Product systems from a bicategorial point of view and duality theory for Hopf C*-algebras

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Introduction

“EVENTUALLY PEOPLE WILL SEE THAT GROUP
REPRESENTATION THEORY IS NOT SUCH A BIG DEAL;
WHAT REALLY MATTERS IS REPRESENTATION OF CATEGORIES.”
(Minhyong Kim to John Baez, see [3])

This thesis consists of two parts. In the first part, containing chapters one to four, we want to take a look at product systems from a new point of view. Product systems were first introduced in 1989 by Arveson in [1] to develop an index theory for continuous semigroups of *-endomorphisms of $\mathcal{L}(\mathcal{H})$. Later, they were studied by Dinh [8] in the discrete case and Fowler generalized the concept of Dihn by using Hilbert bimodules. Fowler's discrete product systems of Hilbert bimodules consist of a family of Hilbert bimodules $\{X_s \colon s \in \mathcal{S}\}$ over a C^*-algebra A indexed by a countable semigroup $\mathcal{S}$ together with a family of unitary bimodule mappings

$$\Phi_{s,t} \colon X_s \otimes_A X_t \to X_{st}, \qquad s, t \in \mathcal{S}.$$

We want to reveal the structure that lies behind product systems using bicategory theory. Similar to a category, a bicategory consists of objects and arrows between these objects, but contrary to categories, bicategories possess an extra structure, namely arrows between the arrows that are called 2-cells. Our main example for a bicategory will be C*ARR, the bicategory with objects C^*-algebras, arrows C^*-arrows (based on Hilbert bimodules, see below) and with adjointable, isometric bimodule mappings as 2-cells.

We will introduce the “functors” between bicategories, which are called morphisms, and we will see that Fowler's product systems are nothing but special morphisms from the semigroup $\mathcal{S}$ (viewed as a bicategory) to the bicategory C*ARR. Thus, we can give a more elegant definition of the notion of a product system by defining them as morphisms from an index category J (viewed as a bicategory) to the bicategory C*ARR. Hence, the product systems that originated from semigroups of *-endomorphisms in Arveson's paper [1] can now be described in a very natural way using the concept of morphisms between bicategories. Moreover, we get a much bigger class of examples taking index categories instead of semigroups to index our product systems.

Next, we will associate two C^*-algebras to every given product system (F, Φ), namely the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F, \Phi)$.

Studying various special cases shows that our method of constructing the reduced Toeplitz and Cuntz-Pimsner algebras generalizes many other constructions of C^*-algebras.

Moreover, we will introduce the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$ and the universal Cuntz-Pimsner algebra $\mathcal{O}(F,\Phi)$. We recall the notion of the bicategorial colimit for a morphism and we finish the first part of this thesis by showing that for certain product systems (F,Φ), the universal Toeplitz algebra can be viewed as the bicategorial colimit object for the morphism (F,Φ).

In the second part of the thesis, containing chapters five and six, we develop a duality theory for locally compact semigroups using the concept of Hopf C^*-algebras. A Hopf C^*-algebra is a C^*-algebra H together with a comultiplication, i.e., a nondegenerate, injective *-homomorphism $\delta_H\colon H \to M(H \otimes H)$. The standard example for a Hopf C^*-algebra is $C_0(\mathcal{S})$, the C^*-algebra of complex functions on a locally compact semigroup $\mathcal{S}$ vanishing at infinity. The multiplication on $\mathcal{S}$ induces a comultiplication on $C_0(\mathcal{S})$. Thus, Hopf C^*-algebras can be viewed as generalized locally compact semigroups.

We will develop a sufficient condition on the Hopf C^*-algebra H that allows us to construct a corepresentation of H on a distinguished Hilbert space, similar to the regular representation of a locally compact group G on the Hilbert space $L^2(G,\mu)$, where μ is the right Haar measure on G. Using this regular corepresentation, we can define the reduced dual C^*-algebra of a Hopf C^*-algebra and we will show that the classic Toeplitz algebra $C^*(\mathbb{N})$ is the reduced dual C^*-algebra of the Hopf C^*-algebra $c_0(\mathbb{N})$. We will also see that $c_0(\mathbb{N})$ is the reduced dual C^*-algebra of the Hopf C^*-algebra $C^*(\mathbb{N})$. This corresponds to the well known fact that for a locally compact group G, the C^*-algebra $C_0(G)$ and the full group C^*-algebra $C^*(G)$ are in duality, which can be viewed as an analogue of Pontryagin's duality theorem.

Finally, we will deal with Takai's duality theorem [28], which is one of most fundamental theorems in the theory of crossed products. It states that for a C^*-dynamical system (A,G,α), the double crossed product $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is strongly Morita equivalent to A. In [25], Schweizer treated an analogue of Takai's duality theorem for crossed products by equivalence bimodules. He showed that for an equivalence bimodule X over a C^*-algebra A, there exists an action γ of $\hat{\mathbb{Z}}$ on $A \rtimes_X \mathbb{Z}$ such that $(A \rtimes_X \mathbb{Z}) \rtimes_\gamma \hat{\mathbb{Z}}$ is strongly Morita equivalent to A.

We want to transfer Schweizer's statement to the situation when E is a C^*-arrow over a C^*-algebra A. Therefore, we define the crossed product $A \rtimes_E \mathbb{N}$ as the reduced Toeplitz algebra of (A,E), where (A,E) is a certain product system over $\mathbb{N}$ that consist of the powers of E. Next, we define the reduced crossed product of a dynamical cosystem and finally, we construct a coaction δ of $C^*(\mathbb{N})$ on $A \rtimes_E \mathbb{N}$ and show that the double crossed product $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ is strongly Morita equivalent to A.

The following chapter summaries will give a more detailed description of this thesis:

Chapter 1 is devoted to the historical development of product systems. We give a short overview over the work of Arveson [1], who introduced product systems, Dinh [8], who first studied discrete product systems, and Fowler [10], who introduced product systems of Hilbert bimodules.

In Chapter 2 we recall the notion of a bicategory and provide several examples. We will introduce the concept of C^*-arrows which play a central role in our thesis. A C^*-arrow is a Hilbert B-module that also possesses an A-B-bimodule structure, where A and B are C^*-algebras. C^*-arrows are the arrows in our motivating example for a bicategory, the bicategory C*ARR. The objects of C*ARR are C^*-algebras and the 2-cells are adjointable, isometric bimodule mappings. Moreover, we will recall the notion of a morphism between bicategories, that generalizes the concept of functors between categories. We will define a product system over a category to be a morphism from an index category J to the bicategory C*ARR. Hence, a product system over an index category J will consist of a family of C^*-algebras A_i, $i \in \mathrm{Ob}(J)$, a family of C^*-arrows F_r, $r \in \mathrm{Arr}(J)$, and a family of isometric, adjointable bimodule mappings $\Phi_{s,r} \colon F_r \otimes F_s \to F_{sr}$ indexed by pairs (r, s) of composable arrows of J. Finally, we will see that Fowler's discrete product systems of Hilbert bimodules are a special case of our definition.

Chapter 3 deals with the construction of the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F, \Phi)$ associated to every given product system (F, Φ). First, we introduce the notion of a Toeplitz representation from a product system (F, Φ) to a C^*-algebra and we provide some technical results about homomorphisms between Hilbert C^*-modules that we will need later. Then we introduce the Fock correspondence $\mathcal{F}(F, \Phi)$ of a product system (F, Φ) and we construct one particular Toeplitz representation, the reduced Toeplitz representation from (F, Φ) to the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$, which is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}(F, \Phi))$. We define the reduced Cuntz-Pimsner algebra to be the quotient of $\mathcal{T}_r(F, \Phi)$ modulo the ideal of generalized compact operators in $\mathcal{T}_r(F, \Phi)$. Next, we provide various examples which show that our method of constructing the reduced Toeplitz and Cuntz-Pimsner algebras generalizes many other constructions of C^*-algebras. Depending on how we choose our product system (F, Φ), the resulting C^*-algebras $\mathcal{T}_r(F, \Phi)$ and $\mathcal{O}_r(F, \Phi)$, respectively, are isomorphic to the direct sum of a family of C^*-algebras, the direct limit of a direct system of C^*-algebras or the crossed product of a C^*-algebra by a group or a semigroup.

In Chapter 4 we will introduce the universal Toeplitz algebra $\mathcal{T}(F, \Phi)$ and the universal Cuntz-Pimsner algebra $\mathcal{O}(F, \Phi)$ together with their corresponding Toeplitz representations. First, we show that given a product system (F, Φ) over an index category J, there is a C^*-algebra that is universal for Toeplitz representations over (F, Φ) and we will call it the universal Toeplitz algebra $\mathcal{T}(F, \Phi)$. Then we recall the notion of Cuntz-Pimsner covariant Toeplitz representations from [10] and introduce the universal Cuntz-Pimsner algebra, which will be universal for Cuntz-Pimsner covariant Toeplitz representations over (F, Φ). Finally, we will recall the notion of a bicategorial colimit for a morphism (F, Φ) from a bicategory $\mathcal{B}$ to a bicategory $\mathcal{B}'$ and we will show that for certain product systems (F, Φ) the universal Toeplitz algebra $\mathcal{T}(F, \Phi)$ can be viewed as the colimit object for the morphism (F, Φ) in the bicategory C*ARR.

In Chapter 5 we first recall the concept of Hopf C^*-algebras, which can be viewed as a generalization of locally compact semigroups. We also recall the notions of corepresentations of Hopf C^*-algebras, coactions of Hopf C^*-algebras on C^*-algebras and covariant representations of dynamical cosystems, which generalize the corresponding notions for semigroups, namely representations of semigroups, actions of semigroups on C^*-algebras and covariant representations of semigroup dynamical systems. Then we show that the existence of an invariant weight τ on the Hopf C^*-algebra H makes it possible to construct a specific covariant representation of the dynamical cosystem (H, H, δ_H) inspired by the right regular covariant representation of $(C_0(G), G, \alpha)$ on $L^2(G, \mu)$, where μ is a right Haar measure on a locally compact group G and α is the action of G on $C_0(G)$ by right translation. For Hopf C^*-algebras H that are equipped with an invariant weight τ, we construct the reduced and universal dual C^*-algebras $C^*_r(H, \delta_H)$ and $C^*(H, \delta_H)$, respectively. We show that the reduced dual C^*-algebra of the Hopf C^*-algebra $(c_0(\mathbb{N}), \alpha_{\mathbb{N}})$ is isomorphic to $C^*(\mathbb{N})$ and that the reduced dual C^*-algebra of the Hopf C^*-algebra $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ is isomorphic to the C^*-algebra $c_0(\mathbb{N})$. This result can be viewed as a generalization of Pontryagin's duality theorem for locally compact, abelian groups to the semigroup $\mathbb{N}$. Finally, we construct the reduced crossed product $A \rtimes_\delta H$ for a dynamical cosystem (A, H, δ), which we will need in the next chapter.

Chapter 6 deals with an analogue of Takai's duality theorem [28] in the setting of crossed products by C^*-arrows. We show that for specific product systems (A, E) over the natural numbers, where A is a C^*-algebra and E a C^*-arrow over A, the double crossed product $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ is strongly Morita equivalent to A, where $A \rtimes_E \mathbb{N}$ is the reduced Toeplitz algebra of (A, E) and δ can be viewed as the dual coaction of $C^*(\mathbb{N})$ on $A \rtimes_E \mathbb{N}$. As a minor result we also get that a product system (A, E) over $\mathbb{N}$ is always strongly Morita equivalent to a product system induced by a *-endomorphism.

Notation

When we talk of a semigroup, we shall always mean a semigroup with a unit element. This also means that the natural numbers $\mathbb{N}$ always include zero and hence, $(\mathbb{N}, +)$ is a semigroup with unit element zero.

If E is a subset of a linear space L, then we write $\operatorname{span} E$ for the linear span of E. If L is a normed linear space, then $\overline{\operatorname{span}} E$ denotes the norm closure of the linear span of E in L. If E and F are subsets of an algebra A, then $EF := \operatorname{span}\{ef : e \in E, f \in F\}$.

For a C^*-algebra A, we denote the positive elements in A by A^+ and the multiplier algebra of A by $M(A)$. The unique unital extension of a *-homomorphism $\sigma\colon A \to M(B)$ to $M(A)$ is denoted by $\overline{\sigma}$. All tensor products of C^*-algebras are minimal.

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Part I

Product systems from a bicategorial point of view

Chapter 1

An introduction to product systems

In this first chapter, we want to sketch the development of product systems. They were first introduced in 1989 by Arveson in [1] to develop an index theory for continuous semigroups of *-endomorphisms of $\mathcal{L}(\mathcal{H})$. Arveson's product systems consist of a continuous family $\{E_t : t \in \mathbb{R}^+\}$ of Hilbert spaces together with a tensoring operation.

Extending Arveson's work, in [8], Dinh studied discrete product systems over G^+ where G is a discrete, countable and dense subgroup of $\mathbb{R}$. He associated to each discrete product system $E = \{E_t : t \in G^+\}$ a separable C^*-algebra $\mathcal{O}_E(G)$ and examined the structure of this C^*-algebra. The family of these C^*-algebras may be regarded as analogues of Cuntz algebras or CAR algebras.

Finally, Fowler considered discrete product systems of Hilbert bimodules in his paper [10], i.e., he replaced the Hilbert spaces E_t by Hilbert bimodules X_t. Moreover, he allowed arbitrary semigroups $\mathcal{S}$ to index his family of Hilbert bimodules. Given such a product system $X = \{X_s : s \in \mathcal{S}\}$, he defined a corresponding generalized Cuntz-Pimsner algebra $\mathcal{O}_X$. His work was motivated by the example when X comes from a semigroup dynamical system $(A, \mathcal{S}, \alpha)$. In this case he showed that $\mathcal{O}_X$ is canonically isomorphic to the semigroup crossed product $A \rtimes_\alpha \mathcal{S}$.

1.1 Arveson's continuous product systems

In [21], [22] and [23], Powers and Robinson introduced an index theory for E_0-semigroups. An *E_0-semigroup* is a continuous semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of normal *-endomorphisms of the C^*-algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$, such that $\alpha_t(1) = 1$ for all $t \geq 0$. Two E_0-semigroups α and β of $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$, respectively, are *paired* if there is a continuous group $\Gamma = \{\Gamma_t : t \in \mathbb{R}\}$ of *-automorphisms of $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ such that $\Gamma_t(A \otimes 1) = \alpha_t(A) \otimes 1$ and $\Gamma_{-t}(1 \otimes B) = 1 \otimes \beta_t(B)$ for all $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K})$ and $t \geq 0$. Two E_0-semigroups have the *same index* if there is an E_0-semigroup σ such that α and σ are paired and σ and β are paired.

In [1], Arveson gave an equivalent characterization of the index by introducing the notion of continuous product systems. A continuous product system E^α arises naturally from each E_0-semigroup α and Arveson showed that two E_0-semigroups α and β have the same Powers-Robinson index if and only if their associated product systems E^α and E^β are isomorphic. Now we want to recall Arveson's definition of a continuous product system and then show in Example 1.2 how a continuous product system E^α can be associated with an E_0-semigroup α.

Definition 1.1 (Arveson's continuous product systems)
Let E be a standard Borel space and p a measurable function from E onto $(0, +\infty)$ such that each fiber

$$E(t) := p^{-1}(t), \qquad t > 0,$$

is a separable infinite dimensional Hilbert space and such that the inner product is measurable if we consider it to be a complex-valued function defined on the following Borel subset of $E \times E$:

$$\{(x, y) \in E \times E : p(x) = p(y)\}.$$

We also require that there is a Hilbert space H_0 such that E is isomorphic to the trivial family $(0, +\infty)\times H_0$, i.e., that there is a Borel isomorphism $\theta : E \to (0, +\infty)\times H_0$ such that for every $t > 0$, θ restricts to a unitary isomorphism of Hilbert spaces $\theta : E(t) \to \{t\} \times H_0$.

Finally, we require that there be given a jointly measurable binary associative operation $(x, y) \in E \times E \mapsto xy \in E$ satisfying the conditions

(i) $p(xy) = p(x) + p(y)$ and

(ii) for every $s, t > 0$, $E(s)E(t)$ is dense in $E(s+t)$ and we have $\langle xy, x'y'\rangle = \langle x, x'\rangle\langle y, y'\rangle$ for all $x, x' \in E(s)$, $y, y' \in E(t)$.

Notice that (i) means that $E(s)E(t) \subseteq E(s + t)$, while (ii) asserts that there is a unique unitary operator $W_{s,t} : E(s) \otimes E(t) \to E(s + t)$ defined by

$$W_{s,t}(x \otimes y) = xy, \qquad x \in E(s), y \in E(t).$$

The structure $p : E \to (0, +\infty)$ satisfying all of the above conditions is called a *continuous tensor product system of Hilbert spaces* or just a *continuous product system.*

Example 1.2 (Continuous product systems associated with E_0-semigroups)
Let α be an E_0-semigroup of $\mathcal{L}(\mathcal{H})$. For every positive real number t, we consider the linear space of operators

$$E^\alpha_t := \{T \in \mathcal{L}(\mathcal{H}) : \alpha_t(A)T = TA \text{ for all } A \in \mathcal{L}(\mathcal{H})\}.$$

The family of vector spaces $\{E_t^\alpha : t \geq 0\}$ has three important properties. First of all, each E_t^α is a Hilbert space. To see this, let S and T be two operators in E_t^α. Then

$$T^*SA = T^*\alpha_t(A)S = (\alpha_t(A^*)T)^*S = (TA^*)^*S = AT^*S$$

for all $A \in \mathcal{L}(\mathcal{H})$, i.e., T^*S commutes with every operator in $\mathcal{L}(\mathcal{H})$, and hence, it must be a scalar multiple of the identity operator:

$$T^*S = \langle S, T\rangle 1.$$

This identity defines an inner product $\langle \cdot, \cdot \rangle$ on E_t^α which turns E_t^α into a Hilbert space.

Now we let E^α be the set of ordered pairs

$$E^\alpha := \{(t, T) \colon t > 0, T \in E_t^\alpha\},$$

and we let $p^\alpha \colon E^\alpha \to (0, +\infty)$ be the projection $p^\alpha((t, T)) = t$. Then the structure $p^\alpha \colon E^\alpha \to (0, +\infty)$ is a family of Hilbert spaces having fibers $(p^\alpha)^{-1}(t) = E_t^\alpha$, $t > 0$. Our second observation is that one can make E^α into an associative semigroup by using operator multiplication:

$$(s, S)(t, T) := (s + t, ST).$$

Then we have $p^\alpha((s, S)(t, T)) = s + t = p^\alpha((s, S)) + p^\alpha((t, T))$, i.e., the projection becomes a homomorphism of E^α onto the additive semigroup of positive real numbers.

Finally, this multiplication acts like tensoring in the sense that it defines a natural unitary operator $W_{s,t}^\alpha$ from the tensor product of Hilbert spaces $E_s^\alpha \otimes E_t^\alpha$ onto E_{s+t}^α, for every $s, t > 0$. $W_{s,t}^\alpha$ is defined by

$$W_{s,t}^\alpha(S \otimes T) := ST,$$

for all $S \in E_s^\alpha$, $T \in E_t^\alpha$. The fact that $W_{s,t}^\alpha$ is unitary is a consequence of the following two properties relating multiplication and the inner products defined in the fiber spaces:

(i) If $S, S' \in E_s^\alpha$ and $T, T' \in E_t^\alpha$, then $\langle ST, S'T'\rangle = \langle S, S'\rangle\langle T, T'\rangle$, and

(ii) $E_{s+t}^\alpha = \overline{\operatorname{span}}\{ST : S \in E_s^\alpha, T \in E_t^\alpha\}$.

Property (i) follows directly from the definition of the inner product. For property (ii), see [1, Proposition 2.2.], where Arveson also shows that E^α satisfies all of the remaining conditions for a continuous product system. So we see that every E_0-semigroup α gives rise to an associated continuous product system E^α.

1.2 Dinh's discrete product systems

In [2], Arveson associated a C^*-algebra to each continuous product system. This C^*-algebra is a separable C^*-algebra whose nondegenerate *-representations correspond bijectively to representations of the continuous product system. Dinh extended the work of Arveson and studied this C^*-algebra in the case when the parameter t varies over a discrete subsemigroup of $\mathbb{R}^+$. We first want to provide Dinh's definition of a discrete product system and then show how the associated C^*-algebra can be constructed from a given discrete product system E. We note that this construction will be similar to the construction of the Cuntz algebra $\mathcal{O}_n$ from the Fock space or the CAR algebra from the canonical anti-commutation relations.

Definition 1.3 (Dinh's discrete product systems)
Let G be a countable dense subgroup of the real line equipped with the discrete topology and let G^+ denote the semigroup of strictly positive elements of G. A *discrete product system* over G^+ is defined to be a disjoint union $E = \bigcup_{t\in G^+} E_t$, where each fiber E_t is a separable Hilbert space. Moreover, there is an associative tensoring operation on E satisfying

(i) For each $s, t \in G^+$, there is a bilinear map $(u, v) \in E_s \times E_t \mapsto uv \in E_{s+t}$. Moreover, $E_s E_t$ is dense in E_{s+t}.

(ii) $\langle uv, u'v'\rangle = \langle u, u'\rangle\langle v, v'\rangle$ for every $u, u' \in E_s$ and $v, v' \in E_t$, $s, t \in G^+$.

Note that (i) and (ii) imply that the mapping $u \otimes v \in E_s \otimes E_t \mapsto uv \in E_{s+t}$, $u \in E_s$, $v \in E_t$, $s, t \in G^+$, extends to a unitary operator from $E_s \otimes E_t$ to E_{s+t}.

Moreover, Dinh defines a *representation* of a discrete product system E on a Hilbert space $\mathcal{H}$ to be a mapping $\phi\colon E \to \mathcal{L}(\mathcal{H})$ satisfying

1. $\phi(\xi)^*\phi(\eta) = \langle \eta, \xi\rangle\, 1$ for $\xi, \eta \in E_t$ and $t \in G^+$,
2. $\phi(\xi)\phi(\eta) = \phi(\xi\eta)$ for $\xi, \eta \in E$.

Then $\mathcal{O}_E(G)$ is defined to be the C^*-algebra generated by the range of ϕ in $\mathcal{L}(\mathcal{H})$. Dinh shows that $\mathcal{O}_E(G)$ does not depend on the representation ϕ, i.e., if $\phi_1\colon E \to \mathcal{L}(\mathcal{H}_1)$ and $\phi_2\colon E \to \mathcal{L}(\mathcal{H}_2)$ are two representations, then the map $\phi_1(u) \mapsto \phi_2(u)$ extends to a *-isomorphism from $\mathcal{O}^1_E(G)$ onto $\mathcal{O}^2_E(G)$.

He also presents one particular representation, the *Fock representation* l on the *Fock space* $\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{t\in G^+} E_t$, where Ω is the unit *vacuum vector*. The Fock representation $l\colon E \to \mathcal{L}(\mathcal{F})$ is defined by

$$l(u)\Omega := u \qquad \text{and} \qquad l(u)w := uw$$

for $u \in E_t$, $w \in E_s$. The operator $l(u)$ is called the *left creation operator*. Its adjoint is called the *left annihilation operator* and is uniquely characterized by the requirements $l(u)^*w = 0$ for $w \in \mathbb{C}\Omega \oplus \bigoplus_{s\in G^+\cap(0,t)} E_s$, $l(u)^*w = \langle w, u\rangle\Omega$ for $w \in E_t$ and $l(u)^*wv = \langle w, u\rangle v$ for $w \in E_t$, $v \in E_s$. It is easy to see that this defines a representation and Dinh shows that the corresponding *-representation of $\mathcal{O}_E(G)$ on $\mathcal{F}$ is irreducible. Now we take a look at two examples for discrete product systems.

Example 1.4 (Discrete product systems over $\mathbb{Z}^+$)
For the sake of simplicity we drop the condition of G being dense in $\mathbb{R}$ in this first example and consider the case $G = \mathbb{Z}$. Let $\mathcal{H} = E_1$. Then $E_2 \cong E_1 \otimes E_1 = \mathcal{H}\otimes\mathcal{H}$, $E_3 \cong E_1 \otimes E_2 \cong \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$, etc. Thus $E = \bigcup_{n\in\mathbb{Z}^+} \mathcal{H}^{\otimes n}$ and the Fock space $\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{n\in\mathbb{Z}^+} \mathcal{H}^{\otimes n}$ is the full Fock space.

The following example is a natural generalization of the above construction for $\mathbb{Z}$, namely when G is countable and dense in $\mathbb{R}$. It was first studied by von Neumann in [19] and so Dinh calls it a von Neumann discrete product system.

Example 1.5 (Von Neumann discrete product systems)
Let $\mathcal{H}$ be a fixed Hilbert space of dimension $N = 1, 2, 3, \dots$ or $\aleph_0$. We fix a unit vector $a \in \mathcal{H}$ and for $t \in G^+$, we define $E_t = \bigotimes_{G^+\cap(0,t]} \mathcal{H}$ as follows. Let $\mathcal{G}$ denote the collection of all finite subsets of $G^+ \cap (0, t]$. Then $\mathcal{G}$ is directed by inclusion. For $\mathcal{T} \in \mathcal{G}$, we form the finite tensor product $\bigotimes_{\mathcal{T}} \mathcal{H}$ and if $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{G}$, $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we define an embedding $\alpha_{\mathcal{T}_2\mathcal{T}_1}\colon \bigotimes_{\mathcal{T}_1} \mathcal{H} \to \bigotimes_{\mathcal{T}_2} \mathcal{H}$ by

$$\alpha_{\mathcal{T}_2\mathcal{T}_1}(x) := x \otimes a^{\otimes \mathcal{T}_2\setminus\mathcal{T}_1} \in (\bigotimes_{\mathcal{T}_1} \mathcal{H}) \otimes (\bigotimes_{\mathcal{T}_2\setminus\mathcal{T}_1} \mathcal{H}) \cong \bigotimes_{\mathcal{T}_2} \mathcal{H}$$

for $x \in \bigotimes_{\mathcal{T}_1} \mathcal{H}$. Then we define E_t to be the Hilbert space direct limit $\varinjlim \bigotimes_{\mathcal{T}\in\mathcal{G}} \mathcal{H}$.

Now let $\{f_1 = a, f_2, \dots, f_N\}$ be an orthonormal basis of $\mathcal{H}$. Then an orthonormal basis of E_t consists of all vectors of the form $\bigotimes_{r\in G^+\cap(0,t]} x_r$ where $x_r \in \{f_1, f_2, \dots, f_N\}$ and $x_r = a$ for all but finitely many r. Note that E_t is separable since $\mathcal{H}$ is separable and $G^+ \cap (0, t]$ is countable. For $u \in E_s$ and $v \in E_t$, we define $uv := u \otimes v \in E_s \otimes E_t \cong E_{s+t}$, i.e., the operation on E is tensoring.

1.3 Fowler's discrete product systems of Hilbert bimodules

The next step in the generalization of discrete product systems is to replace the Hilbert spaces E_t by Hilbert bimodules. This is the concept of Fowler in [10]. In order to present this concept we first have to provide some theory about Hilbert C^*-modules:

Definition 1.6 (Hilbert C^*-modules)
Let A be a C^*-algebra and E a right A-module with an A-valued inner product

$$(\,.\,|\,.\,)_A : E \times E \to A,$$

such that the following identities hold for $\xi, \eta, \eta_1, \eta_2 \in E$ and $a, b \in A$:

1. $(\,\xi\,|\,\eta_1 + \eta_2\,)_A = (\,\xi\,|\,\eta_1\,)_A + (\,\xi\,|\,\eta_2\,)_A$,
2. $(\,\xi \cdot a\,|\,\eta \cdot b\,)_A = a^*(\,\xi\,|\,\eta\,)_A b$,
3. $(\,\xi\,|\,\eta\,)_A = (\,\eta\,|\,\xi\,)_A^*$,
4. $(\,\xi\,|\,\xi\,)_A \geq 0,\ \ (\,\xi\,|\,\xi\,)_A = 0 \implies \xi = 0$,
5. E is complete with respect to the norm $\|.\|_E := \sqrt{\|(\,.\,|\,.\,)_A\|_A}$.

Then we call E a *Hilbert A-module* or a *Hilbert C^*-module over the C^*-algebra A*.

Now let E, F be Hilbert A-modules. A mapping $\Phi\colon E \to F$ is called *adjointable* if there is a mapping $\Phi^*\colon F \to E$ such that

$$(\,\Phi(\xi)\,|\,\eta\,)_A = (\,\xi\,|\,\Phi^*(\eta)\,)_A$$

for all $\xi \in E$, $\eta \in F$. Note that an adjointable mapping $\Phi\colon E \to F$ is always right A-linear and bounded, i.e., $\Phi(\xi \cdot a) = \Phi(\xi) \cdot a$ for all $\xi \in E$, $a \in A$ and there is an $M > 0$ with $\|\Phi(\xi)\| \leq M\|\xi\|$ for all $\xi \in E$. The set of all adjointable mappings from E to F is denoted by $\mathcal{L}(E, F)$. Instead of $\mathcal{L}(E, E)$ we just write $\mathcal{L}(E)$. It is easy to see that $\mathcal{L}(E)$ is a C^*-algebra, see [15, page 8]. $\mathcal{L}(E)$ contains the ideal of *generalized compact operators* $\mathcal{K}(E)$, which is the closed linear span of all operators $\theta_{\xi,\eta}$, $\xi, \eta \in E$, where $\theta_{\xi,\eta}(\zeta) = \xi \cdot (\,\eta\,|\,\zeta\,)_A$.

Definition 1.7 (Hilbert bimodules)
Let A be a separable C^*-algebra. A *Hilbert bimodule over A* is a Hilbert A-module X together with a *-homomorphism $\lambda\colon A \to \mathcal{L}(X)$. We use λ to define a left action of A on X via $a \cdot \xi := \lambda(a)(\xi)$ for $a \in A$ and $\xi \in X$. Since $\lambda(a)$ is adjointable, the following identity holds for the left multiplication of A on X:

$$(\,a \cdot \xi\,|\,\eta\,)_A = (\,\xi\,|\,a^* \cdot \eta\,)_A,$$

for all $\xi, \eta \in X$ and $a \in A$.

We notice that every Hilbert space $\mathcal{H}$ can be viewed as a Hilbert bimodule over the C^*-algebra $\mathbb{C}$.

Example 1.8 (Directed graphs)
Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, s, r)$ is a directed graph with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and mappings $s, r : \mathcal{E} \to \mathcal{V}$, which describe the *s*ource and *r*ange of edges. Let $E_{\mathcal{G}}$ be the vector space of all complex valued functions ξ on $\mathcal{E}$ for which the function

$$v \in \mathcal{V} \mapsto \sum_{\substack{e \in \mathcal{E} \\ r(e)=v}} |\xi(e)|^2$$

belongs to the C^*-algebra $A_{\mathcal{G}} := c_0(\mathcal{V})$. For $\xi, \eta \in E_{\mathcal{G}}, a \in A_{\mathcal{G}}, e \in \mathcal{E}, v \in \mathcal{V}$, we define the operations

$$\begin{aligned} (\xi \cdot a)(e) &:= \xi(e)a(r(e)), \\ (a \cdot \xi)(e) &:= a(s(e))\xi(e), \\ (\, \xi \,|\, \eta \,)_{A_{\mathcal{G}}}(v) &:= \sum_{\substack{e \in \mathcal{E} \\ r(e)=v}} \overline{\xi(e)}\eta(e), \end{aligned}$$

which make $E_{\mathcal{G}}$ a Hilbert bimodule over $A_{\mathcal{G}}$.

Example 1.9 (Endomorphisms)
Let A be a C^*-algebra and α an endomorphism on A. We set ${}_{\alpha}A := A$ and define the inner product and the left and right module multiplications by

$$\begin{aligned} (\, \xi \,|\, \eta \,)_A &:= \xi^*\eta, \\ a \cdot \xi &:= \alpha(a)\xi, \\ \xi \cdot a &:= \xi a, \end{aligned}$$

for $\xi, \eta \in {}_{\alpha}A, a \in A$. It is easy to see, that ${}_{\alpha}A$ is a Hilbert bimodule over A. Note that the norm on ${}_{\alpha}A$ is just the norm on A, since $\|\xi\|^2_{{}_{\alpha}A} = \|(\, \xi \,|\, \xi \,)_A\|_A = \|\xi^*\xi\|_A = \|\xi\|^2_A$. Considering the case when $\alpha = \mathrm{id}$, we see that a C^*-algebra A can always be viewed as a Hilbert bimodule ${}_{\mathrm{id}}A$ over A.

Now let X be a Hilbert bimodule over A. A *Toeplitz representation* of X in a C^*-algebra B is a pair (ψ, π) consisting of a linear mapping $\psi \colon X \to B$ and a *-homomorphism $\pi \colon A \to B$ such that

$$\psi(a \cdot \xi \cdot b) = \pi(a)\psi(\xi)\pi(b) \qquad \text{and} \qquad \psi(\xi)^*\psi(\eta) = \pi((\, \xi \,|\, \eta \,)_A)$$

for $\xi, \eta \in X$ and $a, b \in A$. Given such a representation, there is a *-homomorphism $\pi^{(1)} \colon \mathcal{K}(X) \to B$ which satisfies $\pi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$ for all $\xi, \eta \in X$. We say that the Toeplitz representation (ψ, π) is *Cuntz-Pimsner covariant* if $\pi^{(1)}(\lambda(a)) = \pi(a)$ for all $a \in \lambda^{-1}(\mathcal{K}(X))$, where $\lambda \colon A \to \mathcal{L}(X)$ is the *-homomorphism that defines the left action of A on X. The *Toeplitz algebra* of X is the C^*-algebra $\mathcal{T}_X$ which is universal for Toeplitz representations of X, and the *Cuntz-Pimsner algebra* of X is the C^*-algebra $\mathcal{O}_X$ which is universal for Toeplitz representations of X that are Cuntz-Pimsner covariant. Pimsner showed in [20] that every crossed product by $\mathbb{Z}$ and every Cuntz-Krieger algebra can be realized as $\mathcal{O}_X$ for a suitable Hilbert bimodule X.

Fowler extends this concept to the case when X is a product system of Hilbert bimodules. But in order to define the notion of a discrete product system of Hilbert bimodules we first have to explain how the internal tensor product of two Hilbert bimodules is constructed and what we mean by saying that two Hilbert bimodules are unitarily equivalent. So let X and Y be two Hilbert bimodules over a C^*-algebra A. A mapping $\Phi\colon X \to Y$ is called a *bimodule mapping* if $\Phi(a \cdot \xi \cdot b) = a \cdot \Phi(\xi) \cdot b$ for all $a, b \in A$ and $\xi \in X$. It is called *adjointable* if it is adjointable as a mapping from X to Y, where X and Y are considered as Hilbert A-modules.

An adjointable bimodule mapping $U\colon X \to Y$ is called *unitary* if $UU^* = \mathrm{id}_Y$ and $U^*U = \mathrm{id}_X$. If there exists a unitary bimodule mapping $U\colon X \to Y$, we say that X and Y are *unitarily equivalent*, $X \approx Y$.

Definition 1.10 (The internal tensor product of Hilbert bimodules)
In order to obtain the *(internal) tensor product* $X \otimes_A Y$ of two Hilbert bimodules X, Y over a C^*-algebra A, we first establish an A-valued inner product on the vector space tensor product $X \otimes Y$ given on simple tensors by

$$(\, \xi_1 \otimes \eta_1 \,|\, \xi_2 \otimes \eta_2 \,)_A := (\, \eta_1 \,|\, (\, \xi_1 \,|\, \xi_2 \,)_A \cdot \eta_2 \,)_A,\ \ \xi_1, \xi_2 \in X, \eta_1, \eta_2 \in Y.$$

Then we separate the kernel of this inner product and complete with respect to the norm induced by $(\,.\,|\,.\,)_A$ to get the (internal) tensor product $X \otimes_A Y$, cf. [29, page 266]. We define the bimodule operations by

$$a \cdot (\xi \otimes \eta) \cdot b := (a \cdot \xi) \otimes (\eta \cdot b), \quad a, b \in A, \xi \in X, \eta \in Y,$$

making $X \otimes_A Y$ a Hilbert bimodule over A. We note that the relation $\xi \cdot a \otimes \eta = \xi \otimes a \cdot \eta$ holds in $X \otimes_A Y$ for $a \in A$, $\xi \in X$ and $\eta \in Y$.

Now we are able to present Fowler's discrete product systems of Hilbert bimodules that he introduced in [10].

Definition 1.11 (Fowler's discrete product systems of Hilbert bimodules)
Let $\mathcal{S}$ be a countable semigroup with identity e and let A be a C^*-algebra. A *discrete product system* over $\mathcal{S}$ is defined to be a disjoint union $X = \bigcup_{s\in\mathcal{S}} X_s$, where each fiber X_s is a Hilbert bimodule over A. Moreover, there is an associative tensoring operation on X such that for each $s, t \in \mathcal{S}$, there is a mapping $(\xi, \eta) \in X_s \times X_t \mapsto \xi\eta \in X_{st}$ that extends to a unitary bimodule mapping from the Hilbert bimodule $X_s \otimes_A X_t$ to the Hilbert bimodule X_{st}. Moreover, Fowler requires that $X_e = {}_{\mathrm{id}}A$, see [10] for more details.

So Fowler generalizes the discrete product systems of Dinh by replacing the Hilbert spaces E_t by Hilbert bimodules X_t. This is a generalization since we have already seen that a Hilbert space $\mathcal{H}$ can always be viewed as a Hilbert bimodule over the complex numbers $\mathbb{C}$. Moreover, he admits arbitrary countable semigroups.

Fowler's work was motivated by discrete semigroup dynamical systems. A *discrete semigroup dynamical system* $(A, \mathcal{S}, \alpha)$ consists of a C^*-algebra A and a discrete semigroup $\mathcal{S}$ acting on A via endomorphisms α, i.e., for all $s \in \mathcal{S}$ there is an endomorphism α_s on A such that $\alpha_s \circ \alpha_t = \alpha_{st}$ for all $s, t \in \mathcal{S}$. Given such a discrete semigroup dynamical system $(A, \mathcal{S}, \alpha)$, Fowler constructs a discrete product system $X(A, \mathcal{S}, \alpha)$ of Hilbert bimodules. The following example shows how this can be done in the case when A is unital.

Example 1.12 (A semigroup acting on a C^*-algebra)
Let A be a unital C^*-algebra and let $(A, \mathcal{S}, \alpha)$ be a discrete semigroup dynamical system. We want to construct the corresponding discrete product system $X(A, \mathcal{S}, \alpha)$ of Hilbert bimodules. So we set $X_s := {}_{\alpha_s}A$, $s \in \mathcal{S}$, see Example 1.9. Then $X(A, \mathcal{S}, \alpha) := \bigcup_{s\in\mathcal{S}} X_s$ is a disjoint union of Hilbert bimodules over A. Since $\alpha_e = \mathrm{id}$, we have that $X_e = {}_{\mathrm{id}}A$. We define a mapping $\Phi_{s,t} : X_t \times X_s \to X_{st}$ by setting

$$\Phi_{s,t}(\xi, \eta) := \alpha_s(\xi)\eta, \qquad \xi \in X_t, \eta \in X_s.$$

We claim that $\Phi_{s,t}$ extends to a unitary bimodule mapping from $X_t \otimes_A X_s$ to X_{st} that we also denote by $\Phi_{s,t}$. So let $\xi \in X_t$, $\eta \in X_s$ and $a \in A$ be arbitrary. Then we have $\Phi_{s,t}(a \cdot (\xi \otimes \eta)) = \Phi_{s,t}((a \cdot \xi) \otimes \eta) = \Phi_{s,t}(\alpha_t(a)\xi \otimes \eta) = \alpha_s(\alpha_t(a)\xi)\eta = \alpha_{st}(a)\alpha_s(\xi)\eta = a \cdot \alpha_s(\xi)\eta = a \cdot \Phi_{s,t}(\xi \otimes \eta)$ and $\Phi_{s,t}((\xi \otimes \eta) \cdot a) = \Phi_{s,t}(\xi \otimes (\eta \cdot a)) = \Phi_{s,t}(\xi \otimes (\eta a)) = \alpha_s(\xi)(\eta a) = (\alpha_s(\xi)\eta)a = (\alpha_s(\xi)\eta) \cdot a = \Phi_{s,t}(\xi \otimes \eta) \cdot a$, which shows that $\Phi_{s,t}$ is an A-A-bimodule mapping. Next we define a mapping $\Psi_{s,t} : X_{st} \to X_t \otimes_A X_s$ by

$$\Psi_{s,t}(\zeta) := 1 \otimes \zeta.$$

Then we compute for $\xi \in X_t$, $\eta \in X_s$ and $\zeta \in X_{st}$:

$$\begin{aligned} (\, \Phi_{s,t}(\xi \otimes \eta) \,|\, \zeta \,)_A &= (\, \alpha_s(\xi)\eta \,|\, \zeta \,)_A = \eta^*\alpha_s(\xi^*)\zeta = (\, \eta \,|\, \alpha_s(\xi^*)\zeta \,)_A = (\, \eta \,|\, \xi^* \cdot \zeta \,)_A \\ &= (\, \eta \,|\, (\, \xi \,|\, 1 \,)_A \cdot \zeta \,)_A = (\, \xi \otimes \eta \,|\, 1 \otimes \zeta \,)_A = (\, \xi \otimes \eta \,|\, \Psi_{s,t}(\zeta) \,)_A \end{aligned}$$

and so we see that $\Phi_{s,t}$ is adjointable with $\Phi^*_{s,t} = \Psi_{s,t}$. Finally, we get that $\Phi_{s,t}(\Phi^*_{s,t}(\zeta)) = \Phi_{s,t}(1 \otimes \zeta) = \alpha_s(1)\zeta = \zeta$ and $\Phi^*_{s,t}(\Phi_{s,t}(\xi \otimes \eta)) = \Phi^*_{s,t}(\alpha_s(\xi)\eta) = 1 \otimes \alpha_s(\xi)\eta = 1 \otimes \xi \cdot \eta = 1 \cdot \xi \otimes \eta = \xi \otimes \eta$. Hence, we conclude that $\Phi_{s,t}\Phi^*_{s,t} = \mathrm{id}_{X_{st}}$ and $\Phi^*_{s,t}\Phi_{s,t} = \mathrm{id}_{X_t \otimes X_s}$, which shows that $\Phi_{s,t}$ is unitary. So we have seen that $\Phi_{s,t}$ extends to a unitary bimodule mapping and hence, $X(A, \mathcal{S}, \alpha)$ is a discrete product system of Hilbert bimodules over the opposite semigroup $\mathcal{S}^0$.

Now suppose that $X = \bigcup_{s\in\mathcal{S}} X_s$ is a discrete product system of Hilbert bimodules over a C^*-algebra A and let ψ be a mapping from X to a C^*-algebra B. Let ψ_s denote the restriction of ψ to X_s, $s \in \mathcal{S}$. We call ψ a *Toeplitz representation* of X if

1. for each $s \in \mathcal{S}$, (ψ_s, ψ_e) is a Toeplitz representation of X_s, and
2. $\psi(\xi\eta) = \psi(\xi)\psi(\eta)$ for all $\xi, \eta \in X$.

We say that ψ is *Cuntz-Pimsner covariant*, if in addition each (ψ_s, ψ_e) is Cuntz-Pimsner covariant.

Then Fowler constructs the *Fock representation* $l\colon X \to \mathcal{L}(\mathcal{F}(X))$ on the Hilbert A-module $\mathcal{F}(X) = \bigoplus_{s\in\mathcal{S}} X_s$. This construction is similar to the construction of Dinh's Fock representation on the Fock space $\mathcal{F}$ and it easy to see that it is a Toeplitz representation.

Moreover, he shows that there is a C^*-algebra $\mathcal{T}_X$, called the *Toeplitz algebra* of the product system X, which is universal for Toeplitz representations of X. To be more precise, he shows that given a product system X over a semigroup $\mathcal{S}$, there is a C^*-algebra $\mathcal{T}_X$ and a Toeplitz representation $i_X\colon X \to \mathcal{T}_X$, such that for every Toeplitz representation ψ of X, there is a unique *-homomorphism ψ_* of $\mathcal{T}_X$ such that $\psi_* \circ i_X = \psi$.

Similarly, Fowler shows that there is a C^*-algebra $\mathcal{O}_X$, called the *Cuntz-Pimsner algebra* of the product system X, which is universal for Cuntz-Pimsner covariant Toeplitz representations of X.

Now let $(A, \mathcal{S}, \alpha)$ be a discrete semigroup dynamical system. A *(partial) isometric representation* of $\mathcal{S}$ on a Hilbert space $\mathcal{H}$ is a mapping $V\colon \mathcal{S} \to \mathcal{L}(\mathcal{H})$ such that V_s is a (partial) isometry for all $s \in \mathcal{S}$ and $V_sV_t = V_{st}$ for all $s, t \in \mathcal{S}$. A *covariant representation* of $(A, \mathcal{S}, \alpha)$ on a Hilbert space $\mathcal{H}$ is a pair (π, V) consisting of a nondegenerate representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$ and an isometric representation of $\mathcal{S}$ on $\mathcal{H}$ such that

$$\pi(\alpha_s(a)) = V_s\pi(a)V_s^* \qquad \text{for all } s \in \mathcal{S} \text{ and } a \in A. \tag{1.1}$$

A crossed product for $(A, \mathcal{S}, \alpha)$ is a triple $(B, i_A, i_{\mathcal{S}})$ consisting of a C^*-algebra B, a nondegenerate *-homomorphism $i_A\colon A \to B$, and a mapping $i_{\mathcal{S}}\colon \mathcal{S} \to M(B)$ such that

1. if σ is a nondegenerate representation of B, then $(\sigma \circ i_A, \overline{\sigma} \circ i_{\mathcal{S}})$ is a covariant representation of $(A, \mathcal{S}, \alpha)$ and
2. for every covariant representation (π, V) of $(A, \mathcal{S}, \alpha)$, there is a unique representation $\pi \times V$ of B such that $(\pi \times V) \circ i_A = \pi$ and $\overline{\pi \times V} \circ i_{\mathcal{S}} = V$.

It can be shown that a crossed product exists and that it is unique up to canonical isomorphism. We denote the crossed product by $A \rtimes_\alpha \mathcal{S}$.

Now Fowler shows that given a discrete semigroup dynamical system $(A, \mathcal{S}, \alpha)$ and the corresponding discrete product system $X = X(A, \mathcal{S}, \alpha)$, the Cuntz-Pimsner algebra $\mathcal{O}_X$ is a crossed product for $(A, \mathcal{S}, \alpha)$. Moreover he shows that the Toeplitz algebra $\mathcal{T}_X$ also has a crossed product structure: it is universal for pairs (π, V) satisfying Equation (1.1) in which π is a nondegenerate representation of A and V is a partial isometric representation such that

$$V_s^*V_s\pi(a) = \pi(a)V_s^*V_s \text{ for all } s \in \mathcal{S} \text{ and } a \in A.$$

Chapter 2

A bicategorial view on product systems

In the present chapter we want to provide new insights into product systems by analyzing them from a bicategorial point of view. We start with the category C*ALG whose objects are C^*-algebras and whose arrows are *-homomorphisms. Then we replace the *-homomorphisms by C^*-arrows (based on Hilbert bimodules, see below) and we examine if this new structure C*ARR still is a category. We will see that this is not the case but that C*ARR is an example of what is known as a bicategory. We will recall the concept of a bicategory and we are going to provide several examples.

Following our overview of bicategories, we will introduce the "functors" between bicategories, which are called morphisms. They allow us to give a very short and elegant definition of the notion of a product system. Therefore, the product systems that originated from semigroups of *-endomorphisms in Arveson's paper [1] can now be described in a very natural way using the concept of morphisms between bicategories.

2.1 The bicategory of C^*-arrows

Before we start to deal with bicategories in this section, we first want to recall the notion of a category to show the formal similarity between categories and bicategories.

Definition 2.1 (Categories)
A *category* $\mathcal{C}$ consists of the following data:

(A) a class $\mathrm{Ob}(\mathcal{C})$ of *objects* A, B, $C, \ldots$,

(B) for every two objects A and B a set $\mathcal{C}(A, B)$ of *arrows* from A to B and

(C) for every triple (A, B, C) of objects a mapping called *composition*

$$\begin{array}{rcl} \circ_{A,B,C} \colon \mathcal{C}(B,C) \times \mathcal{C}(A,B) & \to & \mathcal{C}(A,C) \\ (g,f) & \longmapsto & g \circ f = gf \end{array}$$

such that

- the composition is associative and
- for each object A there is an arrow $I_A \in \mathcal{C}(A,A)$, called the *identity arrow* on A satisfying $fI_A = f$ and $I_B f = f$ for every $f \in \mathcal{C}(A,B)$.

We write $f \colon A \to B$ to indicate that f is an arrow from the object A to the object B. By $\mathrm{Arr}(\mathcal{C})$ we denote the collection of all arrows between objects of $\mathcal{C}$ whereas the set $\{(f,g) : \ f \colon A \to B,\ g \colon B \to C,\ A,B,C \in \mathrm{Ob}(\mathcal{C})\}$ of all composable pairs of arrows will be denoted by $\mathrm{Arr}(\mathcal{C}) \circ \mathrm{Arr}(\mathcal{C})$. A category $\mathcal{C}$ is called *small* if $\mathrm{Ob}(\mathcal{C})$ is a set.

An arrow $f \in \mathcal{C}(A,B)$ is called *invertible*, if there is an arrow $\tilde{f} \in \mathcal{C}(B,A)$ such that $\tilde{f}f = I_A$ and $f\tilde{f} = I_B$. If such an arrow $\tilde{f}$ exists, it is unique and we write $\tilde{f} = f^{-1}$. In this case the objects A and B are called isomorphic and we write $A \cong B$.

An arrow $m \in \mathcal{C}(A,B)$ is called *monic* if for arbitrary arrows $f_1, f_2 \in \mathcal{C}(C,A)$ the equation $mf_1 = mf_2$ holds only if $f_1 = f_2$. In other words, m is left-cancellative. An arrow $e \in \mathcal{C}(A,B)$ is called *epi* if for arbitrary arrows $g_1, g_2 \in \mathcal{C}(B,C)$ the equation $g_1 e = g_2 e$ holds only if $g_1 = g_2$, i.e., e is right-cancellative.

Example 2.2 (Semigroups)
Let $\mathcal{S}$ be a small category with only one object A. Such a category is also called a *monoid*. Then $\mathrm{Arr}(\mathcal{S}) = \mathrm{Arr}(\mathcal{S}) \circ \mathrm{Arr}(\mathcal{S})$, i.e., all arrows are composable. The composition $\circ$ is associative and $fI_A = I_A f = f$ for all $f \in \mathrm{Arr}(\mathcal{S})$. Hence, $(\mathrm{Arr}(\mathcal{S}), \circ)$ is a semigroup. The unit element of $(\mathrm{Arr}(\mathcal{S}), \circ)$ is I_A.

On the other hand, if $\mathcal{S}$ is a semigroup with unit element e, we can view each element $s \in \mathcal{S}$ as an arrow $f_s \colon A \to A$ of a monoid $\mathcal{S}'$ with object A. The composition in $\mathcal{S}'$ is defined by $f_s \circ f_t := f_{st}$. It is clear that $\mathcal{S}'$ is a small category. The identity arrow of $\mathcal{S}'$ is f_e.

Example 2.3 (Partial order)
Let $\mathcal{O}$ be a small category with the property that $|\mathcal{O}(A,B) \cup \mathcal{O}(B,A)| \leq 1$ for all objects $A, B \in \mathrm{Ob}(\mathcal{O})$, i.e., given two objects A, B of $\mathcal{O}$, there is at most one arrow between them. Then we can introduce a partial order on the objects of $\mathcal{O}$ by defining that $A \leq B$ if and only if $|\mathcal{O}(A,B)| = 1$, i.e., if and only if there is an arrow going from A to B.

Since for every object A, there is the identity arrow $I_A \in \mathcal{O}(A,A)$, we have that $A \leq A$ for all objects A. If $A \leq B$ and $B \leq C$, there is an arrow $f \colon A \to B$ and an arrow

$g\colon B \to C$. Hence, there is an arrow $gf\colon A \to C$ and thus $A \leq C$. Finally, if $A \leq B$ and $B \leq A$ we have an arrow $f\colon A \to B$ and an arrow $g\colon B \to A$. If we assume that $A \neq B$, we have that $|\mathcal{O}(A,B) \cup \mathcal{O}(B,A)| \geq 2$, which contradicts the basic property of $\mathcal{O}$. Hence, we get $A = B$ and so we have shown that the arrows of $\mathcal{O}$ define a partial order on the set of objects of $\mathcal{O}$.

On the other hand, given a partially ordered set M, we can always define a corresponding small category $\mathcal{O}_M$ in the following way. Let $\mathrm{Ob}(\mathcal{O}_M) := M$ and let there be an arrow $f\colon A \to B$, $A, B \in \mathrm{Ob}(\mathcal{O}_M)$, if and only if $A \leq B$ as elements of M. Now if $f\colon A \to B$ and $g\colon B \to C$ are two arrows of $\mathcal{O}_M$, we know that $A \leq B$ and $B \leq C$. Hence, $A \leq C$ since M is partially ordered and so there is exactly one arrow $h\colon A \to C$ and we set $g \circ f := h$. This defines a composition on $\mathcal{O}_M$ and it is easy to see that $\mathcal{O}_M$ is a small category with $|\mathcal{O}_M(A,B) \cup \mathcal{O}_M(B,A)| \leq 1$ for all $A, B \in \mathrm{Ob}(\mathcal{O}_M)$.

Examples 2.2 and 2.3 show that small categories are a generalization of semigroups on the one hand as well as partially ordered sets on the other. These examples are complementary in a sense, because in Example 2.2 we have only one object but arbitrarily many arrows over this object, whereas in Example 2.3, we have arbitrarily many objects but at most one arrow between two of these objects. So these are the two extreme cases that we always want to keep in mind when we study product systems over small categories. Here are two examples for categories that are not small:

Example 2.4 (Relations)
Let X and Y be sets. A *relation* $\mathcal{R}$ from X to Y is a subset of $X \times Y$. Now let $\mathcal{R}\colon X \to Y$ and $\mathcal{S}\colon Y \to Z$ be relations. We define the composition of $\mathcal{R}$ and $\mathcal{S}$ to be

$$\mathcal{S}\mathcal{R} := \{(x,z) \in X \times Z : \exists\, y \in Y \text{ such that } (x,y) \in \mathcal{R} \wedge (y,z) \in \mathcal{S}\}.$$

This composition is clearly associative and given a set X, the relation $\mathcal{I}_X\colon X \to X$, $\mathcal{I}_X = \{(x,x) : x \in X\}$ satisfies $\mathcal{R}\mathcal{I}_X = \mathcal{R}$ and $\mathcal{I}_X\mathcal{S} = \mathcal{S}$ for relations $\mathcal{R}\colon X \to Y$ and $\mathcal{S}\colon W \to X$. So if we take sets as objects and relations as arrows between these objects, we get a category.

Example 2.5 (The category C*ALG)
Let C*ALG be the category with objects C^*-algebras and arrows *-homomorphisms. It is clear that this is a category since the composition of *-homomorphisms is associative and for every C^*-algebra A we have id_A as identity arrow.

Definition 2.6 (C^*-arrows)
Let A and B be C^*-algebras. A *C^*-arrow from A to B* is a Hilbert B-module E together with a *-homomorphism $\lambda\colon A \to \mathcal{L}(E)$.

We will write $E\colon A \to B$ to indicate that E is a C^*-arrow from A to B and we will consider E as an A-B-bimodule and hence, write $a \cdot \xi$ instead of $\lambda(a)(\xi)$, $a \in A$, $\xi \in E$, except when we discuss properties of λ or $\lambda(a)$. We will say that E is a C^*-arrow *over* A if E is a C^*-arrow from A to A. Hence, a C^*-arrow over a C^*-algebra A is nothing but a Hilbert bimodule over A. We introduce this new name to emphasize that C^*-arrows will be the arrows of the bicategory C*ARR that we will present in a moment.

The following example shows that C^*-arrows are a generalization of *-homomorphisms between C^*-algebras.

Example 2.7 (*-homomorphisms as C^*-arrows)
Let A and B be C^*-algebras and suppose $\lambda\colon A \to B$ is a *-homomorphism. We set ${}_\lambda B := \overline{\lambda(A)B} \subseteq B$ and define module operations and an inner product on ${}_\lambda B$ by

$$\begin{aligned} a \cdot \xi &:= \lambda(a)\xi, \\ \xi \cdot b &:= \xi b, \\ (\,\xi \,|\, \eta\,)_B &:= \xi^*\eta, \end{aligned}$$

for all $\xi, \eta \in {}_\lambda B$, $a \in A$ and $b \in B$. It is easy to see, that ${}_\lambda B$ becomes a C^*-arrow from A to B.

Now let $E, F\colon A \to B$ be two C^*-arrows. A mapping $\Phi\colon E \to F$ is called an *A-B-bimodule mapping* if $\Phi(a \cdot \xi \cdot b) = a \cdot \Phi(\xi) \cdot b$ for all $a \in A$, $b \in B$ and $\xi \in E$. The notions of *adjointable* and *unitary* A-B-bimodule mappings are defined similar to those of adjointable and unitary bimodule mappings in section 1.3. We also say that E and F are *unitarily equivalent*, $E \approx F$, if there exists a unitary A-B-bimodule mapping $U\colon E \to F$.

An adjointable A-B-bimodule mapping $V\colon E \to F$ is called *isometric* if $V^*V = \mathrm{id}_E$. We note that this yields $(\,V\xi \,|\, V\eta\,)_B = (\,\xi \,|\, V^*V\eta\,)_B = (\,\xi \,|\, \eta\,)_B$ for all $\xi, \eta \in E$.

The *(internal) tensor product* $E \otimes_B F$ of two C^*-arrows $E\colon A \to B$ and $F\colon B \to C$, is also defined similar to the case in section 1.3, when E and F were Hilbert bimodules over a C^*-algebra A. Here, $E \otimes_B F$ turns out to be a C^*-arrow from A to C.

Now we use this internal tensor product to introduce the *composition* $F \circ E$ or just FE of two C^*-arrows $E\colon A \to B$ and $F\colon B \to C$ by setting

$$FE := E \otimes_B F.$$

This composition is not associative though, since for C^*-arrows $E\colon A \to B$, $F\colon B \to C$ and $G\colon C \to D$, the C^*-arrows $(GF)E$ and $G(FE)$ are not identical, but only unitarily equivalent, i.e., there is a unitary A-D-bimodule mapping $\alpha_{G,F,E}\colon (GF)E \to G(FE)$ given on simple tensors by $\xi \otimes (\eta \otimes \zeta) \mapsto (\xi \otimes \eta) \otimes \zeta$, $\xi \in E$, $\eta \in F$, $\zeta \in G$. Thus we have $(GF)E \approx G(FE)$ instead of equality.

Given a C^*-algebra B, Example 1.9 tells us that ${}_{\mathrm{id}}B$ is a C^*-arrow from B to B. We set $I_B := {}_{\mathrm{id}}B$. Then, given C^*-arrows $E\colon A \to B$ and $F\colon B \to C$, there exists a unitary A-B-bimodule mapping $\lambda_E\colon I_B E \to E$ as well as a unitary B-C-bimodule mapping $\rho_F\colon F I_B \to F$ given on simple tensors by

$$\lambda_E(\eta \otimes \xi) := \eta \cdot \xi \qquad \text{and} \qquad \rho_F(\xi \otimes \zeta) := \xi \cdot \zeta,$$

where $\xi \in I_B$, $\eta \in E$ and $\zeta \in F$. Thus we have $FI_B \approx F$ and $I_B E \approx E$.

So if we take C^*-algebras as objects and C^*-arrows as arrows we do not get a category anymore, because associativity only holds up to unitary equivalence and also the identity C^*-arrows I_B are only identity arrows up to unitary equivalence. We notice however, that for fixed C^*-algebras A and B, we get a category whose objects are C^*-arrows $E\colon A \to B$ and whose arrows are adjointable A-B-bimodule mappings between the C^*-arrows. What we get when we take C^*-algebras as objects, C^*-arrows as arrows and adjointable bimodule mappings as arrows between the arrows is a so called bicategory, whose definition we want to recall.

Definition 2.8 (Bicategories)
A *bicategory* $\mathcal{B}$ consists of the following data:

(A) A class $\mathrm{Ob}(\mathcal{B})$ of *objects* A, B, $C, \ldots$

(B) For every 2 objects A and B a category $\mathcal{B}(A, B)$ with objects $f, g, h, \ldots$ and arrows $\alpha, \beta, \gamma, \ldots$. The objects of $\mathcal{B}(A, B)$ are called *arrows* from A to B and the arrows of $\mathcal{B}(A, B)$ are called *2-cells* between the arrows from A to B.

(C) For every triple (A, B, C) of objects a functor called *composition*

$$\circ_{A,B,C}\colon \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$$

(where we denote the composition of two arrows f, g by $g \circ f = gf$ and the composition of two 2-cells α, β by $\beta * \alpha$) such that

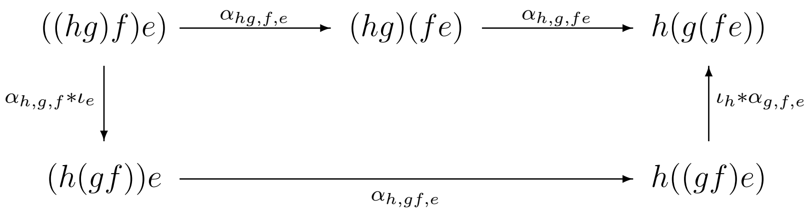
- for every triple f, g, h of arrows $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ there is an invertible *associator 2-cell*

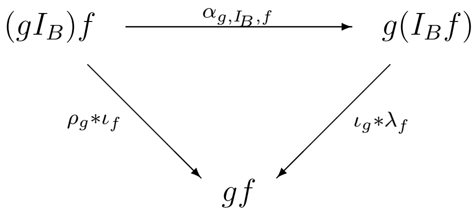
$$\alpha_{h,g,f}\colon (hg)f \Rightarrow h(gf),$$

- for every object A of $\mathcal{B}$ there is an arrow $I_A\colon A \to A$ and for every arrow $f\colon A \to B$ there are invertible 2-cells

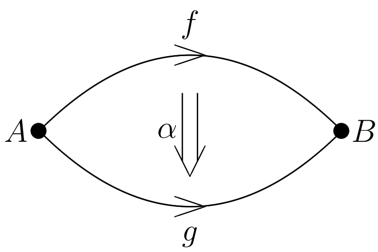
$$\rho_f\colon fI_A \Rightarrow f, \qquad \lambda_f\colon I_B f \Rightarrow f,$$

and such that the following diagrams commute:

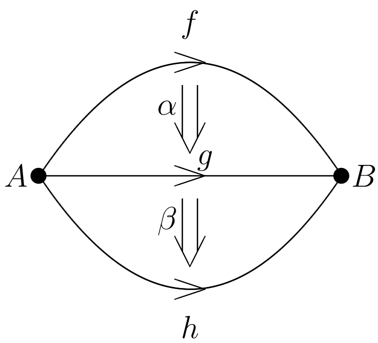




Before we provide examples of bicategories, we first want to give a more detailed description of the notion of a bicategory. Like a category, a bicategory consists of objects and arrows but contrary to a category, there are also arrows between the arrows, called 2-cells.

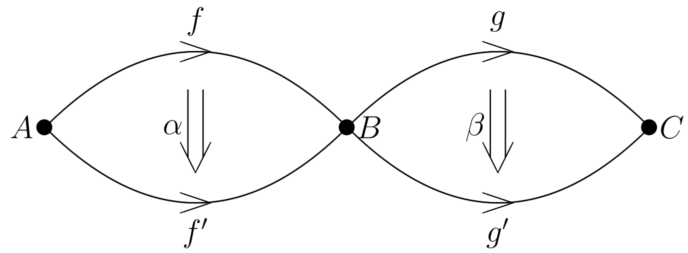


As in a category, arrows can be composed. For 2-cells however, there are two kinds of compositions. First, given arrows $f, g, h\colon A \to B$ and 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, we can compose α and β to get a 2-cell $\beta\alpha\colon f \Rightarrow h$, since $\mathcal{B}(A, B)$ is a category. We call this composition the vertical composition.



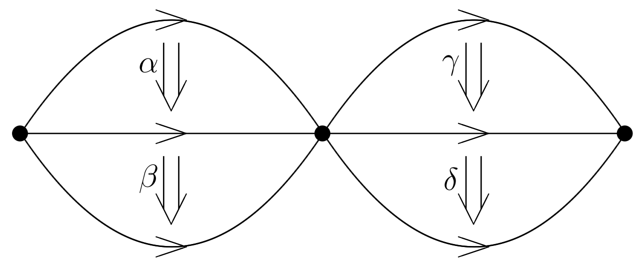
It is associative and for every arrow f there is an identity 2-cell $\iota_f\colon f \Rightarrow f$, since $\mathcal{B}(A,B)$ is a category.

Secondly, given 2-cells $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ as below, we can horizontally compose them to obtain a 2-cell $\beta * \alpha\colon gf \Rightarrow g'f'$:



Since the composition $\circ_{A,B,C}$ is a functor, the following *interchange identity* relating vertical and horizontal composition holds in the situation below:

$$(\delta\gamma) * (\beta\alpha) = (\delta * \beta)(\gamma * \alpha).$$



Now for arbitrary objects A, B, C, D of $\mathcal{B}$ and arbitrary arrows $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ between these objects, we can form the arrows $(hg)f$ and $h(gf)$ from A to D. If $\mathcal{B}$ was a category, then $(hg)f$ and $h(gf)$ would be equal. In a bicategory they are not equal but there is an invertible *associator 2-cell*

$$\alpha_{h,g,f}\colon (hg)f \Rightarrow h(gf).$$

Furthermore, for any object A of $\mathcal{B}$ there is an arrow $I_A\colon A \to A$ and for any arrow $f\colon A \to B$ there are invertible 2-cells

$$\rho_f\colon fI_A \Rightarrow f, \qquad\qquad \lambda_f\colon I_B f \Rightarrow f.$$

Example 2.9 (The bicategory C*ARR)
Now we take a look at our motivating example of C^*-arrows between C^*-algebras again, which was already introduced in a similar way by Landsman in [16]. We claim that we are dealing with a bicategory that we denote by C*ARR. The objects of C*ARR are

C^*-algebras, the arrows are C^*-arrows and the 2-cells are adjointable, isometric bimodule mappings. The invertible 2-cells then turn out to be unitary bimodule mappings.

We have already seen that given C^*-arrows $E\colon A \to B$, $F\colon B \to C$ and $G\colon C \to D$, there is an invertible 2-cell (a unitary A-D-bimodule mapping) $\alpha_{G,F,E} : (GF)E \Rightarrow G(FE)$. We have also seen that given C^*-arrows $E\colon A \to B$ and $F\colon B \to C$, there exists an invertible 2-cell (a unitary A-B-bimodule mapping) $\lambda_E\colon I_B E \Rightarrow E$ as well as an invertible 2-cell (a unitary B-C-bimodule mapping) $\rho_F\colon F I_B \Rightarrow F$.

Now if $E, F, G\colon A \to B$ are C^*-arrows and $V\colon E \Rightarrow F$ and $W\colon F \Rightarrow G$ are 2-cells, we let $WV\colon E \to G$ be the usual composition of A-B-bimodule mappings. Then it is easy to see that WV is an adjointable, isometric bimodule mapping and hence, a 2-cell $WV\colon E \Rightarrow G$ and it is clear that this vertical composition is associative. The identity 2-cell $\iota_E\colon E \Rightarrow E$ is the identity mapping on E.

If $E, E'\colon A \to B$ and $F, F'\colon B \to C$ are C^*-arrows and $V\colon E \Rightarrow E'$ and $W\colon F \Rightarrow F'$ are 2-cells, we define $W * V\colon FE \Rightarrow F'E'$ by

$$(W * V)(\xi \otimes \eta) := V(\xi) \otimes W(\eta)$$

for $\xi \in E$ and $\eta \in F$. It is easy to see that $W * V$ is an isometric A-C-bimodule mapping with adjoint $W^* * V^*$ and so it is clear that $W * V$ is a 2-cell. Moreover, it is rather obvious that C*ARR satisfies all of the remaining conditions of a bicategory.

Now we want to take a closer look at what it means that two C^*-algebras A and B are isomorphic in C*ARR. Therefore, we recall the notion of strong Morita equivalence from [24]. Let X be a Hilbert B-module and an A-B-bimodule equipped with an A-valued inner product ${}_A(\,\cdot\,|\,\cdot\,)$, linear in the first variable, such that

$${}_A(\,\xi\,|\,\eta\,)\cdot\zeta = \xi\cdot(\,\eta\,|\,\zeta\,)_B \qquad \text{for all} \quad \xi, \eta, \zeta \in X$$

and such that ${}_A(\,X\,|\,X\,)$ and $(\,X\,|\,X\,)_B$ are dense in A and B, respectively. Then we call X an *A-B-equivalence bimodule*. A and B are called *strongly Morita equivalent* if there exists an A-B-equivalence bimodule.

If E is a C^*-arrow from A to B it is easy to see that ${}_{\mathcal{K}(E)}(\,\xi\,|\,\eta\,) := \theta_{\xi,\eta}$, $\xi, \eta \in E$, defines a $\mathcal{K}(E)$-valued inner product which is linear in the first variable and satisfies ${}_{\mathcal{K}(E)}(\,\xi\,|\,\eta\,)\cdot\zeta = \theta_{\xi,\eta}(\zeta) = \xi\cdot(\,\eta\,|\,\zeta\,)_B$ for all $\xi, \eta, \zeta \in E$. Now if $\mathcal{K}(E) \subseteq \lambda(A)$ and λ is faithful on $\lambda^{-1}(\mathcal{K}(E))$, we can define an A-valued inner product on E by setting

$${}_A(\,\xi\,|\,\eta\,) := \lambda^{-1}({}_{\mathcal{K}(E)}(\,\xi\,|\,\eta\,)) \qquad \text{for all} \quad \xi, \eta \in E.$$

Moreover, if λ is an isomorphism from A onto $\mathcal{K}(E)$ and if $(\,E\,|\,E\,)_B$ is dense in B then E is an A-B-equivalence bimodule.

By definition, two C^*-algebras A and B are isomorphic in C*ARR if there is an invertible C^*-arrow $E\colon A \to B$. We notice that this means that there is also a C^*-arrow $F\colon B \to A$

such that $E \otimes_B F = FE \approx A$ and $F \otimes_A E = EF \approx B$. Schweizer shows in [25] that $E\colon A \to B$ is an invertible C^*-arrow if and only if E is an A-B-equivalence bimodule. Hence, two C^*-algebras are isomorphic in C*ARR if and only if they are strongly Morita equivalent in the sense of Rieffel.

Example 2.10 (Diagrams)
Let X and Y be sets. A *diagram* $\mathcal{D} : X \to Y$ is a triple (D, s, r) consisting of a set of edges D and *s*ource and *r*ange maps $s\colon D \to X$, $r\colon D \to Y$. We define the composition of diagrams $\mathcal{D}_1\colon X \to Y$, $\mathcal{D}_2\colon Y \to Z$ to be $\mathcal{D}_2\mathcal{D}_1 := (D_3, s_3, r_3)$ with

$$D_3 := \{(e, e') \in D_1 \times D_2 : r_1(e) = s_2(e')\},$$

$$s_3((e, e')) = s_1(e), \qquad r_3((e, e')) = r_2(e').$$

Now this composition is not associative. To see this, let $\mathcal{D}_1\colon W \to X$, $\mathcal{D}_2\colon X \to Y$ and $\mathcal{D}_3\colon Y \to Z$ be diagrams and let D and D' be the sets of edges of $\mathcal{D}_3(\mathcal{D}_2\mathcal{D}_1)$ and $(\mathcal{D}_3\mathcal{D}_2)\mathcal{D}_1$, respectively. Then $D = \{\big((e_1, e_2), e_3\big) : e_i \in D_i,\ i = 1, 2, 3,\ r_1(e_1) = s_2(e_2),\ r_2(e_2) = s_3(e_3)\}$ whereas $D' = \{\big(e_1, (e_2, e_3)\big) : e_i \in D_i,\ i = 1, 2, 3,\ r_1(e_1) = s_2(e_2),\ r_2(e_2) = s_3(e_3)\}$. Hence, the sets of edges D and D' are not equal, but the mapping

$$\alpha_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3}\colon D \to D',\ \big((e_1, e_2), e_3\big) \mapsto \big(e_1, (e_2, e_3)\big)$$

is obviously invertible and respects the sources and ranges of the edges, i.e.,

$$s(e) = s'(\alpha_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3}(e)) \text{ and } r(e) = r'(\alpha_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3}(e)) \text{ for all } e \in D.$$

Moreover, for any set X there is a diagram $\mathcal{I}_X = (I_X, s_X, r_X) : X \to X$ with set of edges $I_X = X$ and $s_X(x) = r_X(x) = x$ for all $x \in I_X$. Then for any diagram $\mathcal{D}\colon X \to Y$ we get $\mathcal{D}\mathcal{I}_X = (D_\rho, s_\rho, r_\rho)$ with

$$D_\rho = \{(x, e) \in I_X \times D : s(e) = x\}$$

and hence, there is an invertible mapping $\rho_{\mathcal{D}}\colon D_\rho \to D$, $(x, e) \mapsto e$, which respects the sources and ranges of the edges. Analogously, we have $\mathcal{I}_Y\mathcal{D} = (D_\lambda, s_\lambda, r_\lambda)$ with $D_\lambda = \{(e, y) \in D \times I_Y : r(e) = y\}$ and an invertible mapping $\lambda_{\mathcal{D}}\colon D_\lambda \to D$, $(e, y) \mapsto e$ that respects the sources and ranges of the edges. It is easy to check that the mappings $\alpha_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3}$, $\rho_{\mathcal{D}}$ and $\lambda_{\mathcal{D}}$ fulfill all the requirements in Definition 2.8.

So we have a bicategory DIAG with objects sets, arrows diagrams between sets and 2-cells mappings between sets of edges that respect the sources and ranges of the edges.

Example 2.11 (Relations)
In Example 2.4 we have seen that if we take sets as objects and relations as arrows between these objects, we get a category $\mathcal{C}$. We want to add 2-cells to $\mathcal{C}$ to get a bicategory $\mathrm{SET}_{\leq}$.

We let two sets X and Y be fixed and consider the set $\mathcal{C}(X,Y)$ of relations $\mathcal{R},\mathcal{S},\mathcal{T},\dots$ from X to Y. These relations are by definition subsets of $X\times Y$ and so $\mathcal{C}(X,Y)$ is partially ordered by inclusion. Now every partially ordered set gives rise to a category as we know from Example 2.3. We let $\mathrm{SET}_{\leq}(X,Y)$ be this category, i.e., $\mathrm{SET}_{\leq}(X,Y)$ is a category whose objects are relations $\mathcal{R},\mathcal{S},\mathcal{T},\dots$ from X to Y and whose arrows are the arrows that come from the partial order on the set of relations.

Now we let $\mathrm{SET}_{\leq}$ be the bicategory whose objects are sets, whose arrows are relations between sets and whose 2-cells are the arrows that come from the partial order on the set of relations. It is easy to see that $\mathrm{SET}_{\leq}$ really is a bicategory. We notice that the associator 2-cells and also $\rho_{\mathcal{R}}$ and $\lambda_{\mathcal{R}}$ are the identity 2-cells, since the structure that remains if we take $\mathrm{SET}_{\leq}$ and forget about all the 2-cells is the category $\mathcal{C}$. Such a bicategory is also called a *2-category*.

2.2 Product systems over small categories

In this section, after recalling the notion of a functor, we introduce morphisms, which are the “functors” between bicategories. We give various examples for morphisms and finally, we use a morphism to define the notion of a product system over a small category.

Definition 2.12 (Functors)
A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{C}'$ is a map from the class of objects of $\mathcal{C}$ to the class of objects of $\mathcal{C}'$ together with a map from the set $\mathcal{C}(A,B)$ for any objects A, B of $\mathcal{C}$ to the set $\mathcal{C}'(F(A),F(B))$ satisfying the following conditions:

- $F(I_A)=I'_{F(A)}$ for all objects A of $\mathcal{C}$,
- $F(gf)=F(g)F(f)$ for all arrows $f\in\mathcal{C}(A,B)$, $g\in\mathcal{C}(B,C)$.

Since bicategories are a generalization of categories that possess a more complex structure, we expect the corresponding mappings between bicategories to be a generalization of functors that also possess an extra structure. These mappings were introduced by Bénabou in [4], who called them morphisms. They are defined as follows:

Definition 2.13 (Morphisms)
Let $\mathcal{B}$ and $\mathcal{B}'$ be bicategories. A *morphism* (F,Φ) consists of a function F sending objects of $\mathcal{B}$ to objects of $\mathcal{B}'$ and for every pair (A,B) of objects in $\mathcal{B}$ a functor $F_{A,B}$ from the category $\mathcal{B}(A,B)$ to the category $\mathcal{B}'(F(A),F(B))$. Furthermore, given any triple (A,B,C) of objects of $\mathcal{B}$ and arrows $f: A\to B$, $g: B\to C$, there is a 2-cell

$$\Phi_{g,f}: F_{B,C}(g)F_{A,B}(f)\Rightarrow F_{A,C}(gf).$$

Finally, for every object A of $\mathcal{B}$, there is a 2-cell $\Phi_A : I'_{F(A)} \Rightarrow F_{A,A}(I_A)$ such that for every arrow $f : A \to B$, $g : B \to C$ and $h : C \to D$ the following diagrams commute, where we omit the indices and just write $F(f)$ instead of $F_{A,B}(f)$:

$$
\begin{array}{ccccc}
(F(h)F(g))F(f) & \xrightarrow{\Phi_{h,g} * \iota'_{F(f)}} & F(hg)F(f) & \xrightarrow{\Phi_{hg,f}} & F((hg)f) \\
\Big\downarrow{\scriptstyle \alpha'_{F(h),F(g),F(f)}} & & & & \Big\downarrow{\scriptstyle F(\alpha_{h,g,f})} \\
F(h)(F(g)F(f)) & \xrightarrow[\iota'_{F(h)} * \Phi_{g,f}]{} & F(h)F(gf) & \xrightarrow[\Phi_{h,gf}]{} & F(h(gf))
\end{array}
$$

$$
\begin{array}{ccc}
F(f)I'_{F(A)} & \xrightarrow{\rho'_{F(f)}} & F(f) \\
\Big\downarrow{\scriptstyle \iota'_{F(f)} * \Phi_A} & & \Big\uparrow{\scriptstyle F(\rho_f)} \\
F(f)F(I_A) & \xrightarrow[\Phi_{f,I_A}]{} & F(fI_A)
\end{array}
\qquad
\begin{array}{ccc}
I'_{F(B)}F(f) & \xrightarrow{\lambda'_{F(f)}} & F(f) \\
\Big\downarrow{\scriptstyle \Phi_B * \iota'_{F(f)}} & & \Big\uparrow{\scriptstyle F(\lambda_f)} \\
F(I_B)F(f) & \xrightarrow[\Phi_{I_B,f}]{} & F(I_Bf)
\end{array}
$$

If all the 2-cells $\Phi_{g,f}$ and Φ_A are invertible, so that $F_{B,C}(g)F_{A,B}(f) \cong F_{A,C}(gf)$ and $I'_{F(A)} \cong F_{A,A}(I_A)$, then (F, Φ) is called a *homomorphism*. If all $\Phi_{g,f}$ and Φ_A are identities, so that $F_{B,C}(g)F_{A,B}(f) = F_{A,C}(gf)$ and $I'_{F(A)} = F_{A,A}(I_A)$, then (F, Φ) is called a *strict homomorphism*.

Example 2.14 (Unital algebras)
Let $\mathcal{B}$ be the bicategory 1 with only one object A, one arrow I_A and one 2-cell ι_{I_A} and let $\mathcal{B}'$ be the bicategory K-VECT. K-VECT has only one object, a field K. The arrows of K-VECT are K-vector spaces and the 2-cells are K-linear mappings between these vector spaces. The composition of two arrows is the vector space tensor product and the identity arrow I'_K is K itself, viewed as a K-vector space.

Now let (F, Φ) be a morphism from $\mathcal{B}$ to $\mathcal{B}'$, i.e., we have $F(A) = K$ and $F_{A,A}(I_A) = V$ for some K-vector space V. Moreover, there are K-linear mappings $\Phi_{I_A,I_A} : V \otimes V \to V$ and $\Phi_A : K \to V$. We define a multiplication on V by setting $u \cdot v := \Phi_{I_A,I_A}(u \otimes v)$ and we set $1_V := \Phi_A(1_K)$. Then it follows from the axioms for morphisms above that the multiplication is associative and that $(\lambda 1_V) \cdot v = v \cdot (\lambda 1_V) = \lambda v$ for $\lambda \in K$, $v \in V$ and hence, V becomes a unital K-algebra with respect to this multiplication and with 1_V as a unit.

On the other hand, given a unital K-algebra V, we can define a morphism (F, Φ) from 1 to K-VECT by setting $F(A) := K$, $F_{A,A}(I_A) := V$, $\Phi_{I_A,I_A}(u \otimes v) := u \cdot v$ and $\Phi_A(\lambda) := \lambda \cdot 1_V$. Thus there is a 1-1 correspondence between morphisms from 1 to K-VECT on the one hand and unital K-algebras on the other.

Example 2.15 (Relations and diagrams)
In this example we want to provide a morphism (F, Φ) from $\mathcal{B} = \mathrm{SET}_{\leq}$ (Example 2.11) to $\mathcal{B}'$ =DIAG (Example 2.10). Of course, we set $F(X) := X$ for a set X. Now let $\mathcal{R} : X \to Y$ be a relation. Then we set $F_{X,Y}(\mathcal{R}) := \mathcal{D}_{\mathcal{R}} := (D_{\mathcal{R}}, s_{\mathcal{R}}, r_{\mathcal{R}})$ with $D_{\mathcal{R}} = \mathcal{R}$, $s_{\mathcal{R}}((x, y)) = x$ and $r_{\mathcal{R}}((x, y)) = y$.

Suppose $\mathcal{R}, \mathcal{S}$ are relations from X to Y with $\mathcal{R} \subseteq \mathcal{S}$. Then there is a 2-cell $\beta : \mathcal{R} \Rightarrow \mathcal{S}$. We let $F_{X,Y}(\beta)$ be the embedding of $D_{\mathcal{R}}$ into $D_{\mathcal{S}}$ which is surely a 2-cell from $\mathcal{D}_{\mathcal{R}}$ to $\mathcal{D}_{\mathcal{S}}$, since it respects the sources and ranges of the edges. It is easy to check that $F_{X,Y}$ is a functor from $\mathcal{B}(X, Y)$ to $\mathcal{B}'(X, Y)$.

Now let $\mathcal{R} : X \to Y$ and $\mathcal{S} : Y \to Z$ be relations. We are looking for a 2-cell

$$\Phi_{\mathcal{S},\mathcal{R}} : \mathcal{D}_{\mathcal{S}}\mathcal{D}_{\mathcal{R}} \Rightarrow \mathcal{D}_{\mathcal{S}\mathcal{R}}.$$

By the definition of $\mathcal{D}_{\mathcal{S}}$, $\mathcal{D}_{\mathcal{R}}$ and the composition of diagrams it is clear that $\mathcal{D}_{\mathcal{S}}\mathcal{D}_{\mathcal{R}} = (D, s, r)$ with $D = \{\big((x, y), (y, z)\big) : (x, y) \in \mathcal{R}, (y, z) \in \mathcal{S}\}$, $s\big(\big((x, y), (y, z)\big)\big) = x$ and $r\big(\big((x, y), (y, z)\big)\big) = z$. Moreover, we have $\mathcal{D}_{\mathcal{S}\mathcal{R}} = (D_{\mathcal{S}\mathcal{R}}, s_{\mathcal{S}\mathcal{R}}, r_{\mathcal{S}\mathcal{R}})$ with $D_{\mathcal{S}\mathcal{R}} = \mathcal{S}\mathcal{R} = \{(x, z) \in X \times Z : \exists\, y \in Y \text{ such that } (x, y) \in \mathcal{R} \wedge (y, z) \in \mathcal{S}\}$, $s_{\mathcal{S}\mathcal{R}}((x, z)) = x$, $r_{\mathcal{S}\mathcal{R}}((x, z)) = z$. So the only mapping from D to $D_{\mathcal{S}\mathcal{R}}$ that respects the source and range of the edges is defined by

$$\Phi_{\mathcal{S},\mathcal{R}}\big(\big((x, y), (y, z)\big)\big) = (x, z).$$

It is not hard to check that this mapping is suitable to make the first diagram in Definition 2.13 commute.

Finally, given a set X, we are looking for a 2-cell $\Phi_X : \mathcal{I}'_X \Rightarrow F_{X,X}(\mathcal{I}_X)$. But this is easy, we just have to remember that $\mathcal{I}'_X = (I'_X, s'_X, r'_X)$ with $I'_X = X$ and $F_{X,X}(\mathcal{I}_X) = \mathcal{D}_{\mathcal{I}_X} = (D_{\mathcal{I}_X}, s_{\mathcal{I}_X}, r_{\mathcal{I}_X})$ with $D_{\mathcal{I}_X} = \{(x, x) : x \in X\}$ and so there is only one way to define a mapping from I'_X to $D_{\mathcal{I}_X}$ that respects the sources and ranges of the edges and this is

$$\Phi_X(x) = (x, x).$$

Again it is easy to see that the remaining two diagrams of Definition 2.13 commute with Φ_X defined like this and hence, (F, Φ) is a morphism from $\mathrm{SET}_{\leq}$ to DIAG.

Example 2.16 (The embedding of C*ALG in C*ARR)
We have already studied the category C*ALG as well as the bicategory C*ARR. Now we want to provide a morphism that embeds C*ALG in C*ARR. We make C*ALG a bicategory (in fact it is a 2-category) by adding identity 2-cells to every arrow. For a C^*-algebra A, we set $F(A) := A$ and for a *-homomorphism $\lambda\colon A \to B$, we set $F_{A,B}(\lambda) := {}_{\lambda}B$, see Example 2.7. Now, given C^*-algebras A, B, C and *-homomorphisms $\lambda\colon A \to B$ and $\mu\colon B \to C$, we define a 2-cell $\Phi_{\mu,\lambda}\colon {}_{\mu}C_{\lambda}B \Rightarrow {}_{\mu\circ\lambda}C$ by setting

$$\Phi_{\mu,\lambda}(\xi \otimes \eta) := \mu(\xi)\eta$$

for $\xi \in {}_\lambda B = \overline{\lambda(A)B}$, $\eta \in {}_\mu C = \overline{\mu(B)C}$. To see that $\Phi_{\mu,\lambda}(\xi \otimes \eta) \in {}_{\mu\circ\lambda}C$ let $\xi = \lambda(a)b$ with $a \in A$, $b \in B$, and let $\eta \in {}_\mu C$ be arbitrary. Then we have $\Phi_{\mu,\lambda}(\xi \otimes \eta) = \mu(\xi)\eta = \mu(\lambda(a)b)\eta = (\mu \circ \lambda)(a)\mu(b)\eta \in \overline{(\mu \circ \lambda)(A)C} = {}_{\mu\circ\lambda}C$, since $\mu(b)\eta \in C$. The following computation shows that $\Phi_{\mu,\lambda}$ is an A-C-bimodule mapping:

$$\Phi_{\mu,\lambda}(a \cdot (\xi \otimes \eta) \cdot c) = \mu(a \cdot \xi)(\eta \cdot c) = \mu(\lambda(a)\xi)\eta c = (\mu \circ \lambda)(a)(\mu(\xi)\eta)c = a \cdot \Phi_{\mu,\lambda}(\xi \otimes \eta) \cdot c$$

for $a \in A$, $\xi \in {}_\lambda B$, $\eta \in {}_\mu C$ and $c \in C$. It is also easy to see that $\Phi_{\mu,\lambda}$ is isometric. Moreover, we have $\Phi_{\mu,\lambda}({}_\lambda B \otimes {}_\mu C) = \Phi_{\mu,\lambda}(\overline{\lambda(A)B} \otimes \overline{\mu(B)C}) = \overline{\mu(\lambda(A)B)\mu(B)C} = \overline{\mu(\lambda(A))\mu(B)C} = \overline{\mu(\lambda(A))C} = {}_{\mu\circ\lambda}C$, since $\mu(B)C \subseteq C$ yields $\overline{\mu(\lambda(A))\mu(B)C} \subseteq \overline{\mu(\lambda(A))C}$ and $\mu(\lambda(A)) = \overline{\mu(\lambda(A))\mu(\lambda(A))} \subseteq \overline{\mu(\lambda(A))\mu(B)}$ yields $\overline{\mu(\lambda(A))C} \subseteq \overline{\mu(\lambda(A))\mu(B)C}$. This shows that $\Phi_{\mu,\lambda}$ is isometric and surjective and hence unitary. To see that the first diagram in Definition 2.13 commutes, suppose that A, B, C, D are C^*-algebras and that $\lambda\colon A \to B$, $\mu\colon B \to C$, $\nu\colon C \to D$ are *-homomorphisms. Furthermore, let $\xi \in {}_\lambda B$, $\eta \in {}_\mu C$ and $\zeta \in {}_\nu D$ be arbitrary. Note that $F(\alpha_{\nu,\mu,\lambda}) = \mathrm{id}$ since $\alpha_{\nu,\mu,\lambda} = \mathrm{id}$, C*ALG being a category. So we have

$$F(\alpha_{\nu,\mu,\lambda}) \circ \Phi_{\nu\mu,\lambda} \circ (\Phi_{\nu,\mu} * \iota_{\lambda E})(\xi \otimes (\eta \otimes \zeta)) = \Phi_{\nu\mu,\lambda}(\xi \otimes \nu(\eta)\zeta) = (\nu \circ \mu)(\xi)\nu(\eta)\zeta \quad \text{and}$$

$$\begin{aligned} \Phi_{\nu,\mu\lambda} \circ (\iota'_{\nu E} * \Phi_{\mu,\lambda}) \circ \alpha'_{\nu E,\mu E,\lambda E}(\xi \otimes (\eta \otimes \zeta)) &= \Phi_{\nu,\mu\lambda} \circ (\iota'_{\nu E} * \Phi_{\mu,\lambda})((\xi \otimes \eta) \otimes \zeta) \\ &= \Phi_{\nu,\mu\lambda}(\mu(\xi)\eta \otimes \zeta) \\ &= \nu(\mu(\xi)\eta)\zeta = (\nu \circ \mu)(\xi)\nu(\eta)\zeta. \end{aligned}$$

Having provided all of the above examples, we come to the main reason why we introduced morphisms. It gives us the possibility to define the notion of a product system over a small category in a very short and elegant way. By an *index category* J we shall mean a small category with objects $i, j, k, \ldots$ and arrows $r, s, t, \ldots$. Notice that an index category J can always be considered as a bicategory: just add an identity 2-cell to each arrow. The 2-cells $\alpha_{r,s,t}$ and ρ_r, λ_r are the identity 2-cells, of course, since J is a category.

Now it is easy to define the notion of a product system in one sentence:

Definition 2.17 (Product systems over a small category)
A *product system* (F, Φ) over an index category J is a morphism from J to the bicategory C*ARR.

But let us repeat what this means exactly. For every object i in J there is a C^*-algebra $F(i)$ in C*ARR, which we denote by A_i, and for every arrow $r\colon i \to j$ there is a C^*-arrow $F_{i,j}(r)$ from A_i to A_j, which we denote by F_r. We also write F_i for the C^*-arrow $F_{i,i}(I_i)$. Moreover, given any triple (i, j, k) of objects of J and arrows $r\colon i \to j$, $s\colon j \to k$, there is an isometric, adjointable A_i-A_k-bimodule mapping

$$\Phi_{s,r}\colon F_sF_r \Rightarrow F_{sr},$$

such that the following *coherence condition* holds for the family $\Phi_{s,r}$:

$$\Phi_{t,sr} \circ (\Phi_{s,r} \otimes \iota'_{F_t})\big((\xi_r \otimes \xi_s) \otimes \xi_t\big) = \Phi_{ts,r} \circ (\iota'_{F_r} \otimes \Phi_{t,s})\big(\xi_r \otimes (\xi_s \otimes \xi_t)\big), \tag{2.1}$$

where $\xi_r \in F_r$, $\xi_s \in F_s$, $\xi_t \in F_t$ and $r\colon i \to j$, $s\colon j \to k$, $t\colon k \to l$. This coherence condition corresponds to the commutativity of the first diagram in Definition 2.13. Note that $\alpha_{t,s,r} = \iota_{tsr}$, since J is a category and hence, $F_{i,l}(\alpha_{t,s,r}) = \iota'_{F_{i,l}(tsr)}$, since $F_{i,l}$ is a functor. We get a multiplication on the composable parts of $\bigcup_{r\in\mathrm{Arr}(J)} F_r$ by setting $\xi_r \cdot \xi_s := \Phi_{s,r}(\xi_r \otimes \xi_s)$ for $\xi_r \in F_r$ and $\xi_s \in F_s$. We note that Equation (2.1) yields the associativity of this multiplication.

Finally, given any object i of J, there is an isometric, adjointable A_i-A_i-bimodule mapping $\Phi_i : I'_{A_i} \Rightarrow F_i$ such that for any arrow $r : i \to j$ the following identities hold:

$$\Phi_{r,I_i} \circ (\Phi_i \otimes \iota'_{F_r}) = \rho'_{F_r}, \qquad \Phi_{I_j,r} \circ (\iota'_{F_r} \otimes \Phi_j) = \lambda'_{F_r}. \tag{2.2}$$

These identities correspond directly to the commutativity of the second and third diagram in Definition 2.13. Notice that $\rho_r = \lambda_r = \iota_r$, since J is a category and thus $F_{i,j}(\rho_r) = F_{i,j}(\lambda_r) = \iota'_{F_{i,j}(r)}$, since $F_{i,j}$ is a functor.

So there are two major differences between Fowler's discrete product systems and our product systems. The first difference is that Fowler only studies product systems over semigroups whereas we allow product systems over arbitrary small categories. The other difference is that Fowler requires that $E_s \otimes_A E_t$ and E_{st} be unitarily equivalent. We only require that there be an isometric, adjointable bimodule mapping $\Phi_{s,t}\colon F_sF_t \to F_{st}$. In fact, Fowler's definition is a special case of our definition as the following example shows.

Example 2.18 (Fowler's discrete product systems of Hilbert bimodules)
Let $\mathcal{S}$ be a countable semigroup with identity e and let A be a C^*-algebra. By $(\mathcal{S}^0)'$ we denote the monoid with object i that corresponds to the opposite semigroup $\mathcal{S}^0$ (see Example 2.2). We take $(\mathcal{S}^0)'$ as an index category and we let (F, Φ) be a product system over $(\mathcal{S}^0)'$ such that (F, Φ) is a homomorphism with $F(i) = A$ and $\Phi_i = \mathrm{id}$. Then for every $f_s \in \mathrm{Arr}((\mathcal{S}^0)')$ we get a C^*-arrow (and hence a Hilbert bimodule) $X_s := F_{i,i}(f_s)$ over A. Moreover there are unitary bimodule mappings

$$\Phi_{f_t,f_s}\colon X_tX_s = X_s \otimes_A X_t \to X_{st}$$

for every $f_s, f_t \in \mathrm{Arr}((\mathcal{S}^0)')$, since (F, Φ) is a homomorphism. We also have that $X_e = F_{i,i}(f_e) = F_i = I_A = {}_{\mathrm{id}}A$, since $\Phi_i = \mathrm{id}$. Hence, $X = \bigcup_{s\in\mathcal{S}} X_s$ is a discrete product system of Hilbert bimodules over the semigroup $\mathcal{S}$ in the sense of Fowler.

Chapter 3

The reduced Toeplitz and Cuntz-Pimsner algebras

In this chapter we associate two C^*-algebras to every given product system (F, Φ), namely the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F, \Phi)$, and we study various special cases.

First, we introduce the notion of a Toeplitz representation from a product system (F, Φ) to a C^*-algebra and we provide some technical results about homomorphisms between Hilbert C^*-modules that we will need later. Then we introduce the Fock correspondence $\mathcal{F}(F, \Phi)$ of a given product system (F, Φ) and we construct one particular Toeplitz representation, the reduced Toeplitz representation from (F, Φ) to the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$, which is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}(F, \Phi))$. We introduce the reduced crossed product $A \rtimes_E \mathbb{N}$ of a C^*-algebra A by a C^*-arrow E as the reduced Toeplitz algebra $\mathcal{T}_r(F, \Phi)$ for a specific product system (F, Φ) over the natural numbers $\mathbb{N}$. Moreover, we define the reduced Cuntz-Pimsner algebra to be the quotient of $\mathcal{T}_r(F, \Phi)$ modulo the ideal of compact operators in $\mathcal{T}_r(F, \Phi)$.

Finally, we provide various examples, which show that our method of constructing the reduced Toeplitz and Cuntz-Pimsner algebras generalizes many other constructions of C^*-algebras. Our findings show that depending on how we choose our product system (F, Φ), the resulting C^*-algebras $\mathcal{T}_r(F, \Phi)$ and $\mathcal{O}_r(F, \Phi)$, respectively, are isomorphic to the direct sum of a family of C^*-algebras, the direct limit of a direct system of C^*-algebras or the crossed product of a C^*-algebra by a group or a semigroup.

3.1 Toeplitz representations

In [10], Fowler introduced the notion of a Toeplitz representation of a discrete product system X over a semigroup $\mathcal{S}$. The following definition generalizes this concept to our situation.

Definition 3.1 (Toeplitz representations)
Let J be an index category and let (F, Φ) be a product system over J. A *Toeplitz representation* π from (F, Φ) into a C^*-algebra B consists of a family of *-homomorphisms $\{\pi_i\colon A_i \to B, i \in \mathrm{Ob}(J)\}$ and linear mappings $\{\pi_r\colon F_r \to B, r \in \mathrm{Arr}(J)\}$ such that the following identities hold:

$$\begin{aligned}
\pi_r(a_i \cdot \xi_r \cdot a_j) &= \pi_i(a_i)\pi_r(\xi_r)\pi_j(a_j),\\
\pi_r(\xi_r)^*\pi_r(\eta_r) &= \pi_j\big(\langle\, \xi_r \,|\, \eta_r \,\rangle_{A_j}\big),\\
\pi_{sr}\big(\Phi_{s,r}(\xi_r \otimes \xi_s)\big) &= \pi_r(\xi_r)\pi_s(\xi_s),\\
\pi_{I_i} \circ \Phi_i &= \pi_i,\\
\pi_i(a_i)\pi_j(a_j) &= 0 \quad \text{for } i \neq j,
\end{aligned}$$

where $\xi_r, \eta_r \in F_r$, $a_i \in A_i$, $a_j \in A_j$ and $\xi_s \in F_s$, $r\colon i \to j$, $s\colon j \to k \in \mathrm{Arr}(J)$. For a Toeplitz representation π from a product system (F, Φ) into a C^*-algebra B we just write $\pi\colon (F, \Phi) \to B$.

Example 3.2 (Covariant semigroup homomorphisms)
Let $(A, \mathcal{S}, \alpha)$ be the discrete semigroup dynamical system from Example 1.12, in particular, let A be unital again. We set $F_s := {}_{\alpha_s}A$ and define $\Phi_{s,t}\colon F_sF_t \to F_{st}$ by $\Phi_{s,t}(\xi_t \otimes \xi_s) := \alpha_s(\xi_t)\xi_s$ for $\xi_t \in F_t$ and $\xi_s \in F_s$. Then (F, Φ) is a product system over the monoid that comes from the semigroup $\mathcal{S}$. Now let (π, V) be a covariant homomorphism from $(A, \mathcal{S}, \alpha)$ to a unital C^*-algebra B, i.e., π is a nondegenerate *-homomorphism from A to B and V is a semigroup homomorphism from $\mathcal{S}$ to the semigroup of isometric elements of B such that

$$\pi(\alpha_s(a)) = V_s\pi(a)V_s^*$$

for all $a \in A$ and $s \in \mathcal{S}$. Then we set $\pi_i := \pi$ and $\pi_r(\xi_r) := V_r^*\pi(\xi_r)$ for $\xi_r \in F_r$ and we claim that this defines a Toeplitz representation from (F, Φ) into B. So we let $a, b \in A$, $\xi_r, \eta_r \in F_r$ and $\xi_s \in F_s$ be arbitrary and compute

$$\begin{aligned}
\pi_r(a \cdot \xi_r \cdot b) &= \pi_r(\alpha_r(a)\xi_r b) = V_r^*\pi(\alpha_r(a)\xi_r b) = V_r^*\pi(\alpha_r(a))\pi(\xi_r)\pi(b)\\
&= \pi(a)V_r^*\pi(\xi_r)\pi(b) = \pi_i(a)\pi_r(\xi_r)\pi_i(b),
\end{aligned}$$

$$\begin{aligned}
\pi_r(\xi_r)^*\pi_r(\eta_r) &= \pi(\xi_r)^*V_rV_r^*\pi(\eta_r) = \pi(\xi_r)^*V_r\pi(1)V_r^*\pi(\eta_r) = \pi(\xi_r)^*\pi(\alpha_r(1))\pi(\eta_r)\\
&= \pi(\xi_r^*\alpha_r(1)\eta_r) = \pi(\xi_r^*\eta_r) = \pi(\langle\, \xi_r \,|\, \eta_r \,\rangle_A) = \pi_i(\langle\, \xi_r \,|\, \eta_r \,\rangle_A) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\pi_r(\xi_r)\pi_s(\xi_s) &= V_r^*\pi(\xi_r)V_s^*\pi(\xi_s) = V_r^*V_s^*V_s\pi(\xi_r)V_s^*\pi(\xi_s) = (V_sV_r)^*\pi(\alpha_s(\xi_r))\pi(\xi_s)\\
&= V_{sr}^*\pi(\alpha_s(\xi_r)\xi_s) = \pi_{sr}(\alpha_s(\xi_r)\xi_s) = \pi_{sr}(\Phi_{s,r}(\xi_r \otimes \xi_s)).
\end{aligned}$$

Let e be the unit element in $\mathcal{S}$. Then $\alpha_e = \mathrm{id} \in \mathrm{End}(A)$ and $V_e = 1_B$. Hence, $F_e = {}_{\mathrm{id}}A = I'_A$ and so $\Phi_i = \mathrm{id}$, which yields that $\pi_{I_i} \circ \Phi_i(\xi) = \pi_e(\xi) = V_e^*\pi_i(\xi) = 1_B^*\pi_i(\xi) = \pi_i(\xi)$ for $\xi \in I'_A = A$. The last identity holds trivially, since there is only one object $i \in \mathrm{Ob}(J)$ and so we have shown that π in fact is a Toeplitz representation.

Definition 3.3 (Homomorphisms between Hilbert C^*-modules)
Let E be a Hilbert A-module and F be a Hilbert B-module. A *homomorphism* from (E, A) to (F, B) is a pair (λ_E, λ_A) consisting of a linear mapping $\lambda_E \colon E \to F$ and a *-homomorphism $\lambda_A \colon A \to B$ such that

$$\lambda_E(\xi \cdot a) = \lambda_E(\xi) \cdot \lambda_A(a) \qquad \text{and} \qquad (\, \lambda_E(\xi_1) \,|\, \lambda_E(\xi_2) \,)_B = \lambda_A((\, \xi_1 \,|\, \xi_2 \,)_A)$$

for all $\xi, \xi_1, \xi_2 \in E$ and $a \in A$. We write $(\lambda_E, \lambda_A) \colon (E, A) \to (F, B)$ to indicate that (λ_E, λ_A) is a homomorphism from the Hilbert A-module E to the Hilbert B-module F and $(\lambda_E, \lambda_A) \colon (E, A) \to B$ if (λ_E, λ_A) is a homomorphism from (E, A) to the Hilbert B-module ${}_{\mathrm{id}}B$. For homomorphisms $(\lambda_E, \lambda_A) \colon (E, A) \to (F, B)$ and $(\mu_F, \mu_B) \colon (F, B) \to (G, C)$ we set

$$(\mu_F, \mu_B) \circ (\lambda_E, \lambda_A) := (\mu_F \circ \lambda_E, \mu_B \circ \lambda_A)$$

and it is easy to see that this makes $(\mu_F, \mu_B) \circ (\lambda_E, \lambda_A)$ a homomorphism from (E, A) to (G, C).

Example 3.4 Let A and B be C^*-algebras and let $\pi \colon A \to B$ be a *-homomorphism. We set $E := {}_{\mathrm{id}}A$ and $F := {}_{\mathrm{id}}B$. Then (π, π) is a homomorphism from (E, A) to (F, B), because $\pi(\xi \cdot a) = \pi(\xi a) = \pi(\xi)\pi(a) = \pi(\xi) \cdot \pi(a)$ and $(\, \pi(\xi_1) \,|\, \pi(\xi_2) \,)_B = \pi(\xi_1)^*\pi(\xi_2) = \pi(\xi_1^*\xi_2) = \pi((\, \xi_1 \,|\, \xi_2 \,)_A)$ for all $\xi, \xi_1, \xi_2 \in E$ and $a \in A$.

Example 3.5 Let (F, Φ) be a product system over J and let π be a Toeplitz representation from (F, Φ) into a C^*-algebra B. If $r \colon i \to j$ is an arrow of J, then it is easy to see that the pair (π_r, π_j) is a homomorphism from (F_r, A_j) to B. This is the main reason why we recall the concept of homomorphisms between Hilbert C^*-modules.

Lemma 3.6 *Let $(\lambda_E, \lambda_A) \colon (E, A) \to (F, B)$ be a homomorphism of Hilbert C^*-modules. Then the identity*

$$\lambda_{\mathcal{K}(E)}(\theta_{\xi,\eta}) := \theta_{\lambda_E(\xi), \lambda_E(\eta)} \qquad \xi, \eta \in E$$

uniquely defines a *-homomorphism $\lambda_{\mathcal{K}(E)} \colon \mathcal{K}(E) \to \mathcal{K}(F)$.

Proof: From [14, Lemma 2.1] we know that if X is a Hilbert C^*-module over a C^*-algebra C then the following identity holds for $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in X$:

$$\Big\| \sum_{i=1}^{n} \theta_{\xi_i, \eta_i} \Big\| = \| ((\, \xi_i \,|\, \xi_j \,)_C)_{ij}^{1/2} ((\, \eta_i \,|\, \eta_j \,)_C)_{ij}^{1/2} \|,$$

where the norm on the right hand side is the C^*-norm on $M_n(C)$. Now it follows easily that

$$
\begin{aligned}
\Big\| \sum_{i=1}^{n} \theta_{\lambda_E(\xi_i),\lambda_E(\eta_i)} \Big\| &= \|((\, \lambda_E(\xi_i) \,|\, \lambda_E(\xi_j) \,)_B)_{ij}^{1/2} ((\, \lambda_E(\eta_i) \,|\, \lambda_E(\eta_j) \,)_B)_{ij}^{1/2}\| \\
&= \|(\lambda_A((\, \xi_i \,|\, \xi_j \,)_A))_{ij}^{1/2} (\lambda_A((\, \eta_i \,|\, \eta_j \,)_A))_{ij}^{1/2}\| \\
&= \|\lambda_A^{(n)}(((\, \xi_i \,|\, \xi_j \,)_A)_{ij})^{1/2} \lambda_A^{(n)}(((\, \eta_i \,|\, \eta_j \,)_A)_{ij})^{1/2}\| \\
&= \|\lambda_A^{(n)}(((\, \xi_i \,|\, \xi_j \,)_A)_{ij}^{1/2} ((\, \eta_i \,|\, \eta_j \,)_A)_{ij}^{1/2})\| \\
&\le \|((\, \xi_i \,|\, \xi_j \,)_A)_{ij}^{1/2} ((\, \eta_i \,|\, \eta_j \,)_A)_{ij}^{1/2}\| = \Big\| \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \Big\|,
\end{aligned}
$$

where $\lambda_A^{(n)}\colon M_n(A) \to M_n(B)$ is the *-homomorphism given by $\lambda_A^{(n)}((a_{ij})_{ij}) = (\lambda_A(a_{ij}))_{ij}$. This shows that the mapping $\lambda_{\mathcal{K}(E)}\colon \mathcal{K}(E) \to \mathcal{K}(F)$ given by

$$
\lambda_{\mathcal{K}(E)}\Big(\sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \Big) := \sum_{i=1}^{n} \lambda_{\mathcal{K}(E)}(\theta_{\xi_i,\eta_i}) = \sum_{i=1}^{n} \theta_{\lambda_E(\xi_i),\lambda_E(\eta_i)}
$$

is well defined and contractive on the linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in E\}$. For $\xi_1, \xi_2, \eta_1, \eta_2 \in E$ we have

$$
\begin{aligned}
\lambda_{\mathcal{K}(E)}(\theta_{\xi_1,\eta_1}\theta_{\xi_2,\eta_2}) &= \lambda_{\mathcal{K}(E)}(\theta_{\xi_1 \cdot (\, \eta_1 \,|\, \xi_2 \,)_A, \eta_2}) = \theta_{\lambda_E(\xi_1 \cdot (\, \eta_1 \,|\, \xi_2 \,)_A), \lambda_E(\eta_2)} \\
&= \theta_{\lambda_E(\xi_1) \cdot \lambda_A((\, \eta_1 \,|\, \xi_2 \,)_A), \lambda_E(\eta_2)} = \theta_{\lambda_E(\xi_1) \cdot (\, \lambda_E(\eta_1) \,|\, \lambda_E(\xi_2) \,)_B, \lambda_E(\eta_2)} \\
&= \theta_{\lambda_E(\xi_1),\lambda_E(\eta_1)} \theta_{\lambda_E(\xi_2),\lambda_E(\eta_2)} = \lambda_{\mathcal{K}(E)}(\theta_{\xi_1,\eta_1}) \lambda_{\mathcal{K}(E)}(\theta_{\xi_2,\eta_2})
\end{aligned}
$$

since (λ_E, λ_A) is a homomorphism. From this it easily follows that $\lambda_{\mathcal{K}(E)}$ is multiplicative on the linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in E\}$. Moreover, we have $\lambda_{\mathcal{K}(E)}((\sum_{i=1}^{n} \theta_{\xi_i,\eta_i})^*) = \lambda_{\mathcal{K}(E)}((\sum_{i=1}^{n} \theta_{\eta_i,\xi_i})) = \sum_{i=1}^{n} \theta_{\lambda_E(\eta_i),\lambda_E(\xi_i)} = (\sum_{i=1}^{n} \theta_{\lambda_E(\xi_i),\lambda_E(\eta_i)})^* = \lambda_{\mathcal{K}(E)}(\sum_{i=1}^{n} \theta_{\xi_i,\eta_i})^*$. Now, since $\lambda_{\mathcal{K}(E)}$ is a contraction on the linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in E\}$, we know that there is a unique extension of $\lambda_{\mathcal{K}(E)}$ to all of $\mathcal{K}(E)$ and that this extension is a *-homomorphism from $\mathcal{K}(E)$ to $\mathcal{K}(F)$. □

Corollary 3.7 *Let $(\lambda_E, \lambda_A)\colon (E, A) \to B$ be a homomorphism of Hilbert C^*-modules. Then the identity*

$$
\lambda_{\mathcal{K}(E)}(\theta_{\xi,\eta}) := \lambda_E(\xi)\lambda_E(\eta)^* \qquad \xi, \eta \in E
$$

uniquely defines a *-homomorphism $\lambda_{\mathcal{K}(E)}\colon \mathcal{K}(E) \to B$.

Proof: If we view B as Hilbert C^*-module over itself, we have $\theta_{\xi,\eta}\zeta = \xi \cdot (\, \eta \,|\, \zeta \,)_B = \xi\eta^*\zeta$ for $\xi, \eta, \zeta \in B$ and so wet get that $\mathcal{K}(B) \cong B$ by identifying $\theta_{\xi,\eta}$ with the operation of left multiplication by $\xi\eta^*$. Hence, the assertion follows directly from Lemma 3.6. □

3.2 The reduced Toeplitz algebra

Now we want to present the construction of the reduced Toeplitz algebra. First, we recall the notions of the direct sum of a family of C^*-algebras and the inner and outer direct sums of families of Hilbert C^*-modules, which we need to define the Fock correspondence $\mathcal{F}(F,\Phi)$ of a given product system (F,Φ) over an index category J. $\mathcal{F}(F,\Phi)$ is a C^*-arrow over the direct sum of the family $\{A_i \colon i \in \mathrm{Ob}(J)\}$. Hence, we are able to define generalized shift operators $\pi_r(\xi_r), \xi_r \in F_r, r \in \mathrm{Arr}(J)$, and multiplication operators $\pi_i(a_i), a_i \in A_i, i \in \mathrm{Ob}(J)$, on $\mathcal{F}(F,\Phi)$ and we show that π is a Toeplitz representation from (F,Φ) to $\mathcal{L}(\mathcal{F}(F,\Phi))$. Finally, we define the reduced Toeplitz algebra $\mathcal{T}_r(F,\Phi)$ to be the C^*-algebra generated by the image of (F,Φ) under π.

Definition 3.8 (The direct sum of C^*-algebras)
The *direct sum* of a family $\{A_i \colon i \in I\}$ of C^*-algebras is defined by

$$A_I := \bigoplus_{i\in I} A_i := \{(a_i) \in \prod_{i\in I} A_i \colon \lim_{i\to\infty} \|a_i\| = 0\}.$$

Here $\lim_{i\to\infty}\|a_i\| = 0$ means that for every $\varepsilon > 0$ there is a finite subset $L(\varepsilon) \subset I$ such that $\|a_i\| < \varepsilon$ for all $i \in I \backslash L(\varepsilon)$. We define addition, multiplication, multiplication by scalars and involution componentwise. Moreover, we define a norm on A_I by setting $\|(a_i)\| := \sup_{i\in I}\|a_i\|$. It is not hard to see that this makes A_I a C^*-algebra.

Definition 3.9 (The inner and outer direct sum of Hilbert C^*-modules)
Let $\{F_j \colon j \in I\}$ be a family of Hilbert A-modules. Then we define the *inner direct sum* of $\{F_j \colon j \in I\}$ by

$$\bigoplus_{j\in I}^{(i)} F_j := \{(\xi_j) \in \prod_{j\in I} F_j \colon \sum_{j\in I} (\, \xi_j \,|\, \xi_j \,)_A \text{ converges in } A\}.$$

We define the right multiplication by A componentwise and an A-valued inner product by $(\,(\xi_j)\,|\,(\eta_j)\,)_A := \sum_{j\in I}(\,\xi_j\,|\,\eta_n\,)_A$ for $(\xi_j),(\eta_j) \in \bigoplus_{j\in I}^{(i)} F_j$. It is easy to see that this makes $\bigoplus_{j\in I}^{(i)} F_j$ a Hilbert A-module, see [15, pages 5-6].

Now let $\{(F_j, A_j) \colon j \in I\}$ be a family of Hilbert C^*-modules. We define the *outer direct sum* of $\{F_j \colon j \in I\}$ by

$$\bigoplus_{j\in I}^{(o)} F_j := \{(\xi_j) \in \prod_{j\in I} F_j \colon ((\,\xi_j\,|\,\xi_j\,)_{A_j})_{j\in I} \in \bigoplus_{j\in I} A_j\}.$$

Again, we define a right multiplication by elements of $\bigoplus_{j\in I} A_j$ componentwise and an inner product that takes values in $\bigoplus_{j\in I} A_j$ by $(\,(\xi_j)\,|\,(\eta_j)\,)_{\bigoplus A_j} := ((\,\xi_j\,|\,\eta_j\,)_{A_j})_{j\in I}$ for $(\xi_j),(\eta_j) \in \bigoplus_{j\in I}^{(o)} F_j$. It is not hard to see that this inner product is well defined and that it makes $\bigoplus_{j\in I}^{(o)} F_j$ a Hilbert C^*-module over $\bigoplus_{j\in I} A_j$.

Proposition 3.10 *Let $\{(F_n, A_n)\colon n \in I\}$ be a family of Hilbert C^*-modules. Then*

$$\mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n) \cong \prod_{n\in I} \mathcal{L}(F_n) \qquad \text{and} \qquad \mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n) \cong \bigoplus_{n\in I} \mathcal{K}(F_n).$$

Proof: Let $T \in \mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$ be arbitrary. Then it is easy to see that $T(F_n) \subset F_n$, where we view F_n as a submodule of $\bigoplus_{n\in I}^{(o)} F_n$. We let $T|_{F_n} \in \mathcal{L}(F_n)$ denote the restriction of T to F_n and define a mapping $\vartheta\colon \mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n) \to \prod_{n\in I} \mathcal{L}(F_n)$ by setting

$$\vartheta(T) := (T|_{F_n})_{n\in I}.$$

It is clear that $\vartheta(T) \in \prod_{n\in I} \mathcal{L}(F_n)$, since $\|T|_{F_n}\| \leq \|T\|$ for all $n \in I$ and it is easy to see that ϑ is an injective *-homomorphism. To see that ϑ is surjective, let $(T_n) \in \prod_{n\in I} \mathcal{L}(F_n)$ be arbitrary. We have to find a $T \in \mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$ with $T|_{F_n} = T_n$ for all $n \in I$. Since $\mathcal{L}(F_n) \subset \mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$ we set $T := \sum_{n\in I} T_n$. If we can show that this sum defines an element of $\mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$ then it is clear that T is the element we are looking for. We have to show that $(\sum_{n\in I} T_n)(\xi_n) = (T_n\xi_n) \in \bigoplus_{n\in I}^{(o)} F_n$ for $(\xi_n) \in \bigoplus_{n\in I}^{(o)} F_n$. But this is easy to see, since $\|T_n\xi_n\| \leq \|T_n\|\|\xi_n\| \leq M\|\xi_n\|$ and so $\lim_{n\to\infty} \|(T_n\xi_n \,|\, T_n\xi_n)_{A_n}\| = \lim_{n\to\infty} \|(\xi_n \,|\, \xi_n)_{A_n}\| = 0$, since $(\xi_n) \in \bigoplus_{n\in I}^{(o)} F_n$. So we have shown that ϑ is a *-isomorphism from $\mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$ to $\prod_{n\in I} \mathcal{L}(F_n)$.

Since $\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)$ is a C^*-subalgebra of $\mathcal{L}(\bigoplus_{n\in I}^{(o)} F_n)$, we can restrict ϑ to $\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)$ to get a *-isomorphism from $\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)$ to $\vartheta(\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n))$. So it remains to prove that $\vartheta(\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)) = \bigoplus_{n\in I} \mathcal{K}(F_n)$. We let $(\xi_n), (\eta_n) \in \bigoplus_{n\in I}^{(o)} F_n$. Then it is easy to see that

$$\vartheta(\theta_{(\xi_n),(\eta_n)}) = (\theta_{\xi_n,\eta_n})_{n\in I}$$

and $(\theta_{\xi_n,\eta_n})_{n\in I} \in \bigoplus_{n\in I} \mathcal{K}(F_n)$, since $0 \leq \lim_{n\to\infty} \|\theta_{\xi_n,\eta_n}\| \leq \lim_{n\to\infty} \|\xi_n\|\|\eta_n\| = 0$. Hence, $\vartheta(\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)) \subseteq \bigoplus_{n\in I} \mathcal{K}(F_n)$. Now let $(K_n)_{n\in I} \in \bigoplus_{n\in I} \mathcal{K}(F_n)$ be arbitrary. Then $K_n \in \mathcal{K}(F_n) \subset \mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)$ and so $\sum_{n\in I} K_n \in \mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)$ by the same argument as above. Now it is clear that $\vartheta(\sum_{n\in I} K_n) = (K_n)_{n\in I}$ which shows that $\vartheta(\mathcal{K}(\bigoplus_{n\in I}^{(o)} F_n)) = \bigoplus_{n\in I} \mathcal{K}(F_n)$. □

From now on, let J be a fixed index category such that all arrows of J are epi and let (F, Φ) be a fixed product system over J.

Definition 3.11 (The Fock correspondence $\mathcal{F}(F, \Phi)$)
Let A_J denote the direct sum of the family $\{A_i\colon i \in \mathrm{Ob}(J)\}$ of C^*-algebras. We set

$$\mathcal{F}(F, \Phi) := \bigoplus_{j\in \mathrm{Ob}(J)}^{(o)} \Big(\bigoplus_{r\colon i\to j}^{(i)} F_r \Big)$$

We recall that $\mathcal{F}(F,\Phi)$ is a Hilbert A_J-module and that we can also write

$$\mathcal{F}(F,\Phi) = \{(\xi_r) \in \prod_{r\in \mathrm{Arr}(J)} F_r \colon \Big(\sum_{r\colon i\to j} (\,\xi_r \,|\, \xi_r\,)_{A_j} \Big)_{j\in \mathrm{Ob}(J)} \in A_J\}.$$

Especially, this means that for every $j \in \mathrm{Ob}(J)$ the sum $\sum_{r\colon i\to j}(\,\xi_r \,|\, \xi_r\,)_{A_j}$ converges in A_j. We also recall that the A_J-valued inner product and the right multiplication with elements of A_J are defined by

$$\begin{aligned} (\,(\xi_r)\,|\,(\eta_r)\,)_{A_J} &:= \Big(\sum_{r\colon i\to j} (\,\xi_r \,|\, \eta_r\,)_{A_j} \Big)_{j\in \mathrm{Ob}(J)} \quad \text{and} \\ (\xi_r)\cdot(a_k) &:= (\zeta_r) \text{ with } \zeta_r = \xi_r \cdot a_j, \text{ for } r\colon i \to j, \end{aligned}$$

$(a_k) \in A_J$, $(\xi_r), (\eta_r) \in \mathcal{F}(F,\Phi)$. Moreover, we define a left multiplication by elements of A_J by setting

$$(a_k)\cdot(\xi_r) := (\zeta_r) \text{ with } \zeta_r = a_i \cdot \xi_r, \text{ for } r\colon i \to j,$$

$(a_k) \in A_J$, $(\xi_r) \in \mathcal{F}(F,\Phi)$, which makes $\mathcal{F}(F,\Phi)$ a C^*-arrow over A_J.

Now we are going to define operators T_{ξ_r} on $\mathcal{F}(F,\Phi)$ and we want to show that these operators are adjointable. But first, we want to talk about notation. By (η_r) we shall mean an element in $\mathcal{F}(F,\Phi)$, whereas η_r denotes an element in F_r, which can of course also be thought of as the element in $\mathcal{F}(F,\Phi)$ that has only one nonzero entry η_r at 'position' $r \in \mathrm{Arr}(J)$.

So let $\xi_r \in F_r$ and $\eta_s \in F_s$ be arbitrary. We define the operator T_{ξ_r} on the single components of $\mathcal{F}(F,\Phi)$ by setting

$$T_{\xi_r}(\eta_s) := \begin{cases} \Phi_{s,r}(\xi_r \otimes \eta_s) & \text{if } (r,s) \in \mathrm{Arr}(J) \circ \mathrm{Arr}(J) \\ 0 & \text{otherwise.} \end{cases}$$

To show that T_{ξ_r} is well defined, we have to make sure that $T_{\xi_r}((\eta_s)) \in \mathcal{F}(F,\Phi)$ for arbitrary $(\eta_s) \in \mathcal{F}(F,\Phi)$. So first we have to show that $\sum_{t\colon i\to j}(\,(T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t\,)_{A_j}$ converges in A_j and then we must check that

$$\Big(\sum_{t\colon i\to j} (\,(T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t\,)_{A_j} \Big)_{j\in \mathrm{Ob}(J)} \in A_J.$$

To simplify notation, we set $M_j := \{r \in \mathrm{Arr}(J) |\ r\colon i \to j,\ i \in \mathrm{Ob}(J)\}$. Now the sum $\sum_{t\in M_j}(\,(T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t\,)_{A_j}$ converges in A_j if and only if for all $\varepsilon > 0$ there is a finite subset $L(\varepsilon) \subseteq M_j$ such that for all finite subsets $K \subseteq M_j \backslash L(\varepsilon)$ we have

$$\Big\| \sum_{t\in K} (\,(T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t\,)_{A_j} \Big\| < \varepsilon.$$

So let $\varepsilon > 0$ be arbitrary. Since $(\eta_s) \in \mathcal{F}(F, \Phi)$, we know that $\sum_{s\in M_j} (\, \eta_s \,|\, \eta_s \,)_{A_j}$ converges in A_j and hence, there is a finite subset $L'(\varepsilon) \subseteq M_j$ such that for all finite subsets $K' \subseteq M_j \backslash L'(\varepsilon)$ we have

$$\|\xi_r\|^2 \Big\| \sum_{s\in K'} (\, \eta_s \,|\, \eta_s \,)_{A_j} \Big\| < \varepsilon.$$

We set $L(\varepsilon) := \{t \in M_j \colon \exists s \in L'(\varepsilon) \text{ such that } t = s \circ r\}$. It is clear that $L(\varepsilon)$ is a finite subset of M_j. Now let K be an arbitrary finite subset of $M_j \backslash L(\varepsilon)$. Then

$$\begin{aligned}
\sum_{t\in K} (\, (T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t \,)_{A_j} &= \sum_{\substack{s\in M_j \\ s\circ r\in K}} (\, T_{\xi_r}(\eta_s) \,|\, T_{\xi_r}(\eta_s) \,)_{A_j} \\
&= \sum_{\substack{s\in M_j \\ s\circ r\in K}} (\, \Phi_{s,r}(\xi_r \otimes \eta_s) \,|\, \Phi_{s,r}(\xi_r \otimes \eta_s) \,)_{A_j} \\
&= \sum_{\substack{s\in M_j \\ s\circ r\in K}} (\, \xi_r \otimes \eta_s \,|\, \xi_r \otimes \eta_s \,)_{A_j} \leq \|\xi_r\|^2 \sum_{\substack{s\in M_j \\ s\circ r\in K}} (\, \eta_s \,|\, \eta_s \,)_{A_j}.
\end{aligned}$$

We set $K' := \{s \in M_j \colon s \circ r \in K\}$. Then K' is a finite subset of M_j, since K is a finite set and r is epi and so for every $t \in K$ there is at most one $s \in M_j$ with $t = s \circ r$. Moreover, it follows easily that $K' \subseteq M_j \backslash L'(\varepsilon)$. Thus we have

$$\Big\| \sum_{t\in K} (\, (T_{\xi_r}(\eta_s))_t \,|\, (T_{\xi_r}(\eta_s))_t \,)_{A_j} \Big\| \leq \|\xi_r\|^2 \Big\| \sum_{s\in K'} (\, \eta_s \,|\, \eta_s \,)_{A_j} \Big\| < \varepsilon,$$

which shows that the sum $\sum_{t\in M_j} (\, (T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t \,)_{A_j}$ converges in A_j. The computation above also yields that

$$\Big\| \sum_{t\in M_j} (\, (T_{\xi_r}(\eta_s))_t \,|\, (T_{\xi_r}(\eta_s))_t \,)_{A_j} \Big\| \leq \|\xi_r\|^2 \Big\| \sum_{s\in M_j} (\, \eta_s \,|\, \eta_s \,)_{A_j} \Big\|$$

and so it is clear that

$$\Big(\sum_{t\in M_j} (\, (T_{\xi_r}((\eta_s)))_t \,|\, (T_{\xi_r}((\eta_s)))_t \,)_{A_j} \Big)_{j\in \mathrm{Ob}(J)} \in A_J$$

since $\big(\sum_{s\in M_j} (\, \eta_s \,|\, \eta_s \,)_{A_j} \big)_{j\in \mathrm{Ob}(J)} \in A_J$. Hence, we have shown that $T_{\xi_r}((\eta_s)) \in \mathcal{F}(F, \Phi)$ and that $\|T_{\xi_r}((\eta_s))\| \leq \|\xi_r\| \|(\eta_s)\|$, and so T_{ξ_r} is a well defined bounded operator on $\mathcal{F}(F, \Phi)$.

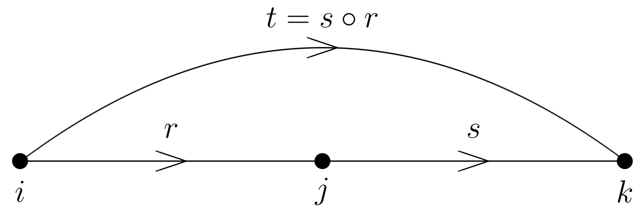
We already claimed that T_{ξ_r} is adjointable, so now we have to provide the adjoint operator $T_{\xi_r}^*$. Suppose $(r, s) \in \mathrm{Arr}(J) \circ \mathrm{Arr}(J)$, $r : i \to j$, $s : j \to k$, and let $K_{\xi_r}^{s,r} : F_r \otimes F_s \to F_s$ be the mapping that is given on simple tensors by

$$K_{\xi_r}^{s,r}(\eta_r \otimes \eta_s) := (\, \xi_r \,|\, \eta_r \,)_{A_j} \cdot \eta_s.$$

Now we define the operator $T^*_{\xi_r}$ componentwise by

$$T^*_{\xi_r}(\eta_t) := \begin{cases} K^{s,r}_{\xi_r} \circ \Phi^*_{s,r}(\eta_t) & \text{if there exists an } s \in \operatorname{Arr}(J) \text{ with } t = s \circ r \\ 0 & \text{otherwise.} \end{cases}$$

To show that $T^*_{\xi_r}$ is the adjoint of T_{ξ_r}, we have to prove the equation $(\,(\eta_t)\,|\,T_{\xi_r}((\zeta_t))\,)_{A_J} = (\,T^*_{\xi_r}((\eta_t))\,|\,(\zeta_t)\,)_{A_J}$ for all (η_t), $(\zeta_t) \in \mathcal{F}(F,\Phi)$. We do that componentwise and consider only the component of the inner product, that arises from the fixed arrow $t \in \operatorname{Arr}(J)$.



If there exists an $s \in \operatorname{Arr}(J)$ with $t = s \circ r$ (such an s is unique then, since all arrows of J are epi), we want to assume that $\Phi^*_{s,r}(\eta_t) = x_r \otimes y_s$, $x_r \in F_r$, $y_s \in F_s$. Then we have

$$\begin{aligned} (\,\eta_t\,|\,T_{\xi_r}(\zeta_s)\,)_{A_k} &= (\,\eta_t\,|\,\Phi_{s,r}(\xi_r \otimes \zeta_s)\,)_{A_k} = (\,\Phi^*_{s,r}(\eta_t)\,|\,\xi_r \otimes \zeta_s\,)_{A_k} \\ &= (\,x_r \otimes y_s\,|\,\xi_r \otimes \zeta_s\,)_{A_k} = (\,y_s\,|\,(\,x_r\,|\,\xi_r\,)_{A_j} \cdot \zeta_s\,)_{A_k} \\ &= (\,(\,\xi_r\,|\,x_r\,)_{A_j} \cdot y_s\,|\,\zeta_s\,)_{A_k} = (\,K^{s,r}_{\xi_r} \circ \Phi^*_{s,r}(\eta_t)\,|\,\zeta_s\,)_{A_k} \\ &= (\,T^*_{\xi_r}(\eta_t)\,|\,\zeta_s\,)_{A_k}. \end{aligned}$$

The left hand side of the equation is zero if there is no $s \in \operatorname{Arr}(J)$ with $t = s \circ r$. But in this case the right hand side of the equation is zero, too, and so we have shown that $T_{\xi_r} \in \mathcal{L}(\mathcal{F}(F,\Phi))$.

We set $\pi_r(\xi_r) := T_{\xi_r}$ to get a mapping π_r from F_r into the C^*-algebra $\mathcal{L}(\mathcal{F}(F,\Phi))$. π_r is surely linear, since the tensor product is linear in the first component and the mappings $\Phi_{s,r}$ are linear.

For $a_i \in A_i$, $i \in \operatorname{Ob}(J)$, we let $\pi_i(a_i)$ be the operator on $\mathcal{F}(F,\Phi)$ that acts componentwise by multiplication with a_i from the left, where we consider a_i as an element of A_J, thus

$$\pi_i(a_i)(\eta_s) := \begin{cases} a_i \cdot \eta_s & \text{if } s\colon i \to j \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that π_i is a C^*-homomorphism from A_i to $\mathcal{L}(\mathcal{F}(F,\Phi))$. (This is a direct consequence of the fact that each F_s is a C^*-arrow.)

Proposition 3.12 *π is a Toeplitz representation from (F,Φ) to $\mathcal{L}(\mathcal{F}(F,\Phi))$.*

Proof: Let $\xi_r, \eta_r \in F_r$, $a_i \in A_i$, $a_j \in A_j$ and $\xi_s \in F_s$ be arbitrary, $r\colon i \to j$, $s\colon j \to k$. We prove the five identities

$$\begin{array}{lrcl} (1) & \pi_r(a_i \cdot \xi_r \cdot a_j)(\zeta_t) & = & \pi_i(a_i)\pi_r(\xi_r)\pi_j(a_j)(\zeta_t), \\ (2) & \pi_r(\xi_r)^*\pi_r(\eta_r)(\zeta_t) & = & \pi_j\big((\xi_r \,|\, \eta_r)_{A_j}\big)(\zeta_t), \\ (3) & \pi_{sr}\big(\Phi_{s,r}(\xi_r \otimes \xi_s)\big)(\zeta_t) & = & \pi_r(\xi_r)\pi_s(\xi_s)(\zeta_t), \\ (4) & (\pi_{I_i} \circ \Phi_i)(a_i)(\zeta_t) & = & \pi_i(a_i)(\zeta_t), \\ (5) & \pi_i(a_i)\pi_j(a_j)(\zeta_t) & = & 0 \text{ for } i \neq j \end{array}$$

for an arbitrary component $\zeta_t \in F_t$.

1) If the source of t is not j then both sides of Equation (1) are zero. So we suppose that $t\colon j \to l$ and we compute

$$\begin{aligned} \pi_i(a_i)\pi_r(\xi_r)\pi_j(a_j)(\zeta_t) &= \pi_i(a_i)T_{\xi_r}(a_j \cdot \zeta_t) = \pi_i(a_i)\Phi_{t,r}(\xi_r \otimes (a_j \cdot \zeta_t)) \\ &= a_i \cdot \Phi_{t,r}((\xi_r \cdot a_j) \otimes \zeta_t) = \Phi_{t,r}((a_i \cdot \xi_r \cdot a_j) \otimes \zeta_t) \\ &= T_{a_i \cdot \xi_r \cdot a_j}(\zeta_t) = \pi_r(a_i \cdot \xi_r \cdot a_j)(\zeta_t). \end{aligned}$$

2) With the same argument as in 1), we suppose that $t\colon j \to l$ and we have

$$\begin{aligned} \pi_r(\xi_r)^*\pi_r(\eta_r)(\zeta_t) &= T^*_{\xi_r}T_{\eta_r}(\zeta_t) = T^*_{\xi_r}(\Phi_{t,r}(\eta_r \otimes \zeta_t)) = K^{t,r}_{\xi_r}(\Phi^*_{t,r}(\Phi_{t,r}(\eta_r \otimes \zeta_t))) \\ &= K^{t,r}_{\xi_r}(\eta_r \otimes \zeta_t) = (\xi_r \,|\, \eta_r)_{A_j} \cdot \zeta_t = \pi_j\big((\xi_r \,|\, \eta_r)_{A_j}\big)(\zeta_t). \end{aligned}$$

3) If the source of t is not k then both sides of Equation (3) are zero. So we suppose that $t\colon k \to l$ and we compute

$$\begin{aligned} \pi_r(\xi_r)\pi_s(\xi_s)(\zeta_t) &= T_{\xi_r}(T_{\xi_s}(\zeta_t)) = T_{\xi_r}(\Phi_{t,s}(\xi_s \otimes \zeta_t)) = \Phi_{ts,r}(\xi_r \otimes (\Phi_{t,s}(\xi_s \otimes \zeta_t))) \\ &= \Phi_{ts,r} \circ (\iota_{F_r} \otimes \Phi_{t,s})\big(\xi_r \otimes (\xi_s \otimes \zeta_t)\big) \\ &= \Phi_{t,sr} \circ (\Phi_{s,r} \otimes \iota_{F_t})\big((\xi_r \otimes \xi_s) \otimes \zeta_t\big) \quad \text{(see Equation (2.1))} \\ &= \Phi_{t,sr}(\Phi_{s,r}(\xi_r \otimes \xi_s) \otimes \zeta_t) = T_{\Phi_{s,r}(\xi_r \otimes \xi_s)}(\zeta_t) \\ &= \pi_{sr}\big(\Phi_{s,r}(\xi_r \otimes \xi_s)\big)(\zeta_t). \end{aligned}$$

4) If the source of t is not i then both sides of Equation (4) are zero. So we suppose that $t\colon i \to j$ and we compute

$$\begin{aligned} (\pi_{I_i} \circ \Phi_i)(a_i)(\zeta_t) &= T_{\Phi_i(a_i)}(\zeta_t) = \Phi_{t,I_i}(\Phi_i(a_i) \otimes \zeta_t) = \Phi_{t,I_i} \circ (\Phi_i \otimes \iota'_{F_t})(a_i \otimes \zeta_t) \\ &= \rho'_{F_t}(a_i \otimes \zeta_t) \quad \text{(see Equation (2.2))} \\ &= a_i \cdot \zeta_t = \pi_i(a_i)(\zeta_t). \end{aligned}$$

5) We have

$$\pi_i(a_i)\pi_j(a_j)(\zeta_t) = \left\{ \begin{array}{cl} \pi_i(a_i)(a_j \cdot \zeta_t) & \text{if } t\colon j \to k \\ 0 & \text{otherwise.} \end{array} \right\} = \left\{ \begin{array}{cl} (a_i a_j) \cdot \zeta_t & \text{if } t\colon j \to k \wedge t\colon i \to k \\ 0 & \text{otherwise.} \end{array} \right.$$

and so it is clear that $\pi_i(a_i)\pi_j(a_j)(\zeta_t) = 0$ for $i \neq j$. □

Definition 3.13 (The reduced Toeplitz algebra)
The *reduced Toeplitz algebra* $\mathcal{T}_r(F,\Phi)$ is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}(F,\Phi))$ generated by the operators $\pi_r(\xi_r)$ and $\pi_i(a_i)$,

$$\mathcal{T}_r(F,\Phi) := C^*(\{\pi_r(\xi_r)\colon \xi_r \in F_r,\, r \in \mathrm{Arr}(J)\} \cup \{\pi_i(a_i)\colon a_i \in A_i,\, i \in \mathrm{Ob}(J)\}).$$

Of course, π is a Toeplitz representation from (F,Φ) into the reduced Toeplitz algebra $\mathcal{T}_r(F,\Phi)$. We call π the *reduced Toeplitz representation.*

Example 3.14 (The reduced crossed product $A \rtimes_E \mathbb{N}$)
Let (F,Φ) be a product system over the natural numbers $\mathbb{N}$ such that (F,Φ) is a homomorphism and $\Phi_i = \mathrm{id}$. Thus our product system (F,Φ) consists of a C^*-algebra A and C^*-arrows F_n, $n \in \mathbb{N}$, from A to A such that $F_{n+m} \approx F_n \otimes F_m$ for all $n,m \in \mathbb{N}$ and so $F_n \approx F_1^n$. Moreover, we have $F_0 = {}_{\mathrm{id}}A$ and $\Phi_{k,0} = \rho'_{F_k}$ since $\Phi_i = \mathrm{id}$. We set $E := F_1$, $E^0 := F_0 = {}_{\mathrm{id}}A$ and we denote the product system (F,Φ) by (A,E). Moreover, we set

$$\mathcal{F}(E) := \mathcal{F}(F,\Phi) = \{(\xi_n) \in \prod_{n\in\mathbb{N}} E^n\colon \sum_{n\in\mathbb{N}} (\,\xi_n \,|\, \xi_n\,)_A \quad \text{converges in } A\}.$$

Now let $\pi\colon (F,\Phi) \to \mathcal{T}_r(F,\Phi)$ be the reduced Toeplitz representation of (F,Φ) on $\mathcal{T}_r(F,\Phi)$. We set $\pi_A := \pi_0$ and $\pi_E := \pi_1$. Since $F_n \approx E^n$, we have $\pi_n(\xi_1\otimes\cdots\otimes\xi_n) = \pi_E(\xi_1)\cdots\pi_E(\xi_n)$ for $\xi_1,\dots,\xi_n \in E$ and thus, it is clear that the corresponding reduced Toeplitz algebra $\mathcal{T}_r(F,\Phi)$, which we denote by $A \rtimes_E \mathbb{N}$, is generated by $\pi_A(A)$ and $\pi_E(E)$:

$$A \rtimes_E \mathbb{N} := \mathcal{T}_r(F,\Phi) = C^*(\{\pi_E(\xi)\colon \xi \in E\} \cup \{\pi_A(a)\colon a \in A\}).$$

We call $A \rtimes_E \mathbb{N}$ the *reduced crossed product of A by the C^*-arrow E.*

3.3 The reduced Cuntz-Pimsner algebra

Let J be an index category such that all arrows of J are epi and let (F,Φ) be a product system over J. Moreover, let $\pi\colon (F,\Phi) \to \mathcal{T}_r(F,\Phi)$ be the reduced Toeplitz representation. Then $\mathcal{K}(\mathcal{F}(F,\Phi)) \cap \mathcal{T}_r(F,\Phi)$ is an ideal in $\mathcal{T}_r(F,\Phi)$. We set

$$\mathcal{O}_r(F,\Phi) := \frac{\mathcal{T}_r(F,\Phi)}{\mathcal{K}(\mathcal{F}(F,\Phi)) \cap \mathcal{T}_r(F,\Phi)}$$

and $\psi := q \circ \pi$, where $q\colon \mathcal{T}_r(F,\Phi) \to \mathcal{O}_r(F,\Phi)$ is the canonical quotient mapping. It is clear that ψ is a Toeplitz representation from (F,Φ) to $\mathcal{O}_r(F,\Phi)$. We call ψ the *reduced Toeplitz representation* from (F,Φ) into the *reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F,\Phi)$*.

In the following sections we will study the reduced Toeplitz algebra $\mathcal{T}_r(F,\Phi)$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F,\Phi)$ for concrete product systems. If we let (F,Φ) be a product system over a discrete small category J, then we will see that the reduced

Toeplitz algebra $\mathcal{T}_r(F,\Phi)$ is isomorphic to the direct sum of C^*-algebras. Then, for a given discrete C^*-dynamical system (A,G,α), we construct a product system (F,Φ) over the monoid J coming from the group G, such that $\mathcal{T}_r(F,\Phi)$ is isomorphic to the reduced crossed product $A \rtimes_{\alpha r} G$. And finally, given a direct system $\{A_i\colon i \in I\}$ of C^*-algebras, we construct a product system (F,Φ) over the index category J coming from the directed set I, such that $\mathcal{O}_r(F,\Phi)$ is strongly Morita equivalent to the direct limit $\varinjlim A_i$.

3.4 The direct sum of C^*-algebras

Let J be a discrete small category, i.e., $I := \mathrm{Ob}(J)$ is a set and the only arrows of J are the identity arrows. Let (F,Φ) be a product system over J with $\Phi_i = \mathrm{id}$ for all $i \in I$. Hence, we have a family of C^*-algebras A_i, $i \in I$, and a family of C^*-arrows ${}_{\mathrm{id}}(A_i)\colon A_i \to A_i$, $i \in I$. For $i \in I$, let $\Phi_{I_i,I_i}\colon {}_{\mathrm{id}}(A_i) \otimes {}_{\mathrm{id}}(A_i) \to {}_{\mathrm{id}}(A_i)$ be defined by

$$\Phi_{I_i,I_i}(a \otimes b) := ab$$

for $a,b \in {}_{\mathrm{id}}(A_i) = A_i$. Then we have $A_J = \bigoplus_{i\in I} A_i$ and

$$\mathcal{F}(F,\Phi) = \bigoplus_{j\in\mathrm{Ob}(J)}^{(o)} \Big(\bigoplus_{r\colon i\to j}^{(i)} F_r \Big) = \bigoplus_{j\in I}^{(o)} {}_{\mathrm{id}}A_j = {}_{\mathrm{id}}\Big(\bigoplus_{j\in I} A_j \Big) = {}_{\mathrm{id}}A_J.$$

For $(a_i) \in A_J$, let $M_{(a_i)} \in \mathcal{K}({}_{\mathrm{id}}(A_J)) = \mathcal{K}(\mathcal{F}(F,\Phi))$ be the multiplication operator $M_{(a_i)}(b_i) := (a_i b_i)$. Then the mapping $(a_i) \mapsto M_{(a_i)}$ is a *-isomorphism from $A_J = \bigoplus_{i\in I} A_i \to \mathcal{K}({}_{\mathrm{id}}(A_J)) = \mathcal{K}(\mathcal{F}(F,\Phi))$. Now for $a \in {}_{\mathrm{id}}(A_i)$, $b \in {}_{\mathrm{id}}(A_j)$, we have

$$\pi_{I_i}(a)(b) = T_a(b) = \delta_{i,j}\Phi_{I_i,I_i}(a\otimes b) = \delta_{i,j}ab \quad \in {}_{\mathrm{id}}(A_i)$$

and so we see that $\pi_{I_i}(a)$ is the multiplication operator M_a, where we view $a \in {}_{\mathrm{id}}(A_i) = A_i \subset A_J$. Since $\pi_i(a) = \pi_{I_i}(a)$ for $a \in A_i = {}_{\mathrm{id}}(A_i)$, we have that $\pi_i(a) = \pi_{I_i}(a) = M_a \in \mathcal{K}(\mathcal{F}(F,\Phi))$ and thus $\mathcal{T}_r(F,\Phi) \subset \mathcal{K}(\mathcal{F}(F,\Phi)) \cong A_J$. But now it is easy to see that the linear span of $\{T_a\colon a \in A_i, i \in I\}$ is dense in $\mathcal{K}(\mathcal{F}(F,\Phi))$ and so we have

$$\mathcal{T}_r(F,\Phi) = \mathcal{K}(\mathcal{F}(F,\Phi)) \cong A_J = \bigoplus_{i\in I} A_i.$$

3.5 Crossed products by groups and semigroups

Let (A,G,α) be a discrete C^*-dynamical system (see Appendix A) and let A be unital. G gives rise to a monoid J with object i and with set of arrows $\{f_s\colon s \in G\}$, such that all arrows are invertible, see Example 2.2. For the sake of convenience we write s instead of f_s. We define a product system (F,Φ) over J by setting $F_s := {}_{\alpha_s}A$ for all $s \in G$

and $\Phi_{s,t}\colon F_sF_t \to F_{st}$, $\Phi_{s,t}(\xi_t \otimes \xi_s) = \alpha_s(\xi_t)\xi_s$. Let $\pi\colon (F,\Phi) \to \mathcal{T}_r(F,\Phi)$ be the reduced Toeplitz representation. We claim that

$$\mathcal{T}_r(F,\Phi) \cong A \rtimes_{\alpha r} G.$$

To prove this, we use the fact, that $A \rtimes_{\alpha r} G$ is isomorphic to the C^*-algebra B described in Appendix A and so we show that $\mathcal{T}_r(F,\Phi) \cong B$. Let Ψ be the universal representation of A on $\mathcal{H}_u$ and let $(\tilde{\Psi}_\rho, \tilde{\rho})$ be the corresponding right regular representation of (A,G,α) on $\ell^2(G,\mathcal{H}_u)$. For $\eta \in \mathcal{H}_u$, we let η_s denote the element of $\ell^2(G,\mathcal{H}_u)$ with $\eta_s(t) = \delta_{s,t}\eta$, $s,t \in G$. Then we have

$$\tilde{\rho}_t(\eta_s) = \eta_{st^{-1}} \qquad \text{and} \qquad \tilde{\Psi}_\rho(a)(\eta_s) = (\Psi(\alpha_s(a))\eta)_s.$$

We also recall from Appendix A that the linear span of $\{\tilde{\Psi}_\rho(a)\tilde{\rho}_s\colon a \in A, s \in G\}$ is dense in B. Now we want to examine $\mathcal{T}_r(F,\Phi)$. We have

$$\mathcal{F}(F,\Phi) = \bigoplus_{j\in \mathrm{Ob}(J)}^{(o)} \Big(\bigoplus_{r\colon i\to j}^{(i)} F_r \Big) = \bigoplus_{r\in G}^{(i)} F_r = \bigoplus_{r\in G}^{(i)} {}_{\alpha_r}A$$

and so $\mathcal{F}(F,\Phi) = \bigoplus_{r\in G}^{(i)} A = \ell^2(G,A)$ as Hilbert A-modules. For $\xi \in F_t$, $\eta \in F_s$ we have

$$\begin{aligned}
\pi_t(\xi)(\eta) &= T_\xi(\eta) = \Phi_{s,t}(\xi\otimes\eta) = \alpha_s(\xi)\eta \in F_{st},\\
\pi_t(\xi)^*(\eta) &= T_\xi^*(\eta) = K_\xi^{st^{-1},t} \circ \Phi^*_{st^{-1},t}(\eta) = K_\xi^{st^{-1},t}(1\otimes\eta)\\
&= (\,\xi \,|\, 1\,)_A \cdot \eta = \alpha_{st^{-1}}(\xi^*)\eta \in F_{st^{-1}} \quad \text{and}\\
\pi_i(a)(\eta) &= a\cdot\eta = \alpha_s(a)\eta \in F_s.
\end{aligned}$$

Now we define $\pi_A := \pi_i$ and $\pi_G(t) := \pi_t(1)^*$ for all $t \in G$. Like above, for $\eta \in A$, we let η_s denote the element of $\ell^2(G,A)$ with $\eta_s(t) = \delta_{s,t}\eta$, $s,t \in G$. Then the above computation yields

$$\pi_G(t)(\eta_s) = \eta_{st^{-1}}, \qquad \pi_G(t)^*(\eta_s) = \eta_{st} \qquad \text{and} \qquad \pi_A(a)(\eta_s) = (\alpha_s(a)\eta)_s$$

for all $t,s \in G$ and $a \in A$. Hence, π_G is a group homomorphism from G to the group of unitary element in $\mathcal{T}_r(F,\Phi)$. Moreover, using the fact that π is a Toeplitz representation, we can show that (π_A,π_G) is a covariant homomorphism:

$$\begin{aligned}
\pi_G(t)\pi_A(a)\pi_G(t)^* &= \pi_t(1)^*\pi_i(a)\pi_t(1) = \pi_t(1)^*\pi_t(a\cdot 1) = \pi_t(1)^*\pi_t(\alpha_t(a))\\
&= \pi_i((\,1\,|\,\alpha_t(a)\,)_A) = \pi_A(\alpha_t(a))
\end{aligned}$$

for $a \in A$ and $t \in G$ arbitrary. To see that $\mathcal{T}_r(F,\Phi) \cong B$, we use the fact that the Hilbert spaces $\ell^2(G,A)\otimes_\Psi \mathcal{H}_u$ and $\ell^2(G,\mathcal{H}_u)$ are unitarily equivalent. The unitary linear mapping $U\colon \ell^2(G,A)\otimes_\Psi \mathcal{H}_u \to \ell^2(G,\mathcal{H}_u)$ is given by $U((a_s)\otimes\xi) := (\Psi(a_s)\xi)$ for $(a_s) \in \ell^2(G,A)$ and $\xi \in \mathcal{H}_u$. Now we define a *-homomorphism $\vartheta\colon \mathcal{L}(\ell^2(G,A)) \to \mathcal{L}(\ell^2(G,\mathcal{H}_u))$ by setting

$$\vartheta(T)(\xi) := U \circ (T\otimes \mathrm{id}) \circ U^*(\xi)$$

for $T \in \mathcal{L}(\ell^2(G, A))$ and $\xi \in \ell^2(G, \mathcal{H}_u)$. ϑ is injective, since Ψ is injective, see [15, p.42], and so it remains to prove that $\vartheta(\mathcal{T}_r(F, \Phi)) = B$. For $\eta \in \mathcal{H}_u$ we compute

$$\begin{aligned}
\vartheta(\pi_G(t))(\eta_s) &= U \circ (\pi_G(t) \otimes \mathrm{id}) \circ U^*((\Psi(1)\eta)_s) = U \circ (\pi_G(t) \otimes \mathrm{id})(1_s \otimes \eta) \\
&= U(1_{st^{-1}} \otimes \eta) = (\Psi(1)\eta)_{st^{-1}} = \eta_{st^{-1}} \qquad \text{and} \\
\vartheta(\pi_A(a))(\eta_s) &= U \circ (\pi_A(a) \otimes \mathrm{id}) \circ U^*((\Psi(1)\eta)_s) = U \circ (\pi_A(a) \otimes \mathrm{id})(1_s \otimes \eta) \\
&= U((\alpha_s(a))_s \otimes \eta) = (\Psi(\alpha_s(a))\eta)_s
\end{aligned}$$

and so we see that $\vartheta(\pi_G(t)) = \tilde{\rho}_t$ and $\vartheta(\pi_A(a)) = \tilde{\Psi}_\rho(a)$. Now $\mathcal{T}_r(F, \Phi)$ is generated by the set $\{\pi_s(a) \colon a \in A, s \in G\}$ and we have $\pi_s(a) = \pi_s(\alpha_s^{-1}(a) \cdot 1) = \pi_i(\alpha_s^{-1}(a))\pi_s(1) = \pi_A(\alpha_s^{-1}(a))\pi_G(s)$. Hence, $\vartheta(\pi_s(a)) = \tilde{\Psi}_\rho(\alpha_s^{-1}(a))\tilde{\rho}_s \in A \rtimes_{\alpha r} G$ and since the linear span of the set $\{\tilde{\Psi}_\rho(a)\tilde{\rho}_s \colon a \in A, s \in G\}$ is dense in B, we get that $\vartheta(\mathcal{T}_r(F, \Phi)) = B$.

The reduced semigroup crossed product can be treated similar to the discussion above. We refer to [10, chapter 3] for a precise description of the universal semigroup crossed product and its connection with the universal Toeplitz and Cuntz-Pimsner algebra, see also section 1.3.

We want to close this section with a special case. We consider the discrete semigroup dynamical system $(\mathbb{C}, \mathcal{S}, \mathrm{id})$, i.e., $\mathcal{S}$ acts by the identity on the complex numbers. Then the corresponding product system (F, Φ) consists of the C^*-algebra $A_i = \mathbb{C}$ and the C^*-arrows $F_r = {}_{\mathrm{id}}\mathbb{C}$, $r \in \mathcal{S}$, over $\mathbb{C}$. Moreover, all of the isometric adjointable $\mathbb{C}$-$\mathbb{C}$-bimodule mappings $\Phi_{s,r}$ are just the canonical mappings from $\mathbb{C} \otimes \mathbb{C}$ to $\mathbb{C}$ given by $z_1 \otimes z_2 \mapsto z_1 z_2$. Then it is clear that $\mathcal{F}(F, \Phi) = \ell^2(\mathcal{S})$. Let e_r be the element of $\ell^2(\mathcal{S})$ with $e_r(s) = \delta_{r,s}$, $r, s \in \mathcal{S}$. Then we have

$$T_{e_r}(e_s) := \Phi_{s,r}(e_r \otimes e_s) = e_{sr}.$$

Moreover, $T_{e_r}^*(e_t) := 0$ if there exists no $s \in \mathcal{S}$ with $t = sr$ and

$$T_{e_r}^*(e_{sr}) := K_{e_r}^{s,r} \circ \Phi_{s,r}^*(e_{sr}) = K_{e_r}^{s,r}(e_r \otimes e_s) = (\, e_r \,|\, e_r \,)_{\mathbb{C}} \cdot e_s = e_s.$$

It is easy to see, that $\pi_i(z) = z \cdot \mathrm{id}$, $z \in \mathbb{C}$, and so $\mathcal{T}_r(F, \Phi) = C^*(\{T_{e_r} \colon r \in \mathcal{S}\}) \subseteq \mathcal{L}(\ell^2(\mathcal{S}))$. In this special case, we denote $\mathcal{T}_r(F, \Phi)$ by $C_r^*(\mathcal{S})$ and hence,

$$C_r^*(\mathcal{S}) := C^*(\{T_{e_r} : r \in \mathcal{S}\}) \subseteq \mathcal{L}(\ell^2(\mathcal{S})).$$

Note that for $\mathcal{S} = \mathbb{N}$ we have that T_{e_1} is the forward unilateral shift operator S on $\ell^2(\mathbb{N})$ and that $T_{e_l} = S^l$ for $l \in \mathbb{N}$. Hence, $C_r^*(\mathbb{N}) = C^*(\{T_{e_l} : l \in \mathbb{N}\}) = C^*(S)$, which is the classical Toeplitz algebra.

3.6 The direct limit

An *upward-directed set* I is a partially ordered set such that for every $i, j \in I$ there is an element $k \in I$ with $i \leq k$ and $j \leq k$. A *direct system of C^*-algebras*, indexed by an upward-directed set I, is a family of C^*-algebras A_i, $i \in I$, together with a family of *-homomorphisms $\varphi_{ij}\colon A_i \to A_j$ for $i \leq j \in I$ such that

$$\varphi_{ii} = \mathrm{id}_{A_i} \quad \text{for all } i \in I \qquad \text{and} \qquad \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

for all $i \leq j \leq k \in I$. The *direct limit* (see also [9, p.34]) of the direct system (A_i, φ_{ij}) is a pair $(\varinjlim A_i, \varphi_i)$ consisting of a C^*-algebra $\varinjlim A_i$ and a system of *-homomorphisms $\varphi_i\colon A_i \to \varinjlim A_i$, $i \in I$, with

$$\varphi_j \circ \varphi_{ij} = \varphi_i \quad \text{for all} \quad i \leq j \in I$$

such that $(\varinjlim A_i, \varphi_i)$ is universal in the sense that if (B, ψ_i) is another pair consisting of a C^*-algebra B and a system of *-homomorphisms $\psi_i\colon A_i \to B$ with $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j \in I$, there exists a unique *-homomorphism $\psi\colon \varinjlim A_i \to B$ with $\psi \circ \varphi_i = \psi_i$ for all $i \in I$. Since $(\varinjlim A_i, \varphi_i)$ is universal, $\varinjlim A_i$ is unique up to isomorphism. Thus we can speak of *the* direct limit $\varinjlim A_i$.

One way to construct $\varinjlim A_i$ is the following: Let A^0 be the *-subalgebra of the product $\prod A_i$ consisting of all $a = (a_i)$ for which there exists an $i_0 \in I$ such that $\varphi_{ij}(a_i) = a_j$ for all $i_0 < i < j \in I$. Then $\|a\| = \lim \|a_i\|$ exists since the φ_{ij} are norm-decreasing. Moreover, $\|\cdot\|$ is a seminorm that satisfies the C^*-identity. Let A^1 be the quotient of A^0 modulo the ideal of elements of norm zero. The direct limit $\varinjlim A_i$ is the completion of A^1. To construct the *-homomorphism φ_i, let $a \in A_i$ be arbitrary. Let $b = (b_j)$ be the element of $\prod A_j$ with $b_j = \varphi_{ij}(a)$ if $j \geq i$ and $b_j = 0$ otherwise. Then $b \in A^0$ and we define $\varphi_i(a)$ to be the image of b under the canonical *-homomorphism from A^0 to $A^1 \subset \varinjlim A_i$. It is easy to see that φ_i becomes a *-homomorphism and that the system $\varphi_i, i \in I$, satisfies the required conditions. The union $\bigcup_{i\in I} \varphi_i(A_i)$ is dense in $\varinjlim A_i$ and hence, given another pair (B, ψ_i) consisting of a C^*-algebra B and a system of *-homomorphisms $\psi_i\colon A_i \to B$ with $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j \in I$, the identity

$$\psi(\varphi_i(a)) := \psi_i(a) \qquad a \in A_i, i \in I$$

uniquely defines a *-homomorphism $\psi\colon \varinjlim A_i \to B$.

For the rest of this section, let (A_i, φ_{ij}) be a fixed direct system of unital C^*-algebras A_i and unital *-homomorphisms φ_{ij} indexed by an upward-directed set I. We set $F_{ij} := {}_{\varphi_{ij}}A_j\colon A_i \to A_j$ for $i \leq j$, i.e., $F_{ij} = A_j$ and for $\xi, \eta \in F_{ij}$, $a_i \in A_i$, $a_j \in A_j$ we have

$$(\,\xi \mid \eta\,)_{A_j} = \xi^*\eta, \quad a_i \cdot \xi = \varphi_{ij}(a_i)\xi \quad \text{and} \quad \xi \cdot a_j = \xi a_j,$$

see Example 2.7. For $i \leq j \leq k \in I$ we define $\Phi_{jk,ij}\colon F_{ij} \otimes F_{jk} \to F_{ik}$ by

$$\Phi_{jk,ij}(\xi \otimes \eta) := \varphi_{jk}(\xi)\eta$$

for $\xi \in F_{ij}$ and $\eta \in F_{jk}$. As in Example 2.16 we get that $\Phi_{jk,ij}$ is a unitary A_i-A_k-bimodule mapping. Since the C^*-algebras A_i are unital, we can even provide the adjoint bimodule mapping $\Phi^*_{jk,ij}\colon F_{ik} \to F_{ij} \otimes F_{jk}$ explicitly:

$$\Phi^*_{jk,ij}(\zeta) = 1 \otimes \zeta,$$

which makes sense, because $\zeta \in F_{ik} = F_{jk} = A_k$. Moreover, it is easy to see that the family $\{\Phi_{jk,ij} \colon i \leq j \leq k \in I\}$ satisfies the coherence condition (2.1) and finally, we define $\Phi_j := \mathrm{id}_{F_{jj}}$ for all $j \in I$. Hence, we get a product system $(F, \Phi) = (A_i, F_{ij}, \Phi_{ij}, \Phi_i)$ over the index category $J = \mathcal{O}_I$, coming from the partial order on I, see Example 2.3. In the following, we want to examine the corresponding reduced Cuntz-Pimsner algebra $\mathcal{O}_r(F, \Phi)$.

We let $F_j := \bigoplus^{(i)}_{i \leq j} F_{ij} = \bigoplus^{(i)}_{i \leq j} A_j$ be the inner direct sum of the family $\{F_{ij} \colon i \leq j\}$ of Hilbert A_j-modules. Then F_j is a Hilbert A_j-module, too, and the definition of the Fock correspondence yields

$$\mathcal{F}(F, \Phi) = \bigoplus^{(o)}_{j \in \mathrm{Ob}(J)} \Big(\bigoplus^{(i)}_{i \leq j} F_{ij} \Big) = \bigoplus^{(o)}_{j \in \mathrm{Ob}(J)} F_j = \bigoplus^{(o)}_{j \in I} F_j .$$

For $a \in A_i$, $\xi \in F_{ij}$, we have $a \cdot \xi = \varphi_{ij}(a)\xi = \theta_{\varphi_{ij}(a),1}\xi$, where we view $\varphi_{ij}(a)$ and ξ as elements of A_j when we write $\varphi_{ij}(a)\xi$ whereas we view $\varphi_{ij}(a)$, 1 and ξ as elements of F_{ij} when we write $\theta_{\varphi_{ij}(a),1}\xi$. Hence, the left action of A_i on F_{ij} is given by the *-homomorphism φ_{ij} and $\varphi_{ij}(a) \in \mathcal{K}(F_{ij})$ for all $a \in A_i$.

For $n \leq m \in I$ we define a mapping $\lambda_{nm} \colon F_n \to F_m$ by setting

$$\lambda_{nm}(\oplus_{i \leq n} \xi_{in}) := \oplus_{i \leq n} \varphi_{nm}(\xi_{in}),$$

i.e., for $\oplus_{j \leq m} \eta_{jm} = \lambda_{nm}(\oplus_{i \leq n} \xi_{in})$ we have $\eta_{jm} = \varphi_{nm}(\xi_{jn})$ if $j \leq n$ and $\eta_{jm} = 0$ otherwise.

Proposition 3.15 *Let $n \leq m \in I$ be fixed. Then $(\lambda_{nm}, \varphi_{nm}) \colon (F_n, A_n) \to (F_m, A_m)$ is a homomorphism of Hilbert C^*-modules.*

Proof: It is clear that λ_{nm} is additive. So let $\xi = \oplus_{i \leq n} \xi_{in}, \eta = \oplus_{i \leq n} \eta_{in} \in F_n$ and $a \in A_n$ be arbitrary. Then we compute

$$\begin{aligned} \lambda_{nm}(\xi \cdot a) &= \lambda_{nm}(\oplus_{i \leq n}(\xi_{in} a)) = \oplus_{i \leq n} \varphi_{nm}(\xi_{in} a) = \oplus_{i \leq n}(\varphi_{nm}(\xi_{in}) \varphi_{nm}(a)) \\ &= (\oplus_{i \leq n} \varphi_{nm}(\xi_{in})) \cdot \varphi_{nm}(a) = \lambda_{nm}(\xi) \cdot \varphi_{nm}(a) \quad \text{and} \end{aligned}$$

$$\begin{aligned} (\, \lambda_{nm}(\xi) \,|\, \lambda_{nm}(\eta) \,)_{A_m} &= (\, \lambda_{nm}(\oplus_{i \leq n} \xi_{in}) \,|\, \lambda_{nm}(\oplus_{i \leq n} \eta_{in}) \,)_{A_m} \\ &= (\, \oplus_{i \leq n} \varphi_{nm}(\xi_{in}) \,|\, \oplus_{i \leq n} \varphi_{nm}(\eta_{in}) \,)_{A_m} = \sum_{i \leq n} \varphi_{nm}(\xi_{in})^* \varphi_{nm}(\eta_{in}) \\ &= \varphi_{nm}(\sum_{i \leq n} \xi_{in}^* \eta_{in}) = \varphi_{nm}((\, \oplus_{i \leq n} \xi_{in} \,|\, \oplus_{i \leq n} \eta_{in} \,)_{A_n}) \\ &= \varphi_{nm}((\, \xi \,|\, \eta \,)_{A_n}). \end{aligned}$$

□

It is clear from the definition that $\lambda_{nn} = \mathrm{id}_{F_n}$ for all $n \in I$ and that $\lambda_{ml} \circ \lambda_{nm} = \lambda_{nl}$ for all $n \leq m \leq l \in I$. Hence, the system $((F_n, A_n), (\lambda_{nm}, \varphi_{nm}))$ is a *direct system of Hilbert C^*-modules* indexed by the directed set I. The *direct limit* of the direct system $((F_n, A_n), (\lambda_{nm}, \varphi_{nm}))$ is a pair $((F, A), (\lambda_n, \varphi_n))$ consisting of a Hilbert A-module F together with a system of homomorphisms $(\lambda_n, \varphi_n)\colon (F_n, A_n) \to (F, A)$ with

$$(\lambda_m, \varphi_m) \circ (\lambda_{nm}, \varphi_{nm}) = (\lambda_n, \varphi_n) \qquad \text{for all } n \leq m \in I,$$

such that $((F, A), (\lambda_n, \varphi_n))$ is universal in the sense that if $((E, B), (\mu_n, \psi_n))$ is another pair consisting of a Hilbert B-module E and a system of homomorphisms $(\mu_n, \psi_n)\colon (F_n, A_n) \to (E, B)$ satisfying $(\mu_m, \psi_m) \circ (\lambda_{nm}, \varphi_{nm}) = (\mu_n, \psi_n)$ for all $n \leq m \in I$, there is a uniquely determined homomorphism $(\mu, \psi)\colon (F, A) \to (E, B)$ satisfying

$$(\mu, \psi) \circ (\lambda_n, \varphi_n) = (\mu_n, \psi_n)$$

for all $n \in I$. Since $((F, A), (\lambda_n, \varphi_n))$ is universal, (F, A) is unique up to isomorphism and we can speak of *the* direct limit $\varinjlim(F_n, A_n) := (F, A)$. It remains to prove that the direct limit $\varinjlim(F_n, A_n)$ exists.

Proposition 3.16 *Let $((F_n, A_n), (\lambda_{nm}, \varphi_{nm}))$ be a direct system of Hilbert C^*-modules over the upward-directed set I. Then the Banach space direct limit $(\varinjlim F_n, \lambda_n)$ can be turned into a Hilbert C^*-module over $\varinjlim A_n$, where $(\varinjlim A_n, \varphi_n)$ is the direct limit of the C^*-algebra direct system (A_n, φ_{nm}), such that $((\varinjlim F_n, \varinjlim A_n), (\lambda_n, \varphi_n))$ is a direct limit of the direct system $((F_n, A_n), (\lambda_{nm}, \varphi_{nm}))$ of Hilbert C^*-modules.*

Before we start to prove the proposition, we recall some facts about the Banach space direct limit. First of all, we notice that every Hilbert C^*-module is also a Banach space. For the mappings $\lambda_{nm}\colon F_n \to F_m$ we have

$$\|\lambda_{nm}(\xi)\|^2 = \|(\, \lambda_{nm}(\xi) \,|\, \lambda_{nm}(\xi) \,)_{A_m}\| = \|\varphi_{nm}((\, \xi \,|\, \xi \,)_{A_n})\| \leq \|(\, \xi \,|\, \xi \,)_{A_n}\| = \|\xi\|^2$$

and thus λ_{nm} is a contraction. Hence, we have a system of Banach spaces F_n, $n \in I$, together with a system of contractions λ_{nm} for $n \leq m \in I$ such that $\lambda_{nn} = \mathrm{id}_{F_n}$ and that $\lambda_{ml} \circ \lambda_{nm} = \lambda_{nl}$ for all $n \leq m \leq l \in I$. We call this system a *direct system of Banach spaces* indexed by I.

The *direct limit* of the direct system (F_n, λ_{nm}) is a pair $(\varinjlim F_n, \lambda_n)$ consisting of a Banach space $\varinjlim F_n$ and a system of contractions $\lambda_n\colon F_n \to \varinjlim F_n$, $n \in I$, with

$$\lambda_m \circ \lambda_{nm} = \lambda_n \quad \text{for all} \quad n \leq m \in I$$

such that $(\varinjlim F_n, \lambda_n)$ is universal in the sense that if (E, μ_n) is another pair consisting of a Banach space E and a system of contractions $\mu_n\colon F_n \to E$ with $\mu_m \circ \lambda_{nm} = \mu_n$ for all $n \leq m \in I$, there exists a unique contraction $\mu\colon \varinjlim F_n \to E$ with $\mu \circ \lambda_n = \mu_n$ for all

$n \in I$. Since $(\varinjlim F_n, \lambda_n)$ is universal, $\varinjlim F_n$ is unique up to isomorphism. Thus, we can speak of *the* direct limit $\varinjlim F_n$. It is possible to construct the direct limit $\varinjlim F_n$ in a way similar to the construction of the direct limit of a direct system of C^*-algebras.

Proof of Proposition 3.16: Since (F_n, λ_{nm}) is a direct system of Banach spaces, we also have that $(F_n \times F_n, \lambda_{nm} \times \lambda_{nm})$ is a direct system of Banach spaces. It is clear that $(\varinjlim F_n \times \varinjlim F_n, \lambda_n \times \lambda_n)$ is a direct limit for this direct system. Now we define mappings $\sigma_n \colon F_n \times F_n \to \varinjlim A_n$ by setting

$$\sigma_n(\xi, \eta) := \varphi_n((\,\xi \,|\, \eta\,)_{A_n})$$

for $\xi, \eta \in F_n$. Then for $n \leq m$ we have $\sigma_m \circ (\lambda_{nm} \times \lambda_{nm})(\xi, \eta) = \varphi_m((\,\lambda_{nm}(\xi) \,|\, \lambda_{nm}(\eta)\,)_{A_m}) = \varphi_m(\varphi_{nm}((\,\xi \,|\, \eta\,)_{A_n})) = \varphi_n((\,\xi \,|\, \eta\,)_{A_n}) = \sigma_n(\xi, \eta)$. Since $\varinjlim F_n \times \varinjlim F_n$ is universal, there exists a uniquely defined mapping $(\,.\,|\,.\,)_{\varinjlim A_n} \colon \varinjlim F_n \times \varinjlim F_n \to \varinjlim A_n$ with

$$(\,\lambda_n(\xi) \,|\, \lambda_n(\eta)\,)_{\varinjlim A_n} = \sigma_n(\xi, \eta) = \varphi_n((\,\xi \,|\, \eta\,)_{A_n})$$

for $\xi, \eta \in F_n$. In a similar way one can show that there is a uniquely defined multiplication $\cdot \colon \varinjlim F_n \times \varinjlim A_n \to \varinjlim F_n$ that satisfies

$$\lambda_n(\xi) \cdot \varphi_n(a) = \lambda_n(\xi \cdot a)$$

for $\xi \in F_n$, $a \in A_n$. It is easy to see that the inner product and multiplication make $\varinjlim F_n$ a Hilbert C^*-module over $\varinjlim A_n$ and the above identities show that $(\lambda_n, \varphi_n) \colon (F_n, A_n) \to (\varinjlim F_n, \varinjlim A_n)$ is a homomorphism of Hilbert C^*-modules. Since it is clear that the system $\{(\lambda_n, \varphi_n) \colon n \in I\}$ is compatible with $\{(\lambda_{nm}, \varphi_{nm}) \colon n \leq m \in I\}$, it only remains to prove that $((\varinjlim F_n, \varinjlim A_n), (\lambda_n, \varphi_n))$ is universal. To see this, let $((E, B), (\mu_n, \psi_n))$ be another pair consisting of a Hilbert B-module E and a system of homomorphisms $(\mu_n, \psi_n) \colon (F_n, A_n) \to (E, B)$ satisfying $(\mu_m, \psi_m) \circ (\lambda_{nm}, \varphi_{nm}) = (\mu_n, \psi_n)$ for all $n \leq m \in I$. Then, in particular, we have $\psi_m \circ \varphi_{nm} = \psi_n$ for the system of *-homomorphisms ψ_n, $n \in I$, and $\mu_m \circ \lambda_{nm} = \mu_n$ for the system of contractions μ_n, $n \in I$. Since $\varinjlim A_n$ and $\varinjlim F_n$ are universal, there is a uniquely defined *-homomorphism $\psi \colon \varinjlim A_n \to B$ as well as a uniquely defined contraction $\mu \colon \varinjlim F_n \to E$ satisfying

$$\psi(\varphi_n(a)) = \psi_n(a) \qquad \text{and} \qquad \mu(\lambda_n(\xi)) = \mu_n(\xi)$$

for all $n \in I$ and $a \in A_n$, $\xi \in F_n$. Now we have $\mu(\lambda_n(\xi) \cdot \varphi_n(a)) = \mu(\lambda_n(\xi \cdot a)) = \mu_n(\xi \cdot a) = \mu_n(\xi) \cdot \psi_n(a) = \mu(\lambda_n(\xi)) \cdot \psi(\varphi_n(a))$, since (λ_n, φ_n) and (μ_n, ψ_n) are homomorphisms. For the same reason we get that $(\,\mu(\lambda_n(\xi)) \,|\, \mu(\lambda_n(\eta))\,)_B = (\,\mu_n(\xi) \,|\, \mu_n(\eta)\,)_B = \psi_n((\,\xi \,|\, \eta\,)_{A_n}) = \psi(\varphi_n((\,\xi \,|\, \eta\,)_{A_n})) = \psi((\,\lambda_n(\xi) \,|\, \lambda_n(\eta)\,)_{\varinjlim A_n})$. Hence, we have shown that (μ, ψ) is a uniquely defined homomorphism from $(\varinjlim F_n, \varinjlim A_n)$ to (E, B), which shows that $(\varinjlim F_n, \varinjlim A_n)$ is universal. □

Proposition 3.17 $\varinjlim \mathcal{K}(F_n) \cong \mathcal{K}(\varinjlim F_n)$

Proof: Let $l \leq m \in I$ be arbitrary. Then it follows from Proposition 3.15 that $(\lambda_{lm}, \varphi_{lm})\colon (F_l, A_l) \to (F_m, A_m)$ is a homomorphism of Hilbert C^*-modules. Hence, by Lemma 3.6 it follows that there is a *-homomorphism $\tau_{lm}\colon \mathcal{K}(F_l) \to \mathcal{K}(F_m)$ that satisfies $\tau_{lm}(\theta_{\xi,\eta}) = \theta_{\lambda_{lm}(\xi),\lambda_{lm}(\eta)}$ for all $\xi, \eta \in F_l$. Now for $k \leq l \leq m \in I$ and $\xi, \eta \in F_k$ arbitrary we have

$$\tau_{lm} \circ \tau_{kl}(\theta_{\xi,\eta}) = \theta_{\lambda_{lm}\circ\lambda_{kl}(\xi),\lambda_{lm}\circ\lambda_{kl}(\eta)} = \theta_{\lambda_{km}(\xi),\lambda_{km}(\eta)} = \tau_{km}(\theta_{\xi,\eta}),$$

which shows that $(\mathcal{K}(F_m), \tau_{lm})$ is a direct system of C^*-algebras indexed by I. Let $(\varinjlim \mathcal{K}(F_n), \tau_n)$ be the direct limit of this system.

For $l \in I$ arbitrary, we know from Proposition 3.16 that $(\lambda_l, \varphi_l)\colon (F_l, A_l) \to (\varinjlim F_n, \varinjlim A_n)$ is a homomorphism of Hilbert C^*-modules. Hence, there is a unique *-homomorphism $\omega_l\colon \mathcal{K}(F_l) \to \mathcal{K}(\varinjlim F_n)$ that satisfies $\omega_l(\theta_{\xi,\eta}) = \theta_{\lambda_l(\xi),\lambda_l(\eta)}$ for all $\xi, \eta \in F_l$. For $l \leq m \in I$ and $\xi, \eta \in F_l$ arbitrary we have

$$\omega_m \circ \tau_{lm}(\theta_{\xi,\eta}) = \theta_{\lambda_m\circ\lambda_{lm}(\xi),\lambda_m\circ\lambda_{lm}(\eta)} = \theta_{\lambda_l(\xi),\lambda_l(\eta)} = \omega_l(\theta_{\xi,\eta}).$$

Hence, there exists a uniquely defined *-homomorphism $\omega\colon \varinjlim \mathcal{K}(F_n) \to \mathcal{K}(\varinjlim F_n)$ with $\omega \circ \tau_n = \omega_n$ for all $n \in I$, since $(\varinjlim \mathcal{K}(F_n), \tau_n)$ is universal. To see that ω is isometric, we let $\xi_i, \eta_i \in F_n$ be arbitrary and compute for $K = \tau_n\Big(\sum_{i=1}^k \theta_{\xi_i,\eta_i}\Big) \in \varinjlim \mathcal{K}(F_n)$

$$\|K\| = \Big\|\tau_n\Big(\sum_{i=1}^k \theta_{\xi_i,\eta_i}\Big)\Big\| = \lim_{m\to\infty} \Big\|\tau_{nm}\Big(\sum_{i=1}^k \theta_{\xi_i,\eta_i}\Big)\Big\| = \lim_{m\to\infty} \Big\|\sum_{i=1}^k \theta_{\lambda_{nm}(\xi_i),\lambda_{nm}(\eta_i)}\Big\| \quad \text{and}$$

$$\begin{aligned}
\|\omega(K)\| &= \Big\|\omega\Big(\tau_n\Big(\sum_{i=1}^k \theta_{\xi_i,\eta_i}\Big)\Big)\Big\| = \Big\|\sum_{i=1}^k \omega_n(\theta_{\xi_i,\eta_i})\Big\| = \Big\|\sum_{i=1}^k \theta_{\lambda_n(\xi_i),\lambda_n(\eta_i)}\Big\| \\
&= \Big\|\big((\lambda_n(\xi_i) \mid \lambda_n(\xi_j))_{\varinjlim A_n}\big)_{i,j}^{1/2} \big((\lambda_n(\eta_i) \mid \lambda_n(\eta_j))_{\varinjlim A_n}\big)_{i,j}^{1/2}\Big\| \\
&= \lim_{m\to\infty} \Big\|\big((\lambda_{nm}(\xi_i) \mid \lambda_{nm}(\xi_j))_{A_m}\big)_{i,j}^{1/2} \big((\lambda_{nm}(\eta_i) \mid \lambda_{nm}(\eta_j))_{A_m}\big)_{i,j}^{1/2}\Big\| \\
&= \lim_{m\to\infty} \Big\|\sum_{i=1}^k \theta_{\lambda_{nm}(\xi_i),\lambda_{nm}(\eta_i)}\Big\|,
\end{aligned}$$

where we use that $M_k(\varinjlim A_n) \cong \varinjlim M_k(A_n)$. Since the union $\bigcup_{n\in I} \tau_n(\mathcal{K}(F_n))$ is dense in $\varinjlim \mathcal{K}(F_n)$, this shows that ω is an isometry and hence injective. To see that ω is surjective it suffices to show that elements of the form $\theta_{\lambda_n(\xi),\lambda_n(\eta)}$, $\xi, \eta \in F_n$, are in the image of ω, since $\bigcup_{n\in I} \lambda_n(F_n)$ is dense in $\varinjlim F_n$. But this is obvious, since

$$\theta_{\lambda_n(\xi),\lambda_n(\eta)} = \omega_n(\theta_{\xi,\eta}) = \omega(\tau_n(\theta_{\xi,\eta})).$$

□

Proposition 3.18 $\mathcal{O}_r(F,\Phi) \cong \varinjlim \mathcal{K}(F_n)$

Proof: We will work with the concrete realization of the direct limit $\varinjlim \mathcal{K}(F_n)$ described above. So let K^0 be the *-subalgebra of the product $\prod_{n\in I}\mathcal{K}(F_n)$ consisting of all $k=(k_n)$ for which there exists an $n_0\in I$ such that $\tau_{nm}(k_n)=k_m$ for all $n_0<n<m\in I$. Let K^1 be the quotient of K^0 modulo $\bigoplus_{n\in I}\mathcal{K}(F_n)$. The direct limit $\varinjlim \mathcal{K}(F_n)$ is the completion of K^1. To construct the *-homomorphisms $\tau_n\colon \mathcal{K}(F_n)\to \varinjlim \mathcal{K}(F_n)$, let $k\in\mathcal{K}(F_n)$ be arbitrary. Let $\kappa=(\kappa_m)$ be the element of $\prod \mathcal{K}(F_n)$ with $\kappa_m=\tau_{nm}(k)$ if $m\ge n$ and $\kappa_m=0$ otherwise. Then $\kappa\in K^0$ and we define $\tau_n(k)$ to be the image of κ under the canonical quotient mapping p from K^0 to $K^1\subset \varinjlim \mathcal{K}(F_n)$.

Now let $\vartheta\colon \mathcal{L}(\mathcal{F}(F,\Phi))=\mathcal{L}(\bigoplus^{(o)}_{n\in I}F_n)\to\prod_{n\in I}\mathcal{L}(F_n)$ be the mapping defined in the proof of Proposition 3.10 by setting

$$\vartheta(T):=(T|_{F_n})_{n\in I},$$

where $T|_{F_n}$ is the restriction of T to F_n. Since $\mathcal{T}_r(F,\Phi)$ is generated by the generalized shift operators $T_{\xi_{ij}}$, $\xi_{ij}\in F_{ij}$, we want to compute $\vartheta(T_{\xi_{ij}})$. So let $\xi_{ij}\in F_{ij}$ and $(\eta_{kn})_{k\le n}\in F_n$ be arbitrary. Then for $j\le n$ we have

$$T_{\xi_{ij}}(\eta_{kn})_{k\le n}=T_{\xi_{ij}}\eta_{jn}=\Phi_{jn,ij}(\xi_{ij}\otimes\eta_{jn})=\varphi_{jn}(\xi_{ij})\eta_{jn}\in F_{in}.$$

This yields that $T_{\xi_{ij}}|_{F_n}=\theta_{\varphi_{jn}(\xi_{ij}),1}\in\mathcal{K}(F_n)$, with $\varphi_{jn}(\xi_{in})\in F_{in}$ and $1\in F_{jn}$ if $j\le n$ and $T_{\xi_{ij}}|_{F_n}=0$ otherwise. Hence, $\vartheta(T_{\xi_{ij}})\in\prod_{n\in I}\mathcal{K}(F_n)$. Moreover, for all $j\le n\le m\in I$ we have

$$\begin{aligned}\tau_{nm}(T_{\xi_{ij}}|_{F_n}) &= \tau_{nm}(\theta_{\varphi_{jn}(\xi_{ij}),1})=\theta_{\lambda_{nm}(\varphi_{jn}(\xi_{ij})),\lambda_{nm}(1)}=\theta_{\varphi_{nm}(\varphi_{jn}(\xi_{ij})),\varphi_{nm}(1)}\\ &= \theta_{\varphi_{jm}(\xi_{ij}),1}=T_{\xi_{ij}}|_{F_m}\end{aligned}$$

and thus $\vartheta(T_{\xi_{ij}})\in K^0$. Hence, ϑ restricts to a *-homomorphism from $\mathcal{T}_r(F,\Phi)$ to K^0. Let $q\colon\mathcal{T}_r(F,\Phi)\to\mathcal{O}_r(F,\Phi)$ be the canonical quotient mapping. We define a *-homomorphism $\tilde\vartheta\colon\mathcal{O}_r(F,\Phi)\to\varinjlim\mathcal{K}(F_n)$ by setting

$$\tilde\vartheta(q(T)):=p(\vartheta(T))$$

for all $T\in\mathcal{T}_r(F,\Phi)$. To see that $\tilde\vartheta$ is well-defined, let $T\in\mathcal{T}_r(F,\Phi)$ and $K\in\mathcal{K}(\mathcal{F}(F,\Phi))\cap\mathcal{T}_r(F,\Phi)$ be arbitrary. Then $q(T+K)=q(T)$ and so we have to show that $p(\vartheta(T+K))=p(\vartheta(T))$. But $p(\vartheta(T+K))=p(\vartheta(T))+p(\vartheta(K))=p(\vartheta(T))$, since $\vartheta(K)\in\bigoplus_{n\in I}\mathcal{K}(F_n)$ by Proposition 3.10. Next, we show that $\tilde\vartheta$ is injective by computing its kernel:

$$\begin{aligned}\ker(\tilde\vartheta) &= \{q(T)\in\mathcal{O}_r(F,\Phi)\colon \tilde\vartheta(q(T))=0\}=\{q(T)\in\mathcal{O}_r(F,\Phi)\colon p(\vartheta(T))=0\}\\ &= \{q(T)\in\mathcal{O}_r(F,\Phi)\colon \vartheta(T)\in\bigoplus_{n\in I}\mathcal{K}(F_n)\}\\ &= \{q(T)\in\mathcal{O}_r(F,\Phi)\colon T\in\mathcal{K}(\mathcal{F}(F,\Phi))\}=0.\end{aligned}$$

To see that $\tilde{\vartheta}$ is surjective, it suffices for arbitrary $(\xi_{in})_{i\leq n}, (\eta_{in})_{i\leq n} \in F_n$ to find an operator $T \in \mathcal{T}_r(F,\Phi)$ with $T|_{F_n} = \theta_{(\xi_{in}),(\eta_{in})}$. We claim that

$$\Big(\sum_{i,j\leq n} T_{\xi_{in}} T^*_{\eta_{jn}}\Big)\Big|_{F_n} = \theta_{(\xi_{in}),(\eta_{in})}.$$

So let $(\zeta_{in})_{i\leq n} \in F_n$ be arbitrary. Then we have

$$\begin{aligned}
\theta_{(\xi_{in})_{i\leq n},(\eta_{jn})_{j\leq n}}(\zeta_{kn})_{k\leq n} &= (\xi_{in})_{i\leq n} \cdot (\, (\eta_{jn})_{j\leq n} \,|\, (\zeta_{kn})_{k\leq n} \,)_{A_n} = (\xi_{in} \cdot \sum_{j\leq n} (\, \eta_{jn} \,|\, \zeta_{jn} \,)_{A_n})_{i\leq n} \\
&= \Big(\sum_{j\leq n} \xi_{in}\eta^*_{jn}\zeta_{jn}\Big)_{i\leq n} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\Big(\sum_{i,j\leq n} T_{\xi_{in}} T^*_{\eta_{jn}}\Big)(\zeta_{kn})_{k\leq n} &= \sum_{i,j\leq n} T_{\xi_{in}} T^*_{\eta_{jn}} \zeta_{jn} = \sum_{i,j\leq n} T_{\xi_{in}}((\, \eta_{jn} \,|\, 1 \,)_{A_n} \cdot \zeta_{jn}) \\
&= \sum_{i,j\leq n} T_{\xi_{in}}(\varphi_{nn}(\eta^*_{jn})\zeta_{jn}) = \Big(\sum_{j\leq n} \varphi_{nn}(\xi_{in})\eta^*_{jn}\zeta_{jn}\Big)_{i\leq n} \\
&= \Big(\sum_{j\leq n} \xi_{in}\eta^*_{jn}\zeta_{jn}\Big)_{i\leq n}
\end{aligned}$$

and thus $\tilde{\vartheta}$ is a *-isomorphism from $\mathcal{O}_r(F,\Phi)$ onto $\varinjlim \mathcal{K}(F_n)$. □

Theorem 3.19 *$\mathcal{O}_r(F,\Phi)$ and $\varinjlim A_n$ are strongly Morita equivalent.*

Proof: From Propositions 3.17 and 3.18 we know that the C^*-algebras $\mathcal{O}_r(F,\Phi)$ and $\mathcal{K}(\varinjlim F_n)$ are isomorphic. Moreover, it is easy to see that $(\, \varinjlim F_n \,|\, \varinjlim F_n \,)_{\varinjlim A_n}$ is dense in $\varinjlim A_n$. Hence, it is clear (see Example 2.9) that $\varinjlim F_n$ is an $\mathcal{O}_r(F,\Phi)$-$\varinjlim A_n$-equivalence bimodule. □

Chapter 4

The universal Toeplitz and Cuntz-Pimsner algebras

In the present chapter we introduce the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$ and the universal Cuntz-Pimsner algebra $\mathcal{O}(F,\Phi)$ together with their corresponding Toeplitz representations. First, we show that given a product system (F,Φ) over an index category J, there is a C^*-algebra that is universal for Toeplitz representations over (F,Φ). This C^*-algebra will be defined by a universal condition and consequently, it will be unique up to isomorphism and we will call it the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$.

Secondly, we recall the notion of Cuntz-Pimsner covariant Toeplitz representations from [10] and introduce the universal Cuntz-Pimsner algebra, which is universal for Cuntz-Pimsner covariant Toeplitz representations over (F,Φ).

Finally, we provide some facts about bicategorial colimits that allow us to reveal the bicategorial structure behind the universal Toeplitz algebra. Given a morphism (F,Φ) from a bicategory $\mathcal{B}$ to a bicategory $\mathcal{B}'$, we recall the notion of the colimit for (F,Φ) and we show that in the bicategory C*ARR, the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$ can be viewed as the colimit object for the product system (F,Φ).

4.1 The universal Toeplitz algebra

Proposition 4.1 *For every product system (F,Φ) over J, there is a C^*-algebra $\mathcal{T}(F,\Phi)$ and a Toeplitz representation $i\colon (F,\Phi)\to\mathcal{T}(F,\Phi)$ such that every Toeplitz representation $\pi\colon (F,\Phi)\to B$ factors uniquely through i, i.e., there is a unique *-homomorphism $\tilde{\pi}\colon \mathcal{T}(F,\Phi)\to B$, such that $\pi_r=\tilde{\pi}\circ i_r$ and $\pi_k=\tilde{\pi}\circ i_k$ for all $r\in\operatorname{Arr}(J)$, $k\in\operatorname{Ob}(J)$.*

Proof: Let $\mathcal{P}$ be the universal *-algebra generated by symbols

$$\{t_{\xi_r}\colon \xi_r\in F_r,\ r\in\operatorname{Arr}(J)\}\cup\{t_{a_i}\colon a_i\in A_i,\ i\in\operatorname{Ob}(J)\}$$

and relations such that for all objects i of J the mapping $A_i \ni a_i \mapsto t_{a_i}$ becomes a *-homomorphism and that the identities

$$\begin{array}{rclrclrcl} t_{\xi_r+\eta_r} &=& t_{\xi_r}+t_{\eta_r}, & t_{a_i\cdot\xi_r\cdot a_j} &=& t_{a_i}t_{\xi_r}t_{a_j}, & t^*_{\xi_r}t_{\eta_r} &=& t_{(\,\xi_r\,|\,\eta_r\,)_{A_j}} \\ t_{\Phi_{s,r}(\xi_r\otimes\xi_s)} &=& t_{\xi_r}t_{\xi_s}, & t_{\Phi_i(a_i)} &=& t_{a_i}, & t_{a_i}t_{a_j} &=& 0 \text{ for } i\neq j \end{array}$$

hold, where $\xi_r, \eta_r \in F_r$, $\xi_s \in F_s$, $a_i \in A_i$, $a_j \in A_j$, $i, j, k \in \mathrm{Ob}(J)$, $r\colon i \to j$, $s\colon j \to k$. We now define a maximal C^*-seminorm on $\mathcal{P}$ by setting

$$\|x\| := \sup\{\nu(x)\colon \nu \text{ is a } C^*\text{-seminorm on } \mathcal{P}\}, \quad x \in \mathcal{P}.$$

To make sure that this supremum is actually bounded, we let ν be an arbitrary C^*-seminorm on $\mathcal{P}$, $i, j \in \mathrm{Ob}(J)$ and $r\colon i \to j$ be arbitrary. Then $a_i \mapsto \nu(t_{a_i})$ defines a C^*-seminorm on A_i and hence, $\nu(t_{a_i}) \le \|a_i\|_{A_i}$, where $\|.\|_{A_i}$ denotes the C^*-norm on A_i. Moreover, for an arbitrary $\xi_r \in F_r$ we have

$$\nu(t_{\xi_r})^2 = \nu(t^*_{\xi_r}t_{\xi_r}) = \nu(t_{(\,\xi_r\,|\,\xi_r\,)_{A_j}}) \le \|(\,\xi_r\,|\,\xi_r\,)_{A_j}\|_{A_j} = \|\xi_r\|^2_{F_r}.$$

Since C^*-seminorms are submultiplicative, it follows that $\|.\|$ is bounded on every element of $\mathcal{P}$. By separating and completing with respect to $\|.\|$ we get a C^*-algebra which we denote by $\mathcal{T}(F,\Phi)$. We write $i_r(\xi_r)$, $\xi_r \in F_r$, and $i_j(a_j)$, $a_j \in A_j$, for the images of t_{ξ_r} and t_{a_j} in $\mathcal{T}(F,\Phi)$. Because of the relations that hold in $\mathcal{P}$ and hence in $\mathcal{T}(F,\Phi)$, it is clear that $i : (F,\Phi) \to \mathcal{T}(F,\Phi)$ becomes a Toeplitz representation.

Now we suppose that $\pi\colon (F,\Phi) \to B$ is a Toeplitz representation. By $\theta(t_{\xi_r}) := \pi_r(\xi_r)$, $\xi_r \in F_r$, $r \in \mathrm{Arr}(J)$ and $\theta(t_{a_i}) := \pi_i(a_i)$, $a_i \in A_i$, $i \in \mathrm{Ob}(J)$, we define a mapping θ that extends to a *-homomorphism from $\mathcal{P}$ to B. By setting $\mu(x) := \|\theta(x)\|$, we get a C^*-seminorm on $\mathcal{P}$ and thus θ is a contraction, since

$$\|\theta(x)\| = \mu(x) \le \sup\{\nu(x) : \nu \text{ is a } C^*\text{-seminorm on } \mathcal{P}\} = \|x\|.$$

Hence, θ extends to a C^*-homomorphism $\tilde{\pi}\colon \mathcal{T}(F,\Phi) \to B$. It is obvious that $\tilde{\pi}(i_r(\xi_r)) = \theta(t_{\xi_r}) = \pi_r(\xi_r)$ and $\tilde{\pi}(i_j(a_j)) = \theta(t_{a_j}) = \pi_j(a_j)$ for all $\xi_r \in F_r$, $r \in \mathrm{Arr}(J)$, and $a_j \in A_j$, $j \in \mathrm{Ob}(J)$. Finally, $\tilde{\pi}$ is unique, since θ is determined by its values on the generating symbols of $\mathcal{P}$. □

Definition 4.2 (The universal Toeplitz algebra)
Since $\mathcal{T}(F,\Phi)$ is defined by a universal condition, it is unique up to isomorphism. Thus we call $\mathcal{T}(F,\Phi)$ the *universal Toeplitz algebra* and we say that i is the *universal Toeplitz representation* from (F,Φ) into $\mathcal{T}(F,\Phi)$.

Remark: ${}_{i_j}\mathcal{T}(F,\Phi) = \overline{i_j(A_j)\mathcal{T}(F,\Phi)}$, $j \in \mathrm{Ob}(J)$, is a family of Hilbert C^*-submodules of the Hilbert $\mathcal{T}(F,\Phi)$-module ${}_{\mathrm{id}}\mathcal{T}(F,\Phi)$. We have that $i_j(a_j) \in {}_{i_j}\mathcal{T}(F,\Phi)$ for $a_j \in A_j$, as well as $i_r(\xi_r) \in {}_{i_j}\mathcal{T}(F,\Phi)$ for $\xi_r \in F_r$, if $r\colon j \to k \in \mathrm{Arr}(J)$. Moreover, for $j \neq k$ we have ${}_{i_j}\mathcal{T}(F,\Phi) \perp {}_{i_k}\mathcal{T}(F,\Phi)$, because for $a_j \in A_j$, $a_k \in A_k$ and $\xi, \eta \in \mathcal{T}(F,\Phi)$ we compute $(\, i_j(a_j)\xi \,|\, i_k(a_k)\eta \,)_{\mathcal{T}(F,\Phi)} = \xi^* i_j(a_j^*) i_k(a_k)\eta = 0$, since $i_j(a_j^*) i_k(a_k) = 0$. Hence, ${}_{\mathrm{id}}\mathcal{T}(F,\Phi)$ is the inner direct sum of the family $\{{}_{i_j}\mathcal{T}(F,\Phi) \colon j \in \mathrm{Ob}(J)\}$,

$$ {}_{\mathrm{id}}\mathcal{T}(F,\Phi) = \bigoplus_{j\in\mathrm{Ob}(J)}^{(i)} {}_{i_j}\mathcal{T}(F,\Phi). $$

4.2 The universal Cuntz-Pimsner algebra

In [10], Fowler introduced the notion of Cuntz-Pimsner covariant Toeplitz representations over product systems that are indexed by a semigroup. We want to carry this notion over to our situation to introduce the universal Cuntz-Pimsner algebra $\mathcal{O}(F,\Phi)$ that will be universal for Cuntz-Pimsner covariant Toeplitz representations over a given product system (F,Φ).

Let $\pi\colon (F,\Phi) \to B$ be a Toeplitz representation. Then for every $r\colon i \to j \in \mathrm{Arr}(J)$, $(\pi_r, \pi_j)\colon (F_r, A_j) \to B$ is a homomorphism of Hilbert C^*-modules. Hence, Corollary 3.7 implies that there is a *-homomorphism $\tilde{\pi}_r\colon \mathcal{K}(F_r) \to B$, which satisfies

$$ \tilde{\pi}_r(\theta_{\xi,\eta}) = \pi_r(\xi)\pi_r(\eta)^* \qquad \text{for all } \xi,\eta \in F_r. $$

We say that the Toeplitz representation $\pi\colon (F,\Phi) \to B$ is *Cuntz-Pimsner covariant* if

$$ \tilde{\pi}_r(\lambda_r(a)) = \pi_i(a) $$

for all $a \in \lambda_r^{-1}(\mathcal{K}(F_r))$, $r\colon i \to j \in \mathrm{Arr}(J)$, where $\lambda_r\colon A_i \to \mathcal{L}(F_r)$ is the *-homomorphism that defines the A_i-left multiplication on F_r.

Proposition 4.3 *For every product system (F,Φ) over J, there is a C^*-algebra $\mathcal{O}(F,\Phi)$ and a Toeplitz representation $j\colon (F,\Phi) \to \mathcal{O}(F,\Phi)$, which is Cuntz-Pimsner covariant, such that every Cuntz-Pimsner covariant Toeplitz representation $\pi\colon (F,\Phi) \to B$ factors uniquely through j, i.e., there is a unique *-homomorphism $\tilde{\pi}\colon \mathcal{O}(F,\Phi) \to B$, such that $\pi_r = \tilde{\pi} \circ j_r$ and $\pi_k = \tilde{\pi} \circ j_k$ for all $r \in \mathrm{Arr}(J)$, $k \in \mathrm{Ob}(J)$.*

Proof: Let i be the universal Toeplitz representation from (F,Φ) into the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$ and let $\mathcal{I}$ be the ideal in $\mathcal{T}(F,\Phi)$ generated by

$$ \{i_k(a) - \tilde{i}_r(\lambda_r(a)) \colon a \in \lambda_r^{-1}(\mathcal{K}(F_r)), r\colon k \to l \in \mathrm{Arr}(J)\}. $$

We define $\mathcal{O}(F,\Phi) := \mathcal{T}(F,\Phi)/\mathcal{I}$ and $j := q \circ i$, where $q\colon \mathcal{T}(F,\Phi) \to \mathcal{O}(F,\Phi)$ is the canonical projection. It is easy to see that j is a Toeplitz representation from (F,Φ) to $\mathcal{O}(F,\Phi)$ and for $a \in \lambda_r^{-1}(\mathcal{K}(F_r)), r\colon k \to l \in \mathrm{Arr}(J)$, we have $\tilde{j}_r(\lambda_r(a)) = q(\tilde{i}_r(\lambda_r(a))) =$

$q(i_k(a)) = j_k(a)$, which shows that j is Cuntz-Pimsner covariant. If ψ is another Cuntz-Pimsner covariant Toeplitz representation from (F, Φ) to a C^*-algebra B, there is a unique *-homomorphism $\tilde{\psi}\colon \mathcal{T}(F, \Phi) \to B$ that satisfies $\tilde{\psi} \circ i_r = \psi_r$ and $\tilde{\psi} \circ i_k = \psi_k$ for all $r \in \mathrm{Arr}(J)$ and $k \in \mathrm{Ob}(J)$, since ψ is a Toeplitz representation. Now for $a \in \lambda_r^{-1}(\mathcal{K}(F_r)), r\colon k \to l \in \mathrm{Arr}(J)$ we have

$$\tilde{\psi}(i_k(a) - \tilde{i}_r(\lambda_r(a))) = \psi_k(a) - \tilde{\psi}_r(\lambda_r(a)) = 0,$$

since ψ is Cuntz-Pimsner covariant and hence, $\tilde{\psi}$ descends to a unique *-homomorphism $\overline{\psi}\colon \mathcal{O}(F, \Phi) \to B$ that satisfies $\overline{\psi} \circ q = \tilde{\psi}$. Hence, we have $\overline{\psi} \circ j_r = \overline{\psi} \circ q \circ i_r = \tilde{\psi} \circ i_r = \psi_r$ and $\overline{\psi} \circ j_k = \overline{\psi} \circ q \circ i_k = \tilde{\psi} \circ i_k = \psi_k$ for all $r \in \mathrm{Arr}(J)$ and $k \in \mathrm{Ob}(J)$. □

Definition 4.4 (The universal Cuntz-Pimsner algebra)
$\mathcal{O}(F, \Phi)$ is defined by a universal condition and thus it is unique up to isomorphism. We call j the *universal Cuntz-Pimsner covariant Toeplitz representation* from (F, Φ) into the *universal Cuntz-Pimsner algebra* $\mathcal{O}(F, \Phi)$.

4.3 Bicategorial colimits

The aim of this section is to show that for certain product systems (F, Φ), the universal Toeplitz algebra $\mathcal{T}(F, \Phi)$ can be viewed as the colimit object for (F, Φ) in the bicategory C*ARR. Therefore, we first have to recall the notion of a colimit for a morphism (F, Φ) from a bicategory $\mathcal{B}$ to another bicategory $\mathcal{B}'$ from [13]. To define the notion of a bicategorial colimit, we need to provide some other basic notions from bicategory theory, namely optransformations, the diagonal morphism and modifications, see also [17].

Definition 4.5 (Optransformations)
Let $\mathcal{B}, \mathcal{B}'$ be bicategories and let $(F, \Phi), (G, \Psi)\colon \mathcal{B} \to \mathcal{B}'$ be morphisms. An *optransformation* $\sigma\colon (F, \Phi) \to (G, \Psi)$ consists of a family of arrows $\sigma_0(A)\colon F(A) \to G(A)$, $A \in \mathrm{Ob}(\mathcal{B})$ and a family of 2-cells

$$\sigma_1(f)\colon \sigma_0(B) \circ F_{A,B}(f) \Rightarrow G_{A,B}(f) \circ \sigma_0(A),$$

$f\colon A \to B \in \mathrm{Arr}(\mathcal{B})$, such that the following diagrams commute for $f\colon A \to B$ and $g\colon B \to C \in \mathrm{Arr}(\mathcal{B})$:

$$
\begin{array}{ccccc}
\sigma_0(C)(F(g)F(f)) & \xrightarrow{(\alpha')^{-1}} & (\sigma_0(C)(F(g))F(f) & \xrightarrow{\sigma_1(g)*\iota'_{F(f)}} & (G(g)\sigma_0(B))F(f) \\
\Big\downarrow{\scriptstyle \iota'_{\sigma_0(C)}*\Phi_{g,f}} & & & & \Big\downarrow{\scriptstyle \alpha'} \\
\sigma_0(C)F(gf) & & & & G(g)(\sigma_0(B)F(f)) \\
\Big\downarrow{\scriptstyle \sigma_1(gf)} & & & & \Big\downarrow{\scriptstyle \iota'_{G(g)}*\sigma_1(f)} \\
G(gf)\sigma_0(A) & \xleftarrow[\Psi_{g,f}*\iota'_{\sigma_0(A)}]{} & (G(g)G(f))\sigma_0(A) & \xleftarrow[(\alpha')^{-1}]{} & G(g)(G(f)\sigma_0(A))
\end{array}
$$

$$
\begin{array}{ccccc}
\sigma_0(A)I'_{F(A)} & \xrightarrow{\iota'_{\sigma_0(A)}*\Phi_A} & \sigma_0(A)F(I_A) & \xrightarrow{\sigma_1(I_A)} & G(I_A)\sigma_0(A) \\
\Big\downarrow{\scriptstyle \rho'_{\sigma_0(A)}} & & & & \Big\uparrow{\scriptstyle \Psi_A*\iota'_{\sigma_0(A)}} \\
\sigma_0(A) & & \xrightarrow[(\lambda'_{\sigma_0(A)})^{-1}]{} & & I'_{G(A)}\sigma_0(A)
\end{array}
$$

Definition 4.6 (The diagonal morphism ΔX)
Let $\mathcal{B}, \mathcal{B}'$ be bicategories and let $X \in \mathrm{Ob}(\mathcal{B}')$. Then we define a morphism $\Delta X = (\Delta X, \Psi_X)\colon \mathcal{B} \to \mathcal{B}'$ by setting

$$\Delta X(A) := X \qquad \Delta X_{A,B}(f) := I'_X \qquad \Delta X_{A,B}(\beta) := \iota'_{I'_X}$$

for all $A, B \in \mathrm{Ob}(\mathcal{B})$, $f \in \mathrm{Ob}(\mathcal{B}(A,B))$ and $\beta \in \mathrm{Arr}(\mathcal{B}(A,B))$. It is clear that $\Delta X_{A,B}$ is a functor from $\mathcal{B}(A,B)$ to $\mathcal{B}'(X,X)$. Furthermore, given any triple (A,B,C) of objects of $\mathcal{B}$ and arrows $f\colon A \to B$, $g\colon B \to C$, we set

$$(\Psi_X)_{g,f} := \rho'_{I'_X} : I'_X \circ I'_X \Rightarrow I'_X \qquad \text{and} \qquad (\Psi_X)_A := \iota'_{I'_X} : I'_X \Rightarrow I'_X.$$

One can show that this definition makes $(\Delta X, \Psi_X)$ a morphism from $\mathcal{B}$ to $\mathcal{B}'$, which we call the *diagonal morphism*.

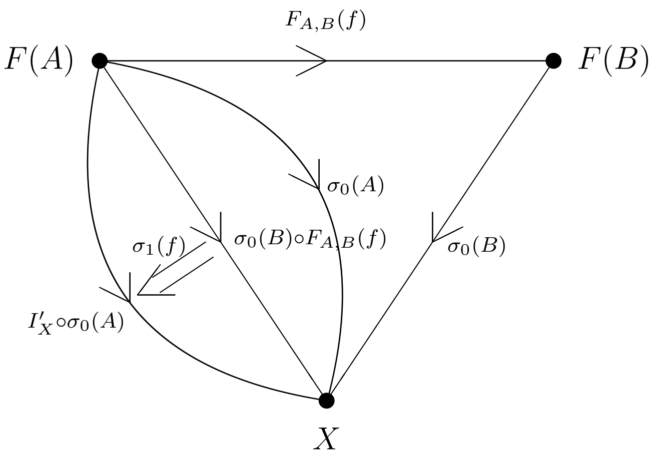
Remark: In fact, Δ is a morphism from $\mathcal{B}'$ to the bicategory $(\mathcal{B}')^{\mathcal{B}}$, whose objects are morphisms from $\mathcal{B}$ to $\mathcal{B}'$, whose arrows are optransformations and whose 2-cells are modifications (see below).

Now that we are familiar with optransformations and the diagonal morphism, we are able to introduce lax cones, which are of great importance, since the definition of bicategorial colimits is based on them.

Definition 4.7 (Lax cones)
Let $\mathcal{B}, \mathcal{B}'$ be bicategories, let $(F, \Phi)\colon \mathcal{B} \to \mathcal{B}'$ be a morphism and let $X \in \mathrm{Ob}(\mathcal{B}')$. A *lax cone from* (F, Φ) *to* X is an optransformation

$$\sigma\colon (F, \Phi) \to (\Delta X, \Psi_X).$$

Hence, a lax cone from (F, Φ) to X consist of a family of arrows $\sigma_0(A)\colon F(A) \to X$, $A \in \mathrm{Ob}(\mathcal{B})$, and a family of 2-cells $\sigma_1(f)\colon \sigma_0(B) \circ F_{A,B}(f) \Rightarrow I'_X \circ \sigma_0(A)$, $f\colon A \to B$,



such that the following diagrams commute for $A, B, C \in \mathrm{Ob}(\mathcal{B})$, $f \in \mathrm{Ob}(\mathcal{B}(A, B))$ and $g \in \mathrm{Ob}(\mathcal{B}(B, C))$ arbitrary:

$$\begin{array}{ccccc}
\sigma_0(C)(F(g)F(f)) & \xrightarrow{(\alpha')^{-1}} & (\sigma_0(C)(F(g))F(f) & \xrightarrow{\sigma_1(g)*\iota'_{F(f)}} & (I'_X\sigma_o(B))F(f) \\
\downarrow{\scriptstyle \iota'_{\sigma_0(C)}*\Phi_{g,f}} & & & & \downarrow{\scriptstyle \alpha'} \\
\sigma_0(C)F(gf) & & & & I'_X(\sigma_0(B)F(f)) \\
\downarrow{\scriptstyle \sigma_1(gf)} & & & & \downarrow{\scriptstyle \iota'_{I'_X}*\sigma_1(f)} \\
I'_X\sigma_0(A) & \xleftarrow[\rho'_{I'_X}*\iota'_{\sigma_0(A)}]{} & (I'_XI'_X)\sigma_0(A) & \xleftarrow[(\alpha')^{-1}]{} & I'_X(I'_X\sigma_0(A))
\end{array}$$

$$\begin{array}{ccc}
\sigma_0(A)I'_{F(A)} & \xrightarrow{\iota'_{\sigma_0(A)}*\Phi_A} & \sigma_0(A)F(I_A) \\
\downarrow{\scriptstyle \rho'_{\sigma_0(A)}} & & \downarrow{\scriptstyle \sigma_1(I_A)} \\
\sigma_0(A) & \xrightarrow[(\lambda'_{\sigma_0(A)})^{-1}]{} & I'_X\sigma_0(A)
\end{array}$$

Before we are able to define the notion of a bicategorial colimit we have to provide a way to construct a new lax cone $f \circ \sigma$ from a given lax cone σ and we need to introduce the notion of a modification between optransformations.

If σ is a lax cone from (F, Φ) to X and $f\colon X \to Y$ is an arrow in $\mathcal{B}'$ then we define a lax cone $f \circ \sigma$ from (F, Φ) to Y as follows. For $A \in \mathrm{Ob}(\mathcal{B})$, we set

$$(f \circ \sigma)_0(A) := f \circ \sigma_0(A)\colon F(A) \to Y$$

and for an arrow $r \in \mathrm{Ob}(\mathcal{B}(A, B))$, we define a 2-cell

$$(f \circ \sigma)_1(r)\colon (f \circ \sigma)_0(B) \circ F_{A,B}(r) \Rightarrow I'_Y \circ (f \circ \sigma)_0(A)$$

by setting

$$(f \circ \sigma)_1(r) := \alpha' \circ ((\lambda'_f)^{-1} * \iota'_{\sigma_0(A)}) \circ (\rho'_f * \iota'_{\sigma_0(A)}) \circ (\alpha')^{-1} \circ (\iota'_f * \sigma_1(r)) \circ \alpha',$$

see diagram below:

$$\begin{array}{ccccc}
(f\sigma_0(B))F(r) & \xrightarrow{\alpha'} & f(\sigma_0(B)F(r)) & \xrightarrow{\iota'_f * \sigma_1(r)} & f(I'_X\sigma_0(A)) \\
 & & & & \downarrow{\scriptstyle (\alpha')^{-1}} \\
{\scriptstyle (f\circ\sigma)_1(r)}\downarrow & & & & (fI'_X)\sigma_0(A) \\
 & & & & \downarrow{\scriptstyle \rho'_f * \iota'_{\sigma_0(A)}} \\
I'_Y(f\sigma_0(A)) & \xleftarrow{\alpha'} & (I'_Y f)\sigma_0(A) & \xleftarrow{(\lambda'_f)^{-1} * \iota'_{\sigma_0(A)}} & f\sigma_0(A)
\end{array}$$

One can show that this definition in deed makes $f \circ \sigma$ a lax cone from (F, Φ) to Y.

Definition 4.8 (Modifications)
Let $\mathcal{B}, \mathcal{B}'$ be bicategories and let $(F, \Phi), (G, \Psi)\colon \mathcal{B} \to \mathcal{B}'$ be morphisms. Moreover, let $\sigma, \tilde{\sigma}\colon (F, \Phi) \to (G, \Psi)$ be optransformations. A *modification* $\Gamma\colon \sigma \to \tilde{\sigma}$ consists of a family of 2-cells

$$\Gamma_A\colon \sigma_0(A) \Rightarrow \tilde{\sigma}_0(A),$$

$A \in \mathrm{Ob}(\mathcal{B})$, such that the following diagram commutes for all $f\colon A \to B \in \mathrm{Arr}(\mathcal{B})$:

$$\begin{array}{ccc}
\sigma_0(B)F(f) & \xrightarrow{\Gamma_B * \iota'_{F(f)}} & \tilde{\sigma}_0(B)F(f) \\
{\scriptstyle \sigma_1(f)}\downarrow & & \downarrow{\scriptstyle \tilde{\sigma}_1(f)} \\
G(f)\sigma_0(A) & \xrightarrow{\iota'_{G(f)} * \Gamma_A} & G(f)\tilde{\sigma}_0(A)
\end{array}$$

Definition 4.9 (Colimit for a morphism (F,Φ))
Let $\mathcal{B},\mathcal{B}'$ be bicategories and let $(F,\Phi)\colon \mathcal{B}\to\mathcal{B}'$ be a morphism. A *colimit* for (F,Φ) is a lax cone σ from (F,Φ) to the *colimit object* X such that for each lax cone τ from (F,Φ) to Y there is a pair (f,β), where $f\colon X\to Y$ is an arrow and $\beta\colon \tau\to f\circ\sigma$ is a modification, which is universal among such pairs, i.e., given another such pair (f',β') there is a unique 2-cell $\chi\colon f\Rightarrow f'$ such that $\beta'_A=(\chi*\iota'_{\sigma_0(A)})\circ\beta_A$ for all $A\in\mathrm{Ob}(\mathcal{B})$.

First, we want to describe what it means that there is a modification $\beta\colon\tau\to f\circ\sigma$. It means that there are 2-cells $\beta_A\colon \tau_0(A)\Rightarrow f\circ\sigma_0(A)$, $A\in\mathrm{Ob}(\mathcal{B})$, such that the following equation holds for $r\in\mathrm{Ob}(\mathcal{B}(A,B))$:

$$\begin{aligned}&(\iota'_{I'_Y}*\beta_A)\circ\tau_1(r)=(f\circ\sigma)_1(r)\circ(\beta_B*\iota'_{F(r)})\\ =\;&\alpha'\circ((\lambda'_f)^{-1}*\iota'_{\sigma_0(A)})\circ(\rho'_f*\iota'_{\sigma_0(A)})\circ(\alpha')^{-1}\circ(\iota'_f*\sigma_1(r))\circ\alpha'\circ(\beta_B*\iota'_{F(r)}).\end{aligned}$$

Now that we have provided the bicategorial background, we want to show that for certain product systems (F,Φ), the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$ can be viewed as the colimit object for (F,Φ) in the bicategory C*ARR. So for the rest of this section, let (F,Φ) be a product system over an index category J such that all C^*-arrows F_r, $r\in\mathrm{Arr}(J)$, are finitely generated and all C^*-algebras A_i, $i\in\mathrm{Ob}(J)$, are unital.

First of all, we want to show that a Toeplitz representation $\pi\colon(F,\Phi)\to B$ can always be viewed as a lax cone $\sigma(\pi)$ from (F,Φ) to B. We set

$$\sigma(\pi)_0(i):={}_{\pi_i}B\qquad\text{and}\qquad\sigma(\pi)_1(r)(\xi\otimes\eta):=\pi_r(\xi)\otimes\eta$$

for $i\in\mathrm{Ob}(J)$, $r\colon i\to j\in\mathrm{Arr}(J)$, $\xi\in F_r$ and $\eta\in{}_{\pi_j}B$.

Lemma 4.10 *$\sigma(\pi)$ is a lax cone from (F,Φ) to B.*

Proof: It is clear that $\sigma(\pi)_0(i)={}_{\pi_i}B$ is a C^*-arrow from A_i to B. To see that $\sigma(\pi)_1(r)$ is a 2-cell from $\sigma(\pi)_0(j)\circ F_r=F_r\otimes_{\pi_j}B$ to $I'_B\circ{}_{\pi_i}B={}_{\pi_i}B\otimes_{\mathrm{id}}B$ we first show that $\sigma(\pi)_1(r)$ is isometric:

$$\begin{aligned}(\,\sigma(\pi)_1(r)(\xi\otimes\eta)\,|\,\sigma(\pi)_1(r)(\tilde\xi\otimes\tilde\eta)\,)_B&=(\,\pi_r(\xi)\otimes\eta\,|\,\pi_r(\tilde\xi)\otimes\tilde\eta\,)_B\\&=(\,\eta\,|\,(\,\pi_r(\xi)\,|\,\pi_r(\tilde\xi)\,)_B\cdot\tilde\eta\,)_B=(\,\eta\,|\,\pi_r(\xi)^*\pi_r(\tilde\xi)\tilde\eta\,)_B\\&=(\,\eta\,|\,\pi_j((\,\xi\,|\,\tilde\xi\,)_{A_j})\tilde\eta\,)_B=(\,\eta\,|\,(\,\xi\,|\,\tilde\xi\,)_{A_j}\cdot\tilde\eta\,)_B\\&=(\,\xi\otimes\eta\,|\,\tilde\xi\otimes\tilde\eta\,)_B,\end{aligned}$$

where $\xi,\tilde\xi\in F_r$, $\eta,\tilde\eta\in{}_{\pi_j}B$. Since A_j is unital, it follows easily that ${}_{\pi_j}B$ is finitely generated as a Hilbert B-module by $\pi_j(1)$. Since F_r is a finitely generated C^*-arrow, we also have that $F_r\otimes_{\pi_j}B$ is finitely generated. Hence, the image of $F_r\otimes_{\pi_j}B$ under $\sigma(\pi)_1(r)$ is finitely generated. Schweizer shows in [26] that a finitely generated Hilbert C^*-module is always self dual. From this it follows easily that $\sigma(\pi)_1(r)(F_r \otimes {}_{\pi_j}B)$ is a complemented submodule of ${}_{\pi_i}B \otimes {}_{\mathrm{id}}B$, which yields that $\sigma(\pi)_1(r)$ is adjointable and hence, a 2-cell.

Now let $r\colon i \to j, s\colon j \to k \in \mathrm{Arr}(J)$ be arbitrary. We have to show that

$$\sigma(\pi)_1(sr)(\iota'_{{}_{\pi_k}B} * \Phi_{s,r}) = (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}(\iota'_{{}_{\mathrm{id}}B} * \sigma(\pi)_1(r))\alpha'(\sigma(\pi)_1(s) * \iota'_{F_r})(\alpha')^{-1}.$$

So let $\xi_r \in F_r$, $\xi_s \in F_s$ and $\eta \in {}_{\pi_k}B$ be arbitrary. Then we have

$$\begin{aligned}\sigma(\pi)_1(sr)(\iota'_{{}_{\pi_k}B} * \Phi_{s,r})((\xi_r \otimes \xi_s) \otimes \eta) &= \sigma(\pi)_1(sr)(\Phi_{sr}(\xi_r \otimes \xi_s) \otimes \eta) \\ &= \pi_{sr}(\Phi_{sr}(\xi_r \otimes \xi_s)) \otimes \eta = \pi_r(\xi_r)\pi_s(\xi_s) \otimes \eta \quad \text{and}\end{aligned}$$

$$\begin{aligned} & (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}(\iota'_{{}_{\mathrm{id}}B} * \sigma(\pi)_1(r))\alpha'(\sigma(\pi)_1(s) * \iota'_{F_r})(\alpha')^{-1}((\xi_r \otimes \xi_s) \otimes \eta) \\ = \; & (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}(\iota'_{{}_{\mathrm{id}}B} * \sigma(\pi)_1(r))\alpha'(\sigma(\pi)_1(s) * \iota'_{F_r})(\xi_r \otimes (\xi_s \otimes \eta)) \\ = \; & (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}(\iota'_{{}_{\mathrm{id}}B} * \sigma(\pi)_1(r))\alpha'(\xi_r \otimes (\pi_s(\xi_s) \otimes \eta)) \\ = \; & (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}(\iota'_{{}_{\mathrm{id}}B} * \sigma(\pi)_1(r))((\xi_r \otimes \pi_s(\xi_s)) \otimes \eta) \\ = \; & (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\alpha')^{-1}((\pi_r(\xi_r) \otimes \pi_s(\xi_s)) \otimes \eta) = (\rho'_{{}_{\mathrm{id}}B} * \iota'_{{}_{\pi_i}B})(\pi_r(\xi_r) \otimes (\pi_s(\xi_s) \otimes \eta)) \\ = \; & \pi_r(\xi_r) \otimes \pi_s(\xi_s)\eta = \pi_r(\xi_r)\pi_s(\xi_s) \otimes \eta. \end{aligned}$$

Finally, it remains to prove that $(\lambda'_{{}_{\pi_i}B})^{-1} \circ \rho'_{{}_{\pi_i}B} = \sigma(\pi)_1(I_i) \circ (\iota'_{{}_{\pi_i}B} * \Phi_i)$ which is equivalent to $\rho'_{{}_{\pi_i}B} = \lambda'_{{}_{\pi_i}B} \circ \sigma(\pi)_1(I_i) \circ (\iota'_{{}_{\pi_i}B} * \Phi_i)$. So we let $\xi \in I'_{A_i} = {}_{\mathrm{id}}A_i$ and $\eta \in {}_{\pi_i}B$ be arbitrary and compute

$$\begin{aligned}\lambda'_{{}_{\pi_i}B} \circ \sigma(\pi)_1(I_i) \circ (\iota'_{{}_{\pi_i}B} * \Phi_i)(\xi \otimes \eta) &= \lambda'_{{}_{\pi_i}B} \circ \sigma(\pi)_1(I_i)(\Phi_i(\xi) \otimes \eta) \\ &= \lambda'_{{}_{\pi_i}B}(\pi_{I_i}(\Phi_i(\xi)) \otimes \eta) = \pi_i(\xi)\eta = \rho'_{{}_{\pi_i}B}(\xi \otimes \eta).\end{aligned}$$

□

From Proposition 4.1 we know that the universal Toeplitz representation $i\colon (F,\Phi) \to \mathcal{T}(F,\Phi)$ is universal for Toeplitz representations over (F,Φ) and now we have seen in Lemma 4.10 that Toeplitz representations can be viewed as lax cones. Hence, the lax cone $\sigma(i)$ is universal for lax cones over (F,Φ) that come from Toeplitz representations. The following theorem states that $\sigma(i)$ is universal for arbitrary lax cones over (F,Φ) or in other words, that $\sigma(i)$ is a colimit for (F,Φ). Hence, Theorem 4.11 extends Proposition 4.1. In what follows, we will denote the universal Toeplitz representation by π in order not to confuse the universal Toeplitz representation i with an object $i \in \mathrm{Ob}(J)$.

Theorem 4.11 *Let (F,Φ) be a product system over an index category J such that all C^*-arrows F_r, $r \in \mathrm{Arr}(J)$, are finitely generated and all C^*-algebras A_i, $i \in \mathrm{Ob}(J)$, are unital. Moreover, let $\pi\colon (F,\Phi) \to \mathcal{T}(F,\Phi)$ be the universal Toeplitz representation. Then $\sigma(\pi)$ is a colimit for (F,Φ) and $\mathcal{T}(F,\Phi)$ is the corresponding colimit object.*

Proof: From Lemma 4.10 we know that $\sigma(\pi)$ is a lax cone from (F, Φ) to $\mathcal{T}(F, \Phi)$. Now let τ be an arbitrary lax cone from (F, Φ) to a C^*-algebra Y. We have to construct a C^*-arrow $f\colon \mathcal{T}(F, \Phi) \to Y$ and a modification $\beta\colon \tau \to f \circ \sigma(\pi)$. $\tau_0(j)$ is a C^*-arrow from A_j to Y, $j \in \mathrm{Ob}(J)$. Especially, all $\tau_0(j)$ are Hilbert Y-modules. Hence, we can set

$$f := \bigoplus_{j \in \mathrm{Ob}(J)}^{(i)} \tau_0(j),$$

which makes f a Hilbert Y-module. We want to make f a C^*-arrow from $\mathcal{T}(F, \Phi)$ to Y and thus we have to provide f with a left multiplication by $\mathcal{T}(F, \Phi)$, i.e., we are looking for a *-homomorphism $\psi\colon \mathcal{T}(F, \Phi) \to \mathcal{L}(f)$. Now the idea is to construct a Toeplitz representation $\psi\colon (F, \Phi) \to \mathcal{L}(f)$ which yields the desired *-homomorphism. For $i \in \mathrm{Ob}(J)$ let

$$\psi_i\colon A_i \to \mathcal{L}(\tau_0(i)) \subset \mathcal{L}(f)$$

be the *-homomorphism that defines the left multiplication of A_i on $\tau_0(i)$ ($\tau_0(i)$ is a C^*-arrow from A_i to Y). For $r\colon i \to j \in \mathrm{Arr}(J)$ we define a linear mapping

$$\psi_r\colon F_r \to \mathcal{L}(\tau_0(j), \tau_0(i)) \subset \mathcal{L}(f) \qquad \text{by} \qquad \psi_r(\xi_r)(\eta_j) := \lambda'_{\tau_0(i)}(\tau_1(r)(\xi_r \otimes \eta_j))$$

for $\xi_r \in F_r$, $\eta_j \in \tau_0(j) \subset f$. The adjoint operator $\psi_r(\xi_r)^* \in \mathcal{L}(\tau_0(i), \tau_0(j))$ is given by

$$\psi_r(\xi_r)^* = K^{j,r}_{\xi_r} \circ \tau_1(r)^* \circ (\lambda'_{\tau_0(i)})^{-1}$$

where $K^{j,r}_{\xi_r}\colon F_r \otimes \tau_0(j) \to \tau_0(j)$ is the mapping given by $K^{j,r}_{\xi_r}(\eta_r \otimes \eta_j) := (\, \xi_r \,|\, \eta_r \,)_{A_j} \cdot \eta_j$. The following computations show that ψ is a Toeplitz representation from (F, Φ) to $\mathcal{L}(f)$. Let $r\colon i \to j$, $s\colon j \to k \in \mathrm{Arr}(J)$ and let $a_i \in A_i$, $a_j \in A_j$, $\xi_r, \eta_r \in F_r$, $\xi_s \in F_s$, $\eta_j \in \tau_0(j)$ and $\eta_k \in \tau_0(k)$ be arbitrary. Then we compute

$$\begin{aligned} \psi_r(a_i \cdot \xi_r \cdot a_j)(\eta_j) &= \lambda'_{\tau_0(i)}(\tau_1(r)(a_i \cdot \xi_r \cdot a_j \otimes \eta_j)) = a_i \cdot \lambda'_{\tau_0(i)}(\tau_1(r)(\xi_r \otimes a_j \cdot \eta_j)) \\ &= \psi_i(a_i)\psi_r(\xi_r)(a_j \cdot \eta_j) = \psi_i(a_i)\psi_r(\xi_r)\psi_j(a_j)(\eta_j), \end{aligned}$$

$$\begin{aligned} \psi_r(\xi_r)^*\psi_r(\eta_r)(\eta_j) &= \psi_r(\xi_r)^*\lambda'_{\tau_0(i)}(\tau_1(r)(\eta_r \otimes \eta_j)) \\ &= K^{j,r}_{\xi_r} \circ \tau_1(r)^* \circ (\lambda'_{\tau_0(i)})^{-1}\lambda'_{\tau_0(i)}(\tau_1(r)(\eta_r \otimes \eta_j)) \\ &= (\, \xi_r \,|\, \eta_r \,)_{A_j} \cdot \eta_j = \psi_j((\, \xi_r \,|\, \eta_r \,)_{A_j})(\eta_j), \end{aligned}$$

$$\begin{aligned} &\psi_{s \circ r}(\Phi_{s,r}(\xi_r \otimes \xi_s))(\eta_k) = \lambda'_{\tau_0(i)}(\tau_1(s \circ r)(\Phi_{s,r}(\xi_r \otimes \xi_s) \otimes \eta_k)) \\ = \; &\lambda'_{\tau_0(i)} \circ (\mathrm{id}_{\tau_0(i)} \otimes \rho'_{\mathrm{id}Y}) \circ (\alpha')^{-1} \circ (\tau_1(r) \otimes \mathrm{id}_{\mathrm{id}Y}) \circ \alpha' \circ (\mathrm{id}_{F_r} \otimes \tau_1(s))(\xi_r \otimes (\xi_s \otimes \eta_k)) \\ = \; &\lambda'_{\tau_0(i)} \circ (\mathrm{id}_{\tau_0(i)} \otimes \rho'_{\mathrm{id}Y}) \circ (\alpha')^{-1} \circ (\tau_1(r) \otimes \mathrm{id}_{\mathrm{id}Y}) \circ \alpha'(\xi_r \otimes \tau_1(s)(\xi_s \otimes \eta_k)) \\ = \; &\lambda'_{\tau_0(i)} \circ \tau_1(r) \circ (\mathrm{id}_{F_r} \otimes \lambda'_{\tau_0(j)})(\xi_r \otimes \tau_1(s)(\xi_s \otimes \eta_k)) \\ = \; &\lambda'_{\tau_0(i)} \circ \tau_1(r)(\xi_r \otimes \lambda'_{\tau_0(j)}\tau_1(s)(\xi_s \otimes \eta_k)) = \lambda'_{\tau_0(i)} \circ \tau_1(r)(\xi_r \otimes \psi_s(\xi_s)(\eta_k)) \\ = \; &\psi_r(\xi_r)\psi_s(\xi_s)(\eta_k), \end{aligned}$$

$$\begin{aligned}(\psi_{I_j}\circ\Phi_j)(a_j)(\eta_j) &= \psi_{I_j}(\Phi_j(a_j))(\eta_j) = \lambda'_{\tau_0(j)}(\tau_1(I_j)(\Phi_j(a_j)\otimes\eta_j))\\ &= \lambda'_{\tau_0(j)}\circ\tau_1(I_j)\circ(\Phi_j\otimes \mathrm{id}_{\tau_0(j)})(a_j\otimes\eta_j) = \lambda'_{\tau_0(j)}(\lambda'_{\tau_0(j)})^{-1}\rho'_{\tau_0(j)}(a_j\otimes\eta_j)\\ &= \rho'_{\tau_0(j)}(a_j\otimes\eta_j) = a_j\cdot\eta_j = \psi_j(a_j)(\eta_j)\end{aligned}$$

and it is clear by the construction of f that $\psi_i(a_i)\psi_j(a_j) = 0$ for $i \neq j$.

Hence, there exists a unique *-homomorphism $\psi\colon \mathcal{T}(F,\Phi)\to\mathcal{L}(f)$ with

$$\psi_i = \psi\circ\pi_i \qquad \text{and} \qquad \psi_r = \psi\circ\pi_r$$

for all $i\in \mathrm{Ob}(J)$ and $r\in\mathrm{Arr}(J)$. We use ψ to define a left multiplication of $\mathcal{T}(F,\Phi)$ on f, which makes f a C^*-arrow from $\mathcal{T}(F,\Phi)$ to Y. Next, we define mappings

$$\beta_i := (\iota_{\tau_0(i)} * \pi_i)\circ(\rho_{\tau_0(i)})^{-1}\colon \tau_0(i)\Rightarrow {}_{\pi_i}\mathcal{T}(F,\Phi)\otimes f, \qquad i\in\mathrm{Ob}(J).$$

Hence, for $\xi = a_i\cdot\eta = \rho_{\tau_0(i)}(a_i\otimes\eta)$, $a_i\in A_i$, $\eta\in\tau_0(i)$ we have $\beta_i(\xi) = \pi_i(a_i)\otimes\eta$. We want to show that the family $\{\beta_i\colon i\in\mathrm{Ob}(J)\}$ is a modification from τ to $f\circ\sigma(\pi)$. First we show that each β_i is a 2-cell from $\tau_0(i)$ to $f\circ\sigma(\pi)_0(i) = {}_{\pi_i}\mathcal{T}(F,\Phi)\otimes f$. It is easy to see that β_i is an A_i-Y-bimodule mapping. To see that it is isometric we let $a_i, b_i\in A_i$ and $\xi,\eta\in\tau_0(i)$ arbitrary and compute

$$\begin{aligned}(\,\beta_i(a_i\cdot\xi)\,|\,\beta_i(b_i\cdot\eta)\,)_Y &= (\,\pi_i(a_i)\otimes\xi\,|\,\pi_i(b_i)\otimes\eta\,)_Y = (\,\xi\,|\,(\,\pi_i(a_i)\,|\,\pi_i(b_i)\,)_{\mathcal{T}(F,\Phi)}\cdot\eta\,)_Y\\ &= (\,\xi\,|\,\psi(\pi_i(a_i^*b_i))(\eta)\,)_Y = (\,\xi\,|\,\psi_i(a_i^*b_i)(\eta)\,)_Y\\ &= (\,\psi_i(a_i)(\xi)\,|\,\psi_i(b_i)(\eta)\,)_Y = (\,a_i\cdot\xi\,|\,b_i\cdot\eta\,)_Y.\end{aligned}$$

Next, we want to show that β_i is adjointable by providing the adjoint mapping β_i^* explicitly. We define

$$\beta_i^*\colon {}_{\pi_i}\mathcal{T}(F,\Phi)\otimes f\Rightarrow\tau_0(i) \qquad \text{by setting} \qquad \beta_i^*(\xi\otimes\eta) := \psi(\xi)(\eta).$$

To see that $\beta_i^*({}_{\pi_i}\mathcal{T}(F,\Phi)\otimes f)\subseteq\tau_0(i)$, we let $\pi_i(a_i)\xi$, $a_i\in A_i$, $\xi\in\mathcal{T}(F,\Phi)$, be a typical element of ${}_{\pi_i}\mathcal{T}(F,\Phi)$ and we let $\eta\in f$ be arbitrary. Then we have $\beta_i^*(\pi_i(a_i)\xi\otimes\eta) = \psi(\pi_i(a_i)\xi)(\eta) = \psi_i(a_i)\psi(\xi)(\eta)\in\tau_0(i)$, since $\psi_i(a_i)\in\mathcal{L}(\tau_0(i))$. It is easy to see that β_i^* is an A_i-Y-bimodule mapping. The following computation shows that β_i^* in deed is the adjoint of β_i:

$$\begin{aligned}(\,\beta_i(a_i\cdot\xi)\,|\,\pi_i(b_i)\eta\otimes\zeta\,)_Y &= (\,\pi_i(a_i)\otimes\xi\,|\,\pi_i(b_i)\eta\otimes\zeta\,)_Y\\ &= (\,\xi\,|\,(\,\pi_i(a_i)\,|\,\pi_i(b_i)\eta\,)_{\mathcal{T}(F,\Phi)}\cdot\zeta\,)_Y = (\,\xi\,|\,\psi(\pi_i(a_i^*)\pi_i(b_i)\eta)(\zeta)\,)_Y\\ &= (\,\xi\,|\,\psi_i(a_i)^*\psi(\pi_i(b_i)\eta)(\zeta)\,)_Y = (\,\psi_i(a_i)(\xi)\,|\,\psi(\pi_i(b_i)\eta)(\zeta)\,)_Y\\ &= (\,a_i\cdot\xi\,|\,\beta_i^*(\pi_i(b_i)\eta\otimes\zeta)\,)_Y\end{aligned}$$

for $a_i, b_i\in A_i$, $\xi\in\tau_0(i)$, $\eta\in\mathcal{T}(F,\Phi)$ and $\zeta\in f$ arbitrary. Hence, β_i is an adjointable, isometric A_i-Y-bimodule mapping. We claim that β_i is a unitary, i.e., we have to show that $\beta_i \circ \beta_i^* = \mathrm{id}_{\pi_i \mathcal{T}(F,\Phi) \otimes f}$. So we let $a_i \in A_i$, $\xi \in \mathcal{T}(F,\Phi)$ and $\eta \in f$ be arbitrary and compute

$$\begin{aligned}\beta_i \circ \beta_i^*(\pi_i(a_i)\xi \otimes \eta) &= \beta_i(\psi(\pi_i(a_i)\xi)(\eta)) = \beta_i(\psi_i(a_i)(\psi(\xi)(\eta))) = \pi_i(a_i) \otimes \psi(\xi)(\eta) \\ &= \pi_i(a_i)\xi \otimes \eta,\end{aligned}$$

which shows that β_i is a unitary.

In order to prove that the family $\{\beta_i \colon i \in \mathrm{Ob}(J)\}$ is a modification from τ to $f \circ \sigma(\pi)$, it remains to show that

$$\begin{aligned}(\beta_i \otimes \mathrm{id}_{I'_Y}) \circ \tau_1(r) &= \alpha' \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes (\lambda'_f)^{-1}) \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes \rho'_f) \circ (\alpha')^{-1} \\ &\quad \circ (\sigma(\pi)_1(r) \otimes \mathrm{id}_f) \circ \alpha' \circ (\mathrm{id}_{F_r} \otimes \beta_j), \text{ which is equivalent to} \\ \lambda'_{\tau_0(i)} \circ \tau_1(r) \circ (\mathrm{id}_{F_r} \otimes \beta_j^*) &= \lambda'_{\tau_0(i)} \circ (\beta_i^* \otimes \mathrm{id}_{I'_Y}) \circ \alpha' \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes (\lambda'_f)^{-1}) \\ &\quad \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes \rho'_f) \circ (\alpha')^{-1} \circ (\sigma(\pi)_1(r) \otimes \mathrm{id}_f) \circ \alpha'.\end{aligned}$$

So for $\xi \in F_r$, $\eta \in {}_{\pi_j}\mathcal{T}(F,\Phi)$ and $\zeta \in f$ we compute

$$\lambda'_{\tau_0(i)} \circ \tau_1(r) \circ (\mathrm{id}_{F_r} \otimes \beta_j^*)(\xi \otimes (\eta \otimes \zeta)) = \lambda'_{\tau_0(i)} \circ \tau_1(r)(\xi \otimes \psi(\eta)(\zeta)) = \psi_r(\xi)(\psi(\eta)(\zeta)),$$

$$\begin{aligned}&\lambda'_{\tau_0(i)} \circ (\beta_i^* \otimes \mathrm{id}_{I'_Y}) \circ \alpha' \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes (\lambda'_f)^{-1}) \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes \rho'_f) \circ (\alpha')^{-1} \\ &\circ (\sigma(\pi)_1(r) \otimes \mathrm{id}_f) \circ \alpha'(\xi \otimes (\eta \otimes \zeta)) \\ = \ &\lambda'_{\tau_0(i)} \circ (\beta_i^* \otimes \mathrm{id}_{I'_Y}) \circ \alpha' \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes (\lambda'_f)^{-1}) \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes \rho'_f) \circ (\alpha')^{-1} \\ &((\pi_r(\xi) \otimes \eta) \otimes \zeta) \\ = \ &\lambda'_{\tau_0(i)} \circ (\beta_i^* \otimes \mathrm{id}_{I'_Y}) \circ \alpha' \circ (\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes (\lambda'_f)^{-1})(\pi_r(\xi) \otimes (\eta \cdot \zeta)) \\ = \ &\beta_i^*(\pi_r(\xi) \otimes (\eta \cdot \zeta)) = \psi(\pi_r(\xi))(\eta \cdot \zeta) = \psi_r(\xi)(\psi(\eta)(\zeta)),\end{aligned}$$

which shows that β is a modification from τ to $f \circ \sigma(\pi)$.

Now let (f', β') be another pair consisting of a C^*-arrow $f' \colon \mathcal{T}(F,\Phi) \to Y$ and a modification $\beta' \colon \tau \to f' \circ \sigma(\pi)$. We have to provide an isometric, adjointable $\mathcal{T}(F,\Phi)$-Y-bimodule mapping $\chi \colon f \to f'$ such that $(\mathrm{id}_{\pi_i \mathcal{T}(F,\Phi)} \otimes \chi) \circ \beta_i = \beta'_i$ for all $i \in \mathrm{Ob}(J)$ and we have to show that this χ is unique.

Let $\rho'_{f'}$ be the canonical $\mathcal{T}(F,\Phi)$-Y-bimodule mapping from ${}_{\mathrm{id}}\mathcal{T}(F,\Phi) \otimes f'$ to f'. We let U_i denote the restriction of $\rho'_{f'}$ to ${}_{\pi_i}\mathcal{T}(F,\Phi) \otimes f'$ and we set $f'_i := U_i({}_{\pi_i}\mathcal{T}(F,\Phi) \otimes f') \subset f'$. Then f'_i is C^*-arrow from A_i to Y, the left multiplication is given by $a_i \cdot \xi := \pi_i(a_i) \cdot \xi$, and U_i is a unitary A_i-Y-bimodule mapping from ${}_{\pi_i}\mathcal{T}(F,\Phi) \otimes f'$ to f'_i. Moreover, due to the remark following Definition 4.2, we know that ${}_{\mathrm{id}}\mathcal{T}(F,\Phi) = \bigoplus^{(i)}_{i \in \mathrm{Ob}(J)} {}_{\pi_i}\mathcal{T}(F,\Phi)$ and thus it follows easily that $f' = \bigoplus^{(i)}_{i \in \mathrm{Ob}(J)} f'_i$.

Next, we define isometric, adjointable A_i-Y-bimodule mappings $\chi_i \colon \tau_0(i) \to f'_i$ by setting

$$\chi_i := U_i \circ \beta'_i, \qquad i \in \mathrm{Ob}(J),$$

and finally, we define a mapping $\chi\colon f = \bigoplus^{(i)}_{i\in \mathrm{Ob}(J)} \tau_0(i) \to f' = \bigoplus^{(i)}_{i\in \mathrm{Ob}(J)} f'_i$ by setting

$$\chi((\xi_i)_{i\in\mathrm{Ob}(J)}) := (\chi_i(\xi_i))_{i\in\mathrm{Ob}(J)}.$$

Then χ is an isometric $\mathcal{T}(F,\Phi)$-Y-bimodule mapping, because for $\xi_i, \eta_i \in \tau_0(i)$, $i \in \mathrm{Ob}(J)$, we have

$$\begin{aligned}
(\,\chi((\xi_i)_{i\in\mathrm{Ob}(J)}) \,|\, \chi((\eta_i)_{i\in\mathrm{Ob}(J)})\,)_Y &= (\,(\chi_i(\xi_i))_{i\in\mathrm{Ob}(J)} \,|\, (\chi_i(\eta_i))_{i\in\mathrm{Ob}(J)}\,)_Y \\
&= \sum_{i\in\mathrm{Ob}(J)} (\,\chi_i(\xi_i) \,|\, \chi_i(\eta_i)\,)_Y = \sum_{i\in\mathrm{Ob}(J)} (\,\xi_i \,|\, \eta_i\,)_Y \\
&= (\,(\xi_i)_{i\in\mathrm{Ob}(J)} \,|\, (\eta_i)_{i\in\mathrm{Ob}(J)}\,)_Y.
\end{aligned}$$

Moreover, χ is adjointable, because each χ_i, $i \in \mathrm{Ob}(J)$, is adjointable and so the adjoint $\chi^*\colon f' \to f$ is given by $\chi^*((\xi_i)_{i\in\mathrm{Ob}(J)}) = (\chi_i^*(\xi_i))_{i\in\mathrm{Ob}(J)}$.

Next, we show that $\beta'_i = (\mathrm{id}_{\pi_i\mathcal{T}(F,\Phi)} \otimes \chi) \circ \beta_i$ for all $i \in \mathrm{Ob}(J)$. For $\xi \in \tau_0(i)$, we suppose that $\beta'_i(\xi) = (\pi_i(b_i)\eta) \otimes \zeta$ for $b_i \in A_i$, $\eta \in \mathcal{T}(F,\Phi)$ and $\zeta \in f'$. Then we have

$$\begin{aligned}
(\mathrm{id}_{\pi_i\mathcal{T}(F,\Phi)} \otimes \chi) \circ \beta_i(a_i \cdot \xi) &= (\mathrm{id}_{\pi_i\mathcal{T}(F,\Phi)} \otimes \chi)(\pi_i(a_i) \otimes \xi) = \pi_i(a_i) \otimes \chi_i(\xi) \\
&= \pi_i(a_i) \otimes U_i(\beta'_i(\xi)) = \pi_i(a_i) \otimes U_i((\pi_i(b_i)\eta) \otimes \zeta) \\
&= \pi_i(a_i) \otimes (\pi_i(b_i)\eta) \cdot \zeta = \pi_i(a_i)\pi_i(b_i)\eta \otimes \zeta \\
&= a_i \cdot (\pi_i(b_i)\eta \otimes \zeta) = a_i \cdot \beta'_i(\xi) = \beta'_i(a_i \cdot \xi),
\end{aligned}$$

which shows that χ is the desired bimodule mapping. Finally, it is easy to see that the bimodule mapping χ is uniquely determined, since every χ_i is uniquely determined by the identity $(\mathrm{id}_{\pi_i\mathcal{T}(F,\Phi)} \otimes \chi) \circ \beta_i = \beta'_i$. □

Part II

Duality theory

Chapter 5

Crossed products by Hopf C*-algebras

In this chapter, we want to make a first step towards the duality theory for locally compact semigroups using Hopf C^*-algebras, which can be viewed as a generalization of locally compact semigroups. A Hopf C^*-algebra is a C^*-algebra H together with a comultiplication δ_H. The standard example for a Hopf C^*-algebra is $C_0(\mathcal{S})$, the C^*-algebra of continuous functions on a locally compact semigroup $\mathcal{S}$ that vanish at infinity. The multiplication in $\mathcal{S}$ corresponds to the comultiplication in $C_0(\mathcal{S})$.

We introduce the notions of corepresentations of Hopf C^*-algebras, coactions of Hopf C^*-algebras on C^*-algebras and covariant representations of dynamical cosystems, which generalize the corresponding notions for semigroups, namely representations of semigroups, actions of semigroups on C^*-algebras and covariant representations of semigroup dynamical systems.

Following our overview of Hopf C^*-algebras, we are looking for a condition on the Hopf C^*-algebra H that makes it possible to construct a specific covariant representation of the dynamical cosystem (H, H, δ_H) inspired by the right regular covariant representation of $(C_0(G), G, \alpha)$ on $L^2(G, \mu)$, where μ is a right Haar measure on a locally compact group G and α is the action of G on $C_0(G)$ by right translation. This condition will be the existence of an invariant weight τ on H and we will see that for the Hopf C^*-algebra $C_0(\mathcal{S})$, this condition is equivalent to the existence of a right invariant Radon measure μ on $\mathcal{S}$.

For Hopf C^*-algebras H that are equipped with an invariant weight τ, we construct the reduced and universal dual C^*-algebras $C^*_r(H, \delta_H)$ and $C^*(H, \delta_H)$, respectively. We show that the reduced dual C^*-algebra of the Hopf C^*-algebra $(c_0(\mathbb{N}), \alpha_{\mathbb{N}})$ is isomorphic to $C^*(\mathbb{N})$ and that the reduced dual C^*-algebra of the Hopf C^*-algebra $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ is isomorphic to the C^*-algebra $c_0(\mathbb{N})$. This result can be viewed as a generalization of Pontryagin's duality theorem for locally compact, abelian groups to the semigroup of natural numbers.

Finally, we construct the reduced crossed product $A \rtimes_\delta H$ for a dynamical cosystem (A, H, δ) and examine the reduced crossed product $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$, which will be of great importance in the next chapter.

5.1 Hopf C^*-algebras

In this section we define the notion of a Hopf C^*-algebra and show that Hopf C^*-algebras can be viewed as a generalization of locally compact semigroups by introducing the Hopf C^*-algebra $C_0(\mathcal{S})$ for a locally compact semigroup $\mathcal{S}$.

Locally compact semigroups can be represented on a Hilbert space, they can act on a C^*-algebra and given a dynamical system $(A, \mathcal{S}, \alpha)$, where α is an action of a locally compact semigroup $\mathcal{S}$ on a C^*-algebra A, there may be covariant representations of $(A, \mathcal{S}, \alpha)$ on a Hilbert space. Since Hopf C^*-algebras generalize locally compact semigroups, we are interested in the generalizations of these notions and so we study the concepts of corepresentations, coactions, dynamical cosystems and covariant representations of dynamical cosystems.

Definition 5.1 (Hopf C^*-algebras)
A *Hopf C^*-algebra* is a C^*-algebra H together with a nondegenerate injective *-homomorphism $\delta_H \colon H \to M(H \otimes H)$ such that

$$(\delta_H \otimes \mathrm{id}) \circ \delta_H = (\mathrm{id} \otimes \delta_H) \circ \delta_H \tag{5.1}$$

holds. δ_H is called a *comultiplication* on H and Equation (5.1) is the *coassociativity identity.*

Definition 5.2 (Locally compact semigroups)
Let $\mathcal{S}$ be a semigroup as well as a locally compact topological space such that the multiplication is continuous. Then we call $\mathcal{S}$ a *locally compact semigroup.* We say that $\mathcal{S}$ is *right-cancellative* if $mp = np$ yields $m = n$ for all $m, n, p \in \mathcal{S}$.

Example 5.3 (Locally compact semigroups)

- Locally compact groups are of course also locally compact semigroups. Since they are groups, they are also right-cancellative. Examples are $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$ and $(S^1, \cdot)$, where $S^1 := \{z \in \mathbb{C} : \ |z| = 1\}$.
- Subsemigroups of locally compact groups that are the intersection of an open and a closed subset are locally compact semigroups. Since they are subsemigroups of a group, they are also right-cancellative. Examples are $(\mathbb{R}^+, +)$ and $(\mathbb{N}, +)$.

As already mentioned above, the following example is our standard example for a Hopf C^*-algebra, which shows that Hopf C^*-algebras generalize locally compact semigroups.

Example 5.4 (The Hopf C^*-algebra $C_0(\mathcal{S})$)
Let $\mathcal{S}$ be an arbitrary locally compact semigroup and let $C_0(\mathcal{S})$ be the C^*-algebra of complex functions on $\mathcal{S}$ that vanish at infinity. We define a comultiplication $\alpha_{\mathcal{S}} : C_0(\mathcal{S}) \to M(C_0(\mathcal{S}) \otimes C_0(\mathcal{S})) \cong C_b(\mathcal{S} \times \mathcal{S})$ by the use of the multiplication in $\mathcal{S}$. We set

$$\alpha_{\mathcal{S}}(f)(s,t) := f(st) \tag{5.2}$$

for $f \in C_0(\mathcal{S})$ and $s, t \in \mathcal{S}$. Then $\alpha_{\mathcal{S}}$ is a nondegenerate injective $*$-homomorphism and the associativity of the multiplication in $\mathcal{S}$ yields the coassociativity of $\alpha_{\mathcal{S}}$.

As a special case of this example we consider the situation when $\mathcal{S}$ is the semigroup of natural numbers.

Example 5.5 (The Hopf C^*-algebra $c_0(\mathbb{N})$)
Let $c_0(\mathbb{N})$ be the C^*-algebra of all complex sequences $a = (a(n))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a(n) = 0$, with addition, multiplication and involution defined componentwise and with the norm $\|a\| = \sup_{n\in\mathbb{N}} |a(n)|$. For $n \in \mathbb{N}$, let $e_n \in c_0(\mathbb{N})$ be the sequence with $e_n(k) = \delta_{n,k}$. Then the defining Equation (5.2) for the comultiplication $\alpha_{\mathbb{N}}$ on $c_0(\mathbb{N})$ yields

$$\alpha_{\mathbb{N}}(e_n) = \sum_{m=0}^{n} e_m \otimes e_{n-m}.$$

Example 5.6 (The Hopf C^*-algebra $C^*(\mathbb{N})$)
Let $C^*(\mathbb{N})$ be the Toeplitz algebra, which is the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by the forward unilateral shift operator S. S is an isometry and the linear span of $\{S^m(S^*)^n : m, n \in \mathbb{N}\}$ is a dense subset of $C^*(\mathbb{N})$. We can define a comultiplication $\delta_{\mathbb{N}} : C^*(\mathbb{N}) \to C^*(\mathbb{N}) \otimes C^*(\mathbb{N})$ by setting

$$\delta_{\mathbb{N}}(S^n) := S^n \otimes S^n.$$

Since $C^*(\mathbb{N})$ is the universal C^*-algebra generated by a non-unitary isometry (see [5] for details), the above identity yields a well defined *-homomorphism that is nondegenerate and injective. It is easy to see that $\delta_{\mathbb{N}}$ satisfies the comultiplication identity (5.1) and so $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ is a Hopf C^*-algebra.

Definition 5.7 (Corepresentations)
A *corepresentation* of a Hopf C^*-algebra H on a Hilbert space $\mathcal{H}$ is a partial isometry $V \in M(\mathcal{K}(\mathcal{H}) \otimes H)$ such that

$$(\mathrm{id} \otimes \delta_H)(V) = V_{12}V_{13} \tag{5.3}$$

as elements of $M(\mathcal{K}(\mathcal{H}) \otimes H \otimes H)$. By V_{12}, we denote the element of $M(\mathcal{K}(\mathcal{H}) \otimes H \otimes H)$ that is V acting on the first and second factors, i.e., $V_{12} = V \otimes 1$. V_{13} is V acting on the first and third factors, that is $V_{13} = \sigma_{23}(V \otimes 1)\sigma_{23}$, where σ_{23} is the flip automorphism on $M(\mathcal{K}(\mathcal{H}) \otimes H \otimes H)$ that satisfies $\sigma_{23}(K \otimes a \otimes b) = K \otimes b \otimes a$.

The following example demonstrates that corepresentations of Hopf C^*-algebras generalize the notion of representations of semigroups. It shows that there is a 1-1-correspondence between the representations of a locally compact semigroup $\mathcal{S}$ and the corepresentations of $C_0(\mathcal{S})$.

Example 5.8 (Corepresentations of $C_0(\mathcal{S})$)
Let $\mathcal{S}$ be a locally compact semigroup that is right cancellative and let $\{V_s : s \in \mathcal{S}\}$ be a family of partial isometries on a Hilbert space $\mathcal{H}$ such that $V_s V_t = V_{st}$ for all $s, t \in \mathcal{S}$ and such that the mapping $s \mapsto V_s\xi$ is continuous for all $\xi \in \mathcal{H}$. We say that the family $\{V_s : s \in \mathcal{S}\}$ is a *semigroup representation* of $\mathcal{S}$ on $\mathcal{H}$. Since the strong topology on $\mathcal{L}(\mathcal{H})$ is the same as the strict topology on $M(\mathcal{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$, we know that the function $V := (s \mapsto V_s)$ is continuous from $\mathcal{S}$ to $M(\mathcal{K}(\mathcal{H}))$ in the strict topology and hence, $V \in C_b(\mathcal{S}, M(\mathcal{K}(\mathcal{H}))) \subseteq M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathcal{S}))$. Now

$$V_{12}V_{13} = ((s,t) \mapsto V_s V_t) = ((s,t) \mapsto V_{st}) = (\mathrm{id} \otimes \alpha_{\mathcal{S}})(V)$$

and

$$VV^*V = (s \mapsto V_s)(s \mapsto V_s^*)(s \mapsto V_s) = (s \mapsto V_s V_s^* V_s) = (s \mapsto V_s) = V.$$

This shows that V is a partial isometry and thus a corepresentation of $C_0(\mathcal{S})$ on $\mathcal{H}$. Hence, each semigroup representation $\{V_s : s \in \mathcal{S}\}$ of $\mathcal{S}$ on a Hilbert space $\mathcal{H}$ gives rise to a corepresentation V of $C_0(\mathcal{S})$ on $\mathcal{H}$.

On the other hand, let $V \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathcal{S}))$ be a corepresentation of $C_0(\mathcal{S})$ on a Hilbert space $\mathcal{H}$ and let $\varepsilon_s \colon C_0(\mathcal{S}) \to \mathbb{C}$ be the evaluation at s, i.e., $\varepsilon_s(f) = f(s)$. For $s \in \mathcal{S}$, we set

$$V_s := (\mathrm{id} \otimes \varepsilon_s)(V) \in \mathcal{L}(\mathcal{H}).$$

Since $(\mathrm{id} \otimes \varepsilon_s) \colon M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathcal{S})) \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, we have

$$V_s V_s^* V_s = (\mathrm{id} \otimes \varepsilon_s)(V)(\mathrm{id} \otimes \varepsilon_s)(V)^*(\mathrm{id} \otimes \varepsilon_s)(V) = (\mathrm{id} \otimes \varepsilon_s)(VV^*V) = (\mathrm{id} \otimes \varepsilon_s)(V) = V_s,$$

which shows that V_s is a partial isometry. Moreover, we have

$$\begin{aligned} V_s V_t &= (\mathrm{id} \otimes \varepsilon_s)(V)(\mathrm{id} \otimes \varepsilon_t)(V) = (\mathrm{id} \otimes \varepsilon_s \otimes \varepsilon_t)(V_{12}V_{13}) \\ &= (\mathrm{id} \otimes \varepsilon_s \otimes \varepsilon_t)(\mathrm{id} \otimes \alpha_{\mathcal{S}})(V) = (\mathrm{id} \otimes \varepsilon_{st})(V) = V_{st}, \end{aligned}$$

since $(\varepsilon_s \otimes \varepsilon_t)(\alpha_{\mathcal{S}}(f)) = \alpha_{\mathcal{S}}(f)(s,t) = f(st) = \varepsilon_{st}(f)$. It is easy to see that the mapping $s \mapsto V_s(\xi) = (\mathrm{id} \otimes \varepsilon_s)(V)(\xi)$ is continuous, since $V \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(\mathcal{S}))$. Hence, each corepresentation V of $C_0(\mathcal{S})$ on a Hilbert space $\mathcal{H}$ gives rise to a semigroup representation $\{V_s : s \in \mathcal{S}\}$ of $\mathcal{S}$ on $\mathcal{H}$ and so we see that there is a 1-1-correspondence between the semigroup representations of $\mathcal{S}$ and the corepresentations of $C_0(\mathcal{S})$.

A locally compact semigroup $\mathcal{S}$ can act on a C^*-algebra A via an action α and in this situation we call the triple $(A, \mathcal{S}, \alpha)$ a dynamical system. The following definition generalizes these concepts to the Hopf C^*-algebra situation.

Definition 5.9 (Coactions, dynamical cosystems)
A *coaction* of a Hopf C^*-algebra H on a C^*-algebra A is a nondegenerate injective *-homomorphism $\delta : A \to M(A \otimes H)$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_H) \circ \delta. \tag{5.4}$$

In this situation, we call the triple (A, H, δ) a *dynamical cosystem*.

Example 5.10 **(a)** Any Hopf C^*-algebra coacts trivially on the complex numbers $\mathbb{C}$ via $z \mapsto z \otimes 1$.

(b) The comultiplication δ_H on a Hopf C^*-algebra H is a coaction of H on itself, which follows directly from Equation (5.1).

The following example shows that coactions of Hopf C^*-algebras are in fact a generalization of the concept of actions of locally compact semigroups.

Example 5.11 (Coactions of $C_0(\mathcal{S})$)
Let $\mathcal{S}$ be a locally compact semigroup, let A be a C^*-algebra and let $\{\alpha_s \colon s \in \mathcal{S}\}$ be an action of $\mathcal{S}$ on A, i.e., $\{\alpha_s \colon s \in \mathcal{S}\}$ is a family of unital *-endomorphisms $\alpha_s \in \mathrm{End}(M(A))$ such that $\alpha_s \circ \alpha_t = \alpha_{st}$ and such that the mappings $s \mapsto \alpha_s(a)$ are continuous for all $a \in A$. In other words, we have that $(s \mapsto \alpha_s(a)) \in C_b(\mathcal{S}, M(A)) \subseteq M(A \otimes C_0(\mathcal{S}))$. Thus, we get an injective and nondegenerate *-homomorphism $\alpha : A \to M(A \otimes C_0(\mathcal{S}))$ by setting

$$\alpha(a) := (s \mapsto \alpha_s(a)).$$

Moreover, we have

$$(\mathrm{id} \otimes \alpha_{\mathcal{S}}) \circ \alpha(a) = ((s,t) \mapsto \alpha_{st}(a)) = ((s,t) \mapsto \alpha_s(\alpha_t(a))) = (\alpha \otimes \mathrm{id}) \circ \alpha(a).$$

So the action $\{\alpha_s \colon s \in \mathcal{S}\}$ of $\mathcal{S}$ on A yields a coaction α of $C_0(\mathcal{S})$ on A. On the other hand, let α be a coaction of $C_0(\mathcal{S})$ on a C^*-algebra A. For $s \in \mathcal{S}$ and $a \in A$, we set

$$\alpha_s(a) := (\mathrm{id} \otimes \varepsilon_s)(\alpha(a)) \in M(A).$$

Then $\alpha_s \colon A \to M(A)$ is a nondegenerate *-homomorphism, since α and $(\mathrm{id} \otimes \varepsilon_s)$ are nondegenerate *-homomorphisms. Hence, α_s can be extended to a unital *-endomorphism in End($M(A)$) that we also denote by α_s. To see that the mapping $s \mapsto \alpha_s$ is a semigroup homomorphism, we compute

$$\begin{aligned}\alpha_s \circ \alpha_t(a) &= \alpha_s \circ (\mathrm{id} \otimes \varepsilon_t) \circ \alpha(a) = (\mathrm{id} \otimes \varepsilon_s) \circ \alpha \circ (\mathrm{id} \otimes \varepsilon_t) \circ \alpha(a) \\ &= (\mathrm{id} \otimes \varepsilon_s \otimes \varepsilon_t)((\alpha \otimes \mathrm{id}) \circ \alpha(a)) = (\mathrm{id} \otimes \varepsilon_s \otimes \varepsilon_t)((\mathrm{id} \otimes \alpha_{\mathcal{S}}) \circ \alpha(a)) \\ &= (\mathrm{id} \otimes \varepsilon_{st})(\alpha(a)) = \alpha_{st}(a).\end{aligned}$$

It is clear that the mapping $s \mapsto \alpha_s(a) = (\mathrm{id}\otimes\varepsilon_s)(\alpha(a))$ is continuous and thus, the family $\{\alpha_s \colon s \in \mathcal{S}\}$ is an action of $\mathcal{S}$ on A. Hence, each coaction of $C_0(\mathcal{S})$ on a C^*-algebra A gives rise to an action of $\mathcal{S}$ on A and so there is a 1-1-correspondence between the actions of $\mathcal{S}$ on A and the coactions of $C_0(\mathcal{S})$ on A.

Definition 5.12 (Covariant homomorphisms and covariant representations)
Let (A, H, δ) be a dynamical cosystem and let B be C^*-algebra. A *covariant homomorphism* of (A, H, δ) into $M(B)$ is a pair (ϕ, v) where $\phi : A \to M(B)$ is a nondegenerate *-homomorphism and $v \in M(B \otimes H)$ is a partial isometry such that

$$(\mathrm{id} \otimes \delta_H)(v) = v_{12}v_{13} \quad \text{and} \tag{5.5}$$

$$(\phi \otimes \mathrm{id})(\delta(a))v = v(\phi(a) \otimes 1) \quad \text{for all } a \in A. \tag{5.6}$$

A *covariant representation* of (A, H, δ) on a Hilbert space $\mathcal{H}$ is a covariant homomorphism of (A, H, δ) into $M(\mathcal{K}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H})$. Hence, it is a pair (π, V), where $\pi : A \to \mathcal{L}(\mathcal{H})$ is a nondegenerate *-representation and $V \in M(\mathcal{K}(\mathcal{H}) \otimes H)$ is a corepresentation of H, satisfying

$$(\pi \otimes \mathrm{id})(\delta(a))V = V(\pi(a) \otimes 1) \quad \text{for all } a \in A. \tag{5.7}$$

In the following example we want to examine how the covariant representations of dynamical cosystems generalize the concept of covariant representations of dynamical systems.

Example 5.13 (Covariant representations of $(A, C_0(\mathcal{S}), \alpha)$)
Let $\{\alpha_s \colon s \in \mathcal{S}\}$ be an action of $\mathcal{S}$ on a C^*-algebra A. We say that $(\pi, \{V_s \colon s \in \mathcal{S}\})$ is a covariant representation of $(A, \mathcal{S}, \{\alpha_s \colon s \in \mathcal{S}\})$ if π is a *-representation of A on a Hilbert space $\mathcal{H}$ and if $\{V_s \colon s \in \mathcal{S}\}$ is a semigroup representation of $\mathcal{S}$ on $\mathcal{H}$ such that

$$\pi(\alpha_s(a))V_s = V_s\pi(a) \qquad \text{for all} \quad a \in A.$$

Let α be the coaction of $C_0(\mathcal{S})$ on A coming from the action $\{\alpha_s \colon s \in \mathcal{S}\}$ of $\mathcal{S}$ on A (see Example 5.11) and let V be the corepresentation of $C_0(\mathcal{S})$ coming from the semigroup representation $\{V_s \colon s \in \mathcal{S}\}$ of $\mathcal{S}$ (see Example 5.8). Then, for $a \in A$ we have

$$\begin{aligned}(\pi \otimes \mathrm{id})(\alpha(a))V &= (s \mapsto \pi(\alpha_s(a)))(s \mapsto V_s) = (s \mapsto \pi(\alpha_s(a))V_s) \\ &= (s \mapsto V_s\pi(a)) = (s \mapsto V_s)(s \mapsto \pi(a)) \\ &= V(\pi(a) \otimes 1),\end{aligned}$$

which shows that the pair (π, V) is a covariant representation of the triple $(A, C_0(\mathcal{S}), \alpha)$ on the Hilbert space $\mathcal{H}$.

On the other hand, let (π, V) be a covariant representation of $(A, C_0(\mathcal{S}), \alpha)$ on a Hilbert space $\mathcal{H}$ and let $\{V_s\colon s \in \mathcal{S}\}$ and $\{\alpha_s\colon s \in \mathcal{S}\}$ be the corresponding semigroup representation of $\mathcal{S}$ on $\mathcal{H}$ and action of $\mathcal{S}$ on A, respectively. Then we compute

$$\begin{aligned}\pi(\alpha_s(a))V_s &= \pi((\mathrm{id}\otimes\varepsilon_s)(\alpha(a)))(\mathrm{id}\otimes\varepsilon_s)(V)\\ &= (\mathrm{id}\otimes\varepsilon_s)((\pi\otimes\mathrm{id})(\alpha(a))V) = (\mathrm{id}\otimes\varepsilon_s)(V(\pi(a)\otimes 1)) = V_s\pi(a),\end{aligned}$$

which shows that $(\pi, \{V_s\colon s \in \mathcal{S}\})$ is a covariant representation of $(A, \mathcal{S}, \{\alpha_s\colon s \in \mathcal{S}\})$. Hence, there is a 1-1-correspondence between the covariant representations of $(A, C_0(\mathcal{S}), \alpha)$ and the covariant representations of $(A, \mathcal{S}, \{\alpha_s\colon s \in \mathcal{S}\})$.

Example 5.14 (Covariant representations of $(\mathbb{C}, H, \delta)$)
Let δ be the trivial coaction of a Hopf C^*-algebra H on the complex numbers introduced in Example 5.10 (a) and let V be any corepresentation of H. Since any nondegenerate *-representation π of the complex numbers is of the form $\pi(z) = z\cdot\mathrm{id}$ for all $z \in \mathbb{C}$, Equation (5.7) holds trivially and so (π, V) is a covariant representation of $(\mathbb{C}, H, \delta)$. Thus, we have seen that the covariant representations of $(\mathbb{C}, H, \delta)$ are just the corepresentations of H.

5.2 The regular covariant representation

Given a locally compact group G, it is a classical result (see [12]) that there exists a right invariant linear functional I on $C_c(G)$, the linear space of all continuous functions on G with compact support. Right invariance means that $I(f_s) = I(f)$ for all $f \in C_c(G)$ and $s \in G$, where $f_s(t) = f(ts)$. This functional is unique up to a multiplicative constant and it is called the right Haar integral on $C_c(G)$. Corresponding to I, there is a right invariant measure μ on G, i.e., $\mu(Ms) = \mu(M)$ for all $M \subset G$ and $s \in G$. μ is unique up to a multiplicative constant, too, and it is called the right Haar measure. Using the right invariance of the right Haar measure μ, it is possible to construct an important group representation of G on the Hilbert space $L^2(G, \mu)$, the right regular representation ρ. Let π be the representation of $C_0(G)$ on $L^2(G, \mu)$ by multiplication operators, i.e., $\pi(f)(g) := fg$ for $f \in C_0(G)$ and $g \in L^2(G, \mu)$. Then it is easy to see that the pair (π, ρ) is a covariant representation of the dynamical system $(C_0(G), G, \alpha)$, where α is the action of G on $C_0(G)$ by right translation. We call (π, ρ) the right regular covariant representation of $(C_0(G), G, \alpha)$ on $L^2(G, \mu)$.

Now for a locally compact semigroup $\mathcal{S}$, it is not true in general that there exists a right invariant measure μ on $\mathcal{S}$. Thus, it is not always possible to construct the regular representation of $\mathcal{S}$ or the regular covariant representation of $(C_0(\mathcal{S}), \mathcal{S}, \alpha)$. So we cannot expect to be able to construct a regular corepresentation of an arbitrary Hopf C^*-algebra

H or a regular covariant representation of the dynamical cosystem (H, H, δ_H). The aim of this section is to find a condition on the Hopf C^*-algebra H that makes it possible to construct a regular covariant representation of (H, H, δ_H). This condition will be the existence of an invariant weight τ on H and we will see that for the Hopf C^*-algebra $C_0(\mathcal{S})$, this condition is equivalent to the existence of a right invariant Radon measure μ on $\mathcal{S}$ with $\operatorname{supp}(\mu) = \mathcal{S}$.

We begin this section by giving some information about weights on C^*-algebras. The standard reference for lower semi-continuous weights is [6].

Definition 5.15 (Weights on C^*-algebras)
Let A be a C^*-algebra and $\tau : A^+ \to [0, \infty]$ a function such that

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in A^+$ and
2. $\tau(rx) = r\tau(x)$ for all $r \in \mathbb{R}^+$ and $x \in A^+$.

Then we call τ a *weight* on A.

Let τ be a weight on a C^*-algebra A. We will use the following standard notations:

- $\mathcal{M}_\tau^+ = \{a \in A^+ : \tau(a) < \infty\}$,
- $\mathcal{N}_\tau = \{a \in A : \tau(a^*a) < \infty\}$,
- $\mathcal{M}_\tau = \operatorname{span} \mathcal{M}_\tau^+ = \mathcal{N}_\tau^* \mathcal{N}_\tau$.

$\mathcal{N}_\tau$ is a left ideal in A and hence, $\mathcal{M}_\tau \subseteq \mathcal{N}_\tau$. Furthermore, $\mathcal{M}_\tau$ is a sub *-algebra of A and $\mathcal{M}_\tau^+ = \mathcal{M}_\tau \cap A^+$.

There exists a unique linear map $\varphi : \mathcal{M}_\tau \to \mathbb{C}$ such that $\varphi(x) = \tau(x)$ for all $x \in \mathcal{M}_\tau^+$. Hence, we get a unique extension of τ to all of $\mathcal{M}_\tau$ by setting $\tau(x) := \varphi(x)$ for all $x \in \mathcal{M}_\tau$.

We say that τ is *densely defined* if $\mathcal{M}_\tau^+$ is dense in A^+. This is the case if and only if $\mathcal{M}_\tau$ is dense in A and if and only if $\mathcal{N}_\tau$ is dense in A. We call τ *lower semi-continuous* if we have for every $\lambda \in \mathbb{R}^+$ that the set $\{a \in A^+ : \tau(a) \leq \lambda\}$ is closed.

Note that for a locally compact topological space X there is a 1-1 correspondence between the densely defined, lower semi-continuous weights on $C_0(X)$ and the Radon measures on X, since the integral corresponding to a Radon measure on X is a densely defined, lower semi-continuous weight on $C_0(X)$. We recall that a *Radon measure* μ on a locally compact topological space X is a Borel measure on X that is locally finite and inner regular, i.e.,

- for each $x \in X$ there is an open neighborhood V_x such that $\mu(V_x) < \infty$,

- $\mu(B) = \sup\{\mu(K) \colon K \subset B, K \text{ compact}\}$ for all Borel sets $B \subseteq X$.

Definition 5.16 (GNS-construction for a weight)
Let τ be a weight on a C^*-algebra A. A *GNS-construction* for τ is a triple $(\mathcal{H}_\tau, \pi_\tau, \Lambda_\tau)$ such that

- $\mathcal{H}_\tau$ is a Hilbert space;
- Λ_τ is a linear map from $\mathcal{N}_\tau$ into $\mathcal{H}_\tau$ such that
  - $\Lambda_\tau(\mathcal{N}_\tau)$ is dense in $\mathcal{H}_\tau$ and
  - we have for every $a, b \in \mathcal{N}_\tau$, that $(\Lambda_\tau(a) \mid \Lambda_\tau(b)) = \tau(a^*b)$;
- π_τ is a *-representation of A on $\mathcal{H}_\tau$ such that $\pi_\tau(a)\Lambda_\tau(b) = \Lambda_\tau(ab)$ for every $a \in A$ and $b \in \mathcal{N}_\tau$.

It is easy to produce such a GNS-construction for any weight on a C^*-algebra. Moreover, this construction is unique up to a unitary transformation. If τ is lower semi-continuous and densely defined, we have that π_τ is nondegenerate. If τ is a lower semi-continuous and densely defined weight on the C^*-algebra $C_0(X)$ for a locally compact topological space X, then π_τ is faithful if and only if the corresponding Radon measure μ on X satisfies $\mathrm{supp}(\mu) = X$.

For the rest of this section, let H be a Hopf C^*-algebra, τ a densely defined, lower semi-continuous weight on H and $(\mathcal{H}_\tau, \pi_\tau, \Lambda_\tau)$ its GNS-construction. We define an operator $V : \mathcal{N}_\tau \otimes_{\mathrm{alg}} H \to M(H \otimes H)$ by setting

$$V(\sum x_i \otimes a_i) := \sum \delta_H(x_i)(1 \otimes a_i) \tag{5.8}$$

on finite sums $\sum x_i \otimes a_i$ with $x_i \in \mathcal{N}_\tau$ and $a_i \in H$.

Proposition 5.17 *If the operator V defined in Equation (5.8) extends to a partial isometry $V_\tau \in \mathcal{L}(\mathcal{H}_\tau \otimes H)$, then V_τ is a corepresentation of H on $\mathcal{H}_\tau$.*

To prove this proposition, we need the following lemma:

Lemma 5.18 *The C^*-arrows $(\mathcal{H}_\tau \otimes H) \otimes_{H\ \delta_H} (H \otimes H)$ and $\mathcal{H}_\tau \otimes_{\mathbb{C}} (H \otimes H)$ are unitarily equivalent.*

Proof: We define $U : (\mathcal{H}_\tau \otimes H) \otimes_{H\ \delta_H} (H \otimes H) \to \mathcal{H}_\tau \otimes_{\mathbb{C}} (H \otimes H)$ by setting

$$U((\xi \otimes x) \otimes (a \otimes b)) := \xi \otimes (\delta_H(x)(a \otimes b))$$

for $\xi \in \mathcal{H}_\tau, x, a, b \in H$ and we want to show that U is a unitary bimodule mapping. We start by proving that U is an isometry. Let $\xi, \eta \in \mathcal{H}_\tau$ and $x, y, a, b, c, d \in H$ arbitrary. Then we have

$$\begin{aligned} & (\,(\xi \otimes x) \otimes (a \otimes b) \,|\, (\eta \otimes y) \otimes (c \otimes d)\,)_{H \otimes H} \\ = \;& (\, a \otimes b \,|\, \delta_H((\, \xi \otimes x \,|\, \eta \otimes y\,)_H)(c \otimes d)\,)_{H \otimes H} \\ = \;& (\, a \otimes b \,|\, \delta_H((\, x \,|\, (\,\xi \,|\, \eta\,)_{\mathbb{C}} \cdot y\,)_H)(c \otimes d)\,)_{H \otimes H} = (a^* \otimes b^*)(\,\xi \,|\, \eta\,)_{\mathbb{C}} \delta_H(x^* y)(c \otimes d) \end{aligned}$$

and

$$\begin{aligned} & (\,U((\xi \otimes x) \otimes (a \otimes b)) \,|\, U((\eta \otimes y) \otimes (c \otimes d))\,)_{H \otimes H} \\ = \;& (\,\xi \otimes (\delta_H(x)(a \otimes b)) \,|\, \eta \otimes (\delta_H(y)(c \otimes d))\,)_{H \otimes H} \\ = \;& (\,\delta_H(x)(a \otimes b) \,|\, (\,\xi \,|\, \eta\,)_{\mathbb{C}} \cdot \delta_H(y)(c \otimes d)\,)_{H \otimes H} = (a^* \otimes b^*)\delta_H(x)^*(\,\xi \,|\, \eta\,)_{\mathbb{C}} \delta_H(y)(c \otimes d) \\ = \;& (a^* \otimes b^*)(\,\xi \,|\, \eta\,)_{\mathbb{C}} \delta_H(x^* y)(c \otimes d). \end{aligned}$$

Moreover, using an approximate unit, it is easy to see that U is surjective, since δ_H is nondegenerate. Hence, U is a surjective isometry and thus a unitary and this would finish the proof. But we also want to show that U is a unitary in the case that H is unital, by providing the adjoint U^* explicitly and showing that $UU^* = \mathrm{id}_{\mathcal{H}_\tau \otimes (H \otimes H)}$. So we suppose that H is unital. Then we have that $U^* : \mathcal{H}_\tau \otimes (H \otimes H) \to (\mathcal{H}_\tau \otimes H) \otimes_{\delta_H} (H \otimes H)$ is given by

$$U^*(\xi \otimes (a \otimes b)) = (\xi \otimes 1) \otimes (a \otimes b)$$

for $\xi \in \mathcal{H}_\tau, a, b \in H$ as the following computation shows for $\xi, \eta \in \mathcal{H}_\tau, x, a, b, c, d \in H$:

$$\begin{aligned} (\,\eta \otimes (c \otimes d) \,|\, U((\xi \otimes x) \otimes (a \otimes b))\,)_{H \otimes H} &= (\,\eta \otimes (c \otimes d) \,|\, \xi \otimes \delta_H(x)(a \otimes b)\,)_{H \otimes H} \\ &= (\,c \otimes d \,|\, (\,\eta \,|\, \xi\,)_{\mathbb{C}} \cdot \delta_H(x)(a \otimes b)\,)_{H \otimes H} \\ &= (\,\eta \,|\, \xi\,)_{\mathbb{C}}(c^* \otimes d^*)\delta_H(x)(a \otimes b) \quad \text{and} \end{aligned}$$

$$\begin{aligned} (\,U^*(\eta \otimes (c \otimes d)) \,|\, (\xi \otimes x) \otimes (a \otimes b)\,)_{H \otimes H} &= (\,(\eta \otimes 1) \otimes (c \otimes d) \,|\, (\xi \otimes x) \otimes (a \otimes b)\,)_{H \otimes H} \\ &= (\,c \otimes d \,|\, \delta_H((\,\eta \otimes 1 \,|\, \xi \otimes x\,)_H)(a \otimes b)\,)_{H \otimes H} \\ &= (c^* \otimes d^*)\delta_H((\,1 \,|\, (\,\eta \,|\, \xi\,)_{\mathbb{C}} \cdot x\,)_H)(a \otimes b) \\ &= (\,\eta \,|\, \xi\,)_{\mathbb{C}}(c^* \otimes d^*)\delta_H(x)(a \otimes b). \end{aligned}$$

Since U is an isometry, we have that $U^*U = \mathrm{id}_{(\mathcal{H}_\tau \otimes H) \otimes_{\delta_H} (H \otimes H)}$ and so it remains to prove that $UU^* = \mathrm{id}_{\mathcal{H}_\tau \otimes (H \otimes H)}$. But for $\xi \in \mathcal{H}_\tau, a, b \in H$ we get

$$\begin{aligned} UU^*(\xi \otimes (a \otimes b)) &= U((\xi \otimes 1) \otimes (a \otimes b)) = \xi \otimes \delta_H(1)(a \otimes b) = \xi \otimes (1 \otimes 1)(a \otimes b) \\ &= \xi \otimes (a \otimes b). \end{aligned}$$

Hence, U is a unitary bimodule mapping and so the C^*-arrows $(\mathcal{H}_\tau \otimes H) \otimes_{H\ \delta_H} (H \otimes H)$ and $\mathcal{H}_\tau \otimes_{\mathbb{C}} (H \otimes H)$ are unitarily equivalent. $\square$

Proof of Proposition 5.17: We have to show that $(\mathrm{id} \otimes \delta_H)(V_\tau) = (V_\tau)_{12}(V_\tau)_{13}$. To do so, we use the fact that $M(\mathcal{K}(\mathcal{H}_\tau) \otimes H) \cong \mathcal{L}(\mathcal{H}_\tau \otimes H)$. We suppose that H is unital to avoid having to deal with approximate units. First we want to show that, for $T \in \mathcal{L}(\mathcal{H}_\tau \otimes H)$, the identity

$$(\mathrm{id} \otimes \delta_H)(T) = U(T \otimes \mathrm{id})U^*$$

holds, where U is the unitary bimodule mapping constructed in the preceding lemma. Let $T = K \otimes h$ for $K \in \mathcal{K}(\mathcal{H}_\tau)$, $h \in H$ and let $\xi \in \mathcal{H}_\tau$, $a, b \in H$. Then we have

$$(\mathrm{id} \otimes \delta_H)(T)(\xi \otimes a \otimes b) = (K \otimes \delta_H(h))(\xi \otimes a \otimes b) = K(\xi) \otimes \delta_H(h)(a \otimes b) \quad \text{and}$$

$$\begin{aligned} U(T \otimes \mathrm{id})U^*(\xi \otimes a \otimes b) &= U(K \otimes h \otimes \mathrm{id})(\xi \otimes 1 \otimes a \otimes b) = U(K(\xi) \otimes h \otimes a \otimes b) \\ &= K(\xi) \otimes \delta_H(h)(a \otimes b). \end{aligned}$$

Since $(\mathrm{id} \otimes \delta_H)(V_\tau)$ and $(V_\tau)_{12}(V_\tau)_{13}$ are elements of $M(\mathcal{K}(\mathcal{H}_\tau) \otimes H \otimes H) \cong \mathcal{L}(\mathcal{H}_\tau \otimes H \otimes H)$ and thus $\mathbb{C} - (H \otimes H)$-bimodule mappings it suffices to prove that they agree on elements of the form $\xi \otimes 1 \otimes 1$. We compute as follows:

$$\begin{aligned} (\mathrm{id} \otimes \delta_H)(V_\tau)(\xi \otimes 1 \otimes 1) &= U(V_\tau \otimes \mathrm{id})U^*(\xi \otimes 1 \otimes 1) = U(V_\tau \otimes \mathrm{id})(\xi \otimes 1 \otimes 1 \otimes 1) \\ &= U(\delta_H(\xi)(1 \otimes 1) \otimes 1 \otimes 1) = U(\delta_H(\xi) \otimes 1 \otimes 1) \\ &= (\mathrm{id} \otimes \delta_H)(\delta_H(\xi)) \quad \text{and} \end{aligned}$$

$$\begin{aligned} (V_\tau)_{12}(V_\tau)_{13}(\xi \otimes 1 \otimes 1) &= (V_\tau)_{12}(\sigma_{2,3}((V_\tau \otimes \mathrm{id})(\xi \otimes 1 \otimes 1))) = (V_\tau)_{12}(\sigma_{2,3}(\delta_H(\xi) \otimes 1)) \\ &= (V_\tau \otimes \mathrm{id})(\sigma_{2,3}(\delta_H(\xi) \otimes 1)) = (\delta_H \otimes \mathrm{id})(\delta_H(\xi)) \end{aligned}$$

and equality follows from the coassociativity identity (5.1). $\square$

Now we want to take a closer look at our standard example, the Hopf C^*-algebra $C_0(\mathcal{S})$. We will see that the condition of V extending to a partial isometry $V_\tau \in \mathcal{L}(\mathcal{H}_\tau \otimes C_0(\mathcal{S}))$ corresponds to the right invariance of the Radon measure μ on $\mathcal{S}$ that induces the weight τ on $C_0(\mathcal{S})$.

Definition 5.19 (Invariant Radon measures on semigroups)
Let $\mathcal{S}$ be locally compact semigroup and let μ be a Radon measure on $\mathcal{S}$. We say that μ is *right invariant* if

$$\mu(M) = \mu(Ms)$$

for all Borel sets $M \subseteq \mathcal{S}$ and all $s \in \mathcal{S}$.

Example 5.20 The counting measure on $(\mathbb{N},+)$ and the Lebesgue measure on $(\mathbb{R}^+,+)$ are examples of right invariant Radon measures on locally compact semigroups.

Let $\mathcal{S}$ be a locally compact semigroup and let μ be a right invariant Radon measure on $\mathcal{S}$ with $\operatorname{supp}(\mu)=\mathcal{S}$. Then

$$\tau(f) := \int_{\mathcal{S}} f(s)d\mu(s)$$

defines a densely defined, lower semi-continuous weight on $C_0(\mathcal{S})$ and the GNS-representation for τ is $(L^2(\mathcal{S},\mu),\pi_\tau,\Lambda_\tau)$, where π_τ is the representation of $C_0(\mathcal{S})$ on $L^2(\mathcal{S},\mu)$ by multiplication operators and Λ_τ is the canonical embedding of $C_0(\mathcal{S})\cap L^2(\mathcal{S},\mu)$ into $L^2(\mathcal{S},\mu)$. It is clear that π_τ is a nondegenerate *-representation and it is faithful since $\operatorname{supp}(\mu)=\mathcal{S}$. We want to construct a coisometry $V_\tau\in\mathcal{L}(L^2(\mathcal{S},\mu)\otimes C_0(\mathcal{S}))$ that extends the operator V from Equation (5.8). For $t\in\mathcal{S}$ we set

$$(V_t f)(s) := f(st)$$

with $f\in L^2(\mathcal{S},\mu)$ and $s\in\mathcal{S}$ arbitrary. We claim that $V_t\in\mathcal{L}(L^2(\mathcal{S},\mu))$ and that V_t^* is given by

$$(V_t^* g)(u) = \begin{cases} g(s) & \text{if there exists an } s\in\mathcal{S} \text{ with } u=st \\ 0 & \text{otherwise.}\end{cases}$$

To see that V_t^* in deed is the adjoint operator of V_t we compute

$$\begin{aligned}(\, V_t f \mid g\,) &= \int_{\mathcal{S}} \overline{(V_t f)(s)}g(s)d\mu(s) = \int_{\mathcal{S}} \overline{f(st)}g(s)d\mu(s)\\ &= \int_{\mathcal{S}} \overline{f(st)}(V_t^* g)(st)d\mu(s) = \int_{\mathcal{S}} \overline{f(st)}(V_t^* g)(st)d\mu(st)\\ &= \int_{\mathcal{S}} \overline{f(u)}(V_t^* g)(u)d\mu(u) = (\, f \mid V_t^* g\,)\end{aligned}$$

for $f,g\in L^2(\mathcal{S},\mu)$ arbitrary, which shows V_t^* is the adjoint of V_t and thus $V_t\in\mathcal{L}(L^2(\mathcal{S},\mu))$. The identity $(V_tV_t^*f)(s)=(V_t^*f)(st)=f(s)$ shows that V_t is a coisometry. Moreover, it is easy to see that $V_sV_t=V_{st}$ for all $s,t\in\mathcal{S}$ and that the mapping $s\mapsto V_sf$ is continuous for all $f\in L^2(\mathcal{S},\mu)$. Hence, the family $\{V_s\colon s\in\mathcal{S}\}$ is a semigroup representation of $\mathcal{S}$ on $L^2(\mathcal{S},\mu)$. We know from Example 5.8 that $V_\tau := (t\mapsto V_t)$ defines a corepresentation of $C_0(\mathcal{S})$.

Now we want to show that V_τ extends the operator V from Equation (5.8). For $f\in L^2(\mathcal{S},\mu)\cap C_0(\mathcal{S})$, $g\in C_0(\mathcal{S})$ and $s,t\in\mathcal{S}$ we compute

$$\begin{aligned}V_\tau(f\otimes g)(s,t) &= (u\mapsto V_u f g(u))(s,t) = (V_t f g(t))(s) = (V_t f)(s)g(t)\\ &= f(st)g(t) = \alpha_{\mathcal{S}}(f)(s,t)(1\otimes g)(s,t) = V(f\otimes g)(s,t).\end{aligned}$$

So V_τ extends V and since V_τ is a coisometry (and thus a partial isometry) it also follows from Proposition 5.17 that V_τ is a corepresentation of $C_0(\mathcal{S})$ on $L^2(\mathcal{S},\mu)$.

Hence, we have seen that a right invariant Radon measure on a locally compact semigroup $\mathcal{S}$ with $\text{supp}(\mu) = \mathcal{S}$ yields a densely defined, lower semi-continuous weight τ on $C_0(\mathcal{S})$ such that the GNS-representation π_τ is faithful and such that the operator V defined in Equation 5.8 extends to a coisometry. The following Proposition shows that the converse is true as well.

Proposition 5.21 *Let $\mathcal{S}$ be a locally compact semigroup that is right-cancellative and let τ be a densely defined, lower semi-continuous weight on $C_0(\mathcal{S})$ such that the GNS-representation π_τ on $\mathcal{H}_\tau$ is faithful. If the operator V defined in Equation (5.8) extends to a partial isometry $V_\tau \in \mathcal{L}(\mathcal{H}_\tau \otimes C_0(\mathcal{S}))$, then the corresponding Radon measure μ on $\mathcal{S}$ is right invariant and V_τ is a coisometry.*

Proof: Since π_τ is faithful, we know that $\text{supp}(\mu) = \mathcal{S}$. Moreover, $\mathcal{H}_\tau = L^2(\mathcal{S}, \mu)$. We let $f \in L^2(\mathcal{S}, \mu) \cap C_0(\mathcal{S})$ and $g \in C_0(\mathcal{S})$ be arbitrary. Then Equation (5.8) yields

$$V(f \otimes g)(s,t) = \alpha_{\mathcal{S}}(f)(s,t)(1 \otimes g)(s,t) = f(st)g(t) = (f \otimes g)(st,t).$$

Now V extends to a partial isometry $V_\tau \in \mathcal{L}(L^2(\mathcal{S},\mu) \otimes C_0(\mathcal{S}))$ and for $f \in L^2(\mathcal{S},\mu) \otimes C_0(\mathcal{S})$ we have $(V_\tau f)(s,t) = f(st,t)$. Hence, V_τ can be written as $V_\tau = (t \mapsto V_t)$ with $(V_t f)(s) := f(st)$ and since V_τ is a partial isometry, we know that V_t must be a partial isometry in $\mathcal{L}(L^2(\mathcal{S},\mu))$ for all $t \in \mathcal{S}$.

Thus, there is a closed subspace $U_t \subset L^2(\mathcal{S},\mu)$ with $\|V_t f\| = \|f\|$ for $f \in U_t$ and $V_t f = 0$ for $f \in U_t^\perp$. It is easy to see that $U_t^\perp = \{f \in L^2(\mathcal{S},\mu) : f|_{\mathcal{S}t} = 0\, \mu\text{-almost everywhere}\}$ and thus $U_t = \{f \in L^2(\mathcal{S},\mu) : f|_{(\mathcal{S}t)^c} = 0\, \mu\text{-almost everywhere}\}$.

Now let $M \subset \mathcal{S}$ be a Borel set such that the indicator function χ_M is an element of $L^2(\mathcal{S},\mu)$. Then we have $\chi_{Mt} \in U_t$ and $V_t(\chi_{Mt}) = \chi_M$ lies in the range of V_t, which shows that the range of V_t is dense in $L^2(\mathcal{S},\mu)$. But since the range of a partial isometry is always closed, we see that V_t is onto. Hence, V_t^* is an isometry and so V_t is a coisometry.

Since $\chi_{Mt} \in U_t$, we have that $V_t^* V_t \chi_{Mt} = \chi_{Mt}$ and since $V_t(\chi_{Mt}) = \chi_M$, we get $V_t^* \chi_M = \chi_{Mt}$. But V_t^* is an isometry and thus $\mu(M) = \|\chi_M\| = \|V_t^* \chi_M\| = \|\chi_{Mt}\| = \mu(Mt)$, which shows that μ is right-invariant. □

Hence, for a locally compact semigroup $\mathcal{S}$ that is right-cancellative, there is a 1-1-correspondence between right invariant Radon measures μ on $\mathcal{S}$ with $\text{supp}(\mu) = \mathcal{S}$ and densely defined, lower semi-continuous weights τ on $C_0(\mathcal{S})$ such that the GNS-representation π_τ is faithful and such that the operator V defined in Equation (5.8) extends to a partial isometry. This motivates the idea to call a weight τ on a Hopf C^*-algebra H with the properties above invariant:

Definition 5.22 (Invariant weights and the regular corepresentation)
Let H be a Hopf C^*-algebra, τ a densely defined, lower semi-continuous weight on H and $(\mathcal{H}_\tau, \pi_\tau, \Lambda_\tau)$ its GNS-construction. If π_τ is faithful and if the operator V defined in Equation (5.8) extends to a partial isometry $V_\tau \in \mathcal{L}(\mathcal{H}_\tau \otimes H)$, we say that τ is an *invariant weight* on the Hopf C^*-algebra H and we call V_τ the *regular corepresentation* of H on $\mathcal{H}_\tau$.

In what follows, the term "an invariant weight τ on a Hopf C^*-algebra H" shall always mean that τ is a densely defined, lower semi-continuous weight on H with faithful GNS-representation π_τ such that the operator V defined in Equation (5.8) extends to a partial isometry $V_\tau \in \mathcal{L}(\mathcal{H}_\tau \otimes H)$. The following proposition shows that the existence of an invariant weight on H is the condition that makes it possible to construct a covariant representation of (H, H, δ_H) on $\mathcal{H}_\tau$.

Proposition 5.23 *Let τ be an invariant weight on a Hopf C^*-algebra H. Then the pair (π_τ, V_τ) is a covariant representation of (H, H, δ_H) on $\mathcal{H}_\tau$.*

Proof: We have to prove that Equation (5.7) holds for $\pi = \pi_\tau$, $V = V_\tau$ and $a \in H$. So we let $a, x, y \in H$ be arbitrary and compute

$$V_\tau(\pi_\tau(a) \otimes 1)(\Lambda_\tau(x) \otimes y) = V_\tau(\Lambda_\tau(ax) \otimes y) = (\Lambda_\tau \otimes \mathrm{id})(\delta_H(ax)(1 \otimes y)),$$

$$\begin{aligned}(\pi_\tau \otimes \mathrm{id})(\delta_H(a))V_\tau(\Lambda_\tau(x) \otimes y) &= (\pi_\tau \otimes \mathrm{id})(\delta_H(a))(\Lambda_\tau \otimes \mathrm{id})(\delta_H(x)(1 \otimes y)) \\ &= (\Lambda_\tau \otimes \mathrm{id})(\delta_H(a)\delta_H(x)(1 \otimes y)) \\ &= (\Lambda_\tau \otimes \mathrm{id})(\delta_H(ax)(1 \otimes y)).\end{aligned}$$

□

Definition 5.24 (The regular covariant representation)
Let τ be an invariant weight on a Hopf C^*-algebra H. Then we call the pair (π_τ, V_τ) the *regular covariant representation* of (H, H, δ_H) on $\mathcal{H}_\tau$.

Example 5.25 (The Hopf C^*-algebra $C_0(\mathcal{S})$)
If τ is an invariant weight on the Hopf C^*-algebra $C_0(\mathcal{S})$, $\mathcal{S}$ a right-cancellative locally compact semigroup, we know from Proposition 5.21 that τ is the integral corresponding to a right-invariant Radon measure μ on $\mathcal{S}$ with $\mathrm{supp}(\mu) = \mathcal{S}$. $\mathcal{H}_\tau = L^2(\mathcal{S}, \mu)$ and π_τ is the representation of $C_0(\mathcal{S})$ as multiplication operators on $L^2(\mathcal{S}, \mu)$. The partial isometry V_τ turns out to be a coisometry with $V_\tau = (t \mapsto V_t)$, where V_t is the coisometry on $L^2(\mathcal{S}, \mu)$ that satisfies $(V_t f)(s) = f(st)$ for $f \in L^2(\mathcal{S}, \mu)$. We want to show that (π_τ, V_τ) in deed is a covariant representation of $(C_0(\mathcal{S}), C_0(\mathcal{S}), \alpha_{\mathcal{S}})$ on $L^2(\mathcal{S}, \mu)$, i.e.,

$$(\pi_\tau \otimes \mathrm{id})(\alpha_{\mathcal{S}}(f))V_\tau = V_\tau(\pi_\tau(f) \otimes 1)$$

for all $f \in C_0(\mathcal{S})$. We identify $(\pi_\tau \otimes \mathrm{id})(\alpha_{\mathcal{S}}(f))V_\tau$ with the mapping

$$(t \mapsto \pi_\tau(s \mapsto f(st)))(t \mapsto V_t) = (t \mapsto \pi_\tau(s \mapsto f(st))V_t)$$

and $V_\tau(\pi_\tau(f) \otimes 1)$ with the mapping $(t \mapsto V_t\pi_\tau(f))$. Then for $g \in L^2(\mathcal{S}, \mu)$ arbitrary we have

$$(\pi_\tau(s \mapsto f(st))V_t)(g)(s) = f(st)g(st) = fg(st) = \pi_\tau(f)(g)(st) = (V_t\pi_\tau(f))(g)(s),$$

which yields the desired identity.

Example 5.26 (The Hopf C^*-algebra $c_0(\mathbb{N})$)
In Example 5.5 we have already described the comultiplication $\alpha_{\mathbb{N}}$ on the Hopf C^*-algebra $c_0(\mathbb{N})$. An invariant weight on $c_0(\mathbb{N})$ comes from a right-invariant Radon measure μ on $\mathbb{N}$ with $\mathrm{supp}(\mu) = \mathbb{N}$. So $\mu(n) = \mu(0) > 0$ for all $n \in \mathbb{N}$ and we have that μ is a multiple of the counting measure on $\mathbb{N}$. Without loss of generality, we may assume that μ is the counting measure on $\mathbb{N}$. Then $\mathcal{H}_\tau = \ell^2(\mathbb{N})$ and π_τ is the representation of $c_0(\mathbb{N})$ as multiplication operators on $\ell^2(\mathbb{N})$, i.e., $\pi_\tau(e_n) = P_n$, where P_n is the projection onto the n-th component of $\ell^2(\mathbb{N})$.

It is clear that $V_k = (S^*)^k$, where S is the forward unilateral shift on $\ell^2(\mathbb{N})$ and V_τ can be written as $V_\tau = \sum_{n\in\mathbb{N}}(S^*)^n \otimes e_n \in M(\mathcal{K}(\ell^2(\mathbb{N}))\otimes c_0(\mathbb{N}))$. We already know that (π_τ, V_τ) is a covariant representation – it is the regular covariant representation of $(c_0(\mathbb{N}), c_0(\mathbb{N}), \alpha_{\mathbb{N}})$ on $\ell^2(\mathbb{N})$ – but we wish to show how this can be proved explicitly in the discrete case.

First we show that V_τ is a coisometry:

$$\begin{aligned} V_\tau V_\tau^* &= \big(\sum_{n\in\mathbb{N}}(S^*)^n \otimes e_n\big)\big(\sum_{m\in\mathbb{N}} S^m \otimes e_m\big) = \sum_{m,n\in\mathbb{N}} (S^*)^n S^m \otimes e_n e_m \\ &= \sum_{n\in\mathbb{N}}(S^*)^n S^n \otimes e_n = \mathrm{id} \otimes \sum_{n\in\mathbb{N}} e_n = \mathrm{id} \otimes 1 = 1. \end{aligned}$$

The following shows that V_τ satisfies equation (5.3):

$$\begin{aligned} (V_\tau)_{12}(V_\tau)_{13} &= \big(\sum_{m\in\mathbb{N}}(S^*)^m \otimes e_m\big)_{12}\big(\sum_{n\in\mathbb{N}}(S^*)^n \otimes e_n\big)_{13} = \sum_{m,n\in\mathbb{N}} (S^*)^m(S^*)^n \otimes e_m \otimes e_n \\ &= \sum_{m,n\in\mathbb{N}} (S^*)^{m+n} \otimes e_m \otimes e_n = \sum_{n\in\mathbb{N}}\sum_{m=0}^{n}(S^*)^n \otimes e_m \otimes e_{n-m} \\ &= (\mathrm{id} \otimes \alpha_{\mathbb{N}})\big(\sum_{n\in\mathbb{N}}(S^*)^n \otimes e_n\big) = (\mathrm{id} \otimes \alpha_{\mathbb{N}})(V_\tau). \end{aligned}$$

It remains to be proven that $(\pi_\tau \otimes \mathrm{id})(\alpha_{\mathbb{N}}(a))V_\tau = V_\tau(\pi_\tau(a) \otimes 1)$ for all $a \in c_0(\mathbb{N})$. Since π_τ and $\alpha_{\mathbb{N}}$ are continuous, it suffices to show that this holds for the elements e_n, $n \in \mathbb{N}$:

$$\begin{aligned}
(\pi_\tau \otimes \mathrm{id})(\alpha_{\mathbb{N}}(e_n))V_\tau &= (\pi_\tau \otimes \mathrm{id})\big(\sum_{l=0}^{n} e_l \otimes e_{n-l}\big)\big(\sum_{m\in\mathbb{N}} (S^*)^m \otimes e_m\big) \\
&= \big(\sum_{l=0}^{n} P_l \otimes e_{n-l}\big)\big(\sum_{m\in\mathbb{N}} (S^*)^m \otimes e_m\big) = \sum_{l=0}^{n}\sum_{m\in\mathbb{N}} P_l (S^*)^m \otimes e_{n-l}e_m \\
&= \sum_{l=0}^{n} P_l (S^*)^{n-l} \otimes e_{n-l} = \sum_{l=0}^{n} (S^*)^{n-l} P_n \otimes e_{n-l} \\
&= \sum_{l=0}^{n} (S^*)^{l} P_n \otimes e_{l} = \sum_{l\in\mathbb{N}} (S^*)^{l} P_n \otimes e_{l} \\
&= \big(\sum_{l\in\mathbb{N}} (S^*)^{l} \otimes e_{l}\big)(P_n \otimes 1) = V_\tau(\pi_\tau(e_n) \otimes 1).
\end{aligned}$$

Example 5.27 (The Hopf C^*-algebra $C^*(\mathbb{N})$)

We have already seen in Example 5.6 that $C^*(\mathbb{N})$ together with the comultiplication $\delta_{\mathbb{N}}$ is a Hopf C^*-algebra. Now we introduce a weight τ on $C^*(\mathbb{N})$ and we show that τ is an invariant weight. This allows us to construct the regular covariant representation of $(C^*(\mathbb{N}), C^*(\mathbb{N}), \delta_{\mathbb{N}})$ on $\mathcal{H}_\tau$.

$C^*(\mathbb{N})$ is a C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$. Let $\Omega = (1, 0, 0, \ldots)$ denote the vacuum vector in $\ell^2(\mathbb{N})$. We define τ by setting

$$\tau(a) := (\,\Omega \,|\, a\Omega\,)$$

for $a \in C^*(\mathbb{N})$. It is obvious that τ is a positive functional on $C^*(\mathbb{N})$ and thus τ is also a densely defined, lower semi-continuous weight on $C^*(\mathbb{N})$. We define an inner product by $(\,a \,|\, b\,) := \tau(a^*b)$. It is easy to see that $\mathcal{H}_\tau \cong \ell^2(\mathbb{N})$ and that $\Lambda_\tau(S^n) = e_n$, while $\Lambda_\tau(S^n(S^*)^m) = 0$ for $m > 0$. Finally, we get that π_τ is the identity map on $C^*(\mathbb{N})$ and hence, in particular, π_τ is faithful.

Now we claim that $V_\tau = \sum_{m\in\mathbb{N}} P_m \otimes S^m$. To see this, we compare $\sum_{m\in\mathbb{N}} P_m \otimes S^m$ to the operator V defined in Equation (5.8) on elements of the form $S^n \otimes S^l(S^*)^k$:

$$\begin{aligned}
\big(\sum_{m\in\mathbb{N}} P_m \otimes S^m\big)(\Lambda_\tau(S^n) \otimes S^l(S^*)^k) &= \sum_{m\in\mathbb{N}} P_m(e_n) \otimes S^{m+l}(S^*)^k = e_n \otimes S^{n+l}(S^*)^k \\
&= (\Lambda_\tau \otimes \mathrm{id})(S^n \otimes S^{n+l}(S^*)^k) \quad \text{and}
\end{aligned}$$

$$V(S^n \otimes S^l(S^*)^k) = \delta_{\mathbb{N}}(S^n)(1 \otimes S^l(S^*)^k) = (S^n \otimes S^n)(1 \otimes S^l(S^*)^k) = S^n \otimes S^{n+l}(S^*)^k.$$

To show that τ is an invariant weight, we have to prove that $V_\tau = \sum_{m\in\mathbb{N}} P_m \otimes S^m$ is a partial isometry. In fact, it is an isometry:

$$\begin{aligned} V_\tau^* V_\tau &= \big(\sum_{n\in\mathbb{N}} P_n \otimes (S^*)^n\big)\big(\sum_{m\in\mathbb{N}} P_m \otimes S^m\big) = \sum_{m,n\in\mathbb{N}} P_n P_m \otimes (S^*)^n S^m \\ &= \sum_{n\in\mathbb{N}} P_n \otimes (S^*)^n S^n = \sum_{n\in\mathbb{N}} P_n \otimes \mathrm{id} = \mathrm{id}\otimes\mathrm{id} = 1. \end{aligned}$$

From Proposition 5.23 we know that the pair (π_τ, V_τ) is a covariant representation of $(C^*(\mathbb{N}), C^*(\mathbb{N}), \delta_{\mathbb{N}})$ on $\ell^2(\mathbb{N})$, the regular covariant representation with respect to τ, but we also want to show this explicitly. Hence, we show that Equations (5.3) and (5.7) hold:

$$\begin{aligned} (V_\tau)_{12}(V_\tau)_{13} &= \big(\sum_{n\in\mathbb{N}} P_n \otimes S^n\big)_{12}\big(\sum_{m\in\mathbb{N}} P_m \otimes S^m\big)_{13} = \sum_{m,n\in\mathbb{N}} P_n P_m \otimes S^n \otimes S^m \\ &= \sum_{n\in\mathbb{N}} P_n \otimes S^n \otimes S^n = (\mathrm{id}\otimes\delta_{\mathbb{N}})\big(\sum_{n\in\mathbb{N}} P_n \otimes S^n\big) = (\mathrm{id}\otimes\delta_{\mathbb{N}})(V_\tau); \end{aligned}$$

$$\begin{aligned} V_\tau(\pi_\tau(S^n(S^*)^m)\otimes 1) &= \big(\sum_{l\in\mathbb{N}} P_l \otimes S^l\big)(S^n(S^*)^m \otimes 1) = \sum_{l\in\mathbb{N}} P_l S^n (S^*)^m \otimes S^l \\ &= \sum_{l\in\mathbb{N}} P_{l+n} S^n (S^*)^m \otimes S^{l+n} \quad \text{and} \end{aligned}$$

$$\begin{aligned} (\pi_\tau\otimes\mathrm{id})(\delta_{\mathbb{N}}(S^n(S^*)^m))V_\tau &= (S^n(S^*)^m \otimes S^n(S^*)^m)\big(\sum_{l\in\mathbb{N}} P_l \otimes S^l\big) \\ &= \sum_{l\in\mathbb{N}} S^n(S^*)^m P_l \otimes S^n(S^*)^m S^l \\ &= \sum_{l\geq m} S^n P_{l-m}(S^*)^m \otimes S^n(S^*)^m S^l \\ &= \sum_{l\geq m} P_{l-m+n} S^n(S^*)^m \otimes S^n(S^*)^m S^l \\ &= \sum_{l\in\mathbb{N}} P_{l+n} S^n(S^*)^m \otimes S^n(S^*)^m S^{l+m} \\ &= \sum_{l\in\mathbb{N}} P_{l+n} S^n(S^*)^m \otimes S^{l+n}. \end{aligned}$$

5.3 The dual C^*-algebra

Given a locally compact group G, there is a pair of C^*-algebras in duality: the C^*-algebra $C_0(G)$ and the full group C^*-algebra $C^*(G)$. Since for locally compact, abelian groups G we have that $C^*(G) \cong C_0(\hat{G})$, this duality can be viewed as an extension of Pontryagin's duality theorem for locally compact, abelian groups, which states that G and $\hat{\hat{G}}$ are topologically isomorphic.

In this section, we want to extend this duality concept to Hopf C^*-algebras equipped with an invariant weight. So in the following, we let H be a Hopf C^*-algebra equipped with an invariant weight τ. We will construct the reduced and the universal dual C^*-algebra of H and we will show that the reduced dual C^*-algebra of the Hopf C^*-algebra $(c_0(\mathbb{N}), \alpha_\mathbb{N})$ is isomorphic to $C^*(\mathbb{N})$ whereas the reduced dual C^*-algebra of the Hopf C^*-algebra $(C^*(\mathbb{N}), \delta_\mathbb{N})$ is isomorphic to the C^*-algebra $c_0(\mathbb{N})$.

Given a functional $f \in H^*$ and a corepresentation $V \in M(\mathcal{K}(\mathcal{H}) \otimes H)$, we set

$$\vartheta_V(f) := (\mathrm{id} \otimes f)(V)$$

to get a bounded linear mapping ϑ_V from H^* to $\mathcal{L}(\mathcal{H})$ that satisfies $\|\vartheta_V(f)\|_{\mathrm{op}} \leq \|f\|\|V\|$, where $\|.\|_{\mathrm{op}}$ is the operator norm on $\mathcal{L}(\mathcal{H})$. For $a, b \in \mathcal{N}_\tau$, let $f_{a,b} \in H^*$ be the functional defined by

$$f_{a,b}(x) := \tau(a^*xb), \quad x \in H.$$

Notice that $\tau(a^*xb) = (\Lambda_\tau(a)|\pi_\tau(x)\Lambda_\tau(b))$, which shows that $f_{a,b}$ is in deed a bounded linear functional on H. We now define Z to be the closure in H^* of the linear span of all functionals of the form $f_{a,b}$, $a, b \in \mathcal{N}_\tau$:

$$Z := \overline{\mathrm{span}}\{f_{a,b} : a, b \in \mathcal{N}_\tau\} \subseteq H^*.$$

Finally, we define the *reduced dual C^*-algebra* $C^*_r(H, \delta_H)$ by

$$C^*_r(H, \delta_H) := C^*(\vartheta_{V_\tau}(Z)) \subseteq \mathcal{L}(\mathcal{H}_\tau),$$

where V_τ is the regular corepresentation of H on $\mathcal{H}_\tau$.

We notice that we have not defined a comultiplication on the reduced dual C^*-algebra, i.e., $C^*_r(H, \delta_H)$ is only a C^*-algebra and not a Hopf C^*-algebra.

The following two examples show that $C^*(\mathbb{N})$ is the reduced dual C^*-algebra of the Hopf C^*-algebra $(c_0(\mathbb{N}), \alpha_\mathbb{N})$ and that $c_0(\mathbb{N})$ is the reduced dual C^*-algebra of the Hopf C^*-algebra $(C^*(\mathbb{N}), \delta_\mathbb{N})$.

Example 5.28 $(c_0(\mathbb{N}), \alpha_\mathbb{N})$
Let τ be the invariant weight on $c_0(\mathbb{N})$ given by $\tau((a(n))_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} a(n)$, i.e., τ is the weight that comes from the counting measure on $\mathbb{N}$. It is easy to see that $\mathcal{N}_\tau = \ell^2(\mathbb{N})$ and

$\mathcal{M}_\tau = \ell^1(\mathbb{N})$. Since $c_0(\mathbb{N})$ is commutative, we have that $f_{a,b}(x) = \tau(a^*xb) = \tau(a^*bx)$ for $a, b \in \ell^2(\mathbb{N})$ and $x \in c_0(\mathbb{N})$. But $a^*b \in \ell^1(\mathbb{N})$ and hence, $Z = \overline{\text{span}}\{f_{a,b} : a, b \in \ell^2(\mathbb{N})\} = \{f_c : c \in \ell^1(\mathbb{N})\} \cong \ell^1(\mathbb{N})$, where f_c is the functional defined by $f_c(x) := \tau(cx)$, $x \in c_0(\mathbb{N})$.

The linear span of the sequences e_n, $n \in \mathbb{N}$, is dense in $\ell^1(\mathbb{N})$ and so we compute

$$\vartheta_{V_\tau}(f_{e_n}) = (\text{id} \otimes f_{e_n})(V_\tau) = (\text{id} \otimes f_{e_n})(\sum_{l\in\mathbb{N}} (S^*)^l \otimes e_l) = (S^*)^n,$$

since $f_{e_n}(e_l) = \tau(e_n e_l) = \delta_{n,l}\tau(e_n) = \delta_{n,l}$. Hence, $C_r^*(c_0(\mathbb{N}), \alpha_\mathbb{N})$ is the C^*-algebra that is generated by the left shift S^* on $\ell^2(\mathbb{N})$ and thus, it is isomorphic to $C^*(\mathbb{N})$:

$$C_r^*(c_0(\mathbb{N}), \alpha_\mathbb{N}) \cong C^*(\mathbb{N}).$$

Example 5.29 $(C^*(\mathbb{N}), \delta_\mathbb{N})$
Let τ be the invariant weight on $C^*(\mathbb{N})$ introduced in Example 5.27. Here we have $\mathcal{M}_\tau = \mathcal{N}_\tau = C^*(\mathbb{N})$ and we compute

$$\begin{aligned} f_{S^m(S^*)^n, S^k(S^*)^l}(S^p(S^*)^q) &= \tau(S^n(S^*)^m S^p(S^*)^q S^k(S^*)^l) = (S^m(S^*)^n\Omega | S^p(S^*)^q S^k(S^*)^l\Omega) \\ &= \begin{cases} 1 & \text{if } l = n = 0,\ p - q = m - k,\ q \leq k \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, we have that $f_{S^m(S^*)^n, S^k(S^*)^l} = 0$ if $n > 0$ or $l > 0$ and

$$f_{S^m, S^n}(S^p(S^*)^q) = \begin{cases} 1 & \text{if } p - q = m - n,\ q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we see that

$$Z = \overline{\text{span}}\{f_{S^m, S^n} : m, n \in \mathbb{N}\}$$

and since the regular corepresentation of $C^*(\mathbb{N})$ on $\ell^2(\mathbb{N})$ is $V_\tau = \sum_{l\in\mathbb{N}} P_l \otimes S^l$, we compute

$$\vartheta_{V_\tau}(f_{S^m, S^n}) = (\text{id} \otimes f_{S^m, S^n})(\sum_{l\in\mathbb{N}} P_l \otimes S^l) = \sum_{l\in\mathbb{N}} P_l f_{S^m, S^n}(S^l) = \begin{cases} P_{m-n} & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases}$$

But now it is easy to see that $C_r^*(C^*(\mathbb{N}), \delta_\mathbb{N})$ is the C^*-algebra that is generated by the one dimensional projections P_l, $l \in \mathbb{N}$, on $\ell^2(\mathbb{N})$ and so

$$C_r^*(C^*(\mathbb{N}), \delta_\mathbb{N}) \cong c_0(\mathbb{N}).$$

The rest of this section is dedicated to the construction of the universal dual C^*-algebra $C^*(H, \delta_H)$. In what follows, we suppose that (H, δ_H) is a Hopf C^*-algebra equipped with an invariant weight τ that satisfies the following condition:

$$(f \otimes g) \circ \delta_H \in Z \text{ for all } f, g \in Z. \tag{5.9}$$

This extra condition allows us to define a multiplication $*$ on Z by setting

$$f * g := (f \otimes g) \circ \delta_H.$$

We want to examine if our standard examples $(c_0(\mathbb{N}), \alpha_{\mathbb{N}})$ and $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ satisfy this condition and what the definition above for the multiplication on Z yields in the concrete examples.

Example 5.30 $(c_0(\mathbb{N}), \alpha_{\mathbb{N}})$
We have seen in Example 5.28 that $Z = \{f_c : c \in \ell^1(\mathbb{N})\}$ and hence, we compute

$$\begin{aligned}(f_{e_n} \otimes f_{e_m})(\alpha_{\mathbb{N}}(e_l)) &= (f_{e_n} \otimes f_{e_m})(\sum_{k=0}^{l} e_k \otimes e_{l-k}) = \sum_{k=0}^{l} f_{e_n}(e_k) f_{e_m}(e_{l-k}) \\ &= \left\{ \begin{array}{ll} 1 & \text{if } l = m+n \\ 0 & \text{otherwise} \end{array} \right\} = f_{e_{n+m}}(e_l).\end{aligned}$$

So we see that $(f_{e_n} \otimes f_{e_m}) \circ \alpha_{\mathbb{N}} = f_{e_{n+m}} \in Z$ and it is easy to see that $(f_a \otimes f_b) \circ \alpha_{\mathbb{N}} \in Z$ for arbitrary $a, b \in \ell^1(\mathbb{N})$. This means that $(c_0(\mathbb{N}), \alpha_{\mathbb{N}})$ satisfies Equation (5.9) and so we can define the multiplication $*$ on $c_0(\mathbb{N})$ to get

$$f_{e_n} * f_{e_m} = f_{e_{n+m}}$$

for all $n, m \in \mathbb{N}$. Hence, the multiplication $*$ on $Z \cong \ell^1(\mathbb{N})$ is the usual convolution product.

Example 5.31 $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$
From Example 5.29 we already know that for the Hopf C^*-algebra $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$, we have $Z = \overline{\text{span}}\{f_{S^m,S^n} : m, n \in \mathbb{N}\}$. To see if $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ satisfies condition (5.9), we compute

$$\begin{aligned}(f_{S^l,S^k} \otimes f_{S^m,S^n})(\delta_{\mathbb{N}}(S^p(S^*)^q)) &= f_{S^l,S^k}(S^p(S^*)^q) f_{S^m,S^n}(S^p(S^*)^q) \\ &= \left\{ \begin{array}{ll} 1 & \text{if } k \geq q, n \geq q \text{ and } l-k = m-n = p-q \\ 0 & \text{otherwise.} \end{array} \right.\end{aligned}$$

So now it is easy to see that

$$(f_{S^l,S^k} \otimes f_{S^m,S^n}) \circ \delta_{\mathbb{N}} = \left\{ \begin{array}{ll} 0 & \text{if } l-k \neq m-n \\ f_{S^l,S^k} & \text{if } l-k = m-n \text{ and } k \leq n \\ f_{S^m,S^n} & \text{if } l-k = m-n \text{ and } k > n. \end{array} \right.$$

In any case $(f_{S^l,S^k} \otimes f_{S^m,S^n}) \circ \delta_{\mathbb{N}} \in Z$ and so we see that $(C^*(\mathbb{N}), \delta_{\mathbb{N}})$ satisfies condition (5.9). Thus we can define the multiplication $*$ on $C^*(\mathbb{N})$ to get

$$f_{S^l,S^k} * f_{S^m,S^n} = \left\{ \begin{array}{ll} 0 & \text{if } l-k \neq m-n \\ f_{S^l,S^k} & \text{if } l-k = m-n \text{ and } k \leq n \\ f_{S^m,S^n} & \text{if } l-k = m-n \text{ and } k > n. \end{array} \right.$$

Proposition 5.32 *The multiplication $*$ on Z is associative.*

Proof: Let $f, g, h \in Z$ and $x \in H$ be arbitrary. Then

$$\begin{aligned}(f*(g*h))(x) &= (f\otimes(g*h))(\delta_H(x)) = (f\otimes g\otimes h)\big((\mathrm{id}\otimes\delta_H)(\delta_H(x))\big)\\ &\overset{(5.1)}{=} (f\otimes g\otimes h)\big((\delta_H\otimes\mathrm{id})(\delta_H(x))\big) = ((f*g)\otimes h)(\delta_H(x))\\ &= ((f*g)*h)(x).\end{aligned}$$

and we see that the associativity of $*$ follows from the coassociativity of δ_H. □

Hence, $(Z, +, *)$ is an associative algebra. Now let $V \in M(\mathcal{K}(\mathcal{H})\otimes H)$ be a corepresentation of H on a Hilbert space $\mathcal{H}$. The following computation shows that the bounded linear mapping $\vartheta_V : Z \to \mathcal{L}(\mathcal{H})$ is multiplicative. For $f, g \in Z$ we have

$$\begin{aligned}\vartheta_V(f)\vartheta_V(g) &= ((\mathrm{id}\otimes f)(V))((\mathrm{id}\otimes g)(V)) = (\mathrm{id}\otimes f\otimes g)(V_{12}V_{13})\\ &= (\mathrm{id}\otimes f\otimes g)((\mathrm{id}\otimes\delta_H)(V)) = (\mathrm{id}\otimes(f*g))(V) = \vartheta_V(f*g).\end{aligned}$$

Next, we want to construct the universal dual C^*-algebra $C^*(H, \delta_H)$ and we proceed in the following way. Let L be the universal enveloping *-algebra of Z, i.e., L is a *-algebra together with an injective algebra homomorphism $\varphi : Z \to L$ that is universal in the sense that if there is another algebra homomorphism ψ from Z into another *-algebra K, there exists a *-homomorphism $\widetilde{\psi} : L \to K$ such that $\psi = \widetilde{\psi}\circ\varphi$.

Given a corepresentation V of H on a Hilbert space $\mathcal{H}$, we know that $\vartheta_V : Z \to \mathcal{L}(\mathcal{H})$ is an algebra homomorphism and thus, there exists a *-homomorphism $\widetilde{\vartheta_V} : L \to \mathcal{L}(\mathcal{H})$. This allows us to define a norm on L. For $a \in L$ we set

$$\|a\| := \sup\{\|\widetilde{\vartheta_V}(a)\|_{\mathrm{op}} : V \text{ corepresentation of } H\},$$

where $\|.\|_{\mathrm{op}}$ denotes the operator norm on $\mathcal{L}(\mathcal{H})$. To see that this supremum exists, note that the linear span of alternating products of elements from $\varphi(Z)$ and $\varphi(Z)^*$ is a dense subset of L. For example, we let $a := \varphi(z_1)\varphi(z_2)^*\varphi(z_3)\varphi(z_4)^*\ldots\varphi(z_{n-1})\varphi(z_n)^*$ be a typical alternating product in L and we let V be an arbitrary corepresentation of H on a Hilbert space $\mathcal{H}$. Then we have

$$\begin{aligned}\|\widetilde{\vartheta_V}(a)\|_{\mathrm{op}} &= \|\widetilde{\vartheta_V}(\varphi(z_1)\varphi(z_2)^*\varphi(z_3)\varphi(z_4)^*\ldots\varphi(z_{n-1})\varphi(z_n)^*)\|_{\mathrm{op}}\\ &= \|\widetilde{\vartheta_V}(\varphi(z_1))\widetilde{\vartheta_V}(\varphi(z_2))^*\widetilde{\vartheta_V}(\varphi(z_3))\widetilde{\vartheta_V}(\varphi(z_4))^*\ldots\widetilde{\vartheta_V}(\varphi(z_{n-1}))\widetilde{\vartheta_V}(\varphi(z_n))^*\|_{\mathrm{op}}\\ &= \|\vartheta_V(z_1)\vartheta_V(z_2)^*\vartheta_V(z_3)\vartheta_V(z_4)^*\ldots\vartheta_V(z_{n-1})\vartheta_V(z_n)^*\|_{\mathrm{op}}\\ &\le \|\vartheta_V(z_1)\|_{\mathrm{op}}\|\vartheta_V(z_2)\|_{\mathrm{op}}\|\vartheta_V(z_3)\|_{\mathrm{op}}\|\vartheta_V(z_4)\|_{\mathrm{op}}\ldots\|\vartheta_V(z_{n-1})\|_{\mathrm{op}}\|\vartheta_V(z_n)\|_{\mathrm{op}}\\ &\le \|z_1\|\|V\|\|z_2\|\|V\|\|z_3\|\|V\|\|z_4\|\|V\|\ldots\|z_{n-1}\|\|V\|\|z_n\|\|V\|\\ &\le \|z_1\|\|z_2\|\|z_3\|\|z_4\|\ldots\|z_{n-1}\|\|z_n\|\end{aligned}$$

and so $\sup\{\|\widetilde{\vartheta_V}(\varphi(z_1)\varphi(z_2)^*\varphi(z_3)\varphi(z_4)^*\ldots\varphi(z_{n-1})\varphi(z_n)^*)\|_{\mathrm{op}} : V$ corepresentation of $H\}$ is bounded by $\|z_1\|\|z_2\|\|z_3\|\|z_4\|\ldots\|z_{n-1}\|\|z_n\|$.

Now $\|.\|$ is a C^*-seminorm on L. Let $N := \{a \in L : \|a\| = 0\}$. We define the *universal dual* C^**-algebra* $C^*(H, \delta_H)$ to be the completion of L/N. Again, we notice that we have not defined a comultiplication on $C^*(H, \delta_H)$ and thus $C^*(H, \delta_H)$ is only a C^*-algebra but not a Hopf C^*-algebra.

5.4 The reduced crossed product

In the present section, we want to construct the reduced crossed product $A \rtimes_\delta H$ for a dynamical cosystem (A, H, δ). We will show that for the dynamical cosystem $(\mathbb{C}, H, \mathrm{id})$, where id is the trivial coaction of a Hopf C^*-algebra H on the complex numbers, the reduced crossed product $\mathbb{C} \rtimes_{\mathrm{id}} H$ corresponds to the reduced dual C^*-algebra $C^*_r(H, \delta_H)$. Finally, we will introduce a coaction δ of the Hopf C^*-algebra $C^*(\mathbb{N})$ on the crossed product $A \rtimes_E \mathbb{N}$ and we will examine the reduced crossed product $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$. This example will be of great importance in the next chapter.

In the following, let τ be an invariant weight on a Hopf C^*-algebra H and let (π_τ, V_τ) be the corresponding regular covariant representation of (H, H, δ_H) on $\mathcal{H}_\tau$. Moreover, let δ be a coaction of H on a C^*-algebra A. We set

$$V := \sigma_{23}(V_\tau \otimes \mathrm{id}_A) \in \mathcal{L}(\mathcal{H}_\tau \otimes A \otimes H)$$

and we define a nondegenerate *-homomorphism $\psi\colon A \to \mathcal{L}(\mathcal{H}_\tau \otimes A)$ by setting

$$\psi := \sigma \circ (\mathrm{id} \otimes \pi_\tau) \circ \delta.$$

Moreover, we define the C^*-algebra $A \rtimes_\delta H \subset \mathcal{L}(\mathcal{H}_\tau \otimes A)$ by

$$A \rtimes_\delta H := C^*(\{\psi(a)(\mathrm{id} \otimes \mathrm{id} \otimes f)(V)\colon a \in A, f \in Z\})$$

and we call $A \rtimes_\delta H$ the *reduced crossed product* for (A, H, δ).

Proposition 5.33 *(ψ, V) is a covariant homomorphism from (A, H, δ) into $M(A \rtimes_\delta H)$.*

Proof: It is clear that $\psi\colon A \to M(A \rtimes_\delta H)$ is a nondegenerate *-homomorphism and that $V \in M(A \rtimes_\delta H \otimes H)$. The fact that V is a partial isometry follows directly from the fact that V_τ is a partial isometry. Moreover, we have

$$\begin{aligned}(\mathrm{id} \otimes \delta_H)(V) &= (\mathrm{id} \otimes \delta_H)(\sigma_{23}(V_\tau \otimes \mathrm{id}_A)) = \sigma_{23}\sigma_{34}(\mathrm{id} \otimes \delta_H \otimes \mathrm{id})(V_\tau \otimes \mathrm{id}_A) \\ &= \sigma_{23}\sigma_{34}((V_\tau)_{12}(V_\tau)_{13} \otimes \mathrm{id}_A) = V_{12}V_{13}\end{aligned}$$

and for $a \in A$ arbitrary we get

$$\begin{aligned} V(\psi(a)\otimes 1) &= (\sigma_{23}(V_\tau \otimes \mathrm{id}_A))(\sigma_{12}(\mathrm{id}_A \otimes \pi_\tau \otimes \mathrm{id}_H)(\delta(a)\otimes 1)) \\ &= \sigma_{12}\big((\mathrm{id}_A \otimes V_\tau)\big((\mathrm{id}_A \otimes \pi_\tau \otimes \mathrm{id}_H)(\delta(a)\otimes 1)\big)\big) \\ &\overset{(\star)}{=} \sigma_{12}\big(\big((\mathrm{id}_A \otimes \pi_\tau \otimes \mathrm{id}_H)(\mathrm{id}_A \otimes \delta_H)(\delta(a))\big)(\mathrm{id}_A \otimes V_\tau)\big) \\ &= (\sigma_{12}(\mathrm{id}_A \otimes \pi_\tau \otimes \mathrm{id}_H)(\delta \otimes \mathrm{id}_H)(\delta(a)))(\sigma_{23}(V_\tau \otimes \mathrm{id}_A)) \\ &= (\psi \otimes \mathrm{id})(\delta(a))V, \end{aligned}$$

where $(\star)$ holds because (π_τ, V_τ) is a covariant representation of (H, H, δ_H) on $\mathcal{H}_\tau$ and hence, $(\pi_\tau \otimes \mathrm{id})(\delta_H(h))V_\tau = V_\tau(\pi_\tau(h)\otimes 1)$ for all $h \in H$. □

Example 5.34 (The reduced dual C^*-algebra)
Let H be a Hopf C^*-algebra equipped with an invariant weight τ and let (π_τ, V_τ) be the corresponding regular covariant representation on $\mathcal{H}_\tau$. Moreover, let id be the trivial coaction of H on the complex numbers $\mathbb{C}$ introduced in Example 5.10 (a). Since $\mathcal{H}_\tau \otimes \mathbb{C} \approx \mathcal{H}_\tau$, we can identify V with V_τ and ψ is just the trivial representation of $\mathbb{C}$ on $\mathcal{H}_\tau$. Finally, we have

$$\begin{aligned} \mathbb{C} \rtimes_{\mathrm{id}} H &= C^*(\{\psi(z)(\mathrm{id}\otimes f)(V_\tau) \colon z \in \mathbb{C}, f \in Z\}) = C^*(\{(\mathrm{id}\otimes f)(V_\tau) \colon f \in Z\}) \\ &= C^*(\vartheta_{\mathcal{V}_\tau}(Z)) = C^*_r(H, \delta_H) \end{aligned}$$

and so the reduced crossed product for $(\mathbb{C}, H, \mathrm{id})$ is the reduced dual C^*-algebra $C^*_r(H, \delta_H)$.

Example 5.35 (The double crossed product $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$)
Let $A \rtimes_E \mathbb{N}$ be the reduced crossed product of A by E from Example 3.14, i.e.,

$$A \rtimes_E \mathbb{N} = C^*(\{\pi_E(\xi) \colon \xi \in E\} \cup \{\pi_A(a) \colon a \in A\}) \subseteq \mathcal{L}(\mathcal{F}(E)).$$

We define a Toeplitz representation $\psi \colon (F, \Phi) \to M((A \rtimes_E \mathbb{N}) \otimes C^*(\mathbb{N}))$ by setting

$$\psi_n(\xi^n) := \pi_n(\xi^n) \otimes S^n \quad \text{for } n \geq 1 \quad \text{and} \quad \psi_0(a) := \pi_0(a) \otimes 1$$

for $\xi^n \in E^n$ and $a \in A$ arbitrary. To see that ψ is a Toeplitz representation from (F, Φ) into $M((A \rtimes_E \mathbb{N}) \otimes C^*(\mathbb{N}))$, we let $a, b \in A$, $\xi^n, \eta^n \in E^n$ and $\xi^m \in E^m$ be arbitrary and compute

$$\begin{aligned} \psi_n(a \cdot \xi^n \cdot b) &= \pi_n(a \cdot \xi^n \cdot b) \otimes S^n = \pi_0(a)\pi_n(\xi^n)\pi_0(b) \otimes S^n \\ &= (\pi_0(a)\otimes 1)(\pi_n(\xi^n)\otimes S^n)(\pi_0(b)\otimes 1) = \psi_0(a)\psi_n(\xi^n)\psi_0(b), \\ \psi_n(\xi^n)^*\psi_n(\eta^n) &= (\pi_n(\xi^n)^* \otimes (S^*)^n)(\pi_n(\eta^n)\otimes S^n) = \pi_n(\xi^n)^*\pi_n(\eta^n) \otimes (S^*)^n S^n \\ &= \pi_0((\, \xi^n \,|\, \eta^n \,)_A) \otimes 1 = \psi_0((\, \xi^n \,|\, \eta^n \,)_A) \quad \text{and} \\ \psi_{n+m}(\Phi_{n,m}(\xi^m \otimes \xi^n)) &= \pi_{n+m}(\Phi_{n,m}(\xi^m \otimes \xi^n)) \otimes S^{n+m} = \pi_m(\xi^m)\pi_n(\xi^n) \otimes S^m S^n \\ &= (\pi_m(\xi^m)\otimes S^m)(\pi_n(\xi^n)\otimes S^n) = \psi_m(\xi^m)\psi_n(\xi^n). \end{aligned}$$

In [11, Corollary 2.2.], Fowler and Raeburn showed that $A \rtimes_E \mathbb{N} = \mathcal{T}_r(F,\Phi)$ is isomorphic to the universal Toeplitz algebra $\mathcal{T}(F,\Phi)$. Hence, there exists a unique *-homomorphism $\delta\colon A \rtimes_E \mathbb{N} \to M((A \rtimes_E \mathbb{N}) \otimes C^*(\mathbb{N}))$ that satisfies

$$\delta(\pi_n(\xi^n)) = \pi_n(\xi^n) \otimes S^n \qquad \text{and} \qquad \delta(\pi_0(a)) = \pi_0(a) \otimes 1$$

for $\xi^n \in E^n$, $n \geq 1$ and $a \in A$ arbitrary. We claim that δ is a coaction of the Hopf C^*-algebra $C^*(\mathbb{N})$ on $A \rtimes_E \mathbb{N}$. It is clear that δ is nondegenerate and injective. To see that it satisfies the coaction identity (5.4), we compute

$$\begin{aligned}(\delta \otimes \mathrm{id}) \circ \delta(\pi_n(\xi^n)) &= (\delta \otimes \mathrm{id})(\pi_n(\xi^n) \otimes S^n) = \pi_n(\xi^n) \otimes S^n \otimes S^n \\ &= (\mathrm{id} \otimes \delta_{\mathbb{N}})(\pi_n(\xi^n) \otimes S^n) = (\mathrm{id} \otimes \delta_{\mathbb{N}}) \circ \delta(\pi_n(\xi^n)).\end{aligned}$$

From Example 5.27 we know that the regular covariant representation (π_τ, V_τ) of the dynamical system $(C^*(\mathbb{N}), C^*(\mathbb{N}), \delta_{\mathbb{N}})$ on the Hilbert space $\ell^2(\mathbb{N})$ with respect to the weight τ consists of the identity representation π_τ of $C^*(\mathbb{N})$ on $\ell^2(\mathbb{N})$ together with the isometry $V_\tau = \sum_{n\in\mathbb{N}} P_n \otimes S^n \in \mathcal{L}(\ell^2(\mathbb{N}) \otimes C^*(\mathbb{N}))$. Thus we get

$$V = \sum_{n\in\mathbb{N}} P_n \otimes \mathrm{id}_{A\rtimes_E\mathbb{N}} \otimes S^n \in \mathcal{L}(\ell^2(\mathbb{N}) \otimes A \rtimes_E \mathbb{N} \otimes C^*(\mathbb{N})) \quad \text{and} \quad \psi(\pi_n(\xi^n)) = S^n \otimes \pi_n(\xi^n)$$

for $\xi^n \in E^n$ and $n \in \mathbb{N}$. Since $(\mathrm{id} \otimes \mathrm{id} \otimes f_{S^l,S^k})(V) = P_{l-k} \otimes \mathrm{id}_{A\rtimes_E\mathbb{N}}$, we finally get that

$$\begin{aligned}(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N}) &= C^*(\{\psi(\pi_m(\xi^m))(\mathrm{id} \otimes \mathrm{id} \otimes f)(V) \colon \xi^m \in E^m, m \in \mathbb{N}, f \in Z\}) \\ &= C^*(\{(S^m \otimes \pi_m(\xi^m))(P_n \otimes \mathrm{id}_{A\rtimes_E\mathbb{N}}) \colon \xi^m \in E^m, m, n \in \mathbb{N}\}) \\ &= C^*(\{S^m P_n \otimes \pi_m(\xi^n) \colon \xi^m \in E^m, m, n \in \mathbb{N}\}) \\ &\subseteq \mathcal{L}(\ell^2(\mathbb{N}) \otimes A \rtimes_E \mathbb{N}).\end{aligned}$$

Chapter 6

Takai duality

For a C^*-dynamical system (A, G, α), Takai's duality theorem [28], which is the most fundamental theorem in the theory of crossed products, states that the double crossed product $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is strongly Morita equivalent to A. In [25], Schweizer provides an analog of Takai duality for crossed products by equivalence bimodules. He shows that there is an action γ of $\hat{\mathbb{Z}}$ on $A \rtimes_X \mathbb{Z}$, where X is an equivalence bimodule over a C^*-algebra A, such that $(A \rtimes_X \mathbb{Z}) \rtimes_\gamma \hat{\mathbb{Z}}$ is strongly Morita equivalent to A.

In this chapter our objective is to transfer this statement to the situation when E is a C^*-arrow over a C^*-algebra A, i.e., we want to show that the C^*-algebra $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ introduced in Example 5.35 is strongly Morita equivalent to A. Thus, we concentrate on the product systems (A, E) over the natural numbers from Example 3.14. First, we give a general definition of strong Morita equivalence for product systems and then we discuss what this notion means for our special product systems over $\mathbb{N}$. Thirdly, we show that an arbitrary product system (A, E) over $\mathbb{N}$ is always strongly Morita equivalent to the product system induced by a *-endomorphism α_E on the C^*-algebra $\mathcal{K}(\mathcal{F}(E))$. In the main theorem of this chapter we show that $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ is isomorphic to $\mathcal{K}(\mathcal{F}(E))$. Since $\mathcal{K}(\mathcal{F}(E))$ is strongly Morita equivalent to A, this implies that $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ and A are strongly Morita equivalent.

Definition 6.1 (Strong Morita equivalence for product systems)
We say that two product systems (F, Φ) and (G, Ψ) over an index category J are *strongly Morita equivalent* if there is an optransformation $\sigma\colon (F, \Phi) \to (G, \Psi)$ such that all C^*-arrows $\sigma_0(i)\colon F_i \to G_i$, $i \in \mathrm{Ob}(J)$, are invertible and all F_i-G_j-bimodule mappings $\sigma_1(r)\colon F_r \otimes \sigma_0(j) \Rightarrow \sigma_0(i) \otimes G_r$, $r\colon i \to j \in \mathrm{Arr}(J)$ are unitary.

For the rest of this chapter we will concentrate on the products systems (A, E) over the natural numbers from Example 3.14. We recall that these product systems (A, E) consist of a C^*-algebra A and a family of C^*-arrows $\{E^n\colon n \in \mathbb{N}\}$ over A, where we set $E^0 := {}_{\mathrm{id}}A$.

We also recall that we denoted the Fock correspondence for (A, E) by $\mathcal{F}(E)$, i.e.,

$$\mathcal{F}(E) = \{(\xi_n) \in \prod_{n\in\mathbb{N}} E^n \colon \sum_{n\in\mathbb{N}} (\, \xi_n \,|\, \xi_n \,)_A \quad \text{converges in } A\}.$$

Moreover, we suppose that

$$\varphi(A) \subseteq \mathcal{K}(E),$$

where $\varphi\colon A \to \mathcal{L}(E)$ is the *-homomorphism that implies the left multiplication of A on E. In what follows the term "(A, E) is a product system over $\mathbb{N}$" shall always mean that (A, E) is of the form described above.

Now, we want to discuss what it means for two product systems (A, E) and (B, F) over $\mathbb{N}$ to be strongly Morita equivalent. By the definition above, (A, E) and (B, F) are strongly Morita equivalent if there is an optransformation $\sigma\colon (A, E) \to (B, F)$ such that the C^*-arrow $\sigma_0\colon A \to B$ is invertible and all 2-cells $\sigma_1(m)\colon E^m \otimes \sigma_0 \Rightarrow \sigma_0 \otimes F^m$, $m \in \mathbb{N}$, are unitary. But it is easy to see that in this situation, all of the 2-cells $\sigma_1(m)$, $m > 1$, are uniquely determined by $\sigma_1 := \sigma_1(1)$ and so (A, E) and (B, F) are strongly Morita equivalent if and only if there is an invertible C^*-arrow $\sigma_0\colon A \to B$ and a unitary bimodule mapping $\sigma_1\colon E \otimes \sigma_0 \Rightarrow \sigma_0 \otimes F$. We notice that the first condition is the strong Morita equivalence of the C^*-algebras A and B.

In the following, we want to show that every product system (A, E) over $\mathbb{N}$ is strongly Morita equivalent to a product system induced by a *-endomorphism α_E on the C^*-algebra $\mathcal{K}(\mathcal{F}(E))$. So let (A, E) be an arbitrary but fixed product system over the natural numbers. By $M \subset \mathcal{F}(E)$ we denote the bisubmodule defined by

$$M := \{\xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \mathcal{F}(E) \colon \xi_0 = 0\}.$$

It is clear that M is a C^*-arrow from A to A. We define a mapping $V\colon \mathcal{F}(E) \otimes_A E \to M$ by setting

$$V((\xi_n)_{n\in\mathbb{N}} \otimes \xi) = (0, \Phi_{1,0}(\xi_0 \otimes \xi), \Phi_{1,1}(\xi_1 \otimes \xi), \Phi_{1,2}(\xi_2 \otimes \xi), \ldots).$$

It is easy to see that V is an A-A-bimodule mapping and the following computation shows that it is isometric:

$$\begin{aligned}
&(\, V((\xi_n) \otimes \xi) \,|\, V((\eta_n) \otimes \eta) \,)_A \\
= \;& (\, (0, \Phi_{1,0}(\xi_0 \otimes \xi), \Phi_{1,1}(\xi_1 \otimes \xi), \ldots) \,|\, (0, \Phi_{1,0}(\eta_0 \otimes \eta), \Phi_{1,1}(\eta_1 \otimes \eta), \ldots) \,)_A \\
= \;& \sum_{n\in\mathbb{N}} (\, \Phi_{1,n}(\xi_n \otimes \xi) \,|\, \Phi_{1,n}(\eta_n \otimes \eta) \,)_A = \sum_{n\in\mathbb{N}} (\, \xi_n \otimes \xi \,|\, \eta_n \otimes \eta \,)_A \\
= \;& \sum_{n\in\mathbb{N}} (\, \xi \,|\, (\, \xi_n \,|\, \eta_n \,)_A \cdot \eta \,)_A = (\, \xi \,|\, (\sum_{n\in\mathbb{N}} (\, \xi_n \,|\, \eta_n \,)_A) \cdot \eta \,)_A \\
= \;& (\, \xi \,|\, (\, (\xi_n) \,|\, (\eta_n) \,)_A \cdot \eta \,)_A = (\, (\xi_n) \otimes \xi \,|\, (\eta_n) \otimes \eta \,)_A.
\end{aligned}$$

Moreover, it is rather obvious that V is surjective (we notice that all the $\Phi_{1,n}$ are unitaries) and hence, V is a unitary. Next, we define a *-homomorphism $\alpha_E\colon \mathcal{K}(\mathcal{F}(E)) \to \mathcal{K}(M)$ by setting

$$\alpha_E(k) := V(k \otimes 1)V^*$$

for $k \in \mathcal{K}(\mathcal{F}(E))$. To see that $V(k \otimes 1)V^* \in \mathcal{K}(M)$, we notice that it follows from [15, Proposition 4.7.] that $k \otimes 1 \in \mathcal{K}(\mathcal{F}(E) \otimes_A E)$. Now we equip M with a $\mathcal{K}(\mathcal{F}(E))$ left action by setting $k \cdot \xi := \alpha_E(k)(\xi)$ for $k \in \mathcal{K}(\mathcal{F}(E))$ and $\xi \in M$. Thus, M is a C^*-arrow from $\mathcal{K}(\mathcal{F}(E))$ to A. Of course, $\mathcal{F}(E) \otimes_A E$ can also be viewed as a C^*-arrow from $\mathcal{K}(\mathcal{F}(E))$ to A and the following computation shows that V is a $\mathcal{K}(\mathcal{F}(E))$-$A$-bimodule mapping:

$$V(k\cdot((\xi_n)\otimes\xi)) = V(k\cdot(\xi_n)\otimes\xi) = V(k\otimes 1)((\xi_n)\otimes\xi) = \alpha_E(k)(V((\xi_n)\otimes\xi)) = k\cdot V((\xi_n)\otimes\xi),$$

where $k \in \mathcal{K}(\mathcal{F}(E))$, $(\xi_n) \in \mathcal{F}(E)$ and $\xi \in E$. This yields that M and $\mathcal{F}(E) \otimes_A E$ are unitarily equivalent as C^*-arrows from $\mathcal{K}(\mathcal{F}(E))$ to A.

Now $\alpha_E\colon \mathcal{K}(\mathcal{F}(E)) \to \mathcal{K}(M) \subset \mathcal{K}(\mathcal{F}(E))$ can be viewed as a *-endomorphism on $\mathcal{K}(\mathcal{F}(E))$ and thus α_E induces another product system over $\mathbb{N}$ that we denote by $(\mathcal{K}(\mathcal{F}(E)), \alpha_E)$. This product system consists of the C^*-algebra $\mathcal{K}(\mathcal{F}(E))$ and the family ${}_{\alpha_E^n}\mathcal{K}(\mathcal{F}(E))$, $n \in \mathbb{N}$, of C^*-arrows over $\mathcal{K}(\mathcal{F}(E))$. The family of unitary $\mathcal{K}(\mathcal{F}(E))$-$\mathcal{K}(\mathcal{F}(E))$-bimodule mappings $\Psi_{n,m}\colon {}_{\alpha_E^m}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} {}_{\alpha_E^n}\mathcal{K}(\mathcal{F}(E)) \to {}_{\alpha_E^{n+m}}\mathcal{K}(\mathcal{F}(E))$, $n, m \in \mathbb{N}$, is defined by setting

$$\Psi_{n,m}(\xi \otimes \eta) := \alpha_E^n(\xi)\eta.$$

Similar to Example 2.16, we can show that $\Psi_{n,m}$ is in deed a unitary $\mathcal{K}(\mathcal{F}(E))$-$\mathcal{K}(\mathcal{F}(E))$-bimodule mapping and that the family $\{\Psi_{n,m}\colon n, m \in \mathbb{N}\}$ satisfies the necessary identities for $(\mathcal{K}(\mathcal{F}(E)), \alpha_E)$ being a product system.

Proposition 6.2 (A, E) *and* $(\mathcal{K}(\mathcal{F}(E)), \alpha_E)$ *are strongly Morita equivalent.*

Proof: By the discussion above, it suffices to provide an invertible C^*-arrow σ_0 from $\mathcal{K}(\mathcal{F}(E))$ to A and a unitary bimodule mapping $\sigma_1\colon {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} \sigma_0 \Rightarrow \sigma_0 \otimes_A E$. We set $\sigma_0 := \mathcal{F}(E)$. $\mathcal{F}(E)$ clearly is a C^*-arrow from $\mathcal{K}(\mathcal{F}(E))$ to A and it is easy to see that $\mathcal{F}(E)$ is a $\mathcal{K}(\mathcal{F}(E))$-$A$-equivalence bimodule and hence invertible. It remains to show that ${}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} \mathcal{F}(E)$ and $\mathcal{F}(E) \otimes_A E$ are unitarily equivalent. We have already shown that $\mathcal{F}(E) \otimes_A E$ and M are unitarily equivalent. So now we define a mapping $W\colon {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} \mathcal{F}(E) \to M$ by setting

$$W(k \otimes \xi) := k(\xi)$$

for $k \in {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E))$ and $\xi \in \mathcal{F}(E)$. We notice that $k(\xi) \in M$ since $k \in {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) = \overline{\alpha_E(\mathcal{K}(\mathcal{F}(E)))\mathcal{K}(\mathcal{F}(E))}$ and $\alpha_E(\mathcal{K}(\mathcal{F}(E)) \subseteq \mathcal{K}(M)$. Moreover, we claim that W is a unitary $\mathcal{K}(\mathcal{F}(E))$-$A$-bimodule mapping. First we show that W is an isometry:

$$\begin{aligned} (\, k_1 \otimes \xi_1 \,|\, k_2 \otimes \xi_2 \,)_A &= (\, \xi_1 \,|\, (\, k_1 \,|\, k_2 \,)_{\mathcal{K}(\mathcal{F}(E))} \cdot \xi_2 \,)_A = (\, \xi_1 \,|\, k_1^* k_2(\xi_2) \,)_A \\ &= (\, k_1(\xi_1) \,|\, k_2(\xi_2) \,)_A = (\, W(k_1 \otimes \xi_1) \,|\, W(k_2 \otimes \xi_2) \,)_A \end{aligned}$$

for $k_1, k_2 \in {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E))$ and $\xi_1, \xi_2 \in \mathcal{F}(E)$. Now let $k \in \mathcal{K}(F(E))$, $k' \in {}_{\alpha_E}\mathcal{K}(\mathcal{F}(E))$, $\xi \in \mathcal{F}(E)$ and $a \in A$. Then

$$\begin{aligned} W(k \cdot (k' \otimes \xi) \cdot a) &= W((k \cdot k') \otimes (\xi \cdot a)) = W(\alpha_E(k)k' \otimes \xi \cdot a) \\ &= \alpha_E(k)k'(\xi \cdot a) = \alpha_E(k)(k'(\xi) \cdot a) \\ &= k \cdot (k'(\xi)) \cdot a = k \cdot W(k' \otimes \xi) \cdot a, \end{aligned}$$

which shows that W is a $\mathcal{K}(\mathcal{F}(E))$-$A$-bimodule mapping. It remains to prove that W is surjective. It suffices to prove that $\Phi_{1,k}(\xi \otimes \eta) \in \operatorname{ran}(W)$ for $\xi \in E^k$, $\eta \in E$, where we view E^n as a subset of $\mathcal{F}(E)$. So let (a_i) be an approximate unit in A. Then we have

$$\begin{aligned} W(\alpha_E(\theta_{\xi,a_i})\theta_{a_i\cdot\eta,a_i} \otimes a_i) &= \alpha_E(\theta_{\xi,a_i})\theta_{a_i\cdot\eta,a_i}(a_i) = V(\theta_{\xi,a_i} \otimes 1)V^*(a_i \cdot \eta \cdot (\, a_i \,|\, a_i \,)_A) \\ &= V(\theta_{\xi,a_i} \otimes 1)(a_i \otimes \eta \cdot (\, a_i \,|\, a_i \,)_A) \\ &= V(\xi \cdot (\, a_i \,|\, a_i \,)_A \otimes \eta \cdot (\, a_i \,|\, a_i \,)_A) \\ &= \Phi_{1,k}(\xi \cdot (\, a_i \,|\, a_i \,)_A \otimes \eta \cdot (\, a_i \,|\, a_i \,)_A), \end{aligned}$$

which converges to $\Phi_{1,k}(\xi \otimes \eta)$, since (a_i) is an approximate unit. Hence, W is a unitary $\mathcal{K}(\mathcal{F}(E))$-$A$-bimodule mapping and so M and ${}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} \mathcal{F}(E)$ are unitarily equivalent as C^*-arrows from $\mathcal{K}(\mathcal{F}(E))$ to A. Together with the result above we get that

$${}_{\alpha_E}\mathcal{K}(\mathcal{F}(E)) \otimes_{\mathcal{K}(\mathcal{F}(E))} \mathcal{F}(E) \approx \mathcal{F}(E) \otimes_A E$$

as C^*-arrows from $\mathcal{K}(\mathcal{F}(E))$ to A. □

Theorem 6.3 $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N}) \cong \mathcal{K}(\mathcal{F}(E))$

To simplify our notation we put $B := A \rtimes_E \mathbb{N}$. Then $\ell^2(\mathbb{N}, B) \approx \ell^2(\mathbb{N}) \otimes B$ is a Hilbert B-module. Moreover, $\pi_A \colon A \to B$ is a *-homomorphism and we define a linear mapping $\pi_{\mathcal{F}(E)} \colon \mathcal{F}(E) \to \ell^2(\mathbb{N}, B)$ by setting

$$\pi_{\mathcal{F}(E)}(\xi_0, \xi_1, \xi_2, \ldots) := (\pi_0(\xi_0), \pi_1(\xi_1), \pi_2(\xi_2), \ldots).$$

Lemma 6.4 *$(\pi_{\mathcal{F}(E)}, \pi_A)$ is a homomorphism from $(\mathcal{F}(E), A)$ to $(\ell^2(\mathbb{N}, B), B)$.*

Proof: For $\xi = (\xi_0, \xi_1, \xi_2, \ldots), \eta = (\eta_0, \eta_1, \eta_2, \ldots) \in \mathcal{F}(E)$ and $a \in A$ we have

$$\begin{aligned} (\, \pi_{\mathcal{F}(E)}(\xi) \,|\, \pi_{\mathcal{F}(E)}(\eta) \,)_B &= \sum_{n\in\mathbb{N}} \pi_n(\xi_n)^*\pi_n(\eta_n) = \sum_{n\in\mathbb{N}} \pi_0((\, \xi_n \,|\, \eta_n \,)_A) \\ &= \pi_A(\sum_{n\in\mathbb{N}} (\, \xi_n \,|\, \eta_n \,)_A) = \pi_A((\, \xi \,|\, \eta \,)_A) \qquad \text{and} \end{aligned}$$

$$\begin{aligned}\pi_{\mathcal{F}(E)}(\xi \cdot a) &= \pi_{\mathcal{F}(E)}(\xi_0 \cdot a, \xi_1 \cdot a, \xi_2 \cdot a, \ldots) = (\pi_0(\xi_0 \cdot a), \pi_1(\xi_1 \cdot a), \pi_2(\xi_2 \cdot a), \ldots)\\ &= (\pi_0(\xi_0)\pi_0(a), \pi_1(\xi_1)\pi_0(a), \pi_2(\xi_2)\pi_0(a), \ldots)\\ &= (\pi_0(\xi_0), \pi_1(\xi_1), \pi_2(\xi_2), \ldots) \cdot \pi_0(a) = \pi_{\mathcal{F}(E)}(\xi) \cdot \pi_0(a).\end{aligned}$$

□

Proof of Theorem 6.3: Since π_A is injective, it is isometric and hence, $\pi_{\mathcal{F}(E)}$ is also isometric. We let L denote the image of $\mathcal{F}(E)$ under $\pi_{\mathcal{F}(E)}$ and C the image of A under π_A,

$$L := \pi_{\mathcal{F}(E)}(\mathcal{F}(E)), \qquad C := \pi_A(A).$$

Hence, $\pi_{\mathcal{F}(E)} \colon \mathcal{F}(E) \to L$ is a bijective linear mapping, $\pi_A \colon A \to C$ is a *-isomorphism and $(\pi_{\mathcal{F}(E)}, \pi_A)$ is a homomorphism from $(\mathcal{F}(E), A)$ to (L, C). Let $\pi^{-1}_{\mathcal{F}(E)} \colon L \to \mathcal{F}(E)$ and $\pi_A^{-1} \colon C \to A$ denote the corresponding inverse mappings. Then it is easy to see that $(\pi^{-1}_{\mathcal{F}(E)}, \pi_A^{-1}) \colon (L, C) \to (\mathcal{F}(E), A)$ is a homomorphism. By Lemma 3.6 it follows that there are *-homomorphisms $\pi_{\mathcal{K}(\mathcal{F}(E))} \colon \mathcal{K}(\mathcal{F}(E)) \to \mathcal{K}(L)$ and $\pi^{-1}_{\mathcal{K}(\mathcal{F}(E))} \colon \mathcal{K}(L) \to \mathcal{K}(\mathcal{F}(E))$ that are uniquely defined by

$$\pi_{\mathcal{K}(\mathcal{F}(E))}(\theta_{\xi_1,\xi_2}) := \theta_{\pi_{\mathcal{F}(E)}(\xi_1),\pi_{\mathcal{F}(E)}(\xi_2)} \qquad \pi^{-1}_{\mathcal{K}(\mathcal{F}(E))}(\theta_{\eta_1,\eta_2}) := \theta_{\pi^{-1}_{\mathcal{F}(E)}(\eta_1),\pi^{-1}_{\mathcal{F}(E)}(\eta_2)}$$

for $\xi_1, \xi_2 \in \mathcal{F}(E)$ and $\eta_1, \eta_2 \in L$. Since $\pi^{-1}_{\mathcal{F}(E)}$ is the inverse mapping of $\pi_{\mathcal{F}(E)}$, it is clear that $\pi_{\mathcal{K}(\mathcal{F}(E))}$ is a *-isomorphism with inverse $\pi^{-1}_{\mathcal{K}(F(E))}$ and hence, the C^*-algebras $\mathcal{K}(\mathcal{F}(E))$ and $\mathcal{K}(L)$ are isomorphic.

Next, we define a mapping $W \colon \mathcal{L}(\mathcal{F}(E)) \to \mathcal{L}(L)$ by setting

$$W(T)(\xi) := \pi_{\mathcal{F}(E)} \circ T \circ \pi^{-1}_{\mathcal{F}(E)}(\xi)$$

for $T \in \mathcal{L}(\mathcal{F}(E))$ and $\xi \in L$. To see that $W(T) \in \mathcal{L}(L)$, we let $\xi, \eta \in L$ arbitrary and compute

$$\begin{aligned}(\, W(T)(\xi) \,|\, \eta \,)_C &= (\, \pi_{\mathcal{F}(E)} \circ T \circ \pi^{-1}_{\mathcal{F}(E)}(\xi) \,|\, \eta \,)_C = \pi_A((\, T \circ \pi^{-1}_{\mathcal{F}(E)}(\xi) \,|\, \pi^{-1}_{\mathcal{F}(E)}(\eta) \,)_A)\\ &= \pi_A((\, \pi^{-1}_{\mathcal{F}(E)}(\xi) \,|\, T^* \circ \pi^{-1}_{\mathcal{F}(E)}(\eta) \,)_A) = (\, \xi \,|\, \pi_{\mathcal{F}(E)} \circ T^* \circ \pi_{\mathcal{F}(E)^{-1}}(\eta) \,)_C\\ &= (\, \xi \,|\, W(T^*)(\eta) \,)_C,\end{aligned}$$

which shows that $W(T) \in \mathcal{L}(L)$ with $W(T)^* = W(T^*)$. It is also easy to see that W is linear and multiplicative and hence, a *-homomorphism from $\mathcal{L}(\mathcal{F}(E))$ to $\mathcal{L}(L)$.

Analogously, we define a mapping $W^{-1} \colon \mathcal{L}(L) \to \mathcal{L}(\mathcal{F}(E))$ by setting

$$W^{-1}(T)(\eta) := \pi^{-1}_{\mathcal{F}(E)} \circ T \circ \pi_{\mathcal{F}(E)}(\eta)$$

for $T \in \mathcal{L}(L)$ and $\eta \in \mathcal{F}(E)$. It is clear that W^{-1} is a *-homomorphism from $\mathcal{L}(L)$ to $\mathcal{L}(\mathcal{F}(E))$ and since $\pi^{-1}_{\mathcal{F}(E)}$ is the inverse mapping of $\pi_{\mathcal{F}(E)}$, it is also easy to see that W^{-1} is the inverse *-homomorphism of W. Hence, we see that the C^*-algebras $\mathcal{L}(L)$ and $\mathcal{L}(\mathcal{F}(E))$ are isomorphic. The following computation shows that $\pi_{\mathcal{K}(\mathcal{F}(E))}$ is the restriction of W to $\mathcal{K}(\mathcal{F}(E))$, because for $\xi, \eta \in \mathcal{F}(E)$ and $\zeta \in L$ we have

$$\begin{aligned} W(\theta_{\xi,\eta})(\zeta) &= \pi_{\mathcal{F}(E)} \circ \theta_{\xi,\eta} \circ \pi_{\mathcal{F}(E)}^{-1}(\zeta) = \pi_{\mathcal{F}(E)}(\xi \cdot (\,\eta \,|\, \pi_{\mathcal{F}(E)}^{-1}(\zeta)\,)_A) \\ &= \pi_{\mathcal{F}(E)}(\xi) \cdot \pi_A((\,\eta \,|\, \pi_{\mathcal{F}(E)}^{-1}(\zeta)\,)_A) = \pi_{\mathcal{F}(E)}(\xi) \cdot (\,\pi_{\mathcal{F}(E)}(\eta) \,|\, \zeta\,)_C \\ &= \theta_{\pi_{\mathcal{F}(E)}(\xi),\pi_{\mathcal{F}(E)}(\eta)}(\zeta) = \pi_{\mathcal{K}(\mathcal{F}(E))}(\theta_{\xi,\eta})(\zeta). \end{aligned}$$

Now let S be the forward shift operator on $\ell^2(\mathbb{N})$, i.e., $S(z_0, z_1, z_2, \ldots) = (0, z_0, z_1, z_2, \ldots)$ and let $P_n \in \mathcal{L}(\ell^2(\mathbb{N}))$ be the projection on the n-th component of $\ell^2(\mathbb{N})$. We consider the operators

$$S^m P_n \otimes \pi_m(\xi^m) \in \mathcal{L}(\ell^2(\mathbb{N}) \otimes B) \cong \mathcal{L}(\ell^2(\mathbb{N}, B)) \qquad (\xi^m \in E^m, m, n \in \mathbb{N})$$

that generate $B \rtimes_\delta C^*(\mathbb{N})$, see Example 5.35. We claim that $L \subset \ell^2(\mathbb{N}, B)$ is an invariant submodule for $S^m P_n \otimes \pi_m(\xi^m)$. So let $\eta = (\eta_0, \eta_1, \eta_2, \ldots) \in \mathcal{F}(E)$ be arbitrary. Then $\pi_{\mathcal{F}(E)}(\eta) = (\pi_0(\eta_0), \pi_1(\eta_1), \pi_2(\eta_2), \ldots) \in L$ and we compute

$$\begin{aligned} & (S^m P_n \otimes \pi_m(\xi^m))(\pi_0(\eta_0), \pi_1(\eta_1), \pi_2(\eta_2), \ldots) \\ = \;& (0, 0, \ldots, 0, \pi_m(\xi^m)\pi_n(\eta_n), 0, \ldots) = (0, 0, \ldots, 0, \pi_{m+n}(\Phi_{n,m}(\xi^m \otimes \eta_n)), 0, \ldots) \\ = \;& \pi_{\mathcal{F}(E)}(0, 0, \ldots, 0, \Phi_{n,m}(\xi^m \otimes \eta_n), 0, \ldots). \end{aligned}$$

Now $(0, 0, \ldots, 0, \Phi_{n,m}(\xi^m \otimes \eta_n), 0, \ldots) \in \mathcal{F}(E)$, since the $n+m$-th component $\Phi_{n,m}(\xi^m \otimes \eta_n)$ is in E^{n+m} and thus $(S^m P_n \otimes \pi_m(\xi^m))(\pi_0(\eta_0), \pi_1(\eta_1), \pi_2(\eta_2), \ldots) \in L$, which shows that $S^m P_n \otimes \pi_m(\xi^m)$ leaves L invariant. Analogously, we can show that $(S^m P_n \otimes \pi_m(\xi^m))^*$ leaves L invariant and hence, $B \rtimes_\delta C^*(\mathbb{N})$ leaves L invariant and can be viewed as a C^*-subalgebra of $\mathcal{L}(L)$. The next computation shows that $W^{-1}(S^m P_n \otimes \pi_m(\xi^m)) = \pi_m(\xi^m)\tilde{P}_n = T_{\xi^m}\tilde{P}_n$, where $\tilde{P}_n$ is the projection onto the n-th component of $\mathcal{F}(E)$. For $\eta = (\eta_0, \eta_1, \eta_2, \ldots) \in \mathcal{F}(E)$ we have

$$\begin{aligned} W^{-1}(S^m P_n \otimes \pi_m(\xi^m))(\eta) &= \pi_{\mathcal{F}(E)}^{-1} \circ (S^m P_n \otimes \pi_m(\xi^m)) \circ \pi_{\mathcal{F}(E)}(\eta) \\ &= \pi_{\mathcal{F}(E)}^{-1} \circ (S^m P_n \otimes \pi_m(\xi^m))(\pi_0(\eta_0), \pi_1(\eta_1), \pi_2(\eta_2), \ldots) \\ &= \pi_{\mathcal{F}(E)}^{-1}(\pi_{\mathcal{F}(E)}(0, 0, \ldots, 0, \Phi_{n,m}(\xi^m \otimes \eta_n), 0, \ldots)) \\ &= (0, 0, \ldots, 0, \Phi_{n,m}(\xi^m \otimes \eta_n), 0, \ldots) \\ &= T_{\xi^m}(0, 0, \ldots, 0, \eta_n, 0, \ldots) = T_{\xi^m}\tilde{P}_n(\eta). \end{aligned}$$

We claim that $T_{\xi^m}\tilde{P}_n \in \mathcal{K}(\mathcal{F}(E))$. So let $(u_\alpha)_\alpha$ be an approximate unit for A and let $\phi^n \colon A \to \mathcal{K}(E^n)$ be the *-homomorphism that implies the left multiplication on E^n. We note that $\phi^n(A) \subset \mathcal{K}(E^n)$ since $\phi(A) \subset \mathcal{K}(E)$. Then there exist $\eta_i^\alpha \in E^n$ such that

$\phi^n(u_\alpha) = \sum_i \theta_{\eta_i^\alpha,\eta_i^\alpha}$ and we compute

$$\begin{aligned}
\sum_i \theta_{\Phi_{n,m}(\xi^m\otimes\eta_i^\alpha),\eta_i^\alpha}(\zeta) &= \sum_i \Phi_{n,m}(\xi^m \otimes \eta_i^\alpha)\cdot(\,\eta_i^\alpha \mid \zeta_n\,)_A = \sum_i \Phi_{n,m}(\xi^m \otimes \eta_i^\alpha \cdot (\,\eta_i^\alpha \mid \zeta_n\,)_A) \\
&= \sum_i \Phi_{n,m}(\xi^m \otimes \theta_{\eta_i^\alpha,\eta_i^\alpha}(\zeta_n)) = \Phi_{n,m}(\xi^m \otimes (\sum_i \theta_{\eta_i^\alpha,\eta_i^\alpha})(\zeta_n)) \\
&= \Phi_{n,m}(\xi^m \otimes u_\alpha \cdot \zeta_n) = \Phi_{n,m}(\xi^m \otimes u_\alpha \cdot \tilde{P}_n(\zeta)) \\
&= \Phi_{n,m}(\xi^m \cdot u_\alpha \otimes \tilde{P}_n(\zeta)) = T_{\xi^m\cdot u_\alpha}\tilde{P}_n(\zeta)
\end{aligned}$$

for $\zeta \in \mathcal{F}(E)$ arbitrary. Hence, $T_{\xi^m\cdot u_\alpha}\tilde{P}_n = \sum_i \theta_{\Phi_{n,m}(\xi^m\otimes\eta_i^\alpha),\eta_i^\alpha} \in \mathcal{K}(\mathcal{F}(E))$. But $T_{\xi^m\cdot u_\alpha}$ converges in norm to T_{ξ^m} and thus also $T_{\xi^m}\tilde{P}_n \in \mathcal{K}(\mathcal{F}(E))$.

Hence, we get that $W^{-1}(S^m P_n \otimes \pi_m(\xi^m)) = T_{\xi^m}\tilde{P}_n \in \mathcal{K}(\mathcal{F}(E))$, which implies that $W^{-1}(B \rtimes_\delta C^*(\mathbb{N})) \subseteq \mathcal{K}(\mathcal{F}(E))$. Finally, we observe that

$$W^{-1}((S^m P_0 \otimes \pi_m(\xi^m))(S^n P_0 \otimes \pi_n(\eta^n))^*) = T_{\xi^m}\tilde{P}_0(T_{\eta^n}\tilde{P}_0)^* = T_{\xi^m}T_{\eta^n}^*\tilde{P}_n = \theta_{\xi^m,\eta^n}$$

for all $\xi^m \in E^m$, $\eta^n \in E^n$, $n, m \in \mathbb{N}$. But it is clear that the linear span of these operators is dense in $\mathcal{K}(\mathcal{F}(E))$ and hence, $W^{-1}(B \rtimes_\delta C^*(\mathbb{N})) = \mathcal{K}(\mathcal{F}(E))$, which yields that

$$B \rtimes_\delta C^*(\mathbb{N}) \cong \mathcal{K}(\mathcal{F}(E)).$$

□

Corollary 6.5 *A and $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ are strongly Morita equivalent.*

Proof: By Theorem 6.3, $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ and $\mathcal{K}(\mathcal{F}(E))$ are isomorphic and hence, strongly Morita equivalent. Moreover, Proposition 6.2 implies that A and $\mathcal{K}(\mathcal{F}(E))$ are strongly Morita equivalent and thus, it is clear that A and $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ are strongly Morita equivalent. □

Appendix A

Crossed products by discrete groups

Let G be a discrete group, i.e., a group that carries the discrete topology, and let A be a C^*-algebra. Moreover, let α be an action of G on A, i.e., the mapping $g \in G \mapsto \alpha_g \in \mathrm{Aut}(A)$ is a group homomorphism. Then we call the triple (A, G, α) a *discrete C^*-dynamical system*.

A *unitary representation* of a discrete group G is a group homomorphism U from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. A *covariant representation* of a discrete C^*-dynamical system (A, G, α) is a pair (π, U), where π is a nondegenerate representation of A on a Hilbert space $\mathcal{H}$ and U is a unitary representation on the same Hilbert space such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*$$

for all $a \in A$ and $s \in G$.

Given a discrete C^*-dynamical system (A, G, α) and a nondegenerate representation π of A on a Hilbert space $\mathcal{H}$, we can always construct covariant representations $(\tilde{\pi}_\lambda, \tilde{\lambda})$ and $(\tilde{\pi}_\rho, \tilde{\rho})$ of (A, G, α) on $\ell^2(G, \mathcal{H})$ using the left and the right regular representation λ and ρ of G on $\ell^2(G)$, respectively :

$$(\lambda_s \xi)(t) := \xi(s^{-1}t) \qquad \text{and} \qquad (\rho_s \xi)(t) := \xi(ts) \qquad \xi \in \ell^2(G), s, t \in G.$$

For $a \in A$, $\xi \in \ell^2(G, \mathcal{H})$ and $s, t \in G$ we define

$$(\tilde{\pi}_\lambda(a)\xi)(t) := \pi(\alpha_{t^{-1}}(a))\xi(t) \quad \text{and} \quad (\tilde{\lambda}_s \xi)(t) := \xi(s^{-1}t).$$

Then one can easily check that $(\tilde{\pi}_\lambda, \tilde{\lambda})$ is a covariant representation, the *left regular representation* corresponding to π. Similarly, we define

$$(\tilde{\pi}_\rho(a)\xi)(t) := \pi(\alpha_t(a))\xi(t) \quad \text{and} \quad (\tilde{\rho}_s \xi)(t) := \xi(ts)$$

and again it is easy to see that $(\tilde{\pi}_\rho, \tilde{\rho})$ is a covariant representation of (A, G, α), the *right regular representation* corresponding to π.

Definition A.1 A *crossed product* corresponding to a discrete C^*-dynamical system (A, G, α) is a C^*-algebra B together with a *-homomorphism $i_A \colon A \to M(B)$ and a group homomorphism $i_G \colon G \to \mathcal{U}M(B)$ such that

- $i_A(\alpha_s(a)) = i_G(s) i_A(a) i_G(s)^*$ for all $a \in A$, $s \in G$ and
- for every covariant representation (π, U) of (A, G, α) there is a unique nondegenerate representation $\pi \times U$ of B with $\pi = \overline{(\pi \times U)} \circ i_A$ and $U = \overline{(\pi \times U)} \circ i_G$.

One can show that given a discrete C^*-dynamical system (A, G, α), a crossed product exists and that it is unique up to isomorphism. Hence, we can talk about *the* crossed product of (A, G, α) and we denote it by $A \rtimes_\alpha G$. By definition, there is a 1-1-correspondence between the representations of $A \rtimes_\alpha G$ and the covariant representations of (A, G, α).

We want to describe one way how the crossed product of a discrete C^*-dynamical system (A, G, α) can be constructed. Let $C_c(G, A)$ be the vector space of all A-valued functions on G with compact support. Hence, a function $f \in C_c(G, A)$ takes nonzero values only on a finite subset of G. We define pointwise addition and a convolution product by

$$(f * g)(t) := \sum_{s \in G} f(s) \alpha_s(g(s^{-1}t))$$

for $f, g \in C_c(G, A)$ and $t \in G$. Moreover, we define an involution and a norm on $C_c(G, A)$ by

$$f^*(t) := \alpha_t(f(t^{-1})^*) \qquad \text{and} \qquad \|f\|_1 := \sum_{s \in G} \|f(s)\|,$$

where $f \in C_c(A, G)$ and $t \in G$. This makes $C_c(G, A)$ a normed algebra with isometric involution. The completion of $C_c(G, A)$, which we denote by $L^1(G, A)$, is an involutive Banach algebra but in general not a C^*-algebra since the norm $\|.\|_1$ is not a C^*-norm.

Now the *full crossed product* $A \rtimes_\alpha G$ is the completion of $C_c(G, A)$ with respect to the maximal C^*-norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is a } ^*\text{-representation of } L^1(G, A)\}$$

and the *reduced crossed product* $A \rtimes_{\alpha r} G$ is the norm closure of $((\tilde{\pi}_u)_\lambda \times \tilde{\lambda})(A \rtimes_\alpha G)$, where $\pi_u \colon A \to \mathcal{L}(\mathcal{H}_u)$ is the universal representation of A on. We notice that the linear span of the set of products $\{(\tilde{\pi}_u)_\lambda(a)\tilde{\lambda}_s \colon a \in A, s \in G\}$ is dense in $A \rtimes_{\alpha r} G$. In other words, the reduced crossed product $A \rtimes_{\alpha r} G$ is the C^*-subalgebra of $\mathcal{L}(\ell^2(G, \mathcal{H}_u))$ generated by the products $\{(\tilde{\pi}_u)_\lambda(a)\tilde{\lambda}_s \colon a \in A, s \in G\}$.

Remark: Let B be the C^*-subalgebra of $\mathcal{L}(\ell^2(G, \mathcal{H}_u))$ generated by the set of products $\{(\tilde{\pi}_u)_\rho(a)\tilde{\rho}_s \colon a \in A, s \in G\}$. We claim that $A \rtimes_{\alpha r} G$ and B are isomorphic. To see this, we use the unitary operator $U \in \mathcal{L}(\ell^2(G, \mathcal{H}_u))$ defined by $(U\xi)(t) := \xi(t^{-1})$ and we define an automorphism $\vartheta \in \mathrm{Aut}(\mathcal{L}(\ell^2(G, \mathcal{H}_u)))$ by $\vartheta(T) := UTU^* = UTU$. An easy calculation yields that $\vartheta(\tilde{\lambda}_s) = \tilde{\rho}_s$ and $\vartheta((\tilde{\pi}_u)_\lambda(a)) = (\tilde{\pi}_u)_\rho(a)$ for all $s \in G$ and $a \in A$ and so it is clear that $\vartheta(A \rtimes_{\alpha r} G) = B$. So one could have also used the right regular representation corresponding to π_u to define the reduced crossed product.

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Zusammenfassung in deutscher Sprache

Diese Arbeit lässt sich in zwei Teile gliedern. Im ersten Teil, der die Kapitel eins bis vier umfasst, wollen wir Produktsysteme von einem höheren Standpunkt aus betrachten. Produktsysteme wurden zuerst von Arveson [1] im Jahre 1989 eingeführt, um eine Indextheorie für stetige Halbgruppen von *-Endomorphismen in $\mathcal{L}(\mathcal{H})$ zu entwickeln. Später betrachtete dann Dinh [8] den diskreten Fall und Fowler [10] wiederum verallgemeinerte Dinhs Konzept, indem er Hilbert-Bimoduln anstelle von Hilberträumen benutzte. Fowlers diskrete Produktsysteme von Hilbert-Bimoduln bestehen aus einer Familie von Hilbert-Bimoduln $\{X_s \colon s \in \mathcal{S}\}$ über einer C^*-Algebra A, die von einer abzählbaren Halbgruppe $\mathcal{S}$ indiziert wird, und einer Familie von unitären Bimodulabbildungen

$$\Phi_{s,t} \colon X_s \otimes_A X_t \to X_{st}, \qquad s,t \in \mathcal{S}.$$

Wir wollen mit Hilfe der Bikategorientheorie die Struktur offenlegen, die sich hinter den Produktsystemen verbirgt. Eine Bikategorie besteht, ebenso wie eine Kategorie, aus Objekten und Pfeilen zwischen diesen Objekten. Im Unterschied zu Kategorien besitzen Bikategorien jedoch noch eine zusätzliche Struktur, nämlich Pfeile zwischen den Pfeilen, die 2-Zellen genannt werden. Unser Hauptbeispiel für Bikategorien ist die Bikategorie C*ARR. Die Objekte von C*ARR sind C^*-Algebren, die Pfeile sind C^*-Pfeile (ein Begriff, der auf Hilbert-Bimoduln basiert und auf den wir später noch näher eingehen werden) und die 2-Zellen sind adjungierbare, isometrische Bimodulabbildungen.

Wir werden die "Funktoren" zwischen Bikategorien einführen – die sogenannten Morphismen – und wir werden sehen, dass die Fowlerschen Produktsysteme im Grunde nur spezielle Morphismen von der Halbgruppe $\mathcal{S}$ (als Bikategorie betrachtet) in die Bikategorie C*ARR darstellen. Somit können wir eine natürlichere und elegantere Definition für Produktsysteme angeben, indem wir definieren, dass ein Produktsystem ein Morphismus von einer Indexkategorie J (die wir als Bikategorie betrachten) in die Bikategorie C*ARR ist. Wir können damit die Produktsysteme, die ursprünglich von Arveson für die Indextheorie für Halbgruppen von *-Endomorphismen entwickelt worden waren, auf eine sehr natürliche Art und Weise beschreiben, indem wir die Bikategorientheorie bemühen. Außerdem ergibt sich dadurch, dass wir unsere Produktsysteme nicht mehr durch Halbgruppen sondern durch Indexkategorien indizieren lassen, eine viel größere Klasse von Beispielen.

Im weiteren Verlauf der Arbeit ordnen wir jedem Produktsystem (F, Φ) zwei C^*-Algebren zu, und zwar die reduzierte Toeplitz-Algebra $\mathcal{T}_r(F, \Phi)$ und die reduzierte Cuntz-Pimsner-Algebra $\mathcal{O}_r(F, \Phi)$. Desweiteren untersuchen wir diverse Spezialfälle, die zeigen, dass unsere Konstruktionsmethoden für die reduzierten Toeplitz- bzw. Cuntz-Pimsner-Algebren viele andere Konstruktionen von C^*-Algebren verallgemeinern.

Anschließend werden wir die universelle Toeplitz-Algebra $\mathcal{T}(F, \Phi)$ und die universelle Cuntz-Pimsner-Algebra $\mathcal{O}(F, \Phi)$ einführen. Wir wiederholen den Begriff des bikategoriellen Kolimes für einen Morphismus und beschließen den ersten Teil dieser Arbeit, indem wir zeigen, dass für gewisse Produktsysteme (F, Φ) die universelle Toeplitz-Algebra als das bikategorielle Kolimesobjekt des Morphismus (F, Φ) betrachtet werden kann.

Im zweiten Teil der Arbeit, der die Kapitel fünf und sechs beinhaltet, entwickeln wir eine Dualitätstheorie für lokalkompakte Halbgruppen und greifen dabei auf das Konzept der Hopf C^*-Algebren zurück. Eine Hopf C^*-Algebra ist eine C^*-Algebra H, auf der ein nicht-degenerierter, injektiver *-Homomorphismus $\delta_H \colon H \to M(H \otimes H)$ – die sogenannte Komultiplikation – definiert ist. Das Standardbeispiel für eine Hopf C^*-Algebra ist $C_0(\mathcal{S})$, die C^*-Algebra der komplexen Funktionen auf einer lokalkompakten Halbgruppe $\mathcal{S}$, die im Unendlichen verschwinden. Die Multiplikation auf $\mathcal{S}$ induziert eine Komultiplikation auf $C_0(\mathcal{S})$. Insofern kann man Hopf C^*-Algebren als verallgemeinerte lokalkompakte Halbgruppen betrachten.

Wir werden dann eine hinreichende Bedingung an die Hopf C^*-Algebra H entwickeln, die es uns ermöglicht, eine Kodarstellung von H auf einem ausgezeichneten Hilbertraum zu konstruieren, ähnlich der regulären Darstellung einer lokalkompakten Gruppe G auf dem Hilbertraum $L^2(G, \mu)$, wobei μ das rechte Haarmaß auf G bezeichnet. Mit Hilfe dieser regulären Kodarstellung können wir dann die reduzierte duale C^*-Algebra von einer Hopf C^*-Algebra definieren und wir werden zeigen, dass die klassische Toeplitz-Algebra $C^*(\mathbb{N})$ die reduzierte duale C^*-Algebra der Hopf C^*-Algebra $c_0(\mathbb{N})$ ist. Ebenso ergibt sich, dass $c_0(\mathbb{N})$ die reduzierte duale C^*-Algebra der Hopf C^*-Algebra $C^*(\mathbb{N})$ ist. Dies entspricht der Tatsache, dass für eine lokalkompakte Gruppe G die beiden C^*-Algebren $C_0(G)$ und $C^*(G)$ in Dualität zueinander stehen und kann als Analogon zum Dualitätssatz von Pontryagin betrachtet werden.

Schließlich werden wir uns noch einem weiteren Dualitätssatz zuwenden, dem Dualitätssatz von Takai [28]. Dies ist einer der grundlegendsten Sätze aus dem Bereich der verschränkten Produkte. Er besagt, dass für ein C^*-dynamisches System (A, G, α) das doppelte verschränkte Produkt $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ stark Morita-äquivalent zu A ist. Schweizer behandelte in seiner Arbeit [25] ein Analogon zu Takais Dualitätssatz für verschränkte Produkte durch Äquivalenzbimoduln. Er zeigte, dass es eine Wirkung γ von $\hat{\mathbb{Z}}$ auf $A \rtimes_X \mathbb{Z}$ gibt – wobei X ein Äquivalenzbimodul über der C^*-Algebra A ist – so dass $(A \rtimes_X \mathbb{Z}) \rtimes_\gamma \hat{\mathbb{Z}}$ stark Morita-äquivalent zu A ist.

Wir wollen Schweizers Aussage auf verschränkte Produkte durch C^*-Pfeile übertragen. Dazu definieren wir zunächst das verschränkte Produkt $A \rtimes_E \mathbb{N}$ eines C^*-Pfeils E über einer C^*-Algebra A als die reduzierte Toeplitz-Algebra von (A, E), wobei (A, E) ein bestimmtes

Produktsystem über $\mathbb{N}$ ist, das aus den Potenzen von E besteht. Anschließend definieren wir das reduzierte verschränkte Produkt zu einem dynamischen Kosystem und schließlich konstruieren wir eine Kowirkung δ von $C^*(\mathbb{N})$ auf $A \rtimes_E \mathbb{N}$ und zeigen, dass das doppelte verschränkte Produkt $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ stark Morita-äquivalent zu A ist.

Nach diesem eher allgemein gehaltenen Überblick wollen wir nun eine etwas präzisere Inhaltsangabe machen:

Das erste Kapitel dient dazu, die historische Entwicklung der Produktsysteme zu skizzieren. Wir geben einen kurzen Überblick über die Arbeiten von Arveson [1], der Produktsysteme erstmals eingeführt hat, Dinh [8], der als erster diskrete Produktsysteme untersuchte, und Fowler [10], der Dinhs diskrete Produktsysteme verallgemeinert hat, indem er Hilberträume durch Hilbert-Bimoduln ersetzte und beliebige abzählbare Halbgruppen als Indexmengen zuließ.

Im zweiten Kapitel wiederholen wir den Begriff der Bikategorie und liefern einige Beispiele für Bikategorien. Wir werden das Konzept der C^*-Pfeile einführen, das eine wichtige Rolle in unserer Arbeit spielt. Ein C^*-Pfeil ist ein Hilbert B-Modul, der auch die Struktur eines A-B-Bimoduls besitzt, wobei A und B C^*-Algebren sind. C^*-Pfeile sind die Pfeile in unserem wichtigsten Beispiel für eine Bikategorie, der Bikategorie C*ARR. Die Objekte von C*ARR sind C^*-Algebren und die 2-Zellen sind adjungierbare, isometrische Bimodulabbildungen. Wir wiederholen dann den Begriff des Morphismus zwischen Bikategorien, welcher das Konzept von Funktoren zwischen Kategorien verallgemeinert. Diesen Begriff werden wir dazu benutzen, um ein Produktsystem über einer Kategorie als Morphismus von einer Indexkategorie J in die Bikategorie C*ARR zu definieren. Nach dieser Definition bestehen unsere Produktsysteme dann aus einer Familie von C^*-Algebren A_i, $i \in \mathrm{Ob}(J)$, einer Familie von C^*-Pfeilen F_r, $r \in \mathrm{Arr}(J)$, und einer Familie von adjungierbaren, isometrischen Bimodulabbildungen $\Phi_{s,r}\colon F_r \otimes F_s \to F_{sr}$ für komponierbare Pfeile s und r in J. Schließlich werden wir sehen, dass die Fowlerschen diskreten Produktsysteme von Hilbert-Bimoduln einen Spezialfall unserer Definition darstellen.

Kapitel 3 behandelt die Konstruktion der reduzierten Toeplitz-Algebra $\mathcal{T}_r(F,\Phi)$ und der reduzierten Cuntz-Pimsner-Algebra $\mathcal{O}_r(F,\Phi)$, die man jedem gegebenen Produktsystem (F,Φ) zuordnen kann. Zunächst führen wir den Begriff der Toeplitz-Darstellung von einem Produktsystem (F,Φ) in eine C^*-Algebra ein und geben einige technische Resultate über Homomorphismen zwischen Hilbert C^*-Moduln an, die wir im späteren Verlauf der Arbeit noch gebrauchen werden. Dann führen wir die Fock-Korrespondenz $\mathcal{F}(F,\Phi)$ eines Produktsystems (F,Φ) als einen C^*-Pfeil über der direkten Summe der A_i, $i \in \mathrm{Ob}(J)$, ein. Wir benutzen $\mathcal{F}(F,\Phi)$, um eine spezielle Toeplitz-Darstellung zu konstruieren: die reduzierte Toeplitz-Darstellung von (F,Φ) in die reduzierte Toeplitz-Algebra $\mathcal{T}_r(F,\Phi)$, bei der es sich um eine C^*-Unteralgebra von $\mathcal{L}(\mathcal{F}(F,\Phi))$ handelt. Wir definieren die reduzierte Cuntz-Pimsner-Algebra $\mathcal{O}_r(F,\Phi)$ als den Quotienten von $\mathcal{T}_r(F,\Phi)$ modulo dem Ideal der kompakten Operatoren in $\mathcal{T}_r(F,\Phi)$. Anschließend liefern wir mehrere Beispiele, die zeigen, dass unsere Methode zur Konstruktion der reduzierten Toeplitz- bzw. Cuntz-Pimsner-Algebren viele andere Methoden zur Konstruktion von C^*-Algebren verallgemeinert. Je nachdem, wie wir unser Produktsystem (F, Φ) wählen, erhalten wir als resultierende Toeplitz- bzw. Cuntz-Pimsner-Algebren C^*-Algebren, die isomorph sind zur direkten Summe einer Familie von C^*-Algebren, zum direkten Limes eines direkten Systems von C^*-Algebren oder zum verschränkten Produkt einer C^*-Algebra mit einer Gruppe oder Halbgruppe.

Im vierten Kapitel werden wir die universelle Toeplitz-Algebra $\mathcal{T}(F, \Phi)$ und die universelle Cuntz-Pimsner-Algebra $\mathcal{O}(F, \Phi)$ zusammen mit ihren jeweiligen Toeplitz-Darstellungen einführen. Zuerst zeigen wir, dass es zu jedem Produktsystem (F, Φ) über einer Indexkategorie J eine C^*-Algebra gibt, die universell ist für Toeplitz-Darstellungen über (F, Φ). Diese C^*-Algebra nennen wir die universelle Toeplitz-Algebra $\mathcal{T}(F, \Phi)$. Dann wiederholen wir den Begriff der Cuntz-Pimsner kovarianten Toeplitz-Darstellungen, den Fowler bereits für seine Produktsysteme benutzt hat, und wir führen die universelle Cuntz-Pimsner-Algebra ein, die universell ist für Cuntz-Pimsner kovariante Toeplitz-Darstellungen über (F, Φ). Schließlich werden wir den Begriff des bikategoriellen Kolimes für einen Morphismus (F, Φ) von einer Bikategorie $\mathcal{B}$ in eine Bikategorie $\mathcal{B}'$ wiederholen und zeigen, dass man für gewisse Produktsysteme (F, Φ) die universelle Toeplitz-Algebra $\mathcal{T}(F, \Phi)$ als zum Morphismus (F, Φ) gehöriges Kolimesobjekt in der Bikategorie C*ARR betrachten kann.

In Kapitel 5 behandeln wir zunächst Hopf C^*-Algebren und wiederholen dabei auch einige Begriffe aus deren Umfeld, die die entsprechenden Begriffe aus der Theorie der lokalkompakten Halbgruppen verallgemeinern. Dabei handelt es sich um Kodarstellungen, Kowirkungen von Hopf C^*-Algebren auf C^*-Algebren und kovariante Darstellungen von dynamischen Kosystemen. Anschließend zeigen wir, dass die Existenz eines invarianten Gewichtes auf einer Hopf C^*-Algebra H es ermöglicht, eine spezielle kovariante Darstellung des dynamischen Kosystems (H, H, δ_H) zu konstruieren. Diese kovariante Darstellung hat ihr Vorbild in der rechtsregulären kovarianten Darstellung von $(C_0(G), G, \alpha)$ auf $L^2(G, \mu)$ – wobei μ das rechte Haarmaß auf einer lokalkompakten Gruppe G und α die Gruppenwirkung von G auf $C_0(G)$ durch Rechtstranslation bezeichnet – und wir nennen sie deshalb reguläre kovariante Darstellung. Wir benutzen diese im folgenden, um die duale C^*-Algebra zu einer Hopf C^*-Algebra mit invariantem Gewicht τ zu konstruieren und geben Beispiele für duale C^*-Algebren. Schließlich konstruieren wir noch das reduzierte verschränkte Produkt $A \rtimes_\delta H$ zu einem dynamischen Kosystem (A, H, δ), das wir in Kapitel 6 brauchen werden.

In diesem sechsten Kapitel übertragen wir Takais Dualitätssatz [28] auf verschränkte Produkte mit C^*-Pfeilen. Wir zeigen, dass für spezielle Produktsysteme (A, E) über den natürlichen Zahlen – wobei E ein C^*-Pfeil über einer C^*-Algebra A ist – gilt, dass das doppelte verschränkte Produkt $(A \rtimes_E \mathbb{N}) \rtimes_\delta C^*(\mathbb{N})$ stark Morita-äquivalent zu A ist. Dabei ist $A \rtimes_E \mathbb{N}$ die reduzierte Toeplitz-Algebra und δ kann als die duale Kowirkung von $C^*(\mathbb{N})$ auf $A \rtimes_E \mathbb{N}$ betrachtet werden. Als weiteres Resultat erhalten wir hierbei noch, dass ein Produktsystem (A, E) über $\mathbb{N}$ stets stark Morita-äquivalent ist zu einem Produktsystem, das von einem *-Endomorphismus herkommt.

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Zu meinen akademischen Lehrern gehörten

V. Batyrev, H. Fischer, A. Greenbaum, H. Heyer, C. Hering, R. Irving, C. Lubich, R. Nagel, G. Scheja, H. Smith, M. Voit, J. Wellner und M. Wolff in Mathematik sowie
M. Baumann, H. Clement, F. Gönnenwein, P. Grabmayr, W.-D. Hartmann, O. Lutz, H. Pfister, A. Schwenk und G. J. Wagner in Physik.