Viscosity Solutions of Hamilton-Jacobi Equations of Eikonal Type on Ramified Spaces

DISSERTATION

der Fakultät für
Mathematik und Physik
der Eberhard-Karls-Universität Tübingen
zur Erlangung des Grades eines

Doktors der Naturwissenschaften

vorgelegt von

Dirk Schieborn

aus Stuttgart

Tübingen, im Juni 2006

Tag der mündlichen Prüfung:	27. Juli 2006
Dekan:	Prof. Dr. P. Schmid
Erster Berichterstatter:	Prof. Dr. K. P. Hadeler
Zweiter Berichterstatter:	Prof. Dr. G. Huisken

Contents

Zusammenfassung in deutscher Sprache

Inhalt dieser Arbeit ist die Betrachtung von nichtlinearen Hamilton-Jacobi-Gleichungen erster Ordnung auf sogenannten verzweigten Räumen. Während die Theorie linearer und quasilinearer Interaktionsprobleme zwischen verschiedenen physikalischen Medien gut entwickelt ist, erscheinen die hier betrachteten nichtlinearen Probleme und insbesondere deren Lösungsmethoden als recht neuartig; dies auch insofern, als unsere physikalische Interpretation weniger im Bereich interagierender Medien, als vielmehr innerhalb der Theorie granularer Medien angesiedelt ist. Das Ziel der vorliegenden Arbeit besteht darin, einerseits die Theorie der Viskositätslösungen nichtlinearer Randwertprobleme auf verzweigte Räume zu übertragen, um damit ein Werkzeug zur Behandlung solcherlei Probleme zur Verfügung zu stellen, und andererseits die Struktur dieser Lösungen zu untersuchen.

Der Begriff der Viskositätslösung geht auf M. C. Crandall und P.-L. Lions zurück, die ihrerseits in ihrer fundamentalen Arbeit "Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi de premier order" [CL81] auf Ideen des russischen Mathematikers S.N. Kružkov zurückgreifen. Dieser entwickelte bereits im Jahre 1975 [Kru75] eine globale Theorie für Lösungen "eikonalartiger" Hamilton-Jacobi-Gleichungen auf Gebieten des $\mathbb{R}^n$. Sein Ansatz wiederum ist eng verknüpft mit dem Konzept der *vanishing viscosity*, welches nichtlineare Probleme auf Konvergenzprobleme verwandter quasilinearer Probleme zurückführt. Crandalls und Lions' Verdienst ist es, aus Kružkovs vorhandenen Konzepten die Charakterisierung von Viskositätslösungen mit Hilfe eines Test- oder Vergleichsfunktionenansatzes herausgeschält zu haben. Dieses Verfahren erlaubt es, die unmittelbaren Bedingungen an einen Lösungskandidaten recht gering zu halten, während die eigentlichen Forderungen, die die Erfüllung der Differentialgleichung betreffen, indirekt an sogenannte Testfunktionen gestellt werden. Grob gesprochen sind dies differenzierbare Funktionen, die die zu testende Funktion von oben bzw. unten berühren. Es stellt sich heraus (und ist wohlbekannt), dass dieser knapp zu formulierende, elegante Ansatz eine beträchtliche Schar von Vergleichs- und Existenzresultaten mit sich bringt und es insbesondere erlaubt, eine globale, allgemeine Theorie skalarer nichtlinearer Gleichungen zu entwickeln.

Eine Motivation für die vorliegende Arbeit und regelmäßig wiederkehrendes Beispiel stellt in diesem Zusammenhang die aus der Optik bekannte Eikonalgleichung

$$|Du| = 1$$

dar, die aus gutem Grund als die einfachste Hamilton-Jacobi-Gleichung angesehen werden darf. Die (eindeutige) Viskositätslösung der Eikonalgleichung mit Nullrandwerten auf einem beschränkten Gebiet ist durch die Distanzfunktion zum Gebietsrand gegeben, welche nun ihrerseits den Bezug zur Theorie der granularen Medien herstellt. Tatsächlich beschreibt die Distanzfunktion die Oberflächenform derjenigen stabilen Konfiguration eines homogenen granularen Materials, die entsteht, wenn man ein maximales Volumen dieses Materials auf besagtem Gebiet deponiert [HK99]. Die Eikonalgleichung gewährleistet hierbei, daß der materialspezifische (und hier einfachheitshalber normierte) Böschungswinkel eingehalten wird. Diesen Zusammenhang kann man sich leicht anhand von Gebieten einfacher Geometrie veranschaulichen. Genauer stellt man fest, dass das Selektionsprinzip des maximalen Volumens unter allen stabilen Konfigurationen gerade die Viskositätslösung der Eikonalgleichung auswählt. Im Kapitel 3 untersuchen wir die Äquivalenz beider Auswahlprinzipien genauer und stellen dazu eine Verbindung zwischen Existenzbeweisen aus der Theorie der Viskositätslösungen und der von Hadeler und anderen in [HK99] entwickelten Perronmethode her, die sogenannte Subeikonallösungen verwendet.

Ausgangspunkt unserer Überlegungen zu verzweigten Räumen ist nun, die Interpretation der Distanzfunktion als Materialkonfiguration maximalen Volumens auf eben diese zu erweitern. Als wichtiges Beispiel eines verzweigten Raumes betrachten wir hierbei vorerst eindimensionale topologische Netzwerke. Diese stellen, vereinfacht gesprochen, Graphen dar, deren Kanten als glatte Kurven im $\mathbb{R}^n$ realisiert sind. Unterteilt man nun die Knoten eines solchen Graphen in Rand- und Verzweigungsknoten, so erhält man ein einfaches eindimensionales Modell, indem man sich vorstellt, die Kantenkurven seien beidseitig von hohen Glaswänden gesäumt, in deren Zwischenraum Sand gefüllt wird. An Verzweigungsknoten wird Sand zwischen den inzidenten Kanten ausgetauscht, während er an Randknoten durch kleine Löcher am Boden abfließen kann. Man sieht schnell, dass auch in diesem Falle die Distanzfunktion zu den Randknoten die (eindeutige) Konfiguration maximalen Volumens beschreibt. Es stellt sich somit die Frage, ob ein erweiterter Viskositätslösungsbegriff auf verzweigten Räumen derart entwickelt werden kann, dass ein ähnlich enger Zusammenhang zu granularen Medien besteht wie in nichtverzweigten Räumen. Hierbei ist leicht einzusehen, dass die Schwierigkeit hauptsächlich darin besteht, eine korrekte Beschreibung einer solchen erweiterten Viskositätslösung an den Verzweigungsknoten zu finden. Als hauptsächlichen Schwerpunkt dieser Arbeit haben wir uns zur Aufgabe gemacht, diesen erweiterten Viskositätslösungsbegriff auf verzweigten Räumen zu entwickeln und darauf basierend Eindeutigkeits- und Existenzbeweise zu führen (Kapitel 5). Es stellt sich heraus, dass dies für eine allgemeine Klasse von sogenannten eikonalartigen Hamilton-Jacobi-Gleichungen möglich ist, die in enger Beziehung zu der von Kružkov in [Kru75] untersuchten Klasse steht. Bei unseren Betrachtungen beschränken wir uns zudem nicht nur auf topologische Netzwerke, sondern betrachten in Kapitel 7 auch höherdimensionale verzweigte Räume.

Ein weiteres Hauptaugenmerk richten wir ferner auf die Frage, inwieweit unser neu entwickelter Viskositätslösungsbegriff mit der namensgebenden Methode der *vanishing viscosity* in Einklang steht. Dazu übertragen wir dieses Verfahren auf verzweigte Räume und beweisen ein entsprechendes Konvergenzresultat für eine Subfamilie der eikonalartigen Hamilton-Jacobi-Gleichungen (Kapitel 4). Später werden wir dann allgemeiner zeigen, dass jede durch *vanishing viscosity* erhaltene Grenzfunktion mit der eindeutigen Viskositätslösung für das zugehörige Problem übereinstimmt (Konsistenz). Diese

Erkenntnis rechtfertigt unseren Lösungsbegriff in weiterer Hinsicht.

Die Methode der *vanishing viscosity* wird zudem in Kapitel 2 anhand der Eikonalgleichung auf nichtverzweigten Gebieten einfacher Geometrie direkt untersucht. In diesen Beispielfällen kann man über die Lösungen der zugehörigen Schar von quasilinearen Problemen (wir nennen dies die *viskose* Eikonalgleichung) genügend Informationen gewinnen, um konkrete Konvergenzaussagen direkt beweisen zu können.

Nach der allgemeinen Herleitung der Theorie von Viskositätslösungen auf topologischen Netzwerken untersuchen wir in Kapitel 6 als Beispiel schließlich eine etwas konkretere Klasse von eikonalartigen Randwertproblemen auf Netzwerken. Diese nennen wir *anisotrope* Eikonalgleichungen und stellen einen Zusammenhang zu den bereits erwähnten Maximalkonfigurationen granularer Medien her. Speziell interessiert uns die Struktur von Viskositätslösungen anisotroper Eikonalgleichungen, und davon ganz besonders die Menge der singulären Punkte. Wie sich herausstellt, ist die Mächtigkeit dieser Singularitätenmenge eine endliche Zahl, die invariant sowohl gegenüber der konkreten anisotropen Eikonalgleichung, als auch gegenüber der exakten Gestalt des Netzwerkes ist. Tatsächlich hängt sie lediglich von der Anzahl der Kanten und Verzweigungsknoten ab—ein Resultat, das wir auf verschiedene Arten interpretieren.

CHAPTER 1

Introduction

In the wide field of partial differential equations, many contributions of new, ingenious approaches have led to the great variety of methods and techniques which it enjoys today. In particular the theory of nonlinear equations, which is far more scattered and less comprehensive than the linear theory, depends upon the development of new methods helping to gain deeper insights and to establish new points of view.

In general, boundary or initial value problems of nonlinear equations, among which we highlight the Hamilton-Jacobi equations as prominent examples, fail to have smooth solutions on a given domain or for all times. A method to overcome this problem is to soften the demands and to introduce appropriate concepts of *weak solutions*. The present thesis is concerned with a particularly important contribution in this spirit: the *theory of viscosity solutions*, which was initiated by S. N. Kružkov and first established in its present form by M. C. Crandall and P.-L. Lions. Similar to the idea of weakly differentiable functions known from Sobolev's theory, the concept of viscosity solutions is a generalization of classical solutions, allowing for not necessarily differentiable functions to be possible solution candidates. Roughly speaking, to be continuous is the only immediate condition a viscosity solution has to satisfy, whereas the crucial requirement—the "test function condition"—is of indirect nature: The solution has to be resistant against each smooth test function touching it from above or below, in the sense that the latter (rather than the solution itself) has to satisfy a corresponding differential inequality. Obviously, the use of test functions parallels the Sobolev theory of weak solutions of linear equations in divergence form. Obviously, the latter employs integration by parts in order to "shift" the derivatives to test functions, whereas the theory of viscosity solutions exploits the maximum principle for the same purpose. It turns out that this concept provides existence and comparison results for a broad class of nonlinear partial differential equations which in general do not possess classical solutions.

The theory of viscosity solutions has been extensively studied and refined by many authors, and among the numerous contributions in the literature one can find different adaptations to more general settings. In the present thesis we refrain from a new discussion of the old subject. In contrast, we suggest a meaningful transfer of the existing theory to a setting which has not yet been treated in this context: *ramified spaces*.

Several physical phenomena such as interaction of different media can be translated into mathematical problems involving differential equations which are not defined on connected manifolds as usual, but instead on so-called ramified spaces. The latter can be roughly visualized as a collection of different manifolds of the same dimension (*branches*) with certain parts of their boundaries identified (*ramification space*). The simplest examples are *topological networks*, which basically are graphs embedded in Euclidean space. Interaction problems can be modeled by a collection of differential equations describing the behavior of physical quantities on the branches, which is additionally controlled by certain *transition conditions* governing the interaction of the quantities across the ramification spaces. From a mathematical point of view, transition conditions are an essential new aspect when searching for solutions. Since the year 1980, many works have been published treating different kinds of interaction problems involving linear and quasilinear differential equations (confer for instance Lagnese and Leugering [LL91], [LL93], Lagnese, Leugering, and Schmidt [LLG94], von Below and Nicaise [vBN96], Ali-Mehmeti [AMN93], and Nicaise [Nic93]). However, as far as we know, fully nonlinear equations such as Hamilton-Jacobi equations have not yet been examined to a similar extent on ramified spaces.

A major goal of the present thesis is to establish a theory of viscosity solutions of first order Hamilton-Jacobi equations on ramified spaces, where the main emphasis will be placed on topological networks. In doing so, mathematical *models for granular matter* applied to ramified spaces will serve us as an motivational and illustrating example.

A closely related aim of interest in this context is the so-called *method of vanishing viscosity*, which origins in fluid dynamics and has eventually led to the modern notion of viscosity solutions. Although the latter has been extended to second order equations—under replacement of the test functions condition by an alternative involving set-valued generalized differential operators (semi-jets)—, its original nature is of *first order type*. In this case the theory is strongly related to the origin of the concept, which gave rise to the terminology: the idea of converting a nonlinear first order equation

$$H(Du(x), u(x), x) = 0$$

into a semilinear second order equation by adding a "viscosity term" $\varepsilon \Delta u$, followed by a passage to the limit $\varepsilon \to 0$ ("vanishing viscosity"). Heuristically speaking, the viscosity term prevents the solution from immediately responding to the equation, but causes it to display a smoothed behavior. The subsequent reduction of viscosity then gradually decreases this smoothening effect and makes the solution react more quickly. The mathematical motivation is to replace the original nonlinear problem by a family of semilinear problems which can be treated with the standard semilinear theory. The difficulty of the problem is thus transferred to the question, whether the ε-family of solutions converges. In many cases this question can be positively answered by means of compactness arguments, the essence of which consists in certain *a priori*-estimates. This so-called method of vanishing viscosity in general acts as a *selection principle* in the following sense: Whereas classical solutions of boundary value problems involving nonlinear equations do not exist in general, the situation is different if the demands are relaxed and weak (i.e., "almost everywhere") solutions are admitted, in which case solutions do exist but may not be unique. Hence in either case the problem is unsatisfactory. A possible solution to this dilemma is provided by the method of vanishing viscosity, as it selects a unique weak solution. In fact it turns out that what at first might seem to be a purely formal restriction to a certain solution motivated by a technical selection procedure will end up being the "correct" solution in several other aspects, both physical and mathematical. For instance, the viscosity term appears naturally in fluid dynamics as the physical viscosity of the liquid, and the vanishing viscosity method describes limiting cases where this viscosity approaches zero. It also can be interpreted as a gradual reduction of the effect of diffusion in reaction-diffusion scenarios. On the other hand, a mathematical justification is given by the very fact, that the limit function selected by the vanishing viscosity method coincides with the viscosity solution of the original problem. Conversely, the characterization by test functions is nothing else than an appropriate intrinsic characterization of the vanishing viscosity limit. In the present thesis we will encounter the vanishing viscosity method in different contexts, starting from explicit calculations in the exemplary case of the eikonal equation as the most prominent Hamilton-Jacobi equation, and ending with general convergence results on networks.

As already mentioned, in order to transfer the test function concept to ramified spaces, we will repeatedly invoke a physical interpretation: mathematical models for granular matter. In fact, we consider the problem of determining the contours of maximal volume configurations of homogeneous (or spatially inhomogeneous) granular material placed upon ramified domains. Let us briefly elaborate on this idea. Equilibrium configurations of dry and homogeneous granular matter can only form "heaps" with local steepness not exceeding a certain *angle of repose* α specific to the respective material. Consequently, the function describing the contours of maximal volume configurations on non-ramified domains without rim (such as tables) is contained in the class of the "almost everywhere" solutions of the eikonal equation

$$H(Du(x), u(x), x) = |Du(x)| - \tan\alpha = 0$$

satisfying $u \equiv 0$ on the boundary. In this case another selection criterion is employed: namely the additional constraint that the volume functional be maximized (cf. [HKG02]). It can be shown that this "maximal volume solution" coincides—up to multiplication with a constant—with the distance function to the boundary.

The connection to the theory of viscosity solutions is given by the observation that the distance function, on the other hand, is the unique viscosity solution of the above boundary value problem and thus is selected by the method of vanishing viscosity. In fact we demonstrate that the construction methods for maximal volume solutions introduced in [HKG02] are closely related to similar methods for viscosity solutions. Having in mind both the granular matter and the viscosity solution interpretation of the distance function on non-ramified domains, the passage to viscosity solutions on ramified spaces suggests itself. As an example let us consider the simplest case of a ramified space: a one-dimensional topological network or a graph. In this setting, the distance function to a given collection of *boundary vertices* in fact describes the maximal volume configuration of granular matter for the following scenario: The network can be pictured as a planar maze with paths connected at the vertices. Now think of the paths to be bounded on both sides by thin, sufficiently high glass walls perpendicular to the plane. Let us then uniformly pour as much sand as possible into the space between the glass walls, assuming that sand can run out of the maze at the boundary vertices. At the other vertices (called *transition vertices*), sand is interchanged between the incident paths. Several sand heaps will grow, each two of them separated by at least one boundary vertex. Finally the heaps stop growing and reach an equilibrium state. By this time each additional sand portion locally violates the angle of repose and is thus forced to leave the maze at the boundary vertices. The contours of the equilibrium configuration are mathematically described by a continuous function defined on the network which vanishes at the boundary points, maximizes the volume functional, and satisfies the eikonal equation almost everywhere on the edges. Analogously to the non-ramified case, the distance function satisfies all these conditions. Hence it is selected by the maximal volume problem among all other weak solutions of the eikonal equation on the network.

The equivalence of the two selection principles (maximal volume and viscosity solution) in the case of the eikonal equation on non-ramified domains gives reason to the question if there is a *generalized test function condition* which is satisfied by the distance function on networks and which might be the key to transferring the theory of viscosity solutions to ramified spaces for a more general class of Hamilton-Jacobi equations.

As a main result of the present thesis it will turn out that indeed there is a class of first order Hamilton-Jacobi equations of *eikonal type* for which the concept of viscosity solutions can be appropriately extended to networks and even higher dimensional ramified spaces. In fact, we propose an intrinsic test function characterization for viscosity solutions on ramified spaces and justify it in different aspects: The reason why viscosity solutions are so convincing in the non-ramified case is the fact that they coincide with vanishing viscosity limits and that they entail a variety of technical advantages such as elegant comparison, uniqueness, and existence results. We show that our theory preserves all these features. In order to emphasize the generality of our concept, we also present an adaptation of the theory to certain higher dimensional ramified spaces, so-called LEP-spaces (locally elementary polygonal ramified spaces).

A corresponding problem is the extension of the method of vanishing viscosity to ramified spaces, which we will treat in detail for the case of networks. It requires special care at the ramification spaces, as after adding the viscosity term the solution of the corresponding semilinear boundary value problem is not unique, unless we impose a further condition at the ramification spaces: the classical Kirchhoff condition, establishing a relation between the outer normal derivatives of the solutions on adjacent branches. The Kirchhoff condition can be thought of as an extension of the "averaging effect" of the viscosity term on the branches to the ramification spaces. This averaging effect can also be observed in granular matter experiments due to local perturbations caused by small spontaneous avalanches.

Another issue of the present thesis is the investigation of certain properties of the distance function on networks. It is well known (cf. [AG96]) that for a convex two-dimensional domain the length of its boundary is related to the curvature functional of the distance function. In fact they are equal up to a normalizing factor independent of the choice of the domain. The distance function thus plays an important role as a link between the eikonal equation and the topology of the domain by connecting local and global concepts, a phenomenon which, as we will show, also appears in the context of networks. We examine the curvature functional (in the sense of geometric measure theory) of the distance function to boundary vertices, which is given by the number of singularities of the distance function, where singularities have to be suitably counted at transition vertices. It turns out that this number is equal to a purely topological quantity depending only on basic graph theoretical properties of the underlying network.

As will turn out, the number of singularities is even invariant under replacement of the eikonal equation by a more general class of what we call *anisotropic eikonal equations*. We hence obtain a rather clear picture of the shape and regularity properties of the corresponding viscosity solutions. This gives rise to different interpretations, in particular within granular matter scenarios.

1.1 Logical organization and chapter summary

The logical relations between the different topics presented in this thesis are manifold. However, we have decided to basically arrange the chapters according to an increasing complexity of the underlying domains, which are: one-, two-, and n-dimensional *non-ramified* domains, one-dimensional *ramified* domains (networks), and, finally, n-dimensional ramified domains (LEP spaces). For a given type of domain, we have tried to structure the material with respect to logical and/or historical consequence. We think that the possible drawback of certain topics such as the vanishing viscosity method being revisited at different stages is compensated by the coherence regarding the complexity.

In **chapter 2** we give a brief historical overview of the concept of viscosity solutions as well as the definitions of Kružkov's generalized solutions and (classical) viscosity solutions. We examine the vanishing viscosity method for the eikonal equation on several domains (interval, square, arbitrary convex domains with smooth boundary) and show its convergence to the distance function.

In **chapter 3** we consider the eikonal equation on two-dimensional domains and connect the granular matter methods developed in [HKG02] to techniques from the theory of viscosity solutions, establishing an equivalence of both approaches in the case of the eikonal equation.

In **chapter 4** we provide a general definition of ramified spaces and investigate the Dirichlet problem of the eikonal equation on topological networks. Furthermore, by means of *a priori* estimates we show the convergence of a generalized vanishing viscosity method on networks for a class of Hamilton-Jacobi equations of eikonal type.

In **chapter 5** we present an extension of the theory of viscosity solutions to networks and prove uniqueness and existence results for solutions of Hamilton-Jacobi equations of eikonal type. We also show the consistency of the extended theory with the network version of the method of vanishing viscosity.

In **chapter 6** we examine the structure of viscosity solutions of the class of so-called *anisotropic* eikonal equations on topological networks, and give an explicit formula for the number of singularities of such solutions. In the special case of the eikonal equation, this result connects the curvature functional of the eikonal equation to the topology of the graph. The result is related to the concept of cycle rank of graphs and is also interpreted in terms of computer scientific aspects.

In **chapter 7** we generally introduce higher dimensional ramified manifolds and define the so-called LEP spaces, to which we extend the results of chapter 5.

1.2 Acknowledgments

I would like to express my gratitude to Prof. Hadeler for giving me the opportunity and the support to write the present thesis, for his interest in the subject, and for giving me great freedom in choosing the main foci. I also want to thank my friend Helge Herr for motivational support and valuable discussions, as well as David Pricking, Prof. Frank Loose, Dr. Joel Braun, and Dr. Christina Kuttler. Furthermore I would like to thank everyone else in the institute of biomathematics for providing an inspiring and warm working atmosphere. Finally I thank all of my friends and my parents for their patience and care.

Monday, June 26, 2006

CHAPTER 2

Viscosity solutions: history and examples

Summary. The purpose of the present chapter is first to discuss two important concepts which led to the modern notion of viscosity solutions: the method of vanishing viscosity and Kružkov's generalized solutions. Secondly, we investigate the method of vanishing viscosity for the eikonal equation in various domains and show that it yields the distance function in the limit.

2.1 Introduction

In the year 1981 the notion of viscosity solutions of nonlinear first order equations, or Hamilton-Jacobi equations, appeared in the literature for the first time, when Michael G. Crandall and Pierre-Louis Lions published their papers "Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi de premier order" [CL81] and "Viscosity solutions of Hamilton-Jacobi equations" [CL83]. Although the definition given in these publications reads simple and elegant, it is nevertheless important to point out that it represents the essence of a development over a long time.

Already in 1975 S. N. Kružkov proposed a concept of generalized solutions of Hamilton-Jacobi equations of eikonal type [Kru75], emanating from the observation that a general theory for these equations entails a twofold difficulty already mentioned in the preceding chapter: Whereas a classical theory fails as general existence of solutions cannot be guaranteed, a weak theory cannot ensure uniqueness. Kružkov solved this problem by imposing a further, physically meaningful, constraint on possible weak solutions. Essentially, he demanded the existence of a uniform lower bound of the second order difference quotients of a solution candidate. And indeed, this additional requirement enabled him to overcome the problem of uniqueness and to pave the way to a general theory for a large class of Hamilton-Jacobi equations of "eikonal type". As a matter of fact, Kružkov's theory is not only justified by existence and uniqueness results, but his notion of a generalized solution also allows physical interpretations related to the classical principles of Fermat and Huygens in geometric optics (cf. [Kru75]). These principles, in turn, are strongly connected to the eikonal equation, reflecting the fact that Kružkov's theory is restricted to what he calls *Hamilton-Jacobi equations of eikonal type.*

As stated in [CIL92], "analogies of S. N. Kružkov's theory of scalar conservation laws ([Kru70]) provided guidance for the notion [of viscosity solutions] and its presentation". In fact, inspired by some of the essential ideas in Kružkov's work, Crandall and Lions found a strikingly simple intrinsic representation of his generalized solutions, leading to the notion of viscosity solutions of first order equations [CL83]. Its publication initiated avid research activities, triggering the discovery of a chain of related and much more general results. Later on, Lions discovered a possibility to extend the concept to second order equations, modifying the definitions in a way such that they bear only little resemblance to their original version. His key achievement was to prove a maximum principle and a corresponding uniqueness result for viscosity solutions of convex nonlinear second order Hamilton-Jacobi equations by means of stochastic control theory, a result which was extended to fully nonlinear second order elliptic equations by R. Jensen [Jen88] five years later.

However, as powerful (and abstract) the recent general definitions of viscosity solutions might be, one basic feature always plays a fundamental role: the possibility of approximating the solution by the method of vanishing viscosity. We begin with a description of this method, followed by three concrete examples for its application to the eikonal equation. After that we elaborate on Kružkov's solutions and end with the modern definition of viscosity solutions in the spirit of Crandall and Lions.

2.2 The idea of vanishing viscosity

As mentioned above, the method of vanishing viscosity selects a certain weak solution of a first order nonlinear problem which in general has no classical solution. The idea is to slightly modify the original problem to get a semilinear problem, whereby the extent of the modification is controlled by a parameter ε. By means of existence and uniqueness results from the standard semilinear theory one obtains a unique, sufficiently regular solution u_ε for each value $\varepsilon > 0$. Afterwards the modification is gradually "undone" by passing to the limit $\varepsilon \to 0$, leaving the question if and in what sense the family of functions u_ε converges to a limit function, called the "vanishing viscosity solution".

Let us put the idea in concrete mathematical terms: Let Ω be a bounded domain in $\mathbb{R}^n$ and consider a first order Hamilton-Jacobi equation of the form

$$\begin{cases} H(Du(x), u(x), x) = 0 & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a nonlinear function, the so-called *Hamiltonian*. The "viscous" modification of the problem, depending on the parameter $\varepsilon > 0$, is given by

$$\begin{cases} \varepsilon \Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon) = 0 & \text{on } \Omega \\ u_\varepsilon = \varphi & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Under the assumption that H satisfies some standard regularity and monotonicity conditions, the solutions u_ε exist uniquely, and it remains to prove the convergence of the functions u_ε to a limit function u as ε approaches 0. This can be achieved by establishing suitable *a priori* estimates. We will elaborate on this in the case of topological networks in chapter 4.

2.3 Vanishing viscosity and the eikonal equation

At this stage we dispense with a general approach and instead illustrate the method of vanishing viscosity by means of several exemplary boundary value problems of the eikonal equation. Accordingly, on a bounded domain $\Omega \subset \mathbb{R}^n$ we consider the boundary value problem

$$\begin{cases} |Du| - 1 = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Clearly a classical solution does not exist. However, there are infinitely many Lipschitz continuous functions which satisfy the boundary condition and whose modulus of gradient—existing almost everywhere by the theorem of Rademacher—equals 1, possibly except for a set of measure zero. Obviously the distance function $\boldsymbol{d}$ to the boundary is contained in this class of weak or *almost everywhere* solutions, and we will demonstrate by means of explicit calculations how $\boldsymbol{d}$ is selected by the method of vanishing viscosity.

Both for technical reasons and in order to stay compatible with the theory in the subsequent chapters, we do not apply the method of vanishing viscosity to the eikonal equation itself, but to the equivalent and "more regular" equation

$$|Du|^2 - 1 = 0.$$

Then for $\varepsilon > 0$ the corresponding semilinear *viscous* problem reads

$$\begin{cases} \varepsilon \Delta u_\varepsilon + |Du_\varepsilon|^2 - 1 = 0 & \text{on } \Omega \\ u \equiv 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

and the semilinear theory can be applied. In fact, it is possible to attack this problem by means of standard linear theory via a transformation approach as used in the following

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying an exterior sphere condition at every boundary point. Then the boundary value problem 2.4 has a unique solution $u_\varepsilon \in C(\bar{\Omega}) \cap C^2(\Omega)$. If $\partial\Omega$ is of class C^∞, we have $u_\varepsilon \in C^\infty(\bar{\Omega})$.*

Proof. Let $\varepsilon > 0$ and set $a := 1/\varepsilon$. According to theorem 6.13 in [GT77] there is a unique solution $w_a \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the boundary value problem

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Furthermore note that the maximum principle implies $w_a > -1$, whence we infer that $u_\varepsilon \in C(\bar{\Omega}) \cap C^2(\Omega)$, where

$$u_\varepsilon(x) := -\frac{1}{a} \log(w_a(x) + 1), \quad x \in \Omega. \tag{2.6}$$

Straightforward calculation shows that u_ε satisfies (2.4). Moreover, the solution u_ε is unique, as the inverse transformation of (2.6) applied to a different solution of (2.4) would contradict to the uniqueness of w_a.

If $\partial\Omega$ additionally is of class C^∞, we have $w_a \in C^\infty(\bar{\Omega})$ according to theorem 6, p. 326, in [Eva98]. Hence $u_\varepsilon \in C^\infty(\bar{\Omega})$. □

We now consider three cases: an interval, a square, and an arbitrary convex domain in $\mathbb{R}^n$ with a smooth boundary.

2.3.1 Convergence on the interval

Lemma 2.1. *Let $\Omega := (0,1) \subset \mathbb{R}$ and let $\boldsymbol{d} : \Omega \to \mathbb{R}$ be the distance function on Ω. Then for each $\varepsilon > 0$ there is a unique solution $u_\varepsilon \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the boundary value problem*

$$\begin{cases} \varepsilon u_\varepsilon''(x) - (u_\varepsilon'(x))^2 + 1 = 0 & \text{on } \Omega, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0\,. \end{cases} \tag{2.7}$$

Furthermore, the functions u_ε converge pointwise to $\boldsymbol{d}$ on $\bar{\Omega}$ as $\varepsilon \to 0$.

Proof. Let $\varepsilon > 0$. The existence and uniqueness of the solution u_ε is a consequence of theorem 2.1. Set $a := 1/\varepsilon$ and define

$$w_a(x) := \exp(-\varepsilon^{-1} u_\varepsilon) - 1.$$

From theorem 2.1 it follows that w_a is a solution of

$$\begin{cases} w_a''(x) - a^2 w_a(x) = a^2 & \text{on } \Omega \\ w_a(0) = w_a(1) = 0. \end{cases} \tag{2.8}$$

Hence it can be represented by the formula

$$w_a(x) = a^2 \int_0^1 g(x,t)\, dt, \tag{2.9}$$

where $g(x,t)$ is the Green's function of the homogeneous equation

$$u''(x) - a^2 u(x) = 0$$

on Ω vanishing at the boundary of Ω.

In other words, for each $t \in \Omega$ the function $g(x,t)$ satisfies the following equation in the distribution sense

$$\begin{cases} \frac{d^2}{dx^2} g(x,t) - a^2 g(x,t) = \delta(x-t) & \text{on } \Omega \\ g(0,t) = g(1,t) = 0\,, \end{cases}$$

where δ denotes the Dirac delta function. It can be readily verified that $g(x,t)$ is given by

$$g(x,t) = \begin{cases} -\sinh(ax)\sinh(a(1-t))(a\sinh(a))^{-1}, & 0 \le x < t \\ -\sinh(a(1-x))\sinh(at)(a\sinh(a))^{-1}, & t < x \le 1. \end{cases}$$

Plugging this into (2.9), we obtain

$$\begin{aligned} w_a(x) &= -a^2 \int_x^1 \frac{\sinh ax \sinh a(1-t)}{a \sinh a} dt - a^2 \int_0^x \frac{\sinh a(1-x) \sinh at}{a \sinh a} dt \\ &= -1 + \frac{\sinh a(1-x) + \sinh ax}{\sinh a}. \end{aligned}$$

Applying the transformation (2.6), we obtain an explicit formula for the solution of (2.7), which reads

$$u_\varepsilon(x) = -\frac{1}{a}\log(w_a(x)+1) = -\frac{1}{a}\log\left(\frac{\sinh a(1-x) + \sinh ax}{\sinh a}\right). \tag{2.10}$$

We now consider the behavior of u_ε for $\varepsilon \to 0$, or, equivalently, $a = \varepsilon^{-1} \to \infty$. For this purpose we express the hyperbolic functions in (2.10) in terms of exponential functions and obtain

$$u_\varepsilon(x) = -\frac{1}{a}\log\left(\frac{e^{a(1-x)} - e^{a(x-1)} + e^{ax} - e^{-ax}}{e^a - e^{-a}}\right) = x - \frac{1}{a}\log\left(\underbrace{\frac{(e^a-1)(e^{a(2x-1)}+1)}{e^a - e^{-a}}}_{(1)}\right).$$

If $x \le 0 \le 1/2$, it is easy to see that the term (1) is bounded from above by 2 and from below by 1/2 for all sufficiently large $a > 0$. On the other hand, by choosing a different representation we get

$$u_\varepsilon(x) = -x + \frac{1}{a}\log\left(\underbrace{\frac{e^a - e^{-a}}{e^{a(1-2x)} - e^{-a} + 1 - e^{-2ax}}}_{(2)}\right).$$

Now, for $1/2 < x \leq 1$, expression (2) obviously behaves like e^a as $a \to \infty$. Altogether it follows

$$\lim_{a\to\infty} u_a(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2 \\ 1-x & \text{if } 1/2 < x \leq 1, \end{cases}$$

and the assertion is proved. $\square$

Lemma 2.2. *Using the notation of lemma (2.1), we have*

$$\lim_{\varepsilon\to 0} u'_\varepsilon(x) = \begin{cases} 1 & \textit{if } 0 \leq x < 1/2 \\ 0 & \textit{if } x = 1/2 \\ -1 & \textit{if } 1/2 < x \leq 1. \end{cases}$$

Proof. Let $\varepsilon > 0$ and set $a := 1/\varepsilon$. Using (2.10) we compute

$$\begin{aligned} u'_\varepsilon(x) &= -\frac{-\cosh(a(1-x)) + \cosh(ax)}{\sinh(a(1-x)) + \sinh(ax)} \\ &= -\frac{-e^{a(1-x)} - e^{a(x-1)} + e^{ax} + e^{-ax}}{e^{a(1-x)} - e^{a(x-1)} + e^{ax} - e^{-ax}} = \frac{e^{a(1-2x)} + e^{-a} - 1 - e^{-2ax}}{e^{a(1-2x)} - e^{-a} + 1 - e^{-2ax}}. \end{aligned}$$

The assertion is immediate when letting $a \to \infty$ in the last term. $\square$

2.3.2 Convergence on a square

We now apply the method of vanishing viscosity to the eikonal equation on a square and prove an analogous statement.

We first provide the following auxiliary result.

Proposition 2.1. *Let $\Omega := (0,1) \times (0,1)$. Let $a > 0$ and set*

$$\sigma_n := \sqrt{\pi^2 n^2 + a^2} \tag{2.11}$$

for each $n \in \mathbb{N}$. We define the function $u_a : \Omega \to \mathbb{R}$ by

$$u_a(x,y) = -\frac{1}{a}\log\left[\sum_{n=1,3,\ldots} \frac{4a^2\sin(n\pi x)}{\pi\sigma_n^2 n}\left(\frac{\pi^2 n^2}{a^2} + \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}\right)\right],$$

where the summation index n runs through the odd natural numbers. Then

$$\lim_{a\to\infty} u_a(x,y) = x \quad \textit{for all } 0 < y \leq 1/2 \textit{ and all } 0 < x \leq y.$$

Proof. Let $0 < y \leq 1/2$ and $0 < x \leq y$. We compute

$$\begin{aligned} u_a(x,y) = {} & -\frac{1}{a}\log\left[e^{-ay}\left(\sum_{n=1,3,\ldots}\frac{4e^{ay}\pi n\sin(n\pi x)}{\sigma_n^2}\right.\right. \\ & \left.\left. + \sum_{n=1,3,\ldots}\frac{4e^{ay}a^2\sin(n\pi x)}{\pi\sigma_n^2 n}\cdot\frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}\right)\right] \\ = {} & y - \frac{1}{a}\log[S_1(a,x,y) + S_2(a,x,y)], \end{aligned} \tag{2.12}$$

where

$$S_1(a,x,y) := \sum_{n=1,3,\dots} \frac{4e^{ay}\pi n \sin(n\pi x)}{\sigma_n^2}$$

and

$$S_2(a,x,y) := \sum_{n=1,3,\dots} \frac{4e^{ay}a^2 \sin(n\pi x)}{\pi\sigma_n^2 n} \cdot \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}.$$

By (2.11) we furthermore have $\lim_{a\to\infty}(\sigma_n - a) = 0$ for fixed n. Invoking this and the relation $0 < y \le 1/2$, we have for any fixed $n \in \mathbb{N}$

$$\begin{aligned} & \lim_{a\to\infty} \frac{e^{\sigma_n + y(a-\sigma_n)} - e^{-\sigma_n + y(a+\sigma_n)} + e^{(a+\sigma_n)y} - e^{(a-\sigma_n)y}}{e^{\sigma_n} - e^{-\sigma_n}} \\ = & \lim_{a\to\infty} \frac{e^a - e^{-a+2ay} + e^{2ay} - 1}{e^a} = 1 + \delta_{1/2}(y), \end{aligned} \tag{2.13}$$

where

$$\delta_{1/2}(y) := \begin{cases} 1 & \text{if } y = 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Since transition to the limit and summation can be interchanged in this case, we compute

$$\begin{aligned} \lim_{a\to\infty} S_2(a,x,y) &= \sum_{n=1,3,\dots} \lim_{a\to\infty} \frac{4e^{ay}a^2 \sin(n\pi x)}{\pi\sigma_n^2 n} \cdot \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)} \\ &= \sum_{n=1,3,\dots} \lim_{a\to\infty} \frac{4a^2 \sin(n\pi x)}{\pi\sigma_n^2 n} \cdot e^{ay} \cdot \frac{e^{\sigma_n(1-y)} - e^{-\sigma_n(1-y)} + e^{\sigma_n y} - e^{-\sigma_n y}}{e^{\sigma_n} - e^{-\sigma_n}} \\ &= \sum_{n=1,3,\dots} \lim_{a\to\infty} \frac{4a^2 \sin(n\pi x)}{\pi\sigma_n^2 n} \cdot \frac{e^{\sigma_n + y(a-\sigma_n))} - e^{-\sigma_n + y(a+\sigma_n)} + e^{(a+\sigma_n)y} - e^{(a-\sigma_n)y}}{e^{\sigma_n} - e^{-\sigma_n}} \\ &= \sum_{n=1,3,\dots} \frac{4 \sin(n\pi x)}{\pi n} \cdot \lim_{a\to\infty} \frac{e^{\sigma_n + y(a-\sigma_n))} - e^{-\sigma_n + y(a+\sigma_n)} + e^{(a+\sigma_n)y} - e^{(a-\sigma_n)y}}{e^{\sigma_n} - e^{-\sigma_n}} \\ &= 1 + \delta_{1/2}(y). \end{aligned} \tag{2.14}$$

Note that we have taken advantage of (2.13) and of the well-known relation

$$\sum_{n=1,3,\dots} \frac{\sin(n\pi x)}{n} = \frac{\pi}{4}. \tag{2.15}$$

Next observe that we can write

$$S_1(a,x,y) = e^{ay}\mathcal{S}_a(x), \tag{2.16}$$

where the function $\mathcal{S}_a : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{S}_a(t) := \sum_{n=1,3,\dots} \frac{4\pi n \sin(n\pi t)}{\pi^2 n^2 + a^2},\ t \in \mathbb{R}. \tag{2.17}$$

For fixed $a > 0$ we consider the 2-periodic function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) := \begin{cases} -e^{-a(t+1)} - e^{-a(1-(t+1))}, & \text{if } -1 \le t \le 0, \\ e^{-at} + e^{-a(1-t)}, & \text{if } 0 \le t \le 1, \end{cases}$$

on the interval $[-1, 1]$. Its Fourier expansion has the form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t), \tag{2.18}$$

where

$$b_n = \int_{-1}^{1} f(s) \sin(n\pi s)\, ds = \begin{cases} \frac{4\pi n(e^{-a}+1)}{\pi^2 n^2 + a^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{2.19}$$

By (2.17), (2.18), and (2.19) it then follows

$$(e^{-a} + 1)\mathcal{S}_a(t) = f(t).$$

From (2.16) we therefore obtain

$$S_1(a, x, y) = e^{ay} \cdot \frac{e^{-ax} + e^{-a(1-x)}}{e^{-a} + 1}. \tag{2.20}$$

For large $a > 0$ we thus derive by virtue of (2.12), (2.14), and (2.20)

$$\begin{aligned} u_a(x, y) &\simeq y - \frac{1}{a} \log \left(e^{ay} \cdot \frac{e^{-ax} + e^{-a(1-x)}}{e^{-a} + 1} + 1 + \delta_{1/2}(y) \right) \\ &= y - \frac{1}{a} \log \left(\frac{e^{-a(x-y)} + e^{-a(1-x-y)}}{e^{-a} + 1} + 1 + \delta_{1/2}(y) \right). \end{aligned}$$

By $0 < y \le 1/2$ and $0 < x \le y$ it follows $\lim_{a\to\infty} u_a(x, y) = x$. □

Lemma 2.3. *For each $\varepsilon > 0$ there is a unique solution $u_\varepsilon \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the boundary value problem*

$$\begin{cases} \varepsilon \Delta u_\varepsilon - |Du_\varepsilon|^2 + 1 = 0 & \text{on } \Omega := (0, 1) \times (0, 1) \\ u_\varepsilon \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

Furthermore, the functions u_ε converge pointwise to the distance function $\boldsymbol{d}$ on Ω as $\varepsilon \to 0$.

For the proof of lemma 2.3 we need the following well-known facts about the spectral representation of Green's functions for the Dirichlet problem on arbitrary domains, associated with a self-adjoint linear operator.

Lemma 2.4. *The Green's function G for the Dirichlet problem associated with the self-adjoint linear operator L on a bounded domain $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}$, has the form*

$$G(p, q) = \sum_{n=1}^{\infty} \frac{u_n(p)\, u_n(q)}{\lambda_n}, \quad p, q \in \Omega,$$

where u_n, $n \geq 1$, are orthonormal eigenfunctions of L corresponding to the eigenvalues λ_n defined by

$$\begin{aligned} Lu_n(x) &= \lambda_n u_n(x), \quad x \in \Omega \\ u_n(x) &= 0, \qquad\qquad x \in \partial\Omega. \end{aligned}$$

Lemma 2.5. *Let $\kappa \in \mathbb{R}$. If L is replaced by $L - \kappa$ in lemma 2.4, the Green's function G takes the form*

$$G(p,q) = \sum_{n=1}^{\infty} \frac{u_n(p)\, u_n(q)}{\lambda_n - \kappa},$$

provided κ is not an eigenvalue of L.

Proof. (of lemma 2.4 and lemma 2.5) Confer theorem 9.4 and its corollary in [Roa70]. □

Proof. (of lemma 2.3) Let $\varepsilon > 0$. The existence and uniqueness of u_ε immediately follows from theorem 2.1. Set $a := 1/\varepsilon$ and define

$$w_a(x) := \exp(-\varepsilon^{-1} u_\varepsilon) - 1.$$

From theorem 2.1 it follows that w_a is a solution of (2.5). Therefore it may be represented by the formula

$$w_a(p) = a^2 \int_\Omega G(p,q)\, dq,$$

where G is the Green's function associated with the linear operator

$$Lu := -\Delta u + a^2 u$$

on Ω. Next note that the functions

$$u_{m,n}(x,y) := 2\sin(m\pi x)\sin(n\pi y), \quad m,n \in \mathbb{N},$$

form a complete orthonormal system of eigenfunctions of $-\Delta$ vanishing on the boundary of Ω, corresponding to the eigenvalues

$$\lambda_{m,n} := \pi^2(m^2 + n^2).$$

Accordingly, setting $L := -\Delta$ and $\kappa := -a^2$, we obtain by virtue of lemma 2.5

$$G(p,q) = G(x_1,y_1,x_2,y_2) = 4 \sum_{m,n=1}^{\infty} \frac{\sin(m\pi x_1)\sin(n\pi y_1)\sin(m\pi x_2)\sin(n\pi y_2)}{\pi^2(m^2+n^2) + a^2}, \qquad (2.22)$$

where $p = (x_1,y_1), q = (x_2,y_2) \in \Omega$. We now represent the solution w_a of the boundary value problem (2.5) by means of the standard integral representation formula (cf. for instance theorem 9.6 in [Roa70]) and obtain

$$\begin{aligned} w_a(x_1,y_1) &= -4a^2 \sum_{m,n=1}^{\infty} \int_0^1 \int_0^1 \frac{\sin(m\pi x_1)\sin(n\pi y_1)\sin(m\pi x_2)\sin(n\pi y_2)}{\pi^2(m^2+n^2)+a^2}\, dx_2\, dy_2 \\ &= -4a^2 \sum_{m,n=1,3,\ldots} \frac{4}{mn\pi^2} \frac{\sin(m\pi x_1)\sin(n\pi y_1)}{\pi^2(m^2+n^2)+a^2}, \quad (x_1,y_1) \in \Omega. \end{aligned}$$

However, this representation is not suitable to gain information about the behavior of w_a as $a \to \infty$. We therefore choose an alternative way to represent the Green's function, which will turn out to be more fruitful. In fact, according to [Roa70], p. 270, problem 11, the above Green's function (2.22) may be expressed in the form

$$G(p,q) = G(x_1,y_1,x_2,y_2) = 2\sum_{n=1}^{\infty} \frac{\sinh(\sigma_n y_2)\,\sinh(\sigma_n(y_1-1))}{\sigma_n\,\sinh(\sigma_n)} \sin(n\pi x_1)\,\sin(n\pi x_2)$$

for the case $0 < y_2 < y_1 < 1$, with $\sigma_n^2 = a^2 + n^2\pi^2$, $p = (x_1, y_1)$ and $q = (x_2, y_2)$.

Applying the integral representation formula of theorem 9.6 in [Roa70] once more, the solution w_a takes the form

$$\begin{aligned} w_a(x_1,y_1) &= 2a^2 \sum_{n=1}^{\infty} \int_0^1 \int_0^{y_1} \frac{\sinh(\sigma_n) y_2 \sinh(\sigma_n(y_1-1))}{\sigma_n \sinh(\sigma_n)} \sin(n\pi x_1) \sin(n\pi x_2) dy_2 dx_2 \\ &+ 2a^2 \sum_{n=1}^{\infty} \int_0^1 \int_{y_1}^1 \frac{\sinh(\sigma_n(y_2-1)) \sinh(\sigma_n y_1)}{\sigma_n \sinh(\sigma_n)} \sin(n\pi x_2) \sin(n\pi x_1) dy_2 dx_2. \end{aligned} \tag{2.23}$$

Evaluating the respective integrals yields

$$\begin{aligned} &\int_0^1 \int_0^{y_1} \frac{\sinh(\sigma_n y_2) \sinh(\sigma_n(y_1-1))}{\sigma_n \sinh(\sigma_n)} \sin(n\pi x_1) \sin(n\pi x_2) dy_2 dx_2 \\ =\;& \frac{\sinh(\sigma_n(y_1-1))\,\sin(n\pi x_1)}{\sigma_n \sinh(\sigma_n)} \int_0^1 \sin(n\pi x_2)\,dx_2 \int_0^{y_1} \sinh(\sigma_n y_2)\,dy_2 \\ =\;& \frac{\sinh(\sigma_n(y_1-1))\,\sin(n\pi x_1)}{\sigma_n \sinh(\sigma_n)} \cdot \frac{((-1)^{n+1}+1)}{n\pi} \cdot \frac{1}{\sigma_n}(\cosh(\sigma_n y_1) - 1) \\ =\;& \begin{cases} \frac{2\sinh(\sigma_n(y_1-1))\,\sin(n\pi x_1)}{n\pi\sigma_n^2 \sinh(\sigma_n)} \cdot (\cosh(\sigma_n y_1) - 1) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\int_0^1 \int_{y_1}^1 \frac{\sinh(\sigma_n(y_2-1)) \sinh(\sigma_n y_1)}{\sigma_n \sinh(\sigma_n)} \sin(n\pi x_2) \sin(n\pi x_1) dy_2 dx_2 \\ =\;& \frac{\sinh(\sigma_n y_1)\,\sin(n\pi x_1)}{\sigma_n \sinh(\sigma_n)} \cdot \frac{((-1)^{n+1}+1)}{n\pi} \cdot \frac{1}{\sigma_n}(1 - \cosh(\sigma_n(y_1-1))) \\ =\;& \begin{cases} \frac{2\sinh(\sigma_n y_1)\,\sin(n\pi x_1)}{n\pi\sigma_n^2 \sinh(\sigma_n)} \cdot (1 - \cosh(\sigma_n(y_1-1))) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{2.25}$$

In order to simplify the presentation, let us define the quantity

$$K(n,x,a) := \frac{4a^2 \sin(n\pi x)}{n\pi\sigma_n^2}.$$

If we insert (2.24) and (2.25) into equation (2.23) and simplify, we obtain that the solution $w_a(p) = w_a(x, y)$ is equal to the quantity

$$\begin{aligned} & \sum_{n=1,3,\ldots} K(n,x,a) \frac{\sinh(\sigma_n(y-1))(\cosh(\sigma_n y)-1) - \sinh(\sigma_n y)(\cosh(\sigma_n(y-1))-1)}{\sinh(\sigma_n)} \\ = & -\sum_{n=1,3,\ldots} K(n,x,a) \frac{\sinh(\sigma_n(1-y))(\cosh(\sigma_n y)-1) + \sinh(\sigma_n y)(\cosh(\sigma_n(1-y))-1)}{\sinh(\sigma_n)} \\ = & -\sum_{n=1,3,\ldots} K(n,x,a) \left(1 - \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}\right). \end{aligned}$$

Applying the transformation (2.6) yields

$$u_\varepsilon(x,y) = -\frac{1}{a} \log\left[1 - \sum_{n=1,3,\ldots} K(n,x,a) \left(1 - \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}\right)\right].$$

Taking advantage of the well-known relation

$$\sum_{n=1,3,\ldots} \frac{\sin(n\pi x)}{n} = \frac{\pi}{4} \tag{2.26}$$

for all $0 < x < 1$, we obtain using $\sigma_n^2 = a^2 + n^2\pi^2$

$$u_\varepsilon(x,y) = -\frac{1}{a} \log\left[\sum_{n=1,3,\ldots} K(n,x,a) \left(\frac{\pi^2 n^2}{a^2} + \frac{\sinh(\sigma_n(1-y)) + \sinh(\sigma_n y)}{\sinh(\sigma_n)}\right)\right].$$

Now proposition 2.1 implies

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,y) = x \quad \text{for all } 0 < y \leq 1/2 \text{ and all } 0 < x \leq y.$$

Note that for each $\varepsilon > 0$ it follows from the uniqueness of the function u_ε as a solution of (2.21) that u_ε is symmetric with respect to the lines $\{x = 1/2\}$, $\{y = 1/2\}$, $\{x - y = 0\}$, and $\{x + y = 1\}$. Hence the assertion is proved. □

2.3.3 L^1-Convergence on convex domains with smooth boundary

As for more general domains the Green's function is not explicitly known, the method of characterizing the solutions u_ε of the *viscous* eikonal equation by means of the Green's function cannot be employed successfully.

However, it turns out that we still can prove an L^1-convergence result for the case that the domain is convex and has a smooth boundary. We start with the following

Lemma 2.6. *Let $\Omega \in \mathbb{R}^n$ be a domain with smooth boundary. Furthermore, for any $\varepsilon > 0$ let $u_\varepsilon \in C^\infty(\bar{\Omega})$ be the unique solution of the boundary value problem*

$$\begin{cases} \varepsilon \Delta u_\varepsilon - |Du_\varepsilon|^2 + 1 = 0 & \text{on } \Omega \\ u_\varepsilon \equiv 0 & \text{on } \partial\Omega, \end{cases} \tag{2.27}$$

which exists according to theorem 2.1. Then we have

$$\lim_{\varepsilon \to 0} \int_\Omega (|Du_\varepsilon|^2 - 1)\, dx = 0.$$

The proof of lemma 2.6 is based upon the following

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. For each $a > 0$ let $w_a \in C(\bar{\Omega}) \cap C^2(\Omega)$ be the unique solution of the boundary value problem*

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

Then we have

$$\lim_{a \to \infty} \int_\Omega (w_a + 1)\, dx = 0.$$

Proof. (of proposition 2.2). Let $a > 0$ and define

$$\Omega_a := \{x \in \mathbb{R}^n \,|\, x/a \in \Omega\}.$$

Then the function

$$v_a(x) := w_a(x/a) + 1$$

satisfies

$$\begin{cases} \Delta v_a - v_a = 0 & \text{on } \Omega_a \\ v_a \equiv 1 & \text{on } \partial\Omega_a. \end{cases} \tag{2.29}$$

Note that we have $v_a \leq 1$ on $\bar{\Omega}_a$ by the maximum principle.

Now define for $\beta > 0$

$$\Omega^\beta := \{x \in \Omega \,|\, \boldsymbol{d}(x) > \beta\},$$

where $\boldsymbol{d}$ is the distance function to the boundary, and set

$$\Omega_a^\beta := \{x \in \mathbb{R}^n \,|\, x/a \in \Omega^\beta\}.$$

Fix $\beta > 0$. We show that there is a positive function $S : \mathbb{R} \to \mathbb{R}$ with $S(a) \to 0$ as $a \to \infty$, such that

$$v_a(x) < S(a) \quad \text{for all } x \in \Omega_a^\beta. \tag{2.30}$$

To this end we define for any $r > 0$ the function v^r to be the solution of boundary value problem (2.29), where the domain Ω_a is replaced by the open ball $B_r(0)$ with radius

$r > 0$. As is well known, v^r is given by a suitable Bessel function depending on the radius r. Hence one easily verifies that

$$v^r(0) \to 0 \quad \text{as } r \to \infty. \tag{2.31}$$

Next fix $x \in \Omega_a^\beta$. Then $B_{a\beta}(x) \subset \Omega_a$. Define the function

$$\varphi^x : y \mapsto v^{a\beta}(y - x).$$

Then we have

$$1 = \varphi^x(y) \geq v_a(y) \quad \text{for all } y \in \partial B_{a\beta}(x),$$

which implies

$$v^{a\beta}(0) = \varphi^x(x) \geq v_a(x)$$

by the maximum principle. As $x \in \Omega_a^\beta$ has been chosen arbitrarily, it follows

$$S(a) \geq v_a(x) \quad \text{for all } x \in \Omega_a^\beta,$$

where we have set $S(a) := v^{a\beta}(0)$. Furthermore, by means of relation (2.31) we have

$$S(a) \to 0 \quad \text{for } a \to \infty, \tag{2.32}$$

and (2.30) is proved.

Now fix $\varepsilon > 0$. As the boundary $\partial\Omega$ is smooth, its curvature is bounded, whence there is a constant $C > 0$ independent of β such that

$$|\Omega \backslash \Omega^\beta| < C \cdot \beta.$$

We now choose $\beta > 0$ such that $C\beta \leq \varepsilon/2$ and compute

$$\begin{aligned} \int_\Omega (w_a(x) + 1)\, dx &= \frac{1}{a^n} \int_{\Omega_a} v_a(y)\, dy \leq \frac{1}{a^n} |\Omega_a \backslash \Omega_a^\beta| + \frac{1}{a^n} \int_{\Omega_a^\beta} v_a(y)\, dy \\ &\leq \; |\Omega \backslash \Omega^\beta| + \frac{1}{a^n} |\Omega_a^\beta| \cdot S(a) < \varepsilon/2 + |\Omega^\beta| \cdot S(a). \end{aligned}$$

Hence by (2.32) we have

$$\int_\Omega (w_a(x) + 1)\, dx < \varepsilon,$$

if a is large enough. The assertion follows. □

Proof. (of lemma 2.6). We apply the same transformation as in the proof of lemma 2.1. Let $\varepsilon > 0$, let $u_\varepsilon \in C^\infty(\bar{\Omega})$ be the solution of (2.27), and set $a := 1/\varepsilon$. Then the function

$$w_a := \exp(-\varepsilon^{-1} u_\varepsilon) - 1$$

satisfies the linear boundary value problem

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.33}$$

From the inverse transformation

$$u_\varepsilon := -\frac{1}{a}\log(w_a+1)$$

it follows

$$|Du_\varepsilon|^2 = \frac{1}{a^2}\sum_{i=1}^n \frac{(D_i w_a)^2}{(w_a+1)^2}.$$

Using

$$D_i\left(\frac{w_a}{w_a+1}\right) = \frac{D_i w_a}{(w_a+1)^2},$$

we get by partial integration

$$\begin{aligned} \int_\Omega |Du_\varepsilon|^2\,dx &= \frac{1}{a^2}\int_\Omega \sum_{i=1}^n \frac{(D_i w_a)^2}{(w_a+1)^2}\,dx = \frac{1}{a^2}\int_\Omega \sum_{i=1}^n \frac{D_i w_a}{(w_a+1)^2} D_i w_a\,dx \\ &= \frac{1}{a^2}\int_\Omega \sum_{i=1}^n D_i\left(\frac{w_a}{w_a+1}\right) D_i w_a\,dx = -\frac{1}{a^2}\int_\Omega \frac{w_a}{w_a+1}\Delta w_a\,dx, \end{aligned}$$

where the boundary terms vanish due to $w_a \equiv 0$ on $\partial\Omega$.

On the other hand, (2.33) implies

$$a^2(w_a+1) = \Delta w_a,$$

whence it follows

$$\int_\Omega |Du_\varepsilon|^2\,dx = -\int_\Omega w_a\,dx.$$

We obtain

$$\int_\Omega (|Du_\varepsilon|^2 - 1)\,dx = -\int_\Omega (w_a+1)\,dx,$$

and the assertion follows by proposition 2.2. □

Theorem 2.2. *Suppose that the domain Ω in lemma 2.6 be convex. Then $|Du_\varepsilon|^2 \to 1$ with respect to the L^1-norm on Ω as $\varepsilon \to 0$.*

Proof. Let $\boldsymbol{d} : \bar\Omega \to \mathbb{R}$ be the distance function to the boundary $\partial\Omega$. We show that $u_\varepsilon \le \boldsymbol{d}$ on $\bar\Omega$ for all $\varepsilon > 0$. For this purpose assume this were not the case. Then there is an $\varepsilon > 0$ and a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) > \boldsymbol{d}(x_0)$. Let $y \in \partial\Omega$ satisfy $|x_0 - y| = \min_{z\in\partial\Omega}|z - x_0|$ and let ν_y be the inward pointing unit normal of Ω at y. For the function

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(x) := \langle \nu_y, x - y\rangle$$

it then follows that

$$\varphi(x_0) = \boldsymbol{d}(x_0) < u_\varepsilon(x_0). \tag{2.34}$$

Define the quasilinear differential operator

$$Q(u) := \varepsilon \Delta u - |Du|^2 + 1, \quad u \in C^2(\Omega),$$

and observe that we have

$$Q(\varphi) = 0 \quad \text{on } \Omega.$$

As Ω is convex, we furthermore have $\varphi \geq 0$ on $\partial\Omega$. On the other hand, we have $Q(u_\varepsilon) = 0$ on Ω as well as $u_\varepsilon \equiv 0$ on $\partial\Omega$. Then the quasilinear comparison principle (cf. theorem 9.2 in [GT77]) implies $u_\varepsilon \leq \varphi$ on $\bar{\Omega}$, a contradiction to (2.34).

The quasilinear comparison principle also implies that we have $u_\varepsilon \geq 0$ on $\bar{\Omega}$, whence altogether it follows

$$0 \leq u_\varepsilon \leq \boldsymbol{d} \quad \text{on } \bar{\Omega}.$$

As $u_\varepsilon \in C^\infty(\bar{\Omega})$, this implies

$$\left| \frac{\partial}{\partial v} u_\varepsilon(x) \right| \leq 1 \tag{2.35}$$

for all $x \in \partial\Omega$ and for any direction $v \in \mathbb{R}^n$, $|v| = 1$. Now fix $v \in \mathbb{R}^n$ with $|v| = 1$. Differentiating (2.27) with respect to v yields

$$\varepsilon \Delta w - 2\langle Du_\varepsilon, Dw \rangle = 0$$

where $w := \frac{\partial}{\partial v} u_\varepsilon$. By (2.35) the linear maximum principle then implies $|w| \leq 1$ on $\bar{\Omega}$. As the choice of v was arbitrary, it follows $|Du_\varepsilon| \leq 1$ on $\bar{\Omega}$. Hence

$$\lim_{\varepsilon \to 0} \int_\Omega \left| |Du_\varepsilon|^2 - 1 \right| \, dx = - \lim_{\varepsilon \to 0} \int_\Omega |Du_\varepsilon|^2 - 1 \, dx = 0$$

by lemma 2.6. This completes the proof. □

Remark 2.1. Observe that the convexity condition is not required until theorem 2.2, in the proof of which, however, it plays an essential role.

2.4 Generalized solutions in the sense of Kružkov

In view of the historical development of the concept of viscosity solutions, it is worth while elaborating on the work of S. N. Kružkov, especially on his achievements in developing a comprising theory of generalized solutions of "Hamilton-Jacobi equations of eikonal type". The corresponding paper [Kru75] provided valuable inspiration for the present thesis, and we therefore outline the essential ideas of Kružkov's theory in order to draw a complete picture.

The theory applies to Hamilton-Jacobi equations of the form

$$H(Du,u,x) = 0 \quad \text{with} \quad H(Du,u,x) = f(Du,u,x) - n^2(u,x), \tag{2.36}$$

with $x \in \Omega$ for some domain $\Omega \subset \mathbb{R}^n$. The functions n and f are subject to several constraints, among which the most important are

$$\begin{array}{ll} (i) & f(0,u,x) = 0 \quad \text{for all } (u,x) \in \mathbb{R} \times \Omega \\ (ii) & H(p,u,x) \text{ is convex with respect to } p \\ (iii) & H(p,u,x) \text{ is non-decreasing with respect to } u. \end{array} \tag{2.37}$$

These equations are related to the eikonal equation, whereby in the geometrical optics interpretation the function $n(u,x)$ corresponds to the index of refraction of light rays determined by the properties of the medium.

Classically, nonlinear problems like this can be attacked by the method of characteristics, which usually yields a unique classical solution in the neighborhood of a given manifold of dimension $n-1$, provided that this manifold is non-characteristic and sufficiently smooth. However, in the case of the above eikonal type equations, uniqueness cannot be expected for these local classical solutions, as the original equation may split into two distinct equations, each of them corresponding to a different local solution. Apart from that, the projections of the characteristics onto the underlying space intersect in general, with the consequence that at a point of intersection each of them corresponds to a different value of the gradient brought to it from the initial manifold. With this in mind it suggests itself to dispense with classical solutions and to allow for weak solutions satisfying the equation only almost everywhere. Then unicity is not guaranteed in the first stage—even in case of problems where the method of characteristics yields global solutions—, and one has to introduce an extra condition. Kružkov's condition can be most easily outlined in the case of the eikonal equation on an interval, that is

$$\begin{cases} |u'(x)| = 1 \quad \text{on } [0,1] \\ u(0) = u(1) = 0. \end{cases} \tag{2.38}$$

In the class of Lipschitz continuous functions satisfying (2.38) almost everywhere, the most obvious are those piecewise linear functions which are composed by sections of lines of slope 1 or -1. Among these solutions the distance function seems to be the most natural one, as it possesses several extremal properties: It maximizes the volume functional, it minimizes the curvature functional (suitably weakly defined), and it is the only concave solution. In fact, if any of these extra conditions is demanded, the distance function will be uniquely singled out. However, only (a modification of) the concavity constraint turns out to be powerful enough to still provide uniqueness if the boundary conditions are more general, the dimension of the space is higher, or if the equation itself deviates from the eikonal equation in a sense which is referred to as "eikonal-type" by Kružkov. Whereas both volume and curvature functionals in connection to the eikonal equation will be discussed later on, let us at this stage elaborate on the modified concavity condition that has been introduced by Kružkov. In fact it is sufficient to ensure that the weakly defined second derivatives are bounded by above.

Definition 2.1. A function $u : \Omega \to \mathbb{R}$ is called a *generalized solution of* (2.36) *in the sense of Kružkov* , if it has the following properties:

(i) u is locally Lipschitz continuous on Ω
(ii) u satisfies (2.36) almost everywhere
(iii) for each $r > 0$ and $x \in \Omega$ such that $B_r(x) \subseteq \Omega$ we have

$$\frac{\Delta^2 u}{|\Delta x|^2} := \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{|\Delta x|^2} \leq C(x, r)$$

for all $\Delta x \in \mathbb{R}^n$ with $0 < |\Delta x| \leq r$.

Kružkov showed that if such a generalized solution exists, it will be unique. The proof is essentially based on property (iii) in combination with a certain transformation of the problem. Furthermore, he made explicit use of the method of vanishing viscosity to show that generalized solutions exist, a fact which reflects the connection to the concept of viscosity solutions designed by Crandall and Lions several years later. In fact, Kružkov's theory coincides with their theory in the case of eikonal-type equations. All this has been exhaustively discussed in the literature.

2.5 Viscosity solutions in the sense of Crandall and Lions

Michael G. Crandall and Pierre-Louis Lions were the first to notice that Kružkov's additional constraint—the local upper bound of the second derivatives—can be seen as the manifestation in a special case of a by far more general approach applicable to a broad class of Hamilton-Jacobi equations. Here we will consider the historically oldest definition, which captures the essential features in the best way. We also restrict ourselves to first order problems of the form

$$H(Du, u, x) = 0 \quad \text{in } \Omega, \tag{2.39}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$. Moreover we demand the fundamental monotonicity condition

$$H(p, r, x) \leq H(p, s, x) \quad \text{whenever } r \leq s \text{ for all } (p, x) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.40}$$

which is essential for the theory (cf. [CIL92]).

We now provide the important notion of upper and lower test functions.

Definition 2.2. Let $u : \Omega \to \mathbb{R}$ and let $x \in \Omega$. A function $\varphi \in C(\Omega)$ which is differentiable at x and for which $u - \varphi$ attains a local maximum (minimum) at x is called *upper (lower) test function of u at x.*

A viscosity solution of (2.39) is then defined via a test function condition according to the following

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $x \in \Omega$. For problem (2.39), a function $u : \Omega \to \mathbb{R}$ is said to satisfy the *viscosity subsolution, viscosity supersolution, or viscosity solution condition at x*, if respectively the first, the second, or both of the following conditions are satisfied.

(i) u is upper semicontinuous on $\bar{\Omega}$ and for all upper test functions φ of u at x we have $H(D\varphi(x), u(x), x) \leq 0$.

(ii) u is lower semicontinuous on $\bar{\Omega}$ and for all lower test functions φ of u at x we have $H(D\varphi(x), u(x), x) \geq 0$.

If u satisfies the viscosity subsolution, viscosity supersolution, or viscosity solution condition for all $x \in \Omega$, then u is respectively called a *viscosity subsolution, viscosity supersolution, or viscosity solution* of (2.39).

As a matter of fact, this definition of viscosity solution allows the derivation of various existence and uniqueness results for boundary value problems which can be proven in a far more elegant way compared with Kružkov's methods (cf. [CIL92]). It can also be shown that the vanishing viscosity limit coincides with the unique viscosity solution, a fact which we will refer to as *consistency with the method of vanishing viscosity.*

As we have announced in chapter 1, we will extend this concept to ramified spaces in chapter 5 and chapter 7.

CHAPTER 3

Perron methods for the eikonal equation

Summary. The present chapter relates viscosity solutions of the eikonal equation on non-ramified domains to maximum volume solutions of granular matter problems. We show the equivalence of the two selection principles induced by the test function condition and the maximum volume constraint. In fact, we characterize viscosity solutions of the eikonal equation as pointwise suprema over a certain class of subsolutions, the so-called *subeikonal* functions (*Perron method*). We also consider the *viscous version* of the eikonal equation and characterize its viscosity solutions as suprema of *subharmoneikonal* functions, which also form a special class of viscosity subsolutions.

3.1 Introduction

In [HKG02], the authors discuss the analogy between the Dirichlet problem of the Laplacian and the Dirichlet problem of the eikonal equation under a maximal volume constraint on bounded domains. They point out that similar to the Laplace equation one can construct solutions for the eikonal equation via a Perron method. Whereas the Perron method for the Laplacian is well known, for the maximal value problem they introduce a special class of subfunctions (the so-called *subeikonal* functions), and show that solutions are given by pointwise suprema of all subeikonal functions staying below given boundary values.

On the other hand, as has been shown by Ishii in [Ish87b], a general Perron method can be designed to construct viscosity solutions for a general class of Hamilton-Jacobi equations. In fact, a viscosity solution is given by the pointwise supremum over all viscosity subsolutions staying below given boundary values.

We demonstrate that Ishii's method applied to the eikonal equation is equivalent to the method given in [HKG02] (at least in most cases). As we will see, the class of subeikonal functions is contained in the class of viscosity subsolutions and has the property that its pointwise supremum as constructed in [HKG02] is indeed a viscosity solution.

We discuss a possible interpretation of this equation in terms of granular matter theory. The *viscous* version of the eikonal equation, which reads

$$\varepsilon\Delta u - |Du| + 1 = 0, \tag{3.1}$$

can be considered as a mixture or combination of the Laplace and the eikonal equation. We show that solutions can be constructed via a special Perron method which in a certain sense is a mixture of the two different Perron methods belonging to the eikonal and the Laplace equation, respectively. In fact, we introduce a class of *subharmoneikonal functions* (*SHE functions*, for short), which is a modification of the class of subeikonal functions. Once more we show that a Perron method based on these functions yields viscosity solutions.

3.2 The eikonal equation and subeikonal functions

Throughout this chapter let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the eikonal equation

$$H(Du) = |Du| - 1 = 0 \tag{3.2}$$

on Ω. The following terminology is introduced in [HKG02].

Definition 3.1. Let $x \in \Omega$. We call a function $u \in C(\bar{\Omega})$ *subeikonal at* x, whenever there is a radius $r_0 > 0$ with $B_{r_0}(x) \subseteq \Omega$ such that

$$u(x) \leq \inf_{y \in S_r(x)} u(y) + r \tag{3.3}$$

for all $0 < r \leq r_0$, where $S_r(x) := \partial B_r(x)$. The function u is called *subeikonal*, if it is subeikonal at each point $x \in \Omega$.

Suppose the boundary of the domain Ω be Lipschitz. For a given boundary data function $\phi : \partial\Omega \to \mathbb{R}$ we define X to be the set of all subeikonal functions u with $u \leq \phi$ on $\partial\Omega$. It has been shown in [HKG02] that the function $\tilde{u} : \bar{\Omega} \to \mathbb{R}$ defined by

$$\tilde{u}(x) := \sup_{u \in X} u(x), \quad x \in \bar{\Omega}, \tag{3.4}$$

is also contained in X (consistency). Furthermore, $\tilde{u}$ is the unique function among all Lipschitz continuous functions with Lipschitz constant 1, which maximizes the volume functional

$$V(u) := \int_\Omega u(x)\, dx.$$

The construction of the maximum volume solution $\tilde{u}$ as a pointwise supremum of subeikonal functions is what we refer to as *Perron's method*. We now show that the *Perron solution* $\tilde{u}$ is in fact a viscosity solution of (3.2) according to definition 2.3. We start with the following

Lemma 3.1. *Let $v \in C(\bar{\Omega})$ be subeikonal. Then v is a viscosity subsolution of (3.2).*

Proof. Let $x \in \Omega$ and let φ be an upper test function of v at x. We have to show $|D\varphi(x)| \leq 1$. First note that we can without loss of generality assume that we have $v(x) = \varphi(x)$. As v is subeikonal and as $v - \varphi$ attains a local maximum at x, it follows that there is a number $\tilde{r} > 0$ such that

$$\varphi(x) = v(x) \leq \inf_{y \in S_r(x)} v(y) + r \leq \inf_{y \in S_r(x)} \varphi(y) + r \tag{3.5}$$

for all $0 < r \leq \tilde{r}$.

As φ is differentiable at x, we obtain by Taylor expansion

$$\varphi(x + re) = \varphi(x) + r\langle D\varphi(x), e\rangle + O(r^2)$$

for any $e \in S^{n-1}$. Choosing $e := -D\varphi(x)/|D\varphi(x)|$, we estimate

$$\inf_{y \in S_r(x)} \varphi(y) = \inf_{z \in S^{n-1}} \varphi(x + rz) \leq \varphi(x + re) = \varphi(x) - r|D\varphi(x)| + O(r^2). \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$0 \leq 1 - |D\varphi(x)| + O(r),$$

which implies $|D\varphi(x)| \leq 1$ upon letting $r \to 0$. □

It has been shown in [HKG02] that $\tilde{u}$ is subeikonal. Hence the above lemma implies that $\tilde{u}$ is a viscosity subsolution of (3.2). Next we show that it also is a viscosity supersolution.

Lemma 3.2. *The function $\tilde{u}$ as defined in (3.4) is a viscosity supersolution of (3.2).*

Proof. Suppose the contrary were the case. Then there is a point $x_0 \in \Omega$ and a lower test function φ of $\tilde{u}$ at x_0 such that we have $|D\varphi(x_0)| < 1$. Without loss of generality we may assume $\tilde{u}(x_0) = \varphi(x_0)$ and that φ be C^2 in an open neighborhood U of x_0. Furthermore we can assume that the local minimum of $\tilde{u} - \varphi$, which by definition is attained at x_0, be *strict*, by possibly adding to φ a quadratic function of the form

$$y \mapsto -\alpha||x_0 - y||^2, \quad \alpha > 0.$$

For reasons of continuity it follows that there are small numbers $\eta, \xi > 0$ such that $\bar{B}_\xi(x_0) \subset U$ and such that for $\tilde{\varphi} := \varphi + \eta$ we have $\tilde{\varphi}(y) < \tilde{u}(y)$ for all $y \in \partial B_\xi(x_0)$ as well as $|D\tilde{\varphi}(y)| < 1$ for all $y \in B_\xi(x_0)$. We show that the function $\bar{u} \in C(\bar{\Omega})$ given by

$$\bar{u}(x) := \begin{cases} \max\{\tilde{u}(x), \tilde{\varphi}(x)\} & \text{if } x \in \bar{B}_\xi(x_0) \\ \tilde{u}(x) & \text{if } x \in \bar{\Omega} \backslash \bar{B}_\xi(x_0) \end{cases}$$

is contained in X. To this end first note that $\bar{u}$ clearly is subeikonal for all $x \in \Omega \backslash \bar{B}_\xi(x_0)$. Now let $x \in \bar{B}_\xi(x_0)$. As φ is C^2 in U and as $x \in U$, there is a radius $r_0 > 0$ such that by Taylor expansion we have

$$\begin{aligned}\varphi(x+re) &> \varphi(x) + r\langle D\varphi(x), e\rangle - Cr^2 \geq \varphi(x) - r\left\langle D\varphi(x), \frac{D\varphi(x)}{|D\varphi(x)|}\right\rangle - Cr^2 \\ &= \varphi(x) - r|D\varphi(x)| - Cr^2\end{aligned}$$

for all $e \in S^{n-1}$ and all $0 < r \leq r_0$, where $C > 0$ is a constant independent of r and e. Consequently, as $|D\varphi(x)| < 1$, there is a $\delta > 0$ such that for all $e \in S^{n-1}$ and all $0 < r \leq r_0$ we have

$$\varphi(x+re) + r > \varphi(x) + \delta r - Cr^2.$$

Hence for all $0 < r < \delta/C := r_1$ we have

$$\inf_{y \in S_r(x)} \varphi(y) + r = \inf_{e \in S^{n-1}} \varphi(x+re) + r > \varphi(x). \tag{3.7}$$

The same holds if φ is replaced by $\tilde{\varphi}$.

On the other hand, $\tilde{u}$ is subeikonal at x according to [HKG02], whence there is an $r_2 > 0$ such that we have

$$\tilde{u}(x) \leq \inf_{y \in S_r(x)} \tilde{u}(y) + r \tag{3.8}$$

for all $0 < r < r_2$. Combining (3.7) and (3.8) we find

$$\max\{\tilde{u}(x), \tilde{\varphi}(x)\} \leq \inf_{y \in S_r(x)} \max\{\tilde{u}(y), \tilde{\varphi}(y)\} + r$$

for all $0 < r < \min\{r_1, r_2\}$, implying that $\bar{u}$ is subeikonal at x. Thus we have $\bar{u} \in X$. Since by construction we have $\bar{u}(x_0) > \tilde{u}(x_0)$, we obtain a contradiction to the definition of $\tilde{u}$. This completes the proof. □

The combination of lemmas 3.1 and 3.2 yields the following

Corollary 3.1. *The function $\tilde{u}$ as defined in (3.4) is a viscosity solution of (3.2).*

As the above lemmas have shown, the Perron method designed in [HKG02] yields a viscosity solution of the eikonal equation. Note that the behavior of both the boundary $\partial\Omega$ and the boundary data ϕ is not relevant in the above proofs. However, as it is also the case for the classical Perron method for the Dirichlet problem, it is not clear if the boundary data are assumed. We thus need an extra *barrier* condition for the boundary values. Such a barrier condition is given in [HKG02].

3.3 An example from granular matter theory

We now turn to the discussion of the viscous eikonal equation (3.1). Let us start with recovering it within a granular matter scenario. In fact, (3.1) can serve as a continuum model for the behavior of the height of a sandpile under the influence of a weak, continuously pouring source, as we will point out in the sequel.

Models for the deposition of homogeneous granular matter on domains or obstacles have been proposed by various authors. As an example we consider an extended version of the BCRE model (due to Bouchaud, Cates, Ravi Prakash, and Edwards [BCRPE95]) as presented in [HKG02]. It reads

$$\begin{aligned} v_t &= \beta \operatorname{div}(vDu) - \gamma(\alpha - |Du|)v + f \\ u_t &= \gamma(\alpha - |Du|)v \end{aligned} \tag{3.9}$$

on a domain $\Omega \subset \mathbb{R}^n$ and provides a continuum description of the dynamics of sandpile surfaces, which involves two populations of grains: standing u and rolling v. The latter move downhill with a velocity depending on the slope of the standing layer, where we assume that u and v represent masses rather than heights. The parameter $\alpha > 0$ is the tangens of the angle of repose specific to the respective granular material, $\beta > 0$ governs the speed of the rolling layer, and $\gamma > 0$ is a measure for the exchange rate between the standing and the rolling layer. The model also incorporates a source term f which may vary in time and space. Furthermore it is mass-conserving, as for functions with compact support in Ω we have

$$\frac{\partial}{\partial t}\int_\Omega u + v\,dx = \int_\Omega f\,dx.$$

We now restrict our considerations to the special situation that new granular matter is being poured uniformly, steadily, and slowly from above, i.e. we assume $f \equiv f_0$ for some small positive number $f_0 << 1$. It is then plausible to make the further simplifying assumption that v be constant in both time and space, i.e. we assume $v \equiv v_0$ for some positive number $v_0 << 1$.

Next observe that the variables u and v represent masses rather than heights. For granular matter in general there is no fixed ratio between mass and volume or height, not even at rest. For example it is known that in dunes the packing of grains is rather dense on the luff side but can be extremely loose on the lee side. Here we assume that the relation of mass and volume equals 1 everywhere for the standing layer, i.e., we interpret u as the height of the standing heap. For the rolling layer we assume that it consists of grains rolling at various speeds and at various heights but that the rolling layer contributes to the perceived height only with a factor $0 < \kappa < 1$. In other words we assume that the upper part of the rolling layer has low volume density and produces microscopic surface roughness such that the perceived height is

$$w = u + \kappa v.$$

From the model equations (3.9) we then obtain

$$w_t = \kappa\beta \operatorname{div}(vDu) + \gamma(1-\kappa)(\alpha - |Du|)v + \kappa f.$$

With $v \equiv v_0$ and $f \equiv f_0$ this simplifies to

$$w_t = \kappa\beta v_0 \Delta w + \gamma(1-\kappa)(\alpha - |Dw|)v_0 + \kappa f_0.$$

Putting $a := \gamma(1-\kappa)v_0$ and $\varepsilon := \kappa\beta v_0$, we obtain

$$w_t = a(\alpha - |Dw|) + \varepsilon\Delta w + \kappa f_0.$$

Changing the scales of time and space appropriately we arrive at

$$w_t = 1 - |Dw| + \varepsilon\Delta w + \kappa f_0,$$

for suitably adjusted constants $\varepsilon > 0$ and $\kappa > 0$. In the special case $f_0 = 0$ we get

$$w_t = \varepsilon\Delta w + 1 - |Dw|.$$

Now (3.1) is the corresponding stationary problem.

Remark 3.1. An alternative, immediate interpretation of the term $\varepsilon\Delta u$ in (3.1) is a sort of diffusion effect caused by small avalanches occurring at hilltops or similar nonsmooth areas of a sand pile.

Let us now examine the structure of solutions of (3.1) in detail.

3.4 The viscous eikonal equation and SHE functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and fix $\varepsilon > 0$. For $u \in C^2(\Omega)$ we define the semilinear operator Q by

$$Q(u) := \varepsilon\triangle u - |Du| + 1.$$

We call the equation

$$Q(u) = 0 \quad \text{on } \Omega \tag{3.10}$$

viscous eikonal equation, where we emphasize that the term $|Du|$ is not squared here – in contrast to the previous chapter.

We have seen that in the case of the eikonal equation viscosity solutions can be constructed as pointwise suprema of subeikonal functions. On the other hand, classical solutions (and thus viscosity solutions) of the Laplace equation can be constructed as pointwise suprema of subharmonic functions. We now examine the question, whether viscosity solutions of the viscous eikonal equation (3.10), which is a semilinear second order equation, can be constructed using a similarly intuitive class of "subsolutions". We will show that in many cases viscosity solutions of (3.10) can indeed be built by taking the pointwise supremum over a suitable class of "subfunctions". In a certain sense, equation (3.10) is a linear combination of the Laplace equation and the eikonal equation, and it is reasonable to expect that the corresponding subfunctions reflect properties of both of them. In fact, subeikonal functions are characterized by a local "subextremal" property, whereas subharmonic functions satisfy a local "subaverage" condition. Note that both of these properties are incorporated in the following definition.

Definition 3.2. Let $x \in \Omega$. We call a function $u \in C(\Omega)$ *subharmoneikonal (SHE) at* x, if there is an $r_0 > 0$ such that $B_{r_0}(x) \subseteq \Omega$ and such that for all $0 < r \leq r_0$ we have

$$u(x) \leq \frac{r}{r + 2n\varepsilon}\left(\inf_{y \in S_r(x)} u(y) + r\right) + \frac{2n\varepsilon}{r + 2n\varepsilon} \fint_{S_r(x)} u(y)\, dS(y), \tag{3.11}$$

where $\fint$ denotes the average integral; $S_r(x) := \partial B_r(x)$.

Furthermore let $A \subseteq \Omega$. If u is SHE at all $x \in A$, we say that u is subharmoneikonal (SHE) on A and write $u \in \mathrm{SHE}(A)$.

Definition 3.3. Let $u, v \in C(\Omega)$ and $x \in \Omega$. We say that v *touches* u *by above (by below)* at x, if $u(x) = v(x)$ and if there is a neighbourhood U around x such that $v(y) \geq\ (\leq)\ u(y)$ for all $y \in U$.

The following two propositions are immediate implications of definition 3.2.

Proposition 3.1. *Let $u \in C(\Omega)$ be SHE at $x \in \Omega$ and assume that $v \in C(\Omega)$ touches u by above at x. Then v is SHE at x.*

Proposition 3.2. *Let $u, v \in C(\Omega)$ be SHE at $x \in \Omega$. Then $w := \max\{u, v\}$ is SHE at x.*

Lemma 3.3. *Let $x \in \Omega$ and let $u : \Omega \to \mathbb{R}$ be twice differentiable at x.*
(i) If u is SHE at x, then $Q(u)(x) \geq 0$.
(ii) If $Q(u)(x) > 0$, then u is SHE at x.

Proof. Expanding u around x yields for small $r > 0$

$$u(x + re) = u(x) + r\langle Du(x), e\rangle + \frac{r^2}{2} e^t D^2 u(x) e + O(r^3) \tag{3.12}$$

for any $e \in S^{n-1}$. We integrate and obtain

$$\fint_{y \in S_r(x)} u(y)\, dS(y) = \fint_{e \in S^{n-1}} u(x + re)\, dS(e) = u(x) + \frac{r^2}{2n}\Delta u(x) + O(r^3), \tag{3.13}$$

where we have used the relations

$$\int_{e \in S^{n-1}} \langle Du(x), e\rangle\, dS(e) = 0$$

and

$$\int_{e\in S^{n-1}} e^t Ae \, dS(e) = \frac{1}{n} \cdot |S^{n-1}| \cdot \mathrm{tr}A$$

for any symmetric $n \times n$-matrix A.

Setting $e = -Du/|Du|$ in (3.12), we obtain the relation

$$\inf_{y\in S_r(x)} u(y) \le u(x) - r|Du(x)| + O(r^2). \tag{3.14}$$

(i) Let u be SHE at x. By (3.11), (3.13), and (3.14) we estimate

$$\begin{aligned} 0 &\le r\left[\inf_{y\in S_r(x)} u(y) + r - u(x)\right] + 2n\varepsilon\left[\oint u(y)\, dS(y) - u(x)\right] \\ &\le -r^2|Du(x)| + r^2\varepsilon\Delta u(x) + r^2 + O(r^3). \end{aligned}$$

Dividing by r^2 and letting $r \to 0$ yields $Q(u)(x) \ge 0$.

(ii) Conversely, suppose that we have $Q(u)(x) > 0$. It follows that there is a number $\delta > 0$ such that $Q(u)(x) - \delta > 0$. Plugging $e = -Du/|Du|$ into (3.12) and multiplying by r yields that for all sufficiently small $r > 0$ we have

$$r \inf_{y\in S_r(x)} u(y) \ge ru(x) - r^2|Du(x)| + O(r^3) - r^2\delta/2. \tag{3.15}$$

Hence we have by (3.13), (3.14), and (3.15)

$$\begin{aligned} 0 &< r^2\varepsilon\Delta u(x) - r^2|Du(x)| + r^2 - r^2\delta \\ &\le r\left[\inf_{y\in S_r(x)} u(y) + r - u(x)\right] + 2n\varepsilon\left[\oint u(y)\, dS(y) - u(x)\right] - r^2\delta/2 + O(r^3) \\ &\le r\left[\inf_{y\in S_r(x)} u(y) + r - u(x)\right] + 2n\varepsilon\left[\oint u(y)\, dS(y) - u(x)\right] \end{aligned}$$

for all sufficiently small $r > 0$, implying that u is SHE at x. □

We now adapt the Perron method applied to the (classical) eikonal equation to the *viscous* eikonal equation. For this purpose we require the concept of viscosity solutions to be extended to second order equations. There are different ways to accomplish this; the easiest is to simply replace the C^1-test functions in definition 2.3 by C^2-test functions. We remark, however, that viscosity solutions of second order will not appear in the subsequent chapters, which is why we dispense with further details. For the general theory we refer to [CIL92]. Here we only need the following

Definition 3.4. A function $u : \Omega \to \mathbb{R}$ is said to be a *viscosity subsolution (supersolution)* of

$$Q(u) = 0, \tag{3.16}$$

on Ω, if the following holds: The function u is upper (lower) semicontinuous and for any $\varphi \in C^2(\Omega)$ and any $y \in \Omega$ such that $u - \varphi$ has a local maximum (minimum) at y we have $Q(\varphi)(y) \geq \; (\leq)\; 0$.

The function u is called *viscosity solution of (3.16)*, if it is both a sub- and a supersolution of (3.16). We denote the class of all viscosity subsolutions, supersolutions, solutions of (3.16) on Ω by $\underline{S}(\Omega)$, $\bar{S}(\Omega)$, and $S(\Omega)$, respectively.

Lemma 3.4. *We have* $\mathrm{SHE}(\Omega) \subset \underline{S}(\Omega)$.

Proof. Let $u \in \mathrm{SHE}(\Omega)$, $\varphi \in C^2(\Omega)$, and let $u - \varphi$ attain a local maximum at some point $y \in \Omega$. We may also assume $u(y) = \varphi(y)$, whence it follows that φ touches u by above. According to proposition 3.1, φ is SHE at y, implying $Q(\varphi)(y) \geq 0$ by lemma 3.3. As u is continuous, we have $u \in \underline{S}(\Omega)$. □

For technical reasons we now require the following

Definition 3.5. Let $u : \Omega \to \mathbb{R}$. Define the functions $u^\star, u_\star : \Gamma \to [-\infty, +\infty]$ by

$$u^\star(x) := \lim_{r \to 0} \sup_{y \in B_r(x)} u(y) \quad \text{and} \quad u_\star(x) := \lim_{r \to 0} \inf_{y \in B_r(x)} u(y)$$

We respectively call $u^\star$ and $u_\star$ the *upper and lower semicontinuous envelope* of u.

Now we choose a continuous boundary data function $\phi : \partial\Omega \to \mathbb{R}$ and define the set

$$X := \{u \in \mathrm{SHE}(\Omega) \cap C(\bar{\Omega}) \,|\, u \leq \phi \text{ on } \partial\Omega\}.$$

As before we define the Perron solution $\bar{u}$ to be the pointwise supremum over X, i.e.

$$\bar{u}(x) := \sup_{u \in X} u(x), \quad x \in \bar{\Omega}, \tag{3.17}$$

and prove that it is indeed a viscosity solution of (3.16).

Lemma 3.5. *We have* $\bar{u}(x) < \infty$ *for all* $x \in \Omega$.

Proof. As $\partial\Omega$ is compact and as ϕ is continuous, there is an $m > 0$ such that $\phi \leq m$ on $\partial\Omega$. Consequently, $u \leq m$ on $\partial\Omega$ for all $u \in X$.

Choose a point $x \in \Omega$ and set $d := \sup\{||x - z||, z \in \partial\Omega\} > 0$. Define the function $v \in C^2(\Omega)$ by

$$v(y) := -\alpha||y - x||^2 + K, \quad y \in \Omega,$$

where $\alpha > (2\varepsilon n)^{-1}$ and $K > \alpha d^2 + m$. It follows that $v > m$ on $\partial\Omega$. Furthermore we have

$$Q(v)(y) = -2\varepsilon n\alpha - 2\alpha||y - x|| + 1 \quad \text{for all } y \in \Omega.$$

By the choice of α, we have $Q(v) < 0$ in Ω. Hence v is a classical supersolution of $Q(u) = 0$. Thus it is also a viscosity supersolution. Let $u \in X$. By lemma 3.4 it follows that u is a viscosity subsolution of $Q(u) = 0$. As $u < v$ on $\partial\Omega$ by construction, we invoke the comparison theorem for viscosity solutions (theorem 3.3 in [CIL92]) and conclude that $u \leq v$ in Ω. We obtain $\bar{u} \leq v$ in Ω. The assertion follows. □

Lemma 3.6. *The function $\bar{u}$ is a viscosity supersolution of $Q(u) = 0$.*

Proof. First note that $\bar{u}$ is lower semicontinuous on Ω, as it is the pointwise supremum of continuous functions. Let us suppose that $\bar{u}$ is *not* a viscosity supersolution of $Q(u) = 0$. Then by definition 3.4 there is a point $y \in \Omega$ and a function $\varphi \in C^2(\Omega)$, such that $\bar{u} - \varphi$ has a local minimum at y and $Q(\varphi)(y) > 0$. We may assume $\bar{u}(y) = \varphi(y)$.

Now choose $\delta > 0$ small enough such that $Q(\varphi)(y) - n\varepsilon\delta > 0$ and define the paraboloid

$$\psi(x) := \varphi(y) + D\varphi(y)(x - y) + \frac{1}{2}(x - y)^t D^2\varphi(y)(x - y) - \frac{\delta}{2}||x - y||^2, \quad x \in \Omega.$$

Then we have $\triangle\psi(y) = \triangle\varphi(y) - n\delta$, whence $Q(\psi)(y) > 0$. By construction of ψ and by the properties of φ there is a radius $s > 0$ such that $\bar{B}_s(y) \subset \Omega$ and

$$\psi < \varphi < \bar{u} \tag{3.18}$$

on $B_s(y)\backslash\{y\}$. Continuity implies that there is a radius t with $s \geq t > 0$ such that $Q(\psi) > 0$ on $\bar{B}_t(y)$. It follows that $\psi \in \text{SHE}(\bar{B}_t(y))$ according to lemma 3.3 (ii).

Let $0 < \xi := \min_{\partial B_t(y)}(\varphi - \psi)$. Define $\bar{\psi} := \psi + \frac{\xi}{3}$. Then we have $\bar{\psi}(y) > \bar{u}(y)$ and $\bar{\psi} \in \text{SHE}(\bar{B}_t(y))$.

By (3.18) we have $\bar{u} \geq \psi + \xi$ on $\partial B_t(y)$. Hence for each $x \in \partial B_t(y)$ there is a function $v_x \in X$ such that $v_x(x) > \psi(x) + \frac{2\xi}{3}$. By the continuity of the functions ψ and v_x, $x \in \partial B_t(y)$, and by compactness of $\partial B_t(y)$ there are finitely many $x_i \in \partial B_t(y)$ such that $v := \max v_{x_i} > \psi + \frac{2\xi}{3}$ on $\partial B_t(y)$. It follows $v \in X$ by proposition 3.2.

By reasons of continuity we conclude that for each $x \in \partial B_t(y)$ we have $w \equiv v$ on a neighborhood of x, where w is defined by

$$w := \begin{cases} \max\{\bar{\psi}, v\} & \text{on } \bar{B}_t(y) \\ v & \text{on } \Omega\backslash\bar{B}_t(y). \end{cases}$$

We obtain that w is SHE at each $x \in \partial B_t(y)$. Furthermore, by proposition 3.2 we have $w \in \text{SHE}(B_t(y))$, as $v, \bar{\psi} \in \text{SHE}(B_t(y))$. Clearly, $w \in \text{SHE}(\Omega\backslash\bar{B}_t(y))$. Furthermore we have $w \in C(\Omega)$. Hence it follows $w \in \text{SHE}(\Omega)$ and also $w \in X$, since $v \equiv w$ on $\partial\Omega$. As $w(y) > \bar{u}(y)$, we get a contradiction to the supremal property of $\bar{u}$. □

It remains to show that $\bar{u}$ is a viscosity subsolution. As it is not clear if $\bar{u}$ is upper semicontinuous, we first show that $\bar{u}^\star$ is a viscosity subsolution. For this purpose we invoke an alternative definition of viscosity (sub-, super-) solutions of second order equations, which, however, is equivalent to definition 3.4. The idea is to replace upper and lower test functions by super- and subdifferentials (or *semi-jets*) according to the following

Definition 3.6. Let $u : \Omega \to \mathbb{R}$ and $x \in \Omega$. The *super- (sub-) differential* $D^+u(x)$ $(D^-u(x))$ *of* u *at* x is the set of all pairs (p, M) of a vector $p \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix M such that

$$u(y) \leq \ (\geq)\ u(x) + \langle p, y - x\rangle + \frac{1}{2}(y - x)^t M(y - x) + o(||y - x||^2)$$

as $y \to x$.

The possibility to characterize viscosity solutions by means of semi-jets instead of test functions is expressed by the following

Lemma 3.7. *We have* $u \in \underline{S}(\Omega)$ $(\in \bar{S}(\Omega))$*, if and only if*

$$\tilde{Q}(p, M) := \varepsilon trM - |p| + 1 \geq \ (\leq)\ 0 \text{ for all } x \in \Omega \text{ and all } (p, M) \in D^+u(x)\, (\in D^-u(x)).$$

Proof. Confer [CIL92], for example. □

Lemma 3.8. *Let* $\mathcal{S} \subseteq \mathrm{SHE}(\Omega)$ *be an arbitrary set of SHE-functions and define the function* $u(x) := \sup_{v\in\mathcal{S}} v(x)$ *for all* $x \in \Omega$*. Assume* $u^\star(x) < \infty$ *for all* $x \in \bar{\Omega}$*. Then* $u^\star \in \underline{S}(\Omega)$*.*

Proof. First note that $u^\star$ is upper semicontinuous by definition. Then let $x \in \Omega$ and $(p, M) \in D^+u^\star(x)$. We show $\tilde{Q}(p, M) \geq 0$. To this end define the paraboloid

$$P : \mathbb{R}^n \to \mathbb{R}, \quad P(y) := \langle p, y\rangle + \frac{1}{2}\langle y, My\rangle,$$

and observe that from definition 3.6 it follows that for any $\delta > 0$ there is a radius $r > 0$ such that

$$u^\star(y) \leq u^\star(x) + P(y - x) + \delta||y - x||^2 \text{ for all } y \in \bar{B} := \bar{B}_r(x). \tag{3.19}$$

Next let $(r_l)_{l\in\mathbb{N}}$ be a sequence with $0 < r_l < r$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty} r_l = 0$. We choose a sequence $(x_l)_{l\in\mathbb{N}}$ with $x_l \in B_l := \bar{B}_{r_l}(x)$ and $\sup_{B_l} u - u(x_l) < 1/l$ for all $l \in \mathbb{N}$. Moreover, for each $l \in \mathbb{N}$ by the definition of u there is a function $u_l \in \mathcal{S}$ such that $u(x_l) - u_l(x_l) < 1/l$. It follows

$$\sup_{B_l} u \geq u_l(x_l) > u(x_l) - 1/l > \sup_{B_l} u - 2/l,$$

whence

$$u^\star(x) = \lim_{l\to\infty} \sup_{B_l} u = \lim_{l\to\infty} u_l(x_l). \tag{3.20}$$

As $\bar{B}$ is compact, for any $l \in \mathbb{N}$ the upper semicontinuous function

$$\psi_l : y \mapsto u_l(y) - P(y-x) - 2\delta||y-x||^2 \tag{3.21}$$

attains a global maximum with respect to $\bar{B}$ at some point $y_l \in \bar{B}$. By extracting a subsequence, if necessary, we assume that $y_l \to z$ as $l \to \infty$ for some $z \in \bar{B}$. By $x_l \in \bar{B}$ for all $l \in \mathbb{N}$ and by the maximum property of the points y_l, $l \in \mathbb{N}$, we obtain

$$u_l(x_l) - P(x_l - x) - 2\delta||x_l - x||^2 \leq u_l(y_l) - P(y_l - x) - 2\delta||y_l - x||^2 \tag{3.22}$$

for all $l \in \mathbb{N}$. By the choice of the points x_l we have $\lim_{l\to\infty} x_l = x$, whence we get by taking the limes inferior of (3.22) and invoking (3.20)

$$u^\star(x) = \liminf_{l\to\infty} u_l(x_l) \leq \liminf_{l\to\infty} u_l(y_l) - P(z-x) - 2\delta||z-x||^2. \tag{3.23}$$

By (3.19) we have

$$u^\star(z) - P(z-x) - \delta||z-x||^2 \leq u^\star(x). \tag{3.24}$$

Adding (3.23) and (3.24) yields

$$u^\star(z) \leq \liminf_{l\to\infty} u_l(y_l) - \delta||z-x||^2. \tag{3.25}$$

Moreover, definition 3.5 implies that we have

$$\liminf_{l\to\infty} u_l(y_l) \leq \limsup_{l\to\infty} u_l(y_l) \leq u^\star(z). \tag{3.26}$$

By (3.25) and (3.26) it follows $||z-x|| = 0$, and thus

$$\lim_{l\to\infty} y_l = x. \tag{3.27}$$

Consequently, we may truncate the sequence $(y_l)_{l\in\mathbb{N}}$ such that all y_l lie in the *interior* of $\bar{B}$. Therefore, setting

$$\varphi_\delta(y) := P(y-x) + 2\delta||y-x||^2,$$

we conclude from (3.21) that for all $l \in \mathbb{N}$ the function $\varphi_\delta + u_l(y_l)$ touches u_l by above. As u_l is SHE at y_l, it follows that $\varphi_\delta + u_l(y_l)$ and thus φ_δ is SHE at y_l by proposition 3.1. Lemma 3.3 implies $Q(\varphi_\delta)(y_l) \geq 0$. Now observe that we have

$$Q(\varphi_\delta)(y_l) = \tilde{Q}(p, M) + 4n\varepsilon\delta \quad \text{for all } l \in \mathbb{N},$$

whence $-4n\varepsilon\delta \leq \tilde{Q}(p, M)$. Since the choice of $\delta > 0$ was arbitrary, we conclude

$$\tilde{Q}(p, M) \geq 0.$$

□

Remark 3.2. Some authors (e.g. [Ish87b]) use a slightly more general definition for viscosity solutions: They omit the semicontinuity condition and call a function u viscosity subsolution, if it is locally bounded and if its upper semicontinuous envelope $u^\star$ satisfies the test function condition; an analogous definition is given for viscosity supersolutions. Observe that lemma 3.6 and 3.8 also hold for this alternative definition. Then, these lemmas and lemma 3.5 would immediately imply that $\bar{u}$ as defined in (3.17) is a viscosity solution in this alternative sense. An additional barrier condition could be introduced to guarantee that $\bar{u}$ indeed attains the boundary values ϕ. Such a barrier condition is for instance to assume the existence of a function $v \in X$ with $v \equiv \phi$ on $\partial\Omega$.

Remark 3.3. We emphasize that the basic ideas of the proofs of lemma 3.6 and lemma 3.8 will also be employed in the examination of viscosity solutions on ramified spaces in chapter 5 and chapter 7. Then, however, the correct treatment of what happens at ramification points will be of crucial importance.

Defining viscosity solutions as in definition 2.3 we have

Theorem 3.1. *Assume that there is a function $W \in C(\Omega)$ and a function $w \in \mathrm{SHE}(\Omega)$ such that $v \leq W$ on Ω for all $v \in X$ and*

$$w \equiv W \equiv \phi \quad \text{on } \partial\Omega. \tag{3.28}$$

Then $\bar{u}$ as defined in (3.17) is a viscosity solution of $Q(u) = 0$.

Proof. Setting $\mathcal{S} = X$ in lemma 3.8 and invoking lemma 3.5, we conclude $\bar{u}^\star \in \underline{S}(\Omega)$. Furthermore we have

$$w \equiv w_\star \leq \bar{u}_\star \leq \bar{u} \leq \bar{u}^\star \leq W^\star \equiv W \quad \text{on } \Omega,$$

and by (3.28) we obtain

$$\bar{u} \equiv \bar{u}^\star \quad \text{on } \partial\Omega.$$

As $\bar{u} \in \bar{S}(\Omega)$ by lemma 3.6, the comparison theorem for viscosity solutions (e.g. [CIL92], theorem 3.3) implies

$$\bar{u} \equiv \bar{u}^\star \quad \text{on } \Omega.$$

Hence $\bar{u} \in S(\Omega)$ and the assertion follows. □

CHAPTER 4

Vanishing viscosity on networks

Summary. In this chapter we generally introduce the notion of ramified spaces and, in particular, networks, to which we extend the vanishing viscosity method for first order Hamilton-Jacobi equations. As an important example, we discuss existence and uniqueness of solutions of the viscous eikonal equation on networks satisfying the Kirchhoff condition at the vertices. Finally, we prove the convergence of the vanishing viscosity method for a general class of Hamilton-Jacobi equations on networks.

4.1 Introduction

The concept of ramified spaces has originally been introduced by Gunter Lumer [Lum80] and has later been refined and specified by various authors, e. g., J. v. Below and S. Nicaise [Nic93]. It serves as an appropriate setting for problems of interaction between different media which are governed by partial differential equations on the branches and transition conditions on the ramification spaces. Interaction problems find various applications in physics, chemistry, and biology – confer for example [SH72] and [Cam80]. Also, well-known models based on scalar equations appear in a new light when being embedded into the ramified space setting, such as the description of the behavior of chemical substances by reaction-diffusion equations [vBN96]. As far as the analysis of these models is concerned, the applicability of various mathematical methods does not only depend on the structure of the differential equations on the branches, but also and particularly on the properties of the transition conditions. In fact, the latter have an considerable effect on existence, uniqueness, and regularity of solutions. Possible aspects regarding which the transition conditions may vary are linearity or nonlinearity, being dynamical or static, dissipative or non-dissipative, etc.

Many well-known elliptic and parabolic concepts have been transferred from the classical non-ramified situation to ramified spaces, providing results for boundary or initial value problems, now formulated for families of media with transition conditions. In many cases, the one-dimensional version of a ramified space, the so-called *topological network*, is of major importance, as the core of the problems often evolves fully already in this simple setting.

However, as far as we know, research has been restricted to linear and semilinear elliptic (stationary) equations [Nic88] on the one hand, and, on the other hand, to evolution equations such as (nonlinear) scalar reaction-diffusion equations [vBN96]. Typically, the first demand adaption of Sobolev space methods and the theory of elliptic operators, whereas the latter are usually attacked by means of semigroup theory and related functional analytical tools such as fixed point theorems. However, fully nonlinear stationary problems on ramified spaces have not enjoyed similar attention, and according to our knowledge there has not been done any work in this area so far.

As in this thesis it is our goal to extend the theory of viscosity solutions to ramified spaces, we consider it natural to start with an extension of the method of vanishing viscosity. First we focus on the Dirichlet problem of the viscous eikonal equation on networks and discuss existence and uniqueness of classical solutions for each $\varepsilon > 0$. In doing so, the necessity of introducing an extra condition at transition vertices will become clear. In a second section we show that the ε-family of solutions converges. This convergence result does not only apply to the eikonal equation, but also to a wide class of Hamilton-Jacobi equations of eikonal type, as we will see.

The fact that the extended method of vanishing viscosity converges provides guidelines for the extension of the theory of viscosity solutions, as the two concepts should coincide in the same way as they do in the non-ramified case. In particular, observing the behavior of the converging family at transition vertices should give hints to how to formulate a correct transition condition for viscosity solutions. We elaborate on this in chapter 5.

4.2 Ramified spaces

We start with the general definition of ramified spaces originally given by Lumer [Lum80].

Definition 4.1. Let $\Omega^\star$ be a non-empty, separable, locally compact space with a countable topological basis. Let $\mathcal{L} = \{\Omega_j\}_{j\in J}$ be a countable family of non-empty open subsets Ω_j of $\Omega^\star$. Furthermore let $N_E^\star$ be a closed (possibly empty) subset of the set $N^\star := \Omega^\star \setminus \cup_{j\in J}\,\Omega_j$ with the property that it contains each point of $N^\star$ which is contained in the boundary of exactly one Ω_j, $j \in J$.

We call $\Omega := \Omega^\star \setminus N_E^\star$ a *ramified space (induced by the triple* $(\Omega^\star, \mathcal{L}, N_E^\star)$*)*, whenever we have

$$(i) \qquad \bar{\Omega}_j \cap \bar{\Omega}_k \subseteq \partial\Omega_j \cap \partial\Omega_k \quad \text{for all } j,k \in J,\ j \neq k$$

$$(ii) \qquad \Omega^\star = \cup_{j\in J}\bar{\Omega}_j$$
$$(iii) \qquad \{\Omega_j\}_{j\in J} \text{ is locally finite in } \Omega^\star$$
$$(iv) \qquad \Omega \text{ is connected.}$$

$N_R := N^\star \backslash N_E^\star$ is then called *ramification space* of Ω.

We now introduce topological networks as important instances of ramified spaces. Higher dimensional ramified spaces and other examples will be discussed in chapter 7.

4.3 Graphs, topological graphs, and networks

In this chapter we distinguish between (abstract) graphs, topological graphs, and networks. We recall the definition of abstract graphs.

Definition 4.2. An (abstract) *graph* G is a pairing $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of *vertices* and *edges*, respectively. An edge $e \in E$ is an unordered pair $\{v_1, v_2\}$ of vertices $v_1, v_2 \in V$; we write $e = v_1v_2$. A *path* (with *endpoints* v_1 and v_n) in G is a formal sequence $v_1v_2 \ldots v_n$ of vertices $v_1, \ldots, v_n \in V$, $n \in \mathbb{N}$, such that $v_iv_{i+1} \in E$ for all $i = 1, \ldots, n-1$.

We also provide the basic graph theoretical notions we will require in the sequel.

Definition 4.3. Let $G = G(V, E)$ be a graph.

- (i) We say that two vertices $v, w \in V$ are *adjacent*, whenever $vw \in E$. We write $v \operatorname{adj} w$.
- (ii) We say that a vertex $v \in V$ and an edge $e \in E$ are *incident*, whenever there is a vertex $w \in V$ such that $vw = e$. We write $v \operatorname{inc} e$.
- (iii) G is called *connected*, if there is a path with endpoints v and w for each pair of vertices $v, w \in V$.
- (iv) For each vertex $v \in V$ we call the quantity $\deg v := |\{e \in E \mid v \operatorname{inc} e\}|$ the *degree* of v.

In the sequel we assume each graph to be non-empty, finite, simple, and free of loops, which is expressed by the conditions

$$(i) \qquad 0 < |V| < \infty$$
$$(ii) \qquad |\{e \in E \mid e = vw\}| = 1 \quad \text{for all } v, w \in V,\ v \neq w$$
$$(iii) \qquad v \neq w \text{ for all } e = vw \in E.$$

For further graph theoretical terminology we refer to [Har69].

Next we introduce the notion of a topological graph.

Definition 4.4. Let $V = \{v_i, i \in I\}$ be a finite collection of pairwise different points in $\mathbb{R}^n$. Furthermore, let $\{\pi_j, j \in J\}$ be a finite collection of continuous, non-self-intersecting curves in $\mathbb{R}^n$ given by

$$\pi_j : [0, l_j] \to \mathbb{R}^n, \quad l_j > 0,\, j \in J.$$

We set $e_j := \pi_j((0, l_j))$, $\bar{e}_j := \pi_j([0, l_j])$, and $E := \{e_j, j \in J\}$. Assume furthermore the following conditions to be satisfied:

$$\begin{array}{ll} (i) & \pi_j(0), \pi_j(1) \in V \text{ for all } j \in J \\ (ii) & |\bar{e}_j \cap V| = 2 \text{ for all } j \in J \\ (iii) & \bar{e}_j \cap \bar{e}_k \subset V \text{ and } |\bar{e}_j \cap \bar{e}_k| \leq 1 \text{ for all } j, k \in J,\, j \neq k. \end{array}$$

Then $G := (V, E)$ is called a *(finite) topological graph.*

Obviously we can consider G not only as a subset of $\mathbb{R}^n$, but also as an abstract graph $G := (V, E)$ according to definition 4.2. As long as confusions are ruled out, we will switch between both interpretations without explicitly mentioning.

Observe that the parametrizations π_j induce an orientation on the edges, which can be expressed by the *signed incidence matrix*

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} := \begin{cases} 1 & \text{if} \quad v_i \operatorname{inc} e_j \quad \text{and} \quad \pi_j(0) = v_i \\ -1 & \text{if} \quad v_i \operatorname{inc} e_j \quad \text{and} \quad \pi_j(l_j) = v_i \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Finally, we introduce the notion of a topological network.

Definition 4.5. Let $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 1$. Let $G = (V, E)$ be a connected topological graph in $\mathbb{R}^n$, and assume $\pi_j \in C^k([0, l_j]; \mathbb{R}^n)$ for all $j \in J$. Then the union

$$\Gamma := \bigcup_{j \in J} \bar{e}_j \subset \mathbb{R}^n$$

is called *the (topological)* c^k*-network* Γ *belonging to* G.

Observe that a topological c^k-network Γ is a compact topological subspace of $\mathbb{R}^n$. Moreover, as the edge parametrizations π_j, $j \in J$, are at least C^1, it is clear that the topology induced by the path metric d on Γ is equivalent to the subspace topology.

In the sequel let Γ always denote a topological c^∞-network with underlying topological graph $G = (V, E)$, $V = \{v_i, i \in I\}$, $E = \{e_j, j \in J\}$.

4.4 Boundary value problems on topological networks

We want to study boundary value problems on Γ. For this purpose we first specify what we mean by the *boundary* $\partial\Gamma$ of Γ. In fact, we single out a non-empty index subset $I_B \subset I$ and define $\partial\Gamma := \{v_i, i \in I_B\} \subseteq V$ to be the set of *boundary vertices*. In contrast, we set $I_T := I \backslash I_B$ and call $\{v_i, i \in I_T\}$ the set of *transition vertices*.

In the case of an Dirichlet interaction problem, the value of solutions is prescribed at boundary vertices, whereas at transition vertices the restrictions of the solutions to the different incident edges are put into relation by a transition condition. Therefore transition conditions at vertices with only one incident edge do not make sense, and it is reasonable to demand $i \in I_B$ for each $i \in I$ with $\deg(v_i) = 1$.

For any function $u : \Gamma \to \mathbb{R}$ and each $j \in J$ we from now on denote by u^j the restriction of u to $\bar{e}_j$, i. e.,

$$u^j := u \circ \pi_j : [0, l_j] \to \mathbb{R}.$$

The C^∞-regularity of the parametrizations π_j of Γ allows to differentiate along the edges, where differentiation along e_j will be denoted by ∂_j, $j \in J$, that is we define

$$\partial_j^\alpha u(x) = \partial_j^\alpha u^j(\pi_j^{-1}(x)) := \left(\frac{\partial}{\partial x}\right)^\alpha u^j(\pi_j^{-1}(x))$$

for all $x \in e_j$ and all $\alpha \in \mathbb{N}$. At a given vertex v_i, $i \in I$, we furthermore define

$$\mathrm{Inc}_i := \{j \in J \,|\, v_i \,\mathrm{inc}\, e_j\}$$

and

$$\partial_j u(v_i) = \partial_j u^j(\pi_j^{-1}(v_i)) := \frac{\partial}{\partial x} u^j(\pi_j^{-1}(v_i))$$

for $j \in \mathrm{Inc}_i$.

Let us introduce the function spaces on Γ we are mainly going to work with.

Definition 4.6. Let $u : \Gamma \to \mathbb{R}$.

We call u *continuous*, if u is continuous with respect to the subspace topology of Γ induced by $\mathbb{R}^n$. We write $u \in C(\Gamma)$.

We call u k *times differentiable*, $k \geq 1$, if $u^j \in C^k([0, l_j])$ for all $j \in J$. We write $u \in C^k(\Gamma)$.

Remark. The sufficient condition that a collection of continuous functions

$$u^j : [0, l_j] \to \mathbb{R}, \quad j \in J,$$

constitutes a function $u \in C(\Gamma)$ is given by

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{whenever} \quad v_i \,\mathrm{inc}\, e_j \quad \text{and} \quad v_i \,\mathrm{inc}\, e_k.$$

In this sense, continuity can be regarded as a *transition condition*. In fact, it is the basic transition condition we require in the sequel.

As has already been announced, the *Kirchhoff condition* known from electrical circuits theory plays a fundamental role in the following considerations. In a way, C^1-differentiability of a function along the edges means that left- and right-sided derivatives are related in a fashion that cusps are ruled out. In other words, the slopes in outward (or inward) direction with respect to each given point add up to zero. At vertices, this condition naturally generalizes to the Kirchhoff condition, the simplest form of a linear transition condition.

Definition 4.7. Let $u \in C^1(\Gamma)$, $i \in I$, and $j \in \mathrm{Inc}_i$. We then set

$$s_{ij}(u) := a_{ij}\partial_j u(v_i),$$

where (a_{ij}) as defined in (4.1) is the incidence matrix of the topological graph belonging to Γ.

Furthermore we define the linear mapping $S_i : C^1(\Gamma) \to \mathbb{R}$ by

$$S_i(u) := \sum_{j\in\mathrm{Inc}_i} s_{ij}(u).$$

We say that u *satisfies the Kirchhoff condition* at v_i, $i \in I$, if $S_i(u) = 0$. We say that u *satisfies the Kirchhoff condition* and write $u \in C^1_K(\Gamma)$, if u satisfies the Kirchhoff condition at all $i \in I_T$. For $k > 1$ we moreover set

$$C^k_K(\Gamma) := C^k(\Gamma) \cap C^1_K(\Gamma).$$

4.5 Maximum and comparison principles for Kirchhoff functions

If we ask the Kirchhoff condition to be satisfied at the transition vertices, several standard maximum principles for linear and semilinear equations can be carried over to networks. The following lemma states a maximum principle for certain linear operators on networks.

Lemma 4.1. *Let $L := (L^j)_{j\in J}$ be a collection of linear differential operators given by*

$$L^j(f) := a^j\partial_j^2 f + b^j\partial_j f, \quad f \in C^2((0, l_j)),$$

with coefficient functions $a^j, b^j : (0, l_j) \to \mathbb{R}$, $j \in J$. Assume L to be uniformly elliptic in the sense that there are constants $\lambda > 0$, $\Lambda > 0$ such that $\lambda \le a^j \le \Lambda$ on $(0, l_j)$ for all $j \in J$. Furthermore assume that there is a constant $C(\lambda)$ such that $|b^j| \le C(\lambda)$ on $(0, l_j)$ for all $j \in J$.

Let $u \in C^2(\Gamma)$ such that

(i) $L^j(u^j) \geq 0$ *(*$L^j(u^j) \leq 0$*) on* $(0, l_j)$ *for all* $j \in J$ *and*
(ii) $S_i(u) \geq 0$ *(*$S_i(u) \leq 0$*) for all* $i \in I_T$.

Then we have

$$\max_{\partial\Gamma} u = \max_{\Gamma} u \quad (\min_{\partial\Gamma} u = \min_{\Gamma} u), \tag{4.2}$$

where $\partial\Gamma := \{v_i \,|\, i \in I_B\}$.

Remark 4.1. In particular, (4.2) holds for all $u \in C^2_K(\Gamma)$ with $L^j u^j \leq 0$ for all $j \in J$.

Remark 4.2. Observe that the coefficient functions do not need to satisfy a transition condition at the transition vertices.

For the proof of lemma 4.1 we need the following proposition.

Proposition 4.1. *Let L be as in lemma 4.1. Then there is a function $f \in C^2(\Gamma)$ satisfying*

$$\begin{cases} L^j(f^j) > 0 & \text{on } (0, l_j) \text{ for all } j \in J, \\ S_i(f) > 0 & \text{for all } i \in I_T. \end{cases}$$

Proof. (of proposition 4.1). Let $\gamma > 0$ such that we have $\lambda\gamma^2 - C(\lambda)\gamma > 0$. Define the set

$$M := \{\xi \in \mathbb{R}^I \text{ with } \xi_i \neq \xi_j \text{ for all } i, j \in I \text{ with } v_i \operatorname{adj} v_j\}. \tag{4.3}$$

Let $\xi \in M$. Fix $k \in J$ and let $i, j \in I$ such that $e_k = v_i v_j$. Since $\xi_i \neq \xi_j$, we can find unique numbers $\sigma, \eta, c \in \mathbb{R}$ with $|\sigma| = 1$ such that the function

$$u^k : [0, l_k] \to \mathbb{R}, \quad u^k(x) := e^{\sigma\gamma(x-\eta)} + c$$

satisfies $u^k(\pi_k^{-1}(v_i)) = \xi_i$ and $u^k(\pi_k^{-1}(v_j)) = \xi_j$.

We then compute

$$L^k(u^k)(x) = (a^k(x)\lambda^2 + b^k(x)\sigma\lambda)e^{\sigma\lambda(x-\eta)} \quad \text{for all } x \in (0, l_k).$$

By the choice of γ it then follows

$$L^k(u^k)(x) > (\lambda\gamma^2 - C(\lambda)\gamma)e^{\sigma\lambda(x-\eta)} > 0 \quad \text{for all } x \in (0, l_k).$$

If we repeat this for all other choices $k \in J$, we obtain an injective mapping

$$\Phi : M \to D \quad \text{with} \quad D := \{u \in C^2(\Gamma) \text{ with } L^j(u^j) > 0 \text{ on } (0, l_j) \text{ for all } j \in J\},$$

satisfying $\Phi(\xi)(v_i) = \xi_i$ for all $i \in I$ and all $\xi \in M$.

Now we show that we can choose $\xi = (\xi_i) \in M$ such that $S_i(\Phi(\xi)) > 0$ is satisfied for all $i \in I_T$. To this end observe that for $i \in I_T$ the mapping

$$T_i : M \to \mathbb{R}, \quad T_i := S_i \circ \Phi, \tag{4.4}$$

is a continuous, strictly decreasing and unbounded function in the component ξ_i. Furthermore observe that T_i is continuous, unbounded, and strictly increasing in each component ξ_j, $j \in A_i$, where

$$A_i := \{j \in I \,|\, v_j \text{ adj } v_i\}.$$

Finally, T_i is independent of the component ξ_j for any $j \in I \backslash (\{i\} \cup A_i)$.

We construct $\xi \in M$ such that $T_i(\xi) > 0$ for all $i \in I_T$. Let dist $: I \times I \to \mathbb{N}$ be the metric given by the smallest number $\text{dist}(i,j)$ of edges a path connecting v_i and v_j can consist of. It induces the partition $I_l := \{i \in I \,|\, \text{dist}(i, I_B) = l\}$, $l \in \mathbb{N}_0$. Observe that $I_0 = I_B$. Let $m := \max\{l \in \mathbb{N}_0 \,|\, I_l \neq \emptyset\}$. Furthermore note that for $i \in I_l$, $1 \leq l \leq m$, there is by construction at least one $j \in I_{l-1}$ such that $j \in A_i$. Moreover, T_i is constant in the component ξ_j for all $j \in I \backslash (I_{l-1} \cup I_l \cup I_{l+1})$.

We first choose pairwise different numbers $\xi_j \in \mathbb{R}$ for all $j \in I$. Let $i \in I_m$. Due to the fact that T_i is unbounded, continuous, and strictly increasing in each ξ_j, $j \in A_i$, and by the fact that $I_{m-1} \cap A_i \neq \emptyset$ for each $i \in I_m$, we may increase the value of the components ξ_j, $j \in I_{m-1}$, such that we have $T_i(\xi) > 0$ for all $i \in I_m$ and such that all ξ_j, $j \in J$, remain pairwise different. Analogously we can increase ξ_j, $j \in I_{m-2}$, such that $T_i(\xi) > 0$ for all $i \in I_{m-1}$ and such that all ξ_j, $j \in J$, remain pairwise different. For $k = 3, \ldots, m$ we continue this procedure by sufficiently increasing ξ_j, $j \in I_{m-k}$, in order to ensure that $T_i(\xi) > 0$ for all $i \in I_{m-k+1}$, ending up with a choice for $\xi \in M$ such that $T_i(\xi) > 0$ for all $i \in \cup_{l=1}^m I_l = I_T$. Setting $f := \Phi(\xi)$ completes the proof. □

Proof. (of lemma 4.1). First assume $L^j(u^j) > 0$ on $(0, l_j)$ for all $j \in J$ and $S_i(u) > 0$ for all $i \in I_T$. Suppose that there be some $j \in J$ and some $x_0 \in (0, l_j)$ such that u^j attains a local maximum at x_0. It follows $\partial_j u^j(x_0) = 0$ and $a^j \partial_j^2 u^j(x_0) \leq 0$, which contradicts the assumption $L^j(u^j)(x_0) > 0$. Now assume that there be some $i \in I_T$ such that u attains a local maximum at v_i. Then $a_{ij} \partial_j u^j(\pi_j^{-1}(v_i)) \leq 0$ for all $j \in \text{Inc}_j$, whence we have $S_i(u)(v_i) \leq 0$, a contradiction to our assumption $S_i(u) > 0$ for all $i \in I_T$.

Now assume $L^j(u^j) \geq 0$ on $(0, l_j)$ for all $j \in J$ as well as $S_i(u) \geq 0$ for all $i \in I_T$. For $\delta > 0$ we then set $w_\delta := u + \delta f \in C^2(\Gamma)$, where f is the function constructed in proposition 4.1. By linearity of L^j, $j \in J$, and S_i, $i \in I_T$, we have $L^j(w_\delta^j) > 0$ on $(0, l_j)$ for all $j \in J$ as well as $S_i(w_\delta) > 0$ for all $i \in I_T$. By the arguments above it follows that $\max_\Gamma(u + \delta f) = \max_{\partial\Gamma}(u + \delta f)$, whence by passing to the limit $\delta \to 0$ we obtain $\max_\Gamma u = \max_{\partial\Gamma} u$.

Finally, assume $L^j(u^j) \leq 0$ on $(0, l_j)$ for all $j \in J$ as well as $S_i(u) \leq 0$ for all $i \in I_T$. By linearity it follows $L^j(-u) \geq 0$ on $(0, l_j)$ for all $j \in J$ as well as $S_i(-u) \geq 0$ for all $i \in I_T$. We obtain $\min_\Gamma u = \max_\Gamma -u = \max_{\partial\Gamma} -u = \min_{\partial\Gamma} u$. □

From lemma 4.1 we now derive a comparison result for certain semilinear operators on networks.

Lemma 4.2. *Let $Q = (Q_j)_{j\in J}$ be a collection of semilinear operators given by*

$$Q^j(f)(x) := a^j(x)\partial_j^2 f(x) + b^j(\partial_j f^j(x), f^j(x), x), \quad x \in (0, l_j), f \in C^2((0, l_j)),$$

with coefficient functions $a^j : (0, l_j) \to \mathbb{R}$ and $b^j : \mathbb{R} \times \mathbb{R} \times (0, l_j) \to \mathbb{R}$, $j \in J$. For all $j \in J$ assume furthermore that $b^j(\cdot, z, x) \in C^1(\mathbb{R})$ for all $(z, x) \in \mathbb{R} \times (0, l_j)$, and that $b^j(p, \cdot, x)$ is non-increasing for all $(p, x) \in \mathbb{R} \times (0, l_j)$.

Let $u, v \in C^2(\Gamma)$ satisfy

$$Q^j(u^j) \geq Q^j(v^j)$$

for all $j \in J$ as well as $S_i(u) \geq S_i(v)$ for all $i \in I_T$. Furthermore suppose $u \leq v$ on $\partial\Gamma$. Then we have $u \leq v$ on Γ.

Proof. For all $j \in J$ and all $x \in (0, l_j)$ we have

$$\begin{aligned} & Q^j(u^j)(x) - Q^j(v^j)(x) \\ = \; & a^j(x)\partial_j(u^j(x) - v^j(x)) + b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), v^j(x), x) \\ = \; & a^j(x)\partial_j(u^j(x) - v^j(x)) + b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), u^j(x), x) \\ & + b^j(\partial_j v^j(x), u^j(x), x) - b^j(\partial_j v^j(x), v^j(x), x) \geq 0. \end{aligned}$$

As $b^j(p, \cdot, x)$ is non-increasing, it follows

$$a^j(x)\partial_j(u^j(x) - v^j(x)) + b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), u^j(x), x) \geq 0, \tag{4.5}$$

for all $x \in A := \{u > v\} \subset \Gamma$. Since $b^j(\cdot, z, x)$ is continuously differentiable, for each $j \in J$ there is a locally bounded function $\tilde{b}^j : (0, l_j) \to \mathbb{R}$, such that

$$b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), u^j(x), x) = \tilde{b}^j(x)\partial_j(u^j(x) - v^j(x))$$

by the mean value theorem. Defining $w \in C^2(\Gamma)$ by $w := u - v$, from (4.5) we derive

$$L^j(w^j) := a^j\partial_j^2 w^j + \tilde{b}^j\partial_j w^k \geq 0 \quad \text{on } (0, l_j) \cap A, \, j \in J.$$

By linearity of S_i we furthermore have $S_i(w) \geq 0$ for all $i \in I_T$. By the proof of lemma 4.1 it follows that w cannot attain a local maximum on the (open) set A, and as we have $A \cap \partial\Gamma = \emptyset$, it follows $A = \emptyset$ and thus $u \leq v$ on Γ. □

4.6 The viscous eikonal equation on networks

Having introduced the framework of topological networks and basic maximum principles, let us now return to the track we have outlined above - the idea of applying the method of vanishing viscosity to nonlinear first order boundary value problems on topological networks. In the present section, before considering the general case, we treat the exemplary case of the viscous eikonal equation and show existence and uniqueness of solutions satisfying the Kirchhoff condition. The proofs are elementary, but serve as an illustration of the importance of the Kirchhoff condition in order to make solutions unique. Later it will turn out that the Kirchhoff condition is sufficient to ensure the convergence of the vanishing viscosity method.

Accordingly, we are interested in existence and uniqueness of solutions $u \in C^2_K(\Gamma)$ of the following boundary value problem on Γ.

$$\begin{cases} \varepsilon\partial_j^2 u - (\partial_j u)^2 + 1 = 0 & \text{on } e_j \text{ for all } j \in J, \\ u(v_i) = g_i & \text{for all } i \in I_B, \quad \varepsilon > 0, \end{cases} \tag{4.6}$$

where $g_i \in \mathbb{R}$, $i \in I_B$.

Theorem 4.1. *There is a unique function $u \in C^2_K(\Gamma)$ solving problem (4.6).*

The proof is given by the following collection of results.

Lemma 4.3. *Let $u, v \in C^2(\Gamma)$ be solutions of boundary value problem (4.6). Then $u \equiv v$ on Γ.*

Proof. Observe that $Q = (Q^j)_{j\in J}$ with

$$Q^j(u^j) := \varepsilon\partial_j^2 u^j - (\partial_j u^j)^2 + 1, \quad j \in J,$$

satisfies the conditions of lemma 4.2. Then the assertion is an immediate consequence of this lemma. □

Proposition 4.2. *Let $l \in \mathbb{R}$, $l > 0$. Then there is an injective mapping*

$$\Psi : \mathbb{R}^2 \to C^2([-l, l]),$$

such that for each pair $(s, t) \in \mathbb{R}^2$ we have $u(-l) = s$, $u(l) = t$, and

$$\varepsilon u'' - (u')^2 + 1 = 0 \quad on\ (-l, l),$$

where $u := \Psi(s, t)$.

Furthermore, define the functions

$$\psi : \mathbb{R}^2 \to \mathbb{R}, \quad \psi(s, t) := \frac{\partial}{\partial x}\Psi(s, t)(-l)$$

and

$$\pi : \mathbb{R}^2 \to \mathbb{R}, \quad \pi(s, t) := \frac{\partial}{\partial x}\Psi(s, t)(l).$$

Then for all $s \in \mathbb{R}$ the functions $\psi(s, \cdot) : \mathbb{R} \to \mathbb{R}$ and $\pi(s, \cdot) : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing. Furthermore for all $t \in \mathbb{R}$ the functions $\psi(\cdot, t) : \mathbb{R} \to \mathbb{R}$ and $\pi(\cdot, t) : \mathbb{R} \to \mathbb{R}$ are continuous and strictly decreasing.

Proof. Using the transformation method from theorem 2.1 we find that Ψ is given by

$$\Psi(s,t) = -\varepsilon \log(w(x)+1),$$

where

$$w(x) := \frac{1}{4}\left(\frac{t-s}{\sinh(\varepsilon^{-1}l)} + \frac{t+s}{\cosh(\varepsilon^{-1}l)}\right)e^{-\frac{1}{\varepsilon}x} + \frac{1}{4}\left(-\frac{t-s}{\sinh(\varepsilon^{-1}l)} + \frac{t+s}{\cosh(\varepsilon^{-1}l)}\right)e^{\frac{1}{\varepsilon}x}$$

The assertion follows from straightforward calculation. □

Proposition 4.3. *There is a function $f \in C^2(\Gamma)$ such that there is a vector $(a_j)_{j\in J}$ with $a_j \neq 0$ for all $j \in J$ and such that*

$$\begin{cases} \partial_j f \equiv a_j & \text{on } e_j \text{ for all } j \in J, \\ S_i(f) > 0 & \text{for each } i \in I_T. \end{cases}$$

Proof. We proceed similarly to the proof of proposition 4.1. Define the set M as in (4.3) and observe that there is a canonical injective mapping

$$\Phi : M \to D := \{u \in C^2(\Gamma) \,|\, \exists\, (a_j)_{j\in J} \text{ such that } a_j \neq 0 \text{ and } \partial_j f \equiv a_j \text{ on } e_j,\, j \in J.\}$$

with $\Phi(\xi)(v_i) = \xi_i$, $i \in I$. It then suffices to show that there is a $\xi \in M$ such that $S_i(\Phi(\xi)) > 0$ for all $i \in I_T$. For this purpose define for all $i \in I_T$ the functions T_i as in (4.4) and observe that they have the same properties as described in the proof of proposition 4.1. We then proceed exactly as in this proof to construct ξ. □

Lemma 4.4. *There exists a solution $u \in C^2_K(\Gamma)$ for the boundary value problem 4.6.*

Proof. By proposition 4.2 there is an injective mapping

$$\Phi : \mathbb{R}^I \to C^2(\Gamma)$$

such that for each $\xi = (\xi_i)_{i\in I}$ and $u := \Phi(\xi)$ we have $u(v_i) = \xi_i$ for all $i \in I$ as well as

$$\varepsilon \partial_j^2 u - (\partial_j u)^2 + 1 = 0 \quad \text{on } e_j \text{ for all } j \in J.$$

Fix $i \in I_T$. Then the mapping

$$T_i : \mathbb{R}^I \to \mathbb{R}, \quad T_i := S_i \circ \Phi = \sum_{j\in \mathrm{Inc}_i} (s_{ij} \circ \Phi), \tag{4.7}$$

is continuous, as the mappings $s_{ij} \circ \Phi : \mathbb{R}^I \to \mathbb{R}$, $j \in \mathrm{Inc}_i$, are continuous by proposition 4.2 (cf. definition 4.7). It follows that the set

$$C := \{\xi \in \mathbb{R}^p \,|\, T_i(\xi) \geq 0 \;\forall\, i \in I_T,\; \xi_i \leq g_i \;\forall\, i \in I_B\}$$

is closed. Furthermore C is non-empty, as we clearly have $\xi^0 = (\xi_i^0)_{i \in I} \in C$, where $\xi_i^0 := \min_{k \in I_B} g_k$ for all $i \in I$. We show that

$$\sup_{\xi \in C} \max_{i \in I} |\xi_i| < \infty. \tag{4.8}$$

For this purpose let f and $(a_j)_{j \in J}$ be the function and the corresponding vector as constructed in proposition 4.3 and let $a := \min_{j \in J} |a_j| > 0$. Furthermore set $g := \max_{i \in I_B} g_i$. For the function

$$\tilde{f} := -1/a \cdot (f - \min_{i \in I_B} f(v_i)) + g \in C^2(\Gamma)$$

we then have

$$Q^j(\tilde{f}^j) := \varepsilon \partial_j^2 \tilde{f}^j + 1 - (\partial_j \tilde{f}^j)^2 = 1 - (a^j/a)^2 \leq 0$$

for all $j \in J$ as well as $S_i(\tilde{f}) \leq 0$ for all $i \in I_T$ by proposition 4.3. Furthermore we have $\tilde{f}(v_i) \geq g_i$ for all $i \in I_B$. Now let $\xi \in C$ and $u := \Phi(\xi)$. By the properties of u and by lemma 4.2 we conclude $u \leq \tilde{f}$ on Γ. Hence $\xi_i \leq \tilde{f}(v_i)$ for all $i \in I$ and (4.8) is proved.

Define $\tilde{\xi} = (\tilde{\xi}_i)_{i \in I}$ by $\tilde{\xi}_i := \sup_{\xi \in C} \xi_i$. We have $\tilde{\xi} \in \mathbb{R}^I$, since $\tilde{\xi}_i < \infty$ for each $i \in I$ by (4.8). We show $\tilde{\xi} \in C$. As C is closed, it suffices to show that there is a sequence $(\xi^n)_{n \in \mathbb{N}}$ in C converging to $\tilde{\xi}$. For this purpose choose for each $i \in I$ a sequence $(\xi^{i;n})_{n \in \mathbb{N}}$ in C such that $\lim_{n \in \mathbb{N}} \xi_i^{i;n} \to \tilde{\xi}_i$. Then for each $n \in \mathbb{N}$ define $\xi^n \in \mathbb{R}^I$ to be the componentwise maximum of the vectors $\xi^{i;n}$, $i \in I$. It follows $\lim_{n \to \infty} \xi^n = \tilde{\xi}$, whence it remains to show that $\xi^n \in C$ for all $n \in \mathbb{N}$. To this end we fix two vectors $\xi, \bar{\xi} \in C$ and verify that their componentwise maximum ζ is contained in C. Fix $i \in I_T$ and assume without restriction that $\zeta_i = \xi_i$. Observe that for all $j \in A_i := \{j \in I \mid v_j \operatorname{adj} v_i\}$ the function $s_{ij} \circ \Phi : \mathbb{R}^I \to \mathbb{R}$ is strictly increasing in the component ξ_j by proposition 4.2. Hence the function $\xi \mapsto T_i(\xi)$ (as defined in (4.7)) is strictly increasing in each component ξ_j, $j \in A_i$. Therefore, as $\xi_i = \zeta_i$ and $\xi_j \leq \zeta_j$ for all $j \in A_i$, it follows $T_i(\zeta) \geq T_i(\xi) \geq 0$. As $i \in I_T$ has been chosen arbitrarily, we obtain $\zeta \in C$.

Now suppose that there is some $i \in I_T$ with $T_i(\tilde{\xi}) > 0$. By continuity of T_i there is a $\xi \in \mathbb{R}^I$ with $\xi_j = \tilde{\xi}_j$ for all $j \in I \backslash \{i\}$ and $\xi_i > \tilde{\xi}_i$, such that $T_i(\xi) > 0$. Furthermore, for all $j \in I_T \cap A_i$ we have $T_j(\xi) \geq 0$, since T_j is strictly increasing in ξ_i. Moreover, $T_j(\tilde{\xi}) = T_j(\xi)$ for all $j \in I_T \backslash A_i$. It follows $\xi \in C$, a contradiction to the definition of $\tilde{\xi}$.

One derives a similar contradiction in the case that there is an $i \in I_B$ with $\tilde{\xi}_i < g_i$. Thus it follows $T_i(\tilde{\xi}) = 0$ for all $i \in I_T$ as well as $\tilde{\xi}_i = g_i$ for all $i \in I_B$. Consequently, $u := \Phi(\tilde{\xi})$ solves the boundary value problem 4.6. □

Remark 4.3. Theorem 4.1 can alternatively be attacked by means of the transformation

$$w_a(x) := \exp(-\varepsilon^{-1} u_\varepsilon) - 1$$

with $a := 1/\varepsilon$ (confer the methods in chapter 2), which preserves the Kirchhoff condition, as can be easily seen. The resulting linear problem can then be treated by the well-known methods for linear interaction problems.

4.7 Convergence of vanishing viscosity on networks

Denoting the unique solution of boundary value problem (4.6) by u_ε, we now ask whether and in which sense the functions u_ε converge to a limit function as $\varepsilon \to 0$, and which properties this limit function will possess. In the special case $g_i = 0$ for all $i \in I_B$, we expect them to converge to the distance function $\boldsymbol{d}$ on Γ, analogously to the non-ramified case treated in chapter 2.

In the present section we will extend our point of view from the special case of the eikonal equation towards a general class of first order Hamilton-Jacobi equations *of eikonal type*. Namely we assume that for any $\varepsilon > 0$ the function $u_\varepsilon \in C^2_K(\Gamma)$ be a solution of the boundary value problem

$$\begin{cases} \varepsilon \partial_j^2 u_\varepsilon^j(x) - H^j(\partial_j u_\varepsilon^j(x), u_\varepsilon^j(x), x) = 0 & \text{for all } x \in (0, l_j) \text{ and } j \in J \\ u_\varepsilon(v_i) = g_i & \text{for all } i \in I_B. \end{cases} \tag{4.9}$$

Here the collection $H = (H^j)_{j \in J}$ with $H^j : \mathbb{R} \times \mathbb{R} \times [0, l_j] \to \mathbb{R}$ is called the *Hamiltonian*.

Definition 4.8. A Hamiltonian $H = (H^j)_{j \in J}$ is said to be of *eikonal type*, if it satisfies the following conditions.

$$\begin{array}{ll} (i) & H^j(0, z, x) < 0 \text{ for all } (z, x) \in \mathbb{R} \times [0, l_j],\ j \in J \\ (ii) & H^j \in C^2(\mathbb{R} \times \mathbb{R} \times [0, l_j]),\ j \in J \\ (iii) & H^j(p, \cdot, x) \text{ is non-decreasing for all } (p, x) \in \mathbb{R} \times [0, l_j],\ j \in J \\ (iv) & H^j(p, z, x) \to \infty \text{ as } |p| \to \infty \text{ for all } (z, x) \in \mathbb{R} \times [0, l_j],\ j \in J \\ (v) & H^j(p, z, x) \text{ is strictly convex in } p \text{ for all fixed } (z, x) \in \mathbb{R} \times [0, l_j],\ j \in J \\ (vi) & H^j(p, z, \pi_j^{-1}(v_i)) = H^j(-p, z, \pi_j^{-1}(v_i))\ \forall\, i \in I,\ j \in \mathrm{Inc}_i,\ (p, z) \in \mathbb{R} \times \mathbb{R} \\ (vii) & H^j(p, z, \pi_j^{-1}(v_i)) = H^k(p, z, \pi_k^{-1}(v_i))\ \forall\, i \in I,\ j, k \in \mathrm{Inc}_i,\ (p, z) \in \mathbb{R} \times \mathbb{R}. \end{array} \tag{4.10}$$

Remark 4.4. Conditions (4.10) (i) - (v) are the basic properties prescribed by Kružkov in [Kru75]. The additional conditions (vi) and (vii) concern the compatibility of the H^j at the vertices, which are continuity (condition (vii)) and independence of the orientation of the incident edges (condition (vi)).

Definition 4.9. Let $H = (H^j)_{j \in J}$ be a Hamiltonian of eikonal type. We call H *isotropic*, if we have $H^j_x(p, z, x) = 0$ for all $(p, z, x) \in \mathbb{R} \times \mathbb{R} \times [0, l_j]$, $j \in J$.

For the remainder of this chapter we will assume each Hamiltonian to be of eikonal type and isotropic, unless otherwise stated.

Example 4.1. The Hamiltonian of the eikonal equation given by

$$H^j(p, z, x) := p^2 - 1, \quad j \in J,$$

is of eikonal type and isotropic, as one can easily verify.

We dispense with general existence considerations for solutions of boundary value problem (4.9) and focus instead on the proof that the family (u_ε) of solutions converges, *provided* it exists. The following lemma is a generalization of theorem 2.3 in [Kru75].

Before presenting it, however, we mention the fact that the limit function will not necessarily attain the boundary values g_i. From now on we therefore assume the following sufficient condition on g_i, $i \in I_B$, to be satisfied, ensuring the assumption of the boundary values.

> *There is a constant $\delta > 0$ and a function $\psi : \Gamma \to \mathbb{R}$ with $\psi^j \in C^2([0, l_j])$ for all $j \in J$ such that we have*

$$\begin{cases} \psi(v_i) = g_i \text{ for all } i \in I_B, \\ H^j(\partial_j \psi^j, \psi^j, x) < -\delta \quad \text{on } (0, l_j) \text{ for all } j \in J, \\ S_i(\psi) \geq 0 \text{ for all } i \in I_T. \end{cases} \tag{4.11}$$

Lemma 4.5. *Assume that for each $\varepsilon > 0$ we have a solution $u_\varepsilon \in C^2_K(\Gamma)$ of boundary value problem (4.9). Then there is a number $\tilde{\varepsilon} > 0$ such that the functions u_ε, $0 < \varepsilon < \tilde{\varepsilon}$, are uniformly bounded in ε and equicontinuous on Γ.*

Theorem 4.2. *Under the conditions of lemma 4.5 there is a sequence $\varepsilon_n \to 0$ such that the functions u_{ε_n} uniformly converge to a limit function $u \in C(\Gamma)$ as $n \to \infty$.*

Remark 4.5. The uniqueness of the limit function u, i. e., its independence of the choice of the sequence ε_n, will be shown in chapter 5.

Proof. (of theorem 4.2). Recall that Γ endowed with the induced topology of $\mathbb{R}^n$ is a compact space. The sets $B(x) := \{u_\varepsilon(x) \,|\, 0 < \varepsilon < \tilde{\varepsilon}\}$, $x \in \Gamma$, are relatively compact by lemma 4.5. Then the assertion follows by the equicontinuity of u_ε and by the theorem of Arzelà-Ascoli. □

Before we prove lemma 4.5, we provide a short

Proposition 4.4. *Let H be a Hamiltonian of eikonal type, not necessarily isotropic. Let $\theta, \eta \in \mathbb{R}$, $\theta > 0$. Then there is a number $M_{\theta;\eta} > 0$ such that*

$$H^j(p, z, x) > \theta \quad \text{for all } p \in \mathbb{R},\ |p| > M_{\theta;\eta},\ z \geq \eta,\ x \in [0, l_j],\ j \in J. \tag{4.12}$$

Proof. By (4.10) (i), (ii), and (iv) there is for each $j \in J$ and for each $x \in [0, l_j]$ a maximal number $M^+_{\theta;\eta}(x) > 0$ such that $H^j(M^+_{\theta;\eta}(x), \eta, x) = \theta$ and $H^j(p, \eta, x) > \theta$ for all $p > M^+_{\theta;\eta}(x)$. Similarly, for each $j \in J$ and for each $x \in [0, l_j]$ there is a minimal number $M^-_{\theta;\eta}(x) < 0$ such that $H^j(M^-_{\theta;\eta}(x), \eta, x) = \theta$ and $H^j(p, \eta, x) > \theta$ for all $p < M^-_{\theta;\eta}(x)$. Observe that for each $j \in J$ the functions $[0, l_j] \ni x \mapsto M^+_{\theta;\eta}(x)$ and $[0, l_j] \ni x \mapsto -M^-_{\theta;\eta}(x)$ are upper semicontinuous by (4.10) (ii). Hence we have

$$M_{\theta;\eta} := \max_{j \in J} \max_{x \in [0, l_j]} \max\{M^+_{\theta;\eta}(x), -M^-_{\theta;\eta}(x)\} < \infty, \tag{4.13}$$

and the assertion follows by means of (4.10) (iii). □

Proof. (of lemma 4.5). We proceed in several steps.

Step 1. Bounding $|u_\varepsilon|$ on Γ uniformly in ε.

We construct functions $v \in C^2(\Gamma)$ and $w \in C^2(\Gamma)$ as uniform lower and upper bounds, respectively, of the functions u_ε, $\varepsilon > 0$. In fact, we choose $v :\equiv m := \min_{i\in I_B} g_i$. Then it follows for all $x \in (0, l_j)$, $j \in J$, and $\varepsilon > 0$

$$Q^j_\varepsilon(v^j)(x) := \varepsilon\partial^2_j v^j(x) - H^j(\partial_j v^j(x), v^j(x), x) = -H^j(0, m, x) > 0 = Q^j_\varepsilon(u_\varepsilon)(x)$$

by property (4.10) (i). Moreover, we have $S_i(v) = 0 = S_i(u_\varepsilon)$ for all $i \in I_T$ as well as $v(v_i) \leq g_i = u_\varepsilon(v_i)$ for all $i \in I_B$. By virtue of (4.10) (ii) we can apply lemma 4.2, implying $u_\varepsilon \geq v$ on Γ.

Now let $f \in C^2(\Gamma)$ and $(a_j)_{j\in J}$ be the function and the corresponding vector as constructed in proposition 4.3 and let $a := \min_{j\in J} |a_j| > 0$. Choose $C > 0$ such that we have

$$w > \max\{0, \max_{i\in I_B} g_i\} > 0 \tag{4.14}$$

for the function

$$w := -M_{0;0}f/a + C \in C^2(\Gamma),$$

where $M_{0;0} > 0$ is defined in proposition 4.4. By construction we have

$$\left|\partial_j w^j(x)\right| > M_{0;0} \tag{4.15}$$

for all $x \in (0, l_j)$, $j \in J$. Let $\varepsilon > 0$. By (4.12), (4.14), and (4.15) it follows

$$Q^j_\varepsilon(w^j)(x) = -H^j(\partial_j w^j(x), w(x), x) < 0$$

for all $x \in (0, l_j)$, $j \in J$. Furthermore, by construction of w we have $S_i(w) < 0$ for all $i \in I_T$ as well as $w(v_i) > g_i$ for all $i \in I_B$. By the properties of u_ε we then have $u_\varepsilon \leq w$ on Γ by virtue of lemma 4.2.

Altogether it follows that there is a constant $C_1 > 0$ with

$$|u_\varepsilon| < C_1 \tag{4.16}$$

for all $\varepsilon > 0$.

Step 2. Bounding $|\partial^j u_\varepsilon(v_i)|$, $j \in \mathrm{Inc}_i$, uniformly in ε at boundary vertices v_i, $i \in I_B$.

Let $\boldsymbol{d} : \Gamma \to \mathbb{R}$ be the distance function to the boundary $\partial\Gamma = \{v_i \,|\, i \in I_B\}$. For $\beta > 0$ let $\Gamma_\beta := \{x \in \Gamma \,|\, \boldsymbol{d}(x) \leq \beta\}$. We show that there are constants $\kappa > 0$, $\beta > 0$, and $\tilde{\varepsilon} > 0$ such that

$$\psi \leq u_\varepsilon \leq \psi + \kappa\boldsymbol{d} \quad \text{on } \Gamma_\beta \text{ for all } 0 < \varepsilon < \tilde{\varepsilon}, \tag{4.17}$$

where ψ is the function whose existence has been assumed in (4.11). As $\psi^j \in C^2([0, l_j])$ for all $j \in J$, it follows that there is a constant $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}\partial^2_j\psi^j(x) > -\delta$ for all

$x \in (0, l_j)$, $j \in J$, where δ is the constant defined in (4.11). Invoking (4.11) once more we then derive

$$Q^j_\varepsilon(\psi^j)(x) = \varepsilon\partial^2_j\psi^j(x) - H^j(\partial_j\psi^j(x), \psi^j(x), x) > 0 = Q^j_\varepsilon(u^j_\varepsilon)(x)$$

for all $x \in (0, l_j)$, $j \in J$, and for all $0 < \varepsilon \le \tilde{\varepsilon}$, $j \in J$. Furthermore we have $S_i(\psi) \ge 0 = S_i(u_\varepsilon)$ for all $i \in I_T$ as well as $\psi(v_i) = g_i = u_\varepsilon(v_i)$ for all $i \in I_B$, and the comparison lemma 4.2 yields $u_\varepsilon \ge \psi$ on Γ for all $0 < \varepsilon \le \tilde{\varepsilon}$. The first part of inequality (4.17) is established.

In order to derive the second part of (4.17) first observe that there is a constant $\beta > 0$ such that the distance function $\boldsymbol{d}$ does not attain a local maximum on Γ_β. Furthermore we assume β to be sufficiently small such that there is no $i \in I_T$ with $v_i \in \Gamma_\beta$. Fix arbitrary indices $i \in I_B$ and $j \in \mathrm{Inc}_i$. Assuming without loss of generality that the edge e_j be parametrized with $\pi_j(0) = v_i$, it follows that $|\partial_j \boldsymbol{d}^j| \equiv 1$ and $\partial^2_j \boldsymbol{d}^j \equiv 0$ on $[0, \beta]$. Let

$$\theta := \tilde{\varepsilon} \max_{j\in J} \max_{x\in[0,l_j]} \partial^2_j\psi^j(x) > 0, \quad \eta := \min_{j\in J} \min_{x\in[0,l_j]} \psi^j(x),$$

and define $M_{\theta;\eta}$ as in proposition (4.4). For

$$\kappa := M_{\theta;\eta} + \max_{j\in J} \max_{x\in[0,l_j]} \left|\partial_j\psi^j(x)\right| \quad \text{and} \quad \tilde{\psi} := \psi + \kappa\boldsymbol{d}$$

we then have $|\partial_j\tilde{\psi}^j(x)| > M_{\theta;\eta}$ for all $x \in [0, l_j]$. By (4.12) it follows

$$\begin{aligned} Q^j_\varepsilon(\tilde{\psi}^j)(x) &= \varepsilon\partial^2_j\psi^j(x) - H^j(\partial_j\tilde{\psi}^j(x), \tilde{\psi}^j(x), x) \\ &\le \theta - H^j(\partial_j\tilde{\psi}^j(x), \tilde{\psi}^j(x), x) < 0 = Q^j_\varepsilon(u^j_\varepsilon)(x) \end{aligned} \tag{4.18}$$

for all $x \in [0, \beta]$, $0 < \varepsilon \le \tilde{\varepsilon}$. By possibly enlarging κ we can additionally arrange that we have

$$\tilde{\psi}^j(\beta) \ge C_1, \tag{4.19}$$

where C_1 is the constant in (4.16). Then (4.19) and (4.11) imply

$$\tilde{\psi}^j(\beta) \ge u^j_\varepsilon(\beta) \quad \text{and} \quad \tilde{\psi}^j(0) \ge u^j_\varepsilon(0) \quad \text{for all } 0 < \varepsilon < \tilde{\varepsilon}.$$

By this and by (4.12) the comparison lemma 4.2 implies

$$\tilde{\psi}^j \ge u^j_\varepsilon \quad \text{on } [0, \beta]. \tag{4.20}$$

We finally choose κ large enough such that (4.18) - (4.20) hold for all $j \in \mathrm{Inc}_i$ and $i \in I_B$, whence the second part of (4.17) follows.

Now inequality (4.17) implies that there is a constant $C_2 > 0$ with

$$|\partial_j u_\varepsilon(v_i)| < C_2 \tag{4.21}$$

for all $j \in \mathrm{Inc}_i$, $i \in I_B$, $0 < \varepsilon < \tilde{\varepsilon}$.

Step 3. Bounding $|\partial_j u_\varepsilon(v_i)|$, $j \in \mathrm{Inc}_i$, uniformly in ε for all $i \in I_T$.

We show that there is a constant $C_3 > 0$ such that

$$|\partial_j u_\varepsilon(v_i)| < C_3 \tag{4.22}$$

for all $j \in \mathrm{Inc}_i$, $i \in I_T$, $0 < \varepsilon < \tilde{\varepsilon}$. For this purpose we assume the contrary. Then there is an index $i \in I_T$ and an index $k \in \mathrm{Inc}_i$ along with a sequence $\varepsilon_n \to 0$ such that for $u_n := u_{\varepsilon_n}$ we have

$$\lim_{n\to\infty} |\partial_k u_n(v_i)| = \infty. \tag{4.23}$$

As for each $n \in \mathbb{N}$ the Kirchhoff condition

$$S_i(u_n) = \sum_{j \in \mathrm{Inc}_i} a_{ij} \partial_j u_n(v_i) = 0$$

is satisfied, it follows from (4.23) that there is an index $j \in \mathrm{Inc}_i$ such that

$$\lim_{n\to\infty} a_{ij} \partial_j u_n(v_i) = \infty.$$

Consequently there is a sequence $x_n \in e_j$ with $x_n \to v_i$ such that

$$\lim_{n\to\infty} a_{ij} \partial_j u_n(x_n) = \infty. \tag{4.24}$$

Let $y_n := \pi_j^{-1}(x_n)$ and fix $t_0 > 0$ such that $y_n + a_{ij} t \in [0, l_j]$ for all $t \in [0, t_0]$ and all $n \in \mathbb{N}$. Define the functions $f_n \in C^2([0, t_0])$ by

$$f_n(t) := u_n^j(y_n + a_{ij} t), \quad n \in \mathbb{N}.$$

Then (4.24) reads

$$\lim_{n\to\infty} f_n'(0) = \infty. \tag{4.25}$$

Using $f_n''(t) = \partial_j^2 u_n^j(y_n + a_{ij} t)$, we conclude by (4.9) that we have for all $t \in [0, t_0]$ and all $n \in \mathbb{N}$

$$\varepsilon_n f_n''(t) - H^j(a_{ij} f_n'(t), f_n(t), y_n + a_{ij} t) = 0$$

or equivalently

$$f_n''(t) = \varepsilon_n^{-1} H^j(a_{ij} f_n'(t), f_n(t), y_n + a_{ij} t). \tag{4.26}$$

Now set

$$\theta := 2C_1/t_0^2 > 0 \quad \text{and} \quad \eta := -C^1, \tag{4.27}$$

where C_1 is the constant in (4.16). Let $M_{\theta;\eta} > 0$ be the constant defined in proposition 4.4. Then (4.25) implies that there is a number $n \in \mathbb{N}$ such that $|a_{ij} f_n'(0)| = f_n'(0) > M_{\theta;\eta}$. Consequently, by (4.26), (4.16), and proposition 4.4 we have

$$f_n''(0) > \theta, \tag{4.28}$$

provided that $\varepsilon_n \leq 1$, which we from now on assume.

We now show that we have

$$f_n''(t) \geq \theta \quad \text{for all } t \in [0, t_0]. \tag{4.29}$$

For this purpose let $A := (f_n'')^{-1}(\{x \geq \theta\}) \subseteq [0, t_0]$. By (4.28) there is a connected component A_0 of A which contains 0. From $f_n \in C^2([0, t_0])$ it follows that A_0 is closed, whence there is a maximal $t \in A_0$. Assume (4.29) to be false. Then $t < t_0$. As $f_n'(0) > M_{\theta;\eta}$ and as $f''(s) \geq \theta > 0$ for all $s \in A_0$, it follows by continuity that there is a neighborhood $U \subseteq [0, t_0]$ of t such that $f_n'(s) > M_{\theta;\eta}$ for all $s \in U$. Then proposition 4.4 and (4.26) imply $f_n''(s) > \theta$ for all $s \in U$, contradicting the maximality of t.

From (4.29) it follows that the inequality

$$f_n(t) \geq \theta t^2 + f_n'(0)t + f_n(0)$$

holds on $[0, t_0]$. As $f_n'(0) \geq 0$, we estimate

$$u_n^j(y_n + a_{ij}t_0) = f_n(t_0) \geq f_n(0) + \theta t_0^2 > -C_1 + \theta t_0^2,$$

where C_1 is the constant in (4.16). By (4.27) we obtain

$$u_n^j(y_n + a_{ij}t_0) > C_1,$$

a contradiction to (4.16). Hence (4.22) is proved.

Step 4. Bounding $|\partial_j u_\varepsilon^j|$, $j \in J$, uniformly in ε

Fix $j \in J$ and set $w^j := \partial_j u_\varepsilon^j$. By (4.10) (ii) we may differentiate equation (4.9) and obtain

$$\varepsilon \partial_j^2 w^j - H_p^j(\partial_j u_\varepsilon^j, u_\varepsilon^j, x)\partial_j w^j - H_z^j(\partial_j u_\varepsilon^j, u_\varepsilon^j, x)w^j - H_x^j(\partial_j u_\varepsilon^j, u_\varepsilon^j, x) = 0. \tag{4.30}$$

Observe moreover that we have $H_z^j(\partial_j u_\varepsilon^j, u_\varepsilon^j, x) \geq 0$ due to (4.10) (iii). We also have

$$H_x^j(\partial_j u_\varepsilon^j, u_\varepsilon^j, x) = 0,$$

as H is isotropic. Then (4.30) simplifies to

$$a\partial_j^2 w^j + b\partial_j w^j + cw^j = 0$$

with bounded coefficient functions $a, b, c \in C(\mathbb{R} \times \mathbb{R} \times (0, l_j))$ and $c \leq 0$. Consequently, (4.21) and (4.22) and the classical maximum principle (cf. [GT77], corollary 3.2) imply that there is a constant $C > 0$ such that $|\partial_j u_\varepsilon| < C$ on $[0, l_j]$ for all $0 < \varepsilon < \tilde{\varepsilon}$. Repeating this argument for all $j \in J$ shows that there is a constant $C_4 > 0$ such that

$$|\partial_j u_\varepsilon| < C_4 \quad \text{on } [0, l_j]$$

for all $j \in J$ and for all $0 < \varepsilon < \tilde{\varepsilon}$. This completes the proof. □

Remark 4.6. Observe that in the above proof we have not invoked the conditions (4.10) (v)-(vii). In fact they are not necessary for the convergence statement to hold. However, as it comes to uniqueness questions in the following chapter, they play an eminent role. Then, on the other hand, we may dispense with the isotropy condition and allow for Hamiltonians depending on x.

CHAPTER 5

Viscosity solutions on networks

Summary. In this chapter we propose an extension of the theory of viscosity solutions of first order Hamilton-Jacobi equations of eikonal type to networks and derive comparison, uniqueness, and existence results. Moreover we show that this notion of viscosity solutions is consistent with the extended method of vanishing viscosity discussed in chapter 4.

5.1 Introduction

As we have seen, a possible approach for selecting a solution of first order Hamilton-Jacobi equations on networks (and on unramified domains) is the method of vanishing viscosity. However, two problems arise in this context: the problem of convergence and the question of uniqueness. Under the condition that H be isotropic, the convergence problem has been solved in the previous chapter. However, the question of uniqueness has not yet been answered. An adaptation of the original uniqueness proof given by Kružkov in [Kru75] entails certain difficulties depending on the topology of the network. In fact, in case that the graph corresponding to the network contains cycles, a direct adaptation of Kružkov's method is not fruitful.

In the present chapter we attack the problem by extending the theory of viscosity solutions based on test functions to networks, a method turning out to be both elegant and powerful. The major task in this context is to establish the correct transition conditions solutions are subjected to at transition vertices. As a matter of fact, these transition conditions make up the core of our theory, as they constitute the major difference of our theory from the classical theory of viscosity solutions. Let us elaborate here on an interesting phenomenon concerning the transition conditions: Among other topics, we will verify the consistency of our viscosity solution concept on networks with the method of vanishing viscosity, which in other words means that we show that any limit function u of a sequence

u_ε converging according to theorem 4.2 is a viscosity solution. As the functions u_ε satisfy the Kirchhoff condition at transition vertices, one might expect that this also holds for the limit function u. This, however, is not the case: whereas the Kirchhoff condition has a certain *averaging* effect on the functions u_ε, the correct transition condition for the limit function u will thoroughly lose this averaging property in favor of another principle which only involves two incident branches at a given vertex instead of all of them. Although this might seem unexpected at a first glance, it is nevertheless coherent with the idea of vanishing viscosity (which can equivalently be interpreted as "vanishing averages"). Compare this for instance with the results of chapter 2, where we have seen that the distance function on an interval is the vanishing viscosity limit of C^2-functions (which all satisfy a trivial Kirchhoff condition due their C^1-regularity), but possesses a singular point (its peak), at which the Kirchhoff condition is violated.

In order to give an impression of the transition condition of the limit function, let us once again consider the eikonal equation on a network with zero boundary conditions. Of course the distance function $\boldsymbol{d}$ to the boundary is an obvious candidate for a solution. However, it generally does not satisfy a Kirchhoff condition at transition vertices. Instead, it displays a behavior which can be described to be governed by "finding the shortest way to the boundary" rather than "building local averages", a characterization which also describes the general case, as we will see. In fact, imagine an additional edge being inserted at a given transition vertex v_i. Then the distance function at v_i and on the incident edges will only be affected, if the new edge permits a shorter connection to a boundary vertex. It is remarkable how much this behavior deviates from the Kirchhoff condition.

Any generalization of existing concepts to new scenarios has of course to be justified by checking if all features the success of the existing theory relies on are preserved. In the case of the theory of viscosity solutions, these features are uniqueness, existence, and consistency with related concepts, i. e., method of vanishing viscosity. Giving this justification will be the main issue of the present chapter. In fact, our generalization of viscosity solution to networks will be just as weak to yield existence, while being sufficiently "selective" in order to ensure uniqueness. We will also demonstrate that our generalization of viscosity solutions is not only a technical construction, but arises as a natural selection principle, which in particular selects the distance function as the unique viscosity solution of the Dirichlet problem for the eikonal equation on networks.

5.2 Hamilton-Jacobi equations on networks

Throughout this chapter let Γ be a topological network with boundary index set $I_B \neq \emptyset$ and boundary $\partial\Gamma := \{v_i,\, i \in I_B\}$. Our objective is to establish a weak theory of solutions of Dirichlet problems on Γ of the form

$$\begin{cases} H^j(\partial_j u(x), u(x), x) = 0 & \text{for all } x \in e_j,\ j \in J, \\ u(v_i) = g_i & \text{for all } i \in I_B. \end{cases} \tag{5.1}$$

Here we make use of the following simplified notation

$$H^j(\partial_j u(x), z, x) := H^j(\partial_j u^j(\pi_j^{-1}(x)), z, \pi_j^{-1}(x)) \tag{5.2}$$

for all $x \in \bar{e}_j$, $j \in J$, and all $z \in \mathbb{R}$, which we will use in the sequel whenever confusion is ruled out. Moreover, throughout this chapter we assume that the Hamiltonian $H = (H^j)_{j\in J}$ be of eikonal type, i.e. that it satisfies the conditions (4.10).

Example 5.1. Consider the eikonal equation with zero boundary data on Γ given by

$$\begin{cases} (\partial_j u)^2 - 1 = 0 & \text{on } e_j,\ j \in J, \\ u(v_i) = g_i & \text{for all } i \in I_B. \end{cases}$$

Here we have $H^j(p, z, x) := p^2 - 1$, and $H = (H^j)_{j\in J}$ clearly satisfies (4.10).

Remark 5.1. In the case of the eikonal equation it is obvious that a smooth solution of (5.1) will not exist in general. Furthermore it is equally unlikely that the Kirchhoff condition is satisfied. This condition rather does not seem to do justice to the structural properties of viscosity solutions in the majority of cases. So by now, *continuity* is the only property of a possible solution candidate for (5.1) which is reasonable to expect. All other properties will be established in the sequel.

5.3 Preliminaries and definitions

Before giving the definition of viscosity solutions, let us provide some useful terminology capturing and simplifying the test function technique we are going to apply in the sequel. Recall that $e_j := \{\pi_j((0, l_j))\}$, $j \in J$, and $\bar{e}_j := \{\pi_j([0, l_j])\}$, $j \in J$, denote the *open* and *closed* edges, respectively (see definition 4.4).

Definition 5.1. Let $\varphi \in C(\Gamma)$.

(i) Let $j \in J$ and $x \in e_j$. We say that φ is *differentiable at* x, if φ^j is differentiable at $\pi_j^{-1}(x)$.

(ii) Let $i \in I_T$ and $j, k \in \mathrm{Inc}_i$, $j \neq k$. We say that φ is (j,k)-*differentiable at* v_i, if we have

$$a_{ij}\partial_j\varphi(v_i) + a_{ik}\partial_k\varphi(v_i) = 0,$$

where (a_{ij}) is the oriented incidence matrix of Γ as defined by (4.1).

Definition 5.2. Let $u : \Gamma \to \mathbb{R}$ and $\varphi \in C(\Gamma)$.

(i) Let $j \in J$ and $x \in e_j$. We call φ an *upper (lower) test function of* u *at* x, if φ is differentiable at x and if $u - \varphi$ attains a local maximum (minimum) at x.

(ii) Let $i \in I_T$ and $j, k \in \mathrm{Inc}_i$. We call φ an *upper (lower)* (j,k)-*test function of* u *at* v_i, if φ is (j,k)-differentiable at v_i and if $u - \varphi$ attains a local maximum (minimum) at v_i.

Now we are ready to introduce viscosity solutions on networks. Note that the condition on the open edges e_j, $j \in J$, is identical with the classical test function condition given in definition 2.3, whereas its extension to the transition vertices is the essential new aspect. Let $\mathrm{USC}(\Gamma)$ and $\mathrm{LSC}(\Gamma)$ denote the set of all upper and lower semicontinuous functions $u : \Gamma \to \mathbb{R}$, respectively.

Definition 5.3. Let $f : \Gamma \to \mathbb{R}$ and let $u \in \mathrm{USC}(\Gamma)$.
We say that u satisfies the *viscosity subsolution condition (associated with H and f)* at $x \in \Gamma_0$, if the following conditions (i) and (ii) hold.
(i) If $x \in e_j$, $j \in J$, we have

$$H^j(\partial_j \varphi(x), u(x), x) \leq f(x) \tag{5.3}$$

for all upper test functions φ of u at x.
(ii) If $x = v_i$, $i \in I_T$, then for any $j, k \in \mathrm{Inc}_i$, $j \neq k$, the inequality (5.3) holds for all upper (j,k)-test functions φ of u at v_i.
Alternatively, we say that u satisfies the formal relation

$$H(\partial u(x), u(x), x) \leq f(x) \tag{5.4}$$

in the viscosity sense.

Now let $u \in \mathrm{LSC}(\Gamma)$.
We say that u satisfies the *viscosity supersolution condition (associated with H and f)* at $x \in \Gamma_0$, if the following conditions (iii) and (iv) hold.
(iii) If $x \in e_j$, $j \in J$, we have

$$H^j(\partial_j \varphi(x), u(x), x) \geq f(x) \tag{5.5}$$

for all lower test functions φ of u at x.
(iv) If $x = v_i$, $i \in I_T$, then for each $j \in \mathrm{Inc}_i$ there is an index $k \in \mathrm{Inc}_i$, $k \neq j$, such that inequality (5.5) holds for all lower (j,k)-test functions φ of u at v_i. We call k an *i-feasible index for j (with respect to u).*
Alternatively, we say that u satisfies the formal relation

$$H(\partial u(x), u(x), x) \geq f(x) \tag{5.6}$$

in the viscosity sense.

We call $u \in \mathrm{USC}(\Gamma)$ ($u \in \mathrm{LSC}(\Gamma)$) a *viscosity sub- (super-) solution* of

$$H(\partial u(x), u(x), x) = f(x) \tag{5.7}$$

on Γ, if it respectively satisfies (5.4) and (5.6) in the viscosity sense for all $x \in \Gamma_0$. Furthermore, $u \in C(\Gamma)$ is said to be a *viscosity solution* of (5.7), if it is both a viscosity sub- and a supersolution of (5.7).

Remark 5.2. Note the asymmetry of the definition of viscosity sub- and supersolutions at transition vertices and compare it with the properties of the distance function on networks. The existence of an i-feasible index k for each index $j \in \mathrm{Inc}_i$ reflects the idea "that there is always a shortest way to the boundary".

Remark 5.3. Let $i \in I_T$, $j, k \in \mathrm{Inc}_i$, and let $\varphi \in C(\Gamma)$ be (j,k)-differentiable at v_i. By definition 5.1 and by (4.10) (vi) and (vii) we then have

$$H^j(\partial^j \varphi(v_i), s, v_i) = H^k(\partial^k \varphi(v_i), s, v_i)$$

for all $s \in \mathbb{R}$. This symmetry will play a crucial role in the sequel whenever a function satisfies a viscosity sub- or supersolution condition at transition vertices according to definition 5.3 (ii) and (iv).

The following result states that the pointwise maximum (minimum) over a finite set of viscosity subsolutions (supersolutions) is itself a viscosity subsolution (supersolution).

Proposition 5.1. *Let $x \in \Gamma_0$ and let $u_1, u_2 \in \mathrm{USC}(\Gamma)$ ($u_1, u_2 \in \mathrm{LSC}(\Gamma)$) satisfy the viscosity sub- (super-) solution condition at x. Then $v := \max\{u_1, u_2\}$ ($v := \min\{u_1, u_2\}$) satisfies the viscosity sub- (super-) solution condition at x.*

Proof. We only treat the case $x = v_i$, $i \in I_T$, as the case $x \in e_j$, $j \in J$ follows from similar (and simpler) arguments. First let $u_1, u_2 \in \mathrm{USC}(\Gamma)$ satisfy the viscosity subsolution condition at x and set $v := \max\{u_1, u_2\}$. Observe that $v \in \mathrm{USC}(\Gamma)$. Now let $j, k \in \mathrm{Inc}_i$, $j \neq k$, and let φ be an upper (j,k)-test function of v at x. Then $v(x) = u_l(x)$ for $l = 1$ or $l = 2$, whence φ is an upper (j,k)-test function of u_l at x. It follows

$$H^j(\partial_j \varphi(x), v(x), x) = H^j(\partial_j \varphi(x), u_l(x), x) \leq 0, \tag{5.8}$$

whence v satisfies the viscosity subsolution condition at x.

Secondly, let $u_1, u_2 \in \mathrm{LSC}(\Gamma)$ satisfy the viscosity supersolution condition at x and set $v := \min\{u_1, u_2\}$. Observe that $v \in \mathrm{LSC}(\Gamma)$. Next note that there is an index $l \in \{1, 2\}$ such that $v(x) = u_l(x)$. Let $j \in \mathrm{Inc}_i$ and let $k \in \mathrm{Inc}_i \backslash \{j\}$ be an i-feasible index for j with respect to u_l. Furthermore, let φ be a lower (j,k)-test function of v (and thus of u_l) at x. It follows

$$H^j(\partial_j \varphi(x), v(x), x) = H^j(\partial_j \varphi(x), u_l(x), x) \geq 0, \tag{5.9}$$

whence k is i-feasible for j also with respect to v, whence v satisfies the viscosity supersolution condition at x. □

We now concretize boundary value problem (5.1) in terms of definition 5.3 and pose the question if there is a unique function $u \in C(\Gamma)$ such that

$$\begin{cases} H(\partial u(x), u(x), x) = 0 & \text{on } \Gamma_0 \text{ in the viscosity sense} \\ u(v_i) = g_i & \text{for all } i \in I_B. \end{cases} \tag{5.10}$$

Such a function will be called *solution* of (5.10).

Our concerns in the following sections are with showing both existence and uniqueness of solutions of (5.10). Moreover, we will demonstrate that the notion of viscosity solutions on network arises naturally as limits of the vanishing viscosity method discussed in chapter 4.

5.4 Uniqueness

We show that the notion of viscosity solution defined above is sufficiently strong to ensure that a solution of (5.10) is unique, whenever it exists. We remark that the major part of the proof of this statement consists in establishing a comparison result, which can be considered as the core of a variety of different uniqueness proofs. In fact, once we have proven the comparison result, then the derivation of the uniqueness statement is a matter of a straightforward adaptation of existing techniques in the literature, among which we have decided to present a modified uniqueness proof due to H. Ishii for the sake of completeness.

Before getting started we have to impose an additional constraint on the Hamiltonian H. Observe that by virtue of the conditions (4.10) for each $p \in \mathbb{R}$ and for each $z \in \mathbb{R}$ the mapping

$$h_{p,z} : x \mapsto H^j(p, z, \pi_j^{-1}(x)) \quad \text{for all } x \in \bar{e}_j,\, j \in J,$$

constitutes a well-defined function $h_{p,z} \in C^2(\Gamma)$. Now we demand that there is a positive constant $C_0 < \infty$ such that

$$|\partial_j h^j_{p,z}(x)| \le C_0 \text{ for all } x \in e_j,\, j \in J,\, p \in \mathbb{R}, \text{ and } z \in \mathbb{R}. \tag{5.11}$$

This condition is assumed to hold throughout the remainder of the present chapter.

Lemma 5.1. *Let $f \in C(\Gamma)$ with $f(x) < 0$ for all $x \in \Gamma$ and suppose that we have two functions $u \in \mathrm{USC}(\Gamma)$ and $v \in \mathrm{LSC}(\Gamma)$ such that*

$$H(\partial u(x), u(x), x) \le f(x) \quad \textit{and} \quad H(\partial v(x), v(x), x) \ge 0 \tag{5.12}$$

in the viscosity sense for all $x \in \Gamma_0$. Assume $u \le v$ on $\partial\Gamma$. Then we have $u \le v$ on Γ.

Proof. Assume the contrary, i.e. assume that there exists a point $z \in \Gamma_0$ with $u(z) > v(z)$. We derive a contradiction.

As u and v are upper and lower semicontinuous, respectively, we have

$$M := \max\{\sup_\Gamma u, -\inf_\Gamma v, 1\} < \infty.$$

Choose a function $\beta \in C^\infty(\mathbb{R})$ with $0 \le \beta \le 1$, $\beta(0) = 1$, $\beta'(0) = 0$, and such that $\beta(x) = 0$ if $|x| \ge 1$. Recall that $d(\cdot,\cdot)$ denotes the path metric on Γ. For $\varepsilon > 0$ we then define the function

$$\Phi : \Gamma \times \Gamma \to \mathbb{R} \quad \text{by} \quad \Phi(x, y) := u(x) - v(y) + 3M\beta(\varepsilon^{-1} d(x, y)).$$

Note that Φ is upper semicontinuous on the compact set $\Gamma^2 = \Gamma \times \Gamma$. Therefore it attains its maximum at some point $(p, q) \in \Gamma^2$. We claim that we have

$$d(p, q) \le \varepsilon. \tag{5.13}$$

In fact, if this were not the case, we would have

$$\Phi(z, z) - \Phi(p, q) = u(z) - v(z) + 3M - u(p) + v(q) \ge 3M - 2M = M > 0$$

by the definition of M, a contradiction to the choice of (p, q).

For any $\varepsilon > 0$ we define the quantity

$$m_\varepsilon := \sup\{u(x) - v(y) \mid d(x, y) \le \varepsilon, (x, y) \in \Gamma^2 \backslash \Gamma_0^2\}.$$

As the function $(x, y) \mapsto u(x) - v(y)$, $(x, y) \in \Gamma^2$, is upper semicontinuous, and as we have $u \le v$ on $\partial\Gamma$, it follows $\lim_{\varepsilon\to 0} m_\varepsilon \le 0$. Hence we can arrange $m_\varepsilon < u(z) - v(z)$ by choosing $\varepsilon > 0$ sufficiently small. Then the definition of m_ε implies $\Phi(z, z) > \Phi(x, y)$ for any choice $(x, y) \in \Gamma^2\backslash\Gamma_0^2$ with $d(x, y) \le \varepsilon$. Consequently, $(p, q) \in \Gamma_0^2$.

Next choose $m \in \mathbb{R}$ with $0 < m < -\max_{x\in\Gamma} f(x)$. If necessary, we now decrease $\varepsilon > 0$ such that we have $\varepsilon C_0 \le m/2$, where C_0 is the constant defined in (5.11), and such that there is a unique path P of length $d(p, q)$ in Γ connecting p and q, which runs through at most one vertex v_i, $i \in I$. This is possible by (5.13). Then the situation may be described by one of the following cases.

Case 1. There are indices $i \in I$ and $j, k \in \mathrm{Inc}_i$ such that $p \in e_j$ and $q \in e_k$ and such that P runs through v_i. Set $d_y := d(\cdot, y)$ for any $y \in \Gamma$ and define the functions $\varphi_q, \varphi_p \in C(\Gamma)$ by

$$\varphi_q : x \mapsto 3M\beta(\varepsilon^{-1} d_q(x)) \quad \text{and} \quad \varphi_p : x \mapsto 3M\beta(\varepsilon^{-1} d_p(x)). \tag{5.14}$$

Clearly, d_q and d_p (and hence φ_q and φ_p) are differentiable at p and q, respectively. In fact, setting

$$\tilde{p} := \pi_j^{-1}(p) \quad \text{and} \quad \tilde{q} := \pi_k^{-1}(q) \tag{5.15}$$

we have

$$\partial_j d_q^j(\tilde{p}) = a_{ij} \quad \text{and} \quad \partial_k d_p^k(\tilde{q}) = a_{ik},$$

where (a_{ij}) is the oriented incidence matrix as defined in (4.1). Consequently we have

$$\partial_j \varphi_q^j(\tilde{p}) = 3M\partial_j[\beta(\varepsilon^{-1} d_q^j)]|_{x=\tilde{p}} = a_{ij}\eta \tag{5.16}$$

and

$$\partial_k \varphi_p^k(\tilde{q}) = 3M \partial_k[\beta(\varepsilon^{-1} d_p^k)]|_{x=\tilde{q}} = a_{ik}\eta, \tag{5.17}$$

where

$$\eta := 3M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)). \tag{5.18}$$

Now observe that by the choice of p and q the function $u+\varphi_q$ attains a local maximum at p, whence $-\varphi_q$ is an upper test function of u at p. By (5.12) and (5.16) we thus deduce

$$f(p) \geq H^j(-\partial_j \varphi_q^j(\tilde{p}), u^j(\tilde{p}), \tilde{p}) = H^j(-a_{ij}\eta, u^j(\tilde{p}), \tilde{p}). \tag{5.19}$$

Similarly observe that $-v+\varphi_p$ attains a local maximum at q, implying that $v-\varphi_p$ attains a local minimum at q. Hence φ_p is a lower test function of v at q, and by (5.12) and (5.17) we conclude

$$0 \leq H^k(\partial_k \varphi_p^k(\tilde{q}), v^k(\tilde{q}), \tilde{q}) = H^k(a_{ik}\eta, v^k(\tilde{q}), \tilde{q}). \tag{5.20}$$

Observe that by (5.19) and by the definition of m we have

$$0 > f(p) + m \geq H^j(-a_{ij}\eta, u^j(\tilde{p}), \tilde{p}) + m =: T_1. \tag{5.21}$$

As we have $|\tilde{p} - \pi_j^{-1}(v_i)| = d(p, v_i) \leq \varepsilon$ and $\varepsilon C_0 < m/2$, relation (5.11) and the mean value theorem imply

$$T_1 \geq H^j(-a_{ij}\eta, u^j(\tilde{p}), \pi_j^{-1}(v_i)) + m/2 =: T_2. \tag{5.22}$$

Since $|a_{ij}| = |a_{ik}|$ and $\pi_j^{-1}(v_i) = \pi_k^{-1}(v_i)$, we furthermore have by (4.10) (vi) and (vii)

$$T_2 = H^k(a_{ik}\eta, u^j(\tilde{p}), \pi_k^{-1}(v_i)) + m/2. \tag{5.23}$$

Applying the mean value theorem once more, we estimate

$$T_2 \geq H^k(a_{ik}\eta, u^j(\tilde{p}), \tilde{q}) =: T_3. \tag{5.24}$$

By the definition of p and q we have

$$u^j(\tilde{p}) - v^k(\tilde{q}) = u(p) - v(q) \geq u(z) - v(z) > 0,$$

and we invoke (4.10) (iii) to conclude

$$T_3 \geq H^k(a_{ik}\eta, v^k(\tilde{q}), \tilde{q}) \geq 0. \tag{5.25}$$

The last inequality follows by (5.20). Then the successive combination of (5.21) - (5.25) yields a contradiction.

Case 2. There are indices $i \in I$ and $j \in \mathrm{Inc}_i$ such that $q = v_i$ and $p \in e_j$. As $q \in \Gamma_0$, we have $i \in I_T$. Setting $\tilde{p}$ and $\tilde{q}$ as in (5.15), we have

$$\partial_j d_p^j(\tilde{q}) = -a_{ij} \quad \text{and} \quad \partial_k d_p^k(\tilde{q}) = a_{ik} \quad \text{for all } k \in \mathrm{Inc}_i,\ k \neq j. \tag{5.26}$$

Define φ_q and φ_p as in (5.14) and η as in (5.18). Similar as in (5.16) we compute the one-sided differentials

$$\partial_j \varphi_p^j(\tilde{q}) = -a_{ij}\eta \quad \text{and} \quad \partial_k \varphi_p^k(\tilde{q}) = a_{ik}\eta \quad \text{for all } k \in \mathrm{Inc}_i,\ k \neq j.$$

for all $l \in \mathrm{Inc}_i$. Invoking (5.26) we obtain

$$a_{ij}\partial_j \varphi_p(\tilde{q}) + a_{ik}\partial_k \varphi_p(\tilde{q}) = (-a_{ij}^2 + a_{ik}^2)\eta = 0 \tag{5.27}$$

for all $k \in \mathrm{Inc}_i$, $k \neq j$. Hence φ_p is (j,k)-differentiable at q for all $k \in \mathrm{Inc}_i$, $k \neq j$. Exactly as in (5.16) we moreover conclude

$$\partial_j \varphi_q^j(\tilde{p}) = 3M\partial_j[\beta(\varepsilon^{-1} d_q^j)]|_{x=\tilde{p}} = a_{ij}\eta, \tag{5.28}$$

and, by (5.12),

$$f(p) \geq H^j(-a_{ij}\eta, u^j(\tilde{p}), \tilde{p}), \tag{5.29}$$

as $-\varphi_q$ is an upper test function of u at $p \in e_j$. On the other hand, by (5.12) and according to definition 5.3 (iv) there is an i-feasible index $k_0 \in \mathrm{Inc}_i$, $k_0 \neq j$, for j. As $v - \varphi_p$ attains a local minimum at q, it follows by (5.27) that φ_p is a lower (j, k_0)-test function of v at q, whence we have

$$0 \leq H^j(\partial_j \varphi_p^j(\tilde{q}), v^j(\tilde{q}), \tilde{p}) = H^j(-a_{ij}\eta, v^j(\tilde{q}), \tilde{p}). \tag{5.30}$$

Using the mean value theorem, condition (4.10) (iii), and (5.11) we derive a contradiction from (5.29) and (5.30) similar to case 1.

Case 3. There are indices $i \in I$ and $j \in \mathrm{Inc}_i$ such that $p = v_i$ and $q \in e_j$. We proceed as in case 2, with the difference that definition 5.3 (ii) has to be invoked instead of definition 5.3 (iv). Observe, however, that the first is more restrictive than the latter, whence no extra arguments are required.

Case 4. There is an index $j \in J$ such that $p, q \in e_j$, $p \neq q$. With $\tilde{p}$ and $\tilde{q}$ as in (5.15) we have

$$\partial_j d_q^j(\tilde{p}) = -\partial_j d_p^j(\tilde{q}), \quad \text{implying} \quad \partial_j \varphi_q^j(\tilde{p}) = -\partial_j \varphi_p^j(\tilde{q}).$$

As the functions $-\varphi_q$ and φ_p are upper and lower test functions for u and v at p and q, respectively, we obtain

$$H^j(\partial_j \varphi_q^j(\tilde{p}), u^j(\tilde{p}), \tilde{p}) < f(p) \tag{5.31}$$

and

$$0 \leq H^j(-\partial_j \varphi_p^j(\tilde{q}), v^j(\tilde{q}), \tilde{q}) = H^j(\partial_j \varphi_q^j(\tilde{p}), v^j(\tilde{q}), \tilde{q}) \tag{5.32}$$

by definition 5.3. Using the mean value theorem, condition (4.10) (iii), and (5.11), we derive a contradiction from (5.31) and (5.32) similar as in the previous cases.

Case 5. In this last case we assume that p and q coincide. If they coincide on a vertex v_i, $i \in I$, it follows $i \in I_T$, and by the fact that we have $\beta'(0) = 0$ it follows

$$\partial_j \varphi_p^j(\pi_j^{-1}(v_i)) = \partial_j \varphi_q^j(\pi_j^{-1}(v_i)) = 0$$

for all $j \in \mathrm{Inc}_i$. In particular, for each choice of $j, k \in \mathrm{Inc}_i$, $j \neq k$, both $-\varphi_q$ and φ_p are upper and lower (j,k)-test functions of u and v at v_i, respectively, and a contradiction similar to the previous cases can be derived.

The case $p = q \in e_j$, $j \in J$, can be treated analogously. □

As we have already mentioned, a uniqueness result based upon the comparison lemma 5.1 can be established by various methods, among which we have chosen to present an adaptation of an elegant proof due to H. Ishii [Ish87a] making explicit use of the convexity condition (4.10) (v).

Lemma 5.2. *Let* $u \in \mathrm{USC}(\Gamma)$*,* $v \in \mathrm{LSC}(\Gamma)$ *be a viscosity sub- and supersolution, respectively, of*

$$H(\partial u(x), u(x), x) = 0 \quad \text{on } \Gamma, \tag{5.33}$$

with $u(v_i) \leq v(v_i)$ *for all* $i \in I_B$*. Furthermore assume that there is a lower bound* $M \in \mathbb{R}$ *such that* $M < u$ *on* Γ*. Then* $u \leq v$ *on* Γ*.*

Proof. Define for each $\theta \in [0,1]$ the function $u_\theta \in \mathrm{USC}(\Gamma)$ by

$$u_\theta(x) := \theta u(x) + (1-\theta)M, \quad x \in \Gamma.$$

Clearly we have

$$u_\theta < u \quad \text{on } \Gamma \text{ for all } 0 \leq \theta < 1. \tag{5.34}$$

Now choose a compact set $K \subset \mathbb{R}$ such that $u_\theta(x) \in K$ for all $x \in \Gamma$ and all $\theta \in [0,1]$, which is possible as u is bounded from below. Due to conditions (4.10) (i), (ii), and (vii) we then can indicate a function $h \in C(\Gamma)$ with $h < 0$ on Γ and

$$H^j(0, z, x) \leq h^j(x) \tag{5.35}$$

for all $x \in [0, l_j]$, $j \in J$, $z \in K$.

Fix $\theta \in (0,1)$. We show that we have

$$H(\partial u_\theta(x), u_\theta(x), x) \leq (1-\theta)h(x) \quad \text{in the viscosity sense for all } x \in \Gamma_0. \tag{5.36}$$

First assume $x \in e_j$ for some $j \in J$. Let $\varphi \in C(\Gamma)$ be an upper test function of u at x. Setting $\varphi_\theta := \theta\varphi + (1-\theta)M$ and $\tilde{x} := \pi_j^{-1}(x)$ we obtain by means of the convexity condition (4.10) (v), and by virtue of (5.34), (5.35), and (4.10) (iii)

$$\begin{aligned} H^j(\partial_j \varphi_\theta^j(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x}) &\leq \theta H^j(\partial_j \varphi^j(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x}) + (1-\theta) H^j(0, u_\theta^j(\tilde{x}), \tilde{x}) \\ &\leq \theta H^j(\partial_j \varphi^j(\tilde{x}), u^j(\tilde{x}), \tilde{x}) + (1-\theta) h^j(\tilde{x}). \end{aligned} \tag{5.37}$$

As u is a viscosity subsolution and as φ is an upper test function of u at x, we have

$$H^j(\partial_j \varphi^j(\tilde{x}), u^j(\tilde{x}), \tilde{x}) \leq 0,$$

whence we further conclude from (5.37)

$$H^j(\partial_j \varphi_\theta(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x})) \leq (1-\theta) h^j(\tilde{x}).$$

Secondly, assume $x = v_i$ for some $i \in I_T$. Fix any two indices $j, k \in \mathrm{Inc}_i$, $j \neq k$, let φ be an upper (j,k)-test function of u at x, and set $\tilde{x} := \pi_j^{-1}(x)$. As u is a viscosity subsolution, we have

$$H^j(\partial_j \varphi^j(\tilde{x}), u^j(\tilde{x}), \tilde{x}) \leq 0.$$

Moreover, we derive (5.37) as above. It follows

$$H^j(\partial_j \varphi_\theta^j(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x}) \leq (1-\theta) h^j(\tilde{x}),$$

whence, as the choice of j and k was arbitrary, (5.36) is shown.

By (5.34) we have $u_\theta \leq v$ on $\partial\Gamma$. As v is a viscosity supersolution of (5.33) and as (5.36) holds, we now apply lemma 5.1 with $f := (1-\theta)h$ and obtain $u_\theta \leq v$ for all $\theta \in (0,1)$. Letting θ tend to 1 completes the proof. □

Corollary 5.1. *Let u, v be solutions of boundary value problem (5.10). Then $u \equiv v$.*

5.5 Existence

Having established the uniqueness of viscosity solutions on networks, we now turn to the question of their existence. The key idea is to apply a Perron method (cf. [Ish87b]) as in chapter 3 to construct a viscosity solution. Roughly speaking, a viscosity solution is given by the pointwise supremum over a suitable class of viscosity subsolutions of (5.10). Unfortunately, the pointwise supremum over a class of upper semicontinuous functions is not automatically upper semicontinuous. Hence we have to take a technical detour by considering the upper semicontinuous envelope of the supremum function.

The proof consists of two parts: for any given class of viscosity subsolutions it first has to be shown that (the upper semicontinuous envelope of) the pointwise supremum is contained in this class. Afterward it has to be verified that the pointwise supremum over the class of those viscosity subsolutions, which stay below the boundary data, is a viscosity supersolution, too. The latter case is treated indirectly, by assuming the contrary and showing that one then can construct a strictly larger function within this class, contradicting to the pointwise supremal property.

We recall the definition of semicontinuous envelopes.

Definition 5.4. Let $u : \Gamma \to \mathbb{R}$. Define the functions $u^\star, u_\star : \Gamma \to [-\infty, +\infty]$ by

$$u^\star(x) := \lim_{r\to 0} \sup\{u(y) \mid d(x,y) \le r\} \quad \text{and} \quad u_\star(x) := \lim_{r\to 0} \inf\{u(y) \mid d(x,y) \le r\}.$$

We respectively call $u^\star$ and $u_\star$ the *upper and lower semicontinuous envelope* of u.

Remark 5.4. Observe that $u^\star$ and $u_\star$ are upper and lower semicontinuous, respectively. Furthermore, $u_\star(x) \le u \le u^\star(x)$ for all $x \in \Gamma$. However, as $u^\star$ and $u_\star$ may attain the values $+\infty$ and $-\infty$, they are in general not contained in $\mathrm{USC}(\Gamma)$ and $\mathrm{LSC}(\Gamma)$, respectively.

Lemma 5.3. *Let V be an arbitrary set of viscosity subsolutions of*

$$H(\partial u(x), u(x), x) = 0 \quad on\ \Gamma. \tag{5.38}$$

Define the function $u(x) := \sup_{v\in V} v(x)$ for all $x \in \Gamma$ and assume $u^\star(x) < \infty$ for all $x \in \Gamma$. Then $u^\star$ is a viscosity subsolution of (5.38).

Proof. Observe that the assumption $u^\star < \infty$ implies $u^\star \in \mathrm{USC}(\Gamma)$. The main issue of the proof is to verify that $u^\star$ satisfies

$$H(\partial u^\star(x), u^\star(x), x) \le 0$$

in the viscosity sense for all $x \in \Gamma_0$. We will, however, only treat the case $x = v_i$, $i \in I_T$, as the simpler case $x \in e_j$, $j \in J$, is essentially based upon the same (or simpler) arguments.

Accordingly, assume $x = v_i$, $i \in I_T$, fix any $j, k \in \mathrm{Inc}_i$, $j \ne k$, and suppose that $\varphi \in C(\Gamma)$ be an upper (j,k)-test function of $u^\star$ at x. Observe that we may assume without restriction that φ be continuously differentiable in a neighborhood of x. We have to show

$$H^j(\partial_j\varphi(x), u^\star(x), x) \le 0. \tag{5.39}$$

For this purpose we introduce the auxiliary function

$$\varphi_\delta : \Gamma \to \mathbb{R} \quad \text{with } \varphi_\delta(y) := \varphi(y) + \delta(d(x,y))^2,\ \delta > 0. \tag{5.40}$$

Since φ is an upper (j,k)-test function of $u^\star$ at x, there is a radius $r > 0$ such that $u^\star - \varphi$ attains a global maximum on $\bar{B}_r(x)$ at x. Moreover, observe that this maximum is also global with respect to $\bar{B} := \bar{B}_r(x) \cap (\bar{e}_j \cup \bar{e}_k)$. Then the function φ_δ also attains a maximum at x, which is global with respect to $\bar{B}$. Next let $(r_l)_{l\in\mathbb{N}}$ be a sequence with $0 < r_l < r$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty} r_l = 0$. Note that we can indicate a sequence $(x_l)_{l\in\mathbb{N}}$ with $x_l \in B_l := \bar{B}_{r_l}(x)$ for all $l \in \mathbb{N}$ such that $\sup_{B_l} u - u(x_l) < 1/l$. By the definition of u, for each $l \in \mathbb{N}$ there is a function $u_l \in V$ such that $u(x_l) - u_l(x_l) < 1/l$. It follows

$$\sup_{B_l} u \ge u_l(x_l) > u(x_l) - 1/l > \sup_{B_l} u - 2/l,$$

whence

$$u^\star(x) = \lim_{l\to\infty} \sup_{B_l} u = \lim_{l\to\infty} u_l(x_l). \tag{5.41}$$

As $\bar{B}$ is compact, for any $l \in \mathbb{N}$ the upper semicontinuous function

$$y \mapsto u_l(y) - \varphi(y) - 2\delta(d(x,y))^2$$

attains a global maximum with respect to $\bar{B}$ at some point $y_l \in \bar{B}$. By extracting a subsequence, if necessary, we assume that $y_l \to z$ as $l \to \infty$ for some $z \in \bar{B}$. By $x_l \in \bar{B}$ for all $l \in \mathbb{N}$ and by the maximum property of the points y_l, $l \in \mathbb{N}$, we obtain

$$u_l(x_l) - \varphi(x_l) - 2\delta(d(x,x_l))^2 \le u_l(y_l) - \varphi(y_l) - 2\delta(d(x,y_l))^2 \tag{5.42}$$

for all $l \in \mathbb{N}$. By the choice of the points x_l we have $\lim_{l\to\infty} x_l = x$, whence we get by taking the limes inferior of (5.42) and invoking (5.41)

$$u^\star(x) = \liminf_{l\to\infty} u_l(x_l) \le \liminf_{l\to\infty} u_l(y_l) - \varphi(z) + \varphi(x) - 2\delta(d(x,z))^2. \tag{5.43}$$

As $u^\star - \varphi_\delta$ attains a global maximum with respect to $\bar{B}$ at x, we have

$$u^\star(z) - \varphi(z) - \delta(d(x,z))^2 \le u^\star(x) - \varphi(x). \tag{5.44}$$

Adding (5.43) and (5.44) yields

$$u^\star(z) \le \liminf_{l\to\infty} u_l(y_l) - \delta(d(x,z))^2. \tag{5.45}$$

Moreover, definition 5.4 implies that we have

$$\liminf_{l\to\infty} u_l(y_l) \le \limsup_{l\to\infty} u_l(y_l) \le u^\star(z). \tag{5.46}$$

By (5.45) and (5.46) it follows $d(x,z) = 0$, and thus

$$\lim_{l\to\infty} y_l = x. \tag{5.47}$$

Consequently, we may truncate the sequence $(y_l)_{l\in\mathbb{N}}$ such that all y_l lie in the *interior* of $\bar{B}$. Furthermore, (5.45) and (5.46) imply

$$\limsup_{l\to\infty} u_l(y_l) \le u^\star(x) \le \liminf_{l\to\infty} u_l(y_l),$$

and, consequently,

$$\lim_{l\to\infty} u_l(y_l) = u^\star(x). \tag{5.48}$$

Now fix $l \in \mathbb{N}$. We distinguish two cases.

Case 1. We have $y_l \neq x$. Then $y_l \in e_m$ with either $m = j$ or $m = k$. By the fact that y_l lies in the interior of $\bar{B}$ and by the definition of y_l, the function $u_l - \varphi_{2\delta}$ attains a local maximum at y_l, where

$$\varphi_{2\delta} : y \to \varphi + 2\delta(d(x,y))^2.$$

Note that $\varphi_{2\delta}$ is differentiable at y_l with

$$\partial_m \varphi_{2\delta}(y_l) = \partial_m \varphi(y_l) + 4\delta a_{im} d(x, y_l).$$

Hence it is an upper test function of u_l at y_l. As u_l satisfies the viscosity subsolution condition at y_l, it follows

$$H^m(\partial_m \varphi(y_l) + 4\delta a_{im} d(x, y_l), u_l(y_l), y_l) \leq 0. \tag{5.49}$$

Case 2. We have $y_l = x$. Then we have

$$\partial_m \varphi_{2\delta}(y_l) = \partial_m \varphi(y_l) \quad \text{for } m = j \text{ and } m = k.$$

As φ is (j,k)-differentiable at y_l, it follows that $\varphi_{2\delta}$ is (j,k)-differentiable at y_l. Hence $\varphi_{2\delta}$ is an upper (j,k)-test function of u_l at y_l. As u_l satisfies the viscosity subsolution condition at y_l, it follows

$$H^j(\partial_j \varphi(y_l), u_l(y_l), y_l) \leq 0. \tag{5.50}$$

Now let $l \to \infty$. By virtue of (4.10) (i) and (vii), the relations (5.49) and (5.50) in combination with (5.47), (5.48) yield

$$H^j(\partial_j \varphi(x), u^\star(x), x) \leq 0.$$

As $j, k \in \mathrm{Inc}_i$, $j \neq k$, have been chosen arbitrarily, it follows that $u^\star$ satisfies the viscosity subsolution condition at x. Hence $u^\star$ is a viscosity subsolution of (5.38). □

Theorem 5.1. *Assume that there is a viscosity subsolution* $w \in \mathrm{USC}(\Gamma)$ *and a viscosity supersolution* $W \in \mathrm{LSC}(\Gamma)$ *of*

$$H(\partial u(x), u(x), x) = 0 \quad \text{on } \Gamma \tag{5.51}$$

satisfying the boundary condition $w_\star(v_i) = w(v_i) = W^\star(v_i) = W(v_i) = g_i$ *for all* $i \in I_B$. *Furthermore assume that* w *is uniformly bounded by below on* Γ.

Define the function $u : \Gamma \to \mathbb{R}$ *by* $u(x) := \sup_{v \in X} v(x)$, *where*

$$X = \{v \in \mathrm{USC}(\Gamma) \text{ is a viscosity subsolution of (5.51) with } w \leq v \leq W \text{ on } \Gamma\}.$$

Then u *is a solution of (5.10).*

Proof. We first show that u is a viscosity subsolution of (5.51) with $u(v_i) = g_i$ for all $i \in I_B$. For this purpose observe that we have

$$g_i = w_\star(v_i) \le u_\star(v_i) \le u(v_i) \le u^\star(v_i) \le W^\star(v_i) = g_i \quad \forall\, i \in I_B, \tag{5.52}$$

implying $u_\star(v_i) \le u(v_i) \le u^\star(v_i) = g_i$ for all $i \in I_B$. By lemma 5.3 it follows that $u^\star$ is a viscosity subsolution of (5.51). Furthermore, as the viscosity subsolution $u^\star$ and the viscosity supersolution W coincide on $\partial\Gamma$ and as $u^\star$ is uniformly bounded by below, we conclude $u^\star \le W$ on Γ by lemma 5.2. Consequently, we have $u^\star \in X$ and therefore $u^\star \le u$ on Γ by the definition of u. As we furthermore have $u^\star \ge u$ by the definition of upper semicontinuous envelope, we find

$$u = u^\star \quad \text{on } \Gamma. \tag{5.53}$$

Suppose that $u_\star$ is a viscosity supersolution of (5.51). Then we apply lemma 5.2 to conclude $u_\star \ge u$, implying $u_\star = u$ by means of the definition of lower semicontinuous envelope, and the theorem is proved.

It therefore remains to show that $u_\star$ is a viscosity supersolution of (5.51). We apply an indirect argument: If $u_\star$ does *not* satisfy the supersolution condition at some point $y \in \Gamma_0$, then we construct a function $v \in X$ with $v(y^\star) > u(y^\star)$ for some $y^\star \in \Gamma_0$, contradicting the definition of u. As in the proof of lemma 4.2, we restrict ourselves to the case $y = v_i$ for some $i \in I_T$, as the case $y \in e_j$, $j \in J$, can be treated by similar (and simpler) arguments.

Accordingly, assume that $u_\star$ does not satisfy the supersolution condition at some point $y = v_i$ for some $i \in I_T$. Then by definition 5.3 (iv) there is an index $j \in \mathrm{Inc}_i$ for which there does *not* exist any i-feasible index $k \in \mathrm{Inc}_i$, $k \ne j$. Hence for each $k \in K := \mathrm{Inc}_i \backslash \{j\}$ there is a lower (j,k)-test function φ_k of $u_\star$ at y with

$$H^j(\partial_j \varphi_k(y), u_\star(y), y) < 0. \tag{5.54}$$

Observe that without loss of generality we may assume

$$\varphi_k(y) = u_\star(y) \tag{5.55}$$

for all $k \in K$. We also may assume that the functions φ_k, $k \in K$, are continuously differentiable in a neighborhood of y. In view of remark 5.3, relations (5.54) and (5.55) imply

$$H^j(\partial_j \varphi_k(y), \varphi_k(y), y) = H^k(\partial_j \varphi_k(y), \varphi_k(y), y) < 0. \tag{5.56}$$

Furthermore, as for each $k \in K$ the function φ_k is a lower (j,k)-test function of $u_\star$ at y, the function $u_\star - \varphi_k$ attains a local minimum at y. Note that we may assume these minima to be *strict* by possibly adding to each φ_k a quadratic function of the form $x \mapsto \alpha_k (d(x,y))^2$ for some $\alpha_k < 0$. Then there is a $t > 0$ with $\bar{B}_t(y) \subset (\cup_{j \in \mathrm{Inc}_i} e_j) \cup \{y\}$ such that

$$(u_\star - \varphi_k)(x) > 0 \quad \text{for all } k \in K \text{ and all } x \in \bar{B}_t(y) \backslash \{y\}. \tag{5.57}$$

Furthermore we can assume $t > 0$ to be sufficiently small such that for all $k \in K$ we have

$$H^j(\partial_j \varphi_k(x), \varphi_k(x), x) < 0 \quad \text{and} \quad H^k(\partial_k \varphi_k(x), \varphi_k(x), x) < 0 \quad \forall\, x \in \bar{B}_t(y). \tag{5.58}$$

This is possible by (5.56) and by the continuity of H, φ_k, $\partial_k \varphi_k$, and $\partial_j \varphi_k$ for all $k \in K$. By compactness of the set $\partial B_t(y)$ it then follows from (5.57) that there is a $\xi > 0$ such that

$$u_\star(x) - \varphi_k(x) > \xi \quad \text{for all } k \in K \text{ and all } x \in \partial B_t(y). \tag{5.59}$$

By (5.58) and the continuity of H we may furthermore assume $\xi > 0$ to be sufficiently small to make the relations

$$H^j(\partial_j \varphi_k(x), \varphi_k(x) + \xi, x) < 0 \quad \text{and} \quad H^k(\partial_k \varphi_k(x), \varphi_k(x) + \xi, x) < 0 \tag{5.60}$$

hold for all $x \in \bar{B}_t(y)$ and all $k \in K$. For the functions $\tilde{\varphi}_k := \varphi_k + \xi$, $k \in K$, it then follows

$$\tilde{\varphi}_k(y) = \varphi_k(y) + \xi = u_\star(y) + \xi > u_\star(y). \tag{5.61}$$

By definition of lower semicontinuous envelope there is a sequence $y_n \to y$, $y_n \in \Gamma_0$, such that $\lim_{n\to\infty} u(y_n) = u_\star(y)$. Therefore, by (5.61) and by the continuity of $\tilde{\varphi}_k$, $k \in K$, we can indicate a point $y^\star \in B_t(y)$ such that

$$\tilde{\varphi}_k(y^\star) > u(y^\star). \tag{5.62}$$

Observe also that by (5.59) we have

$$u(x) \geq u_\star(x) > \tilde{\varphi}_k(x) \quad \forall\, x \in \partial B_t(y). \tag{5.63}$$

We now define the function

$$\tilde{v} : \{y\} \cup \bigcup_{k \in \mathrm{Inc}_i} e_k \to \mathbb{R}, \quad \tilde{v}(x) := \begin{cases} \max_{k\in K} \tilde{\varphi}_k(x) & \text{if } x \in \bar{e}_j \\ \tilde{\varphi}_k(x) & \text{if } x \in e_k,\ k \in K. \end{cases}$$

Let us verify the viscosity subsolution condition of $\tilde{v}$ for all $x \in \bar{B}_t(y)$. First observe that $\tilde{v}$ is continuous at each $x \in \bar{B}_t(y)$. Then we distinguish two cases.

Case 1. Let $x \in \bar{B}_t(y) \cap e_l$ for some $l \in \mathrm{Inc}_i$. If $l \in K$, by (5.60) we have

$$H^l(\partial_l \tilde{\varphi}_l(x), \tilde{\varphi}_l(x), x) < 0. \tag{5.64}$$

As $\tilde{\varphi}_l$ is continuously differentiable at x, (5.64) immediately implies the viscosity subsolution condition of $\tilde{\varphi}_l$ at x. By the definition of $\tilde{v}$ it follows that $\tilde{v}$ satisfies the viscosity subsolution condition at x. On the other hand, if $l = j$, by (5.60) we have

$$H^l(\partial_l \tilde{\varphi}_k(x), \tilde{\varphi}_k(x), x) < 0$$

for each $k \in K$, which immediately implies the viscosity subsolution condition of $\tilde{\varphi}_k$ at x for each $k \in K$. The viscosity subsolution condition of $\tilde{v}$ at x then follows from proposition 5.1 and the definition of $\tilde{v}$.

Case 2. Let $x = y$. First let $l, m \in K$, $l \neq m$. Assume that there is an upper (l,m)-test function ψ of $\tilde{v}$ at x, as otherwise there is nothing to show. Set

$$d_l := a_{il}\partial_l\psi(x),\ d_m := a_{im}\partial_m\psi(x),\ e_l := a_{il}\partial_l\tilde{\varphi}_l(x),\ e_m := a_{im}\partial_m\tilde{\varphi}_m(x).$$

As ψ is (l,m)-differentiable at x, we have $d_l + d_m = 0$. If $d_l \leq 0$, we have $d_l \geq e_l$ by the definition of $\tilde{v}$ and by the fact that $\tilde{v} - \psi$ attains a local maximum at x. Consequently, $|d_l| \leq |e_l|$. Similarly, if $d_m \leq 0$, we have $d_m \geq e_m$, implying $|d_m| \leq |e_m|$. Altogether we have

$$|\partial_l\psi(x)| = |\partial_m\psi(x)| \leq \max\{|\partial_l\tilde{\varphi}_l(x)|, |\partial_m\tilde{\varphi}_m(x)|\}. \tag{5.65}$$

By (4.10) (i), (v), (vi), and (vii) the function

$$h : \mathbb{R} \to \mathbb{R}, \quad h(p) := H^s(p, \tilde{v}(x), x)), \quad s \in \mathrm{Inc}_i,$$

is independent of s, symmetric at $p = 0$ and strictly increasing in $|p|$. By (5.60) we have

$$h(\partial_l\tilde{\varphi}_l(x)) < 0 \quad \text{and} \quad h(\partial_m\tilde{\varphi}_m(x)) < 0.$$

Then by (5.65) it follows

$$H^l(\partial_l\psi(x), \tilde{v}(x), x) = h(\partial_l\psi(x)) \leq \max\{h(\partial_l\tilde{\varphi}_l(x)), h(\partial_m\tilde{\varphi}_m(x))\} < 0.$$

Now let $l \in K$, let ψ be an upper (j,l)-test function of $\tilde{v}$ at x, and set

$$d_j := a_{ij}\partial_j\psi(x),\ d_l := a_{il}\partial_l\psi(x),\ e_j := \max_{k\in K} a_{ij}\partial_j\tilde{\varphi}_k(x),\ e_l := a_{il}\partial_l\tilde{\varphi}_l(x).$$

As above we derive $|d_j| \leq |e_j|$ whenever $d_j \leq 0$, as well as $|d_l| \leq |e_l|$ whenever $d_l \leq 0$. Hence we arrive at

$$|\partial_j\psi(x)| = |\partial_l\psi(x)| \leq \max\{\max_{k\in K} |\partial_j\tilde{\varphi}_k(x)|, |\partial_l\tilde{\varphi}_l(x)|\}. \tag{5.66}$$

Then (5.60) and (5.66) imply

$$H^j(\partial_j\psi(x), \tilde{v}(x), x) = h(\partial_j\psi(x)) \leq \max\{\max_{k\in K} h(\partial_j\tilde{\varphi}_k(x)), h(\partial_l\tilde{\varphi}_l(x))\} < 0.$$

Hence we have shown that $\tilde{v}$ satisfies the viscosity subsolution condition for all $x \in \bar{B}_t(y)$. We now define the function $v : \Gamma \to \mathbb{R}$ by

$$v(x) := \begin{cases} \max\{\tilde{v}(x), u(x)\} & \text{if } x \in \bar{B}_t(y) \\ u(x) & \text{if } x \in \Gamma\backslash\bar{B}_t(y) \end{cases}$$

and show that $v \in X$. For this purpose first observe that the maximum of finitely many functions, which are upper semicontinuous at a given point, is also upper semicontinuous at this point. As the function $\tilde{v}$ is continuous on $\bar{B}_t(y)$, and as u is upper semicontinuous by (5.53), it follows that v is upper semicontinuous on $B_t(y)$. By (5.63) and by the definition of $\tilde{v}$ we then conclude that v is upper semicontinuous on Γ. Moreover, from proposition 5.1 it follows that v satisfies the viscosity subsolution condition at each $x \in \Gamma_0$, as both $\tilde{v}$ and u satisfy the viscosity subsolution condition on $\bar{B}_t(y)$. Furthermore we have $v(g_i) = u(g_i) = W(g_i)$ by construction of v, and lemma 5.2 implies $v \leq W$. Finally we clearly have $v \geq w$. Altogether we conclude $v \in X$. However, by (5.62) we have $v(y^\star) > u(y^\star)$, a contradiction to the definition of u. This completes the proof. □

5.6 Consistency with vanishing viscosity

The purpose of the present section is to establish a link of the concept of viscosity solutions on networks to the convergence result of chapter 4 by verifying the consistency of viscosity solutions with the method of vanishing viscosity. In fact, we show that the limit function of any uniformly converging class of solutions of the *viscous* boundary value problem (4.9) is a viscosity solution. Again we assume the Hamiltonian $H = (H^j)_{j \in J}$ to be of eikonal type, i. e., to satisfy the conditions (4.10).

Theorem 5.2. *Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence with $\varepsilon_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$ let $u_n \in C^2_K(\Gamma)$ be a solution of the boundary value problem (4.9) with $\varepsilon := \varepsilon_n$. Assume that the functions u_n converge uniformly to a limit function $u \in C(\Gamma)$. Furthermore assume that the functions u_n and their first derivatives be uniformly bounded in $n \in \mathbb{N}$ on Γ by a constant $0 \leq C < \infty$. Then u is a (viscosity) solution of (5.10).*

Remark 5.5. Observe that the uniform boundedness of u_n, $n \in \mathbb{N}$, and of their first derivatives is a consequence of the proof of lemma 4.5, whenever H is isotropic.

The proof of theorem 5.2 is given by the following collection of results. We first provide a short statement on the behavior of the Hamiltonian H at transition vertices. In fact, the following proposition is an immediate consequence of (4.10).

Proposition 5.2. *Let $i \in I_T$ and define the function $h : \mathbb{R} \to \mathbb{R}$ by*

$$h_i(p) := H^j(p, 0, v_i), \quad j \in \mathrm{Inc}_i.$$

Then h_i is independent of the choice of $j \in \mathrm{Inc}_i$. We have $h_i \in C^2(\mathbb{R})$, $h(0) < 0$, and $h_i(-p) = h_i(p)$ for all $p \in \mathbb{R}$. Furthermore, h_i is symmetric and strictly convex on $\mathbb{R}$. In particular, it is strictly increasing on $[0, \infty)$, and strictly decreasing on $(-\infty, 0]$. Finally, there is a unique number $a > 0$ such that $h_i(a) = h_i(-a) = 0$.

In the following propositions and lemmas let the assumptions and notations be as in theorem 5.2.

Lemma 5.4. *Let $i \in I_T$, $j \in \mathrm{Inc}_i$, and let $\xi > 0$. Furthermore let the sequence $(x_m)_{m\in\mathbb{N}}$, $x_m \in e_j$, converge to v_i. Then there is a number $m_\xi \in \mathbb{N}$ such that*

$$H^j(p_m, u(v_i), v_i) \leq \xi$$

for all $m \geq m_\xi$, where

$$p_m := \frac{u(x_m) - u(v_i)}{d(x_m, v_i)}.$$

Proof. Suppose without loss of generality that we have $a_{ij} = 1$ and $u(v_i) = u^j(0) = 0$. Setting $v := u^j$ and $y_m := \pi_j^{-1}(x_m)$ we then have to show that we have

$$h(v(y_m)/y_m) \leq \xi$$

for all sufficiently large $m \in \mathbb{N}$, where $h := h_i$ is the function defined in proposition 5.2.

Now set $v_n := u_n^j$ and plug $x := y_m z$ into (4.9) for $z \in [0, 1]$. Integration over z yields

$$\int_0^1 H^j(\partial_j v_n(y_m z), v_n(y_m z), y_m z)\, dz = \varepsilon_n \int_0^1 \partial_j^2 v_n(y_m z)\, dz.$$

Moreover we have

$$\left| \int_0^1 \partial_j^2 v_n(y_m x)\, dx \right| \leq 2 \max_{x\in[0,y_m]} |\partial_j v_n(x)| < 2C.$$

It follows

$$\lim_{n\to\infty} \int_0^1 H^j(\partial_j v_n(y_m x), v_n(y_m x), y_m x)\, dx = 0 \tag{5.67}$$

for all $m \in \mathbb{N}$.

As h is convex we may apply Jensen's inequality to compute for all $m, n \in \mathbb{N}$

$$\begin{aligned} h\left(\frac{v_n(y_m) - v_n(0)}{y_m}\right) &= h\left(\frac{1}{y_m}\int_0^{y_m} \partial_j v_n(x)\, dx\right) \\ &= h\left(\int_0^1 \partial_j v_n(y_m x)\, dx\right) \leq \int_0^1 h(\partial_j v_n(y_m x))\, dx \\ &\leq \int_0^1 H^j(\partial_j v_n(y_m x), v_n(y_m x), y_m x)\, dx + C_1 y_m \end{aligned} \tag{5.68}$$

for some constant $C_1 < \infty$. Here the last inequality follows by a Taylor expansion of H^j in the second and third argument as in combination with the uniform boundedness of v_n and $\partial_j v_n$ in n. Letting $n \to \infty$ in (5.68) yields by means of (5.67)

$$h\left(\frac{v(y_m)}{y_m}\right) \leq C_1 y_m$$

for all $m \in \mathbb{N}$. As $\lim_{m\to\infty} y_m = 0$, there is a number $m_\xi \in \mathbb{N}$ such that

$$h\left(\frac{v(y_m)}{y_m}\right) \leq \xi$$

for all $m \geq m_\xi$. The assertion follows. □

Proposition 5.3. *Let $j \in J$ and let $y \in (0, l_j)$. Assume there is a function $f \in C^2((0, l_j))$ such that $u^j - f$ attains a local maximum (minimum) at y. Then*

$$H^j(\partial_j f(y), u^j(y), y) \leq (\geq)\, 0.$$

Proof. Assume that $u^j - f$ attains a local maximum at y. Let $g \in C^2((0, l_j))$ be defined by $g(x) := |x-y|^2$. Then $u^j - f - g$ attains a strict local maximum at y. As the functions u_n converge uniformly to u, it follows that there is a sequence $(y_n)_{n\in\mathbb{N}}$, $y_n \in (0, l_j)$ for all $n \in \mathbb{N}$, and a number $n_0 \in \mathbb{N}$, such that $\lim_{n\to\infty} y_n = y$ and such that $u_n^j - f - g$ attains a local maximum at y_n for all $n \geq n_0$. We then have

$$\partial_j u_n^j(y_n) = \partial_j f(y_n) + \partial_j g(y_n) \quad \text{and} \quad \partial_j^2 u_n^j(y_n) \leq \partial_j^2 f(y_n) + \partial_j^2 g(y_n)$$

for all $n \geq n_0$. It follows

$$\begin{aligned} 0 &= \varepsilon_n \partial_j^2 u_n^j(y_n) - H^j(\partial_j u_n^j(y_n), u_n^j(y_n), y_n) \\ &\leq \varepsilon_n(\partial_j^2 f(y_n) + \partial_j^2 g(y_n)) - H^j(\partial_j f(y_n) + \partial_j g(y_n), u_n^j(y_n), y_n). \end{aligned}$$

By the continuity of H^j we obtain

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \left[\varepsilon_n(\partial_j^2 f^j(y_n) + \partial_j^2 g(y_n)) - H^j(\partial_j f^j(y_n) + \partial_j g(y_n), u_n^j(y_n), y_n)\right] \\ &= -H^j(\partial_j f^j(y), u^j(y), y) \end{aligned} \tag{5.69}$$

for all $n \geq n_0$. The case that $u^j - f^j$ attains a local minimum at y can be treated similarly. □

Lemma 5.5. *The limit function u satisfies $H(\partial u(x), u(x), x) \leq 0$ in the viscosity sense on Γ.*

Proof. The proof is divided into two parts. We verify the viscosity subsolution condition of u at points $x \in e_j$, $j \in J$, in the first part, and at transition vertices $x = v_i$, $i \in I_T$, in the second.

Part 1. Let $j \in J$ and let $x \in e_j$. Let φ be an upper test function of u at x. Without loss of generality we may assume that φ be continuously differentiable within a neighborhood of x. Then there is an open set $U \subset (0, l_j)$ with $x \in U$, and a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in C^2((0, l_j))$, such that $f_n|_U \to \varphi^j|_U$ with respect to the C^1-topology on U. Moreover, we can assume the local maximum of $u - \varphi$ at x to be strict by possibly adding a parabola centered at x. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in (0, l_j)$, and a number $n_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} z_n = z := \pi_j^{-1}(x)$ and such that $u^j - f_n$ attains a local maximum at z_n for all $n \geq n_1$. For each fixed $n \geq n_0$ we now apply proposition 5.3 with $f := f_n$ and conclude

$$H^j(\partial_j f_n(z), u^j(z), z) \leq 0.$$

The assertion follows from the fact that we have $\lim_{n\to\infty} \partial_j f_n(z_n) = \partial_j \varphi^j(z)$.

Part 2. Let $x = v_i$, $i \in I_T$. Let $j, k \in \mathrm{Inc}_i$, $j \neq k$. Furthermore let φ be an upper (j,k)-test function of u at x and assume $u(x) = 0$ without restriction. Then

$$h(\partial_j \varphi(x)) = H^j(\partial_j \varphi(x), u(x), x),$$

where $h := h_i$ is the function defined in proposition 5.2. Suppose

$$h(\partial_j \varphi(x)) > 0. \tag{5.70}$$

As φ is (j,k)-differentiable at x, it follows by proposition 5.2 that for one of the indices j, k, say for j, there is a number $\delta_0 > 0$ such that

$$a_{ij} \partial_j \varphi(v_i) = -(a + \delta_0), \tag{5.71}$$

with $a > 0$ as defined in proposition 5.2. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence with $x_m \in e_j$, $m \in \mathbb{N}$, and $\lim_{m\to\infty} x_m = x$. As $u - \varphi$ attains a local maximum at x, we obtain by (5.71)

$$p_m := \frac{u(x_m) - u(x)}{d(x_m, x)} < -(a + \delta_0/2)$$

for all sufficiently large $m \in \mathbb{N}$. By the properties of h (see proposition 5.2) it follows that there is a number $\delta_1 > 0$ such that

$$\delta_1 < h(p_m) = H^j(p_m, u(x), x)$$

for all sufficiently large $m \in \mathbb{N}$, a contradiction to lemma 5.4. Hence

$$H^j(\partial_j \varphi(x), u(x), x) = h(\partial_j \varphi(x)) \leq 0.$$

□

It remains to check that u is a viscosity supersolution. The following proposition provides the key result.

Proposition 5.4. *Let $i \in I_T$ and assume that there is an index $j \in \mathrm{Inc}_i$ such that $a_{ij} \partial_j u_n(v_i) \leq 0$ for infinitely many $n \in \mathbb{N}$. Furthermore assume that we have a function $f \in C^2(\Gamma)$ such that $u - f$ attains a local minimum at v_i. Then*

$$H^j(\partial_j f(v_i), u(v_i), v_i) \geq 0.$$

Proof. Without restriction we assume $u(v_i) = f(v_i) = 0$ and $a_{ij} = 1$. By assumption we can choose a subsequence of $(u_n)_{n\in\mathbb{N}}$ (which we again denote by $(u_n)_{n\in\mathbb{N}}$) such that $\partial_j u_n^j(0) \le 0$ for all $n \in \mathbb{N}$. By virtue of proposition 5.2 it then suffices to show $\partial_j f(v_i) \le -a$. To this end we assume the contrary, i.e. that there is a number $\delta > 0$ such that

$$\partial_j f(v_i) = -a + \delta. \tag{5.72}$$

First we assert that for each $n \in \mathbb{N}$ there is a $r > 0$ such that $u_n^j(x) < 0$ and $\partial_j u_n^j(x) < 0$ for all $0 < x < r$. This is clear if $\partial_j u_n(0) < 0$. If $\partial_j u_n(0) = 0$, the assertion follows from (4.9) and (4.10) (i). Next, as a direct consequence of (4.9) and (4.10) (i) we conclude that u_n^j does not attain a local minimum on $(0, l_j)$. Consequently,

$$u_n^j(x) \le u_n^j(0) \quad \text{for all } x \in [0, l_j] \text{ and all } n \in \mathbb{N}. \tag{5.73}$$

It follows

$$u^j(y) = \lim_{n\to\infty} u_n^j(y) \le \lim_{n\to\infty} u_n^j(0) = 0 \quad \text{for all } y \in [0, l_j]. \tag{5.74}$$

As $u - f$ attains a local minimum at v_i, (5.74) implies that we may restrict our considerations to the case $\delta \le a$. In this case we have

$$H^j(\partial_j f^j(0), 0, 0) = h(-a + \delta) < 0 \tag{5.75}$$

by (5.72) and by proposition 5.2. By continuity of H^j, by (4.10) (iii), and by proposition 5.2 it follows from (5.75) that there are numbers $\eta, \gamma > 0$ with $\eta < \min\{\delta, l_j\}$, such that

$$H^j(p, z, x) \le -\gamma \quad \text{for all } p \in [-\beta, 0], \text{ all } z \in (-\infty, \eta], \text{ and all } x \in [0, \eta], \tag{5.76}$$

where $\beta := a - \delta + \eta$. Choose a number $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0}\beta/\gamma < l_j$ and such that $u_n^j(0) < \eta$ for all $n \ge n_0$. Fix $n \ge n_0$. Setting $v_n := \partial_j u_n^j$, (4.9), (5.73), and (5.76) imply

$$\partial_j v_n(x) = H^j(v_n(x), u_n(x), x)/\varepsilon_n \le -\gamma/\varepsilon_n \tag{5.77}$$

for all $x \in [0, \eta]$ satisfying $-\beta \le v_n(x) \le 0$. In particular, as we have $v_n(0) \le 0$, we derive from (5.77) that there is a number x_n with

$$0 \le x_n \le \varepsilon_n\beta/\gamma \le \varepsilon_{n_0}\beta/\gamma < l_j, \tag{5.78}$$

such that

$$v_n(x_n) = -\beta. \tag{5.79}$$

We furthermore claim that

$$v_n(x) \le -\beta \quad \text{for all } x_n < x \le \eta. \tag{5.80}$$

For if this were not the case, there would be an x_0 with $x_n < x_0 < \eta$, such that $v_n(x_0) = -\beta$ and $\partial^j v_n(x_0) \ge 0$. This, however, contradicts to (5.77).

Now (5.80) and (5.73) imply

$$u_n^j(y) = u_n^j(x_n) + \int_{x_n}^{y} v_n(s)\,ds \leq u_n^j(x_n) - (y - x_n)\beta \leq u_n^j(0) - (y - x_n)\beta$$

for all y with $x_n \leq y \leq \eta$. Using (5.78) we conclude

$$u^j(y) = \lim_{n\to\infty} u_n^j(y) \leq -y\beta = y(-a + \delta - \eta)$$

for all $0 \leq y \leq \eta$. As $u^j - f^j$ attains a local minimum at 0, it follows that there is a radius $r > 0$ such that

$$f^j(y) \leq y(-a + \delta - \eta)$$

for all $0 \leq y \leq r$, a contradiction to (5.72). □

Lemma 5.6. *The limit function u satisfies $H(\partial u(x), u(x), x) \geq 0$ in the viscosity sense on Γ.*

Proof. The proof splits into two parts, the first part treating points $x \in e_j$, $j \in J$, the second part treating transition vertices $x = v_i$, $i \in I_T$. We skip the presentation of the first part, as it is in perfect analogy with part 1 of the proof of lemma 5.5.

Accordingly, let $x = v_i$, $i \in I_T$. As the functions u_n, $n \in \mathbb{N}$, satisfy the Kirchhoff condition at x, there is an index $j \in \mathrm{Inc}_i$ such that

$$a_{ij}\partial_j u_n(x) \leq 0 \tag{5.81}$$

for infinitely many $n \in \mathbb{N}$. We show that u satisfies the viscosity supersolution condition at x by verifying that j is an i-feasible index for each $k \in K := \mathrm{Inc}_i \backslash \{j\}$. Fix $k \in K$ and assume that there is a lower (j,k)-test function φ of u at x (if not, then there is nothing to prove). Without restriction we can assume $a_{ij} = 1$ and that there is a number $r > 0$ such that φ^j is continuously differentiable on $[0, r)$. Then there is a sequence $(f_m)_{m\in\mathbb{N}}$ of functions $f_m \in C^2([0, l_j])$, $m \in \mathbb{N}$, converging to φ^j with respect to the C^1-topology of $[0, r]$. Moreover we can assume the (right-sided) minimum of $u^j - \varphi^j$ at 0 to be strict. Then there is a number $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ there is a $z_m \in [0, r)$ where $u^j - f_m$ attains a local minimum and such that $\lim_{m\to\infty} z_m = 0$. For each fixed $m \geq m_0$ we now apply proposition 5.3 (if $z_m > 0$) or proposition 5.4 (if $z_m = 0$) to conclude

$$H^j(\partial_j f_m(z_m), u^j(z_m), z_m) \geq 0.$$

As we have $\lim_{m\to\infty} \partial_j f_m(z_m) = \partial_j \varphi^j(0)$, we obtain

$$\begin{aligned} H^j(\partial_j\varphi(x), u(x), x) &= H^j(\lim_{m\to\infty} \partial_j f_m(z_m), \lim_{m\to\infty} u^j(z_m), \lim_{m\to\infty} z_m) \\ &= \lim_{m\to\infty} H^j(\partial_j f_m(z_m), u^j(z_m), z_m) \geq 0. \end{aligned}$$

Hence j is an i-feasible index for k at x, and, by symmetry, k is i-feasible for j at x. As the choice of $k \in K$ was arbitrary, the assertion follows. □

Corollary 5.2. *The limit function u in theorem 4.2 is independent of the choice of the sequence ε_n.*

Proof. This is a consequence of theorem 5.2 and corollary 5.1. □

5.7 Example: the eikonal equation

We return to our standard example, the eikonal equation, and its Dirichlet problem

$$\begin{aligned} (\partial_j u)^2 - 1 = 0 \qquad & \text{on } e_j,\ j \in J, \\ u(v_i) = g_i \qquad & \text{for all } i \in I_B. \end{aligned} \tag{5.82}$$

Lemma 5.7. *Assume $d(v_i, v_j) \geq |g_i - g_j|$ for all $i, j \in I_B$. Then the function*

$$f \in C(\Gamma),\ f(x) := \min_{i \in I_B} (d(x, v_i) + g_i),$$

is the (unique) viscosity solution of (5.82).

Proof. Fix $j \in I_B$. Observe that we have $f(v_j) \neq g_j$ if and only if there is no $i \in I_B$, $i \neq j$, such that $g_j > d(v_j, v_i) + g_i$. This, however, is ruled out by assumption, whence $f(v_i) = g_i$ for all $i \in I_B$. It remains to verify the viscosity sub- and supersolution condition everywhere on Γ_0. For this purpose observe that f is continuously differentiable with $|\partial_j f^j(x)| = 1$ (and thus trivially satisfies the viscosity sub- and supersolution conditions) for almost every $x \in (0, l_j)$). Let $x \in (0, l_j)$ be a singular point. From the definition of f it follows $\partial_j^- f^j(x) = +1$ for the left-sided derivative and $\partial_j^+ f^j(x) = -1$ for the right-sided derivative. Consequently, we have $(\partial_j \varphi(x))^2 - 1 \leq 0$ for all upper test functions φ at x. Lower test functions do not exist.

Now let $x = v_i$ for some $i \in I_T$. Observe that the definition of f implies that there are disjoint sets I^- and I^+ with $I^- \cup I^+ = \mathrm{Inc}_i$, such that we have $a_{ij} \partial_j f(x) \in= -1$ for all $j \in I^-$ and $a_{ij} \partial_j f(x) \in= +1$ for all $j \in I^+$. Furthermore we have $|I^-| \geq 1$. Now let $j, k \in \mathrm{Inc}_i$. If $j, k \in I^-$, we conclude $(\partial_j \varphi(x))^2 - 1 \leq 0$ for all upper (j,k)-test functions φ of f at x. If $j \in I^-$, $k \in I^+$, then f is (j,k)-differentiable at x and the viscosity subsolution condition is trivially satisfied. If $j, k \in I^+$, then there is no upper (j,k)-test function. Furthermore, let $j \in \mathrm{Inc}_i$. If $j \in I^-$, then each index $k \in \mathrm{Inc}_i$, $k \neq j$, is i-feasible for j: If $k \in I^+$, then f is (j,k)-differentiable at x; if $k \in I^-$, then there is no lower (j,k)-test function at x. Finally, if $j \in I^+$, then each index $k \in I^-$ is i-feasible for j, as then f is (j,k)-differentiable at x. As $|I^-| > 0$, such an index exists. □

5.8 Optimal path integrals

The (classical) theory of viscosity solutions is closely related to the theory of optimal control of differential equations and to the theory of dynamic programming. The basic idea of dynamic programming is to assign to each fixed point (x, t) in time and space a *least cost value* $C(x, t)$, representing the infimum of all appropriately defined cost integrals associated with certain *control functions*. The control functions can be thought of as steering devices which impact the dynamics of the underlying differential equation, whereas the corresponding cost integral expresses the expenses produced by the trajectory which is governed by the differential equation steered by the control function. The plan is to choose for (x, t) the control function producing the least cost $C(x, t)$. It turns out that the *value function* $(x, t) \to C(x, t)$ is a viscosity solution of a certain *Hamilton-Jacobi-Bellman equation* associated with the problem, where for this purpose viscosity solutions have to be extended to Hamilton-Jacobi equations involving time.

The idea of viscosity solutions expressing optimality with respect to certain cost (or path) integrals is also reflected when time is not involved. In the case of the eikonal equation this is clear: The distance function on a domain or on a network tells us the shortest way to the boundary. A similar principle holds for a more general class of stationary Hamilton-Jacobi equations, a class which we call *anisotropic eikonal equations.*

A closer investigation of the structure of this class is the topic of the following chapter.

CHAPTER 6

Singularities of viscosity solutions

Summary. In the present chapter we study the structure of viscosity solutions of a special class of Hamilton-Jacobi equations on networks, the so-called *anisotropic eikonal equations*, and compute their number of singular points. For illustrating reasons we interpret these solutions as maximal volume equilibrium configurations of granular matter (sand) placed upon a network, the singular points corresponding to the number of "hilltops" of the configuration. We also interpret the solutions in terms of optimality of certain corresponding path integrals, establishing a relation to optimal control theory. Finally we discuss connections to abstract graph theory and shortest path algorithms appearing in computer scientific contexts.

6.1 Introduction

The behavior of homogeneous granular matter, when being poured onto objects from sources above, is rather well understood and has been described by various models. Several one- and two-dimensional models have been proposed by Hadeler et al ([HK99], [HK01], [HKG02]). In particular, situations have been examined where a maximal amount of homogeneous granular material is deposited on top of a given object or domain. As has been pointed out in chapter 3, the shape of these maximal volume configurations on flat and bounded domains (such as tables) is described by the graph of the unique viscosity solution u (in the classical sense) of the eikonal equation $|\nabla u| = \tan\alpha$, where α is the angle of repose of the respective granular material. The function u has an almost everywhere existing gradient of length $\tan\alpha$ and coincides with (a multiple of) the distance function $\boldsymbol{d}$.

We now focus on the granular matter interpretation of the distance function $\boldsymbol{d}$ on a topological network Γ. In fact, it describes (up to multiplication by a constant) the shape of the "sand heap" formed by a maximal amount of homogeneous granular matter placed upon Γ. Moreover, it is the unique viscosity solution of the eikonal equation on Γ. In a more general approach we assume the angle of repose to be a function of the local position within the network and of the local height of the heap, taking account of the possibility that the properties of the granular material may vary with position and height. Then it is not hard to see that the corresponding maximal volume configuration is given by the viscosity solution of an appropriate *anisotropic eikonal equation* to be introduced below.

In any case, an important feature of these maximal volume configurations is their number of local maxima. Our concerns of the present chapter are with deriving a simple formula for this number. In the special case of the eikonal equation, this result is related to a theorem of Aviles and Giga ([AG96], lemma 2.6), basically stating that the suitably defined curvature functional of the distance function on an n-dimensional domain Ω equals the $n-1$-dimensional Hausdorff measure of the boundary $\partial\Omega$.

6.2 Maximal volume configurations

As we have outlined in chapter 1, we can consider a topological network Γ as a maze in the plane and think of its edges e_j, $j \in J$, as a set of slim "paths" whose end points are connected at the vertices v_i, $i \in I$. Let us suppose the paths on both sides to be bounded by thin, sufficiently high glass walls perpendicular to the plane. As before we split the vertex set into a nonempty set of boundary vertices $\partial\Gamma = \{v_i, i \in I_B\}$ and a set of transition vertices $\{v_i, i \in I_T\}$, assuming that sand may run out of the maze through a hole at the bottom at boundary points, whereas at transition vertices sand may be interchanged between the different incident paths. Let us then uniformly pour as much sand as possible from above into the space between the glass walls. Several sand heaps will grow, every two of them separated by at least one boundary point. Finally the shape of the sand heaps will stop growing and reach an equilibrium configuration. Similarly to the case of non-ramified domains it is easy to understand that in this maze model the contours of the maximal volume equilibrium are described by (a multiple of) the distance function $\boldsymbol{d}$ to the boundary—under the constraint that the angle of repose is constant.

It is, however, also plausible to assume that the angle of repose is *not* constant, but depends on the position within the network as well as on the local height of the sand pile. In fact, in the sequel we focus our interest on solutions of what we call the *anisotropic eikonal equation.*

Let Γ be a topological network as in chapter 5 and let $f : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

$$\begin{array}{ll} (i) & f^j \in C^2([0,l_j] \times \mathbb{R}), \text{ with } f^j(y,z) := f(\pi_j(y),z) \ \forall\ (y,z) \in [0,l_j] \times \mathbb{R} \\ (ii) & f(x,z) \text{ is non-increasing in } z \in \mathbb{R} \text{ for all } x \in \Gamma \\ (iii) & f(x,z) > 0 \text{ for all } (x,z) \in \Gamma \times \mathbb{R} \\ (iv) & f^j(\pi_j^{-1}(v_i),z) = f^k(\pi_k^{-1}(v_i),z) \text{ for all } i \in I,\ j,k \in \mathrm{Inc}_i. \end{array} \qquad (6.1)$$

Then the Hamiltonian $H = (H^j)_{j \in J}$ given by

$$H^j(p, z, x) := p^2 - (f^j(x, z))^2 \quad \text{for all } p \in \mathbb{R},\, z \in \mathbb{R},\, x \in [0, l_j],\, j \in J, \tag{6.2}$$

is of eikonal type, i. e., it satisfies the conditions (4.10).

Making use of the simplified notation introduced in (5.2), we call the boundary value problem

$$\begin{cases} H^j(\partial_j u(x), u(x), x) = 0 & \text{for all } x \in e_j,\, j \in J, \\ u(v_i) = 0 & \text{for all } i \in I_B \end{cases} \tag{6.3}$$

the *Dirichlet problem of the anisotropic eikonal equation*.

Lemma 6.1. *There is a unique viscosity solution $u \in C(\Gamma)$ of boundary value problem (6.3).*

Proof. Observe that the function $w :\equiv 0 \in C(\Gamma)$ is a viscosity subsolution, whereas the function $W := \kappa \boldsymbol{d} \in C(\Gamma)$ with $\kappa := \max_\Gamma f(x, 0)$ is a viscosity supersolution of (6.3). Then there is a unique viscosity solution u of (6.3) according to theorem 5.1 and lemma 5.2. □

Let us explain in what sense the function u describes as a maximal volume configuration of granular matter: Let $v \in C(\Gamma)$ be a (continuous) viscosity subsolution of (6.3) with $v \geq 0$ on Γ, $v \equiv 0$ on $\partial\Gamma$. Let $x \in e_j$ for some $j \in J$. Note that the viscosity subsolution condition does not permit the existence of a differentiable function φ touching v from above such that the local gradient of φ at x is greater than $f(x, v(x))$. If we interpret $f(x, z)$ as the angle of repose the material possesses at position x and height z, we conclude that the granular matter configuration described by v does not violate the angle of repose, and thus is stable. The same holds if $x = v_i$, $i \in I_T$: Sand deposited at x is only stable if it cannot run downhill along neither of the incident edges e_j, $j \in \mathrm{Inc}_i$. This is indeed the case, as the viscosity subsolution condition ensures that the angle of repose is not violated *for any combination of two different incident edges*. Now lemma 5.3 basically states that u itself also satisfies the viscosity subsolution and thus describes a stable configuration. By the construction of u as the pointwise supremum over viscosity subsolutions, we have

$$\int_\Gamma u\, dx := \sum_{j \in J} \int_0^{l_j} u^j(x)\, dx \geq \int_\Gamma v\, dx$$

for all such stable configurations v. This means that u maximizes the volume functional among all stable configurations described by nonnegative viscosity subsolutions of (6.3) with zero boundary data.

6.3 The order of singularities

A characteristic feature of a maximal volume configuration is its number of peaks or hilltops, which—in a specified sense—we will call the *order of singularities*. As will turn out, this quantity is exclusively determined by the number of edges and transition vertices and thus is independent of the choice of f or the precise shape of Γ.

Let u be the viscosity solution of (6.3).

Proposition 6.1. *The function u does not attain a local minimum on Γ_0.*

Proof. This is an immediate consequence of the fact that u is a viscosity solution of (6.3). □

Lemma 6.2. *Let $j \in J$ and let $k, l \in I$, $k \neq l$, such that $j \in \mathrm{Inc}_l \cap \mathrm{Inc}_k$. Moreover let*

$$\sigma_{ij} := sign(a_{ij}\partial_j u(\pi_j^{-1}(v_i))) \in \{-1, 1\} \quad \text{for all } j \in \mathrm{Inc}_i,\ i \in I.$$

(i) u^j attains a local maximum at $x \in (0, l_j)$, if and only if x is a singular point of u^j.

(ii) We have either $s_j = 0$ or $s_j = 1$, where

$$s_j := |\{x \in (0, l_j) \,|\, x \text{ is a singular point of } u^j\}|.$$

(iii) We have $s_j = 1$, if and only if $\sigma_{kj} = \sigma_{lj} = 1$.

(iv) We have $s_j = 0$, if and only if $\sigma_{kj} + \sigma_{lj} = 0$.

Before we prove lemma 6.2, we provide the following

Proposition 6.2. *Let $a < b$ and let $w \in C([a, b])$ be a viscosity solution of*

$$(w'(x))^2 - (g(x, w(x)))^2 = 0 \tag{6.4}$$

on $[a, b]$ with $g \in C^2([a, b] \times \mathbb{R})$ and $g(x, z) > 0$ for all $x \in [a, b]$, $z \in \mathbb{R}$. Furthermore assume that $g(x, z)$ is non-increasing in z for all $x \in [a, b]$. Then one of the following cases is true:

(i) $w \in C^1([a, b])$ and either $w' > 0$ or $w' < 0$ everywhere on $[a, b]$

(ii) w has exactly one singular point $x \in (a, b)$ and we have $w' > 0$ on (a, x) and $w' < 0$ on (x, b).

Proof. (of proposition 6.2). Consider the initial value problems

$$w'(x) - g(x, w(x)) = 0 \quad \text{on } [a, b] \tag{6.5}$$

and

$$w'(x) + g(x, w(x)) = 0 \quad \text{on } [a, b]. \tag{6.6}$$

As $g \in C^2([a,b] \times \mathbb{R})$ and as g is non-increasing in the second variable, standard ODE theory ensures the existence of two functions $w_a, w_b \in C^1([a,b])$ satisfying (6.5) and (6.6), respectively, as well as $w_a(a) = w(a)$ and $w_b(b) = w(b)$, respectively. Furthermore, we have

$$w_a'(x) = g(x, w_a(x)) > 0 \text{ and } w_b'(x) = -g(x, w_b(x)) < 0 \quad \text{on } [a,b].$$

Moreover, both w_a and w_b satisfy (6.4) in the classical and thus in the viscosity sense. We set $\tilde{w} := \min\{w_a, w_b\}$ and distinguish the following cases:

(i) $w(a) < w_b(a)$ and $w(b) < w_a(b)$. Then $\tilde{w}$ has exactly one singular point $x \in (a,b)$. Observe that lower test functions of $\tilde{w}$ do not exist at x, whereas we have $|\varphi'(x)| \in [-g(x, \tilde{w}(x)), g(x, \tilde{w}(x)]$ for all upper test functions φ of $\tilde{w}$ at x. We conclude that $\tilde{w}$ satisfies (6.4) in the viscosity sense at x. The same is trivially the case everywhere else on (a,b). As we have $\tilde{w}(a) = w(a)$ and $\tilde{w}(b) = w(b)$, the comparison theorem for viscosity solutions (e.g. [CIL92], theorem 3.3) implies $\tilde{w} = w$.

(ii) $w(a) = w_b(a)$ or $w(b) = w_a(b)$. It is immediately clear that we either have $\tilde{w} = w_a$ or $\tilde{w} = w_b$, as well as $\tilde{w}(a) = w(a)$ and $\tilde{w}(b) = w(b)$. Hence we have $\tilde{w} = w_a$ or $\tilde{w} = w_b$ by the comparison theorem for viscosity solutions. In both cases we conclude $w \in C^1([a,b])$ and either $w' < 0$ or $w' > 0$ everywhere on $[a,b]$.

(iii) $w(a) > w_b(a)$ or $w(b) > w_a(b)$. This case does not occur (as the two cases are symmetric, we consider only the second one): Let $c \in [a,b]$ be the maximal number satisfying $w(c) = \max_{x \in [a,b]} w(x)$. From $w(a) > w_b(a)$, $w(b) = w_b(b)$, and from the fact that w_b is decreasing on $[a,b]$, it follows $c < b$ and $w(c) > w_b(c)$. Furthermore, w is non-increasing on $[b,c]$, as otherwise w would attain a local minimum at some point $x \in (c,b)$ by the choice of c. This, in turn, would imply the existence of a lower test function φ of w at x with $\varphi'(x) = 0$, a contradiction.

Since w is the viscosity solution of (6.4), it satisfies (6.4) in the classic sense almost everywhere on $[c,b]$. Hence there is a set $N \subset [c,b]$ of measure zero such that (6.4) is satisfied in the classic sense on $[c,b] \backslash N$. Observe that we have $w'(x) = -g(x, w(x)) < 0$ for all $x \in [c,b] \backslash N$, as w is non-increasing. Now note that we can choose a number $\gamma \in \mathbb{R}$ with

$$0 < \gamma < \min\{1, \min_{x \in [c,b]} g(x, w_b(c) + b - c)\},$$

such that the unique point $z \in (c,b)$ with $v(z) = w(z)$ is not contained in N, where we set

$$v(x) := w_b(c) + \gamma x \quad \text{for all } x \in [c,b].$$

By the choice of γ the function v is a viscosity subsolution of (6.4) on $[c,b]$. Now set $\tilde{v}(x) := \min\{v(x), w(x)\}$ for all $x \in [c,b]$. We show that $\tilde{v}$ satisfies

$$(w'(x))^2 - (g(x, w(x)))^2 \leq 0 \tag{6.7}$$

at z in the viscosity sense. For this purpose let φ be an upper test function of $\tilde{v}$ at z. By $w'(z) = -g(z, w(z) < 0$ and $v'(z) = \gamma > 0$, we conclude

$$-g(z, w(z)) \leq \varphi'(z) \leq \gamma \leq g(z, w(z)),$$

which implies the assertion. Furthermore, $\tilde{v}$ trivially satisfies (6.7) everywhere else on (c, b). As we have $w_b(c) = \tilde{v}(c)$ and $w_b(b) = \tilde{v}(b)$, the comparison theorem for viscosity solutions implies $w_b \geq \tilde{v}$ on $[c, b]$, a contradiction. □

Proof. (of lemma 6.2). For each $j \in J$ observe that u^j is a (classical) viscosity solution of

$$H^j(\partial_j u^j(y), u^j(y), y) = 0 \quad \text{on } [0, l_j].$$

Then the lemma follows from proposition 6.2 and the properties of H. □

Lemma 6.2 basically states that edges either contain no or exactly one singular point, inducing a partition $J = J_R \dot{\cup} J_S$, where J_R and J_S consist of the indices of the *regular* and *singular* edges, respectively.

Definition 6.1. The function $\kappa^E : J \to \{0, 1\}$ given by

$$\kappa^E(j) := \begin{cases} 1 & \text{if } j \in J_S \\ 0 & \text{if } j \in J_R \end{cases}$$

is called the *order of singularity of the edges.*

Let $i \in I_T$. Lemma 6.2 tells us that on each incident edge v_j, $j \in \text{Inc}_i$, the graph of u leaves v_i either "uphill" or "downhill". Informally speaking, the more incident edges lead "downhill", the more v_i assumes the character of a local maximum and the higher v_i should be weighted when counting the singularities. As will turn out, the following definition of the order of singularity at transition vertices is correct, since it keeps the total number of singularities invariant under small transformations of Γ.

Definition 6.2. Let $i \in I$. We define the sets

$$\text{Inc}_i^+ := \{j \in \text{Inc}_i \,|\, \sigma_{ij} = 1\} \quad \text{and} \quad \text{Inc}_i^- := \{j \in \text{Inc}_i \,|\, \sigma_{ij} = -1\}.$$

Definition 6.3. The function $\kappa^V : I \to \mathbb{N}$ given by

$$\kappa^V(i) := |\text{Inc}_i^-|$$

is called the *order of singularity of the vertices.*

Remark 6.1. Observe that $\kappa^V(i) = 0$ for all $i \in I_B$. Furthermore, proposition 6.1 implies $\kappa^V(i) \geq 1$ for all $i \in I_T$.

Definition 6.4. The quantity

$$S := \sum_{i \in I} \kappa^V(i) + \sum_{j \in J} \kappa^E(j)$$

is called the *order of singularity.*

The following theorem states the main result of the present chapter.

Theorem 6.1. *We have*

$$S = |J|.$$

Proof. Observe that we have

$$\kappa^V(i) = \frac{1}{2} \sum_{j \in \mathrm{Inc}_i} (1 - \sigma_{ij}) \quad \text{and} \quad \kappa^E(j) = \frac{1}{2}(\sigma_{k(j)j} + \sigma_{l(j)j})$$

where $k(j), l(j) \in I$ such that $j \in \mathrm{Inc}_{k(j)} \cap \mathrm{Inc}_{l(j)}$. Using this, we compute by means of definition 6.4

$$S = \frac{1}{2} \left[\sum_{i \in I} \sum_{j \in \mathrm{Inc}_i} (1 - \sigma_{ij}) + \sum_{j \in J} (\sigma_{k(j)j} + \sigma_{l(j)j}) \right] = \frac{1}{2} \sum_{i \in I} \deg v_i = |J|.$$

□

An immediate consequence is

Corollary 6.1. *Assume*

$$\kappa^V(i) = 1 \quad \textit{for all } i \in I_T \tag{6.8}$$

and let M be the number of local maxima of u on Γ. Then we have

$$M = |J| - |I_T|.$$

In a way, condition (6.8) represents the *generic case*, which we will elaborate on in the next section. We emphasize that the fundamental consequence of corollary 6.1 is given by the fact that the cardinal number of the singular set of the viscosity solution of (6.3) only depends on the number of edges and the number of transition vertices of Γ (provided that (6.8) is satisfied). In particular, neither the edge lengths nor the choice of the function f have to be taken into account. Both quantities, however, have an substantial impact on the location of the singular points.

6.4 Cost integrals

As already mentioned in section 5.8, the viscosity solution of boundary value problem (6.3) allows an alternative interpretation in terms of optimal path integrals. In this section we even admit arbitrary boundary values, i.e., we consider the boundary value problem of the form

$$\begin{cases} H^j(\partial_j u(x), u(x), x) = 0 & \text{for all } x \in e_j,\ j \in J, \\ u(v_i) = g_i & \text{for all } i \in I_B. \end{cases} \tag{6.9}$$

Let $\gamma : [0,1] \to \Gamma$ be a piecewise differentiable path in the sense that there are finally many numbers

$$x_0 := 0 < x_1 < \cdots < x_m < 1 =: x_{m+1}$$

such that for all $l = 0, \ldots, m$ we have $\gamma([x_l, x_{l+1}]) \subseteq \bar{e}_{j_l}$ for suitable $j_l \in J$ and $\pi_{j_l}^{-1} \circ \gamma \in C^1((x_l, x_{l+1}))$. Let $x \in \Gamma$ and let $\mathcal{C}_x$ be the set of all such paths γ with $\gamma(0) = x$ and $\gamma(1) = v_{i_\gamma}$ for some $i_\gamma \in I_B$. In other words, $\mathcal{C}_x$ consists of all paths leading from x to the boundary.

By the properties (6.1) of f it is now easy to see that for each $x \in \Gamma$ and each $\gamma \in \mathcal{C}_x$ there is a unique function $\phi^\gamma : [0,1] \to \mathbb{R}$ satisfying $\phi^\gamma(1) = g_{i_\gamma}$ and

$$\frac{d}{ds}\phi^\gamma(s) = -f(\gamma(s), \phi^\gamma(s)) \left| \frac{d}{ds}\gamma(s) \right|$$

for all $s \in [0,1]$ with $x_l < s < x_{l+1}$ for some $l = 1, \ldots, m$, where we set

$$\left| \frac{d}{ds}\gamma(s) \right| := \left| \frac{d}{ds}(\pi_{j_l} \circ \gamma)(s) \right|$$

for $x_l < s < x_{l+1}$.

In view of the previous results it is not hard to verify the following fact.

Lemma 6.3. *The function*

$$u(x) = \inf_{\gamma \in \mathcal{C}_x} \phi^\gamma(0)$$

is a viscosity solution of (6.9), which, however, does not necessarily attain all boundary values. The smallest boundary value is attained in any case.

The value $\phi^\gamma(0)$ can be interpreted as the expenses arising when one moves on γ from x to the boundary point v_{i_γ}, composed by the *terminal cost* g_{i_γ} and the *running cost per space unit* f. Hence $u(x)$ can be considered as the minimal cost arising when one is allowed to choose among all possible paths leading to arbitrary boundary points.

Example 6.1. An elegant visualization is obtained when the travelling direction is inverted: Think of a traveller starting at the boundary $\partial\Gamma$ and intending to reach the point

$x \in \Gamma$ on the cheapest way possible. Again the cost is composed by a fixed cost g_i associated with the chosen starting boundary vertex v_i, $i \in I_B$, and the running cost f. Observe that the conditions (6.1) include that $f(x, z)$ is non-increasing in z. This corresponds with the fact that the farther the traveller has moved away from the boundary, the less running expenses arise. This circumstance may be related to a scenario where the traveller has to carry the fuel needed for the journey by himself. The farther he gets, the less fuel is to carry, the easier he can move along his way.

This example implies a sufficient condition for the attaining of the boundary values g_i, $i \in I_B$: A boundary point v_{i_0} will never be chosen as a starting point by the traveller, if its fixed cost g_{i_0} exceeds the cost for the journey from another boundary point to v_{i_0}, as then each journey starting from v_{i_0} can be replaced by a cheaper journey starting from somwhere else. Hence the boundary value of v_{i_0} will not be attained by u. On the other hand, if it can be ruled out that there exists any journey between two boundary points which is cheaper than the fixed cost of the target point, all boundary values will be attained by u. A special case of this condition is given in lemma 5.7.

6.5 Degree of freedom

The generality of theorem 6.1 and the rather abstract definition of order of singularity at transition vertices (definition 6.3) allow us to embed them into the least cost interpretation above. Let u be the viscosity solution of boundary value problem 6.3 and set

$$D(x) := \lim_{r \to 0} |\partial B_r(x) \cap \{y \in \Gamma \,|\, u(y) < u(x)\}|.$$

We interpret $D(x)$ as the number of different possible directions leading from x to the boundary with minimal cost. Observe that we have $D(x) \geq 1$ for all $x \in \Gamma_0$ by proposition 6.1. On the other hand, at any $x \in \Gamma_0$ with $D(x) > 1$ one has the freedom to choose among $D(x)$ different directions, whence we may call $D(x) - 1$ the *degree of freedom of* x. Accordingly, we call the quantity $\sum_{x \in \Gamma_0} (D(x) - 1)$ the *total degree of freedom of* Γ.

Lemma 6.4. *We have* $\sum_{x \in \Gamma_0} (D(x) - 1) = |J| - |I_T|$.

Proof. Observe that we have

$$\sum_{x \in e_j} (D(x) - 1) = \kappa^E(j) \quad \text{for all } j \in J$$

and

$$D(x) = \kappa^V(i) \quad \text{for all } i \in I_T.$$

Then the assertion follows from theorem 6.1. □

6.6 Non-uniqueness of shortest path algorithms

As graphs are persuasively simple and versatile objects, graph theoretical problems subsume many concrete problems arising in the theory of computer algorithms. A prominent question is how to efficiently detect shortest paths connecting a given vertex with prescribed *source* vertices in a weighted graph. Dijkstra's classical algorithm [Dij59] basically was the first managing the situation where there is exactly one source vertex (single-source shortest path problem), and is followed by a long list of modifications or more specific approaches to the same problem (see for instance [AMOT90], [Bel58], [FF62], and [Tho99]).

Dijkstra's algorithm successively lists all shortest paths from the single source to the other vertices, starting at the source vertex. In terms of a topological network Γ with exactly one boundary vertex $i_0 \in I_B$ it does the following: Starting at the source vertex v_{i_0}, it determines the level sets $L_t := \{x \in \Gamma \mid \boldsymbol{d}(x) = t\}$ for continuously increasing $t \geq 0$. As soon as a set L_t contains one or more vertices, these vertices are assigned the shortest distance t to the source, along with the way "back downhill" as shortest path to the source (which is not necessarily unique). This procedure is continued until a shortest path is assigned to each vertex. Whereas Dijkstra's algorithm is originally restricted to abstract weighted graphs (that is, detection of shortest ways between *vertices*), the level set idea described here may of course be extended to all points $x \in \Gamma$, particularly to edge points.

The relation to the previous section is now easily established: At a point $x \in \Gamma_0$ with $D(x) > 1$, the single-source shortest path problem cannot be solved uniquely in the sense, that the direction in which shortest ways leave x is unique.

Of course one can also consider multiple-source shortest path problems, represented by networks with more than one boundary vertices. Again, the singular points are exactly those points for which this problem cannot be solved uniquely in the above sense. Thus, theorem 6.4 captures and quantifies the "non-uniqueness" of such shortest path problems.

6.7 Singularities of the distance function

In this section we discuss theorem 6.1 in the special case of the eikonal equation on Γ with its unique viscosity solution $\boldsymbol{d}$, the distance function. Accordingly, assume from now on $f \equiv 1$ in (6.2) and set $u = \boldsymbol{d}$.

In this special case, condition (6.8) of corollary 6.1 can be made true by slightly varying the edge lengths l_j, $j \in J$, while keeping the order of singularity S constant. As this fact is easy to see, we dispense with a rigorous proof and explain the idea in loose terms:

Assume $\kappa^V(i) > 1$ for some $i \in I_T$ and choose any $j \in \mathrm{Inc}_i^-$. Clearly, $\kappa^E(j) = 0$. Assume now that l_j be increased by an arbitrarily small number $\delta > 0$. Obviously, this does not affect the values of $\boldsymbol{d}$ at the endpoints of e_j, since the "upper" endpoint v_i possesses at least one more shortest way to the boundary, whereas the "lower" endpoint's shortest way to the boundary does not contain e_j. As a consequence, $\kappa^E(j)$ is increased by 1, whereas $\kappa^V(i)$ is decreased by 1, whence S remains unaffected. Repeating this procedure will finally make (6.8) true.

This observation justifies the following

Definition 6.5. The network Γ (and its underlying graph) is called *generic*, if it satisfies condition (6.8).

Then the reformulation of corollary 6.2 reads:

Lemma 6.5. *Assume that Γ is generic. Then the number M of local maxima of the distance function $\boldsymbol{d}$ is given by*

$$M = |J| - |I_T|.$$

As we have outlined above, this result is not obvious from a physical point of view, as one can imagine many possible deformations of Γ, which preserve $|J|$ and $|I_T|$, but change the lengths of the edges dramatically. As an effect of such a deformation, the local maxima may move along the edges, possibly jumping from one edge to another at a transition vertex, whereas others may vanish, melt together, or be newly created.

6.8 Singular points and cycle rank

It is worth mentioning the close relation of lemma 6.5 to the concept of *cycle rank* of abstract graphs. Let $G = (V, E)$ with $V = \{v_1, \ldots, v_p\}$ and $E = \{e_1, \ldots, x_q\}$ be a graph. A *cycle* in G is a path whose endpoints coincide. Analogously to techniques in algebraic topology a 0-*chain* of G is a formal linear combination $\sum \varepsilon_i v_i$, where $\varepsilon_i \in F_2$ (the field with two elements), whereas 1-*chains* are formal sums $\sum \varepsilon_j e_j$ of edges. The *boundary operator* ∂ sends 1-chains to 0-chains in such a way that ∂ is linear and that we have $\partial e = v + w$ for edges of the form $e = vw$. A 1-chain with boundary 0 is called *cycle vector* and can be regarded as a set of edge-disjoint cycles. The collection of all cycle vectors forms a vector space over F_2 called the *cycle space* of G. A *cycle basis* is defined as a basis for the cycle space of G consisting entirely of cycles. A cycle vector Z is said to be dependent on the cycles $Z_1, \ldots, Z_k$, if it can be written as $\sum_{i=1}^k \varepsilon_i Z_i$, $\varepsilon_i \in F_2$. Thus a cycle basis of G is a maximal collection of independent cycles of G, or a minimal collection of cycles on which all cycles depend. The number $m(G)$ of cycles in a cycle basis of G is called *cycle rank*, and it can be shown (see for instance [Har69], pp. 37-39) that we have

$$m(G) = q - p + k, \tag{6.10}$$

where k is the number of *components* of G, that is, the maximal number of subsets of V with the property that any two elements of two different subsets are not connected by a path.

The connection of this result to lemma 6.5 is established as follows. We first recall

Definition 6.6. A *subgraph* $H = (V_H, E_H)$ of the graph $G = (V_G, E_G)$ is a graph with $V_H \subset V_G$ and $E_H \subset E_G$. A subgraph H of G is said to be *spanned* by $V_H \subset V_G$, if $vw \in E_H$ for each edge $vw \in E_G$ with $v, w \in V_H$.

Let Γ be generic according to definition 6.5, and let G be the underlying graph. The genericity ensures that local maxima of $\boldsymbol{d}$ only occur on edges; let us call those edges *maximal edges*. We then "disconnect" G by deleting certain edges in the following way: For each boundary point $u \in \partial G$ we form the subgraph G_u spanned by

$$V_u := \{v \in V \mid d(u,v) = \min_{w \in \partial\Gamma} d(w,v)\}.$$

Observe that the sets V_u form a partition of V, i.e. they are pairwise disjoint and $\dot\bigcup_{u\in\partial\Gamma} V_u = V$. The union of the graphs $G_u, u \in \partial\Gamma$, forms the graph $\tilde{G}$ which is obtained by deleting the set E_0 of exactly those edges from G, whose respective endpoints are not contained in the same vertex set V_u. Observe that the deleted edges are all maximal, which is easy to see by the definition of the subgraphs G_u. The remaining graph $\tilde{G}$ has exactly $|\partial G|$ components. According to the formula for the cycle rank (6.10), its cycle rank $m = m(\tilde{G})$ satisfies $m = |E \backslash E_0| - |V| + |\partial G|$.

Now consider G_u for any fixed $u \in \partial G$. As G_u has only one component, the cycle rank m_u of G_u is $m_u = |E_u| - |V_u| + 1$, where E_u is the edge set of G_u. Clearly we have $m = \sum_{u\in\partial\Gamma} m_u$. We now claim that E_u contains exactly m_u maximal edges – in other words, each element of the cycle space basis of G_u contributes one maximal edge. The argument is the following: It is clear that whenever G_u has at least one cycle, it must also contain at least one maximal edge. Deleting all maximal edges from G_u thus leaves a *tree*, that is, a cycle-free graph. Successively re-adding the deleted maximal edges then yields one new independent cycle per maximal edge. Any other cycle or cycle vector depends on these cycles; the maximal number of independent cycles – the cycle rank m_u – thus exactly equals the number of maximal edges in G_u.

Altogether we obtain for the total number M of maxima of $\boldsymbol{d}$:

$$M = |E_0| + \sum_{u\in\partial\Gamma} m_u = |E_0| + m = |E_0| + |E\backslash E_0| - |V| + |\partial G| = |E| - |V| + |\partial\Gamma|.$$

We thus have recovered lemma 6.5.

CHAPTER 7

Viscosity solutions on LEP spaces

Summary. In the present chapter we generalize the results obtained in chapter 5 to higher dimensional ramified spaces. For this purpose we introduce ramified manifolds and, as special cases, locally elementary polygonal ramified spaces (LEP spaces). On LEP spaces we develop a theory of viscosity solutions of Hamilton-Jacobi equations, including existence and uniqueness results.

7.1 Introduction

The notion of viscosity solutions introduced in chapter 5 differs from its classical origin by the transition conditions we have additionally imposed at transition vertices. The concept of (j,k)-test functions allows us to “ignore” the transition vertex by treating two edges as one connected edge. In this chapter we shall study a generalization of this approach: The idea of (j,k)-differentiability is to link the two “normal” derivatives of a function with respect to a given pair of edges which are incident to the same vertex. It suggests itself to apply this pattern in case of manifolds of dimension n which have a certain manifold of dimension $n-1$ in common, as long as this manifold is smooth enough to ensure that we have well-defined normal derivatives with respect to each incident “branch manifold”. In fact we can establish an analogous theory on a suitable class of higher dimensional ramified spaces.

In the literature there are many different ways of introducing “ramified spaces” (cf. [Lum80], [Nic88], [Nic88], [vBN96], [AMN93]) or “branched manifolds” (cf. [Wil67]). The definitions vary in different aspects, depending on the kind of theory to be developed. In a general approach, subsets of classic differentiable manifolds are glued together along parts of their boundaries by means of the topological gluing operation. Another, more specific definition, demands the uniqueness of the ”tangent space” at ramification points (cf. [Wil67]) by describing how the branches should be situated relatively to each other in the ambient space. Here we choose an approach which is very similar to the concept of a manifold with boundary. The basic idea is that in contrast to classic topological manifolds, a *ramified* topological manifold should not only contain points at which it is locally homeomorphic to Euclidean space (*simple points*), but should also allow for *ramification points* at which it is locally homeomorphic to some kind of *ramified* Euclidean space. The latter is visualized as a collection of closed Euclidean half spaces glued together at their boundary hyperplanes and will be called *elementary ramified space.* Consequently, small neighborhoods of a given ramification point split up into different *branches* corresponding to the branches of the homeomorphic elementary ramified space. If we endow these ramified topological manifolds with suitable differentiable structures, we end up with an extension of the concept of tangent space at ramification points. This generalization should have the property that a real function defined in a neighborhood of a ramification point can be differentiated in direction of all different branches. Put in other terms, each branch "contributes" a different tangent space. In particular, we dispense with the uniqueness of the tangent space at ramification points.

Once we have introduced the differentiable structure, we will see that for each of the branches emanating from a fixed ramification point $x \in \Sigma$, a *normal direction* at x on this branch with respect to Σ is well-defined. The possibility to differentiate into normal directions at ramification points is crucial for our theory, as it will turn out that a general definition of viscosity solutions on ramified manifolds depends on this very possibility. In fact, at ramification points we will make use of test functions whose normal derivatives are related (the generalization of (j,k)-differentiable functions mentioned above).

In order not to get lost in too general approaches, we restrict ourselves to a rather simple kind of ramified manifolds, the so-called *locally elementary polygonal ramified spaces (LEP spaces)*, which are characterized by two main criteria: On the one hand, LEP spaces are ramified spaces in the sense of Lumer (cf. definition 4.1) meeting the additional requirement that each branch is a flat n-dimensional submanifold of $\mathbb{R}^{n+1}$. On the other hand, they are ramified manifolds in the sense described above. Hence they can be visualized as polygonal subsets of hyperplanes in $\mathbb{R}^{n+1}$ which are glued together along certain edges, with the restriction that "corner points" cannot occur. The term "locally elementary" refers to the fact described above: that they are locally homeomorphic to an open subset either of Euclidean space or of an elementary ramified space.

Once the notion of viscosity solutions on LEP spaces has been correctly extended to LEP spaces, the development of the theory consists of a rather direct translation of the theory of chapter 5. However, technically we will proceed somewhat differently, as we replace the test function technique by the equivalent, but more convenient concept of upper and lower semijets.

7.2 Ramified manifolds

Our first aim is to define ramified manifolds as objects which locally resemble either (non-ramified) Euclidean space or some ramified parameter space. We begin with the introduction of the latter, the *n-dimensional elementary ramified spaces (of ramification order r)*.

Definition 7.1. Let $n \geq 1$. We respectively define the n-dimensional open and closed Euclidean half-space by

$$\mathbb{R}^n_{>0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 > 0\} \quad \text{and} \quad \mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}.$$

Definition 7.2. Let $n \geq 1$ and let $r \geq 2$ be an integer. Set

$$\tilde{\mathcal{R}}^n_r := \mathbb{R}^n_{\geq 0} \times \{1, \ldots, r\} \quad \text{and} \quad \mathcal{R}^n_r := \tilde{\mathcal{R}}^n_r / R,$$

where R be the equivalence relation on $\tilde{\mathcal{R}}^n_r$ which for each choice of $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ identifies all points $((0, x_2, \ldots, x_n), j) \in \tilde{\mathcal{R}}^n_r$, $1 \leq j \leq r$. Equivalence classes with respect to R are denoted by an upper bar.

Let $\tilde{\mathcal{R}}^n_r$ carry the product topology formed by the Euclidean topology on $\mathbb{R}^n_+$ and the discrete topology on $\{1, \ldots, r\}$. The quotient topology of this product topology with respect to the quotient mapping induced by R turns $\mathcal{R}^n_r$ into a topological space which we denote by the *n-dimensional elementary ramified space of order r*.

Furthermore define the *ramification space*

$$\Sigma^n_r := \{\overline{((0, x_2, \ldots, x_n), j)} \mid (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, 1 \leq j \leq r\} \subset \mathcal{R}^n_r$$

and the open and closed *branches*

$$\mathcal{R}^n_{r,j} := \{\overline{(x, j)} \in \mathcal{R}^n_r, x \in \mathbb{R}^n_{>0}\} \quad \text{and} \quad \bar{\mathcal{R}}^n_{r,j} := \{\overline{(x, j)} \in \mathcal{R}^n_r, x \in \mathbb{R}^n_{\geq 0}\}, \quad 1 \leq j \leq r.$$

Remark 7.1. Note that for $n \geq 2$ we can identify $\mathcal{R}^n_r$ with $\mathcal{R}^1_r \times \mathbb{R}^{n-1}$, $\bar{\mathcal{R}}^n_r$ with $\bar{\mathcal{R}}^1_r \times \mathbb{R}^{n-1}$, as well as Σ^n_r with $\Sigma^1_r \times \mathbb{R}^{n-1} = \{0\} \times \mathbb{R}^{n-1}$, where $0 = \overline{(0, j)}$. Moreover we can identify $\mathcal{R}^n_r$ with $\mathbb{R}^n$, if $r = 2$. Furthermore observe that $\Omega := \mathcal{R}^n_r$ is a ramified space according to definition 4.1 with $\Omega_j = \mathcal{R}^n_{r,j}$, $1 \leq j \leq r$, and $N_R = \Sigma^n_r$.

We obviously can interpret $\mathcal{R}^n_r$ as the subset of $\mathbb{R}^{n+1}$ given by

$$\left(\cup_{j=0}^{r-1} \{(t \cos(2\pi j/r), t \sin(2\pi j/r)) \in \mathbb{R}^2, t \geq 0\}\right) \times \mathbb{R}^{n-1},$$

and it is easy to verify that the subset topology on this subset coincides with the topology on $\mathcal{R}^n_k$ given by the definition above. Both topologies in turn coincide with the topology on $\mathcal{R}^n_k$ induced by the path metric $d(\cdot, \cdot)$.

When introducing ramified manifolds below, elementary ramified spaces will be the corresponding parameter spaces. Beforehand we provide a fundamental lemma about homeomorphisms from $\mathcal{R}^n_k$ to itself.

Proposition 7.1. *Let $n \geq 1$ and let r, s be integers with $r, s \geq 3$. Let $x \in \Sigma_r^n$, let $U \subset \mathcal{R}_r^n$ be an open connected set with $x \in U$, and let $V \subset \mathcal{R}_s^n$ be an open set. Furthermore let $\varphi : U \to V$ be a homeomorphism. Then the following holds:*
(i) $\varphi(x) \in \Sigma_s^n$
(ii) $s = r$
(iii) For each $1 \leq j \leq r$ there is a $1 \leq k \leq r$ such that $\varphi(U \cap \mathcal{R}_{r,j}^n) \subset \mathcal{R}_{r,k}^n$.

Proof. Let $n = 1$. Observe that if (i) or (ii) were violated, then $U \backslash \{x\}$ and $V \backslash \{\varphi(x)\}$ would possess different numbers of connected components.

Let $n \geq 2$. Theorem 11.2.2 in [SZ94] implies

$$H_n(U, U \backslash \{x\}) \cong H_n(V, V \backslash \{\varphi(x)\}),$$

where $H_q(X, X \backslash \{x_0\})$ denotes the qth local homology group of a topological space at a point $x_0 \in X$. The theorem also implies that

$$H_n(B_1(x), B_1(x) \backslash \{x\}) \cong H_n(B_1(\varphi(x)), B_1(\varphi(x)) \backslash \{\varphi(x)\}),$$

where $B_1(y)$ is the open ball of radius 1 around a point y with respect to the path metric. Observe now that if (i) or (ii) was violated, we would obtain a contradiction to lemma 7.1 below.

Assertion (iii) is immediately clear. □

Lemma 7.1. *Let $n \geq 2$ and let S^{n-1} be the $(n-1)$-dimensional sphere. Let $r \in \mathbb{N}$, $r \geq 2$, and define*

$$\mathcal{S}_r^{n-1} := S_+^{n-1} \times \{1, \ldots, r\} / \mathcal{E},$$

where $S_+^{n-1} := S^{n-1} \cap \mathbb{R}_+^n$ and $\mathcal{E}$ be the equivalence relation which for each

$$y \in S_+^{n-1} \cap P^{n-1} \quad \text{with} \quad P^{n-1} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid x_1 = 0\}$$

identifies the points $(y, i) \in S_+^{n-1} \times \{1, \ldots, r\}$, $1 \leq i \leq r$.
Then we have $H_{n-1}(\mathcal{S}_r^{n-1}) = \mathbb{Z}^r$.

Proof. Observe that $\mathcal{S}_r^{n-1}$ is topologically equivalent to the space A_r, where A_r is formed by $(r-1)$ copies $B_1, \ldots, B_{r-1}$ of the boundary of the n-dimensional half-ball of radius 1, which are identified along their flat sides $F_1, \ldots, F_{r-1}$. Observe that A_2 is homologically equivalent to the $(n-1)$-dimensional sphere, whence we have $H_{n-1}(A_2) = H_{n-1}(S^{n-1}) = \mathbb{Z}$. Assume $r > 2$. For each $1 \leq i \leq r-1$ let $C_i \subset A_r$ be a closed circular cap with $C_i \subset B_i$ and $C_i \cap F_i = \emptyset$. Then the sets $U := A_r \backslash C_{r-1}$ and $V := A_r \backslash \cup_{i=1}^{r-2} C_i$ are open and satisfy $U \cup V = A_r$. Furthermore, it is easy to see that U is homotopically equivalent to A_{r-1}, whereas V is homotopically equivalent to the sphere S^{n-1}. Moreover, $U \cap V$ is homotopically equivalent to the open $(n-1)$-dimensional ball, whence we have

$$H_{n-1}(U \cap V) = H_{n-2}(U \cap V) = 0. \tag{7.1}$$

Now we take advantage of the corresponding Mayer-Vietoris sequence yielding an exact sequence of group homomorphisms

$$\ldots \to H_{n-1}(U \cap V) \to H_{n-1}(U) \oplus H_{n-1}(V) \to H_{n-1}(A_r) \to H_{n-2}(U \cap V) \to \ldots$$

Relation (7.1) implies that $H_{n-1}(U) \oplus H_{n-1}(V)$ and $H_{n-1}(A_r)$ are isomorphic. As $H_{n-1}(U) \cong H_{n-1}(A_{r-1})$ and $H_{n-1}(V) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, we conclude

$$H_{n-1}(A_r) \cong H_{n-1}(A_{r-1}) \oplus \mathbb{Z}.$$

The assertion follows by induction. □

Definition 7.3. Let $m, n \geq 1$, $r \geq 2$ be integers, let $U \subseteq \mathcal{R}_r^n$ be an open set, and let $f : U \to \mathbb{R}^m$ be continuous. Then f is said to be *C^l-differentiable at $x \in U$* $(1 \leq l \leq \infty)$, if the following holds.

(i) If $x \in \Sigma_n^k$, then for each $1 \leq j \leq r$ there is a domain $V_j \subset \mathbb{R}^n$ and an l times continuously differentiable function $f_j : V_j \to \mathbb{R}^m$ such that, if $\bar{\mathcal{R}}_{r,j}^n$ is interpreted as the closed half-space $\mathbb{R}_{\geq 0}^n$, we have $x \in V_j$ and $f_j \equiv f$ on $V_j \cap \mathbb{R}_{\geq 0}^n$.

(ii) If $x \in \mathcal{R}_{r,j}^n \simeq \mathbb{R}_{>0}^n$ for some $1 \leq j \leq r$, then f is l times continuously differentiable at x in the classic sense.

Definition 7.4. Let $r \geq 3$ and let $U, V \subset \mathcal{R}_r^n$ be open sets. Let $\varphi : U \to V$ be a homeomorphism. Then φ is called *diffeomorphism*, if for all $1 \leq j \leq r$ the respective restrictions of φ and φ^{-1} to $\bar{\mathcal{R}}_{r,j}^n \cap U$ and $\bar{\mathcal{R}}_{r,j}^n \cap V$, are C^∞-differentiable in the sense of definition 7.3 (note that each connected component of the images of these restrictions can be thought of a subset of $\mathbb{R}_{\geq 0}^n$ according to proposition 7.1).

We are now ready to define topological and differentiable ramified manifolds.

Definition 7.5. Let $n \geq 1$. A set M is called *n-dimensional topological ramified manifold*, if it is endowed with a Hausdorff topology and if for each point $x \in M$ there is an open set $U \subset M$ with $x \in U$ such that there are an integer $r = r(x) \geq 2$, an open set $V \subset \mathcal{R}_r^n$ with $V \cap \Sigma_r^n \neq \emptyset$, and a homeomorphism $\mathbf{x} : U \to V$ with $\mathbf{x}(x) \in \Sigma_r^n$. The number $r(x)$ is called *ramification order of x* and is independent of $\mathbf{x}$ according to proposition 7.1.

A point $x \in M$ is called *simple point* if $r(x) = 2$ and *ramification point* if $r(x) \geq 3$. The set of all ramification points is denoted by Σ and is called *ramification space of M*.

Remark 7.2. In view of remark 7.1 observe that topological ramified manifolds are locally homeomorphic to Euclidean space at simple points.

Definition 7.6. Let $n \geq 1$ and let M be an n-dimensional topological ramified manifold with ramification space Σ. We call M *differentiable* if the following conditions are the satisfied:

(i) There is a family of charts $(U_\alpha, \mathbf{x}_\alpha)$, i.e. of open sets $U_\alpha \subset M$ and injective mappings $\mathbf{x}_\alpha : U_\alpha \to \mathcal{R}^n_{r(\alpha)}$ with the following properties:
For any pair α, α' with $V := U_\alpha \cap U_{\alpha'} \neq \emptyset$, the sets $\mathbf{x}_\alpha(V)$ and $\mathbf{x}_{\alpha'}(V)$ are open sets in $\mathcal{R}^n_{r(\alpha)}$ and $\mathcal{R}^n_{r(\alpha')}$, respectively. Furthermore, the mapping $\varphi : \mathbf{x}_\alpha(V) \to \mathbf{x}_{\alpha'}(V)$ given by $\varphi := \mathbf{x}_{\alpha'} \circ \mathbf{x}_\alpha^{-1}$ is a diffeomorphism in the sense of definition 7.4. (Note that we have $r = r(\alpha) = r(\alpha')$ according to proposition 7.1.)
(ii) $\bigcup_\alpha U_\alpha = M$.
(iii) The family $\{(U_\alpha, x_\alpha)\}$ is maximal with respect to the conditions (i) and (ii).

Example 7.1. Let Γ be a topological network. Set $\tilde{\Gamma} := \Gamma \backslash \{v \in V \mid \deg(v) = 1\}$. Then for $n \geq 2$ the set $M := \tilde{\Gamma} \times \mathbb{R}^{n-1}$ is an n-dimensional topological ramified manifold. The sets M_r of points of ramification order r are given by

$$M_r = \begin{cases} \cup_{\{v \in V \mid \deg(v) = r\}} (\{v\} \times \mathbb{R}^{n-1}) & \text{if } r > 2 \\ M \backslash \Sigma(M) & \text{if } r = 2, \end{cases}$$

where $\Sigma(M) = \cup_{\{r>2\}} M_r$. We call M an *n-dimensional network* (cf. [Nic93]).

Next we extend the notion of tangent space to differentiable ramified manifolds. In fact, the idea of interpreting tangent vectors as equivalence classes of curves in M can easily be transferred to ramification points.

Definition 7.7. Let M be a differentiable ramified manifold and let $1 \leq l \leq \infty$. A continuous function $f : M \to \mathbb{R}$ is said to be *C^l-differentiable at $x \in M$*, if for any chart $(U, \mathbf{x})$ around x the function $f \circ \mathbf{x}^{-1}$ is C^l-differentiable at x according to definition 7.3. f is called *C^l-differentiable in $V \subset M$*, if it is C^l-differentiable at all $x \in V$.

In the sequel let M always be an n-dimensional differentiable ramified manifold with ramification space Σ. Unless otherwise noted, let $x \in \Sigma$ and let $r := r(x)$ be the ramification order of x.

Remark 7.3. In order to avoid indexation problems, let us from now on assume without restriction of generality that for any two charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ around x the mapping $\mathbf{y} \circ \mathbf{x}^{-1} : \mathcal{R}^n_r \to \mathcal{R}^n_r$ maps $\mathcal{R}_{r,j}$ to itself for any $1 \leq j \leq r$. Observe that the following definition will then be independent of the choice of the respective chart.

Definition 7.8. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = x$ be a continuous curve. Let $1 \leq j \leq r$. We say that *α reaches x from the branch j*, whenever there exists a chart $(U, \mathbf{x})$ with $x \in U$ such that

$$\tilde{\alpha}(t) := (\mathbf{x} \circ \alpha)(t) \in \bar{\mathcal{R}}^n_{r,j} \quad \text{for all } t \in (-\delta, 0) \text{ and some } \delta > 0. \tag{7.2}$$

Let $\mathcal{C}_j(x)$ be the set of all curves reaching x from the branch j and set

$$\mathcal{C}(x) := \bigcup_{1 \leq j \leq r} \mathcal{C}_j(x).$$

Remark 7.4. Note that if (7.2) holds for some chart $(U, \mathbf{x})$ and some $1 \leq j \leq r$, then it automatically holds for all charts $(V, \mathbf{y})$ with $x \in V$ and the same j by proposition 7.1 and remark 7.3. Hence definition 7.8 is independent of the choice of the specific chart.

Now we subdivide the set $\mathcal{C}(x)$ of all curves reaching x from a unique branch into equivalence classes representing the tangent vectors.

Definition 7.9. Let $\alpha, \beta \in \mathcal{C}(x)$. We call α and β *equivalent*, if for all functions $f : M \to \mathbb{R}$ which are C^∞-differentiable at x (in the sense of definition 7.7) we have

$$(f \circ \alpha)'(0) = (f \circ \beta)'(0),$$

where both derivatives are left-sided. We denote the set of all equivalence classes by the *tangent space* T_xM *of* M *at* x and say that $\xi \in T_xM$ is a j*-tangent vector at* x, $1 \leq j \leq r$, if ξ contains a curve reaching x from the branch j. We set

$$\xi(f) := (f \circ \alpha)'(0), \ \alpha \in \xi.$$

The set of all j-tangent vectors at x (the j*-tangent space at* x) is denoted by T_x^jM, and $T_x\Sigma := \cap_j T_x^jM$ is called the Σ*-tangent space at* x; any $\xi \in T_x\Sigma$ is called a Σ*-tangent vector at* x.

Remark 7.5. Clearly, T_xM is not a vector space. Instead it can be identified with the elementary ramified space $\mathcal{R}_r^n$, where

$$T_x^jM \equiv \bar{\mathcal{R}}_{r,j}^n, \ 1 \leq j \leq r, \text{ and } T_x\Sigma \equiv \Sigma_r^n.$$

Hence $T_x\Sigma \subset T_x^jM$ can be identified with $\mathbb{R}^{n-1}$, whereas T_x^jM can be identified with $\mathbb{R}^n_{\geq 0}$. For the purpose of shorter presentation, let us for each $1 \leq j \leq r$ agree to extend T_x^jM to the entire Euclidean space $\mathbb{R}^n$ via the relation $\xi(f) = -\xi(-f)$ for any $f \in C^\infty(M)$ and any $\xi \in T_x^jM$. The advantage of this extension is that each j-tangent space T_x^jM at x carries a vector space structure.

An important tangent vector is the gradient of a function at a given point.

Definition 7.10. Let $1 \leq j \leq r$. Let $f : M \to \mathbb{R}$ be continuously differentiable at x and let $\xi_1, \ldots, \xi_n$ be a basis of T_x^jM. We define the j*-gradient* by

$$D^j f(x) := \sum_{i=1}^n \xi_i(f)\xi_i \in T_x^jM.$$

7.3 LEP spaces

Having introduced the concept of ramified manifolds, we have a rather general framework at our disposal in order to extend the theory of chapter 5. However, a theory of viscosity solutions on general ramified Riemannian manifolds is not within the scope of this work. In order to understand the crucial mechanisms it rather is sufficient to focus on the phenomena occurring in the neighborhood of the ramification space. We therefore have decided to consider flat ramified manifolds in the following sense.

Definition 7.11. Let Ω be a ramified space induced by the triple $(\Omega^\star, (\Omega_j)_{j\in J}, N_E^\star)$ according to definition 4.1 and let $n \geq 1$. We call Ω an *(n-dimensional) polygonal ramified space* if the following conditions are satisfied.

(i) $\Omega^\star$ is a subset of $\mathbb{R}^{n+1}$ and is endowed with the induced subset topology.

(ii) For each $j \in J$ there is a hyperplane P_j of $\mathbb{R}^{n+1}$ such that Ω_j is a bounded subset of P_j.

(iii) All P_j, $j \in J$, are pairwise different.

For each $j \in J$ we set

$$\partial_E\Omega_j := \partial\Omega_j \cap N_E^\star \quad \text{and} \quad \partial_R\Omega_j := \partial\Omega_j \cap N_R.$$

The set $N_E^\star$ is also called *boundary* $\partial\Omega$ *of* Ω.

Remark 7.6. Observe that $\bar{\Omega} = \Omega^\star$, where the closure is taken with respect to the topology of $\Omega^\star$. Furthermore note that $\Omega^\star$ is a compact space.

Remark 7.7. Let Ω be an n-dimensional polygonal ramified space. In view of definition 4.1 note that the boundary $\partial\Omega$ of Ω is a closed subset of $\Omega^\star$. Note moreover that $\partial\Omega$ is a subset of $\Omega^\star\backslash\Omega$ meeting the requirement that it contains each point which is contained in the boundary $\partial\Omega_j$ of exactly one branch Ω_j. As such, it is not uniquely determined; we rather have some freedom in choosing $\partial\Omega$. In order to emphasize this circumstance, we will explicitly speak of *the polygonal ramified space* Ω *with boundary* $\partial\Omega$ in the sequel. As we will see below, the fact that $\partial\Omega$ is not uniquely defined will be of importance when we turn to the question of existence of solutions of boundary value problems on Ω. The corresponding uniqueness results, however, remain unaffected.

Example 7.2. Let $n \geq 1$. Let $\Omega^\star$ be the surface of the $(n+1)$-dimensional cube $C^{n+1} \subset \mathbb{R}^{n+1}$ endowed with the subset topology and let Ω_j, $j \in J$, be the $2(n+1)$ open faces. Furthermore let $\partial\Omega = N_E^\star$ be any closed (possibly empty) subset of the union of the edges with the property that $\Omega := \Omega^\star\backslash N_E^\star$ is connected. Then Ω is an n-dimensional polygonal ramified space with $2(n+1)$ branches and boundary $\partial\Omega$.

The possible occurrence of "corner points" in the above example shows that a polygonal ramified space is in general not locally homeomorphic to an elementary ramified space at any point $x \in N_R$. As we do require this property, we strengthen the concept in the following definition.

Definition 7.12. An n-dimensional polygonal ramified space $\Omega \subset \mathbb{R}^{n+1}$ is called *locally elementary*, if it is also a differentiable ramified manifold. We call locally elementary polygonal ramified spaces *LEP spaces*, for short.

Remark 7.8. Note that the ramification space Σ coincides with N_R for any LEP space Ω.

In order to establish our theory, we need an additional convexity condition.

Definition 7.13. Let Ω be an n-dimensional LEP space, $n \geq 2$. Ω is called *convex*, if Ω_j is convex as a subset of $\mathbb{R}^n$ for each $j \in J$.

Example 7.3. Any n-dimensional network as defined in example 7.1 is a convex LEP space.

Example 7.4. Let Ω be as in example 7.2 and assume that the 2^n corner points are contained in $\partial\Omega$. Then Ω is a convex LEP space.

In the sequel let Ω always denote an n-dimensional convex LEP space, $n \geq 2$, with finitely many branches Ω_j, $j \in J$ and non-empty boundary $\partial\Omega$. Furthermore let Σ denote the ramification space of Ω.

Remark 7.9. The convexity condition will play a technical role in the proof of the comparison lemma 7.2.

Definition 7.14. For each $j \in J$ we set

$$\tilde{\Omega}_j := \Omega_j \cup \partial_R \Omega_j.$$

For any $x \in \Sigma$ we set $\mathrm{Inc}_x := \{j \in J \,|\, x \in \partial\Omega_j\}$.

As the branches of Ω can be thought of as subsets of $\mathbb{R}^n$, LEP spaces have the convenient property that around any given point we can always choose a chart induced by the canonical identification with the Euclidean or a suitable elementary ramified space (the inclusion mapping). Although this property is obvious, we state the following proposition.

Proposition 7.2. *For any $x \in \Omega$ there is a neighborhood V_x of x and a canonical identification*

$$\boldsymbol{i}_x : V_x \to \boldsymbol{i}_x(V_x),$$

where $\boldsymbol{i}_x(V_x) \subset \mathbb{R}^n$, $\boldsymbol{i}_x(x) = 0$, if $x \notin \Sigma$, and $\boldsymbol{i}_x(V_x) \subset \mathcal{R}^n_{r(x)}$, $\boldsymbol{i}_x(x) = 0$, if $x \in \Sigma$. In the latter case, $\boldsymbol{i}_x$ induces a bijective mapping $\mathcal{I}_x$ between the index set Inc_x *and the set* $\{1, \ldots, r(x)\}$.

7.4 Hamilton-Jacobi equations on LEP spaces

We want to study first order Hamilton-Jacobi equations and corresponding boundary value problems on LEP spaces. For this purpose we suitably extend the notion of the Hamiltonian introduced in (4.9) and (4.10).

Definition 7.15. A *Hamiltonian* $H = (H^j)_{j\in J}$ is a family of mappings H^j, such that for fixed $j \in J$ the mapping H^j assigns to each $x \in \tilde{\Omega}_j$ a function

$$H^j(\cdot,\cdot,x) : T^j_x\Omega \times \mathbb{R} \to \mathbb{R}.$$

We furthermore assume the mappings H^j to be twice continuously differentiable in the following sense: Let $j \in J$. By means of the canonical identification map $\mathbf{i}_x$ around some fixed $x \in \tilde{\Omega}_j$, locally around x we can think of H^j as a mapping

$$H^j : \mathbb{R}^n \times \mathbb{R} \times V \to \mathbb{R}, \tag{7.3}$$

where V is a neighborhood of $0 \in \mathbb{R}^n$ or $0 \in \mathbb{R}^n_{\geq 0}$, provided $x \in \Omega_j$ or $x \in \Sigma$, respectively. We then assume this mapping to be C^2 in each argument, where in the latter case we mean C^2 in the sense of definition 7.7. In the sequel we will speak of H^j *under the canonical identification (around x)*, whenever we refer to H^j in the sense of (7.3).

Let us assume from now on that H be a Hamiltonian such that each H^j, $j \in J$, has the following properties under the canonical identification around any fixed $x \in \Omega$:

$$\begin{array}{ll} (i) & H^j(0,z,0) < 0 \text{ for all } z \in \mathbb{R} \\ (ii) & H^j(p,\cdot,0) \text{ is non-decreasing for all } p \in \mathbb{R}^n \\ (iii) & H^j(p,z,0) \to \infty \text{ as } |p| \to \infty \text{ for all } (p,z) \in \mathbb{R}^n \times \mathbb{R} \\ (iv) & H^j(p,z,0) \text{ is convex in } p \text{ for all fixed } z \in \mathbb{R}. \end{array} \tag{7.4}$$

In addition, at the ramification space Σ we require that the H^j are continuous across the ramification space, as well as rotationally symmetric in p. In fact, under the canonical identification we demand at each $x \in \Sigma$

$$\begin{array}{ll} (v) & H^j(p,z,0) = H^k(p,z,0) \;\forall\; z \in \mathbb{R},\, p \in \mathbb{R}^n,\, j,k \in \mathrm{Inc}_x \\ (vi) & H^j(p,z,0) = H^j(\tilde{p},z,0) \;\forall\; p,\tilde{p} \in \mathbb{R}^n \text{ with } |\tilde{p}| = |p|,\, \forall\; z \in \mathbb{R},\, j \in \mathrm{Inc}_x. \end{array} \tag{7.5}$$

Observe that the conditions (7.4) and (7.5) are analogous to conditions (4.10).

As in chapter 5, we require a last additional assumption on H: the uniform boundedness of the spatial derivative of H in p and z. Accordingly, we assume that there is a constant $C_0 < \infty$ that for each $x \in \tilde{\Omega}^j$, $j \in J$, we have

$$(vii) \quad |D^j h_{p,z}(x)| \leq C_0 \text{ for all } p \in \mathbb{R}^n \text{ and all } z \in \mathbb{R}, \tag{7.6}$$

where under the canonical identification around x we set

$$h_{p,z} : y \mapsto H^j(p,z,y) \quad \text{for all } y \in V_x.$$

Accordingly, we define

Definition 7.16. A Hamiltonian H is said to be *of eikonal type*, if it satisfies conditions (7.4) and (7.5).

We study boundary value problems associated with H on the LEP space Ω. Consequently, let $\phi : \partial\Omega \to \mathbb{R}$ be a prescribed boundary value function. We then consider the Dirichlet problem

$$\begin{cases} H^j(D^j u(x), u(x), x) = 0 & \text{for all } x \in \Omega_j,\ j \in J, \\ u(x) = \phi(x) & \text{for all } x \in \partial\Omega, \end{cases} \tag{7.7}$$

and establish a viscosity solution theory for it.

7.5 Test functions and semijets

We transfer the test function technique employed in chapter 5 to LEP spaces. For this purpose we extend the necessary concepts, such as differentiability across the ramification space.

Throughout this and the following sections, for each $x \in \Omega$ we always fix a canonical identification chart $(V_x, \mathbf{i}_x)$ as defined in proposition 7.2. All concepts to be discussed will be expressed in terms of $\mathbf{i}_x$ for the sake of simplicity. However, in each case it will be easy to verify that they in fact do not depend on the choice of any specific chart.

Definition 7.17. Let $r \geq 3$, $n \geq 2$, and let $x \in \Sigma_r^n \subset \mathcal{R}_r^n$. Furthermore let $u : \mathcal{R}_r^n \to \mathbb{R}$ be a function which is continuously differentiable at x in the sense of definition 7.3. We then denote by $\partial_1 u(x), \ldots, \partial_{n-1} u(x)$ the directional derivatives of u at x with respect to the canonical basis vectors $e_1, \ldots, e_{n-1}$ of $\Sigma_r^n \equiv \mathbb{R}^{n-1}$. For $1 \leq j \leq r$ we furthermore denote by $\partial_{\nu_j} u(x)$ the directional derivative of u at x with respect to the inward pointing unit normal ν_i of $\mathcal{R}_{r,j}^n \equiv \mathbb{R}_{\geq 0}^n$ at x.

Definition 7.18. Let $x \in \Sigma$, $r := r(x)$. Furthermore let $V \subset \Omega$ be a neighborhood of x and let $u : V \to \mathbb{R}$ be a function which is continuously differentiable at x in the sense of definition 7.7. With $\mathcal{I}_x$ as defined in proposition 7.2, we set

$$\partial_i u(x) := \partial_i (u \circ \mathbf{i}_x^{-1})(0),\ 1 \leq i \leq n-1, \quad \text{and} \quad \partial_{\nu_j} u(x) := \partial_{\nu_{\mathcal{I}_x(j)}} (u \circ \mathbf{i}_x^{-1})(0),\ j \in \mathrm{Inc}_x.$$

Remark 7.10. Note that for each $i \in \mathrm{Inc}_x$ the collection $\{\partial_1, \ldots, \partial_{n-1}, \partial_{\nu_j}\}$ forms a basis of $T_x^j \Omega$.

Let us define some function classes on Ω.

Definition 7.19. Let $u : \Omega \to \mathbb{R}$. We denote the restriction of u to Ω_j by u^j, $j \in J$.

Definition 7.20. The space of all continuous functions on Ω is denoted by $C(\Omega)$.

Definition 7.21. Let $1 \leq l \leq \infty$. A function $u \in C(\Omega)$ is said to be l *times continuously differentiable*, if it is C^l-differentiable at each $x \in \Omega$ in the sense of definition 7.7. We write $u \in C^l(\Omega)$.

The natural generalization of (k,l)-differentiability and (k,l)-test functions introduced in chapter 5 is given by the following definitions.

Definition 7.22. Let $\varphi \in C(\Omega)$, $x \in \Sigma$, and $k, l \in \mathrm{Inc}_x$, $k \neq l$. Then φ is said to be (k,l)*-differentiable at* x, if φ is C^1-differentiable at x, and if we have

$$\partial_{\nu_k}\varphi(x) + \partial_{\nu_l}\varphi(x) = 0.$$

Definition 7.23. Let $u : \Omega \to \mathbb{R}$ and let $\varphi \in C(\Omega)$.

Let $j \in J$ and let $x \in \Omega_j$. We call φ an *upper (lower) test function of* u *at* x, if $u - \varphi$ attains a local maximum (minimum) at x and if φ is C^1-differentiable at x.

Let $x \in \Sigma$ and let $k, l \in \mathrm{Inc}_x$, $k \neq l$. We call φ an *upper (lower)* (k,l)*-test function of* u *at* x, if $u - \varphi$ attains a local maximum (minimum) at x with respect to $\Omega_{k,l} := \bar{\Omega}_k \cup \bar{\Omega}_l$ and if φ is (k,l)-differentiable at x.

Additionally to the test function technique, we now invoke another way of characterizing viscosity solutions. In fact, as the characterization by test functions is of local nature, only their behavior around the point x is of interest. To be more precise, we only require the knowledge of their first derivatives along with the fact that their difference function with the tested function u attains an extremum at x. Hence we dispense with the test function concept and replace it by a formulation employing a generalized concept of differentials of continuous functions, the so-called *semijets*.

The basic idea of semijets is to arrange all first derivatives of possible upper and lower test functions of a function at a given point in two sets, called the *upper and lower semijet*, respectively. Although this concept is equivalent to the test function approach, it is more elegant, as the essential properties of test functions are distilled (the first derivatives), whereas all unnecessary information (the actual shape of the test function) is omitted.

The generalization of semijets to (j,k)-semijets at a ramification point $x \in \Sigma$ with respect to two incident branches $k, l \in \mathrm{Inc}_x$ is carried out analogously by taking into account all possible (j,k)-test functions at x.

Definition 7.24. Let $u : \Omega \to \mathbb{R}$.

(i) Let $x \in \Omega_j$, $j \in J$.

We define the *super- (sub-) semijet* $J_j^+ u(x)$ $(J_j^- u(x))$ *of* u *at* x to be the set of all $p \in T_x^j\Omega$ such that

$$u(x) \geq \ (\leq)\, u(y_n) + \langle D^j \mathbf{i}_x(x)(p), \mathbf{i}_x(y_n) - \mathbf{i}_x(x)\rangle + o(d(x, y_n))$$

for every sequence $y_n \in \Omega_j \cap V_x$ with $y_n \to x$.

(ii) Let $x \in \Sigma$ and let $k, l \in Inc_x$, $k \neq l$.
We define the *super- (sub-) (k,l)-semijet $J^+_{k,l}u(x)$ ($J^-_{k,l}u(x)$) of u at x* to be the set of all $p \in T^k_x\Omega$ such that

$$u(x) \geq \ (\leq)\, u(y_n) + \langle D^k \mathbf{i}_x(x)(p), \mathbf{i}_x(y_n) - \mathbf{i}_x(x)\rangle + o(d(x, y_n))$$

for every sequence $y_n \in \Omega_k \cap V_x$ with $y_n \to x$ and such that

$$u(x) \geq \ (\leq)\, u(y_n) + \langle D^l \mathbf{i}_x(x)(S^l_k(p)), \mathbf{i}_x(y_n) - \mathbf{i}_x(x)\rangle + o(d(x, y_n))$$

for every sequence $y_n \in \Omega_l \cap V_x$ with $y_n \to x$, where

$$S^l_k : T^k_x\Omega \to T^l_x\Omega$$

is defined as $S^l_k(p) := -p_1\partial_{\nu_l} + \sum_{m=1}^{n-1} p_m\partial_m$ with $p := p_1\partial_{\nu_k} + \sum_{m=1}^{n-1} p_m\partial_m$ (cf. remark 7.10).

We collect straightforward properties of semijets.

Proposition 7.3. *Let $\varphi : \Omega \to \mathbb{R}$.*
(i) We have $J^+_{k,j}\varphi(x) = \{S^k_j(p),\ p \in J^+_{j,k}\varphi(x)\}$ and $J^-_{k,j}\varphi(x) = \{S^k_j(p),\ p \in J^-_{j,k}\varphi(x)\}$ for any $x \in \Sigma$, $j, k \in \mathrm{Inc}_x$, $j \neq k$.
(ii) If φ is differentiable at $x \in \Omega_j$, $j \in I$, then we have $J^+_j\varphi(x) = J^-_j\varphi(x) = \{D^j\varphi(x)\}$.
(iii) If φ is (j,k)-differentiable at $x \in \Sigma$, $j, k \in \mathrm{Inc}_x$, $j \neq k$, then we have $J^+_{j,k}\varphi(x) = J^-_{j,k}\varphi(x) = \{D^j\varphi(x)\}$.

Proof. The proof is an easy consequence of the Taylor expansion theorem and the definition of (j,k)-differentiability. □

The following proposition expresses the equivalence of test functions and semijets.

Proposition 7.4. *Let $u : \Omega \to \mathbb{R}$.*
(i) Assume $\varphi \in C(\Omega)$ is an upper (lower) test function of u at $x \in \Omega_j$ for some $j \in J$. Then $D^j\varphi \in J^+_j u(x)$ ($D^j\varphi \in J^-_j u(x)$). On the other hand, assume $p \in J^+_j u(x)$ ($p \in J^-_j u(x)$). Then there is an upper (lower) test function $\varphi \in C(\Omega)$ of u at x, which is C^1-differentiable in an open neighborhood $U \subset \Omega_j$ of x, and which satisfies $p = D^j\varphi(x)$.

(ii) Assume $\varphi \in C(\Omega)$ is an upper (lower) (k,l)-test function of u at $x \in \Sigma$ for some $k, l \in \mathrm{Inc}_x$, $k \neq l$. Then $D^k\varphi \in J^+_{k,l}u(x)$ ($D^k\varphi \in J^-_{k,l}u(x)$). On the other hand, assume $p \in J^+_{k,l}u(x)$ ($p \in J^-_{k,l}u(x)$). Then there is an upper (lower) (k,l)-test function $\varphi \in C(\Omega)$ of u at x and an open neighborhood $U \subset \bar{\Omega}_k \cup \bar{\Omega}_l$ with the following properties:
(a) φ is C^1-differentiable on U in the sense of definition 7.7
(b) φ is (k,l)-differentiable on $\Sigma \cap U$
(c) $D^k\varphi(x) = p$.

Proof. This is a direct consequence of the definition of semijets and test functions. □

7.6 Viscosity solutions on LEP spaces

By means of the semijet concept we now define viscosity solutions for Hamilton-Jacobi equations on LEP spaces. By USC($\bar{\Omega}$) and LSC($\bar{\Omega}$) we respectively denote the set of all upper and lower semicontinuous functions $u : \bar{\Omega} \to \mathbb{R}$.

Definition 7.25. Let $f : \Omega \to \mathbb{R}$ and $u \in \mathrm{USC}(\bar{\Omega})$.

We say that u satisfies the *viscosity subsolution condition (associated with H and f) at* $x \in \Omega$ (or, alternatively, that it formally satisfies

$$H(Du(x), u(x), x) \leq f(x)$$

in the viscosity sense), if the following conditions (i) and (ii) are satisfied.

(i) If $x \in \Omega_j$ for some $j \in J$, then

$$H^j(p, u(x), x) \leq f(x) \quad \text{for all } p \in J_j^+ u(x).$$

(ii) If $x \in \Sigma$, then for all $k, l \in Inc_x$, $k \neq l$,

$$H^k(p, u(x), x) \leq f(x) \quad \text{for all } p \in J_{k,l}^+ u(x).$$

Let $u \in \mathrm{LSC}(\bar{\Omega})$.

We say that u satisfies the *viscosity supersolution condition (associated with H and f) at* $x \in \Omega$ (or, alternatively, that it formally satisfies

$$H(Du(x), u(x), x) \geq f(x)$$

in the viscosity sense), if the following conditions (iii) and (iv) are satisfied.

(iii) If $x \in \Omega_j$ for some $j \in J$, then

$$H^j(p, u(x), x) \geq f(x) \quad \text{for all } p \in J_j^- u(x).$$

(iv) If $x \in \Sigma$, then for all Σ-tangent vectors $p^\top \in T_x\Sigma$ at x (cf. definition 7.9) and for each $k \in \mathrm{Inc}_x$ there is an index $l \in \mathrm{Inc}_x$, $l \neq k$, such that

$$H^k(p, u(x), x) \geq f(x) \quad \text{for all } p \in J_{k,l}^- u(x) \text{ with } \pi_k^\top(p) = p^\top.$$

Here $\pi_k^\top : T_x^k\Omega \to T_x\Sigma$ is the projection given by

$$\pi_k^\top(p) := \sum_{m=1}^{n-1} p_m \partial_m \quad \text{for} \quad p = p_n \partial_{\nu_k} + \sum_{m=1}^{n-1} p_m \partial_m.$$

We say that $u \in \mathrm{USC}(\bar{\Omega})$ ($u \in \mathrm{LSC}(\bar{\Omega})$) is a *viscosity sub- (super-) solution* of

$$H(Du(x), u(x), x) = f(x) \tag{7.8}$$

on Ω, if it satisfies the viscosity sub- (super-) solution condition at each $x \in \Omega$. We call $u \in C(\bar{\Omega})$ a *viscosity solution* of (7.8) on Ω, if it is both a viscosity sub- and supersolution of (7.8) on Ω.

Remark 7.11. Observe that in the above definition we have

$$H^k(p, u(x), x) = H^l(S_k^l(p), u(x), x)$$

for all $x \in \Sigma$, $k, l \in \mathrm{Inc}_x$, $k \neq l$, in view of (7.5).

The proof of the following proposition is similar to the proof of proposition 5.1.

Proposition 7.5. *Let $x \in \Omega$ and let $u_1, u_2 \in \mathrm{USC}(\bar{\Omega})$ ($u_1, u_2 \in \mathrm{LSC}(\bar{\Omega})$) satisfy the viscosity sub- (super-) solution condition at x. Then $v := \max\{u_1, u_2\}$ ($v := \min\{u_1, u_2\}$) satisfies the viscosity sub- (super-) solution condition at x.*

We now seek to solve Dirichlet problem (7.7) in the sense that we look for a function $u \in C(\bar{\Omega})$ such that

$$\begin{cases} H(Du(x), u(x), x) = 0 & \text{in the viscosity sense for all } x \in \Omega_j,\ j \in J, \\ u(x) = \phi(x) & \text{for all } x \in \partial\Omega. \end{cases} \tag{7.9}$$

Such a function is called a *solution* of (7.9).

In the following sections we prove uniqueness and existence results similar to those stated in chapter 5. The structure of the proofs is also quite similar, whence we dispense with the presentation of details whenever they can be deduced from the proofs in chapter 5. Note, however, that here we work with a semijet characterization of viscosity solution (instead of the test function characterization used in chapter 5) causing the arguments to take different forms. In this case we stick to the detailed presentation.

7.7 Uniqueness

The central idea of the proof of the comparison lemma 5.1 is the fact that the two partial derivatives of the distance function between two points equal in modulus. We can take advantage of a similar circumstance in the LEP space Ω.

In this section we understand the sets Ω_j, $j \in J$, as subsets of $\mathbb{R}^n$, whose boundaries are (partly) identified according to the equivalence relation $\sim$ defined by the way the sets Ω_j are situated as subsets of Ω.

Definition 7.26. Let $x, y \in \bar{\Omega}$. A continuous curve $\gamma : [0, 1] \to \bar{\Omega}$ is called a *connection of x and y*, if the following conditions are satisfied:

(i) $\gamma(0) = x$ and $\gamma(1) = y$.

(ii) $\gamma((0, 1)) \subset \Omega$.

(iii) There are finitely many numbers $0 = a_0 \leq \cdots \leq a_s = 1$, such that for each $l = 1, \ldots, s$ the number a_l is maximal with the property that there is an index $j_l \in J$ with

$\gamma([a_{l-1}, a_l]) \in \bar{\Omega}_{j_l}$. Furthermore we demand that $\gamma_l : [a_{l-1}, a_l] \to \bar{\Omega}_{j_l} \subset \mathbb{R}^n$ given by $\gamma_l(x) := \gamma(x)$ is continuously differentiable for all $l = 1, \ldots, s$.

We define the *length* of γ by

$$\mathcal{L}(\gamma) := \sum_{l=1}^{s} \mathcal{L}(\gamma_l),$$

where $\mathcal{L}(\gamma_l)$ is the usual length of a continuously differentiable curve in $\mathbb{R}^n$.

Definition 7.27. Let $x, y \in \bar{\Omega}$. We define the *distance* $d(x, y)$ between x, y by

$$d(x, y) := \inf\{\mathcal{L}(\gamma),\ \gamma \text{ is a connection of } x \text{ and } y\}.$$

The following proposition follows from the simple structure of LEP spaces.

Proposition 7.6. *Let $y \in \Omega$ and define $d_y(x) := d(x, y)$ for all $x \in \Omega$. Then there is a number $\Lambda > 0$, which is independent of the choice of y, such that for all $x \in \Omega$ with $d(x, y) < \Lambda$ and $x \neq y$ the following is true.*

(i) If $x \in \Omega_j$ for some $j \in J$, then d_y is continuously differentiable at x.

(ii) If $x \in \Sigma$, then for each index $j \in \mathrm{Inc}_x$ there is an index $k \in \mathrm{Inc}_x$ with $k \neq j$, such that d_y is (j, k)-differentiable at x.

(iii) In both cases (i) and (ii) we have $|D^j d_y(x)| = 1$.

Proof. The assertions (i) and (ii) follow from the observation that the fact that Ω is an LEP space implies that we can choose $\Lambda > 0$ in such a way that the distance of any given point $x \in \bar{\Omega}$ to its cut locus is larger than Λ. Assertion (iii) is immediately clear. □

We state the following comparison result.

Lemma 7.2. *Let $f \in C(\bar{\Omega})$ with $f(x) < 0$ for all $x \in \bar{\Omega}$. Furthermore let $u \in \mathrm{USC}(\bar{\Omega})$ and $v \in \mathrm{LSC}(\bar{\Omega})$ respectively satisfy*

$$H(Du(x), u(x), x) \leq f(x) \quad \text{and} \quad H(Dv(x), v(x), x) \geq 0$$

in the viscosity sense for all $x \in \Omega$ and assume $u \leq v$ on $\partial\Omega$. Then $u \leq v$ on Ω.

Proof. We assume the contrary, i.e., that there is a point $z \in \Omega$ with $u(z) > v(z)$ and derive a contradiction.

First note that we have

$$M := \max\{\sup_{\bar{\Omega}} u, -\inf_{\bar{\Omega}} v, 1\} < \infty,$$

as the functions u and $-v$ are upper semicontinuous, and as $\bar{\Omega}$ is compact. Next observe that the function $d : \bar{\Omega}^2 \to \mathbb{R}$, $d : (x, y) \mapsto d(x, y)$ is continuous by definition 7.27. Hence for each $\varepsilon > 0$ the function

$$\Phi : \bar{\Omega}^2 \to \mathbb{R} \quad \text{by} \quad \Phi(x, y) := u(x) - v(y) + 3M\beta(\varepsilon^{-1} d(x, y))$$

is upper semicontinuous on $\bar{\Omega}^2$, where β is chosen as in the proof of lemma 5.1. Therefore there is a point $(p, q) \in \bar{\Omega}^2$, where Φ attains a global maximum.

Borrowing arguments from the proof of lemma 5.1, we derive $(p, q) \in \Omega^2$ and

$$d(p, q) \leq \varepsilon. \tag{7.10}$$

Now let $m := -\max_{\bar{\Omega}} f > 0$ and choose

$$\varepsilon < C_1 := \min\{m/C_0, \Lambda\} \tag{7.11}$$

where $C_0 > 0$ is the constant defined in (7.6) and Λ is the constant defined in proposition 7.6. Then by (7.11) there is a connection γ of p and q with $\mathcal{L}(\gamma) < C_1$. According to definition 7.26 we may assume that there are an integer $s \geq 1$ and numbers

$$0 = a_0 \leq \ldots \leq a_s = 1$$

such that for each $1 \leq l \leq s$ the number a_l is maximal with the property that there is an index $j_l \in J$ with $\gamma([a_{l-1}, a_l]) \in \bar{\Omega}_{j_l}$. We set $j_p := j_0$ and $j_q := j_s$.

Next observe that by (7.6) and by the mean value theorem we have the estimate

$$|H^{j_l}(\xi, z, \gamma(a_l)) - H^{j_l}(\xi, z, \gamma(a_{l+1}))| \leq C_1 \mathcal{L}(\gamma_{l+1}) \tag{7.12}$$

for all $\xi \in \mathbb{R}^n$, $z \in \mathbb{R}$, $0 \leq l \leq s-1$, where γ_l is defined in definition 7.26

By (7.5) (vi) we furthermore have

$$H^{j_l}(\xi, z, \gamma(a_l)) = H^{j_{l+1}}(\zeta, z, \gamma(a_l)) \quad \text{for all } 1 \leq l \leq s-1 \tag{7.13}$$

and all $\xi, \zeta \in \mathbb{R}^n$ with $|\xi| = |\zeta|$.

Consequently, if $s > 1$, then from (7.12) and (7.13) we conclude

$$|H^{j_0}(\xi, z, p) - H^{j_s}(\zeta, z, q)| \leq C_1 \sum_{l=1}^{s-1} \mathcal{L}(\gamma_l) < m, \tag{7.14}$$

for all $z \in \mathbb{R}$ and all $\xi, \zeta \in \mathbb{R}^n$ with $|\xi| = |\zeta|$.

Next we define the functions $d_q : \Omega \to \mathbb{R}$ and $d_p : \Omega \to \mathbb{R}$ by $d_q(x) := d(x, q)$ and $d_p(x) := d(x, p)$, respectively. As in the proof of lemma 5.1 we also define the functions

$$\varphi_q : \Omega \to \mathbb{R}, \quad \varphi_q(x) := 3M\beta(\varepsilon^{-1} d_q(x)) \quad \text{and} \quad \varphi_p : \Omega \to \mathbb{R}, \quad \varphi_p : x \mapsto 3M\beta(\varepsilon^{-1} d_p(x)).$$

If $s > 1$, then proposition 7.6 implies

$$|D^{j_p} d_q(p)| = |D^{j_q} d_p(q)|.$$

Consequently, we have

$$|D^{j_p}\varphi_q(p)| = |D^{j_q}\varphi_p(q)|.$$

Observe furthermore that by the choice of p and q the function $u+\varphi_q$ attains a local maximum at p, whereas the function $v-\varphi_p$ attains a local minimum at q. Hence $-\varphi_q$ is an upper test function of u at p and φ_p is a lower test function of v at q. It follows

$$-D^{j_p}\varphi_q(p) = -\eta D^{j_p}d_q(p) \in J^+_{j_p}u(p) \quad \text{and} \quad D^{j_q}\varphi_p(q) = \eta D^{j_q}d_p(q) \in J^-_{j_q}v(q), \tag{7.15}$$

where $\eta := 3M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q))$.

By proposition 7.6 and by the properties of u and v we respectively obtain

$$f(p) > H^{j_p}(-D^{j_p}\varphi_q(p), u(p), p) \tag{7.16}$$

and

$$H^{j_q}(D^{j_q}\varphi_p(q), v(q), q) > 0. \tag{7.17}$$

The definition of Φ yields

$$u(p) - v(q) + 3M\beta(\varepsilon^{-1}d(p,q)) = \Phi(p,q) \geq \Phi(z,z) = u(z) - v(z) + 3M > 3M,$$

whence we have $u(p) > v(q)$. By virtue of (7.4) (ii) and by (7.17) we then obtain

$$H^{j_q}(D^{j_q}\varphi_p(q), u(p), q) > 0. \tag{7.18}$$

First assume $s > 1$. Then (7.18), (7.16), and (7.14) imply a contradiction.

Secondly assume $s = 1$. As Ω is a convex LEP space, $\Omega_{j_p} = \Omega_{j_p}$ is convex, implying

$$D^{j_p}d_q(p) = -D^{j_q}d_p(q), \quad \text{and thus} \quad D^{j_p}\varphi_q(p) = -D^{j_q}\varphi_p(q),$$

which yields a contradiction in combination with (7.18), (7.16), and (7.12). □

As in chapter 5 we obtain:

Lemma 7.3. *Let $u, v \in C(\bar{\Omega})$ be a viscosity sub- and supersolution of*

$$H(Du(x), u(x), x) = 0 \quad \text{on } \Omega, \tag{7.19}$$

respectively, with $u \leq v$ on $\partial\Omega$. Furthermore assume that there is an $M \in \mathbb{R}$ such that $M < u$ on $\bar{\Omega}$. Then $u \leq v$ on $\bar{\Omega}$.

Proof. The proof is completely analogous to the proof of lemma 5.2, where we invoke lemma 7.2 instead of lemma 5.1. □

7.8 Existence

In the present section we show that we can find a solution of (7.9), provided that certain barrier functions exist. For this purpose we define the upper and lower semicontinuous envelopes $u^\star$ and $u_\star$ as in definition 5.4.

Lemma 7.4. *Let V be an arbitrary set of viscosity subsolutions of*

$$H(Du(x), u(x), x) = 0 \quad \text{on } \Omega. \tag{7.20}$$

Define the function $u(x) := \sup_{v \in V} v(x)$ for all $x \in \Omega$ and assume $u^\star(x) < \infty$ for all $x \in \Omega$. Then $u^\star$ is a viscosity subsolution of (7.20).

Proof. We have to show that $u^\star$ satisfies the viscosity subsolution condition for all $x \in \Omega$. As in the proof of lemma 5.3 we restrict our considerations to ramification points $x \in \Sigma$, since the case $x \notin \Sigma$ is based upon similar, but easier, arguments.

Accordingly, let $x \in \Sigma$. We show that $u^\star$ satisfies the viscosity subsolution condition at x. By definition, $u^\star$ is upper semicomtinuous at x. Let $k, l \in \mathrm{Inc}_x$, $k \neq l$, and fix $p \in J^+_{k,l} u^\star(x)$. We have to show

$$H^k(p, u^\star(x), x) \leq 0.$$

According to proposition 7.4 we can choose an upper (k,l)-test function φ of $u^\star$ at x with $D^k\varphi(x) = p$ and such that φ is both C^1-differentiable in an open neighborhood U of x and (k,l)-differentiable for all $x \in \Sigma \cap U$. Furthermore assume $\varphi(x) = u(x)$ without restriction. Consequently, the function $u^\star - \varphi$ attains a local maximum at x, whence there is a radius $r > 0$ with $B_r(x) \subset U$ such that this maximum is global with respect to $\bar{B}$, where $B := B_r(x) \cap (\bar{\Omega}_k \cup \bar{\Omega}_l)$. Since Ω is an n-dimensional LEP space, we can assume $r > 0$ to be sufficiently small such that $B \cap \Sigma$ is contained in an $(n-1)$-dimensional affine linear subspace of $\mathbb{R}^{n+1}$.

As in the proof of lemma 5.3 we now indicate a sequence $(x_m)_{m \in \mathbb{N}}$, $x_m \in B$ for all $m \in \mathbb{N}$, and a sequence $(u_m)_{m \in \mathbb{N}}$, $u_m \in V$ for all $m \in \mathbb{N}$, such that

$$\lim_{m \to \infty} x_m = x \quad \text{and} \quad u^\star(x) = \lim_{m \to \infty} u_m(x_m). \tag{7.21}$$

Now let $y_m \in \bar{B}$ for each $m \in \mathbb{N}$ be a point at which the upper semicontinuous function

$$\varphi_{2\delta} : \Omega \to \mathbb{R}, \quad y \mapsto u_m(y) - \varphi(y) - 2\delta(d_x(y))^2$$

attains its maximum with respect to $\bar{B}$, where we have set

$$d_x(y) := d(x, y) \quad \text{for } y \in \Omega.$$

As in the proof in lemma 5.3 we then conclude $d(x, z) = 0$, implying

$$\lim_{m\to\infty} y_m = x \quad \text{and} \quad \lim_{m\to\infty} u_m(y_m) = u^\star(x). \tag{7.22}$$

Consequently, we may truncate the sequence $(y_m)_{m\in\mathbb{N}}$ such that all y_m lie in the *interior* of $\bar{B}$. Fixing $m \in \mathbb{N}$, we distinguish two cases.

Case 1. There are infinitely many y_m with $y_m \notin \Sigma$. Then there is a subsequence – also denoted by $(y_m)_{m\in\mathbb{N}}$ –, which is completely contained in either Ω_k or Ω_l. Without restriction we assume $y_m \in \Omega_k$ for all $m \in \mathbb{N}$. As the function $u_m - \varphi_{2\delta}$ attains a local maximum at y_m with respect to B and as the function $\varphi_{2\delta}$ is C^1-differentiable at y_m, the latter is an upper test function of u_m at y_m, implying $D^k\varphi_{2\delta}(y_m) \in J_k^+u_m(y_m)$. As u_m satisfies the viscosity subsolution condition at y_m, we conclude

$$H^k(D^k\varphi_{2\delta}(y_m), u_m(y_m), y_m) \le 0. \tag{7.23}$$

Now, since φ is C^1-differentiable in B, we have

$$\lim_{m\to\infty} D^k\varphi_{2\delta}(y_m) = D^k\varphi(x) = p. \tag{7.24}$$

Hence the relations (7.23), (7.24), and (7.22) imply

$$H^k(p, u^\star(x), x) = \lim_{m\to\infty} H^k(D^k\varphi(x_m), u^\star(x_m), x_m) \le 0,$$

as H is continuous.

Case 2. By possibly truncating the sequence $(y_m)_{m\in\mathbb{N}}$ we can assume that $y_m \in \Sigma \cap B$ for all $m \in \mathbb{N}$. As $\Sigma \cap B$ is contained in a linear subspace of $\mathbb{R}^{n+1}$, we have

$$\partial_{\nu_j} d_x(y) = 0 \quad \text{for all } y \in \Sigma \cap B \text{ and } j \in \text{Inc}_x;$$

in particular it follows that d_x is (k, l)-differentiable on $\Sigma \cap B$. By assumption the same holds for φ as well, whence we conclude that $\varphi_{2\delta}$ is (k, l)-differentiable on $\Sigma \cap B$. Thus $\varphi_{2\delta}$ is an upper (k, l)-test function of u_m at y_m for all $m \in \mathbb{N}$. As u_m satisfies the viscosity subsolution condition at y_m, it follows

$$H^k(D^k\varphi_{2\delta}(y_m), u_m(y_m), y_m) \le 0 \tag{7.25}$$

for all $m \in \mathbb{N}$. We furthermore have

$$\lim_{m\to\infty} D^k\varphi_{2\delta}(y_m) = D^k\varphi(x) = p, \tag{7.26}$$

since φ is C^1-differentiable on B. Now (7.22), (7.25), and (7.26), as well as the continuity of H^k imply

$$H^k(p, u^\star(x), x) = \lim_{m\to\infty} H^k(D^k\varphi_{2\delta}(y_m), u_m(y_m), y_m) \le 0.$$

As the choice of k and l was arbitrary, it follows that $u^\star$ satisfies the viscosity subsolution condition at x. □

Returning to the Dirichlet problem (7.9), we now state the following existence result.

Theorem 7.1. *Assume that there is a viscosity subsolution w and a viscosity supersolution W of*

$$H(Du(x), u(x), x) = 0 \quad \textit{on } \Omega. \tag{7.27}$$

satisfying the boundary condition $w_\star \equiv w \equiv W^\star \equiv W \equiv \phi$ on $\partial\Omega$. Furthermore assume that w is uniformly bounded by below on Γ.

Define the function $u : \Omega \to \mathbb{R}$ by $u(x) := \sup_{v \in X} v(x)$, where

$$X := \{v \textit{ is a viscosity subsolution of (7.27) with } w \le v \le W \textit{ on } \bar{\Omega}\}.$$

Then u is a solution of (7.9).

Proof. As in the proof of lemma 5.1 we first conclude by means of lemma 7.3 that we have

$$u \in X \quad \text{and} \quad u \equiv u^\star \text{ on } \Omega. \tag{7.28}$$

Then suppose that $u_\star$ be a viscosity supersolution of (7.27). By lemma 7.3 we conclude $u_\star \ge u$, implying $u_\star = u$ by means of the definition of lower semicontinuous envelope, and the theorem is proved.

It therefore remains to show that $u_\star$ is a viscosity supersolution of (7.27), where, as in the previous proofs, we only treat the case $x \in \Sigma$. Accordingly, let $x \in \Sigma$ and assume that $u_\star$ does *not* satisfy the viscosity supersolution condition at x. Then—according to definition 7.25—, there is a Σ-tangent vector $p^\top \in T_x\Sigma$ and an index $k \in \text{Inc}_x$ such that for each $l \in K := \text{Inc}_x \backslash \{k\}$ there is a tangent vector $p_l \in J^-_{k,l} u_\star(x)$ with $\pi^\top(p_l) = p^\top$ and

$$H^k(p_l, u_\star(x), x) < 0. \tag{7.29}$$

Then, according to proposition 7.4, there is an open neighborhood U of x and a family of functions $\varphi_l \in C(\Omega)$, $l \in K$, such that for all $l \in K$ we have

$$\begin{array}{ll} (i) & \varphi_l \text{ is } C^1\text{-differentiable on } U \cap (\Omega_k \cup \Omega_l) \\ (ii) & \varphi_l \text{ is } (k,l)\text{-differentiable on } U \cap \Sigma \\ (iii) & D^k\varphi_l(x) = p_l \\ (iv) & \varphi_l(x) = u_\star(x) \\ (v) & \varphi_l(y) < u_\star(y) \text{ for all } y \in U \backslash \{x\} \\ (vi) & \exists \varphi_\Sigma \in C(U \cap \Sigma) \text{ such that } \varphi_l \equiv \varphi_\Sigma \text{ on } U \cap \Sigma. \end{array} \tag{7.30}$$

Note that we are allowed to assume property (vi), as we have $\pi^\top(D^k\varphi_l) = p^\top$ for all $l \in K$.

Due to relation (7.29) and by the continuity of H there is a small number $\delta > 0$ such that for all $l \in K$ and all $m \in \text{Inc}_x$ we have

$$H^m(D^m\varphi_l(y), \varphi_l(y), y) < 0$$

for all $y \in B_m := \bar{B}_\delta(x) \cap \bar{\Omega}_m$. We furthermore assume δ to be sufficiently small such that $B_\delta(x) \subset U$.

Hence, as B_m is compact, there is a number $\varepsilon_0 > 0$ such that for all $l \in K$ and all $m \in \mathrm{Inc}_x$ we have

$$H^m(D^m \varphi_l(y), \varphi_l(y) + \varepsilon, y) < 0 \tag{7.31}$$

for all $0 \leq \varepsilon < \varepsilon_0$ and all $y \in B_m$.

We now construct a function $v \in X$ such that $v(x^\star) > u(x^\star)$ for some $x^\star \in \Omega$, contradicting the supremal property of u. In fact, for $y \in \bar{B}_\delta(x)$ we define

$$\tilde{v}(y) := \begin{cases} \max_{l \in K} \varphi_l(y) & \text{if } y \in B_k \\ \varphi_l(y) & \text{if } y \in B_l \backslash \Sigma \text{ for some } l \in K \end{cases}$$

and $\tilde{v}_\varepsilon := \tilde{v} + \varepsilon$, where ε with $0 < \varepsilon < \varepsilon_0$ is yet to be determined. Observe that $\tilde{v}$ is continuous on $\bar{B}_\delta(x)$, as the functions φ_l coincide on $\bar{B}_\delta(x) \cap \Sigma$ according to (7.30) (vi). We show that $\tilde{v}_\varepsilon$ satisfies the viscosity subsolution condition at each $y \in B_\delta(x)$.

Case 1. Let $y \in B_\delta(x) \cap \Sigma$. For each choice of $i, j \in \mathrm{Inc}_y$, $i \neq j$, we have to show

$$H^i(p, \tilde{v}(y), y) \leq 0 \tag{7.32}$$

for all $p \in J^+_{i,j} \tilde{v}(y)$. Choose $p \in J^+_{i,j} \tilde{v}(y)$ arbitrarily, represented by

$$p = p_n \partial_{\nu_i} + \sum_{m=1}^{n-1} p_m \partial_m,$$

where $\partial_{\nu_i}, \partial_1 \ldots, \partial_{n-1}$ are defined by means of the canonical identification around y as in definition 7.18. Let us first assume the special case $i \neq k$ and $j \neq k$. By the definition of $\tilde{v}$ and by the definition of superdifferential it follows

$$p_m = \partial_m \varphi_i(y) \text{ for all } m = 1, \ldots, n-1 \text{ and } p_n \geq \partial_{\nu_i} \varphi_i(y).$$

This implies

$$|p| \leq |D^i \varphi_i(y)|, \text{ provided we have } p_n \leq 0. \tag{7.33}$$

Applying the same arguments to

$$S_i^j(p) = -p_n \partial_{\nu_j} + \sum_{m=1}^{n-1} p_m \partial_m \in J^+_{j,i} \tilde{v}(y),$$

we derive $-p_n \geq \partial_{\nu_j} \varphi_j(y)$, implying

$$|S_i^j(p)| \leq |D^j \varphi_j(y)|, \text{ provided we have } -p_n \leq 0. \tag{7.34}$$

Then (7.33) and (7.34) yield

$$|p| = |S_i^j(p)| \leq \max\{|D^i\varphi_i(y)|, |D^j\varphi_j(y)|\}. \tag{7.35}$$

Now observe that conditions (7.4) and (7.5) imply that for each $l \in \{i, j\}$ the function

$$h^l : T_y^l\Omega \to \mathbb{R},\ q \mapsto H^l(q, \tilde{v}_\varepsilon(y), y)$$

is strictly convex and rotationally symmetric in q. Consequently, $h^l(q)$ is strictly isotonic in $|q|$ for both $l = i$ and $l = j$. By (7.35) we thus have

$$H^i(p, \tilde{v}_\varepsilon(y), y) = H^j(S_i^j(p), \tilde{v}_\varepsilon(y), y) \leq \max\{H^i(D^i\varphi_i(y), \tilde{v}_\varepsilon(y), y), H^j(D^j\varphi_j(y), \tilde{v}_\varepsilon(y), y)\}.$$

Hence (7.32) follows upon choosing $l = m = i$ and $l = m = j$, respectively, in (7.31).

In order to prove the general case it remains to examine the situation where one of the indices i, j, say i, coincides with k. Then by the definition of $\tilde{v}$ and by the definition of superdifferential we have

$$p_n \geq \max\{\partial_{\nu_i}\varphi_i(y), \partial_{\nu_i}\varphi_j(y)\} \text{ and } p_m = \partial_m\varphi_i(y) = \partial_m\varphi_j(y) \text{ for all } m = 1, \ldots, n-1.$$

This implies

$$|p| \leq \min\{|D^i\varphi_i(y)|, |D^i\varphi_j(y)|\}, \text{ provided } p_n \leq 0.$$

We combine this with (7.34), which we derive as above, in order to obtain (7.35). We then proceed as above to show (7.32).

Case 2. Let $y \in B_\delta(x)\backslash\Sigma$. If $y \in \Omega_l$, $l \neq k$, we have to show

$$H^i(p, \tilde{v}_\varepsilon(y), y) \leq 0 \quad \text{for all } p \in J_l^+\tilde{v}(y). \tag{7.36}$$

As φ is differentiable at y, this follows immediately from (7.31) and the definition of $\tilde{v}_\varepsilon$. If $y \in \Omega_k$, (7.31) implies that each $\varphi_l + \varepsilon$, $l \in K$, satisfies the viscosity subsolution condition at y. Then (7.36) follows by the definition of $\tilde{v}_\varepsilon$ and proposition 7.5.

By (7.30) (v) we may now choose the number ε with $0 < \varepsilon < \varepsilon_0$ such that $\tilde{v}_\varepsilon(y) < u_\star(y)$ for all $y \in \partial B_\delta(x)$. As $u_\star \leq u$ on Ω, we have $\tilde{v}_\varepsilon < u$ on $\partial B_\delta(x)$. Note that $\tilde{v}_\varepsilon$ is upper semicontinuous on $B_\delta(x)$. Furthermore, u is upper semicontinuous on Ω by (7.28). Hence the function $v : \Omega \to \mathbb{R}$ given by

$$v(y) := \begin{cases} \max\{u(y), \tilde{v}_\varepsilon(y)\} & \text{if } y \in B_\delta(x) \\ u(y) & \text{if } y \notin B_\delta(x) \end{cases}$$

is upper semicontinuous on Ω. Furthermore, by the fact that $\tilde{v}_\varepsilon$ satisfies the viscosity subsolution on B_δ(x) and by (7.28), we may invoke proposition 7.5 to conclude that v satisfies the viscosity subsolution condition on $B_\delta(x)$. Hence, as $v \equiv u$ on $\Omega\backslash B_\delta(x)$, we conclude $v \in X$. Finally observe that by (7.30) (iv) we have $v(x) = u_\star(x) + \varepsilon$, implying that there is a point $x^\star \in \Omega$ with $v(x^\star) > u(x^\star)$. Thus we have derived a contradiction to the definition of u, which completes the proof. □

Bibliography

[AG96] P. Aviles and Y. Giga. The distance function and defect energy. *Proceedings of the Royal Society of Edinburgh*, 126A:923–938, 1996.

[AMN93] F. Ali Mehmeti and S. Nicaise. Nonlinear interaction problems. *Nonlinear Analysis, Theory, Meth., and Appl.*, 20:27–61, 1993.

[AMOT90] R. K. Ahuja, K. Mehlhorn, J. B. Orlin, and R. E. Tarjan. Faster algorithms for the shortest way problem. *Journal of the ACM*, 37:213–223, 1990.

[BCRPE95] J.-P. Bouchaud, M. E. Cates, J. Ravi Prakash, and S. F. Edwards. Hysteresis and metastability in a continuum sandpile model. *Phys. Rev. Lett.*, 74:1982–1985, 1995.

[Bel58] R. Bellman. On a routing problem. *Quarterly of applied mathematics*, 16(1):87–90, 1958.

[Cam80] H. Camerer. *Die elektrotonische Ausbreitung im Soma, Dendritenbaum und Axon von Nervenzellen.* Dissertation, Tübingen, 1980.

[CIL92] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[CL81] M. G. Crandall and P.-L. Lions. Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi de premier order. *C. R. Acad. Sci. Paris*, 292:183–186, 1981.

[CL83] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.

[Dij59] E. W. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1:269–271, 1959.

[Eva98] L. C. Evans. *Partial differential equations.* American Mathematical Society, 1998.

[FF62] L. R. Ford and D. R. Fulkerson. *Flows in networks.* Princeton University Press, 1962.

[GT77] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer Verlag, 1977.

[Har69] F. Harary. *Graph theory.* Addison-Wesley Publishing Company, 1969.

[HK99] K. P. Hadeler and C. Kuttler. Dynamical models for granular matter. *Granular Matter*, 2:9–18, 1999.

[HK01] K. P. Hadeler and C. Kuttler. Granular matter in a silo. *Granular Matter*, 3:193–197, 2001.

[HKG02] K. P. Hadeler, C. Kuttler, and I. Gergert. Dirichlet and obstacle problems for granular matter. *SFB 382, Univ. Tübingen, Report*, 173, 2002.

[Ish87a] H. Ishii. A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type. *Proc. Amer. Math. Soc.*, 100,2:247–251, 1987.

[Ish87b] H. Ishii. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.*, 55:369–384, 1987.

[Jen88] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rat. Mech. Anal.*, 101:1–27, 1988.

[Kru70] S. N. Kružkov. First order quasilinear equations in several independent variables. *Math. USSR Sbornik*, 10:217–243, 1970.

[Kru75] S. N. Kružkov. Generalized solutions of the Hamilton-Jacobi equations of eikonal type. I. Formulation of the problems; existence, uniqueness and stability theorems; some properties of the solutions. *Math. USSR Sbornik*, 27:406–446, 1975.

[LL91] J.E. Lagnese and G. Leugering. Modelling and controllability of networks of thin beams. *Proc. of the 15th IFIP conference*, 1991.

[LL93] J.E. Lagnese and G. Leugering. Modelling of dynamic networks of thin elastic plates. *Math. Methods Appl. Sci.*, 6:379–407, 1993.

[LLG94] J.E. Lagnese, G. Leugering, and Schmidt E. J. P. G. On the analysis and control of hyperbolic systems associated with vibrating networks. *Proc. R. Soc. Edinb.*, 1:77–104, 1994.

[Lum80] G. Lumer. Espaces ramifiés et diffusions sur les réseaux topologiques. *C. R. Acad. Sc. Paris, Série A*, 291:627–630, 1980.

[Nic88] S. Nicaise. Elliptic operators on elementary ramified spaces. *Integral Equations and Operator Theory*, 11:230–257, 1988.

[Nic93] S. Nicaise. *Polygonal interface problems, volume 39 of "Methoden und Verfahren der Mathematischen Physik"*. Peter Lang GmbH, Europäischer Verlag der Wissenschaften, 1993.

[Roa70] G. F. Roach. *Green's functions - Introductory theory with applications*. Van Nostrand Reinhold Company, 1970.

[SH72] M. G. Slinko and K. Hartmann. *Methoden und Programme zur Berechnung chemischer Reaktoren*. Akademie-Verlag, Berlin, 1972.

[SZ94] R. Stöcker and H. Zieschang. *Algebraische Topologie*. B. G. Teubner, Stuttgart, 1994.

[Tho99] M. Thorup. Undirected single-source shortest paths with positive integer weights in linear time. *Journal of the ACM*, 46(3):362–394, 1999.

[vBN96] J. von Below and S. Nicaise. Dynamical interface transition in ramified media with diffusion. *Comm. in Part. Diff. Eq.*, 21:255–279, 1996.

[Wil67] R. F. Williams. One-dimensional non-wandering sets. *Topology*, 6:473–487, 1967.

Curriculum Vitae

May 19, 1975	born in Stuttgart
1981 - 1985	attendance at the elementary school in Stuttgart
1985 - 1994	attendance at the Leibniz-Gymnasium in Stuttgart
June 1994	graduation from high school (*Abitur*)
1994 - 1995	two semesters of undergraduate studies of Psychology at the Eberhard-Karls University of Tübingen
1995 - 2001	studies of mathematics and computer science at the Eberhard-Karls University in Tübingen
1996 - 1999	studies of classical piano at the Musikakademie Tübingen
September 1997	graduation (*Vordiplom*) in Mathematics and Computer Science
1999 - 2001	participation in the *International Studies of Mathematics* program of the University of Tübingen
March-November 1999	attendance at the honors year program in mathematics at the Australian National University in Canberra, Australia, participation in the final exams
August 2001	completion of diploma thesis entitled *Convexity of Surfaces Moving by Mean Curvature Flow* supervised by Prof. G. Huisken
November 2001	graduation (*Diplom*) in Mathematics and Computer Science at the University of Tübingen
2002-2006	PhD program with major in partial differential equations and granular matter theory at the University of Tübingen
2002-2006	position as a scientific assistant at the University of Tübingen
June 2006	completion of the doctoral thesis entitled *Viscosity Solutions of Hamilton-Jacobi Equations of Eikonal Type on Ramified Spaces*

Academic teachers

Eberhard-Karls University of Tübingen
Mathematics: K.P. Hadeler, G. Huisken, M. Wolff, G. Scheja,
R. Nagel, H. Heyer, H. Kaul, V. Batyrev
Computer Science: R. Loos, A. Zell, K.-J. Lange

Australian National University
Mathematics: N. Trudinger, J. Hutchinson, B. Andrews