Smooth projective planes, smooth generalized quadrangles, and isoparametric hypersurfaces

DISSERTATION

DER MATHEMATISCHEN FAKULTÄT
DER EBERHARD-KARLS-UNIVERSITÄT TÜBINGEN
ZUR ERLANGUNG DES GRADES EINES DOKTORS
DER NATURWISSENSCHAFTEN

VORGELEGT VON

STEFAN IMMERVOLL

IM SOMMERSEMESTER 2001

Tag der mündlichen Prüfung:	4. Juli 2001
Dekan:	Professor Dr. Ch. Lubich
1. Berichterstatter	Professor Dr. V. Batyrev
2. Berichterstatter	Professor Dr. H. Salzmann

Diese Dissertation ist Dem gewidmet,
der mir am meisten dabei geholfen hat.

Contents

Zusammenfassung

Die vorliegende Dissertation besteht aus zwei Teilen, die durch ein gemeinsames Prinzip verbunden sind: die Charakterisierung glatter Geometrien mittels (differential-)topologischer Bedingungen.

In Teil I dieser Arbeit verwenden wir eine solche Charakterisierung von Tits-Gebäuden vom Typ C_2 (verallgemeinerte Vierecke), um ein differentialgeometrisches Problem weitgehend zu lösen, das von Thorbergsson vor ca. 10 Jahren in seinen Arbeiten [35] und [36] formuliert wurde. Wir beweisen den folgenden

Satz. *Sei $\mathfrak{I}$ eine kompakte, zusammenhängende isoparametrische Hyperfläche in einer Sphäre. Falls $\mathfrak{I}$ vier verschiedene Hauptkrümmungen besitzt, ist die mit $\mathfrak{I}$ assoziierte Inzidenzstruktur ein Tits-Gebäude vom Typ* C_2.

Die Konzentration auf kompakte, zusammenhängende isoparametrische Hyperflächen mit vier verschiedenen Hauptkrümmungen ist keineswegs so speziell, wie sie vielleicht erscheinen mag. Nach [27] ist jede zusammenhängende isoparametrische Hyperfläche in einer Sphäre eine offenene Teilmenge einer kompakten, zusammenhängenden isoparametrischen Hyperfläche $\mathfrak{I}$. Ferner können nach [28] nur 1, 2, 3, 4 oder 6 verschiedene Hauptkrümmungen auftreten. In den ersten beiden Fällen ist $\mathfrak{I}$ selbst eine geometrische Sphäre bzw. ein Produkt zweier geometrischer Sphären, und die Fokalmannigfaltigkeiten sind Punkte bzw. geometrische Sphären, siehe [31]. In den verbleibenden Fällen ist mit der isoparametrischen Hyperfläche $\mathfrak{I}$ und ihren Fokalmannigfaltikeiten $\mathfrak{P}$ und $\mathfrak{L}$ eine interessante Inzidenzstruktur verbunden. Man erhält sie, indem man $\mathfrak{I}$ vermöge der Projektionen auf

$\mathcal{P}$ und $\mathcal{L}$ als Fahnenraum $\mathcal{F}$ in $\mathcal{P} \times \mathcal{L}$ einbettet.

Im Fall dreier verschiedener Hauptkrümmungen läßt sich relativ einfach zeigen, daß die assoziierten Inzidenzstrukturen $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ verallgemeinerte Dreiecke, d.h. projektive Ebenen, sind. Bereits E. Cartan klassifizierte die kompakten, zusammenhängenden isoparametrischen Hyperflächen mit drei verschiedenen Hauptkrümmungen, siehe [7]. Nach [18] sind die assoziierten Inzidenzstrukturen genau die klassischen projektiven Ebenen über den vier reellen alternativen Divisionsalgebren.

Im Fall von sechs verschiedenen Hauptkrümmungen ist bekannt, daß alle Hauptkrümmungen entweder die Vielfachheit 1 oder die Vielfachheit 2 besitzen. Ferner existiert jeweils genau eine maximale Familie homogener isoparametrische Hyperflächen mit diesen Parametern. Es gibt keine weitere isoparametrische Hyperfläche mit sechs verschiedenen Hauptkrümmungen der Vielfachheit 1 in einer Sphäre, vergleiche [37]. Bis auf die Frage, ob eine inhomogene isoparametrische Hyperfläche mit sechs verschiedenen Hauptkrümmungen der Vielfachheit 2 existiert, sind somit auch die isoparametrischen Hyperflächen mit sechs verschiedenen Hauptkrümmungen in Sphären klassifiziert. Im homogenen Fall sind die auftretenden Inzidenzstrukturen verallgemeinerte Sechsecke, vergleiche [36].

Die bisher noch offene Frage, ob mit isoparametrischen Hyperflächen mit vier verschiedenen Hauptkrümmungen in Sphären ebenfalls stets Tits-Gebäude assoziiert sind, wird durch den obigen Satz beantwortet. Der Fall isoparametrischer Hyperflächen mit vier verschiedenen Hauptkrümmungen ist wesentlich reichhaltiger als die anderen Fälle. Insbesondere sind solche Hyperflächen bis jetzt noch nicht klassifiziert. Zu ihnen gehören die in [10] mittels reeller Darstellungen von Cliffordalgebren konstruierten Beispiele, die in Kapitel 2 untersucht werden. Wir beweisen dort innerhalb der Theorie der Cliffordalgebren, daß die assoziierten verallgemeinerten Vierecke (siehe [36]) *glatte* verallgemeinerte Vierecke sind. Die isoparametrischen Hyperflächen dieses Typs umfassen bis auf zwei homogene Beispiele alle bekannten isoparametrischen Hyperflächen mit vier verschiedenen Hauptkrümmungen in Sphären, siehe [10]. Kapitel 3 ist dem Beweis des obigen Satzes mit Hilfe der Theorie isoparametrischer Tripelsysteme gewidmet. Auf dem Weg zu einem Beweis dieses Satzes erhalten wir weitere neue Resultate über die Geometrie isoparametrischer Hyperflächen. In Kapitel 1 untersuchen wir detailliert das Wechselspiel zwischen verschiedenen Glattheitseigenschaften verallgemeinerter Vierecke. Insbesondere beweisen wir eine zu Beginn angesprochene Charakterisierung glatter verallgemeinerter Vierecke, die die Grundlage für die Hauptresultate der Kapitel 2 und 3 bildet.

In Teil II dieser Arbeit lösen wir ein klassisches Problem der topologischen Geometrie. Es war bisher nicht bekannt, ob es nicht-klassische projektive Ebenen gibt, deren Punkt- und Geradenraum reell analytische Mannigfaltigkeiten sind, so daß Schneiden und Verbinden durch reell analytische Abbildungen gegeben sind. Wir konstruieren Beispiele solcher Ebenen in den Dimensionen 2, 4 und 8. Ferner sind diese Ebenen die ersten Beispiele nicht-klassischer glatter projektiver Ebenen mit großen Automorphismengruppen. In Dimension 2 stimmen sie mit einer von Segre entdeckten Klasse projektiver Ebenen überein, siehe [33]. Eine ausführlicher Einführung in diesen Problemkreis geben wir in der Einleitung zu Kapitel 5. Grundlegend für den Nachweis, daß es sich tatsächlich um projektive Ebenen handelt, ist eine Charakterisisierung glatter projektiver Ebenen in Kapitel 4. In Analogie zu Kapitel 1 werden in diesem Kapitel glatte Inzidenzstrukturen untersucht, die eine Verallgemeinerung glatter projektiver Ebenen darstellen.

Wenn ich auch über die Themen dieser Dissertation nicht mehr mit Richard Bödi diskutieren konnte, so habe ich doch sehr von den "Nachwirkungen" der endlosen Gespräche mit ihm während meines Studiums profitiert. Ich möchte ihm dafür herzlich danken. Kapitel 4 entspricht im wesentlichen einem Teil einer gemeinsamen Publikation mit ihm, siehe [4]. Linus Kramer danke ich für hilfreiche Gespräche, die mir wertvolle Anregungen für meine Dissertation gegeben haben. Dem Referenten meiner Arbeit [16] habe ich die Idee zu verdanken, meine Resultate aus dieser Arbeit mit der Theorie isoparametrischer Tripelsysteme zu verbinden. Bei der Familie Parantainen, und insbesondere bei Annika Parantainen, möchte ich mich für die Gastfreundschaft während eines Aufenthalts in Finnland bedanken, wo ein wesentlicher Teil des zweiten Teils dieser Dissertation entstanden ist.

Meinem Betreuer, Victor Batyrev, danke ich für die Freiheit, die er mir in der mathematischen Forschung gelassen hat, und für das Vertrauen, das er mir dabei stets entegengebracht hat. Abschließend möchte ich Reiner Salzmann danken für zahllose interessante Gespräche und Anregungen.

Introduction

This thesis consists of two parts, which are joined by a common principle: the characterization of smooth geometries by means of (differential-) topological conditions.

In Part I we use a caracterization of Tits buildings of type C_2 (generalized quadrangles) in order to solve to a large extent a differential-geometric problem which was posed by Thorbergsson ca. 10 years ago in his papers [35] und [36]. To be more precise, we prove the following

Theorem. *Let $\mathcal{I}$ be a compact, connected isoparametric hypersurface with four distinct principal curvatures in a sphere. Then the incidence structure associated with $\mathcal{I}$ and its focal manifolds is a Tits building of type* C_2.

The concentration on compact, connected isoparametric hypersurfaces with four distinct principal curvatures is not as special as it might seem. By [27], every connected isoparametric hypersurface in a sphere is an open subset of a compact, connected isoparametric hypersurface $\mathcal{I}$. Moreover, by [28], the number g of distinct principal curvatures is equal to 1, 2, 3, 4 or 6. In the first two cases, $\mathcal{I}$ is a geometric sphere or a product of two geometric spheres, and the focal manifolds are points or geometric spheres, respectively, see [31]. In the other three cases, an interesting incidence structure is associated with $\mathcal{I}$ and the focal manifolds $\mathcal{P}$ and $\mathcal{L}$. It is obtained by embedding $\mathcal{I}$ into $\mathcal{P} \times \mathcal{L}$ by means of the projections onto $\mathcal{P}$ und $\mathcal{L}$. The image of $\mathcal{I}$ in $\mathcal{P} \times \mathcal{L}$ is denoted by $\mathcal{F}$.

In the case of three distinct principal curvatures it is not difficult to show that the incidence structures $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ are generalized triangles, i.e.

projective planes. The isoparametric hypersurfaces with three distinct principal curvatures have been classified by E. Cartan, see [7]. The associated incidence structures are precisely the classical projective planes over the four real alternative division algebras, see [18].

In the case $g = 6$, it is known that all principal curvatures either have multiplicity 1 or multiplicity 2. Moreover, in both cases there exists precisely one maximal family of isoparametric hypersurfaces with these parameters, and in the case of multiplicity 1, this family is unique up to isometries of the sphere, cf. [37]. Hence, also the isoparametric hypersurfaces with six distinct principal curvatures in spheres are classified, except for the question whether there is an inhomogeneous hypersurface of this type whose principal curvatures have multiplicity 2. In the two homogeneous cases, the incidence structures associated with these isoparametric hypersurfaces are generalized hexagons, cf. [36].

The question, whether Tits buildings are associated also with isoparametric hypersurfaces with four distinct principal curvatures in spheres, is answered affirmatively in the theorem above. The class of isoparametric hypersurfaces with four distinct principal curvatures in spheres is much wider than in the other cases. In particular, these isoparametric hypersurfaces have not been classified so far. Except for two homogeneous hypersurfaces, all known examples may be described by means of real representations of Clifford algebras, see [10]. In Chapter 2, we will prove within the theory of Clifford algebras that the generalized quadrangles associated with these isoparamatric hypersurfaces of Clifford type are *smooth* generalized quadrangles. Chapter 3 is dedicated to a proof of the above theorem by means of the theory of isoparametric triple systems. On the way to a proof of this theorem we will obtain further new results on the geometry of isoparametric hypersurfaces. In Chapter 1 we will investigate in detail the interplay between various smoothness properties of generalized quadrangles. In particular, we will prove a characterization of smooth generalized quadrangles mentioned at the beginning of the introduction, which will serve as a basis for the main results in the Chapters 2 and 3.

The theorem above is related to Thorbergsson's papers [35] and [36]. In [35], he shows that isoparametric submanifolds in spheres of codimension at least 2 give rise to Tits buildings of rank at least 3. In his proof, he needs the fact that homogeneous isoparametric hypersurfaces with 2, 3 or 4 distinct principal curvatures in spheres give rise to generalized g-gons. This is more or less obvious for $g = 2$, and for $g = 3$ it follows by a geometric argument due to Thorbergsson, see [18], 3.3 and 3.4. The case $g = 4$ is a special case of the theorem above. In [35], Thorbergsson uses instead of direct geometric arguments the homogeneity of the isoparametric hypersurfaces in order to apply results of [13]. Our main result together with the elementary arguments in the cases $g = 2$, 3 provides a short cut for this step in Thorbergsson's proof.

In Part II of this thesis we will solve a classical problem in topological geometry. We will prove that there exist non-classical projective planes whose point space and line space are real analytic manifolds such that the geometric operations of joining points and intersecting lines are real analytic maps on their respective domains. Our examples of these real analytic projective planes have the dimensions 2, 4, or 8. Furthermore, these planes are the first examples of non-classical smooth projective planes with large automorphism groups. In dimension 2, they correspond to a class of projective planes discovered by Segre, see [33]. We will give a comprehensive introduction to this problem area at the beginning of Chapter 5. The basis of our proof that the incidence structures constructed in this chapter are indeed projective planes is a characterization of smooth projective planes in Chapter 4. Analogously to Chapter 1, we will investigate in this chapter smooth incidence structures, which are generalizations of smooth projective planes.

Part I

Chapter 1

Smooth Generalized Quadrangles

Introduction

In this chapter we investigate the interplay between various smoothness properties of generalized quadrangles. In the next chapters we will apply our results (in particular, Theorems 1.10 and 1.17) in the context of isoparametric hypersurfaces in spheres.

After giving the basic definitions and introducing some notation in the first section, we will consider generalized quadrangles whose sets of vertices are smooth manifolds. Our major results in the second section are Theorems 1.10 and 1.17. These results characterize the smoothness of certain maps associated with a generalized quadrangle in a natural way in terms of transversality conditions involving point rows, line pencils, etc. We present Theorem 1.10 here as an example for the character of these results since it can be stated without introducing too much notation at this point. For the definition of smooth generalized quadrangles, see the first section.

Theorem. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized quadrangle which satisfies the following conditions:*

(SGQ1) *There are positive integers a, b such that $\mathcal{P}$ is a smooth manifold of dimension $2a + b$ and $\mathcal{L}$ is a smooth manifold of dimension $a + 2b$.*

(SGQ2) *The flag space $\mathcal{F}$ is a $(2a + 2b)$-dimensional submanifold of $\mathcal{P} \times \mathcal{L}$ such that the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.*

If moreover for each antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$ the submanifolds $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$, then $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a smooth generalized quadrangle.

Here, $\mathcal{P}_L$ denotes the point row corresponding to the line L, and $\mathcal{L}_p$ denotes the line pencil associated with the point p, see the first section. As a consequence of (SGQ1) and (SGQ2), point rows are a-dimensional submanifolds of $\mathcal{P}$ and line pencils are b-dimensional submanifolds of $\mathcal{L}$, cf. Lemma 1.6. Hence, the transversality condition in the above theorem makes sense and means that for each $(q, K) \in (\mathcal{P}_L \times \mathcal{L}_p) \cap \mathcal{F}$ the tangent spaces satisfy

$$\mathrm{T}_{(q,K)}(\mathcal{P}_L \times \mathcal{L}_p) \cap \mathrm{T}_{(q,K)}\mathcal{F} = \{0\}.$$

In the third section we will establish differential-topological properties of smooth generalized quadrangles. In this way we will see that the conditions imposed in the preceding section are indeed satisfied by every smooth generalized quadrangle. In particular, smooth generalized quadrangles have the properties used as conditions in the above theorem. In the fourth section we will characterize smooth generalized quadrangles among incidence structures which have similar differential-topological properties. The main result of this section, Theorem 1.17, is particularly useful for applications. It may be stated as follows:

Theorem. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence structure which satisfies the following conditions ($a, b \in \mathbb{N}$):*

- (SIS1) *The point space $\mathcal{P}$ and the line space $\mathcal{L}$ are compact, connected smooth manifolds. The dimensions of $\mathcal{P}$ and $\mathcal{L}$ are $2a + b$ and $a + 2b$, respectively.*
- (SIS2) *The flag space $\mathcal{F}$ is a $(2a + 2b)$-dimensional closed submanifold of $\mathcal{P} \times \mathcal{L}$, and the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.*

(SIS3) *For every antiflag* $(p, L) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$, *the submanifolds* $\mathcal{P}_L \times \mathcal{L}_p$ *and* $\mathcal{F}$ *intersect transversally in* $\mathcal{P} \times \mathcal{L}$.

Then there is a finite number n *such that* $\mathcal{P}_L \times \mathcal{L}_p$ *and* $\mathcal{F}$ *intersect in precisely* n *points for each antiflag* $(p, L) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$.

In particular, an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ satisfying the conditions of this theorem is a smooth generalized quadrangle provided that there is at least one antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$ such that $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect in exactly one point. Note that in contrast to conditions (SGQ1) and (SGQ2) we assume here that $\mathcal{P}$ and $\mathcal{L}$ are compact and connected and that $\mathcal{F}$ is closed in $\mathcal{P} \times \mathcal{L}$.

Preliminaries

Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence structure, where $\mathcal{P}$ denotes the set of *points*, $\mathcal{L}$ the set of *lines* and $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ the set of *flags*. The sets $\mathcal{P}$ and $\mathcal{L}$ are assumed to be disjont. We call a point x and a line y *incident* if $(x, y) \in \mathcal{F}$. The elements of $V = \mathcal{P} \cup \mathcal{L}$ are called *vertices*. A k*-chain* ($k \in \mathbb{N}$) is a sequence $(v_0, v_1, \ldots, v_k)$ of vertices such that v_i is incident with v_{i+1} for $0 \leq i < k$ and $v_i \neq v_{i+2}$ for $0 \leq i < k-1$. We say that the vertices v_0 and v_k are *joined* by this k-chain. Two vertices x and y are said to have *distance* $d(x, y) = k$ if they are joined by a k-chain and k is minimal with respect to this property.

For $x \in \mathcal{P}$ we denote by $\mathcal{L}_x = \{y \in V \mid d(x, y) = 1\}$ the *line pencil* through x. Analogously, we call the set $\mathcal{P}_z = \{y \in V \mid d(y, z) = 1\}$ the *point row* associated with $z \in \mathcal{L}$. For $x \in V$ we set $\mathrm{D}_2 x = \{y \in V \mid d(x, y) = 2\}$.

1.1 Definition. A *generalized quadrangle* is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ which satisfies the following axioms:

(1) $d(x, y) \leq 4$ for any x, $y \in V$.

(2) If $d(x,y) = k < 4$, then the k-chain joining x and y is unique.
(3) For any $x \in \mathcal{P}$, $y \in \mathcal{L}$ we have $|\mathcal{P}_x|, |\mathcal{L}_y| \geq 3$.

According to axiom (2) we can define maps

$$f_k : \{(v_0, v_k) \in V^2 \mid d(v_0, v_k) = k\} \to V^{k-1}$$

for $k = 2$ and $k = 3$ with $f_2(v_0, v_2) = v_1$ and $f_3(v_0, v_3) = (v_1, v_2)$, where $(v_0,\, v_1,\, v_2)$ and $(v_0,\, v_1,\, v_2,\, v_3)$, respectively, are the uniquely determined k-chains joining the vertices v_0 and v_k. Analogously, we define a map

$$g : \{(v_0, v_3) \in V^2 \mid d(v_0, v_3) = 3\} \to V : (v_0, v_3) \mapsto v_2.$$

1.2 Definition. A generalized quadrangle is called a *topological quadrangle* if the point space and the line space are endowed with topologies which are neither discrete nor anti-discrete such that the map f_3 is continuous.

Remark. In a topological quadrangle, the map f_2 is continuous on its respective domain, see [12], Proposition 2.3.

For the sake of simplicity, the words "differentiable" and "smooth" will be used in the sense of C^∞. By a "submanifold" we will always mean a smoothly embedded submanifold of a smooth manifold.

1.3 Definition. A topological quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is called *smooth* if the point space $\mathcal{P}$ and the line space $\mathcal{L}$ carry smooth structures such that the map f_3 is smooth.

Remark. In a topological generalized quadrangle the domain of the map f_3 is open in $V \times V$ since the flag space $\mathcal{F}$ is closed in $\mathcal{P} \times \mathcal{L}$ (see [12], Proposition 2.4). Therefore, if $\mathcal{P}$ and $\mathcal{L}$ are smooth manifolds, it makes sense to require the map f_3 to be smooth. Note that all the above definitions are self-dual.

If $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a topological generalized quadrangle, then the spaces $\mathcal{P}$ and $\mathcal{L}$ as well as point rows and line pencils are either all connected or all totally disconnected, cf. [12], Proposition 3.3. Hence, all these spaces are connected if $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a smooth generalized quadrangle. In this case, the topologies on $\mathcal{P}$ and $\mathcal{L}$ are even compact, see [12], Proposition 3.4. Topological quadrangles whose sets of vertices are compact spaces will be called *compact quadrangles* in the following. Among the generalized quadrangles with compact topologies on the point space and the line space, the compact quadrangles may be characterized as follows (see [14], 2.1 (a)).

1.4 Proposition. *A generalized quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ with compact topologies on $\mathcal{P}$ and $\mathcal{L}$ is topological if and only if $\mathcal{F}$ is closed in $\mathcal{P} \times \mathcal{L}$.*

Remark. Proposition 1.4 is not true without the compactness assumptions, cf. [14], 1.2 (5). In the next section, we will give an analogous characterization for smooth generalized quadrangles (see Theorem 2.5).

The following Proposition follows directly from Lemma 2.7 in [17].

1.5 Proposition. *If $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a compact quadrangle such that point rows and line pencils are manifolds, then $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are also manifolds, and we have* $\dim \mathcal{P} = 2p + q$, $\dim \mathcal{L} = p + 2q$ *and* $\dim \mathcal{F} = 2p + 2q$, *where p and q denote the dimensions of the point rows and the line pencils, respectively.*

Remark. It can be seen by means of the geometric operations that any two point rows (line pencils) of a topological generalized quadrangle are homeomorphic (see [12], Lemma 2.2). Hence, the definition of p and q makes sense. If $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a smooth generalized quadrangle, then by [5], 4.2 and 4.4, point rows and line pencils are submanifolds of $\mathcal{P}$ and $\mathcal{L}$, respectively, which are homeomorphic to spheres. Thus Proposition 1.5 can be applied in this case. Any two point rows (line pencils) of a smooth generalized quadrangle are even diffeomorphic. The proof is the same as in the topological case.

Smoothness Properties of Generalized Quadrangles

The aim of this section to investigate various relations between different smoothness properties of generalized quadrangles. We show that under weak differential-topological assumptions point rows and line pencils are submanifolds of the point space and the line space, respectively. The operations of intersecting lines and joining points are smooth on their respective domains, provided that any two intersecting point rows and any two intersecting line pencils intersect "weakly transversally", see Definition 1.7 and Theorem 1.12. Generalized quadrangles which satisfy even stronger transversality conditions are shown to be smooth, see Theorem 1.10 and Corollary 1.11.

The generalized quadrangles $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ considered in this section are assumed to satisfy the following two additional axioms:

(SGQ1) There are positive integers p, q such that $\mathcal{P}$ is a smooth manifold of dimension $2p+q$ and $\mathcal{L}$ is a smooth manifold of dimension $p+2q$.

(SGQ2) The flag space $\mathcal{F}$ is a $(2p+2q)$-dimensional submanifold of $\mathcal{P} \times \mathcal{L}$ such that the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.

Remark. Note that the axioms (SGQ1) and (SGQ2) are self-dual. Since we do not require that $\mathcal{P}$ and $\mathcal{L}$ are compact and that $\mathcal{F}$ is closed in $\mathcal{P} \times \mathcal{L}$, a generalized quadrangle satisfying the above axioms is not known to be a topological one. In the case of a compact quadrangle, the dimension assumptions above are satisfied automatically by Proposition 1.5, which can be applied since, as a consequence of the proof of the next lemma (without using the dimension assumptions in (SGQ1) and (SGQ2)), point rows and line pencils are submanifolds of $\mathcal{P}$ and of $\mathcal{L}$, respectively.

1.6 Lemma. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be an incidence structure which satisfies axioms* (SGQ1) *and* (SGQ2)*. Then (non-empty) point rows are p-dimensional submanifolds of $\mathcal{P}$ and (non-empty) line pencils are q-dimensional submanifolds of $\mathcal{L}$.*

Proof. By duality it suffices to give a proof for point rows only. The point row $\mathcal{P}_x$ associated with some line $x \in \mathcal{L}$ is the set $\pi_{\mathcal{P}}(\pi_{\mathcal{L}}^{-1}(x))$. Since $\pi_{\mathcal{L}}$ is a submersion, the set $\pi_{\mathcal{L}}^{-1}(x)$ is a submanifold of $\mathcal{F}$. Moreover, we have $\dim \pi_{\mathcal{L}}^{-1}(x) = \dim \mathcal{F} - \dim \mathcal{L} = p$. The map $\rho_x : \mathcal{P} \to \mathcal{P} \times \mathcal{L} : z \mapsto (z, x)$ is smooth and we have $\rho_x \circ \pi_{\mathcal{P}}|_{\pi_{\mathcal{L}}^{-1}(x)} = \mathrm{id}_{\pi_{\mathcal{L}}^{-1}(x)}$. This proves that $\pi_{\mathcal{P}}|_{\pi_{\mathcal{L}}^{-1}(x)} : \pi_{\mathcal{L}}^{-1}(x) \to \mathcal{P}$ is an embedding. Hence, $\mathcal{P}_x$ is a p-dimensional submanifold of $\mathcal{P}$. □

1.7 Definition. Two submanifolds N_1, N_2 of a smooth manifold M are said to intersect *transversally* in $x \in N_1 \cap N_2$ if the tangent spaces satisfy $\mathrm{T}_x N_1 + \mathrm{T}_x N_2 = \mathrm{T}_x M$. They are said to intersect transversally if they intersect transversally in each common point. We say that two point rows $\mathcal{P}_y$, $\mathcal{P}_z$ intersect *weakly transversally* in some point $x \in \mathcal{P}$, if we have $\mathrm{T}_x \mathcal{P}_y \cap \mathrm{T}_x \mathcal{P}_z = \{0\}$. Weakly transversal intersection of line pencils is defined dually.

Remark. By Lemma 1.6 and the dimension assumptions in (SGQ1), two point rows (line pencils) can never intersect transversally in the usual sense. Therefore we have introduced this notion. It is astonishing that for smooth generalized quadrangles the dimension of the span of the spaces $\mathrm{T}_x \mathcal{P}_z$, $z \in \mathcal{L}_x$, may vary with the point $x \in \mathcal{P}$. This is a consequence of [10], 5.8, and [36], as was pointed out in [5], Section 5.

1.8 Proposition. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a compact generalized quadrangle which satisfies axioms* (SGQ1) *and* (SGQ2) *and has the property that any two intersecting point rows (line pencils) intersect weakly transversally. Then for each* $x \in \mathcal{L}$ *(each* $x \in \mathcal{P}$*) the set* $\mathrm{D}_2 x$ *is a* $(p+q)$*-dimensional submanifold of* $\mathcal{L}$ *(of* $\mathcal{P}$*).*

Proof. By duality, we need to consider only the case in which x is a point. Since $\pi_{\mathcal{L}}$ is a submersion, the set

$$\pi_{\mathcal{L}}^{-1}(\mathcal{L}_x) = \{(y,z) \in \mathcal{F} \mid d(x,y) = 2,\ z \in \mathcal{L}_x\} \cup \{x\} \times \mathcal{L}_x$$

is a $(p+q)$-dimensional submanifold of $\mathcal{F}$. The projection $\pi_{\mathcal{P}}$ maps the submanifold $\mathcal{F}_x = \{(y,z) \in \mathcal{F} \mid d(x,y) = 2, z \in \mathcal{L}_x\}$ onto $\mathrm{D}_2 x$. A continuous inverse is given by the map $\mathrm{D}_2 x \to \mathcal{F}_x : y \mapsto (y, f_2(x,y))$. Note that f_2 is continuous by the Remark after Definition 1.2. Thus it is sufficient to show that the restriction of $\pi_{\mathcal{P}}$ to $\mathcal{F}_x$ is an immersion. Choose $(y,z) \in \mathcal{F}_x$ arbitrarily. Because of $\pi_{\mathcal{P}}^{-1}(y) = \{y\} \times \mathcal{L}_y$ we get $\{0\} \times \mathrm{T}_z \mathcal{L}_y \subseteq \ker((\mathrm{D}\pi_{\mathcal{P}})_{(y,z)})$. Since $\pi_{\mathcal{P}}$ is a submersion, we even have $\{0\} \times \mathrm{T}_z \mathcal{L}_y = \ker((\mathrm{D}\pi_{\mathcal{P}})_{(y,z)})$ for reasons of dimension. Hence, in order to show that $(\mathrm{D}\pi_{\mathcal{P}})_{(y,z)}$ restricted to $\mathrm{T}_{(y,z)}\mathcal{F}_x$ is one-to-one, we only have to prove that $\mathrm{T}_{(y,z)}\mathcal{F}_x \cap (\{0\} \times \mathrm{T}_z \mathcal{L}_y) = \{(0,0)\}$. By definition of $\mathcal{F}_x$, we have $\mathcal{F}_x \subseteq \mathcal{P} \times \mathcal{L}_x$ and therefore $\mathrm{T}_{(y,z)}\mathcal{F}_x \subseteq \mathrm{T}_y \mathcal{P} \times \mathrm{T}_z \mathcal{L}_x$. Because $\mathcal{L}_x$ and $\mathcal{L}_y$ intersect weakly transversally in $z \in \mathcal{L}$, we have $\mathrm{T}_z \mathcal{L}_x \cap \mathrm{T}_z \mathcal{L}_y = \{0\}$. This proves the above equality. Hence, the map $\pi_{\mathcal{P}}|_{\mathcal{F}_x}$ is an embedding, and $\mathrm{D}_2 x$ is a submanifold of the point space $\mathcal{P}$. □

The next lemma is formulated in a more general way than necessary for the proof of Theorem 1.10. We will use it also in the fourth section, where we will characterize smooth generalized quadrangles within a more general class of incidence structures.

1.9 Lemma. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be an incidence structure which satisfies axioms* (SQG1) *and* (SGQ2)*. Assume that* $\mathcal{F}$ *and* $\mathcal{P}_{x_0} \times \mathcal{L}_{x_3}$ *intersect transversally* *in* $(x_1, x_2) \in \mathcal{P} \times \mathcal{L}$ *for some antiflag* $(x_3, x_0) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$. *Then there are open neighbourhoods* U *of* (x_0, x_3) *in* $\mathcal{L} \times \mathcal{P}$ *and* U' *of* (x_1, x_2) *in* $\mathcal{P} \times \mathcal{L}$ *such that for any* $(y_0, y_3) \in U$ *there is exactly one intersection point* (y_1, y_2) *of* $\mathcal{F}$ *and* $\mathcal{P}_{y_0} \times \mathcal{L}_{y_3}$ *in* U'. *Furthermore, the map* $U \to U' : (y_0, y_3) \mapsto (y_1, y_2)$ *defined in this way is smooth.*

Remark. We do not assume here that $(x_1, x_2) \in \mathcal{P} \times \mathcal{L}$ is the only intersection point of $\mathcal{P}_{x_0} \times \mathcal{L}_{x_3}$ and $\mathcal{F}$.

Proof. We prove this lemma by means of the implicit function theorem. By $\mathcal{F}^*$ we denote the submanifold of $\mathcal{L} \times \mathcal{P}$ obtained from $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ by interchanging points and lines: $(x, y) \in \mathcal{F} \iff (y, x) \in \mathcal{F}^*$. Since $\mathcal{F}^*$ is a smoothly embedded submanifold of $\mathcal{L} \times \mathcal{P}$, there is an open neighbourhood $W_1 \subseteq \mathcal{L} \times \mathcal{P}$ of (x_0, x_1) and a submersion $\psi_1 : W_1 \to \mathbb{R}^{p+q}$ which vanishes exactly on $\mathcal{F}^* \cap W_1$. Analogously, we find submersions $\psi_2 : W_2 \to \mathbb{R}^{p+q}$, $\psi_3 : W_3 \to \mathbb{R}^{p+q}$ on open neighbourhoods $W_2 \subseteq \mathcal{P} \times \mathcal{L}$, $W_3 \subseteq \mathcal{L} \times \mathcal{P}$ of (x_1, x_2) and (x_2, x_3), respectively, such that $\psi_2^{-1}(0) = \mathcal{F} \cap W_2$ and $\psi_3^{-1}(0) = \mathcal{F}^* \cap W_3$. We put

$$\varphi : (\mathcal{L} \times \mathcal{P})^2 \to (\mathcal{L} \times \mathcal{P}) \times (\mathcal{P} \times \mathcal{L}) \times (\mathcal{L} \times \mathcal{P}) :$$
$$(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, y_1, y_2, y_2, y_3),$$
$$\psi : W_1 \times W_2 \times W_3 \to \mathbb{R}^{3p+3q} : (u, v, w) \mapsto (\psi_1(u), \psi_2(v), \psi_3(w)),$$

and set $W = \varphi^{-1}(W_1 \times W_2 \times W_3)$. In this way we define a smooth map

$$F : W \to \mathbb{R}^{3p+3q} : (y_0, y_1, y_2, y_3) \mapsto \psi \circ \varphi(y_0, y_1, y_2, y_3),$$

which vanishes if and only if $(y_0, y_1) \in \mathcal{F}^* \cap W_1$, $(y_1, y_2) \in \mathcal{F} \cap W_2$ and $(y_2, y_3) \in \mathcal{F}^* \cap W_3$. For $(y_0, y_1, y_2, y_3) \in W$, this is equivalent to $(y_1, y_2) \in (\mathcal{P}_{y_0} \times \mathcal{L}_{y_3}) \cap \mathcal{F}$. In order to apply the implicit function theorem, we have to show that the differential of the map

$$\{(y_1, y_2) \in \mathcal{P} \times \mathcal{L} \mid (x_0, y_1, y_2, x_3) \in W\} \to \mathbb{R}^{3p+3q} :$$
$$(y_1, y_2) \mapsto F(x_0, y_1, y_2, x_3)$$

at (x_1, x_2) is regular. Since $\mathrm{D}F_x = \mathrm{D}\psi_{\varphi(x)}\mathrm{D}\varphi_x$ and $\mathrm{D}\psi_{\varphi(x)}$ vanishes precisely on $\mathrm{T}_{(x_0,x_1)}\mathcal{F}^* \times \mathrm{T}_{(x_1,x_2)}\mathcal{F} \times \mathrm{T}_{(x_2,x_3)}\mathcal{F}^*$, this means that

$$\mathrm{D}\varphi_x(\{0\}\times\mathrm{T}_{x_1}\mathcal{P}\times\mathrm{T}_{x_2}\mathcal{L}\times\{0\})\cap(\mathrm{T}_{(x_0,x_1)}\mathcal{F}^*\times\mathrm{T}_{(x_1,x_2)}\mathcal{F}\times\mathrm{T}_{(x_2,x_3)}\mathcal{F}^*) = \{0\}.$$

Choose $(u, v) \in \mathrm{T}_{x_1}\mathcal{P} \times \mathrm{T}_{x_2}\mathcal{L}$ such that

$$\mathrm{D}\varphi_x(0, u, v, 0) \in \mathrm{T}_{(x_0,x_1)}\mathcal{F}^* \times \mathrm{T}_{(x_1,x_2)}\mathcal{F} \times \mathrm{T}_{(x_2,x_3)}\mathcal{F}^*.$$

By definition of φ we have $\mathrm{D}\varphi_x(0, u, v, 0) = (0, u, u, v, v, 0)$. Hence, we get $(0, u) \in \mathrm{T}_{(x_0,x_1)}\mathcal{F}^*$, $(u, v) \in \mathrm{T}_{(x_1,x_2)}\mathcal{F}$, and $(v, 0) \in \mathrm{T}_{(x_2,x_3)}\mathcal{F}^*$. Since $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{L}}$ are submersions, we have the identities $(\{0\} \times \mathrm{T}_{x_1}\mathcal{P}) \cap \mathrm{T}_{(x_0,x_1)}\mathcal{F}^* = \{0\} \times \mathrm{T}_{x_1}\mathcal{P}_{x_0}$ and $(\mathrm{T}_{x_2}\mathcal{L} \times \{0\}) \cap \mathrm{T}_{(x_2,x_3)}\mathcal{F}^* = \mathrm{T}_{x_2}\mathcal{L}_{x_3} \times \{0\}$, cf. the proof of Proposition 1.8. We conclude that

$$(u, v) \in (\mathrm{T}_{x_1}\mathcal{P}_{x_0} \times \mathrm{T}_{x_2}\mathcal{L}_{x_3}) \cap \mathrm{T}_{(x_1,x_2)}\mathcal{F} = \{0\},$$

since for reasons of dimension the transversal intersection of $\mathcal{P}_{x_0} \times \mathcal{L}_{x_3}$ and $\mathcal{F}$ in $(x_1, x_2) \in \mathcal{P} \times \mathcal{L}$ is equivalent to $(\mathrm{T}_{x_1}\mathcal{P}_{x_0} \times \mathrm{T}_{x_2}\mathcal{L}_{x_3}) \cap \mathrm{T}_{(x_1,x_2)}\mathcal{F} = \{0\}$. By the implicit function theorem, we conclude that there exist open neighbourhoods U of (x_0, x_3) in $\mathcal{L} \times \mathcal{P}$ and U' of (x_1, x_2) in $\mathcal{P} \times \mathcal{L}$ and a smooth map $f : U \to U'$ such that for any $(y_0, y_3) \in U$ and $(y_1, y_2) \in U'$ we have $(y_0, y_1, y_2, y_3) \in W$ and

$$(y_1, y_2) \in (\mathcal{P}_{y_0} \times \mathcal{L}_{y_3}) \cap \mathcal{F} \iff F(y_0, y_1, y_2, y_3) = 0 \iff (y_1, y_2) = f(y_0, y_3).$$

Hence, for any $(y_0, y_3) \in U$ there is precisely one $(y_1, y_2) \in U'$, namely $f(y_0, y_3)$, such that $(y_1, y_2) \in (\mathcal{P}_{y_0} \times \mathcal{L}_{y_3}) \cap \mathcal{F}$. This completes the proof. $\square$

Remark. Lemma 1.9 shows that incidence structures which satisfy the above conditions admit locally defined maps similar to the map f_3 in the case of a generalized quadrangle.

In Theorem 1.10 and Corollary 1.11 we present two implicit characterizations of smooth generalized quadrangles. Since we do not assume the generalized quadrangles considered in these results to be compact, it is not even obvious that they are topological. The smoothness of the map f_3 will follow directly from the preceding lemma without using continuity assumptions.

1.10 Theorem. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a generalized quadrangle which satisfies axioms* (SGQ1) *and* (SGQ2). *If for each antiflag* $(y, z) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$ *the submanifolds* $\mathcal{P}_z \times \mathcal{L}_y$ *and* $\mathcal{F}$ *intersect transversally in* $\mathcal{P} \times \mathcal{L}$, *then* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *is a smooth generalized quadrangle.*

Proof. The map f_3 assigns to any antiflag $(y, z) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$ the unique intersection point of $\mathcal{F}$ and $\mathcal{P}_z \times \mathcal{L}_y$. By the previous lemma, the domain of f_3 is open and f_3 is a smooth map. □

Remark. This theorem yields a very useful criterion for the smoothness of generalized quadrangles. In the next chapter, we will use it as an essential tool for our proof that generalized quadrangles arising from isoparametric hypersurfaces of Clifford type are smooth quadrangles.

1.11 Corollary. *Let* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a generalized quadrangle. Assume that in addition to* (SGQ1) *and* (SGQ2) *the following two conditions are satisfied:*

(i) *For every* $x \in V$, *the set* $\mathrm{D}_2 x$ *is a submanifold of* $\mathcal{P}$ *or* $\mathcal{L}$, *respectively.*

(ii) *For every antiflag* $(y, z) \in (\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$, *the point row* $\mathcal{P}_z$ *and* $\mathrm{D}_2 y$ *intersect transversally in* $\mathcal{P}$. *The dual statement also holds.*

Then $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *is a smooth generalized quadrangle.*

Remark. Though the quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is not known to be compact, the map $\pi_{\mathcal{P}}$ restricted to $\mathcal{F}_x$ (see the proof of Proposition 1.8) induces a smooth bijection between the $(p+q)$-dimensional submanifolds $\mathcal{F}_x$ and $\mathrm{D}_2 x$ for every $x \in \mathcal{P}$, provided condition (i) is satisfied. By Sard's Theorem (or by invariance of domain) we conclude that D_2x has dimension $p+q$ for each $x \in \mathcal{P}$.

In Proposition 1.8, we have seen that for compact quadrangles weakly transversal intersection of intersecting point rows and line pencils implies that for each $x \in V$ the set D_2x is a submanifold of $\mathcal{P}$ or $\mathcal{L}$, respectively. Conversely, if the sets D_2x are submanifolds of $\mathcal{P}$ for each $x \in \mathcal{P}$ (see (i)) and if they intersect point rows transversally in the sense of (ii), then intersecting point rows intersect weakly transversally: for any two distinct point rows $\mathcal{P}_y$ and $\mathcal{P}_z$ which intersect in a point $x \in \mathcal{P}$ we choose a point $w \in \mathcal{P}_z \backslash \{x\}$. Since $T_x\mathcal{P}_y + T_xD_2w = T_x\mathcal{P}$, we get $T_x\mathcal{P}_y \cap T_xD_2w = \{0\}$ for reasons of dimension. Because of $\mathcal{P}_z \backslash \{w\} \subseteq D_2w$, we conclude that $T_x\mathcal{P}_y \cap T_x\mathcal{P}_z = \{0\}$, i.e. the two point rows $\mathcal{P}_y$ and $\mathcal{P}_z$ intersect weakly transversally. The dual statements are also true.

Proof of Corollary 1.11. By Theorem 1.10, we only have to show that for every antiflag $(y, z) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$ the submanifolds $\mathcal{P}_z \times \mathcal{L}_y$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$. So, let $(x_3, x_0) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$ and $(x_1, x_2) = f(x_0, x_3)$. For reasons of dimension, it suffices to show that

$$(T_{x_1}\mathcal{P}_{x_0} \times T_{x_2}\mathcal{L}_{x_3}) \cap T_{(x_1,x_2)}\mathcal{F} = \{0\}. \qquad (*)$$

The $(p+q)$-dimensional submanifolds $\pi_{\mathcal{P}}^{-1}(\mathcal{P}_{x_0})$ and $\pi_{\mathcal{L}}^{-1}(\mathcal{L}_{x_3})$ of $\mathcal{F}$ intersect in $(x_1, x_2) \in \mathcal{F}$. For their tangent spaces we have

$$T_{(x_1,x_2)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_{x_0}) \subseteq (T_{x_1}\mathcal{P}_{x_0} \times T_{x_2}D_2x_0) \cap T_{(x_1,x_2)}\mathcal{F} \qquad (**)$$

and

$$T_{(x_1,x_2)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_{x_3}) \subseteq (T_{x_1}D_2x_3 \times T_{x_2}\mathcal{L}_{x_3}) \cap T_{(x_1,x_2)}\mathcal{F}. \qquad (***)$$

According to (ii), the point row $\mathcal{P}_{x_0}$ intersects D_2x_3 transversally in x_1 and the line pencil $\mathcal{L}_{x_3}$ intersects D_2x_0 transversally in x_2. Therefore we have

$\mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_{x_3}) \cap \mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_{x_0}) = \{0\}$. For reasons of dimension we get

$$\mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_{x_0}) \oplus \mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_{x_3}) = \mathrm{T}_{(x_1,x_2)}\mathcal{F}.$$

In order to verify $(*)$, we take two vectors $(u_1, u_2) \in \mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_{x_0})$, $(v_1, v_2) \in \mathrm{T}_{(x_1,x_2)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_{x_3})$ and assume that $(u_1 + v_1, u_2 + v_2) \in \mathrm{T}_{x_1}\mathcal{P}_{x_0} \times \mathrm{T}_{x_2}\mathcal{L}_{x_3}$. Because of $u_1 \in \mathrm{T}_{x_1}\mathcal{P}_{x_0}$ (see $(**)$) we get $v_1 = (u_1 + v_1) - u_1 \in \mathrm{T}_{x_1}\mathcal{P}_{x_0}$. On the other hand we have $v_1 \in \mathrm{T}_{x_1}\mathrm{D}_2 x_3$ (see $(***)$), which forces v_1 to be 0. So, we get $(0, v_2) \in (\{0\}\times\mathrm{T}_{x_2}\mathcal{L}_{x_3})\cap\mathrm{T}_{(x_1,x_2)}\mathcal{F}$ by equation $(***)$. Because of $(\{0\}\times\mathrm{T}_{x_2}\mathcal{L})\cap\mathrm{T}_{(x_1,x_2)}\mathcal{F} = \{0\}\times\mathrm{T}_{x_2}\mathcal{L}_{x_1}$ we conclude that $v_2 = 0$, since the line pencils $\mathcal{L}_{x_1}$ and $\mathcal{L}_{x_3}$ intersect transversally in $x_2 \in \mathcal{L}$ (see the remark preceding this proof). In the same way, we get $u_1 = 0$ and $u_2 = 0$. This proves equation $(*)$ and completes the proof. □

The following theorem shows that under the assumptions of Proposition 1.8 the operations of joining points and intersecting lines are smooth on their respective domains.

1.12 Theorem. *Assume that $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ satisfies the conditions of Proposition 1.8. Then the set $\mathcal{P}_2 = \{(x, y) \in \mathcal{P}^2 \mid d(x, y) = 2\}$ is a $(3p + 2q)$-dimensional submanifold of $\mathcal{P}^2$. Dually, the set $\mathcal{L}_2$ (defined analogously) is a $(2p + 3q)$-dimensional submanifold of $\mathcal{L}^2$. Furthermore, the maps $f_2|_{\mathcal{P}_2}$ and $f_2|_{\mathcal{L}_2}$ are smooth.*

Proof. We show first that $\mathcal{P}_2$ is a submanifold of $\mathcal{P}^2$. The smoothness of the restriction of f_2 to $\mathcal{P}_2$ will then be an immediate consequence. The preimage of the submanifold $\{(L, L) \mid L \in \mathcal{L}\}$ of $\mathcal{L} \times \mathcal{L}$ under the submersion

$$\chi_{\mathcal{L}} : \mathcal{F}^2 \to \mathcal{L}^2 : (x_1, y_1, x_2, y_2) \mapsto (\pi_{\mathcal{L}}(x_1, y_1), \pi_{\mathcal{L}}(x_2, y_2)) = (y_1, y_2)$$

is the submanifold $\{(x_1, y, x_2, y) \in \mathcal{F} \times \mathcal{F}\}$ of $\mathcal{F}^2$. Hence,

$$\mathcal{F}_2 = \{(x_1, y, x_2, y) \in \mathcal{F} \times \mathcal{F} \mid x_1 \neq x_2\}$$

is a submanifold of $\mathcal{F}^2$ with dimension $(p+2q)+2p = 3p+2q$. The set $\mathcal{P}_2$ is the image of the smooth map

$$\chi_{\mathcal{P}} : \mathcal{F}_2 \to \mathcal{P}^2 : (x_1, y, x_2, y) \mapsto (\pi_{\mathcal{P}}(x_1, y), \pi_{\mathcal{P}}(x_2, y)) = (x_1, x_2)$$

with continuous inverse

$$\chi_{\mathcal{P}}^{-1} : \mathcal{P}_2 \to \mathcal{F}_2 : (x_1, x_2) \mapsto (x_1, f_2(x_1, x_2), x_2, f_2(x_1, x_2)).$$

Thus $\chi_{\mathcal{P}}$ maps $\mathcal{F}_2$ homeomorphically onto $\mathcal{P}_2$. It remains to show that the differential of $\chi_{\mathcal{P}}$ is injective at each point of its domain. Choose $(x_1, y, x_2, y) \in \mathcal{F}_2$ and let v be a vector in the kernel of $(\mathrm{D}\chi_{\mathcal{P}})_{(x_1,y,x_2,y)}$. Then we have $v = (0, u, 0, u) \in \mathrm{T}_{(x_1,y)}\mathcal{F} \times \mathrm{T}_{(x_2,y)}\mathcal{F}$ for some $u \in \mathrm{T}_y\mathcal{L}$. Because of $(\{0\} \times \mathrm{T}_y\mathcal{L}) \cap \mathrm{T}_{(x_1,y)}\mathcal{F} = \{0\} \times \mathrm{T}_y\mathcal{L}_{x_1}$ and $(\{0\} \times \mathrm{T}_y\mathcal{L}) \cap \mathrm{T}_{(x_2,y)}\mathcal{F} = \{0\} \times \mathrm{T}_y\mathcal{L}_{x_2}$ (cf. proof of Proposition 1.8), we get $u \in \mathrm{T}_y\mathcal{L}_{x_1} \cap \mathrm{T}_y\mathcal{L}_{x_2}$. Since intersecting line pencils intersect weakly transversally, we conclude that $u = 0$. Thus we have $v = 0$, and the injectivity of the differential $(\mathrm{D}\chi_{\mathcal{P}})_{(x_1,y,x_2,y)}$ is proved. Hence, $\chi_{\mathcal{P}}$ is an embedding and $\mathcal{P}_2$ is a submanifold of $\mathcal{P}^2$. The restriction of f_2 to $\mathcal{P}_2$ is the composition of the diffeomorphism $\chi_{\mathcal{P}}^{-1} : \mathcal{P}_2 \to \mathcal{F}_2$ and the smooth map $\mathcal{F}_2 \to \mathcal{L} : (x_1, y, x_2, y) \to y$. Thus the map $f_2|_{\mathcal{P}_2}$ is smooth. By duality, this completes the proof. □

Properties of Smooth Generalized Quadrangles

In the preceding section, we established some smoothness properties of generalized quadrangles under various differential-topological assumptions. In this section, we want to show that, vice versa, every smooth generalized quadrangle satisfies those conditions. Hence, Theorem 1.10 and Corollary 1.11 yield characterizations of smooth generalized quadrangles. Similar results for smooth projective planes can be found in [3].

1.13 Proposition. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a smooth generalized quadrangle. Then $\mathcal{F}$ is a submanifold of $\mathcal{P} \times \mathcal{L}$ and the natural projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.*

Proof. We choose $y \in \mathcal{L}$ and set $W_y = \{x \in \mathcal{P} \mid d(x,y) = 3\} \times \{z \in \mathcal{L} \mid d(y,z) = 4\}$. On the open set $W_y \subseteq \mathcal{P} \times \mathcal{L}$ we define a smooth map $\pi : W_y \to \mathcal{F} \cap W_y : (x,z) \mapsto (g(g(x,y),z),z)$ (for the definition of g, see the first section). It is easily checked that this map is actually well-defined. For each $(x,z) \in \mathcal{F} \cap W_y$ we have $\pi(x,z) = (x,z)$. Hence, the map π is a smooth retraction onto $\mathcal{F} \cap W_y$. By [6], Theorem 5.13, the set $\mathcal{F} \cap W_y$ is a submanifold of W_y. Since $\mathcal{P} \times \mathcal{L}$ is covered by the open sets W_y, $y \in \mathcal{L}$, we conclude that $\mathcal{F}$ is a submanifold of $\mathcal{P} \times \mathcal{L}$.

It remains to show that the projections $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{L}}$ are submersions. Choose $(x,y) \in \mathcal{F}$. By duality, it suffices to prove that the differential $(\mathrm{D}\pi_{\mathcal{P}})_{(x,y)} : \mathrm{T}_{(x,y)}\mathcal{F} \to \mathrm{T}_x\mathcal{P}$ is surjective. Let $z \in \mathcal{L}$ be a line which intersects the line y in a point different from x. Then we have $g(z,x) = y$. We set $\tau_z : \mathcal{P} \backslash \mathcal{P}_z \to \mathcal{F} : w \mapsto (w, g(z,w))$. The map τ_z is smooth and we have $\pi_{\mathcal{P}} \circ \tau_z = \mathrm{id}|_{\mathcal{P} \backslash \mathcal{P}_z}$. Hence, $(\mathrm{D}\pi_{\mathcal{P}})_{(x,y)} \mathrm{D}(\tau_z)_x$ is the identity on $\mathrm{T}_x\mathcal{P}$ and the surjectivity of $(\mathrm{D}\pi_{\mathcal{P}})_{(x,y)}$ follows. □

Remark. Due to this proposition and Lemma 1.6 (or by [5], Proposition 4.2), point rows and line pencils are submanifolds of $\mathcal{P}$ and $\mathcal{L}$, respectively. As in the preceding section, the dimensions of point rows and line pencils are denoted by p and q, respectively. The second statement of Proposition 1.13 is also a consequence of Proposition 1.16 (f1). In the proof of this proposition, however, we will already use the fact that point rows and line pencils are submanifolds.

1.14 Proposition. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a smooth generalized quadrangle. Then for every $x \in \mathcal{P}$ the set D_2x is a submanifold of $\mathcal{P}$. Furthermore, for every antiflag $(y,z) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$ the submanifolds $\mathcal{P}_z$ and D_2y intersect transversally in $\mathcal{P}$. The dual statements also hold.*

Proof. We use a construction taken from [5], Definition 1.6 (cf. [17], Definition 2.4). We choose an arbitrary antiflag $(z_1, z_4) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ and set $f_3(z_1, z_4) = (z_2, z_3)$. Let $v_0 \in \mathcal{L}_{z_1}\backslash\{z_2\}$, $v_1 \in \mathcal{P}_{v_0}\backslash\{z_1\}$ and $(v_2, v_3) = f_3(v_1, v_4)$, where $v_4 = z_4$. Then the sequence $\mathbf{v} = (v_0, v_1, v_2, v_3, v_4)$ is a 4-chain. By [5], Definition 1.6, we have a bijection

$$\begin{aligned}\xi_{\mathbf{v}} : \{x \in \mathcal{P} \mid d(x, v_1) = 4\} \to (\mathcal{P}_{v_4}\backslash\{v_3\}) \times (\mathcal{L}_{v_3}\backslash\{v_2\}) \times (\mathcal{P}_{v_2}\backslash\{v_1\}) :\\ x \mapsto (g(g(x, v_0), v_4), g(g(v_0, x), v_3), g(x, v_2))\end{aligned}$$

which can be expressed in both directions in terms of the map g. Since the map g is smooth, the map $\xi_{\mathbf{v}}$ is a diffeomorphism. In [5], the map $\xi_{\mathbf{v}}$ is described in a more implicit way and the open set $U_{v_1} = \{x \in \mathcal{P} \mid d(x, v_1) = 4\}$ corresponds to $\mathcal{P}_3(v_1, v_0)$ in the notation used there, where it is called a *big cell.* The sets $\mathrm{D}_2 z_1\backslash\mathcal{P}_{v_0}$ and $\mathcal{P}_{z_4}\backslash\{v_3\}$ are contained in U_{v_1} and intersect in $z_3 \in U_{v_1}$. For $x \in \mathrm{D}_2 z_1\backslash\mathcal{P}_{v_0}$ we get $g(x, v_0) = z_1$ and $g(z_1, v_4) = z_3$. Thus we have

$$\xi_{\mathbf{v}}(\mathrm{D}_2 z_1\backslash\mathcal{P}_{v_0}) \subseteq \{z_3\} \times (\mathcal{L}_{v_3}\backslash\{v_2\}) \times (\mathcal{P}_{v_2}\backslash\{v_1\}). \tag{1}$$

Now let $x \in \mathcal{P}_{z_4}\backslash\{v_3\}$. Then we have $g(g(v_0, x), v_3) = v_4$ and $g(x, v_2) = v_3$. Hence we get

$$\xi_{\mathbf{v}}(\mathcal{P}_{z_4}\backslash\{v_3\}) \subseteq \mathcal{P}_{v_4}\backslash\{v_3\} \times \{v_4\} \times \{v_3\}. \tag{2}$$

Since $\mathcal{P}_{z_2}\backslash\{z_1\}$ is contained in $\mathrm{D}_2 z_1\backslash\mathcal{P}_{v_0}$ and $\xi_{\mathbf{v}}$ is a diffeomorphism, we see that the lines z_2 and z_4 intersect weakly transversally in $z_3 \in \mathcal{P}$. As the antiflag $(z_1, z_4) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ was chosen arbitrarily, we conclude that any two intersecting point rows (and, dually, any two intersecting line pencils) intersect weakly transversally. Hence, for every $x \in V$ the set $\mathrm{D}_2 x$ is a $(p+q)$-dimensional submanifold of $\mathcal{P}$ or $\mathcal{L}$, respectively, see Proposition 1.8. Moreover, equations (1) and (2) show that $\mathrm{D}_2 z_1$ and $\mathcal{P}_{z_4}$ intersect transversally in $z_3 \in \mathcal{P}$. Since $(z_1, z_4) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ was chosen arbitrarily, this completes the proof. □

Remarks. (i) In order to prove that the sets D_2x, $x \in V$, are submanifolds of $\mathcal{P}$ or $\mathcal{L}$, respectively, we have used Proposition 1.8 of the preceding section. Here, we sketch a direct proof: let $z \in \mathcal{P}$ and choose $w \in \mathcal{L}_z$ arbitrarily. It suffices to show that $\mathrm{D}_2z\backslash\mathcal{P}_w$ is a submanifold of $\mathcal{P}$. For this purpose we choose $v \in \mathcal{P}_w\backslash\{z\}$, $y \in \mathcal{L}_v\backslash\{w\}$ and define a smooth map $\rho : \{x \in \mathcal{P} \mid d(x,v) = 4\} \to \mathcal{P} : x \mapsto g(z, g(y,x))$. For every $x \in \mathrm{D}_2z\backslash\mathcal{P}_w$, we have $\rho(x) = x$ and, in fact, ρ is a smooth retraction onto $\mathrm{D}_2z\backslash\mathcal{P}_w$. Hence, by [6], Theorem 5.13, the set $\mathrm{D}_2z\backslash\mathcal{P}_w$ is a submanifold of $\mathcal{P}$.

(ii) As mentioned in the proof of Proposition 1.14, any two intersecting point rows (line pencils) intersect weakly transversally. Hence, the conditions of Theorem 1.12 are satisfied. We conclude that for smooth generalized quadrangles $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ the sets $\mathcal{P}_2$ and $\mathcal{L}_2$ (in the notation of that theorem) are smooth submanifolds of $\mathcal{P}^2$ and $\mathcal{L}^2$, respectively and that $f_2|_{\mathcal{P}_2}$ and $f_2|_{\mathcal{L}_2}$ are smooth maps.

1.15 Corollary. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a smooth generalized quadrangle. Then for every antiflag $(y,z) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$, the submanifolds $\mathcal{P}_z \times \mathcal{L}_y$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$.*

Proof. We take $(x_3, x_0) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ and put $(x_1, x_2) = f_3(x_0, x_3)$. By Propositions 1.13 and 1.14, the conditions required in Corollary 1.11 are satisfied. In the proof of that corollary, we showed that

$$(\mathrm{T}_{x_1}\mathcal{P}_{x_0} \times \mathrm{T}_{x_2}\mathcal{L}_{x_3}) \cap \mathrm{T}_{(x_1,x_2)}\mathcal{F} = \{0\},$$

cf. equation $(*)$. Thus, for reasons of dimension, the submanifolds $\mathcal{F}$ and $\mathcal{P}_{x_0} \times \mathcal{L}_{x_3}$ intersect transversally in $(x_1, x_2) \in \mathcal{P} \times \mathcal{L}$. □

According to Proposition 1.13 and the fibration theorem of Ehresmann, see [6], Theorem 8.12, the flag space $\mathcal{F}$ is a smooth locally trivial fibre bundle over $\mathcal{P}$ and over $\mathcal{L}$, because $\mathcal{F}$ is compact by Proposition 1.4 and the remark preceding that proposition. Nevertheless, it is interesting to note that the local trivializations may be realized by the maps f_2 and f_3 (or g, respectively). For the results of the following proposition in the topological case, see [17], Lemma 2.2, and [19], Lemma 2.1.6 and Proposition 2.1.8. For convenience, we restate the proofs in our notation in the smooth case.

1.16 Proposition. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a smooth generalized quadrangle. Then the following two statements hold:*

(f1) *The flag space $\mathcal{F}$ is a smooth locally trivial fibre bundle over $\mathcal{P}$ and over $\mathcal{L}$ with projections $\pi_{\mathcal{P}}$ and $\pi_{\mathcal{L}}$, respectively.*

(f2) *For every $x \in \mathcal{P}$, the set $\mathrm{D}_2 x$ is a smooth locally trivial fibre bundle over $\mathcal{L}_x$ with projection $\mathrm{D}_2 x \to \mathcal{L}_x : z \mapsto f_2(x, z)$. The dual statement holds, too.*

In both cases, the local trivializations may be expressed in terms of geometric operations.

Proof. By duality, it suffices to show that $\mathcal{F}$ is a smooth locally trivial fibre bundle over $\mathcal{P}$. So let $x \in \mathcal{P}$ and $U_x = \{y \in \mathcal{P} \mid d(x, y) = 4\}$. Then the map

$$\varphi_x : \pi_{\mathcal{P}}^{-1}(U_x) \to U_x \times \mathcal{L}_x : (y, z) \mapsto (y, g(z, x))$$

is a diffeomorphism whose inverse is given by

$$\varphi_x^{-1} : U_x \times \mathcal{L}_x \to \pi_{\mathcal{P}}^{-1}(U_x) : (y, z) \mapsto (y, g(z, y)).$$

The projection $\pi_{\mathcal{P}}$ corresponds to the projection onto the first component under this diffeomorphism. Since any two line pencils are diffeomorphic (cf. remark after Proposition 1.5) and $\mathcal{P}$ is covered by the open sets U_x, $x \in \mathcal{P}$, the claim (f1) follows.

Now choose $x_0 \in \mathcal{P}$, $x_1 \in \mathcal{L}_{x_0}$. Let $x_3 \in \mathcal{L}$ be a line which intersects the line x_1 in some point $x_2 \neq x_0$. Then we have a smooth map

$$\tau : \mathrm{D}_2 x_0 \backslash \mathcal{P}_{x_1} \to (\mathcal{L}_{x_0} \backslash \{x_1\}) \times (\mathcal{P}_{x_3} \backslash \{x_2\}) : z \mapsto (f_2(x_0, z), g(z, x_3))$$

with smooth inverse

$$\tau^{-1} : (\mathcal{L}_{x_0}\setminus\{x_1\}) \times (\mathcal{P}_{x_3}\setminus\{x_2\}) \to \mathrm{D}_2 x_0 \setminus \mathcal{P}_{x_1} : (y, z) \mapsto g(z, y).$$

As above, we see that $\mathrm{D}_2 x_0$ is a smooth locally trivial fibre bundle over $\mathcal{L}_{x_0}$. By duality, the claim (f2) follows. □

Similar Smooth Incidence Structures

In this section, we prove the following

1.17 Theorem. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence structure which satisfies the following conditions ($p, q \in \mathbb{N}$):*

- (SIS1) *The point space $\mathcal{P}$ and the line space $\mathcal{L}$ are compact, connected smooth manifolds. The dimensions of $\mathcal{P}$ and $\mathcal{L}$ are $2p + q$ and $p + 2q$, respectively.*
- (SIS2) *The flag space $\mathcal{F}$ is a $(2p + 2q)$-dimensional closed submanifold of $\mathcal{P} \times \mathcal{L}$, and the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.*
- (SIS3) *For every antiflag $(y, z) \in (\mathcal{P} \times \mathcal{L})\setminus\mathcal{F}$ the submanifolds $\mathcal{P}_z \times \mathcal{L}_y$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$.*

Then there is a finite number n such that $\mathcal{P}_z \times \mathcal{L}_y$ and $\mathcal{F}$ intersect in precisely n points for each antiflag $(y, z) \in (\mathcal{P} \times \mathcal{L})\setminus\mathcal{F}$.

Remark. Note that two submanifolds which intersect transversally need not have a common point. The canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are surjective since $\mathcal{F}$ is compact, submersions are open maps, and $\mathcal{P}$, $\mathcal{L}$ are connected. Hence, by Lemma 1.6, point rows $\mathcal{P}_y$ and line pencils $\mathcal{L}_x$ are submanifolds of $\mathcal{P}$ and $\mathcal{L}$, respectively, where $\dim \mathcal{P}_y = p$ and $\dim \mathcal{L}_x = q$. Furthermore, point rows and line pencils are compact.

The conditions above are satisfied for every smooth generalized quadrangle, see the first and the third section. Theorem 1.17 shows that incidence structures which have (differential-) topological properties similar to those of smooth generalized quadrangles are quite close to generalized quadrangles from the incidence geometric point of view. The following corollary characterizes smooth quadrangles among similar incidence structures. It will be used in Chapter 3 in order to prove that isoparametric hypersurfaces with four distinct principal curvatures in spheres give rise to smooth generalized quadrangles.

1.18 Corollary. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence structure which satisfies the conditions of Theorem 1.17. If there is an antiflag $(y, z) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$ such that $\mathcal{P}_y \times \mathcal{L}_x$ and $\mathcal{F}$ intersect in exactly one point, then $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a smooth generalized quadrangle.*

By the fibration theorem of Ehresmann ([6], Theorem 8.12) we get the following

1.19 Proposition. *Assume that $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is an incidence structure which satisfies the conditions of Theorem 1.17. Then $\mathcal{F}$ is a smooth locally trivial fibre bundle over $\mathcal{P}$ and over $\mathcal{L}$. As a consequence, any two point rows (and any two line pencils) are diffeomorphic.*

The proof of Theorem 1.17 follows from Lemma 1.9 by purely topological arguments.

Proof of Theorem 1.17. Since $\mathcal{F}$ is compact, the set

$$\mathcal{O} = \{(y, z) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F} \mid (\mathcal{P}_z \times \mathcal{L}_y) \cap \mathcal{F} = \emptyset\}$$

is open in $\mathcal{P} \times \mathcal{L}$. Choose an antiflag $(x_3, x_0) \in (\mathcal{P} \times \mathcal{L}) \backslash (\mathcal{F} \cup \mathcal{O})$ (if it exists). By (SIS3), the submanifolds $\mathcal{P}_{x_0} \times \mathcal{L}_{x_3}$ and $\mathcal{F}$ intersect transversally in

$\mathcal{P}\times\mathcal{L}$. Hence, for reasons of dimension, the set $(\mathcal{P}_{x_0}\times\mathcal{L}_{x_3})\cap\mathcal{F}$ is a compact 0-dimensional submanifold of $\mathcal{P}\times\mathcal{L}$. Thus the set $(\mathcal{P}_{x_0}\times\mathcal{L}_{x_3})\cap\mathcal{F}$ is finite and we have a well-defined map

$$\eta : (\mathcal{L}\times\mathcal{P})\backslash\mathcal{F}\to\mathbb{N} : (x,y)\mapsto|(\mathcal{P}_x\times\mathcal{L}_y)\cap\mathcal{F}|.$$

We set $k=|(\mathcal{P}_{x_0}\times\mathcal{L}_{x_3})\cap\mathcal{F}|$ and denote the intersection points of $\mathcal{P}_{x_0}\times\mathcal{L}_{x_3}$ and $\mathcal{F}$ by $(x_1^{(i)},x_2^{(i)})$, $i=1,\ldots,k$. By Lemma 1.9, there is an open neighbourhood W of (x_0,x_3) in $\mathcal{L}\times\mathcal{P}$ and there are pairwise disjoint open neighbourhoods W_i of $(x_1^{(i)},x_2^{(i)})$ in $\mathcal{P}\times\mathcal{L}$ such that for any $(y_0,y_3)\in W$ there is exactly one intersection point $(y_1^{(i)},y_2^{(i)})\in W_i$ of $\mathcal{P}_{x_0}\times\mathcal{L}_{x_3}$ and $\mathcal{F}$, $i=1,\ldots,k$. Hence, we have $\eta(y_0,y_3)\geq\eta(x_0,x_3)$ for each $(y_0,y_3)\in W$. We want to show that there is a neighbourhood $U\subseteq W$ of (x_0,x_3) in $\mathcal{L}\times\mathcal{P}$ such that η is constant on U. Assume to the contrary that such a neighbourhood does not exist. Then there is a sequence $(\tilde{y}_0^{(n)},\tilde{y}_3^{(n)})_n\subseteq W$ converging to (x_0,x_3) such that $\eta(\tilde{y}_0^{(n)},\tilde{y}_3^{(n)})>k$ for each $n\in\mathbb{N}$. Hence, for every $n\in\mathbb{N}$ we can find an element $(\tilde{y}_1^{(n)},\tilde{y}_2^{(n)})\in(\mathcal{P}_{\tilde{y}_0^{(n)}}\times\mathcal{L}_{\tilde{y}_3^{(n)}})\cap(\mathcal{F}\backslash\bigcup_{i=1}^k W_i)$. Since $\mathcal{F}$ is compact and $\bigcup_{i=1}^k W_i$ is open, there is a subsequence of $((\tilde{y}_1^{(n)},\tilde{y}_2^{(n)}))_n$ which converges to some element $(\tilde{y}_1,\tilde{y}_2)\in(\mathcal{P}_{x_0}\times\mathcal{L}_{x_3})\cap(\mathcal{F}\backslash\bigcup_{i=1}^k W_i)$. This is a contradiction to $(\mathcal{P}_{x_0}\times\mathcal{L}_{x_3})\cap\mathcal{F}=\{(x_1^{(i)},x_2^{(i)})\mid i=1,\ldots,k\}$. Hence, the map η is constant on some open neighbourhood $U\subseteq W$ of (x_0,x_3). Since $(x_3,x_0)\in(\mathcal{P}\times\mathcal{L})\backslash(\mathcal{F}\cup\mathcal{O})$ was chosen arbitrarily and $\mathcal{O}$ is open in $\mathcal{P}\times\mathcal{L}$, we conclude that η is a locally constant map. By (SIS1), the manifolds $\mathcal{P}$ and $\mathcal{L}$ and therefore also the manifold $(\mathcal{P}\times\mathcal{L})\backslash\mathcal{F}$ are connected. Hence the map η is even constant. This completes the proof. $\square$

Chapter 2

Isoparametric Hypersurfaces of Clifford Type

Introduction

In [10], Ferus, Karcher and Münzner constructed examples of isoparametric hypersurfaces in spheres by means of representations of Clifford algebras on $\mathbb{R}^l$ or, equivalently (cf. [10], 3.3), by means of Clifford systems on $\mathbb{R}^{2l}$. For short, we will call them isoparametric hypersurfaces of Clifford type or of FKM-type, as in [5]. Except for two homogeneous examples in dimensions 8 and 18, their examples include all known isoparametric hypersurfaces in spheres with four distinct principal curvatures. Note that by a remarkable result of Münzner ([27], Theorem A, and [28]) only 1, 2, 3, 4 or 6 distinct principal curvatures can occur for isoparametric hypersurfaces in spheres. The possible hypersurfaces in the first three cases have already been classified by E. Cartan, see [18] for a different approach in the case of three distinct principal curvatures. In particular, it turns out that all these hypersurfaces are pieces of orbits of subgroups of orthogonal groups under the standard operation.

In the case of four distinct principal curvatures, isoparametric hypersufaces in spheres are not necessarily homogeneous in this sense, and the geometries associated with such isoparametric hypersurfaces and their focal manifolds can be more complicated. In [36], Thorbergsson showed that isoparametric hypersurfaces of FKM-type and their focal manifolds give rise to generalized quadrangles and he claimed that, as a consequence of his proof, they were even *smooth* generalized quadrangles. In the next section, we will prove the smoothness of these generalized quadrangles by means of Theorem 1.10. Even more, we will see in the last section that our approach also yields an elementary proof for the result that the incidence structures associated with isoparametric hypersurfaces of FKM-type are generalized quadrangles.

Main Theorem

In this section we will prove and make precise the following theorem, which contains the main result of this chapter.

2.1 Theorem. *Generalized quadrangles associated with isoparametric hypersurfaces of FKM-type are smooth generalized quadrangles.*

Before giving a proof of this theorem, we will first discuss some general properties of isoparametric hypersurfaces in spheres, and we will associate to each such hypersurface with at least two distinct principal curvatures an incidence structure. Then we will specialize to hypersurfaces of FKM-type. In this case, some properties of these incidence structures, which are obtained by the general theory, can be verified easily by explicit calculations. As references for our brief account of the general theory of isoparametric hypersurfaces we mention [7], [8], [27], and [31]. For a comprehensive survey on isoparametric hypersurfaces, see [37].

2.2 Definition. An orientable hypersurface with constant principal curvatures in the sphere is called an *isoparametric hypersurface.*

We will always identify the sphere in the above definition with the unit sphere in some euclidean vector space. Furthermore, we will consider only connected hypersurfaces. Hypersurfaces in the sphere which are parallel to an isoparametric hypersurface $\mathcal{I}$ are isoparametric again. In this way, an isoparametric hypersurface gives rise to an *isoparametric family* of such hypersurfaces. Besides these parallel hypersurfaces there are precisely two parallel submanifolds of lower dimension, the so-called *focal manifolds*, which we denote by $\mathcal{P}$ and $\mathcal{L}$, respectively. There are two projections $\rho_{\mathcal{P}} : \mathcal{I} \to \mathcal{P}$ and $\rho_{\mathcal{L}} : \mathcal{I} \to \mathcal{L}$ along great circles normal to $\mathcal{I}$. These projections are submersions, and $\mathcal{I}$ may be chosen in the isoparametric family in such a way that for each $z \in \mathcal{I}$ the images $\rho_{\mathcal{P}}(z)$ and $\rho_{\mathcal{L}}(z)$ have spherical distance $\frac{\pi}{2g}$ from z, where g denotes the number of distinct principal curvatures, see [27], Section 6. For $g > 1$, we may embed $\mathcal{I}$ into $\mathcal{P} \times \mathcal{L}$ by means of the map $z \mapsto (\rho_{\mathcal{P}}(z), \rho_{\mathcal{L}}(z))$. The differential of this map is actually injective at every point of $\mathcal{I}$ since the kernels of $\mathrm{D}\rho_{\mathcal{P}}$ and $\mathrm{D}\rho_{\mathcal{L}}$ are eigendistributions of the Weingarten map which belong to different eigenvalues. The image of $\mathcal{I}$ in $\mathcal{P} \times \mathcal{L}$ is denoted by $\mathcal{F}$. In this way we obtain an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$. The canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions, since even the projections $\rho_{\mathcal{P}}$ and $\rho_{\mathcal{L}}$ have this property. A point $x \in \mathcal{P}$ and a line $y \in \mathcal{L}$ are incident if and only if the spherical distance of x and y is equal to $\frac{\pi}{g}$, cf. [18], Proposition 3.2.

We will now specialize to isoparametric hypersurfaces of FKM-type. For this purpose we need the following

2.3 Definition. For positive integers l, m, we call an $(m+1)$-tuple of symmetric matrices $P_0, \ldots, P_m \in \mathbb{R}^{2l \times 2l}$ a *Clifford system* if we have for all $i, j \in \{0, \ldots, m\}$

$$P_i P_j + P_j P_i = 2\delta_{ij}\mathrm{id}.$$

We repeat some notions and constructions from [10]. The subspace of symmetric matrices in $\mathbb{R}^{2l \times 2l}$ is endowed with the scalar product given by $\langle A, B \rangle = \frac{1}{2l}\mathrm{trace}(AB)$. The unit sphere in the span of $P_0, \ldots, P_m$ is called the *Clifford sphere* determined by this system and is denoted by

$\Sigma(P_0, \ldots, P_m)$ or simply by Σ. Every orthonormal basis of $\mathbb{R}\Sigma(P_0, \ldots, P_m)$ is again a Clifford system, and the function

$$H : \mathbb{R}^{2l} \to \mathbb{R} : x \mapsto \sum_{i=0}^{m} \langle P_i x, x \rangle^2$$

depends only on $\Sigma(P_0, \ldots, P_m)$. Moreover, we have $\langle Px, Qx \rangle = \langle P, Q \rangle \langle x, x \rangle$ for all P, Q in the span of $P_0, \ldots, P_m$ and all $x \in \mathbb{R}^{2l}$, where $\mathbb{R}^{2l}$ is endowed with the standard scalar product.

The following theorem contains some important results of [10], Section 4.

2.4 Theorem. *Let $P_0, \ldots, P_m$ denote a Clifford system on $\mathbb{R}^{2l}$ such that $l - m - 1 > 0$. We define*

$$F : \mathbb{R}^{2l} \to \mathbb{R} : x \mapsto \langle x, x \rangle^2 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2.$$

Then the intersection $\mathcal{I}$ of $F^{-1}(0)$ with the unit sphere $\mathbb{S}$ of $\mathbb{R}^{2l}$ is a connected isoparametric hypersurface in $\mathbb{S}$ with four distinct principal curvatures of multiplicities m, $l - m - 1$. The focal manifolds are

$$\mathcal{P} = F^{-1}(1) \cap \mathbb{S} = \{x \in \mathbb{S} \mid \langle P_i x, x \rangle = 0, \ i = 0, \ldots, m\}$$

and

$$\mathcal{L} = F^{-1}(-1) \cap \mathbb{S} = \Big\{ y \in \mathbb{S} \,\Big|\, \sum_{i=0}^{m} \langle P_i y, y \rangle P_i y = y \Big\},$$

where $\mathcal{P}$ has dimension $2l - m - 2$ and $\mathcal{L}$ has dimension $l + m - 1$.

That the sets $\mathcal{I} = F^{-1}(0) \cap \mathbb{S}$ in Theorem 2.4 are indeed isoparametric hypersurfaces in $\mathbb{S}$ is shown by means of Münzner's differential equations, see [27], Theorem 3, or the next chapter. By [36], the incidence structures $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ constructed from these hypersurfaces of FKM-type as above are generalized quadrangles. They will be called *Clifford quadrangles* in the sequel. Line pencils are m-dimensional submanifolds and point rows are submanifolds of dimension $l-m-1$, cf. Lemma 1.6.

Setting $a = l-m-1$ and $b = m$, we see that conditions (SGQ1) and (SGQ2) in the introduction of Chapter 1 are satisfied, cf. Theorem 1.10. So, in order to prove Theorem 2.1, it remains to show that for Clifford quadrangles $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ the flag space $\mathcal{F}$ and $\mathcal{P}_L \times \mathcal{L}_p$ intersect transversally for every antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$. For the rest of this section, $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ will always denote a Clifford quadrangle.

In our summary of properties of general isoparametric hypersurfaces in spheres we have already mentioned that a point $x \in \mathcal{P}$ and a line $y \in \mathcal{L}$ are incident if and only if the spherical distance of x and y is equal to $\frac{\pi}{g}$, i.e. $\langle x, y\rangle = \frac{1}{\sqrt{2}}$ for $g = 4$. We need, however, a better description of the incidence relation which contains more information on the position of the tangent spaces of $\mathcal{P}$ and $\mathcal{L}$ (see also [22], 10.7).

2.5 Lemma. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a Clifford quadrangle with Clifford system $P_0, \ldots, P_m$. Then $x \in \mathcal{P}$ and $y \in \mathcal{L}$ are incident if and only if*

$$y = \frac{1}{\sqrt{2}}\Big(\sum_{i=0}^{m} \langle P_i y, y\rangle P_i x + x\Big).$$

Proof. Assume that the equation above is satisfied. Then $\langle x, y\rangle = \frac{1}{\sqrt{2}}$ is an immediate consequence since $\langle P_i x, x\rangle = 0$ for all $x \in P$. Now let $\langle x, y\rangle = \frac{1}{\sqrt{2}}$. Then the euclidean norm of $\sqrt{2}y - x$ is equal to 1. Since, by the definition of $\mathcal{L}$, the vector $\sum_{i=0}^{m} \langle P_i y, y\rangle P_i x$ has norm 1, too, it suffices to show that

$$\Big\langle \sqrt{2}y - x, \sum_{i=0}^{m} \langle P_i y, y\rangle P_i x \Big\rangle = 1.$$

This is correct because of $\langle P_i x, x\rangle = 0$ and

$$\left\langle y, \sum_{i=0}^{m} \langle P_i y, y\rangle P_i x \right\rangle = \left\langle \sum_{i=0}^{m} \langle P_i y, y\rangle P_i y, x \right\rangle = \langle y, x\rangle = \frac{1}{\sqrt{2}}.$$

□

For $L \in \mathcal{L}$, we set $P_L = \sum_{i=0}^{m} \langle P_i L, L\rangle P_i$. Note that P_L is the unique element $P \in \Sigma$ such that $PL = L$: by setting $P = \sum_{i=0}^{m} \nu_i P_i$ ($\nu_i \in \mathbb{R}$) we obtain $L = \sum_{i=0}^{m} \nu_i P_i L = \sum_{i=0}^{m} \langle P_i L, L\rangle P_i L$. The $P_i L$ are linearly independent, so we have $\nu_i = \langle P_i L, L\rangle$ for $i = 0, \ldots, m$, which shows that $P = P_L$. Conversely, if L and K are two lines having a common point $q \in \mathcal{P}$ such that $P_L = P_K$, then we have $L = \frac{1}{\sqrt{2}}(P_L q + q) = \frac{1}{\sqrt{2}}(P_K q + q) = K$ by Lemma 2.5.

The proof of Theorem 2.1 is now completed by the following

2.6 Lemma. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a Clifford quadrangle. Then $\mathcal{F}$ and $\mathcal{P}_L \times \mathcal{L}_p$ intersect transversally for every antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$.*

Proof. Let $(p, L) \in (\mathcal{P}\times\mathcal{L})\backslash\mathcal{F}$ be an antiflag and let (q, K) be the intersection point of $\mathcal{P}_L\times\mathcal{L}_p$ and $\mathcal{F}$ in $\mathcal{P}\times\mathcal{L}$. For reasons of dimension, it suffices to prove that $(\mathrm{T}_q\mathcal{P}_L \times \mathrm{T}_K\mathcal{L}_p) \cap \mathrm{T}_{(q,K)}\mathcal{F} = \{0\}$ in order to show that $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect transversally in (q, K). So let $(u, v) \in (\mathrm{T}_q\mathcal{P}_L \times \mathrm{T}_K\mathcal{L}_p) \cap \mathrm{T}_{(q,K)}\mathcal{F}$. A priori, we have $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L} \subseteq \mathbb{R}^{2l} \times \mathbb{R}^{2l}$ and hence $(u, v) \in \mathbb{R}^{2l} \times \mathbb{R}^{2l}$, but we may identify these two vector spaces and consider u, v as elements of the same vector space $\mathbb{R}^{2l}$. The following equations for u and v have to be understood in this sense. Note that in our description of incidence in Lemma 2.5, from which these equations are derived, $\mathcal{P}$ and $\mathcal{L}$ are also considered as subsets of the same vector space $\mathbb{R}^{2l}$. By differentiating the equation for the incidence relation as given in Lemma 2.5, we see that $u \in \mathrm{T}_q P_L$ implies that $P_L u + u = 0$. Analogously, $v \in \mathrm{T}_K\mathcal{L}_p$ implies that $v = \sqrt{2}\sum_{i=0}^{m}\langle P_i K, v\rangle P_i p$. Finally, $(u, v) \in \mathrm{T}_{(q,K)}\mathcal{F}$ shows that $v = \frac{1}{\sqrt{2}}(P_K u + u) + \sqrt{2}\sum_{i=0}^{m}\langle P_i K, v\rangle P_i q$. Since F and hence the Clifford quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ depend only on the Clifford sphere $\Sigma(P_0, \ldots, P_m)$ we may assume that $P_K = P_0$. Then we have $\langle P_0 K, v\rangle = \langle K, v\rangle = 0$ because of $v \in T_K\mathcal{L} \subseteq T_K\mathbb{S}$. So, the last two equations above reduce to

$$v = \sqrt{2}\sum_{i=1}^{m}\langle P_i K, v\rangle P_i p$$

and

$$v = \frac{1}{\sqrt{2}}(P_0 u + u) + \sqrt{2}\sum_{i=1}^{m}\langle P_i K, v\rangle P_i q.$$

For simplicity we set $\lambda_i = \langle P_i K, v\rangle$ for $i = 1, \ldots, m$. Subtracting those two equations yields $P_0 u + u = 2\sum_{i=1}^{m}\lambda_i P_i(p-q)$. By applying the map $\mathrm{id} - P_L$ and using the identity $P_0 P_L + P_L P_0 = 2\langle P_0 L, L\rangle \mathrm{id}$, we get

$$2(1 - \langle P_0 L, L\rangle)u = 2\sum_{i=1}^{m}\lambda_i(P_i - P_L P_i)(p-q)$$

because of $P_L u = -u$. By differentiating the equations defining $\mathcal{P}$ (see Theorem 2.4), we see that $\langle u, P_j q\rangle = 0$ $(j = 1, \ldots, m)$. So we have $\sum_{i=1}^{m}\lambda_i\langle (P_i - P_L P_i)(p-q), P_j q\rangle = 0$ for $j = 1, \ldots, m$. We set $\lambda = (\lambda_1, \ldots, \lambda_m)$, $c_{ij} = \langle (P_i - P_L P_i)(p-q), P_j q\rangle$, and $C = (c_{ij})_{i,j} \in \mathbb{R}^{m\times m}$. Then we have $\lambda C = 0$.

In the following calculations we will need several times that for $i = 1, \ldots, m$ we have $\langle P_i p, q\rangle = 0$: since p and q are incident with K we obtain $P_0(p-q) + (p-q) = 0$ as a consequence of Lemma 2.5, hence $\langle P_i(p-q), (p-q)\rangle = -\langle P_i(p-q), P_0(p-q)\rangle = 0$ for $i \in \{1, \ldots, m\}$. Then we get $\langle P_i p, q\rangle = 0$ because of $\langle P_i p, p\rangle = \langle P_i q, q\rangle = 0$.

Let us now have a closer look at the entries c_{ij} of the matrix C. For the diagonal elements we have $c_{ii} = \langle (\mathrm{id} - P_i P_L P_i)(p-q), q\rangle$. Using the identity $P_i P_L + P_L P_i = 2\langle P_i L, L\rangle \mathrm{id}$, we get $P_i P_L P_i = 2\langle P_i L, L\rangle P_i - P_L$, hence $c_{ii} = \langle p-q, q\rangle + \langle P_L(p-q), q\rangle$. Here we have used that $\langle P_i(p-q), q\rangle = 0$ for $i = 1, \ldots, m$. For the same reason we have $\langle P_L(p-q), q\rangle = \langle P_0 L, L\rangle\langle P_0(p-$

$q),q\rangle = -\langle P_0L,L\rangle\langle p-q,q\rangle$. This shows that $c_{ii} = (1-\langle P_0L,L\rangle)(\langle p,q\rangle -1)$, independently of $i \in \{1,\dots,m\}$. We set $c = (1-\langle P_0L,L\rangle)(\langle p,q\rangle - 1)$. Assume that $c = 0$. Then we have $P_0L = L$ because of $p \neq q$. By the remarks preceding this lemma we conclude that $P_L = P_0 = P_K$ and hence $L = K$, a contradiction. So we have $c \neq 0$. Now let us consider the other entries of C. For $i \neq j$ we have

$$\begin{aligned} c_{ij} + c_{ji} =& \langle (P_iP_j - P_iP_LP_j)(p-q),q\rangle + \langle P_jP_i - P_jP_LP_i)(p-q),q\rangle \\ =& \langle (P_iP_LP_j + P_jP_LP_i)(p-q),q\rangle. \end{aligned}$$

Using again the identity $P_kP_L + P_LP_k = 2\langle P_kL,L\rangle\mathrm{id}$ for $k = i,j$ we see that

$$P_iP_LP_j + P_jP_LP_i = 2\langle P_jL,L\rangle P_i + 2\langle P_iL,L\rangle P_j.$$

Because of $\langle P_i(p-q),q\rangle = \langle P_j(p-q),q\rangle = 0$, we conclude that $c_{ij}+c_{ji} = 0$.

This shows that $C = B + c\,\mathrm{id}$, where $B \in \mathbb{R}^{m\times m}$ is skew symmetric and $c \neq 0$. In particular, C is regular. So we have $\lambda = 0$ and hence $(1 - \langle P_0L,L\rangle)u = 0$. As mentioned above, $\langle P_0L,L\rangle = 1$ is impossible, which shows that $u = 0$. Moreover, $v = \sqrt{2}\sum_{i=1}^m \lambda_iP_ip$ implies that $v = 0$. This proves the lemma. $\square$

Consequences

A closer look at the proof of Theorem 2.1 in the preceding section shows that (in virtue of Corollary 1.18) we do not actually need Thorbergsson's result that the incidence structures $(P,\mathfrak{L},\mathfrak{F})$ coming from isoparametric hypersurfaces of FKM-type are generalized quadrangles. In the proof of Lemma 2.6, e.g., it was not important that (q,K) is the unique intersection point of $\mathfrak{F}$ and $P_L \times \mathfrak{L}_p$. Essentially, only the following properties of these incidence structures were used, which are independent of Thorbergsson's work:

(SIS1) $\mathcal{P}$ and $\mathcal{L}$ are compact, connected smooth manifolds of dimensions $2a+b$ and $a+2b$, respectively, where $a = l - m - 1$ and $b = m$.

(SIS2) $\mathcal{F}$ is a closed submanifold of $\mathcal{P} \times \mathcal{L}$ of dimension $2a+2b$, and the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to P$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.

Moreover, by Lemma 2.6 also the following condition is satisfied:

(SIS3) For every antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$, the submanifolds $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect transversally in $P \times \mathcal{L}$.

Hence, by Theorem 1.17 there is a finite number n such that $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect in precisely n points for each antiflag $(p, L) \in (P \times \mathcal{L}) \backslash \mathcal{F}$. In order to prove that $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a smooth generalized quadrangle it remains to show that $n = 1$, cf. Corollary 1.18. But this is easily seen to be true for all incidence strucures coming from arbitrary isoparametric (compact, connected) hypersurfaces in spheres with four distinct principal curvatures: let S be a great circle in $\mathbb{S}$ normal to the isoparametric submanifold $\mathcal{I}$. Note that by [27], Section 6, every great circle which intersects $\mathcal{I}$ orthogonally in one point intersects $\mathcal{I}$ and the focal manifolds $\mathcal{P}$ and $\mathcal{L}$ orthogonally in each intersection point. Moreover, the points of $\mathcal{P} \cap S$ and $\mathcal{L} \cap S$ follow on S alternatingly at spherical distance $\frac{\pi}{g}$, cf. also [18], Proposition 3.2. Now, in the case $g = 4$ choose $p \in \mathcal{P} \cap S$ and $L \in \mathcal{L} \cap S$ with spherical distance equal to $\frac{3\pi}{4}$. Then there is exactly one pair $(q, K) \in \mathbb{S} \times \mathbb{S}$ such that $\operatorname{dist}(p, K) = \operatorname{dist}(K, q) = \operatorname{dist}(q, L) = \frac{\pi}{4}$, and we have $q \in \mathcal{P} \cap S$, $K \in \mathcal{L} \cap S$, and $(q, K) \in \mathcal{F}$. Hence, the flag space $\mathcal{F}$ and $\mathcal{P}_L \times \mathcal{L}_p$ intersect in precisely one point of $P \times \mathcal{L}$, i.e. we have $n = 1$. So, our approach yields an elementary proof for the following

2.7 Theorem. *The incidence structures associated with isoparametric hypersurfaces of FKM-type are smooth generalized quadrangles.*

In [22], Theorem 10.9, L. Kramer has given a purely algebraic proof that these incidence structures are generalized quadrangles without proving that they are smooth quadrangles. Our proof here is of a different nature: the algebraic structure of Clifford systems is only used in order to establish the transversality condition (SIS3). In fact, conditions (SIS1) and (SIS2) are satisfied for incidence structures $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ coming from arbitrary (compact, connected) isoparametric hypersurfaces $\mathcal{I}$ in spheres with four distinct principal curvatures. Here, a and b are nothing else but the multiplicities of the four distinct principal curvatures of $\mathcal{I}$, see [27], Section 1, cf. [18], Section 1. Note that by [27], Theorem 1, there are at most two different values for these multiplicities.

So, besides the calculations in Lemma 2.6, which yield condition (SIS3), we need nothing from the special situation of isoparametric hypersurfaces of FKM-type. However, the proof of the transversal intersection of $P_L \times \mathcal{L}_p$ and $\mathcal{F}$ for every antiflag $(p, L) \in (P \times \mathcal{L}) \backslash \mathcal{F}$ as required in (SIS3) made essential use of the structure of Clifford systems.

Chapter 3

Isoparametric Triple Systems and Geometry

Introduction

The main result of this chapter essentially solves a problem posed by G. Thorbergsson in his papers [35], [36]. It may be stated as follows:

Theorem. *Incidence structures arising from isoparametric hypersurfaces with four distinct principal curvatures in spheres are Tits buildings of type* C_2, *also called generalized quadrangles.*

Moreover, as a direct consequence of our proof, these generalized quadrangles are *smooth* quadrangles. In fact, they are real analytic and even Nash quadrangles, see the end of the third section. On the way to our main theorem, we will prove further new results on generalized quadrangles which are associated with isoparametric hypersurfaces in spheres. By Corollary 3.7, e.g., the join map $\vee$ which assigns to any two distinct points $p, q \in \mathcal{P}$ with $\mathcal{L}_p \cap \mathcal{L}_q \neq \emptyset$ the unique line $p \vee q$ joining them is the restriction of a rational map, and this map can be described explicitly.

Throughout this chapter, we will make essential use of the algebraic approach to isoparametric hypersurfaces developed by Dorfmeister und Neher in [9]. Their theory of isoparametric triple systems turns out to be well-suited for the description of geometric properties of incidence structures derived from isoparametric hypersurfaces. As an example, we mention that point rows and line pencils have a natural description in terms of eigenspaces of linear operators which are defined by the triple product.

We have decided to carry out the calculations in the third section explicitly, in order to show that only few calculations remain to be done if the theory of isoparametric triple is used effectively. In the next section we will give an introduction to this theory which presents the results that are relevant for the proof of our main result. We will also give proofs if they are not given explicitly in [9] or if they are essential for a better understanding of the third section.

Isoparametric Triple Systems

In Chapter 2 we explained how incidence structures can be associated to isoparametric hypersurfaces with g distinct principal curvatures in spheres for $g \geq 2$. For our proof that these incidence structures are smooth generalized quadrangles for $g = 4$, we will use Corollary 1.18, which characterizes these objects by implicit conditions. As remarked at the end of Chapter 2, conditions (SIS1) and (SIS2) in Theorem 1.17 are satisfied by incidence structures $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ derived from arbitrary compact, connected isoparametric hypersurfaces $\mathcal{I}$ with four distinct principal curvatures in spheres. Moreover, we showed that in these incidence structures there always exist anti-flags $(p, L) \in (\mathcal{P} \times \mathcal{L}) \backslash \mathcal{F}$ such that $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect in precisely one flag. So, in order to prove the main result of this chapter, it only remains to prove condition (SIS3) for these incidence structures. This is a subtle property. It will be verified in the next section by means of the theory of isoparametric triple systems developed by Dorfmeister and Neher in [9]. Here, we present the parts of this theory which are relevant for the next section.

Let $\mathcal{I}$ denote a compact, connected isoparametric hypersurface with g distinct principal curvatures in the unit sphere $\mathbb{S}$ of a euclidean vector space

V. By the second section of Chapter 2, the isoparametric hypersurface $\mathcal{I}$ may be chosen in the corresponding isoparametric family in such a way that for each $z \in \mathcal{I}$ the images $\rho_{\mathcal{P}}(z)$ and $\rho_{\mathcal{L}}(z)$ have spherical distance $\frac{\pi}{2g}$ from z. By [27], we then have $\mathcal{I} = F^{-1}(0) \cap \mathbb{S}$, where $F : V \to \mathbb{R}$ is a homogeneous polynomial function of degree g which satisfies the two differential equations

$$\langle \operatorname{grad} F(x), \operatorname{grad} F(x) \rangle = g^2 \langle x, x \rangle^{g-1},$$
$$\Delta F(x) = (1/2)(m_2 - m_1) g^2 \langle x, x \rangle^{g/2-1}.$$

The polynomial F is called a *Cartan-Münzner polynomial*. The parameters m_1 and m_2 denote the multiplicities of the four distinct principal curvatures of $\mathcal{I}$. Recall that there are at most two different values for these multiplicities, and they are equal if g is odd, see [27], Theorem 1. The two focal manifolds $\mathcal{P}$ and $\mathcal{L}$ are given by $\mathcal{P} = F^{-1}(1) \cap \mathbb{S}$ and $\mathcal{L} = F^{-1}(-1) \cap \mathbb{S}$.

The starting point of the theory of isoparametric triple systems is the observation that every homogeneous polynomial of degree 4 on V may be written in the form $3\langle x, x\rangle^2 - \frac{2}{3}\langle \{x, x, x\}, x\rangle$, where $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$ is a trilinear map which satisfies the identies $\{x_1, x_2, x_3\} = \{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\}$ and $\langle \{x_1, x_2, x_3\}, x_4\rangle = \langle x_1, \{x_2, x_3, x_4\}\rangle$ for all $x_1, x_2, x_3, x_4 \in V$ and all permutations σ of the set $\{1, 2, 3\}$. We put $T(x, y) : V \to V : z \mapsto \{x, y, z\}$ and $T(x) = T(x, x)$ for $x, y \in V$. For $g = 4$, the two partial differential equations above translate into identities of the triple system $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ associated with the polynomial F.

3.1 Lemma. *Let $F : V \to \mathbb{R}$ be a homogeneous polynomial of degree 4 and let $\{\cdot,\cdot,\cdot\}$ be defined as above. Assume further that m_1 and m_2 are positive integers such that $\dim V = 2(m_1 + m_2 + 1)$. Then F satisfies the two differential equations above with $g = 4$ if and only if*

$$\langle \{x, x, x\}, \{x, x, x\}\rangle - 9\langle x, x\rangle\langle \{x, x, x\}, x\rangle + 18\langle x, x\rangle^3 = 0$$

and

$$\operatorname{trace} T(x, y) = 2(2m_1 + m_2 + 3)\langle x, y\rangle$$

for all $x, y \in V$.

Proof. Because of $\operatorname{grad} F(x) = 12\langle x, x\rangle x - \frac{8}{3}\{x, x, x\}$, an easy calculation shows that the first differential equation is equivalent to the first identity of this lemma. Let us assume that F satisfies the second differential equation. The second-order derivative of F at $z \in V$ is given by

$$H_F(z) : V \times V \to V : (x, y) \mapsto 24\langle x, z\rangle\langle y, z\rangle + 12\langle x, y\rangle\langle z, z\rangle - 8\langle\{x, y, z\}, z\rangle.$$

We choose $x, y \in V$ arbitrarily and set $Q_1 : V \to \mathbb{R} : z \mapsto H_F(z)(x, y)$, $Q_2 : V \to \mathbb{R} : z \mapsto \langle T(x, y)z, z\rangle$. Then we have $\Delta Q_2(z) = 2 \operatorname{trace} T(x, y)$ and, because of the second differential equation, $\Delta Q_1(z) = H_{\Delta F}(z)(x, y) = 16(m_2 - m_1)\langle x, y\rangle$ for any $z \in V$. Hence, the equation $\Delta Q_1(z) = 48\langle x, y\rangle + 24 \dim V\langle x, y\rangle - 8\,\Delta Q_2(z)$ implies that

$$(m_2 - m_1)\langle x, y\rangle = 3\langle x, y\rangle + 3(m_1 + m_2 + 1)\langle x, y\rangle - \operatorname{trace} T(x, y),$$

which is equivalent to $\operatorname{trace} T(x, y) = 2(2m_1 + m_2 + 3)\langle x, y\rangle$.

Conversely, if this equation holds for $x, y \in V$, then we see by means of the same calculation that $H_{\Delta F}(z)(x, y) = 16(m_2 - m_1)\langle x, y\rangle$ for all $z \in V$. Since ΔF is a homogeneous polynomial of degree 2, we have $H_{\Delta F}(z)(x, x) = 2\Delta F(x)$. This proves the second differential equation. □

The definition of isoparametric triple systems is motivated by Lemma 3.1.

3.2 Definition. An *isoparametric triple system* is a triple system $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$, where $(V, \langle\cdot,\cdot\rangle)$ is a euclidean vector space and $\{\cdot,\cdot,\cdot\}$ is a triple product on V which satisfies the following axioms:

- (ISO1) $\{\cdot,\cdot,\cdot\}$ is totally symmetric.
- (ISO2) $\langle\{x, y, z\}, w\rangle = \langle x, \{y, z, w\}\rangle$ for all $x, y, z, w \in V$.
- (ISO3) $\langle\{x, x, x\}, \{x, x, x\}\rangle - 9\langle x, x\rangle\langle\{x, x, x\}, x\rangle + 18\langle x, x\rangle^3 = 0$ for all $x \in V$.

(ISO4) There exist positive integers m_1 and m_2 such that $\dim V = 2(m_1 + m_2 + 1)$ and $\operatorname{trace} T(x,y) = 2(2m_1 + m_2 + 3)\langle x, y\rangle$ for all $x, y \in V$.

By Lemma 3.1, we see that the polynomials describing compact, connected isoparametric hypersurfaces with four distinct principal curvatures in the sense of [27] are in one-to-one correspondence with isoparametric triple systems. By differentiating the equation in (ISO3) and dividing by 6, we obtain the identity

$$\{x, x\{x,x,x\}\} - 6\langle x,x\rangle\{x,x,x\} - 3\langle x, \{x,x,x\}\rangle x + 18\langle x,x\rangle^2 x = 0. \quad (*)$$

The successive linearizations of this identity are very important in the theory of isoparametric triple systems. They are, however, quite complicated, see [9], p. 193. We will use explicitly only the first linearization:

$$\begin{gathered} 3T(x)^2 y + 2T(x, \{x,x,x\})y - 18\langle x,x\rangle T(x)y - 3\langle x, \{x,x,x\}\rangle y \\ -12(x\langle\{x,x,x\}, y\rangle + \{x,x,x\}\langle x,y\rangle) + 18\langle x,x\rangle^2 y + 72x\langle x,x\rangle\langle x,y\rangle = 0. \end{gathered}$$

Note that in this setting linearization is the same as differentiation (up to a constant factor).

The two focal manifolds $\mathcal{P}$ and $\mathcal{L}$ have an easy description in the context of isoparametric triple systems: they coincide with the sets of *maximal tripotents* and *minimal tripotents*, respectively. By definition, a minimal tripotent is an element $x \in \mathbb{S}$ with $\{x,x,x\} = 6x$, see [9], the comments after Theorem 2.3 and Remark 2.4 (b). Analogously, a maximal tripotent is a vector $y \in \mathbb{S}$ with $\{y,y,y\} = 3y$, see [9], the comments after Theorem 2.5. These descriptions of $\mathcal{P}$ and $\mathcal{L}$ may be proved as follows: let z be an extremal point of F restricted to $\mathbb{S}$. Then $\operatorname{grad} F(z) = 12\langle z,z\rangle z - \frac{8}{3}\{z,z,z\}$, and hence $\{z,z,z\}$, is a multiple of z. Choose $\lambda \in \mathbb{R}$ such that $\{z,z,z\} = \lambda z$. By means of equation $(*)$, we get $(\lambda - 3)(\lambda - 6) = 0$. We have $F(z) = 1$ for $\lambda = 3$, and $F(z) = -1$ for $\lambda = -1$. This proves the above description of $\mathcal{P} = F^{-1}(1) \cap \mathbb{S}$ and $\mathcal{L} = F^{-1}(-1) \cap \mathbb{S}$ in terms of tripotents.

As in the theory of Jordan triple systems, the operators $T(x)$ play an important rôle in the theory of isoparametric triple systems. For $x \in V$, $\mu \in \mathbb{R}$, we define

$$V_\mu(x) = \{y \in V \mid T(x)(y) = \mu y, \langle x, y\rangle = 0\}.$$

An essential feature of isoparametric triple systems are *Peirce decompositions* of V relative to maximal or minimal tripotents. For $p \in \mathcal{P}$, $L \in \mathcal{L}$, these Peirce decompositions are given by $V = \mathbb{R}p \oplus V_1(p) \oplus V_3(p)$ and $V = \mathbb{R}L \oplus V_0(L) \oplus V_2(L)$, respectively, see [9], Theorem 2.2. Here, we sketch a proof of these statements: by the first linearization of identity $(*)$, we have $3T(p)^2 - 12T(p) + 9\mathrm{id} = 0$. Hence, for each eigenvalue λ of $T(p)$ we have $\lambda^2 - 4\lambda + 3\lambda = (\lambda - 1)(\lambda - 3) = 0$. This proves the Peirce decomposition of V relative to p. In the case of L, the first linearization of equation $(*)$ yields $3T(L)^2y - 6T(L)y - 72L\langle L, y\rangle = 0$. The self-adjoint operator $T(L)$ leaves the orthogonal complement $L^\perp$ of L invariant. Hence, each eigenvalue of the restriction of $T(L)$ to $L^\perp$ satisfies the equation $\lambda^2 - 2\lambda = 0$, and the Peirce decomposition of V relative to L follows.

It can be shown that $\dim V_3(p) = m_1 + 1$, $\dim V_1(p) = m_1 + 2m_2$, $\dim V_0(L) = m_2 + 1$, and $\dim V_2(L) = 2m_1 + m_2$, see [9], Theorem 2.2. We prove this statement for the Peirce spaces relative to L. By Lemma 3.1, we have have $\operatorname{trace} T(L) = 2(2m_1 + m_2 + 3)$. On the other hand, we have $\operatorname{trace} T(L) = 2\dim V_2(L) + 6$, which shows that $\dim V_2(L) = 2m_1 + m_2$. Hence, we get $\dim V_0(L) = \dim V - \dim V_2(L) - 1 = m_2 + 1$.

For vectors in the Peirce spaces $V_0(L)$ and $V_2(L)$, the quite complicated identities corresponding to the successive linearizations of equation $(*)$ reduce to much simpler identities, see [9], Theorem 2.3. An analogous statement holds for vectors in $V_i(p)$ $(i = 1, 3)$, see [9], Theorem 2.5. In particular, we mention the following four identities which will be used in the next section $(u_0, v_0, w_0 \in V_0(L),\ u_3, v_3, w_3 \in V_3(p))$:

$$(1) \qquad \{u_0, L, v_0\} = 0$$

(2) $\{u_0, v_0, w_0\} = 2(\langle u_0, v_0\rangle w_0 + \langle v_0, w_0\rangle u_0 + \langle w_0, u_0\rangle v_0)$

(3) $\{u_3, p, v_3\} = 3\langle u_3, v_3\rangle p$

(4) $\{u_3, v_3, w_3\} = \langle u_3, v_3\rangle w_3 + \langle v_3, w_3\rangle u_3 + \langle w_3, u_3\rangle v_3$

These identities correspond to equations (2.3), (2.6), (2.10), and (2.13) in [9]. In the next section, we will see that the Peirce spaces $V_0(L)$, $V_2(L)$, $V_1(p)$, and $V_3(p)$ have a precise geometrical meaning.

The Geometry of Isoparametric Hypersurfaces

The following Proposition establishes a close relationship between point rows and line pencils on the one hand, and eigenspaces of the operators $T(x)$ on the other hand. For definitions, see the preceding section.

3.3 Proposition. *Let $q \in \mathcal{P}$ and $K \in \mathcal{L}$. Then we have*

$$\mathcal{P}_K = \mathbb{S} \cap \Big(\frac{1}{\sqrt{2}}K + V_0(K)\Big) \quad \textit{and} \quad \mathcal{L}_q = \mathbb{S} \cap \Big(\frac{1}{\sqrt{2}}q + V_3(q)\Big).$$

Proof. First, we mention that point rows and line pencils are geometric spheres of dimensions m_2 and m_1 respectively. This can be seen as follows: by comparing the notation of Theorem 2 in [27] and of equations (1.2) and (1.3) in [9] (Münzner's differential equations as presented in the preceding section), we see that m_0 and m_1 in [27] correspond to m_1 and m_2 (in this order) in the notation of [9], which we have adopted as our notation. By the proof of parts a) and b) of Theorem 4 in [27], we conclude that $\mathcal{P} = \mathbb{S} \cap F^{-1}(1)$ has dimension $m_1 + 2m_2$ and that $\mathcal{L} = \mathbb{S} \cap F^{-1}(-1)$ has dimension $2m_1 + m_2$. So, we have $\dim \mathcal{P}_K = \dim \pi_{\mathcal{L}}^{-1}(K) = \dim \mathcal{F} - \dim \mathcal{L} = m_2$ and $\dim \mathcal{L}_q = m_1$. The well-known fact that point rows and line pencils are geometric spheres is, e.g., an immediate consequencee of [27], Section 6, see also [31], Theorem 6.2.9 (iii).

On the other hand, the geometric spheres $\mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K))$ and $\mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q))$ have dimensions m_2 and m_1, respectively, since $V_0(K)$ has dimension $m_2 + 1$ and $V_3(q)$ has dimension $m_1 + 1$, see the preceding section. Hence, it suffices to prove the inclusions $\mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K)) \subseteq \mathcal{P}_K$ and $\mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q)) \subseteq \mathcal{L}_q$. For this purpose, choose $p \in \mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K))$ arbitrarily. Then we have $p = \frac{1}{\sqrt{2}}K + p_0$ with $p_0 \in V_0(K)$ and $\|p_0\| = \frac{1}{\sqrt{2}}$. So, we get

$$\begin{aligned}\{p,p,p\} &= \Big\{\frac{1}{\sqrt{2}}K + p_0, \frac{1}{\sqrt{2}}K + p_0, \frac{1}{\sqrt{2}}K + p_0\Big\} \\ &= \frac{3}{\sqrt{2}}K + \frac{3}{\sqrt{2}}\{p_0, K, p_0\} + \{p_0, p_0, p_0\}\end{aligned}$$

because of $\{K,K,K\} = 6K$ and $\{K,K,p_0\} = 0$. By using the identites (1) and (2) in the preceding section, we see that $\{p_0, K, p_0\} = 0$ and that $\{p_0,p_0,p_0\} = 6\langle p_0,p_0\rangle p_0 = 3p_0$. Hence, we obtain $\{p,p,p\} = \frac{3}{\sqrt{2}}K + 3p_0 = 3p$. This shows that $p \in \mathcal{P}$. Furthermore, we get $\langle p, K\rangle = \langle \frac{1}{\sqrt{2}}K + p_0, K\rangle = \frac{1}{\sqrt{2}}$, i.e. p and K are incident. Hence, we have $p \in \mathcal{P}_K$, and the inclusion $\mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K)) \subseteq \mathcal{P}_K$ is proved.

The other inclusion is proved analogously: for $L \in \mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q))$, we have $L = \frac{1}{\sqrt{2}}q + L_3$ with $\|L_3\| = \frac{1}{\sqrt{2}}$. We obtain

$$\begin{aligned}\{L,L,L\} &= \Big\{\frac{1}{\sqrt{2}}q + L_3, \frac{1}{\sqrt{2}}q + L_3, \frac{1}{\sqrt{2}}q + L_3\Big\} \\ &= \frac{1}{2\sqrt{2}}\{q,q,q\} + \frac{3}{2}\{q,q,L_3\} + \frac{3}{\sqrt{2}}\{q,L_3,L_3\} + \{L_3,L_3,L_3\} \\ &= \frac{3}{2\sqrt{2}}q + \frac{9}{2}L_3 + \frac{9}{2\sqrt{2}}q + \frac{3}{2}L_3 = 6L.\end{aligned}$$

Here, we have used that because of the identities (3) and (4) we have $\{q, L_3, L_3\} = \frac{3}{2}q$ and $\{L_3,L_3,L_3\} = 3\langle L_3, L_3\rangle L_3 = \frac{3}{2}L_3$. Hence, we have $L \in \mathcal{L}$, and since $\langle q, L\rangle = \frac{1}{\sqrt{2}}$ is an immediate consequence of $L = \frac{1}{\sqrt{2}}q + L_3$, we get $L \in \mathcal{L}_q$. This proves the inclusion $\mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q)) \subseteq \mathcal{L}_q$ and completes the proof. □

3.4 Corollary. *A point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are incident if and only if*

$$\{p, L, L\} = 3\sqrt{2}L.$$

If p and L are incident, then we also have $\{p, p, L\} = 3L$.

Proof. If p and L satisfy the equation $\{p, L, L\} = 3\sqrt{2}L$, then we have $6\langle p, L\rangle = \langle p, \{L, L, L\}\rangle = \langle \{p, L, L\}, L\rangle = \langle 3\sqrt{2}L, L\rangle = 3\sqrt{2}$, i.e. $\langle p, L\rangle = \frac{1}{\sqrt{2}}$, which shows that p and L are incident. Conversely, if p and L are incident, then we have $p \in \mathcal{P}_L$, which implies that $p = \frac{1}{\sqrt{2}}L + p_0$ with $p_0 \in V_0(L)$ (see Proposition 3.3). Hence, we get $\{p, L, L\} = \{\frac{1}{\sqrt{2}}L + p_0, L, L\} = 3\sqrt{2}L$. Analogously, by Proposition 3.3 we have $L = \frac{1}{\sqrt{2}}p + L_3$ with $L_3 \in V_3(p)$. Thus, we obtain $\{p, p, L\} = \{p, p, \frac{1}{\sqrt{2}}p + L_3\} = \frac{3}{\sqrt{2}}p + 3L_3 = 3L$. This completes the proof. □

Remarks. (i) It is not true that $\{p, p, L\} = 3L$ implies that p and L are incident. This may be seen as follows: let $q \in \mathcal{P}$ and $L \in \mathcal{L}$ be incident. For $p = -q$ we have $p \in \mathcal{P}$ (since $\{p, p, p\} = 3p$, for example) and $\{p, p, L\} = 3L$, but p and L are not incident because of $\{p, L, L\} = -3\sqrt{2}L$ or $\langle p, L\rangle = -\frac{1}{\sqrt{2}}$.

However, it is true that $\{p, p, L\} = 3L$ implies that either p and L, or $-p$ and L are incident. A proof of this statement can be obtained in the following way: as mentioned in the previous section, the euclidean vector space V may be decomposed as an orthogonal sum $\mathbb{R}p \oplus V_1(p) \oplus V_3(p)$. Hence, we have $L = \langle p, L\rangle p + L_1 + L_3$ with $L_1 \in V_1(p)$, $L_3 \in V_3(p)$, and $\langle p, L\rangle^2 + \langle L_1, L_1\rangle + \langle L_3, L_3\rangle = 1$. In this notation, we get $\{p, p, L\} = 3\langle p, L\rangle p + L_1 + 3L_3$, and we see that $\{p, p, L\} = 3L$ implies $L_1 = 0$. Using the identities (3) and (4) of the preceding section, a short computation shows that

$$\begin{aligned} 6L = \{L, L, L\} &= \{\langle p, L\rangle p + L_3, \langle p, L\rangle p + L_3, \langle p, L\rangle p + L_3\} \\ &= 3\langle p, L\rangle^3 p + 9\langle p, L\rangle^2 L_3 + 9\langle p, L\rangle\langle L_3, L_3\rangle p + 3\langle L_3, L_3\rangle L_3. \end{aligned}$$

By replacing L_3 by $L - \langle p, L\rangle p$ and $\langle L_3, L_3\rangle$ by $1 - \langle p, L\rangle^2$, we obtain

$$2\langle p, L\rangle(1 - 2\langle p, L\rangle^2)p = (1 - 2\langle p, L\rangle^2)L$$

after an easy calculation. Since p and L are linearly independent, we conclude that $\langle p, L\rangle^2 = \frac{1}{2}$. So, we have $\langle p, L\rangle = \frac{1}{\sqrt{2}}$ or $\langle -p, L\rangle = \frac{1}{\sqrt{2}}$, and the above statement is proved.

(ii) There is another possibility to prove Propositions 3.3 and 3.4 in reverse order and in an essentially different way. Using this approach, we do not need a priori the facts that point rows and line pencils are geometric spheres and that their dimensions are m_2 and m_1, respectively. In contrast, these properties were required in the proof of Proposition 3.3 presented above. We sketch here this different approach.

As we have seen in the preceding proof, the identity $\{p, L, L\} = 3\sqrt{2}L$ for $p \in \mathcal{P}$, $L \in \mathcal{L}$ implies that $\langle p, L\rangle = \frac{1}{\sqrt{2}}$. We want to prove directly that, conversely, we have $\{p, L, L\} = 3\sqrt{2}L$, provided that p and L are incident. In this case, we have $p = \rho_{\mathcal{P}}(z)$ and $L = \rho_{\mathcal{L}}(z)$ for some $z \in \mathcal{I}$, i.e. p and L lie on a great circle normal to the isoparametric hypersurface $\mathcal{I}$, and both have the same distance $\frac{\pi}{8}$ from z, see the preceding section. Hence, we have $\frac{p+L}{\|p+L\|} = z \in \mathcal{I}$ (where $\|p + L\| = 2\cos\frac{\pi}{8}$). Since the polynomial F is homogeneous, we conclude that $F(p + L) = 0$. Using the description of F via $\{\cdot,\cdot,\cdot\}$, we get

$$3\langle p + L, p + L\rangle^2 - \frac{2}{3}\langle\{p + L, p + L, p + L\}, p + L\rangle = 0.$$

An easy calculation, which uses the identites $\{p, p, p\} = 3p$, $\{L, L, L\} = 6L$, and $\langle p, L\rangle = \frac{1}{\sqrt{2}}$, shows that this equation is equivalent to $\langle\{p, L, L\}, p\rangle = 3$. By applying the first linearization of equation $(*)$ with $x = L$ and $y = p$, we get

$$3\{L, L, \{L, L, p\}\} - 6\{L, L, p\} - 36\sqrt{2}L = 0.$$

The scalar product with p yields $\langle\{L, L, p\}, \{L, L, p\}\rangle = 18$ because of $\langle\{p, L, L\}, p\rangle = 3$. Finally, we obtain

$$\begin{aligned}&\langle\{p, L, L\} - 3\sqrt{2}L, \{p, L, L\} - 3\sqrt{2}L\rangle\\ =&\langle\{p, L, L\}, \{p, L, L\}\rangle - 6\sqrt{2}\langle\{p, L, L\}, L\rangle + 18\\ =&18 - 36 + 18 = 0,\end{aligned}$$

i.e. $\{p, L, L\} = 3\sqrt{2}L$. So, we have given a direct proof for the description of incidence in Corollary 3.4, which uses only the most basic parts of the theory of Dorfmeister and Neher. The second result of that corollary may be derived in a similar way by using the equation $\langle\{p, L, L\}, p\rangle = 3$ and the first linearization of equation $(*)$ with $x = p$ and $y = L$.

The descriptions of point rows and line pencils given in Proposition 3.3 may be derived easily by means of these results: in the proof of Proposition 3.3, we have already proved directly that $\mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K)) \subseteq \mathcal{P}_K$ and that $\mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q)) \subseteq \mathcal{L}_q$ for $q \in \mathcal{P}$, $K \in \mathcal{L}$. In order to prove the other inclusions, choose $p \in \mathcal{P}_K$ and $L \in \mathcal{L}_q$ arbitrarily. Then we have $T(K)(p - \frac{1}{\sqrt{2}}K) = \{p - \frac{1}{\sqrt{2}}K, K, K\} = 0$, because of $\{p, K, K\} = 3\sqrt{2}K$ and $\{K, K, K\} = 6K$. This shows that $p - \frac{1}{\sqrt{2}}K \in V_0(K)$ or, eqivalently, that $p \in \frac{1}{\sqrt{2}}K + V_0(K)$. Analogously, we get $T(q)(L - \frac{1}{\sqrt{2}}q) = \{q, q, L - \frac{1}{\sqrt{2}}q\} = 3(L - \frac{1}{\sqrt{2}}q)$, since $\{q, q, L\} = 3L$ and $\{q, q, q\} = 3q$. Because of $\langle L - \frac{1}{\sqrt{2}}q, q\rangle = 0$, we conclude that $L - \frac{1}{\sqrt{2}}q \in V_3(q)$, i.e. that $L \in \frac{1}{\sqrt{2}}q + V_3(q)$. This proves the equations for point rows and line pencils in Proposition 3.3.

The following corollary shows that the eigenspaces of the operators $T(x)$ have a precise geometrical meaning for $x \in \mathcal{P}$ or $x \in \mathcal{L}$. For $x \in V$, we denote by $x^\perp$ the orthogonal complement of x in V.

3.5 Corollary. *Let $q \in \mathcal{P}$ and $K \in \mathcal{L}$. Then the tangent spaces of the point space $\mathcal{P}$ in q and the line space $\mathcal{L}$ in K are given by $\mathrm{T}_q\mathcal{P} = V_1(q)$ and $\mathrm{T}_K\mathcal{L} = V_2(K)$. Moreover, for the tangent spaces of the point row $\mathcal{P}_K$ in p and the line pencil $\mathcal{L}_q$ in L we have $\mathrm{T}_p\mathcal{P}_K = V_0(K) \cap p^\perp$ and $\mathrm{T}_L\mathcal{L}_q = V_3(q) \cap L^\perp$.*

Proof. The statements on the tangent spaces of $\mathcal{P}_K$ and $\mathcal{L}_q$, respectively, are an immediate consequence of Proposition 3.3. As already mentioned at the beginning of Proposition 3.3, we have $\dim \mathcal{P} = m_1 + 2m_2$ and $\dim \mathcal{L} = 2m_1 + m_2$. By the preceding section, we also have $\dim V_1(q) = m_1 + 2m_2$

and $\dim V_2(K) = 2m_1 + m_2$. Hence, it suffices to prove the inclusions $\mathrm{T}_q\mathcal{P} \subseteq V_1(q)$ and $\mathrm{T}_K\mathcal{L} \subseteq V_2(K)$. Since the maps

$$V \to V : x \mapsto \{x, x, x\} - 3x \quad \text{and} \quad V \to V : y \mapsto \{y, y, y\} - 6y$$

vanish on $\mathcal{P}$ and $\mathcal{L}$, respectively, the differentials of these maps vanish on $\mathrm{T}_q\mathcal{P}$ and $\mathrm{T}_K\mathcal{L}$, respectively. So, we get $\{q, q, u\} - u = 0$ and $\{K, K, v\} - 2v = 0$, i.e. $T(q)(u) = u$ and $T(K)(v) = 2v$, for all $u \in \mathrm{T}_q\mathcal{P}$, $v \in \mathrm{T}_K\mathcal{L}$. This proves the required inclusions. □

Remark. In the proofs of Lemma 3.9 and Theorem 3.10 we will use the fact that every tangent vector $u \in \mathrm{T}_q\mathcal{P}$ ($v \in \mathrm{T}_K\mathcal{L}$) and every line $L \in \mathcal{L}_q$ (point $p \in \mathcal{P}_K$) are orthogonal. This is a direct consequence of the geometric properties of isoparametric hypersurfaces in spheres as they were presented at the beginning of the preceding section. Of course, this statement is also a consequence of Proposition 3.3 and the preceding corollary. Because of $u \in \mathrm{T}_q\mathcal{P}$, we have $u \in V_1(q)$, and $L \in \mathcal{L}_q$ implies that L lies in the span of q and $V_3(q)$. Analogously, $v \in \mathrm{T}_K\mathcal{L}$ and $p \in \mathcal{P}_K$ yield $v \in V_2(K)$ and $p \in \operatorname{span}\{K, V_0(K)\}$. Hence, we get $\langle L, u\rangle = \langle p, v\rangle = 0$.

The identities (ii) and (iv) in the next proposition are crucial for the following results.

3.6 Proposition. *Let $p, q \in \mathcal{P}$ be incident with $K \in \mathcal{L}$. Then the following two identities hold:*

(i) $\{p, K, q\} = 3K$
(ii) $\{p, p, q\} = 2\sqrt{2}(1 - \langle p, q\rangle)K + 2(2\langle p, q\rangle - 1)p + q$

Let $L, K \in \mathcal{L}$ be incident with $q \in \mathcal{P}$. Then we have the following two identities:

(iii) $\{L, q, K\} = 3(\langle L, K\rangle - 1)q + \frac{3}{\sqrt{2}}(L + K)$
(iv) $\{L, L, K\} = 2\sqrt{2}(\langle L, K\rangle - 1)q + 2(\langle L, K\rangle + 1)L + 2K$

Proof. By Proposition 3.3, we have $\mathcal{P}_K = \mathbb{S} \cap (\frac{1}{\sqrt{2}}K + V_0(K))$ and $\mathcal{L}_q = \mathbb{S} \cap (\frac{1}{\sqrt{2}}q + V_3(q))$. So, we may write $p = \frac{1}{\sqrt{2}}K + p_0$, $q = \frac{1}{\sqrt{2}}K + q_0$, $L = \frac{1}{\sqrt{2}}q + L_3$, and $K = \frac{1}{\sqrt{2}}q + K_3$ with $p_0, q_0 \in V_0(K)$ and $L_3, K_3 \in V_3(q)$. In order to prove (i), we use identity (1) in the preceding section, which states that $\{p_0, K, q_0\} = 0$. We obtain

$$\begin{aligned}
\{p, K, q\} &= \Big\{\frac{1}{\sqrt{2}}K + p_0, K, \frac{1}{\sqrt{2}}K + q_0\Big\} \\
&= \frac{1}{2}\{K, K, K\} + \frac{1}{\sqrt{2}}\{K, K, q_0\} + \frac{1}{\sqrt{2}}\{p_0, K, K\} + \{p_0, K, q_0\} \\
&= 3K.
\end{aligned}$$

In order to prove (ii), we first see that

$$\begin{aligned}
\{p, p, q\} &= \Big\{\frac{1}{\sqrt{2}}K + p_0, \frac{1}{\sqrt{2}}K + p_0, \frac{1}{\sqrt{2}}K + q_0\Big\} \\
&= \frac{1}{2\sqrt{2}}\{K, K, K\} + \frac{1}{2}\{K, K, q_0\} + \{K, K, p_0\} \\
&\quad + \sqrt{2}\{K, p_0, q_0\} + \frac{1}{\sqrt{2}}\{p_0, p_0, K\} + \{p_0, p_0, q_0\} \\
&= \frac{3}{\sqrt{2}}K + \{p_0, p_0, q_0\},
\end{aligned}$$

since $\{p_0, K, q_0\} = \{p_0, K, p_0\} = 0$. By equation (2), we have $\{p_0, p_0, q_0\} = 2(\langle p_0, p_0\rangle q_0 + \langle p_0, q_0\rangle p_0 + \langle q_0, p_0\rangle p_0) = q_0 + 4\langle p_0, q_0\rangle p_0$ because of $\|p_0\| = \frac{1}{\sqrt{2}}$. After replacing p_0 by $p - \frac{1}{\sqrt{2}}K$ and q_0 by $q - \frac{1}{\sqrt{2}}K$, we get

$$\begin{aligned}
\{p, p, q\} &= \frac{3}{\sqrt{2}}K + \Big(q - \frac{1}{\sqrt{2}}K\Big) + 4\Big\langle p - \frac{1}{\sqrt{2}}K, q - \frac{1}{\sqrt{2}}K\Big\rangle\Big(p - \frac{1}{\sqrt{2}}K\Big) \\
&= \sqrt{2}K + q + 4\Big(\langle p, q\rangle - \frac{1}{2}\Big)\Big(p - \frac{1}{\sqrt{2}}K\Big) \\
&= 2\sqrt{2}(1 - \langle p, q\rangle)K + 2(2\langle p, q\rangle - 1)p + q.
\end{aligned}$$

Equations (iii) and (iv) are proved in an analagous way. By identity (3), we have $\{L_3, q, K_3\} = 3\langle L_3, K_3\rangle q$. Hence, we get

$$\begin{aligned}
\{L, q, K\} &= \Big\{\frac{1}{\sqrt{2}}q + L_3, q, \frac{1}{\sqrt{2}}q + K_3\Big\} \\
&= \frac{1}{2}\{q, q, q\} + \frac{1}{\sqrt{2}}\{q, q, K_3\} + \frac{1}{\sqrt{2}}\{L_3, q, q\} + \{L_3, q, K_3\} \\
&= \frac{3}{2}q + \frac{3}{\sqrt{2}}K_3 + \frac{3}{\sqrt{2}}L_3 + 3\langle L_3, K_3\rangle q.
\end{aligned}$$

Now, we obtain identity (iii) by replacing in this equation L_3 by $L - \frac{1}{\sqrt{2}}q$ and K_3 by $K - \frac{1}{\sqrt{2}}q$:

$$\begin{aligned}
\{L, q, K\} &= \frac{3}{2}q + \frac{3}{\sqrt{2}}\Big(K - \frac{1}{\sqrt{2}}q\Big) + \frac{3}{\sqrt{2}}\Big(L - \frac{1}{\sqrt{2}}q\Big) \\
&\quad + 3\Big\langle L - \frac{1}{\sqrt{2}}q, K - \frac{1}{\sqrt{2}}q\Big\rangle q \\
&= -\frac{3}{2}q + \frac{3}{\sqrt{2}}K + \frac{3}{\sqrt{2}}L + 3\Big(\langle L, K\rangle - \frac{1}{2}\Big)q \\
&= 3(\langle L, K\rangle - 1)q + \frac{3}{\sqrt{2}}(L + K)
\end{aligned}$$

For a proof of equation (iv), we use the identities $\{L_3, q, K_3\} = 3\langle L_3, K_3\rangle q$, $\{L_3, q, L_3\} = \frac{3}{2}q$, and $\{L_3, L_3, K_3\} = \frac{1}{2}K_3 + 2\langle L_3, K_3\rangle L_3$, which are derived from equations (3) and (4) in the second section. By means of these identities, we get

$$\begin{aligned}
\{L, L, K\} &= \Big\{\frac{1}{\sqrt{2}}q + L_3, \frac{1}{\sqrt{2}}q + L_3, \frac{1}{\sqrt{2}}q + K_3\Big\} \\
&= \frac{1}{2\sqrt{2}}\{q, q, q\} + \frac{1}{2}\{q, q, K_3\} + \{q, q, L_3\} \\
&\quad + \sqrt{2}\{q, L_3, K_3\} + \frac{1}{\sqrt{2}}\{L_3, L_3, q\} + \{L_3, L_3, K_3\} \\
&= \frac{3}{2\sqrt{2}}q + \frac{3}{2}K_3 + 3L_3 + 3\sqrt{2}\langle L_3, K_3\rangle q + \frac{3}{2\sqrt{2}}q \\
&\quad + \frac{1}{2}K_3 + 2\langle L_3, K_3\rangle L_3.
\end{aligned}$$

Again, we replace L_3 by $L - \frac{1}{\sqrt{2}}q$ and K_3 by $K - \frac{1}{\sqrt{2}}q$. In this way, we obtain

$$\begin{aligned}\{L, L, K\} &= \frac{3}{\sqrt{2}}q + 2\Big(K - \frac{1}{\sqrt{2}}q\Big) + 3\Big(L - \frac{1}{\sqrt{2}}q\Big) \\ &\quad + 3\sqrt{2}\Big(\langle L, K\rangle - \frac{1}{2}\Big)q + 2\Big(\langle L, K\rangle - \frac{1}{2}\Big)\Big(L - \frac{1}{\sqrt{2}}q\Big) \\ &= 2\sqrt{2}(\langle L, K\rangle - 1)q + 2(\langle L, K\rangle + 1)L + 2K. \qquad \square\end{aligned}$$

The first statement of the following corollary has been proved by Thorbergsson by a direct geometric argument, see [18], Lemma 3.3. Of course, it is also a consequence of Theorem 3.10.

3.7 Corollary. *Any two distinct points are joined by at most one line, and any two distinct lines intersect in at most one point. The join map* $\vee$ *and the intersecting map* $\wedge$ *defined in this way are restrictions of rational functions, and are explicitly given by*

$$\vee : \{(p, q) \in \mathcal{P} \times \mathcal{P} \mid p \neq q,\ \mathcal{L}_p \cap \mathcal{L}_q \neq \emptyset\} \to \mathcal{L} :$$

$$(p, q) \mapsto \frac{\{p, p, q\} + 2(1 - 2\langle p, q\rangle)p - q}{2\sqrt{2}(1 - \langle p, q\rangle)}$$

and

$$\wedge : \{(L, K) \in \mathcal{L} \times \mathcal{L} \mid L \neq K,\ \mathcal{P}_L \cap \mathcal{P}_K \neq \emptyset\} \to \mathcal{P} :$$

$$(L, K) \mapsto \frac{\{L, L, K\} - 2(\langle L, K\rangle + 1)L - 2K}{2\sqrt{2}(\langle L, K\rangle - 1)}.$$

$\square$

The next corollary shows how the distances of p, q, L, and K are related in the situation of Theorem 1.17. It will be used in the proof of Theorem 3.10.

3.8 Corollary. *Let* $(p, L) \in \mathcal{P} \times \mathcal{L}$ *and* $(q, K) \in (\mathcal{P}_L \times \mathcal{L}_p) \cap \mathcal{F}$. *Then we have*

$$1 - \sqrt{2}\langle p, L\rangle = 2(1 - \langle p, q\rangle)(1 - \langle L, K\rangle).$$

Proof. The points p and q are both incident with K. Hence, by Proposition 3.6, we have

$$\{p, q, q\} = 2\sqrt{2}(1 - \langle p, q\rangle)K + 2(2\langle p, q\rangle - 1)q + p.$$

Since q is incident with L, we have $\langle \{p, q, q\}, L\rangle = \langle p, \{q, q, L\}\rangle = 3\langle p, L\rangle$ by Corollary 3.4. So, the scalar product with L yields

$$3\langle p, L\rangle = 2\sqrt{2}(1 - \langle p, q\rangle)\langle L, K\rangle + \sqrt{2}(2\langle p, q\rangle - 1) + \langle p, L\rangle.$$

By rearranging terms and dividing by $-\sqrt{2}$, we obtain

$$-\sqrt{2}\langle p, L\rangle = (1 - 2\langle p, q\rangle) - 2(1 - \langle p, q\rangle)\langle L, K\rangle$$

and hence

$$1 - \sqrt{2}\langle p, L\rangle = 2(1 - \langle p, q\rangle)(1 - \langle L, K\rangle).$$

$\square$

3.9 Lemma. *Let* $p \in \mathcal{P}$, $(q, K) \in \pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$, *and* $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$. *Then we have*

$$\text{(i)} \quad \langle u, v\rangle = \sqrt{2}(1 - \langle p, q\rangle)\langle v, v\rangle.$$

Dually, for $L \in \mathcal{L}$, $(q, K) \in \pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$, *and* $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$ *we have*

$$\text{(ii)} \quad \langle u, v\rangle = \sqrt{2}(1 - \langle L, K\rangle)\langle u, u\rangle.$$

Remark. In this lemma, we consider u and v as vectors in the same euclidean vector space V. The scalar product $\langle u, v\rangle$ has to be understood in this sense.

Proof. By Proposition 3.6 (ii), the map

$$\mathcal{P} \times \mathcal{L} \to V : (x, y) \mapsto 2\sqrt{2}(1 - \langle p, x\rangle)y + 2(2\langle p, x\rangle - 1)p + x - \{p, p, x\}$$

vanishes identically on $\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$. Hence, for $(q, K) \in \pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$ and $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$, we get by differentiating this map

$$-2\sqrt{2}\langle p, u\rangle K + 4\langle p, u\rangle p + u - \{p, p, u\} + 2\sqrt{2}(1 - \langle p, q\rangle)v = 0.$$

Because of $v \in \mathrm{T}_K\mathcal{L}_p$, we have $\langle K, v\rangle = \langle p, v\rangle = 0$. Thus, the scalar product with v yields

$$\langle u, v\rangle - \langle\{p, p, u\}, v\rangle + 2\sqrt{2}(1 - \langle p, q\rangle)\langle v, v\rangle = 0.$$

Moreover, by Corollary 3.5, we have $v \in V_3(p)$ and hence $\langle\{p, p, u\}, v\rangle = \langle u, \{p, p, v\}\rangle = 3\langle u, v\rangle$. This implies equation (i).

Now, let $(q, K) \in \pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$ and let $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$. The map

$$\mathcal{P} \times \mathcal{L} \to V : (x, y) \mapsto 2\sqrt{2}(\langle L, y\rangle - 1)x + 2(1 + \langle L, y\rangle)L + 2y - \{L, L, y\}$$

is constant on $\pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$ by Proposition 3.6 (iv). As above, we see by differentiating this map that

$$2\sqrt{2}(\langle L, K\rangle - 1)u + 2\sqrt{2}\langle L, v\rangle q + 2\langle L, v\rangle L + 2v - \{L, L, v\} = 0.$$

By forming the scalar product with u, we get

$$2\sqrt{2}(\langle L, K\rangle - 1)\langle u, u\rangle + 2\langle u, v\rangle - \langle u, \{L, L, v\}\rangle = 0$$

because of $\langle q, u\rangle = \langle L, u\rangle = 0$. Furthermore, as a consequence of $u \in \mathrm{T}_q\mathcal{P}_L$, we have $\{L, L, u\} = 0$, see Corollary 3.5. This implies that $\langle u, \{L, L, v\}\rangle = \langle\{L, L, u\}, v\rangle = 0$, and equation (ii) follows. □

Remark. As a consequence of Lemma 3.9 (i), we have $\mathrm{T}_K\mathcal{L}_p \cap \mathrm{T}_K\mathcal{L}_q = \{0\}$ for any two distinct line pencils $\mathcal{L}_p$ and $\mathcal{L}_q$ which intersect in a common line K, i.e. they intersect weakly transversally in the sense of Definition 1.6. Dually, any two distinct intersecting point rows $\mathcal{P}_L$ and $\mathcal{P}_K$ intersect weakly transversally. This may be seen as follows: let $u \in \mathrm{T}_q\mathcal{P}_L \cap \mathrm{T}_q\mathcal{P}_K$, where $\mathcal{P}_L \cap \mathcal{P}_K = \{q\}$. Since $u \in \mathrm{T}_q\mathcal{P}_L$ and $\pi_{\mathcal{P}}$ is a submersion, there is some $v \in \mathrm{T}_K\mathcal{L}$ such that $(u,v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$. Because of $u \in \mathrm{T}_q\mathcal{P}_K$ and $v \in \mathrm{T}_K\mathcal{L}$, we have $u \in V_0(K)$ and $v \in V_2(K)$, see Corollary 3.5. So we get $\langle u, v\rangle = 0$. By Lemma 3.9 (ii), we conclude that $u = 0$, since L and K are distinct.

The following theorem presents the main result of this chapter.

3.10 Theorem. *Incidence structures* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *which arise from (compact, connected) isoparametric hypersurfaces with four distinct principal curvatures in spheres are smooth generalized quadrangles.*

Proof. Let $(p, L) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ be an arbitrary antiflag. As mentioned in the preceding section, we only have to show that $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$. Choose $(q, K) \in \mathcal{F} \cap (\mathcal{P}_L \times \mathcal{L}_p)$ arbitrarily (if this intersection is non-empty). For reasons of dimension, it suffices to show that the intersection of $\mathrm{T}_{(q,K)}\mathcal{F}$ and $\mathrm{T}_{(q,K)}(\mathcal{P}_L \times \mathcal{L}_p)$ is trivial. So, let $(u, v) \in \mathrm{T}_{(q,K)}\mathcal{F} \cap (\mathrm{T}_q\mathcal{P}_L \times \mathrm{T}_K\mathcal{L}_p)$. We have $\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p) = \mathcal{F} \cap (\mathcal{P} \times \mathcal{L}_p)$, and since the projection $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ is a submersion, this intersection is transversal. By [6], Theorem 5.12, we get $\mathrm{T}_{(q,K)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p) = \mathrm{T}_{(q,K)}\mathcal{F} \cap (\mathrm{T}_q\mathcal{P} \times \mathrm{T}_K\mathcal{L}_p)$. In particular, we have $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$. The fact that $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{P}}^{-1}(\mathcal{P}_L)$ is proved analogously. In the following, we will consider u and v as elements of the same vector space V. We adopt this point of view, since in Proposition 3.6 we assumed $\mathcal{P}$ and $\mathcal{L}$ to be contained in the same euclidean space V, and the following equations for u and v are derived from identities given in that lemma. As in the proof of the preceding Lemma, we see by differentiating the map

$$\mathcal{P} \times \mathcal{L} \to V : (x, y) \mapsto 2\sqrt{2}(1 - \langle p, x\rangle)y + 2(2\langle p, x\rangle - 1)p + x - \{p, p, x\}$$

that

$$-2\sqrt{2}\langle p, u\rangle K + 4\langle p, u\rangle p + u - \{p, p, u\} + 2\sqrt{2}(1 - \langle p, q\rangle)v = 0$$

because of $(u, v) \in \mathrm{T}_{(q,K)}\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$. Recall that this map vanishes identically on $\pi_{\mathcal{L}}^{-1}(\mathcal{L}_p)$ by Proposition 3.6 (ii). In the sequel we will assume that u has norm 1. Then the scalar product with u yields

$$4\langle p, u\rangle^2 + 1 - \langle\{p, p, u\}, u\rangle + 2\sqrt{2}(1 - \langle p, q\rangle)\langle u, v\rangle = 0.$$

Here, we have used that $\langle K, u\rangle = 0$ by the remark after Corollary 3.5. By Lemma 3.9 (ii), we have $\langle u, v\rangle = \sqrt{2}(1 - \langle L, K\rangle)$. Because of Corollary 3.8 we obtain

$$4\langle p, u\rangle^2 + 1 - \langle\{p, p, u\}, u\rangle + 2(1 - \sqrt{2}\langle p, L\rangle) = 0. \tag{$\star$}$$

By identity (2) or [9], Theorem 3.11 (a), the vector u is a a minimal tripotent. In the sequel, we will use the Peirce decomposition of V relative to u. So, we may write $p = \langle p, u\rangle u + p_0 + p_2$ with $p_i \in V_i(u)$, $i = 0, 2$. Then we have $T(u)(p) = 6\langle p, u\rangle u + 2p_2$ and hence $\langle\{p, p, u\}, u\rangle = \langle p, \{p, u, u\}\rangle = 6\langle p, u\rangle^2 + 2\langle p_2, p_2\rangle$. By equation (1), we have $T(u)(L) = \{u, L, u\} = 0$, since $u \in V_0(L)$ by Corollary 3.5. Thus we get $L \in V_0(u)$ and $\langle p, L\rangle = \langle p_0, L\rangle$. Using these identities, we obtain from equation ($\star$) that

$$4\langle p, u\rangle^2 + 1 - (6\langle p, u\rangle^2 + 2\langle p_2, p_2\rangle) + 2(1 - \sqrt{2}\langle p_0, L\rangle) = 0.$$

Because of $\|p\|^2 = \langle p, u\rangle^2 + \langle p_0, p_0\rangle + \langle p_2, p_2\rangle = 1$, we get $1 + 2\langle p_0, p_0\rangle - 2\sqrt{2}\langle p_0, L\rangle = 0$. By the Cauchy-Schwarz inequality, we conclude that

$$0 = 1 - 2\sqrt{2}\langle p_0, L\rangle + 2\langle p_0, p_0\rangle \geq 1 - 2\sqrt{2}\,\|p_0\| + 2\|p_0\|^2 = (1 - \sqrt{2}\,\|p_0\|)^2,$$

which implies that $\|p_0\| = \frac{1}{\sqrt{2}}$ and $1 - 2\sqrt{2}\langle p_0, L\rangle + 1 = 0$. Hence, we get $\langle p, L\rangle = \langle p_0, L\rangle = \frac{1}{\sqrt{2}}$, which contradicts $(p, L) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$. This contradiction comes from our assumption that $\|u\| = 1$. Since the condition $(u, v) \in \mathrm{T}_{(q,K)}\mathcal{F} \cap (\mathrm{T}_q\mathcal{P}_L \times \mathrm{T}_K\mathcal{L}_p)$ is linear, we conclude that $u = 0$. By Lemma 3.9 (i), we then get $v = 0$ since $p \neq q$. So, we have shown that the intersection of $\mathrm{T}_{(q,K)}\mathcal{F}$ and $\mathrm{T}_{(q,K)}(\mathcal{P}_L \times \mathcal{L}_p)$ is trivial for arbitrary $(q, K) \in \mathcal{F} \cap (\mathcal{P}_L \times \mathcal{L}_p)$. This completes the proof. □

It can be shown that for these generalized quadrangles the map f_3 (see Chapter 1) is not only smooth, but real analytic and even Nash, i.e. these generalized quadrangles are real analytic and Nash quadrangles. For the definition of Nash functions and maps, see [1], 2.9.3 and 2.9.9, cf. also Section 8.1 in that book. The proof of this statement is based on an appropriate modification of Theorem 1.10, which may be stated as follows.

3.11 Theorem. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized quadrangle which satisfies the following conditions:*

(SGQ1) *There are positive integers a, b such that $\mathcal{P}$ is a real analytic (or Nash) manifold of dimension $2a + b$ and $\mathcal{L}$ is a real analytic (or Nash) manifold of dimension $a + 2b$.*

(SGQ2) *The flag space $\mathcal{F}$ is a $(2a+2b)$-dimensional real analytic (or Nash) submanifold of $\mathcal{P}\times\mathcal{L}$ such that the canonical projections $\pi_{\mathcal{P}} : \mathcal{F} \to \mathcal{P}$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L}$ are submersions.*

If moreover for each antiflag $(p, L) \in (\mathcal{P} \times \mathcal{L})\backslash\mathcal{F}$ the submanifolds $\mathcal{P}_L \times \mathcal{L}_p$ and $\mathcal{F}$ intersect transversally in $\mathcal{P} \times \mathcal{L}$, then $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a real analytic quadrangle (or Nash quadrangle, respectively).

Note that the transversality condition in this theorem makes sense, since condition (SGQ2) implies that point rows and line pencils are submanifolds of $\mathcal{P}$ and $\mathcal{L}$, respectively, compare Lemma 1.6. In order to prove this theorem, we only need to copy the proof of Theorem 1.10. This is possible without any alterations by using the implicit function theorem for real analytic maps or for Nash maps, respectively, see, e.g., [23], Theorem 1.8.3 and [1], Theorem 2.9.8.

It remains to check that the conditions of the above theorem are actually satisfied for generalized quadrangles $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ arising from isoparametric hypersurfaces with four distinct prinicipal curvatures in spheres. For this purpose, we explain first how proofs of real analytic or Nash versions of well-known theorems in elementary differential topology may be obtained.

By the proof of the constant rank theorem as given in [6], Theorem 5.4, we see that this theorem is a consequence of the inverse function theorem, which holds true also in the real analytic or Nash setting, see [23], Theorem 1.8.1 and [1], Theorem 2.9.7. So, we have an analogous statement for real analytic (or Nash) maps of constant rank between real analytic (or Nash) manifolds. In the following we concentrate on the Nash case for simplicity of formulation. As a direct consequence of the constant rank theorem, we obtain the following version of Theorem 5.7 in [6]: the image of an injective Nash immersion $f : M \to N$ which maps M homeomorphically onto $f(M) \subseteq N$ (where M and N are Nash manifolds) is a Nash submanifold of N. A further consequence is a Nash version of the standard result on preimages of regular values, see [6], Lemma 5.9.

Now we explain, how the conditions in the above theorem may be verified by means of these results from elementary differentiable topology. First, the isoparametric hypersurface $\mathcal{I} = F^{-1}(0) \cap \mathbb{S} = F|_{\mathbb{S}}^{-1}(0)$ is a Nash submanifold of V, because $\mathbb{S}$ is a Nash submanifold of V and 0 is a regular value of the polynomial map F restricted to $\mathbb{S}$, see [27], Theorem 3. Choose $x \in \mathcal{I}$ arbitrarily. Since F is a homogeneous polynomial of degree 4, we have $\langle \mathrm{grad} F(x), x \rangle = 4F(x) = 0$, where $\|\mathrm{grad} F(x)\| = 4$ by the first of Münzner's differential equations. As mentioned at the beginning of the preceding section, the images $\rho_{\mathcal{P}}(x) \in F^{-1}(1) \cap \mathbb{S}$ and $\rho_{\mathcal{L}}(x) \in F^{-1}(-1) \cap \mathbb{S}$

lie on a great circle normal to $\mathcal{I}$ and have distance $\frac{\pi}{8}$ from x. Hence, we get

$$\rho_{\mathcal{P}}(x) = \cos\frac{\pi}{8}\, x + \frac{1}{4}\sin\frac{\pi}{8}\,\mathrm{grad}\, F(x)$$

and

$$\rho_{\mathcal{L}}(x) = \cos\frac{\pi}{8}\, x - \frac{1}{4}\sin\frac{\pi}{8}\,\mathrm{grad}\, F(x).$$

Thus, the projections $\rho_{\mathcal{P}}$ and $\rho_{\mathcal{L}}$ are described by polynomial maps. Since they are submersions onto the focal manifolds, we conclude by the Nash version of the constant rank theorem that $\mathcal{P}$ and $\mathcal{L}$ are Nash submanifolds of V. Finally, by the embedding theorem presented above, we conclude that the image $\mathcal{F}$ of the Nash submanifold $\mathcal{I}$ under the even polynomial embedding $\rho_{\mathcal{P}} \times \rho_{\mathcal{L}} : \mathcal{I} \mapsto \mathcal{P} \times \mathcal{L}$ is a Nash submanifold of $\mathcal{P} \times \mathcal{L}$. So, we have proved the remaining conditions in the above theorem. Hence, generalized quadrangles derived from isoparametric hypersurfaces with four distinct prinicipal curvatures in spheres Nash quadrangles and, in particular, real analytic quadrangles.

Part II

Chapter 4

Characterizations of Smooth Projective Planes

Introduction

Compact projective planes play a prominent rôle in topological geometry. The underlying incidence structure is a projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$, where P denotes the set of points, $\mathcal{L}$ the set of lines, and $\mathcal{F} \subseteq P \times \mathcal{L}$ the flag space or incidence relation. Any two distinct points are joined by a unique line and, dually, any two distinct lines intersect in a unique point. A projective plane $\mathcal{P}$ is called a topological plane if P and $\mathcal{L}$ are topological spaces such that the join map and the intersection map are continuous. A topological projective plane $\mathcal{P}$ is said to be a *compact (connected) projective plane* if P (or equivalently $\mathcal{L}$) is a compact (connected) topological space. Although the continuity of the geometric operations is a natural postulate, it is sometimes inconvenient. A characterization of compact projective planes which replaces continuity by other conditions is given in [32], 43.1: a projective plane with compact topologies on P and $\mathcal{L}$ is a topological plane if and only if the flag space $\mathcal{F}$ is a closed subset of $P \times \mathcal{L}$.

From topological geometry it is only a small step to *smooth geometry.* For a *smooth projective plane* $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ we require the sets P and $\mathcal{L}$ to be smooth manifolds and the geometric operations to be smooth on their respective domains. *Real analytic* or *Nash projective planes* may be defined analogously, see the next chapter. In view of the flag space characterization of compact projective planes it is quite a natural question if there is an analogous characterization for smooth projective planes. We will answer this question in Corollary 4.4.

Implicit Characterizations of Smooth Geometries

In this section we want to show how mild differential-topological assumptions affect the geometry which is encoded in incidence structures. We will work with the following set of axioms:

Definition. An incidence structure $\mathfrak{I} = (P, \mathfrak{L}, \mathfrak{F})$ is called a *smooth generalized plane*, if there is a positive integer l such that the following two axioms are satisfied:

(SGP1) P and $\mathfrak{L}$ are $2l$-dimensional smooth manifolds.

(SGP2) the flag space $\mathfrak{F}$ is a $3l$-dimensional submanifold of $P \times \mathfrak{L}$, and the canonical projections $\pi_P : \mathfrak{F} \to P : (p, L) \mapsto p$ and $\pi_{\mathfrak{L}} : \mathfrak{F} \to \mathfrak{L} : (p, L) \mapsto L$ are submersions.

In Proposition 4.2 we will see that the notion of a smooth generalized plane is adequate. Note that axioms (SGP1) and (SGP2) are self-dual and hence smooth generalized planes satisfy the duality principle, i.e. every valid statement remains true when the rôles of P and $\mathfrak{L}$ are interchanged.

4.1 Lemma. *Let $\mathfrak{I} = (P, \mathfrak{L}, \mathfrak{F})$ be a smooth generalized plane such that the natural projections π_P and $\pi_{\mathfrak{L}}$ are surjective. Then every (non-empty) point row is an l-dimensional submanifold of P, and every (non-empty) line pencil is an l-dimensional submanifold of $\mathfrak{L}$.*

Proof. Because of our last remark, it suffices to prove the claim for line pencils only. So let $p \in P$. Then the inverse image $\pi_P^{-1}(p)$ is a smoothly embedded submanifold of $\mathfrak{F}$, since by definition π_P is a submersion. Moreover, $\dim \pi_P^{-1}(p) = \dim \mathfrak{F} - \dim P = 3l - 2l = l$. The map $\delta_p : \mathfrak{L} \to P \times \mathfrak{L} : L \mapsto (p, L)$ is smooth and we have $\delta_p \circ \pi_{\mathfrak{L}}|_{\pi_P^{-1}(p)} = \mathrm{id}_{\pi_P^{-1}(p)}$. This proves that $\pi_{\mathfrak{L}}|_{\pi_P^{-1}(p)} : \pi_P^{-1}(p) \to \mathfrak{L}$ is a smooth embedding and thus $\mathfrak{L}_p$ is a smoothly embedded l-dimensional submanifold of $\mathfrak{L}$. □

Definition. Two lines L_1 and L_2 of a smooth generalized plane are said to intersect *transversally* in some point p, if the associated point rows P_{L_1} and P_{L_2} intersect transversally in p as submanifolds of P, i.e. their tangent spaces in p span the tangent space T_pP, or, equivalently, the intersection of their tangent spaces in p is trivial. They are said to intersect transversally if they intersect transversally in each common point. Note that two lines which intersect transversally need not have a common point. Transversal intersection of line pencils is defined dually.

The transversal intersection of two lines in some point implies that the intersection map is "locally" well defined and smooth. In more detail, we have the following

4.2 Proposition. *Let $\mathfrak{I} = (P, \mathfrak{L}, \mathfrak{F})$ be a smooth generalized plane. Assume that the lines L_1, $L_2 \in \mathfrak{L}$ intersect transversally in $p \in P$. Then there are disjoint open neighborhoods U_i of L_i in $\mathfrak{L}$, $i = 1, 2$, and V of p in P such that any two lines $K_i \in U_i$, intersect in precisely one point $K_1 \wedge K_2 \in V$. Moreover, the intersection map $\wedge : U_1 \times U_2 \to V : (K_1, K_2) \mapsto K_1 \wedge K_2$ defined in this way is smooth.*

Proof. Since the flag space $\mathfrak{F}$ is a submanifold of $P \times \mathfrak{L}$, for $i = 1, 2$ there is an open neighborhood V_i of (p, L_i) in $P \times \mathfrak{L}$ as well as a submersion $\psi_i : V_i \to \mathbb{R}^l$, which vanishes exactly on the set $\mathfrak{F} \cap V_i$. We set

$$\psi : V_1 \times V_2 \to \mathbb{R}^l \times \mathbb{R}^l : (x_1, x_2) \mapsto (\psi_1(x_1), \psi_2(x_2))$$

$$\varphi : \mathfrak{L} \times \mathfrak{L} \times P \to (P \times \mathfrak{L})^2 : (K_1, K_2, q) \mapsto (q, K_1, q, K_2)$$

and for $W = \varphi^{-1}(V_1 \times V_2)$ we put

$$F : W \to \mathbb{R}^l \times \mathbb{R}^l : (K_1, K_2, q) \mapsto \psi \circ \varphi(K_1, K_2, q).$$

By definition of F we have $F(K_1, K_2, q) = 0$ if and only if q is a common point of K_1 and K_2. In order to prove the assertions of the proposition by using the implicit function theorem it suffices to check that the differential of the map

$$\{q \in P \mid (L_1, L_2, q) \in W\} \to \mathbb{R}^l \times \mathbb{R}^l : q \mapsto F(L_1, L_2, q)$$

is regular at p. So let $v \in \mathrm{T}_p P$ be in the kernel of this differential. The differentials $\mathrm{D}_{(p,L_1)}\psi_1$ and $\mathrm{D}_{(p,L_2)}\psi_2$ vanish exactly on $\mathrm{T}_{(p,L_1)}\mathfrak{F}$ and $\mathrm{T}_{(p,L_2)}\mathfrak{F}$, respectively. Using the chain rule and the definition of φ we thus get

$$(v, 0, v, 0) \in \mathrm{T}_{(p,L_1)}\mathfrak{F} \times \mathrm{T}_{(p,L_2)}\mathfrak{F}.$$

Since the projection $\pi_{\mathfrak{L}}$ is a submersion, the submanifolds $P \times \{L_1\}$ and $\mathfrak{F}$ intersect transversally in $P \times \mathfrak{L}$. Hence we have

$$(\mathrm{T}_p P \times \{0\}) \cap \mathrm{T}_{(p,L_1)}\mathfrak{F} = \mathrm{T}_p P_{L_1} \times \{0\}.$$

From $(v, 0) \in \mathrm{T}_{(p,L_1)}\mathfrak{F}$ we infer that $v \in \mathrm{T}_p P_{L_1}$, and analogously we get $v \in \mathrm{T}_p P_{L_2}$. Since the lines L_1 and L_2 intersect transversally in p, this implies that $v = 0$, and we have proved the proposition. □

A common notion in topological geometry is that of a stable plane, see [25] and [26] for details. Before we are going to relate Proposition 4.2 to stable planes we need two more definitions.

Definition. An incidence structure $(P, \mathfrak{L}, \mathfrak{F})$ is called a *linear space*, if each two distinct points $p, q \in P$ can be joined by exactly one line L, i.e. $(p, L), (q, L) \in \mathfrak{F}$, and every line is incident with at least one point.

Definition. A *smooth stable plane* $\mathcal{S}$ is a linear space $(P, \mathcal{L}, \mathcal{F})$ which satisfies the following axioms:

(SSP1) The domain $\mathcal{O}$ of the intersection map is an open subset of $\mathcal{L} \times \mathcal{L}$ (*stability axiom*). P and $\mathcal{L}$ are smooth manifolds such that the join map $\vee$ and the intersection map $\wedge$ are smooth.

(SSP2) Every line is incident with at least 3 points and dually.

4.3 Corollary. *Let $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ be a smooth generalized plane as well as a linear space. Assume that any two distinct lines and any two distinct line pencils intersect transversally. Then $\mathcal{I}$ is a smooth stable plane.*

Proof. By the preceding proposition, the maximal domain of the intersection map is an open subset of $\mathcal{L} \times \mathcal{L}$ and both the join map and the intersection map are smooth. (Remember that the dual statement of Proposition 4.2 is also true.) The validity of axiom (SSP2) is a direct consequence of Lemma 4.1 and (SGP1). □

If $\mathcal{I}$ is a stable plane, then the dimension assumptions in axioms (SGP1) and (SGP2) are automatically satisfied and the integer l is one of the numbers 1, 2, 4, or 8, see [25]. By [3], a smooth stable plane is a smooth generalized plane, and any two distinct lines (line pencils) intersect transversally. Thus, Corollary 4.3 yields a characterization of smooth stable planes. For the particularly interesting case of smooth projective planes we formulate this characterization as another corollary.

4.4 Corollary. *Let $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ be a projective plane. Then the following statements are equivalent:*

(i) *$\mathcal{P}$ is a smooth projective plane.*

(ii) *$\mathcal{P}$ is a smooth generalized plane such that any two distinct lines and any two distinct line pencils intersect transversally.*

This result may considerably facilitate the verification that a given projective plane is smooth. For example, the proof that the classical projective planes $\mathcal{P}_2\mathbb{F}$ over the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions or Caley numbers) are smooth is now immediate.

The assumptions on the flag space contained in Corollary 4.4 (ii) are indispensable. There are examples of non-smooth projective planes whose point rows (and line pencils) are submanifolds, which intersect pairwise transversally in P (in $\mathfrak{L}$), such that (SGP2) is not satisfied. Moreover, these planes cannot be turned into smooth projective planes by changing the smooth structures on P and $\mathfrak{L}$, see [15].

We proceed with another characterization of smooth projective planes, which does not start with an abstract projective plane but which uses additional assumptions on the topology of the plane instead. The next theorem is the key to this characterization.

4.5 Theorem. *Let $\mathcal{I} = (P, \mathfrak{L}, \mathcal{F})$ be a smooth generalized plane such that any two distinct lines and any two line pencils associated with distinct points intersect transversally. Assume that P and $\mathfrak{L}$ are compact and connected and that $\mathcal{F}$ is closed in $P \times \mathfrak{L}$. Then there are positive integers m, n such that any two distinct points are joined by exactly m lines and any two distinct lines intersect in exactly n points. Furthermore, any two points rows (line pencils) are diffeomorphic.*

Proof. The natural projections π_P and $\pi_{\mathfrak{L}}$ are surjective since $\mathcal{F}$ is compact, submersions are open maps, and P, $\mathfrak{L}$ are connected. Thus point rows and line pencils are l-dimensional manifolds, and there are two distinct intersecting lines L_1, L_2. By Proposition 4.2, the set of intersection points is discrete and also compact, because the point rows P_{L_1}, P_{L_2} are compact. Thus, we have a finite set $\{p_1, \ldots, p_n\}$ of intersection points. By Proposition 4.2 there exist disjoint neighborhoods U_1 of L_1 and U_2 of L_2 in $\mathfrak{L}$ and pairwise disjoint open neighborhoods $V_1, \dots, V_n$ of $p_1, \dots, p_n$ such that any two lines $K_1 \in U_1$ and $K_2 \in U_2$ intersect in a unique point of V_i, $i = 1, \dots, n$. By using the compactness of P we may pass to smaller neighborhoods such that any two lines $K_1 \in U_1$, $K_2 \in U_2$ intersect in exactly n points. Hence, each of the sets $\mathcal{O}_k = \{(K_1, K_2) \in \mathcal{L} \times \mathcal{L} \mid |P_{L_1} \cap P_{L_2}| = k\}$, $k = 1, 2, \dots$ is open in $\mathcal{L} \times \mathcal{L}$. Obviously, the set $\mathcal{O}_0 = \{(K_1, K_2) \in \mathcal{L} \times \mathcal{L} \mid P_{L_1} \cap P_{L_2} = \emptyset\}$ is open, too. The connected set $\{(K_1, K_2) \in \mathcal{L} \times \mathcal{L} \mid K_1 \neq K_2\}$ is covered by the pairwise disjoint open sets $\mathcal{O}_k$ $(k = 0, 1, 2, \dots)$. We conclude that only one of the sets $\mathcal{O}_k$ is non-empty, namely $\mathcal{O}_n$. This proves that any two distinct lines intersect in precisely n points. By duality, there is a positive integer m such that any two distinct line pencils intersect in exactly m lines. Equivalently, any two distinct points are joined by exactly m lines. In order to prove the last assertion, we use the fact that the projection $\pi_P : \mathcal{F} \to P$ is a smooth locally trivial fibration (by the fibration theorem of Ehresmann, see [6], 8.12). Since $\pi_P^{-1}(p) = \{p\} \times \mathcal{L}_p$ for $p \in P$, we infer that any two line pencils are diffeomorphic. Analogously, any two point rows are diffeomorphic. $\square$

Remark. It would be interesting to know whether $n, m \neq 1$ can actually occur.

4.6 Corollary. *Assume that $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ satisfies the conditions of Theorem 4.5. If there are two lines whose intersection consists of at most one point, or if there are two points which are joined by at most one line, then $\mathcal{I}$ is a smooth projective plane.*

This corollary will be used in the next chapter in order to construct the first examples of non-classical smooth projective planes with large automorphism groups.

Chapter 5
Examples of Smooth Projective Planes

Introduction

In [33], B. Segre constructed examples of non-desarguesian smooth projective planes, whose lines are real algebraic curves in the real projective plane with its usual real algebraic structure. The construction of these planes was motivated by a prize-question posed by *Het Wiskundig Genootschap* in 1955. However, as mentioned in [32], 75.6, he did 'not consider the question whether the planes are, for example, real analytic or algebraic planes, that is, whether the geometric operations belong to one of these categories'. In this chapter we show that the geometric operations of joining points and intersecting lines are in fact real analytic and even Nash. Furthermore, we present the first examples of projective planes with these properties in dimensions 4 and 8. These results are obtained by real analytic or Nash versions of Corollaries 4.4 and 4.6, respectively. In fact, our approach also yields a new proof for Segre's result that the incidence structures constructed by him are projective planes. Note in this context that finite-dimensional, compact, connected projective planes in general have dimension 2, 4, 8 or 16, cf. [32], 52.5. It should be possible to prove an analogous result in the 16-dimensional setting by using Veronese coordinates instead of homogeneous coordinates (see [32], 16.1). Homogeneous coordinates cannot be used in this case because of the non-associativity of the octonions.

The projective planes considered in this chapter are constructed as follows: the point space P and the line space $\mathcal{L}$ are copies of the point space and the line space of $\mathcal{P}_2\mathbb{K}$ with their standard smooth, real analytic, and real algebraic structure ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). Hence, points and lines may be described by means of homogeneous coordinates in the usual way. A point $(x, y, z)^t \in P$ (where t denotes transposition) and a line $(a, b, c) \in \mathcal{L}$ are called *incident* if

$$(|a|^2 + |b|^2 + |c|^2)(ax + by + cz)(|x|^2 + |y|^2 + |z|^2) + \lambda |c|^2 cz|z|^2 = 0.$$

Here, $\lambda \in \mathbb{R}$ is a fixed parameter. For $\lambda = 0$ we get the incidence relation of the classical projective plane $\mathcal{P}_2\mathbb{K}$. The flag space $\mathcal{F}_\lambda$ is the set of incident point-line-pairs. The incidence structures $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ defined in this way are self-dual. A polarity is given by the map $P \times \mathcal{L} \to P \times \mathcal{L}$: $((x, y, z)^t, (a, b, c)) \mapsto ((\overline{a}, \overline{b}, \overline{c})^t), (\overline{x}, \overline{y}, \overline{z}))$, where "$-$" denotes conjugation. Of course, the incidence structures $\mathcal{P}_\lambda$ cannot be expected to be projective planes in general. In this chapter we prove that they are real analytic and even Nash projective planes for $|\lambda|$ sufficiently small. To be more precise, our proof yields that $|\lambda| < \frac{1}{9}$ is sufficient. In [33], Segre proves in two different ways that the planes $\mathcal{P}_\lambda$ are non-desarguesian for $\lambda \neq 0$ and $\mathbb{K} = \mathbb{R}$. In Section 1 (pp. 36/37) he shows this by a theoretical argument, and in Section 4 (pp. 39/40) he verifies directly that Desargues' theorem fails in $\mathcal{P}_\lambda$ for $\lambda \neq 0$ sufficiently small. A projective plane $\mathcal{P}_\lambda$ with $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$ has a 2-dimensional subplane equal to the projective plane constructed by Segre with the same parameter λ and hence is not desarguesian for $\lambda \neq 0$.

The projective planes presented in this chapter are the first examples of non-classical smooth projective planes with large automorphism groups and the first examples of non-classical real analytic projective planes. Note that by [32], Theorem 75.1, or by [20], every holomorphic projective plane is isomorphic to $\mathcal{P}_2\mathbb{C}$ with its usual holomorphic structure, and by [34] or [21], every algebraic projective plane over an algebraically closed field is Pappian.

Before we proceed, let us first recall some basic results on automorphisms of compact or smooth projective planes. The *automorphism group* Σ of a compact (smooth) projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ is the group of all automorphisms of $\mathcal{P}$ as an incidence structure which induce homeomorphisms (diffeomorphisms) on P and $\mathcal{L}$. These automorphisms are called continuous automorphisms (smooth automorphisms). Note that by [5], 4.7, a continuous automorphism of a smooth projective plane is smooth. The automorphism group Σ of $\mathcal{P}$ is endowed with the compact-open topology derived from its action on P or $\mathcal{L}$, respectively. These two topologies coincide by [32], 44.2. In this way, Σ becomes a locally compact topological group with a countable basis, see [32], 44.3. Hence it makes sense to define the dimension of Σ, compare [32], 93.5 and 6. By a group of automorphisms of $\mathcal{P}$ we mean a subgroup of Σ endowed with the induced topology.

By [2], the dimension of the automorphism group of a 2l-dimensional non-classical smooth projective plane is at most 2, 6 or 16 for $l = 1$, 2 or 4, respectively. These bounds are lower by 2 then the corresponding bounds in the case of compact projective planes, but it is not known if they are sharp. The projective planes $\mathcal{P}_\lambda$ admit Lie groups of smooth automorphisms of dimension 1, 4 or 13 for $l = 1$, 2 or 4, respectively. In particular, our examples show that the bounds found by Bödi are not far from the truth. The Lie groups of smooth automorphisms of the projective planes $\mathcal{P}_\lambda$ mentioned above are in fact *compact* groups. This shows that, in contrast to the automorphism groups of smooth projective planes, the bounds for the dimensions of compact groups of automorphisms of non-classical compact projective planes are the same as those in the smooth setting for $l \in \{1, 2, 4\}$, see Theorem 5.9.

There are some further interesting results on smooth projective planes. In [30], Otte proves that there are no smooth projective translation planes besides the classical ones. The situation is different for differentiable *affine* planes: in [29], Otte constructs examples of non-classical smooth affine translation planes. In contrast, the only smooth affine planes of Lenz type V are the classical planes, see [11].

An independent solution of the prize-question mentioned above was presented by N. H. Kuiper, see [24]. He also constructed a non-desarguesian projective plane whose point space is the real projective plane with its ordinary real algebraic structure. The point rows of this plane, however, are in general only semi-algebraic and not algebraic. The polynomial equations by which the lines in general are defined describe real algebraic curves with one isolated point, which is not part of the point rows. But after deleting a suitable "line at infinity" one obtains in fact an affine plane with real algebraic point rows. In the point set $\mathbb{R}^2$ with coordinates (x, y), the lines of these affine planes are the horizontal lines $\mathbb{R} \times \{-\gamma\}$ and the curves described by equations of the form

$$x = -\mu y - \gamma - \rho\Big(\frac{\mu}{\mu^2+1}\Big)\Big(\frac{y^2}{y^2+1}\Big),$$

where $\mu, \gamma \in \mathbb{R}$ and $\rho = 0,01$.

The projective plane constructed by Kuiper is similar to the planes constructed as projective closures of *smooth affine generalized shift planes* in [15]. Except for the classical plane of the corresponding dimension, it turned out that these projective planes are not smooth projective planes with respect to arbitrary differentiable structures on the point space and the line space. Motivated by this result, we conjecture that also the projective plane found by Kuiper is not a smooth projective plane.

Proofs and Details

We first want to show that the incidence structures $\mathcal{P}_\lambda$ in general ($\lambda \in \mathbb{R}$ arbitrary) admit non-trivial groups of smooth automorphisms which are compact Lie groups.

5.1 Lemma. *Let $\mathbb{K} = \mathbb{R}$. Then $O_2\mathbb{R}$ acts on the incidence structure $\mathcal{P}_\lambda$ as a group of smooth automorphisms.*

Proof. Let Γ be the subgroup of $O_3\mathbb{R}$ which fixes $(0,0,1) \in \mathbb{R}^3$. This subgroup is isomorphic to $O_2\mathbb{R}$. The standard action of Γ on $\mathbb{R}^3$ induces an effective smooth action of Γ on the point space $\mathcal{L}$. Analogously, we define an effective smooth action of Γ on P by $\Gamma \times P \to P : (\gamma, (a,b,c)^t) \mapsto \gamma^{-1}(a,b,c)^t$. By definition of the incidence relation in $\mathcal{P}_\lambda$ we see that the induced action of Γ on $P \times \mathcal{L}$ leaves $\mathcal{F}_\lambda$ invariant, i.e. Γ acts on $\mathcal{P}_\lambda$ as a group of smooth automorphisms. $\square$

5.2 Lemma. *Let $\mathbb{K} = \mathbb{C}$. Then $\mathcal{P}_\lambda$ admits a group of smooth automorphisms isomorphic to the unitary group $U_2\mathbb{C}$. Also complex conjugation induces a smooth automorphism of $\mathcal{P}_\lambda$.*

Proof. Let Γ be the subgroup of $U_3\mathbb{C}$ which fixes $(0,0,1) \in \mathbb{C}^3$ with its usual unitary structure. As in the preceding Lemma we see that Γ acts on $\mathcal{P}_\lambda$ as a group of smooth automorphisms. The second statement is also a direct consequence of the definition of the incidence relation in $\mathcal{P}_\lambda$. $\square$

5.3 Lemma. *Let $\mathbb{K} = \mathbb{H}$. Then $\mathcal{P}_\lambda$ admits a group of smooth automorphisms isomorphic to the product of $\mathrm{Spin}_3\mathbb{R}$ and $\mathrm{Spin}_5\mathbb{R}$ with amalgamated centers.*

Proof. Let Γ be the subgroup of the unitary group $U_3\mathbb{H}$ isomorphic to $U_2\mathbb{H} \times U_1\mathbb{H}$, which acts on the first two components of $(x,y,z) \in \mathbb{H}^3$ as $U_2\mathbb{H}$ and on the last component as $U_1\mathbb{H}$. Note that $U_1\mathbb{H}$ is isomorphic to $\mathrm{Spin}_3\mathbb{R}$ and that $U_2\mathbb{H}$ is isomorphic to $\mathrm{Spin}_5\mathbb{R}$, cf. [32], p. 624. The action of Γ on $\mathbb{H}^3$ induces an effective smooth action of the product of $U_2\mathbb{H}$ and $U_1\mathbb{H}$ with amalgamated centers on the line space $\mathcal{L}$. As in the two preceding lemmas we see that this group acts on $\mathcal{P}_\lambda$ by smooth automorphisms. $\square$

By means of these three lemmas we will be able to choose appropriate coordinates in the proof of the main result of this section.

5.4 Theorem. *The incidence structures $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ are smooth projective planes for $|\lambda| < \frac{1}{9}$.*

The next lemma presents the most difficult part of the proof of this theorem. In the sequel, we will use the description of the point space P and the line space $\mathcal{L}$ of the incidence structure $\mathcal{P}_\lambda = (P, \mathcal{L}, \mathcal{F}_\lambda)$ by means of the standard charts: for the point space P, the corresponding open sets U_1, U_2, and U_3 are given by $x \neq 0$, $y \neq 0$, and $z \neq 0$, respectively, and these sets are identified with $\mathbb{K}^2$ in the usual way. In the latter case, for example, we use the map $U_3 \to \mathbb{K}^2 : (x, y, z)^t \mapsto (x/z, y/z)$. Analogously we define open sets V_1, V_2, and V_3 by $a \neq 0$, $b \neq 0$, and $c \neq 0$, respectively, which cover the line space $\mathcal{L}$. Sometimes it will be convenient to identify $\mathbb{K}$ with $\mathbb{R}^l$ by choosing $\{1\}$, $\{1, i\}$ or $\{1, i, j, k\}$, respectively, as a basis of $\mathbb{K}$ over $\mathbb{R}$. In this way, left multiplication by some element $c \in \mathbb{K}$ gives rise to a linear map $L_c : \mathbb{R}^l \to \mathbb{R}^l$, and right multiplication by c induces a linear map $R_c : \mathbb{R}^l \to \mathbb{R}^l$.

In order to avoid a too cumbersome notation we will sometimes use the same names for different variables in the following two proofs, if such a choice is natural, facilitates reading, and no confusion is possible.

5.5 Lemma. *Let $|\lambda| < \frac{1}{9}$. Then the set $\mathcal{F}^\circ = \mathcal{F} \cap (U_3 \times V_3)$ is a smooth $3l$-dimensional submanifold of $\mathcal{P} \times \mathcal{L}$. The restrictions of the natural projections π_P and $\pi_\mathcal{L}$ to $\mathcal{F}^\circ$ are submersions. The sets $P_L \cap U_3$ and $\mathcal{L}_p \cap V_3$ with $p \in \pi_P(\mathcal{F}^\circ)$ and $L \in \pi_\mathcal{L}(\mathcal{F}^\circ)$ are smooth l-dimensional submanifolds of P and $\mathcal{L}$, respectively. If two distinct lines $L, L' \in V_3$ intersect in a point $p \in U_3$ then the submanifolds $P_L \cap U_3$ and $P_{L'} \cap U_3$ intersect transversally in p. Also the dual statement holds.*

Proof. We identify the open subsets $U_3 \subseteq P$ and $V_3 \subseteq \mathcal{L}$ with two distinct copies of $\mathbb{K}^2$. By means of these identifications, the set $\mathcal{F}^\circ$ corresponds to

$$\big\{(x, y, a, b) \in \mathbb{K}^2 \times \mathbb{K}^2 \;\big|\; (|a|^2 + |b|^2 + 1)(ax + by + 1)(|x|^2 + |y|^2 + 1) + \lambda = 0\big\}.$$

For any $(a,b) \in V_3$ we define

$$g_{(a,b)} : \mathbb{K}^2 \to \mathbb{K} : (x,y) \mapsto (|a|^2 + |b|^2 + 1)(ax + by + 1)(|x|^2 + |y|^2 + 1) + \lambda.$$

We want to prove the technical result that the kernels of the differentials $\mathrm{D}_{(x,y)} g_{(a,b)}$ and $\mathrm{D}_{(x,y)} g_{(a',b')}$ have trivial intersection for any two distinct quadruples $(x,y,a,b), (x,y,a',b') \in \mathcal{F}^\circ$. By using transitivity properties of the group of smooth automorphisms of $\mathcal{P}_\lambda$ (see the preceding lemmas) we may assume that $y = 0$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. The above incidence relation then shows that $ax \in \mathbb{R}\backslash\{0\}$ (because of $|\lambda| < 1$) and hence that $a \in \mathbb{R}$. Analogously we see that $a' \in \mathbb{R}$. For simplicity of formulation we will assume in the following that $\mathbb{K} = \mathbb{H}$. Sometimes we will identify $\mathbb{H}$ with $\mathbb{R}^4$ and associate to any element $w \in \mathbb{H}$ a vector $(w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. In this way, the differential of the map $\vartheta : \mathbb{H} \to \mathbb{H} : t \mapsto |t|^2$ at some point $t \in \mathbb{H}$ corresponds to the map $\mathrm{D}_t\vartheta : \mathbb{R}^4 \to \mathbb{R}^4 : (w_1, w_2, w_3, w_4) \mapsto (2(w_1t_1 + w_2t_2 + w_3t_3 + w_4t_4), 0, 0, 0)$. Now let $(u,v) \in \ker \mathrm{D}_{(x,0)} g_{(a,b)} \cap \ker \mathrm{D}_{(x,0)} g_{(a',b')}$ and assume that $(u,v) \neq (0,0)$. By differentiating $g_{(a,b)}$ at $(x,0)$ we get

$$\begin{aligned} &(R_{x^2+1}L_a + L_{ax+1}\mathrm{D}_x\vartheta)u + (R_{x^2+1}L_b + L_{ax+1}\mathrm{D}_0\vartheta)v \\ &= (x^2+1)au + (ax+1)2xu_1 + (x^2+1)bv = 0. \end{aligned} \tag{1}$$

Here we have considered u and v as elements of $\mathbb{R}^4$ in the first line and as elements of $\mathbb{H}$ in the second line. Analogously, we get

$$(x^2+1)a'u + (a'x+1)2xu_1 + (x^2+1)b'v = 0. \tag{2}$$

We multiply equation (1) by b' from the left and equation (2) by b. Subtracting the two equations obtained in this way yields

$$(ab' - a'b)(x^2+1)u + ((ax+1)b' - (a'x+1)b)2xu_1 = 0$$

and hence

$$(ab' - a'b)((x^2+1)u + 2x^2u_1) + (b' - b)2xu_1 = 0. \tag{3}$$

As a next step we want to prove that $b \neq b'$. If we have $b' = b \in \mathbb{R}$, then a and a' are zeros of the real polynomial function $p : \mathbb{R} \to \mathbb{R} : s \mapsto (s^2 + b^2 + 1)(sx + 1)(x^2 + 1) + \lambda$. Let $s \in \mathbb{R}$ with $p'(s) = 0$ (if it exists). Then we have $2s(sx + 1) + (s^2 + b^2 + 1)x = 0$ which implies that

$$sx + 1 = 1 - \frac{2s^2}{3s^2 + b^2 + 1} > \frac{1}{3}.$$

Hence, we get $p(s) = (s^2+b^2+1)(sx+1)(x^2+1)+\lambda > \frac{1}{3}+\lambda > 0$ because of $|\lambda| < \frac{1}{9}$. Since p ist a real polynomial function of degree 3, this shows that p has precisely one real zero. We conclude that $a = a'$, a contradiction. So, we have $b \neq b'$ and therefore also $u \neq 0$ by equations (1) and (2). Equation (3) then yields

$$(b' - b)^{-1}(ab' - a'b) = -2xu_1((x^2 + 1)u + 2x^2u_1)^{-1} \tag{4}$$

and

$$\begin{aligned} \frac{|ab' - a'b|^2}{|b' - b|^2} &= \frac{(2x)^2 u_1^2}{(3x^2 + 1)^2 u_1^2 + (x^2 + 1)^2 (u_2^2 + u_3^2 + u_4^2)} \\ &\leq \frac{(2x)^2}{(x^2 + 1)^2} \; \frac{u_1^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2} \\ &\leq \left(\frac{2x}{x^2 + 1}\right)^2 \leq 1. \end{aligned}$$

Thus we get

$$|(b' - b)^{-1}(ab' - a'b)| \leq 1. \tag{5}$$

On the other hand, equation (4) implies that

$$\begin{aligned} (b' - b)^{-1}(ab' - a'b)x + 1 &= 1 - 2x^2u_1((x^2 + 1)u + 2x^2u_1)^{-1} \\ &= (x^2 + 1)u((x^2 + 1)u + 2x^2u_1)^{-1}. \end{aligned}$$

We conclude that

$$\begin{aligned} |(b' - b)^{-1}(ab' - a'b)x + 1|^2 &= \frac{(x^2 + 1)^2 |u|^2}{(3x^2 + 1)^2 u_1^2 + (x^2 + 1)^2 (u_2^2 + u_3^2 + u_4^2)} \\ &\geq \frac{(x^2 + 1)^2}{(3x^2 + 1)^2} \; \frac{|u|^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2}. \end{aligned}$$

Because of $\left(\frac{x^2+1}{3x^2+1}\right) > \frac{1}{3}$, we get the inequality

$$|(b'-b)^{-1}(ab'-a'b)x+1| > \frac{1}{3}. \tag{6}$$

Since $(x,0,a,b) \in \mathcal{F}^\circ$ implies $(a^2+b^2+1)(ax+1)(x^2+1)+\lambda = 0$, we have $ax+1 = -\lambda(a^2+b^2+1)^{-1}(x^2+1)^{-1}$ and, analogously, $a'x+1 = -\lambda(|a'|^2+|b'|^2+1)^{-1}(x^2+1)^{-1}$. We multiply the first of these two equations by b' and the second by b. After subtracting these two equations we obtain

$$(ab'-a'b)x+(b'-b) = -\frac{\lambda}{x^2+1}\frac{(|a'|^2+|b'|^2+1)b'-(a^2+b^2+1)b}{(a^2+b^2+1)(|a'|^2+|b'|^2+1)}. \tag{7}$$

We have

$$(|a'|^2+|b'|^2+1)b'-(a^2+b^2+1)b = (|a'|^2b'-a^2b)+(|b'|^2b'-b^3)+(b'-b),$$

where $|a'|^2b'-a^2b = (b'-b)(a^2+aa'+|a'|^2)-(ab'-a'b)(a+a')$ because of $a' \in \mathbb{R}$, and $|b'|^2b'-b^3 = (b'-b)(|b'|^2+\overline{b'}b+b^2)-(b'-\overline{b'})b^2$ with

$$\frac{|b'-\overline{b'}|^2}{|b'-b|^2} = \frac{4(b_2'^2+b_3'^2+b_4'^2)}{(b_1'-b)^2+b_2'^2+b_3'^2+b_4'^2} \le 4.$$

Hence, we get

$$\begin{aligned} |(b'-b)^{-1}(|b'|^2b'-b^3)| &\le |b'|^2+|b'b|+b^2+|(b'-b)^{-1}(b'-\overline{b'})|\,b^2 \\ &\le |b'|^2+|b'b|+3b^2 \end{aligned}$$

and

$$\begin{aligned} &|(b'-b)^{-1}(|a'|^2b'-a^2b)| \\ &\le a^2+|aa'|+|a'|^2+|(b'-b)^{-1}(ab'-a'b)|(|a|+|a'|) \\ &\le a^2+|aa'|+|a'|^2+|a|+|a'| \end{aligned}$$

by equation (5). By putting together these inequalities with (6) and (7), we obtain

$$\begin{aligned} \frac{1}{3} &< |(b'-b)^{-1}(ab'-a'b)x+1| \\ &\leq |\lambda| \frac{a^2+|aa'|+|a'|^2+|a|+|a'|+|b'|^2+|b'b|+3b^2+1}{(a^2+b^2+1)(|a'|^2+|b'|^2+1)}. \end{aligned} \tag{8}$$

Obviously, we have

$$\frac{a^2+|a'|^2+|b'|^2+b^2+1}{(a^2+b^2+1)(|a'|^2+|b'|^2+1)} \leq 1.$$

Because of $s(1-s) \leq \frac{1}{4}$, i.e. $s \leq s^2 + \frac{1}{4}$ for $s \in \mathbb{R}$, we get

$$\begin{aligned} |a|+|a'|+|aa'|+|bb'|+b^2 &\leq a^2+|a'|^2+|aa'|^2+|b'b|^2+1+b^2 \\ &\leq (a^2+b^2+1)(|a'|^2+|b'|^2+1). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &(a^2+|a'|^2+|b'|^2+b^2+1)+(|a|+|a'|+|aa'|+|b'b|+b^2)+b^2 \\ &\leq 3(a^2+b^2+1)(|a'|^2+|b'|^2+1), \end{aligned}$$

which shows together with inequality (8) that $\frac{1}{3} < 3|\lambda|$, in contradiction to $|\lambda| < \frac{1}{9}$. Thus the kernels of the differentials $\mathrm{D}_{(x,y)}g_{(a,b)}$ and $\mathrm{D}_{(x,y)}g_{(a',b')}$ intersect trivially for any two distinct quadruples $(x,y,a,b),(x,y,a',b') \in \mathcal{F}^\circ$.

We want to show next that there are infinitely many lines in V_3 through any point $(x,y) \in \pi_P(\mathcal{F}^\circ)$. By means of the transitivity properties of the automorphism group of $\mathcal{P}_\lambda$ we may assume again that $(x,y) = (x,0)$ with $x \in \mathbb{R}$. Because of $(x,0) \in \pi_P(\mathcal{F}^\circ)$ there is a line $(a',b') \in V_3$ incident with the point $(x,0)$. We then have $(|a'|^2+|b'|^2+1)(a'x+1)(x^2+1)+\lambda = 0$, which shows that $x \neq 0$. Hence the real polynomial function $q_b : \mathbb{R} \to \mathbb{R} : s \mapsto (s^2+b^2+1)(sx+1)(x^2+1)+\lambda$ has degree 3 for every $b \in \mathbb{R}$. Thus, for any $b \in \mathbb{R}$ there exists $a \in \mathbb{R}$ such that $(a^2+b^2+1)(ax+1)(x^2+1)+\lambda = 0$, i.e. such that $(x, 0, a, b) \in \mathcal{F}^\circ$.

By (x, y) we denote again an arbitrary point of $\pi_P(\mathcal{F}^\circ)$. Choose two distinct lines $(a, b), (a', b') \in V_3$ through (x, y). By definition of $g_{(a,b)}$ and $g_{(a',b')}$, the dimensions of the kernels of the two differentials $\mathrm{D}_{(x,y)}g_{(a,b)}$ and $\mathrm{D}_{(x,y)}g_{(a',b')}$ are at least l. Since they intersect trivially, their dimension is precisely l and hence these differentials are surjective. In particular, also the total differential of the map

$$f : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} : (x, y, a, b) \mapsto (|a|^2+|b|^2+1)(ax+by+1)(|x|^2+|y|^2+1)+\lambda$$

is surjective at every point of $\mathcal{F}^\circ$. Therefore $\mathcal{F}^\circ$ is a $3l$-dimensional submanifold of $U_3 \times V_3$ and hence of $P \times \mathfrak{L}$.

Now we want to show that the restriction of the natural projection $\pi_\mathfrak{L}$ to $\mathcal{F}^\circ$ is a submersion. Choose $(x, y, a, b) \in \mathcal{F}^\circ$ arbitrarily. An element of $\ker \mathrm{D}_{(x,y,a,b)}\pi_\mathfrak{L}$ has the form $(u, v, 0, 0)$ with $(u, v) \in \mathbb{K}^2$. By definition of $\mathcal{F}^\circ$ we have $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = \mathrm{D}_{(x,y,a,b)}f(u, v, 0, 0) = 0$. Since the kernel of $\mathrm{D}_{(x,y)}g_{(a,b)}$ is l-dimensional, we conclude that the dimension of $\ker \mathrm{D}_{(x,y,a,b)}\pi_\mathfrak{L}$ is at most l. Thus the differential $\mathrm{D}_{(x,y,a,b)}\pi_\mathfrak{L}$ is surjective. Hence the restriction of $\pi_\mathfrak{L}$ to $\mathcal{F}^\circ$ and, for reasons of symmetry, also the restriction of π_P to $\mathcal{F}^\circ$ are submersions. By Lemma 4.1 it follows that the sets $P_L \cap U_3$ and $\mathfrak{L}_p \cap V_3$ are smooth l-dimensional submanifolds of P and $\mathfrak{L}$, respectively, for any $p \in \pi_P(\mathcal{F}^\circ)$, $L \in \pi_\mathfrak{L}(\mathcal{F}^\circ)$.

It remains to show that any two distinct lines $L = (a, b)$ and $L' = (a', b')$ in V_3 which intersect in a point $p = (x, y) \in U_3$ intersect transversally in p. The dual statement then follows for reasons of symmetry. Choose (u, v) in the intersection of the tangent spaces of the point rows P_L and $P_{L'}$ in p. Since $g_{(a,b)}$ vanishes on $P_L \cap U_3$, we conclude that $\mathrm{D}_{(x,y)}g_{(a,b)}(u, v) = 0$. Analogously, we get $\mathrm{D}_{(x,y)}g_{(a',b')}(u, v) = 0$ and hence $(u, v) = (0, 0)$, since the kernels of $\mathrm{D}_{(x,y)}g_{(a,b)}(u,v) = 0$ and $\mathrm{D}_{(x,y)}g_{(a,b)}(u,v) = 0$ have trivial intersection. This completes the proof. $\square$

Proof of Theorem 5.4. As in the classical projective plane $\mathcal{P}_0 = \mathcal{P}_2\mathbb{K}$, the point rows of the lines $(1,0,0), (0,1,0) \in \mathcal{L}$ intersect precisely in the point $(0,0,1)^t \in P$. Hence, by Corollary 4.6, it suffices to verify the conditions of Theorem 4.5. We first show that the flag space $\mathcal{F}_\lambda$ is a $3l$-dimensional submanifold of $P \times \mathcal{L}$ such that $\pi_{\mathcal{L}}$ is a submersion. Then also the natural projection π_P is a submersion for reasons of symmetry. By the previous lemma, it remains to prove these properties in neighbourhoods of flags (p, L) in $P \times \mathcal{L}$, where the last coordinate of p or L is 0. By using transitivity properties of the group of smooth automorphisms of $\mathcal{P}_\lambda$ (see lemmas 5.1–5.3), we see that it is sufficient to consider the following cases:

(F1) $p = (x, y, 1)^t$, $L = (1, 0, 0)$
(F2) $p = (x, 1, 0)^t$, $L = (1, 0, 0)$
(F3) $p = (1, 0, 0)^t$, $L = (a, b, 1)$

Note that the point $(1,0,0)^t$ and the line $(1,0,0)$ are not incident. Moreover, the condition that (p, L) is a flag implies that $x = 0$ in the first two cases and that $a = 0$ in (F3). As in the proof of the previous lemma we introduce appropriate inhomogeneous coordinates. In the case (F1) we identify U_3 and V_1 with two copies of $\mathbb{K}^2$. In this way, the point p corresponds to $(0, y) \in \mathbb{K}^2$ and the line L corresponds to $(0,0) \in \mathbb{K}^2$. The set $\mathcal{F}_\lambda \times (U_3 \times V_1)$ is then given by $(f^{(1)})^{-1}(\{0\})$, where $f^{(1)}$ is defined by

$$f^{(1)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(x, y, b, c) \mapsto (1 + |b|^2 + |c|^2)(x + by + c)(|x|^2 + |y|^2 + 1) + \lambda |c|^2 c.$$

For any $(b, c) \in \mathbb{K}^2$ we define $g^{(1)}_{(b,c)} : \mathbb{K}^2 \to \mathbb{K} : (x, y) \mapsto f(x, y, b, c)$. We have $\mathrm{D}_{(0,y)}g^{(1)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) = (|y|^2 + 1)(u + bv)$, which shows that $\mathrm{D}_{(0,y)}g^{(1)}_{(0,0)}$ is surjective. Hence, the total differential of $f^{(1)}$ in $(0, y, 0, 0)$

is also surjective. Thus there exists an open neighbourhood W of (p, L) in $P \times \mathfrak{L}$, such that $\mathcal{F}_\lambda \cap W$ is a $3l$-dimensional submanifold of W. Moreover, we see as in the proof of Lemma 5.5 that the restriction of the natural projection $\pi_{\mathfrak{L}}$ to $\mathcal{F}_\lambda \cap W$ is a submersion if the neighbourhood W of (p, L) is so small such that the differential of $g^{(1)}_{(0,0)}$ is surjective at all points of W.

For (F2) we identify U_2 and V_1 with $\mathbb{K}^2$ such that (p, L) corresponds to $(0, 0, 0, 0) \in \mathbb{K}^2 \times \mathbb{K}^2$. We define

$$f^{(2)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(x, z, b, c) \mapsto (1 + |b|^2 + |c|^2)(x + b + cz)(|x|^2 + 1 + |z|^2) + \lambda |c|^2 cz|z|^2$$

such that $\mathcal{F}_\lambda \cap (U_2 \times V_1)$ is identified with the set $(f^{(2)})^{-1}(\{0\})$. The differential of

$$g^{(2)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (x, z) \mapsto x(|x|^2 + 1 + |z|^2),$$

defined in anology to $g^{(1)}_{(b,c)}$, at $(0, 0)$ is given by $\mathrm{D}_{(0,0)} g^{(2)}_{(0,0)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto u$ and hence surjective. As in the previous case we conclude that there is an open neighbourhood W of (p, L) in $P \times \mathfrak{L}$ such that $\mathcal{F}_\lambda \cap W$ is a submanifold of W and $\pi_{\mathfrak{L}}$ restricted to $\mathcal{F}_\lambda \cap W$ is a submersion.

In (F3) we identify $U_1 \times V_3$ with $\mathbb{K}^2 \times \mathbb{K}^2$ such that the flag (p, L) corresponds to $(0, 0, 0, b) \in \mathbb{K}^2$. The set $\mathcal{F}_\lambda \cap (U_1 \times V_3)$ is then identified with $(f^{(3)})^{-1}(\{0\})$, where

$$f^{(3)} : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K} :$$
$$(y, z, a, b) \mapsto (|a|^2 + |b|^2 + 1)(a + by + z)(1 + |y|^2 + |z|^2) + \lambda z|z|^2.$$

We define $g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (y, z) \mapsto f^{(3)}(y, z, 0, b)$. Then we have

$$\mathrm{D}_{(0,0)} g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto (|b|^2 + 1)(bu + v),$$

which shows that $\mathrm{D}_{(0,0)}g^{(3)}_{(0,b)}$ is surjective. The case (F3) is then completed as the previous two cases above. Hence, $\mathcal{F}_\lambda$ is a $3l$-dimensional submanifold of $P \times \mathcal{L}$ and the natural projections π_P and $\pi_\mathcal{L}$ are submersions. Note that $\mathcal{F}$ is obviously closed in $P \times \mathcal{L}$. Thus π_P and $\pi_\mathcal{L}$ are surjective since $\mathcal{F}$ is compact, submersions are open maps, and $\mathcal{P}$, $\mathcal{L}$ are connected.

By Lemma 4.1, point rows and line pencils are l-dimensional submanifolds of P and $\mathcal{L}$, respectively. In order to complete this proof, it suffices for reasons of symmetry to show that any two distinct lines L, L' intersect transversally. By using transitivity properties of the group of smooth automorphisms of $\mathcal{P}_\lambda$, the different possibilities of pairs $(L, L') \in \mathcal{L} \times \mathcal{L}$ reduce to the following three cases:

(L1) $L = (a, b, 1)$, $L' = (a', b', 1)$
(L2) $L = (a, b, 1)$, $L' = (1, 0, 0)$
(L3) $L = (a, 1, 0)$, $L' = (1, 0, 0)$

In the first case, we may use the group of smooth automorphisms of $\mathcal{P}_\lambda$ in order to choose appropriate coordinates for possible intersection points of L and L'. We may assume that these two lines intersect in the point $(1, 0, 0)^t$ or in a point $(x, y, 1)^t$. Since the second case has been treated already in Lemma 5.5, we assume that the intersection point of L and L' is $(1, 0, 0)^t$. Then we have $a, a' = 0$ and hence $b \neq b'$ since L and L' are distinct. We identify the open sets U_1 and V_3 with two disjoint copies of $\mathbb{K}^2$ such that L and L' are identified with $(0, b)$ and $(0, b')$, respectively, and $(1, 0, 0)^t$ is identified with $(0, 0)$. The map $g^{(3)}_{(0,b)}$ of the previous paragraph vanishes on $P_L \cap U_1$. Thus the differential

$$\mathrm{D}_{(0,0)}g^{(3)}_{(0,b)} : \mathbb{K}^2 \to \mathbb{K} : (u, v) \mapsto (|b|^2 + 1)(bu + v)$$

vanishes on the tangent space of $P_L \cap U_1$ in $(0, 0)$, and an analogous statement holds for the line L'. Since the kernels of the differentials $\mathrm{D}_{(0,0)}g^{(3)}_{(0,b)}$

and $D_{(0,0)}g^{(3)}_{(0,b')}$ have trivial intersection, we conclude that L and L' intersect transversally in $(1,0,0)^t$.

In the case (L2), let $(x,y,z)^t$ denote an intersection point of L and L'. Then we have $x=0$ and hence $(x,y,z)^t=(0,y,1)^t$ or $(x,y,z)^t=(0,1,0)^t$. Let us first assume that $(0,y,1)^t$ is an intersection point of L and L'. By means of the transitivity properties of the group of smooth automorphisms acting on $\mathcal{P}_\lambda$ we may assume that $y\in\mathbb{R}$. After identifying U_3 with $\mathbb{K}^2$, the intersection point corresponds to $(0,y)$ and the submanifolds $P_L\cap U_3$ and $P'_L\cap U_3$ correspond to $g^{-1}_{(a,b)}(0)$ and $\{0\}\times\mathbb{K}$ with $g_{(a,b)}$ as in the proof of Lemma 5.5. Choose (u,v) in the intersection of the tangent spaces of $P_L\cap U_3$ and $P'_L\cap U_3$ in $(0,y)$. Then we get

$$(y^2+1)au+(y^2+1)bv+(by+1)2yv_1=0$$

by differentiating $g_{(a,b)}$ (compare the proof of Lemma 5.5) and $u=0$. Thus we have $(y^2+1)bv+(by+1)2yv_1=0$ and hence

$$|b|=2|by+1|\frac{|y|}{y^2+1}\frac{|v_1|}{|v|}$$

provided that $v\neq 0$. We obtain that

$$|by|\leq 2|by+1|\frac{y^2}{y^2+1}\leq 2|by+1|.$$

Because of

$$g_{(a,b)}(0,y)=(|a|^2+|b|^2+1)(by+1)(y^2+1)+\lambda=0$$

we have $|by+1|\leq|\lambda|$, which implies that $1\leq|by|+|by+1|\leq 3|\lambda|$, a contradiction. Thus we have $v=0$. This proves the transversal intersection of L and L' in $(0,y,1)^t$. Let us now consider the case that L and L' intersect in the point $(0,1,0)^t$. Then we have $b=0$. We identify U_2 with $\mathbb{K}^2$ such that $(0,1,0)^t$ corresponds to $(0,0) \in \mathbb{K}^2$. The submanifolds $P_L \cap U_3$ and $P_{L'} \cap U_3$ are identifed with the submanifolds $(g^{(4)}_{(a,0)})^{-1}(\{0\})$ and $\{0\} \times \mathbb{K}$, respectively, where

$$g^{(4)}_{(a,0)} : \mathbb{K}^2 \to \mathbb{K} : (x,z) \mapsto (|a|^2+1)(ax+z)(|x|^2+1+|z|^2) + \lambda z|z|^2.$$

The differential $\mathrm{D}_{(0,0)} g^{(4)}_{(a,0)} : (u,v) \mapsto (|a|^2+1)(au+v)$ vanishes on the tangent space of $P_L \cap U_3$ in $(0,0)$. This proves the transversal intersection of L and L' in $(0,1,0)^t$, since $\ker \mathrm{D}_{(0,0)} g^{(4)}_{(a,0)}$ and $\{0\} \times \mathbb{K}$ have trivial intersection.

In the third case, both point rows P_L and $P_{L'}$ are equal to point rows of the classical projective plane $\mathcal{P}_0 = \mathcal{P}_2\mathbb{K}$. Hence they intersect transversally. □

For the projective planes $\mathcal{P}_\lambda$, where $|\lambda| < \frac{1}{9}$, the join map $\vee$ and the intersection map $\wedge$ are not only smooth but real analytic and even Nash, i.e. they are *real analytic* or *Nash projective planes*, respectively. This statement can be proved by an appropriate modification of Corollary 4.4.

5.6 Theorem *Let $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ be a projective plane which satisfies the following conditions:*

(APP1) *There is a positive integer l such that P and $\mathcal{L}$ are real analytic (or Nash) $2l$-dimensional manifolds.*

(APP2) *The flag space $\mathcal{F}$ is a real analytic (or Nash) $3l$-dimensional submanifold of $P \times \mathcal{L}$ such that the canonical projections π_P and $\pi_{\mathcal{L}}$ are submersions.*

Suppose, moreover, that any two distinct point rows and any two distinct line pencils intersect transversally. Then the join map $\vee$ and the intersection map $\wedge$ are real analytic (or Nash, respectively).

Note that point rows and line pencils are submanifolds of P and $\mathcal{L}$, respectively, by Lemma 4.1. Hence the transversality condition makes sense.

This theorem can be proved by simply copying the proof of Corollary 4.4 and using a real analytic or Nash version, respectively, of the implicit function theorem, see, e.g., [23], Theorem 1.8.3 and [1], Theorem 2.9.8.

It remains to check the conditions of the above theorem for the projective planes $\mathcal{P}_\lambda$ with $|\lambda| < \frac{1}{9}$. For simplicity of formulation, we concentrate on the Nash setting in the sequel. First, the point space P and the line space $\mathcal{L}$ are copies of the point space and the line space of the classical projective plane $\mathcal{P}_2\mathbb{K}$ with their usual algebraic structure. Hence, P and $\mathcal{L}$ are Nash manifolds. In the proofs of Theorem 5.4 and Lemma 5.5 we have shown that for each flag there is an open neighbourhood W in $P \times \mathcal{L}$ (identified with an open subset of $\mathbb{K}^2 \times \mathbb{K}^2$) and a real polynomial submersion $f_W : W \to \mathbb{K}$ such $\mathcal{F}_\lambda \cap W = f_W^{-1}(0)$. By a Nash version of the standard result on preimages of regular values we conclude that $\mathcal{F}_\lambda$ is a Nash submanifold of $P \times \mathcal{L}$, cf. the end of Chapter 3. The other conditions required in Theorem 5.6 have already been verified above. Hence, the join map $\vee$ and the intersection map $\wedge$ are Nash and, in particular, real analytic. So, we have proved the following

5.7 Theorem. The incidence structures $\mathcal{P}_\lambda$ are Nash projective planes and, in particular, real analytic projective planes for $|\lambda| < \frac{1}{9}$.

The following theorem contains results on the dimensions of the automorphism groups of the planes $\mathcal{P}_\lambda$, which are direct consequences of the lemmas 5.1, 5.2, and 5.3.

5.8 Theorem. *The smooth projective planes $\mathcal{P}_\lambda$ admit groups of smooth automorphisms which are Lie groups of dimension* 1, 4 *or* 13 *for* $l = 1, 2$ *or* 4, *respectively. Furthermore, these groups are compact.*

By the main result of [2], the dimension of the automorphism group of a $2l$-dimensional, non-classical smooth projective plane is at most 2, 6 or

16 for $l = 1$, 2 or 4, respectively. By Theorem 5.8, the smooth projective planes $\mathcal{P}_\lambda$ admit Lie groups of smooth automorphisms whose dimensions come close to these bounds. The dimensions of automorphism groups of non-classical compact projective planes of dimension $2l$ can be higher than in the smooth case, see [32], Section 65. The maximal dimensions of *compact* groups of automorphisms of non-classical compact projective planes (with $l = 1$, 2 or 4), however, are the *same* as the dimensions of the Lie groups in Theorem 5.8, i.e. in this respect there is no difference between compact projective planes and smooth projective planes: by [32], 32.21 and 22, a compact group of automorphisms of a 2-dimensional, non-classical compact projective plane is a Lie group of dimension at most 1. In the 4-dimensional case, 71.9 and 72.6 in [32] imply that the dimension of a compact group of automorphisms acting on a non-classical compact projective plane is at most 4. Finally, in dimension 8 a compact group of automorphisms acting on a non-classical compact projective plane is at most 13-dimensional, see [32], 84.9. Even more, the connected component of such a group is necessarily isomorphic to $\mathrm{SO}_2\mathbb{R}$ for $l = 1$, to $\mathrm{U}_2\mathbb{C}$ for $l = 2$, and to the product of $\mathrm{Spin}_3\mathbb{R}$ and $\mathrm{Spin}_5\mathbb{R}$ with amalgamated centers for $l = 4$. The following theorem summarizes the general information obtained in this way.

5.9 Theorem. *The bounds for the dimensions of compact groups of automorphisms of $2l$-dimensional, non-classical smooth projective planes are the same as those in the case of $2l$-dimensional, non-classical compact projective planes for $l \in \{1, 2, 4\}$.*

References

[1] Bochnak, J., Coste, M., Roy, M.-Fr., *Real algebraic geometry*, Berlin, etc.: Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 36, **1998**

[2] Bödi, R., *Smooth stable and projective planes*, Tübingen: Habilitationsschrift **1996**

[3] Bödi, R., *Smooth stable planes*, Result. Math. **31** (1997), 300–321

[4] Bödi, R., Immervoll, St., *Implicit characterizations of smooth incidence geometries*, Geom. Dedicata **83** (2000), 63–76

[5] Bödi, R., Kramer, L., *On homomorphisms between generalized polygons*, Geom. Dedicata **58** (1995), 1–14

[6] Bröcker, Th., Jänich, K., *Introduction to differential topology*, Cambridge University Press **1987**

[7] Cartan, E., *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45** (1939), 335–367

[8] Cecil, T. E., Ryan, P. J., *Tight and taut immersions of manifolds*, Boston: Pitman, Res. Notes in Math. 107, **1985**

[9] Dorfmeister, J., Neher, E., *An algebraic approach to isoparametric hypersurfaces in spheres I*, Tôhoko Math. J. **35** (1983), 187–224

[10] Ferus, D., Karcher, H., Münzner, H.-F., *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502

[11] Grundhöfer, T., Hähl, H., *Fibrations of spheres by great spheres over division algebras and their differentiability*, J. Differential Geom. **31** (1990), 357–363

[12] Grundhöfer, T., Knarr, N., *Topology in generalized quadrangles*, Topology Appl. **34** (1990), 139–152

[13] Hsiang, W.-Y., Lawson, H. B., *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971), 1–38

[14] Grundhöfer, T., Van Maldeghem, H., *Topological polygons and affine buildings of rank three*, Atti. Sem. Mat. Fis. Univ. Modena **38** (1990), 459–479

[15] Immervoll, St., *Glatte affine Ebenen mit großer Automorphismengruppe und ihr projektiver Abschluß*, Tübingen: Diplomarbeit **1998**

[16] Immervoll, St., *Smooth generalized quadrangles and isoparametric hypersurfaces of Clifford type*, Forum Math. (to appear)

[17] Knarr, N., *The nonexistence of certain topological polygons*, Forum Math. **2** (1990), 603–612

[18] Knarr, N., Kramer, L., *Projective planes and isoparametric hypersurfaces*, Geom. Dedicata **58** (1995), 192–202

[19] Kramer, L., *Compact polygons*, Tübingen: Dissertation **1994**

[20] Kramer, L., *Holomorphic polygons*, Math. Z. **223** (1996), 333–341

[21] Kramer, L., Tent, K., *Algebraic Polygons*, J. Algebra **182** (1996), 435–447

[22] Kramer, L., *Compact ovoids in quadrangles III: Clifford algebras and isoparametric hypersurfaces*, Geom. Dedicta **79** (2000), 321–339

[23] Krantz, St. G., Parks, H. R., *A primer of real analytic functions*, Basel, etc.: Birkhäuser, Basler Lehrbücher, Vol. 4, **1992**

[24] Kuiper, N. H., *A real analytic non-desarguesian plane*, Nieuw. Arch. Wisk. **3** (1957), 19–24

[25] Löwen, R., *Vierdimensionale stabile Ebenen*, Geom. Dedicata **5** (1976), 239–294

[26] Löwen, R., *Topology and dimension of stable planes: On a conjecture of H. Freudenthal*, J. Reine Angew. Math. **343** (1983), 108–122

[27] Münzner, H. F., *Isoparametrische Hyperflächen in Sphären I*, Math. Ann. **251** (1980), 57–71

[28] Münzner, H. F., *Isoparametrische Hyperflächen in Sphären II*, Math.

Ann. **256** (1981), 215–232

[29] Otte, J., *Differenzierbare Ebenen*, Kiel: Dissertation **1992**

[30] Otte, J., *Smooth projective translation planes*, Geom. Dedicata **58** (1995), 203–212

[31] Palais, R. S., Terng, C.-I., *Critical point theory and submanifold geometry*, Berlin, etc.: Springer, Lecture Notes in Math. 1353, **1988**

[32] Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R., Stroppel, M., *Compact projective planes*, Berlin: de Gruyter, Expositions in Mathematics, **1995**

[33] Segre, B., *Plans graphiques algébriques réels non-désarguesiens et correspondances crémoniennes topologiques*, Rev. Math. Pures Appl. **3** (1956), 35–50

[34] Strambach, K., *Algebraische Geometrien*, Rend. Sem. Mat. Univ. Padova **53** (1975), 165–210

[35] Thorbergsson, G., *Isoparametric foliations and their buildings*, Ann. Math. **133** (1991), 429–446

[36] Thorbergsson, G., *Clifford algebras and polar planes*, Duke Math. J. **67** (1992), 627–632

[37] Thorbergsson, G., *A survey on isoparametric hypersurfaces and their generalizations,* in *Handbook of differential geometry* (ed. by Dillen, F. J. E., Verstraelen, L. C. A.), pp. 963–995, Elsevier Science B. V. **2000**

[38] Van Maldeghem, H., *Generalized polygons*, Basel, etc.: Birkhäuser **1998**

Lebenslauf

Persönliche Daten

Geburtsdatum	14. Juli 1973
Familienstand	verheiratet
Nationalität	deutsch

Studium und Schule

seit 04/99	Promotion in Mathematik an der Universität Tübingen
	Wissenschaftlicher Angestellter bei Professor V. Batyrev
10/93 – 02/99	Studium der Mathematik (Diplom) mit Nebenfach Physik an der Universität Tübingen
03/95 – 02/99	Förderung durch die Studienstiftung des deutschen Volkes
10/96 – 06/97	Studium der Mathematik an der Scuola Normale Superiore in Pisa
09/84 – 05/93	Besuch des Rosenstein Gymnasiums in Heubach

Meine akademischen Lehrer waren u.a. die Professoren und Dozenten V. Batyrev, R. Bödi, G. van der Geer, F. Hirzbebruch, R. Löwen, R. Nagel, F. Räbiger, H. Salzmann, U. Schlotterbeck, P. Schmid.