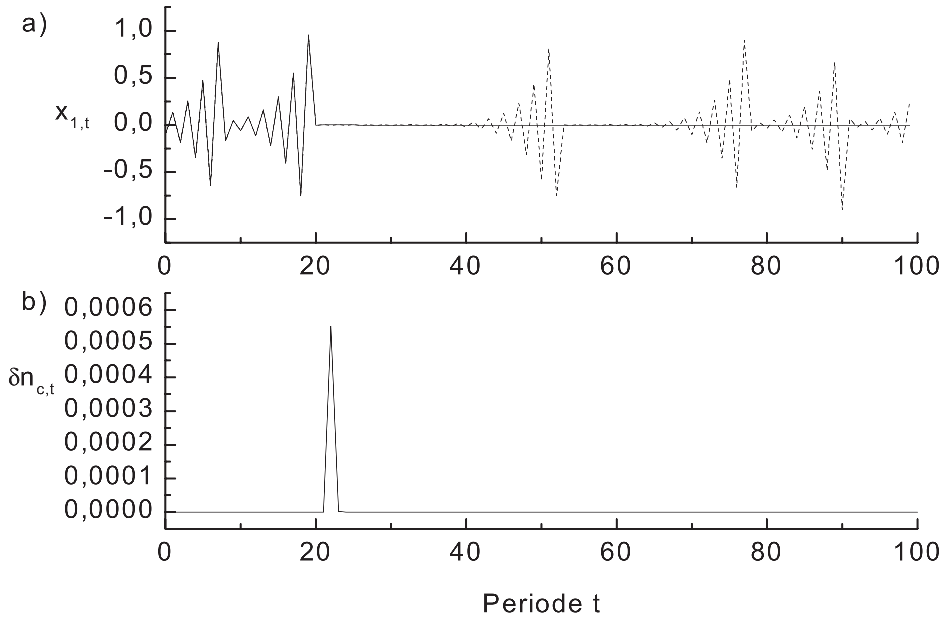

Figure 2: a) The controlled and uncontrolled trajectory with the initial value $\mathbf{x}_0 = (-0.1 \;\; 0.1 \;\; 0)'$ b) The control perturbation of the market authority

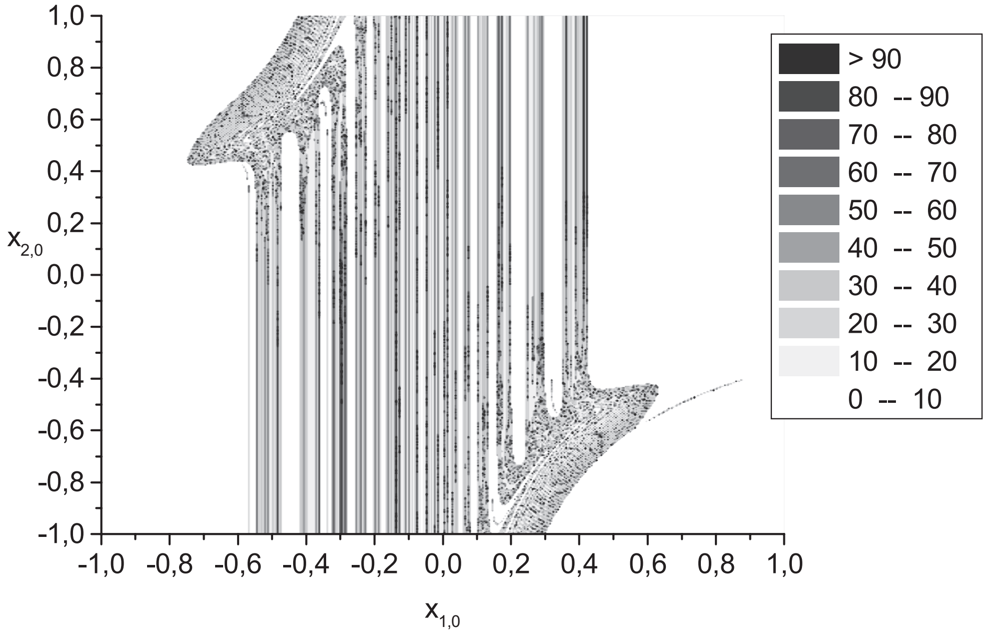

Figure 3: Control speed for different initial values

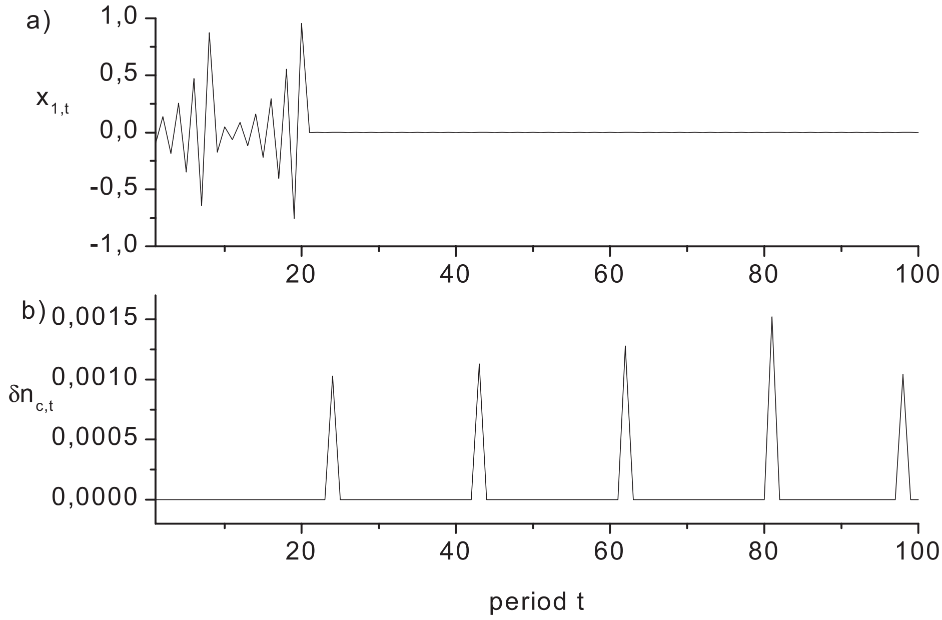

Figure 4: Control with a minimum and maximum boundary of $\delta n_{\min} = 0.001$ and $\delta n_{\max} = 0.01$

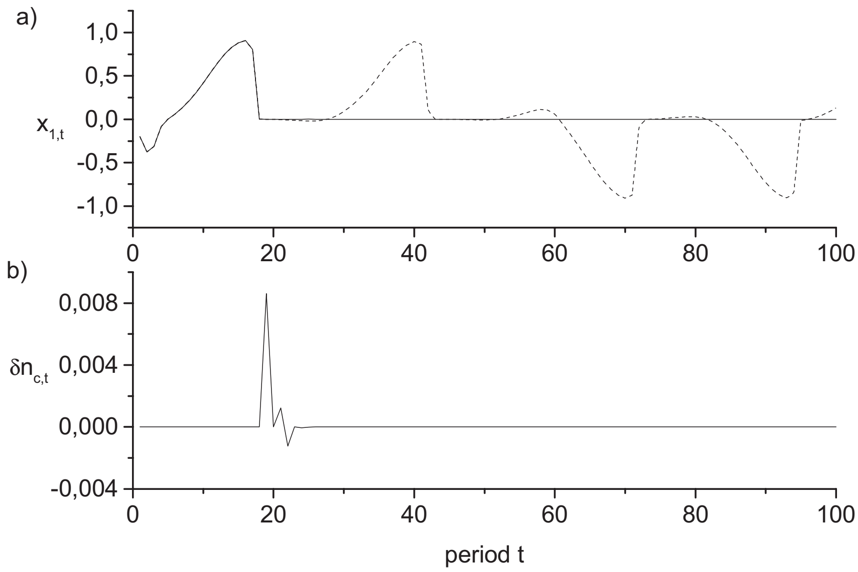

Figure 5: Four belief types: a) The controlled and uncontrolled trajectory with the initial value $\mathbf{x}_0 = (-0.2 \;\; 0.2 \;\; 0)'$ b) The control perturbation of the market authority

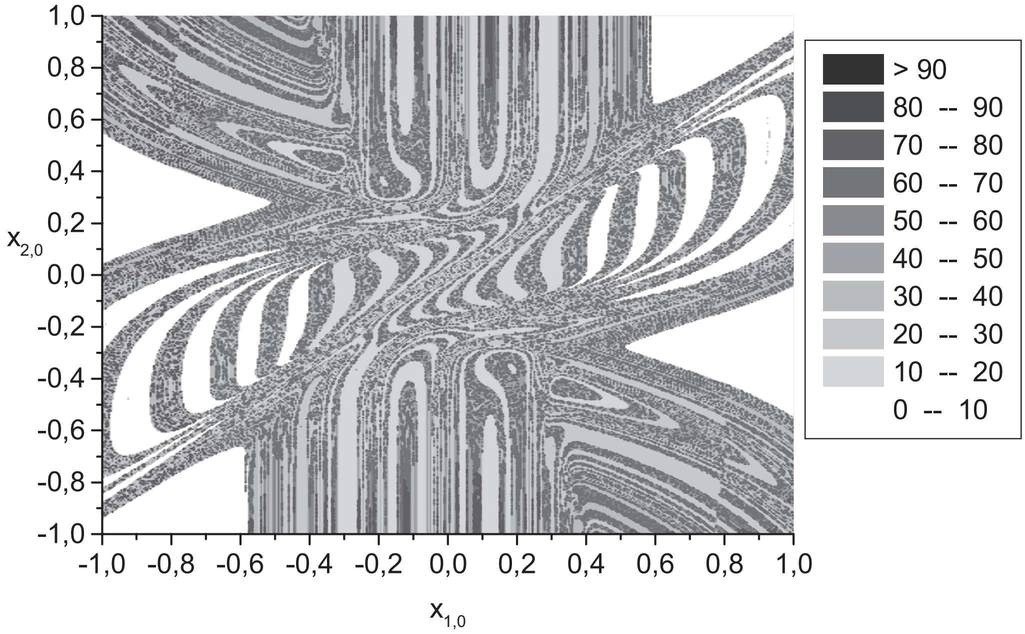

Figure 6: Four belief types: Control speed for different initial values

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