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Delay Equations with Nonautonomous Past

DISSERTATION
der Mathematischen Fakultät der
Eberhard–Karls–Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

Vorgelegt von
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2002

Tag der mündlichen Qualifikation:	11. Februar 2002
Dekan:	Professor Dr. Ch. Lubich
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To my family

Zusammenfassung in deutscher Sprache

Ziel dieser Dissertation ist die Entwicklung einer Theorie für eine Klasse von abstrakten Differentialgleichungen mit unendlicher Verzögerung und nicht-autonomer Vergangenheit. Die Motivation für das Studium solcher Gleichungen kommt daher, daß in vielen Fällen die Verzögerung auf eine in der Vergangenheit modifizierte "history function" wirkt. Dabei ist die Modifikation der "history function" durch eine Evolutionsfamilie gegeben, die ein nicht-autonomes Cauchyproblem löst, das zum Beispiel mit einer Diffusion in der Vergangenheit assoziiert ist.

In Kapitel 1 geben wir zwei Beispiele, die zeigen, daß die üblichen retardierten partiellen Differentialgleichung nicht ausreichen. Insbesondere analysieren wir eine Populationsgleichung mit Diffusion (siehe [22] oder [70]) und ein Zellmodell, das von Mahaffy, Pao, Busenberg in [12], [13], [41] und [42] untersucht wurde. Wir erklären, warum die benutzten Verzögerungsterme nicht exakt sind und schlagen modifizierte Verzögerungsterme vor.

Kapitel 2 ist in drei Abschnitte gegliedert. Im ersten stellen wir die Definition und fundamentale Eigenschaften einer Evolutionsfamilie auf einem Banachraum X und der assoziierten Evolutionshalbgruppe auf $L^p(\mathbb{R}_-, X)$ vor. Im zweiten Abschnitt diskutieren wir die Wohlgestelltheit des nichtautonomen Cauchyproblems, das mit mit der Existenz einer Evolutionsfamilie in Verbindung gebracht werden kann. Im letzten Abschnitt erinnern wir an die Defintionen des kritischen Spektrums und der kritischen Wachstumsschranke einer Halbgruppe. Insbesondere zeigen wir, daß für eine Evolutionshalbgruppe das kritische Spektrum beziehungsweise seine kritische Wachstumsschranke mit dem gewöhnlichen Spektrum beziehungsweise Wachstumsschranke zusammenfallen.

Im Kapitel 3 diskutieren wir die Wohlgestelltheit der retardierten partiellen Differentialgleichung mit nicht-autonomer Vergangenheit und zeigen, daß diese äquivalent zu einem Cauchyproblem

$$(CP) \quad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{C}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f} \end{cases}$$

für einen linearen Operator $(\mathcal{C}, D(\mathcal{C}))$ auf dem Produktraum $\mathcal{E} := X \times L^p(\mathbb{R}_-, X)$ ist. Anschließend geben wir Bedingungen an, die garantieren, daß $(\mathcal{C}, D(\mathcal{C}))$ eine starkstetige Halbgruppe $(\mathcal{T}(t))_{t\geq 0}$ erzeugt. In diesem Fall sind die Lösungen der ursprünglichen Gleichung durch

$$\binom{u(t)}{\widetilde{u}_t} = \mathcal{T}(t)\binom{x}{f},$$

gegeben, wobei $\widetilde{u}_t$ die modifizierte "history function" ist.

Im vierten Kapitel untersuchen wir die Stabilität der Halbgruppe $(\mathcal{T}(t))_{t\geq 0}$ mittels Spektralmethoden. Im ersten Abschnitt betrachten wir den Hilbertraumfall. Unter Verwendung des Satzes von Gearhart-Greiner -Prüss geben wir Bedingungen dafür an, daß $(\mathcal{T}(t))_{t\geq 0}$ gleichmäßg exponentiell stabil ist. Im zweiten Abschnitt beweisen wir mittels des kritischen Spektrums den spektralen Abbildungssatz für $(\mathcal{T}(t))_{t\geq 0}$. Im letzten Abschnitt zeigen mit Hilfe dieses Satzes, daß die Wachstumsschranke von $(\mathcal{T}(t))_{t\geq 0}$ durch

$$\omega_0(\mathcal{T}) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}$$

gegeben ist.

In Kapitel 5 wenden wir diese Theorie auf die beiden im ersten Kapitel vorgestellten Beispiele an und geben Bedingungen dafür an, daß Lösungen für die Populationsgleichung und für das Zellmodell existieren. Ferner geben wir Bedingungen an, die garantieren, daß diese Lösungen gleichmäßg exponentiell stabil sind.

Introduction

Nothing changes more consistently than the past...
the past that influences our lives (is) not what actually
happened but what (we) believe happened.
[Gerald W. Johnson, 1890-1980]

Why do we need differential equations with delay? The answer is that many processes in biology, physics, chemistry, engineering, economics, etc. involve time delays. They occur so often that to ignore them is to ignore reality.

The original motivation for studying differential equations with delay came from the theory of feedback control theory. In fact, in the 40's Minorsky (see [45]) pointed out very clearly that in feedback mechanisms a finite time is required to receive information and then react to it. The importance of such problems has contributed to the rapid development of the theory of partial differential equations with dependence on the past.

J. Hale in [32] and G. Webb in [68] were among the first who applied semigroup theory to delay equations. Such equations can be written in an abstract way on a Banach space X, using the standard notation (see, e.g., [38] or [70]), as

$$(DE) \quad \begin{cases} \dot{u}(t) = Bu(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where $(B, D(B))$ is a (unbounded) operator on X and the delay operator Φ is supposed to belong to $\mathcal{L}(W^{1,p}([-1,0],X),X)$. Recently, many authors, e.g., A. Bátkai, O. Diekmann, K.J. Engel, S.M. Verduyn Lunel, R. Nagel, S. Piazzera, A. Rhandi, J. Wu (see [5], [6], [7], [8], [20], [22], [33], [55], [57] or [70]) have continued the study of (DE) in this spirit obtaining well-posedness and qualitative results.

The aim of this thesis is to study a class of differential equations with *infinite* delay which we call *delay equations with nonautonomous past.* The motivation for these equations is that in general the previous delay equations are lacking an important feature frequently appearing in reality. In fact, as we show in Chapter 1 for a population equation and for a model on genetic repression, the history function describing a given system will be submitted to a modification, e.g., by a diffusion process, while time is passing. Therefore, the delay should act on this modified history function.
Since this modification is assumed to be governed by a *nonautonomous Cauchy problem* which again is solved by an *evolution family* on a Banach space, these two concepts play an essential role in this thesis. However, both of them have to be considered on the halfline $\mathbb{R}_-$ only, and in order to define a corresponding *evolution semigroup* we need a boundary condition at $s = 0$. That is where the *delay operator* enters the scene. The interplay of these four concepts is the underlying idea of this thesis and allows to obtain a semigroup solving the original equations. Moreover, spectral methods enable us to determine the asymptotic behavior of these solutions.

We now describe more precisely the content of the present thesis.
In Chapter 1 we give two examples to explain the motivation why delay equations are not "good enough" in some cases. In particular, we analyze a population equation with diffusion presented by K.J.Engel and R. Nagel in [22] or by J. Wu in [70] and the genetic repression studied, e.g., by Mahaffy, Pao, Busenberg in [12], [13], [41] or in [42]. We explain why the term on which the delay acts is not realistic, and we propose a modified term.
Chapter 2 is organized in three sections. In the first one we state definitions and fundamental properties of a backward evolution family $\mathcal{U} := (U(t,s))_{t\le s\le 0}$ on a Banach space X and of the associated backward evolution semigroup $(T_0(t))_{t\ge 0}$ defined as

$$(T_0(t)f)(s) = \begin{cases} U(s,s+t)f(t+s), & s+t\le 0,\\ 0, & s+t>0, \end{cases}$$

on $L^p(\mathbb{R}_-,X)$. Such backward evolution families $\mathcal{U}$ arise as the solutions of backward nonautonomous Cauchy problems of the form

$$(NCP)\qquad \begin{cases} \dot u(t) = -A(t)u(t), & t\le s,\\ u(s) = x\in X, & s\le 0, \end{cases}$$

where $(A(t), D(A(t)))_{t\leq 0}$ are (unbounded) linear operators on a Banach space X. In Section 2.2 we discuss the well-posedness of (NCP) and characterize it by properties of the generator $(G_0, D(G_0))$ of the backward evolution semigroup $(T_0(t))_{t\geq 0}$ associated to $\mathcal{U}$.
In the last section we recall the definitions of the critical spectrum and the critical growth bound of a semigroup which will be used in an essential way in Chapter 4. Moreover, we prove that for a backward evolution semigroup its spectrum and its growth bound coincide with its critical spectrum and its critical growth bound, respectively (see Theorem 2.20 and Corollary 2.21).

In Chapter 3 we discuss the well-posedness of the *delay equations with nonautonomous past*. Such equations can be written as

$$(NDE) \quad \begin{cases} \dot{u}(t) = Bu(t) + \Phi\widetilde{u}_t, & t \geq 0, \\ u(0) = x \in X, \\ \widetilde{u}_0 = f \in L^p(\mathbb{R}_-, X), \end{cases}$$

where $(B, D(B))$ is a closed, densely defined operator, the delay operator $\Phi : D(\Phi) \subseteq L^p(\mathbb{R}_-, X) \to X$ is a linear operator and $\widetilde{u}_t$ is the modified history function (see Definition 3.2). In Section 3.1, following the approach of A. Bátkai and S. Piazzera (see [5], [6], [7], [8] or [55]), we prove the equivalence of (NDE) to an abstract Cauchy problem

$$(CP) \quad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{C}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f} \end{cases}$$

for a linear operator $(\mathcal{C}, D(\mathcal{C}))$ on the product space $\mathcal{E} := X \times L^p(\mathbb{R}_-, X)$. This is shown by proving that the following relation between a classical solution $\mathcal{U} : \mathbb{R}_+ \to \mathcal{E}$ of (CP) and a solution $u : \mathbb{R} \to X$ of (NDE) holds

$$\binom{u(t)}{\widetilde{u}_t} = \mathcal{U}(t), \quad t \geq 0,$$

where

$$\widetilde{u}_t(s) := \begin{cases} U(s,0)u(t+s), & t+s \geq 0, \\ U(s,t+s)f(t+s), & t+s \leq 0. \end{cases}$$

In Section 3.2 the well-posedness of (CP) is studied. In particular, we prove that $(\mathcal{C}, D(\mathcal{C}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ by rewriting $\mathcal{C}$ as a sum of two matrices, $\mathcal{C} = \mathcal{C}_0 + \mathcal{F}$, where $\mathcal{C}_0$ generates a C_0-semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{E}$ and $\mathcal{F}$ is the operator corresponding to the delay Φ. Using the perturbation theory of Miyadera-Voigt, we show that $(\mathcal{C}, D(\mathcal{C}))$ is a generator. In the last section, using the technique of our paper [24], we show that this approach also gives classical solutions to a two variable version of (NDE), proposed by S. Brendle and R. Nagel in [10], of the form

$$\begin{aligned}\frac{\partial}{\partial t}u(t,s) &= \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \quad s \leq 0,\ t \geq 0,\\ \frac{\partial}{\partial t}u(t,0) &= Bu(t,0) + \Phi u(t,\cdot), \quad t \geq 0.\end{aligned}$$

In Chapter 4 we study the stability of the solution semigroup using spectral methods. We assume, in the first section, that the product space $\mathcal{E}$ is a Hilbert space and that the backward evolution semigroup $(T_0(t))_{t\geq 0}$ and the semigroup $(S(t))_{t\geq 0}$ generated by B are stable. The theorem of Gearhart-Greiner -Prüss (see, for example, [22, Theorem V.1.11]) allows us to say how the delay operator influences the stability of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ and hence under which assumptions the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable. In Section 4.2, using the critical spectrum of a semigroup, we prove that, under appropriate assumptions on B, the spectral mapping theorem holds for $(\mathcal{T}(t))_{t\geq 0}$. The proof of this theorem is based on the perturbation results on the critical spectrum due to S. Brendle, R. Nagel and J. Poland (see [11]) and on Theorem 4.14 where we prove that

$$\sigma_{\text{crit}}(\mathcal{T}_0(t)) = \sigma(T_0(t)).$$

In Section 4.3 we obtain, as a consequence of the above spectral mapping theorem,

$$\omega_0(\mathcal{T}) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}.$$

In the last chapter we apply this theory to the two examples presented in the first chapter and obtain conditions such that there exist solutions for the population equation and for the genetic repression. Moreover, we give conditions which guarantee that these solutions are uniformly exponentially stable.

Some of the results in Chapters 3 and 4 are in collaboration with G. Nickel (see [25], [26]), while the investigation of population equation (Chapter 5) will be continued jointly with L. Tonetto (see [27]).

Acknowledgments

It is my pleasure to acknowledge the support I obtained from many people.

First of all I want to thank my advisor Rainer Nagel, who has followed me with human care, friendly support and invaluable mathematical suggestions, and my professors in Lecce, Giorgio Metafune and Diego Pallara, who have introduced me to semigroup theory and encouraged me to go to Tübingen.

I am in debt with András Bátkai, Gregor Nickel, Roland Schnaubelt and in particular with Susanna Piazzera for many helpful discussions and with Lorenza Tonetto for her collaboration and her hospitality in Padova.

I am grateful to Elena Babini for some biological comments and for her moral support.

I would like to thank all the members of the *Arbeitsgemeinschaft Funktionalanalysis* (AGFA) of the Univesity of Tübingen and in particular Franziska Kühnemund and Marcella Winter for the beautiful moments that we have shared.

I also thank Luca Lorenzi for some mathematical suggestions and for his support during the last year and Brigitta Bekesi and Stefan Immervoll for their constant help when I arrived in Tübingen.

The Internet Seminar allowed me to know the AGFA, and during the Summer School in Cortona (July 2001) and the Autumn School on Evolution Equations and Semigroups in Levico Terme (October 2001) I could learn much mathematics and meet interesting people. Thanks to the organizers.

I am in debt also with Klaus J. Engel who has made many useful comments and remarks.

I thank Dimitri, my sun, who is close to me in every moment with his infinite patience, his encouragement and his constant love.

I want to thank Lorenzo, Livianna and my parents, who have always believed in me and encouraged me during all difficult periods. I wish to dedicate this thesis to them.

Finally, I thank *Istituto Nazionale di Alta Matematica* for financial support during these years.

Contents

Chapter 1

Two Motivating Examples

The purpose of this chapter is to present two examples in order to justify the introduction and investigation of a new class of delay equations, the so-called *delay equations with nonautonomous past.*

1.1 A Population Equation with Diffusion

Since the 80's, the use of functional analysis and, in particular, semigroup methods for the study of population equations is well established and documented in the monographs by O. Diekmann and J.A.J. Metz [19] and by G.F. Webb [69]. Among the recent papers, we mention [35], [53], [58], [59], where the authors study populations depending on time, age, and spatial diffusion.

We look at the following population equation which is derived from biological assumptions in [70, Introduction] and discussed by semigroup methods in [22, Example VI.6.19].

The population density $u(t,x)$ at time t and position $x \in [0,1]$ satisfies

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - d(x)u(t,x) + b(x)\int_r^0 u(t+s,x)ds, \tag{1.1}$$

for $t \geq 0,\ r < 0,$ and $x \in [0,1]$ with the initial condition

$$u(s,x) = f(s,x), \quad x \in [0,1], s \in [r,0],$$

and boundary conditions

$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,1) = 0, \quad t \geq 0.$$

In particular in [22, Example VI.6.19] or in [70, Introduction] the authors consider a point delay, i.e.,

$$b(x)u(t+r,x)$$

instead of

$$b(x)\int_r^0 u(t+s,x)ds.$$

Here, the functions d and b, defined in $[0,1]$ with values in $\mathbb{R}_+$, represent the mortality and the birth rate, respectively, while r is the delay due to pregnancy.

The meaning of equation (1.1) is that the variation of the population density is given by the diffusion, i.e., by the migration of the population, minus the contribution due to deaths plus the contribution due to births (depending on the delay).

In [22, Section VI.6], K. Engel and R. Nagel rewrite (1.1) as an abstract delay equation, and apply semigroup techniques.

However, according to equation (1.1) the newborns at time t at position x depend only on pregnant females that where at time $t+r$ at the *same* position x. Clearly, this is unrealistic since also the pregnant individuals move during the time between $t+r$ and t. Due to such a migration process in the past, the value $u(t+s,x)$ should be replaced by a modified value $\widetilde{u}(t+s,x)$, for $s \in [r,0]$.

In this way our population equation becomes

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - d(x)u(t,x) + b(x)\int_r^0 \widetilde{u}(t+s,x)ds, \tag{1.2}$$

for $t \geq 0,\ r < 0,\ x \in [0,1]$ and some "modified history function" $\widetilde{u}(t+\cdot,x)$.

Our aim in this thesis is to make this precise and then solve the above equation. In fact, we will study a large class of abstract equations including this example.

1.2 Genetic Repression

Systems of delayed reaction diffusion equations have been used frequently in modelling genetic repression, (see, e.g., J. Wu [70, Introduction]). The study of these equations goes back to the 60's with B.C. Goodwin, F. Jacob and J. Monod (see, e.g.,[29], [30] or [36]).

In particular, B.C. Goodwin suggested that time delays caused by the processes of transcription and translation as well as spatial diffusion of reactants could play a role in the behavior of this system.

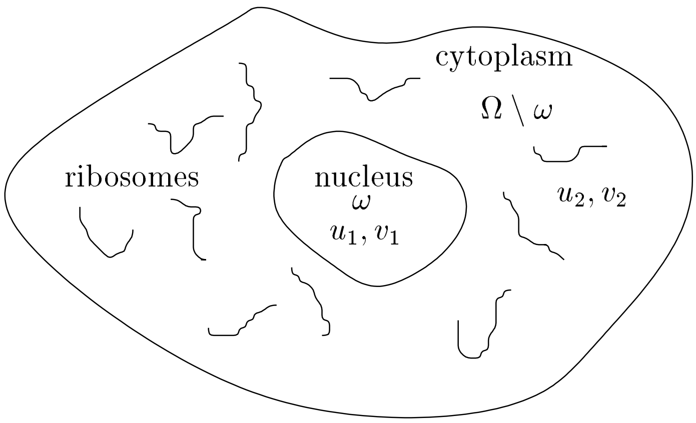

Figure 1.1: cell

The later studies of these models included either time delays (see, e.g., [4], [40] or [65]) or the spatial diffusion (see, e.g. [42]). We now explain the model from [12] and [41], which includes spatial diffusion and time delay (see Figure 1.1).

The eucharyotic cell consists of two compartments where the most important chemical reactions happen. Such compartments are enclosed within the cell wall, unpermeable to the mRNA (messanger ribo nuclein acid) and to the repressor, and separated by the permeable nuclear membrane. The first compartment ω is the nucleus where mRNA is produced. The second compartment, denoted by $\Omega \setminus \omega$, is the cytoplasm in which the ribosomes are randomly dispersed. The process of translation and the production of the repressor occurs here.

We denote by u_i and v_i the concentrations of mRNA and of the repressor, respectively, in ω if $i = 1$ and in $\Omega \setminus \omega$ if $i = 2$. These two species interact to control each other's production. In the nucleus ω, mRNA is transcribed from the gene at a rate depending on the concentration of the repressor v_1. The mRNA leaves ω and enters the cytoplasm $\Omega \setminus \omega$ where it diffuses and reacts with ribosomes. Through the delayed process of translation, a sequence of enzymes is produced which in turn produces a repressor v_2. This repressor comes back to ω where it inhibits the production of u_1. This process can be written as the following system of equations.

$$
(1.3) \quad \begin{cases}
\dfrac{du_1(t)}{dt} = h(v_1(t+r_1)) - b_1 u_1(t) + a_1 \displaystyle\int_{\partial\omega} [u_2(t,x) - u_1(t)] dS_\omega, \\
\dfrac{dv_1(t)}{dt} = -b_2 v_1(t) + a_2 \displaystyle\int_{\partial\omega} [v_2(t,x) - v_1(t)] dS_\omega, \\
\dfrac{\partial u_2(t,x)}{\partial t} = D_1 \Delta u_2(t,x) - b_1 u_2(t,x), & x \in \Omega \setminus \omega, \\
\dfrac{\partial v_2(t,x)}{\partial t} = D_2 \Delta v_2(t,x) - b_2 v_2(t,x) + c_0 u_2(t+r_2,x), & x \in \Omega \setminus \omega,
\end{cases}
$$

with boundary conditions

$$
(1.4) \quad \begin{cases}
\dfrac{\partial u_2(t,x)}{\partial n} = -\beta_1 [u_2(t,x) - u_1(t)], & x \in \partial\omega, \\
\dfrac{\partial v_2(t,x)}{\partial n} = -\beta_1^* [v_2(t,x) - v_1(t)], & x \in \partial\omega, \\
\dfrac{\partial u_2(t,x)}{\partial n} = \dfrac{\partial v_2(t,x)}{\partial n} = 0, & x \in \partial\Omega.
\end{cases}
$$

The constants b_i are *kinetic rates of decay*, a_i are *rates of transfer* between ω e $\Omega \setminus \omega$ and are directly proportional to the concentration gradient. The constants D_i are the diffusivity coefficients, and the constant c_0 is the *production rate* for the repressor. The function h is a decreasing function in v_1 and represents the production of mRNA. It is of the form $\dfrac{1}{1+k(v_1(t+r_1))^\rho}$, where k is a kinetic constant and ρ is the Hill coefficient. The delay $-r_1 \geq 0$ is the transcription time, i.e., the time necessary to the transcription reaction, and $-r_2 \geq 0$ is the translation time. The constants β_1 and β_1^* are the constants of Fick's law (see, e.g., [2, Chapter VI] or [64, Chapter V]). In a one dimensional model, as in [41], the interval $(0,1)$ corresponds to the cytoplasm and the nucleus is localized in 0 (see Figure 1.2). The equations (1.3) become

$$
(1.5) \quad \begin{cases}
\dfrac{du_1(t)}{dt} = h(v_1(t+r_1)) - b_1 u_1(t) + a_1 (u_2(t,0) - u_1(t)), \\
\dfrac{dv_1(t)}{dt} = -b_2 v_1(t) + a_2 (v_2(t,0) - v_1(t)), \\
\dfrac{\partial u_2(t,x)}{\partial t} = D_1 \dfrac{\partial^2 u_2(t,x)}{\partial x^2} - b_1 u_2(t,x), & x \in [0,1], \\
\dfrac{\partial v_2(t,x)}{\partial t} = D_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} - b_2 v_2(t,x) + c_0 u_2(t+r_2,x), & x \in [0,1],
\end{cases}
$$

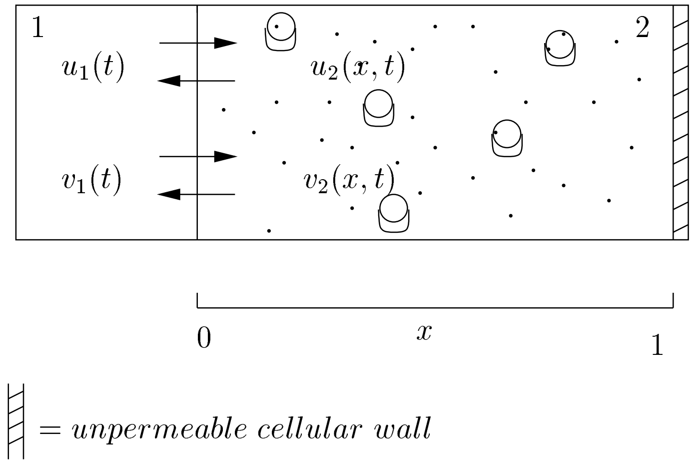

Figure 1.2: cell in one dimension

with boundary conditions

$$\text{(1.6)} \qquad \begin{cases} \dfrac{\partial u_2(t,0)}{\partial x} = -\beta_1(u_2(t,0) - u_1(t)), \\ \dfrac{\partial v_2(t,0)}{\partial x} = -\beta_1^*(v_2(t,0) - v_1(t)), \\ \dfrac{\partial u_2(t,1)}{\partial x} = \dfrac{\partial v_2(t,1)}{\partial x} = 0. \end{cases}$$

However, according to this model, the variation of the repressor $v_2(t,x)$ at time t and position x depends on the mRNA which was at time $t + r_2$ at the *same* position x. As in Section 1.1, this is unrealistic because the mRNA is submitted to a diffusion process between $t + r_2$ and t.

Thus, also in this case, $u_2(t + r_2, x)$ must be modified. If $\widetilde{u}_2(t + r_2, x)$ is this modification, the last equation in (1.5) becomes

$$\frac{\partial v_2(t,x)}{\partial t} = D_2 \frac{\partial^2 v_2(t,x)}{\partial x^2} - b_2 v_2(t,x) + c_0 \widetilde{u}_2(t + r_2, x).$$

Again, we will develop a theory to treat such delay equations.

Chapter 2

Tools

Life can only be understood backward
but it must first be lived forward.
[S. Kierkegaard, 1813-1855]

Most problems in this thesis will have the form of (or will be written as) an **abstract Cauchy problem**

$$(CP) \qquad \begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x \in X, \end{cases}$$

on a Banach space X, where $A : D(A) \subset X \to X$ is a linear operator (see, for instance, [22], [23], [28], [49], [54]).

A function $u : \mathbb{R}_+ \to X$ is then called a **classical solution** of (CP) if u is continuously differentiable on $\mathbb{R}_+$, $u(t) \in D(A)$ for all $t \geq 0$ and (CP) holds. Moreover, we say that (CP) is **well-posed** if for every $x \in D(A)$ there exists a unique solution $u(\cdot, x)$ of (CP), $D(A)$ is dense in X, and for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ satisfying $\lim_{n\to\infty} x_n = 0$, one has $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly on compact intervals of $\mathbb{R}_+$.

The well-posedness of (CP) can be characterized using the concept of a C_0-semigroup $(T(t))_{t\geq 0}$ and of its generator. In fact, (CP) is well-posed if and only if the operator A is the generator of a C_0- semigroup $(T(t))_{t\geq 0}$. In this case, the semigroup yields the unique solution of the associated abstract Cauchy problem, i.e., for all $x \in D(A)$, the function $t \mapsto u(t) := T(t)x$ is the unique solution of (CP) (see [22, Proposition II.6.2]).

Replacing the operator A by time dependent operators $A(t)$ leads to the **nonautonomous abstract Cauchy problem**

$$(NCP)_s \qquad \begin{cases} \dot{u}(t) = A(t)u(t), & t \in \mathbb{R}, \\ u(s) = x \in X, & t \geq s, \end{cases}$$

which is treated, e.g., in [16], [22, Chapter IV.9], [51], [52], [54], [61], [60], [62] or [63]. In order to study this $(NCP)_s$, we use the concept of an **evolution family**, to which we can associate a strongly continuous semigroup on $X-$valued function spaces. These semigroups characterize many features of the evolution family and are called **evolution semigroups** (see [1], [39] [51], [52], [60], [61], [63]).

For our purposes, we have to study **backward nonautonomous abstract Cauchy problems** of the form

$$(NCP)_s \qquad \begin{cases} \dot{u}(t) = -A(t)u(t), & t \leq s, \\ u(s) = x \in X, & s \leq 0, \end{cases}$$

on a Banach space X. This leads to **backward evolution families** and **backward evolution semigroups**.

2.1 Backward Evolution Families and Backward Evolution Semigroups

In this section, we state the definitions, notations and some of the fundamental properties of a backward evolution family on the Banach space X and of the associated backward evolution semigroup on $E := L^p(\mathbb{R}_-, X)$, respectively.

Definition 2.1. A family $(U(t,s))_{t \leq s \leq 0}$ of bounded, linear operators on a Banach space X is called an (exponentially bounded, backward) **evolution family** on $\mathbb{R}_-$ if

(i) $U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = Id$ for all $t \leq r \leq s \leq 0$,

(ii) the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous,

(iii) $\|U(t,s)\| \leq Me^{\omega(s-t)}$ for some $M \geq 1, \omega \in \mathbb{R}$ and all $t \leq s \leq 0$.

Remarks 2.2. (1) We observe that if $\mathcal{U} := (U(t,s))_{t \le s \le 0}$ is a backward evolution family, then the family $(V(\tau,\sigma))_{\tau \ge \sigma \ge 0}$, defined by

$$V(\tau,\sigma) := U(-\tau,-\sigma) \quad \text{for} \quad \tau \ge \sigma \ge 0,$$

is a forward evolution family (see, e.g., [22, Definition VI.9.2] or [60, Proposition 1.2] for the definition). So all results on forward evolution families can be transferred to backward evolution families as well.

(2) The exponential bound in (iii) is needed in order to obtain strongly continuous evolution semigroups.

We define the **growth bound** of $(U(t,s))_{t \le s \le 0}$by

$$\omega_0(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \exists\, M_\omega \; \ge 1 \text{ with } \|U(t,s)\| \le M_\omega e^{\omega(s-t)} \text{ for } t \le s \le 0\}.$$

We will use evolution semigroup techniques as developed in [16], [22, Section VI.9], [47], [60], [61] or [63]. To that purpose, we extend the backward evolution family $(U(t,s))_{t \le s \le 0}$ to an evolution family $(\widetilde{U}(t,s))_{t \le s}$ on all of $\mathbb{R}$.

Definition 2.3. (1) The evolution family $(U(t,s))_{t \le s \le 0}$ on X is extended to an evolution family $(\widetilde{U}(t,s))_{t \le s}$ on $\mathbb{R}$ by setting

$$\widetilde{U}(t,s) := \begin{cases} U(t,s) & \text{for } t \le s \le 0, \\ U(t,0) & \text{for } t \le 0 \le s, \\ U(0,0) = Id & \text{for } 0 \le t \le s. \end{cases}$$

(2) On the space $\widetilde{E} := L^p(\mathbb{R}, X)$, we then define the corresponding **evolution semigroup** $(\widetilde{T}(t))_{t \ge 0}$ by

$$(\widetilde{T}(t)\widetilde{f})(s) := \widetilde{U}(s,s+t)\widetilde{f}(s+t) = \begin{cases} U(s,s+t)\widetilde{f}(s+t) & \text{for } s \le s+t \le 0, \\ U(s,0)\widetilde{f}(s+t) & \text{for } s \le 0 \le s+t, \\ \widetilde{f}(s+t) & \text{for } 0 \le s \le s+t, \end{cases}$$

for all $\widetilde{f} \in \widetilde{E}, s \in \mathbb{R}, t \ge 0$.

It is easy to prove that the semigroup $(\widetilde{T}(t))_{t \ge 0}$ is strongly continuous on $\widetilde{E}$ (see [22, Lemma VI.9.10]). We denote its generator by $(\widetilde{G}, D(\widetilde{G}))$. Remark that we did not assume any differentiability for $(\widetilde{U}(t,s))_{t \le s}$, and hence the precise description of the domain $D(\widetilde{G})$ is difficult (compare Section 2.2 below, or see [47]). However, in [56, Proposition 2.1], the following important property of $D(\widetilde{G})$ is proved.

Lemma 2.4. *The domain $D(\widetilde{G})$ of the generator $\widetilde{G}$ of the evolution semigroup $(\widetilde{T}(t))_{t\geq 0}$ in $L^p(\mathbb{R}, X)$ is a dense subspace of $C_0(\mathbb{R}, X)$.*

Moreover, we can make the following remark.

Remark 2.5. The evolution semigroup $(\widetilde{T}(t))_{t\geq 0}$ can be written as

$$\widetilde{T}(t)\widetilde{f} = \widetilde{U}(\cdot, \cdot + t)\widetilde{T}_l(t)\widetilde{f}, \quad \widetilde{f} \in \widetilde{E},$$

where $(\widetilde{T}_l(t))_{t\geq 0}$ is the left translation semigroup on the space $\widetilde{E}$ with generator $\widetilde{C}\widetilde{f} := \widetilde{f}'$ on the domain

$$D(\widetilde{C}) := W^{1,p}(\mathbb{R}, X) := \{\widetilde{f} \in \widetilde{E} : \widetilde{f} \text{ absolutely continuous and } \widetilde{f}' \in \widetilde{E}\}.$$

Since $(\widetilde{G}, D(\widetilde{G}))$ is a local operator ([56, Theorem 2.4]), we can restrict it to the space $E := L^p(\mathbb{R}_-, X)$ by the following definition.

Definition 2.6. Take

$$D(G) := \{\widetilde{f}_{|\mathbb{R}_-} \, : \, \widetilde{f} \in D(\widetilde{G})\}$$

and define

$$Gf := (\widetilde{G}\widetilde{f})_{|\mathbb{R}_-} \qquad \text{for } f = \widetilde{f}_{|\mathbb{R}_-} \in D(\widetilde{G}).$$

Remark 2.7. Recently, H. Nguyen Thieu proved that $(G, D(G))$ coincides with an operator $(I, D(I))$ defined analogously to [44, Introduction and Preliminaries].

This operator G with its maximal domain $D(G)$ is not a generator on E. However, if we identify E with the subspace $Y := \{f \in \widetilde{E} : f(s) = 0 \; \forall s > 0\}$, then E remains invariant under $(\widetilde{T}(t))_{t\geq 0}$ yielding, using Remark 2.5, the semigroup described in the following lemma.

Lemma 2.8. *The semigroup $(T_0(t))_{t\geq 0}$ induced by $(\widetilde{T}(t))_{t\geq 0}$ on E is*

$$(T_0(t)f)(s) = U(s, s+t)(T_l(t)f)(s) = \begin{cases} U(s, s+t)f(t+s), & s+t \leq 0, \\ 0, & s+t > 0, \end{cases}$$

for $f \in E$, where the left translation semigroup $(T_l(t))_{t\geq 0}$ on E is defined by

$$(T_l(t)f)(\tau) := \begin{cases} f(t+\tau) & \text{for } t+\tau \leq 0, \\ 0 & \text{for } t+\tau \geq 0. \end{cases}$$

The following lemma characterizes the generator of this semigroup.

Lemma 2.9. *The generator* $(G_0, D(G_0))$ *of* $(T_0(t))_{t\geq 0}$ *is given by*

$$D(G_0) = \{f \in D(\widetilde{G}) \cap E \, : \, f(0) = 0\}, \qquad G_0 f = Gf.$$

Proof. Since $(T_0(t))_{t\geq 0}$ is the restriction of $(\widetilde{T}(t))_{t\geq 0}$ to the invariant, closed subspace E, where we identify E with Y, the generator of $(T_0(t))_{t\geq 0}$ is the part of $\widetilde{G}$ in Y, i.e.,

$$\widetilde{G}_{|_Y} f := \widetilde{G} f,$$

with domain

$$D(\widetilde{G}_{|_Y}) := \{f \in D(\widetilde{G}) \cap Y : \widetilde{G} f \in Y\}.$$

Since $\widetilde{G}$ is a local operator, then $(\widetilde{G}f)(s) = 0$ for all $s \geq 0$. Thus we can identify $D(G_0)$ with $D(\widetilde{G}_{|_Y})$, and the thesis follows.

□

We thus end up with operators

$$(G_0, D(G_0)) \subset (G, D(G)) \subset (\widetilde{G}, D(\widetilde{G})),$$

where only the first and the third are generators on E and $\widetilde{E}$, respectively.

2.2 Well-posedness of Backward Nonautonomous Cauchy Problems

In this section we adapt the concept of well-posedness of nonautonomous Cauchy problems (see, e.g., [47], [51] [52]) to our situation, i.e., we replace $\mathbb{R}$ by $\mathbb{R}_-$ and consider

$$(NCP)_s \qquad \begin{cases} \dot{u}(t) = -A(t)u(t), & t \leq s \leq 0, \\ u(s) = x \in X, \end{cases}$$

on a Banach space X for a family $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ of (unbounded) linear operators. Such nonautonomous Cauchy problems on a half line have been studied, e.g., by D. Henry, H. Nguyen Thieu, N. Van Minh, F. Räbiger, R. Schnaubelt (see, e.g., [34], [43] or [44]).

Definition 2.10. A continuous function $u : (-\infty, s] \to X$ is called a **classical solution** of $(NCP)_s$ if $u_s(\cdot, x) = u(\cdot) \in C^1((-\infty, s], X), u(t) \in D(A(t))$ for all $t \le s \le 0$, $u(s) = x$, and $\dot{u}(t) = -A(t)u(t)$ for $t \le s \le 0$.

Definition 2.11. For a family $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ of linear operators on the Banach space X, the nonautonomous Cauchy problem $(NCP)_s$ is called **well-posed** with **regularity subspaces** $(Y_s)_{s\in\mathbb{R}_-}$ if the following holds.

(i) **(Existence)** For each $s \in \mathbb{R}_-$ the subspace

$$Y_s := \{x \in X \;:\; \text{ there exists a classical solution for } (NCP)_s\} \subset D(A(s))$$

is dense in X.

(ii) **(Uniqueness)** For every $x \in Y_s$ the solution $u_s(\cdot, x)$ is unique.

(iii) **(Continuous dependence)** The solution depends continuously on s and x, i. e., if $s_n \to s \in \mathbb{R}_-$, $x_n \to x \in Y_s$ with $x_n \in Y_{s_n}$, then

$$\|\hat{u}_{s_n}(t, x_n) - \hat{u}_s(t, x)\| \to 0$$

uniformly for t in compact subsets of $\mathbb{R}_-$, where

$$\hat{u}_s(t, x) := \begin{cases} u_s(t, x) & \text{if } s \ge t, \\ x & \text{if } s < t. \end{cases}$$

If, in addition, there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that

$$\|u_s(t, x)\| \le Me^{\omega(s-t)}\|x\|$$

for all $x \in Y_s$ and $t \ge s$, then $(NCP)_s$ is called **well-posed with exponentially bounded solutions**.

The conditions required to obtain a well-posed Cauchy problem $(NCP)_s$, when $A(\cdot)$ is a family of *bounded* operators, are well understood (see [18] or [23] for details).

In the general case, when the operators are unbounded, it is a very delicate matter to prove that a nonautonomous abstract Cauchy problem is well-posed.

We connect well-posedness to the existence of a backward evolution family solving the nonautonomous Cauchy problem.

Definition 2.12. A backward evolution family $(U(t,s))_{t\le s\le 0}$ is called a **backward evolution family solving** $(NCP)_s$ if there are dense subspaces Y_s of X such that the function $t \mapsto u(t) = U(t,s)x$ is a classical solution of $(NCP)_s$ for $s \in \mathbb{R}_-$ and $x \in Y_s$.

Remark 2.13. We observe that the well-posedness of $(NCP)_s$ on *regularity subspaces* $(Y_s)_{s\in\mathbb{R}_-}$, see [22, Section IV.9], is equivalent to the well-posedness of the following forward Cauchy problem

$$(CP) \qquad \begin{cases} \dot{v}(\tau) = B(\tau)v(\tau), & \tau \ge \sigma \ge 0, \\ v(\sigma) = x \end{cases}$$

on *regularity subspaces* $Z_\tau := Y_{-\tau}$ with solutions $v(\tau) := V(\tau,\sigma)x$, where the family $(V(\tau,\sigma))_{\tau\ge\sigma\ge 0}$ is the forward evolution family corresponding to $(U(t,s))_{t\le s\le 0}$ (see Remark 2.2.1) and the operators $B(\tau)$ are defined as $B(\tau) := A(-\tau)$, $\tau \in \mathbb{R}_+$.

As in [52, Proposition 2.5], we can show that for each well-posed $(NCP)_s$ there exists a unique backward evolution family $(U(t,s))_{t\le s\le 0}$ solving $(NCP)_s$, i.e., the function $t \mapsto u(t) := U(t,s)x$ is a classical solution of $(NCP)_s$ for $s \in \mathbb{R}_-$ and $x \in Y_s$.

Lemma 2.14. *If $(NCP)_s$ is well-posed and the backward evolution family $(U(t,s))_{t\le s\le 0}$ solves it, then*

$$U(t,s)Y_s \subseteq Y_t \subseteq D(A(t))$$

for all $t \le s \le 0$.

Moreover, the well-posedness of the backward nonautonomous Cauchy problem $(NCP)_s$ can be characterized by properties of the generator $(G_0, D(G_0))$ of $(T_0(t))_{t\ge 0}$. As in [52, Theorem 2.9], one can prove the following theorem.

Theorem 2.15. *Let X be a Banach space and $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ be a family of linear operators on X. The following assertions are equivalent.*

(i) The backward nonautonomous Cauchy problem $(NCP)_s$ for the family $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ is well-posed with exponentially bounded solutions.

(ii) There exists a unique evolution semigroup $(T_0(t))_{t\geq 0}$ with generator $(G_0, D(G_0))$ and an invariant core $\mathcal{D}$ such that

$$G_0 f = A(\cdot)f + f' \quad a.e.$$

for $f \in \mathcal{D}$.

In particular, we have the following result which can be shown in the same way as in [60, Proposition 1.13].

Proposition 2.16. *Let $(NCP)_s$ be well-posed with regularity subspaces $(Y_s)_{s\in\mathbb{R}_-}$ solved by an evolution family $(U(t,s))_{t\leq s\leq 0}$. Let $(G_0, D(G_0))$ be the generator of the corresponding evolution semigroup $(T_0(t))_{t\geq 0}$ on E. Then the set*

$$D_0 := \{f \in W^{1,p}(\mathbb{R}_-, X) : f(0) = 0, f(s) \in Y_s, s \mapsto A(s)f(s) \in E\}$$

is a core of G_0. Moreover

$$G_0 f = f' + A(\cdot)f \quad a.e.$$

for $f \in D_0$.

Finally, we state a lemma which will be useful for Propositions 3.7 and 3.8.

Lemma 2.17. *(a) If a continuous function $\widetilde{f} \in L^p(\mathbb{R}, X)$ satisfies $f := \widetilde{f}_{|\mathbb{R}_-} \in D(G)$ and $\widetilde{f}_{|\mathbb{R}_+} \in W^{1,p}(\mathbb{R}_+, X)$, then $\widetilde{f} \in D(\widetilde{G})$ and*

$$(\widetilde{G}\widetilde{T}(t)\widetilde{f})(s) = \begin{cases} U(s,s+t)(Gf)(s+t) & \text{for } s \leq s+t \leq 0, \\ U(s,0)\widetilde{f}'(s+t) & \text{for } s \leq 0 \leq s+t, \\ \widetilde{f}'(s+t) & \text{for } 0 \leq s \leq s+t. \end{cases}$$

(b) If $\widetilde{f}_{|\mathbb{R}_-} \in D(G)$ and $\widetilde{f}_{|\mathbb{R}_+} \in C^1(\mathbb{R}_+, X)$, then

$$(\widetilde{T}(t)\widetilde{f})_{|\mathbb{R}_-} := (\widetilde{U}(\cdot, \cdot + t)\widetilde{f}(\cdot + t))_{|\mathbb{R}_-} \in D(G)$$

and

$$\frac{\partial}{\partial t}(\widetilde{T}(t)\widetilde{f})_{|\mathbb{R}_-} = G(\widetilde{T}(t)\widetilde{f})_{|\mathbb{R}_-}.$$

Proof. (a) By definition, $\widetilde{T}(t)\widetilde{f}$ is given by

$$(\widetilde{T}(t)\widetilde{f})(s) = \begin{cases} U(s, s+t)\widetilde{f}(s+t) & \text{for } s \le s+t \le 0, \\ U(s,0)\widetilde{f}(s+t) & \text{for } s \le 0 \le s+t, \\ \widetilde{f}(s+t) & \text{for } 0 \le s \le s+t. \end{cases}$$

For $s \geq 0$ and $h > 0$, we thus obtain that

$$\frac{1}{h}[\widetilde{T}(t+h)\widetilde{f}(s) - \widetilde{T}(t)\widetilde{f}(s)] = \frac{1}{h}(\widetilde{f}(s+t+h) - \widetilde{f}(s+t)),$$

which converges to $\widetilde{f}'(s+t)$ as $h \downarrow 0$.

For $s \leq 0 \leq s+t$ and $h > 0$, we have

$$\frac{1}{h}[\widetilde{T}(t+h)\widetilde{f}(s) - \widetilde{T}(t)\widetilde{f}(s)] = U(s,0)\left[\frac{\widetilde{f}(s+t+h) - \widetilde{f}(s+t)}{h}\right],$$

which converges to $U(s,0)\widetilde{f}'(s+t)$ as $h \downarrow 0$. Since $\widetilde{f}_{|\mathbb{R}_+} \in W^{1,p}(\mathbb{R}_+, X)$, this implies L^p-convergence.

Finally, for $s \leq s+t < 0$, we obtain the desidered result by extending $\widetilde{f}_{|\mathbb{R}_-}$ to a function in $D(\widetilde{G})$ and restricting it afterwards.

(b) Take $\alpha \in C^1(\mathbb{R})$ such that $\alpha(s) = 1$ for $s \leq t+1$ and $\alpha(s) = 0$ for $s \geq t+2$. Then $\alpha\widetilde{f}$ satisfies the assumption in (a), i.e., $\alpha\widetilde{f} \in L^p(\mathbb{R}, X)$, $f := (\alpha\widetilde{f})_{|\mathbb{R}_-} \in D(G)$ and $(\alpha\widetilde{f})_{|\mathbb{R}_+} \in W^{1,p}(\mathbb{R}_+, X)$. By the locality of G the assertion follows. □

2.3 The Critical Spectrum and the Critical Growth Bound for a Backward Evolution Semigroup

It is wellknown that the spectral mapping theorem does not hold for all strongly continuous semigroups. In general, only one inclusion is true, i.e., if $(e^{tA})_{t\geq 0}$ is the strongly continuous semigroup generated by A, then

$$e^{t\sigma(A)} \subseteq \sigma(e^{tA}), \quad t \geq 0.$$

In [48], J. Poland and R. Nagel introduced the *critical spectrum* in order to deal with this problem. Following their paper we recall their definitions.

Let X be a Banach space and $\mathcal{T} := (T(t))_{t\geq 0}$ be a strongly continuous semigroup on X. We can extend this semigroup to a semigroup $\widetilde{\mathcal{T}} :=(\widetilde{T}(t))_{t\geq 0}$ on $\widetilde{X} := \{(x_n)_{n\in\mathbb{N}} \subset X : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\}$, no longer strongly continuous, by

$$\widetilde{T}(t)(x_n) := (T(t)x_n)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in \widetilde{X}.$$

Now, the subspace

$$\widetilde{X}_{\mathcal{T}} := \{(x_n)_{n\in\mathbb{N}} \in \widetilde{X} : \lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} \|T(t)x_n - x_n\| = 0\}$$

is closed and $(\widetilde{T}(t))_{t\geq 0}$ -invariant. Therefore, on the quotient space

$$\hat{X} := \widetilde{X}/\widetilde{X}_{\mathcal{T}}$$

the quotient operators

$$\hat{T}(t)(\widetilde{x} + \widetilde{X}_{\mathcal{T}}) := \widetilde{T}(t)\widetilde{x} + \widetilde{X}_{\mathcal{T}}, \quad \widetilde{x} + \widetilde{X}_{\mathcal{T}} \in \hat{X},$$

are well defined and yield a semigroup $\hat{\mathcal{T}} := (\hat{T}(t))_{t\geq 0}$ of bounded operators on $\hat{X}$.

Definition 2.18. The **critical spectrum** of the semigroup $(T(t))_{t\geq 0}$ is defined as

$$\sigma_{\text{crit}}(T(t)) := \sigma(\hat{T}(t)), \quad t \geq 0,$$

while the **critical growth bound** is

$$\omega_{\text{crit}}(\mathcal{T}) := \inf\{\omega \in \mathbb{R} : \exists\, M \geq 1 \text{ such that } \|\hat{T}(t)\| \leq Me^{\omega t} \text{for all } t \geq 0\} = \omega_0(\hat{T}(\cdot)) .$$

For our purposes it will be important to determine the critical spectrum and the critical growth bound of a backward evolution semigroup.

Let $\mathcal{U} :=(U(t,s))_{t\leq s\leq 0}$ be a backward evolution family on $\mathbb{R}_-$ and $(T_0(t))_{t\geq 0}$ the corresponding evolution semigroup on E, i.e.,

$$(T_0(t)f)(s) := \begin{cases} U(s,s+t)f(s+t), & s+t \leq 0, \\ 0, & s+t > 0, \end{cases} \tag{2.1}$$

for $f \in E$. Then, following [16, Theorem 3.22], [22, Section VI.9], [43, Corollary 2.4] or [44, Theorem 2.3] one can prove that the spectral mapping theorem holds for $(T_0(t))_{t\geq 0}$.

Theorem 2.19. *Let $(G_0, D(G_0))$ be the generator of $(T_0(t))_{t\geq 0}$ on E. Then $\sigma(T_0(t))$ is a disk centered at the origin and the spectrum $\sigma(G_0)$ is a halfplane. Moreover, $(T_0(t))_{t\geq 0}$ satisfies the spectral mapping theorem*

$$\sigma(T_0(t)) \setminus \{0\} = e^{t\sigma(G_0)}, \qquad t \geq 0.$$

In particular, $s(G_0) = \omega_0(T_0(\cdot)) = \omega_0(\mathcal{U})$.

Based on this result we can compute the critical spectrum for $(T_0(t))_{t\geq 0}$.

Theorem 2.20. *The critical spectrum of $T_0 := (T_0(t))_{t\geq 0}$ coincides with its spectrum, i.e.,*

$$\sigma_{crit}(T_0(t)) = \sigma(T_0(t)), \quad t \geq 0. \tag{2.2}$$

Proof. " $\supseteq$ " It is obvious that $\sigma_{\text{crit}}(T_0(t)) \subseteq \sigma(T_0(t))$.
" $\subseteq$ " Using rescaling, the inclusion follows if we can show that

$$2\pi i\mathbb{Z} \in \sigma(G_0) \Rightarrow 1 \in \sigma_{\text{crit}}(T_0(1)). \tag{2.3}$$

Since the spectrum $\sigma(G_0)$ is the union of the approximate point spectrum $A\sigma(G_0)$ and the residual spectrum $R\sigma(G_0)$ (see, e.g., [22, Section IV.1]), it follows from $2\pi i\mathbb{Z} \subset \sigma(G_0)$ that at least one of the sets

$$A\sigma(G_0) \cap 2\pi i\mathbb{Z} \qquad \text{or} \qquad R\sigma(G_0) \cap 2\pi i\mathbb{Z}$$

is unbounded. In the first case the assertion follows from [9, Proposition 4]. Assume now that $2\pi i k_n \in R\sigma(G_0)$ for some unbounded sequence $(k_n)_{n\in\mathbb{N}}$.

Observe next (see [22, Proposition IV.2.18]) that

$$\sigma_{\text{crit}}(T_0(t)) = \sigma(\hat{T}_0(t)) = \sigma(\hat{T}_0{}'(t)) \tag{2.4}$$

on

$$(\hat{X})' = \left(\widetilde{X}/X_{T_0}\right)' \cong (X_{T_0})^\circ \subset (\widetilde{X})'. \tag{2.5}$$

By [22, Proposition IV.2.18], one has

$$2\pi i k_n \in R\sigma(G_0) = P\sigma(G_0') \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.6}$$

Therefore, there exists $x_n' \in X'$, $\|x_n'\| = 1$, such that $T_0'(t)x_n' = e^{2\pi i k_n t}x_n'$ for $t \geq 0$ and all $n \in \mathbb{N}$.

We define

$$y_n' := x_n' - T_0'\left(\frac{1}{2k_n}\right)x_n' = 2x_n' \quad \text{for } n \in \mathbb{N}.$$

It holds that $T_0'(1)y_n' = y_n'$ and

$$\lim_{n\to\infty}\langle y_n', x_n\rangle = \lim_{n\to\infty}\langle x_n', x_n - T_0\left(\frac{1}{2k_n}\right)x_n\rangle = 0. \tag{2.7}$$

for all $(x_n)_{n\in\mathbb{N}} \in X_{T_0}$.

Define $y' \in (\widetilde{X})'$ such that

$$\langle y', (x_n)\rangle := \psi(\langle y_n', x_n\rangle) \quad \text{for all } (x_n) \in \widetilde{X},$$

where ψ is a Banach limit on l^∞. By (2.7),

$$y' \in (X_{T_0})^\circ \quad \text{and} \quad \widetilde{T}_0(1)'y' = (T_0'(1)y') = y'.$$

Thus $\hat{T_0}'(1)y' = y'$ and $1 \in \sigma(\hat{T}_0(1))$. □

Corollary 2.21. *The critical growth bound of a backward evolution semigroup is equal to the growth bound of the corresponding evolution family, i.e.,*

$$\omega_{crit}(T_0(\cdot)) = \omega_0(\mathcal{U}) = \omega_0(T_0(\cdot)).$$

Proof. By Theorem 2.20, we have

$$\omega_{\text{crit}}(T_0(\cdot)) = \omega_0(T_0(\cdot)),$$

while Theorem 2.19 implies

$$\omega_0(T_0(\cdot)) = \omega_0(\mathcal{U}).$$

□

Chapter 3

Well-posedness

The distinction between the past, present, and future is only an illusion, however persistent.
[A. Einstein, 1879-1955]

3.1 Delay Equations with Nonautonomous Past as Abstract Cauchy Problems

Motivated by the examples in Chapter 1, we now introduce a new type of delay equations. We start from linear partial differential equations with delay and recall that they can be written in an abstract way and using standard notation as

$$(DE) \qquad \begin{cases} \dot{u}(t) = Bu(t) + \Phi u_t, & t \geq 0, \\ u(0) = x \in X, \\ u_0 = f \in L^p([-1,0], X), \end{cases}$$

for some Banach space X, where $(B, D(B))$ is a closed, densely defined operator on X, the **delay operator** $\Phi : W^{1,p}([-1,0], X) \to X$ is a linear operator, and the **history function** $u_t : [-1,0] \to X$ is defined as $u_t(\tau) := u(t+\tau)$ for $\tau \in [-1,0]$. Many authors, e.g., O. Diekmann, J. Hale, S.M.V. Lunel, I. Miyadera, J. Wu studied such delay equations using semigroups and we refer to [20], [32], [33], [38] or [70] for this theory.

Recently, A. Bátkai and S. Piazzera in [5], [6], [7], [8], [55] studied (DE) with semigroup techniques in an L^p-setting. In particular, they showed that solving (DE) is "equivalent" to solving the abstract Cauchy problem

$$(CP) \qquad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{B}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}, \end{cases}$$

on the product space $\mathcal{E} := X \times L^p([-1,0],X)$, where $\mathcal{B}$ is defined by the operator matrix

$$\mathcal{B} := \begin{pmatrix} B & \Phi \\ 0 & d/d\tau \end{pmatrix}$$

on the domain

$$D(\mathcal{B}) := \left\{ \binom{x}{f} \in D(B) \times W^{1,p}([-1,0],X) \, : \, f(0) = x \right\}.$$

Applying perturbation theory for C_0-semigroups they proved the generator property of $\mathcal{B}$, hence well-posedness of (DE). In addition, they obtained results on the asymptotics of the solutions of (CP) and hence of (DE) (see [15]).

In this thesis we also allow infinite delay, so we substitute the compact interval $[-1,0]$ by $\mathbb{R}_-$.

Using this semigroup approach, the meaning of (DE) is that if we start with the history function f, this function is shifted to the left by $-t$, and for values greater than $-t$ the value of the solution is given by the delay operator Φ applied to the shifted function (see Figure 3.1).

As we have seen in Chapter 1, there are cases in which the function f is not only shifted but also modified by an evolution family.

For this reason, we replace the delay equations (DE) by equations that we call **delay equations with nonautonomous past**

$$(NDE) \qquad \begin{cases} \dot{u}(t) = Bu(t) + \Phi\tilde{u}_t, & t \geq 0, \\ u(0) = x \in X, \\ \tilde{u}_0 = f \in L^p(\mathbb{R}_-,X). \end{cases}$$

Here, the delay operators acts on a **modified history function** $\widetilde{u}_t$ (see below). We now fix the notations and assumptions to be used in the rest of this thesis.

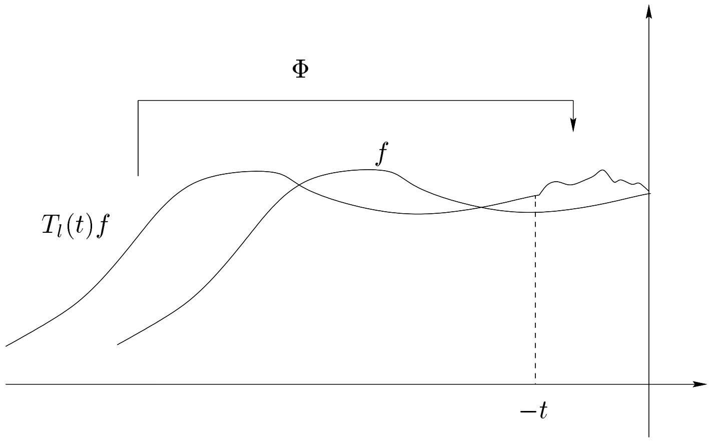

Figure 3.1: history function

General assumptions 3.1.

1. The operator $(B, D(B))$ is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X.
2. The (linear) delay operator $\Phi : C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X) \subseteq D(\Phi) \to X$ is bounded with respect to $\|\cdot\|_p$ or $\|\cdot\|_\infty$.
3. The evolution family $(\widetilde{U}(t,s))_{t\leq s}$ is the extension (as in Definition 2.3) of an evolution family $(U(t,s))_{t\leq s\leq 0}$ solving a backward nonautonomous Cauchy problem for a family of operators $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ on regularity subspaces Y_t.

Definition 3.2. The **modified history function** (see Figure 3.2) $\tilde{u}_t$ in (NDE) is defined as

$$\begin{aligned}\tilde{u}_t(\tau) :&= \begin{cases} \widetilde{U}(\tau, t+\tau)u(t+\tau) & \text{for } t+\tau \geq 0, \\ \widetilde{U}(\tau, t+\tau)f(t+\tau) & \text{for } t+\tau \leq 0, \end{cases} \\ &= \begin{cases} U(\tau, 0)u(t+\tau) & \text{for } t+\tau \geq 0 \geq \tau, \\ U(\tau, t+\tau)f(t+\tau) & \text{for } 0 \geq t+\tau \geq \tau \end{cases}\end{aligned}$$

for $\tau \leq 0$.

Remark 3.3. In the definition of the modified history function $\widetilde{u}_t$ two time variables t and τ appear. The variable t can be interpreted as the "absolute time" and τ as the "relative time".

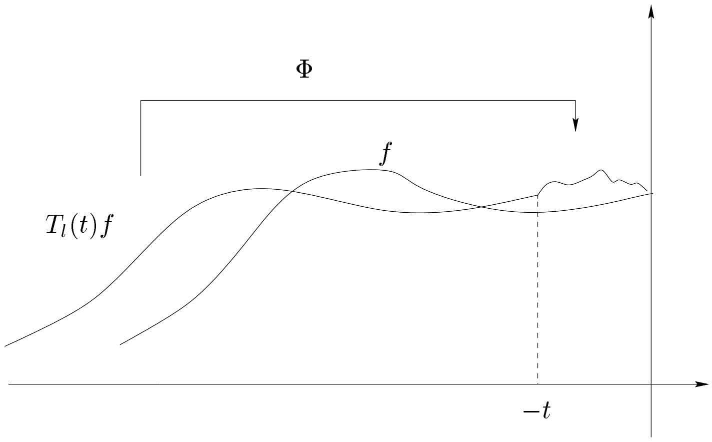

Figure 3.2: modified history function

Definition 3.4. 1. We call a function $u : \mathbb{R} \to X$ a **classical solution** of (NDE) if

(i) $u \in C(\mathbb{R}, X) \cap C^1(\mathbb{R}_+, X)$,

(ii) $u(t) \in D(B)$, $\tilde{u}_t \in D(\Phi)$, $t \geq 0$,

(iii) u satisfies (NDE) for all $t \geq 0$.

2. We call (NDE) **well-posed** if

(i) for every $\binom{x}{f}$ in a dense subspace $\mathcal{S} \subseteq X \times L^p(\mathbb{R}_-, X)$ there is a unique (classical) solution $u(x, f, \cdot)$ of (NDE) and

(ii) the solutions depend continuously on the initial values, i.e., if a sequence $\binom{x_n}{f_n}$ in $\mathcal{S}$ converges to $\binom{x}{f} \in \mathcal{S}$, then $u(x_n, f_n, t)$ converges to $u(x, f, t)$ uniformly for t in compact intervals.

It is now our purpose to investigate existence and uniqueness of the solutions of (NDE). To do this we can use the same approach of A. Bátkai and S. Piazzera from [7], i.e., we reformulate equation (NDE) as an abstract Cauchy problem on the space $\mathcal{E} := X \times L^p(\mathbb{R}_-, X)$ using, instead of the derivative $\frac{d}{d\sigma}$, the operator G from Definition 2.6.

Definition 3.5. Consider the operator given by the matrix

$$\mathcal{C} := \begin{pmatrix} B & \Phi \\ 0 & G \end{pmatrix}$$

on the domain

$$D(\mathcal{C}) := \left\{ \binom{x}{f} \in D(B) \times D(G) \,:\, f(0) = x \right\} \quad \text{in } \mathcal{E} = X \times L^p(\mathbb{R}_-, X)$$

and the corresponding abstract Cauchy problem

$$(CP) \qquad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{C}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}. \end{cases}$$

It is easy to show that $\mathcal{C}$ is closed and densely defined on $\mathcal{E}$. We now prove that (NDE) and the abstract Cauchy problem (CP) are "equivalent", i.e., (NDE) has a unique solution for $\binom{x}{f} \in D(\mathcal{C})$ continuously depending on the initial value if and only if (CP) is well-posed (in the usual sense). In analogy to [7, Theorem 2.8], we thus obtain well-posedness of (NDE) by proving well-posedness of (CP) for the operator $(\mathcal{C}, D(\mathcal{C}))$.

Theorem 3.6. *The delay equation (NDE) is well-posed if and only if the operator $(\mathcal{C}, D(\mathcal{C}))$ defined above is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$. In this case, (NDE) has a unique solution u for every $\binom{x}{f} \in D(\mathcal{C})$ given by*

$$u(t) = \begin{cases} \pi_1\left(\mathcal{T}(t)\binom{x}{f}\right), & t \geq 0, \\ f(t), & a.e.\ t \leq 0, \end{cases} \tag{3.1}$$

where π_1 is the projection onto the first component of $\mathcal{E}$.

The main part of the assertion is a consequence of the following two propositions, while the continuous dependence follows from the relation between the solutions of (NDE) and (CP).

Proposition 3.7. *Consider $\binom{x}{f} \in D(\mathcal{C})$ and let $\mathbb{R}_+ \ni t \mapsto \mathcal{U}(t) := \binom{z(t)}{v(t)} \in \mathcal{E}$ a classical solution of (CP) with initial value $\binom{x}{f}$. Then for the function $u : \mathbb{R} \to X$ defined by*

$$u(t) := \begin{cases} z(t), & t \geq 0, \\ f(t), & a.e.\ t < 0, \end{cases}$$

we have $\widetilde{u}_t = v(t)$ for every $t \geq 0$ and u is a classical solution of (NDE).

Proof. Since $\mathcal{U}$ is a classical solution of (CP), we obtain that $v \in C^1(\mathbb{R}_+, L^p(\mathbb{R}_-, X))$ solves the equation

$$\begin{cases} \dot{y}(t) = Gy(t), & t \geq 0, \\ y(t)(0) = z(t), & t \geq 0, \\ y(0) = f, \end{cases} \tag{3.2}$$

in the space $L^p(\mathbb{R}_-, X)$. On the other hand, also the function $t \mapsto \widetilde{U}(\cdot, \cdot+t)u(\cdot+t) = \widetilde{u}_t(\cdot)$ solves equation (3.2) since $\widetilde{u}_t(0) = \widetilde{U}(0,t)u(t) = z(t)$ for $t \geq 0$, $\widetilde{u}_0 = u_{|\mathbb{R}_-} = f \in D(G)$, and, by Lemma 2.17 (b), we have $u \in D(\widetilde{G})$. Thus $\widetilde{u}_t = (\widetilde{T}(t)u)_{|\mathbb{R}_-}$ is differentiable with derivative $G\widetilde{u}_t$. We now define $t \mapsto w(t) := v(t) - \widetilde{u}_t$ which solves (3.2) but with $f = 0$ and $z(\cdot) \equiv 0$. In this way it is the solution of the abstract Cauchy problem for the generator G_0 with initial value 0 which implies $w(t) = 0$. Therefore, we obtain

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ \widetilde{u}_t \end{pmatrix},$$

and thus u is a solution of (NDE). □

The converse of Proposition 3.7 is also true.

Proposition 3.8. *Let* $\binom{x}{f} \in D(\mathcal{C})$ *and let* $u : \mathbb{R} \to X$ *be a classical solution of* (NDE)*. Then the function*

$$\mathbb{R}_+ \ni t \mapsto \begin{pmatrix} u(t) \\ \widetilde{u}_t \end{pmatrix}$$

is a classical solution of (CP) *with initial value* $\binom{x}{f}$.

Proof. Since the function u is a solution of (NDE), it remains to show that the function $t \mapsto \widetilde{u}_t$ is continuously differentiable with derivative $G\widetilde{u}_t$. In order to show this, observe that $u_{|[0,\infty)} \in C^1(\mathbb{R}_+, X)$ and $u_{|\mathbb{R}_-} = f \in D(G)$. Thus, by Lemma 2.17 (b), we obtain $u \in D(\widetilde{G})$ and thus

$$\frac{d}{dt}\widetilde{T}(t)u = \frac{d}{dt}\widetilde{U}(\cdot, \cdot + t)u(\cdot + t) = \widetilde{G}\widetilde{T}(t)u = \widetilde{G}\widetilde{U}(\cdot, \cdot + t)u(\cdot + t).$$

Restricting this equation to $\mathbb{R}_-$ yields the desired result. □

Proof of Theorem 3.6. We assume that $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$. Then the Cauchy problem (CP) has a unique classical solution $\mathcal{U}$ for all $\binom{x}{f} \in D(\mathcal{C})$ and

$$\mathcal{U}(t) = \mathcal{T}(t)\binom{x}{f}, \quad t \geq 0.$$

Thus, the function u defined in (3.1) is a solution of (NDE) by Proposition 3.7. Uniqueness follows from Proposition 3.8.

Conversely, if (NDE) is well-posed, then for every $\binom{x}{f} \in D(\mathcal{C})$ equation (NDE) has a unique solution u. Then Proposition 3.7 yields that for every $\binom{x}{f} \in D(\mathcal{C})$ the abstract Cauchy problem (CP) has a classical solution which is unique by Proposition 3.8. Since (NDE) is well-posed, these solutions depend continuously on the initial values. Finally, since $(\mathcal{C}, D(\mathcal{C}))$ is a closed operator by [22, Theorem II.6.7], $(\mathcal{C}, D(\mathcal{C}))$ generates a strongly continuous semigroup on $L^p(\mathbb{R}_-, X)$. □

Remark 3.9. We consider

$$u(t) := \begin{cases} \pi_1\left(\mathcal{T}(t)\binom{x}{f}\right), & t \geq 0, \\ f(t), & \text{a.e. } t \leq 0, \end{cases} \tag{3.3}$$

as a **mild solution** of (NDE) for every $\binom{x}{f} \in \mathcal{E}$.

Our reason for this terminology is the following corollary.

Corollary 3.10. *If $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a C_0−semigroup $(\mathcal{T}(t))_{t\geq 0}$, then the function $u : \mathbb{R} \to X$ defined in (3.3) for every $\binom{x}{f} \in \mathcal{E}$ satisfies the integral equation*

$$u(t) = \begin{cases} x + B\int_0^t u(s)ds + \phi\int_0^t \widetilde{u}_s ds, & t \geq 0, \\ f(t), & a.e.\ t \in \mathbb{R}_-, \end{cases}$$

where $\widetilde{u}_s$ is as in Definition 3.2.

Proof. Let π_2 be the projection onto the second component of $\mathcal{E}$, i.e., $\pi_2\binom{x}{f} := f$ for all $\binom{x}{f} \in \mathcal{E}$.
First Step: We prove that

$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\binom{x}{f}) \quad \text{a.e.}. \tag{3.4}$$

Indeed, (3.4) holds by Proposition 3.7 for $\binom{x_n}{f_n} \in D(\mathcal{C})$. Take now $\binom{x}{f} \in \mathcal{E}$ and a sequence $\binom{x_n}{f_n} \in D(\mathcal{C})$ converging to $\binom{x}{f}$. Since the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous, the sequence $\mathcal{T}(t)\binom{x_n}{f_n}$ converges to $\mathcal{T}(t)\binom{x}{f}$ in $\mathcal{E}$.

Let

$$u_n(t) := \begin{cases} \pi_1(\mathcal{T}(t)\binom{x_n}{f_n}), & t \geq 0, \\ f_n(t), & \text{a.e. } t \leq 0. \end{cases}$$

Since $\binom{x_n}{f_n} \in D(\mathcal{C})$, we have $(\widetilde{u}_n)_t = \pi_2(\mathcal{T}(t)\binom{x_n}{f_n})$.

Moreover,

$$(\widetilde{u}_n)_t(s) = \widetilde{U}(s, s+t)u_n(s+t) = \widetilde{U}(s, s+t)\pi_1(\mathcal{T}(t+s)\tbinom{x_n}{f_n})$$

holds for $-t \leq s \leq 0$. By our assumptions, it follows that $\|(\widetilde{u}_n)_t - \widetilde{u}_t\|_p \to 0$, as $n \to +\infty$. Thus, there exists a subsequence $(\widetilde{u}_{n_k})_t$ of $(\widetilde{u}_n)_t$ such that $(\widetilde{u}_{n_k})_t(s) \to (\widetilde{u}_t)(s)$ a.e..

Since

$$(\widetilde{u}_{n_k})_t(s) = \pi_2(\mathcal{T}(t)\tbinom{x_{n_k}}{f_{n_k}})(s) \to \pi_2(\mathcal{T}(t)\tbinom{x}{f})(s),$$

we can conclude that

$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\tbinom{x}{f}) \quad \text{a.e..}$$

If $s \leq -t$, one has

$$(\widetilde{u}_n)_t(s) = \widetilde{U}(s, s+t)u_n(s+t) = \widetilde{U}(s, s+t)f_n(s+t).$$

Since $\|f_n - f\|_p \to 0$, there exists a subsequence f_{n_k} of f_n such that $f_{n_k}(s) \to f(s)$ a.e.. Thus

$$\begin{aligned}(\widetilde{u}_{n_k})_t(s) &= \widetilde{U}(s, s+t)u_{n_k}(s+t) = \widetilde{U}(s, s+t)f_{n_k}(s+t)\\ &\to \widetilde{U}(s, s+t)f(s+t) = (\widetilde{u}_t)(s) \quad \text{a.e. for } s \leq -t.\end{aligned}$$

Proceeding as before, we have

$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\tbinom{x}{f}) \quad \text{a.e..}$$

Second step: Taking the first component of the identity

$$\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{C}\int_0^t \mathcal{T}(s)\begin{pmatrix} x \\ f \end{pmatrix} ds,$$

one has

$$\begin{aligned} u(t) - x &= \pi_1\left[\mathcal{C}\begin{pmatrix} \int_0^t \pi_1(\mathcal{T}(s)\binom{x}{f})ds \\ \int_0^t \pi_2(\mathcal{T}(s)\binom{x}{f})ds \end{pmatrix}\right] \\ &= \pi_1\left[\begin{pmatrix} B & \Phi \\ 0 & G \end{pmatrix}\begin{pmatrix} \int_0^t u(s)ds \\ \int_0^t \widetilde{u}_s ds \end{pmatrix}\right] = B\int_0^t u(s)ds + \Phi\int_0^t \widetilde{u}_s ds \end{aligned}$$

for all $t \geq 0$. □

3.2 The Generator Property

In view of Theorem 3.4 we now give sufficient conditions in order that $\mathcal{C}$ generates a strongly continuous semigroup on $\mathcal{E}$. First, we write $\mathcal{C}$ in the form

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{F} := \begin{pmatrix} B & 0 \\ 0 & G \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

with domain $D(\mathcal{C}_0) = D(\mathcal{C})$ and $\mathcal{F} \in \mathcal{L}(D(\mathcal{C}_0), \mathcal{E})$.

We then show that $\mathcal{C}_0$ generates a strongly continuous semigroup and finally apply perturbation theory to $\mathcal{C}_0 + \mathcal{F}$.

As a first step, we compute the resolvent $R(\lambda, \mathcal{C}_0)$ of $\mathcal{C}_0$ for $\Re\lambda$ larger than the growth bound $\omega_0(T_0(\cdot))$ of the semigroup $(T_0(t))_{t\geq 0}$ from Lemma 2.8.

Lemma 3.11. *(i) For each $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(T_0(\cdot))$, we define the bounded operator $\epsilon_\lambda : X \to E$ by*

$$(\epsilon_\lambda x)(s) := e^{\lambda s} U(s, 0)x, \quad s \leq 0, x \in X. \tag{3.5}$$

Then $\epsilon_\lambda x$ is an eigenvector of G with eigenvalue λ for every $x \in X$.

(ii) For each $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(T_0(\cdot))$ and $\lambda \in \rho(B)$, we have $\lambda \in \rho(\mathcal{C}_0)$, and the resolvent $R(\lambda, \mathcal{C}_0)$ is given by

$$R_\lambda := \begin{pmatrix} R(\lambda, B) & 0 \\ \epsilon_\lambda R(\lambda, B) & R(\lambda, G_0) \end{pmatrix}. \tag{3.6}$$

Proof. (i): Since $\Re\lambda > \omega_0(T_0(\cdot))$, we take $\Re\lambda > \omega > \omega_0(T_0(\cdot))$ and $M_\omega \geq 1$ such that $\|U(r,s)\| \leq M_\omega e^{\omega(s-r)}$. It follows that $\|\epsilon_\lambda\|_{\mathcal{L}(X,E)} \leq \frac{M_\omega}{[p(\Re\lambda - \omega)]^{\frac{1}{p}}}$.

Consider $\alpha \in C^1(\mathbb{R})$ such that $\alpha(s) = 1$ for $s \leq 1$ and $\alpha(s) = 0$ for $s \geq 2$ and define

$$\tilde{f}(s) := \begin{cases} e^{\lambda s} U(s,0)x & \text{for } s \leq 0, \\ e^{\lambda s} x & \text{for } s > 0, \end{cases}$$

for arbitrary $x \in X$. Then $\widetilde{f}_\lambda := \alpha \tilde{f} \in \widetilde{E}$ is continuously differentiable for $s > 0$ and $\widetilde{f}(s) = 0$ for $s \geq 2$. For $\widetilde{f}_\lambda$ we obtain

$$\widetilde{T}(t)\widetilde{f}_\lambda(s) = e^{\lambda t}\widetilde{f}_\lambda(s)$$

for all $0 \le t \le 1$ and $s \le 0$, where $(\widetilde{T}(t))_{t\ge 0}$ is defined as in Definition 2.3.2. Together with the differentiability of $\tilde{f}$ for $s \ge 0$ this implies $\tilde{f}_\lambda \in D(\widetilde{G})$ and

$$(\widetilde{G}\tilde{f}_\lambda)(s) = \lambda(\tilde{f}_\lambda)(s)$$

for every $s \le 0$. By the definition of G this implies $\epsilon_\lambda x \in D(G)$ and

$$G\epsilon_\lambda x = \lambda\epsilon_\lambda x.$$

(ii): Let $\lambda \in \rho(B)$ with $\Re\lambda > \omega_0(T_0(\cdot))$, hence $\lambda \in \rho(G_0)$. Then the matrix in (3.6) is a bounded operator in $\mathcal{E}$ defining the inverse of $(\lambda - \mathcal{C}_0)$. In fact, for $\binom{x}{f} \in \mathcal{E}$ we have

$$R_\lambda \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} R(\lambda, B)x \\ \epsilon_\lambda R(\lambda, B)x + R(\lambda, G_0)f \end{pmatrix} \in D(\mathcal{C}_0)$$

since $R(\lambda, B)x \in D(B)$, $\epsilon_\lambda R(\lambda, B)x \in D(G)$ and $(\epsilon_\lambda R(\lambda, B)x + R(\lambda, G_0)f)(0) = R(\lambda, B)x$. Moreover, by (i),

$$(\lambda - \mathcal{C}_0)R_\lambda \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} x \\ (\lambda - G)(\epsilon_\lambda R(\lambda, B)x + R(\lambda, G_0)f) \end{pmatrix} = \begin{pmatrix} x \\ f \end{pmatrix}$$

since $(\lambda - G)\epsilon_\lambda R(\lambda, B)x = 0$. As a result we obtain $(\lambda - \mathcal{C}_0)R_\lambda = Id_{\mathcal{E}}$.

On the other hand, consider $\binom{x}{f} \in D(\mathcal{C}_0)$, thus $x \in D(B), f \in D(G)$, and $f(0) = x$. It follows that

$$R_\lambda(\lambda - \mathcal{C}_0) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} x \\ \epsilon_\lambda x + R(\lambda, G_0)(\lambda - G)f \end{pmatrix} = \begin{pmatrix} x \\ f \end{pmatrix}$$

since

$$\begin{aligned} \epsilon_\lambda x + R(\lambda, G_0)(\lambda - G)f &= \epsilon_\lambda f(0) + R(\lambda, G_0)(\lambda - G)[f - \epsilon_\lambda f(0)] \\ &= \epsilon_\lambda f(0) + R(\lambda, G_0)(\lambda - G_0)[f - \epsilon_\lambda f(0)] \\ &= \epsilon_\lambda f(0) + f - \epsilon_\lambda f(0) = f. \end{aligned} \tag{3.7}$$

□

As the next step, we determine explicitly the semigroup generated by $\mathcal{C}_0$.

Proposition 3.12. *The operator $(\mathcal{C}_0, D(\mathcal{C}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{E}$ given by*

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \quad t \geq 0,$$

where $S_t : X \to L^p(\mathbb{R}_-, X)$ is

$$(S_t x)(\tau) := \begin{cases} U(\tau,0)S(t+\tau)x, & \tau + t > 0, \\ 0, & \tau + t \leq 0. \end{cases}$$

Proof. In order to prove that $(\mathcal{T}_0(t))_{t\geq 0}$ is a semigroup, it is suffices to verify that

$$S_{t+s} = S_t S(s) + T_0(t) S_s \qquad \text{for all } t, s \geq 0.$$

In fact, we have
(3.8)

$$\begin{aligned}
(S_t S(s)x)(\tau) + (T_0(t)S_s x)(\tau) &= \begin{cases} U(\tau,0)S(t+\tau)S(s)x, & \tau + t > 0, \\ 0, & \tau + t \leq 0, \end{cases} \\
&+ \left(T_0(t) \begin{cases} U(\cdot,0)S(s+\cdot)x, & \cdot + s > 0, \\ 0, & \cdot + s \leq 0, \end{cases} \right)(\tau) \\
&= \begin{cases} U(\tau,0)S(t+s+\tau)x, & \tau + t > 0, \\ 0, & \tau + t \leq 0, \end{cases} \\
r &+ \begin{cases} U(\tau,0)S(s+\tau+t)x, & t \leq -\tau \leq (t+s), \\ 0, & \text{elsewhere}, \end{cases} \\
&= (S_{t+s}x)(\tau)
\end{aligned}$$

for all $x \in X$ and $\tau \leq 0$.

Now, denote the generator of $(\mathcal{T}_0(t))_{t\geq 0}$ by $(\mathcal{H}, D(\mathcal{H}))$.Using the definition of the function $t \mapsto S_t$ and a result of R. Schnaubelt (see [61, Lemma 2.11]), one can prove $\int_0^{+\infty} e^{-\lambda t} S_t dt = \epsilon_\lambda R(\lambda, B)$. Thus, for λ sufficiently large,

$$R(\lambda, \mathcal{H}) = \int_0^{+\infty} e^{\lambda t} \mathcal{T}_0(t) dt = \begin{pmatrix} R(\lambda, B) & 0 \\ \int_0^{+\infty} e^{-\lambda t} S_t dt & R(\lambda, G_0) \end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix} R(\lambda, B) & 0 \\ \epsilon_\lambda R(\lambda, B) & R(\lambda, G_0) \end{pmatrix} = R(\lambda, \mathcal{C}_0). \tag{3.10}$$

It follows that $(\mathcal{C}_0, D(\mathcal{C}_0))$ is the generator of $(\mathcal{T}_0(t))_{t\geq 0}$. □

By applying the bounded perturbation theorem we obtain the following immediate consequence.

Theorem 3.13. *Let $(\mathcal{C}_0, D(\mathcal{C}_0))$ be the generator of a strongly continuous semigroup on $\mathcal{E}$ and let $\Phi : L^p(\mathbb{R}_-, X) \to X$ be bounded. Then the operator $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}$, thus (NDE) is well-posed.*

In the case of unbounded Φ (e.g. for point delays), we apply the theorem of Miyadera-Voigt (see [46] or [66]), which we quote from [22, Corollary III.3.16].

Theorem 3.14. *Let $(W_0(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X with generator $(H_0, D(H_0))$ and $F : D(H_0) \to X$ be a linear operator on X. Suppose that there exist $t_0 > 0$ and $0 \leq q < 1$ such that*

$$\int_0^{t_0} \|FW_0(t)x\| dt \leq q\|x\| \tag{3.11}$$

for all $x \in D(H_0)$. Then $(H_0+F, D(H_0))$ generates a strongly continuous semigroup $(W(t))_{t\geq 0}$ on $\mathcal{E}$ satisfying

$$W(t)x = W_0(t)x + \int_0^t W(t-s)FW_0(s)x ds$$

for all $x \in D(H_0), t \geq 0$.

If we take in Theorem 3.14 the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ as $(W_0(t))_{t\geq 0}$ and the operator $\mathcal{F}$ as F, we obtain the following result.

Theorem 3.15. *Let $(\mathcal{C}_0, D(\mathcal{C}_0))$ be the generator of the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ defined as above and consider a linear operator $\Phi : C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X) \to X$ satisfying*

$$\int_0^{t_0} \|\Phi(S_r x + T_0(r)f)\| dr \leq q \left\| \binom{x}{f} \right\| \tag{M}$$

for some $0 < t_0$, $0 \leq q < 1$ and for all $\binom{x}{f} \in D(\mathcal{C}_0)$. Then $(\mathcal{C}, D(\mathcal{C}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$ satisfying

$$\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{T}_0(t)\begin{pmatrix} x \\ f \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\mathcal{F}\mathcal{T}_0(s)\begin{pmatrix} x \\ f \end{pmatrix} ds$$

for all $\binom{x}{f} \in D(\mathcal{C}_0), t \geq 0$. Thus (NDE) is well-posed.

Proof. We only have to observe that

$$\begin{aligned}\|\mathcal{F}\mathcal{T}_0(t)\left(\begin{smallmatrix}x\\ f\end{smallmatrix}\right)\| &= \left\|\begin{pmatrix}0 & \Phi\\ 0 & 0\end{pmatrix}\begin{pmatrix}S(t) & 0\\ S_t & T_0(t)\end{pmatrix}\begin{pmatrix}x\\ f\end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix}\Phi(S_t x + T_0(t)f)\\ 0\end{pmatrix}\right\| = \|\Phi(S_t x + T_0(t)f)\|\end{aligned}$$

for all $\left(\begin{smallmatrix}x\\ f\end{smallmatrix}\right) \in D(\mathcal{C}_0)$. Therefore, (3.11) becomes (M). □

Example 3.16. Let $1 < p < \infty$ and let $\eta : \mathbb{R}_- \to \mathcal{L}(X)$ be of bounded variation such that $|\eta|(\mathbb{R}_-) < +\infty$, where $|\eta|$ is the positive Borel measure in $\mathbb{R}_-$ defined by the total variation on η. Let $\Phi : C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X) \to X$ be the linear operator given by the Riemann-Stieltjes integral

$$\Phi f := \int_{-\infty}^{0} f\, d\eta \quad \text{for all } f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X). \tag{3.12}$$

By [3, Proposition 1.9.4], this integral is well-defined. We now show that Φ fulfills condition (M), thus (NDE) is well-posed.

In fact, using the exponential boundedness $\|U(t,s)\| \leq M_\omega e^{\omega(s-t)}$ and putting $M := \sup_{r\in[0,1]} \|S(r)\| < \infty$, we obtain for $0 < t < 1$

$$\begin{aligned}&\int_0^t \|\Phi(S_r x + T_0(r)f)\| dr \\ &= \int_0^t \left\| \int_{-r}^{0} U(\sigma,0)S(\sigma+r)x\, d\eta(\sigma) + \int_{-\infty}^{-r} U(\sigma,\sigma+r)f(\sigma+r)\, d\eta(\sigma)\right\| dr \\ &\leq \|x\| \int_0^t \int_{-r}^{0} \|U(\sigma,0)\|\, \|S(\sigma+r)\|\, d|\eta|(\sigma) dr \\ &\quad + \int_0^t \int_{-\infty}^{-r} \|U(\sigma,\sigma+r)\|\, \|f(\sigma+r)\| d|\eta|(\sigma) dr \\ &\leq \|x\| \int_0^t \int_{-r}^{0} M_\omega e^{-\omega\sigma} \|S(\sigma+r)\|\, d|\eta|(\sigma) dr + \int_0^t \int_{-\infty}^{-r} M_\omega e^{\omega r} \|f(\sigma+r)\| d|\eta|(\sigma) dr\end{aligned}$$

$$
\begin{aligned}
&\leq \|x\| \int_0^t M_\omega e^{|\omega| r} \int_{-r}^0 \|S(\sigma + r)\|\, d|\eta|(\sigma) dr \; + \int_0^t M_\omega e^{\omega r} \int_{-\infty}^{-r} \|f(\sigma + r)\| d|\eta|(\sigma) dr \\
&\leq \|x\| M_\omega e^{|\omega| t} \left[\int_0^t M \left(\int_{-r}^0 d|\eta|(\sigma) \right) dr \right. \\
&\qquad \left. + \int_{-t}^0 \left(\int_\sigma^0 \|f(s)\| ds \right) d|\eta|(\sigma) + \int_{-\infty}^{-t} \left(\int_\sigma^{\sigma+t} \|f(s)\| ds \right) d|\eta|(\sigma) \right] \\
&\leq M_\omega e^{|\omega| t} \left[tM |\eta|(\mathbb{R}_-) \|x\| + \int_{-t}^0 \sigma^{\frac{1}{p'}} \|f\|_p d|\eta|(\sigma) + \int_{-\infty}^{-t} t^{\frac{1}{p'}} \|f\|_p d|\eta|(\sigma) \right] \\
&\leq M_\omega e^{|\omega| t} \left[tM |\eta|(\mathbb{R}_-) \|x\| + \int_{-\infty}^0 t^{\frac{1}{p'}} \|f\|_p d|\eta|(\sigma) \right] \\
&\leq M_\omega e^{|\omega| t} |\eta|(\mathbb{R}_-) (tM \|x\| + t^{\frac{1}{p'}} \|f\|_p)
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus we can conclude

$$
\int_0^t \|\Phi(S_r x + T_0(r) f)\| dr \quad \leq \quad M M_\omega e^{|\omega| t} |\eta|(\mathbb{R}_-) t^{\frac{1}{p'}} (\|f\|_p + \|x\|)
$$

for all $0 < t < 1$. Choose now t_0 so small that $M M_\omega e^{|\omega| t_0} |\eta|(\mathbb{R}_-) t_0^{\frac{1}{p'}} < 1$. Then condition (M) is satisfied with $q := M M_\omega e^{|\omega| t_0} |\eta|(\mathbb{R}_-) t_0^{\frac{1}{p'}}$.

Remark 3.17. As concrete delay operators Φ having the form (3.12) we can take, for all $f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X)$,

$$
\Phi f := \int_{-\infty}^0 \phi(s) f(s) ds,
$$

where $\phi(\cdot) \in L^1(\mathbb{R}_-)$, or

$$
\Phi f := \delta_s f,
$$

where δ_s is the Dirac measure for some $s < 0$.

3.3 Classical Solutions

By Theorem 3.6 we obtained classical and mild solutions of (NDE). We now show that our approach also yields solutions to the following system of equations introduced by S. Brendle and R. Nagel in [10]:

$$
\frac{\partial}{\partial t} u(t,s) \;=\; \frac{\partial}{\partial s} u(t,s) + A(s) u(t,s), \quad s \leq 0,\ t \geq 0, \tag{3.13}
$$

$$
\frac{\partial}{\partial t} u(t,0) \;=\; Bu(t,0) + \Phi u(t,\cdot), \qquad t \geq 0, \tag{3.14}
$$

where $A(s)$ are (unbounded) operators on a Banach space X for which the associated nonautonomous Cauchy problem $(NCP)_s$ is well-posed, B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X and Φ, the delay operator, is a linear operator from a space of $X-$valued functions on $\mathbb{R}_-$ into X. They find mild solutions of the above equations by constructing an appropriate semigroup on $C_0(\mathbb{R}_-, X)$, while we proved in [24] that, under appropriate assumptions, their semigroup also yields the classical solutions of (3.13) and (3.14).

With a similar technique we now find classical solutions of (3.13) and (3.14) using our semigroup on $\mathcal{E}$.

To this purpose we fix the additional assumptions to be used in this section.

General assumptions 3.18. (1) The generator $(B, D(B))$ is such that $D(B) \hookrightarrow Y_0$.

(2) The operator $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ are such that the function $s \mapsto A(s)(\epsilon_\lambda x)(s) \in L^p(\mathbb{R}_-, X)$ for $x \in D(B)$.

(3) The operator $(\mathcal{C}, D(\mathcal{C}))$, defined in Definition 3.5, is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$.

Moreover we recall that the function ϵ_λ and the space D_0 are defined, respectively, as in Lemma 3.11 and Proposition 2.16.

Proposition 3.19. *The set*

$$D := D_0 \oplus \{\epsilon_\lambda y : y \in D(B)\} \tag{3.15}$$

is a core of G. Moreover

$$Gf = f' + A(\cdot)f \quad a.e.$$

for $f \in D$.

Proof. We first prove that D is dense in $D(G)$ with respect to the graph-norm, i.e.,

$$\forall\ f\ \in\ D(G) \text{ and } \forall\ \epsilon\ >\ 0 \text{ there exists } g\ \in\ D \text{ such that } \|f-g\|_G\ <\ \epsilon.$$

Let λ be such that $\Re\lambda > \omega_0(T_0(\cdot))$, hence $\lambda \in \rho(G_0)$. Then, by [31, Lemma 1.2], we obtain

$$D(G) = D(G_0) \oplus \mathrm{Ker}(\lambda - G),$$

i.e., for every $f \in D(G)$ there exists $f_0 \in D(G_0)$ and $x \in X$ such that $f = f_0 + \mu\epsilon_\lambda x$, where μ is a constant.

Since ϵ_λ is bounded, there exists a positive constant M such that $\|\epsilon_\lambda\|_{\mathcal{L}(X,E)} \leq M$. Let $k_\lambda := 1 + M(1 + |\lambda|)$ and $\epsilon' := \frac{\epsilon}{k_\lambda}$.

Since $D(B)$ and D_0 are dense in X and $D(G_0)$, respectively, (see Proposition 2.16), there exist $x_0 \in D(B)$ and $g_0 \in D_0$ such that

$$\|x - x_0\|_X < \epsilon'$$

and

$$\|f_0 - g_0\|_{G_0} < \epsilon'.$$

Let $g := g_0 + \mu\epsilon_\lambda x_0$. Then $g \in D$ and

$$\begin{aligned}\|f - g\|_G = \|f_0 - g_0\|_G + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_G &= \|f_0 - g_0\|_{G_0} + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_G \\ &\leq \epsilon' + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_E + \|G\epsilon_\lambda x - G\epsilon_\lambda x_0\|_E \\ &\leq \epsilon' + \|\epsilon_\lambda\|_{\mathcal{L}(X,E)}\|x - x_0\|_X + |\lambda|\|\epsilon_\lambda\|_{\mathcal{L}(X,E)}\|x - x_0\|_X \\ &\leq \epsilon' + M(1 + |\lambda|)\epsilon' = k_\lambda\epsilon' = \epsilon.\end{aligned}$$

Moreover, $Gf = f' + A(\cdot)f$ for $f \in D$. In fact, write $f \in D$ as $f = f_0 + \mu\epsilon_\lambda x_0$ for $f_0 \in D_0$ and $x_0 \in D(B)$.

Then

$$Gf = G(f_0 + \mu\epsilon_\lambda x_0) = f_0' + A(\cdot)f_0 + \mu G\epsilon_\lambda x_0 = f_0' + A(\cdot)f_0 + \mu\lambda\epsilon_\lambda x_0$$

and

$$f' + A(\cdot)f = f_0' + (\mu\epsilon_\lambda x_0)' + A(\cdot)f_0 + \mu A(\cdot)\epsilon_\lambda x_0.$$

Since

$$(\mu\epsilon_\lambda x_0)'(s) = \mu\lambda(\epsilon_\lambda x_0)(s) + \mu e^{\lambda s}\frac{\partial}{\partial s}U(s,0)x_0 = \mu\lambda(\epsilon_\lambda x_0)(s) - \mu A(s)(\epsilon_\lambda x_0)(s),$$

it follows that $Gf = f' + A(\cdot)f$ for $f \in D$.

□

The following lemma gives an other expression for D.

Lemma 3.20. *The core D of $D(G)$, defined in (3.15), coincides with*

$$C := \{f \in W^{1,p}(\mathbb{R}_-, X) : f(0) \in D(B), f(s) \in Y_s, s \mapsto A(s)f(s) \in L^p(\mathbb{R}_-, X)\}.$$

Proof. " $\supseteq$ " Let $f \in C$ and put

$$g := f - \epsilon_\lambda f(0).$$

Using General assumptions 3.18.2 on the operators $A(s)$, it is easy to prove that $g \in D_0$. In fact, $g(0) = 0$, $g \in W^{1,p}(\mathbb{R}_-, X)$ and the function $s \mapsto A(s)g(s) = A(s)f(s) + A(s)(\epsilon_\lambda f(0))(s) \in L^p(\mathbb{R}_-, X)$. Since $U(s,0)Y_0 \subseteq Y_s$, then $g(s) \in Y_s$. Thus $f = g + \epsilon_\lambda f(0) \in D$.
" $\subseteq$ " Let $f \in D$, then $\exists\ f_0 \ \in D_0$ and $x \in D(B)$ such that $f = f_0 + \mu\epsilon_\lambda x$.

One has $f(0) = \mu x \in D(B)$, and since $U(s,0)Y_0 \subseteq Y_s$, it follows that $f(s) \in Y_s$. Moreover, by General assumtpions 3.18.2 on the operators $A(s)$, we have that the function $s \mapsto A(s)f(s) \in L^p(\mathbb{R}_-, X)$. □

The proof of the following proposition is an immediate consequence of the Theorem 3.15 and of the definitions of $(T_0(t))_{t\geq 0}$ and of the function $t \mapsto S_t$ (see Lemma 2.8 and Proposition 3.12, respectively).

Proposition 3.21. *The projections of $(\mathcal{T}(t))_{t\geq 0}$ onto the first and the second component on $\widetilde{\mathcal{E}}$ satisfy the following identities*

$$\pi_1(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) = e^{tB}x + \int_0^t e^{(t-\tau)B}\Phi\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))\, d\tau, \tag{3.16}$$

$$\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(s) = U(s, s+t)f(s+t) \tag{3.17}$$

if $s + t \leq 0$, and

$$\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(s) = U(s,0)e^{(s+t)B}x + \int_0^{s+t} U(s,0)e^{(s+t-\tau)B}\Phi\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))\, d\tau \tag{3.18}$$

if $s + t \geq 0$.

In order to have solutions of (3.13) and (3.14) in a classical sense, we consider

$$\mathcal{D} := \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times D : f(0) = x\}$$

as a subspace of $D(\mathcal{C})$.

Lemma 3.22. *If the functions*

$$s \mapsto A(s)U(s,s+t)f(s+t)$$

and

$$s \mapsto A(s)U(s,0)g(s)$$

belong to $\in L^p([-t,0],X)$ *for all* $f(\cdot) \in L^p(\mathbb{R}_-,X)$ *and* $g(\cdot) \in C([-t,0],D(B))$*, then the space* $\mathcal{D}$ *defined above is a* $\mathcal{T}-$*invariant subspace of* $D(\mathcal{C})$.

Proof. Let $\binom{x}{f} \in \mathcal{D}$, then $\mathcal{T}(t)\binom{x}{f} \in D(\mathcal{C})$. Thus $\pi_1(\mathcal{T}(t)\binom{x}{f}) \in D(B)$, $\pi_2(\mathcal{T}(t)\binom{x}{f}) \in D(G)$ and $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) = \pi_1(\mathcal{T}(t)\binom{x}{f})$. It remains to prove that $\pi_2(\mathcal{T}(t)\binom{x}{f}) \in D$. To this end we consider two cases.
First case: For $s \geq -t$, by Proposition 3.21, we can write $\pi_2(\mathcal{T}(t)\binom{x}{f})(s)$ as

$$\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = U(s,0)g_t(s),$$

where $g_t(s) := e^{(s+t)B}x + \int_0^{s+t} e^{(s+t-\tau)B}\Phi\pi_2(\mathcal{T}(t)\binom{x}{f})(\tau)f\,d\tau$. Since $\binom{x}{f} \in D(\mathcal{C})$, $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) \in D(B)$ and the function

$$\mathbb{R}_+ \ni \tau \mapsto \Phi\pi_2(\mathcal{T}(t)\tbinom{x}{f})(\tau) \in X \tag{3.19}$$

is continuously differentiable. It follows that $g_t(\cdot) \in C^1([-t,0],X) \cap C([-t,0],D(B))$. Hence $g_t(s) \in Y_0$ for $s \in [-t,0]$ because $D(B) \subseteq Y_0$. By assumption, we have that $U(s,0)Y_0 \subseteq Y_s$, so $\pi_2(\mathcal{T}(t)\binom{x}{f})(s) \in Y_s$ and

$$\begin{aligned}\partial_s(\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = &- A(s)U(s,0)g_t(s) + U(s,0)Bg_t(s)\\ &+ U(s,0)\Phi\pi_2(\mathcal{T}(t+s)\tbinom{x}{f}),\end{aligned} \tag{3.20}$$

hence the map $s \mapsto (\pi_2(\mathcal{T}(t)\binom{x}{f}))(s)$ is differentiable. In order to prove the assertion, it remains to show that the functions

(i) $[-t,0] \ni s \mapsto (\pi_2(\mathcal{T}(t)\binom{x}{f}))'(s)$,

(ii) $[-t,0] \ni s \mapsto (A(\cdot)\pi_2(\mathcal{T}(t)\binom{x}{f}))(s)$

are in $L^p(\mathbb{R}_-,X)$.

First of all we prove that the function in (ii) belongs to $L^p(\mathbb{R}_-,X)$. It is obvious that $(A(\cdot)\pi_2(\mathcal{T}(t)\binom{x}{f}))(s) \in X$ because $(A(\cdot)\pi_2(\mathcal{T}(t)\binom{x}{f}))(s) = A(s)(\pi_2(\mathcal{T}(t)\binom{x}{f}))(s)$ and $(\pi_2(\mathcal{T}(t)\binom{x}{f}))(s) \in Y_s \subseteq D(A(s)) \in X$. Since $(\pi_2(\mathcal{T}(t)\binom{x}{f}))(s) = U(s,0)g_t(s)$, it follows by the assumption in the theorem that $s \mapsto A(s)(\pi_2(\mathcal{T}(t)\binom{x}{f}))(s)$ is in $L^p(\mathbb{R}_-, X)$.

Now, since $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) \in D(B)$ and the function in (ii) is in $L^p(\mathbb{R}_-, X)$, by (3.20) it is an immediate consequence that the function in (i) is also in $L^p(\mathbb{R}_-, X)$.

Second case: For $s < -t$, by Proposition 3.21, we can write $(\mathcal{T}(t)\binom{x}{f})(s)$ as

$$\pi_2(\mathcal{T}(t)\binom{x}{f})(s) = U(s, s+t)f(s+t)$$

and obtain

(3.21)
$$\begin{aligned}\partial_s(\pi_2(\mathcal{T}(t)\binom{x}{f})(s) = &- A(s)U(s, s+t)f(t+s) + U(s, s+t)A(s+t)f(s+t)\\ &+ U(s, s+t)f'(s+t).\end{aligned}$$

As before and using the assumption on $A(s)$, we can show that the functions in (i) and (ii) are in $L^p(\mathbb{R}_-, X)$ for $s < -t$.
Combining the two cases we conclude that $\mathcal{D}$ is $\mathcal{T}-$invariant. □

Using the previous proposition we can prove the following theorem.

Theorem 3.23. *Consider $\binom{x}{f} \in D(\mathcal{C})$ such that $\mathbb{R}_+ \ni t \mapsto \mathcal{U}(t) := \binom{z(t)}{v(t)} \in \mathcal{E}$ is a classical solution of (CP) with initial value $\binom{x}{f}$. Under the assumptions of the previous lemma, the function*

$$(t, s) \mapsto v(t, s) := \pi_2\left(\mathcal{T}(t)\binom{x}{f}\right)(s)$$

is the unique classical solution of (3.13) and (3.14) whenever $\binom{x}{f} \in \mathcal{D}$.

Proof. By the previous lemma we have that $\mathcal{T}(t)\binom{x}{f} \in \mathcal{D}$ if $\binom{x}{f} \in \mathcal{D}$. Since $\mathbb{R}_+ \ni t \mapsto \mathcal{U}(t) := \binom{z(t)}{v(t)} \in \mathcal{E}$ is a classical solution of (CP) with initial value $\binom{x}{f}$, we obtain that $v(t, s)$ is continuously differentiable with respect to t and s such that

$$\begin{aligned}Gv(t, \cdot) &= \frac{d}{dt}v(t, \cdot),\\ Gv(t, 0) &= Bv(t, 0) + \Phi v(t, \cdot)\end{aligned}$$

and

$$Gv(t, s) = \frac{\partial}{\partial s}v(t, s) + A(s)v(t, s).$$

So, $v(t, s)$ satisfies the two equations (3.13) and (3.14).

For the uniqueness, we assume that $v(\cdot,\cdot)$ is a solution of (3.13) and (3.14) for the initial value $v(0,\cdot) = f \in D$. From (3.14) and using the fact that $(U(t,s))_{t\le s\le 0}$ solves a nonautonomous Cauchy problem, we obtain

$$\begin{aligned} \frac{\partial}{\partial s}U(r,s)v(t,s) &= U(r,s)\left[A(s)v(t,s) + \frac{\partial}{\partial s}v(t,s)\right] \\ &= U(r,s)\frac{\partial}{\partial t}v(t,s) = \frac{\partial}{\partial t}U(r,s)v(t,s) \end{aligned} \tag{3.22}$$

for $r \le s \le 0$. Consequently, the expression

$$U(r,s)v(t,s)$$

can be written as a function of r and $s+t$. From this it follows that

$$U(r,s)v(t,s) = \begin{cases} U(r,s+t)v(0,s+t), & s+t \le 0, \\ U(r,0)v(s+t,0), & s+t \ge 0, \end{cases} \tag{3.23}$$

for $r \le s \le 0$. Putting $r = s$, we obtain

$$v(t,s) = \begin{cases} U(s,s+t)v(0,s+t), & s+t \le 0, \\ U(s,0)v(s+t,0), & s+t \ge 0. \end{cases} \tag{3.24}$$

By equation (3.14) we have

$$\frac{d}{dt}v(t,0) = Bu(t,0) + \Phi v(t,\cdot).$$

Therefore, using the fact that $v(0,\cdot) = f$, we obtain

$$v(t,0) = e^{tB}f(0) + \int_0^t e^{(t-\tau)B}\Phi v(\tau,\cdot)d\tau. \tag{3.25}$$

Thus, by (3.23) and (3.24), we have

$$v(t,s) = \begin{cases} U(s,s+t)f(s+t), & s+t \le 0, \\ U(s,0)e^{(t+s)B}f(0) + \int_0^{t+s} e^{(t+s-\tau)B}\Phi v(\tau,\cdot)d\tau, & s+t \ge 0. \end{cases} \tag{3.26}$$

Let now $f \equiv 0$. Using Gronwall's inequality (see [21, Lemma 2.A]), we see that $v(t,s) \equiv 0$. □

Chapter 4

Stability

4.1 Stability on Hilbert Spaces

As we have seen in the previous chapter, the solutions of (NDE) are given, under appropriate assumptions, by a semigroup $(\mathcal{T}(t))_{t\geq 0}$ and are of the form

$$u(t) = \begin{cases} \pi_1\left(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right), & t \geq 0, \\ f(t), & \text{a.e. } t \leq 0. \end{cases}$$

So, in order to obtain information about the stability of the solutions of (NDE), it suffices to study the semigroup $(\mathcal{T}(t))_{t\geq 0}$.

To that purpose, we use spectral methods as developed in [22, Chapter IV] and calculate first the resolvent set $\rho(\mathcal{C})$ and the resolvent $R(\lambda, \mathcal{C})$ of the operator $(\mathcal{C}, D(\mathcal{C}))$ on the space $\mathcal{E}$.

Lemma 4.1. *For $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(T_0(\cdot))$, we have that*

$$\lambda \in \rho(\mathcal{C}) \text{ if and only if } \lambda \in \rho(B + \Phi\epsilon_\lambda).$$

Moreover, for these $\lambda \in \rho(\mathcal{C})$ the resolvent $R(\lambda, \mathcal{C})$ is given by

$$R_\lambda := \begin{pmatrix} r_\lambda & r_\lambda \Phi R(\lambda, G_0) \\ \epsilon_\lambda r_\lambda & (\epsilon_\lambda r_\lambda \Phi + Id) R(\lambda, G_0) \end{pmatrix} \tag{4.1}$$

with $r_\lambda := R(\lambda, B + \Phi\epsilon_\lambda)$.

Proof. " $\Rightarrow$ " Take $\lambda \in \rho(\mathcal{C})$. Then, for all $\binom{y}{g} \in \mathcal{E}$ there exists a unique $\binom{x}{f} \in D(\mathcal{C})$ such that

$$(\lambda - \mathcal{C})\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} y \\ g \end{pmatrix}. \tag{4.2}$$

The second component in this equality is equivalent to

$$(\lambda - G)f = g.$$

In particular, for $g = 0$ one has

$$(\lambda - G)f = 0, \tag{4.3}$$

which implies $f = \epsilon_\lambda x$. In fact, by Lemma 3.11, $\hat{f} := \epsilon_\lambda x$ solves (4.3) and $(f - \hat{f})(0) = x - x = 0$. Thus $f - \hat{f} \in D(G_0)$ and, since $(\lambda - G)(f - \hat{f}) = (\lambda - G_0)(f - \hat{f}) = 0$, we conclude that $f = \hat{f}$.

Thus, the first component in (4.2) turns into

$$(\lambda - B)x - \Phi f = (\lambda - B - \Phi\epsilon_\lambda)x = y. \tag{4.4}$$

Since this equation has a unique solution for each $y \in X$, it follows that $\lambda \in \rho(B + \Phi\epsilon_\lambda)$.

" $\Leftarrow$ " For $\lambda \in \rho(B + \Phi\epsilon_\lambda)$, the operator $R_\lambda : \mathcal{E} \to \mathcal{E}$, defined above, is bounded. To show that it is the inverse of $(\lambda - \mathcal{C})$, we proceed as in Lemma 3.11.(ii). If $\binom{x}{f} \in \mathcal{E}$, then

$$R_\lambda \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} r_\lambda x + r_\lambda \Phi R(\lambda, G_0) f \\ \epsilon_\lambda r_\lambda x + (\epsilon_\lambda r_\lambda \Phi + Id) R(\lambda, G_0) f \end{pmatrix} \in D(\mathcal{C})$$

since $\Phi\epsilon_\lambda$ is a bounded operator and

$$(\epsilon_\lambda r_\lambda x + (\epsilon_\lambda r_\lambda \Phi + Id) R(\lambda, G_0) f)(0) = r_\lambda x + r_\lambda \Phi R(\lambda, G_0) f.$$

Moreover, it follows that

$$(\lambda - \mathcal{C}) R_\lambda \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} x \\ f \end{pmatrix}$$

since

$$\begin{aligned} (\lambda &- B)(r_\lambda x + r_\lambda \Phi R(\lambda, G_0) f) - \Phi(\epsilon_\lambda r_\lambda x + (\epsilon_\lambda r_\lambda \Phi + Id) R(\lambda, G_0) f) \\ &= (\lambda - B - \Phi\epsilon_\lambda) r_\lambda \Phi R(\lambda, G_0) f - \Phi R(\lambda, G_0) f + (\lambda - B - \Phi\epsilon_\lambda) r_\lambda x = x \end{aligned}$$

and

$$\begin{aligned}(\lambda &- G)(\epsilon_\lambda r_\lambda x + (\epsilon_\lambda r_\lambda \Phi + Id)R(\lambda, G_0)f)\\ &= (\lambda - G)\epsilon_\lambda r_\lambda x + (\lambda - G)\epsilon_\lambda r_\lambda \Phi R(\lambda, G_0)f + (\lambda - G)R(\lambda, G_0)f = f.\end{aligned}$$

As a result, we obtain $(\lambda - \mathcal{C})R_\lambda = Id_{\mathcal{E}}$.

On the other hand, take $\binom{x}{f} \in D(\mathcal{C})$, thus $f(0) = x \in D(B)$ and $f \in D(G)$. It follows that

$$R_\lambda(\lambda - \mathcal{C})\binom{x}{f} = R_\lambda \begin{pmatrix}(\lambda - B)x - \Phi f\\ (\lambda - G)f\end{pmatrix} = \binom{x}{f}$$

since

$$\begin{aligned}r_\lambda(\lambda &- B)x - r_\lambda \Phi f + r_\lambda \Phi R(\lambda, G_0)(\lambda - G)f\\ &= r_\lambda(\lambda - B)f(0) - r_\lambda \Phi f + r_\lambda \Phi R(\lambda, G_0)(\lambda - G)(f - \epsilon_\lambda f(0))\\ &= r_\lambda((\lambda - B)f(0) - \Phi f)) + r_\lambda \Phi f - r_\lambda \Phi \epsilon_\lambda f(0)\\ &= r_\lambda(\lambda - B - \Phi \epsilon_\lambda)f(0) = x\end{aligned}$$

and

$$\begin{aligned}\epsilon_\lambda r_\lambda(\lambda &- B)x + R(\lambda, G_0)(\lambda - G)f\\ &= \epsilon_\lambda r_\lambda(\lambda - B)f(0) + R(\lambda, G_0)(\lambda - G)(f - \epsilon_\lambda r_\lambda(\lambda - B)f(0))\\ &= \epsilon_\lambda r_\lambda(\lambda - B)f(0) + f - \epsilon_\lambda r_\lambda(\lambda - B)f(0) = f.\end{aligned}$$

□

We now generalize a result of A. Bátkai and S. Piazzera (see [7, Lemma 4.3]) using the following definition (see [3, Section V.1] or [50, Section I.3]).

Definition 4.2. Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach space X. The **abscissa of uniform boundedness** of the resolvent of A is

$$s_0(A) := \inf\{\omega \in \mathbb{R} : \{\Re\lambda > \omega\} \subset \rho(A) \text{ and } \sup_{\Re\lambda > \omega} \|R(\lambda, A)\| < \infty\}.$$

It is shown in [50, Sections 1.2, 4.1] that

$$-\infty \leq s(A) \leq s_0(A) \leq \omega_0(A) < \infty.$$

In the following we assume that Φ is as in Example 3.16, i.e,

$$\Phi f := \int_{-\infty}^{0} f\, d\eta$$

for all $f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X)$, $1 < p < +\infty$, where $\eta : \mathbb{R}_- \to \mathcal{L}(X)$ is a function of bounded variation such that $|\eta|(\mathbb{R}_-) < +\infty$. In addition, we assume that

$$\int_{-\infty}^{0} \sigma^{\frac{1}{p'}} d|\eta|(\sigma) < +\infty \tag{4.5}$$

for $\frac{1}{p} + \frac{1}{p'} = 1$. By Theorem 3.15, our matrix $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on the Banach space $\mathcal{E}$. For this generator we want to estimate $s_0(\mathcal{C})$.

To that purpose we assume that $s_0(B) < 0$ and $\omega_0(T_0(\cdot)) < 0$. Moreover, we take $\alpha \leq 0$ such that

$$\tau := \max\{s_0(B), \omega_0(T_0(\cdot))\} < \alpha$$

and define

$$a_n := \sup_{\omega\in\mathbb{R}} \|(\Phi\epsilon_{\alpha+i\omega} R(\alpha + i\omega, B))^n\| < +\infty$$

for each $n \in \mathbb{N}$.

Theorem 4.3. *If*

$$a_\alpha := \sum_{0}^{\infty} a_n < +\infty, \tag{4.6}$$

then $s_0(\mathcal{C}) < \alpha \leq 0$.

Proof. The proof follows the idea of [5, Theorem 3.1]. Take $\lambda := \beta + i\omega$ with $\beta \geq \alpha$. The functions $\lambda \mapsto \Phi\epsilon_\lambda$ and $\lambda \mapsto (\Phi\epsilon_\lambda R(\lambda, B))^n$ are analytic and, by our assumption on $s_0(B)$, bounded for $\Re\lambda \geq \alpha$.

Since the suprema of bounded analytic functions along vertical lines $\Re\lambda = c$ decrease as c increases (see [17, Chapter 6]), we obtain

$$\sup_{\omega\in\mathbb{R}} \|(\Phi\epsilon_{\beta+i\omega} R(\beta + i\omega, B))^n\| \leq a_n \quad \text{for each } \ n \in \mathbb{N}.$$

It follows that

$$R(\beta + i\omega, B) \sum_{n=0}^{\infty} (\Phi\epsilon_{\beta+i\omega} R(\beta + i\omega, B))^n \in \mathcal{L}(X)$$

for $\beta \geq \alpha$. In fact,

$$\begin{aligned} \|R(\beta+i\omega,B)\sum_{n=0}^{\infty}(\Phi\epsilon_{\beta+i\omega}R(\beta+i\omega,B))^n\| &\leq N\sum_{n=0}^{\infty}\sup_{\omega\in\mathbb{R}}\|(\Phi\epsilon_{\beta+i\omega}R(\beta+i\omega,B))^n\| \\ &\leq N\sum_{n=0}^{\infty}a_n = Na_\alpha \end{aligned}$$

with $N := \sup_{\beta\geq\alpha}\|R(\beta+i\omega,B)\|$.

It is easy to check that

$$R(\beta+i\omega,B)\sum_{n=0}^{+\infty}(\Phi\epsilon_{\beta+i\omega}R(\beta+i\omega,B))^n = R(\beta+i\omega,B+\Phi\epsilon_{\beta+i\omega}).$$

This implies that $\{\beta+i\omega\in\mathbb{C} : \beta\geq\alpha\}\subset\rho(\mathcal{C})$.

We now have to show that $R(\beta+i\omega,\mathcal{C})$ is bounded on the right halfplane determined by α. We already know that $R(\beta+i\omega,B+\Phi\epsilon_{\beta+i\omega})$ is uniformly bounded on $\{\beta\geq\alpha\}$. Since $\alpha>\omega_0(T_0(\cdot))$, the same holds for $R(\beta+i\omega,G_0)$.

Finally, the function $\lambda\mapsto\Phi R(\lambda,G_0)$ is bounded since

$$\begin{aligned} \|\Phi R(\lambda,G_0)f\| &= \left\|\int_{-\infty}^{0}(R(\lambda,G_0)f)(\sigma)d\eta(\sigma)\right\| \leq \int_{-\infty}^{0}\|(R(\lambda,G_0)f)(\sigma)\|d|\eta|(\sigma) \\ &\leq \int_{-\infty}^{0}\left\|\left(\int_0^{\infty}e^{-\lambda s}T_0(t)f\,ds\right)(\sigma)\right\|d|\eta|(\sigma) \\ &\leq \int_{-\infty}^{0}\left(\int_0^{-\sigma}\|e^{-\lambda s}U(\sigma,\sigma+s)f(\sigma+s)\|ds\right)d|\eta|(\sigma) \\ &\leq M_\alpha\int_{-\infty}^{0}\left(\int_0^{-\sigma}e^{(\alpha-\beta)s}\|f(\sigma+s)\|ds\right)d|\eta|(\sigma) \\ &\leq M_\alpha\int_{-\infty}^{0}\left(\int_0^{-\sigma}\|f(\sigma+s)\|ds\right)d|\eta|(\sigma) \\ &\leq M_\alpha\int_{-\infty}^{0}\left(\int_\sigma^{0}\|f(t)\|dt\right)d|\eta|(\sigma) \\ &\leq M_\alpha\int_{-\infty}^{0}\|f\|_{L^p[\sigma,0]}\sigma^{\frac{1}{p'}}d|\eta|(\sigma) \\ &\leq M_\alpha\|f\|_{L^p(\mathbb{R}_-)}\int_{-\infty}^{0}\sigma^{\frac{1}{p'}}d|\eta|(\sigma) < +\infty \end{aligned}$$

for every $\lambda=\beta+i\omega$ with $\beta\geq\alpha$ and every $f\in L^p(\mathbb{R}_-,X)$. Here, M_α is a constant such that $\|U(r,s)\|\leq M_\alpha e^{\alpha(s-r)}$.

From Lemma 4.1 we know that the resolvent of $\mathcal{C}$ is given by (4.1). Since all terms of this matrix remain bounded in the halfplane determined by α, we conclude $\sup_{\beta \geq \alpha} \|R(\beta + i\omega, \mathcal{C})\| < \infty$, hence $s_0(\mathcal{C}) < \alpha \leq 0$. $\square$

Here is a stronger, but simpler condition implying the same conclusion. Assume that $s_0(B) < 0$ and $\omega_0(T_0(\cdot)) < 0$ and take $\alpha \leq 0$ such that

$$\tau := \max\{s_0(B), \omega_0(T_0(\cdot))\} < \alpha.$$

Corollary 4.4. *If*

$$\sup_{\omega \in \mathbb{R}} \|\Phi \epsilon_{\alpha+i\omega} R(\alpha + i\omega, B)\| < 1, \tag{4.7}$$

or, in particular, if

$$\sup_{\omega \in \mathbb{R}} \|\Phi \epsilon_{\alpha+i\omega}\| < \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(\alpha + i\omega, B)\|}, \tag{4.8}$$

then $s_0(\mathcal{C}) < \alpha \leq 0$.

Proof. It holds that

$$a_n := \sup_{\omega \in \mathbb{R}} \|(\Phi \epsilon_{\alpha+i\omega} R(\alpha + i\omega, B))^n\| \leq \sup_{\omega \in \mathbb{R}} \|\Phi \epsilon_{\alpha+i\omega} R(\alpha + i\omega, B)\|^n =: q^n, \tag{4.9}$$

where, by assumption, $q < 1$. Thus

$$\sum_{n=0}^{+\infty} a_n \leq \sum_{n=0}^{+\infty} q^n < +\infty. \tag{4.10}$$

Applying the previous theorem, the assertion follows. $\square$

Finally, using the theorem of Gearhart-Prüss-Greiner, we obtain the followig immediate consequence.

Theorem 4.5. *Consider $E = L^2(\mathbb{R}_-, X)$ for a Hilbert space X. Under the assumptions of Theorem 4.3, the semigroup $(\mathcal{T}(t))_{t \geq 0}$ is uniformly exponentially stable on $\mathcal{E}$, i.e.,*

$$\omega_0(\mathcal{T}(\cdot)) < 0.$$

Similarly, one obtains a Hilbert space version of Corollary 4.4.

4.1.1 Examples

In this subsection, we give two examples, where we always take Φ as in Example 3.16.

In the case that $A(t) \equiv 0$, i.e., $U(t,s) = Id$ for all $t \leq s \leq 0$, we are in the situation amply studied by A. Bátkai and S. Piazzera on the compact interval $[-1,0]$ (see, e.g.,[6] [7], [8]).

Example 1. Take $A(t) \equiv A$ to be the generator of a uniformly exponentially stable C_0- semigroup $(Q(t))_{t\geq 0}$ on a Hilbert space X. Let $E := L^2(\mathbb{R}_-, X)$ and B a normal operator on X such that it is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ with $s_0(B) < 0$. Take $\alpha \leq 0$ such that

$$\tau := \max\{s_0(B), \omega_0(Q(\cdot))\} < \alpha. \tag{4.11}$$

In this example, the regularity subspaces Y_s coincide with $D(A)$, the evolution family $\mathcal{U}$ is given by $U(t,s) = Q(s-t)$, and the operator G_0 generates the strongly continuous semigroup $(T_0(t))_{t\geq 0}$ defined by

$$(T_0(t)f)(\tau) = (Q(t)(T_{l_0}(t)f))(\tau) = \begin{cases} Q(t)f(\tau+t), & \tau+t \leq 0 \\ 0, & \tau+t > 0. \end{cases}$$

With these assumptions, by Theorem 3.15, the matrix $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup, and the delay equation (NDE) is well-posed.

In order to show that this semigroup is uniformly exponentially stable, we have to verify the convergence of the series in (4.6). Using that B is a normal operator on a Hilbert space, we obtain that

$$\sup_{\omega\in\mathbb{R}} \|R(i\omega, B)\| = \frac{1}{dist(i\mathbb{R}, \sigma(B))} = \frac{1}{|s_0(B)|},$$

(see [37, Section V.3.8]). In particular, if B is selfadjoint with compact resolvent, then

$$\sup_{\omega\in\mathbb{R}} \|R(i\omega, B)\| = \frac{1}{|\lambda_1|},$$

where λ_1 is the largest eigenvalue of B.

Now, since $U(t,s) = Q(s-t)$, the growth bound of $\mathcal{U}$ is equal to the growth bound of $(Q(t))_{t\geq 0}$ and, by (4.11), $\alpha > \omega_0(\mathcal{U})$. Thus there exists a positive constant

$M_\alpha \geq 1$ such that $\|U(r,s)\| \leq M_\alpha e^{\alpha(s-r)}$. Moreover for $\lambda = \beta + i\omega$, where $\beta \geq \alpha$, one has

$$\epsilon_\lambda \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X).$$

Hence, it follows from the definition of Φ, that

$$\begin{aligned} \|\Phi\epsilon_{\alpha+i\omega}\| = \sup_{\|y\|=1} \|\Phi\epsilon_{\alpha+i\omega} y\| &= \sup_{\|y\|=1} \| \int_{-\infty}^{0} e^{(\alpha+i\omega)\sigma} U(\sigma,0) y d\eta(\sigma)\| \\ &\leq \int_{-\infty}^{0} e^{\alpha\sigma} M_\alpha e^{-\alpha\sigma} d|\eta|(\sigma) = M_\alpha |\eta|(\mathbb{R}_-). \end{aligned} \tag{4.12}$$

Assume $N := M_\alpha|\eta|(\mathbb{R}_-) < |s_0(B)|$ (or $N < |\lambda_1|$). Then the series in (4.6) converges and, applying Theorem 4.5 or Corollary 4.4, we have that, if $N < |s_0(B)|$ and

$$\int_{-\infty}^{0} \sigma^{\frac{1}{p'}} d|\eta|(\sigma) < +\infty, \tag{4.13}$$

then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable, and the solutions of (NDE) decay exponentially.

Example 2. Consider $E := L^2(\mathbb{R}_-, X)$ for a Hilbert space X. Take $A(t) \equiv a(t)B$, where $a(\cdot) \in C(\mathbb{R}_-)$ with $a(t) > 0$ and $(B, D(B))$ is a normal operator on X such that $s_0(B) < 0$. In this case, the evolution family is given by $U(t,s) = e^{(\int_t^s a(\tau)d\tau)B}$ and the regularity subspaces Y_t coincide with $D(B)$ for all $t \leq 0$ (see [24, Example 4.7]). Moreover, the growth bound of $(U(t,s))_{t\leq s\leq 0}$is given by

$$\omega_0(\mathcal{U}) = \inf_{h\geq 0} \sup_{s+h\leq 0} \left(\frac{1}{h} \int_s^{s+h} a(\sigma) d\sigma \right) \lambda_0,$$

where λ_0 is the largest eigenvalue of B.

Take $\alpha \leq 0$ such that

$$\tau := \max\{s_0(B), \omega_0(\mathcal{U})\} < \alpha.$$

Under the above assumptions, the operator $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$. Thus (NDE) is well-posed.

In order to find conditions implying that this semigroup is uniformly exponentially stable, we can proceed as in Example 1.

4.2 The Spectral Mapping Theorem

In this section we return to general Banach spaces and prove a spectral mapping theorem for the semigroup $(\mathcal{T}(t))_{t\geq 0}$. To do this we use the representation of $(\mathcal{T}(t))_{t\geq 0}$ by the Dyson-Phillips series and make the following assumption slightly stronger than condition (M) in Theorem 3.15 (compare [67]).

Assumption 4.6. There exists a function $q : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{t\to 0} q(t) = 0$ such that

$$\int_0^t \|\Phi(S_r x + T_0(r)f)\| dr \leq q(t) \left\| \binom{x}{f} \right\| \tag{4.14}$$

for $t > 0$ and each $\binom{x}{f} \in D(\mathcal{C}_0)$.

Remark 4.7. The concrete delay operators of Remark 3.17 even satisfy (4.14).

By the perturbation theorem of Miyadera-Voigt we then know that $(\mathcal{C}_0+\mathcal{F}, D(C_0))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$ given by the Dyson-Phillips series

$$\mathcal{T}(t) = \sum_{n=0}^{+\infty} \mathcal{T}_n(t)$$

for all $t \geq 0$, where

$$\mathcal{T}_{n+1}(t)\binom{x}{f} := \int_0^t \mathcal{T}_0(t-s)\mathcal{F}\mathcal{T}_n(s)\binom{x}{f} ds$$

for $t \geq 0$ and each $\binom{x}{f} \in D(\mathcal{C}_0)$. Thus (NDE) is well-posed.

Now, let $R(\cdot)$ be a norm continuous function from $\mathbb{R}_+$ to $\mathcal{L}(X)$ and define the function

$$[0, +\infty) \ni t \mapsto R_t \in \mathcal{L}(X, E)$$

by

$$(R_t x)(s) := \begin{cases} U(s,0)R(t+s)x & \text{for } t+s \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then we can prove the following lemma.

Lemma 4.8. *The function* $[0, +\infty) \ni t \mapsto R_t \in \mathcal{L}(X, E)$ *is norm continuous.*

Proof. Let $t \geq 0$, $M := \sup_{t\in[0,1]} \|R(t)\|$ and $1 > h > 0$. One has

$$\lim_{h\to 0} \|R_{t+h} - R_t\| = \lim_{h\to 0} \sup_{\|x\|\leq 1} \left(\int_{\mathbb{R}_-} \|(R_{t+h}x)\tau - (R_t x)\tau\|_X^p d\tau \right)^{\frac{1}{p}}$$

$$= \lim_{s\to t} \sup_{\|x\|\leq 1} \left(\int_{-(t+h)}^{-t} \|U(\tau,0)R(t+h+\tau)x\|^p d\tau \right.$$

$$\left. + \int_{-t}^{0} \|U(\tau,0)R(t+h+\tau)x - U(\tau,0)R(t+\tau)x\|^p d\tau \right)^{\frac{1}{p}}$$

$$\leq \lim_{h\to 0} \sup_{\|x\|\leq 1} \left(\int_{-(t+h)}^{-t} (M_\omega e^{-\omega\tau} \|R(t+h+\tau)\| \, \|x\|)^p d\tau \right.$$

$$\left. + \|x\|^p \int_{-t}^{0} \|U(\tau,0)\|^p \|R(t+h+\tau) - R(t+\tau)\|^p d\tau \right)^{\frac{1}{p}}$$

$$\leq \lim_{h\to 0} \sup_{\|x\|\leq 1} M_\omega \|x\| e^{|\omega|(t+h)} \left(\int_{-(t+h)}^{-t} \|R(t+h+\tau)\|^p d\tau \right)^{\frac{1}{p}}$$

$$+ \lim_{h\to 0} \sup_{\|x\|\leq 1} M_\omega \|x\| \left(\int_{-t}^{0} e^{-\omega\tau p} \|R(t+h+\tau) - R(t+\tau)\|^p d\tau \right)^{\frac{1}{p}}$$

$$\leq \lim_{h\to 0} M_\omega e^{|\omega|(t+h)} \left(\int_0^h \|R(\sigma)\|^p d\sigma \right)^{\frac{1}{p}}$$

$$+ \lim_{h\to 0} M_\omega e^{|\omega|t} \left(\int_{-t}^{0} \|R(t+h+\tau) - R(t+\tau)\|^p d\tau \right)^{\frac{1}{p}}$$

$$\leq \lim_{h\to 0} M_\omega e^{|\omega|(t+h)} M h$$

$$+ \lim_{h\to 0} M_\omega e^{|\omega|t} \left(\int_{-t}^{0} \|R(t+h+\tau) - R(t+\tau)\|^p d\tau \right)^{\frac{1}{p}}.$$

This last term tends to zero as $h \to 0^+$ since $R(\cdot)$ is uniformly norm continuous on compact intervals. The proof for $h \to 0^-$ is similar.

□

In order to obtain a spectral mapping theorem, we need an assumption on the semigroup without delay, i.e., on the semigroup $(S(t))_{t\geq 0}$ generated by B. The following turns out to be appropriate.

(4.15)

$(B, D(B))$ generates an immediately norm continuous semigroup $(S(t))_{t\geq 0}$ on X.

Let

$$V(t) := \begin{pmatrix} S(t) & 0 \\ 0 & T_0(t) \end{pmatrix}$$

and

$$Q(t) := \begin{pmatrix} 0 & 0 \\ -S_t & 0 \end{pmatrix}$$

for $t \geq 0$, where S_t is defined as in Proposition 3.12.

Remark 4.9. (1) It is easy to prove that $(V(t))_{t\geq 0}$ is a semigroup and

$$V(t) = \mathcal{T}_0(t) + Q(t) \quad \text{for all } \ t \geq 0.$$

(2) Since $(S(t))_{t\geq 0}$ is immediately norm continuous, it follows from Lemma 4.8 that the function $t \mapsto Q(t)$ is norm continuous from $(0, +\infty)$ to $\mathcal{L}(\mathcal{E})$.

In the next step, we extend both semigroups $(\mathcal{T}_0(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ to semigroups $\widetilde{\mathcal{T}_0} := (\widetilde{\mathcal{T}_0}(t))_{t\geq 0}$ and $\widetilde{V} := (\widetilde{V}(t))_{t\geq 0}$ on $\widetilde{\mathcal{E}}$ (for definition see Section 2.3) and show that their spaces of strong continuity coincide.

Lemma 4.10. *We have* $\widetilde{\mathcal{E}}_{\widetilde{V}} = \widetilde{\mathcal{E}}_{\widetilde{\mathcal{T}_0}}$.

Proof. Using the definitions of $(V(t))_{t\geq 0}$ and $(\mathcal{T}_0(t))_{t\geq 0}$, we obtain

$$\begin{aligned}\left\| V(h)\begin{pmatrix} x \\ f \end{pmatrix} - \mathcal{T}_0(h)\begin{pmatrix} x \\ f \end{pmatrix} \right\| &= \left\| \begin{pmatrix} S(h)x - S(h)x \\ T_0(h)f - S_h x - T_0(h)f \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ -S_h x \end{pmatrix} \right\| \\ &\leq \|S_h x\|.\end{aligned}$$

However,

$$\begin{aligned}\|S_h x\|^p_{L^p(\mathbb{R}_-,X)} &= \int_{\mathbb{R}_-} \|(S_h x)(\tau)\|^p d\tau = \int_{-h}^0 \|U(\tau,0)S(h+\tau)x\|^p d\tau \\ &\leq \int_{-h}^0 M_\omega^p e^{-\omega\tau p} M_{\bar{\omega}}^p e^{\bar{\omega}(h+\tau)p} \|x\|^p d\tau \leq C^p e^{ph(|\omega|+|\bar{\omega}|)} h \|x\|^p,\end{aligned} \tag{4.16}$$

where ω and M_ω are such that $\|U(t,s)\| \leq M_\omega e^{\omega(s-t)}$ for $t \leq s \leq 0$, $M_{\bar{\omega}}$ and $\bar{\omega}$ are such that $\|S(t)\| \leq M_{\bar{\omega}} e^{t\bar{\omega}}$ for $t \geq 0$ and $C := M_\omega M_{\bar{\omega}}$. Clearly, the last term tends to zero as $h \searrow 0$. □

The next lemma relates the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ and the operators $Q(t)$. This relation will be used in Proposition 4.12.

Lemma 4.11. *With the above definitions we have*

$$\lim_{h\downarrow 0}\|\mathcal{T}_0(h)Q(t)-Q(t+h)\|=0 \quad \text{for all } t\geq 0.$$

Proof. Using the definitions of $(\mathcal{T}_0(t))_{t\geq 0}$ and $Q(t)$ we have

$$\begin{aligned}\|\mathcal{T}_0(h)Q(t)-Q(t+h)\| &= \sup_{\left\|\binom{x}{f}\right\|\leq 1}\left\|\begin{pmatrix}S(h) & 0\\ S_h & T_0(h)\end{pmatrix}\begin{pmatrix}0 & 0\\ -S_t & 0\end{pmatrix}\begin{pmatrix}x\\ f\end{pmatrix}\right.\\ &\qquad\left.-\begin{pmatrix}0 & 0\\ -S_{t+h} & 0\end{pmatrix}\begin{pmatrix}x\\ f\end{pmatrix}\right\|\\ &= \sup_{\left\|\binom{x}{f}\right\|\leq 1}\left\|\begin{pmatrix}0\\ -T_0(t)S_t x+S_{t+h}x\end{pmatrix}\right\|.\end{aligned}$$

Since, see Proposition 3.12, $S_{t+h}=S_hS(t)+T_0(h)S_t$, we obtain

$$\left\|\begin{pmatrix}0\\ -T_0(t)S_t x+S_{t+h}x\end{pmatrix}\right\|=\left\|\begin{pmatrix}0\\ S_hS(t)x\end{pmatrix}\right\|$$

for all $x\in X$. As in Lemma 4.10 we can prove that

$$\|S_hS(t)x\|_{L^p(\mathbb{R}_-,X)}\leq Ce^{h|\omega|}e^{(h+t)|\bar{\omega}|}h^{\frac{1}{p}}\|x\|,$$

hence

$$\sup_{\|x\|\leq 1}\|S_hS(t)x\|_{L^p(\mathbb{R}_-,X)}\leq Ce^{h|\omega|}e^{(h+t)|\bar{\omega}|}h^{\frac{1}{p}},$$

which tends to zero as $h\searrow 0$. ☐

The following proposition gives a relation between the critical spectrum of $(V(t))_{t\geq 0}$ and $(\mathcal{T}_0(t))_{t\geq 0}$. In the proof we follow the idea of [11, Theorem 4.5].

Proposition 4.12. *The critical spectrum of the semigroup $(V(t))_{t\geq 0}$ is equal to the critical spectrum of $(\mathcal{T}_0(t))_{t\geq 0}$, i.e.,*

$$\sigma_{crit}(V(t))=\sigma_{crit}(\mathcal{T}_0(t))$$

for $t>0$.

Proof. Using the norm continuity of $Q(t)$ and Lemma 4.11, we obtain

$$\lim_{h\downarrow 0}\|\mathcal{T}_0(h)Q(t)-Q(t)\|\le \lim_{h\downarrow 0}(\|\mathcal{T}_0(h)Q(t)-Q(t+h)\|+\|Q(t+h)-Q(t)\|)=0$$

for every $t>0$. This implies that $\widetilde{Q}(t)$ maps $\widetilde{\mathcal{E}}$ into $\widetilde{\mathcal{E}}_{\mathcal{T}_0}$, hence $\hat{Q}(t)=0$ for $t>0$.
Therefore, we have

$$\hat{V}(t)=\hat{\mathcal{T}}_0(t)$$

and hence

$$\sigma_{\text{crit}}(V(t))=\sigma_{\text{crit}}(\mathcal{T}_0(t))$$

for $t>0$. □

We can now relate the critical spectrum of $(\mathcal{T}_0(t))_{t\ge 0}$ to the critical spectrum of $(T_0(t))_{t\ge 0}$.

Theorem 4.13. *The critical spectra of the semigroups $(\mathcal{T}_0(t))_{t\ge 0}$ on $\mathcal{E}$ and $(T_0(t))_{t\ge 0}$ on $L^p(\mathbb{R}_-,X)$ coincide, i.e.,*

$$\sigma_{crit}(\mathcal{T}_0(t))=\sigma_{crit}(T_0(t))$$

for $t>0$.

Proof. By Remark 4.9.1, we know that

$$V(t)=\mathcal{T}_0(t)+Q(t)$$

and, using Proposition 4.12, we have

$$\sigma_{\text{crit}}(V(t))=\sigma_{\text{crit}}(\mathcal{T}_0(t)).$$

By the immediate norm continuity of $(S(t))_{t\ge 0}$, one has

$$\sigma_{\text{crit}}(V(t))=\sigma_{\text{crit}}(T_0(t))\cup\sigma_{\text{crit}}(S(t))=\sigma_{\text{crit}}(T_0(t))\cup\{0\}=\sigma_{\text{crit}}(T_0(t)).$$

Hence, the thesis follows.

□

Using the previous theorem and Theorem 2.20, the following result is immediate.

Corollary 4.14. *The critical spectrum of* $(\mathcal{T}_0(t))_{t\geq 0}$ *is equal to the spectrum of the backward evolution semigroup* $(T_0(t))_{t\geq 0}$ *, i.e.,*

$$\sigma_{crit}(\mathcal{T}_0(t)) = \sigma(T_0(t))$$

for $t > 0$.

We now prove that the first term $\mathcal{T}_1(t)$ of the Dyson-Phillips series of $(\mathcal{T}(t))_{t\geq 0}$ is norm continuous.

Proposition 4.15. *The function*

$$t \mapsto \mathcal{T}_1(t)$$

is norm continuous for $t \geq 0$.

Proof. The first Dyson-Phillips term $\mathcal{T}_1(t)$ applied to $\binom{x}{f} \in D(\mathcal{C}_0)$ yields

$$\begin{aligned}
\mathcal{T}_1(t)\begin{pmatrix} x \\ f \end{pmatrix} &= \int_0^t \mathcal{T}_0(t-s)\mathcal{F}\mathcal{T}_0(s)\begin{pmatrix} x \\ f \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s) & 0 \\ S_{t-s} & T_0(t-s) \end{pmatrix}\begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S(s)x & 0 \\ S_s x & T_0(s)f \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s) & 0 \\ S_{t-s} & T_0(t-s) \end{pmatrix}\begin{pmatrix} \Phi(S_s x + T_0(s)f) \\ 0 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s)\Phi(S_s x + T_0(s)f) \\ S_{t-s}\Phi(S_s x + T_0(s)f) \end{pmatrix} ds.
\end{aligned}$$

We will prove norm continuity of both components separately.
(1): Let $t \geq 0$, $1 > h > 0$. Then

$$\begin{aligned}
&\left\| \int_0^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\,ds - \int_0^t S(t-s)\Phi(S_s x + T_0(s)f)\,ds \right\| \\
&\quad = \left\| \int_0^t S(t+h-s)\Phi(S_s x + T_0(s)f)\,ds \right. \\
&\quad \left. + \int_t^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\,ds - \int_0^1 S(t-s)\Phi(S_s x + T_0(s)f)\,ds \right\| \\
&\quad \leq \int_0^t \|S(t+h-s) - S(t-s)\|\|\Phi(S_s x + T_0(s)f)\|\,ds \\
&\quad + \int_t^{t+h} \|S(t+h-s)\|\|\Phi(S_s x + T_0(s)f)\|\,ds.
\end{aligned}$$

By the change of variable $s := \tau + t$, we obtain that the first term is equal to

$$\begin{aligned}\int_0^t &\|S(t+h-s) - S(t-s)\|\|\Phi(S_s x + T_0(s)f)\|\,ds \\ &+ \int_0^h \|S(h-\tau)\|\|\Phi(S_{\tau+t}x + T_0(\tau+t)f)\|\,d\tau \\ &\le \int_0^t \|S(t+h-s) - S(t-s)\|\|\Phi(S_s x + T_0(s)f)\|\,ds \\ &+ \sup_{0\le r\le 1} \|S(r)\| q(h) \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|.\end{aligned}$$

By condition (4.14), the Lebesgue dominated convergence theorem and by the immediate norm continuity of $(S(t))_{t\ge 0}$, we have that

$$\int_0^t \|S(t+h-s) - S(t-s)\|\|\Phi(S_s x + T_0(s)f)\|ds + \sup_{0\le r\le 1} \|S(r)\| q(h) \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|$$

tends to zero as $h \to 0^+$ uniformly for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{C}_0)$, $\left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| \le 1$.

For $h \to 0^-$, the proof is analogous. Since $D(B)$ is dense in $\mathcal{E}$, it follows that the first component of $\mathcal{T}_1(t)$ is immediately norm continuous.
(2): For the second component we can proceed in a similar way, but we have to use the norm continuity of the function $t \mapsto S_t$ proved in Lemma 4.8.

Hence the map $t \mapsto \mathcal{T}_1(t)$ is norm continuous.

□

Proposition 4.16. *Under the assumption 4.15, the critical spectra of the perturbed semigroup* $(\mathcal{T}(t))_{t\ge 0}$ *and of the unperturbed semigroup* $(\mathcal{T}_0(t))_{t\ge 0}$ *concide, i.e.,*

$$\sigma_{crit}(\mathcal{T}(t)) = \sigma_{crit}(\mathcal{T}_0(t)). \tag{4.17}$$

Proof. Let $R_k(t) := \sum_{j=k}^{\infty} \mathcal{T}_j(t)$. By the previous proposition, the function $t \mapsto \mathcal{T}_1(t)$ is norm continuous. Hence, by [11, Proposition 4.7], the map

$$t \mapsto R_1(t) \tag{4.18}$$

is norm continuous.

Since B satisfies the Assumption 4.6, a result of S. Brendle, R. Nagel and J. Poland (see [11, Theorem 4.5]) implies the equality (4.17). □

We are now ready to prove the spectral mapping theorem for the semigroup $(\mathcal{T}(t))_{t\geq 0}$.

Theorem 4.17. *If B generates an immediately norm continuous semigroup and ϕ satisfies the stronger Miyadera-Voigt condition (see Assumption 4.6), then*

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(\mathcal{C})} \cup \sigma(T_0(t)) \setminus \{0\} \tag{4.19}$$

for $t > 0$.

Proof. By the previous remark, we have that

$$\sigma_{\text{crit}}(\mathcal{T}(t)) = \sigma_{\text{crit}}(\mathcal{T}_0(t)). \tag{4.20}$$

Thus, applying [11, Corollary 4.6], one has

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(\mathcal{C})} \cup \sigma_{\text{crit}}(\mathcal{T}_0(t)) \setminus \{0\}.$$

By Theorem 4.13, we know that

$$\sigma_{\text{crit}}(\mathcal{T}_0(t)) = \sigma_{\text{crit}}(T_0(t))$$

for $t > 0$, thus the assertion follows. □

The right hand side of (4.19) determines $\sigma(\mathcal{T}(t))$ in a very satisfactory way. Indeed, $\sigma(\mathcal{C})$ and $e^{t\sigma(\mathcal{C})}$ can be calculated via Lemma 4.1, while

$$\sigma(T_0(t)) = \{\lambda \in \mathbb{C} : |\lambda| \leq e^{t\omega_0(\mathcal{U})}\}$$

by Theorem 2.19.

4.3 Stability on Banach Spaces

In the first section of this chapter we studied the stability for the semigroup corresponding to the delay equation (NDE) in a Hilbert space. In this section we assume that $\mathcal{E}$ is a Banach space and use the perturbation results due to S. Brendle, R. Nagel and J. Poland on the critical growth bound (see [11]).

Lemma 4.1 is helpful to compute the spectral bound of the operator $\mathcal{C}$. However, in general, one has only

$$s(\mathcal{C}) \leq \omega_0(\mathcal{T}) \tag{4.21}$$

and by [48, Proposition 4.3],

$$\omega_0(\mathcal{T}(\cdot)) = \max\{s(\mathcal{C}), \omega_{\text{crit}}(\mathcal{T}(\cdot))\}. \tag{4.22}$$

It follows from Proposition 4.16 and Theorem 4.13 that $\omega_{\text{crit}}(\mathcal{T}(\cdot)) = \omega_0(\mathcal{U})$, hence we obtain the following result.

Theorem 4.18. *The growth bound of $\mathcal{T}$ is given by*

$$\omega_0(\mathcal{T}(\cdot)) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}.$$

This theorem is nice because $\omega_0(\mathcal{U})$ depends only on the modification in the past, while the spectral bound $s(\mathcal{C})$ of the operator $\mathcal{C}$ can be computed using Lemma 4.1 and depends on B and Φ.
However, the determination of all $\lambda \in \sigma(B + \Phi\epsilon_\lambda)$ remains a very difficult task. In this case the *positivity* is helpful as we can see in the next section.

4.3.1 Example

Let X be the Hilbert space $L^2[0,1]$, $E := L^p(\mathbb{R}_-, X)$ and B the Dirichlet Laplacian, i.e., $Bf := \Delta f$ with domain $D(B) := \{f \in H^2[0,1] : f(0) = f(1) = 0\}$. This operator generates a positive analytic semigroup $(S(t))_{t\geq 0}$ with $\omega_{\text{crit}}(S(t)) = -\infty$. Moreover, the regularity subspaces Y_t (see Remark 2.13) coincide with $D(B)$ (see [24, Example 4.7]). As in [10, Example 5] we define the operators $A(s)$ as

$$A(s) := a(s)B, \quad s \leq 0,$$

where $0 < a(\cdot) \in C(\mathbb{R}_-)$. We recall that these operators generate a backward evolution family $(U(t,s))_{t\leq s\leq 0}$ given by

$$U(t,s) = e^{(\int_t^s a(\sigma)d\sigma)\Delta}, \quad t \leq s \leq 0.$$

Since

$$||U(t,s)|| = e^{(\int_t^s a(\sigma)d\sigma)\lambda_0},$$

where λ_0 is the largest eigenvalue of Δ, we can compute directly the growth bound of $(U(t,s))_{t\leq s\leq 0}$.

Proposition 4.19 (see [10], Example 5). *The growth bound of* $(U(t,s))_{t\leq s\leq 0}$ *is given by*

$$\omega_0(\mathcal{U}) = \inf_{h\geq 0} \sup_{s+h\leq 0} \left(\frac{1}{h} \int_s^{s+h} a(\sigma)d\sigma \right) \lambda_0.$$

Define the delay operator Φ as in Remark 3.17, i.e.,

$$\Phi f := \int_{-\infty}^{0} \phi(s)f(s)ds \tag{4.23}$$

for $f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X)$, where $0 \leq \phi(\cdot) \in L^1(\mathbb{R}_-)$. Then Φ fulfills condition (M) (see [25, Example 4.6]). With these assumptions the operator $(\mathcal{C}, D(\mathcal{C}))$ is well defined and is the generator of a strongly continuous semigroup $\mathcal{T}$ on $\mathcal{E}$. Using the positivity of $(S(t))_{t\geq 0}$ and Φ, we can prove, as in [10], the following result.

Proposition 4.20. *The spectral bound* $s(\mathcal{C})$ *of the generator* $\mathcal{C}$ *is the unique solution of the equation*

$$\lambda_0 + \int_{-\infty}^{0} \phi(s)e^{\lambda s} e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0} ds = \lambda.$$

Proof. By definition, we have

$$B + \Phi\epsilon_\lambda = \Delta + \int_{-\infty}^{0} \phi(s)e^{\lambda s} e^{(\int_s^0 a(\sigma)d\sigma)\Delta} ds.$$

Using the spectral mapping theorem for selfadjoint operators, this implies

$$s(B + \Phi\epsilon_\lambda) = \lambda_0 + \int_{-\infty}^{0} \phi(s)e^{\lambda s} e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0} ds,$$

where λ_0 is the largest eigenvalue of Δ.

Since the function

$$\lambda \mapsto \lambda_0 + \int_{-\infty}^{0} \phi(s)e^{\lambda s} e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0} ds$$

is continuous and strictly decreasing, the spectral bound $s(\mathcal{C})$ is the unique solution of the equation

$$\lambda_0 + \int_{-\infty}^{0} \phi(s)e^{\lambda s} e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0} ds = \lambda.$$

□

Moreover, one shows as in [22, Theorem VI.6.14] that

$$\begin{aligned} s(\mathcal{C}) < 0 &\Leftrightarrow s(B + \Phi\epsilon_0) < 0 \\ &\Leftrightarrow \lambda_0 + \int_{-\infty}^{0} \phi(s) e^{(\int_s^0 a(\sigma) d\sigma)\lambda_0} ds < 0. \end{aligned}$$

By Theorem 4.18, we then obtain that the growth bound $\omega_0(\mathcal{T}(\cdot))$ of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is given as

$$\omega_0(\mathcal{T}(\cdot)) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}.$$

Since $\omega_0(\mathcal{U})$ is less than 0, then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable if and only if

$$\lambda_0 + \int_{-\infty}^{0} \phi(s) e^{(\int_s^0 a(\sigma) d\sigma \lambda_0} ds < 0.$$

Chapter 5

Applications

In this final chapter we apply the theory developed in the previous chapters to the two examples presented in Chapter 1.

5.1 A Population Equation with Diffusion as a Delay Equation with Nonautonomous Past

In Section 1.1 we have seen that the population equation studied by K. J. Engel and R. Nagel in [22] or J. Wu in [70] is not realistic since the delay term $u(t+s,x),\ s \in [r,0]$, is not submitted to a migration process. To include such a phenomenon in our model, we suppose, for simplicity, that this migration is given by a diffusion of the form $e^{t_p\Delta}$, where ${}_p\Delta$ is the Laplacian with Dirichlet boundary conditions (we write ${}_p\Delta$ to underline the fact that this diffusion is in the past).

To be more precise, we take the state space $X := L^1[0,1]$ and the Laplacian ${}_p\Delta$ with domain

$$D({}_p\Delta) := \{f \in W^{2,1}[0,1] : f(0) = f(1) = 0\}.$$

Then the evolution family solving the corresponding Cauchy problem (see [25, Example 6.1]) is

$$U(r,s) := T(s-r),\ r \leq s \leq 0,$$

where $(T(t))_{t\geq 0} = (e^{t_p\Delta})_{t\geq 0}$ is the heat semigroup in $L^1[0,1]$. Observe that $\omega_0(\mathcal{U}) = \omega_0(T(\cdot)) < 0$. Thus the modification of $u(t+s,x)$, s $\in [r,0]$, governed by this evolution family becomes

$$\widetilde{u}(t+s) := \begin{cases} U(s,s+t)f(s+t), & s+t \le 0 \\ U(s,0)u(s+t), & s+t \ge 0 \end{cases} = \begin{cases} T(-s)f(t+s), & s+t \le 0, \\ T(t)u(t+s), & s+t \ge 0. \end{cases}$$

In order to rewrite (1.2) as a delay equation with nonautonomous past we make the following assumptions and definitions.

General assumptions 5.1. (1) The mortality rate d is a nonnegative continuous function from $[0,1]$ to $\mathbb{R}_+$.

(2) The birth rate depends on the state space variable and on the time. Moreover $b(\cdot,\cdot)$ is a positive function such that $b(\cdot,x) \in L^1(\mathbb{R}_-)$ for each $x \in [0,1]$ and $b(s,\cdot) \in C([0,1])$ for each $s \le 0$.

General definitions 5.2. (1) As state space we take $X := L^1[0,1]$.

(2) Let $B := \Delta_x - M_d$, where Δ_x denotes the Laplacian operator on

$$D(\Delta_x) := \{f \in W^{2,1}[0,1] : f'(0) = f'(1) = 0\},$$

and M_d is the multiplication operator induced by the continuous function $d : [0,1] \to \mathbb{R}_+$, with $D(M_d) := X$.

(3) Take $(\Phi f)(s) := \int_{-\infty}^{0} b(s)f(s)ds$, where $f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X)$, and $b(s) := b(s,\cdot)$.

Remark 5.3. The Neumann Laplacian Δ_x in the General definitions 5.2 is different from the Dirichlet Laplacian ${}_p\Delta$ that governs the diffusion in the past.

Reassuming, our population equation becomes

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - d(x)u(t,x) + \int_{-\infty}^{0} b(s,x)\widetilde{u}(t+s,x)ds, \tag{5.1}$$

for $t \ge 0, s \le 0$ and $x \in [0,1]$.

With the General assumptions 5.1 we can rewrite (5.1) as (NDE), in fact it satisfies the General assumptions 3.1, in particular, for the first one, we refer to the following proposition.

Proposition 5.4. *The operator $(B, D(B))$ is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$. Moreover $(S(t))_{t\geq 0}$ is a contractive semigroup.*

Proof. Since the function d is bounded and $D(M_d) = X$, by [22, Proposition I.4.2], M_d is bounded.

Moreover, the Laplacian operator $(\Delta_x, D(\Delta_x))$ generates the heat semigroup $(T(t))_{t\geq 0}$ on X. As we know

$$\|T(t)\| \leq 1 \quad \forall\, t \geq 0. \tag{5.2}$$

Thus, applying the bounded perturbation theorem (see, e.g. [22, Theorem III.1.3]), the operator

$$B = \Delta_x - M_d \quad \text{with} \;\; D(B) = D(\Delta_x) \tag{5.3}$$

generates a C_0–semigroup $(S(t))_{t\geq 0}$. Moreover

$$\|S(t)\| = \|e^{t(\Delta_x - M_d)}\| \leq \|e^{t\Delta_x}\| = \|T(t)\| \leq 1 \quad \forall\, t \geq 0.$$

□

Our aim is to prove that there is a solution of (NDE). According to Theorem 3.6, we have to show the the operator matrix $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$. By Theorem 3.15, it is sufficent to show that the operator Φ satisfies the Miyadera-Voigt condition or the stronger one (see Assumption 4.6). At the same way of Example 3.16, one can prove that there exists a function $q : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $q(t) := \|b\|_1 t^{\frac{1}{p'}}$ such that $\lim_{t\to 0} q(t) = 0$ and

$$\int_0^t \|\Phi(S_r x + T_0(r) f)\| dr \leq q(t) \|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\| \tag{5.4}$$

for $t > 0$ and each $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{C})$, where in this case

$$(S_r x)(\tau) := \begin{cases} T(-\tau) S(r+\tau) x, & r+\tau \geq 0, \\ 0, & \text{elsewhere}, \end{cases} \tag{5.5}$$

$$(T_0(r) f)(\tau) = \begin{cases} T(r) f(r+\tau), & r+\tau \leq 0, \\ 0, & r+\tau > 0, \end{cases}$$

for $f \in L^p(\mathbb{R}_-, X)$, and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Thanks to perturbation theorem of Miyadera-Voigt, we can say that for each initial values $\binom{x}{f} \in D(\mathcal{C})$ there exists a solution of (NDE) and hence a solution of (5.1).

In order to study the stability of this solution, we use a result obtained in Chapter 4, but before we make some remarks.

Remark 5.5. (1) Since the heat semigroup is analytic, the semigroup $(S(t))_{t\geq 0}$ is analytic (see, e.g., [22, Theorem III.1.12]).

(2) Using the positivity $e^{t\Delta_x}$ and e^{-tM_d}, we can prove that the semigroup $(S(t))_{t\geq 0}$ is positive by the Trotter product formula. Thus also the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ and using [22, Theorem VI.6.11], we have the positivity of $(\mathcal{T}(t))_{t\geq 0}$ too.

In Chapter 4, Theorem 4.18 it is proved that if $(S(t))_{t\geq 0}$ is immediately norm continuous then the growth bound of $\mathcal{T}$ is given by

$$\omega_0(\mathcal{T}) = \max\{s(\mathcal{C}), \omega_0(\mathcal{U})\}. \tag{5.6}$$

Since $\omega_0(\mathcal{U}) < 0$, where, we recall, $(U(t,s))_{t\leq s\leq 0}$ is the evolution family associated to ${}_p\Delta$, then, by (4.18)

$$\omega_0(\mathcal{T}) < 0 \quad \text{if and only if} \quad s(\mathcal{C}) < 0. \tag{5.7}$$

Using again the positivity of $(S(t))_{t\geq 0}$ and b, we can prove, as in Proposition 4.20, the following result.

Proposition 5.6. *The spectral bound $s(\mathcal{C})$ of the generator $\mathcal{C}$ is the unique solution of the equation*

$$\lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{(\lambda-\lambda_0)s}ds = \lambda, \tag{5.8}$$

where λ_0 is largest eigenvalue of Δ and γ_0 is the smallest spectral value of M_d.

Moreover

$$\begin{aligned} s(\mathcal{C}) < 0 &\Leftrightarrow s(B + \Phi\epsilon_0) < 0 \\ &\Leftrightarrow \lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{-\lambda_0 s}ds < 0. \end{aligned}$$

So the solution of (NDE) is uniformly exponentially stable if and only if $\lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{-\lambda_0 s}ds < 0$. But, as we known, $\lambda_0 = 0$, thus

$$\lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{-\lambda_0 s}ds = -\gamma_0 + \int_{-\infty}^{0} b(s)ds = -\gamma_0 + \|b\|_1 < 0 \Leftrightarrow \|b\|_1 < \gamma_0.$$

This means that if the norm of the birth rate $\|b\|_1 \leq \inf_{x\in[0,1]} d(x)$, then the solution of (NDE) is uniformly exponentially stable.

5.1.1 A Non Constant Diffusion in the Past

Assume, now, that the diffusion in the past is *not constant*, i.e., it is governed by the operators $A(t) := a(t)({}_p\Delta)$, where $0 < a(\cdot) \in C(\mathbb{R}_-)$ and ${}_p\Delta$ is the Dirichlet Laplacian as before. In this case the evolution family associated to these operators is

$$U(t,s) = e^{(\int_t^s a(\sigma)d\sigma)_p\Delta}. \tag{5.9}$$

Since the norm of the evolution family is

$$\|U(t,s)\| = e^{(\int_t^s a(\sigma)d\sigma)\lambda_{0,p}}, \tag{5.10}$$

where $\lambda_{0,p}$ is the largest eigenvalue of ${}_p\Delta$, we can compute directly the growth bound of $(U(t,s))_{t\leq s\leq 0}$.

Proposition 5.7 (see [10], Example 5). *The growth bound of* $(U(t,s))_{t\leq s\leq 0}$ *is given by*

$$\omega_0(\mathcal{U}) = \inf_{h\geq 0} \sup_{s+h\leq 0} \left(\frac{1}{h}\int_s^{s+h} a(\sigma)d\sigma\right)\lambda_{0,p},$$

Since ${}_p\Delta$ is the Dirichlet Laplacian, then $\lambda_{0,p} = -\pi^2$ and $\omega_0(\mathcal{U}) < 0$.

Under the General assumptions 5.1, we can prove again that the operator Φ satisfies the stronger Miyadera-Voigt condition, where the function $q : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $q(t) := \|b\|_1 e^{-\pi^2 L_t} t^{\frac{1}{p'}}$ and L_t is a positive constant such that $0 < a(\tau) \leq L_t$ for $\tau \in [-t, 0]$.

By the perturbation theorem of Miyadera-Voigt, one has that there exists a solution of (NDE). To discuss the stability we proceed as before.

Using the positivity of $(S(t))_{t\geq 0}$ and Φ, as in Proposition 4.20 we can prove the following result.

Proposition 5.8. *The spectral bound $s(\mathcal{C})$ of the generator $\mathcal{C}$ is the unique solution of the equation*

$$\lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{\lambda s}e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0}ds = \lambda. \tag{5.11}$$

Moreover

$$\begin{aligned} s(\mathcal{C}) < 0 &\Leftrightarrow s(B + \Phi\epsilon_0) < 0 \\ &\Leftrightarrow \lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0}ds < 0. \end{aligned}$$

Since again $\omega_0(\mathcal{U}) < 0$, then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable if and only if $\lambda_0 - \gamma_0 + \int_{-\infty}^{0} b(s)e^{(\int_s^0 a(\sigma)d\sigma)\lambda_0}ds < 0$.

Now, $\lambda_0 = 0$, so

$$s(\mathcal{C}) < 0 \Leftrightarrow -\gamma_0 + \int_{-\infty}^{0} b(s)ds < 0 \Leftrightarrow \|b\|_1 < \gamma_0,$$

as before.

5.2 Genetic Repression as a Delay Equation with Nonautonomous Past

In the first chapter we have seen that the genetic repression proposed by J.M. Mahaffy, C.V.Pao in [41] or J. Wu in [70] is not realistic since the delay term $u_2(t + r_2, x)$, $r_2 < 0$, is submitted to a migration process. Assume, for simplicity, that this diffusion is *constant* (if it is not constant we can procede as in the Section 5.1.1), i.e., it is given by $e^{t({}_p\Delta)}$, where, as in Section 5.1, ${}_p\Delta$ is the Laplacian with domain

$$D({}_p\Delta) := \{f \in W^{2,1}[0,1] : f(0) = f(1) = 0\}.$$

Again we write ${}_p\Delta$ to underline the fact that this diffusion is in the past. As for the population equation, the backward evolution family (see [25, Example 6.1]) associated to this diffusion in the past is given by $U(t,s) = T(s-t)$, $t \leq s \leq 0$, where $(T(t))_{t\geq 0} = (e^{t_p\Delta})_{t\geq 0}$ is the heat semigroup. Thus the modification of $u_2(t + r_2, x)$

governed by $(U(t,s))_{t\le s\le 0}$ becomes

$$\begin{aligned}\widetilde{u}_2(t+r_2) : &= \begin{cases} U(r_2,0)u_2(t+r_2), & 0\le r_2+t,\\ U(r_2,t+r_2)f_2(t+r_2), & r_2+t\le 0,\end{cases}\\ &= \begin{cases} T(-r_2)u_2(t+r_2), & 0\le r_2+t,\\ T(t)f_2(t+r_2), & r_2+t\le 0.\end{cases}\end{aligned} \tag{5.12}$$

Extending the evolution family $(U(t,s))_{t\le s\le 0}$ to all of $\mathbb{R}$, we obtain the same situation as in Definition 3.2.

The problem now is to rewrite the genetic repression as a delay equation of the form (NDE). To this end we make the following definitions.

General definitions 5.9. (1) As state space we choose $X := \mathbb{R}^4 \times (L^1[0,1])^2$.

(2) Take the function $L : (W^{2,1}[0,1])^2 \to \mathbb{R}^4$ defined by

$$L\begin{pmatrix} f\\ g\end{pmatrix} = \begin{pmatrix} f'(1)\\ g'(1)\\ \frac{f'(0)}{\beta_1}+f(0)\\ \frac{g'(0)}{\beta_1^*}+g(0)\end{pmatrix}. \tag{5.13}$$

(3) Let

$$\mathcal{B} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -b_1-a_1 & 0 & a_1\eta_0 & 0\\ 0 & 0 & 0 & -b_2-a_2 & 0 & a_2\eta_0\\ 0 & 0 & 0 & 0 & D_1\Delta - b_1 & 0\\ 0 & 0 & 0 & 0 & 0 & D_2\Delta - b_2\end{pmatrix}, \tag{5.14}$$

with domain

$$D(\mathcal{B}) := \left\{ \begin{pmatrix} a\\ b\\ x\\ y\\ f\\ g\end{pmatrix} \in \mathbb{R}^4 \times (W^{2,1}[0,1])^2 : L\begin{pmatrix} f\\ g\end{pmatrix} = \begin{pmatrix} a\\ b\\ x\\ y\end{pmatrix}\right\}$$

where $\eta_0 f := f(0)$, for $f \in C[0,1]$ and Δ denotes the Laplacian.

(4) Let $\Phi : W^{1,p}([-1,0],X) \mapsto X$ given by

$$\Phi := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h\delta_{r_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0\delta_{r_2} & 0 \end{pmatrix}, \tag{5.15}$$

where we recall that $(h\delta_{r_1})v_{1_t} = \dfrac{1}{1+k(v_1(t-1))^\rho}$, if we assume, for simplicity, that $r_1 = r_2 = -1$.

Let now

$$\mathcal{U}(t) := \begin{pmatrix} a \\ b \\ u_1(t) \\ v_1(t) \\ u_2(t) \\ v_2(t) \end{pmatrix} \tag{5.16}$$

and consider the delay equation

$$\dot{\mathcal{U}}(t) = \mathcal{B}\mathcal{U}(t) + \Phi\widetilde{\mathcal{U}}_t, \quad t \geq 0, \tag{5.17}$$

where

$$\widetilde{\mathcal{U}}_t := \begin{pmatrix} a \\ b \\ u_{1_t} \\ v_{1_t} \\ \widetilde{u}_{2_t} \\ v_{2_t} \end{pmatrix}. \tag{5.18}$$

The equation (5.17) is of the form (NDE) and one can prove easily that, if

$a = b = 0$, a solution of (5.17) is also a solution of the genetic repression

$$(5.19)\qquad \begin{cases} \dfrac{du_1(t)}{dt} = h(v_1(t-1)) - b_1 u_1(t) + a_1(u_2(t,0) - u_1(t)), \\ \dfrac{dv_1(t)}{dt} = -b_2 v_1(t) + a_2(v_2(t,0) - v_1(t)), \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_1 \dfrac{\partial^2 u_2(t,x)}{\partial x^2} - b_1 u_2(t,x), \\ \dfrac{\partial v_2(t,x)}{\partial t} = D_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} - b_2 v_2(t,x) + c_0 \widetilde{u}_2(t-1,x), \end{cases}$$

with boundary conditions

$$(5.20)\qquad \begin{cases} \dfrac{\partial u_2(t,0)}{\partial x} = -\beta_1(u_2(t,0) - u_1(t)), \\ \dfrac{\partial v_2(t,0)}{\partial x} = -\beta_1^*(v_2(t,0) - v_1(t)), \\ \dfrac{\partial u_2(t,1)}{\partial x} = \dfrac{\partial v_2(t,1)}{\partial x} = 0. \end{cases}$$

Moreover assume that the genetic repression satisfies the following initial conditions

$$(5.21)\qquad \begin{cases} u_1(s) = f_1(s), \\ v_1(s) = g_1(s), \\ u_2(s,x) = f_2(s,x), \\ v_2(s,x) = g_2(s,x), \end{cases}$$

for $x \in [0,1]$ and $s \in [-1,0]$.

Now we have to prove that $(\mathcal{B}, D(\mathcal{B}))$ is the generator of a strongly continuous semigroup. To this end we rewrite $\mathcal{B}$ in the form

$$\mathcal{B} = \mathcal{B}_0 + \Theta - \Pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_1\Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & D_2\Delta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1\eta_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2\eta_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 + a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 + a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 \end{pmatrix},$$

with domains $D(\mathcal{B}_0) = D(\mathcal{B})$, $D(\Theta) = \mathbb{R}^4 \times (W^{1,1}[0,1])^2$ and $\Pi \in \mathcal{L}(X)$. Let

$$B_m := \begin{pmatrix} D_1\Delta & 0 \\ 0 & D_2\Delta \end{pmatrix}$$

and $B_0 := B_{m_{|\text{Ker L}}}$. By [22, Chapter VI, Section 4] B_0 generates a strongly continuous positive semigroup $(S_0(t))_{t\geq 0}$.

Applying a result of V. Casarino, K.-J. Engel, R. Nagel and G. Nickel (see [14, Corollary 2.8]), one obtains that $(\mathcal{B}_0, D(\mathcal{B}_0))$ generates an analytic semigroup.

Lemma 5.10. *The operator Θ is $\mathcal{B}_0-$bounded, having $\mathcal{B}_0-$bound $a_0 = 0$.*

Proof. Obviously, $D(\mathcal{B}_0) \subseteq D(\Theta)$ and for

$$\mathcal{V} = \begin{pmatrix} a \\ b \\ x \\ y \\ f \\ g \end{pmatrix} \in D(\mathcal{B}_0),$$

we have

$$\|\Theta\mathcal{V}\| = \left\| \begin{pmatrix} 0 \\ 0 \\ a_1 f(0) \\ a_2 g(0) \\ 0 \\ 0 \end{pmatrix} \right\| \leq \|a_1 f(0)\| + \|a_2 g(0)\|.$$

Since

$$\|\mathcal{B}_0\mathcal{V}\| = \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f'' \\ g'' \end{pmatrix} \right\|,$$

it sufficies to prove that for arbitrary small $a, b > 0$ there exist constants $c,\ d \in \mathbb{R}_+$, such that $\|f(0)\| \leq a\|f''\|_1 + b\|f\|_1$ and $\|g(0)\| \leq c\|f''\|_1 + d\|f\|_1$.

Using the fundamental theorem of calculus and the fact that the operator $\frac{d}{dx}$ is $\frac{d^2}{dx^2}-$bounded with $\frac{d^2}{dx^2}-$bound (see [22, Example III.2.2]), the assertion follows. □

The following proposition is now a consequence of [22, Theorem III.2.10].

Proposition 5.11. *The operator $(\mathcal{B}, D(\mathcal{B}))$ generates a positive analytic semigroup $(S(t))_{t\geq 0}$ on X.*

Proof. From the previous proposition, the operator Θ is $\mathcal{B}_0-$bounded with $\mathcal{B}_0-$bound $a_0 = 0$. Using [22, Theorem III.2.10], one has that $\mathcal{B}_0 + \Theta$ is the generator of an analytic semigroup. Since $\Pi \in \mathcal{L}(X)$, by [22, Theorem III.1.3], the operator $(\mathcal{B}, D(\mathcal{B}))$ is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$.

Moreover, by [22, Proposition III.1.12], $(S(t))_{t\geq 0}$ is analytic, while [14, Proposition 5.2] implies that it is positive as well. □

Remark 5.12. Observe that the delay operator Φ is not linear, so we have to linearize it with respect to the steady-state solutions. To find these solutions it sufficies to set the time derivatives of the genetic repression equal to zero (see [41] for details).

The linearization of Φ about the equilibrium can be written as

$$\bar{\Phi} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{h}\delta_{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0\delta_{-1} & 0 \end{pmatrix},$$

where $\bar{h}$ is the linearization of h about the equilibrium v_1^s and it is given by

$$\bar{h}(v_1(t-1)) = \dot{h}(v_1^s)v_1(t-1). \tag{5.22}$$

Note that $h'(v_1^s) < 0$ since h is a decreasing function.

Thus we can rewrite the linearized genetic repression as a delay equation with nonautonomous past

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{B}\mathcal{U}(t) + \bar{\Phi}\mathcal{U}_t \\ \mathcal{U}(0) = y \in X \\ \widetilde{\mathcal{U}}_0 = f \in L^p([-1,0],X). \end{cases} \tag{5.23}$$

To present the equation (5.23) as an abstract Cauchy problem, we consider the operator matrix

$$\mathcal{C} := \begin{pmatrix} \mathcal{B} & \bar{\Phi} \\ 0 & \mathcal{G} \end{pmatrix}, \tag{5.24}$$

with domain

$$D(\mathcal{C}) := \{ \left(\begin{smallmatrix} y \\ f \end{smallmatrix}\right) \in D(\mathcal{B}) \times D(\mathcal{G}) : f(0) = y \} \tag{5.25}$$

on the product space $\mathcal{E} := X \times L^p([-1,0],X)$, where

$$\mathcal{G} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{d\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{d\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{d\sigma} \end{pmatrix},$$

with domain

$$D(\mathcal{G}) := (L^p([-1,0])^2 \times (W^{1,p}[-1,0])^2 \times D(G) \times W^{1,p}([-1,0],L^1[0,1]).$$

Here the operator $(G, D(G))$ is defined as in Definition 2.6. If we take the operator $\mathcal{G}_0$ defined by

$$\mathcal{G}_0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{d\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{d\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{d\sigma} \end{pmatrix},$$

with $D(\mathcal{G}_0) = (L^p([-1,0]))^2 \times (W^{1,p}([-1,0]))^2 \times D(G_0) \times W^{1,p}([-1,0], L^1[0,1])$, then it generates a strongly continuous semigroup $(\mathcal{W}_0(t))_{t\geq 0}$, given by

$$\mathcal{W}_0(t) := \begin{pmatrix} Id & 0 & 0 & 0 & 0 & 0 \\ 0 & Id & 0 & 0 & 0 & 0 \\ 0 & 0 & T_l(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & T_l(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & T_0(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & T_l(t) \end{pmatrix}.$$

Here the operator $(G_0, D(G_0))$ and the semigroups $(T_0(t))_{t\geq 0}$ and $(T_l(t))_{t\geq 0}$ are defined as in Lemma 2.9 and in Lemma 2.8, respectively. The operator $\mathcal{C}_0$ defined by

$$\mathcal{C}_0 := \begin{pmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{G}_0 \end{pmatrix},$$

with $D(\mathcal{C}_0) = D(\mathcal{C})$, generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$, given by

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & \mathcal{W}_0(t) \end{pmatrix}.$$

Let $0 \leq t < 1$ and $q(t) := Pc_0 t^{\frac{1}{p'}}$, where $P := \|\bar{h}(\pi_4(f(-1)))\|$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, using the fact that $\bar{h}$ is decreasing, one has

$$\begin{aligned}
\int_0^t \|\bar{\Phi}(S_r x + \mathcal{W}_0(r)f)\| dr &\leq \int_0^t \|\bar{h}\delta_{-1}(\pi_4(S_r x) + \pi_4(\mathcal{W}_0(r)f))\| dr \\
&+ \int_0^t \|c_0\delta_{-1}(\pi_5(S_r x) + \pi_5(\mathcal{W}_0(r)f))\| dr \\
&\leq \int_0^t \|\bar{h}(\pi_4(f(r-1)))\| dr \\
&+ c_0 \int_0^t \|U(r, r-1)\| \, \|f(r-1)\| dr \\
&\leq \int_0^t \|\bar{h}(\pi_4(f(r-1)))\| dr + c_0 \int_0^t \|f(r-1)\| dr \\
&\leq \int_0^t \|\bar{h}(\pi_4(f(-1)))\| dr + c_0 \|f\| t^{\frac{1}{p'}} \\
&\leq Pt + c_0 \|f\| t^{\frac{1}{p'}} \\
&\leq Pc_0 t^{\frac{1}{p'}} (\|y\| + \|f\|)
\end{aligned}$$

for all $\binom{y}{f} \in D(\mathcal{C}_0)$.

By the perturbation theorem of Miyadera-Voigt, there exists a solution of (5.23) and hence a solution of (5.19).

For the stability of the solutions of (5.23), one can proceed as in the previous section. Also in this case $\omega_0(\mathcal{U}) < 0$. Thus the semigroup $(\mathcal{T}(t))_{t\geq 0}$ decays exponentially if and only if $s(\mathcal{C}) < 0$.

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List of Symbols and Abbreviations

(CP)	Abstract Cauchy Problem
(NCP)	Nonautonomous Abstract Cauchy Problem
(DE)	Delay Equation
(NDE)	Delay Equation with Nonautonomous Past
$A\sigma(G_0)$	Approximate Point Spectrum of G_0
$\mathcal{C}$, $\mathcal{C}_0$	Operator Matrices on $\mathcal{E}$
Δ_x	Laplacian Operator
Δ_p	Laplacian Operator in the Past
$\mathcal{E}$	Product Space
ϵ_λ	Eigenvector of G_0
$(G_0, D(G_0))$	Generator of $(T_0(t))_{t\geq 0}$
$\mathcal{L}(X)$	Space of Bounded Operators
$L^p(\mathbb{R}_-, X)$	Space of $p-$integrable Functions
M_d	Moltiplicator Operator Associated to d
$\omega_0(\mathcal{U})$	Growth Bound of $(U(t,s))_{t\leq s\leq 0}$
$\omega_0(T_0(\cdot))$	Growth Bound of $(T_0(t))_{t\geq 0}$
$\omega_{\text{crit}}(T_0(\cdot))$	Critical Growth Bound of $(T_0(t))_{t\geq 0}$
π_1, π_2	Projections onto the First and the Second Component
$R\sigma(G_0)$	Residual Spectrum of G_0
$R(\lambda, G_0)$	Resolvent of G_0 in λ
$\rho(G_0)$	Resolvent Set of G_0
$\sigma(G_0)$	Spectrum of G_0
$\sigma(T_0(t))$	Spectrum of $(T_0(t))_{t\geq 0}$
$\sigma_{\text{crit}}(T_0(t))$	Critical Spectrum of $(T_0(t))_{t\geq 0}$
$\sigma(G_0)$	Spectrum of G_0
$s(G_0)$	Spectral Bound of G_0
$s_0(\mathcal{C})$	Abscissa of Uniform Boundedness of the Resolvent of $\mathcal{C}$

$(T_0(t))_{t\geq 0}$	Backward Evolution Semigroup on $L^p(\mathbb{R}_-, X)$
$(\widetilde{T}(t))_{t\geq 0}$	Backward Evolution Semigroup on $L^p(\mathbb{R}, X)$
$(\mathcal{T}(t))_{t\geq 0}$, $(\mathcal{T}_0(t))_{t\geq 0}$	Semigroups on $\mathcal{E}$
$(U(t,s))_{t\leq s\leq 0}$	Backward Evolution Family on $\mathbb{R}_-$
$(\widetilde{U}(t,s))_{t\leq s}$	Backward Evolution Family on $\mathbb{R}$
u_t	History function
$\widetilde{u}_t$	Modified History Function
$W^{k,p}((0,1),X)$	Sobolev Spaces
Y_s	Regularity Subspaces

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