Semigroups applied to transport and queueing processes

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Introduction

Many problems describing the time evolution of a system can be written in the form of an abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dfrac{d}{dt}u(t) = Bu(t), & t \in [0, \infty), \\ u(0) = u_0 \in X, \end{cases}$$

for some closed linear operator $(B, D(B))$ on a suitable Banach space X.

Semigroup theory provides a powerful tool for the treatment of such problems. Indeed, it allows to prove well-posedness of (ACP) via the classical Hille-Yosida theorem. In recent years a systematic theory has been added in order to describe the qualitative and in particular the long term behaviour of the solutions of (ACP). Most of these results are based on a careful analysis of the spectrum of the operator B. In combination with results from the Perron-Frobenius theory of positive semigroups this leads to stability and convergence theorems for the semigroup generated by B.

In this thesis we show how one can effectively apply the above approach to transport and queueing problems. We thus obtain new results on the asymptotic behaviour.

The problems we study are given by partial differential equations including nontrivial boundary conditions. In a first step we rewrite these problems in the form (ACP). Then, in all our problems B turns out to be the generator of a positive and strongly continuous semigroup on some Banach space.

Our main concern is on the asymptotics, i.e. the behaviour of the solutions as t tends to infinity. To this end we study the spectrum of B via a so called *characteristic equation*. This characteristic equation yields a precise description of the spectrum of B using the spectra of simpler operators on a smaller Banach space. We show that in all our examples the eigenvalue 0 is the only spectral value on the imaginary axis. Moreover, the semigroups generated by the corresponding operators share the properties of being bounded, mean ergodic, positive and irreducible.

These are the major ingredients leading to our main result: There exists a one-dimensional equilibrium to which the solutions of our problems converge in time.

We now explain the contents of this thesis in more detail.

In Chapter 1 we outline a general framework developed by G. Greiner [Gre87] into which all our examples fit.

Then, in Chapter 2 we present a transport problem in networks. Originally, such transport equations go back to L. Boltzmann in the 19$^{\text{th}}$ century. Semigroup methods were applied in the fifties of the last century to this kind of problems, see [Bir59]. Now, there exists a vast literature on the application of semigroup theory to transport problems, e.g. [DM79], [KLH82], [MK97].

Usually such flows take place in domains in $\mathbb{R}^n$. Since many transport processes occur in a piping system, it seems relevant to study the transport on networks. Here, the network is modelled by a directed and weighted graph. So also elements from graph theory come into the play, and it is an interesting question how the structure of the graph affects the long term behaviour of the system.

We study a generalisation of the situation from [KS05]. We assume that particles can move between the nodes of the network if they are connected by an edge. Single particles move with constant velocity along the edges. But, unlike the case in [KS05], different particles can have different velocities. When the particles pass a vertex then they are distributed among the outgoing edges. The proportion of the mass flowing into an outgoing edge is given by the weight of the respective edge. Finally, and this is the main feature of our model, the particles are scattered in the vertices, i.e. they change their velocity. We require that in the vertices a Kirchhoff law for the velocity profiles holds.

After formulating this problem as an abstract Cauchy problem, well-posedness is verified as well as certain spectral properties of the generator. If we assume that the scattering is realised by a compact integral operator such that the number of the particles is maintained after scattering, then the spectrum of the corresponding generator is a pure point spectrum and 0 is the only spectral value on the imaginary axis.

The following Chapters 3 and 4 are devoted to problems arising in queueing theory. We shortly sketch the general background. The mathematical investigation of queues started with the Danish mathematician A. Erlang at the beginning of the 20$^{\text{th}}$ century. He studied queues in the context of telephone traffic while he was working in a telephone company. Later in [Cox55], queueing systems were described using partial differential equations. A semigroup approach to queueing theory was proposed by G. Gupur and others, see [GLZ01a].

Here, we consider queueing problems with a time parameter x counting the service time in addition to the system time t. Service time is reset to 0 as soon as a new service starts. The equations then resemble those arising in the description of age-dependent populations in biology.

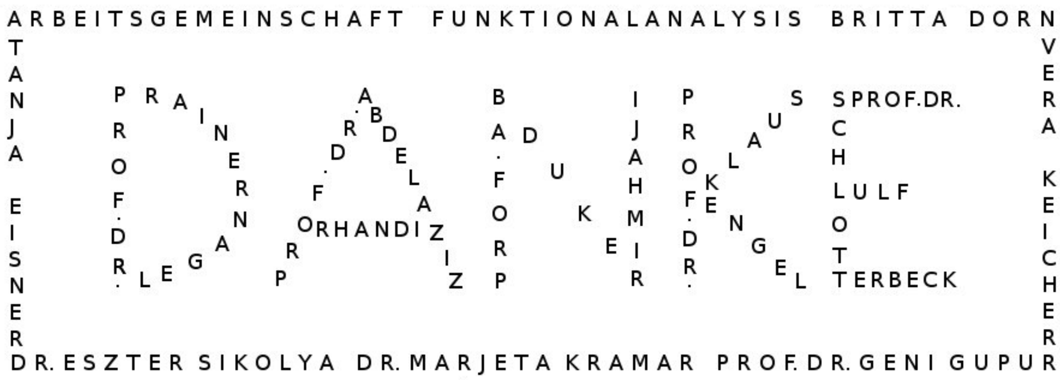
In Chapter 3 the $M/M^{k,B}/1$ queueing model is investigated. In this model there is a single server which can serve B customers simultaneously. The service starts as soon as there are k customers in the queue. The arrival of the customers in the queue is at random. The interarrival times of the customers as well as the service times are exponentially distributed.

In Chapter 4 we consider a simple queueing network consisting of two servers or machines that are separated by a finite storage buffer. The treatment of this problem is inspired by a joint work with A. Haji, see [HR05]. The customers or objects have to pass both machines. The objects enter the system at the first machine and can leave the system only at the second machine. After service at the first machine the objects either have to be reprocessed or they are transferred to the buffer with constant probabilities. The second machine takes the objects from the buffer and again after service they are either reprocessed or leave the system with certain probabilities. The buffer is finite. So if the buffer is full, the first machine is blocked until an object leaves the buffer.

Again, the problems are written as abstract Cauchy problems. Then the well-posedness of these problems is shown and the spectra of the generators are determined. In particular, 0 is an eigenvalue of the respective generator and is the only spectral value on the imaginary axis.

Finally, in Chapter 5 we determine the long term behaviour of the previously studied models. As far as the transport problem is concerned, irreducibility of the corresponding semigroup is obtained under specific conditions imposed on the graph and on the scattering operator. We give examples showing that the semigroup is not necessarily irreducible if we drop one of these assumptions. The irreducibility of the semigroups from queueing theory is proved without additional assumptions. We state the main conclusions on the asymptotic behaviour of the solutions of each of these problems, i.e. the convergence to a one-dimensional equilibrium. The proof is based on the Arendt-Batty-Lyubich-Vũ Theorem [ABHN01, Thm. 5.5.5].

The appendix contains the definitions and results on positive operators and semigroups needed in this thesis.



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CHAPTER 1

Tools from operator theory

Here, we present some of the tools we use to study the semigroups arising in our context. Of particular value is the Characteristic Equation 1.7 below.

We assume the reader to be familiar with basic semigroup theory and refer to [EN00], [EN06], [Gol85] and [Paz83]. Let X be a Banach space called the *state space* and let $(B, D(B))$ be a closed linear operator on X. The *abstract Cauchy problem* associated to $(B, D(B))$ and the initial value $u_0 \in X$ is

$$\text{(ACP)} \qquad \begin{cases} \dfrac{d}{dt}u(t) = Bu(t), & t \in [0,\infty), \\ u(0) = u_0. \end{cases}$$

We call a function $u(\cdot, u_0) : [0,\infty) \to X$ a *classical solution* of (ACP) if

(i) $u(\cdot, u_0)$ is continuously differentiable,
(ii) $u(t, u_0) \in D(B)$ for all $t \geq 0$, and
(iii) (ACP) holds,

see [EN00, Def. II.6.1 (ii)]. According to [EN00, Def. II.6.8], the problem (ACP) is called *well-posed* if

(i) for every initial value $u_0 \in D(B)$ there exists a unique classical solution $u(\cdot, u_0)$ of (ACP),
(ii) $D(B)$ is dense in X, and
(iii) for every sequence $(u_n)_{n\in\mathbb{N}} \subseteq D(B)$ satisfying
$$\lim_{n\to\infty} u_n = 0$$
one has
$$\lim_{n\to\infty} u(t, u_n) = 0$$
uniformly in compact intervals $[0, t_0]$.

The well-posedness of (ACP) is characterised as follows, see [EN00, Cor. II.6.9].

Proposition 1.1. For a closed operator $(B, D(B))$ on X the associated abstract Cauchy problem (ACP) is well-posed if and only if B generates a strongly continuous semigroup on X.

If (ACP) is well-posed, then, by [EN00, Prop. II.6.2], the unique classical solution u is given by the orbit of u_0 under the semigroup $(T(t))_{t\geq 0}$ generated by

B, i.e.

$$u(t) = T(t)u_0, \qquad t \geq 0.$$

We now consider a class of operators $(A, D(A))$ which are constructed in a particular way. We start from a closed linear operator $(A_m, D(A_m))$ on X, called the *maximal operator*. Moreover, we take another Banach space ∂X — the *boundary space* — and use *boundary operators* $L, \Phi \in \mathcal{L}(D(A_m), \partial X)$. In the following we always assume that L is surjective. The operator $(A, D(A))$ is given as follows.

DEFINITION 1.2. The operator $(A, D(A))$ is defined as

$$\begin{aligned} Au &:= A_m u, \\ D(A) &:= \{u \in D(A_m) \,:\, Lu = \Phi u\}. \end{aligned}$$

Under appropriate assumptions, there exists a characterisation of its spectrum $\sigma(A)$ and an explicit representation of its resolvent. The abstract framework for this was developed by G. Greiner in [Gre87]. We sketch these results. The starting point is the operator $(A_0, D(A_0))$ which is the restriction of A_m to the kernel of L, i.e.

$$\begin{aligned} D(A_0) &:= \{u \in D(A_m) \,:\, Lu = 0\}, \\ A_0 u &:= A_m u. \end{aligned}$$

Then by [Gre87, Lemma 1.2] the domain of A_m can be decomposed as follows.

LEMMA 1.3. Let $\gamma \in \rho(A_0)$. Then

$$D(A_m) = D(A_0) \oplus \ker(\gamma - A_m).$$

Since L is supposed to be surjective and $D(A_0) = \ker L$, we conclude from the above decomposition that the restriction $L|_{\ker(\gamma - A_m)}$ of L to $\ker(\gamma - A_m)$ is bijective. Its inverse is even bounded which follows by the closed graph theorem.

DEFINITION 1.4. For $\gamma \in \rho(A_0)$ the operator $D_\gamma := (L|_{\ker(\gamma - A_m)})^{-1}$ is called *Dirichlet operator* corresponding to A_m and L.

Using the operators D_γ and Φ, we can characterise the spectrum and the point spectrum of A. To prove a characteristic equation for the spectrum of A we work on the product space $X \times \partial X$ and extend the given operators, see also [KS05, Sect. 3].

DEFINITION 1.5.

(i) $\mathcal{X} := X \times \partial X$.

(ii) $\mathcal{A}_0 := \begin{pmatrix} A_m & 0 \\ -L & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) := D(A_m) \times \{0\}.$

(iii) $\mathcal{X}_0 := X \times \{0\} = \overline{D(A_m) \times \{0\}} = \overline{D(\mathcal{A}_0)}.$

(iv) $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix}, \quad D(\mathcal{B}) := D(A_m) \times \partial X.$

(v) $\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_m & 0 \\ \Phi - L & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A_m) \times \{0\}.$

REMARK 1.6.

(i) Observe that $\rho(\mathcal{A}_0) \supseteq \rho(A_0)$. Moreover, the resolvent of $\mathcal{A}_0$ for $\gamma \in \rho(A_0)$ is given by

$$R(\gamma, \mathcal{A}_0) = \begin{pmatrix} R(\gamma, A_0) & D_\gamma \\ 0 & 0 \end{pmatrix}.$$

(ii) The part $\mathcal{A}|_{\mathcal{X}_0}$ of $\mathcal{A}$ in $\mathcal{X}_0$ is given by

$$D(\mathcal{A}|_{\mathcal{X}_0}) = D(A) \times \{0\}, \quad \mathcal{A}|_{\mathcal{X}_0} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{A}|_{\mathcal{X}_0}$ can be identified with the operator $(A, D(A))$.

Now we show that the spectrum of A is characterised by the spectrum of operators on the boundary space ∂X.

CHARACTERISTIC EQUATION 1.7. Let $\gamma \in \rho(A_0)$. Then

(i) $$\gamma \in \sigma_p(A) \Leftrightarrow 1 \in \sigma_p(\Phi D_\gamma).$$

Suppose in addition that there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$. Then

(ii) $$\gamma \in \sigma(A) \Leftrightarrow 1 \in \sigma(\Phi D_\gamma).$$

PROOF. First we show for $\mathcal{A}$ from Definition 1.5 the equivalence

(1) $$\gamma \in \sigma(\mathcal{A}) \quad \Leftrightarrow \quad 1 \in \sigma(\Phi D_\gamma),$$

as in [KS05, Prop. 3.3]. Therefore, we decompose

(2) $$\gamma - \mathcal{A} = \gamma - \mathcal{A}_0 - \mathcal{B} = (\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))(\gamma - \mathcal{A}_0).$$

From this we see that $\gamma - \mathcal{A}$ is invertible if and only if $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ is invertible. Since

(3) $$\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0) = \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix},$$

one can easily see that the invertibility of $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ is equivalent to $1 \notin \sigma(\Phi D_\gamma)$ and (1) is shown. From our assumption that $1 \notin \sigma(\Phi D_{\gamma_0})$ it follows now that $\gamma_0 \in \rho(\mathcal{A})$ and therefore, $\rho(\mathcal{A})$ is not empty. Hence we obtain from [EN00, Prop. IV.2.17] that

$$\sigma(\mathcal{A}) = \sigma(A),$$

since A is the part of $\mathcal{A}$ in $\mathcal{X}_0$. This shows (ii).

To prove (i) observe first that the point spectra of $\mathcal{A}$ and A coincide, i.e.,

$$\sigma_p(\mathcal{A}) = \sigma_p(A).$$

Suppose now that $1 \in \sigma_p(\Phi D_\gamma)$. Then there exists $0 \neq f \in \partial X$ such that $(Id_{\partial X} - \Phi D_\gamma)f = 0$. Clearly, $0 \neq \left(\begin{smallmatrix} D_\gamma f \\ 0 \end{smallmatrix}\right) \in D(\mathcal{A})$. So we can compute

$$\begin{aligned}
(\gamma - \mathcal{A})\begin{pmatrix} D_\gamma f \\ 0 \end{pmatrix} &= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix}\begin{pmatrix} (\gamma - A_m)D_\gamma f \\ LD_\gamma f \end{pmatrix} \\
&= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (Id_{\partial X} - \Phi D_\gamma)f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Hence, $\gamma \in \sigma_p(\mathcal{A})$.

Conversely, assume that $\gamma \in \sigma_p(\mathcal{A})$. Then there exists $0 \neq f \in D(A_m)$ such that $(\gamma - \mathcal{A})\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right) = 0$. From

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= (\gamma - \mathcal{A})\begin{pmatrix} f \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix}\begin{pmatrix} (\gamma - A_m)f \\ Lf \end{pmatrix} \\
&= \begin{pmatrix} (\gamma - A_m)f \\ -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf \end{pmatrix}
\end{aligned}$$

we obtain that $f \in \ker(\gamma - A_m)$ and hence

$$0 = -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf = (Id_{\partial X} - \Phi D_\gamma)Lf.$$

It follows from Lemma 1.3 that $Lf \neq 0$ and thus $1 \in \sigma_p(\Phi D_\gamma)$.

□

The operator ΦD_γ is defined on the boundary space ∂X which will be, in most cases, much smaller than the state space X. So we expect that it is easier to determine the spectrum of ΦD_γ than to compute the spectrum of A directly. This allows, by the above characteristic equation, to characterise the spectrum of A in a second step.

For later use, we express the resolvent of A in terms of the operators D_γ, Φ and the resolvent of A_0.

Proposition 1.8. Suppose that there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$ and let $\gamma \in \rho(A_0) \cap \rho(A)$. Then

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma(Id_{\partial X} - \Phi D_\gamma)^{-1}\Phi R(\gamma, A_0)$$

holds.

Proof. Recall from the Characteristic Equation 1.7 and its proof that under our assumptions $1 \notin \sigma(\Phi D_\gamma)$ and that $\gamma - \mathcal{A}$ is invertible with inverse

$$(\gamma - \mathcal{A})^{-1} = (\gamma - \mathcal{A}_0)^{-1}(\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1}.$$

Using the explicit representation (3) for $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ we obtain

$$(\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1} = \begin{pmatrix} Id_X & 0 \\ (Id_{\partial X} - \Phi D_\gamma)^{-1}\Phi R(\gamma, A_0) & (Id_{\partial X} - \Phi D_\gamma)^{-1} \end{pmatrix},$$

and hence

$$R(\gamma, \mathcal{A}) = \begin{pmatrix} R(\gamma) & D_\gamma(Id_{\partial X} - \Phi D_\gamma)^{-1} \\ 0 & 0 \end{pmatrix},$$

where $R(\gamma) = (Id_X + D_\gamma(Id_{\partial X} - \Phi D_\gamma)^{-1}\Phi)R(\gamma, A_0)$. Since A is the part of $\mathcal{A}$ in $\mathcal{X}_0$ and since

$$\left(\begin{smallmatrix} R(\gamma) & 0 \\ 0 & 0 \end{smallmatrix}\right) = R(\gamma, \mathcal{A})|_{\mathcal{X}_0} = R(\gamma, \mathcal{A}|_{\mathcal{X}_0}),$$

it follows that

$$R(\gamma, A) = R(\gamma).$$

□

REMARK 1.9. The problems we investigate in this thesis are formulated by partial differential equations involving nontrivial boundary conditions. These problems will be rewritten as abstract Cauchy problems of the form (ACP) and we will apply semigroup theory to prove existence and qualitative properties of the solutions.

All our operators will arise in the abstract form of Definition 1.2. Here, the maximal operator is a differential operator on its natural maximal domain while the boundary space is a space of functions "on the boundary". The domain $D(A)$ of A incorporates the boundary conditions of the underlying problems.

We will determine the spectra of these operators in detail using the Characteristic Equation 1.7.

CHAPTER 2

Networks

2.1. Introduction

We consider a transport process with absorption and scattering as described by the classical linear Boltzmann equation, see [DM79], [GvdMP87], [KLH82]. As many authors before, e.g. [Vid68], [Vid70], [Gre84a], [Voi84], [Voi85], [MK97], we use the theory of strongly continuous operator semigroups, see [EN00], [Gol85], [Paz83], in particular the theory of positive semigroups on Banach lattices, see [Nag86], to show well-posedness and to discuss the asymptotic behaviour of the solutions in Chapter 5. However, while the problem is usually considered on a domain in $\mathbb{R}^n$, we study the transport process in a network. This seems to be physically relevant, and it is mathematically interesting to discuss how the network structure influences the process. Moreover, we assume that absorption and scattering takes place only in the ramification nodes of the network and that a Kirchhoff law holds in each node. As predecessors we mention papers studying transport equations in slab geometry as e.g. [Bou03b], [Bou03a], [Cha02] and [Lat00]. Closer to our setting is [Bar96] who concentrates on the well-posedness of a similar problem and discusses some applications to physics. Transport on networks is also studied in [KS05], [MS], and [Sik05]. This chapter is mainly inspired by [KS05]. However, these authors assume that all particles move with the same speed in the network. In doing so they developed the semigroup techniques we will use. Some of our results will appear in [Rad].

2.2. Setting

Our network is represented by a simple, directed and weighted graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices (or nodes) and $E = \{e_1, \ldots, e_m\}$ is the set of edges (or arcs). If two vertices are connected by an edge, then the particles can move between the vertices in the direction given by the edge. The velocity of each particle is constant during its motion along an edge. However, for different particles this velocity can vary between a minimal speed $v_{min} > 0$ and a maximal speed $v_{max} > v_{min}$. By the assumption on the minimal speed, each particle will reach a vertex after a finite time. In the vertices the particles are scattered, i.e. they change their velocity, or will be absorbed. Thereafter, they are distributed to the outgoing edges of the vertex according to the (positive) weight of the outgoing edge. We consider only the case that each vertex has at least one incoming and one outgoing edge.

This physical situation will now be modelled in mathematical terms. The edges $e_j, j = 1, \dots, m$, are parameterised over the intervals $[0, l_j]$ where $e_j(0)$ is the tail of the edge e_j and $e_j(l_j)$ is the head of the edge e_j: $e_j(0) \xrightarrow{e_j} e_j(l_j)$.

If edge e_j is an outgoing edge of vertex v_i, then ω_{ij} gives the weight of the edge e_j. In each vertex v_i the weights of the outgoing edges shall sum up to 1, i.e.

$$\sum_{j=1}^{m} \omega_{ij} = 1, \tag{4}$$

for each $i \in \{1, \dots, n\}$.

Our transport process is then described by the equations

$$\text{(N)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u_j(x,v,t) = -v \dfrac{\partial}{\partial x} u_j(x,v,t), & \\ u_j(x,v,0) = f_j(x,v), & (\mathrm{IC_N}) \\ \iota_{ij}^{-} u_j(0,\cdot,t) = \omega_{ij} J \displaystyle\sum_{k=1}^{m} \iota_{ik}^{+} u_k(l_k,\cdot,t), & (\mathrm{BC_N}) \end{cases}$$

where $x \in (0, l_j)$, $v \in [v_{min}, v_{max}]$, $t \geq 0$, and $j = 1, \dots, m$, $i = 1, \dots, n$. Here, $u_j(x,v,t)$ gives the density of the particles on edge e_j depending on the position x, the velocity v and the time t. The first equation is the well-known one-dimensional transport equation without scattering and absorption effects, while $(\mathrm{IC_N})$ is the usual initial condition for $t = 0$.

The equation $(\mathrm{BC_N})$ is a condition in the vertices of the graph and models the scattering, absorption, and redistribution of particles in the vertices. The coefficients ι_{ij}^{-} and ι_{ik}^{+} in $(\mathrm{BC_N})$ arise from matrices coding the structure of the graph and are defined in Definition 2.2.2 below. In this way, the equations in $(\mathrm{BC_N})$ relate the one-dimensional particle transport to the underlying network. The operator J appearing in $(\mathrm{BC_N})$ is called *scattering operator*. It converts, in each vertex v_i, the incoming velocity profile $\sum_{k=1}^{m} \iota_{ik}^{+} u_k(l_k,\cdot,t)$ into an outgoing velocity profile. Then the ω_{ij}^{th} part of this velocity profile is leaving vertex v_i into edge e_j. For the scattering operator J we assume the following.

General Assumption 2.2.1. The operator J is a positive contraction from $Y := L^1[v_{min}, v_{max}]$ to Y.

Since $\|f\|_1 = \int_{v_{min}}^{v_{max}} f(v)\, dv$ gives the total number of particles for positive functions in Y, this assumption means that no particles can enter the system.

The properties of J will play an important role for the asymptotics of the process and we will later make additional assumptions on J, see Sections 2.3, 5.2.1 and 5.2.2, with interesting consequences on the spectral properties and the asymptotic behaviour of the corresponding semigroup.

To describe the graph we use the following matrices, see also [KS05].

DEFINITION 2.2.2.

(i) The *outgoing incidence matrix* $\mathbb{I}^- = (\iota_{ij}^-)_{n\times m}$ is defined by

$$\iota_{ij}^- := \begin{cases} 1, & v_i = e_j(0), \text{ i.e. } v_i \xrightarrow{e_j}, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The *weighted outgoing incidence matrix* $\mathbb{I}_w^- = (\iota_{ij,w}^-)_{n\times m}$ is defined by

$$\iota_{ij,w}^- := \begin{cases} \omega_{ij}, & v_i = e_j(0), \text{ i.e. } v_i \xrightarrow[\omega_{ij}]{e_j}, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) The *incoming incidence matrix* $\mathbb{I}^+ = (\iota_{ij}^+)_{n\times m}$ is defined by

$$\iota_{ij}^+ := \begin{cases} 1, & v_i = e_j(l_j), \text{ i.e. } \xrightarrow{e_j} v_i, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) The *weighted transposed adjacency matrix* $\mathbb{A} = (\alpha_{ij})_{n\times n}$ is defined by $\mathbb{A} := \mathbb{I}^+(\mathbb{I}_w^-)^T$, i.e.

$$\alpha_{ij} = \begin{cases} \omega_{jk}, & \text{if } v_j = e_k(0) \text{ and } v_i = e_k(l_k), \text{ i.e. } v_j \xrightarrow[\omega_{jk}]{e_k} v_i, \\ 0, & \text{otherwise.} \end{cases}$$

(v) The *weighted transposed adjacency matrix* $\mathbb{B} = (\beta_{ij})_{m\times m}$ of the line graph is defined by $\mathbb{B} := (\mathbb{I}_w^-)^T\mathbb{I}^+$, i.e.

$$\beta_{ij} = \begin{cases} \omega_{ki}, & \text{if } e_i(0) = e_j(l_j) = v_k, \text{ i.e. } \xrightarrow{e_j} \overset{v_k}{\cdot} \xrightarrow[\omega_{ki}]{e_i}, \\ 0, & \text{otherwise.} \end{cases}$$

These matrices determine the structure of the graph completely, see [Bol98] and [GR01]. However, we need the following operator version of the above defined (scalar) matrices.

DEFINITION 2.2.3. Let Id_Y denote the identity operator on Y. We introduce the following operator matrices.

(i) $\tilde{\mathbb{I}}^- := (\iota_{ij}^- Id_Y)_{n\times m}$,
(ii) $\tilde{\mathbb{I}}_w^- := (\iota_{ij,w}^- Id_Y)_{n\times m}$,
(iii) $\tilde{\mathbb{I}}^+ := (\iota_{ij}^+ Id_Y)_{n\times m}$,
(iv) $\mathbb{I}_J^+ := (\iota_{ij}^+ J)_{n\times m}$,
(v) $\tilde{\mathbb{A}} := (\alpha_{ij} Id_Y)_{n\times n}$,
(vi) $\tilde{\mathbb{B}} := (\beta_{ij} Id_Y)_{m\times m}$,
(vii) $\mathbb{B}_J := (\beta_{ij} J)_{m\times m}$.

These operator matrices define operators in the canonical way on products of the space Y. Now we make a useful observation, see [KS05, Sect. 2].

REMARK 2.2.4. In each column of $\mathbb{I}^-$, $\mathbb{I}^-_w$, $\tilde{\mathbb{I}}^-$, $\tilde{\mathbb{I}}^-_w$, $\mathbb{I}^+$, $\tilde{\mathbb{I}}^+$ and $\mathbb{I}^+_J$ there is exactly one non-zero entry. Furthermore, an easy computation using condition (4) yields

$$\mathbb{I}^-(\mathbb{I}^-_w)^T = Id_{\mathbb{C}^n}$$

and

$$\tilde{\mathbb{I}}^-(\tilde{\mathbb{I}}^-_w)^T = Id_{Y^n}.$$

Moreover, one can easily show that $\mathbb{A}$ and $\mathbb{B}$ are column stochastic matrices.

Since we want to treat the problem (N) by semigroup methods, we rewrite it as an abstract Cauchy problem on a suitable state space X_N. As the state space for our problem we choose

$$X_N := L^1([0,l_1],Y) \times \cdots \times L^1([0,l_m],Y)$$

which is isomorphic to

$$L^1([0,l_1] \times [v_{min}, v_{max}]) \times \cdots \times L^1([0,l_m] \times [v_{min}, v_{max}]).$$

If all arc lengths are equal to l, then

$$X_N \cong L^1([0,l],Y^m) \cong (L^1([0,l],Y))^m \cong (L^1([0,l] \times [v_{min}, v_{max}]))^m.$$

The space X_N is endowed with the norm

$$\|\cdot\|_1 : X_N \to \mathbb{R}, \qquad \|u\|_1 := \sum_{j=1}^m \int_0^{l_j} \int_{v_{min}}^{v_{max}} |u_j(x,v)|\, dv\, dx,$$

where $u = (u_j)_{1\le j\le m} \in X_N$. In the spirit of [KS05] we choose the abstract boundary space as

$$\partial X_N := Y^n,$$

endowed with the norm

$$\|\cdot\|_1 : \partial X_N \to \mathbb{R}, \qquad \|f\|_1 := \sum_{i=1}^n \int_{v_{min}}^{v_{max}} |f_i(v)|\, dv,$$

where $f = (f_i)_{1\le i\le n} \in \partial X_N$.

Furthermore, we define

$$W := W^{1,1}([0,l_1],Y) \times \cdots \times W^{1,1}([0,l_m],Y)$$

which is a Banach space for the norm

$$\|\cdot\|_W : W \to \mathbb{R}, \qquad u \mapsto \|u\|_W := \|u\|_1 + \|u'\|_1.$$

The trace operators

$$\Gamma_0, \Gamma_l : W \to Y^m$$

are defined by

$$\Gamma_0 u := (u_j(0))_{1\le j\le m},$$

and

$$\Gamma_l u := (u_j(l_j))_{1\le j\le m},$$

respectively, where $u = (u_j)_{1\leq j\leq m} \in W$, and give the velocity profiles at the endpoints of the edges. Both operators are continuous on $(W, \|\cdot\|_W)$.

To formulate (N) as an abstract Cauchy problem we proceed as in [KS05], and start from the following maximal operator on X_N.

DEFINITION 2.2.5. The operator $(A_m^N, D(A_m^N))$ is defined by

$$D(A_m^N) := \{u \in W : \Gamma_0 u \in \operatorname{rg}(\tilde{\mathbb{I}}_w^-)^T\},$$
$$(A_m^N u)_j(x,v) := -v\tfrac{\partial}{\partial x} u_j(x,v), \quad x \in [0,l_j],\ v \in [v_{min}, v_{max}],\ j = 1,\ldots,m.$$

PROPOSITION 2.2.6. *The operator $(A_m^N, D(A_m^N))$ on X_N is closed.*

PROOF. Let $(u_n)_{n\in\mathbb{N}} \subseteq D(A_m^N)$ be a Cauchy sequence with respect to $\|\cdot\|_W$. Since $(W, \|\cdot\|_W)$ is a Banach space, the sequence converges to an element $u \in W$. From the condition appearing in the definition of $D(A_m^N)$ we obtain that for all $n \in \mathbb{N}$ there exists an $f^{(n)} \in Y^n$ sucht that

$$\Gamma_0 u^{(n)} = (\tilde{\mathbb{I}}_w^-)^T f^{(n)}.$$

By Remark 2.2.4 and the continuity of Γ_0 on $(W, \|\cdot\|_W)$ it follows that

$$f^{(n)} = \tilde{\mathbb{I}}^-(\tilde{\mathbb{I}}_w^-)^T f^{(n)} = \tilde{\mathbb{I}}^-\Gamma_0 u^{(n)} \longrightarrow \tilde{\mathbb{I}}^-\Gamma_0 u =: f.$$

Hence,

$$\Gamma_0 u = \lim_{n\to\infty} \Gamma_0 u^{(n)} = \lim_{n\to\infty} (\tilde{\mathbb{I}}_w^-)^T f^{(n)} = (\tilde{\mathbb{I}}_w^-)^T f,$$

i.e.

$$\Gamma_0 u \in \operatorname{rg}((\tilde{\mathbb{I}}_w^-)^T),$$

and thus, $u \in D(A_m^N)$. This means that $D(A_m^N)$ is complete with respect to $\|\cdot\|_W$, which is equivalent to the graph norm of A_m^N. This shows the closedness of $(A_m^N, D(A_m^N))$. ☐

The condition $\Gamma_0 u \in \operatorname{rg}(\tilde{\mathbb{I}}_w^-)^T$ in the definition of $D(A_m^N)$ means that the proportion of the mass leaving vertex v_i over edge e_j is determined by the weight ω_{ij}. However, this does not contain the complete boundary condition ($\mathrm{BC_N}$) from (N). To formulate a condition equivalent to ($\mathrm{BC_N}$) we introduce the following continuous operators on $(W, \|\cdot\|_W)$.

DEFINITION 2.2.7. The *outgoing boundary operator* L_N is defined by

$$L_N : W \to \partial X_N, \quad u \mapsto \tilde{\mathbb{I}}^-\Gamma_0 u,$$

while for the *incoming boundary operator* Φ_N we take

$$\Phi_N : W \to \partial X_N, \quad u \mapsto \mathbb{I}_J^+\Gamma_l u.$$

Note that $(\tilde{\mathbb{I}}^+\Gamma_l u)_i$ gives the velocity profile coming into vertex v_i. Then, $(\Phi_N u)_i$ gives the velocity profile in vertex v_i after the scattering and $(L_N u)_i$ gives the velocity profile leaving vertex v_i. Thus, the condition

$$L_N u = \Phi_N u \tag{5}$$

expresses the Kirchhoff law.

The operator corresponding to our original problem (N) is now given as follows.

DEFINITION 2.2.8. The operator $(A_N, D(A_N))$ is defined by

$$D(A_N) := \{u \in D(A_m^N) : L_N u = \Phi_N u\},$$
$$A_N u := A_m^N u.$$

To show the equivalence of $D(A_N)$ and $(\mathrm{BC_N})$, fix t in $(\mathrm{BC_N})$. Then

$$(u_j(0,\cdot,t))_{1\le j\le m} \in \mathrm{rg}(\tilde{\mathbb{I}}_w^-)^T.$$

Taking the sum over j in $(\mathrm{BC_N})$ yields the Kirchhoff law.

On the other hand, let us require that $L_N v = \Phi_N v$ and $\Gamma_0 v \in \mathrm{rg}(\tilde{\mathbb{I}}_w^-)^T$ for $v \in D(A_m^N)$. Then there exists $d = (d_i)_{1\le i\le n} \in Y^n$ such that $\Gamma_0 v = (\tilde{\mathbb{I}}_w^-)^T d$. Since in each row of $(\tilde{\mathbb{I}}_w^-)^T$ there is exactly one non-zero entry, it follows from the condition $\Gamma_0 v \in \mathrm{rg}(\tilde{\mathbb{I}}_w^-)^T$ that for every $j \in \{1,\dots,m\}$ there exists exactly one $i \in \{1,\dots,n\}$ such that

$$v_j(0,\cdot) = \omega_{ij} d_i. \tag{6}$$

Using this we compute for $i = 1,\dots,n$

$$J\sum_{j=1}^m \iota_{ij}^+ v_j(l_j,\cdot) \overset{\Phi_N v = L_N v}{=} \sum_{j=1}^m \iota_{ij}^- v_j(0,\cdot) \overset{(6)}{=} \sum_{j=1}^m \iota_{ij}^- \omega_{ij} d_i = \sum_{j=1}^m \omega_{ij} d_i \overset{(4)}{=} d_i. \tag{7}$$

Combining (7) and (6) yields

$$v_j(0,\cdot) = \omega_{ij} J \sum_{i=1}^m \iota_{ij}^+ v_j(l_j,\cdot).$$

If we multiply both sides by ι_{ij}^- and remember that $\omega_{ij} \neq 0$ if and only if $\iota_{ij}^- \neq 0$, we see that $(\mathrm{BC_N})$ is fulfilled.

Thus, (N) can be formulated as the abstract Cauchy problem

$$(\mathrm{ACP_N}) \qquad \begin{cases} \dfrac{d}{dt}u(t) = A_N u(t), & t \ge 0, \\ u(0) = u_0, \end{cases}$$

for the operator $(A_N, D(A_N))$ in the Banach space X_N and the initial value $u_0 = (f_j)_{1\le j\le m}$. Following our general philosophy we now deal only with $(\mathrm{ACP_N})$. Note that if $(A_N, D(A_N))$ is the generator of a strongly continuous semigroup $(T_N(t))_{t\ge 0}$, then we can regain the unique solution of (N) with the initial condition $f = (f_j)_{1\le j\le n} \in D(A_m^N)$ by

$$u_j(x,v,t) = (T_N(t)f)_j(x,v).$$

PROPOSITION 2.2.9. *The operator $(A_N, D(A_N))$ is closed and densely defined.*

PROOF. Note that the graph norm of A_m^N is equivalent to $\|\cdot\|_W$. Since $D(A_N) = \{u \in D(A_m^N) \ : \ u \in \ker(L_N - \Phi_N)\}$ and since the operators L_N and Φ_N are continuous with respect to $\|\cdot\|_W$, it follows that $D(A_N)$ is a closed subspace of $D(A_m^N)$. Hence, the operator $(A_N, D(A_N))$ is closed.

Clearly, the set

$$M := \{u \in W : \Gamma_0 u = \Gamma_l u = 0\}$$

is dense in W with respect to the norm $\|\cdot\|_1$. Since $M \subseteq D(A_N) \subseteq W \subseteq X_N$ and since W is dense in X_N, also $D(A_N)$ is dense in X_N. □

This is the basis to prove the generator property of A_N in Section 2.4 below. Before doing so we investigate its spectral properties.

2.3. Spectral properties

We start with the decomposition of $D(A_m^N)$ as in Lemma 1.3. To apply this lemma, it is essential that $L_N|_{D(A_m^N)}$ is surjective.

PROPOSITION 2.3.1. *The operator L_N is surjective from $D(A_m^N)$ to ∂X_N.*

PROOF. Let $f \in \partial X_N$. Then $g = (g_j)_{1 \leq j \leq m} := (\tilde{\mathbb{I}}_w^-)^T f \in Y^m$. Consider now the element $u = (u_j)_{1 \leq j \leq m} \in X_N$ where

$$u_j : [0, l_j] \to Y, \quad x \mapsto g_j$$

is a constant function for $1 \leq j \leq m$. Clearly, we have $u \in D(A_m^N)$. Applying L_N to u yields

$$L_N u = \tilde{\mathbb{I}}^- \Gamma_0 u = \tilde{\mathbb{I}}^- (u_j(0))_{1 \leq j \leq m} = \tilde{\mathbb{I}}^- g = \tilde{\mathbb{I}}^- (\tilde{\mathbb{I}}_w^-)^T f = f.$$

□

Next, we consider the operator A_m^N with homogeneous boundary conditions.

DEFINITION 2.3.2. The operator $(A_0^N, D(A_0^N))$ is defined by

$$D(A_0^N) := \{u \in D(A_m^N) : L_N u = 0\},$$
$$A_0^N u := A_m^N u.$$

LEMMA 2.3.3. *The domain $D(A_0^N)$ coincides with*

$$K := \{u \in W : \Gamma_0 u = 0\}.$$

PROOF. The inclusion $K \subseteq D(A_0^N)$ is clear.

To show the other inclusion, suppose that $u \in D(A_0^N)$. Then, by the condition $\Gamma_0 u \in \operatorname{rg}(\tilde{\mathbb{I}}_w^-)^T$, there exists $f \in \partial X_N$ such that

$$\Gamma_0 u = (\tilde{\mathbb{I}}_w^-)^T f.$$

Therefore, and since $L_N u = 0$, we obtain

$$0 = L_N u = \tilde{\mathbb{I}}^- \Gamma_0 u = \tilde{\mathbb{I}}^- (\tilde{\mathbb{I}}_w^-)^T f = f,$$

hence

$$\Gamma_0 u = (\tilde{\mathbb{I}}_w^-)^T f = (\tilde{\mathbb{I}}_w^-)^T 0 = 0.$$

□

Hence, it is clear that A_0^N can be written as an $m \times m$ operator matrix whose entries in the off-diagonal are 0 and with the same operator in each entry in the diagonal. Its domain is given by the product of the domains of the operators in the diagonal. Each of the diagonal entries is the generator of a strongly continuous semigroup, see [Rha02, Sect. 3.1], and the semigroup $(T_0^N(t))_{t\geq 0}$ generated by A_0^N is just the direct sum of these semigroups. More precisely, it is given by

$$(T_0^N(t)u)_j(x,v) := \chi_j(x,v,t)u_j(x-vt,v),$$

where

$$\chi_j(x,v,t) := \begin{cases} 1, & \text{if } 0 \leq x - vt \leq l_j, \\ 0, & \text{otherwise,} \end{cases}$$

$j = 1, \ldots, m$. Similarly, the resolvent of A_0^N is obtained as

$$(R(\gamma, A_0^N)u)_j(x,v) = \int_0^x \tfrac{1}{v} e^{-\gamma \frac{x-r}{v}} u_j(r,v)\, dr,$$

$j = 1, \ldots, m$. From this representation one can easily see that $T_0^N(t)$ and $R(\gamma, A_0^N)$ are positive for $t \geq 0$ and $\gamma \in \mathbb{R}$, respectively. It is also clear that the semigroup $(T_0^N(t))_{t\geq 0}$ is nilpotent. This implies that the spectrum of A_0^N is empty. Hence, by Lemma 1.3 we can decompose the domain of A_m^N for any $\gamma \in \mathbb{C}$ as

$$D(A_m^N) = D(A_0^N) \oplus \ker(\gamma - A_m^N). \tag{8}$$

By Proposition 2.3.1 the operator L_N is surjective. Therefore, the restriction of L_N to $\ker(\gamma - A_m^N)$ is bijective. By the open mapping theorem, its inverse D_γ^N is bounded for every $\gamma \in \mathbb{C}$. Before we give the explicit form of D_γ^N we first introduce the following notation.

DEFINITION 2.3.4. The operator $\epsilon_\gamma \in \mathcal{L}(Y^m, X_N), \gamma \in \mathbb{C}$, is defined by

$$\epsilon_\gamma : Y^m \to X_N, \quad (\epsilon_\gamma f)_j(x,v) := e^{-\frac{\gamma}{v}x} f_j(v),$$

where $f = (f_j)_{1\leq j\leq m} \in Y^m, \quad x \in [0, l_j], \quad v \in [v_{min}, v_{max}]$.

We now define an operator which turns out to be the inverse of $L_N|_{\ker(\gamma - A_m^N)}$.

DEFINITION 2.3.5. For $\gamma \in \mathbb{C}$ the operator

$$D_\gamma^N : \partial X_N \to \ker(\gamma - A_m^N)$$

is defined by

$$f \mapsto D_\gamma^N f := \epsilon_\gamma (\tilde{\mathbb{I}}_w^-)^T f.$$

It is clear that D_γ^N maps into $\ker(\gamma - A_m^N)$. So it suffices to check that D_γ^N is the inverse of $L_N|_{\ker(\gamma - A_m^N)}$.

PROPOSITION 2.3.6. *For $\gamma \in \mathbb{C}$ we have*

$$L_N D_\gamma^N = Id_{\partial X_N} \tag{9}$$

and

$$D_\gamma^N L_N = Id_{\ker(\gamma - A_m^N)}, \tag{10}$$

i.e., $D_\gamma^N = \left(L_N|_{\ker(\gamma - A_m^N)}\right)^{-1}$.

PROOF. Let $f \in \partial X_N$ and recall that $\tilde{\mathbb{I}}^-(\tilde{\mathbb{I}}_w^-)^T = Id_{\partial X_N}$, see Remark 2.2.4. Thus,

$$L_N D_\gamma^N f = \tilde{\mathbb{I}}^- \Gamma_0 \epsilon_\gamma \tilde{\mathbb{I}}_w^- f = \tilde{\mathbb{I}}^-(\tilde{\mathbb{I}}_w^-)^T f = f,$$

and (9) is satisfied. To show (10) take an element $u = (u_j)_{1\le j\le m} \in \ker(\gamma - A_m^N)$. The functions $w = (w_j)_{1\le j\le m} \in W$ of the form

$$w_j(x, v) = f_j(v) e^{-\frac{\gamma}{v}x},$$

where $x \in [0, l_j], v \in [v_{min}, v_{max}], f_j \in Y$, and $\Gamma_0 w \in \operatorname{rg}(\tilde{\mathbb{I}}_w^-)^T$ compose the kernel of $\gamma - A_m^N$. Therefore, there exists $d \in \partial X_N$ such that $\Gamma_0 u = (\tilde{\mathbb{I}}_w^-)^T d$. Thus, u can be written as $u = \epsilon_\gamma(\tilde{\mathbb{I}}_w^-)^T d$. Hence,

$$D_\gamma^N L_N u = \epsilon_\gamma (\tilde{\mathbb{I}}_w^-)^T \tilde{\mathbb{I}}^- \Gamma_0 u = \epsilon_\gamma (\tilde{\mathbb{I}}_w^-)^T \tilde{\mathbb{I}}^- (\tilde{\mathbb{I}}_w^-)^T d = \epsilon_\gamma (\tilde{\mathbb{I}}_w^-)^T d = u.$$

□

The condition $1 \in \sigma(\Phi_N D_\gamma^N)$ appearing in the Characteristic Equation 1.7 is indeed a condition in ∂X_N, hence in a space much smaller than the state space X_N. To proceed we compute $\Phi_N D_\gamma^N$ as

$$\Phi_N D_\gamma^N = \mathbb{I}_J^+ \begin{pmatrix} Q_{e^{-\frac{\gamma}{\cdot} l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\gamma}{\cdot} l_m}} \end{pmatrix} (\tilde{\mathbb{I}}_w^-)^T.$$

Here, and in the following Q_g denotes the multiplication by a function $g \in L^\infty[v_{min}, v_{max}]$, i.e.

$$Q_g : Y \to Y, \quad f \mapsto Q_g f := gf.$$

This form of $\Phi_N D_\gamma^N$ (and the Characteristic Equation 1.7) immediately allow the following conclusions.

PROPOSITION 2.3.7.

(i) *Let $\gamma \in \mathbb{C}$. If $\Re\gamma > 0$ then $\|\Phi_N D_\gamma^N\| < 1$. Thus, the spectral bound of A_N satisfies $\mathrm{s}(A_N) \le 0$.*
(ii) *If $\|Jf\|_1 = \|f\|_1$ holds for all $f \ge 0$, then $\mathrm{s}(A_N) = 0$.*
(iii) *The resolvent $R(\gamma, A_N)$ is a positive operator for all $\gamma > 0$.*

PROOF. (i), (ii) First, using that J is a contraction, we estimate the norm of $\Phi_N D_\gamma^N$ as

$$\begin{aligned}\|\Phi_N D_\gamma^N\| &= \|\mathbb{I}_J^+ \begin{pmatrix} Q_{e^{-\frac{\gamma}{\cdot}l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\gamma}{\cdot}l_m}} \end{pmatrix} (\tilde{\mathbb{I}}_w^-)^T\| \\ &\le \|\mathbb{I}_J^+\| \left\| \begin{pmatrix} Q_{e^{-\frac{\gamma}{\cdot}l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\gamma}{\cdot}l_m}} \end{pmatrix} \right\| \|(\tilde{\mathbb{I}}_w^-)^T\| \\ &= \|J\| \max_{1\le j\le m} \|Q_{e^{-\frac{\gamma}{\cdot}l_j}}\| \le \max_{1\le j\le m} \|Q_{e^{-\frac{\gamma}{\cdot}l_j}}\|.\end{aligned}$$

Suppose now that $\Re\gamma > 0$. Then

$$\|\Phi_N D_\gamma^N\| \le e^{-\frac{\Re\gamma}{v_{max}} \min_{1\le j\le m} l_j} < 1,$$

and therefore $1 \notin \sigma(\Phi_N D_\gamma^N)$ which is equivalent to $\gamma \notin \sigma(A_N)$ by the Characteristic Equation 1.7. Moreover, if $\gamma = 0$ then

$$\begin{aligned}\sigma(\Phi_N D_0^N) &= \sigma(\mathbb{I}_J^+ \begin{pmatrix} Q_{e^{-\frac{0}{\cdot}l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{0}{\cdot}l_m}} \end{pmatrix} (\tilde{\mathbb{I}}_w^-)^T) \\ &= \sigma(\mathbb{I}_J^+(\tilde{\mathbb{I}}_w^-)^T) = \sigma((\alpha_{ij}J)_{n\times n}).\end{aligned}$$

This can be further decomposed into

$$\sigma(\Phi_N D_0^N) = \sigma(\mathbb{A})\sigma(J),$$

see [Nag85, Sect. 4]. By the assumption in (ii) on J, we have that $\mathrm{r}(J) = 1$ and from the positivity of J we know that $\mathrm{r}(J) \in \sigma(J)$, see [Sch74, Prop. V.4.1]. Since $\mathbb{A}$ is a column stochastic matrix, $1 \in \sigma(\mathbb{A})$ and again by the Characteristic Equation 1.7 it follows that $0 \in \sigma(A_N)$. So we conclude that $\mathrm{s}(A_N) = 0$.

(iii) If $\gamma > 0$ then $R(\gamma, A_0^N)$, D_γ^N, Φ_N, and $\Phi_N D_\gamma^N$ are positive operators. Since $\|\Phi_N D_\gamma^N\| < 1$, the inverse of $Id - \Phi_N D_\gamma^N$ is given by the Neumann series, i.e.

$$(Id - \Phi_N D_\gamma^N)^{-1} = \sum_{n=0}^{\infty} (\Phi_N D_\gamma^N)^n.$$

Thus, we see that it is also a positive operator. So from the representation

$$R(\gamma, A_N) = R(\gamma, A_0^N) + D_\gamma^N (1 - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N)$$

from Proposition 1.8 we see that the resolvent $R(\gamma, A_N)$ is composed of positive operators and is therefore also positive. □

Note that assertion (iii) in the above proposition also follows from Theorem 2.4.5 below. In order to use the Characteristic Equation 1.7 we investigate $\sigma(\Phi_N D_\gamma^N)$ in more detail.

LEMMA 2.3.8. *For $\gamma \in \mathbb{C}$ the following holds.*

(i)

$$\sigma(\Phi_N D_\gamma^N) \setminus \{0\} = \sigma(\begin{pmatrix} Q_{e^{-\gamma l_1}} J & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\gamma l_m}} J \end{pmatrix} \tilde{\mathbb{B}}) \setminus \{0\}$$

$$= \sigma(\tilde{\mathbb{B}} \begin{pmatrix} Q_{e^{-\gamma l_1}} J & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\gamma l_m}} J \end{pmatrix}) \setminus \{0\}.$$

(ii) *If all arc lengths are equal to l, then*

$$\sigma(\Phi_N D_\gamma^N) = \sigma(\mathbb{A})\sigma(JQ_{e^{-\gamma l}}).$$

PROOF. (i) The first assertion follows from the well-known fact that

$$\sigma(EF) \setminus \{0\} = \sigma(FE) \setminus \{0\} \text{ for } E \in \mathcal{L}(X_1, X_2) \text{ and } F \in \mathcal{L}(X_2, X_1),$$

where X_1 and X_2 are arbitrary Banach spaces.

(ii) If all arc lengths are equal to l, then we have

$$\begin{aligned} \Phi_N D_\gamma^N &= \mathbb{I}_J^+ \begin{pmatrix} Q_{e^{-\gamma l}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\gamma l}} \end{pmatrix} (\tilde{\mathbb{I}}_w^-)^T \\ &= \begin{pmatrix} JQ_{e^{-\gamma l}} & & 0 \\ & \ddots & \\ 0 & & JQ_{e^{-\gamma l}} \end{pmatrix} \tilde{\mathbb{I}}^+ (\tilde{\mathbb{I}}_w^-)^T \\ &= \begin{pmatrix} JQ_{e^{-\gamma l}} & & 0 \\ & \ddots & \\ 0 & & JQ_{e^{-\gamma l}} \end{pmatrix} \tilde{\mathbb{A}} \\ &= \tilde{\mathbb{A}} \begin{pmatrix} JQ_{e^{-\gamma l}} & & 0 \\ & \ddots & \\ 0 & & JQ_{e^{-\gamma l}} \end{pmatrix} \\ &= (\alpha_{ij} JQ_{e^{-\gamma l}})_{n \times n}, \end{aligned}$$

where $\mathbb{A} = (\alpha_{ij})_{n \times n}$. The spectrum of operator matrices of this special form is given by

$$\sigma(\Phi_N D_\gamma^N) = \sigma(\mathbb{A})\sigma(JQ_{e^{-\gamma l}}),$$

see [Nag85, Sect. 4]. □

We now make additional assumptions on J and discuss the spectrum of A_N using the Characteristic Equation 1.7 and the above lemma. First, we consider the case that the operator J is compact.

PROPOSITION 2.3.9. *If J is a compact operator, then*

$$\sigma_p(A_N) = \sigma(A_N).$$

PROOF. Since J is a compact operator, also $\Phi_N D_\gamma^N$ is compact and therefore $\sigma(\Phi_N D_\gamma^N) = \sigma_p(\Phi_N D_\gamma^N) \cup \{0\}$. The assertion now follows from the Characteristic Equation 1.7. □

A physically realistic assumption is that the scattering operator J is a compact integral operator with a strictly positive kernel. More precisely, we assume that $J \in \mathcal{L}(Y)$ is given by

$$Jf := \int_{v_{min}}^{v_{max}} k(\cdot, w) f(w)\, dw, \quad f \in Y.$$

The measurable kernel

$$k : [v_{min}, v_{max}] \times [v_{min}, v_{max}] \to \mathbb{R}$$

fulfills $k(v, w) > 0$ for almost all $v, w \in [v_{min}, v_{max}]$. In addition, we assume that

$$\int_{v_{min}}^{v_{max}} k(v, w)\, dv = 1 \text{ for all } w \in [v_{min}, v_{max}] \tag{11}$$

so that our General Assumption 2.2.1 is satisfied. Note that these assumptions imply the irreducibility of J, see [Sch74, Example V.6.4] and Definition A.10 below.

Under these assumptions we can show that 0 is the only spectral value of A_N on the imaginary axis.

THEOREM 2.3.10. *Suppose that all the arc lengths are equal to l and suppose that the scattering operator J is as above. Then*

$$\sigma(A_N) \cap i\mathbb{R} = \{0\}.$$

PROOF. From assumption (11) follows that

$$\|Jf\|_1 = \|f\|_1 \text{ for all } f \geq 0. \tag{12}$$

Hence, the adjoint operator $J' \in \mathcal{L}(Y')$ where $Y' \cong L^\infty[v_{min}, v_{max}]$ satisfies

$$J'\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ denotes the constant one function. By the irreducibility of J and [Sch74, Thm. V.5.2] we then obtain that there exists $g \in Y_+$ such that $Jg = g$ and $g(v) > 0$ for almost all $v \in [v_{min}, v_{max}]$. Consider the Banach space $\tilde{Y} := L^1([v_{min}, v_{max}], g(v)dv)$. The positive operator $\tilde{J} := Q_{g^{-1}} J Q_g \in \mathcal{L}(\tilde{Y})$ is similar to J and satisfies

$$\tilde{J}\mathbf{1} = \mathbf{1}. \tag{13}$$

Since J is irreducible, the same holds for $\tilde{J}$, and also

$$\|\tilde{J}f\|_{\tilde{Y}} = \|f\|_{\tilde{Y}}$$

remains true for $f \in \tilde{Y}_+$. This again implies for the adjoint operator $\tilde{J}' \in \mathcal{L}(Y')$ of $\tilde{J}$ that

$$\tilde{J}'\mathbf{1} = \mathbf{1}. \tag{14}$$

Suppose now that there is a spectral value $\gamma \in i\mathbb{R} \setminus \{0\}$ of A_N. Define the operator $\tilde{J}_\gamma := Q_{g^{-1}} J Q_{e^{-\frac{\gamma}{\cdot}l}} Q_g \in \mathcal{L}(\tilde{Y})$. Note that $\tilde{J}_\gamma$ is similar to $JQ_{e^{-\frac{\gamma}{\cdot}l}} \in \mathcal{L}(Y)$. Therefore, their spectra coincide. We know from the Characteristic Equation 1.7 with the aid of Lemma 2.3.8 (ii) that there must exist an $\alpha \in \sigma(\tilde{J}_\gamma)$ such that $|\alpha| = 1$. Since J is compact, $\alpha \in \sigma_p(\tilde{J}_\gamma)$. So there exists $f \in \tilde{Y}, f \neq 0$, such that

$$\tilde{J}_\gamma f = \alpha f.$$

Since

$$|f| = |\alpha f| = |\tilde{J}_\gamma f| \leq |\tilde{J}_\gamma||f| = \tilde{J}|f|,$$

we have

$$|\tilde{J}|f| - |f|| = \tilde{J}|f| - |f|.$$

From

$$\langle \mathbf{1}, |\tilde{J}|f| - |f|| \rangle = \langle \mathbf{1}, \tilde{J}|f| \rangle - \langle \mathbf{1}, |f| \rangle = \langle \tilde{J}'\mathbf{1}, |f| \rangle - \langle \mathbf{1}, |f| \rangle \overset{(14)}{=} 0,$$

it follows that $\tilde{J}|f| = |f|$. By [Sch74, Thm. V.5.2] the fixed space of $\tilde{J}$ is one-dimensional and by (13) we conclude that it is spanned by $\mathbf{1}$. Therefore, we can assume that $|f| = \mathbf{1}$. Thus, f is a unimodular eigenfunction of $\tilde{J}_\gamma$.

If we take $h \in L^\infty[v_{min}, v_{max}] \subseteq \tilde{Y}$, then

$$0 \leq |\tilde{J}_\gamma h| \leq |\tilde{J}_\gamma||h| = \tilde{J}|h| \leq \tilde{J}(\|h\|_\infty \mathbf{1}) = \|h\|_\infty \tilde{J}\mathbf{1} = \|h\|_\infty \mathbf{1}.$$

Therefore, $\tilde{J}_\gamma(L^\infty[v_{min}, v_{max}]) \subseteq L^\infty[v_{min}, v_{max}]$.

By Gelfand's theorem

$$L^\infty[v_{min}, v_{max}] \cong C(K)$$

holds for a suitable compact space K. So far, we have shown that all the assumptions of [Sch74, Prop. V.7.4] are fulfilled. Hence,

$$\tilde{J}_\gamma|_{L^\infty[v_{min}, v_{max}]} = \alpha Q_f \tilde{J} Q_{f^{-1}}|_{L^\infty[v_{min}, v_{max}]}.$$

This implies

$$k(v,w)e^{-\frac{\gamma}{w}l} = \alpha f(v)k(v,w)\overline{f(w)}$$

for almost all $v, w \in [v_{min}, v_{max}]$. Since k is strictly positive, this means that

$$\overline{f(v)} = \alpha e^{\frac{\gamma}{w}l}\overline{f(w)}$$

has to be fulfilled for almost all $v, w \in [v_{min}, v_{max}]$. Evidently, this is not possible, hence there is no spectral value $\gamma \neq 0$ on the imaginary axis. □

2.4. Well-posedness

In this section we show the generator property of A_N and hence the well-posedness of (N). We first renorm the space X_N and then check that A_N fulfills all the conditions in the Phillips generation theorem, see Theorem A.5. Therefore, A_N is the generator of a positive contraction semigroup on X_N for this norm.

Since J is contractive on Y_+, also $\mathbb{B}_J$ is contractive on Y_+^m as is shown in the following lemma.

LEMMA 2.4.1. *If $f \in Y_+^m$, then*

$$\|\mathbb{B}_J f\|_1 - \|f\|_1 \leq 0.$$

PROOF. Let $f \in Y_+^m$. Then the following computation shows the assertion.

$$\begin{aligned}
\|\mathbb{B}_J f\|_1 - \|f\|_1 &= \sum_{j=1}^{m} \int_{v_{min}}^{v_{max}} (\mathbb{B}_J f - f)_j(v)\, dv \\
&= \sum_{j=1}^{m} \int_{v_{min}}^{v_{max}} [J(\mathbb{B} f)_j - f_j](v)\, dv \\
&= \sum_{j=1}^{m} \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^{m} b_{jk} f_k \right) - f_j \right](v)\, dv \\
&= \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^{m} f_k \sum_{j=1}^{m} b_{jk} \right) - \sum_{j=1}^{m} f_j \right](v)\, dv \\
&\overset{\mathbb{B}\text{ column stochastic}}{=} \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^{m} f_k \right) - \sum_{j=1}^{m} f_j \right](v)\, dv \\
&= \sum_{j=1}^{m} (\|J f_j\|_1 - \|f_j\|_1) \\
&\overset{\text{Gen. Ass. 2.2.1}}{\leq} 0.
\end{aligned}$$

□

There is an alternative way of defining the domain of A_N by using the operator matrix $\mathbb{B}_J$.

PROPOSITION 2.4.2. *The domain of A_N is given by*

$$D(A_N) = \{u \in W : \Gamma_0 u = \mathbb{B}_J \Gamma_l u\}.$$

PROOF. If $u \in D(A_N)$ then $\tilde{\mathbb{I}}^- \Gamma_0 u = \mathbb{I}_J^+ \Gamma_l u$ and there exists $f \in \partial X_N$ such that $\Gamma_0 u = (\tilde{\mathbb{I}}_w^-)^T f$. Using this we compute

$$\mathbb{I}_J^+ \Gamma_l u = \tilde{\mathbb{I}}^- \Gamma_0 u = \tilde{\mathbb{I}}^- (\tilde{\mathbb{I}}_w^-)^T f = f.$$

This implies

$$\Gamma_0 u = (\tilde{\mathbb{I}}_w^-)^T f = (\tilde{\mathbb{I}}_w^-)^T \mathbb{I}_J^+ \Gamma_l u = \mathbb{B}_J \Gamma_l u.$$

On the other hand, if for $u \in W$ the condition $\Gamma_0 u = \mathbb{B}_J \Gamma_l u$ is fulfilled, then $\Gamma_0 u \in \mathrm{rg}(\tilde{\mathbb{I}}_w^-)^T$ holds since $\mathbb{B}_J = (\tilde{\mathbb{I}}_w^-)^T \mathbb{I}_J^+$ Moreover,

$$L_N u = \tilde{\mathbb{I}}^- \Gamma_0 u = \tilde{\mathbb{I}}^- \mathbb{B}_J \Gamma_l u = \tilde{\mathbb{I}}^- (\tilde{\mathbb{I}}_w^-)^T \mathbb{I}_J^+ \Gamma_l u = \mathbb{I}_J^+ \Gamma_l u = \Phi_N u.$$

□

This representation of $D(A_N)$ is needed in Lemma 2.4.4 below to show the dispersivity of A_N if X_N is endowed with the following norm.

DEFINITION 2.4.3. The norm $\|\cdot\|_{1,v}$ on X_N is

$$\|\cdot\|_{1,v} : X_N \to \mathbb{R}, \quad u = (u_j)_{1\le j\le m} \mapsto \|u\|_{1,v} := \sum_{j=1}^{m} \int_0^{l_j} \int_{v_{min}}^{v_{max}} \frac{1}{v} |u_j(x,v)| \, dv \, dx.$$

Since

$$\tfrac{1}{v_{max}} \|\cdot\|_1 \;\le\; \|\cdot\|_{1,v} \;\le\; \tfrac{1}{v_{min}} \|\cdot\|_1,$$

the norm $\|\cdot\|_{1,v}$ is equivalent to the original norm $\|\cdot\|_1$ on X_N.

We now check the dispersivity of A_N, cf. Definition A.4.

LEMMA 2.4.4. *The operator $(A_N, D(A_N))$ is dispersive on the Banach lattice $(X_N, \|\cdot\|_{1,v})$.*

PROOF. The dual space of X_N is

$$\begin{aligned} X_N' &\cong L^\infty([0,l_1], Y') \times \cdots \times L^\infty([0,l_m], Y') \\ &\cong L^\infty([0,l_1] \times [v_{min}, v_{max}]) \times \cdots \times L^\infty([0,l_m] \times [v_{min}, v_{max}]) \end{aligned}$$

where $Y' = L^\infty[v_{min}, v_{max}]$. For $\Psi = (\Psi_k)_{1\le k\le m} \in X_N'$ and $u = (u_k)_{1\le k\le m} \in X_N$ we have

$$\langle u, \Psi \rangle = \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} \tfrac{1}{v} u_k(x,v) \Psi_k(x,v) \, dv \, dx.$$

Let $u \in D(A_N)$ and let $\Psi = (\Psi_k)_{1\le k\le m} \in X_N'$ be defined by

$$\Psi_k(x,\cdot) = \begin{cases} v \mapsto 1, & u_k(x,\cdot) = u_k^+(x,\cdot), \\ v \mapsto 0, & \text{else}, \end{cases}$$

where $x \in [0, l_k]$. Clearly, $\|\Psi\| \le 1$.

Next, we compute

$$\begin{aligned} \langle u, \Psi \rangle &= \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} \tfrac{1}{v} u_k(x,v) \Psi_k(x,v) \, dv \, dx \\ &= \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} \tfrac{1}{v} u_k^+(x,v) \, dv \, dx \\ &= \|u^+\|_{1,v}. \end{aligned}$$

We then obtain

$$
\begin{aligned}
& \langle A_N u, \Psi \rangle \\
= \quad & \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} \tfrac{1}{v}(-v) \tfrac{\partial}{\partial x} u_k^+(x,v)\, dv\, dx \\
= \quad & -\sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \int_0^{l_k} \tfrac{\partial}{\partial x} u_k^+(x,v)\, dx\, dv \\
= \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left(u_k^+(0,v) - u_k^+(l_k,v) \right)\, dv \\
\overset{\text{Prop. 2.4.2}}{=} \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left((\mathbb{B}_J(\Gamma_l u))_k^+ - u_k^+(l_k,\cdot) \right)(v)\, dv \\
\leq \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left((\mathbb{B}_J(\Gamma_l u)^+)_k(v) - u_k^+(l_k,v) \right)\, dv \\
= \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{k=1}^{m} J \left(\sum_{j=1}^{m} \beta_{kj} u_j^+(l_j,\cdot) \right) \right)(v)\, dv \\
& -\sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v)\, dv \\
= \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{j=1}^{m} J \left[\left(\sum_{k=1}^{m} \beta_{kj} \right) u_j^+(l_j,\cdot) \right] \right)(v)\, dv \\
& -\sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v)\, dv \\
\overset{\mathbb{B}\text{ column stochastic}}{=} \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{j=1}^{m} J u_j^+(l_j,\cdot) \right)(v)\, dv - \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v)\, dv \\
= \quad & \sum_{j=1}^{m} \| J u_j^+(l_j,\cdot) \|_1 - \sum_{k=1}^{m} \| u_k^+(l_k,\cdot) \|_1 \\
\overset{\text{Gen. Ass. 2.2.1}}{\leq} \quad & 0.
\end{aligned}
$$

This shows that all the conditions of Definition A.4 are fulfilled, hence A_N is dispersive. □

As our final conclusion we obtain the generator property of A_N.

THEOREM 2.4.5. *The operator $(A_N, D(A_N))$ on X_N is the generator of a positive and bounded strongly continuous semigroup $(T_N(t))_{t\geq 0}$ with bound $\frac{v_{max}}{v_{min}}$.*

PROOF. By Proposition 2.2.9 and Lemma 2.4.4 it follows that A_N is a densely defined, dispersive operator and by Proposition 2.3.7 the operator $\gamma - A_N$ is surjective for $\gamma > 0$. Therefore, the Phillips theorem, see Theorem A.5, implies that A_N is the generator of a positive contraction semigroup on $(X_N, \|\cdot\|_{1,v})$. Returning to our original norm $\|\cdot\|_1$ on X_N we obtain that the semigroup is bounded by $\frac{v_{max}}{v_{min}}$. □

REMARK 2.4.6. The asymptotic behaviour of this semigroup, hence of the solutions of $(\mathrm{ACP_N})$, will be discussed in Section 5.2 below.

CHAPTER 3

Queues

3.1. Introduction

Since its creation by the Danish mathematician A. Erlang, mathematical queueing theory is now a well-established mathematical field and we refer to [Kle75] and [Coh82] for monographs.

Among the many questions to ask, one is interested in *performance measures* of the queue such as e.g. the queue length or the average waiting time for a customer in the queue. Another aspect of interest is the existence of a *steady state solution* to which the system converges as time tends to infinity.

In our queueing models we have a time parameter for the evolution of the system and an additional time parameter giving the time since the service of the last customer began. Queueing problems of this type have first been studied in [Cox55] and [Ken53]. A semigroup approach to the treatment of these problems was established by G. Gupur and others, see [GLZ01b], [GLZ01a], [Gup04].

In the sequel we concentrate on the $M/M^{k,B}/1$ queueing model. This problem has already been studied in [Gup04], where he showed the well-posedness. Here, we give a more detailed analysis and show the existence of a unique positive steady state solution in Chapter 5.

3.2. Setting

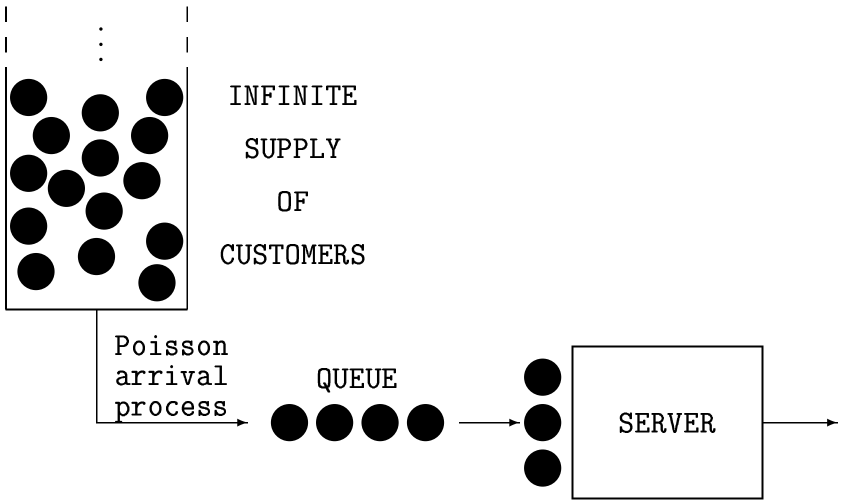
We investigate the dynamical $M/M^{k,B}/1$ queueing system i.e., we consider a queueing system consisting of a single server which can serve at most $B \in \mathbb{N}$ customers simultaneously, see Figure 1.

The server starts service as soon as there are at least k customers in the queue, where $1 \leq k \leq B$ is fixed. There is an infinite supply of customers. The arrival of the customers is at random and obeys a Poisson process with parameter λ. This means in particular that the *interarrival times* X_n, i.e., the time difference between the arrival of the n^{th} and $(n+1)^{st}$ customer are independent and identically distributed random variables and the probability $P(\{X_n \leq t\})$ that the n^{th} interarrival time is less than or equal to t is

$$P(\{X_n \leq t\}) = 1 - e^{-\lambda t}.$$

This explains the first M in the notation $M/M^{k,B}/1$ which stands for Markovian or memoryless. The *mean arrival rate* is given by $\frac{1}{\lambda}$.

FIGURE 1. Single server queue, $B = 3$



The second M in the notation $M/M^{k,B}/1$ means that the *service times* Y_n are exponentially distributed with parameter μ. Thus, the time period Y_n in which the server is busy with the n^{th} customer are also independent and identically distributed random variables. The probability $P(\{Y_n \leq t\})$ that the service time for the n^{th} customer is less than or equal to t is

$$P(\{Y_n \leq t\}) = 1 - e^{-\mu t}.$$

The *mean service rate* is $\frac{1}{\mu}$.

For these parameters we assume the following.

GENERAL ASSUMPTION 3.2.1. We require that

$$0 < \lambda < \mu.$$

The ratio $\rho := \frac{\lambda}{\mu}$ is called *traffic rate*. From the above general assumption it follows that $\rho < 1$. So, intuitively, we expect that the queue will not grow infinitely.

Let $0 \leq r < k, n \in \mathbb{N} \cup \{0\}$ and $t, x \geq 0$. Then $p_{r,0}(t)$ denotes the probability that at time t there are r customers in the queue and therefore the server is not busy. Moreover, we investigate $p_{n,1}(x,t)$, where

$$\int_0^\infty p_{n,1}(x,t)\ dx$$

is the probability that at time t there are n customers in the queue and the server is busy. Note that

$$\sum_{r=0}^{k-1} p_{r,0}(t) + \sum_{n=0}^{\infty} \int_0^\infty p_{n,1}(x,t)\ dx = 1$$

for all $t \geq 0$. The parameter t represents the time since the evolution of the whole system has started, whereas the parameter x refers to the elapsed service time, i.e., the time since the last service has started. It is reset to 0 as soon as the next service starts.

The above queueing model can then be described by the equations

$$(\mathrm{Q}) \qquad \begin{cases} \dfrac{dp_{0,0}(t)}{dt} = -\lambda p_{0,0}(t) + \mu \displaystyle\int_0^\infty p_{0,1}(x,t)\, dx, \\ \dfrac{dp_{r,0}(t)}{dt} = -\lambda p_{r,0}(t) + \lambda p_{r-1,0}(t) + \mu \displaystyle\int_0^\infty p_{r,1}(x,t)\, dx, \\ \qquad 1 \leq r \leq k-1, \\ \dfrac{\partial p_{0,1}(x,t)}{\partial t} = -\dfrac{\partial p_{0,1}(x,t)}{\partial x} - (\lambda + \mu) p_{0,1}(x,t), \\ \dfrac{\partial p_{n,1}(x,t)}{\partial t} = -\dfrac{\partial p_{n,1}(x,t)}{\partial x} - (\lambda + \mu) p_{n,1}(x,t) + \lambda p_{n-1,1}(x,t), \\ \qquad n \geq 1. \end{cases}$$

For $x = 0$ the boundary conditions

$$(\mathrm{BC_Q}) \qquad \begin{cases} p_{0,1}(0,t) = \mu \displaystyle\sum_{i=k}^{B} \int_0^\infty p_{i,1}(x,t)\, dx + \lambda p_{k-1,0}(t), \\ p_{n,1}(0,t) = \mu \displaystyle\int_0^\infty p_{n+B,1}(x,t)\, dx, \quad n \geq 1, \end{cases}$$

are imposed. They correspond to the situation when the server just starts a new process.

As initial condition we choose

$$(\mathrm{IC_Q}) \qquad \begin{cases} p_{r,0} = c_r \in \mathbb{C}, \quad 0 \leq r \leq k-1, \\ p_{n,1}(x,0) = f_n(x), \quad n \geq 0, \end{cases}$$

where $f_n \in L^1[0,\infty)$. But actually one is mainly interested in the initial condition

$$(\mathrm{IC_{Q,0}}) \qquad \begin{cases} p_{0,0}(0) = 1, \\ p_{r,0}(0) = 0, \quad 1 \leq r \leq k-1, \\ p_{n,1}(x,0) = 0, \quad n \geq 1, \end{cases}$$

which means that at time $t = 0$ the server as well as the queue are empty.

The suitable state space to formulate this problem as an abstract Cauchy problem is

$$X_Q := \mathbb{C}^k \times l^1\left(L^1[0,\infty)\right)$$

endowed with the norm

$$\|p\|_{X_Q} = \sum_{i=0}^{k-1} |p_{i,0}| + \sum_{i=0}^{\infty} \|p_{i,1}(\cdot)\|_{L^1[0,\infty)}$$

for $p = (p_{0,0}, \dots, p_{k-1,0}, p_{0,1}(\cdot), p_{1,1}(\cdot), \dots)^T \in X_Q$. In the spirit of the approach sketched in the introduction we define on X_Q the maximal operator as

$$\begin{aligned} D(A_m^Q) &:= \mathbb{C}^k \times l^1\left(W^{1,1}[0,\infty)\right), \\ A_m^Q &:= \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ 0 & \mathcal{D} \end{pmatrix}, \end{aligned}$$

where

$$\mathcal{L} := \begin{pmatrix} -\lambda & 0 & 0 & \cdots & 0 \\ \lambda & -\lambda & \ddots & & \vdots \\ 0 & \lambda & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda & -\lambda \end{pmatrix}$$

has dimension $k \times k$,

$$\mathcal{M} := \begin{pmatrix} \mu\psi & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \mu\psi & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & \cdots \\ 0 & \cdots & 0 & \mu\psi & 0 & \cdots \end{pmatrix}$$

and

$$\mathcal{D} := \begin{pmatrix} D & 0 & \cdots & \cdots & \cdots \\ \lambda & D & 0 & \cdots & \cdots \\ 0 & \lambda & D & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Here,

$$\psi : L^1[0,\infty) \to \mathbb{C}, \qquad f \mapsto \psi(f) := \int_0^\infty f(x)\, dx,$$

and

$$D := -\frac{d}{dx} - \lambda - \mu.$$

Clearly, the operator $(A_m^Q, D(A_m^Q))$ is closed on X_Q.

As boundary space we choose

$$\partial X_Q := l^1$$

and define the boundary operators

$$L_Q : D(A_m^Q) \to \partial X_Q, \qquad \begin{pmatrix} p_{0,0} \\ \vdots \\ p_{k-1,0} \\ p_{0,1} \\ p_{1,1} \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} p_{0,1}(0) \\ p_{1,1}(0) \\ \vdots \end{pmatrix},$$

and

$$\Phi_Q : D(A_m^Q) \to \partial X_Q$$

given by the operator matrix

$$\Phi_Q := \begin{pmatrix} \overbrace{0 & \cdots & 0}^{k-1} & \lambda & \overbrace{0 & \cdots & 0}^{k} & \overbrace{\mu\psi & \cdots & \mu\psi}^{B-k+1} & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \mu\psi & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \mu\psi & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \mu\psi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now obtain the operator $(A_Q, D(A_Q))$ corresponding to the underlying problem by

$$\begin{aligned} A_Q p &:= A_m^Q p, \\ D(A_Q) &:= \{p \in D(A_m^Q) \ : \ L_Q p = \Phi_Q p\}. \end{aligned}$$

The problem (Q), ($\mathrm{BC_Q}$), ($\mathrm{IC_Q}$) can be reformulated as the abstract Cauchy problem

$$(\mathrm{ACP_Q}) \qquad \begin{cases} \dfrac{d}{dt} p(t) = A_Q p(t), \quad t \in [0, \infty), \\ \quad p(0) = (c_0, \ldots, c_{k-1}, f_1, f_2, \ldots)^T \in X_Q. \end{cases}$$

If A_Q generates a strongly continuous semigroup $(T_Q(t))_{t \geq 0}$ and if the initial values in ($\mathrm{IC_Q}$) satisfy $p_0 := (c_0, \ldots, c_{k-1}, f_1, f_2, \ldots)^T \in D(A_Q)$, then the unique solution of (Q), ($\mathrm{BC_Q}$) and ($\mathrm{IC_Q}$) is given by

$$\begin{aligned} p_{r,0}(t) &= (T_Q(t)p_0)_{r+1}, \quad 0 \leq r \leq k-1, \\ p_{n,1}(x,t) &= (T_Q(t)p_0)_{n+k}(x), \quad n \geq 0. \end{aligned}$$

So in the following we concentrate our attention on ($\mathrm{ACP_Q}$).

We use the Characteristic Equation 1.7 to investigate the boundary spectrum of A_Q. For this purpose, we need more information on the resolvent set of the operator $(A_0^Q, D(A_0^Q))$ given by

$$\begin{aligned} D(A_0^Q) &:= \{p \in D(A_m^Q) \ : \ L_Q p = 0\}, \\ A_0^Q p &:= A_m^Q p. \end{aligned}$$

Moreover, we give an explicit formula for the resolvent needed in Lemma 5.3.1 below to prove the irreducibility of the semigroup generated by A_Q.

LEMMA 3.2.2. *For the set* $S := \{\gamma \in \mathbb{C} \ : \ \Re\gamma > -\mu\} \setminus \{-\lambda\}$ *we have*

$$S \subseteq \rho(A_0^Q).$$

Moreover, if $\gamma \in S$*, then*

$$\begin{aligned} R(\gamma, A_0^Q) &= \begin{pmatrix} \gamma - \mathcal{L} & -\mathcal{M} \\ 0 & \gamma - \mathcal{D} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (\gamma-\mathcal{L})^{-1} & (\gamma-\mathcal{L})^{-1}\mathcal{M}(\gamma-\mathcal{D})^{-1} \\ 0 & (\gamma-\mathcal{D})^{-1} \end{pmatrix}, \end{aligned}$$

where

$$(\gamma-\mathcal{L})^{-1} = \begin{pmatrix} \frac{1}{\gamma+\lambda} & 0 & \cdots & \cdots & 0 \\ \frac{\lambda}{(\gamma+\lambda)^2} & \frac{1}{\gamma+\lambda} & \ddots & \cdots & \vdots \\ \frac{\lambda^2}{(\gamma+\lambda)^3} & \frac{\lambda}{(\gamma+\lambda)^2} & \frac{1}{\gamma+\lambda} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{\lambda^{k-1}}{(\gamma+\lambda)^k} & \frac{\lambda^{k-2}}{(\gamma+\lambda)^{k-1}} & \frac{\lambda^{k-3}}{(\gamma+\lambda)^{k-2}} & \cdots & \frac{1}{\gamma+\lambda} \end{pmatrix},$$

$$(\gamma-\mathcal{D})^{-1} = \begin{pmatrix} R(\gamma,D) & 0 & \cdots & \cdots \\ \lambda R(\gamma,D)^2 & R(\gamma,D) & 0 & \cdots \\ \lambda^2 R(\gamma,D)^3 & \lambda R(\gamma,D)^2 & R(\gamma,D) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$(R(\gamma,D)p)(x) = e^{-(\gamma+\lambda+\mu)x}\int_0^x e^{(\gamma+\lambda+\mu)s}p(s)\,ds, \quad p \in L^1[0,\infty).$$

PROOF. Let $C_c[0,\infty)$ denote the space of continuous functions on $[0,\infty)$ with compact support. We estimate for $p \in C_c[0,\infty)$ and $\gamma \in S$

$$\begin{aligned} &\int_0^\infty |(R(\gamma,D)p)(x)|\,dx \\ =&\int_0^\infty \left| e^{-(\gamma+\lambda+\mu)x}\int_0^x e^{(\gamma+\lambda+\mu)s}p(s)\,ds\right|\,dx \\ \le&\int_0^\infty e^{-(\Re\gamma+\lambda+\mu)x}\int_0^x e^{(\Re\gamma+\lambda+\mu)s}|p(s)|\,ds\,dx \\ =&\left[-\frac{1}{\Re\gamma+\lambda+\mu}e^{-(\Re\gamma+\lambda+\mu)x}\int_0^x e^{(\Re\gamma+\lambda+\mu)s}|p(s)|\,ds\right]_0^\infty \\ &+\int_0^\infty \frac{1}{\Re\gamma+\lambda+\mu}e^{-(\Re\gamma+\lambda+\mu)x}e^{(\Re\gamma+\lambda+\mu)x}|p(x)|\,dx \\ =&\frac{1}{\Re\gamma+\lambda+\mu}\|p\|_{L^1[0,\infty)} \end{aligned}$$

Since $C_c[0,\infty)$ is dense in $L^1[0,\infty)$, we conclude from the above estimate that

$$\|R(\gamma, D)\| \leq \frac{1}{\Re\gamma + \lambda + \mu}.$$

Therefore,

$$\begin{aligned} \|(\gamma - \mathcal{D})^{-1}\| &\leq \sum_{n=0}^{\infty} \lambda^n \|R(\gamma, D)\|^{n+1} \\ &\leq \sum_{n=0}^{\infty} \frac{\lambda^n}{(\Re\gamma + \lambda + \mu)^{n+1}} \\ &< \infty \end{aligned}$$

if $\Re\gamma > -\mu$. Clearly, the other components of $R(\gamma, A_0^Q)$ are also bounded, so that $R(\gamma, A_0^Q)$ is a bounded operator for $\Re\gamma > -\mu$.

Now a straightforward calculation shows that the operator

$$\begin{pmatrix} (\gamma - \mathcal{L})^{-1} & (\gamma - \mathcal{L})^{-1}\mathcal{M}(\gamma - \mathcal{D})^{-1} \\ 0 & (\gamma - \mathcal{D})^{-1} \end{pmatrix}$$

is indeed the inverse of $\gamma - A_0^Q$. □

The following consequence will be used in Section 3.4 for the computation of the boundary spectrum of A_Q.

Corollary 3.2.3. *The resolvent set of A_0^Q contains the imaginary axis, i.e.,*

$$i\mathbb{R} \subseteq \rho(A_0^Q).$$

In the sequel we use the abbreviations

$$\Gamma := \gamma + \lambda + \mu$$

and

$$\Lambda := \gamma + \lambda.$$

Note that Γ and Λ both depend on γ which is not stated explicitly.

Now we determine the eigenfunctions of A_m^Q.

LEMMA 3.2.4. *For $\gamma \in \mathbb{C}, \Re\gamma > -\mu$, the following holds.*

$$p \in \ker(\gamma - A_m^Q) \tag{15}$$

$$\Leftrightarrow$$

$$p = (p_{0,0}, \ldots, p_{k-1,0}, p_{0,1}(\cdot), p_{1,1}(\cdot), \ldots)^T \tag{16}$$

where

$$p_{0,0} = \frac{\mu}{\Lambda}\frac{c_1}{\Gamma}, \tag{17}$$

$$p_{r,0} = \frac{1}{\Lambda}\left(\lambda p_{r-1,0} + \mu \sum_{i=1}^{r+1} c_i \frac{\lambda^{r+1-i}}{\Gamma^{r+2-i}}\right), \quad 1 \le r \le k-1, \tag{18}$$

$$p_{n,1}(x) = e^{-\Gamma x} \sum_{i=1}^{n+1} c_i \frac{\lambda^{n+1-i}}{(n+1-i)!} x^{n+1-i} \tag{19}$$

for some $(c_i)_{i\ge 1} \in l^1$ and all $n \ge 0$.

PROOF. First, we verify that each p given as in (16)–(19) is contained in $D(A_m^Q)$. Note that for $0 \ne C \in \mathbb{C}$ and $k \in \mathbb{N}$

$$\int_0^\infty e^{-Cx} x^k \, dx = \frac{k!}{C^{k+1}}.$$

Using this we estimate the norm

$$\begin{aligned}
\|p_{n,1}\|_{L^1[0,\infty)} &= \int_0^\infty \left| e^{-\Gamma x} \sum_{i=1}^{n+1} c_i \frac{\lambda^{n+1-i}}{(n+1-i)!} x^{n+1-i} \right| dx \\
&\le \sum_{i=1}^{n+1} |c_i| \frac{\lambda^{n+1-i}}{(n+1-i)!} \frac{(n+1-i)!}{(\Re\Gamma)^{n+2-i}} \\
&= \sum_{i=0}^{n} |c_{n+1-i}| \frac{\lambda^i}{(\Re\Gamma)^{i+1}}.
\end{aligned}$$

Since $\Re\gamma > -\mu$ the series $\sum_{i=0}^\infty \left(\frac{\lambda}{\Re\Gamma}\right)^k$ converges absolutely. So we can further estimate using the Cauchy product

$$\begin{aligned}
\sum_{n=0}^\infty \|p_{n,1}\|_{L^1[0,\infty)} &\le \sum_{n=0}^\infty \sum_{i=0}^n \frac{1}{\Re\Gamma}\left(\frac{\lambda}{\Re\Gamma}\right)^i |c_{n+1-i}| \\
&= \frac{1}{\Re\Gamma}\left(\sum_{i=0}^\infty \left(\frac{\lambda}{\Re\Gamma}\right)^i\right)\left(\sum_{i=1}^\infty |c_i|\right) \\
&= \frac{1}{\Re\Gamma}\frac{1}{1-\frac{\lambda}{\Re\Gamma}} \|(c_i)_{i\ge1}\|_{l^1} \\
&< \infty.
\end{aligned}$$

Similarly, we can estimate the derivatives $p'_{n,1}$ as

$$\sum_{n=0}^{\infty} \|p'_{n,1}\|_{L^1[0,\infty)} \;<\; \infty.$$

Hence, the graph norm $\|p\|_{D(A_m^Q)}$ of p is finite, and we have $p \in D(A_m^Q)$. We can directly compute that each p as in (16)–(19) satisfies

$$(\gamma - A_m^Q)p = 0.$$

Conversely, assume that $p \in \ker(\gamma - A_m^Q)$. Expressing $(\gamma - A_m^Q)p = 0$ explicitly yields a system of differential equations. Solving this we immediately see that only $(c_i)_{i\geq 1} \in l^1$ has to be checked. From

$$\begin{aligned} \sum_{i=1}^{\infty} |c_i| \;&=\; \sum_{i=1}^{\infty} |p_{i,1}(0)| \;\leq\; \sum_{i=1}^{\infty} \|p_{i,1}\|_\infty \\ &\leq\; \sum_{i=1}^{\infty} \|p_{i,1}\|_{W^{1,1}[0,\infty)} \;\leq\; \|p\|_{D(A_m^Q)} \\ &<\; \infty \end{aligned}$$

we obtain that $(c_i)_{i\geq 1} \in l^1$. □

Observe that the operator L_Q is surjective. Hence,

$$L_Q|_{\ker(\gamma - A_m^Q)} : \ker(\gamma - A_m^Q) \to \partial X_Q$$

is invertible if $\gamma \in \rho(A_0^Q)$, see Chapter 1. Next we compute its inverse D_γ^Q, the Dirichlet operator. For convenience we introduce the following operators.

DEFINITION 3.2.5. For $k \in \mathbb{N}$ we define the operators $\epsilon_k : \mathbb{C} \to L^1[0,\infty)$ as

$$(\epsilon_k(c))(x) := c\frac{\lambda^k}{k!}x^k e^{-(\gamma+\lambda+\mu)x}, \quad c \in \mathbb{C},\; x \in [0,\infty).$$

LEMMA 3.2.6. *Let $\gamma \in \mathbb{C}$, $\Re\gamma > -\mu$. Then*

$$D_\gamma^Q = \begin{pmatrix} d_{1,1} & 0 & \cdots & \cdots & \cdots & \cdots \\ d_{2,1} & d_{2,2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots & \cdots \\ d_{k,1} & \cdots & \cdots & d_{k,k} & 0 & \cdots \\ \epsilon_0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \epsilon_1 & \epsilon_0 & 0 & \cdots & \cdots & \cdots \\ \epsilon_2 & \epsilon_1 & \epsilon_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \end{pmatrix},$$

where

$$d_{k,r} := \frac{\mu\lambda^{k+1-r}}{\Gamma\Lambda^{k+2}} \sum_{i=0}^{k+1-r} \frac{\Lambda^{r+i}}{\Gamma^i}.$$

Once the Dirichlet operator is known, the spectrum $\sigma(A_Q)$ can be determined via the Characteristic Equation 1.7. For this purpose we need the explicit form of $\Phi_Q D_\gamma^Q$.

COROLLARY 3.2.7. *Let* $\gamma \in \mathbb{C}$, $\Re\gamma > -\mu$. *Then*

$$\Phi_Q D_\gamma^Q = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,B+1} & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Gamma}\right)^{B+1} & \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Gamma}\right)^{B} & \cdots & \frac{\mu}{\Gamma}\frac{\lambda}{\Gamma} & \frac{\mu}{\Gamma} & 0 & 0 & \cdots \\ \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Gamma}\right)^{B+2} & \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Gamma}\right)^{B+1} & \cdots & \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Gamma}\right)^{2} & \frac{\mu}{\Gamma}\frac{\lambda}{\Gamma} & \frac{\mu}{\Gamma} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where

$$a_{1,r} := \frac{\mu}{\Gamma}\left(\frac{\lambda}{\Lambda}\right)^{k+1-r}\sum_{i=0}^{k-r}\left(\frac{\Lambda}{\Gamma}\right)^i + \frac{\mu}{\Gamma}\sum_{i=k+1-r}^{B+1-r}\left(\frac{\lambda}{\Gamma}\right)^i \quad \textit{for } 1 \le r \le k,$$

$$a_{1,r} := \frac{\mu}{\Gamma}\sum_{i=0}^{B+1-r}\left(\frac{\lambda}{\Gamma}\right)^i \quad \textit{for } k+1 \le r \le B+1.$$

3.3. Well-posedness

We are now able to prove the generator property of A_Q and thus the well-posedness of ($\mathrm{ACP_Q}$). For that purpose we first show the dispersivity of A_Q using the same proof as in [Gup04, Lemma 1].

LEMMA 3.3.1. *The operator* $(A_Q, D(A_Q))$ *is dispersive.*

PROOF. Since A_Q is a real operator, it suffices to check the properties from Definition A.4 only for real valued elements in X_Q.

For $p = (p_{0,0}, \dots, p_{k-1,0}, p_{0,1}(\cdot), p_{1,1}(\cdot), \dots)^T \in \mathbb{R}^k \times l^1(L^1[0,\infty))$ define $\chi(\cdot) = (\chi_{0,0}, \dots, \chi_{k-1,0}, \chi_{0,1}(\cdot), \chi_{1,1}(\cdot), \dots)^T \in X_Q' = \mathbb{R}^k \times l^\infty(L^\infty[0,\infty))$ by

$$\chi_{r,0} := \begin{cases} 1, & \text{if } p_{r,0} > 0, \\ 0, & \text{else,} \end{cases} \qquad 0 \le r \le k-1,$$

$$\chi_{n,1}(x) := \begin{cases} 1, & \text{if } p_{n,1}(x) > 0, \\ 0, & \text{else,} \end{cases} \qquad n \ge 0.$$

For the following computations it is useful to observe that

$$\chi_{r,0} = \begin{cases} \frac{p_{r,0}^+}{p_{r,0}}, & \text{if } p_{r,0} > 0, \\ 0, & \text{else,} \end{cases} \qquad 0 \le r \le k-1,$$

$$\chi_{n,1}(x) = \begin{cases} \frac{p_{n,1}^+(x)}{p_{n,1}(x)}, & \text{if } p_{n,1}(x) > 0, \\ 0, & \text{else,} \end{cases} \qquad n \ge 0.$$

Note that $\|\chi\| \le 1$ and $\langle p, \chi\rangle = \|p^+\|$. To estimate $\langle A_Q p, \chi\rangle$ for $p \in D(A_Q)$ we need the equality

$$\int_0^\infty \frac{dp_{i,1}}{dx}(x)\chi_{i,1}(x)\,dx \;=\; \int_0^\infty \frac{dp_{i,1}^+}{dx}(x)\,dx = -p_{i,1}^+(0), \quad i \ge 0. \tag{20}$$

and the inequality

$$\begin{aligned}
&\sum_{i=0}^\infty p_{i,1}^+(0) \\
\overset{L_Q p = \Phi_Q p}{=}\;& \left[\mu \sum_{i=k}^B \int_0^\infty p_{i,1}(x)\,dx + \lambda p_{k-1,0}\right]^+ + \sum_{i=1}^\infty \left[\mu \int_0^\infty p_{i+B,1}(x)\,dx\right]^+ \\
\le\;& \lambda p_{k-1,0}^+ + \mu \sum_{i=k}^B \int_0^\infty p_{i,1}^+(x)\,dx + \mu \sum_{i=1}^\infty \int_0^\infty p_{i+B,1}^+(x)\,dx \\
=\;& \lambda p_{k-1,0}^+ + \mu \sum_{i=k}^\infty \int_0^\infty p_{i,1}^+(x)\,dx. \qquad (21)
\end{aligned}$$

Now we can compute

$$\begin{aligned}
\langle A_Q p, \chi\rangle =\;& \sum_{i=0}^{k-1}\left(-\lambda p_{i,0} + \mu \int_0^\infty p_{i,1}(x)\,dx\right)\chi_{i,0} + \sum_{i=1}^{k-1} \lambda p_{i-1,0}\chi_{i,0} \\
&+ \int_0^\infty \left(-\frac{dp_{0,1}}{dx}(x) - (\lambda+\mu)p_{0,1}(x)\right)\chi_{0,1}(x)\,dx \\
&+ \sum_{i=1}^\infty \int_0^\infty \left(-\frac{dp_{i,1}}{dx}(x) - (\lambda+\mu)p_{i,1}(x) + \lambda p_{i-1,1}(x)\right)\chi_{i,1}(x)\,dx \\
=\;& -\lambda \sum_{i=0}^{k-1} p_{i,0}^+ + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^\infty p_{i,1}(x)\,dx + \sum_{i=1}^{k-1} \lambda p_{i-1,0}\chi_{i,0} \\
&- \int_0^\infty \frac{dp_{0,1}}{dx}(x)\chi_{0,1}(x)\,dx - (\lambda+\mu)\int_0^\infty p_{0,1}^+(x)\,dx \\
&+ \sum_{i=1}^\infty \Big\{ -\int_0^\infty \frac{dp_{i,1}}{dx}(x)\chi_{i,1}(x)\,dx - (\lambda+\mu)\int_0^\infty p_{i,1}^+(x)\,dx \\
&\qquad + \lambda \int_0^\infty p_{i-1,1}(x)\chi_{i,1}(x)\,dx \Big\} \\
\overset{(20)}{=}\;& -\lambda \sum_{i=0}^{k-1} p_{i,0}^+ + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^\infty p_{i,1}(x)\,dx + \sum_{i=1}^{k-1} \lambda p_{i-1,0}\chi_{i,0} \\
&+ p_{0,1}^+(0) - (\lambda+\mu)\int_0^\infty p_{0,1}^+(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i=1}^{\infty} \left\{ p_{i,1}^{+}(0) - (\lambda + \mu) \int_0^{\infty} p_{i,1}^{+}(x)\, dx \right. \\
& \qquad \left. + \lambda \int_0^{\infty} p_{i-1,1}(x) \chi_{i,1}(x)\, dx \right\} \\
= \quad & -\lambda \sum_{i=0}^{k-1} p_{i,0}^{+} + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^{\infty} p_{i,1}(x)\, dx + \sum_{i=1}^{k-1} \lambda p_{i-1,0} \chi_{i,0} \\
& + \sum_{i=0}^{\infty} \left(p_{i,1}^{+}(0) - (\lambda + \mu) \int_0^{\infty} p_{i,1}^{+}(x)\, dx \right) \\
& + \lambda \sum_{i=1}^{\infty} \int_0^{\infty} p_{i-1,1}(x) \chi_{i,1}(x)\, dx \\
\leq \quad & -\lambda \sum_{i=0}^{k-1} p_{i,0}^{+} + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^{\infty} p_{i,1}^{+}(x)\, dx + \sum_{i=1}^{k-1} \lambda p_{i-1,0}^{+} \\
& + \sum_{i=0}^{\infty} \left(p_{i,1}^{+}(0) - (\lambda + \mu) \int_0^{\infty} p_{i,1}^{+}(x)\, dx \right) \\
& + \lambda \sum_{i=1}^{\infty} \int_0^{\infty} p_{i-1,1}^{+}(x)\, dx \\
= \quad & -\lambda p_{k-1,0}^{+} + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
& + \sum_{i=0}^{\infty} p_{i,1}^{+}(0) - \mu \sum_{i=0}^{\infty} \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
\overset{(21)}{\leq} \quad & -\lambda p_{k-1,0}^{+} + \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
& + \lambda p_{k-1,0}^{+} + \mu \sum_{i=k}^{\infty} \int_0^{\infty} p_{i,1}^{+}(x)\, dx - \mu \sum_{i=0}^{\infty} \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
= \quad & \mu \sum_{i=0}^{k-1} \chi_{i,0} \int_0^{\infty} p_{i,1}^{+}(x)\, dx - \mu \sum_{i=0}^{k-1} \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
= \quad & \mu \sum_{i=0}^{k-1} (\chi_{i,0} - 1) \int_0^{\infty} p_{i,1}^{+}(x)\, dx \\
\leq \quad & 0.
\end{aligned}
$$

Thus, A_Q is dispersive. □

The dispersivity of A_Q together with the surjectivity of $\gamma - A_Q$ for $\gamma > 0$ leads to the generator property of A_Q by Phillips' theorem, see Theorem A.5.

THEOREM 3.3.2. *The operator $(A_Q, D(A_Q))$ is the generator of a positive strongly continuous contraction semigroup on X_Q.*

PROOF. Let $\mathbb{R} \ni \gamma > 0$. Then all the entries of $\Phi_Q D_\gamma^Q$ are positive or 0 and we can estimate the j^{th} column sum, $j \geq 1$, as

$$\sum_{i=1}^{\infty} (\Phi_Q D_\gamma^Q)_{i,j} \quad < \quad \frac{\mu}{\Gamma} \sum_{i=0}^{\infty} \left(\frac{\Lambda}{\Gamma}\right)^i = \frac{\mu}{\Gamma} \frac{1}{1 - \frac{\Lambda}{\Gamma}} = \frac{\mu}{\Gamma - \Lambda} = 1.$$

Since the column sums are all equal from the $(B+1)^{st}$ column on, it follows that

$$\|\Phi_Q D_\gamma^Q\| = \sup_{1 \leq j} \sum_{i=1}^{\infty} (\Phi_Q D_\gamma^Q)_{i,j} = \max_{1 \leq j \leq B+1} \sum_{i=1}^{\infty} (\Phi_Q D_\gamma^Q)_{i,j} < 1,$$

and thus also

$$r(\Phi_Q D_\gamma^Q) \leq \|\Phi_Q D_\gamma^Q\| < 1.$$

Using the Characteristic Equation 1.7 we conclude that $\gamma \in \rho(A_Q)$ if $\gamma > 0$. Moreover, A_Q is a dispersive operator by Lemma 3.3.1. Now the claim follows from Phillips' theorem, see Theorem A.5. □

3.4. Boundary spectrum

Using the same idea as in [HR06] we can show that 0 is in the point spectrum of A_Q.

LEMMA 3.4.1. *For the operator $(A_Q, D(A_Q))$ we have*

$$0 \in \sigma_p(A_Q).$$

PROOF. By the Characteristic Equation 1.7 it suffices to prove that $1 \in \sigma_p(\Phi_Q D_0^Q)$. Define $p := \frac{\mu}{\mu+\lambda}$ and $q := \frac{\lambda}{\mu+\lambda}$. First, we compute $\Phi_Q D_0^Q : l^1 \longrightarrow l^1$ as

$$\Phi_Q D_0^Q = \begin{pmatrix} \sum_{k=0}^{B} pq^k & \sum_{k=0}^{B-1} pq^k & \sum_{k=0}^{B-2} pq^k & \cdots & p+pq & p & 0 & 0 & \cdots \\ pq^{B+1} & pq^B & pq^{B-1} & \cdots & pq^2 & pq & p & 0 & \cdots \\ pq^{B+2} & pq^{B+1} & pq^B & \cdots & pq^3 & pq^2 & pq & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The equation $\Phi_Q D_0^Q c = c$ for some $0 \neq c \in l^1$ is equivalent to the following system of equations:

$$c_1 = (\sum_{k=0}^{B} pq^k)c_1 + (\sum_{k=0}^{B-1} pq^k)c_2 + (\sum_{k=0}^{B-2} pq^k)c_3 + \cdots + (p+pq)c_B + pc_{B+1},$$
$$c_n = p\sum_{k=1}^{n+B} q^{n+B-k}c_k, \quad n \geq 2,$$

which is again equivalent to

$$c_1 = (\sum_{k=0}^{B} pq^k)c_1 + (\sum_{k=0}^{B-1} pq^k)c_2 + (\sum_{k=0}^{B-2} pq^k)c_3 + \cdots + (p+pq)c_B + pc_{B+1},$$
$$c_{B+n+1} = \frac{c_{n+1} - qc_n}{1-q}, \quad n \geq 2. \qquad (*)$$

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) := q^{(B+1)x} - q^{(B+1)x+1} - q^x + q.$$

Clearly, f is continuously differentiable and

$$f'(x) = (B+1)(1-q)\ln q e^{(B+1)x\ln q} - \ln q e^{x\ln q}, \quad x \in \mathbb{R}.$$

Since the traffic intensity satisfies $\rho = \frac{\lambda}{\mu} < 1$ by our General Assumption 3.2.1, it follows that $q = \frac{\lambda}{\mu+\lambda} < \frac{1}{2}$ and thus $(B+1)(1-q) > 1$. Hence we can estimate

$$f'(0) = (B+1)(1-q)\ln q - \ln q < 0.$$

Therefore, there exists $x_0 > 0$ such that $f'(x) < 0$ for all $x \in (0, x_0)$, hence f is decreasing on $(0, x_0)$. Since $f(0) = 0$ and $\lim_{x\to+\infty} f(x) = q > 0$, there exists $a > 0$ such that $f(a) = 0$ or $q^{na}f(a) = 0$ for all $n \in \mathbb{N}$, respectively. Thus, we obtain that

$$q^{(B+n+1)a} = \frac{q^{(n+1)a} - qq^{na}}{1-q}.$$

We conclude that for $c_n := q^{na}, n \geq 2$, the equations $(*)$ are fulfilled. The first equation of the above system yields

$$\begin{aligned}
q^{B+1}c_1 &= (\sum_{k=0}^{B-1} pq^k)q^{2a} + (\sum_{k=0}^{B-2} pq^k)q^{3a} + \cdots + (p+pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a}\sum_{k=0}^{B-1} q^k + pq^{3a}\sum_{k=0}^{B-2} q^k + \cdots + (p+pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a}\frac{1-q^B}{1-q} + pq^{3a}\frac{1-q^{B-1}}{1-q} + \cdots + pq^{Ba}\frac{1-q^2}{1-q} + pq^{(B+1)a} \\
&= q^{2a}(1-q^B) + q^{3a}(1-q^{B-1}) + \cdots + q^{Ba}(1-q^2) + q^{(B+1)a}(1-q) \\
&= q^{2a}(1 + q^a + q^{2a} + \cdots + q^{(B-1)a}) \\
&\quad - q^{2a+B}(1 + q^{a-1} + q^{2(a-1)} + \cdots + q^{(B-1)(a-1)}) \\
&= q^{2a}\left[\frac{1-q^{Ba}}{1-q^a} - q^B\frac{1-q^{B(a-1)}}{1-q^{(a-1)}}\right] \\
&= q^{2a}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B - q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})},
\end{aligned}$$

and hence

$$c_1 = q^{2a-B-1}\frac{(1-q^{Ba})(1-q^{a-1}) - (q^B - q^{Ba})(1-q^a)}{(1-q^a)(1-q^{a-1})}.$$

Obviously, $c := (c_n)_{n\in\mathbb{N}} \in l^1$ and thus c is a fixed point of $\Phi_Q D_0^Q$. By the Characteristic Equation 1.7 we conclude that $0 \in \sigma_p(A_Q)$. $\square$

Since A_Q generates a contraction semigroup, see Theorem 3.3.2, its spectral bound $\mathrm{s}(A_Q)$ is less than or equal to 0. Together with the above lemma it follows that $\mathrm{s}(A_Q) = 0$ and hence the boundary spectrum $\sigma_b(A_Q)$ of A_Q is located on the imaginary axis. The following lemma describes it completely.

THEOREM 3.4.2. *For the above operator* $(A_Q, D(A_Q))$ *we have*

$$\sigma_p(A_Q) \cap i\mathbb{R} = \sigma(A_Q) \cap i\mathbb{R} = \{0\}.$$

PROOF. From the above Lemma 3.4.1 we know that $0 \in \sigma_p(A_Q)$.

Let now $\gamma = ia,\ a \in \mathbb{R}$, such that $|a| > \lambda^k\mu + 2\mu + 2\lambda + 1$. Then

$$|\Gamma| \geq |a| - (\lambda+\mu) > \lambda + \mu, \tag{22}$$

$$|\Lambda| \geq |a| - \lambda > \lambda^k\mu + 2\mu + \lambda + 1 > 1, \tag{23}$$

$$|\Gamma| \geq |a| - (\lambda+\mu) > \lambda^k\mu + \mu + \lambda + 1. \tag{24}$$

First, we show that $\|\Phi_Q D_\gamma^Q\| < 1$. Therefore, we consider the columns of $|\Phi_Q D_\gamma^Q|$. If $j \geq k$ we estimate the j^{th} column sum as

$$\sum_{i=1}^{\infty} |(\Phi_Q D_\gamma^Q)_{i,j}| \leq \frac{\mu}{|\Gamma|} \sum_{i=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^i = \frac{\mu}{|\Gamma|} \frac{1}{1-\frac{\lambda}{|\Gamma|}} = \frac{\mu}{|\Gamma|-\lambda} \overset{(22)}{<} \frac{\mu}{\lambda+\mu-\lambda} = 1.$$

If $k > 1$ and if $|a|$ is such that $|\Lambda| \geq 1$, we obtain for the first column sum

$$\begin{aligned}
\sum_{i=1}^{\infty} |(\Phi_Q D_\gamma^Q)_{i,1}| &\leq \frac{\mu}{|\Gamma|} \left(\frac{\lambda}{|\Lambda|}\right)^k \sum_{i=0}^{k-1} \left|\frac{\Lambda}{\Gamma}\right|^i + \frac{\mu}{|\Gamma|} \sum_{i=k}^{B} \left(\frac{\lambda}{|\Gamma|}\right)^i + \frac{\mu}{|\Gamma|} \sum_{i=B+1}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^i \\
&\overset{(23)}{\leq} \frac{\mu}{|\Gamma|} \lambda^k \sum_{i=0}^{k-1} \left(\frac{1}{|\Gamma|}\right)^i + \frac{\mu}{|\Gamma|} \sum_{i=k}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^i \\
&= \frac{\mu}{|\Gamma|} \lambda^k \frac{1-\left(\frac{1}{|\Gamma|}\right)^k}{1-\frac{1}{|\Gamma|}} + \frac{\mu}{|\Gamma|} \left(\frac{1}{1-\frac{\lambda}{|\Gamma|}} - \frac{1-\left(\frac{\lambda}{|\Gamma|}\right)^k}{1-\frac{\lambda}{|\Gamma|}} \right) \\
&= \mu\lambda^k \frac{1-\left(\frac{1}{|\Gamma|}\right)^k}{|\Gamma|-1} + \mu \left(\frac{1}{|\Gamma|-\lambda} - \frac{1-\left(\frac{\lambda}{|\Gamma|}\right)^k}{|\Gamma|-\lambda} \right) \\
&= \mu\lambda^k \frac{1-\left(\frac{1}{|\Gamma|}\right)^k}{|\Gamma|-1} + \mu \left(\frac{\lambda}{|\Gamma|}\right)^k \frac{1}{|\Gamma|-\lambda} \\
&\leq \frac{\mu\lambda^k}{|\Gamma|-1} + \frac{\mu}{|\Gamma|-\lambda} \\
&\overset{(24)}{<} \frac{\mu\lambda^k}{\lambda^k\mu+\mu+\lambda} + \frac{\mu}{\lambda^k\mu+\mu+1} \\
&< \frac{\mu\lambda^k}{\lambda^k\mu+\mu} + \frac{\mu}{\lambda^k\mu+\mu} \\
&= 1.
\end{aligned}$$

Similarly, we can show that $\sum_{i=1}^{\infty} |(\Phi_Q D_\gamma^Q)_{i,j}| < 1$ for $1 < j < k$. Altogether we have shown that

$$\|\Phi_Q D_\gamma\| = \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |(\Phi_Q D_\gamma^Q)_{i,j}| < 1,$$

and hence also $\mathrm{r}(\Phi_Q D_\gamma^Q) \leq \|\Phi_Q D_\gamma^Q\| < 1$ if $|a| > \lambda^k\mu + 2\mu + 2\lambda + 1$. By the Characteristic Equation 1.7 this means that $\gamma \in \rho(A_Q)$ if $|a| > \lambda^k\mu + 2\mu + 2\lambda + 1$. In other words, $\sigma(A_Q) \cap i\mathbb{R}$ is bounded.

Since the semigroup is positive, the boundary spectrum of A_Q, i.e. $\sigma(A_Q) \cap i\mathbb{R}$, is cyclic, see [Nag86, Prop. C-III 2.10], which means that if $\alpha + i\beta \in \sigma_b(A_Q)$, $\alpha, \beta \in$

$\mathbb{R}$, then also $\alpha + ik\beta \in \sigma_b(A_Q)$ for all $k \in \mathbb{Z}$. From the boundedness of $\sigma(A_Q) \cap i\mathbb{R}$ we finally obtain that $\sigma(A_Q) \cap i\mathbb{R} = \{0\}$. $\square$

CHAPTER 4

A queueing network

4.1. Introduction

In the previous chapter we considered a queueing problem with only one server. However, in many cases the customers or objects in the system have to be served by several servers or machines. When a service is finished at one machine the object goes to another server with a certain probability. If this machine is currently busy then the object is stored in a buffer. The transfer times of the objects from one machine to the next machine or to the buffer are neglected in this model. An object can enter or leave this network only from predefined servers. For more information on queueing networks see [Kle76], [Kle76], [BB05].

Here we consider such a queueing network with only two machines. Some versions of this model were already studied using semigroup methods in [Gup03] and [HG04]. Again, we are interested in the convergence to a steady state solution. For this purpose, we discuss in this chapter spectral properties of the operator from the corresponding abstract Cauchy problem.

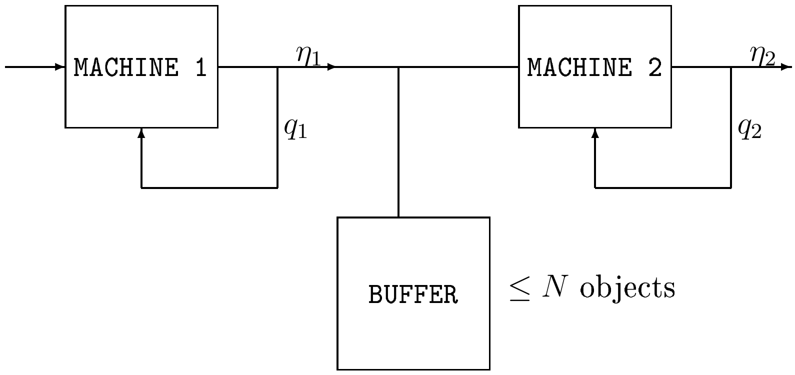
4.2. Setting

We study a queueing network consisting of a finite storage buffer and two machines that have to process certain objects. We assume that machine one takes objects from an infinite store and processes them one by one. When an object is finished here, it either passes from machine one to the buffer with probability $0 < \eta_1 < 1$, or it is reprocessed by machine one with probability $q_1 := 1 - \eta_1$. Machine two takes the objects from the buffer and also processes them one by one. When an object is finished here, it either exits the system with probability $0 < \eta_2 < 1$ or it is reprocessed by machine two with probability $q_2 := 1 - \eta_2$. The buffer can store at most N objects. If the buffer is full, then machine one rests until an object leaves the buffer. A schematic picture of this transfer line is drawn in Figure 1.

We assume that the service times X_n for the n^{th} object passing the first machine and Y_n for the n^{th} object passing the second machine are both independent and identically distributed random variables. For the first machine we require an exponential distribution with parameter $\lambda > 0$, i.e. the probability that the service time for the n^{th} object is less than or equal to t is

$$P(\{X_n \leq t\}) = 1 - e^{-\lambda t}.$$

FIGURE 1. Queueing network



For machine two the service times depend on a function μ which takes the elapsed service time of machine two into account. This function μ is supposed to fulfill the following.

GENERAL ASSUMPTION 4.2.1. The function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded such that $\lim_{x\to\infty} \mu(x)$ exists and $\mu_\infty := \lim_{x\to\infty} \mu(x) > 0$.

So the probability that the service time for the n^{th} object is less than or equal to t is

$$P(\{Y_n \leq t\}) = 1 - e^{-\int_0^t \mu(x)\, dx}.$$

We now consider for $t \geq 0$ the probability $p_0(t)$ that at time t only machine one is processing an object and there are neither objects in the buffer nor in machine two. For $1 \leq n \leq N+1$ and $x, t \geq 0$ we consider $p_n(x,t)$, where

$$\int_0^\infty p_n(x,t)\, dx$$

is the probability that at time t both machines are processing objects and there are $n-1$ objects in the buffer. Finally, for $t, x \geq 0$ we consider $p_{N+2}(x,t)$, where

$$\int_0^\infty p_{N+2}(x,t)\, dx$$

is the probability that at time t machine one rests, machine two is processing an object, the buffer contains N objects, so it is full, and the last object processed by machine one is waiting to be stored in the buffer. Thus,

$$p_0(t) + \sum_{n=1}^{N+2} \int_0^\infty p_n(x,t)\, dx = 1.$$

With these variables the dynamics of the above system can be described by the following equations.

$$\text{(R)}\quad \begin{cases} \dfrac{dp_0(t)}{dt} = -\lambda\eta_1 p_0(t) + \eta_2 \displaystyle\int_0^\infty p_1(x,t)\mu(x)dx, \\ \dfrac{\partial p_1(x,t)}{\partial t} = -\dfrac{\partial p_1(x,t)}{\partial x} - (\lambda\eta_1 + \mu(x))p_1(x,t), \\ \dfrac{\partial p_n(x,t)}{\partial t} = -\dfrac{\partial p_n(x,t)}{\partial x} - (\lambda\eta_1 + \mu(x))p_n(x,t) + \lambda\eta_1 p_{n-1}(x,t), \\ \qquad \text{for } 2 \le n \le N+1, \\ \dfrac{\partial p_{N+2}(x,t)}{\partial t} = -\dfrac{\partial p_{N+2}(x,t)}{\partial x} - \mu(x)p_{N+2}(x,t) + \lambda\eta_1 p_{N+1}(x,t). \end{cases}$$

For $x = 0$ the following boundary conditions

$$(\mathrm{BC_R})\quad \begin{cases} p_1(0,t) = \lambda\eta_1 p_0(t) + q_2 \displaystyle\int_0^\infty p_1(x,t)\mu(x)dx + \eta_2 \int_0^\infty p_2(x,t)\mu(x)dx, \\ p_n(0,t) = q_2 \displaystyle\int_0^\infty p_n(x,t)\mu(x)dx + \eta_2 \int_0^\infty p_{n+1}(x,t)\mu(x)dx, \\ \qquad \text{for } 2 \le n \le N+1, \\ p_{N+2}(0,t) = q_2 \displaystyle\int_0^\infty p_{N+2}(x,t)\mu(x)dx, \end{cases}$$

are prescribed. We consider the usual initial condition

$$(\mathrm{IC_R})\quad \begin{cases} p_0 = c \in \mathbb{C}, \\ p_n(x,0) = f_n(x) \quad \text{for } 1 \le n \le N+2, \end{cases}$$

where $f_n \in L^1[0,\infty)$. Assuming that at the time $t = 0$ there is only one object in machine one and no object in machine two or in the buffer, respectively, leads to the most interesting initial condition

$$(\mathrm{IC_{R,0}})\quad \begin{cases} p_0(0) = 1, \\ p_n(x,0) = 0 \quad \text{for } 1 \le n \le N+2. \end{cases}$$

In [Gup03], G. Gupur converted this model into an abstract Cauchy problem on a suitable Banach space and then proved the existence of a unique positive time-dependent solution by using the theory of strongly continuous semigroups of linear operators. Then, in [HG04], the authors considered the case where μ is constant and obtained asymptotic stability of the time-dependent solutions of this system.

Here, we consider the case where μ is an arbitrary function satisfying General Assumption 4.2.1 and prove the asymptotic stability of the time-dependent solution of this system using spectral theory and semigroup methods.

We first reformulate the underlying problem as an abstract Cauchy problem with an operator $(A_R, D(A_R))$ on the state space

$$X_R := \mathbb{C} \times (L^1[0,\infty))^{N+2}.$$

Clearly, X_R is a Banach space endowed with the norm

$$\|p\| := |p_0| + \sum_{n=1}^{N+2} \|p_n\|_{L^1[0,\infty)},$$

where $p = (p_0, p_1, \dots, p_{N+2})^T \in X_R$.

To arrive at the appropriate operator $(A_R, D(A_R))$ we start from the maximal operator $(A_m^R, D(A_m^R))$ on X_R which we define as

$$\begin{aligned} A_m^R &:= \begin{pmatrix} -\lambda\eta_1 & \eta_2\psi & 0 & 0 & \cdots & 0 & 0 \\ 0 & C & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda\eta_1 & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda\eta_1 & C & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C & 0 \\ 0 & 0 & 0 & 0 & \cdots & \lambda\eta_1 & \tilde{C} \end{pmatrix}, \\ D(A_m^R) &:= \mathbb{C} \times \left(W^{1,1}[0,\infty)\right)^{N+2}. \end{aligned}$$

Here and in the following ψ denotes the linear functional

$$\psi : L^1[0,\infty) \to \mathbb{C}, \qquad f \mapsto \psi(f) := \int_0^\infty \mu(x) f(x)\, dx.$$

Moreover, the operators C and $\tilde{C}$ on $W^{1,1}[0,\infty)$ are defined as

$$Cf := -\frac{d}{dx} f - (\lambda\eta_1 f + \mu f) \text{ and } \tilde{C} f := -\frac{d}{dx} f - \mu f,$$

respectively. Note that $(A_m^R, D(A_m^R))$ is a closed operator on X_R.

To model the boundary conditions ($\mathrm{BC_R}$) we will use the following boundary operators L_R and Φ_R mapping into the boundary space

$$\partial X_R := \mathbb{C}^{N+2}.$$

As the operator L_R we take

$$L_R : D(A_m^R) \to \partial X_R, \qquad \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N+2} \end{pmatrix} \mapsto L_R \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N+2} \end{pmatrix} := \begin{pmatrix} p_1(0) \\ \vdots \\ p_{N+2}(0) \end{pmatrix},$$

while $\Phi_R \in \mathcal{L}(X_R, \partial X_R)$ is

$$\Phi_R := \begin{pmatrix} \lambda\eta_1 & q_2\psi & \eta_2\psi & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & q_2\psi & \eta_2\psi & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & q_2\psi & \eta_2\psi & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & q_2\psi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_2\psi & \eta_2\psi \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q_2\psi \end{pmatrix}.$$

The operator $(A_R, D(A_R))$ on X_R is then given as

$$\begin{aligned} A_R p &:= A_m^R p, \\ D(A_R) &:= \{p \in D(A_m^R) \ : \ L_R p = \Phi_R p\}. \end{aligned}$$

With these definitions, the above equations (R), ($\mathrm{BC_R}$) and ($\mathrm{IC_R}$) become equivalent to the abstract Cauchy problem

$$(\mathrm{ACP_R}) \qquad \begin{cases} \dfrac{d}{dt}p(t) = A_R p(t), & t \in [0, \infty), \\ \quad p(0) = (c, f_1, \ldots, f_n)^T \in X_R. \end{cases}$$

If A_R generates a strongly continuous semigroup $(T_R(t))_{t\geq 0}$ and if the initial values in ($\mathrm{IC_Q}$) satisfy $q := (c_0, f_1, f_2, \ldots, f_{N+2})^T \in D(A_Q)$, then the unique solution of (R), ($\mathrm{BC_R}$) and ($\mathrm{IC_R}$) is given by

$$\begin{aligned} p_0(t) &= (T_R(t)q)_1, \\ p_{n,1}(x,t) &= (T_R(t)q)_{n+1}(x), \quad 1 \leq n \leq N+2. \end{aligned}$$

So it suffices to investigate ($\mathrm{ACP_R}$).

We take the following result from [Gup03].

THEOREM 4.2.2. *The operator $(A_R, D(A_R))$ generates a positive strongly continuous contraction semigroup $(T_R(t))_{t\geq 0}$.*

4.3. Boundary spectrum

In this section we investigate the boundary spectrum $\sigma_b(A_R)$ of A_R using the Characteristic Equation 1.7. Therefore we first need the operator $(A_0^R, D(A_0^R))$ defined by

$$\begin{aligned} D(A_0^R) &:= \{p \in D(A_m^R) \ : \ L_R p = 0\}, \\ A_0^R p &:= A_m^R p. \end{aligned}$$

The resolvent set of this operator is given as follows.

LEMMA 4.3.1. *The resolvent set of A_0^R satisfies*

$$\rho(A_0^R) \supseteq \{\gamma \in \mathbb{C} \ : \ \Re\gamma > -\mu_\infty\} \setminus \{\lambda\eta_1\}.$$

For $\gamma \in \rho(A_0^R)$ the resolvent of A_0^R is obtained as

$$R(\gamma, A_0^R) = \begin{pmatrix} r_{1,1} & r_{1,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_{3,2} & r_{3,3} & 0 & \cdots & 0 & 0 \\ 0 & r_{4,2} & r_{4,3} & r_{4,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & r_{N+2,2} & r_{N+2,3} & r_{N+2,4} & \cdots & r_{N+2,N+2} & 0 \\ 0 & r_{N+3,2} & r_{N+3,3} & r_{N+3,4} & \cdots & r_{N+3,N+2} & r_{N+3,N+3} \end{pmatrix},$$

where

$$\begin{aligned} r_{1,1} &:= \tfrac{1}{\gamma+\lambda\eta_1}, \\ r_{1,2} &:= \tfrac{\eta_2}{\gamma+\lambda\eta_1}\psi R(\gamma, C), \\ r_{j,k} &:= (\lambda\eta_1)^{j-k} R^{j-k+1}(\gamma, C) \quad \textit{for } 2 \leq k \leq j \leq N+2, \\ r_{N+3,k} &:= (\lambda\eta_1)^{N+3-k} R(\gamma, \tilde{C}) R^{N+3-k}(\gamma, C) \quad \textit{for } 2 \leq k \leq N+3. \end{aligned}$$

The resolvent operators of the differential operators C and $\tilde{C}$ are given by

$$(R(\gamma, C)p)(x) = e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)d\xi} \int_0^x e^{(\gamma+\lambda\eta_1)s + \int_0^s \mu(\xi)d\xi} p(s)ds,$$

and

$$(R(\gamma, \tilde{C})p)(x) = e^{-\gamma x - \int_0^x \mu(\xi)d\xi} \int_0^x e^{\gamma s + \int_0^s \mu(\xi)d\xi} p(s)ds$$

for $p \in L^1[0, \infty)$.

Proof. The first assertion follows from a combination of [Gre84b, Prop. 2.1] and [Nag89, Thm. 2.4]. Applying the formula for the inverse of operator matrices from [Nag89, Thm. 2.4] yields the representation of the resolvent. □

Corollary 4.3.2. *The resolvent set of A_0^R contains the imaginary axis, i.e.*

$$i\mathbb{R} \subseteq \rho(A_0^R).$$

The elements in $\ker(\gamma - A_m^R)$ are characterised as follows.

LEMMA 4.3.3. *Let* $\gamma \in \mathbb{C}$.

(25) $p \in \ker(\gamma - A_m^R)$

$\Leftrightarrow$

(26) $p = (p_0, p_1(\cdot), \dots, p_{N+2}(\cdot))^T$,

(27) $p_0 = \dfrac{\eta_2 a_1}{\gamma + \lambda\eta_1} \displaystyle\int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)d\xi} dx,$

(28) $p_n(x) = e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)d\xi} \displaystyle\sum_{k=1}^{n} \frac{(\lambda\eta_1)^{k-1}}{(k-1)!} x^{k-1} a_{n+1-k}$

for $1 \le n \le N+1$,

(29) $p_{N+2}(x) = e^{-\gamma x - \int_0^x \mu(\xi)d\xi} \left[\displaystyle\sum_{i=0}^{N} a_{N+1-i} \left(1 - \sum_{k=0}^{i} \frac{(\lambda\eta_1)^k}{k!} x^k e^{-\lambda\eta_1 x} \right) + a_{N+2} \right]$

where $a_1, \dots, a_{N+2} \in \mathbb{C}$.

PROOF. If for $p \in X_R$ (26)–(29) is fulfilled, then we can easily compute that $p \in \ker(\gamma - A_m^R)$. Conversely, condition (25) gives a system of differential equations. Solving these differential equations, we see that (26)–(29) are necessarily satisfied. □

Observe that L_R is surjective and hence the Dirichlet operator D_γ^R exists, see Chapter 1. For $k \in \mathbb{N}$ we define

$$\delta_k : \mathbb{C} \to L^1[0,\infty), \qquad (\delta_k(c))(x) := c \frac{(\lambda\eta_1)^k}{k!} x^k e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)\, d\xi},$$

$k \in \{0, \dots, N\}$, Now we can give the explicit form of D_γ^R.

LEMMA 4.3.4. *For each* $\gamma \in \rho(A_0^R)$, *the operator* D_γ^R *has the form*

$$D_\gamma^R = \begin{pmatrix} d_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ \delta_0 & 0 & 0 & \cdots & 0 & 0 \\ \delta_1 & \delta_0 & 0 & \cdots & 0 & 0 \\ \delta_2 & \delta_1 & \delta_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_N & \delta_{N-1} & \delta_{N-2} & \cdots & \delta_0 & 0 \\ d_{N+3,1} & d_{N+3,2} & d_{N+3,3} & \cdots & d_{N+3,N+1} & d_{N+3,N+2} \end{pmatrix},$$

where

$$\begin{aligned} d_{1,1} &:= \frac{\eta_2}{\gamma+\lambda\eta_1} \int_0^\infty \mu(x)(\delta_0(1))(x)\, dx, \\ d_{N+3,k} &:= e^{-\gamma\cdot - \int_0^{\cdot} \mu(\xi)d\xi} - \sum_{n=0}^{N+1-k} \delta_n \quad \text{for } 1 \le k \le N+1, \\ d_{N+3,N+2} &:= e^{-\gamma\cdot - \int_0^{\cdot} \mu(\xi)d\xi}. \end{aligned}$$

As explained in Chapter 1 we can characterise the spectrum $\sigma(A_R)$ and the point spectrum $\sigma_p(A_R)$ of A_R using the operators D_γ^R and Φ_R.

LEMMA 4.3.5. *For $\gamma \in \rho(A_0^R)$ the operator $\Phi_R D_\gamma^R$ can be represented by the $(N+2)\times(N+2)$-matrix*

$$\Phi_R D_\gamma^R = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & a_{N,4} & \cdots & a_{N,N+1} & 0 \\ a_{N+1,1} & a_{N+1,2} & a_{N+1,3} & a_{N+1,4} & \cdots & a_{N+1,N+1} & a_{N+1,N+2} \\ a_{N+2,1} & a_{N+2,2} & a_{N+2,3} & a_{N+2,4} & \cdots & a_{N+2,N+1} & a_{N+2,N+2} \end{pmatrix}$$

where

$$\begin{aligned}
a_{k,j} &:= 0 \quad \text{if } 1 \le k \le N \text{ and } j \ge k+2, \\
a_{1,1} &:= \left(\frac{\lambda\eta_1\eta_2}{\gamma+\lambda\eta_1} + q_2\right)\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad + \lambda\eta_1\eta_2\int_0^\infty x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx, \\
a_{k,j} &:= \eta_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx \quad \text{if } 1 \le k \le N \text{ and } j = k+1, \\
a_{N+1,N+2} &:= \eta_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi}dx, \\
a_{k,j} &:= \frac{(\lambda\eta_1)^{k-j}}{(k-j)!}q_2\int_0^\infty x^{k-j}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad + \frac{(\lambda\eta_1)^{k-j+1}}{(k-j+1)!}\eta_2\int_0^\infty x^{k-j+1}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad \text{if } 2 \le k \le N \text{ and } j \le k, \\
a_{N+1,j} &:= \frac{(\lambda\eta_1)^{N+1-j}}{(N+1-j)!}q_2\int_0^\infty x^{N+1-j}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad + \eta_2\int_0^\infty \mu(x)\left(1-\sum_{n=0}^{N+1-j}\frac{(\lambda\eta_1)^n}{n!}x^n e^{-\lambda\eta_1 x}\right)e^{-\gamma x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad \text{if } 1 \le j \le N+1, \\
a_{N+2,j} &:= q_2\int_0^\infty \mu(x)\left(1-\sum_{n=0}^{N+1-j}\frac{(\lambda\eta_1)^n}{n!}x^n e^{-\lambda\eta_1 x}\right)e^{-\gamma x-\int_0^x \mu(\xi)d\xi}dx \\
&\quad \text{if } 1 \le j \le N+1,
\end{aligned}$$

$$a_{N+2,N+2} := q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx.$$

With these operators and based on the Characteristic Equation 1.7 we investigate the boundary spectrum of A_R in more detail. Since by Theorem 4.2.2 the semigroup is bounded, the spectral bound $\mathrm{s}(A_R)$ of A_R is not greater than 0. It indeed coincides with 0 as we can conclude from the following lemma.

LEMMA 4.3.6. *For the above operator we have* $0 \in \sigma_p(A_R)$.

PROOF. Since $0 \in \rho(A_0^R)$ by Lemma 4.3.1, we can use the Characteristic Equation 1.7. So we have to prove that $1 \in \sigma_p(\Phi_R D_0^R)$. A simple calculation using that

$$\int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx = 1$$

shows that $\Phi_R D_0^R$ is column stochastic. Hence $1 \in \sigma_p(\Phi_R D_0^R)$. □

We show in the following lemma that 0 is the only spectral value on the imaginary axis.

THEOREM 4.3.7. *Under the General Assumption 4.2.1*

$$\sigma(A_R) \cap i\mathbb{R} = \{0\}$$

holds.

PROOF. Assume that $ai \in \sigma(A_R)$ for some $0 \neq a \in \mathbb{R}$ and consider $\Phi_R D_0^R = (b_{kj})_{N+2\times N+2}$ and $\Phi_R D_{ai}^R = (c_{kj})_{N+2\times N+2}$. Using the triangle inequality for integrals we see that

$$|c_{kj}| \leq b_{kj}$$

holds for all $k, j \in \{1, \dots, N+2\}$. By our General Assumption 4.2.1, there exists $r \in \mathbb{R}_+$ such that $\mu(x) > 0$ for all $x \in [r, r + \frac{2\pi}{a}]$.

For the lower right entry of $\Phi_R D_{ai}^R$ we compute, using the abbreviation $h(x) := q_2 \mu(x) e^{-\int_0^x \mu(\xi)\, d\xi}$,

$$\begin{aligned} |c_{N+2,N+2}| &= \left| \int_0^\infty e^{-aix} h(x) dx \right| \\ &\leq \left| \int_r^{r+\frac{2\pi}{a}} e^{-aix} h(x)\, dx \right| + \left| \int_0^r e^{-aix} h(x)\, dx + \int_{r+\frac{2\pi}{a}}^\infty e^{-aix} h(x)\, dx \right| \\ &\leq \left| \int_r^{r+\frac{2\pi}{a}} e^{-aix} h(x)\, dx \right| + \int_0^r h(x)\, dx + \int_{r+\frac{2\pi}{a}}^\infty h(x)\, dx. \end{aligned}$$

Next, we estimate the first term on the right hand side of the above inequality as

$$\begin{aligned}
\left|\int_r^{r+\frac{2\pi}{a}} e^{-aix} h(x)\,dx\right| &= \left|\int_r^{r+\frac{\pi}{a}} e^{-aix} h(x)\,dx + \int_{r+\frac{\pi}{a}}^{r+\frac{2\pi}{a}} e^{-aix} h(x)\,dx\right| \\
&= \left|\int_r^{r+\frac{\pi}{a}} e^{-aix} h(x)\,dx + \int_r^{r+\frac{\pi}{a}} e^{-ai\left(x+\frac{\pi}{a}\right)} h\left(x+\tfrac{\pi}{a}\right)\,dx\right| \\
&= \left|\int_r^{r+\frac{\pi}{a}} e^{-aix} h(x)\,dx - \int_r^{r+\frac{\pi}{a}} e^{-aix} h\left(x+\tfrac{\pi}{a}\right)\,dx\right| \\
&= \left|\int_r^{r+\frac{\pi}{a}} e^{-aix} \left(h(x) - h\left(x+\tfrac{\pi}{a}\right)\right)\,dx\right| \\
&\leq \int_r^{r+\frac{\pi}{a}} \left|h(x) - h\left(x+\tfrac{\pi}{a}\right)\right|\,dx \\
&< \int_r^{r+\frac{\pi}{a}} \left(h(x) + h\left(x+\tfrac{\pi}{a}\right)\right)\,dx \\
&= \int_r^{r+\frac{\pi}{a}} h(x)\,dx + \int_{r+\frac{\pi}{a}}^{r+\frac{2\pi}{a}} h(x)\,dx \\
&= \int_r^{r+\frac{2\pi}{a}} h(x)\,dx.
\end{aligned}$$

Note that for the last inequality we used the strict positivity of μ on $\left[r, r+\frac{2\pi}{a}\right]$.

We now obtain

$$\begin{aligned}
|c_{N+2,N+2}| &< \int_r^{r+\frac{2\pi}{a}} h(x) \; + \int_0^r h(x)\,dx + \int_{r+\frac{2\pi}{a}}^{\infty} h(x)\,dx \\
&= \int_0^{\infty} h(x)\,dx = b_{N+2,N+2}.
\end{aligned}$$

From the zero pattern of $\Phi_R D_0^R$ it is clear that $\Phi_R D_0^R$ is an irreducible matrix. Hence we can apply [Sch74, Cor. p. 22] to obtain for the spectral radii

$$\mathrm{r}(\Phi_R D_{ai}^R) < \mathrm{r}(\Phi_R D_0^R).$$

Since $\Phi_R D_0^R$ is column stochastic, we know that $\mathrm{r}(\Phi_R D_0^R) = 1$. Therefore $\mathrm{r}(\Phi_R D_{ai}^R) < 1$. Since $ai \in \rho(A_0^R)$ by Corollary 4.3.2, we can apply the Characteristic Equation 1.7 to obtain that $ai \notin \sigma(A_R)$. □

CHAPTER 5

Asymptotics

5.1. General results

Our main goal in this chapter is to describe the asymptotic behaviour of the solutions of the previous problems using the theory of positive and irreducible semigroups from [Nag86] and [Sch74]. To this end, we first collect some results on this aspect. For notations and results concerning positive operators we refer to Appendix A and the relevant monographs such as [Sch74], [MN91] and [Nag86].

Let E be a Banach lattice and $(B, D(B))$ be the generator of a positive semigroup $(S(t))_{t\geq 0}$ on E. The fixed space of the semigroup $(S(t))_{t\geq 0}$ is

$$\operatorname{fix}(S(t))_{t\geq 0} = \bigcap_{t\geq 0} \operatorname{fix}(S(t)) = \{u \in E : S(t)z = z \text{ for all } t \geq 0\}.$$

By [EN00, Cor. IV.3.8 (i)] the equality

$$\operatorname{fix}(S(t))_{t\geq 0} = \ker B \tag{30}$$

holds.

For our purposes it is enough to consider Banach lattices which are AL-spaces, see [Sch74, Def. II.8.1], i.e., for all $z_1, z_2 \in E_+$ we have

$$\|z_1 + z_2\| = \|z_1\| + \|z_2\|.$$

To treat the asymptotic behaviour of $(S(t))_{t\geq 0}$ the following compactness property is useful.

LEMMA 5.1.1. *Suppose that E is an AL-space and that the positive semigroup $(S(t))_{t\geq 0}$ is irreducible and bounded. Let $0 \in \sigma_p(B)$. Then $\{S(t) : t \geq 0\} \subseteq \mathcal{L}(E)$ is relatively compact for the weak operator topology. In particular, it is mean ergodic, i.e.*

$$\lim_{r\to\infty} \tfrac{1}{r} \int_0^r S(s)z \, ds,$$

exists for all $z \in E$.

PROOF. Since $0 \in \sigma_p(B)$ there exists $0 \neq z \in \operatorname{fix}(S(t))_{t\geq 0}$, see (30). Then, by the positivity of the semigroup, the inequality

$$S(t)^n|z| = S(t)^n|S(t)z| \leq S(t)^{n+1}|z| \tag{31}$$

follows for all $n \in \mathbb{N}$ and $t \geq 0$, see [Sch74, p. 58]. Note that since $(S(t))_{t\geq 0}$ is bounded by assumption, also the sequence $((S(t)^n|z|)_{n\in\mathbb{N}}$ is norm-bounded. By

[Sch74, Prop. II.8.3] the sequence converges to an element $z_0 \gneq 0$. In this step we use that E is an AL-space. From

$$S(t)z_0 = S(t) \lim_{n\to\infty} S(t)^n|z| = \lim_{n\to\infty} S(t)^{n+1}|z| = z_0$$

we obtain that $z_0 \in \operatorname{fix}(S(t))_{t\geq 0}$. Thus, we can assume without loss of generality that $z \gneq 0$.

Since the semigroup is irreducible we obtain from [Nag86, Prop. C-III 3.5 (a)] that z is a quasi-interior point of E which means that

$$E_z := \bigcup_{n\geq 1} [-nz, nz]$$

is dense in E.

Let $n \in \mathbb{N}$ and take $w \in [-nz, nz]$. Then

$$-nz = -nS(t)z \leq S(t)w \leq nS(t)z = nz,$$

for all $t \geq 0$. Since the order interval $[-nz, nz]$ is weakly compact in E, see [Sch74, p. 92], the orbit $\{S(t)w : t \geq 0\}$ is relatively weakly compact in E. So far, we have shown that the orbits of elements w from the dense subset E_z of E are relatively weakly compact. Since the semigroup $(S(t))_{t\geq 0}$ is bounded, it follows that $\{S(t) : t \geq 0\} \subseteq \mathcal{L}(E)$ is relatively compact for the weak operator topology, see [EN00, Lem. V.2.7].

The mean ergodicity of $(S(t))_{t\geq 0}$ follows from [EN00, Lem. V.4.6]. □

The mean ergodicity of the semigroup allows a decomposition of E into the direct sum of $\ker B$ and $\overline{\operatorname{rg} B}$. If the semigroup is irreducible, then $\ker B$ is one-dimensional. If in addition $\sigma(B) \cap i\mathbb{R} = \sigma_p(B) \cap i\mathbb{R} = \{0\}$, then the semigroup converges strongly to a one-dimensional projection onto $\ker B$. This is a consequence of the Arendt-Batty-Lyubich-Vũ Theorem and shown in the next theorem.

THEOREM 5.1.2. *Suppose that E is an AL-space and that the positive semigroup $(S(t))_{t\geq 0}$ is irreducible and bounded. If*

$$\sigma(B) \cap i\mathbb{R} = \sigma_p(B) \cap i\mathbb{R} = \{0\},$$

then E can be decomposed into the direct sum

$$E = E_1 \oplus E_2$$

where $E_1 = \operatorname{fix}(S(t))_{t\geq 0} = \ker B$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{z} \in \ker B$ of B. In addition, the restriction $(S(t)|_{E_2})_{t\geq 0}$ is strongly stable.

PROOF. Since $(S(t))_{t\geq 0}$ is mean ergodic by Lemma 5.1.1, the space E can be decomposed into

$$E = \ker B \oplus \overline{\operatorname{rg}(B)} =: E_1 \oplus E_2$$

where $\ker B = \operatorname{fix}(S(t))_{t\geq 0}$ and E_1 and E_2 are invariant under $(S(t))_{t\geq 0}$, see [EN00, Lem. V.4.4]. There exists $\tilde{z} \in \ker B$ such that $\tilde{z} > 0$, confer the proof of Lemma 5.1.1. Moreover, by the same construction as in the proof of [EN00, Lemma V.2.20 (i)], we find $z' \in E'$ such that $z' > 0$ and $B'z' = 0$. Hence we obtain that

$$\dim \ker B = 1$$

and that $\tilde{z}$ is strictly positive, see [Nag86, Prop. C-III 3.5].

We now consider the generator $(B_2, D(B_2))$ of the restricted semigroup $(S_2(t))_{t\geq 0}$ where

$$B_2 v = Bv, \quad D(B_2) = D(B) \cap E_2$$

and $S_2(t) = S(t)|_{E_2}$. Since by Lemma 5.1.1 every $z \in E$ has a relatively weakly compact orbit, $(S_2(t))_{t\geq 0}$ is totally ergodic by [ABHN01, Prop. 4.3.12], i.e., $(e^{-iat}S(t))_{t\geq 0}$ is mean ergodic for all $a \in \mathbb{R}$. This implies that $\ker(B_2 - iat)$ separates $\ker(B_2' - iat)$ for all $a \in \mathbb{R}$, see [EN00, Thm. V.4.5]. By our assumption $\ker(B_2 - iat) = \{0\}$, thus $\ker(B_2' - iat) = \{0\}$ for all $a \in \mathbb{R}$. Hence, it follows that $\sigma_p(B_2') \cap i\mathbb{R} = \emptyset$. Applying the Arendt-Batty-Lyubich-Vũ Theorem, see [ABHN01, Thm. 5.5.5], we obtain the strong stability of $(S_2(t))_{t\geq 0}$. □

5.2. Networks

5.2.1. Irreducibility. As we have seen in the previous section, irreducibility of the semigroup is a useful property for the asymptotic behaviour. So we are now interested under which conditions the semigroup $(T_N(t))_{t\geq 0}$ from Chapter 2 is irreducible. It turns out that we need both a condition on the structure of the graph and on the scattering operator J in the vertices. In Section 5.2.2, this will lead to a precise description of the asymptotic behaviour of the semigroup. To show irreducibility for our semigroup we need the following concept from graph theory.

DEFINITION 5.2.1. A directed graph is called *strongly connected* if for any two vertices v, w of the graph there exists a path from v to w and from w to v.

We now obtain irreducibility of our semigroup combining the strong connectedness of the graph with the strict positivity of J.

PROPOSITION 5.2.2. *Let G be strongly connected and suppose that*

$$Jf \gg 0 \text{ if } f \gneq 0. \tag{32}$$

Then the semigroup $(T_N(t))_{t\geq 0}$ generated by A_N is irreducible.

PROOF. Suppose that $\gamma > 0$ and let $u \gneq 0$. Then also $R(\gamma, A_0^N)u \gneq 0$ and $\Phi_N R(\gamma, A_0^N)u \gneq 0$. By Proposition 2.3.7 (i) $\|\Phi_N D_\gamma^N\| < 1$, and hence the inverse of $Id_{\partial X_N} - \Phi_N D_\gamma^N$ is given by the Neumann series

$$(Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} = \sum_{n=0}^{\infty} (\Phi_N D_\gamma^N)^n.$$

The operator $\Phi_N D_\gamma^N$ has the same zero pattern as the adjacency matrix $\mathbb{A}$. Observe that $\mathbb{A}^k$ has a non-zero entry at position ij if there is a path from vertex v_j to vertex v_i of length k. Since G is assumed to be strongly connected, for every pair i, j there exists $k \in \mathbb{N}$ such that the entry ij of $\mathbb{A}^k$ and thus of $(\Phi_N D_\gamma^N)^k$ is nonzero. This entry can be written as the composition of J with an operator composed of J and multiplications by strictly positive functions. By assumption (32) we conclude that

$$(Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N) u \gg 0$$

and therefore by the special form of D_γ^N also

$$D_\gamma (Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N) u \gg 0.$$

This implies

$$R(\gamma, A_N) u \gg 0,$$

which by Proposition A.11 is equivalent to the irreducibility of the semigroup. □

In the following examples we show that without an assumption on the graph and on the scattering operator J the semigroup is not necessarily irreducible.

EXAMPLE 5.2.3. If we drop the assumption of the strong connectivity of the graph, then the semigroup need not be irreducible.

To prove this we decompose the graph into its strongly connected components. Assuming the graph not to be strongly connected, there exists a strongly connected component $C = (V', E'), V' \subseteq V, E' \subseteq E$ such that there is no edge $e \in E \setminus E'$ that is an incoming edge for a vertex $v \in V'$. Without loss of generality we can assume that $V' = \{v_r, \ldots, v_n\}$ for some $2 \leq r < n$, and $E' = \{e_s, \ldots, e_m\}$ for some $1 < s < m - 1$. The incidence matrices have the form

$$\tilde{\mathbb{I}}_w^- = \begin{pmatrix} \tilde{\mathbb{I}}_{11}^- & 0 \\ \tilde{\mathbb{I}}_{21}^- & \tilde{\mathbb{I}}_{22}^- \end{pmatrix} \text{ and } \tilde{\mathbb{I}}^+ = \begin{pmatrix} \tilde{\mathbb{I}}_{11}^+ & \tilde{\mathbb{I}}_{12}^+ \\ 0 & \tilde{\mathbb{I}}_{22}^+ \end{pmatrix} \text{ respectively,}$$

where $\tilde{\mathbb{I}}_{11}^-$ and $\tilde{\mathbb{I}}_{11}^+$ are $(r-1) \times (s-1)-$, $\tilde{\mathbb{I}}_{12}^-$ and $\tilde{\mathbb{I}}_{12}^+$ are $(r-1) \times (m-s+1)-$, $\tilde{\mathbb{I}}_{21}^-$ and $\tilde{\mathbb{I}}_{21}^+$ are $(n-r+1) \times (s-1)-$ and $\tilde{\mathbb{I}}_{22}^-$ and $\tilde{\mathbb{I}}_{22}^+$ are $(n-r+1) \times (m-s+1)-$ operator matrices. Moreover, since there is no path from a vertex $v_i, 1 \leq i \leq r-1$, leading into the subgraph C, we have

$$(\Phi_N D_\gamma^N)^n = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}, n \in \mathbb{N},$$

where M_{11} is an $(r-1) \times (r-1)-$, M_{12} is an $(r-1) \times (n-r+1)-$, and M_{22} is an $(n-r+1) \times (n-r+1)-$ operator matrix. Take an element $u \in X_N$ such that $u = (u_j)_{1 \leq j \leq m} \gneq 0$ and $u_j = 0$ for $j \in \{r, \ldots, n\}$. Then, it follows from the special form of the operators appearing in the resolvent of A_N that $(R(\gamma, A_N)u)_j = 0$ for $\gamma > 0$ and $j \in \{r, \ldots, n\}$. By Proposition A.11, the semigroup is not irreducible.

Note that in (32) we require a stronger condition than the irreducibility of J. If G is strongly connected and J is only assumed to be irreducible, then the semigroup generated by A_N is not necessarily irreducible as the following example shows.

EXAMPLE 5.2.4. Define

$$k : [v_{min}, v_{max}] \times [v_{min}, v_{max}] \to \mathbb{R},$$

$$k(v,w) := \begin{cases} c & \text{if } v \in [v_{min}, v'] \text{ and } w \in [v', v_{max}] \\ & \text{or if } v \in (v', v_{max}] \text{ and } w \in [v_{min}, v'], \\ 0 & \text{else,} \end{cases}$$

where $0 \neq c \in \mathbb{R}$ and $v_{min} < v' < v_{max}$. Then the integral operator

$$J : Y \to Y,$$

$$(Jf)(v) := \int_{v_{min}}^{v_{max}} k(v,w) f(w)\, dw = \begin{cases} c \displaystyle\int_{v'}^{v_{max}} f(w)\, dw \text{ if } v_{min} \leq v \leq v', \\ c \displaystyle\int_{v_{min}}^{v'} f(w)\, dw \text{ if } v' < v \leq v_{max}, \end{cases}$$

is irreducible which can be shown by an easy computation.

Consider a graph with the incidence matrices $\tilde{\mathbb{I}}_w^- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\tilde{\mathbb{I}}^+ = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and suppose that both arcs have length l. Let $u = (u_1, u_2) \in X_N$ such that $u_1 \gneq 0, u_1(x,v) = 0$ if $0 \leq x \leq l$ and $v_{min} \leq v \leq v'$ and $u_2 \equiv 0$. Then also $(R(\gamma, A_0^N)u)_2 = 0$ and

$$(f_1, f_2) := \Phi_N R(\gamma, A_0^N) u = (0, J(\int_0^l \tfrac{1}{\cdot} e^{-\gamma \frac{l-r}{\cdot}} u_1(r, \cdot)\, dr)).$$

Observe that if $f \in Y$ with $f|_{[v_{min}, v']} = 0$, then

$$(J^{2k+1} f)|_{[v', v_{max}]} = 0 \tag{33}$$

and

$$(J^{2k} f)|_{[v_{min}, v']} = 0 \text{ for } k \in \mathbb{N}. \tag{34}$$

Therefore, $f_2|_{[v', v_{max}]} = 0$ holds. Suppose that $\gamma > 0$. Then the inverse of $Id_{\partial X_N} - \Phi_N D_\gamma^N$ is given by the Neumann series

$$\begin{aligned} (Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} &= \sum_{k=0}^{\infty} (\Phi_N D_\gamma^N)^k \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} (J Q_{e^{-\frac{\gamma}{\cdot} l}})^{2k} & \sum_{k=0}^{\infty} (J Q_{e^{-\frac{\gamma}{\cdot} l}})^{2k+1} \\ \sum_{k=0}^{\infty} (J Q_{e^{-\frac{\gamma}{\cdot} l}})^{2k+1} & \sum_{k=0}^{\infty} (J Q_{e^{-\frac{\gamma}{\cdot} l}})^{2k} \end{pmatrix}. \end{aligned}$$

For $f \in Y$ the function $Q_{e^{-\frac{\gamma}{\cdot}l}} f$ vanishes on the same set as f, since these operators are multiplications by positive functions. Therefore we conclude, using (33) and (34), that

$$((Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N) u)_1|_{[v', v_{max}]} = 0$$

and, by the definition of D_γ, also

$$(D_\gamma (Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N) u)_1|_{[v', v_{max}]} = 0$$

holds. With these considerations it follows that

$$\begin{aligned} &(R(\gamma, A_N) u)_1|_{[v', v_{max}]} \\ =&(R(\gamma, A_0^N) u)_1 + D_\gamma (Id_{\partial X_N} - \Phi_N D_\gamma^N)^{-1} \Phi_N R(\gamma, A_0^N) u)_1|_{[v', v_{max}]} = 0. \end{aligned}$$

By Proposition A.11 the semigroup $(T_N(t))_{t \geq 0}$ is not irreducible.

5.2.2. Asymptotic behaviour. As our final result we describe the asymptotic behaviour of the solutions of $(\mathrm{ACP_N})$ from Chapter 2. If the semigroup is irreducible, then $\ker A_N$ is one-dimensional. If in addition the scattering operator is as in Theorem 2.3.10, then the semigroup converges strongly to the one-dimensional projection onto $\ker A_N$. This is shown in the next theorem.

THEOREM 5.2.5. *Let G be strongly connected. Then, under the assumptions of Theorem 2.3.10, the space X_N can be decomposed into the direct sum*

$$X_N = X_N^1 \oplus X_N^2$$

where $X_N^1 = \mathrm{fix}(T_N(t))_{t \geq 0} = \ker A_N$ is one-dimensional and spanned by a strictly positive eigenvector $u \in \ker A_N$ of A_N, $u \gg 0$, and $(T_N(t)|_{X_N^2})_{t \geq 0}$ is strongly stable.

PROOF. Note that the norm on X_N fulfills $\|u_1 + u_2\| = \|u_1\| + \|u_2\|, u_1, u_2 \in X_N$, and that the semigroup $(T_N(t))_{t \geq 0}$ is bounded. Moreover, by Theorem 2.3.10 we know that $\sigma(A_N) \cap i\mathbb{R} = \sigma_p(A_N) \cap i\mathbb{R} = \{0\}$. Hence, the assertion follows from Theorem 5.1.2. □

We reformulate the above theorem as our final result.

COROLLARY 5.2.6. *Under the conditions of Theorem 5.2.5 there exist $0 \ll \tilde{w} \in X_N$ and $0 \ll w' \in X_N'$ such that*

$$\lim_{t \to \infty} T_N(t) w = \langle w', w \rangle \tilde{w}$$

for all $w \in X_N$.

5.3. Queues

In this section the asymptotic behaviour of the semigroup $(T_Q(t))_{t\geq 0}$ from Chapter 3 is investigated using the results on positive semigroups collected in Section 5.1.

First we show the irreducibility of the semigroup via the representation of the resolvent of A_Q from Proposition 1.8 in terms of the resolvent of A_0^Q and the operators Φ_Q and D_γ^Q.

LEMMA 5.3.1. *The semigroup $(T_Q(t))_{t\geq 0}$ generated by $(A_Q, D(A_Q))$ is irreducible.*

PROOF. For the proof we show that the condition (ii) from Proposition A.11 is fulfilled. Therefore, recall the representation of the resolvent of A_Q

$$R(\gamma, A_Q) = R(\gamma, A_0^Q) + D_\gamma^Q(Id - \Phi_Q D_\gamma^Q)^{-1}\Phi_Q R(\gamma, A_0^Q)$$

for $\gamma \in \rho(A_0^Q)$ from Proposition 1.8. Let $\gamma > -\lambda$. From the representation of the resolvent of A_0^Q in Lemma 3.2.2 we see that $R(\gamma, A_0^Q)$ is a positive operator. We even have that $R(\gamma, A_0^Q)p > 0$ if $0 < p \in X_Q$. Moreover, it cannot occur that $(R(\gamma, A_0^Q)p)_k = 0$ and $(R(\gamma, A_0^Q)p)_i = 0$ for all $i > 2k$ simultaneously. Hence, also $\Phi_Q R(\gamma, A_0^Q)p > 0$. Since $\|\Phi_Q D_\gamma^Q\| < 1$, cf. the proof of Theorem 3.3.2, the inverse of $Id - \Phi_Q D_\gamma^Q$ is given by the Neumann series, i.e.

$$(Id - \Phi_Q D_\gamma^Q)^{-1} = \sum_{i=0}^{\infty} (\Phi_Q D_\gamma^Q)^i.$$

The special form of $\Phi_Q D_\gamma^Q$, see Corollary 3.2.7, yields that for each $j \in \mathbb{N}$ there exists $i_0 \in \mathbb{N}$ such that

$$((\Phi_Q D_\gamma^Q)^{i_0} q)_j > 0 \quad \text{if } q > 0.$$

Thus, $((Id - \Phi_Q D_\gamma^Q)^{-1}\Phi_Q R(\gamma, A_0^Q)p)_i > 0 \quad$ for all $i \in \mathbb{N}$. Then, also

$$D_\gamma^Q(Id - \Phi_Q D_\gamma^Q)^{-1}\Phi_Q R(\gamma, A_0^Q)p \gg 0.$$

Adding the positive element $R(\gamma, A_0^Q)p$ does not change the strict positivity. So we have indeed

$$R(\gamma, A_Q) \gg 0.$$

□

Now we are ready to prove our main result on the asymptotic behaviour. Combining Lemma 5.3.1 with the results from Section 5.1 we obtain the strong convergence of the semigroup to a one-dimensional equilibrium.

THEOREM 5.3.2. *The space X_Q can be decomposed into the direct sum*

$$X_Q = X_Q^1 \oplus X_Q^2$$

where $X_Q^1 = \mathrm{fix}(T_Q(t))_{t\geq 0} = \ker A_Q$ *is one-dimensional and spanned by a strictly positive eigenvector* $\tilde{p} \in \ker A_Q$ *of* A_Q*. The restricted semigroup* $(T_Q(t)|_{X_Q^2})_{t\geq 0}$ *is strongly stable.*

PROOF. From Theorem 3.3.2, Theorem 3.4.2 and Lemma 5.3.1 it is clear that the assumptions of Theorem 5.1.2 are fulfilled and hence the assertion follows. □

This theorem can be reformulated as follows.

COROLLARY 5.3.3. *There exist* $0 \ll \tilde{p} \in X_Q$ *and* $0 \ll p' \in X_Q'$ *such that*

$$\lim_{t\to\infty} T_R(t)p = \langle p', p\rangle \tilde{p}$$

for all $p \in X_Q$*.*

For our original problem with the initial condition $(\mathrm{IC_{Q,0}})$ this means the following.

COROLLARY 5.3.4. *The time-dependent solution of the system (Q),* (BC_Q) *and* $(IC_{Q,0})$ *converges strongly to the steady-state solution as time tends to infinity.*

5.4. A queueing network

Finally we study the asymptotic behaviour of the solutions of $(\mathrm{ACP_R})$. We know from Proposition 1.8 that the resolvent of A_R can be represented in terms of the resolvent of A_0^R, the Dirichlet operator D_γ^R and the boundary operator and we have computed this resolvent explicitly in Lemma 4.3.1. This representation shows that it is a positive operator for $\gamma > 0$. We need this property in the following lemma to prove the irreducibility of the generated semigroup.

LEMMA 5.4.1. *The semigroup* $(T_R(t))_{t\geq 0}$ *generated by* $(A_R, D(A_R))$ *is irreducible.*

PROOF. It suffices to show that there exists $\gamma > 0$ such that $0 < p \in X_R$ implies $R(\gamma, A_R)p \gg 0$, see Proposition A.11. By Proposition 1.8 we have to prove that there exists $\gamma > 0$ such that $0 < p \in X_R$ implies

$$R(\gamma, A_0^R)p + (Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1}\Phi_R R(\gamma, A_0^R)p \gg 0.$$

Suppose that $\gamma > 0$ and $0 < p \in X_R$. Then also $R(\gamma, A_0^R)p > 0$ and $\Phi_R R(\gamma, A_0^R)p > 0$. Since $\|\Phi_R D_\gamma^R\| < 1$ for any $\gamma > 0$, the inverse of $Id_{\partial X_R} - \Phi_R D_\gamma^R$ is given by the Neumann series

$$(Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1} = \sum_{n=0}^{\infty} (\Phi_R D_\gamma^R)^n.$$

We know from the form of $\Phi_R D_\gamma^R$ that for every $i \in \{1, \ldots, N+2\}$ there exists $k \in \mathbb{N}$ such that the real number $((\Phi_R D_\gamma^R)^k \Phi_R R(\gamma, A_0^R)p)_i > 0$, i.e.,

$$(Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1}\Phi_R R(\gamma, A_0^R)p \gg 0,$$

and by the form of D_γ^R we have

$$D_\gamma^R(Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1}\Phi_R R(\gamma, A_0^R)p \gg 0.$$

This implies

$$R(\gamma, A_R)p \gg 0.$$

Therefore the semigroup $(T_R(t))_{t\geq 0}$ is irreducible. □

We can now show the convergence of the semigroup to a one-dimensional equilibrium point.

THEOREM 5.4.2. *The space X_R can be decomposed into the direct sum*

$$X_R = X_R^1 \oplus X_R^2$$

where $X_R^1 = \mathrm{fix}(T_R(t))_{t\geq 0} = \ker A_R$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A_R$ of A_R. In addition, the restriction $(T_R(t)|_{X_R^2})_{t\geq 0}$ is strongly stable.

PROOF. A combination of Theorem 4.2.2, Lemma 4.3.6, Theorem 4.3.7 and Lemma 5.4.1, Theorem 5.1.2 gives the proof of the theorem. □

We now reformulate the above theorem in the following way.

COROLLARY 5.4.3. *There exist $0 \ll \tilde{p} \in X_R$ and $0 \ll p' \in X_R'$ such that*

$$\lim_{t\to\infty} T_R(t)p = \langle p', p\rangle \tilde{p}$$

for all $p \in X_R$.

Since the semigroup gives the solutions of the original system, we obtain for $(\mathrm{IC}_{\mathrm{R},0})$ the following consequence.

COROLLARY 5.4.4. *The time-dependent solution of the system (R), (BC_R) and ($IC_{R,0}$) converges strongly to the steady-state solution as time tends to infinity.*

APPENDIX A

Positive operators

In this thesis we use the theory of positive operators on Banach lattices. Therefore, we collect the basic definitions and properties needed in our situation. We refer to [Sch74] and [MN91] for an exhaustive treatment of this theory and to [Nag86] for the corresponding semigroup theory.

We start by defining an order relation on vector spaces. Recall that a relation $\geq$ is said to be an *order relation* if it is reflexive, antisymmetric and transitive.

DEFINITION A.1.

(i) A real vector space E is an *ordered vector space* if there is an order relation $\geq$ defined on E such that for $f, g \in E$

$$f \geq g \quad \Rightarrow \quad f + h \geq g + h \quad \text{ for all } h \in E,$$
$$f \geq g \quad \Rightarrow \quad \alpha f \geq \alpha g \quad \text{ for all } \alpha \geq 0.$$

(ii) A *vector lattice* is an ordered vector space such that for any two elements $f, g \in E$ the supremum (i.e. least upper bound)

$$\sup\{f, g\}$$

and infimum (i.e. greatest lower bound)

$$\inf\{f, g\}$$

exists.

Clearly, the notation $g \leq f$ means that $f \geq g$. Moreover, $f > 0$ means that $f \geq 0$ and $f \neq 0$. If $g \leq f$, then the set

$$[g, f] := \{h \in E \ : \ g \leq h \leq f\}$$

is called *order interval.* The *positive cone* E_+ of an ordered vector space E is

$$E_+ = \{f \in E \ : \ f \geq 0\}.$$

If $f \in E_+$, then we say that f is *positive.* If E is a vector lattice, the *positive part* of $f \in E$ is

$$f^+ := \sup\{f, 0\},$$

and the *negative part* of f is

$$f^- := \sup\{-f, 0\},$$

while the *absolute value* or *modulus* of f is

$$|f| := \sup\{f, -f\}.$$

Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

On a vector lattice E we now consider a norm that is compatible with the order.

DEFINITION A.2.

(i) A norm $\|\cdot\|$ on a vector lattice E is a *lattice norm* if

$$|f| \leq |g| \quad \Rightarrow \quad \|f\| \leq \|g\|.$$

(ii) A *normed vector lattice* is a vector lattice endowed with a lattice norm.

(iii) A *Banach lattice* is a normed vector lattice that is complete.

We now extend the concept of a Banach lattice to complex vector spaces and call the complexification

$$E_{\mathbb{C}} := E \times iE$$

with scalar multiplication

$$(\alpha + i\beta)(f, g) = (\alpha f - \beta g, \beta f + \alpha g) \quad \text{for} \quad \alpha, \beta \in \mathbb{R}$$

a *complex Banach lattice*. The space E is the *real part* of $E_{\mathbb{C}}$. For $f, g \in E_{\mathbb{C}}$ we write $f \geq g$ if $f, g \in E$ and if $f \geq g$ holds. The modulus of $(f, g) \in E_{\mathbb{C}}$ is

$$|(f, g)| := \sup_{0 \leq \phi < 2\pi} |(\cos\phi)f + (\sin\phi)g|.$$

It can be shown that the modulus indeed exists, see [Nag86, Sect. C-I 7]. Moreover, the norm on $E_{\mathbb{C}}$ is defined by

$$\|(f, g)\|_{E_{\mathbb{C}}} := \|\, |(f, g)|\, \|_E.$$

The spaces $\mathbb{C}$ and $L^1_{\mathbb{C}}(\Omega, \mu)$ are complex Banach lattices. The underlying real vector lattices are $\mathbb{R}$ endowed with the usual order and $L^1_{\mathbb{R}}(\Omega, \mu)$ endowed with the order

$$f \geq g \quad \text{if} \quad f(x) \geq g(x) \quad \text{for almost all} \quad x \in \Omega.$$

Moreover, for $f \in L^1_{\mathbb{C}}(\Omega, \mu)$ the modulus is

$$|f|(x) = |f(x)|, \quad x \in \Omega.$$

In this thesis, spaces like

$$\mathbb{C}^n \times l^1(L^1_{\mathbb{C}}(\Omega, \mu))$$

occur. They are complex Banach lattices with underlying real spaces

$$\mathbb{R}^n \times l^1(L^1_{\mathbb{R}}(\Omega, \mu)).$$

Their order is given by

$$(f_i)_{i\in\mathbb{N}} \geq (g_i)_{i\in\mathbb{N}} \quad \text{if} \quad f_i \geq g_i \quad \text{for all} \quad i \in \mathbb{N}.$$

The modulus of $(f_i)_{i\in\mathbb{N}} \in \mathbb{C}^n \times l^1(L^1_{\mathbb{C}}(\Omega,\mu))$ is

$$|(f_i)_{i\in\mathbb{N}}| = (|f_i|)_{i\in\mathbb{N}}.$$

We now turn our attention to operators on these spaces. Operators which respect the order structure are called positive.

DEFINITION A.3. Let E be a real Banach lattice.

(i) A linear operator T on E is *positive* (in symbols, $T \geq 0$) if

$$Tf \geq 0 \quad \text{for all} \quad f \geq 0.$$

(ii) A linear operator T on E is called *strictly positive* (in symbols, $T \gg 0$) if

$$Tf > 0 \quad \text{for all} \quad f > 0.$$

(iii) A strongly continuous semigroup $(S(t))_{t\geq 0}$ on E is *positive* if $S(t) \geq 0$ for all $t \geq 0$.

This definition can be extended to operators on complex vector lattices which map the underlying real part into the real part. In this case, positivity or strict positivity means that the restriction of the operator to the real part is positive or strictly positive, respectively.

Note that for a positive operator T on a vector lattice E the inequality

$$|Tf| \;\leq\; T|f|$$

holds for all $f \in E$, see [Sch74, p. 58].

Next, we are interested in generators of positive semigroups. Therefore, we give the following definition from [Nag86, p. 249].

DEFINITION A.4. A linear operator $(B, D(B))$ on a real Banach lattice E is called *dispersive* if for every $z \in D(B)$ there exists a $\chi \in E'_+$ such that $\|\chi\| \leq 1$, $\langle z, \chi\rangle = \|z^+\|$ and $\Re\langle Bz, \chi\rangle \leq 0$.

Now, the generator property for generators of positive contraction semigroups is characterised as follows, see [Nag86, Thm. C-II 1.2]

THEOREM A.5 (Phillips' theorem). Let B be a densely defined operator on a real Banach lattice E. The following assertions are equivalent.

(i) B is the generator of a positive contraction semigroup.
(ii) B is dispersive and $\gamma - B$ is surjective for some $\gamma > 0$.

The following subspaces play an important role in the theory of positive operators.

DEFINITION A.6. An *ideal* in a real or complex Banach lattice E is a linear subspace F such that

$$f \in F,\ |g| \leq |f| \quad \Rightarrow \quad g \in F.$$

REMARK A.7.

(i) The ideals in $\mathbb{C}^n$ are the subspaces
$$J_H := \{x = (x_i)_{1\le i\le n} \in \mathbb{C}^n \;:\; x_i = 0 \text{ for } i \in H\}$$
where H is an arbitrary subset of $\{1,\dots,n\}$, see [Sch74, p. 2].

(ii) Every closed *ideal* in $L^1_{\mathbb{C}}(\Omega,\mu)$ is of the form
$$I_M = \{f \in E \;:\; f(x) = 0 \text{ for almost all } x \in \Omega\}$$
where M is a measurable subset of Ω. Conversely, every set I_M is a closed ideal in $L^1_{\mathbb{C}}(\Omega,\mu)$, see [Sch74, Example III.1.2].

The ideal E_f generated by $f \in E_+$ is the smallest ideal containing f. By [Sch74, Example II.2.1] the equality
$$E_f = \bigcup_{n\ge 1} n[-f,f]$$
holds.

DEFINITION A.8. Let $f \in E_+$. If E_f is dense in E, then we call f a *quasi-interior* point.

REMARK A.9. A function $f \in L^1_{\mathbb{C}}(\Omega,\mu)$ is a quasi-interior point if and only if $f(x) > 0$ for almost all $x \in \Omega$. In this case we write $f \gg 0$.

Irreducibility of the semigroup is a key property in order to determine the asymptotic behaviour. We briefly recall the basic definitions.

DEFINITION A.10.

(i) A positive linear operator B on E is called *irreducible* if there is no closed non-trivial ideal in E which is invariant under B.

(ii) A positive semigroup $(S(t))_{t\ge 0}$ on E is called *irreducible* if there is no closed non-trivial ideal in E which is invariant under $(S(t))_{t\ge 0}$.

The irreducibility of our semigroups on E can be characterised in the following way, cf. [Nag86, Def. C-III 3.1].

PROPOSITION A.11. Let B be the generator of a positive semigroup $(S(t))_{t\ge 0}$. The following assertions are equivalent.

(i) The semigroup $(S(t))_{t\ge 0}$ on E is irreducible.

(ii) If $f \in E$ and $f \gneq 0$, then $R(\gamma, B)f \gg 0$ for (some) all $\gamma > \mathrm{s}(B)$.

Table of symbols

$D(A)$	domain of A
$\mathrm{fix}(S(t))_{t\geq 0}$	fixed space of the semigroup $(S(t))_{t\geq 0}$
$\ker T$	kernel of T
$\mathcal{L}(X)$	space of bounded linear operators on X
$L^1(\Omega, \mu)$	space of complex valued integrable functions on Ω with respect to μ
$L^1_{\mathbb{C}}(\Omega, \mu)$	space of complex valued integrable functions on Ω with respect to μ
$L^1_{\mathbb{R}}(\Omega, \mu)$	space of real valued integrable functions on Ω with respect to μ
$\Re z$	real part of z
$\mathrm{r}(T)$	spectral radius of T
$\mathrm{rg}(T)$	range of T
$\rho(A)$	resolvent set of A
$R(\gamma, A)$	resolvent of A in γ
$\mathrm{s}(A)$	spectral bound of A
$\sigma(A)$	spectrum of A
$\sigma_b(A)$	boundary spectrum of A
$\sigma_p(A)$	point spectrum of A
$\sigma_r(A)$	residual spectrum of A

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Zusammenfassung in deutscher Sprache

In dieser Arbeit wird ein Transportproblem auf Netzwerken und zwei Probleme aus der Theorie der Warteschlangen mit halbgruppentheoretischen Methoden behandelt.

Im ersten Kapitel wird der abstrakte Rahmen, siehe [Gre87], erläutert, in den alle drei Problemstellungen passen.

Das zweite Kapitel befasst sich mit einem Transport-Problem auf Netzwerken, welches eine Verallgemeinerung der Fragestellung in [KS05] ist. Das Netzwerk wird durch einen gewichteten und gerichteten Graphen dargestellt. Wir nehmen an, dass Teilchen zwischen den Knoten fließen können, falls sie durch eine Kante verbunden sind. Einzelne Teilchen bewegen sich mit konstanter Geschwindigkeit, jedoch können unterschiedliche Teilchen unterschiedliche Geschwindigkeiten haben. Wenn die Teilchen einen Knoten passieren, werden sie auf die ausgehenden Kanten entsprechend der Kantengewichte verteilt. In den Knoten werden die Teilchen gestreut, das heißt, sie ändern ihre Geschwindigkeit, und es gilt ein Kirchhoffsches Gesetz.

Nachdem wir dieses Problem als abstraktes Cauchy-Problem auf einem geeigneten Banachraum umgeschrieben haben, wird die Wohlgestelltheit bewiesen. Außerdem wird das Spektrum des zugehörigen Generators näher untersucht. Falls der Streuoperator in den Knoten ein kompakter Integraloperator ist, so dass bei der Streuung die Teilchenzahl erhalten bleibt, so ist das Spektrum des zugehörigen Generators ein reines Punktspektrum, und das Randspektrum besteht nur aus 0.

In den nächsten beiden Kapiteln diskutieren wir Probleme aus der Warteschlangentheorie.

Kapitel 3 beschäftigt sich mit dem $M/M^{k,B}/1$ Warteschlangenmodell. Hierbei gibt es einen Server, der B Kunden gleichzeitig bedienen kann. Der Server fängt jedoch erst an zu arbeiten, sobald sich k Kunden in der Warteschlange befinden. Die Kunden treffen zufällig ein. Die Zeitabstände zwischen der Ankunft zweier Kunden sind ebenso wie die Servicezeiten exponentiell verteilt.

Kapitel 4 behandelt ein einfaches Warteschlangen-Netzwerk bestehend aus zwei Servern, die durch einen Puffer endlicher Kapazität getrennt sind. Die Kunden müssen von beiden Servern bedient werden. Sie können das Netzwerk nur am ersten Server betreten und nur vom zweiten Server aus verlassen. Falls der erste

Server einen Prozess an einem Kunden beendet, muss der Kunde mit einer gewissen Wahrscheinlichkeit erneut zum ersten Server oder in den Puffer. Dort bleibt er, bis der zweite Server frei ist und die vor ihm im Puffer wartenden Kunden bedient worden sind. Nachdem der Kunde vom zweiten Server bedient worden ist, muss er entweder erneut vom zweiten Server bedient werden, oder er verlässt das Netzwerk. Für beide Fälle sind wiederum gewisse Wahrscheinlichkeiten angegeben. Falls der Puffer voll ist, arbeitet der erste Server nicht mehr, bis ein Kunde den Puffer verlässt.

Diese beiden Fragestellungen aus der Warteschlangentheorie werden in Form eines abstrakten Cauchy-Problems formuliert. Anschließend befassen wir uns mit der Wohlgestelltheit der Probleme und untersuchen das Spektrum des jeweiligen Generators. In beiden Fällen is 0 der einzige Eigen- und Spektralwert auf der imaginären Achse.

Abschließend wird im fünften Kapitel das asymptotische Verhalten der Lösungen besprochen. Wir zeigen, dass die zu dem Transportproblem gehörende Halbgruppe unter gewissen Bedingungen an den Graphen sowie an den Streuoperator irreduzibel ist. Wir zeigen, dass die Halbgruppe nicht notwendigerweise irreduzibel ist, falls eine dieser Bedingungen nicht erfüllt ist. Außerdem beweisen wir die Irreduzibilität der Halbgruppen von den Warteschlangenproblemen. In allen oben beschriebenen Fällen wird die Konvergenz gegen ein eindimensionales Gleichgewicht gezeigt.

Im Anhang stellen wir die in dieser Arbeit benötigten Resultate über positive Operatoren und positive Halbgruppen auf Banachverbänden zusammen.

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