A discrete Clark-Ocone formula

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1 Introduction

In this article we develope a discrete version of the Malliavin calculus as it was done by Holden et al. in [8, 7]. A minor modification of the interpretation of some definitions gives us the possibility to formulate and to proof the Clark-Ocone formula in this discrete setting. The notation is much inspired by Meyer's toy Fock space as it is found in [14, 13, 15]. This finite calculus has its analog in the Maassen kernel calculus of quantum stochastics [12, 9]. There the non-causal non-quantum stochastic calculus is contained as a special case [10]. This approach uses the symmetric space over the space of square integrable functions over the Lebesgue space [6]. Fundamental to the discrete calculus is the Wick product of random variables [5]. This allows an easy definition of the Skorohod integral. For this article we had most profit from [8].

2 Basic Definitions, Notations and Facts

Let be $N \in \mathbb{N}$ and set $\triangle t = \frac{1}{N}$. Then we take the set

$$\Lambda = \{0, \triangle t, \ldots, (N-1)\Delta t\}$$

as a discrete version of the finite time line $[0, 1]$. As measure μ on Λ we take the uniform counting measure, i.e. for $A \subset \Lambda$ we have $\mu(A) = \frac{|A|}{N}$. The measure algebra is the potential set of Λ. So the triple $(\Lambda, \mathcal{P}(\Lambda), \mu)$ is our discrete version of the Lebesgue space $([0,1], \mathcal{B}, \lambda)$.

Next we introduce the set

$$\Omega = \{\omega | \omega : \Lambda \to \{-1, +1\}\}$$

and think of each ω as a Bernoulli random variable. On $\mathcal{P}(\Omega)$ we take the uniform probability measure P, i.e. for $S \subset \Omega$ we have $P(S) = \frac{|S|}{|\Omega|} = \frac{|S|}{2^N}$. With respect to P we form $L^2(\Omega, P)$ with the inner product

$$\langle X, Y \rangle_{L^2} = \sum_{\omega \in \Omega} X(\omega) Y(\omega) P(\omega).$$

It is $\dim L^2(\Omega, P) = 2^N$ since we have a basis of characteristic functions to each atom $\omega \in \Omega$ scaled with the factor $\sqrt{2^N}$.

Definition 2.1 *For $A \in \mathcal{P}(\Lambda)$ we define the functions $\chi_A : \Omega \to \mathbb{R}$ by $\chi_A(\omega) = \prod_{s \in A} \omega(s)$.* ◁

Proposition 2.2 *The set $\{\chi_A\}_{a \in \mathcal{P}(\Lambda)}$ is an orthonormal system in $L^2(\Omega, P)$.*

PROOF: First note that for $A, B \in \mathcal{P}(\Lambda)$

$$\chi_A(\omega)\chi_B(\omega) = \prod_{s \in A} \omega(s) \prod_{t \in B} \omega = \prod_{s \in A \triangle B} \omega(s) = \chi_{A \triangle B}(\omega).$$

Thus we see that

$$\langle \chi_A, \chi_B \rangle = \sum_{\omega \in \Omega} \chi_{A \triangle B}(\omega) P(\omega).$$

For $A = B$ we obtain $\sum_{\omega \in \Omega} \chi_\emptyset(\omega) P(\omega) = 1$ and for $A \neq B$ we have to show that $\sum_{\omega \in \Omega} \chi_C(\omega) P(\omega) = 0$ with $C = A \triangle B \neq \emptyset$. But χ_C has as only possible values -1 and +1 depending how often ω has value -1 on C. Summing over ω it is enough to show, that the product $\chi_C(\omega) = \prod_{s \in C} \omega(s)$ is in the half of the cases -1. Suppose that $|C| = n$ then the product is -1 if an odd number of -1's occures in $\prod_{s \in C} \omega(s)$ and +1 for an even number of -1's. Thus we show that $\sum_k \binom{n}{2k} = \sum_k \binom{n}{2k+1}$:

$$0 = (1 + (-1))^n = \sum_k \binom{n}{k} (-1)^k 1^{n-k} = \sum_k \binom{n}{2k} - \sum_k \binom{n}{2k+1}$$

□

Corollary 2.3 *$\{\chi_A\}_{A\in\mathcal{P}(\Lambda)}$ is a basis for $L^2(\Omega, P)$.*

PROOF:

$$\sharp\{\chi_A\}_{A\in\mathcal{P}(\Lambda)} = \sharp\mathcal{P}(\Lambda) = 2^N = \dim L^2(\Omega, P).$$

$\square$

Notation 2.4 As shorthand notation we set

$$\mathcal{P}_n = \{A \in \mathcal{P}(\Lambda) : |A| = n\}, \ \mathcal{P} = \mathcal{P}(\Lambda) = \dot{\cup}_n \mathcal{P}_n.$$

And for $X \in L^2(\Omega, P)$ we call

$$X = \sum_{A\in\mathcal{P}} X(A)\chi_A = \sum_n \sum_{A\in\mathcal{P}_n} X(A)\chi_A$$

the *Walsh decomposition* of X. $\triangleleft$

Proposition 2.5 *Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$. Then $E[X] = X(\emptyset)$.*

PROOF:

$$E[X] = \sum_{\omega\in\Omega}\sum_{A\in\mathcal{P}} X(A)\chi_A(\omega)P(\omega) = \sum_{A\in\mathcal{P}} X(A)\sum_{\omega\in\Omega}\chi_A(\omega)P(\omega) = X(\emptyset)$$

since in the proof of the preceeding proposition we have seen that

$$\sum_{\omega\in\Omega}\chi_A(\omega)P(\omega) = \begin{cases} 0, & A \neq \emptyset \\ 1, & A = \emptyset \end{cases}.$$

$\square$

Definition 2.6 *Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ and $Y = \sum_{A\in\mathcal{P}} Y(B)\chi_B$ random variables. Then the* Wick product *$X \diamond Y$ is defined by*

$$X \diamond Y = \sum_{C\in\mathcal{P}} (\sum_{A\dot{\cup}B} X(A)Y(B))\chi_C.$$

$\triangleleft$

Remark 2.7

(1)

$$\chi_A \diamond \chi_B \quad = \quad \begin{cases} \chi_{A\cup B}, \; if \; A \cap B = \emptyset \\ 0, \; otherwise \end{cases}$$

(2) If $A \cap B = \emptyset$ then $\chi_A \diamond \chi_B = \chi_A \cdot \chi_B$.

Lemma 2.8 *$(L^2(\Omega, P), +, \diamond)$ is a commutative ring with unit $\chi_\emptyset$.*

PROOF: straightforward. □

Definition 2.9 (discrete analoga)

- *A* stochastic process *is a family of random variables $(X_s)_{s\in\Lambda}$, i.e. a map $X : \Omega \times \Lambda \to \mathbb{R}$ such that for each fixed $s \in \Lambda$ the map $X(\,\cdot\,, s)$ is in $L^2(\Omega, P)$.*

- *The* Brownian motion *B is the random walk*

$$B : \Omega \times \Lambda \to \mathbb{R}, \; B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\triangle t}$$

- *The* white noise *W over (Λ, μ) is the map*

$$W : \Omega \times \mathcal{P}(\Lambda) \to \mathbb{R}, \; W(\omega, A) = \sum_{s\in A} \frac{\omega(s)}{\sqrt{\triangle t}}.$$

  For $t \in \Lambda$ we set $W_t(\omega) = W(\omega, \{t\}) = \frac{\omega(t)}{\sqrt{\triangle t}}$.

- *The forward increment of B is defined by*

$$\triangle B_t = \triangle B(\omega, t) = B(\omega, t + \triangle t) - B(\omega, t) = \omega(t)\sqrt{\triangle t}.$$

  Thus the derivative *of the Brownian motion is the white noise:*

$$\frac{\triangle B_t}{\triangle t} = \frac{\omega\sqrt{\triangle t}}{\triangle t} = \frac{\omega t}{\sqrt{\triangle t}} = W_t.$$

- *Let be* $(X_s)_{s\in\Lambda}$ *an adapted (whatsoever this means) stochastic process. Then the* Itô integral *is defined by*

$$\int X dB = \int X_s dB_s = \sum_s X_s \cdot \triangle B_s = \sum_s X_s \cdot W_s \triangle t.$$

$\triangleleft$

Now we will establish a discrete version of the Wiener-Itô decomposition for random variables $X \in L^2(\Omega, P)$. Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ the Walsh decomposition of X. Then we define for $n > 0$ the symmetric function X_n on Λ^n by

$$X_n(t_1, \ldots, t_n) = \begin{cases} (\triangle t^{n/2} n!)^{-1} X(\{t_1, \ldots, t_n\}), \text{ if } t_i \neq t_j \text{ for } i \neq j \\ 0, \text{ otherwise} \end{cases}$$

where $X(\{t_1, \ldots, t_n\})$ is the Walsh component to $A = \{t_1, \ldots, t_n\}$. For $n = 0$ we set $X_0 = X(\emptyset) = E[X]$. Then we get

$$\begin{aligned} X &= \sum_{A\in\mathcal{P}} X(A)\chi_A = \sum_n \sum_{A\in\mathcal{P}_n} X(A)\chi_A \\ &= \sum_n \sum_{\{t_1,\ldots,t_n\}\in\mathcal{P}_n} X(\{t_1, \ldots, t_n\})\omega(t_1) \cdot \ldots \cdot \omega(t_n) \\ &= \sum_n \sum_{\substack{(t_1,\ldots,t_n)\in\Lambda^n \\ t_1<\ldots<t_n}} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n}{2}} \omega(t_1) \cdot \ldots \cdot \omega(t_n) \\ &= \sum_n \sum_{(t_1,\ldots,t_n)\in\Lambda^n} X_n(t_1, \ldots, t_n) \triangle B(t_1) \cdots \triangle B(t_n). \end{aligned}$$

The last term is nothing else than the *discrete Wiener-Itô decomposition.*

3 Conditional expectations

Notation 3.1 For $B \subset \Lambda$ we denote by $\mathcal{F}_B$ the σ-algebra on Ω generated by the random variables $\{\omega(s) : s \in B\}$. $\triangleleft$

For example for each $s \in \Lambda$ we have

$$\mathcal{F}_{\{s\}} = \{\emptyset, \{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\}, \Omega\}.$$

Since this are the atomic σ-algebras we can construct every $\mathcal{F}_B$ out of them:

$$\mathcal{F}_B = \sigma - alg[\{\{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\} | s \in B\}].$$

Proposition 3.2 *Let be $X = \sum_{A \subset \Lambda} X(A)\chi_A$ and $\mathcal{F}_B$ given. Then the conditional expectation of X with respect to $\mathcal{F}_B$ is given by*

$$E[X|\mathcal{F}_B] = \sum_{A \subset B} X(A)\chi_A.$$

PROOF: That $\sum_{A \subset B} X(A)\chi_A$ is $\mathcal{F}_B$-measurable is evident. Further we have to prove that for every $H \in \mathcal{F}_B$ it holds

$$\int_{\omega \in H} E[X|\mathcal{F}_B] dP = \int_{\omega \in H} X dP .$$

The left hand side is

$$\begin{aligned} \int_{\omega \in H} E[X|\mathcal{F}_B] dP &= \sum_{\omega \in H} \sum_{A \subset B} X(A)\chi_A(\omega)P(\omega) \\ &= \sum_{A \subset B} X(A) \sum_{\omega \in H} \chi_A(\omega)P(\omega) \end{aligned}$$

and the right hand side is

$$\begin{aligned} \int_{\omega \in H} X dP &= \sum_{\omega \in H} \sum_{A \subset \Lambda} X(A)\chi_A(\omega)P(\omega) \\ &= \sum_{A \subset \Lambda} X(A) \sum_{\omega \in H} \chi_A(\omega)P(\omega) . \end{aligned}$$

So it is sufficient to show that for $H \in \mathcal{F}_B$ and for every $A \not\subset B$ we have

$$\sum_{\omega \in H} \chi_A(\omega)P(\omega) = 0 .$$

If $A \not\subset B$ then there exists an $s_0 \in A$ with $s_0 \notin B$. But this shows that we can divide the set H into two parts

$$H_{s_0}^- = \{\omega \in H : \omega(s_0) = -1\} \text{ and } H_{s_0}^+ = \{\omega \in H : \omega(s_0) = +1\}$$

and $H = H_{s_0}^- \dot\cup H_{s_0}^+$. Furthermore for each $\omega^- \in H_{s_0}^-$ there exists exactly one $\omega^+ \in H_{s_0}^+$ such that $\omega^-(s) = \omega^+(s)$ for all $s \in A \setminus \{s_0\}$. This shows $\sharp H_{s_0}^- = \sharp H_{s_0}^+$ and therefore $\sum_{\omega \in H} \chi_A(\omega)P(\omega) = 0$. Thus we have proven that

$$\int_{\omega \in H} X dP = \sum_{A \subset B} X(A) \sum_{\omega \in H} \chi_A(\omega) P(\omega) = \int_{\omega \in H} E[X|\mathcal{F}_B] dP$$

for every $H \in \mathcal{F}_B$. □

The formula shows that the conditional expectation of X with respect to $\mathcal{F}_B$ depends only on those Walsh components χ_A such that $A \subset B$.

We observe that

$$\begin{aligned} X \diamond \chi_B &= \left(\sum_{A \subset \Lambda} X(A) \chi_A \right) \diamond \chi_B \\ &= \sum_{A \subset B^c} X(A) \chi_A \cdot \chi_B \quad = \quad E[X|\mathcal{F}_{B^c}] \cdot \chi_B . \end{aligned} \tag{1}$$

With that observation it follows easily

Proposition 3.3 *Let be $X, Y \in L^2(\Omega, P)$. Then*

$$\begin{aligned} X \diamond Y &= \sum_{A \subset \Lambda} Y(A) E[X|\mathcal{F}_{A^c}] \chi_A \\ &= \sum_{A \subset \Lambda} X(A) E[Y|\mathcal{F}_{A^c}] \chi_A \\ &= \frac{1}{2} \sum_{A \subset \Lambda} (X(A) E[Y|\mathcal{F}_{A^c} + Y(A) E[X|\mathcal{F}_{A^c}]) \chi_A . \end{aligned}$$

PROOF: The first and the second term follow immediately from equation (1). The last is just the average of the first two. □

The next observation is implicitly contained in the remark that $\chi_A \diamond \chi_B = \chi_A \cdot \chi_B$ if $A \cap B = \emptyset$, but the interpretation now has another flavour.

Proposition 3.4 *Let be $A, B \subset \Lambda$ and $X, Y \in L^2(\Omega, P)$. Assume $A \cap B = \emptyset$ and that X is $\mathcal{F}_A$-measurable and Y is $\mathcal{F}_B$-measurable. Then*

$$X \diamond Y = X \cdot Y .$$

PROOF: The measurability assumption shows that the walsh decompositions of X and Y are

$$X = \sum_{C \subset A} X(C)\chi_C \text{ and } Y = \sum_{D \subset B} X(D)\chi_D.$$

Thus

$$\begin{aligned} X \diamond Y &= \sum_{C,D} \{X(C)Y(D) : C \subset A, D \subset B, C \cap D = \emptyset\}\chi_{C \cup D} \\ &= \sum_{C,D} \{X(C)Y(D) : C \subset A, D \subset B\}\chi_{C \triangle D} \quad = \quad X \cdot Y\,. \end{aligned}$$

□

Definition 3.5 *For $t \in \Lambda$ we set*

$$\begin{aligned} \mathcal{F}_t &= \sigma - alg[\{\omega(s) | s < t\}] \\ &= \sigma - alg[\{\{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\} | s < t\}] \end{aligned}$$

and call this the past algebra*. (Note that $\omega(t)$ is not contained in the generating set.) A random variable X is said to be $\mathcal{F}_t$-*adapted *if*

$$E[X|\mathcal{F}_t] = X\,.$$

This means that the walsh decomposition of X has the form

$$X = \sum_{A \subset [0,t[} X(A)\chi_A \textit{ with } [0,t[= \{s \in \Lambda : s < t\}\,.$$

A stochastic process $(X_s)_{s \geq 0}$ is adapted *if the random variable X_t is $\mathcal{F}_t$-adapted for each $t \in \Lambda$.* ◁

Thus for a $\mathcal{F}_t$-adapted random variable all walsh coefficients $X(A)$ with $\max A \geq t$ are zero. Also the Itô integral of an adapted process makes sense since the product of the walsh components $X_t(A)\chi_A$ of X_t and the forward increment $\triangle B_t = \chi_{\{t\}}\sqrt{\triangle t}$ of the Brownian motion are well defined.

Corollary 3.6 *For every process* $(X_s)_{s\in\Lambda}$ *with Walsh decomposition* $X_t = \sum_{A\subset\Lambda} X(A;t)\chi_A$ *one has*

$$E[X_t|\mathcal{F}_t] = \sum_{A\subset[0,t[} X(A;t)\chi_A = \sum_{\substack{A\subset\Lambda \\ \max A<t}} X(A;t)\chi_A \,. \tag{2}$$

PROOF: Follows from proposition 3.2 and the definition. □

4 Discrete Skorohod integral

Definition 4.1 *Let be* $X : \Omega\times\Lambda \to \mathbb{R}$ *a stochastic process. The* Skorohod integral *of* X *with respect to the Brownian motion* B *is defined by*

$$\int X\delta B = \int X_s \delta B_s = \sum_{s\in\Lambda} X_s \diamond \triangle B_s.$$

◁

As an easy consequence we have

$$\int X_s\delta B_s = \sum_{s\in\Lambda} X_s \diamond \chi_{\{s\}}\sqrt{\triangle t} = \sum_{s\in\Lambda} X_s \diamond W_s \triangle t\,.$$

So we see that the Skorohod integral is the Lebesgue integral of the transformed process by Wick multiplication with white noise.

Remark 4.2

(1) If the stochastic process X is adapted then the Skorohod integral reduces to the Itô integral.

(2) If $X_s = \sum_{A\in\mathcal{P}} X(A;s)\chi_A$ is the Walsh decomposition of X_s and transforming this into the corresponding discrete Wiener-Itô decomposition then the Skorohod integral is roughly speaking Integration with $\sum_s \cdot \triangle B_s$ over the parameter s.

PROOF of (1): Take $A < s$ as notation for $\max A < s$. Since X is adapted we have the Walsh decomposition $X_s = \sum_{A<s} X(A;s)\chi_A$. Hence A and $\{s\}$ are disjoint and we obtain

$$\begin{aligned}\int X_s \delta B_s &= \sum_s \sum_{A<s} X(A;s)\chi_A \diamond \chi_{\{s\}}\sqrt{\triangle t} \\ &= \sum_s \sum_{A<s} X(A;s)\chi_A \cdot \chi_{\{s\}}\sqrt{\triangle t} \\ &= \sum_s X_s \cdot \triangle B_s \ = \ \int X_s dB_s.\end{aligned}$$

PROOF of (2):

$$\begin{aligned}\int X\delta B &= \sum_s X_s \diamond \chi_{\{s\}}\sqrt{\triangle t} \\ &= \sum_s \left(\sum_n \sum_{A\in\mathcal{P}_n} X(A;s)\chi_A \right) \diamond \chi_{\{s\}}\sqrt{\triangle t}\end{aligned}$$

$$= \sum_s \left(\sum_n \sum_{(t_1,\ldots,t_n)\in\lambda^n} X_n(t_1,\ldots,t_n;s)\chi_{\{t_1,\ldots,t_n\}}\triangle t^{\frac{n}{2}} \right) \diamond \chi_{\{s\}}\sqrt{\triangle t}$$

with $X_n(\cdot;s)$ the symmetric functions in the Wiener-Itô decomposition of X_s. Now we rename the parameter $s = t_n$ and introduce the symmetric functions $\widehat{X}_{n+1}$ of $n+1$ arguments by

$$\begin{aligned}\widehat{X}_{n+1}(t_1,\ldots,t_{n+1}) &= 0 \ \text{ if } \ t_i = t_j \ \text{ for some } \ i \neq j \ \text{ and} \\ \widehat{X}_{n+1}(t_1,\ldots,t_{n+1}) &=\end{aligned}$$

$$\frac{1}{n+1}\Big(X_n(t_1,\ldots,t_n;s) \ + \ \sum_{k=1}^{n} X_n(t_1,\ldots,t_{k-1},s,t_{k+1},\ldots,t_n;t_k)\Big)$$

otherwise.

Then one obtains changing the sum over s inside

$$\begin{aligned}\int X\delta B &= \sum_n \sum_{(t_1,\ldots,t_{n+1})\in\Lambda^{n+1}} \widehat{X}_{n+1}(t_1,\ldots,t_{n+1})\chi_{\{t_1,\ldots,t_{n+1}\}}\triangle t^{\frac{n+1}{2}} \\ &= \sum_n \sum_{(t_1,\ldots,t_{n+1})\in\Lambda^{n+1}} \widehat{X}_{n+1}(t_1,\ldots,t_{n+1})\triangle B(t_1)\cdots\triangle B(t_{n+1}).\end{aligned}$$

□

So one sees that the discrete Skorohod integral recovers formally the properties of the continuous one.

5 Discrete Malliavin derivative

In this section we will develop a discrete version of the Malliavin derivative. For this purpose we introduce an "integrated" Malliavin derivative, called the Malliavin process.

Notation 5.1 The *discrete Cameron-Martin space* CM is the space of stochastic processes $X : \Omega \times \Lambda \to \mathbb{R}$ with inner product

$$\langle X, Y \rangle_{CM} = E\left[\sum_s \left(\frac{\triangle X_s}{\triangle t} \cdot \frac{\triangle Y_s}{\triangle t}\right) \triangle t\right] = \frac{1}{\triangle t} \sum_s E[\triangle X_s \cdot \triangle Y_s].$$

◁

For the definition of the Malliavin process and the Malliavin derivative we need the following notation:

Notation 5.2 For $s \in \Lambda$ and $\omega \in \Omega$ we define ω_s^+ and ω_s^- by

$$\omega_s^{\pm}(t) = \begin{cases} \omega(t) & \text{for } t \neq s \\ \pm 1 & \text{for } t = s \end{cases}.$$

◁

Definition 5.3 *For every random variable* $X \in L^2(\Omega, P)$ *we define the* Malliavin process $(\mathbb{D}_t X)_{t \geq 0}$ *by the family* $(\mathbb{D}_t)_{t \geq 0}$ *of operators on* $L^2(\Omega, P)$

$$\mathbb{D}_t X(\omega) = \frac{1}{2} \sum_{s<t} \left(X(\omega_s^+) - X(\omega_s^-)\right) \sqrt{\triangle t}.$$

The Malliavin derivative $(D_t X)_{t \geq 0}$ *of* X *is the derivative of the Malliavin process:*

$$D_t X(\omega) = \frac{\triangle \mathbb{D}_t X(\omega)}{\triangle t} = \frac{X(\omega_t^+) - X(\omega_t^-)}{2\sqrt{\triangle t}}.$$

◁

This two families of operators can be viewed as two operators

$$\mathbb{D} : L^2(\Omega, P) \to CM \quad \text{and} \quad D : L^2(\Omega, P) \to CM\,.$$

Remark 5.4 The Malliavin derivative acts on the discrete Wiener-Itô decomposition as just leaving aside one of the integrations over B_t.

PROOF:

$$\begin{aligned}
D_t X(\omega) &= D_t \left(\sum_n \sum_{(t_1 < \ldots < t_n) \in \Lambda^n} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n}{2}} \chi_{\{t_1,\ldots,t_n\}}(\omega) \right) \\
&= \sum_n \sum_{(t_1 < \ldots < t_n) \in \Lambda^n} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n}{2}} \frac{\chi_{\{t_1,\ldots,t_n\}}(\omega_t^+) - \chi_{\{t_1,\ldots,t_n\}}(\omega_t^-)}{2\sqrt{\triangle t}} \\
&= \sum_n \sum_{(t_1 < \ldots < t_n) \in \Lambda^n} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n-1}{2}} \cdot \\
&\quad \cdot \frac{1}{2} \left(\prod_{s \in \{t_1,\ldots,t_n\}} \omega_t^+(s) - \prod_{s \in \{t_1,\ldots,t_n\}} \omega_t^-(s) \right) \\
&= \sum_n \sum_{\substack{(t_1 < \ldots < t_n) \in \Lambda^n \\ t \in \{t_1,\ldots,t_n\}}} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n-1}{2}} \chi_{\{t_1,\ldots,t_n\} \setminus \{t\}}(\omega) \\
&= \sum_n \sum_{(t_1 < \ldots < t_{n-1}) \in \Lambda^{n-1}} n! X_n(t_1, \ldots, t_{n-1}, t) \triangle t^{\frac{n-1}{2}} \chi_{\{t_1,\ldots,t_{n-1}\}}(\omega) \\
&= \sum_n \sum_{(t_1,\ldots,t_{n-1}) \in \Lambda^{n-1}} n X_n(t_1, \ldots, t_{n-1}; t) \triangle B(t_1) \cdots \triangle B(t_{n-1})\,.
\end{aligned}$$

□

So the discrete Malliavin derivative acts on the discrete Wiener-Itô decomposition of a random variables as expected from the continuous case.

Remark 5.5 Holden et al. in their article [8] called $\mathbb{D}_t X$ the Malliavin derivative. This seems to be the wrong way around since the Cameron-Martin increment $D_t X = \frac{\triangle \mathbb{D}_t X}{\triangle t}$ of $\mathbb{D}_t X$ acts in the right way on the chaos decomposition. Furthermore with our Malliavin derivative $D_t X$ we establish a discrete version of the Clark Ocone formula.

One sees that $D_t\chi_\emptyset = 0$. In quantum mechanics $\chi_\emptyset$ is the vacuum state and D_t is the one particle creation operator at time t.

Proposition 5.6

$$D_t X = X \cdot W_t + X \diamond W_t\,.$$

PROOF: If $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ then

$$\frac{X(\omega_s^+) - X(\omega_s^-)}{2\sqrt{\triangle t}} \quad = \quad \sum_{\substack{A\in\mathcal{P}\\ t\in A}} X(A)\chi_{A\setminus\{t\}}(\omega)(\triangle t)^{-\frac{1}{2}}\,.$$

But since $\chi_A \cdot W_t = \chi_A \diamond W_t$ if A and $\{t\}$ are disjoint we get

$$\begin{aligned} X \cdot W_t + X \diamond W_t \quad &= \\ \sum_{A\in\mathcal{P}} X(A)\chi_A \cdot \chi_{\{t\}}(\triangle t)^{-\frac{1}{2}} \quad &- \quad \sum_{A\in\mathcal{P}} X(A)\chi_A \diamond \chi_{\{t\}}(\triangle t)^{-\frac{1}{2}} \\ = \sum_{A\in\mathcal{P}} X(A)\chi_{A\triangle\{t\}}(\triangle t)^{-\frac{1}{2}} \quad &- \quad \sum_{\substack{A\in\mathcal{P}\\ t\notin A}} X(A)\chi_{A\cup\{t\}}(\triangle t)^{-\frac{1}{2}} \\ &= \quad \sum_{\substack{A\in\mathcal{P}\\ t\in A}} X(A)\chi_{A\setminus\{t\}}(\triangle t)^{-\frac{1}{2}}\,. \end{aligned}$$

Thus the proposition follows. □

6 Discrete Malliavin divergence

In this section we define the discrete Malliavin divergence and show that the Malliavin divergence is the adjoint operator to the Malliavin process.

Definition 6.1 *The* Malliavin divergence δ *is an operator from* CM *into* $L^2(\Omega, P)$ *defined for a process* $Y = Y_t$ *by*

$$\delta Y = \sum_{t\in\Lambda} \frac{\triangle Y_t}{\triangle t} \diamond \triangle B_t = \int \frac{\triangle Y}{\triangle t}\delta B\,.$$

◁

Remark 6.2 If $Y_t = \sum_{s<t} X_s \triangle t$ for some process X_s then

$$\delta\left(\sum_{s<t} X_s \triangle t\right) = \sum_{t\in\Lambda} \frac{X_t \triangle t}{\triangle t} \diamond \triangle B_t = \int X \delta B\,.$$

This means that the discrete divergence of the discrete Lebesgue integral of a process is nothing else than the discrete Skorohod integral of that process.

Proposition 6.3 *$\mathbb{D}$ and δ are adjoint operator in the following sense:*

$$\langle X, \delta Y\rangle_{L^2(\Omega,P)} = \langle \mathbb{D}X, Y\rangle_{CM} \quad \forall X \in L^2(\Omega, P),\ \forall Y \in CM\,.$$

PROOF: First we note that

$$\sum_{\omega\in\Omega} \chi_A(\omega) P(\omega) = \begin{cases} 0, & A \neq \emptyset \\ 1, & A = \emptyset \end{cases}.$$

We show that the left hand side is equal to the right hand side. Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ and the $\triangle Y_s = \sum_{B\in\mathcal{P}} \triangle Y(B;s)\chi_B$ the walsh decompositions of the random variables X respectively $\triangle Y_s$.

$$\begin{aligned}
\sqrt{\triangle t}\langle X, \delta Y\rangle_{L^2(\Omega,P)} &= \sqrt{\triangle t}\sum_s E[X \cdot (\frac{\triangle Y_s}{\triangle t} \diamond \triangle B_s)] \\
&= \frac{\sqrt{\triangle t}\sqrt{\triangle t}}{\triangle} \sum_s E[X \cdot \sum_{B\in\mathcal{P}} \triangle Y(B;s)\chi_B \diamond \chi_{\{s\}}] \\
&= \sum_s E\left[\left(\sum_{A\in\mathcal{P}} X(A)\chi_A\right) \cdot \left(\sum_{\substack{B\in\mathcal{P}\\ s\notin B}} \triangle Y(B;s)\chi_{B\cup\{s\}}\right)\right] \\
&= \sum_s \sum_{\substack{B\in\mathcal{P}\\ s\notin B}} \sum_{A\in\mathcal{P}} E[X(A)\triangle Y(B;s)\chi_{B\cup\{s\}\triangle A}] \\
&= \sum_s \sum_{\substack{B\in\mathcal{P}\\ s\notin B}} \sum_{A\in\mathcal{P}} \sum_{\omega\in\Omega} X(A)\triangle Y(B;s)\chi_{B\cup\{s\}\triangle A}(\omega) \\
&= \sum_s \sum_{\substack{B\in\mathcal{P}\\ s\notin B}} X(B\cup\{s\})\triangle Y(B;s)\,.
\end{aligned}$$

Now we calculate the right hand side:

$$\begin{aligned}
\sqrt{\triangle t}\langle \mathbb{D}X, Y\rangle_{CM} &= \sqrt{\triangle t}E[\sum_s D_t X \cdot \triangle Y_s] \\
&= E\left[\sum_s \left(\left(\sum_{\substack{A\in\mathcal{P}\\ s\in A}} X(A)\chi_{A\setminus\{s\}}\right) \cdot \left(\sum_{B\in\mathcal{P}} \triangle Y(B;s)\chi_B\right)\right)\right] \\
&= E\left[\sum_s \sum_{\substack{A\in\mathcal{P}\\ s\in A}} \sum_{B\in\mathcal{P}} X(A)\triangle Y(B;s)\chi_{B\triangle(A\setminus\{s\})}\right] \\
&= \sum_{\omega\in\Omega}\sum_s \sum_{\substack{A\in\mathcal{P}\\ s\in A}} \sum_{B\in\mathcal{P}} X(A)\triangle Y(B;s)\chi_{B\triangle(A\setminus\{s\})}(\omega) \\
&= \sum_{\omega\in\Omega}\sum_s \sum_{\substack{A\in\mathcal{P}\\ s\notin A}} \sum_{B\in\mathcal{P}} X(A\cup\{s\})\triangle Y(B;s)\chi_{B\triangle A}(\omega) \\
&= \sum_s \sum_{\substack{A\in\mathcal{P}\\ s\notin A}} X(A\cup\{s\})\triangle Y(A;s)\,.
\end{aligned}$$

Thus we obtain the result. □

7 Discrete Clark-Ocone formula

Now we are prepared to proof the discrete version of the Clark-Ocone formula. The continuous Clark-Ocone formula for random variables F looks like this:

$$F = E[F] + \int E[D_t F|\mathcal{F}_t]dB_t$$

and can be proven under certain conditions for F. The integral here is an Itô integral. In the discrete version we have'nt any condition since there are'nt any convergence problems for sums or integrals. The discrete Clark-Ocone formula reads as follows.

Theorem 7.1 *Let be $X \in L^2(\Omega, P)$. Then it holds*

$$X = E[X] + \sum_{t\in\Lambda} E[D_t X|\mathcal{F}_t] \cdot \triangle B_t\,.$$

PROOF: Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ the Walsh decomposition of X. First remember from proposition 2.5 that $E[X] = X(\emptyset)$. We show that $\sum_{t\in\Lambda} E[D_tX|\mathcal{F}_t]\cdot\triangle B_t$ is equal to $X - X(\emptyset)$. We use the following expression for D_tX:

$$\begin{aligned}
D_tX(\omega) &= \frac{X(\omega_t^+) - X(\omega_t^-)}{2\sqrt{\triangle t}} = \sum_{A\in\mathcal{P}} \frac{X(A)}{2\sqrt{\triangle t}}\left(\chi_A(\omega_t^+) - \chi_A(\omega_t^-)\right) \\
&= \sum_{\substack{A\in\mathcal{P}\\ t\in A}} \frac{X(A)}{\sqrt{\triangle t}}\chi_{A\setminus\{t\}}(\omega) = \sum_{\substack{A\in\mathcal{P}\\ t\notin A}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A(\omega)\,.
\end{aligned}$$

Since the conditional expectation with respect to $\mathcal{F}_t$ cuts the Walsh components χ_A with $A \not\subset [0,t[$, that means it must be $\max A < t$, we obtain

$$E[D_tX|\mathcal{F}_t] = \sum_{\substack{A\in\mathcal{P}\\ t\notin A\wedge\max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A = \sum_{\substack{A\in\mathcal{P}\\ \max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A\,.$$

Now we integrate this with $\sum_t \cdot\triangle B_t$ and since A and $\{t\}$ are disjoint using $\chi_A\cdot\chi_{\{t\}} = \chi_{A\cup\{t\}}$ we get

$$\begin{aligned}
\sum_{t\in\Lambda} E[D_tX|\mathcal{F}_t]\cdot\triangle B_t &= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P}\\ \max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A\cdot\chi_{\{t\}}\sqrt{\triangle t} \\
&= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P}\\ \max A<t}} X(A\cup\{t\})\chi_{A\cup\{t\}} \\
&= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P}\\ \max A=t}} X(A)\chi_A = \sum_{A\in\mathcal{P}\setminus\emptyset} X(A)\chi_A \\
&= X - X(\emptyset)\,.
\end{aligned}$$

Thus the proof of the theorem is done. □

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