McKay correspondence and G-Hilbert schemes

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Introduction

The observation of McKay, published in 1980, relates exceptional curves in the minimal resolution of quotient singularities $\mathbb{A}_{\mathbb{C}}^2/G$ for finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ to the representation theory of the group G:

Observation 0.1. (*"Classical McKay correspondence"*, [McK80]).
There is a bijection between the set of irreducible components of the exceptional divisor E and the set of isomorphism classes of nontrivial irreducible representations of the group G and moreover an isomorphism of graphs between the intersection graph of components of E_{red} and the representation graph of G, both being graphs of ADE type.

Here the intersection resp. representation graph contains information about the configuration of the exceptional curves resp. the decomposition of tensor products of irreducible representations with the given 2-dimensional representation.

Subsequently, to explain this observation, there have been considered several approaches, further this theme has undergone numerous variations and considerable extensions. We try to formulate the fundamental idea, more detailed expositions of this field of research are to be found in [Re97], [Re99].

Let G be a finite group of automorphisms of a smooth variety M over $\mathbb{C}$, for instance $M = \mathbb{A}_{\mathbb{C}}^n$ with a linear operation of a finite subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$. Usually the quotient M/G is singular and one considers resolutions of singularities $Y \to M/G$ with some minimality properties (in dimension 2 there is a minimal resolution unique up to isomorphism, in higher dimensions one has the notion of a crepant resolution). The McKay correspondence in general describes the resolution Y in terms of the representation theory of the group G, the following principle was formulated by Reid:

Principle 0.2. ([Re99, Principle 1.1]). *The answer to any well posed question about the geometry of Y is the G-equivariant geometry of M.*

Realisations of this principle are an isomorphism $K(Y) \cong K^G(M)$ between the K-theory of Y and the G-equivariant K-theory of M ("K-theoretic McKay correspondence") or an equivalence $D(Y) \simeq D^G(M)$ between the derived category of Y and the G-equivariant derived category of M ("derived McKay correspondence").

A method to construct resolutions of quotient singularities is the G-Hilbert scheme G-Hilb$_{\mathbb{C}}\, M$. It parametrises G-clusters, these are G-stable finite closed subschemes $Z \subseteq M$, whose coordinate ring as a representation over $\mathbb{C}$ is isomorphic to the regular representation of G. A free G-orbit is a G-cluster, but in the case of nontrivial stabiliser there might be many nonreduced G-clusters supported by the same orbit, for example for $G \subset \mathrm{SL}(2,\mathbb{C})$ the exceptional divisor $E \subset$ G-Hilb$_{\mathbb{C}}\, \mathbb{A}_{\mathbb{C}}^2$ consists of G-clusters supported by the origin.

For $G \subset \mathrm{SL}(2,\mathbb{C})$ the G-Hilbert scheme G-Hilb$_{\mathbb{C}}\, \mathbb{A}_{\mathbb{C}}^2$ is the minimal resolution of the quotient singularity $\mathbb{A}_{\mathbb{C}}^2/G$. Similar statements are true in dimension 3, in particular for finite subgroups $G \subset \mathrm{SL}(3,\mathbb{C})$ the G-Hilbert scheme is irreducible, nonsingular and G-Hilb$_{\mathbb{C}}\, \mathbb{A}_{\mathbb{C}}^3 \to \mathbb{A}_{\mathbb{C}}^3/G$ a crepant resolution, but all of this may fail in higher dimensions.

The main new results of this thesis are generalisations of the McKay correspondence as well as extensions and improvements in the construction of G-Hilbert schemes. These results are contained in the papers [Bl06a], [Bl06b].

For the first time we consider McKay correspondence over fields that are not necessarily algebraically closed and for finite group schemes instead of simply finite groups. Let $G \subset \mathrm{SL}(2, K)$ be a finite subgroup scheme over a field K of characteristic 0. Over non algebraically closed K there may exist both representations of G and components of the exceptional divisor E that are irreducible over K but split over the algebraic closure. We will see that these two kinds of splittings that arise by extending the ground field are related and we will formulate a McKay correspondence over arbitrary fields K of characteristic 0 relating nontrivial irreducible representations to exceptional prime divisors. In particular the scheme structure of the group scheme G is linked to the scheme structure of the exceptional fiber E. The following will be consequence of more detailed theorems in chapter 6:

Theorem 0.3. *Let K be any field of characteristic 0 and $G \subset \mathrm{SL}(2, K)$ a finite subgroup scheme. Then there is a bijection between the set of irreducible components of the exceptional divisor E and the set of isomorphism classes of nontrivial irreducible representations of G and moreover an isomorphism between the intersection graph of E_{red} and the representation graph of G.*

As preparation for the constructions concerning McKay correspondence and G-Hilbert schemes, some theory of G-equivariant sheaves for group schemes G has to be developed, this is done in chapter 3.

With the aim to generalise the McKay correspondence, we generalise the G-Hilbert scheme construction to finite group schemes. Working with group schemes, things have to be formulated in a strict functorial language, further, properties of G-equivariant sheaves for group schemes G have to be used. In chapter 4 we arrive at the following theorem:

Theorem 0.4. *Let $G = \mathrm{Spec}\,A$ be a finite group scheme over a field K with A cosemisimple. Let X be a G-scheme algebraic over K and assume that a geometric quotient $\pi : X \to X/G$, π affine, of X by G exists. Then the G-Hilbert functor $\underline{\mathrm{G\text{-}Hilb}}_K X$ is represented by an algebraic K-scheme $\mathrm{G\text{-}Hilb}_K X$ and the natural morphism $\tau : \mathrm{G\text{-}Hilb}_K X \to X/G$ is projective.*

Apart from the generalisation to group schemes with cosemisimple Hopf algebra over arbitrary fields we have the following extensions and simplifications in the construction of G-Hilbert schemes: We introduce relative G-Hilbert schemes associated to a scheme with G-operation over another scheme and vary the base scheme of G-Hilbert schemes. This allows to construct the G-Hilbert scheme without using the Hilbert scheme of n points. This new construction works under more natural hypotheses, further, it gives additional information about the morphism from the G-Hilbert scheme to the quotient, which is interpreted as the structure morphism of a relative G-Hilbert scheme.

As an application it is possible to calculate relative tangent spaces of the G-Hilbert scheme over the quotient, these are related to a certain stratification of the G-Hilbert scheme considered in works on the McKay correspondence.

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This thesis is organised as follows:

Part I consists of two expository chapters. In the first chapter we introduce the main themes of this thesis, the McKay correspondence and G-Hilbert schemes. We state the observation of McKay, constructions such as the stratification of the G-Hilbert scheme and the tautological sheaves, we shortly discuss extensions such as K-theoretic and derived McKay correspondence, but we restrict to the case of finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$. The second chapter contains general theory about quotient singularities, about the description of the G-Hilbert scheme for abelian G as a toric variety and other topics as well as several examples for this theory.

Part II about G-sheaves and G-Hilbert schemes forms the technical core of this thesis. In chapter 3 we develop the theory of G-equivariant sheaves for group schemes G needed for the constructions concerning G-Hilbert schemes and McKay correspondence in this thesis. In chapter 4 we review, extend and simplify the construction of G-Hilbert schemes. We construct the G-Hilbert scheme for finite group schemes G and introduce relative G-Hilbert schemes, we find a construction for the G-Hilbert scheme that does not need the Hilbert scheme of n points and works under more natural assumptions. Further, we obtain additional information about the morphism from the G-Hilbert scheme to the quotient.

Part III contains results about McKay correspondence over non algebraically closed fields. Chapter 5 treats the relations, both for representations and components of schemes, between operation of the Galois group and irreducibility with respect to a Galois extension of the ground field, introduces Galois conjugate G-sheaves and describes the Galois operation on the G-Hilbert scheme. In the last chapter we come back to the situation of the first chapter and extend the classical McKay correspondence for finite groups $G \subset \mathrm{SL}(2, \mathbb{C})$ to finite group schemes $G \subset \mathrm{SL}(2, K)$ over fields K of characteristic 0 that are not necessarily algebraically closed.

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We describe the results in this thesis in more detail. More thematically oriented information about the contents can be found in the introductions to the individual chapters, here we concentrate on pointing out the new contributions, the main ones are to be found in chapters 4 and 6:

Chapter 1 (McKay correspondence for $G \subset \mathrm{SL}(2, \mathbb{C})$) and chapter 2 (Quotient singularities and G-Hilbert schemes of higher dimension) are expository and form an introduction to some aspects of McKay correspondence and G-Hilbert schemes. Original are only the presentation, some reformulations and some worked out examples, to mention are example 2.31 and the example in section 2.3.

Chapter 3 (G-sheaves) is a technical chapter in which we develop some theory of G-equivariant sheaves on schemes with an operation of a group scheme G.
Although, at least in the case that G is a finite group, mostly these things might be regarded as common knowledge, there is no suitable reference. Further, for group schemes G instead of simply finite groups things become technically considerably more demanding. Here they are carried out, because they are indispensable for what follows.
The constructions of functors for G-equivariant sheaves, the adjunctions and natural homomorphisms of section 3.3 in this generality can not be found in the literature.
In section 3.4 we treat G-sheaves in the case of trivial G-operation on the underlying scheme and decompositions into isotypic components by using Hopf algebras and comodules and as well achieve a new and more general treatment of this topic.

Chapter 4 (G-Hilbert schemes) develops the theory of G-Hilbert schemes.

We review known constructions, the G-Hilbert scheme as a closed subscheme of the Hilbert scheme of n points and the morphism to the quotient as introduced in [ItNm96], [ItNm99], and provide some proofs that can not be found or remain unclear in the literature, compare also to [Té04]. In doing so we put things in a strict functorial setting which is the adequate language to formulate these things. This allows to extend the construction from the case of finite groups over $\mathbb{C}$ to finite group schemes with cosemisimple Hopf algebra over arbitrary fields. Moreover, relative G-Hilbert schemes are introduced und used to improve the construction. Although the main ideas can be carried over to the more general situation, it requires substantial work and some technical preparations, in particular results of chapter 3 (G-sheaves) are needed.

We consider ways to change the base scheme of G-Hilbert schemes. We show that the G-Hilbert functor can be regarded as a relative G-Hilbert functor over the quotient (corollary 4.21). This allows to construct the G-Hilbert scheme as a scheme over the quotient and gives a description of the morphism G-Hilb $X \to X/G$ as structure morphism of a relative G-Hilbert scheme with respect to the morphism $X \to X/G$. The result is formulated in theorem 4.28. In particular, this new construction does not require the existence of the Hilbert scheme of n points and works without unnecessary and unnatural assumptions on quasiprojectivity of X. These ideas are completely new.

This new construction gives additional information about the morphism G-Hilb $X \to X/G$, for example one obtains a description of fibers of this morphism as G-Hilbert schemes (remark 4.23). Further, relative tangent spaces can be calculated. One knows that the methods of differential study of Quot schemes can be applied to G-Quot schemes by considering G-equivariant sheaves and their deformations instead of ordinary sheaves, our treatment (main result theorem 4.38, applied to G-Hilbert schemes in corollaries 4.41 and 4.43) is inspired and generalises [Gr61, section 5], [HL, Ch. 2.2]. These relative tangent spaces are related to the stratification considered in [ItNm96], [ItNm99], which is an observation not occuring in the literature.

Chapter 5 (Galois operation and irreducibility) considers Galois operation on schemes, G-sheaves and representations and treats the relations between operation of the Galois group and irreducibility with respect to a Galois extension of the ground field.

We introduce Galois conjugate G-sheaves for group schemes G generalising Galois conjugate representations for finite groups (Galois conjugate representations occur for example in [CR, Vol. I, §7B]). We show that for a Galois extension $K \to L$ irreducible components of schemes resp. of isotypic components of G-sheaves over K correspond to Galois orbits (proposition 5.10 resp. corollary 5.23). At least in some cases this seems to be known: concerning components of schemes for example there is a remark in [EH, II.2], for representations of finite groups see e.g. [CR, Vol II, §74], however, new apart from the generality is that we trace back both to the simple statement of Galois descent for vector spaces (proposition 5.6).

Further, the operation of the Galois group on the G-Hilbert scheme is determined.

Chapter 6 (McKay correspondence over non algebraically closed fields) contains an extension of the classical McKay correspondence to finite subgroup schemes $G \subset \mathrm{SL}(2, K)$ over not necessarily algebraically closed fields K of characteristic 0 (main theorems in subsection 6.2.2).

McKay correspondence over non algebraically closed ground fields and the relations between irreducibility of representations and components of the exceptional divisor with respect to field extensions have not been considered before. Further, it is completely new to consider finite group schemes instead of only finite groups and to relate their scheme structure to the scheme structure of the exceptional divisor.

ADE singularities over non algebraically closed fields and the configuration of exceptional curves have been studied in [Li69], but using entirely different methods and not with regard to the representation theory of the corresponding group scheme. There these singularities occur as defined by equations and not as quotients by finite group schemes.

As another new result we obtain criteria for finite subgroups $G \subset \mathrm{SL}(2, C)$, C algebraically closed of characteristic 0, to be realisable as subgroups of $\mathrm{SL}(2, K)$ for subfields $K \subseteq C$ (theorem 6.10).

We consider several examples for McKay correspondence over non algebraically closed fields of characteristic 0.

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Notations and references

This thesis is made of mainly two basic ingredients, representation theory and algebraic geometry, taken from the following sources.
- For the representation theory of finite groups see [Se, LR], [FH], also [CR]. However, the representation theory part here appears reformulated in terms of semisimple algebras [Bour, Algèbre Ch. 8], coalgebras and comodules [Sw], [Abe], to take into account the more general situation of group schemes instead of finite groups. G-equivariant sheaves, introduced as in [Mu, GIT], are used as generalisation of representations.
- As general references for algebraic geometry, sheaves and schemes we have used [EGA], [EGA1], also [Ha, AG], [EH].
Standard results from these sources we will use, sometimes implicitly. Standard notations we mostly have taken over, for clarification the following remarks.

Miscellaneous.
- We denote by $\mathrm{Mod}(X), \mathrm{Qcoh}(X), \mathrm{Coh}(X)$ the categories of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.
- We write $\mathbb{A}_X(\mathscr{F})$ for the affine X-scheme $\mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X} \mathscr{F})$ (denoted by $\mathbf{V}(\mathscr{F})$ in [EGA]).
- We usually use $\mathrm{Hom}_X(\cdot, \cdot)$ for $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \cdot)$ (if no other sheaves of algebras than $\mathcal{O}_X$ on the space X are considered).
- Sometimes we use implicitly canonical isomorphisms like $f^* g^* \mathscr{F} \cong (gf)^* \mathscr{F}$ and write "=".

Functors of points and T-valued points.
- Sometimes we will underline a functor to distinguish it in case of representability from the corresponding scheme, in general the functor corresponding to a scheme will be denoted by the same symbol underlined.
- T-valued points: we sometimes identify $X(T)$ with $X_T(T) = \Gamma(X_T/T)$. For a T-valued point $x \in X(T)$ and a T-scheme $t : T' \to T$ we write $x_{T'}$ for the morphism $T' \to X_{T'}$ as well as for $x \circ t : T' \to X$.

Base extension and restriction.
- In general we write a lower index for base extensions, for example if X, T are S-schemes then X_T denotes the T-scheme $X \times_S T$ or if V is a representation over a field K then V_L denotes the representation $V \otimes_K L$ over the extension field L.
- Likewise for morphisms of schemes: If $\varphi : X \to Y$ is a morphism of S-schemes and $T \to S$ an S-scheme, then write φ_T for the morphism $\varphi \times id_T : X_T \to Y_T$ of T-schemes.
- Let $\psi : T \to S$ be a morphism and $\mathscr{F}$ an $\mathcal{O}_S$-module. Then write $\mathscr{F}_T$ for the $\mathcal{O}_T$-module $\psi^* \mathscr{F}$ and put $\varphi_T = \psi^* \varphi : \mathscr{F}_T \to \mathscr{G}_T$, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_S$-modules.
- More generally let X be an S-scheme, $\mathscr{F}$ an $\mathcal{O}_X$-module and let $\psi : T \to S$ a morphism. Then write $\mathscr{F}_T$ for the $\mathcal{O}_{X_T}$-module $\psi_X^* \mathscr{F}$ and φ_T for $\psi_X^* \varphi : \mathscr{F}_T \to \mathscr{G}_T$, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules.
- Base restriction will be denoted by a left lower index, e.g. if $A \to B$ is a ring homomorphism and M a B-module, then write $_A M$ for M considered as an A-module.

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Part I

Quotient singularities and McKay correspondence

Chapter 1

McKay correspondence for $G \subset \mathrm{SL}(2, \mathbb{C})$

This chapter is expository and introductory, we state the observation of McKay and discuss approaches to explain the McKay correspondence in the original situation for finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. Note that in the following the topics are not always ordered by their historical succession and as well not always grouped by the papers in which they appeared.

The singularities that arise as a quotient of $\mathbb{A}^2_{\mathbb{C}}$ by a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ are the ADE singularities which are well known, classified and have been extensively studied. In the first section we summarise some information about these quotient singularities and their resolutions. We then describe the observation of McKay: It relates the exceptional prime divisors in the minimal resolution $Y \to \mathbb{A}^2_{\mathbb{C}}/G$ as well as their configuration to the representation theory of the group G. This is what will be called "classical McKay correspondence" later.

The second section describes the method of G-Hilbert schemes and discusses approaches to explain the classical McKay correspondence.

The minimal resolution of the ADE singularities $\mathbb{A}^2_{\mathbb{C}}/G$ can be constructed as a G-Hilbert scheme $\text{G-Hilb}_{\mathbb{C}} \, \mathbb{A}^2_{\mathbb{C}}$ that parametrises G-stable finite closed subschemes of $\mathbb{A}^2_{\mathbb{C}}$ with coordinate ring isomorphic to the regular representation.

There are several ways to relate nontrivial irreducible representations to exceptional prime divisors, we describe two of them, the original proofs of both rely on case by case investigations and the classification of finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$:

- It is possible to define a stratification of the minimal resolution $\text{G-Hilb}_{\mathbb{C}} \, \mathbb{A}^2_{\mathbb{C}}$ by attaching a representation to any point of the G-Hilbert scheme. The nontrivial irreducible representations of G then correspond to the 1-dimensional strata and the closures of these are the exceptional prime divisors.

- For any irreducible representation there can be defined a vector bundle $\mathscr{F}_i$ on the minimal resolution (here in principle it is not necessary to have constructed the minimal resolution as a G-Hilbert scheme). These sheaves are called the tautological sheaves. Its highest exterior powers are invertible sheaves which are dual to the exceptional prime divisors with respect to the intersection form.

In the third section we summarise results of K-theoretic and derived McKay correspondence. These extend the original observation, they relate the Grothendieck group resp. derived category of sheaves on the minimal resolution to that of G-equivariant sheaves on $\mathbb{A}^2_{\mathbb{C}}$.

1.1 ADE singularities and the observation of McKay

In this section we introduce the ADE singularities and collect some basic properties of them and their resolutions. We then describe the observation of McKay [McK80].

1.1.1 ADE singularities

Let G be a nontrivial finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. The quotient $X := \mathbb{A}^2_{\mathbb{C}}/G$ has an isolated singularity at the point corresponding to the origin. The singularities obtained this way are the ADE singularities (this term derives from the fact that the exceptional curves in the minimal resolution form Dynkin diagrams of type ADE as explained below), also called Kleinian singularities, Du Val singularities or rational double points and were studied extensively. They have the property that they can be embedded into $\mathbb{A}^3_{\mathbb{C}}$ (equations are listed below), the tangent space of the singular point is 3-dimensional.

Resolutions of singularities and the minimal resolution. A desingularisation or resolution of singularities is a proper birational morphism $Y \to X$ with Y nonsingular. In dimension 2 singularities can be resolved by a sequence of monoidal transformations (i.e. blow-ups of closed points) and there exists a minimal resolution in the sense that any other resolution factors through it. It is also characterised by the property that it has no exceptional (-1)-curves since such curves can be contracted.
The surfaces $X = \mathbb{A}^2_{\mathbb{C}}/G$ are rational (then their resolutions as well), since the extensions of function fields $\mathbb{C}(\mathbb{A}^2_{\mathbb{C}})/\mathbb{C}(X)$ are finite separable [Ha, AG, Remark V.6.2.1].

Construction and explicit description of minimal resolutions of ADE singularities.
- In the abelian case the minimal resolution can be constructed by toric methods. This is also possible for any of these quotient singularities: Any nonabelian finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ is a central extension $0 \to \mathbb{Z}/2\mathbb{Z} \to G \to \overline{G} \to 0$, the factor group $\overline{G}$ operates on the minimal resolution of $\mathbb{A}^2_{\mathbb{C}}/(\mathbb{Z}/2\mathbb{Z})$ such that the stabilisors are abelian.
- In [GV83, Section 6] (see also references therein) explicit descriptions of the minimal resolutions as glueings of copies of $\mathbb{A}^2_{\mathbb{C}}$ can be found (in the abelian case compare to the toric resolution, see also subsections 2.2.1, 2.2.2 and 6.3.1).
- Later, in subsection 1.2.1, we will see, that in some cases resolutions of quotient singularities can be constructed as G-Hilbert schemes. This in particular gives a general method to construct the minimal resolution of ADE singularities.

Multiplicities of the exceptional curves and the fundamental cycle. Let $f : Y \to X = \mathbb{A}^2_{\mathbb{C}}/G$ be the minimal resolution, we call the fiber E over the singular point the exceptional divisor. E is reduced for cyclic groups and nonreduced otherwise, then certain components occur with multiplicities as pictured in the diagrams below (see e.g. [GV83, p. 411 and p. 447]). As a divisor it coincides with the fundamental cycle defined in [Ar66] by a certain property with respect to the intersection form ([Ar66, Thm. 4]).

Properties of the singularities $\mathbb{A}^2_{\mathbb{C}}/G$ and of the minimal resolution. (see also subsections 2.1.1 and 2.1.2).
- The singularities $X = \mathbb{A}^2_{\mathbb{C}}/G$ are rational, i.e. any desingularisation $f : Y \to X$ has the property $f_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^1 f_* \mathcal{O}_Y = 0$. Then $H^1(Y, \mathcal{O}_Y) = 0$ and $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$ by the exponential sequence ([Ha, AG, Appendix B.5]).
- The quotient $\mathbb{A}^2_{\mathbb{C}}/G$ is Gorenstein, the canonical sheaf ω_X is a line bundle.
- The minimal resolution $f : Y \to X$ is crepant, that is $f^* \omega_X \cong \omega_Y$, and, since ω_X is trivial, one has $\omega_Y \cong \mathcal{O}_Y$. Then the components $E_i \cong \mathbb{P}^1_{\mathbb{C}}$ of E_{red} have self-intersection number -2, because for the normal sheaf one has $\mathcal{N}_{E_i/Y} \cong \omega_{E_i} \cong \mathcal{O}_{E_i}(-2)$.

Intersection graphs. The intersection graph for the resolution of $\mathbb{A}^2_{\mathbb{C}}/G$ is defined to be the graph with the irreducible components E_i of E_{red} as vertices and two different vertices E_i and E_j connected by $E_i.E_j$ undirected edges.

Alternatively consider the E_i as simple roots and form the Dynkin diagram with respect to the negative of the intersection form.

The graphs that occur this way are exactly the Dynkin diagrams of ADE type, the intersection matrices are exactly the negatives of the Cartan matrices of the corresponding root systems (these are listed in detail in [Bour, Groupes et algèbres de Lie]).

Here we list the singularities grouped by the corresponding finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, equations for embeddings into $\mathbb{A}^3_{\mathbb{C}}$ and their intersection graphs together with the multiplicities of the components E_i of E_{red} in E. The indices of (A_n), (D_n) and (E_n) always indicate the number of vertices of the intersection graph.

Group, singularity	Intersection graph

1.1.2 The observation of McKay

The subject nowadays known as McKay correspondence originates from the observation of McKay [McK80]: For a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ the graph describing the intersection properties of the irreducible components of the exceptional divisor in the minimal desingularisation of the quotient singularity $\mathbb{A}^2_{\mathbb{C}}/G$ can also be obtained from only the representation theory of the group G.

Representation graphs. Let V be the 2-dimensional representation given by inclusion $G \subset \mathrm{SL}(2, \mathbb{C})$. Define the extended representation graph to be the graph having as vertices the irreducible representations of G over $\mathbb{C}$ and with two different vertices V_i, V_j connected by $a_{ij} = \mathrm{Hom}^G_{\mathbb{C}}(V_i, V \otimes_{\mathbb{C}} V_j)$ edges from V_i to V_j (see also definition 6.1 for more details). Here the integers a_{ij} satisfy $V \otimes_{\mathbb{C}} V_j = \bigoplus_i a_{ij} V_i$ (if the ground field is not assumed to be algebraically closed, this is not necessarily the case, see remark 6.2). Two directed edges of opposed direction form an undirected edge. The representation graph occurs by leaving out the trivial representation and all its edges. It is elementary to show, that for the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ these graphs are connected and undirected.

The representation graph can be described in terms of the $\mathbb{Z}$-bilinear form

$$\langle V_i, V_j \rangle := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}^G(V_i, V \otimes_{\mathbb{C}} V_j) - 2\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}^G(V_i, V_j)$$

on the representation ring $R(G)$ (see definition 6.3 and remark 6.4, this form will naturally occur in subsection 1.3.2). One also obtains the graphs (A_n), (D_n), (E_n) as Dynkin diagrams with respect to the negative of the form $\langle \,\cdot\,, \,\cdot\, \rangle$.

Observation of McKay. The (extended) representation graphs obtained this way are exactly the (extended) graphs of ADE type (A_n), (D_n), (E_6), (E_7), (E_8) corresponding to the cyclic, binary dihedral and the three binary polyhedral groups and one has:

Observation 1.1 ([McK80]). *For any finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ the representation graph for the group G is isomorphic to the intersection graph of the singularity $\mathbb{A}_{\mathbb{C}}^2/G$.*

In particular the nontrivial irreducible representations of G are in bijection with the exceptional prime divisors. The additional vertices corresponding to the trivial representation can be related to additional vertices in the intersection graphs by considering general hyperplane sections (see e.g. [GV83, p. 411], [Mat, 4-6-9]).

List of finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ and their presentations and representation graphs. We list the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, their presentations and extended representation graphs together with the dimensions of the irreducible representations. Write $\circ$ for the trivial representation.

Finite subgroup of $\mathrm{SL}(2,\mathbb{C})$	Graph	
Cyclic groups of order n $(n \geq 2)$ $\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n = 1 \rangle$	(A_{n-1})	
Binary dihedral groups *of order 4n $(n \geq 2)$* $BD_n = \langle a,b,c \mid a^2 = b^2 = c^n = abc \rangle$	(D_{n+2})	
Binary tetrahedral group of order 24 $BT = \langle a,b,c \mid a^2 = b^3 = c^3 = abc \rangle$	(E_6)	
Binary octahedral group of order 48 $BO = \langle a,b,c \mid a^2 = b^3 = c^4 = abc \rangle$	(E_7)	
Binary icosahedral group of order 120 $BI = \langle a,b,c \mid a^2 = b^3 = c^5 = abc \rangle$	(E_8)	

Remark 1.2. There are relations to the Lie groups and algebras of the corresponding Dynkin diagrams as well, see [Sl], but these will not be treated in this thesis.

1.2 G-Hilbert schemes, stratification and tautological sheaves

In this section we introduce the G-Hilbert scheme [ItNm96] and discuss methods, the stratification of the G-Hilbert scheme [ItNm96] and the tautological sheaves [GV83], to relate irreducible representations to prime divisors.

1.2.1 G-Hilbert schemes

The minimal resolution of a 2-dimensional quotient singularity (and in higher dimensions in some cases crepant resolutions) can be constructed as a G-Hilbert scheme. The G-Hilbert scheme has been introduced in [ItNm96], [ItNm99] for 2-dimensional quotient singularities, the higher and especially the 3-dimensional case for abelian subgroups of $\mathrm{SL}(3, \mathbb{C})$ and the relation to the theory of toric varieties have been studied in [Nm01] (see also subsection 2.2.1). Here we use the definition of the G-Hilbert scheme as a moduli space of G-clusters (this seems to have been introduced in [Re97], see also [CrRe02, section 4] for discussion and comparision between the two definitions), i.e. here we specify the functor of points and obtain a possibly not irreducible scheme. Its irreducible component birational to the quotient is the Hilbert scheme of G-orbits in the sense as originally introduced by Nakamura.

For a scheme X over $\mathbb{C}$ with an operation of a finite group G the G-Hilbert scheme is defined to parametrise G-clusters, where a G-cluster is a zero-dimensional closed subscheme $Z \subseteq X$ such that the representation $H^0(Z, \mathcal{O}_Z)$ is isomorphic to the regular representation of G. More precisely, define the G-Hilbert functor $\underline{\mathrm{G\text{-}Hilb}}_{\mathbb{C}} X : (\mathbb{C}\text{-}schemes)^{\circ} \to (sets)$ by

$$\underline{\mathrm{G\text{-}Hilb}}_{\mathbb{C}} X(T) := \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \\ Z \textit{ finite flat over } T, \textit{ for } t \in T: H^0(Z_t, \mathcal{O}_{Z_t}) \textit{ isomorphic} \\ \textit{to the regular representation} \end{array} \right\}$$

(see section 4.2 for more details). If X is quasiprojective one may prove that $\underline{\mathrm{G\text{-}Hilb}}_{\mathbb{C}} X$ is representable by showing that it is a closed subfunctor of the Hilbert functor of n points (see subsection 4.2.1), this functor is representable as a consequence of a more general theorem of Grothendieck [Gr61], in this thesis cited as theorem 4.1.

For a detailed discussion on G-Hilbert schemes see chapter 4. In section 4.3 we prove representability of $\underline{\mathrm{G\text{-}Hilb}}_K X$ for algebraic K-schemes X and finite group schemes G over a field K under the condition that the Hopf algebra of G is cosemisimple and a geometric quotient $\pi : X \to X/G$, π affine, exists. In this proof the existence of the Hilbert scheme of n points will not be needed.

Since the support of any G-cluster is exactly one G-orbit (use that the trivial representation occurs in the regular representation only once), there is a map of closed points of G-Hilb$_{\mathbb{C}} X$ to the quotient X/G. This is part of a projective morphism $\tau : \mathrm{G\text{-}Hilb}_{\mathbb{C}} X \to X/G$, which is an isomorphism over the open subscheme of X/G whose points correspond to free G-orbits (see section 4.3).

In dimension 2 the G-Hilbert scheme is the minimal resolution:

Theorem 1.3 ([ItNm96]). *Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. Then G-Hilb$_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^2$ is irreducible, nonsingular and $\tau : \mathrm{G\text{-}Hilb}_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^2 \to \mathbb{A}_{\mathbb{C}}^2/G$ is the minimal resolution of the quotient singularity $\mathbb{A}_{\mathbb{C}}^2/G$.* □

By [BKR01], under some conditions, the G-Hilbert scheme is a crepant resolution (for the notion "crepant" see subsection 2.1.1), in particular this is the case for finite $G \subset \mathrm{SL}(n, \mathbb{C})$, $n \leq 3$ and thus implies that G-Hilb$_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^2$ for $G \subset \mathrm{SL}(2, \mathbb{C})$ is the minimal resolution. This is also true for finite small subgroups of $\mathrm{GL}(2, \mathbb{C})$, see [Is02].

1.2.2 Stratification of the G-Hilbert scheme

In [ItNm96], [ItNm99] one defines a stratification of the G-Hilbert scheme in order to construct the bijection between nontrivial irreducible representations and irreducible components of the exceptional divisor.

Let G be a nontrivial finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, V the given 2-dimensional representation and let $\pi : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}/G$ the quotient morphism. $\tau : \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_C/G$ is the minimal resolution, define the exceptional divisor E to be the fiber over the singular point, that is $E = \tau^{-1}(\overline{O})$, where $\overline{O} = \pi(O)$, O the origin of $\mathbb{A}^2_{\mathbb{C}}$. Let $\mathfrak{m} = \langle x_1, x_2 \rangle$ be the maximal ideal of $\mathbb{C}[x_1, x_2]$ corresponding to O, $\mathfrak{n}$ the maximal ideal of $\mathbb{C}[x_1, x_2]^G$ corresponding to $\overline{O}$.

The closed points of G-Hilb $\mathbb{A}^2_{\mathbb{C}}$ correspond to ideals $I \subset \mathbb{C}[x_1, x_2]$ such that $\mathbb{C}[x_1, x_2]/I$ is isomorphic to the regular representation. For any G-cluster define the representation

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n}\mathbb{C}[x_1, x_2])$$

Then there is the following theorem (proven in [ItNm99] by case by case investigations):

Theorem 1.4 ([ItNm96]).

(i) *$I \in E$ if and only if $V(I) \neq 0$. In this case $V(I)$ is either nontrivial irreducible or consists of two nontrivial irreducible representations nonisomorphic to each other.*

(ii) *There is a bijection*

$$\{\text{nontrivial irreducible representations of } G\} \;\to\; \{\text{irreducible components of } E\}$$
$$V_i \;\mapsto\; E_i$$

such that for a closed point $I \in \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}}$: $I \in E_i \iff V_i \subseteq V(I)$.

(iii) *For nonisomorphic irreducible representations V_i, V_j it is $E_i.E_j \neq 0$ if and only if $V_i \subseteq V \otimes_{\mathbb{C}} V_j$ (or equivalently $V_j \subseteq V \otimes_{\mathbb{C}} V_i$). In this case there is exactly one closed point $I \in \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}}$ such that $V(I) \cong V_i \oplus V_j$.* $\qquad\square$

An analogous theorem, there restricting to so called special representations, applies to finite small subgroups of $\mathrm{GL}(2,\mathbb{C})$, see [Is02], [It01]. A similar stratification exists for finite subgroups $G \subset \mathrm{SO}(3,\mathbb{R})$ naturally operating on $\mathbb{A}^3_{\mathbb{C}}$, see [GNS04] and also subsection 2.2.3.

A relation to relative tangent spaces of G-Hilb$_{\mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$ over $\mathbb{A}^n_{\mathbb{C}}/G$ is given in subsection 4.4.5.

1.2.3 Tautological sheaves

For a finite subgroup $G \subseteq \mathrm{SL}(2,\mathbb{C})$ let $V_0, V_1, \ldots, V_r$ the isomorphism classes of irreducible representations of G, V_0 the trivial one, of dimensions $d_0, d_1, \ldots, d_r$. Let $Y := \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}}$ and $0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{A}^2_Y} \to \mathcal{O}_Z \to 0$ the universal quotient. There is the commutative diagram

$$
\begin{array}{ccc}
 & Z & \\
{}^{p}\swarrow & & \searrow^{q} \\
Y & & \mathbb{A}^2_{\mathbb{C}} \\
{}_{\tau}\searrow & & \swarrow_{\pi} \\
 & \mathbb{A}^2_{\mathbb{C}}/G &
\end{array}
\qquad\qquad (1.1)
$$

The projection $p : Z \to Y$ is a finite flat morphism, the sheaf $p_* \mathcal{O}_Z$ is a locally free G-sheaf on Y with fibers $p_* \mathcal{O}_Z \otimes_{\mathcal{O}_Y} \kappa(y)$ for $y \in Y$ isomorphic to the regular representation of G over $\kappa(y)$. Decomposing $p_* \mathcal{O}_Z$ into isotypic components for the irreducible representations $V_0, V_1, \ldots V_r$ of G and writing these components as tensor products of a sheaf with the corresponding representation, one has

$$p_* \mathcal{O}_Z \cong \bigoplus_{i=0}^{r} \mathscr{H}om_{\mathcal{O}_Y}^G (\mathcal{O}_Y \otimes_{\mathbb{C}} V_i, p_* \mathcal{O}_Z) \otimes_{\mathbb{C}} V_i$$

(see also section 3.4). This decomposition relates irreducible representations to certain sheaves on Y.

Definition 1.5. *Define the tautological sheaves $\mathscr{F}_0, \ldots, \mathscr{F}_r$ on Y corresponding to the isomorphism classes of irreducible representations $V_0, \ldots, V_r$ by*

$$\mathscr{F}_i := \mathscr{H}om_{\mathcal{O}_Y}^G (\mathcal{O}_Y \otimes_{\mathbb{C}} V_i, p_* \mathcal{O}_Z) \tag{1.2}$$

The sheaves $\mathscr{F}_i$ are locally free, because $p_* \mathcal{O}_Z$ is locally free, and $\mathscr{F}_i$ is of rank $d_i = \dim_{\mathbb{C}} V_i$, since the fibers of $p_* \mathcal{O}_Z$ are isomorphic to the regular representation. For the trivial representation V_0 one has $\mathscr{F}_0 \cong (p_* \mathcal{O}_Z)^G \cong \mathcal{O}_Y$. The sheaves $\mathscr{F}_i$ may defined as well as

$$\mathscr{F}_i = (p_* q^* (\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^2} \otimes_{\mathbb{C}} V_i^\vee))^G \tag{1.3}$$

or

$$\mathscr{F}_i = \tau^* \mathscr{H}om_{\mathbb{A}_{\mathbb{C}}^2/G}^G (V_i \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^2/G}, \pi_* \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^2}) / (\mathcal{O}_Y\text{-}torsion) \tag{1.4}$$

(the equivalence of (1.2) and (1.3) is easy to show, for (1.3) and (1.4) see [GV83, Prop. 2.8]). The tautological sheaves were studied in [GV83], [KaVa00]. Originally, in [GV83], they were introduced as in equations (1.3) and (1.4). They have been studied in higher dimensional cases as well, see e.g. [Re97], [ItNj00], [Cr01], [Cr05].

The tautological sheaves $\mathscr{F}_i$ corresponding to the nontrivial irreducible representations V_i of $G \subset \mathrm{SL}(2, \mathbb{C})$ have the following property that can be used to relate irreducible representations to exceptional prime divisors in the McKay correspondence.

Define line bundles

$$\mathscr{L}_i := \bigwedge\nolimits^{d_i} \mathscr{F}_i$$

as highest exterior powers of the tautological sheaves $\mathscr{F}_i$.

Theorem 1.6 ([GV83, Thm. 2.2], [KaVa00, Lemma 2.1]). *There exists a bijection $V_i \leftrightarrow E_i$ between the nontrivial irreducible representations $V_1, \ldots, V_r$ of G and the components $E_1, \ldots, E_r$ of E_{red} such that*

$$\mathscr{L}_i . E_j = \delta_{ij}$$

It is

$$\mathscr{F}_i|_{E_j} \cong \begin{cases} \mathcal{O}_{E_j}(1) \oplus \mathcal{O}_{E_j}^{\oplus d_i - 1} & i = j \\ \mathcal{O}_{E_j}^{\oplus d_i} & i \neq j \end{cases}$$

$\square$

This bijection coincides with the one defined by the stratification of the G-Hilbert scheme (see [KaVa00]).

1.3   K-theoretic and derived McKay correspondence

In this section we review results on the McKay correspondence as an isomorphism of Grothendieck groups and as an equivalence of derived categories with [GV83] and [KaVa00], [BKR01] as main references.
Let G be a finite group and $V_0, \ldots, V_r$ the isomorphism classes of nontrivial irreducible representations of G of dimensions $d_0, \ldots, d_r$ with V_0 the trivial one.

1.3.1   Equivariant Grothendieck groups

In [GV83] a geometric construction is provided to explain the McKay correspondence. There the McKay correspondence is interpreted as an isomorphism of $\mathbb{Z}$-modules between the Grothendieck group of coherent sheaves on the minimal resolution and the representation ring of the group G. In this subsection we develop some elementary facts concerning equivariant Grothendieck groups on affine spaces with a linear group operation in order to prepare for the formulation of K-theoretic McKay correspondence in the next subsection.

As general references for K-theory we use [FL], [Man]. It is well known that any coherent sheaf on a regular noetherian scheme X with an ample invertible sheaf has a finite locally free resolution, so the Grothendieck groups $K^0(X)$ and $K_0(X)$ of classes of locally free resp. coherent sheaves coincide (see for example [FL, Prop. VI.3.1] or [Man, Thm. 1.9]), in this case we write $K(X)$. Further, on $\mathbb{A}^n_{\mathbb{C}}$ locally free sheaves of finite rank are already free (see e.g. [GM, III.5.15] and references therein), thus $K(\mathbb{A}^n_{\mathbb{C}}) \cong \mathbb{Z}$ by rank.

Let V be a finite dimensional representation of G over $\mathbb{C}$, this determines a linear operation of G on the affine space $\mathbb{A}_{\mathbb{C}}(V)$. Choosing coordinates we have $\mathbb{A}_{\mathbb{C}}(V) = \mathbb{A}^n_{\mathbb{C}}$ with $n = \dim V$. Let $O \in \mathbb{A}^n_{\mathbb{C}}$ be the origin and $\mathfrak{m}$ the corresponding maximal ideal, then O is a G-fixed point and $\mathfrak{m}$ a G-stable ideal.

Denote by $K^G(\mathbb{A}^n_{\mathbb{C}})$ the Grothendieck group of G-equivariant coherent sheaves on $\mathbb{A}^n_{\mathbb{C}}$ (for a treatment of G-equivariant sheaves see chapter 3) and by $R(G) = K^G(\mathrm{Spec}\,\mathbb{C})$ the representation ring or in other words the Grothendieck group of finite dimensional representations of G. Define $K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\})$ to be the Grothendieck group of coherent sheaves supported by $\{O\}$. Using the induction functor (left adjoint to the restriction functor, see [FH], [Se, LR] for the functors Ind and Res for representations) one easily shows, that for X regular with an ample invertible sheaf any coherent G-equivariant sheaf on X has a finite locally free G-equivariant resolution. In particular $K^G(\mathbb{A}^n_{\mathbb{C}})$ coincides with the Grothendieck group of locally free or free G-equivariant sheaves of finite rank.

The following proposition relates the Grothendieck groups of equivariant sheaves to the representation ring (see [GV83, Prop.1.4]).

Proposition 1.7. *The morphisms* $\mathrm{Spec}\,\mathbb{C} = \{O\} \xrightarrow{\;i\;} \mathbb{A}^n_{\mathbb{C}}$, $\mathbb{A}^n_{\mathbb{C}} \xrightarrow{\;p\;} \mathrm{Spec}\,\mathbb{C}$ *induce an isomorphism of λ-rings*

$$R(G) \to K^G(\mathbb{A}^n_{\mathbb{C}}), \ [V] \mapsto [\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \otimes_{\mathbb{C}} V]$$

and an isomorphism of $\mathbb{Z}$-modules

$$R(G) \to K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\}), \ [V] \mapsto [\kappa(O) \otimes_{\mathbb{C}} V]$$

Proof. To prove the first assertion, one shows that the homomorphisms

$$R(G) = K^G(\operatorname{Spec}\mathbb{C}) \;\overset{i^*,p^*}{\longleftrightarrow}\; K^G(\mathbb{A}^n_{\mathbb{C}})$$
$$[W] \;\mapsto\; [p^*W] = [\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \otimes_{\mathbb{C}} W]$$
$$[\mathscr{F} \otimes_{\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}} \kappa(O)] = [i^*\mathscr{F}] \;\leftarrow\; [\mathscr{F}], \quad \text{for free } \mathscr{F}$$

are inverse to each other. They preserve multiplication and λ-operations.

For the second assertion note that for a coherent sheaf $\mathscr{F}$ supported by $\{O\}$ one can use inductively G-equivariant exact sequences $0 \to \mathfrak{m}\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathfrak{m}\mathscr{F} \to 0$ to get an equation $[\mathscr{F}] = \sum_{i\geq 0}[\mathfrak{m}^i\mathscr{F}/\mathfrak{m}^{i+1}\mathscr{F}]$, where the sum is finite, because $\mathfrak{m}^i\mathscr{F} = 0$ for some i (these are standard arguments, see for example [FL, Ch. VI.6]). With this it is easy to see that the following maps are mutually inverse:

$$R(G) = K^G(\operatorname{Spec}\mathbb{C}) \;\overset{p_*,i_*}{\longleftrightarrow}\; K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\})$$
$$[W] \;\mapsto\; [i_*W] = [\kappa(O) \otimes_{\mathbb{C}} W]$$
$$[p_*\mathscr{F}] \;\leftarrow\; [\mathscr{F}]$$

□

There is the natural homomorphism of $K^G(\mathbb{A}^n_{\mathbb{C}})$-modules $K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\}) \to K^G(\mathbb{A}^n_{\mathbb{C}})$ (not injective) with image of $K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\})$ in $K^G(\mathbb{A}^n_{\mathbb{C}})$ the ideal generated by $[\kappa(O)]$, and the isomorphism $K^G(\mathbb{A}^n_{\mathbb{C}}) \to K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\})$ given by multiplication with $[\kappa(O)]$. Taking into account the isomorphisms to the representation ring $R(G)$, there is the diagram of homomorphisms of $\mathbb{Z}$-modules

$$
\begin{array}{ccc}
R(G) & \overset{\sim}{\longleftarrow} & K^G(\mathbb{A}^n_{\mathbb{C}}, \{O\}) \\
{\scriptstyle x\mapsto o\cdot x}\,\big\updownarrow\,{\scriptstyle id} & & \big\updownarrow \\
R(G) & \overset{\sim}{\longleftarrow} & K^G(\mathbb{A}^n_{\mathbb{C}})
\end{array}
\tag{1.5}
$$

with $o \in R(G)$ corresponding to $[\kappa(O)] \in K^G(\mathbb{A}^n_{\mathbb{C}})$.

For the purpose to calculate o we consider a G-equivariant Koszul resolution of $\kappa(O)$, for the following proposition see [GV83, Prop. 1.4] and also [BKR01, Section 9]. Note that the sheaf of differentials $\Omega_{\mathbb{A}^n_{\mathbb{C}}}$ as G-equivariant sheaf is isomorphic to $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \otimes_{\mathbb{C}} V$, so the isomorphism $K^G(\mathbb{A}^n_{\mathbb{C}}) \cong R(G)$ identifies the class $[\Omega_{\mathbb{A}^n_{\mathbb{C}}}]$ with $[V]$, the cotangent space of the origin as a representation is isomorphic to V.

Proposition 1.8. *For an n-dimensional representation V of G and $\mathbb{A}^n_{\mathbb{C}} = \mathbb{A}_{\mathbb{C}}(V)$ with G-operation given by V the isomorphism $K^G(\mathbb{A}^n_{\mathbb{C}}) \cong R(G)$ maps the class of the structure sheaf of the origin $[\kappa(O)] \in K^G(\mathbb{A}^n_{\mathbb{C}})$ to the element $o = \lambda_{-1}([V]) = \sum_i(-1)^i[\bigwedge^i V] \in R(G)$.*

Proof. From the equivariant Koszul resolution of $\kappa(O)$

$$
0 \longrightarrow \bigwedge^n \Omega_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \ldots \longrightarrow \bigwedge^2 \Omega_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \Omega_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \kappa(O) \longrightarrow 0
$$
$$dx_i \;\mapsto\; x_i$$

derives the equation $[\kappa(O)] = \lambda_{-1}([\Omega_{\mathbb{A}^n_{\mathbb{C}}}]) = \sum_i(-1)^i[\bigwedge^i \Omega_{\mathbb{A}^n_{\mathbb{C}}}]$ in $K^G(\mathbb{A}^n_{\mathbb{C}})$, this element corresponds to $o = \lambda_{-1}([V]) = \sum_i(-1)^i[\bigwedge^i V] \in R(G)$. □

Example 1.9. In the case $n = 2$, $G \subset \operatorname{SL}(2, K)$ one has the Koszul resolution

$$0 \longrightarrow \bigwedge^2 \Omega_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \Omega_{\mathbb{A}^n_{\mathbb{C}}} \longrightarrow \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \longrightarrow \kappa(O) \longrightarrow 0$$

and obtains $[\kappa(O)] = [\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}] - [\Omega_{\mathbb{A}^2_{\mathbb{C}}}] + [\omega_{\mathbb{A}^2_{\mathbb{C}}}]$, so $o = 2 - [V]$.

Further, in the case $G \subset \mathrm{SL}(2, K)$ the ideal of $K^G(\mathbb{A}^n_{\mathbb{C}}) = R(G)$ generated by $[\kappa(O)] \leftrightarrow 2 - [V]$ is a free $\mathbb{Z}$-module of rank r with basis $[V_1 \otimes_{\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}} \kappa(O)], \dots [V_r \otimes_{\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}} \kappa(O)]$, there is the exact sequence

$$0 \longrightarrow \mathbb{Z}(\textstyle\sum_{i=0}^r d_i[V_i]) \longrightarrow R(G) \xrightarrow{\ 2 - [V]\ } R(G)$$

(see [GV83, Prop. 1.4]). In particular the regular representation satisfies $[\bigoplus_{i=0}^r V_i^{\oplus d_i}] \cdot (2 - [V]) = 0$ in $R(G)$, there is the relation $[\kappa(O)] = -(\sum_{i=1}^r d_i[V_i \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}]) \cdot [\kappa(O)]$ in $K^G(\mathbb{A}^n_{\mathbb{C}})$.

1.3.2 McKay correspondence as an isomorphism of Grothendieck groups

Assume that G is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and V the natural 2-dimensional representation. $Y := \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}/G$ is the minimal resolution, denote by $K(Y)$ the Grothendieck group of coherent $\mathcal{O}_Y$-modules.

The Grothendieck group $K(Y)$ has the filtration ([FL, Ch. III.1], [Man, §8])

$$K(Y) = F^0 K(Y) \supseteq F^1 K(Y) \supseteq F^2 K(Y)$$

where by definition $F^1 K(Y) = \ker(\mathrm{rk} : K(Y) \to \mathbb{Z})$. One has $K(Y)/F^1 K(Y) \cong \mathbb{Z}$ by rank, $G^1 K(Y) = F^1 K(Y)/F^2 K(Y) \cong \mathrm{Pic}(Y)$ by determinant and $K(Y)/F^2 K(Y) \cong \mathbb{Z} \oplus \mathrm{Pic}(Y)$ ([FL, Thm. III.1.7 and V.3 Remark 1], [Man, Cor. of Thm. 10.8]), further $F^3 K(Y) = 0$ ([FL, Cor. V.3.10], [Man, Thm. 9.1]). Y is regular and quasiprojective, therefore $K(Y)$ is generated by invertible sheaves and $F^i K(Y) = (F^1 K(Y))^i$ (see [FL, p. 49], [Man, Prop. 8.5]).
There is also a lower filtration $F_i K(Y)$ ([FL, Ch. VI.5]). It is known that $F_0 K(Y) = 0$ (see [GV83, Proof of Prop. 1.2]), then it follows that $F^2 K(Y) = 0$ (use [FL, Prop. VI.5.3]) and thus $K(Y) \cong \mathbb{Z} \oplus \mathrm{Pic}(Y)$, $x \leftrightarrow (\mathrm{rk}\, x, \det x)$.

Again we use diagram (1.1) and following [GV83, Thm. 2.2] we define a group homomorphism

$$\begin{aligned} \psi : \quad K^G(\mathbb{A}^2_{\mathbb{C}}) \quad &\to \quad K(Y) \\ [\mathscr{E}] \quad &\mapsto \quad [(p_* q^* \mathscr{E})^G] \quad \textit{for } \mathscr{E} \textit{ locally free} \end{aligned}$$

The classes of irreducible representations $[V_0], [V_1^\vee], \dots, [V_r^\vee] \in R(G) \cong K^G(\mathbb{A}^2_{\mathbb{C}})$ are mapped to the classes of the tautological sheaves $[\mathscr{F}_0 = \mathcal{O}_Y], [\mathscr{F}_1], \dots, [\mathscr{F}_r]$ of definition 1.5.

The main result of K-theoretic McKay correspondence for $G \subset \mathrm{SL}(2, \mathbb{C})$ is the following theorem [GV83, Thm. 2.2], there proven by case by case considerations. For generalisation to $G \subset \mathrm{SL}(3, \mathbb{C})$ abelian see [ItNj00]. The derived McKay correspondence, discussed in the next subsection, implies the isomorphism of K-theory, in particular this applies to finite $G \subset \mathrm{SL}(2, \mathbb{C})$ and $G \subset \mathrm{SL}(3, \mathbb{C})$ without investigations of the individual cases.

Theorem 1.10 ([GV83, Thm. 2.2]). *The map $\psi : K^G(\mathbb{A}^2_{\mathbb{C}}) \to K(Y)$ is an isomorphism of $\mathbb{Z}$-modules, the tautological sheaves $[\mathscr{F}_0] = [\mathcal{O}_Y], [\mathscr{F}_1], \dots, [\mathscr{F}_r]$ form a $\mathbb{Z}$-basis of $K(Y)$.* $\qquad\square$

We describe the map ψ in more detail:

$$\begin{array}{ccccccc} R(G) & \cong & K^G(\mathbb{A}^2_{\mathbb{C}}) & \xrightarrow{\ \psi\ } & K(Y) & \cong & \mathbb{Z} \oplus \mathrm{Pic}(Y) \\ [V_0] & \leftrightarrow & [\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}] & \mapsto & [\mathcal{O}_Y] & \leftrightarrow & (1, \mathcal{O}_Y) \\ [V_i^\vee] & \leftrightarrow & [\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \otimes_{\mathbb{C}} V_i^\vee] & \mapsto & [\mathscr{F}_i] & \leftrightarrow & (d_i, \bigwedge^{d_i} \mathscr{F}_i) \end{array}$$

As stated in subsection 1.2.3 with respect to the intersection form the line bundles $\mathscr{L}_i = \bigwedge^{d_i} \mathscr{F}_i$ are dual to the exceptional prime divisors E_i.

Similarly define a homomorphism $\psi_0 : K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) \to K(Y, E)$, where $K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\})$ resp. $K(Y, E)$ are the Grothendieck groups of coherent sheaves supported by the origin $\{O\} \subset \mathbb{A}^2_{\mathbb{C}}$ resp. by the exceptional divisor $E \subset Y$. There is the diagram

$$
\begin{array}{ccccc}
R(G) & \xleftarrow{\;\sim\;} & K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) & \xrightarrow{\;\psi_0\;} & K(Y, E) \\
{\scriptstyle x \mapsto (2-[V])\cdot x}\downarrow & & \downarrow & & \downarrow \\
R(G) & \xleftarrow{\;\sim\;} & K^G(\mathbb{A}^2_{\mathbb{C}}) & \xrightarrow{\;\psi\;} & K(Y)
\end{array}
$$

For ψ_0 one has (see [GV83, Thm. 2.2.(ii)] and [KaVa00, Thm. 2.3], [Is02, Thm. 5.1])

$$
\begin{array}{ccccccc}
R(G) & \cong & K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) & \xrightarrow{\;\psi_0\;} & K(Y, E) & \longrightarrow & K(Y) \\
[V_0] & \leftrightarrow & [\kappa(O)] & \mapsto & [\mathcal{O}_E] & \mapsto & [\mathcal{O}_E] = \sum_{i=1}^r d_i[\mathcal{O}_{E_i}] \\
[V_i^\vee] & \leftrightarrow & [\kappa(O) \otimes_{\mathbb{C}} V_i^\vee] & \mapsto & -[\mathcal{O}_{E_i}(-1)] & \mapsto & -[\mathcal{O}_{E_i}]
\end{array}
$$

where $E_i \cong \mathbb{P}^1_{\mathbb{C}}$ are the irreducible components of E_{red}, numerated as in subsection 1.2.3.

One may introduce the $\mathbb{Z}$-bilinear forms ([BKR01, section 9])

$$
\begin{array}{rcl}
\chi^G : \quad K^G(\mathbb{A}^2_{\mathbb{C}}) \times K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) & \to & \mathbb{Z} \\
([\mathscr{F}], [\mathscr{G}]) & \mapsto & \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^{G,i}_{\mathbb{A}^2_{\mathbb{C}}}(\mathscr{F}, \mathscr{G})
\end{array}
$$

where $\mathrm{Ext}^{G,i}$ are the derived functors of Hom^G, and

$$
\begin{array}{rcl}
\chi : \quad K(Y) \times K(Y, E) & \to & \mathbb{Z} \\
([\mathscr{F}], [\mathscr{G}]) & \mapsto & \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_Y(\mathscr{F}, \mathscr{G})
\end{array}
$$

Then $[\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \otimes_{\mathbb{C}} V_0], [\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \otimes_{\mathbb{C}} V_1], \dots, [\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}} \otimes_{\mathbb{C}} V_r]$ and $[\kappa(O) \otimes_{\mathbb{C}} V_0], [\kappa(O) \otimes_{\mathbb{C}} V_1], \dots, [\kappa(O) \otimes_{\mathbb{C}} V_r]$ are bases of $K^G(\mathbb{A}^2_{\mathbb{C}})$ resp. $K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\})$ dual with respect to χ^G and $[\mathscr{F}_0] = [\mathcal{O}_Y], [\mathscr{F}_1], \dots, [\mathscr{F}_r]$ and $[\mathcal{O}_E], -[\mathcal{O}_{E_1}(-1)], \dots, -[\mathcal{O}_{E_r}(-1)]$ are bases of $K(Y)$ resp. $K(Y, E)$ dual with respect to χ (this may be shown directly with the explicit description above or using the derived McKay correspondence [BKR01, Section 9]).

The form χ^G on $K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) \cong R(G)$, that is the composition $K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) \times K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) \to K^G(\mathbb{A}^2_{\mathbb{C}}) \times K^G(\mathbb{A}^2_{\mathbb{C}}, \{O\}) \to \mathbb{Z}$, is given by

$$
\begin{array}{rcl}
\chi^G : \quad R(G) \times R(G) & \to & \mathbb{Z} \\
([W], [W']) & \mapsto & 2 \dim_{\mathbb{C}} \mathrm{Hom}^G_{\mathbb{C}}(W, W') - \dim_{\mathbb{C}} \mathrm{Hom}^G_{\mathbb{C}}(W \otimes_{\mathbb{C}} V, W')
\end{array}
$$

and thus describes the representation graph (see subsection 1.1.2). Similarly, the form χ on $K(Y, E)$ corresponds to the negative of the intersection form, i.e. $\chi([\mathcal{O}_{E_i}], [\mathcal{O}_{E_j}]) = -E_i.E_j$, and thus describes the intersection graph (see subsection 1.1.1).

1.3.3 McKay correspondence as an equivalence of derived categories

In [KaVa00] the McKay correspondence for $G \subset \mathrm{SL}(2, \mathbb{C})$ has been realised as an equivalence of derived categories. As general references on derived categories we have used [GM], [Ha, RD]. Of course, an equivalence as stated in theorem 1.11 implies an isomorphism of K-theory as in theorem 1.10.

Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup, let $Y = \text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}}$ and $Z \subseteq \mathbb{A}^2_Y$ the universal family, consider diagram (1.1). We denote by $D(Y)$ the derived category of quasicoherent sheaves on

Y with bounded coherent cohomology and by $D^G(\mathbb{A}^2_{\mathbb{C}})$ the derived category of quasicoherent G-equivariant sheaves on $\mathbb{A}^2_{\mathbb{C}}$ with bounded coherent cohomology etc.. As in [KaVa00] we define functors ($\overline{p}$ is the morphism of ringed spaces $(Z, \mathcal{O}_Z) \to (Y, p_*\mathcal{O}_Z)$, see also subsection 3.1.1)

$$
\begin{array}{rcccccl}
\Phi : & D(Y) & \to & D^G(\mathbb{A}^2_{\mathbb{C}}), & E & \mapsto & Rq_*\overline{p}^*\mathscr{H}om_{\mathcal{O}_Y}(p_*\mathcal{O}_Z, E) \\
\Psi : & D^G(\mathbb{A}^2_{\mathbb{C}}) & \to & D(Y), & F & \mapsto & (p_*Lq^*F)^G
\end{array}
$$

Here Φ is constructed as right-adjoint to Ψ, that there is the adjunction (Ψ, Φ) can be seen as follows:

$$
\begin{array}{rcl}
\mathrm{Hom}_{D(Y)}(\Psi(F), E) & = & \mathrm{Hom}_{D(Y)}((p_*Lq^*F)^G, E) \\
& \cong & \mathrm{Hom}_{D(Y)^G}(p_*Lq^*F, E) \\
& \cong & \mathrm{Hom}_{D(Z)^G}(Lq^*F, \overline{p}^*\mathscr{H}om_{\mathcal{O}_Y}(p_*\mathcal{O}_Z, E)) \\
& \cong & \mathrm{Hom}_{D(\mathbb{A}^2_{\mathbb{C}})^G}(F, Rq_*\overline{p}^*\mathscr{H}om_{\mathcal{O}_Y}(p_*\mathcal{O}_Z, E)) \\
& = & \mathrm{Hom}_{D(\mathbb{A}^2_{\mathbb{C}})^G}(F, \Phi(E))
\end{array}
$$

The isomorphism $\mathrm{Hom}_{D(Y)^G}(p_*Lq^*F, E) \cong \mathrm{Hom}_{D(Z)^G}(Lq^*F, \overline{p}^*\mathscr{H}om_{\mathcal{O}_Y}(p_*\mathcal{O}_Z, E))$ is equivariant Grothendieck duality for the finite flat morphism $p : Z \to Y$ (see [Ha, RD, Ch. III, §6], subsection 3.1.1 and proposition 3.57).

Theorem 1.11 ([KaVa00]).
The functors Φ and Ψ are mutually inverse equivalences of triangulated categories.   □

This theorem also follows from [BKR01], there the derived McKay correspondence has been vastly generalised: These results imply that for finite $G \subset \mathrm{SL}(3,\mathbb{C})$ the G-Hilbert scheme is irreducible, smooth and a crepant resolution of $\mathbb{A}^3_{\mathbb{C}}/G$ and that there is an equivalence of derived categories as in theorem 1.11. They also give an equivalence of derived categories in other cases, e.g. for the crepant resolution $\mathrm{Hilb}^n_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}} \cong \mathrm{S}_n\text{-}\mathrm{Hilb}_{\mathbb{C}}(\mathbb{A}^2_{\mathbb{C}})^n \to (\mathbb{A}^2_{\mathbb{C}})^n/S_n$ (see [Bo06] and the references therein).
The methods of [BKR01] have been applied in [Is02] to finite small subgroups $G \subset \mathrm{GL}(2,\mathbb{C})$, there derived categories of G-equivariant sheaves on $\mathbb{A}^2_{\mathbb{C}}$ are equivalent to subcategories of derived categories of sheaves on the minimal resolution.

Chapter 2

Quotient singularities and G-Hilbert schemes of higher dimension

In this expository chapter we are concerned with higher dimensional quotient singularities, resolutions of these and in particular the construction of resolutions as G-Hilbert schemes. We consider the case of abelian quotient singularities and the toric description of the G-Hilbert scheme. Further, we discuss the case of reflection groups and subgroups of these of index 2.

In the first section we introduce the notion of canonical singularities, discrepancy of a resolution and the notion of a crepant resolution which guarantees some kind of minimality. Our main interest lies in quotient singularities $\mathbb{A}^n_{\mathbb{C}}/G$ for $G \subset \mathrm{SL}(n, \mathbb{C})$, which can be treated this way. It arises the question under what conditions crepant resolutions of $\mathbb{A}^n_{\mathbb{C}}/G$, $G \subset \mathrm{SL}(n, \mathbb{C})$ exist.
In particular we consider the case of abelian groups. Then the quotient singularity $\mathbb{A}^n_{\mathbb{C}}/G$ is a toric variety and is determined by the combinatorial data of a cone with respect to a lattice. Resolutions can be constructed by subdividing this cone and described by combinatorial methods.

In the second section we consider resolutions of quotient singularities constructed as G-Hilbert schemes. In the abelian case the component Y of the G-Hilbert scheme birational to $\mathbb{A}^n_{\mathbb{C}}/G$ is a not necessarily normal toric variety, we describe the method of Nakamura to determine the fan of its normalisation. This is done by introducing local coordinates around torus fixed points. We explicitly calculate some examples for this construction.
A subsection is devoted to reflection groups and subgroups of these of index 2. Reflection groups are generated by quasireflections, equivalently these are those finite subgroups $\widetilde{G} \subset \mathrm{GL}(n, \mathbb{C})$ for which the quotient $\mathbb{A}^n_{\mathbb{C}}/\widetilde{G}$ is isomorphic to $\mathbb{A}^n_{\mathbb{C}}$. For subgroups $G = \widetilde{G} \cap \mathrm{SL}(n, \mathbb{C})$ of index 2 in a reflection group $\widetilde{G}$ one has a relatively simple structure of the fiber over the origin which is determined in several works.

In the last section we discuss a simple example of group schemes of roots of unity in arbitrary dimension in detail, determine the tautological sheaves and consider the K-theoretic McKay correspondence.

2.1 Quotient singularities

In this section we collect some definitions and results concerning higher dimensional singularities and their resolutions aiming at the definition of discrepancy and the notion of a crepant resolution. We use [Mat], [Re80], [Re87] as main references, also look at [Cr01].

2.1.1 Canonical singularities and discrepancy

In the following let X be an irreducible and normal variety quasiprojective over the field $\mathbb{C}$ or any algebraically closed field of characteristic 0.

Remark 2.1. (Some standard definitions and results).
(1) A desingularisation or resolution of singularities is a (surjective) proper birational morphism $f : Y \to X$ with Y nonsingular.
(2) The canonical divisor K_X is the Weil divisor (more precisely one considers divisor classes) $\mathrm{div}(s_X)$ associated to a nonzero rational differential form $s_X \in \bigwedge^{\dim X} \Omega_{\mathbb{C}(X)/\mathbb{C}}$, where $\mathbb{C}(X)$ denotes the function field of X. It need not to be Cartier.
(3) For smooth X the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega_X$ is an invertible sheaf and coincides with the dualising sheaf.
(4) The canonical sheaf ω_X on possibly singular X (see e.g. [Re87, (1.4), p. 349]): One sets $\omega_X := j_* \omega_U$, where $U \to X$ is the inclusion of the smooth locus.
(5) One may associate a sheaf $\mathcal{O}_X(K_X)$ to the canonical divisor K_X, see [Re80, p. 282]. It coincides with the canonical sheaf ω_X as defined above, see [Re80, p. 283].
(6) X is called Cohen-Macaulay, if all its local rings are Cohen-Macaulay local rings. If X is Cohen-Macaulay, then the canonical sheaf ω_X coincides with the dualising sheaf, see [Re80, p. 283], [AlKl, p. 5].
(7) X is called Gorenstein, if all its local rings are Gorenstein local rings. If X is Gorenstein, then ω_X is invertible and K_X is Cartier.

Definition 2.2. (Canonical and terminal singularities). ([Re87, (1.1)], [Mat, 4-1-1, 4-2-1]).
X has canonical singularities, if it satisfies the following conditions:

(i) *For some integer $r \geq 1$ the Weil divisor rK_X is Cartier.*

(ii) *If $\varphi : Y \to X$ is a resolution with $\{E_i \,|\, i \in I\}$ the family of exceptional prime divisors, then*

$$rK_Y = \varphi^*(rK_X) + \sum_{i \in I} a_i E_i \tag{2.1}$$

with $a_i \geq 0$.

If $a_i > 0$ for every E_i, then X has terminal singularities.

Remark 2.3.
(1) Equation (2.1) can be read either as an equation of divisors, then after choosing a nonzero rational differential form $s_X \in \bigwedge^{\dim X} \Omega_{\mathbb{C}(X)/\mathbb{C}}$ on X one sets $K_X = \mathrm{div}(s_X)$, $K_Y = \mathrm{div}(\varphi^* s_X)$, or as an equation of divisor classes, then note that the numbers a_i are independent of choice of representatives (see also [Mat, 4-1-2]).
(2) In definition 2.2 it is sufficient that the conditions are satisfied for one resolution [Mat, 4-1-2, 4-2-2].
(3) If $\dim X = 2$ then X has terminal singularities if and only if it is smooth. The canonical singularities in dimension 2 are exactly the ADE singularities (up to isomorphism of the germs) [Mat, 4-6-5, 4-6-7].

We now introduce the notion of rational singularities (see e.g. [Vw77]):

Definition 2.4. (Rational singularities).
X has rational singularities, if for any resolution $\varphi : Y \to X$ one has $R\varphi_* \mathcal{O}_Y = \mathcal{O}_X$, that is $\varphi_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i \varphi_* \mathcal{O}_Y = 0$ for $i > 0$.

Theorem 2.5. ([Re87, (3.8), p. 363]).
Canonical singularities are rational. ☐

One introduces the notion of a crepant resolution as follows (see e.g. [Re87, p. 360]):

Definition 2.6. (Crepant resolution).
A resolution $\varphi : Y \to X$ is called crepant, if $\varphi^* \omega_X = \omega_Y$.

Remark 2.7. (Discrepancy and crepant resolutions).
(1) Assume that X has canonical singularities. For a resolution $\varphi : Y \to X$ one has the equation of $\mathbb{Q}$-Cartier divisors

$$K_Y = \varphi^* K_X + \sum_i a_i E_i \tag{2.2}$$

The $\mathbb{Q}$-Cartier divisor $\sum_i a_i E_i$ is called the discrepancy of φ. The E_i occuring with multiplicity $a_i = 0$ are called crepant. φ is crepant if and only if $K_Y = \varphi^* K_X$ or equivalently the discrepancy vanishes, that is $a_i = 0$ for all i.
(2) In dimension 2 a crepant resolution of a canonical singularity is the minimal resolution, because blowing up a regular point gives $K_Y = \varphi^* K_X + E$, where E is the exceptional divisor.

2.1.2 Quotient singularities

Let X be a nonsingular quasiprojective variety over $\mathbb{C}$ and let G be a finite group of automorphisms. Then a geometric quotient X/G in the sense of [Mu, GIT] exists and is constructed by locally taking invariants. The quotient variety X/G is always normal [BH, Prop. 6.4.1] and Cohen-Macaulay [BH, Cor. 6.4.6].

Further, there are the following general results concerning quotient singularities by finite groups:

Theorem 2.8. ([Vw77]).
Quotient singularities are rational. ☐

Theorem 2.9. ([Re80, Prop. (1.7), Rem. (1.8), p. 279]).
Gorenstein quotient singularities are canonical. ☐

Consider the case of a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ naturally operating on $\mathbb{A}_{\mathbb{C}}^n$. An element $g \in \mathrm{GL}(n, \mathbb{C})$ of finite order is called a quasireflection, if the eigenspace for the eigenvalue 1 has dimension $n-1$. A finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is called small, if it contains no quasireflections. Since for a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$ generated by quasireflections (such groups are called reflection groups) the quotient $\mathbb{A}_{\mathbb{C}}^n / G$ again is isomorphic to $\mathbb{A}_{\mathbb{C}}^n$ [BH, Thm. 6.4.12], the study of quotients $\mathbb{A}_{\mathbb{C}}^n / G$ reduces to the case of small subgroups. For more on reflection groups see subsection 2.2.3.

For quotients of $\mathbb{A}_{\mathbb{C}}^n$ by finite subgroups $G \subset \mathrm{GL}(n, \mathbb{C})$ one has the following criteria for $\mathbb{A}_{\mathbb{C}}^n / G$ to be Gorenstein resp. canonical:

Theorem 2.10. ([Re80, Rem. (3.2), p. 292], [BH, Thm. 6.4.10]).
Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite small subgroup and $X = \mathbb{A}_{\mathbb{C}}^n / G$. Then X is Gorenstein if and only if $G \subset \mathrm{SL}(n, \mathbb{C})$. If $g \in G$ operates by $g : x_i \mapsto \varepsilon^{a_i} x_i$, ε a primitive r-th root of unity, this means that $\sum_i a_i \equiv 0 \mod r$. ☐

Theorem 2.11. ([Re80, Thm. (3.1), Rem. (3.2), p. 292]).
Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite small subgroup and $X = \mathbb{A}^n_{\mathbb{C}}/G$. Then X is canonical (resp. terminal) if and only if for any $g \in G$, if r is the order of g and with respect to an eigenbasis $x_1, \ldots, x_n$ the operation is given by $g : x_i \mapsto \varepsilon^{a_i} x_i$, ε a primitve r-th root of unity and $a_i \in \{0, \ldots, r-1\}$, then $\sum_i a_i \geq r$ (resp. $\sum_i a_i > r$). □

Example 2.12. Consider $X = \mathbb{A}^n_{\mathbb{C}}/G$ for $G \subset \mathrm{SL}(n, \mathbb{C})$. Then X is Gorenstein, ω_X is a line bundle and for a resolution $\varphi : Y \to X$ the equation (2.2) is an equation of Cartier divisors. The sheaf $\omega_{\mathbb{A}^n_{\mathbb{C}}}$ is trivial as a G-sheaf, a G-invariant global section $s_{\mathbb{A}^n_{\mathbb{C}}}$ that generates $\omega_{\mathbb{A}^n_{\mathbb{C}}}$ gives a global section s_X that generates ω_X and its pull-back s_Y to Y has the discrepancy divisor $\sum_i a_i E_i$, $a_i \geq 0$ as divisor of zeros. If the resolution is crepant, s_Y has no zeros and generates ω_Y.

Assume that for a finite subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$ there exists a crepant resolution $Y \to \mathbb{A}^n_{\mathbb{C}}/G$. Then one has constructed a variety Y having the properties $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$ (because $\mathbb{A}^n_{\mathbb{C}}/G$ has rational singularities) and $\omega_Y \cong \mathcal{O}_Y$ (since $\omega_{\mathbb{A}^n_{\mathbb{C}}} \cong \mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}$ as a G-sheaf and the resolution is crepant).

It arises the question about the existence of crepant resolutions of Gorenstein quotient singularities $\mathbb{A}^n_{\mathbb{C}}/G$, $G \subset \mathrm{SL}(n, \mathbb{C})$. In dimension 2 there always exists a minimal resolution, the minimal resolution is crepant. In dimension 3 there exist crepant resolutions as well, see [Ro96] and the references given there for a proof based on case by case investigations, a proof without using the classification of finite subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$ is given in [BKR01], there it is shown that the G-Hilbert scheme G-Hilb$_{\mathbb{C}} \mathbb{A}^3_{\mathbb{C}}$ is a crepant resolution. In dimension $n \geq 4$ crepant resolutions need not to exist in general. This theme in connection with resolutions constructed as G-Hilbert schemes will further be discussed in the next section.

2.1.3   Abelian quotient singularities and toric methods

Quotients $\mathbb{A}^n_{\mathbb{C}}/G$ for finite abelian subgroups $G \subset \mathrm{GL}(n, \mathbb{C})$ are toric varieties (see construction 2.13 below). In the toric case there is a simple combinatorial description of the above theory, which is the subject of this subsection. As general references for the theory of toric varieties we use [Da], [Fu], [Oda], for toric singularities see [Mat], [Re80]. We assume toric varieties to be normal.

We show how to describe quotients $\mathbb{A}^n/G$ by finite abelian G as toric varieties. The notations introduced here are used throughout this subsection.

Construction 2.13. (Quotients $\mathbb{A}^n_{\mathbb{C}}/G$ by abelian G as toric varieties).
Let $N = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n \cong \mathbb{Z}^n$ and $M = \mathbb{Z}f_1 \oplus \ldots \oplus \mathbb{Z}f_n \cong \mathbb{Z}^n$ be dual lattices, the duality pairing given by $\langle e_i, f_j \rangle = \delta_{ij}$. N resp. M are naturally embedded into $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ resp. $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. For any vector $n \in N_{\mathbb{Q}}$ write $|n| := \sum_i n_i$. The toric variety $\mathbb{A}^n_{\mathbb{C}}$ corresponds to the cone $\sigma = \langle e_1, \ldots, e_n \rangle_{\mathbb{Q} \geq 0} \subset N_{\mathbb{Q}}$ and is given as the spectrum of $\mathbb{C}[\sigma^{\vee} \cap M] = \mathbb{C}[\mathbb{N}f_1 \oplus \ldots \oplus \mathbb{N}f_n] = \mathbb{C}[x_1, \ldots, x_n]$. It contains the open dense torus $T_M = \mathrm{Spec}\,\mathbb{C}[M] = \mathrm{Spec}\,\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $G = \mathrm{Spec}\,A \subset T_M$ be a finite (abelian) subgroup scheme (over an algebraically closed field of characteristic 0 it is discrete). G operates on T_M, the quotient is a torus $T_{M'}$ corresponding to a sublattice $M' \subseteq M$, one has the exact sequence of group schemes

$$0 \longrightarrow G \longrightarrow T_M \longrightarrow T_{M'} \longrightarrow 0$$

of the corresponding algebras

$$0 \longrightarrow \mathbb{C}[M'] \longrightarrow \mathbb{C}[M] \longrightarrow A \cong \mathbb{C}[M/M'] \longrightarrow 0$$

and its character lattices $0 \to M' \to M \to M/M' \to 0$. Since $\mathbb{C}[M/M'] \cong A$ the elements of M/M' naturally correspond to the simple subcoalgebras of the Hopf algebra A resp. to the characters of G. By $M \to M/M'$ the lattice M and the algebra $\mathbb{C}[M]$ are graded by the character group of G with $\mathbb{C}[M'] = \mathbb{C}[M]^G$ the part corresponding to the trivial character.

The dual $N' \subset N_\mathbb{Q}$ of M' is an overlattice of N. There is the natural nondegenerate bilinear pairing $N'/N \times M/M' \to \mathbb{Q}/\mathbb{Z}$ that makes N'/N dual to the character group of G, so after choice of a primitive r-th root of unity ε, where $r = |N'/N|$, one has an isomorphism $N'/N \cong G$ by $n' \leftrightarrow g$ if $\varepsilon^{\langle n',m \rangle} = \chi(g)$ for $m \in M/M'$ corresponding to a character χ. The operation of G is then determined by $gx^m = \varepsilon^{\langle n',m \rangle} x^m$ if $n' \in N'/N$ corresponds to $g \in G$.

Consider again $\mathbb{A}_\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$: The quotient morphism $\mathbb{A}_\mathbb{C}^n \to \mathbb{A}_\mathbb{C}^n/G$ is given by the inclusion $\mathbb{C}[\sigma^\vee \cap M'] = \mathbb{C}[\sigma^\vee \cap M]^G \to \mathbb{C}[\sigma^\vee \cap M]$, $\mathbb{A}_\mathbb{C}^n/G$ is the simplicial affine toric variety given as $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M']$ corresponding to the cone σ with respect to the finer lattice N'.

As to the converse, any simplicial affine toric variety is the quotient of an affine space $\mathbb{A}_\mathbb{C}^n$ by some finite abelian group [Fu, 2.2].

Remark 2.14. (Some generalities about toric varieties).
(1) A toric variety is nonsingular if and only if any of its cones is generated by elements that are part of a basis of the lattice N [Da, §3], [Fu, 2.1].
(2) Resolutions of toric singularities are constructed by subdividing cones (see construction 2.18 below). Singularities of toric varieties are rational [Da, §8], [Fu, p. 76].
(3) Torus-equivariant invertible sheaves resp. Cartier divisors correspond to certain piecewise linear functions on the support of the fan [Da, §6], [Fu, p. 66], [Oda, Ch. 2.1].
(4) Torus-invariant prime divisors correspond to the orbit closures for the 1-dimensional cones [Da, §6], [Fu, 3.3], [Oda, Ch. 2.1].
(5) Simplicial toric varieties are $\mathbb{Q}$-factorial, that is any Weil divisor is $\mathbb{Q}$-Cartier [Fu, p. 65].
(6) For the sheaf of differentials and the canonical sheaf for toric varieties see [Da, §4], [Fu, 4.3, 4.4]: The canonical divisor is given as $K_X = -\sum_i D_i$, where the D_i are the toric invariant prime divisors [Fu, p. 85, p. 89]. It determines a coherent sheaf $\omega_X = \mathcal{O}_X(K_X)$, which is a dualising sheaf in the sense of duality theory [Fu, p. 89]. If $j : U \to X$ is the inclusion of the nonsingular locus, then $j_*(\bigwedge^{\dim X} \Omega_U) \cong \mathcal{O}_X(K_X)$ [Fu, p. 89].
(7) Toric varieties are Cohen-Macaulay [Da, p. 106], [Fu, p. 30].
(8) X is Gorenstein if and only if ω_X is an invertible sheaf. For an affine simplicial toric variety $X = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ determined by an n-dimensional cone $\sigma \subset N_\mathbb{Q} \cong \mathbb{Q}^n$ this condition means: If $e_1, \ldots, e_n \in N \cap \sigma$ are primitive elements that generate $\sigma \subset N_\mathbb{Q}$ over $\mathbb{Q}_{\geq 0}$, and if $f_1, \ldots, f_n \in M_\mathbb{Q}$ is the dual basis, then $\sum_i f_i \in M$ (see also [Re80, p. 294]).

Let $X = \mathbb{A}_\mathbb{C}^n/G$ be the quotient of a finite small abelian subgroup $G \subset \operatorname{GL}(n, \mathbb{C})$, in the notation of construction 2.13 X corresponds to the cone σ with respect to the lattice $N' \subset N_\mathbb{Q}$. Then X is classified according to the theory of the last subsections by the following propositions:

Proposition 2.15. *X is Gorenstein if and only if $\sum_i f_i \in M$ or equivalently $\forall \, n' \in N' \cap \sigma : |n'| \in \mathbb{Z}$.*

Proof. Follows from remark 2.14.(8). $\qquad \qquad \square$

Proposition 2.16. ([Re80, p. 294-295], [Mat, 14-3-1]).
X is canonical (resp. terminal) if and only if $\forall \, n' \in N' \cap \square \setminus \{0\} : |n'| \geq 1, \quad (resp. \ |n'| > 1),$ where $\square = \{(a_1, \ldots, a_n) \in N_\mathbb{Q} \mid 0 \leq a_i < 1\} \subset N_\mathbb{Q}$.

Proof. Since the primitive elements $n' \in N' \cap \square \setminus \{0\}$ correspond to exceptional prime divisors in some resolution (see construction 2.18 below), the claim will follow from the discrepancy calculation in remark 2.19. $\qquad \qquad \square$

Remark 2.17. The relation to theorems 2.10, 2.11 is given as follows (in [Re80] theorem 2.11 is reduced to proposition 2.16): After choice of an r-th root of unity ε, $r = |G|$, there is the isomorphism $G \cong N'/N$ (see construction 2.13), any $g \in G$ has a unique representative $\frac{1}{r}(a_1 \ldots, a_n) \in N' \cap \square$. Then $g \in G$ operates by $g : x_i \mapsto \varepsilon^{a_i} x_i$. The condition "$G$ small" means that the elements e_i are primitive.

Construction 2.18. (Toric method to construct resolutions of abelian quotient singularities). Resolutions of $X = \mathbb{A}^n_{\mathbb{C}}/G = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ are constructed by subdividing the cone σ. Write Σ for the fan determined by σ. The following process leads to a desingularisation (see [Da, §8], [Fu, 2.6]): Adding a new ray $\tau = \mathbb{Q}_{\geq 0} n'$ determined by an element $n' \in (N' \cap \square) \setminus \{0\}$ (exists, if X is singular) and subdividing the cone determines a fan $\widetilde{\Sigma}$ with a morphism (a refinement) of fans $\widetilde{\Sigma} \to \Sigma$ and thus a proper birational morphism of the corresponding toric varieties $\widetilde{X} \to X$. The exceptional divisor E is the toric variety associated to the fan $\operatorname{Star}(\tau)$. Proceeding this way, eventually one is lead to a desingularisation, because certain multiplicities [Da, p. 124], [Fu, p. 48] decrease with every step.

Remark 2.19. (Calculation of discrepancy).
For quotients $\mathbb{A}^n_{\mathbb{C}}/G = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]^G = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M']$, $G \subset \operatorname{GL}(n, \mathbb{C})$ finite small, a canonical divisor is given as $K_{\mathbb{A}^n_{\mathbb{C}}/G} = -D_1 - \ldots - D_n$, where D_i is the prime divisor corresponding to the ray $\mathbb{Q}_{\geq 0} e_i \subset N_{\mathbb{Q}}$ (any e_i is primitive, that is not an integral multiple of some element of $\sigma \cap N'$, because G is small). For r a common multiple of the orders of elements of G the Weil divisor $r K_X$ is Cartier (because $r M \subseteq M'$), the r-th tensor product of the canonical sheaf is the line bundle $\omega_X^{\otimes r} = \mathcal{O}_X(r K_X)$ and via the correspondence between Cartier divisors resp. line bundles and piecewise linear functions on fans ([Da, §6], [Fu, 3.4, p. 66]) $\omega_X^{\otimes r}$ corresponds to the linear functional $f = r \sum_i f_i \in M'$ (that is *(multiplicity of D_i in $r K_X$)* $= -\langle e_i, f \rangle$).
Let $\varphi : Y \to X$ be the partial resolution constructed by adding a ray corresponding to some primitive $n' \in N' \cap \sigma$, let E be its exceptional divisor. In the ramification formula $r K_Y = \varphi^*(r K_X) + aE$ one has $r K_Y = -(r \sum_i D_i + rE)$, and one has $\varphi^*(r K_X) = -r \sum_i D_i - r|n'|E$ since the pull-back of a line bundle is given by the same piecewise linear function on $N_{\mathbb{Q}}$ as the original one, so E occurs with multiplicity $-\langle n', f \rangle = -r|n'|$. Thus $a = r(|n'| - 1)$ and the discrepancy is $(|n'| - 1)E$.

Notation: We say that the cyclic group μ_r of order r operates as $\frac{1}{r}(a_1, \ldots, a_n)$ on $\mathbb{A}^n_{\mathbb{C}}$, if for an eigenbasis $x_1, \ldots, x_n$ and a primitive r-th root of unity ε a generator $g \in \mu_r$ maps $x_i \mapsto \varepsilon^{a_i} x_i$.

Example 2.20. Consider the quotient $X = \mathbb{A}^n_{\mathbb{C}}/\mu_r$, the operation given as $\frac{1}{r}(1, \ldots, 1)$. X is Gorenstein if and only if n is a multiple of r. It is $M' = \{m \in M \mid |m| \in \mathbb{Z}r\}$ and $N' = N + \mathbb{Z}\frac{1}{r}(1, \ldots, 1)$.
A resolution $\varphi : Y \to X$ is obtained by adding the ray $\mathbb{Q}_{\geq 0} n'$ for $n' = \frac{1}{r}(1, \ldots, 1)$, giving a fan consisting of the cones σ_i, $i = 1, \ldots, n$, the monoid $N' \cap \sigma_i$ generated by $e_1, \ldots, e_{i-1}, \frac{1}{r} \sum_j e_j, e_{i+1}, \ldots, e_n$. The monoid $\sigma_i^\vee \cap M'$ is generated by $f_1 - f_i, \ldots, f_{i-1} - f_i, r f_i, f_{i+1} - f_i, \ldots, f_n - f_i$, the corresponding algebra $\mathbb{C}[\sigma_i^\vee \cap M']$ is the polynomial algebra $\mathbb{C}[\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, x_i^r, \frac{x_{i+1}}{x_i} \ldots, \frac{x_n}{x_i}]$.
This corresponds to the blow-up of the singular point, as exceptional divisor one has $E \cong \mathbb{P}^{n-1}_{\mathbb{C}}$. The discrepancy is $(|n'| - 1)E = \frac{n-r}{r}E$, therefore X is canonical, if $n \geq r$, and terminal, if $n > r$. Explicit calculation with differentials: $s_{\mathbb{A}^n_{\mathbb{C}}} = (dx_1 \wedge \ldots \wedge dx_n)^{\otimes r}$ generates the line bundle $\omega_{\mathbb{A}^n_{\mathbb{C}}}^{\otimes r}$, further it is G-invariant, so it comes from some s_X on X that generates $\omega_X^{\otimes r}$. The pull-back s_Y of s_X is given on the affine space $\operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M']$ by $(const.) \cdot (x_i^r)^{n-r} (d(\frac{x_1}{x_i}) \wedge \ldots \wedge d(\frac{x_{i-1}}{x_i}) \wedge dx_i^r \wedge d(\frac{x_{i+1}}{x_i}) \wedge \ldots \wedge d(\frac{x_n}{x_i}))^{\otimes r}$. In this affine space $\operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M']$ the prime divisor E is the zero set of the coordinate function x_i^r, s_Y has a zero (resp. pole) of order $n - r$ in E. This leads to the ramification formula $r K_Y = f^*(r K_X) + (n - r)E$.

2.2 G-Hilbert schemes

2.2.1 The G-Hilbert scheme for abelian groups

As discussed in the last section, quotients $\mathbb{A}^n_{\mathbb{C}}/G$ by finite abelian groups G are toric varieties. In this case the G-Hilbert scheme G-Hilb$_{\mathbb{C}}\mathbb{A}^n_{\mathbb{C}}$ has a natural torus operation, its irreducible component Y birational to $\mathbb{A}^n_{\mathbb{C}}/G$ is a not necessarily normal toric variety and the natural birational morphism $Y \to \mathbb{A}^n_{\mathbb{C}}/G$ is equivariant with respect to the torus operations. In this section we review the method of Nakamura [Nm01] to determine the fan of Y resp. its normalisation.

Let $\mathbb{A}^n_{\mathbb{C}}/G$ be the quotient by a finite diagonal (abelian) subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ of order r. We use the notations of construction 2.13 of subsection 2.1.3, in particular the algebra $\mathbb{C}[M]$ is graded by the characters of G. The operation of T_M on $\mathbb{A}^n_{\mathbb{C}}$ induces operations of $T_M/G = T_{M'}$ on $\mathbb{A}^n_{\mathbb{C}}/G$ and on G-Hilb$_{\mathbb{C}}\mathbb{A}^n_{\mathbb{C}}$.

We identify closed points of G-Hilb$_{\mathbb{C}}\mathbb{A}^n_{\mathbb{C}}$ with G-stable ideals $I \subset \mathbb{C}[\sigma^{\vee} \cap M] = \mathbb{C}[x_1, \ldots, x_n]$ such that $\mathbb{C}[x_1, \ldots, x_n]/I$ is isomorphic to the regular representation of G. Of the closed points $I \in$ G-Hilb$_{\mathbb{C}}\mathbb{A}^n_{\mathbb{C}}$ the ones which are torus fixed points are the ideals generated by monomials. A G-graph is defined to be a subset $\Gamma \subset \sigma^{\vee} \cap M$ corresponding to a set of monomials that arises as the complement of all monomials of $\mathbb{C}[x_1, \ldots, x_n]$ by those which are contained in some ideal $I \in Y \subseteq$ G-Hilb$_{\mathbb{C}}\mathbb{A}^n_{\mathbb{C}}$ corresponding to a torus fixed point ([Nm01, Def. 1.4], note that in addition we require that I lies in the component Y). Write $I(\Gamma)$ for the corresponding ideal. We sometimes also consider a G-graph, here defined as a subset of $\sigma^{\vee} \cap M$, as a set of monomials.

Definition 2.21. ([Nm01, Def. 1.5]). *For a G-graph Γ define a map $M \to \Gamma$, $m \mapsto m_{\Gamma}$ such that both m and m_{Γ} correspond to the same character of G. Define*

$$S(\Gamma) := \left\langle m - m_{\Gamma} \mid m \in \sigma^{\vee} \cap M \right\rangle_{\mathrm{monoid}} \subseteq M' \quad and \quad \sigma(\Gamma) := S(\Gamma)^{\vee} \subset N'_{\mathbb{Q}}$$

$S(\Gamma)$ is a finitely generated monoid, $S(\Gamma)$ generates the lattice M' [Nm01, Le. 1.7]. For the computation it is useful, that if the ideal $I(\Gamma)$ is generated by a set of some monomials x^m, then $S(\Gamma)$ is generated by the elements $m - m_{\Gamma}$ [Nm01, Le. 1.8]. The algebra $\mathbb{C}[\sigma(\Gamma)^{\vee} \cap M']$ is the normalisation of $\mathbb{C}[S(\Gamma)]$ [Nm01, Def. 1.9].

Definition 2.22. ([Nm01, Def. 1.9]). *For a G-graph Γ define*

$$I^{\mathrm{vers}}(\Gamma) = \left\langle y^m - x^{m - m_{\Gamma}} y^{m_{\Gamma}} \mid m \in \sigma^{\vee} \cap M \right\rangle \subset \mathbb{C}[S(\Gamma)][y_1, \ldots, y_n]$$

Again it suffices to consider a set of elements $m \in \sigma^{\vee} \cap M$ such that the monomials x^m generate the ideal $I(\Gamma)$. One shows that the ideals $I^{\mathrm{vers}}(\Gamma)$ define flat families of G-clusters over the affine varieties $V(\Gamma) := \mathrm{Spec}\,\mathbb{C}[S(\Gamma)]$ [Nm01, Le. 2.3.(i)].

Remark 2.23. [CMT06, Ex. 4.12, Rem. 4.13] exhibit an example of a finite abelian subgroup $G \subset \mathrm{GL}(3, \mathbb{C})$ and a torus fixed point $I \in$ G-Hilb$\mathbb{A}^3_{\mathbb{C}}$ that does not lie in Y. This provides a counterexample to [Nm01, Cor. 2.4].

Let Γ be a G-graph with n-dimensional cone $\sigma(\Gamma)$ and let τ be a 1-codimensional face of $\sigma(\Gamma)$. The closure of the orbit corresponding to τ in $\mathrm{Spec}\,\mathbb{C}[S(\Gamma)]$ (see e.g. [Fu, p. 53]) is $\mathrm{Spec}\,\mathbb{C}[x^{m^*}] \cong \mathbb{A}^1_{\mathbb{C}}$ with m^* a generator of $M' \cap \sigma(\Gamma)^{\vee} \cap \tau^{\perp}$ of the form $m^* = m_- - m_+$ for some $m_- \in M \cap \sigma^{\vee}$, $m_+ = (m_-)_{\Gamma}$ satisfying some properties [Nm01, Le. 2.5]. The deformation $I^{\mathrm{vers}}(\Gamma)$ can be restricted to $\mathrm{Spec}\,\mathbb{C}[x^{m^*}]$, one obtains

$$I(\Gamma, m^*) = \left\langle \{y^{m_-} - x^{m^*} y^{m_+}\} \cup \{y^m \mid m \in (\sigma^{\vee} \cap M) \setminus \Gamma, m - m_{\Gamma} \notin \tau^{\perp}\} \right\rangle \subset \mathbb{C}[x^{m^*}][y_1, \ldots, y_n]$$

This deformation can be extended to $\mathbb{P}^1_{\mathbb{C}}$, the fiber over the additional torus fixed point is an ideal corresponding to a G-graph Γ'. One says, that Γ and Γ' are related by a G-igsaw transformation [Nm01, Def. 2.6].

By [Nm01, Rem. 2.10] the affine varieties $V(\Gamma) = \operatorname{Spec}\mathbb{C}[S(\Gamma)]$ for all G-graphs Γ can be glued to a not necessarily normal toric variety W, likewise the ideals $I^{\mathrm{vers}}(\Gamma)$ can be glued to an ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_W[x_1, \ldots, x_n]$ defining a flat family of G-clusters over W.

Theorem 2.24. ([Nm01, Thm. 2.11]).
Let $G \subset \operatorname{GL}(n, \mathbb{C})$ be a finite subgroup and Y the irreducible component of G-Hilb$_{\mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$ birational to $\mathbb{A}^n_{\mathbb{C}}/G$. Then the flat family of G-clusters given by the ideal $\mathscr{I} \subset \mathcal{O}_W[x_1, \ldots, x_n]$ defines isomorphisms $W \xrightarrow{\sim} Y$ and $W_{\mathrm{norm}} \xrightarrow{\sim} Y_{\mathrm{norm}}$. $\qquad\square$

Remark 2.25. Y need not to be normal in general [CMT06, Ex. 5.6, Cor. 5.8].

For $G \subset \operatorname{SL}(3, \mathbb{C})$ it is possible to obtain a rather explicit description of Y which allows to prove the following theorem.

Theorem 2.26. ([Nm01, Section 4]).
Let G be a finite abelian subgroup of $\operatorname{SL}(3, \mathbb{C})$. Then the irreducible component Y of G-Hilb$_{\mathbb{C}} \mathbb{A}^3_{\mathbb{C}}$ birational to $\mathbb{A}^3_{\mathbb{C}}/G$ is smooth and a crepant resolution of $\mathbb{A}^3_{\mathbb{C}}/G$. Its fan consists of exactly $|G|$ 3-dimensional cones. $\qquad\square$

By [BKR01] Y coincides with G-Hilb$\mathbb{A}^3_{\mathbb{C}}$ and G-Hilb$\mathbb{A}^3_C$ is smooth and a crepant resolution of $\mathbb{A}^3_{\mathbb{C}}/G$ for any finite subgroup $G \subset \operatorname{SL}(3, \mathbb{C})$. For the G-Hilbert scheme in the case of abelian subgroups $G \subset \operatorname{SL}(3, \mathbb{C})$ see also [Re97], [Re99], [ItNj00], [CrRe02].

2.2.2 Examples of G-Hilbert schemes for abelian groups

This section gives some examples of fans of G-Hilbert schemes for abelian groups G.

Example 2.27. (see also [ItNm99, Thm. 12.3]). Consider $G = \mu_r \subset \operatorname{SL}(2, \mathbb{C})$ operating on $\mathbb{A}^2_{\mathbb{C}}$ by $\frac{1}{r}(1, -1)$. The quotient $\mathbb{A}^2_{\mathbb{C}}/G$ is given by the cone $\sigma = \mathbb{Q}_{\geq 0}(1, 0) + \mathbb{Q}_{\geq 0}(0, 1)$ with respect to the lattice $N' = \mathbb{Z}(1, 0) + \mathbb{Z}\frac{1}{r}(r - 1, 1) \subset N_{\mathbb{Q}}$, it is $M' = \mathbb{Z}(r, 0) + \mathbb{Z}(1, 1)$. The G-graphs are

$$\Gamma_0 = \{1, x_2, \ldots, x_2^{r-1}\}, \ \ldots \ , \Gamma_i = \{1, x_1, \ldots, x_1^i, x_2, \ldots, x_2^{r-i-1}\}, \ \ldots \ , \Gamma_{r-1} = \{1, x_1 \ldots, x_1^{r-1}\}$$

It is $S(\Gamma_i) = \langle (i + 1, i + 1 - r), (-i, r - i) \rangle$, $\sigma_i := \sigma(\Gamma_i) = \mathbb{Q}_{\geq 0}(r - i - 1, i + 1) + \mathbb{Q}_{\geq 0}(r - i, i)$, the monoids $S(\Gamma_i)$ and $M' \cap \sigma_i^\vee$ coincide, its algebras are polynomial algebras in two variables $\mathbb{C}[S(\Gamma_i)] = \mathbb{C}[M' \cap \sigma_i^\vee] = \mathbb{C}[s_i, t_i]$ where $s_i = x_1^{i+1}/x_2^{r-(i+1)}$, $t_i = x_2^{r-i}/x_1^i$. The deformation over $\operatorname{Spec}\mathbb{C}[S(\Gamma_i)] = \operatorname{Spec}\mathbb{C}[s_i, t_i]$ of the ideal $I(\Gamma_i) = \langle x_1^{i+1}, x_1 x_2, x_2^{r-i} \rangle$ is

$$I^{\mathrm{vers}}(\Gamma_i) = \left\langle y_1^{i+1} - s_i y_2^{r-(i+1)}, y_1 y_2 - s_i t_i, y_2^{r-i} - t_i y_1^i \right\rangle \subset \mathbb{C}[s_i, t_i][y_1, y_2]$$

The fan of Y is obtained by subdividing σ, adding the 1-dimensional cones $\tau_i = \mathbb{Q}_{\geq 0}(r - i, i)$ for $i = 1, \ldots, r - 1$. These correspond to exceptional prime divisors $E_i \cong \mathbb{P}^1_{\mathbb{C}}$, E_i is given by the equation $t_{i-1} = 0$ resp. $s_i = 0$ on the affine chart corresponding to σ_{i-1} resp. σ_i, it is covered by the affine charts $\operatorname{Spec}\mathbb{C}[s_{i-1} = x_1^i/x_2^{r-i}]$ resp. $\operatorname{Spec}\mathbb{C}[t_i = x_2^{r-i}/x_1^i]$ and has coordinates $(a = x_1^i : b = x_2^{r-i})$.

We draw the case $r = 6$:
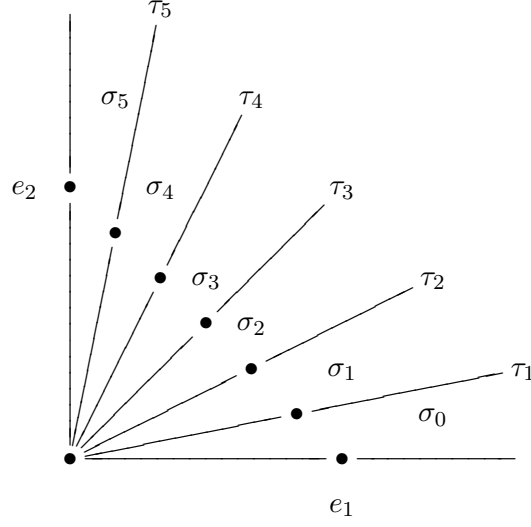
Two G-graphs Γ_{i-1} and Γ_i are related by a G-igsaw transformation with respect to the common face τ_i of $\sigma(\Gamma_{i-1})$ and $\sigma(\Gamma_i)$. The corresponding deformation over $E_i \cong \mathbb{P}^1_{\mathbb{C}}$, parametrised by $(a = x_1^i : b = x_2^{r-i})$, is given by $\langle y_1^i - \frac{a}{b}y_2^{r-i}, y_1 y_2, y_2^{r-i+1} \rangle$ over $\operatorname{Spec} \mathbb{C}[\frac{a}{b}]$ with fiber $I(\Gamma_{i-1})$ over the origin and $\langle y_1^{i+1}, y_1 y_2, y_2^{r-i} - \frac{b}{a}y_1^i \rangle$ over $\operatorname{Spec} \mathbb{C}[\frac{b}{a}]$ with fiber $I(\Gamma_i)$ over the origin. These are the restrictions of $I^{\mathrm{vers}}(\Gamma_{i-1})$ resp. $I^{\mathrm{vers}}(\Gamma_i)$ to $\{t_{i-1} = 0\}$ resp. $\{s_i = 0\}$.

Example 2.28. (see also example 2.20). $G = \mu_r \subset \mathrm{SL}(n, \mathbb{C})$ operating by $\frac{1}{r}(1, \ldots, 1)$ on $\mathbb{A}^n_{\mathbb{C}}$. The G-Hilbert scheme is smooth, it is crepant for $n = r$.
The quotient $\mathbb{A}^n_{\mathbb{C}}/G$ is given by the cone $\sigma = \mathbb{Q}_{\geq 0}e_i + \ldots + \mathbb{Q}_{\geq 0}e_n$ with respect to the lattice $N' = \ldots + \mathbb{Z}e_{i-1} + \mathbb{Z}\frac{1}{r}\sum_j e_j + \mathbb{Z}e_{i+1} + \ldots \subset N_{\mathbb{Q}}$, it is $M' = \{\sum_j m_j f_j \mid \sum_j m_j \in r\mathbb{Z}\} \subseteq M$.
The G-graphs are $\Gamma_i = \{1, x_i, \ldots, x_i^{r-1}\}$ for $i = 1, \ldots, n$.
For a G-graph Γ_i one has $I(\Gamma_i) = \langle \ldots, x_{i-1}, x_i^r, x_{i+1}, \ldots \rangle$, $S(\Gamma_i) = \langle \ldots, f_{i-1} - f_i, r f_i, f_{i+1} - f_i, \ldots \rangle$, $\sigma(\Gamma_i) = \ldots \mathbb{Q}_{\geq 0}e_{i-1} + \mathbb{Q}_{\geq 0}\sum_j e_j + \mathbb{Q}_{\geq 0}e_{i+1} + \ldots$. The algebra $S(\Gamma_i) = \mathbb{C}[\ldots, \frac{x_{i-1}}{x_i}, x_i^r, \frac{x_{i+1}}{x_i}, \ldots]$ is a polynomial algebra and coincides with $\mathbb{C}[M' \cap \sigma(\Gamma_i)]$. It is

$$I^{\mathrm{vers}}(\Gamma_i) = \left\langle \ldots, y_{i-1} - \frac{x_{i-1}}{x_i}y_i, y_i^r - x_i^r, y_{i+1} - \frac{x_{i+1}}{x_i}y_i, \ldots \right\rangle$$

Any pair of G-graphs $\{\Gamma_i, \Gamma_{i'}\}$ is related by a G-igsaw transformation: The cones $\sigma(\Gamma_i), \sigma(\Gamma_{i'})$ have the common face $\tau = \mathbb{Q}_{\geq 0}(\sum_j e_j) + \mathbb{Q}_{\geq 0}e_1 + \ldots + \mathbb{Q}_{\geq 0}e_{i-1} + \mathbb{Q}_{\geq 0}e_{i+1} + \ldots + \mathbb{Q}_{\geq 0}e_{i'-1} + \mathbb{Q}_{\geq 0}e_{i'+1} + \ldots \mathbb{Q}_{\geq 0}e_n$. The G-igsaw transformation from Γ_i to $\Gamma_{i'}$ corresponds to the following deformation over $\operatorname{Spec} \mathbb{C}[x^{m^*}]$ with $m^* = m_- - m_+ = f_{i'} - f_i$:

$$I(\Gamma_i, m^*) = \left\langle y_{i'} - \frac{x_{i'}}{x_i}y_i, y_1, \ldots, y_{i-1}, y_i^r, y_{i+1}, \ldots, y_{i'-1}, y_{i'}^r, y_{i'+1}, \ldots \right\rangle$$

In the case $n = 3$ the fan, more precisely its section with the plane $\{\sum_j n_j e_j \mid \sum_j n_j = 1\} \subset N_{\mathbb{Q}}$ (the generator of $N' \cap \mathbb{Q}_{\geq 0}\sum_j e_j$ lies in this plane if and only if $r = n = 3$) looks as follows
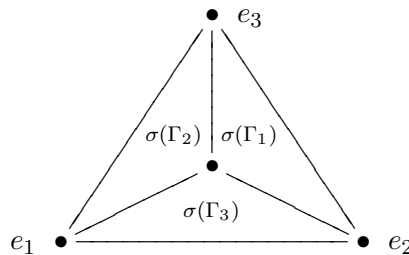
Example 2.29. (see also [Cr01], [ChRo01, Example 4.2]).
Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{g \in \mathrm{SL}(3, \mathbb{C}) \mid g \ diagonal, \ g^2 = 1\} \subset \mathrm{SL}(3, \mathbb{C})$.
The G-Hilbert scheme is smooth and crepant (as always for $G \subset \mathrm{SL}(3, \mathbb{C})$).
The G-graphs are

$$\Gamma_0 = \{1, x_1, x_2, x_3\}, \ \Gamma_1 = \{1, x_2, x_3, x_2 x_3\}, \ \Gamma_2 = \{1, x_3, x_1, x_3 x_1\}, \ \Gamma_1 = \{1, x_1, x_2, x_1 x_2\}$$

Its algebras are $\mathbb{C}[S(\Gamma_0)] = \mathbb{C}[x_1 x_2 / x_3, x_2 x_3 / x_1, x_3 x_1 / x_2]$, $\mathbb{C}[S(\Gamma_1)] = \mathbb{C}[x_1 / x_2 x_3, x_2^2, x_3^2]$, etc..
One has the fan
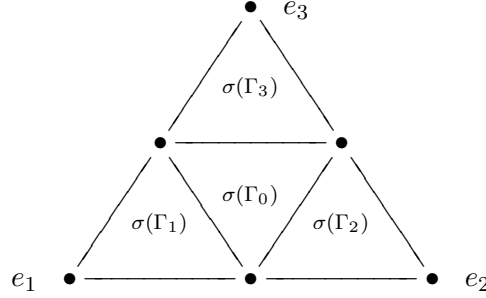
with cones $\sigma(\Gamma_0) = \mathbb{Q}_{\geq 0}(e_1 + e_2) + \mathbb{Q}_{\geq 0}(e_2 + e_3) + \mathbb{Q}_{\geq 0}(e_3 + e_1)$, $\sigma(\Gamma_1) = \mathbb{Q}_{\geq 0}e_1 + \mathbb{Q}_{\geq 0}(e_1 + e_2) + \mathbb{Q}_{\geq 0}(e_1 + e_3)$, etc..

Example 2.30. ([Se05]). Let $G = \mu_7 \subset \mathrm{SL}(3, \mathbb{C})$ operate on $\mathbb{A}^3_{\mathbb{C}}$ by $\frac{1}{7}(1, 2, 4)$.
The G-Hilbert scheme is smooth and crepant.
The G-graphs are

$$\Gamma_1 = \{1, x_1, x_1^2, \ldots, x_1^6\}, \ \Gamma_2 = \{1, x_2, x_2^2, \ldots, x_2^6\}, \ \Gamma_3 = \{1, x_3, x_3^2, \ldots, x_3^6\},$$
$$\Gamma_{12} = \{1, x_1, x_2, x_1 x_2, x_2^2, x_1 x_2^2, x_2^3\}, \ \Gamma_{23} = \{1, x_2, x_3, x_2 x_3, x_3^2, x_2 x_3^2, x_3^3\},$$
$$\Gamma_{31} = \{1, x_3, x_1, x_3 x_1, x_1^2, x_3 x_1^2, x_1^3\},$$
$$\Gamma_{123} = \{1, x_1, x_2, x_1 x_2, x_3, x_1 x_3, x_2 x_3\}$$

The fan arises by adding the rays $\mathbb{Q}_{\geq 0}n_i$ for $n_1 = \frac{1}{7}(1, 2, 4)$, $n_2 = \frac{1}{7}(4, 1, 2)$, $n_3 = \frac{1}{7}(2, 4, 1)$ and subdividing into the cones $\sigma(\Gamma_1) = \mathbb{Q}_{\geq 0}e_2 + \mathbb{Q}_{\geq 0}e_3 + \mathbb{Q}_{\geq 0}n_1$, $\ldots$, $\sigma(\Gamma_{12}) = \mathbb{Q}_{\geq 0}e_3 + \mathbb{Q}_{\geq 0}n_1 + \mathbb{Q}_{\geq 0}n_2$, $\ldots$, $\sigma(\Gamma_{123}) = \mathbb{Q}_{\geq 0}n_1 + \mathbb{Q}_{\geq 0}n_2 + \mathbb{Q}_{\geq 0}n_3$.

Example 2.31. $G = \mu_r \subset \mathrm{Sp}(4, \mathbb{C})$ operating on $\mathbb{A}^4_{\mathbb{C}}$ by $\frac{1}{r}(1, 1, -1, -1)$, $r \geq 3$.
This is an example of a terminal singularity, there does not exist a crepant resolution (there are no elements $n' \in \square \cap N'$ such that $|n'| = 1$ corresponding to crepant exceptional prime divisors), but the G-Hilbert scheme is smooth.
It is $M' = \langle r f_1, f_1 - f_2, f_1 + f_3, f_1 + f_4 \rangle$, $N' = \langle e_1, e_2, e_3, \frac{1}{r}(e_1 + e_2 - e_3 - e_4) \rangle$.
There are the $4r - 4$ G-graphs

$$\Gamma_1 = \{1, x_1, x_1^2, \ldots, x_1^{r-1}\}, \ \ldots \ , \Gamma_4 = \{1, x_4, x_4^2, \ldots, x_4^{r-1}\}$$
$$\Gamma_{13}^i = \{1, x_1, \ldots, x_1^{r-i-1}, x_3, \ldots, x_3^i\}, \quad i = 1, \ldots, r-2$$
$$\Gamma_{14}^i = \{1, x_1, \ldots, x_1^{r-i-1}, x_4, \ldots, x_4^i\}, \quad i = 1, \ldots, r-2$$
$$\Gamma_{23}^i = \{1, x_2, \ldots, x_2^{r-i-1}, x_3, \ldots, x_3^i\}, \quad i = 1, \ldots, r-2$$
$$\Gamma_{24}^i = \{1, x_2, \ldots, x_2^{r-i-1}, x_4, \ldots, x_4^i\}, \quad i = 1, \ldots, r-2$$

For Γ_1 one has (similarly for $\Gamma_2, \Gamma_3, \Gamma_4$):

$$I(\Gamma_1) = \langle x_1^r, x_2, x_3, x_4 \rangle,$$
$$S(\Gamma_1) = \langle r f_1, f_2 - f_1, f_3 - (r-1)f_1, f_4 - (r-1)f_1 \rangle,$$
$$\sigma(\Gamma_1) = \mathbb{Q}_{\geq 0}e_2 + \mathbb{Q}_{\geq 0}e_3 + \mathbb{Q}_{\geq 0}e_4 + \mathbb{Q}_{\geq 0}(e_1 + e_2 + (r-1)(e_3 + e_4)),$$
$$\mathbb{C}[S(\Gamma_1)] = \mathbb{C}[M' \cap \sigma(\Gamma_1)^{\vee}] = \mathbb{C}[x_1 / x_2, x_3 / x_1^{r-1}, x_4 / x_1^{r-1}, x_1^r].$$

For Γ^i_{13} one has (similarly for $\Gamma^i_{14}, \Gamma^i_{23}, \Gamma^i_{24}$):

$$I(\Gamma^i_{13}) = \left\langle x_1^{r-i}, x_1 x_3, x_3^{i+1}, x_2, x_4 \right\rangle,$$
$$S(\Gamma^i_{13}) = \left\langle (r-i)f_1 - if_3, f_2 - f_1, (i+1)f_3 - (r-i-1)f_1, f_4 - f_3 \right\rangle,$$
$$\sigma(\Gamma^i_{13}) = \mathbb{Q}_{\geq 0} e_2 + \mathbb{Q}_{\geq 0} e_4 + \mathbb{Q}_{\geq 0}(i(e_1 + e_2) + (r-i)(e_3 + e_4))$$
$$\qquad + \mathbb{Q}_{\geq 0}((i+1)(e_1 + e_2) + (r-i-1)(e_3 + e_4)),$$
$$\mathbb{C}[S(\Gamma^i_{13})] = \mathbb{C}[M' \cap \sigma(\Gamma^i_{13})^\vee] = \mathbb{C}[x_1^{r-i}/x_3^i, x_2/x_1, x_3^{i+1}/x_1^{r-i-1}, x_4/x_3].$$

The G-graphs are related by G-igsaw transformations as follows:
$\Gamma_1 \rightsquigarrow \Gamma^1_{13}$ by $m^* = f_3 - (r-1)f_1$, $\Gamma^i_{13} \rightsquigarrow \Gamma^{i+1}_{13}$ by $m^* = (i+1)f_3 - (r-i-1)f_1$, $\Gamma^{r-2}_{13} \rightsquigarrow \Gamma_3$ by
$m^* = (r-1)f_3 - f_1$ and similarly for $\Gamma_1, \Gamma^1_{14}, \ldots, \Gamma^i_{14}, \ldots, \Gamma^{r-2}_{14}, \Gamma_4$ and the pairs $\{2,3\}, \{2,4\}$.
$\Gamma_1 \rightsquigarrow \Gamma_2$ by $m^* = f_2 - f_1$, similarly $\Gamma_3 \rightsquigarrow \Gamma_4$ by $m^* = f_4 - f_3$.
$\Gamma^i_{13} \rightsquigarrow \Gamma^i_{14}$ by $m^* = f_4 - f_3$ and similarly $\Gamma^i_{23} \rightsquigarrow \Gamma^i_{24}$, $\Gamma^i_{13} \rightsquigarrow \Gamma^i_{23}$, $\Gamma^i_{14} \rightsquigarrow \Gamma^i_{24}$.
One may draw this as follows:
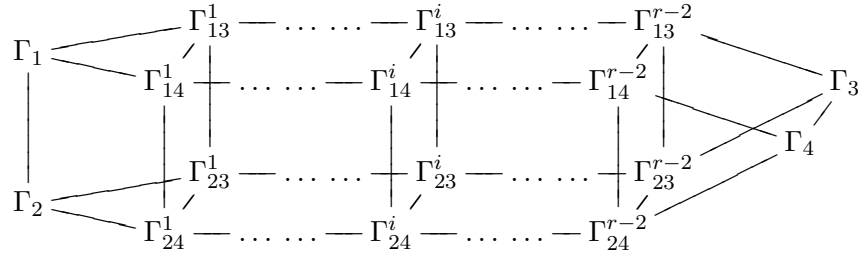
There are $r - 1$ exceptional divisors E_i corresponding to the additional rays

$$\tau_i = \mathbb{Q}_{\geq 0}(i(e_1 + e_2) + (r-i)(e_3 + e_4)), \quad i = 1, \ldots, r-1$$

The generators for $N' \cap \tau_i$ are $n_i := \frac{1}{r}(i(e_1 + e_2) + (r-i)(e_3 + e_4))$, for these $|n_i| = 2$, so for the discrepancy $a_i E_i$ of E_i in the formula $K_Y = f^* K_X + a_i E_i$ one has $a_i = 1$.

Example 2.32. The G-Hilbert scheme for μ_r operating as $\frac{1}{r}(1, p, -1, -p)$ with r, p relatively prime is not in general smooth. For example consider the case $\frac{1}{5}(1, 2, 4, 3)$: The G-cluster $I = \left\langle x_1 x_3, x_2 x_4, x_1^2, x_1 x_2, x_2^2, x_2 x_3, x_3^2, x_3 x_4, x_4^2, x_4 x_1 \right\rangle$ has tangent space (see subsection 4.4.4) $T_I\, G\text{-Hilb}_{\mathbb{C}}\, \mathbb{A}^4_{\mathbb{C}} = \mathrm{Hom}^G_S(I, S/I)$, where $S = \mathbb{C}[x_1, x_2, x_3, x_4]$, of dimension greater than 4.

Remark 2.33. The examples $\frac{1}{r}(1, p, -1, -p)$, $\frac{1}{r}(1, 1, -1, -1)$ considered above are related to the examples $\frac{1}{r}(1, p, -p)$, $\frac{1}{r}(1, 1, -1)$ considered in [Ke04].

Higher dimensional generalisations

... of examples 2.27, 2.29: In [ChRo01, Equation (4.10)] the finite abelian subgroup

$$A_r(n) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g \text{ diagonal}, \ g^{r+1} = 1\} \subset \mathrm{SL}(n, \mathbb{C})$$

is introduced, special cases are $A_r(2) \subset \mathrm{SL}(2, \mathbb{C})$ in example 2.27 and $A_1(3) \subset \mathrm{SL}(3, \mathbb{C})$ in example 2.29, see [ChRo01, Example 4.2] for the group $A_r(3)$ with arbitrary r. In [ChRo01], [ChRo04] the case $G = A_r(4)$ is studied in detail, it is shown that the component Y of G-Hilb$_{\mathbb{C}}\, \mathbb{A}^4_{\mathbb{C}}$ is smooth, but the resolution is not crepant ([ChRo01, Thm. 5.3 for $r = 1$, Thm. 6.1 for general r], [ChRo04, Thm. 3.4 for $r = 1$, Thm. 4.1 for general r]), further it is explicitly described how crepant resolutions are constructed from Y by blowing down certain divisors, this procedure being not unique, and how these different crepant resolutions dominated by Y are connected by flops. In [ChRo03] also the case $A_1(5)$ is investigated.

... of example 2.30: The group $G = \mu_{2^n - 1} \subset \mathrm{SL}(n, \mathbb{C})$ with operation $\frac{1}{2^n - 1}(1, 2, 4, \ldots, 2^{n-1})$ on $\mathbb{A}^n_{\mathbb{C}}$ has a singular point at the origin, for any n the G-Hilbert scheme provides a crepant resolution, see [Se04] for the case $n = 4$ and [Se05] for the general case.

2.2.3   Reflection groups

A family of examples generalising the examples of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are finite subgroups $G \subset \mathrm{SL}(n,\mathbb{C})$ that are subgroups of reflection groups of index 2. For these the G-Hilbert scheme and in particular the structure of the exceptional fiber for certain non abelian groups $G \subset \mathrm{SL}(n,\mathbb{C})$ is investigated in [GNS00], [GNS04], [ChRo04], [Té04], [Té06], [BoSa05].

Definition 2.34. ([Bour, Groupes et algèbres de Lie, Ch. V, §2.1], [Co76], [GNS00], [GNS04]). *A finite subgroup $\widetilde{G} \subset \mathrm{GL}(n,\mathbb{C})$ is called a (complex) reflection group if it is generated by quasireflections (also called complex reflections or pseudo-reflections, these are elements $g \in \mathrm{GL}(n,\mathbb{C})$ whose eigenspace of eigenvalue 1 is $(n-1)$-dimensional, see also subsection 2.1.2). A reflection group is called a real reflection group if there is a G-stable $\mathbb{R}$-subspace $V' \subset \mathbb{C}^n$ of dimension n such that $V'_{\mathbb{C}} = \mathbb{C}^n$.*

Let $\widetilde{G} \subset \mathrm{GL}(n,\mathbb{C})$ be a finite reflection group and $G := \widetilde{G} \cap \mathrm{SL}(n,\mathbb{C})$. The exact sequence $0 \longrightarrow \mathrm{SL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(n,\mathbb{C}) \overset{\det}{\longrightarrow} \mathbb{C}^* \longrightarrow 0$ induces an exact sequence $0 \longrightarrow G \longrightarrow \widetilde{G} \overset{\det}{\longrightarrow} \mu_r \longrightarrow 0$ for some $r \geq 1$. If G is of index 2 in $\widetilde{G}$, elements $g \in \widetilde{G}$ have determinant ± 1 and there is the exact sequence $0 \longrightarrow G \longrightarrow \widetilde{G} \overset{\det}{\longrightarrow} \{\pm 1\} \longrightarrow 0$. This is the case for any nontrivial real reflection group $\widetilde{G}$, since its elements have determinant ± 1.

Example 2.35. Finite subgroups $G \subset \mathrm{SO}(3)$ are subgroups of index 2 of real reflection groups $\widetilde{G}$ such that $G = \widetilde{G} \cap \mathrm{SL}(3,\mathbb{C})$ ([GNS04, Section 2.7] and example 2.36 below), the finite subgroups $G \subset \mathrm{SL}(2,\mathbb{C})$ are subgroups of index 2 of reflection groups $\widetilde{G}$ such that $G = \widetilde{G} \cap \mathrm{SL}(2,\mathbb{C})$ ([GV83, Section 3], [GNS00, 1.8], [GNS04, Section 2.7]).
For example the cyclic group $\mu_r \cong \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle \subset \mathrm{SL}(2,\mathbb{C})$, ξ a primitive r-th root of unity, is subgroup of index 2 of the dihedral group $\left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \subset \mathrm{GL}(2,\mathbb{C})$, which is generated by reflections.

Example 2.36. Let R be a root system in a real vector space $\mathbb{R}^n$ (see [Bour, Groupes et algèbres de Lie]). The Weyl group $W(R)$ of R defines a real reflection group $W \subset \mathrm{GL}(n,\mathbb{C})$, the subgroup $W_+ := W \cap \mathrm{SL}(,\mathbb{C})$ is of index 2 in W.
For example the root system (A_n) has the symmetric group S_{n+1} as Weyl group, the group W_+ is isomorphic to the alternating group A_{n+1}.

Theorem 2.37. ([Bour, Groupes et algèbres de Lie, Ch. V, §5], [Co76], [GNS00, Thm. 1.2], [GNS04, Thm. 1.4]). *Let V an n-dimensional $\mathbb{C}$-vector space and $\widetilde{G} \subset \mathrm{GL}(V)$ be a finite reflection group. Then for the linear operation of $\widetilde{G}$ on the $\mathbb{C}$-algebra $S := \mathrm{Sym}_{\mathbb{C}} V \cong \mathbb{C}[x_1, \ldots, x_n]$ one has:*

(i) *Let $S^{\widetilde{G}} \subseteq S$ be the subalgebra of $\widetilde{G}$-invariant elements. Then the $S^{\widetilde{G}}$-module S is free of rank $|\widetilde{G}|$ and has a basis of homogeneous polynomials. There are n algebraically independent homogeneous polynomials $f_1, \ldots, f_n$ that generate $S^{\widetilde{G}}$ as $\mathbb{C}$-algebra, these satisfy $\prod_i \deg(f_i) = |\widetilde{G}|$.*

(ii) *Let χ be the character of $\widetilde{G}$ given by $g \mapsto \det(g)^{-1}$. Then there is a homogeneous polynomial $f_0 \in S$ of degree $\deg(f_0) = \sum_{i=1}^n (\deg(f_i) - 1)$ such that the isotypic component of S corresponding to χ is $S^{\widetilde{G}} f_0$. The element f_0 is given as the jacobian of $f_1, \ldots, f_n$.* □

By (i), for a reflection group $\widetilde{G}$ the quotient $\mathbb{A}^n_{\mathbb{C}}/\widetilde{G}$ is isomorphic to $\mathbb{A}^n_{\mathbb{C}}$ (see also subsection 2.1.2).

For the quotient morphism $\pi : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}/G$ the general fiber $\mathbb{C}(\mathbb{A}^n_{\mathbb{C}})$ is isomorphic to the regular representation over $\mathbb{C}(\mathbb{A}^n_{\mathbb{C}}/G)$. Thus any fiber of π contains the regular representation. We study the fiber over the origin. Let $\mathfrak{m} \subset S = \mathbb{C}[x_1, \ldots, x_n]$ be the maximal ideal corresponding to the origin $O \in \mathbb{A}^n_{\mathbb{C}}$ and $\mathfrak{n} \subset S^G$ the maximal ideal corresponding to $\pi(O)$. Similarly define $\widetilde{\pi}$ and $\widetilde{\mathfrak{n}}$ for $\widetilde{G}$.

The fiber of $\widetilde{\pi}$ over $\widetilde{\pi}(O)$ is given by the algebra $S/\widetilde{\mathfrak{n}}S$. Because of theorem 2.37.(i) $S/\widetilde{\mathfrak{n}}S$ is isomorphic to the regular representation, that is $S/\widetilde{\mathfrak{n}}S \cong \bigoplus_{i=0}^{\widetilde{r}} \widetilde{V}_i^{\oplus \dim(\widetilde{V}_i)}$ as $\widetilde{G}$-representation, if $\widetilde{V}_0, \ldots, \widetilde{V}_{\widetilde{r}}$ are the irreducible representations of $\widetilde{G}$ up to isomorphism.

Remark 2.38. Assume that $G := \widetilde{G} \cap \mathrm{SL}(n, \mathbb{C})$ is of index 2 in the reflection group $\widetilde{G}$, then there is the exact sequence $0 \to G \to \widetilde{G} \to \{1, \alpha\} \to 0$. The representation theory of these groups are related as follows, write $\mathrm{Res}, \mathrm{Ind}$ for $\mathrm{Res}^{\widetilde{G}}_G, \mathrm{Ind}^{\widetilde{G}}_G$. The proofs will require some elementary representation theory and are left to the reader, see also [GV83, Section 3].
(1) Write $\chi \widetilde{V}$ for the representation $\widetilde{V}$ of $\widetilde{G}$ tensored with the character χ of theorem 2.37.(ii). $\chi \widetilde{V}$ has the property (and can be defined by) $\mathrm{Ind} \, \mathrm{Res} \, \widetilde{V} \cong \widetilde{V} \oplus \chi \widetilde{V}$.
(2) $\widetilde{G}/G = \{1, \alpha\}$ operates on the representation ring of G: To a representation V of G one associates a representation V^α such that $\mathrm{Res} \, \mathrm{Ind} \, V \cong V \oplus V^\alpha \cong \mathrm{Res} \, \mathrm{Ind} \, V^\alpha$. Alternatively V^α is given as the conjugate representation with respect to an element $g \in \widetilde{G} \setminus G$.
(3) Always $\chi(\mathrm{Ind} \, V) \cong \mathrm{Ind} \, V$, $\mathrm{Ind}(V^\alpha) \cong \mathrm{Ind} \, V$ and $\mathrm{Res}(\chi \widetilde{V}) \cong \mathrm{Res} \, \widetilde{V}$, $(\mathrm{Res} \, \widetilde{V})^\alpha \cong \mathrm{Res} \, \widetilde{V}$.
(4) For an irreducible representation V of G the $\widetilde{G}$-representation $\mathrm{Ind} \, V$ is irreducible if and only if $V \not\cong V^\alpha$ (use the adjunction $(\mathrm{Ind}, \mathrm{Res})$). Otherwise $\mathrm{Ind} \, V$ decomposes as $\widetilde{G}$-representation into $\mathrm{Ind} \, V \cong \widetilde{V} \oplus \chi \widetilde{V}$ for some irreducible $\widetilde{G}$-representation $\widetilde{V}$.
(5) For an irreducible representation $\widetilde{V}$ of $\widetilde{G}$ the representation $\mathrm{Res} \, \widetilde{V}$ is irreducible if and only if $\widetilde{V} \not\cong \chi \widetilde{V}$ (again adjunction $(\mathrm{Ind}, \mathrm{Res})$). Otherwise $\mathrm{Res} \, \widetilde{V} \cong V \oplus V^\alpha$ for some irreducible G-representation V such that $V \not\cong V^\alpha$.

As a representation of $\widetilde{G}$ one has $S/\widetilde{\mathfrak{n}}S \cong \mathbb{C}[\widetilde{G}]$ by theorem 2.37.(i). $\mathbb{C}[\widetilde{G}] = \mathrm{Ind} \, \mathbb{C}[G]$ decomposes as G-representation into $\mathrm{Res} \, \mathrm{Ind} \, \mathbb{C}[G] \cong 2\mathbb{C}[G]$, so $S/\widetilde{\mathfrak{n}}S \cong \mathbb{C}1 \oplus \mathbb{C}f_0 \oplus 2\bigoplus_{i=1}^r V_i^{\oplus \dim(V_i)}$, if $V_0, V_1, \ldots, V_r$ are the irreducible representations of G up to isomorphism with V_0 the trivial one.

Theorem 2.39. ([GNS00, Thm. 1.3, Thm. 1.6], [GNS04, Thm. 1.4]).
Let $\widetilde{G} \subset \mathrm{GL}(n, \mathbb{C})$ be a finite reflection group such that $G := \widetilde{G} \cap \mathrm{SL}(n, \mathbb{C})$ is a subgroup of index 2 in $\widetilde{G}$. Let $f_0, f_1, \ldots, f_n$ be as in theorem 2.37. Then, using the notations above, for the linear operation of G on $S = \mathbb{C}[x_1, \ldots, x_n]$ one has:

  (i) *The algebra of G-invariants is the subalgebra of S generated by $f_0, f_1, \ldots, f_n$.*

  (ii) *For the fiber $S/\mathfrak{n}S$ of π over $\pi(O)$ one has $S_k \subseteq \mathfrak{n}S$ for $k > m := \deg(f_0)$ and thus $S/\mathfrak{n}S \cong V_0 \oplus \bigoplus_{i=1}^r V_i^{\oplus 2 \dim(V_i)}$, if $V_0, V_1, \ldots, V_r$ are the irreducible representations of G up to isomorphism with V_0 the trivial one. Further, the components of $S/\mathfrak{n}S$ of degrees k and $m - k$ are isomorphic, i.e. $(S/\mathfrak{n}S)_k \cong (S/\mathfrak{n}S)_{m-k}$ for $k \in \{1, \ldots, m - 1\}$.* $\square$

Quotients of $\mathbb{A}^n_{\mathbb{C}}$ by subgroups $G = \widetilde{G} \cap \mathrm{SL}(n, \mathbb{C})$ of index 2 of reflection groups $\widetilde{G} \subset \mathrm{GL}(n, \mathbb{C})$ are hypersurface singularities, it is $f_0^2 \in S^{\widetilde{G}}$, so $f_0^2 = h(f_1, \ldots, f_n)$ for some polynomial h and $S^G = \mathbb{C}[f_0, f_1, \ldots, f_n] \cong \mathbb{C}[y_0, \ldots, y_n]/\langle y_0^2 - h(y_1, \ldots, y_n) \rangle$ as quotient of a polynomial ring $\mathbb{C}[y_0, \ldots, y_n]$. Equations for the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ are listed in subsection 1.1.1.

Subgroups of index 2 of reflection groups are subject of the following papers:

- In [GNS00] the structure of the fiber over the origin for a trihedral group of order 12 and simple subgroups of $SL(3, \mathbb{C})$ of order 60 and 168 are studied.

- [GNS04] determines the fiber over the origin for the finite subgroups of $SO(3)$.

- In [ChRo04] the G-Hilbert scheme for G the alternating group $A_4 \subset SL(3, \mathbb{C})$ (corresponds to the root system (A_3)) is studied in detail.

- Quotients $\mathbb{A}^n_{\mathbb{C}}/W_+$ and their resolution by $W_+\text{-}\operatorname{Hilb}_{\mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$ for subgroups W_+ of index 2 of Weyl groups corresponding to the root systems $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, $A_1 \times B_2$, $A_1 \times G_2$, A_3, B_3 are considered in [Té04],[Té06] and the structure of the fiber over the origin is determined.

- [BoSa05] relate the McKay correspondences for finite subgroups of $SL(2, \mathbb{C})$ and $SO(3)$.

2.3   An example

In this section we consider again the example of linear operations of cyclic groups μ_r on $\mathbb{A}_{\mathbb{C}}^n$ of type $\frac{1}{r}(1, \ldots, 1)$, but using different methods. It is formulated such that it applies to group schemes of roots of unity in positive characteristic or over non algebraically closed fields as well. We show that the G-Hilbert scheme arises as the blow-up of the singular point, compare to subsection 2.2.1 and example 2.28. Here we use the relative G-Hilbert scheme construction of section 4.3 and relate the G-Hilbert scheme directly to the Proj of a graded algebra without introducing local coordinates.

We determine the tautological sheaves and consider the K-theoretic McKay correspondence in this case.

2.3.1   The G-Hilbert scheme as a blow-up

We will work in the following setting: Let K be a field and $G = \mu_r = \operatorname{Spec} K[y]/\langle y^r - 1 \rangle$ the group scheme of r-th roots of unity. There are r isomorphism classes of irreducible representations $V_0, V_1, \ldots, V_{r-1}$ with V_i corresponding to the subcoalgebra $\langle y^i \rangle_K \subseteq K[y]/\langle y^r - 1 \rangle$.

Let V be an n-dimensional representation of G over K that is isotypic with its simple components isomorphic to V_1. G can be considered as a subgroup scheme of the 1-dimensional torus $T = \operatorname{Spec} K[y, y^{-1}]$ that operates on $\mathbb{A}_K(V) = \operatorname{Spec} K[x_1, \ldots, x_n]$ by $x_i \mapsto y \otimes x_i$.

Let $\pi : \mathbb{A}_K(V) \to \mathbb{A}_K(V)/G$ be the quotient. Let $S := \operatorname{Sym}_K V \cong K[x_1, \ldots, x_n]$, let $O \in \mathbb{A}_K(V)$ be the origin, $\mathfrak{m} \subset S$ the corresponding maximal ideal, $\overline{O} := \pi(O) \in \mathbb{A}_K^n/G$ with corresponding maximal ideal $\mathfrak{n} \subset S^G$. Then $\mathfrak{n}$ is generated by the monomials of degree r.

On S there is the $\mathbb{Z}$-grading by degree $S = \bigoplus_{i \geq 0} S^{(i)}$, which is the same as the grading by the characters of T. After restriction to $G \subset T$ one has a $\mathbb{Z}/r\mathbb{Z}$-grading $S = \bigoplus_{i=0}^{r-1} S^{(i)}$ by the characters of G.

In general we will write $(\cdot)^{(k)}$ for the isotypic part corresponding to the simple representation V_k of G of a representation or a G-sheaf on a G-scheme with trivial operation. Further, we will use the notations $M := \mathbb{A}_K(V)$, $X := M/G$, let $\tau : \text{G-Hilb}_K M \to X$ be the natural morphism, $E := \tau^{-1}(\overline{O})$. Sometimes $\text{G-Hilb}_K M$ is considered as an X-scheme via τ, it then coincides with the relative G-Hilbert scheme $\text{G-Hilb}_X M$ (see section 4.3).

Proposition 2.40. *There are isomorphisms of X-schemes*

$$\text{G-Hilb}_X M \;\cong\; \mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)}) \;\cong\; (\text{Bl}_{\mathfrak{m}} M)/G \;\cong\; \text{Bl}_{\mathfrak{n}} X$$

Proof. We show that there are isomorphisms

$$\text{G-Hilb}_X M \;\cong\; \mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)}) \;\cong\; \text{Bl}_{\mathfrak{n}} X$$

- $\text{G-Hilb}_X M \cong \mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)})$.
Use the description of $\mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)})$ as a Quot scheme (see subsection 4.1.1). The isomorphism $\text{G-Hilb}_X M \to \mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)})$ is given by the maps

$$
\begin{array}{rcl}
\text{G-Hilb}_X M(T) & \to & \mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)})(T) \\
[0 \to \mathscr{I} \to \pi_{T*}\mathcal{O}_{M_T} \to \mathcal{O}_Z \to 0] & \mapsto & [0 \to \mathscr{I}^{(1)} \to (\pi_{T*}\mathcal{O}_{M_T})^{(1)} \to (\mathcal{O}_Z)^{(1)} \to 0]
\end{array}
$$

The inverse maps take $\mathscr{H} \subset (\pi_{T*}\mathcal{O}_{M_T})^{(1)}$ to the ideal $\langle \mathscr{H} \rangle \subset \pi_{T*}\mathcal{O}_{M_T}$ generated by $\mathscr{H}$.

- $\mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)}) \cong \text{Bl}_{\mathfrak{n}} X$.
It is $\mathbb{P}_X((\pi_* \mathcal{O}_M)^{(1)}) = \operatorname{Proj}_X(\operatorname{Sym}_{\mathcal{O}_X}((\pi_* \mathcal{O}_M)^{(1)}))$ and $\text{Bl}_{\mathfrak{n}} X = \operatorname{Proj}_X(\text{Bl}_{\mathfrak{n}} \mathcal{O}_X)$, where the sheaf

of graded $\mathcal{O}_X$-algebras $\mathrm{Bl}_\mathfrak{n}\,\mathcal{O}_X$ is the blow-up algebra $\mathrm{Bl}_\mathfrak{n}\,\mathcal{O}_X = \mathcal{O}_X \oplus \mathfrak{n} \oplus \mathfrak{n}^2 \oplus \dots$. To show the above isomorphism, we show that the part $(\mathrm{Sym}_{\mathcal{O}_X}((\pi_*\mathcal{O}_M)^{(1)}))^{[r]}$ of the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}((\pi_*\mathcal{O}_M)^{(1)})$ of degrees that are multiples of r is isomorphic to $((\mathrm{Bl}_\mathfrak{m}\,\mathcal{O}_M)^{[r]})^G$, which is isomorphic to the blow-up algebra $\mathrm{Bl}_\mathfrak{n}\,\mathcal{O}_X$ (this suffices, see e.g. [EGA, II, (3.1.8)]).
Notation: $\mathrm{Sym}_{\mathcal{O}_X}((\pi_*\mathcal{O}_M)^{(1)})$ can be written as $\mathrm{Sym}_{S^G}(\mathfrak{m}^{(1)})$ etc..
One has $S[y_1,\dots,y_n]/\langle\{x_iy_j - x_jy_i \mid 1 \le i < j \le n\}\rangle \cong \mathrm{Sym}_S\,\mathfrak{m} \cong \mathrm{Bl}_\mathfrak{m}\,S$ (see for example [FL, Thm. IV.2.2]), the natural homomorphism of graded S^G-algebras

$$\mathrm{Sym}_{S^G}(\mathfrak{m}^{(1)}) \to \mathrm{Sym}_S\,\mathfrak{m} \cong \mathrm{Bl}_\mathfrak{m}\,S = S \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \dots \tag{2.3}$$

is injective (verification left to the reader). Taking the subalgebras of degrees that are multiples of r and on the right side in addition G-invariants one has a homomorphism of graded S^G-algebras

$$(\mathrm{Sym}_{S^G}(\mathfrak{m}^{(1)}))^{[r]} \to ((\mathrm{Bl}_\mathfrak{m}\,S)^{[r]})^G = \mathrm{Bl}_\mathfrak{n}(S^G) = S^G \oplus \mathfrak{n} \oplus \mathfrak{n}^2 \oplus \dots$$

which furthermore is surjective.
Note that $(\mathrm{Bl}_\mathfrak{m}\,M)/G \cong \mathrm{Proj}_X(((\mathrm{Bl}_\mathfrak{m}\,\mathcal{O}_M)^{[r]})^G) \cong \mathrm{Proj}_X(\mathrm{Bl}_\mathfrak{n}(\mathcal{O}_X)) = \mathrm{Bl}_\mathfrak{n}\,X$. □

Outside $\overline{O}$ the morphism $\tau : \text{G-Hilb}_K\,M \to X$ is an isomorphism, its fiber $E = \tau^{-1}(\overline{O})$ is isomorphic to $\mathbb{P}_K^{n-1}$.

2.3.2   Tautological sheaves

Let $0 \to \mathscr{H} \to (\pi_*\mathcal{O}_M)_P^{(1)} \to \mathcal{O}_P(1) \to 0$ be the universal quotient of $P := \mathbb{P}_X((\pi_*\mathcal{O}_M)^{(1)})$. It is $P \cong \mathrm{Proj}_X(\mathscr{S})$, where $\mathscr{S}$ is the graded $\mathcal{O}_X$-algebra $\mathscr{S} = \mathrm{Sym}_{\mathcal{O}_X}((\pi_*\mathcal{O}_M)^{(1)})$. Let $\mathscr{J} \subset \mathscr{S}_P$ be the ideal generated by $\mathscr{H}$, then $0 \to \mathscr{H} \to (\pi_*\mathcal{O}_M)_P^{(1)} \to \mathcal{O}_P(1) \to 0$ is the part of degree 1 of the exact sequence $0 \to \mathscr{J} \to \mathscr{S}_P \to \bigoplus_{i\ge 0}\mathcal{O}_P(i) \to 0$.
Let $0 \to \mathscr{I} \to (\pi_*\mathcal{O}_M)_Y \to \mathcal{O}_Z \to 0$ be the universal quotient of $Y = \text{G-Hilb}_X\,M$. By proposition 2.40 one has the isomorphism of X-schemes $Y \xrightarrow{\sim} \mathbb{P}_X((\pi_*\mathcal{O}_M)^{(1)})$ given by the isotypic part $0 \to \mathscr{I}^{(1)} \to (\pi_*\mathcal{O}_M)_Y^{(1)} \to \mathcal{O}_Z^{(1)} \to 0$ of the universal quotient of $Y = \text{G-Hilb}_X\,M$ for the representation V_1. Identifying these schemes via this isomorphism, $0 \to \mathscr{I}^{(1)} \to (\pi_*\mathcal{O}_M)_Y^{(1)} \to \mathcal{O}_Z^{(1)} \to 0$ becomes isomorphic to the universal quotient $0 \to \mathscr{H} \to (\pi_*\mathcal{O}_M)_P^{(1)} \to \mathcal{O}_P(1) \to 0$ of $P = \mathbb{P}_X((\pi_*\mathcal{O}_M)^{(1)})$.
The tautological sheaves are defined as in definition 1.5, here $\mathscr{F}_i$ is given as the isotypical part $\mathcal{O}_Z^{(i)}$ for the representation V_i forgetting the G-sheaf structure.

Proposition 2.41. *The tautological sheaves $\mathscr{F}_0, \mathscr{F}_1, \dots, \mathscr{F}_{r-1}$ on $\text{G-Hilb}_X\,M \cong P = \mathrm{Proj}_X(\mathscr{S})$ corresponding to the isomorphism classes $V_0, V_1, \dots, V_{r-1}$ of irreducible representations of G are $\mathscr{F}_i \cong \mathcal{O}_P(i)$.*

Proof. For $i = 0$ clearly $\mathscr{F}_0 \cong \mathcal{O}_P$, for $i = 1$ the isomorphism $\mathscr{F}_1 \cong \mathcal{O}_P(1)$ follows from the isomorphism between the universal quotients $0 \to \mathscr{I}^{(1)} \to (\pi_*\mathcal{O}_M)_Y^{(1)} \to \mathcal{O}_Z^{(1)} \to 0$ and $0 \to \mathscr{H} \to (\pi_*\mathcal{O}_M)_P^{(1)} \to \mathcal{O}_P(1) \to 0$.
For arbitrary $i \in \{0, \dots, r-1\}$ one has the isomorphism of $\mathcal{O}_X$-modules

$$\mathscr{S}^{(i)} = \mathrm{Sym}_{\mathcal{O}_X}^i((\pi_*\mathcal{O}_M)^{(1)}) \cong (\pi_*\mathcal{O}_M)^{(i)}$$

or equivalently $\mathrm{Sym}_{S^G}^i(\mathfrak{m}^{(1)}) \cong (\mathfrak{m}^i)^{(i)}$, coming from the injective homomorphism (2.3). Since both $\mathscr{I}$ and $\mathscr{J}$ are generated in degree 1, the quotients $0 \to \mathscr{I}^{(i)} \to (\pi_*\mathcal{O}_M)_Y^{(i)} \to \mathcal{O}_Z^{(i)} \to 0$ and $0 \to \mathscr{J}^{(i)} \to \mathscr{S}_P^{(i)} \to \mathcal{O}_P(i) \to 0$ coincide, therefore $\mathscr{F}_i = \mathcal{O}_Z^{(i)} \cong \mathcal{O}_P(i)$. □

It follows in particular, that for the restrictions of the tautological sheaves to $E \cong \mathbb{P}_K^{n-1}$ one has $\mathscr{F}_i|_E \cong \mathcal{O}_E(i)$.

2.3.3 K-theoretic McKay correspondence

In this subsection we explicitly describe the K-theoretic McKay correspondence for the present example, see also [Re97, Ex. 4.3].

It is $Y \cong \mathbb{A}_E(\mathcal{O}_E(r))$: $\mathrm{Bl}_{\mathfrak{m}} M$ is a bundle $\mathrm{Bl}_{\mathfrak{m}} M \cong \mathbb{A}_{\widetilde{E}}(\mathcal{O}_{\widetilde{E}}(1)) = \mathrm{Spec}_{\widetilde{E}}(\bigoplus_{k \geq 0} \mathcal{O}_{\widetilde{E}}(k))$ over the exceptional $\widetilde{E} \cong \mathbb{P}_K^{n-1}$ (see also [EGA, II, (8.7.8)]). $Y = \text{G-Hilb}_K M$ arises as $(\mathrm{Bl}_{\mathfrak{m}} M)/G$ (see proposition 2.40). G operates trivially on $\widetilde{E}$, so $E = \widetilde{E}$. One obtains $Y \cong \mathrm{Spec}_E((\bigoplus_{k \geq 0} \mathcal{O}_E(k))^G)$ $= \mathrm{Spec}_E(\bigoplus_{k \geq 0} \mathcal{O}_E(kr)) = \mathbb{A}_E(\mathcal{O}_E(r))$.

Because in general, for a vector bundle $V = \mathbb{A}_T(\mathscr{E})$, $\mathscr{E}$ locally free of finite type over a noetherian scheme T, one has the isomorphism $K(T) \cong K(V)$ by [SGA6, Exposé IX, Prop. 1.6], it follows that

$$s^* : K(Y) \to K(E)$$

is an isomorphism, where $s : E \to Y$ is the inclusion.
Using the well known description of the K-theory of $E \cong \mathbb{P}_K^{n-1}$ (see e.g. [Man, Ex. 3.11, p. 17 and Thm. 4.5, p. 19], [FL, Thm. V.2.3, p. 115], [SGA6, Exposé VI, Thm. 1.1, and p. 374]) one obtains isomorphism of rings

$$
\begin{array}{ccccc}
K(Y) & \cong & K(E) & \cong & \mathbb{Z}[v, v^{-1}]/\langle (v-1)^n \rangle \\
[\mathcal{O}_Y(1)] & \leftrightarrow & [\mathcal{O}_E(1)] & \leftrightarrow & v
\end{array}
$$

For the representation ring $R(G) = K^G(\mathrm{Spec}\, K) \cong K^G(M)$ one has the isomorphism of rings

$$R(G) \cong \mathbb{Z}[w, w^{-1}]/\langle w^r - 1 \rangle$$

where $w = [V_1]$.

K-theoretic McKay correspondence: There is the homomorphism of $\mathbb{Z}$-modules

$$
\begin{array}{ccccc}
R(G) & \cong & K^G(M) & \to & K(Y) \\
[V_i^\vee] & \leftrightarrow & [\mathcal{O}_M \otimes_K V_i^\vee] & \mapsto & [\mathscr{F}_i = \mathcal{O}_Y(i)]
\end{array}
$$

It is an isomorphism if $r = n$, in which case the resolution $Y \to X$ is crepant. It is not multiplicative.

Remark 2.42. ($\mathbb{P}_K^n$ as G-Hilbert scheme).
We restrict this example to the fiber over the origin. The projective space $\mathbb{P}_K^n$ can be considered as G-Hilbert scheme $\text{G-Hilb}_K M$ for $M = \mathrm{Spec}\, B$, $B = K[x_0, \ldots, x_n]/\langle \prod_i x_i^{a_i} \mid \sum_i a_i = n+1 \rangle$ and $G = \mu_{n+1}$. Then as above the K-theory of $\mathbb{P}_K^n$ can be described via the McKay correspondence, the tautological sheaves are $\mathcal{O}_{\mathbb{P}_K^n}, \mathcal{O}_{\mathbb{P}_K^n}(1), \ldots, \mathcal{O}_{\mathbb{P}_K^n}(n)$ and form a $\mathbb{Z}$-basis of $K(\mathbb{P}_K^n)$. Of course one also may obtain a description of the derived category of $\mathbb{P}_K^n$ via the McKay correspondence.

Part II

G-sheaves and G-Hilbert schemes

Chapter 3

G-sheaves

In this chapter we develop the theory of G-sheaves for group schemes G. A G-sheaf (or G-equivariant sheaf, G-linearised sheaf) on a G-scheme X is a sheaf of modules $\mathscr{F}$ on X with some extra structure that defines something similar to a compatible G-operation on $\mathscr{F}$. It is a generalisation of the notion of a representation.

The concept of a G-sheaf can adequately be formulated using the language of fibered categories, see e.g. [Vi05, Section 3.8]. Here we use the equivalent definition of [Mu, GIT, Ch. 1, §3]. The theory of G-sheaves developed in this chapter forms the essential basis for the constructions of G-Hilbert schemes and McKay correspondence in this thesis. It has to be elaborated in some detail because of lack of a suitable reference.

Many constructions will be made in a very general setting over an arbitrary base scheme S, in some cases we restrict to the case of a base field. For the decomposition of a G-sheaf over a G-scheme with trivial G-operation into isotypic components we will consider affine group schemes $G = \operatorname{Spec} A$ over a field K whose Hopf algebra A is cosemisimple. Although the main interest in this thesis lies in finite group schemes, almost everything in this chapter works for affine group schemes over a field with cosemisimple Hopf algebra.

We always tried come close to the natural conditions necessary for the arguments of the individual constructions. This makes clear, what assumptions are needed, further, restriction to the cases used later would not make things considerably simpler.

One main part of this chapter consists in establishing certain standard constructions, adjunctions and natural isomorphisms for sheaves in the G-equivariant setting.

Another main part is concerned with the case of trivial G-operation on the underlying scheme X. In this case the G-sheaf structure for an affine group scheme $G = \operatorname{Spec} \mathscr{A}$ on $\mathscr{F}$ can equivalently be expressed as an $\mathscr{A}$-comodule structure. As such it can be treated similar to a representation. In particular, for $G = \operatorname{Spec} A$ over a field K, if A is cosemisimple, there is a decomposition of $\mathscr{F}$ into isotypic components.

For a group scheme G over a field K a representation will be defined as a G-sheaf over K (or more generally over an extension field of K), for $G = \operatorname{Spec} A$ this is the same as an A-comodule. This includes the representation theory of finite groups, but as well applies to finite group schemes.

We summarise the contents in more detail:

Preliminary, we review definitions to fix notations and develop some basic properties needed later. This first section can be grouped into three parts:
In the first two subsections the categorical notion of an adjunction (see e.g. [McL]) is introduced in general and in particular the adjunctions (f^*, f_*) and $(f_*, f^!)$ are considered, their behaviour under base extensions and the base change homomorphism related to the adjunction (f^*, f_*).
The next two subsections contain basic definitions concerning group schemes and operations as well as Hopf algebras and comodules taken from [Mu, GIT], [Mu, AV], [Wa] resp. [Sw], [Abe], [Ka]. Here we work over a base scheme S and introduce sheaf-versions.
Forming the last two subsections, we have also included a summary about cosemisimple coalgebras over a fixed base field. These results are essentially known (compare [Sw], [Abe]), however, the results are proven and a self-contained treatment different from the sources used is obtained.

In the second section we begin to develop the theory of G-sheaves starting with the definition of [Mu, GIT, Ch. 1, §3]. We show the equivalence to another definition more similar to the one used in [BKR01] and make standard definitions such as G-subsheaves, G-equivariant homomorphisms. For a group scheme G over a field K, a representation will be defined as a G-sheaf over the spectrum of an extension field of K. G-sheaves on a G-scheme X with equivariant homomorphisms as morphisms form an abelian category.

One section contains some standard constructions for G-sheaves that extend the corresponding constructions for usual sheaves: The bifunctors $\otimes_{\mathcal{O}_X}$ and $\mathscr{H}om_{\mathcal{O}_X}$ for G-sheaves on a fixed G-scheme X are introduced as well as the functors f^*, f_* for equivariant morphisms of G-schemes. Further, relations between these functors are treated: We establish adjunctions (f^*, f_*), $(f_*, f^!)$ between the functors on categories of G-sheaves constructed before and natural isomorphisms $f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$, $f^*(\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{G}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{E}, \mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$ and $\mathscr{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G})$, again extending the corresponding adjunctions and natural isomorphisms for usual sheaves.

The last section is about G-sheaves over schemes with trivial G-operation. Then, for affine group schemes $G = \operatorname{Spec} \mathscr{A}$ a G-sheaf structure is equivalent to an $\mathscr{A}$-comodule structure and can be decomposed as an $\mathscr{A}$-comodule according to a decomposition of $\mathscr{A}$ into a direct sum of subcoalgebras. In particular, for group schemes $G = \operatorname{Spec} A$ over fields K with A cosemisimple the decomposition of A into simple subcoalgebras induces an isotypic decomposition with summands corresponding to the isomorphism classes of simple representations of G.

3.1 Preliminaries

In this preliminary section we remind basic definitions and simple properties to fix notations and to outline on which notions this capter is based. It can be subdivided into three parts consisting of two subsections each.

Subsections 4.1.1 and 4.1.2 are about adjunctions in general and more specifically about the adjunctions (f^*, f_*), $(f_*, f^!)$ between pull-back and push-forward functors with respect to morphisms of schemes and their behaviour under base extensions. Further we consider the base change homomorphism related to the adjunction (f^*, f_*).

Subsections 4.1.3 and 4.1.4 are concerned with basic definitions concerning group schemes and operations as well as Hopf algebras and comodules, here sheaf-versions are introduced, the definitions are made relative to a given base scheme S.

In 4.1.5 and 4.1.6 we go through the theory of simple and cosemisimple coalgebras over a base field, there are references [Sw], [Abe] that include most results, however, we choose our own way giving proofs and obtaining a treatment different from the sources used. Because in this chapter concerning G-sheaves and representations we use formulations in terms of comodules, we will not comment on the dual notions of simple and semisimple modules and algebras, for these see [Bour, Algébre, Ch. VIII].

3.1.1 Adjunctions

Adjunctions. ([McL, Ch. IV]). Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{D} \to \mathcal{C}$ functors. An adjunction (F, H) consists of morphisms of functors

$$\varepsilon : F \circ H \to Id_{\mathcal{D}}, \qquad \eta : Id_{\mathcal{C}} \to H \circ F$$

(ε is called the counit, η the unit of the adjunction), such that the diagrams

$$
\begin{array}{ccc}
F \xrightarrow{\quad Id_F \quad} F & \qquad & H \xrightarrow{\quad Id_H \quad} H \\
F\eta \searrow \quad \nearrow \varepsilon F & & \eta H \searrow \quad \nearrow H\varepsilon \\
FHF & & HFH
\end{array}
\qquad (3.1)
$$

commute. Equivalent is an isomorphism of bifunctors

$$\mathrm{Mor}_{\mathcal{D}}(F\cdot, \cdot) \cong \mathrm{Mor}_{\mathcal{C}}(\cdot, H\cdot)$$

We are mostly concerned with additive functors between abelian categories. Then the bijections $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Hd)$ are isomorphisms of abelian groups. Sometimes the Hom-sets have some extra module structure that is respected by the functors considered, in which case the above bijections are module isomorphisms.

The adjunction (f^*, f_*). ([EGA1, 0, (4.4)]). Let $f : X \to Y$ be a morphism of schemes. Then there is the adjunction (f^*, f_*), consisting of morphisms of functors $\varepsilon : f^* f_* \to Id_{\mathcal{C}(X)}$, $\eta : Id_{\mathcal{C}(Y)} \to f_* f^*$ or a morphism of bifunctors $\mathrm{Hom}_X(f^*\cdot, \cdot) \cong \mathrm{Hom}_Y(\cdot, f_*\cdot)$, where $\mathcal{C} = \mathrm{Mod}$ (or $\mathcal{C} = \mathrm{Qcoh}, \mathrm{Coh}$ if the functors f^*, f_* restrict to these subcategories).

The adjunction $(f_*, f^!)$ for finite flat morphisms. ([Bour, Algebra I, Ch. II, §5.1, Remark (4)], [Ha, AG, Ch. III, Ex. 6.10], [Ha, RD, Ch. III, §6]).

Let $f : X \to Y$ be a finite flat morphism of schemes. We work with categories of quasicoherent sheaves $\mathcal{C}(X) = \mathrm{Qcoh}(X)$, $\mathcal{C}(Y) = \mathrm{Qcoh}(Y)$. As a very special case of Grothendieck duality there is an adjunction $(f_*, f^!)$, consisting of morphisms of functors

$$\varepsilon : f_* f^! \to Id_{\mathcal{C}(Y)}, \quad \eta : Id_{\mathcal{C}(X)} \to f^! f_*$$

or an isomorphism of bifunctors

$$\mathrm{Hom}_{\mathcal{O}_X}(\cdot, f^! \cdot) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f_* \cdot, \cdot)$$

where the functor $f^! : \mathcal{C}(Y) \to \mathcal{C}(X)$ is given by $f^! \mathscr{G} = \bar{f}^* \mathscr{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathscr{G})$ for quasicoherent $\mathcal{O}_Y$-modules $\mathscr{G}$, here $\mathscr{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathscr{G})$ is considered as an $f_* \mathcal{O}_X$-module via the first argument and $\bar{f}$ is the morphism of ringed spaces $\bar{f} : (X, \mathcal{O}_X) \to (Y, f_* \mathcal{O}_X)$.

The morphism f is affine, so $X \cong \mathrm{Spec}_Y \mathscr{A} \xrightarrow{f} Y$, where $\mathscr{A} = f_* \mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_Y$-algebras of finite rank. Then quasicoherent $\mathcal{O}_X$-modules can be identified with quasicoherent $\mathscr{A}$-modules on Y and the above adjunction can be written as isomorphisms functorial in $\mathscr{F}$ and $\mathscr{G}$ for $\mathscr{A}$-modules $\mathscr{F}$ and $\mathcal{O}_Y$-modules $\mathscr{G}$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{G})) & \cong & \mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \\
\varphi & \mapsto & (f \mapsto \varphi(f)(1)) \\
(f \mapsto (a \mapsto \psi(af))) & \leftarrow & \psi
\end{array}
$$

or as homomorphisms functorial in $\mathscr{F}$ resp. $\mathscr{G}$

$$
\begin{array}{ccccc}
\varepsilon(\mathscr{G}) : & f_* f^! \mathscr{G} & = & \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{G}) & \to & \mathscr{G} \\
& & & \varphi & \mapsto & \varphi(1)
\end{array}
$$

$$
\begin{array}{ccccc}
\eta(\mathscr{F}) : & \mathscr{F} & \to & \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{F}) & = & f^! f_* \mathscr{F} \\
& f & \mapsto & (a \mapsto af)
\end{array}
$$

There is also a local version $f_* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, f^! \mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_Y}(f_* \mathscr{F}, \mathscr{G})$. The affine case is already contained in [Bour, Algebra I, Ch. II, §5.1, Remark (4)].

3.1.2   The adjunctions (f^*, f_*), $(f_*, f^!)$ and base change

In this subsection we are concerned with properties of the adjunction (f^*, f_*) and the base change morphism (3.3) $g^* f_* \to g'^* f'_*$ for a commutative diagram (3.2) introduced below.

Remark 3.1. Let

$$
\begin{array}{ccc}
& Y' & \\
f' \swarrow & & \searrow g' \\
X' & & Y \\
g \searrow & & \swarrow f \\
& X &
\end{array}
\tag{3.2}
$$

be a commutative diagram of schemes. One may define a morphism of functors $\mathrm{Mod}(Y) \to \mathrm{Mod}(X')$

$$g^* f_* \to f'_* g'^* \tag{3.3}$$

as follows (see also [SGA4, (3), Exposé XII, 4.], sometimes this is called the base change morphism):

(1) Using the adjunctions (g'^*, g'_*) and (g^*, g_*) :

$$g^* f_* \to g^* f_* g'_* g'^* = g^* g_* f'_* g'^* \to f'_* g'^*$$

(2) Using the adjunctions (f'^*, f'_*) and (f^*, f_*):

$$g^* f_* \to f'_* f'^* g^* f_* \cong f'_* g'^* f^* f_* \to f'_* g'^*$$

The morphisms $g^* f_* \to f'_* g'^*$ constructed by (1) and (2) coincide. This is can be verified using the explicit description of the functors $f^*, f_*, \dots$ and the adjunction homomorphisms. A proof using the language of fibered categories developed in [SGA1, Exposé VI] can be found in [SGA4, (3), Exposé XVII, Prop. 2.1.3]. There one obtains a useful characterisation of the homomorphisms (3.3) $g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ as morphisms of a fibered category making a certain diagram ([SGA4, (3), Exposé XVII, (2.1.3.2)]) commute.

In particular the morphism (3.3) will be considered in the case of base extensions of a morphism f: For a cartesian diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ g'\ } & Y \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{\ g\ } & X
\end{array}
$$

and an $\mathcal{O}_{Y'}$-module $\mathscr{F}$ there is the natural homomorphism of $\mathcal{O}_{X'}$-modules $g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$. Under certain conditions these homomorphisms are isomorphisms, see e.g. [EGA1, (9.3.3)].

Lemma 3.2. *For morphisms of schemes* $\dots \xleftarrow{f'} Y' \xrightarrow{g'} Y \xrightarrow{f} \dots$ *resp.* $\dots \xrightarrow{f'} X' \xrightarrow{g} X \xleftarrow{f} \dots$ *the diagrams*

commute, where η, ε resp. η', ε' are the unit, counit of the adjunctions (f^*, f_*) resp. (f'^*, f'_*).

Proof. Verification left to the reader. □

Proposition 3.3. *For a commutative diagram* (3.2) *the diagrams*

commute, where the horizontal arrows are coming from the morphism (3.3) $g^* f_* \to f'_* g'^*$ *of remark 3.1 and η, ε resp. η', ε' are the unit, counit of the adjunctions (f^*, f_*) resp. (f'^*, f'_*).*

Proof. We use the construction of the horizontal homomorphisms via the adjunctions (f'^*, f'_*) and (f^*, f_*). The commutativity of the above diagrams then follows from commutativity of the diagrams in lemma 3.2. $\qquad\square$

Remark 3.4. Consider the commutative diagram (3.2), let $\mathscr{A}$ be an $\mathcal{O}_Y$-algebra. Then both $g^* f_* \mathscr{A}$ and $f'_* g'^* \mathscr{A}$ have natural $\mathcal{O}_{X'}$-algebra structures (see e.g. [EGA1, 0, (4.2.4), (4.3.4)]) and the natural homomorphism of $\mathcal{O}_{X'}$-modules (3.3) $g^* f_* \mathscr{A} \to f'_* g'^* \mathscr{A}$ is a homomorphism of $\mathcal{O}_{X'}$-algebras: One shows that the diagrams concerning multiplication map

$$
\begin{array}{ccccccc}
g^* f_* \mathscr{A} \otimes_{\mathcal{O}_{X'}} g^* f_* \mathscr{A} \cong & g^*(f_* \mathscr{A} \otimes_{\mathcal{O}_X} f_* \mathscr{A}) & \longrightarrow & g^* f_*(\mathscr{A} \otimes_{\mathcal{O}_Y} \mathscr{A}) & \longrightarrow & g^* f_* \mathscr{A} \\
\downarrow & & & \downarrow & & \downarrow \\
f'_* g'^* \mathscr{A} \otimes_{\mathcal{O}_{X'}} f'_* g'^* \mathscr{A} \to f'_*(g'^* \mathscr{A} \otimes_{\mathcal{O}_{Y'}} g'^* \mathscr{A}) \cong & f'_* g'^*(\mathscr{A} \otimes_{\mathcal{O}_Y} \mathscr{A}) & \longrightarrow & f'_* g'^* \mathscr{A}
\end{array}
$$

and unit map

$$
\begin{array}{ccc}
 & g^* f_* \mathcal{O}_Y \longrightarrow g^* f_* \mathscr{A} \\
\mathcal{O}_{X'} \nearrow\searrow & \downarrow \qquad\qquad \downarrow \\
 & f'_* g'^* \mathcal{O}_Y \longrightarrow f'_* g'^* \mathscr{A}
\end{array}
$$

commute.

For the following proposition see also [SGA4, (3), Exposé XII, Prop. 4.4], it can be verified directly or shown using the language of fibered categories and the characterisation in [SGA4, (3), Exposé XVII, Prop. 2.1.3].

Proposition 3.5. *For a diagram with commutative squares*

$$
\begin{array}{ccccc}
X'' & \xrightarrow{\alpha''} & X' & \xrightarrow{\alpha'} & X \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
Y'' & \xrightarrow{\beta''} & Y' & \xrightarrow{\beta'} & Y
\end{array}
$$

the diagram arising from the base change morphisms as defined in remark 3.1

$$
\begin{array}{ccc}
\beta''^* \beta'^* f_* & \longrightarrow \beta''^* f'_* \alpha'^* \longrightarrow & f''_* \alpha''^* \alpha'^* \\
\wr \updownarrow & & \updownarrow \wr \\
\beta^* f_* & \longrightarrow & f''_* \alpha^*
\end{array}
$$

where $\alpha = \alpha' \circ \alpha''$, $\beta = \beta' \circ \beta''$, commutes. $\qquad\square$

The next proposition is about the behaviour of the adjunction $(f_*, f^!)$ with respect to base extensions, again its verification will be left to the reader.

Proposition 3.6. *Let $f : X = \mathrm{Spec}_Y \mathscr{A} \to Y$ be a finite flat morphism, $\alpha : Y' \to Y$ a morphism and let $f' : X' = \mathrm{Spec}_{Y'} \alpha^* \mathscr{A} \to Y'$ be the base extension of f. Then the diagrams*

$$
\begin{array}{ccc}
\alpha^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{G}) & \longrightarrow & \mathscr{H}om_{\mathcal{O}_{Y'}}(\alpha^* \mathscr{A}, \alpha^* \mathscr{G}) \\
{\scriptstyle \alpha^* \varepsilon(\mathscr{G})} \searrow & & \swarrow {\scriptstyle \varepsilon'(\alpha^* \mathscr{G})} \\
 & \alpha^* \mathscr{G} &
\end{array}
\qquad
\begin{array}{ccc}
 & \alpha^* \mathscr{F} & \\
{\scriptstyle \alpha^* \eta(\mathscr{F})} \swarrow & & \searrow {\scriptstyle \eta'(\alpha^* \mathscr{F})} \\
\alpha^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{A}, \mathscr{F}) & \longrightarrow & \mathscr{H}om_{\mathcal{O}_{Y'}}(\alpha^* \mathscr{A}, \alpha^* \mathscr{F})
\end{array}
$$

commute, where the horizontal arrows are the natural ones and η, ε resp. η', ε' are the unit, counit of the adjunctions $(f_*, f^!)$ resp. $(f'_*, f'^!)$. $\qquad\square$

3.1.3 Group schemes and operations

Group schemes. ([Mu, GIT], [Mu, AV], [Wa]). Let S be a scheme. A group scheme (G, e, m, i) over S is an S-scheme $p : G \to S$ with morphisms e, m, i over S, $e : S \to G$ the unit, $m : G \times_S G \to G$ the multiplication, $i : G \to G$ the inverse, such that the following diagrams commute (products over S):

$$(3.4)$$

As it has been defined here, a group scheme over S is a group object in the category of S-schemes, see e.g. [McL, Ch. III.6], [Vi05, Section 2.2].

Remark 3.7.
(1) For an S-scheme G the structure of a group scheme is equivalent to group structures on the sets $G(T)$ of T-valued points for S-schemes T that are functorial in T, i.e. the functor of points $\underline{G} = \mathrm{Mor}_S(\,\cdot\,, G) : (S\text{-}schemes)^\circ \to (sets)$ factors through the category of groups.
(2) If G is affine over S, then $G = \mathrm{Spec}_S \mathscr{A}$ for a sheaf of $\mathcal{O}_S$-Hopf-algebras $(\mathscr{A}, \eta, \mu, \iota, \varepsilon, \Delta)$ (see definition below), where

Algebra: $\quad \eta = p^\# : \mathcal{O}_S \to \mathscr{A}$, $\mu : \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A} \to \mathscr{A}$.
Antipode: $\quad \iota = p_* i^\# : \mathscr{A} \to \mathscr{A}$.
Coalgebra: $\quad \varepsilon = p_* e^\# : \mathscr{A} \to \mathcal{O}_S$, $\Delta = p_* m^\# : \mathscr{A} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A}$.

Hopf algebras. ([Sw], [Abe], [Ka, Ch. III], [Wa]). Let $\mathscr{A}$ be a sheaf of $\mathcal{O}_S$-algebras, where $\eta : \mathcal{O}_S \to \mathscr{A}$, $\mu : \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A} \to \mathscr{A}$ form the algebra structure, with homomorphisms $\iota : \mathscr{A} \to \mathscr{A}$, $\varepsilon : \mathscr{A} \to \mathcal{O}_S$, $\Delta : \mathscr{A} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A}$ of $\mathcal{O}_S$-algebras. Then $(\mathscr{A}, \eta, \mu, \iota, \varepsilon, \Delta)$ is an $\mathcal{O}_S$-Hopf algebra, if the following diagrams of homomorphisms of $\mathcal{O}_S$-algebras commute (tensor products over $\mathcal{O}_S$):

$$(3.5)$$

Operations. ([Mu, GIT], [Mu, AV]). Let (G, e, m, i) be a group scheme over S and X an S-scheme. A (left-)operation of G on X over S is a morphism $s_X : G \times_S X \to X$ such that the following diagrams commute (products over S):

(i)

$$
\begin{array}{c}
G \times X \\
\end{array}
$$

$e \times id_X$ $\nearrow$ $\qquad$ $\searrow$ s_X

$$
S \times X \xrightarrow{\quad \sim \quad} X
$$

(ii)

$$
G \times G \times X
$$

$m \times id_X$ $\swarrow$ $\qquad$ $\searrow$ $id_G \times s_X$

$$
G \times X \qquad\qquad\qquad G \times X
$$

s_X $\searrow$ $\qquad$ $\swarrow$ s_X

$$
G
$$

(3.6)

An S-scheme X with the structure of an operation of G over S will be called a G-scheme over S. For T an S-scheme, a T-scheme X with operation $G_T \times_T X \to X$ over T we will call a G-scheme over T.

Remark 3.8.

(1) A group scheme operation $s_X : G \times_S X \to X$ over S is equivalent to operations $G(T) \times_{S(T)} X(T) = (G \times_S X)(T) \to X(T)$ of the groups of T-valued points $G(T)$ on the sets $X(T)$ for S-schemes T that are functorial in T.

For a fixed $g \in G(T)$ one has for T-schemes T' bijections $X(T') \to X(T')$ functorial in T' and thus an isomorphism $\varphi_g : X_T \to X_T$ of T-schemes. Note that φ_g is also given by the composition

$$
X_T = T \times_T X_T \xrightarrow{\quad g \times id_{X_T} \quad} G_T \times_T X_T \xrightarrow{\quad s_{X_T} \quad} X_T
$$

The map $g \mapsto \varphi_g$ is a group homomorphism of $G(T)$ to the automorphism group of X_T over T. Later, the symbol g will be used for $\varphi_g : X_T \to X_T$ as well.

(2) In particular, for $T = G$, $g = id_G \in G(G)$ this leads to the isomorphism $\varphi_{id_G} = (\mathrm{pr}_1, s_X) : G \times_S X \to G \times_S X$ of X_G over G, where pr_1 is the projection to the first factor. It makes the diagram

$$
G \times_S X \xrightarrow[\sim]{\quad (\mathrm{pr}_1, s_X) \quad} G \times_S X
$$

s_X $\searrow$ $\quad X \quad$ $\swarrow$ p_X

commute and has the inverse $\varphi_{id_G^{-1}} = \varphi_i = (\mathrm{pr}_1, s_X \circ i_X)$.

Equivariant morphisms. A morphism $f : X \to Y$ of G-schemes is called equivariant, if the diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\; id_G \times f \;} & G \times Y \\
{\scriptstyle s_X} \downarrow & & \downarrow {\scriptstyle s_Y} \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

(3.7)

commutes.

3.1.4   Coalgebras and comodules

In this subsection we consider the coalgebra structure, that is the part $(\mathscr{A}, \varepsilon, \Delta)$ of a Hopf algebra, separatedly. We will work over a fixed base scheme S.

Coalgebras. ([Sw], [Abe], [Ka]). A sheaf of coalgebras over S is an $\mathcal{O}_S$-module $\mathscr{C}$ with homomorphisms of $\mathcal{O}_S$-modules $\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$, $\varepsilon : \mathscr{C} \to \mathcal{O}_S$, such that for $(\mathscr{C}, \Delta, \varepsilon)$ the diagrams (i) and (ii) of (3.5) commute.

A subcoalgebra of an $\mathcal{O}_S$-coalgebra $(\mathscr{C}, \varepsilon, \Delta)$ is an $\mathcal{O}_S$-submodule $\mathscr{B}$ such that $\Delta(\mathscr{B}) \subseteq \mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B}$. In this case $\mathscr{B}$ with the restricted homomorphisms itself is an $\mathcal{O}_S$-coalgebra.

Comodules. ([Sw], [Abe], [Ka]). Let $\mathscr{C}$ be a sheaf of $\mathcal{O}_S$-coalgebras. A (left-)$\mathscr{C}$-comodule on an S-scheme X is an $\mathcal{O}_X$-module $\mathscr{F}$ with a homomorphism of $\mathcal{O}_X$-modules

$$\varrho : \mathscr{F} \to \mathscr{C} \otimes_{\mathcal{O}_S} \mathscr{F} = \mathscr{C}_X \otimes_{\mathcal{O}_X} \mathscr{F}$$

such that the diagrams (tensor products over $\mathcal{O}_S$) of homomorphisms of $\mathcal{O}_X$-modules

(i)

(ii)

(3.8)

commute. We will mostly work with left-comodules, the definitions for right-comodules are made analogously.

Remark 3.9. Let $G = \operatorname{Spec}_S \mathscr{A}$ be a group scheme affine over S and $X = \operatorname{Spec}_S \mathscr{B}$ be an affine S-scheme. Then an operation of G on X over S is equivalent to an $\mathscr{A}$-comodule structure on $\mathscr{B}$: A morphism of S-schemes $s_X : G \times_S X \to X$ making diagrams (i) and (ii) of (3.6) commutative corresponds to a homomorphism of $\mathcal{O}_S$-algebras $\varrho : \mathscr{B} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}$ making diagrams (i) and (ii) of (3.8) commutative.

Some standard definitions and properties, let X be an S-scheme and $\mathscr{C}$ be an $\mathcal{O}_S$-coalgebra, we assume $\mathscr{C}$ to be flat over S:

- A $\mathscr{C}$-subcomodule $\mathscr{F}'$ of a $\mathscr{C}$-comodule $\mathscr{F}$ is an $\mathcal{O}_X$-submodule with the property that $\varrho|_{\mathscr{F}'} : \mathscr{F}' \to \mathscr{C} \otimes \mathscr{F}$ factors through $\mathscr{C} \otimes \mathscr{F}' \to \mathscr{C} \otimes \mathscr{F}$. A subcomodule inherits a comodule structure.

- A homomorphism of $\mathscr{C}$-comodules is a homomorphism $\alpha : \mathscr{E} \to \mathscr{F}$ of $\mathcal{O}_X$-modules such that the diagram (tensor products over $\mathcal{O}_S$)

commutes.

- Kernels and images of homomorphisms of comodules are subcomodules.

- The quotient $\mathscr{F}/\mathscr{F}'$ of a comodule $\mathscr{F}$ by a subcomodule $\mathscr{F}'$ has a unique comodule structure such that $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is a homomorphism of comodules.

- The direct sum of comodules has a natural comodule structure.

- If a bialgebra structure on $\mathscr{C}$ is given, then one can define tensor products of $\mathscr{C}$-comodules.

- $\mathscr{C}$-comodules with homomorphisms of comodules as morphisms form an abelian category. It is $\Gamma(X, \mathcal{O}_X)$-linear in the sense that the Hom-groups carry $\Gamma(X, \mathcal{O}_X)$-module structures satisfying the usual compatibilities.

3.1.5   Simple comodules and coalgebras

In this subsection let the base scheme be the spectrum of a field K, all coalgebras, comodules and tensor products are over K.

Definition 3.10.
A coalgebra C is called simple, if $C \neq 0$ and C has no subcoalgebras other that $\{0\}$ and C.
A C-comodule $V \neq 0$ is called simple, if it has no other subcomodules than $\{0\}$ and V.
A comodule is called isotypic, if it is a direct sum of simple comodules of one isomorphism class.

Remark 3.11.
(1) For a C-comodule $\varrho : F \to C \otimes F$ the commutative diagram (ii) in (3.8)

$$
\begin{array}{ccc}
C \otimes F & \xrightarrow{\;id_C \otimes \varrho\;} & C \otimes C \otimes F \\
{\scriptstyle \varrho}\uparrow & & \uparrow{\scriptstyle \Delta \otimes id_F} \\
F & \xrightarrow{\qquad \varrho \qquad} & C \otimes F
\end{array}
\tag{3.9}
$$

can be interpreted such that $\varrho : F \to C \otimes F$ is a homomorphism of C-comodules. Here the comodule structure of $C \otimes F$ comes from the first factor, e.g. if F is n-dimensional then $C \otimes F \cong C \otimes K^{\oplus n} \cong C^{\oplus n}$. Note that ϱ is injective because of diagram (i) in (3.8). A special case is C considered as a C-comodule by $\Delta : C \to C \otimes C$.
(2) It follows, that any C-comodule is isomorphic to a subcomodule of a direct sum of the comodule C. In particular, any simple comodule has a copy in C.

Remark 3.12. (see also [Mu, GIT, Ch. 1, §1, p. 25,26]).
(1) For a K-subspace S of a C-comodule F write $\langle S \rangle$ for the smallest subcomodule of F that contains S.
(2) One may construct $\langle S \rangle$ as the image of $C^\vee \otimes S$ in F under the composition

$$
C^\vee \otimes F \xrightarrow{\;id_{C^\vee} \otimes \varrho\;} C^\vee \otimes C \otimes F \xrightarrow{\;\langle \cdot,\cdot \rangle \otimes id_F\;} K \otimes F = F
$$

where $\langle \cdot, \cdot \rangle : C^\vee \otimes C \to K$ is the duality pairing. Equivalently, after choice of a basis $(c_i)_{i \in I}$ of C take $\langle S \rangle$ to be generated as a K-subspace by the s_i that occur in $\varrho(s) = \sum_i c_i \otimes s_i$ for $s \in S$. (To check that this is a subcomodule use diagram (3.8).(ii), to check $S \subseteq \langle S \rangle$ use $\varepsilon \in C^\vee$ and diagram (3.8).(i).)
(3) By construction, if S is finite dimensional, then so is $\langle S \rangle$. It follows that any element of an C-comodule F is contained in a finite dimensional subcomodule. In particular, simple comodules are finite dimensional. Further, any comodule contains a simple comodule.

Now we come to the study of coalgebras and their subcoalgebras. Note that the intersection of subcoalgebras again is a subcoalgebra.

Remark 3.13.
(1) An isotypic component C_i of a coalgebra C as a (left- or right-)C-comodule is a subcoalgebra. This follows since the homomorphism of (say left-)C-comodules $C_i \to C \otimes C_i \cong C^{\oplus d}$, where $d = \dim C_i$ (a priori possibly inifinite), factors through $C_i \otimes C_i \cong C_i^{\oplus d} \subseteq C^{\oplus d}$.
(2) In particular, a simple coalgebra C is isotypic as a (left- or right-)C-comodule.
(3) A simple coalgebra has only one isomorphism class of simple C-comodules (left or right). This follows from (2) and remark 3.11.(2).

Definition 3.14. *For a C-comodule F define $C(F)$ to be the smallest K-subspace of C such that $\varrho(F) \subseteq C(F) \otimes F$.*

Remark 3.15.
(1) Obviously $C(F)$ only depends on the isomorphism class of F.

(2) One may construct $C(F)$ as the image of the K-linear map

$$F \otimes F^\vee \xrightarrow{\varrho \otimes id_{F^\vee}} C \otimes F \otimes F^\vee \xrightarrow{id_C \otimes \langle \cdot, \cdot \rangle} C \otimes K = C$$

Equivalently, after choice of a basis $(f_i)_{i \in I}$ of F the space $C(F)$ can be described as the K-subspace of C generated by the coefficients c_i in $\varrho(f) = \sum_i c_i \otimes f_i$ for $f \in F$ (see also [Abe, Ch. 3.1, p. 129]).
(3) By construction, if F is finite dimensional, then so is $C(F)$.
(4) $C(F)$ is a subcoalgebra of C (use diagram (3.8).(ii)).
(5) If $F \subseteq C$ is a subcomodule then $F \subseteq C(F)$ (consider $\varepsilon|_F \in F^\vee$) and $C(F)$ is the smallest subcoalgebra that contains F (that is the intersection over all subcoalgebras that contain F).

Proposition 3.16. *Let $C \neq 0$ be a coalgebra. Then*

$$C \text{ is simple} \iff C \text{ is isotypic as a } C\text{-comodule (left or right)}$$

Proof. "$\Rightarrow$" Remark 3.13.(2).
"$\Leftarrow$" Let $0 \neq D \subseteq C$ be a subcoalgebra of C. If $E \subseteq D$ is a simple subcomodule, then $C(E) \subseteq D$. But C is a direct sum of simple subcomodules $E_i \cong E$ and for each of these $E_i \subseteq C(E_i) = C(E) \subseteq D$, therefore $D = C$. $\square$

Proposition 3.17. *Let $C \neq 0$ be a coalgebra.*

(i) *If E is a simple C-comodule, then $C(E)$ is a simple subcoalgebra.*

(ii) *Any simple subcoalgebra D of C is of the form $D = C(E)$ for some simple C-comodule E.*

Proof. (i) Consider E as a subcomodule of $C(E)$ by remark 3.11.(2). The isotypic component D of E in $C(E)$ is a subcoalgebra by remark 3.13.(1), in particular $\varrho(E) \subseteq D \otimes E$. But $C(E)$ is the smallest subcoalgebra of C with this property (remark 3.15.(5)), so $C(E) = D$ is isotypic and thus simple.
(ii) Take as E a simple subcomodule of D. Then $C(E)$ is a subcoalgebra with $E \subseteq C(E) \subseteq D$ (remark 3.15.(5)), therefore $C(E) = D$. $\square$

Proposition 3.18. *Any simple coalgebra is finite dimensional.*

Proof. Let C be a simple coalgebra. Then $C = C(E)$ for some simple C-comodule E. A simple E is finite dimensional by remark 3.12.(3), then by remark 3.15.(3) this is also true for $C(E) = C$. $\square$

3.1.6 Cosemisimple coalgebras

Again all coalgebras, comodules, tensor products etc. are over a field K.

Dual to the notion of semisimplicity for algebras ([Bour, Algèbre, Ch. VIII]) there is the notion of cosemisimplicity for coalgebras ([Sw, Ch. XIV, p. 287,290], [Abe, Ch. 2.3, p. 80]).

Definition 3.19. *(cosemisimple).*
A coalgebra is called cosemisimple, if it decomposes into a direct sum of simple subcoalgebras.

Remark 3.20. Let C be a coalgebra.
(1) Any sum $F = \sum_{i \in I} F_i$ of simple C-comodules has a partial sum that is direct, that is $F = \bigoplus_{i \in J} F_i$ for some subset $J \subseteq I$.
(2) Since the intersection of two subcoalgebras again is a subcoalgebra, the same is true for a sum $C = \sum_{i \in I} C_i$ of simple subcoalgebras of C.

Remark 3.21.
(1) One may introduce the coradical $\operatorname{corad} C = $ *sum of simple subcoalgebras* dual to the radical of an algebra and express cosemisimplicity as $\operatorname{corad} C = C$ ([Sw, Ch. IX, p. 181], [Abe, Ch. 2.3, p. 80]).
(2) For the meaning of cosemisimplicity for a Hopf algebra A in the case that $G = \operatorname{Spec} A$ is an algebraic group see [Abe, Ch. 4.6].

The following two propositions are proved completely the same way as in the case of modules (see e.g. [Bour, Algébre, Ch. VIII, §3.3]).

Proposition 3.22. *For a comodule F are equivalent:*

(i) *F is a direct sum of simple subcomodules.*

(ii) *Every subcomodule of F is a direct summand.* □

Definition 3.23. *A C-comodule satisfying the equivalent conditions* (i), (ii) *of proposition 3.22 is called* completely reducible.

Proposition 3.24. *If F is completely reducible, then so is any subcomodule and any quotient comodule of F.* □

Proposition 3.25. *A coalgebra C is cosemisimple if and only if every C-comodule is completely reducible.*

Proof. If C is cosemisimple, then C as a C-comodule decomposes into a direct sum of simple subcoalgebras, these are isotypic by remark 3.13.(2), so C decomposes into a direct sum of simple comodules. This is true in general for any C-comodule F, since F can be embedded into a direct sum of copies of C by remark 3.11.(2).
If any C-comodule decomposes into simple components, then in particular this is true for C. The isotypic components in this decomposition are simple subcoalgebras by remark 3.13.(1).   □

Proposition 3.26. *Let C be a cosemisimple coalgebra.*

(i) *The simple subcoalgebras of C are exactly the isotypic components of C as a C-comodule (left or right). In particular, the decomposition of C into simple subcoalgebras is unique.*

(ii) *There is a bijection*

$$\{\textit{isomorphism classes of simple C-comodules}\} \quad \leftrightarrow \quad \{\textit{simple subcoalgebras of C}\}$$
$$E \qquad\qquad \mapsto \qquad\qquad C(E)$$
$$\textit{class of the simple subcomodules of } D \quad \leftarrow\!\shortmid \qquad\qquad D$$

Proof. (i) C is completely reducible as a C-comodule by proposition 3.25, the isotypic components of C are subcoalgebras by remark 3.13.(1), they are simple by proposition 3.16.
Any simple subcoalgebra of C is one of these: It is isotypic by proposition 3.16, so a subcoalgebra of an isotypic component and coincides with it, since this component is a simple subcoalgebra.
(ii) By proposition 3.17.(i) $C(E)$ for simple E is a simple subcoalgebra and by proposition 3.16 (or by (i)) a simple subcoalgebra is isotypic, so the maps are well defined.
They are inverse to each other: Firstly, since $C(E)$ has only one isomorphism class of simple comodules, $E' \cong E$ for any simple $E' \subseteq C(E)$, secondly for a simple subcomodule $E \subseteq D$ one has $C(E) = D$ (same argument as in the proof of proposition 3.17.(ii)).   □

3.2 *G*-sheaves

In this section we treat the basic definitions and examples concerning G-sheaves, we work relative to a fixed base scheme S.

We begin with the definition taken from [Mu, GIT] and then introduce an equivalent formulation closer to the one given in [BKR01]. On any G-scheme the structure sheaf and the sheaf of relative differentials have natural G-sheaf structures. For a group scheme G over a field K, a representation over an extension field L of K will be defined as a G-sheaf over $\mathrm{Spec}\, L$.

We define notions such as G-subsheaves, equivariant homomorphisms of G-sheaves etc.. (Coherent, quasicoherent) G-sheaves on a G-scheme X with equivariant homomorphisms as morphisms form an abelian category.

In the following let a base scheme S be given, products are formed over S if not indicated otherwise.

3.2.1 Definition and examples

Let (G, e, m, i) be a group scheme over a scheme S with structure morphism $p : G \to S$, let X be a G-scheme over S with operation $s_X : G \times X \to X$.

Definition 3.27. ([Mu, GIT, Ch. 1, §3]). *A (quasicoherent, coherent) G-sheaf on X is a (quasicoherent, coherent) $\mathcal{O}_X$-module $\mathscr{F}$ with an isomorphism*

$$\lambda^{\mathscr{F}} : s_X^* \mathscr{F} \ \widetilde{\to} \ p_X^* \mathscr{F}$$

of $\mathcal{O}_{G \times X}$-modules satisfying

(i) *The restriction of $\lambda^{\mathscr{F}}$ to the unit in G_X is the identity, i.e. the following diagram commutes:*

$$
\begin{array}{ccc}
e_X^* s_X^* \mathscr{F} & \xrightarrow{\ e_X^* \lambda^{\mathscr{F}}\ } & e_X^* p_X^* \mathscr{F} \\
\| & & \| \\
\mathscr{F} & \xrightarrow{\ id_{\mathscr{F}}\ } & \mathscr{F}
\end{array}
$$

(ii) $(m \times id_X)^* \lambda^{\mathscr{F}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{F}} \circ (id_G \times s_X)^* \lambda^{\mathscr{F}}$ *on $G \times G \times X$, where $\mathrm{pr}_{23} : G \times G \times X \to G \times X$ is the projection to the factors 2 and 3.*

Now we will introduce another equivalent formulation similar to the one used in [BKR01].

Construction 3.28. For an S-scheme T and $g \in G(T) = G_T(T)$ there is the diagram (see remark 3.8, write g also for φ_g)

Pulling back $(\lambda^{\mathscr{F}})_T : s_{X_T}^* \mathscr{F}_T \to p_{X_T}^* \mathscr{F}_T$ from $G_T \times_T X_T$ to X_T by $g \times id_{X_T}$ leads to an isomorphism $\lambda_g^{\mathscr{F}_T} : g^* \mathscr{F}_T \to \mathscr{F}_T$.

Proposition 3.29. *Let $\mathscr{F}$ be an $\mathcal{O}_X$-module on a G-scheme X over S. Then a G-sheaf structure on $\mathscr{F}$ is equivalent to the following data: For any S-scheme T and any T-valued point $g \in G(T) = G_T(T)$ an isomorphism $\lambda_g^{\mathscr{F}_T} : g^*\mathscr{F}_T \to \mathscr{F}_T$ of $\mathcal{O}_{X_T}$-modules such that $\lambda_{g_{T'}}^{\mathscr{F}_{T'}} = (\lambda_g^{\mathscr{F}_T})_{T'}$ for T-schemes T' and the properties*

(i) *$\lambda_{e_T}^{\mathscr{F}_T} = id_{\mathscr{F}_T}$ for $e_T : T \to G_T$ the identity of $G(T) = G_T(T)$*

(ii) *$\lambda_{hg}^{\mathscr{F}_T} = \lambda_g^{\mathscr{F}_T} \circ g^*\lambda_h^{\mathscr{F}_T}$ for $g, h \in G(T)$*

are satisfied (again write g as well for φ_g, see remark 3.8). The correspondence is given by construction 3.28 and by specialisation to $T = G$, $g = id_G$, that is $\lambda^{\mathscr{F}} = \lambda_{id_G}^{\mathscr{F}_G}$.

Proof. Assume that $\mathscr{F}$ has a G-sheaf structure. Then construction 3.28 gives isomorphisms $\lambda_g^{\mathscr{F}_T} : g^*\mathscr{F}_T \to \mathscr{F}_T$, we show that they satisfy properties (i) and (ii) of the proposition.
(i) $\lambda_{e_T}^{\mathscr{F}_T}$ is constructed using the morphism $(e_X)_T = e_T \times id_{X_T} : X_T = T \times_T X_T \to (G \times X)_T = G_T \times_T X_T$. Condition (i) of definition 3.27 then implies that $\lambda_{e_T}^{\mathscr{F}_T} = id_{\mathscr{F}_T}$.
(ii) For $g, h \in G_T(T)$ use the morphism $(h, g) \times id_{X_T} : X_T = T \times X_T \to G_T \times_T G_T \times_T X_T$ to pull back the base extension of $(m \times id_X)^*\lambda^{\mathscr{F}} = \mathrm{pr}_{23}^*\lambda^{\mathscr{F}} \circ (id_G \times s_X)^*\lambda^{\mathscr{F}}$ on $G_T \times_T G_T \times_T X_T$ to X_T: One obtains the equation (ii) in the proposition.

Assume that isomorphisms $\lambda_g^{\mathscr{F}_T} : g^*\mathscr{F}_T \to \mathscr{F}_T$ with $\lambda_{g_{T'}}^{\mathscr{F}_{T'}} = (\lambda_g^{\mathscr{F}_T})_{T'}$ for T-schemes T' that satisfy properties (i) and (ii) are given. Then $\lambda^{\mathscr{F}}$ arises for $id_G \in G(G)$, that is $\lambda^{\mathscr{F}} = \lambda_{id_G}^{\mathscr{F}_G} : s_X^*\mathscr{F} = (\mathrm{pr}_1, s_X)^*\mathscr{F}_G \to p_X^*\mathscr{F} = \mathscr{F}_G$ (here $(\mathrm{pr}_1, s_X) : G \times X \to G \times X$ is the automorphism of X_G corresponding to $id_G \in G(G)$, see remark 3.8.(2)). We show that $\lambda^{\mathscr{F}}$ satisfies conditions (i) and (ii) of definition 3.27.
(i) $e_X^*\lambda^{\mathscr{F}} = e_X^*\lambda_{id_G}^{\mathscr{F}_G} = \lambda_e^{\mathscr{F}} = id_{\mathscr{F}}$.
(ii) Using the morphisms $p_1 = \mathrm{pr}_1, p_2 = \mathrm{pr}_2 \in G(G \times G)$ as g, h in (ii) of the proposition one obtains the equation

$$\lambda_{p_1 \cdot p_2}^{\mathscr{F}_{G \times G}} = \lambda_{p_2}^{\mathscr{F}_{G \times G}} \circ p_2^*\lambda_{p_1}^{\mathscr{F}_{G \times G}} \qquad \textit{(on } X_{G \times G})$$

that can be converted to

$$\lambda_m^{\mathscr{F}_{G \times G}} = \mathrm{pr}_{23}^*\lambda^{\mathscr{F}} \circ (\mathrm{pr}_1, \mathrm{pr}_2, s_X \circ \mathrm{pr}_{23})^* \mathrm{pr}_{13}^*\lambda^{\mathscr{F}} \qquad \textit{(on } G \times G \times X)$$

and

$$(m \times id_X)^*\lambda^{\mathscr{F}} = \mathrm{pr}_{23}^*\lambda^{\mathscr{F}} \circ (id_G \times s_X)^*\lambda^{\mathscr{F}} \qquad \textit{(on } G \times G \times X)$$

Note that by the base change property $\lambda_{g_{T'}}^{\mathscr{F}_{T'}} = (\lambda_g^{\mathscr{F}_T})_{T'}$ the $\lambda_g^{\mathscr{F}_T}$ are uniquely determined by $\lambda_{id_G}^{\mathscr{F}_G}$. It follows that construction 3.28 and taking $id_G \in G(G)$ are inverse to each other. $\qquad\square$

Remark 3.30. If G is flat, étale, ... over S, then it suffices to consider flat, étale, ... S-schemes T, for G a finite group regarded as a discrete group scheme over K it suffices to consider its K-valued points.

Remark 3.31. The requirement for $\lambda^{\mathscr{F}}$ and for the $\lambda_g^{\mathscr{F}_T} : g^*\mathscr{F}_T \to \mathscr{F}_T$ to be isomorphisms is not necessary, it follows from conditions (i) and (ii): For $\lambda_g^{\mathscr{F}_T}$ this can easily be seen by (i) and (ii) of proposition 3.29. For $\lambda^{\mathscr{F}}$ it follows, since $\lambda^{\mathscr{F}} = \lambda_g^{\mathscr{F}_T}$ for $T = G$ and $g = id_G$. Here we have $\lambda_{id_G}^{\mathscr{F}_G} \circ (\mathrm{pr}_1, s_X)^*\lambda_i^{\mathscr{F}_G} = \lambda_{id_G \cdot i}^{\mathscr{F}_G} = \lambda_{e_G}^{\mathscr{F}_G} = \lambda_{i \cdot id_G}^{\mathscr{F}_G} = \lambda_i^{\mathscr{F}_G} \circ (\mathrm{pr}_1, s_X \circ i_X)^*\lambda_{id_G}^{\mathscr{F}_G}$ and thus $\lambda^{\mathscr{F}} \circ (i \circ \mathrm{pr}_1, s_X)^*\lambda^{\mathscr{F}} = id_{\mathscr{F}_G} = i_X^*\lambda^{\mathscr{F}} \circ (\mathrm{pr}_1, s_X \circ i_X)^*\lambda^{\mathscr{F}}$ on $G \times X$.

Example 3.32. (Structure sheaf).
Let X be a G-scheme over S. Then the structure sheaf $\mathcal{O}_X$ has the structure of a G-sheaf: Let T be an S-scheme. Any T-valued point $g \in G(T)$ defines an isomorphism $g : X_T \to X_T$ (see remark 3.8), define $\lambda_g^{\mathcal{O}_{X_T}}$ to be the corresponding homomorphism $g^* \mathcal{O}_{X_T} \to \mathcal{O}_{X_T}$. These have the property $\lambda_{g_{T'}}^{(\mathcal{O}_{X_T})_{T'}} = (\lambda_g^{\mathcal{O}_{X_T}})_{T'}$ with respect to base extensions $T' \to T$ and satisfy conditions (i) $\lambda_{e_T}^{\mathcal{O}_{X_T}} = id_{\mathcal{O}_{X_T}}$ and (ii) $\lambda_{hg}^{\mathcal{O}_{X_T}} = \lambda_g^{\mathcal{O}_{X_T}} \circ g^* \lambda_h^{\mathcal{O}_{X_T}}$ for $g, h \in G(T)$ of proposition 3.29.
In particular one has $\lambda^{\mathcal{O}_X} = \varphi^{\#} : s_X^* \mathcal{O}_X = \varphi^* \mathcal{O}_{X_G} \to \mathcal{O}_{X_G} = p_X^* \mathcal{O}_X$ where $\varphi = (\mathrm{pr}_1, s_X) : G \times X \to G \times X$ is the isomorphism of X_G over G corresponding to $id_G \in G(G)$ (see remark 3.8).

Example 3.33. (Sheaf of differentials).
Let X be a G-scheme over S. Then the sheaf $\Omega_{X/S}$ of relative differentials has the structure of a G-sheaf: Let T be an S-scheme. Any T-valued point $g \in G(T)$ defines an isomorphism $g : X_T \to X_T$ (see remark 3.8), define $\lambda_g^{(\Omega_{X/S})_T}$ to be the natural isomorphism $g^*(\Omega_{X/S})_T \cong g^* \Omega_{X_T/T} \to \Omega_{X_T/T} \cong (\Omega_{X/S})_T$, where the natural isomorphism $(\Omega_{X/S})_T \to \Omega_{X_T/T}$ for the base extension $X_T \to X$ is used. More generally for any $T' \to T$ and $g \in G(T)$ there is the natural isomorphism $(\Omega_{X_T/T})_{T'} \to \Omega_{X_{T'}/T'}$ and the commutative diagram of isomorphisms

$$
\begin{array}{ccc}
(g^* \Omega_{X_T/T})_{T'} & \longrightarrow & (\Omega_{X_T/T})_{T'} \\
\downarrow & & \downarrow \\
g_{T'}^* \Omega_{X_{T'}/T'} & \longrightarrow & \Omega_{X_{T'}/T'}
\end{array}
$$

thus $\lambda_{g_{T'}}^{(\Omega_{(X/S)})_{T'}} = (\lambda_g^{(\Omega_{(X/S)})_T})_{T'}$. Further conditions (i) and (ii) proposition 3.29 are satisfied because of the functorial properties of $\Omega_{X_T/T}$ with respect to automorphisms of X_T over T.

Definition 3.34. (Representations). *Let G be a group scheme over a field K. A G-sheaf on $\mathrm{Spec}\, L$, L an extension field of K, is also called a representation of G over L.*

Later, in section 3.4 we will see the relation to the more ordinary notion of a representation (remark 3.65).

3.2.2 G-subsheaves

Let G be a group scheme over S and let $\mathscr{F}$ be a G-sheaf on a G-scheme X.

Definition 3.35. (G-subsheaves). *A subsheaf $\mathscr{F}' \subseteq \mathscr{F}$ is called G-stable or a G-subsheaf if*

$$\lambda^{\mathscr{F}}(s_X^* \mathscr{F}') \subseteq p_X^* \mathscr{F}'$$

In the following let G be a group scheme flat over S.

Remark 3.36. (G-sheaf structure on a G-stable subsheaf). Let $\mathscr{F}' \subseteq \mathscr{F}$ be a G-subsheaf. Then the restriction $\lambda^{\mathscr{F}}|_{s_X^* \mathscr{F}'}$ defines a G-sheaf structure on $\mathscr{F}'$: The conditions (i) and (ii) of definition 3.27 remain valid for $\lambda^{\mathscr{F}'} := \lambda^{\mathscr{F}}|_{s_X^* \mathscr{F}'}$ and by remark 3.31 $\lambda^{\mathscr{F}'}$ is an isomorphism.

Remark 3.37. Let $\mathscr{F}$ be a G-sheaf on X and $\mathscr{F}' \subseteq \mathscr{F}$ be a subsheaf. Then $\mathscr{F}'$ is a G-subsheaf if and only if $\lambda_g^{\mathscr{F}_T} : g^* \mathscr{F}_T \to \mathscr{F}_T$ restricts to $g^* \mathscr{F}_T' \to \mathscr{F}_T'$ for all flat S-schemes T and $g \in G(T)$. This directly follows from the constructions in the proof of proposition 3.29, which establish the correspondence between $\lambda^{\mathscr{F}}$ and the $\lambda_g^{\mathscr{F}_T}$, and from remark 3.30.

Remark 3.38. (Quotients of G-sheaves). Let $\mathscr{F}' \subseteq \mathscr{F}$ be a G-subsheaf. Then $\mathscr{F}/\mathscr{F}'$ has a natural G-sheaf structure given by the induced map in the exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & s_X^*\mathscr{F}' & \longrightarrow & s_X^*\mathscr{F} & \longrightarrow & s_X^*(\mathscr{F}/\mathscr{F}') & \longrightarrow & 0 \\
& & \lambda^{\mathscr{F}'}\downarrow & & \lambda^{\mathscr{F}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & p_X^*\mathscr{F}' & \longrightarrow & p_X^*\mathscr{F} & \longrightarrow & p_X^*(\mathscr{F}/\mathscr{F}') & \longrightarrow & 0
\end{array}
$$

3.2.3   Equivariant homomorphisms and categories of G-sheaves

Let G be a group scheme over S and $f : X \to S$ a G-scheme over S.

Let $\mathscr{E}, \mathscr{F}$ be G-sheaves on X. For $\varphi \in \operatorname{Hom}_X(\mathscr{E}, \mathscr{F})$ there is the diagram

$$
\begin{array}{ccc}
s_X^*\mathscr{E} & \xrightarrow{\;s_X^*\varphi\;} & s_X^*\mathscr{F} \\
\lambda^{\mathscr{E}}\downarrow & & \downarrow\lambda^{\mathscr{F}} \\
p_X^*\mathscr{E} & \xrightarrow{\;p_X^*\varphi\;} & p_X^*\mathscr{F}
\end{array}
\tag{3.10}
$$

Definition 3.39. *Define the set of G-equivariant homomorphisms between $\mathscr{E}$ and $\mathscr{F}$ to be*

$$
\operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F}) := \{\varphi \in \operatorname{Hom}_X(\mathscr{E}, \mathscr{F}) \mid \text{the diagram (3.10) commutes for } \varphi\}
$$

Example 3.40. Elements $s \in \Gamma(S, \mathcal{O}_S)$ considered as elements in $\Gamma(X, \mathcal{O}_X)$ via $X \to S$ define G-equivariant homomorphisms $\varphi_s : \mathscr{F} \to \mathscr{F}$, $\varphi_s(U) : \mathscr{F}(U) \to \mathscr{F}(U)$, $f \mapsto s|_U f$.

Remark 3.41.
(1) $\operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F})$ is an abelian group, it is a subgroup of $\operatorname{Hom}_X(\mathscr{E}, \mathscr{F})$.
(2) $\operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F})$ becomes a $\Gamma(S, \mathcal{O}_S)$-module by $\Gamma(S, \mathcal{O}_S) \times \operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F}) \to \operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F})$, $(s, \psi) \mapsto \varphi_s \circ \psi$. It is a $\Gamma(S, \mathcal{O}_S)$-submodule of $\operatorname{Hom}_X(\mathscr{E}, \mathscr{F})$.
(3) Assume G flat over S. Then for $\varphi \in \operatorname{Hom}_X^G(\mathscr{E}, \mathscr{F})$ the kernel and cokernel of φ have canonical G-sheaf structures.

Definition 3.42. *(G-equivariant categories of sheaves).*
For a G-scheme X and $\mathcal{C} = \mathrm{Mod}, \mathrm{Qcoh}, \mathrm{Coh}$ define the category $\mathcal{C}^G(X)$ with G-sheaves as objects and G-equivariant homomorphisms as morphisms.

Remark 3.43. Assume G flat over S. Then $\mathcal{C}^G(X)$ for $\mathcal{C} = \mathrm{Mod}, \mathrm{Qcoh}, \mathrm{Coh}$ is a ($\Gamma(S, \mathcal{O}_S)$-linear) abelian category.

In the case of trivial G-operation (that is $p_X = s_X$, one may assume $S = X$) one can define a sheaf $\mathscr{H}om^G$ of equivariant homomorphisms by imposing for open $U \subseteq X$ and homomorphisms $\alpha : \mathscr{E}_U \to \mathscr{F}_U$ commutativity of the diagrams

$$
\begin{array}{ccc}
p_U^*\mathscr{E}_U & \xrightarrow{\;p_U^*\alpha\;} & p_U^*\mathscr{F}_U \\
\lambda_U^{\mathscr{E}}\downarrow & & \downarrow\lambda_U^{\mathscr{F}} \\
p_U^*\mathscr{E}_U & \xrightarrow{\;p_U^*\alpha\;} & p_U^*\mathscr{F}_U
\end{array}
$$

that is $\alpha \in \operatorname{Hom}_U^G(\mathscr{E}_U, \mathscr{F}_U)$.

Definition 3.44. *For G-sheaves $\mathscr{E}, \mathscr{F}$ on a G-scheme X with trivial G-operation define by*

$$\mathscr{H}om^G_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})(U) := \operatorname{Hom}^G_U(\mathscr{E}_U, \mathscr{F}_U)$$

the $\mathcal{O}_X$-submodule $\mathscr{H}om^G_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \subseteq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ of equivariant homomorphisms.

Remark 3.45. Later we will see that $\mathscr{H}om^G_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ coincides with the G-invariant part $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})^G$ of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ (see remark 3.67) provided that a G-sheaf structure on $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ can be defined by proposition 3.47.

3.3   Constructions, adjunctions and natural isomorphisms

Let G be a group scheme over a scheme S. In this section we define certain functors and bifunctors on categories of G-sheaves on G-schemes over S.

On a fixed G-scheme X we construct the tensor product $\otimes_{\mathcal{O}_X}$ and the $\mathscr{H}\!om_{\mathcal{O}_X}$-sheaf of G-sheaves. For equivariant morphisms $f : X \to Y$ of G-schemes we define the functors f_* and f^*. Essential in these constructions is the commutativity of the corresponding functor or bifunctor (for usual sheaves) with base extension by G.

We then treat relations between these functors on categories of G-sheaves. We establish adjunctions (f^*, f_*), $(f_*, f^!)$ and certain natural isomorphisms.

3.3.1   Sheaf of homomorphisms and tensor product of G-sheaves

In this subsection we construct the tensor product and the $\mathscr{H}\!om$-sheaf of G-sheaves.

Proposition 3.46. *(Tensor product of G-sheaves).*
Let $\mathscr{E}, \mathscr{F}$ be G-sheaves on a G-scheme X. Then the tensor product $\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}$ has a natural G-sheaf structure $\lambda^{\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}}$ given by $\lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} \lambda^{\mathscr{F}}$.

Proof. The bifunctor $(\cdot) \otimes_{\mathcal{O}_{G \times X}} (\cdot)$ applied to the isomorphisms $\lambda^{\mathscr{E}} : s_X^* \mathscr{E} \to p_X^* \mathscr{E}$ and $\lambda^{\mathscr{F}} : s_X^* \mathscr{F} \to p_X^* \mathscr{F}$ gives an isomorphism

$$\lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} \lambda^{\mathscr{F}} : s_X^* \mathscr{E} \otimes_{\mathcal{O}_{G \times X}} s_X^* \mathscr{F} \to p_X^* \mathscr{E} \otimes_{\mathcal{O}_{G \times X}} p_X^* \mathscr{F}$$

Define $\lambda^{\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}}$ by the diagram

$$
\begin{array}{ccc}
s_X^*(\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}) & \xrightarrow{\ \lambda^{\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}}\ } & p_X^*(\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}) \\
\wr \uparrow & & \uparrow \wr \\
s_X^* \mathscr{E} \otimes_{\mathcal{O}_{G \times X}} s_X^* \mathscr{F} & \xrightarrow{\ \lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} \lambda^{\mathscr{F}}\ } & p_X^* \mathscr{E} \otimes_{\mathcal{O}_{G \times X}} p_X^* \mathscr{F}
\end{array}
$$

The vertical isomorphisms: For any morphism $h : Y \to Z$ and $\mathcal{O}_Z$-modules $\mathscr{E}, \mathscr{F}$ the natural homomorphism

$$h^* \mathscr{E} \otimes_{\mathcal{O}_Y} h^* \mathscr{F} \to h^*(\mathscr{E} \otimes_{\mathcal{O}_Z} \mathscr{F})$$

is an isomorphism ([EGA1, 0, (4.3.3)]).

The conditions (i) and (ii) of definition 3.27 are satisfied:
(i): One has $\quad e_X^* \lambda^{\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}} \cong e_X^*(\lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} \lambda^{\mathscr{F}}) \cong (e_X^* \lambda^{\mathscr{E}}) \otimes_{\mathcal{O}_X} (e_X^* \lambda^{\mathscr{F}}) \cong id_{\mathscr{E}} \otimes_{\mathcal{O}_X} id_{\mathscr{F}} \cong id_{\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}}$, since $\otimes$ and e_X^* commute as above.
(ii): Similar to (i), apply $\otimes_{\mathcal{O}_{G \times X \times X}}$ to the equations $(m \times id_X)^* \lambda^{\mathscr{E}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{E}} \circ (id_G \times s_X)^* \lambda^{\mathscr{E}}$ and $(m \times id_X)^* \lambda^{\mathscr{F}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{F}} \circ (id_G \times s_X)^* \lambda^{\mathscr{F}}$.

$\otimes_{\mathcal{O}_X}$ is a bifunctor: Let $\alpha : \mathscr{E} \to \mathscr{E}'$ and $\beta : \mathscr{F} \to \mathscr{F}'$ be homomorphisms of G-sheaves. To see that $\alpha \otimes \beta : \mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{E}' \otimes_{\mathcal{O}_X} \mathscr{F}'$ is a homomorphism of G-sheaves, apply the bifunctor $\otimes_{\mathcal{O}_{G \times X}}$ to the commutative diagrams

$$
\begin{array}{ccc}
s_X^* \mathscr{E} & \xrightarrow{\ \lambda^{\mathscr{E}}\ } & p_X^* \mathscr{E} \\
{\scriptstyle s_X^* \alpha} \downarrow & & \downarrow {\scriptstyle p_X^* \alpha} \\
s_X^* \mathscr{E}' & \xrightarrow{\ \lambda^{\mathscr{E}'}\ } & p_X^* \mathscr{E}'
\end{array}
\qquad
\begin{array}{ccc}
s_X^* \mathscr{F} & \xrightarrow{\ \lambda^{\mathscr{F}}\ } & p_X^* \mathscr{F} \\
{\scriptstyle s_X^* \beta} \downarrow & & \downarrow {\scriptstyle p_X^* \beta} \\
s_X^* \mathscr{F}' & \xrightarrow{\ \lambda^{\mathscr{F}'}\ } & p_X^* \mathscr{F}'
\end{array}
$$

to obtain the commutative diagram

$$
\begin{array}{ccc}
s_X^*(\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}) & \xrightarrow{\ \lambda^{\mathscr{E} \otimes \mathscr{F}}\ } & p_X^*(\mathscr{E} \otimes_{\mathcal{O}_X} \mathscr{F}) \\
{\scriptstyle s_X^*(\alpha \otimes \beta)} \downarrow & & \downarrow {\scriptstyle p_X^*(\alpha \otimes \beta)} \\
s_X^*(\mathscr{E}' \otimes_{\mathcal{O}_X} \mathscr{F}') & \xrightarrow{\ \lambda^{\mathscr{E}' \otimes \mathscr{F}'}\ } & p_X^*(\mathscr{E}' \otimes_{\mathcal{O}_X} \mathscr{F}')
\end{array}
$$

$\square$

Proposition 3.47. (*$\mathscr{H}om$ of G-sheaves*).
Let $\mathscr{E}, \mathscr{F}$ be quasicoherent G-sheaves on a G-scheme X, $\mathscr{E}$ finitely presented. Assume that G is flat over S. Then the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ has a natural G-sheaf structure $\lambda^{\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})}$ given by $\mathscr{H}om_{\mathcal{O}_{G \times X}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}})$.

Proof. The bifunctor $\mathscr{H}om_{\mathcal{O}_{G \times X}}(\cdot, \cdot)$ applied to the isomorphisms $\lambda^{\mathscr{E}} : s_X^* \mathscr{E} \to p_X^* \mathscr{E}$ and $\lambda^{\mathscr{F}} : s_X^* \mathscr{F} \to p_X^* \mathscr{F}$ gives an isomorphism

$$
\mathscr{H}om_{\mathcal{O}_{G \times X}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}}) : \mathscr{H}om_{\mathcal{O}_{G \times X}}(s_X^* \mathscr{E}, s_X^* \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_{G \times X}}(p_X^* \mathscr{E}, p_X^* \mathscr{F})
$$

Define $\lambda^{\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})}$ by the diagram

$$
\begin{array}{ccc}
s_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) & \xrightarrow{\ \lambda^{\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})}\ } & p_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \\
{\scriptstyle \wr} \downarrow & & \downarrow {\scriptstyle \wr} \\
\mathscr{H}om_{\mathcal{O}_{G \times X}}(s_X^* \mathscr{E}, s_X^* \mathscr{F}) & \xrightarrow{\ \mathscr{H}om_{\mathcal{O}_{G \times X}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}})\ } & \mathscr{H}om_{\mathcal{O}_{G \times X}}(p_X^* \mathscr{E}, p_X^* \mathscr{F})
\end{array}
$$

The vertical isomorphisms: Using the assumption "G flat over S", the natural homomorphism

$$
p_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_{G \times X}}(p_X^* \mathscr{E}, p_X^* \mathscr{F})
$$

is an isomorphism by [EGA1, 0, (5.7.6)] or [EGA, 0_I, (6.7.6)], the same is true for s_X, because the morphisms s_X and p_X are isomorphic via $(\mathrm{pr}_1, s_X) : G \times X \to G \times X$.

The conditions (i) and (ii) of definition 3.27 are satisfied: To show this, again we use commutativity of $\mathscr{H}om$ with certain pull-backs.
(i) There is the commutative diagram

$$
\begin{array}{ccc}
e_X^* s_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) & \xrightarrow{\ e_X^* \lambda^{\mathscr{H}om(\mathscr{E}, \mathscr{F})}\ } & e_X^* p_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \\
{\scriptstyle \wr} \downarrow & & \downarrow {\scriptstyle \wr} \\
e_X^* \mathscr{H}om_{\mathcal{O}_{G \times X}}(s_X^* \mathscr{E}, s_X^* \mathscr{F}) & \xrightarrow{\ e_X^* \mathscr{H}om(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}})\ } & e_X^* \mathscr{H}om_{\mathcal{O}_{G \times X}}(p_X^* \mathscr{E}, p_X^* \mathscr{F}) \\
\downarrow & & \downarrow \\
\mathscr{H}om_{\mathcal{O}_X}(e_X^* s_X^* \mathscr{E}, e_X^* s_X^* \mathscr{F}) & \xrightarrow{\ \mathscr{H}om(e_X^* \lambda^{\mathscr{E}}, e_X^* \lambda^{\mathscr{F}})\ } & \mathscr{H}om_{\mathcal{O}_X}(e_X^* p_X^* \mathscr{E}, e_X^* p_X^* \mathscr{F}) \\
{\scriptstyle \wr} \uparrow & & \uparrow {\scriptstyle \wr} \\
\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) & \xrightarrow{\ id_{\mathscr{H}om(\mathscr{E}, \mathscr{F})}\ } & \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})
\end{array}
$$

The vertical homomorphisms in the middle are isomorphisms, since $e_X^* s_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_X}(e_X^* s_X^* \mathscr{E}, e_X^* s_X^* \mathscr{F})$ resp. $e_X^* p_X^* \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_X}(e_X^* p_X^* \mathscr{E}, e_X^* p_X^* \mathscr{F})$ and the upper ones are, use the compatibility of lemma 3.48 below.
(ii): Apply $\mathscr{H}om_{\mathcal{O}_{G \times X \times X}}$ to the equations $(m \times id_X)^* \lambda^{\mathscr{E}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{E}} \circ (id_G \times s_X)^* \lambda^{\mathscr{E}}$ and $(m \times id_X)^* \lambda^{\mathscr{F}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{F}} \circ (id_G \times s_X)^* \lambda^{\mathscr{F}}$.

$\mathscr{H}om$ is a bifunctor: The proof is made the same way as in the case of tensor products. $\square$

Lemma 3.48. *Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be morphisms of schemes and $\mathscr{E}, \mathscr{F}$ quasicoherent $\mathcal{O}_Z$-modules, $\mathscr{E}$ finitely presented. Then the diagram of natural homomorphisms*

$$
\begin{array}{ccccc}
\alpha^*\beta^* \, \mathscr{H}om_{\mathcal{O}_Z}(\mathscr{E}, \mathscr{F}) & \longrightarrow & \alpha^* \, \mathscr{H}om_{\mathcal{O}_Y}(\beta^*\mathscr{E}, \beta^*\mathscr{F}) & \longrightarrow & \mathscr{H}om_{\mathcal{O}_X}(\alpha^*\beta^*\mathscr{E}, \alpha^*\beta^*\mathscr{F}) \\
\wr \uparrow & & & & \uparrow \wr \\
(\beta \circ \alpha)^* \, \mathscr{H}om_{\mathcal{O}_Z}(\mathscr{E}, \mathscr{F}) & & \longrightarrow & & \mathscr{H}om_{\mathcal{O}_X}((\beta \circ \alpha)^*\mathscr{E}, (\beta \circ \alpha)^*\mathscr{F})
\end{array}
$$

commutes.

Proof. Left to the reader. $\qquad\square$

Definition 3.49. *For a locally free G-sheaf of finite rank $\mathscr{E}$ on a G-scheme X define the dual G-sheaf by*

$$
\mathscr{E}^\vee := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)
$$

3.3.2 f_* and f^* for equivariant morphisms

The aim of this subsection is to define functors f^*, f_* on categories of G-sheaves for equivariant morphisms f of G-schemes that are the usual ones for the underlying categories of sheaves. We show for equivariant $f : X \to Y$ that $f^*\mathscr{G}$ for G-sheaves $\mathscr{G}$ on Y and under some conditions $f_*\mathscr{F}$ for G-sheaves $\mathscr{F}$ on X have natural G-sheaf structures.

Let G be a group scheme over S and X, Y be G-schemes over S with operations $s_X : G \times_S X \to X$, $s_Y : G \times_S Y \to Y$.

Remark 3.50. In the following we will use the cartesian diagrams (products over S)

$$
\begin{array}{ccc}
G \times X \xrightarrow{\ p_X\ } X & G \times X \xrightarrow{\ s_X\ } X & X \xrightarrow{\ e_X\ } G \times X \\
{\scriptstyle id_G \times f}\downarrow \quad\quad \downarrow{\scriptstyle f} & {\scriptstyle id_G \times f}\downarrow \quad\quad \downarrow{\scriptstyle f} & {\scriptstyle f}\downarrow \quad\quad \downarrow{\scriptstyle id_G \times f} \\
G \times Y \xrightarrow{\ p_Y\ } Y & G \times Y \xrightarrow{\ s_Y\ } Y & Y \xrightarrow{\ e_Y\ } G \times Y
\end{array}
$$

for an equivariant morphism $f : X \to Y$. Note that the left and middle diagram are isomorphic.

Proposition 3.51. *(f_* and f^* for G-sheaves).*

Let $f : X \to Y$ be an equivariant morphism.

(i) *Assume f to be quasicompact and quasiseparated, further assume G flat over S or f affine. Then for a quasicoherent G-sheaf $\mathscr{F}$ on X the sheaf $f_*\mathscr{F}$ has a natural G-sheaf structure. f defines a functor*

$$
f_* : \mathcal{C}^G(X) \to \mathcal{C}^G(Y)
$$

for $\mathcal{C} = \mathrm{Qcoh}$ and if Y is locally noetherian and f proper also for $\mathcal{C} = \mathrm{Coh}$.

(ii) *For a G-sheaf $\mathscr{G}$ on Y the sheaf $f^*\mathscr{G}$ has a natural G-sheaf structure. f defines a functor*

$$
f^* : \mathcal{C}^G(Y) \to \mathcal{C}^G(X)
$$

for $\mathcal{C} = \mathrm{Mod}, \mathrm{Qcoh}$ and if X, Y are locally noetherian also for $\mathcal{C} = \mathrm{Coh}$.

Proof. The functor f^* transforms quasicoherent sheaves into quasicoherent sheaves, $f^*\mathscr{G}$ is coherent for coherent $\mathscr{G}$, if X, Y are locally noetherian.

For quasicoherent $\mathscr{F}$ the sheaf $f_*\mathscr{F}$ is quasicoherent by [EGA1, I, (6.7.1)], since f is quasicompact and quasiseparated. If Y is locally noetherian and f is proper, then $f_*\mathscr{F}$ is coherent for coherent $\mathscr{F}$ on X by [EGA, III (1), (3.2.2)].

The assumptions "f quasicompact and quasiseparated and further G flat over S or f affine" imply that f_* commutes with base extension by $G \to S$, that is for quasicoherent $\mathscr{F}$ on X the natural homomorphism

$$p_Y^* f_* \mathscr{F} \to (id_G \times f)_* p_X^* \mathscr{F}$$

is an isomorphism. This follows from [EGA1, I, (9.3.3)] or [EGA, IV (2), (2.3.1)], [EGA, II, (1.5.2)]. As well it applies to base extensions of f such as $id_G \times f : G \times X \to G \times X$.

Define $\lambda^{f_*\mathscr{F}}$ and $\lambda^{f^*\mathscr{G}}$ by the commutative diagrams

$$
\begin{array}{ccc}
s_Y^* f_* \mathscr{F} & \xrightarrow{\;\;\lambda^{f_*\mathscr{F}}\;\;} & p_Y^* f_* \mathscr{F} \\
\wr\downarrow & & \downarrow\wr \\
(id_G \times f)_* s_X^* \mathscr{F} & \xrightarrow{(id_G \times f)_* \lambda^{\mathscr{F}}} & (id_G \times f)_* p_X^* \mathscr{F}
\end{array}
$$

and

$$
\begin{array}{ccc}
s_X^* f^* \mathscr{G} & \xrightarrow{\;\;\lambda^{f^*\mathscr{G}}\;\;} & p_X^* f^* \mathscr{G} \\
\| & & \| \\
(id_G \times f)^* s_Y^* \mathscr{G} & \xrightarrow{(id_G \times f)^* \lambda^{\mathscr{G}}} & (id_G \times f)^* p_Y^* \mathscr{G}
\end{array}
$$

The vertical identifications in the case f_*: By assumption f_* commutes with base extension by $G \to S$, therefore $p_Y^* f_* \mathscr{F} \to (id_G \times f)_* p_X^* \mathscr{F}$ is an isomorphism. Then the same is true for s_X, s_Y because the corresponding diagrams (see remark 3.50) are isomorphic.

The isomorphisms $\lambda^{f_*\mathscr{F}}$ and $\lambda^{f^*\mathscr{G}}$ satisfy the conditions (i) and (ii) of definition 3.27:

(i) In the case f_* there is the commutative diagram

$$
\begin{array}{ccc}
e_Y^* s_Y^* f_* \mathscr{F} & \xrightarrow{\;\;e_Y^* \lambda^{f_*\mathscr{F}}\;\;} & e_Y^* p_Y^* f_* \mathscr{F} \\
\wr\downarrow & & \downarrow\wr \\
e_Y^* (id_G \times f)_* s_X^* \mathscr{F} & \xrightarrow{e_Y^* (id_G \times f)_* \lambda^{\mathscr{F}}} & e_Y^* (id_G \times f)_* p_X^* \mathscr{F} \\
\downarrow & & \downarrow \\
f_* e_X^* s_X^* \mathscr{F} & \xrightarrow{f_* e_X^* \lambda^{\mathscr{F}}} & f_* e_X^* p_X^* \mathscr{F} \\
\| & & \| \\
f_* \mathscr{F} & \xrightarrow{\;\;id_{f_*\mathscr{F}}\;\;} & f_* \mathscr{F}
\end{array}
$$

The vertical homomorphisms in the middle are isomorphisms since $e_Y^* s_Y^* f_* \mathscr{F} \to f_* e_X^* s_X^* \mathscr{F}$ resp. $e_Y^* p_Y^* f_* \mathscr{F} \to f_* e_X^* p_X^* \mathscr{F}$ and the upper ones are, use the compatibility of proposition 3.5.

In the case f^*: $e_X^* \lambda^{f^*\mathscr{G}} = e_X^* (id_G \times f)^* \lambda^{\mathscr{G}} = f^* e_Y^* \lambda^{\mathscr{G}} = f^* id_{\mathscr{G}} = id_{f^*\mathscr{G}}$.

(ii) Apply $(id_G \times id_G \times f)_*$ resp. $(id_G \times id_G \times f)^*$ to the equation (ii) of definition 3.27.

These constructions are functorial: In the case f_* let $\alpha : \mathscr{F} \to \mathscr{F}'$ be a homomorphism of G-sheaves on X, to see that $f_* \alpha : f_* \mathscr{F} \to f_* \mathscr{F}'$ is a homomorphism of G-sheaves on Y apply $(id_G \times f)_*$ to the commutative diagram

$$
\begin{array}{ccc}
s_X^* \mathscr{F} & \xrightarrow{\;\;s_X^* \alpha\;\;} & s_X^* \mathscr{F}' \\
\lambda^{\mathscr{F}}\downarrow & & \downarrow\lambda^{\mathscr{F}'} \\
p_X^* \mathscr{F} & \xrightarrow{\;\;p_X^* \alpha\;\;} & p_X^* \mathscr{F}'
\end{array}
$$

to obtain the commutative diagram

$$
\begin{array}{ccc}
s_Y^* f_* \mathscr{F} & \xrightarrow{\ s_Y^* f_* \alpha\ } & s_Y^* f_* \mathscr{F}' \\
\lambda^{f_* \mathscr{F}} \downarrow & & \downarrow \lambda^{f_* \mathscr{F}'} \\
p_Y^* f_* \mathscr{F} & \xrightarrow{\ p_Y^* f_* \alpha\ } & p_Y^* f_* \mathscr{F}'
\end{array}
$$

Similarly for f^*. $\square$

Remark 3.52. For an equivariant morphism of G-schemes $f : X \to Y$ the homomorphism of sheaves of rings $f^* \mathcal{O}_Y \to \mathcal{O}_X$ is equivariant, where $f^* \mathcal{O}_Y$ carries the G-sheaf structure as defined in proposition 3.51.(ii). If the conditions of proposition 3.51.(i) are satisfied, in which case we have defined a G-sheaf structure on $f_* \mathcal{O}_X$, then $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is equivariant as well, because it is adjoint to $f^* \mathcal{O}_Y \to \mathcal{O}_X$ (see proposition 3.55 below).

To show that $f^* \mathcal{O}_Y \to \mathcal{O}_X$ is equivariant, we use the definition of f^* for G-sheaves (proposition 3.51) and the G-sheaf structure on the structure sheaf (example 3.32). Consider the commutative diagram

where $\varphi_X = (\mathrm{pr}_1, s_X) : G \times X \to G \times X$ is the automorphism of X_G over G corresponding to $id_G \in G(G)$ (see remark 3.8.(2)) and similar φ_Y.

The left inner square, the four outer quadrangles and the large outer square of the diagram

commute, therefore the right inner square commutes, that is $f^\# : f^* \mathcal{O}_Y \to \mathcal{O}_X$ is equivariant.

Example 3.53.

(1) Let X be a G-scheme over $S = \mathrm{Spec}\, K$, K a field. The structure morphism $f : X \to \mathrm{Spec}\, K$ is equivariant, for a K-linear representation V, that is a G-sheaf on $\mathrm{Spec}\, K$, there is the G-sheaf $f^* V = V \otimes_K \mathcal{O}_X$ on X.

(2) Let X be a G-scheme with trivial operation and $\mathscr{F}$ a G-sheaf on X. Then for $x \in X$ the inclusion $i : \{x\} \to X$ is equivariant and there is the representation $i^* \mathscr{F} = \mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x)$ on $\{x\} = \mathrm{Spec}\, \kappa(x)$.

(3) Let X be an G-scheme of finite type over a field K with structure morphism $f : X \to \mathrm{Spec}\, K$ and let $\mathscr{F}$ be a G-sheaf on X. Then $f_* \mathscr{F}$ is a G-sheaf on $\mathrm{Spec}\, K$, equivalently $H^0(X, \mathscr{F})$ is a K-linear representation of G. The same is true for the space of sections $\mathscr{F}(U)$ over a G-stable open subset $U \subseteq X$.

3.3.3 Adjunctions for G-sheaves

In this subsection we consider adjunctions between pull-back and push-forward functors on categories of G-sheaves, we establish the adjunction (f^*, f_*) for an equivariant morphism of G-schemes and the adjunction $(f_*, f^!)$ for a finite flat equivariant morphism of G-schemes $f : X \to Y$, Y with trivial G-operation.

Remark 3.54. Let $F : \mathcal{C}^G \to \mathcal{D}^G$, $H : \mathcal{D}^G \to \mathcal{C}^G$ are functors between categories of equivariant sheaves that are adjoint as functors between the underlying categories of sheaves (for example for equivariant f functors f^*, f_* on categories of G-sheaves have been constructed in proposition 3.51 by defining G-sheaf structures in addition to the usual functors f^*, f_* that are adjoint). Then in order to show that this adjunction is also an adjunction in the equivariant setting, it suffices to show that the unit and counit are equivariant, that is any $\varepsilon(\mathscr{F})$ and $\eta(\mathscr{G})$ is an equivariant homomorphism of G-sheaves.

Proposition 3.55. (*The adjunction* (f^*, f_*)).
Let $f : X \to Y$ be an equivariant morphism of G-schemes. Assume that the conditions of proposition 3.51 are satisfied, such that there are functors $f_* : \mathcal{C}^G(X) \to \mathcal{C}^G(Y)$ and $f^* : \mathcal{C}^G(Y) \to \mathcal{C}^G(X)$, for $\mathcal{C} =$ Qcoh or $\mathcal{C} =$ Coh. Then for these functors between categories of equivariant sheaves there is an adjunction (f^*, f_*), whose underlying adjunction for sheaves is the usual adjunction (f^*, f_*). In particular one has isomorphisms

$$\mathrm{Hom}_X^G(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y^G(\mathscr{G}, f_*\mathscr{F})$$

functorial in $\mathscr{F}$ and $\mathscr{G}$.

Proof. One has to show that any $\varepsilon(\mathscr{F})$ and $\eta(\mathscr{G})$ is an equivariant homomorphism of G-sheaves. For ε there is the diagram

The lower quadrangle, where ε_G is the counit of the adjunction (f_G^*, f_{G*}), commutes (ε_G is a morphism of functors). The left and right triangles commute by proposition 3.3 applied to the middle and left diagram of remark 3.50. The vertical maps are isomorphisms, since f_* commutes with base extension by G because of the conditions of proposition 3.51.(i) by [EGA1, I, (9.3.3)]. Therefore the upper quadrangle commutes, that is $\varepsilon(\mathscr{F})$ is equivariant.
Similarly for η. $\qquad\square$

Remark 3.56. These adjunctions can be used for construction of the natural homomorphism of remark 3.1 in the equivariant setting. Let
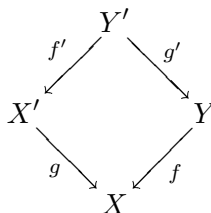
be an equivariant commutative diagram of G-schemes over S. Assume that the considered pull-back and push-forward functors for quasicoherent G-sheaves exist by the construction of proposition 3.51. Then for a quasicoherent G-sheaf $\mathscr{F}$ on Y the natural homomorphism of $\mathcal{O}_{X'}$-modules $g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}$ as defined in remark 3.1 is a homomorphism of G-sheaves. For a quasicoherent $\mathcal{O}_Y$-algebra $\mathscr{A}$ there is the homomorphism $g^* f_* \mathscr{A} \to f'_* g'^* \mathscr{A}$ of G-equivariant $\mathcal{O}_{X'}$-algebras (see also remark 3.4).

––––––––––––––––––––

The next aim is to establish an equivariant version of the adjunction $(f_*, f^!)$ (introduced in subsection 3.1.1) for finite flat morphisms of G-schemes $f : X \to Y$, Y with trivial G-operation, which is a very special case of equivariant Grothendieck duality.

We work with categories of quasicoherent sheaves on X, Y. The functor f_* for quasicoherent G-sheaves is defined by proposition 3.51. Define the functor $f^!$ as in subsection 3.1.1, that is $f^! \mathscr{G} = \mathscr{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathscr{G})$, but carrying a G-sheaf structure as it is given by propositions 3.51 and 3.47 (f is affine, $f_* \mathcal{O}_X$ is locally free of finite rank).

Proposition 3.57. (*The adjunction* $(f_*, f^!)$).
Let $f : X \to Y$ a finite flat equivariant morphism of G-schemes, Y with trivial G-operation. Assume that G is flat over S. Then for the functors $f_*, f^!$ between categories of equivariant quasicoherent sheaves there is an adjunction $(f_*, f^!)$, whose underlying adjunction for sheaves is the usual adjunction $(f_*, f^!)$. In particular one has isomorphisms

$$\mathrm{Hom}_X^G(\mathscr{F}, f^! \mathscr{G}) \cong \mathrm{Hom}_Y^G(f_* \mathscr{F}, \mathscr{G})$$

functorial in $\mathscr{F}$ and $\mathscr{G}$.

Proof. We have trivial operation on Y, that is $s_Y = p_Y$. We use the same notations as in subsection 3.1.1, here $X = \mathrm{Spec}_Y \mathscr{A}$, $\mathscr{A} = f_* \mathcal{O}_X$ a locally free G-sheaf of finite rank.
One has to show that any $\varepsilon(\mathscr{G})$ and $\eta(\mathscr{F})$ is an equivariant homomorphism of G-sheaves.
For ε consider the diagram

The lower quadrangle, where ε_G is the counit of the adjunction (f_G^*, f_{G*}), commutes (ε_G is a morphism of functors). The left and right triangles commute by proposition 3.6. The vertical maps are isomorphisms, since $\mathscr{H}om$ commutes with base extension by G ($\mathscr{A}$ is locally free of finite rank and G is flat over S). Therefore the upper quadrangle commutes, that is $\varepsilon(\mathscr{G})$ is equivariant.
Similarly for η. $\square$

3.3.4   Natural isomorphisms

We show that some natural isomorphisms of sheaves give rise to natural isomorphisms of G-sheaves: We establish isomorphisms between certain bifunctors on categories of G-sheaves that occur by composing bifunctors and functors for G-sheaves constructed before.

Proposition 3.58. *Let $f : X \to Y$ be an equivariant morphism of G-schemes over S and let $\mathscr{E}, \mathscr{F}$ be quasicoherent G-sheaves on Y, assume $\mathscr{E}$ finitely presented and G flat over S and further $\mathscr{E}$ locally free of finite rank or f flat. Then the natural homomorphism of quasicoherent $\mathcal{O}_X$-modules*

$$f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{E}, f^*\mathscr{F})$$

is an isomorphism of G-sheaves.

Proof. Under the assumption "$\mathscr{E}$ locally free of finite rank or f flat" the natural homomorphism of $\mathcal{O}_X$-modules

$$f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{E}, f^*\mathscr{F})$$

is an isomorphism by [EGA1, 0, (5.4.9)] or [EGA, 0_I, (6.7.6)].
Both sides have natural G-sheaf structures as constructed in proposition 3.47, since $\mathscr{E}$ is finitely presented and G flat over S. The above isomorphism is G-equivariant:

$$\lambda^{f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{E}, \mathscr{F})} \cong f_G^* \lambda^{\mathscr{H}om_{\mathcal{O}_Y}(\mathscr{E}, \mathscr{F})} \cong f_G^* \mathscr{H}om_{\mathcal{O}_{G \times Y}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}})$$
$$\cong \mathscr{H}om_{\mathcal{O}_{G \times X}}(f_G^* \lambda^{\mathscr{E}}, f_G^* \lambda^{\mathscr{F}}) \cong \mathscr{H}om_{\mathcal{O}_{G \times X}}(\lambda^{f^*\mathscr{E}}, \lambda^{f^*\mathscr{F}}) \cong \lambda^{\mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{E}, f^*\mathscr{F})}$$

Here, for the equation $f_G^* \mathscr{H}om_{\mathcal{O}_{G \times Y}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}}) \cong \mathscr{H}om_{\mathcal{O}_{G \times X}}(f_G^* \lambda^{\mathscr{E}}, f_G^* \lambda^{\mathscr{F}})$ consider the commutative diagram

$$
\begin{array}{ccc}
f_G^* \mathscr{H}om_{\mathcal{O}_{G \times Y}}(s_Y^*\mathscr{E}, s_Y^*\mathscr{F}) & \xrightarrow{f_G^* \mathscr{H}om_{\mathcal{O}_{G \times Y}}(\lambda^{\mathscr{E}}, \lambda^{\mathscr{F}})} & f_G^* \mathscr{H}om_{\mathcal{O}_{G \times Y}}(p_Y^*\mathscr{E}, p_Y^*\mathscr{F}) \\
\wr \downarrow & & \downarrow \wr \\
\mathscr{H}om_{\mathcal{O}_{G \times X}}(f_G^* s_Y^*\mathscr{E}, f_G^* s_Y^*\mathscr{F}) & \xrightarrow{\mathscr{H}om_{\mathcal{O}_{G \times X}}(f_G^* \lambda^{\mathscr{E}}, f_G^* \lambda^{\mathscr{F}})} & \mathscr{H}om_{\mathcal{O}_{G \times X}}(f_G^* p_Y^*\mathscr{E}, f_G^* p_Y^*\mathscr{F})
\end{array}
$$

where the vertical homomorphisms are isomorphisms, because $p_Y^*\mathscr{E}, s_Y^*\mathscr{E}$ are locally free of finite rank or they are finitely presented and f_G is flat. $\qquad\square$

Proposition 3.59. *Let $f : X \to Y$ be an equivariant morphism of G-schemes and let $\mathscr{E}, \mathscr{F}$ G-sheaves on Y. Then the natural isomorphism of $\mathcal{O}_X$-modules*

$$f^*\mathscr{E} \otimes_{\mathcal{O}_X} f^*\mathscr{F} \to f^*(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{F})$$

is an isomorphism of G-sheaves.

Proof. There is the natural isomorphism of $\mathcal{O}_X$-modules $f^*\mathscr{E} \otimes_{\mathcal{O}_X} f^*\mathscr{F} \to f^*(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{F})$ (see e.g. [EGA1, 0, (4.3.3)]). It is G-equivariant: $\lambda^{f^*\mathscr{E} \otimes_{\mathcal{O}_X} f^*\mathscr{F}} \cong \lambda^{f^*\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} \lambda^{f^*\mathscr{F}} \cong (id_G \times f)^* \lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times X}} (id_G \times f)^* \lambda^{\mathscr{F}} \cong (id_G \times f)^*(\lambda^{\mathscr{E}} \otimes_{\mathcal{O}_{G \times Y}} \lambda^{\mathscr{F}}) \cong (id_G \times f)^*(\lambda^{\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{F}}) \cong \lambda^{f^*(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{F})}$. $\qquad\square$

Similarly, using [Bour, Algebra I, Ch. II, §4.1 and §4.2, Prop. 1 and 2], one obtains:

Proposition 3.60. *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be quasicoherent G-sheaves on a G-scheme X over S, assume $\mathscr{E}, \mathscr{F}$ finitely presented and G flat over S. Then the natural isomorphism*

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{E}, \mathscr{G}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$$

is an isomorphism of G-sheaves. $\qquad\square$

Proposition 3.61. *Let $\mathscr{E}, \mathscr{G}$ be quasicoherent G-sheaves on a G-scheme X over S, assume $\mathscr{E}$ locally free of finite rank and G flat over S. Then the natural isomorphism*

$$\mathscr{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G} \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G})$$

is an isomorphism of G-sheaves. $\qquad\square$

3.4   Trivial operation: G-sheaves as comodules, decomposition

In this section G-sheaves on a G-scheme with trivial operation of an affine group scheme are considered.

We show that in this case G-sheaves are the same as comodules for the Hopf algebra of G.

Further we introduce the invariant subsheaf and show that a direct sum decomposition of the Hopf algebra of the group scheme G into subcoalgebras induce direct sum decompositions of G-sheaves, the components being comodules under the corresponding coalgebras.

In particular the case that G is an affine group scheme over a field K with cosemisimple Hopf algebra is considered.

In the following let $(G = \operatorname{Spec}_S \mathscr{A}, e, m, i)$ be an affine group scheme over a scheme S. Then $(\mathscr{A}, \eta, \mu, \iota, \varepsilon, \Delta)$ is a sheaf of $\mathcal{O}_S$-Hopf algebras. Assume that G operates trivially on an S-scheme X, that is $s_X = p_X : G \times_S X \to X$.

3.4.1   G-sheaves as comodules

Remark 3.62. In this remark we introduce morphisms of functors α and β using the adjunctions (p_X^*, p_{X*}) and (e_X^*, e_{X*}) and describe their relations to the homomorphisms η_X and ε_X of the Hopf algebra structure of $\mathscr{A}_X$.

(1) The adjunction (p_X^*, p_{X*}) and η_X:

Part of the adjunction (p_X^*, p_{X*}) is the natural homomorphism

$$\alpha(\mathscr{F}) : \mathscr{F} \to p_{X*}p_X^*\mathscr{F} = \mathscr{A}_X \otimes_{\mathcal{O}_X} \mathscr{F}$$

Here the tensor product $\mathscr{A}_X \otimes_{\mathcal{O}_X} \mathscr{F}$ is formed by $p_X^{\#} : \mathcal{O}_X \to \mathscr{A}_X$.

For $\mathcal{O}_X$ one has

$$\alpha(\mathcal{O}_X) = \eta_X = p_X^{\#} : \mathcal{O}_X \to p_{X*}p_X^*\mathcal{O}_X = p_{X*}\mathcal{O}_{G \times X} = \mathscr{A}_X$$

and conversely one may describe α in terms of η_X as

$$\alpha(\mathscr{F}) = \alpha(\mathcal{O}_X) \otimes id_{\mathscr{F}} = \eta_X \otimes id_{\mathscr{F}} : \mathscr{F} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{A}_X \otimes_{\mathcal{O}_X} \mathscr{F}$$

(2) The adjunction (e_X^*, e_{X*}) and ε_X:

Part of the adjunction (e_X^*, e_{X*}) is the natural homomorphism $\mathscr{E} \to e_{X*}e_X^*\mathscr{E} = e_{X*}\mathcal{O}_X \otimes_{\mathcal{O}_{G \times X}} \mathscr{E}$, the tensor product formed by $e_X^{\#} : \mathcal{O}_{G \times X} \to e_{X*}\mathcal{O}_X$.

For $\mathscr{E} = p_X^*\mathscr{F}$ one has $p_X^*\mathscr{F} \to e_{X*}e_X^*p_X^*\mathscr{F}$ and after application of p_{X*}:

$$\beta(\mathscr{F}) : p_{X*}p_X^*\mathscr{F} \to p_{X*}e_{X*}e_X^*p_X^*\mathscr{F} = \mathscr{F}$$

For $\mathcal{O}_X$ one has

$$\beta(\mathcal{O}_X) = \varepsilon_X = p_{X*}e_X^{\#} : p_{X*}p_X^*\mathcal{O}_X \to \mathcal{O}_X$$

and conversely

$$\beta(\mathscr{F}) = \beta(\mathcal{O}_X) \otimes id_{\mathscr{F}} = \varepsilon_X \otimes id_{\mathscr{F}} : \mathscr{A}_X \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathscr{F} = \mathscr{F}$$

(3) It is

$$\beta(\mathscr{F}) \circ \alpha(\mathscr{F}) = id_{\mathscr{F}}$$

because $\varepsilon_X \circ \eta_X = id_{\mathcal{O}_X}$, since $p_X \circ e_X = id_X$.

Proposition 3.63. *For an $\mathcal{O}_X$-module $\mathcal{F}$ the following data are equivalent:*

(a) *A G-sheaf structure on $\mathcal{F}$.*

(b) *An $\mathcal{A}$-comodule structure $\varrho : \mathcal{F} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$.*

Proof. Remember that a G-sheaf structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is an isomorphism of $\mathcal{O}_{G \times X}$-modules $\lambda^{\mathcal{F}} : p_X^* \mathcal{F} \xrightarrow{\sim} p_X^* \mathcal{F}$ satisfying the conditions (i) and (ii) of definition 3.27 and an $\mathcal{A}$-comodule structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is a homomorphism of $\mathcal{O}_X$-modules $\varrho : \mathcal{F} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} = \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ such that the diagrams (i) and (ii) in (3.8) commute. Construct the relation between $\lambda^{\mathcal{F}}$ and ϱ by the natural isomorphism

$$\operatorname{Hom}_X(\mathcal{F}, p_{X*} p_X^* \mathcal{F}) \cong \operatorname{Hom}_{G \times X}(p_X^* \mathcal{F}, p_X^* \mathcal{F})$$

coming from the adjunction (p_X^*, p_{X*}).
Then ϱ arises from $\lambda^{\mathcal{F}}$ as

$$\varrho = p_{X*} \lambda^{\mathcal{F}} \circ \alpha(\mathcal{F}) \colon \ \mathcal{F} \ \to \ \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$$

Here

$$p_{X*} \lambda^{\mathcal{F}} \colon \ \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$$

is an isomorphism of $\mathcal{A}_X$-modules on X and

$$\alpha(\mathcal{F}) = \eta \otimes id_{\mathcal{F}} \colon \ \mathcal{F} = \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{F} \ \to \ p_{X*} p_X^* \mathcal{F} = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F}$$

is a natural homomorphism of the adjunction (p_X^*, p_{X*}) (see also remark 3.62).
The conditions (i) *and* (ii) *of definition 3.27 are satisfied if and only if the diagrams* (i) *and* (ii) *in* (3.8) *commute.*
(i). One has the diagram (use notation of remark 3.62)

where the square with vertical maps $\beta(\mathcal{F})$ is commutative and $\beta(\mathcal{F}) \circ \alpha(\mathcal{F}) = id_{\mathcal{F}}$. It follows $e_X^* \lambda^{\mathcal{F}} = \beta(\mathcal{F}) \circ p_{X*} \lambda^{\mathcal{F}} \circ \alpha(\mathcal{F})$. Condition (i) of definition 3.27 means $e_X^* \lambda^{\mathcal{F}} = id_{\mathcal{F}}$. So there are the equivalences

$$\begin{array}{rcl} \text{(i) } \textit{of definition 3.27 is satisfied} & \Longleftrightarrow & id_{\mathcal{F}} = \beta(\mathcal{F}) \circ p_{X*} \lambda^{\mathcal{F}} \circ \alpha(\mathcal{F}) \\ & \Longleftrightarrow & id_{\mathcal{F}} = \beta(\mathcal{F}) \circ \varrho \\ & \Longleftrightarrow & \textit{diagram} \text{ (i) } \textit{of} \text{ (3.8) } \textit{commutes} \end{array}$$

(ii). Condition (ii) of definition 3.27 here becomes (one has $s_X = p_X$, $id_G \times p_X = \operatorname{pr}_{13}$):

$$(m \times id_X)^* \lambda^{\mathcal{F}} = \operatorname{pr}_{23}^* \lambda^{\mathcal{F}} \circ \operatorname{pr}_{13}^* \lambda^{\mathcal{F}}$$

The commutativity of diagram (ii) of (3.8) means

$$(\Delta \otimes id_{\mathcal{F}}) \circ \varrho = (id_{\mathcal{A}} \otimes \varrho) \circ \varrho$$

Left side of these equations: One has (tensor products without index are over $\mathcal{O}_S$)

$$\mathrm{pr}_{3*}(m \times id_X)^* \lambda^{\mathscr{F}} = id_{\mathscr{A} \otimes \mathscr{A}} \otimes p_{X*} \lambda^{\mathscr{F}} : (\mathscr{A} \otimes \mathscr{A})_{\Delta \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F}) \to (\mathscr{A} \otimes \mathscr{A})_{\Delta \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F})$$

Here the tensor product over $\mathscr{A}$ is formed by $\Delta = p_* m^\# : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$. This leads to the commutative diagram

$$\tag{3.11}$$

Right side: One has

$$\mathrm{pr}_{3*} \mathrm{pr}_{13}^* \lambda^{\mathscr{F}} = id_{\mathscr{A} \otimes \mathscr{A}} \otimes p_{X*} \lambda^{\mathscr{F}} : (\mathscr{A} \otimes \mathscr{A})_{i_1 \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F}) \to (\mathscr{A} \otimes \mathscr{A})_{i_1 \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F})$$

the tensor products formed by $i_1 : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$, $a \mapsto a \otimes 1$ and similar

$$\mathrm{pr}_{3*} \mathrm{pr}_{23}^* \lambda^{\mathscr{F}} = id_{\mathscr{A} \otimes \mathscr{A}} \otimes p_{X*} \lambda^{\mathscr{F}} : (\mathscr{A} \otimes \mathscr{A})_{i_2 \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F}) \to (\mathscr{A} \otimes \mathscr{A})_{i_2 \otimes_{\mathscr{A}}} (\mathscr{A} \otimes \mathscr{F})$$

This leads to the commutative diagram

$$\tag{3.12}$$

The numbers $1, 2, 3$ refer to the factors in the tensor products.

The maps of the upper rows of diagram (3.11) and (3.12) are $(\mathscr{A} \otimes \mathscr{A})_X$-linear and as $(\mathscr{A} \otimes \mathscr{A})_X$-linear maps uniquely determined by the maps of the respective lower rows. Therefore the maps of the upper rows of (3.11) and (3.12) coincide (condition (ii) of definition 3.27) if and only if the maps of the lower rows coincide (commutativity of diagram (ii) of (3.8)). $\qquad\square$

Remark 3.64. Let $\mathscr{E}, \mathscr{F}$ be G-sheaves on X.
(1) For a subsheaf $\mathscr{F}' \subseteq \mathscr{F}$: $\mathscr{F}'$ is a G-subsheaf $\iff$ $\mathscr{F}'$ is an $\mathscr{A}$-subcomodule.
(2) For $\varphi \in \mathrm{Hom}_X(\mathscr{E}, \mathscr{F})$: φ is G-equivariant $\iff$ φ is a homomorphism of $\mathscr{A}$-comodules.
(3) Under the above correspondence the tensor product of G-sheaves and the tensor product of $\mathscr{A}$-comodules coincide.
(4) Similarly as for G-sheaves one may define pull-back and under some conditions push-forward of comodules with respect to morphisms and these constructions coincide with those for G-sheaves.

Remark 3.65. For a finite group G the dual A of the group algebra $K[G]$ is a coalgebra. By dualising one sees that a finite dimensional $K[G]$-module is essentially the same as a finite dimensional A-comodule over K. By the proposition this coincides with our definition of the term "representation" (definition 3.34).

3.4.2   The invariant subsheaf, decomposition

Definition 3.66. *(Subsheaf of G-invariant sections).*
Let $\mathscr{F}$ be a G-sheaf with G-sheaf structure equivalent to the $\mathscr{A}$-comodule structure $\varrho : \mathscr{F} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{F}$. Define the G-subsheaf $\mathscr{F}^G$ of $\mathscr{F}$ of G-invariant sections by

$$\mathscr{F}^G(U) := \{ f \in \mathscr{F}(U) \mid \varrho(f) = 1 \otimes f \}$$

for open $U \subseteq X$.

Remark 3.67. Assume that the assumptions of proposition 3.47 are satisfied for G-sheaves $\mathscr{E}, \mathscr{F}$ on X in which case $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})$ has a natural G-sheaf structure. Then

$$\mathscr{H}om^G_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})^G$$

––––––––––––––––––––––

We show that a decomposition of the coalgebra $\mathscr{A}$ into a direct sum of subcoalgebras $\mathscr{A} = \bigoplus_i \mathscr{A}_i$ leads to a decomposition of any G-sheaf on X into a direct sum of G-subsheaves corresponding to the $\mathscr{A}_i$.

Proposition 3.68. *Let $\mathscr{A}$ be a $\mathcal{O}_S$-coalgebra which decomposes into $\mathcal{O}_S$-subcoalgebras $\mathscr{A} = \bigoplus_i \mathscr{A}_i$. Then any $\mathscr{A}$-comodule $\mathscr{F}$ on X has a decomposition*

$$\mathscr{F} = \bigoplus_i \mathscr{F}_i$$

into $\mathscr{A}$-subcomodules $\mathscr{F}_i$ whose comodule structure reduces to that of an $\mathscr{A}_i$-comodule.

Proof. Define $\mathscr{F}_i := \varrho^{-1}(\mathscr{A}_i \otimes \mathscr{F})$.
Then $\mathscr{F}_i$ is a subcomodule, that is $\varrho(\mathscr{F}_i) \subseteq \mathscr{A} \otimes \mathscr{F}_i$: Considering the preimages of $\mathscr{A} \otimes \mathscr{A}_i \otimes \mathscr{F}$ in the commutative diagram (ii) in (3.8) one obtains $\varrho^{-1}(\mathscr{A} \otimes \mathscr{F}_i) = \mathscr{F}_i$.
The comodule structure on $\mathscr{F}$ restricts to an $\mathscr{A}_i$-comodule structure $\mathscr{F}_i \to \mathscr{A}_i \otimes \mathscr{F}_i$ on $\mathscr{F}_i$.
It remains to show that $\mathscr{F}$ is the direct sum of the $\mathscr{F}_i$. $\mathscr{A}$ is the direct sum of the $\mathscr{A}_i$, the injections $j_i : \mathscr{A}_i \to \mathscr{A}$ and projections $q_i : \mathscr{A} \to \mathscr{A}_i$ satisfy $q_i \circ j_i = id_{\mathscr{A}_i}$, $q_{i'} \circ j_i = 0$ for $i' \neq i$ and $\sum_i j_i \circ q_i = id_{\mathscr{A}}$. Construct projections $p_i : \mathscr{F} \to \mathscr{F}_i$ by the diagram (tensor products over $\mathcal{O}_S$)

$$\begin{array}{ccccc} \mathscr{F} & \xrightarrow{\quad \varrho \quad} & \mathscr{A} \otimes \mathscr{F} & \xrightarrow{\varepsilon \otimes id_{\mathscr{F}}} & \mathcal{O}_S \otimes \mathscr{F} = \mathscr{F} \\ & {\scriptstyle p_i} \searrow & \downarrow {\scriptstyle q_i \otimes id_{\mathscr{F}}} & & \\ & & \mathscr{A}_i \otimes \mathscr{F} & \xrightarrow{\varepsilon|_{\mathscr{A}_i} \otimes id_{\mathscr{F}}} & \mathcal{O}_S \otimes \mathscr{F} = \mathscr{F} \end{array}$$

These satisfy the desired properties.   □

Example 3.69. $\mathscr{A}_0 = \eta(\mathcal{O}_S)$ is a subcoalgebra of $\mathscr{A}$ with trivial coalgebra structure. Assume that $\mathscr{A}_0$ is a direct summand, so $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}'$ for some subcoalgebra $\mathscr{A}'$. Then for a G-sheaf $\mathscr{F}$ this decomposition leads to the decomposition $\mathscr{F} = \mathscr{F}_0 \oplus \mathscr{F}'$ with $\mathscr{F}_0 = \mathscr{F}^G$.

Remark 3.70. The decomposition of proposition 3.68 is compatible with functors f^*, f_* for morphisms f of G-schemes with trivial operation, that is

$$f^* \mathscr{F}_i = (f^* \mathscr{F})_i, \quad f_* \mathscr{F}_i = (f_* \mathscr{F})_i$$

if $(\cdot)_i$ denotes the part corresponding to $\mathscr{A}_i$.

3.4.3   Decomposition in the cosemisimple case over a field

For simplicity we assume $S = \operatorname{Spec} K$, K a field. Let $G = \operatorname{Spec} A$ be a group scheme over K and X a G-scheme over K with trivial operation. Assume that A is cosemisimple and let $A = \bigoplus_i A_i$ be its decomposition into simple subcoalgebras.

Corollary 3.71. *(Decomposition into isotypic components).*
Any G-sheaf $\mathscr{F}$ on X has a decomposition

$$\mathscr{F} = \bigoplus_i \mathscr{F}_i$$

into G-subsheaves $\mathscr{F}_i$ whose G-sheaf structure is equivalent to an A_i-comodule structure. $\square$

Remark 3.72. In particular the compatibility of the decomposition into isotypic components of corollary 3.71 with functors f^*, f_* implies that, given a G-sheaf $\mathscr{F}$ on X, for fibers $\mathscr{F} \otimes_{\mathcal{O}_X} \kappa(x)$ resp. spaces of sections $\mathscr{F}(U)$ if $\mathscr{F}_i$ quasicoherent and $U \subseteq X$ affine open, $\mathscr{F}_i \otimes_{\mathcal{O}_X} \kappa(x)$ resp. $\mathscr{F}_i(U)$ are the isotypic components corresponding to A_i.

Proposition 3.73. *Let $\mathscr{F}_i, \mathscr{F}_j$ be quasicoherent G-sheaves on X such that the A-comodule structure of $\mathscr{F}_i$ resp. $\mathscr{F}_j$ reduces to that of an A_i- resp. A_j-comodule. Then $\operatorname{Hom}_X^G(\mathscr{F}_i, \mathscr{F}_j) = 0$ for $i \neq j$.*

Proof. Let $\varphi \in \operatorname{Hom}_X^G(\mathscr{F}_i, \mathscr{F}_j)$. Then for $U \subseteq X$ affine open $\varphi(U) : \mathscr{F}_i(U) \to \mathscr{F}_j(U)$ is a homomorphism of A-comodules over K. These homomorphisms are the zero-homomorphisms since $\mathscr{F}_i(U)$ and $\mathscr{F}_i(U)$ are isotypic with components corresponding to different simple subcoalgebras A_i and A_j. $\square$

Thus, for $\mathcal{C} = \operatorname{Qcoh}, \operatorname{Coh}$ one has a decomposition

$$\mathcal{C}^G(X) \cong \bigoplus_i \mathcal{C}^{A_i}(X)$$

of the equivariant category of sheaves into categories $\mathcal{C}^{A_i}(X)$ of sheaves with A_i-comodule structure. In the following we will investigate the components $\mathcal{C}^{A_i}(X)$ and describe the structure of A_i-comodules on X.

Let W be an A_i-comodule over K. Because A_i is simple, it has only one isomorphism class V of irreducible comodules over K, A_i and V are finite dimensional. W decomposes into a direct sum $W = \bigoplus_{j \in J} V^{(j)}$ of copies of V. There is the natural homomorphism of A-comodules (consider $\operatorname{Hom}_K^G(V, W)$ as a right-$\operatorname{End}_K^G(V)$-module, $\operatorname{Hom}_K^G(V, W) \otimes_{\operatorname{End}_K^G(V)} V$ as A-comodule via its second factor)

$$\begin{array}{ccc} \operatorname{Hom}_K^G(V, W) \otimes_{\operatorname{End}_K^G(V)} V & \to & W \\ \varphi \otimes v & \mapsto & \varphi(v) \end{array}$$

This is an isomorphism: There is the identification $\operatorname{Hom}_K^G(V, W) \otimes_{\operatorname{End}_K^G(V)} V \cong \ldots$
$\ldots \cong \operatorname{Hom}_K^G(V, \bigoplus_{j \in J} V^{(j)}) \otimes_{\operatorname{End}_K^G(V)} V \cong \bigoplus_{j \in J} \operatorname{Hom}_K^G(V, V^{(j)}) \otimes_{\operatorname{End}_K^G(V)} V \cong \bigoplus_{j \in J} V^{(j)}$ which makes the diagram

$$\begin{array}{ccc} \operatorname{Hom}_K^G(V, W) \otimes_{\operatorname{End}_K^G(V)} V & \longrightarrow & W \\ \wr \updownarrow & & \updownarrow \wr \\ \bigoplus_{j \in J} \operatorname{Hom}_K^G(V, V^{(j)}) \otimes_{\operatorname{End}_K^G(V)} V & \xrightarrow{\;\sim\;} & \bigoplus_{j \in J} V^{(j)} \end{array}$$

commutative.

Proposition 3.74. *(Comodules of simple coalgebras).*
Let A_i be a simple subcoalgebra of A, V_i the corresponding simple A_i-comodule over K (up to isomorphism) and let $\mathscr{F}$ be a quasicoherent A_i-comodule on X. Then the natural homomorphism of G-sheaves on X

$$\mathscr{H}om^G_{\mathcal{O}_X}(\mathcal{O}_X \otimes_K V_i, \mathscr{F}) \otimes_{\mathrm{End}^G_K(V_i)} V_i \to \mathscr{F}$$

is an isomorphism.

Proof. The natural homomorphism of $\mathcal{O}_X$-modules

$$\mathscr{H}om^G_{\mathcal{O}_X}(\mathcal{O}_X \otimes_K V_i, \mathscr{F}) \otimes_{\mathrm{End}^G_K(V_i)} V_i \to \mathscr{F}$$

is given for open $U \subseteq X$ by

$$
\begin{array}{ccc}
\mathrm{Hom}^G_U(\mathcal{O}_U \otimes_K V_i, \mathscr{F}|_U) \otimes_{\mathrm{End}^G_K(V_i)} V_i & \to & \mathscr{F}(U) \\
\varphi \otimes v & \mapsto & \varphi(U)(1 \otimes v)
\end{array}
$$

To show that it is an isomorphism of A_i-comodules use the adjunction (f^*, f_*) for the structure morphism $f : X \to \mathrm{Spec}\, K$: For affine open $U \subseteq X$ one has a natural isomorphism $\mathrm{Hom}^G_U(\mathcal{O}_U \otimes_K V_i, \mathscr{F}|_U) = \mathrm{Hom}^G_U(f|^*_U V_i, \mathscr{F}|_U) \cong \mathrm{Hom}^G_K(V_i, f|_{U*}\mathscr{F}|_U) = \mathrm{Hom}^G_K(V_i, \mathscr{F}(U))$ of $\mathrm{End}^G_K(V_i)$-modules, which makes the diagram

$$
\begin{array}{c}
\mathrm{Hom}^G_U(\mathcal{O}_U \otimes_K V_i, \mathscr{F}|_U) \otimes_{\mathrm{End}^G_K(V_i)} V_i \\
\wr \Big\updownarrow \qquad\qquad\qquad \searrow \\
\mathrm{Hom}^G_K(V_i, \mathscr{F}(U)) \otimes_{\mathrm{End}^G_K(V_i)} V_i \qquad\quad \mathscr{F}(U)
\end{array}
$$

commutative. The natural homomorphism $\mathrm{Hom}^G_K(V_i, \mathscr{F}(U)) \otimes_{\mathrm{End}^G_K(V_i)} V_i \to \mathscr{F}(U)$ is an isomorphism of A_i-comodules by the previous discussion. $\qquad\square$

Chapter 4

G-Hilbert schemes

This chapter is devoted to the study of G-Hilbert schemes. The G-Hilbert scheme construction over a base scheme S in general, given an S-scheme X with operation over S of a finite group scheme G, forms under some assumptions an S-scheme G-Hilb$_S X$, that parametrises certain G-stable zero dimensional closed subschemes of X.

This construction, introduced in [ItNm96], [ItNm99], [Nm01] for $\mathbb{A}_{\mathbb{C}}^n$ and more generally for quasiprojective schemes over the base scheme $\operatorname{Spec}\mathbb{C}$, is motivated by the problem to resolve quotient singularities and to describe the properties of such resolutions. It has been made extensive use of it in works concerning the McKay correspondence and is central in the approaches to the McKay correspondence discussed in this thesis.

In this chapter we go through the construction of G-Hilbert schemes, we formulate ideas in a strict functorial setting which allows to extend to finite group schemes with cosemisimple Hopf algebra over arbitrary fields what has been done before for finite groups and over the base field $\mathbb{C}$. The aim of extending these constructions are generalisations of the McKay correspondence to base schemes other than $\mathbb{C}$, the slightly more general case of non algebraically closed fields K of characteristic 0 and finite subgroup schemes $G \subset \operatorname{SL}(2,K)$ is investigated in chapter 6.

As mentioned at the beginning, we will introduce a relative G-Hilbert scheme construction with respect to a morphism $X \to S$, X with G-operation over S. On the level of generality taken here, the group scheme G as well as the scheme S will be defined over a field K. We will construct a morphism of functors $\underline{\text{G-Hilb}}_S X \to X/G$ and consider ways of changing the base scheme of G-Hilbert functors. In particular $\underline{\text{G-Hilb}}_S X$ can be considered as a X/G-functor and then coincides with the relative G-Hilbert functor $\underline{\text{G-Hilb}}_{X/G} X$.

This extended theory leads to some improvements and simplifications, for example the fibration of the G-Hilbert functor over the functor of the quotient leads to a representability proof of G-Hilbert functors for algebraic K-schemes X that works without the theorem on representability of the Hilbert functor of n points and which replaces the assumption "quasiprojective" by the more natural condition that a geometric quotient $\pi : X \to X/G$, π affine, exists.

Regarding the G-Hilbert scheme as relative G-Hilbert scheme G-Hilb$_{X/G} X$, one can carry out the differential study for Quot schemes [Gr61, Section 5] in the equivariant setting to determine relative tangent spaces of the G-Hilbert scheme over X/G. There is a relation to the stratification introduced in [ItNm96], [ItNm99].

———————————————

We summarise the contents in more detail:

Preparatory, we review definitions and results concerning Quot and Hilbert schemes. These schemes are defined by their functor of points to parametrise certain quotient sheaves resp. closed subschemes of a scheme X (quasi)projective over a base scheme S. The reference for this is [Gr61], also see [Ko, Ch. I], [HL, Ch. 2.2], [Ni05], as well [Mu, CS] might be helpful. Language and methods will be taken over to the theory of G-Hilbert schemes.
We also include some general facts about quotients of a scheme by a group scheme, this is taken from [Mu, GIT], [Mu, AV].

Working with categories of G-sheaves on a G-scheme X for a group scheme G over a base scheme S one can carry out the same Quot and Hilbert scheme construction as in the ordinary case in [Gr61]. These equivariant Quot schemes are closed subschemes of the original Quot schemes, they can equivalently be described as fixed point subschemes with respect to the natural G-operation.
We review and extend the representability results of the G-Hilbert functor $\underline{\text{G-Hilb}}_S X$ for quasiprojective S-schemes X using the general representability theorem of [Gr61] in the case of Hilbert functors of points by showing that the G-Hilbert functor is an open and closed subfunctor of the equivariant Hilbert functor which is a closed subfunctor of the ordinary Hilbert functor of $|G|$ points.

The next main theme and main part of this chapter is the relation of the G-Hilbert scheme to the quotient X/G. In this part we consider G-schemes X that are not necessarily quasiprojective over S. We construct the natural morphism $\underline{\text{G-Hilb}}_S X \to \underline{X/G}$. More generally, any equivariant S-morphism $X \to Y$, Y with trivial G-operation, induces a morphism $\underline{\text{G-Hilb}}_S X \to \underline{Y}$. This can be used to relate the Hilbert functor $\underline{\text{G-Hilb}}_S X$ and the relative Hilbert functor $\underline{\text{G-Hilb}}_{X/G} X$, because it allows to consider $\underline{\text{G-Hilb}}_S X$ as a functor over X/G. We pursue the general idea to vary the base scheme of G-Hilbert functors and give some applications.
For finite $X \to S$ Hilbert schemes of points and G-Hilbert schemes can be constructed directly in a simple way as projective schemes over S by showing that a natural embedding into a Grassmannian is closed. Applied to the relative Hilbert functor $\underline{\text{G-Hilb}}_{X/G} X$ for an affine geometric quotient morphism $X \to X/G$ of an algebraic K-scheme X this shows the existence of the projective X/G-scheme G-Hilb$_{X/G} X$. The earlier investigations about base changes then imply that $\underline{\text{G-Hilb}}_K X$ is representable and that there is an isomorphism of K-schemes G-Hilb$_K X \cong {}_K(\text{G-Hilb}_{X/G} X)$, which identifies $\tau : \text{G-Hilb}_K X \to X/G$ with the structure morphism of G-Hilb$_{X/G} X$. In particular, one sees that the morphism τ is projective.
We make some remarks on the case of free operation. Here $\tau : \text{G-Hilb}_K X \to X/G$ is an isomorphism. Thus, if for an irreducible variety X the operation is free on an open dense subscheme, then (G-Hilb$_K X$)$_{\text{red}}$ has a unique irreducible component birational to X/G.

We carry out the differential study for Quot schemes [Gr61, Section 5] (see also [Ko], [HL]) in the equivariant setting. After summarising the relations between differentials, derivations and infinitesimal extensions of morphisms, in subsection 4.4.2 we prove a result about flat deformations of quotient G-sheaves similar to [Gr61, Prop. 5.1], which is applied to the infinitesimal study of equivariant Quot schemes and more specifically of the G-Hilbert scheme in subsequent subsections. This method is known and used to determine tangent spaces of G-Hilbert schemes over base fields but as well it applies to relative G-Hilbert schemes and allows to determine their sheaf of relative differentials and their relative tangent spaces. We observe, that relative tangent spaces of G-Hilb$_\mathbb{C} \mathbb{A}^2_\mathbb{C}$ over $\mathbb{A}^2_\mathbb{C}/G$ are related to the stratification introduced in [ItNm96], [ItNm99] (in this thesis treated in subsection 1.2.2).

4.1 Preliminaries

4.1.1 Quot and Hilbert schemes

Let S be a scheme (later mostly assumed to be noetherian) and $f : X \to S$ an S-scheme.

Quotient sheaves. Let $\mathscr{F}$ be a quasicoherent $\mathcal{O}_X$-module. By a quotient sheaf of $\mathscr{F}$ we mean an exact sequence $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ of quasicoherent $\mathcal{O}_X$-modules with $\mathscr{H}, \mathscr{G}$ specified up to isomorphism, that is either a quasicoherent subsheaf $\mathscr{H} \subseteq \mathscr{F}$ or an equivalence class $[\mathscr{F} \to \mathscr{G}]$ of quasicoherent quotients, where two quotients are defined to be equivalent, if their kernels coincide. Also write $[0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0]$ for the corresponding equivalence class.

Quot and Hilbert functors. For a quasicoherent $\mathcal{O}_X$-module $\mathscr{F}$ the Quot functor for $\mathscr{F}$ on X over S is the functor

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/S} : \begin{array}{ccc} (S\text{-}schemes)^{\circ} & \to & (sets) \\ T & \mapsto & \left\{ \begin{array}{l} Quotient\ sheaves\ [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]\ on\ X_T, \\ \mathscr{G}\ flat\ over\ T \end{array} \right\} \end{array}$$

where for a morphism of S-schemes $\varphi : T' \to T$ the map $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(\varphi) : \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T')$ is defined by application of φ_X^*, i.e. $[\mathscr{F}_T \to \mathscr{G}] \mapsto [\mathscr{F}_{T'} \to \varphi_X^* \mathscr{G}]$ (remember that φ_X^* is right-exact and flatness is stable under base change).
The Hilbert functor arises as the special case $\mathscr{F} = \mathcal{O}_X$:

$$\underline{\mathrm{Hilb}}_{X/S} : \begin{array}{ccc} (S\text{-}schemes)^{\circ} & \to & (sets) \\ T & \mapsto & \left\{ \begin{array}{l} Quotient\ sheaves\ [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0]\ on\ X_T, \\ \mathcal{O}_Z\ flat\ over\ T \end{array} \right\} \end{array}$$

Later, we sometimes only mention the closed subscheme $Z \subseteq X_T$ and write $Z \in \underline{\mathrm{Hilb}}_{X/S}(T)$ instead of the whole quotient.

Decomposition via Hilbert polynomials. In the following assume that S is noetherian, that $f : X \to S$ is projective with $\mathcal{O}_X(1)$ a very ample line bundle relative to f and that $\mathscr{F}$ is coherent.
Then for any locally noetherian S-scheme $T \to S$ the $\mathcal{O}_{X_T}$-module $\mathscr{F}_T$ is coherent and so are its quotients. The morphism $f_T : X_T \to T$ is projective with very ample line bundle $\mathcal{O}_{X_T}(1)$, for any point $t \in T$ with residue field $\kappa(t)$ one has the projective $\kappa(t)$-scheme $f_t : X_t \to \mathrm{Spec}\,\kappa(t)$.
For a coherent $\mathcal{O}_{X_T}$-module $\mathscr{G}$ there is for any point $t \in T$ the Hilbert polynomial $p_{\mathscr{G}_t}$ of the coherent $\mathcal{O}_{X_t}$-module $\mathscr{G}_t$: The Euler-Poincaré characteristic of the twisted sheaves $\mathscr{G}_t(n) = \mathscr{G}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{O}_{X_t}(n)$

$$\chi(\mathscr{G}_t(n)) = \sum_{i \geq 0} (-1)^i h^i(X_t, \mathscr{G}_t(n))$$

is a polynomial in n ([EGA, III, (1), (2.5.3)]), that is $\chi(\mathscr{G}_t(n)) = p_{\mathscr{G}_t}(n)$ for some $p_{\mathscr{G}_t} \in \mathbb{Q}[z]$.
We restrict the functors $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ and $\underline{\mathrm{Hilb}}_{X/S}$ to the category of locally noetherian S-schemes and define subfunctors $\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S} : (locally\ noetherian\ S\text{-}schemes)^{\circ} \to (sets)$ by

$$\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} Quotient\ sheaves\ [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]\ on\ X_T, \\ \mathscr{G}\ flat\ over\ T,\ fibers\ of\ \mathscr{G}\ have\ Hilbert\ polynomial\ p \end{array} \right\}$$

and similarly $\underline{\mathrm{Hilb}}^p_{X/S}$. These are subfunctors, since the Hilbert polynomial is invariant under pull-backs. Note, that in general the assignment of a component to a given polynomial depends on the choice of isomorphism class of a very ample line bundle $\mathcal{O}_X(1)$.

If $\mathscr{G}$ is flat over T, then the Hilbert polynomial of $\mathscr{G}$ is locally constant on T ([EGA, III, (2), (7.9.11)]). It follows, that these subfunctors are open and closed. Further, they cover the original functor, thus there are decompositions

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/S} = \coprod_p \underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}, \qquad \underline{\mathrm{Hilb}}_{X/S} = \coprod_p \underline{\mathrm{Hilb}}^p_{X/S}$$

Representability. The question of representability of the Quot and Hilbert functors reduces to that of the subfunctors for fixed Hilbert polynomials. For these there is the following theorem:

Theorem 4.1. ([Gr61]). *Let S be a noetherian scheme, $f : X \to S$ a projective morphism, $\mathcal{O}_X(1)$ a very ample line bundle relative to f and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then the functors $\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}$: (locally noetherian S-schemes)$^\circ \to$ (sets) are representable by projective S-schemes.* □

This means, that there exists a projective S-scheme $\mathrm{Quot}^p_{\mathscr{F}/X/S}$ (unique up to isomorphism) with an isomorphism $\mathrm{Mor}_S(\,\cdot\,, \mathrm{Quot}^p_{\mathscr{F}/X/S}) \cong \underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}$. The morphism id_Q, where $Q := \mathrm{Quot}^p_{\mathscr{F}/X/S}$, corresponds to a universal quotient $[0 \to \mathscr{H} \to \mathscr{F}_Q \to \mathscr{G} \to 0]$ on X_Q, which determines the above isomorphism.

Base change. Let $\alpha : S' \to S$ be a noetherian S-scheme. Then $\mathscr{F}' := \mathscr{F}_{S'}$ is coherent on $X' := X_{S'}$ and the functor $\underline{\mathrm{Quot}}_{\mathscr{F}'/X'/S'}$ is the restriction of $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ to the category of S'-schemes. In this situation one has the following (see also [EGA1, 0, (1.3.10)]): If $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ is represented by Q with universal quotient $[\mathscr{F}_Q \to \mathscr{G}]$, then $\underline{\mathrm{Quot}}_{\mathscr{F}'/X'/S'}$ is represented by $Q' := Q_{S'}$ with universal quotient $[\mathscr{F}_{Q'} \to \mathscr{G}_{Q'}]$.

Grassmannian. The Grassmannian functor arises in the special case $X = S$. For any scheme S, a quasicoherent $\mathcal{O}_S$-module $\mathscr{F}$ and $n \in \mathbb{N}$ define

$$\underline{\mathrm{Grass}}^n_S(\mathscr{F})(T) = \left\{ \begin{array}{l} \text{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \text{ on } T, \\ \mathscr{G} \text{ locally free of rank } n \end{array} \right\}$$

The proof of representability of general Quot functors uses the representability of Grassmannian functors – it is shown that any Quot functor occurs as a closed subfunctor of a Grassmannian functor. The representability theorem for Grassmannian functors reads as follows:

Theorem 4.2. ([EGA1, I, (9.7.4)]). *For a scheme S, a quasicoherent $\mathcal{O}_S$-module $\mathscr{F}$ and $n \in \mathbb{N}$ the Grassmannian functor $\underline{\mathrm{Grass}}^n_S(\mathscr{F})$ is representable.* □

The Plücker embedding $\mathrm{Grass}^n_S(\mathscr{F}) \to \mathbb{P}_S(\bigwedge^n \mathscr{F})$ shows that for $\mathscr{F}$ quasicoherent of finite type $\mathrm{Grass}^n_S(\mathscr{F})$ is projective over S.

Closed subschemes. If $i : Y \to X$ is a closed embedding over S, then (in case of representability) there is the closed embedding of Quot schemes

$$\mathrm{Quot}_{i^*\mathscr{F}/Y/S} \cong \mathrm{Quot}_{i_*i^*\mathscr{F}/X/S} \to \mathrm{Quot}_{\mathscr{F}/X/S}$$

for example $\mathrm{Hilb}_{Y/S} \to \mathrm{Hilb}_{X/S}$.

Open subschemes. For an open subscheme $j : U \to X$ over S define

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \, \mathrm{supp}(\mathscr{G}) \subseteq U_T \end{array} \right\}$$

This defines an open subfunctor, so (in case of representability) one has the open embedding

$$\mathrm{Quot}_{\mathscr{F}/X/S}|_U \to \mathrm{Quot}_{\mathscr{F}/X/S}$$

Quasi-projective schemes. For a quasiprojective S-scheme U and $\mathscr{E}$ coherent on U define the following variant (which coincides with the original Quot functor for projective U)

$$\underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{E}_T \to \mathscr{G} \to 0] \textit{ on } U_T, \\ \mathscr{G} \textit{ flat over } T, \, \mathrm{supp}(\mathscr{G}) \textit{ proper over } T \end{array} \right\}$$

After choice of an open embedding over S into a projective S-scheme $j : U \to X$ and of a coherent prolongation of $\mathscr{E}$, that is a coherent $\mathcal{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}|_U = \mathscr{E}$ (see [EGA1, I, (6.9.8)]), one has the following proposition.

Proposition 4.3. *There is an isomorphism* $\underline{\mathrm{Quot}}_{\mathscr{E}/U/S} \cong \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U$.

Proof. We show that there are bijections

$$\begin{array}{ccc}
\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U(T) & \leftrightarrow & \underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T) \\
[\mathscr{F}_T \to \mathscr{G}] & \mapsto & [\mathscr{E}_T = j_T^*\mathscr{F}_T \to j_T^*\mathscr{G}] \\
[\mathscr{F}_T \to j_{T*}\mathscr{E}_T \to j_{T*}\mathscr{G}] & \leftarrow\!\shortmid & [\mathscr{E}_T \to \mathscr{G}]
\end{array}$$

functorial in T.

The map $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U(T) \to \underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T)$ is given by application of j_T^* to a quotient sheaf. It is $j_T^*\mathscr{F}_T = \mathscr{E}_T$, since $\mathscr{F}$ is a prolongation of $\mathscr{E}$. Because $\mathrm{supp}(\mathscr{G})$ is proper over T and $\mathrm{supp}(\mathscr{G}) \subseteq U_T$, $\mathrm{supp}(j_T^*\mathscr{G}) = \mathrm{supp}(\mathscr{G})$ is proper over T.

The map $\underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T) \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U(T)$ arises as follows: By the adjunction $\mathrm{Hom}_U(j^*\mathscr{F}, \mathscr{E}) \cong \mathrm{Hom}_X(\mathscr{F}, j_*\mathscr{E})$, corresponding to the isomorphism $j^*\mathscr{F} \to \mathscr{E}$ there is a homomorphism $\mathscr{F} \to j_*\mathscr{E}$ that restricts over U to this isomorphism. For a quotient $[\mathscr{E}_T \to \mathscr{G}] \in \underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T)$, since $\mathrm{supp}(\mathscr{G})$ is proper over T, $\mathrm{supp}(\mathscr{G}) \to X_T$ is proper, so $\mathrm{supp}(\mathscr{G})$ is closed in X_T. It follows that $\mathrm{supp}(j_{T*}\mathscr{G}) \subseteq U_T$. The homomorphism $\mathscr{F}_T \to j_{T*}\mathscr{G}$ is surjective, since it is surjective on U_T and $\mathrm{supp}(j_{T*}\mathscr{G}) \subseteq U_T$.

One verifies that these maps are inverse to each other. $\qquad\square$

So there is an open embedding

$$\underline{\mathrm{Quot}}_{\mathscr{E}/U/S} \cong \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$$

This extends the construction of Quot schemes to quasiprojective S-schemes, in this case the components $\underline{\mathrm{Quot}}^p_{\mathscr{E}/U/S}$ are representable by quasiprojective S-schemes.

Hilbert scheme of n points. Let X be quasiprojective over the noetherian scheme S. The component $\mathrm{Hilb}^n_{X/S}$ of $\mathrm{Hilb}_{X/S}$ for constant Hilbert polynomials $n \in \mathbb{N}$ (it is independent of choice of $\mathcal{O}_X(1)$) is called the Hilbert scheme of n points, we also write $\mathrm{Hilb}^n_S X$ for this scheme. Constant Hilbert polynomial means, that one parametrises 0-dimensional subschemes [EGA, IV, (2), (5.3.1)]. For these the Euler-Poincaré characteristic reduces to h^0. One may rewrite the corresponding functor as

$$\underline{\mathrm{Hilb}}^n_S X(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \textit{ Z flat and} \\ \textit{proper over } T, \textit{ } Z_t \textit{ 0-dimensional with } h^0(Z_t, \mathcal{O}_{Z_t}) = n \textit{ for } t \in T \end{array} \right\} \quad (4.1)$$

The condition "Z flat, proper over T, Z_t 0-dimensional with $h^0(Z_t, \mathcal{O}_{Z_t}) = n$ for $t \in T$" implies
- $H^i(Z_t, \mathcal{O}_{Z_t}) = 0$ for $i > 0$
- $f_{T*}\mathcal{O}_Z$ is locally free of rank n
- the canonical homomorphisms $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathcal{O}_Z \to H^0(Z_t, \mathcal{O}_{Z_t})$ for $t \in T$ are isomorphisms

by the theorem on cohomology and base change [Ha, AG, Thm. 12.11], [EGA, III (2), (7.9.9)], [Mu, GIT, Ch. 0, §5, p. 19]. Moreover, in this situation Z is finite over T by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)] (of course the above cohomology and base change property then also follows). Similar statements are true for the Quot functors $\underline{\mathrm{Quot}}^n_{\mathscr{F}/X/S}$.

Hilbert scheme of 1 point. Under not restrictive assumptions the Hilbert functor $\underline{\mathrm{Hilb}}^1_S X$ as defined in equation (4.1) is represented by the original S-scheme X, we show that there are natural bijections $\underline{\mathrm{Hilb}}^1_S X(T) \leftrightarrow \Gamma(X_T/T) = X(T)$ (see also [AlKl80, Lemma (8.7), p. 108]):

Proposition 4.4. *Let $f : X \to S$ be locally of finite type over the noetherian scheme S. Then there is a morphism $\underline{\mathrm{Hilb}}^1_S X \to \underline{X}$ taking an element $Z \in \underline{\mathrm{Hilb}}^1_S(T)$ to the morphism $T \cong Z \subseteq X_T \to X$. If X is separated over S then this is an isomorphism and determined by $\Delta \leftrightarrow id_X$, $\Delta \subseteq X \times_S X$ the diagonal.*

Proof. For the first statement, the main point is to show that for any $Z \subseteq X_T$, T a locally noetherian S-scheme, defining an element of $\underline{\mathrm{Hilb}}^1_S X(T)$ the restriction $f_T|_Z : Z \to T$ is an isomorphism. Z is flat, proper and quasifinite over T, then by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)] Z is finite over T, so $f_T|_Z : Z \to T$ corresponds to a homomorphism $\mathcal{O}_T \to \mathscr{A}$ of locally free $\mathcal{O}_T$-algebras of finite rank. This is an isomorphism, because $\kappa(t) \to \kappa(t) \otimes_{\mathcal{O}_T} \mathscr{A}$ for any $t \in T$ is an isomorphism. The maps $\underline{\mathrm{Hilb}}^1 X(T) \to \Gamma(X_T/T) = X(T)$ take an element $Z \in \underline{\mathrm{Hilb}}^1_S X(T)$ to the section $T \to Z \subseteq X_T$ inverse to $f_T|_Z : Z \to T$.
If X is separated over S, then there is an inverse morphism $\underline{X} \to \underline{\mathrm{Hilb}}^1_S X$ given by $id_X \mapsto \Delta$ consisting of maps $\underline{X}(T) \to \underline{\mathrm{Hilb}}^1_S X(T)$, $\varphi \mapsto \Gamma_\varphi$ for locally noetherian S-schemes T. Here $\Gamma_\varphi = (id \times \varphi)^{-1}(\Delta) \subseteq X \times_S T$ is the graph of φ, it also arises as the closed subscheme corresponding to the closed embedding $(\varphi, id_T) : T \to X \times_S T$ (closed embedding, because X is separated over S). $\qquad \square$

4.1.2 Group operations and quotients

Let G be a group scheme over a scheme S and $G \times X \overset{p_X}{\underset{s_X}{\rightrightarrows}} X$ a G-scheme over S.

Orbits. ([Mu, GIT]). For a T-valued point $\alpha : T \to X$ the image (as a set of points) of the morphism $\psi_T = (s_X \circ (id_G \times \alpha), p_T) : G \times T \to X \times T$ is called the orbit of α. For $\alpha = id_X$ one has the morphism $\psi = (s_X, p_X) : G \times X \to X \times X$, an arbitrary $\alpha : T \to X$ gives rise to the cartesian diagram

$$
\begin{array}{ccc}
G \times X & \overset{\psi}{\longrightarrow} & X \times X \\
{\scriptstyle id_G \times \alpha} \uparrow & & \uparrow {\scriptstyle id_X \times \alpha} \\
G \times T & \overset{\psi_T}{\longrightarrow} & X \times T
\end{array}
$$

Categorical quotient. ([Mu, GIT], [HL, Ch. 4]). A categorical quotient of X by G is an S-scheme Y with a morphism of S-schemes $\pi : X \to Y$, which makes the diagram

$$
\begin{array}{c}
G \times X \\
{\scriptstyle s_X} \swarrow \quad \searrow {\scriptstyle p_X} \\
X \qquad\qquad X \\
{\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} \\
Y
\end{array}
\tag{4.2}
$$

commutative and satisfies the following universal property: If $\pi' : X \to Y'$ is another morphism such that diagram (4.2) commutes for (Y', π'), then there is a unique morphism $Y \to Y'$ making the whole diagram commute.

Geometric quotient. ([Mu, GIT], [HL, Ch. 4]). A geometric quotient of X by G is an S-scheme Y with a morphism of S-schemes $\pi : X \to Y$ such that

(i) $\pi \circ s_X = \pi \circ p_X$ (i.e. diagram (4.2) commutes).

(ii) π is surjective and $\operatorname{im}(\psi) = X \times_Y X \subseteq X \times X$
(as above $\psi = (s_X, p_X) : G \times X \to X \times X$; the fiber product $X \times_Y X$ is formed by π).

(iii) $U \subseteq Y$ *is open* $\Longleftrightarrow \pi^{-1}(U) \subseteq X$ *is open*

(iv) The homomorphism $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$.

In the case of an affine group scheme G we will in addition assume π to be affine. Note that these conditions are local on Y.

By condition (ii) the orbit for any T-valued point $\alpha : T \to X$ is given by the underlying set of the fiber $X \times_Y T$ of $\pi : X \to Y$ over $T \to Y$, there is the cartesian diagram

$$
\begin{array}{ccccc}
G \times X & \overset{\psi}{\longrightarrow} & X \times_Y X & \hookrightarrow & X \times X \\
{\scriptstyle id_G \times \alpha} \uparrow & & {\scriptstyle id_X \times \alpha} \uparrow & & \uparrow {\scriptstyle id_X \times \alpha} \\
G \times T & \overset{\psi_T}{\longrightarrow} & X \times_Y T & \hookrightarrow & X \times T
\end{array}
$$

It is well known, that any geometric quotient is also a categorical quotient ([Mu, GIT, Prop. 0.1, p.4]), so, provided its existence, it is unique up to isomorphism as well.

Geometric quotients of algebraic K-schemes by finite group schemes. ([Mu, AV], [Mu, GIT]). Set $S = \operatorname{Spec} K$, K a field. Let $G = \operatorname{Spec} A$ be a finite group scheme over K, assume that A is cosemisimple. Let G operate on an algebraic K-scheme X. Then under not very restrictive conditions a geometric quotient $\pi : X \to Y$ exists in the category of algebraic K-schemes and the morphism π is finite:

The affine case: For affine X a geometric quotient can be constructed as $\pi : X = \operatorname{Spec} B \to Y = \operatorname{Spec} B^G$ given by the inclusion $B \to B^G$. The main points are:
- B is finite over B^G. This also implies that B^G is finitely generated over K.
- Different orbits can be separated by an invariant.

The general case: This construction can be applied to not necessarily affine algebraic K-schemes X that can be covered by G-stable affine open subschemes. Conversely, if a geometric quotient $\pi : X \to Y$, π affine, exists, then there is such a G-stable affine open cover. Moreover, the condition (iv) leads to the construction above. Then π is a finite morphism and Y is an algebraic K-scheme as well.

Free operation and G-torsors. ([Mu, GIT]). Let $G = \operatorname{Spec} A$ be a finite group scheme over a field K, and let G operate on an algebraic K-scheme X.
The operation of G on X is said to be free, if $\psi = (s_X, p_X) : G \times X \to X \times X$ is a closed embedding.
If $\pi : X \to Y$ is a geometric quotient of X by G and the operation is free, then by [Mu, AV, Ch. III.12, Thm. 1, p. 111,112], [Mu, GIT, Ch. 0, §4, Prop. 9, p. 16] X is a principal fiber bundle or torsor for G over Y.
For the finite group scheme G over K a G-torsor over a K-scheme Y is a finite flat Y-scheme $X \to Y$ with a G-operation over Y such that $G_Y \times_Y X \to X \times_Y X$ is an isomorphism [Mu, GIT, Ch. 0, §4, Def. 10, p. 16], [Mi, Ch. III, §4].
The fibers $\pi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \kappa(y)$ of a G-torsor $X \to Y$ for $y \in Y$ are isomorphic to the regular representation (because the fibers of $X \times_Y X \cong G \times X$ over X are). A G-torsor is a geometric quotient (see also lemma 4.14 for the more general case of a finite flat morphism with fibers isomorphic to the regular representation).

4.2 Equivariant Quot schemes and the G-Hilbert scheme

4.2.1 Equivariant Quot and Hilbert schemes

Let G be a group scheme flat over a scheme S.

By a quotient G-sheaf of a quasicoherent G-sheaf $\mathscr{F}$ we mean an exact sequence $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ in the category of G-equivariant quasicoherent sheaves with $\mathscr{H}, \mathscr{G}$ specified up to isomorphism or equivalently a quotient of $\mathscr{F}$ by a G-subsheaf, again write $[0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0]$ or sometimes $[\mathscr{F} \to \mathscr{G}]$ for the corresponding equivalence class.

For a G-scheme X over S and a quasicoherent G-sheaf $\mathscr{F}$ on X define the equivariant Quot functor by

$$\underline{\mathrm{Quot}}^{G}_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T \end{array} \right\}$$

In particular, the G-sheaf $\mathcal{O}_X$ gives rise to the equivariant Hilbert functor

$$\underline{\mathrm{Hilb}}^{G}_{X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \\ \mathcal{O}_Z \textit{ flat over } T \end{array} \right\}$$

One also may consider the original Quot functors, where it is not assumed that a subsheaf $\mathscr{H} \subseteq \mathscr{F}_T$ is a G-subsheaf. Here the group operation on X and the G-sheaf structure of $\mathscr{F}$ induce an operation of G on the corresponding Quot schemes as follows:
For any S-scheme T a T-valued point $g \in G(T)$ defines an automorphism $g : X_T \to X_T$ (see remark 3.8), we define maps

$$\begin{array}{ccc} G(T) \times_S \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) & \to & \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) \\ (g, [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]) & \mapsto & [0 \to g_*\mathscr{H} \to \mathscr{F}_T \to g_*\mathscr{G} \to 0] \end{array}$$

by applying g_* and using the isomorphism $\mathscr{F}_T \to g_*\mathscr{F}_T$ coming from the G-sheaf structure of $\mathscr{F}$. These maps are group operations functorial in T, so they form an operation $\underline{G} \times_S \underline{\mathrm{Quot}}_{\mathscr{F}/X/S} \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ of the group functor $\underline{G}$ on the Quot functor $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ and in case of representability an operation of the group scheme G on the scheme $\mathrm{Quot}_{\mathscr{F}/X/S}$ over S.

For any group scheme G operating on a scheme Y over S there is the notion of the fixed point subfunctor $\underline{Y}^G \subseteq \underline{Y}$ defined by (see also [DG, II, §1, Def. 3.4])

$$\underline{Y}^G(T) := \{x \in Y(T) \mid \textit{for all } T\textit{-schemes } T' \to T \textit{ and } g \in G(T')\colon \ gx_{T'} = x_{T'}\}$$

Remark 4.5. If G is flat over S, then it suffices to consider flat T-schemes T' (the case $T' = G_T \to T$ is sufficient): After base extension $T \to S$ one has $(x : T \to Y_T) \in \Gamma(Y_T/T)$ and for any $T' \to T$, $g \in Y_T(T')$ there is the commutative diagram

$$\begin{array}{ccc} g \qquad\qquad x_{T'} & & gx_{T'} \\ G_T(T') \times_T Y_T(T') & \longrightarrow & Y_T(T') \\ \uparrow \qquad\qquad \uparrow & & \uparrow \\ G_T(G_T) \times_T Y_T(G_T) & \longrightarrow & Y_T(G_T) \\ id_{G_T} \qquad\quad x_{G_T} & & id_{G_T}x_{G_T} \end{array}$$

where x_{G_T} is written for $x \circ p_T : G_T \to T \to Y_T$, $x_{T'}$ for the composition $T' \to T \to Y_T$. So $gx_{T'}$ is determined by $id_{G_T}x_{G_T}$. The morphism $G_T \to T$ is flat since G is flat over S.

We show that under not very restrictive assumptions this is a closed subfunctor and thus represented by a closed subscheme $Y^G \subseteq Y$.

Theorem 4.6. ([DG, II, §1, Thm. 3.6]). *Let Y be a G-scheme over a scheme S. Assume that $G = \mathrm{Spec}_S \mathscr{A}$ is affine over S with $\mathscr{A}$ locally free on S (e.g. G finite flat over S or $S = \mathrm{Spec}\, K$, K a field) and that Y is separated over S. Then the fixed point subfunctor $\underline{Y}^G$ is a closed subfunctor of $\underline{Y}$.*

Proof. We use the functors $\underline{\mathrm{Mor}}_S(G,Y) : (S\text{-}schemes)^\circ \to (sets)$ defined by $\underline{\mathrm{Mor}}_S(G,Y)(T) = \mathrm{Mor}_T(G_T, Y_T)$ and $\underline{\mathrm{Mor}}_S(G,Y)(T) \to \underline{\mathrm{Mor}}_S(G,Y)(T')$, $\alpha \mapsto \alpha_{T'}$ for $T' \to T$.
The functor $\underline{Y}^G$ is given as the fiber product in the cartesian square

$$
\begin{array}{ccc}
\underline{Y}^G & \longrightarrow & \underline{\mathrm{Mor}}_S(G,Y) \\
\downarrow & & \downarrow \\
\underline{Y} & \longrightarrow & \underline{\mathrm{Mor}}_S(G,Y) \times_S \underline{\mathrm{Mor}}_S(G,Y)
\end{array}
$$

where $\underline{Y} \to \underline{\mathrm{Mor}}_S(G,Y) \times_S \underline{\mathrm{Mor}}_S(G,Y)$ is given by $y \mapsto ((g \mapsto gy), (g \mapsto y))$ (more precisely, $y \in Y(T)$ defines an element $\varphi \in \underline{\mathrm{Mor}}_S(G,Y)(T) = \mathrm{Mor}_T(G_T, Y_T)$ by $g \mapsto gy_{T'}$ for $T' \to T$, $g \in G_T(T')$, write this as $(g \mapsto gy)$) and $\underline{\mathrm{Mor}}_S(G,Y) \to \underline{\mathrm{Mor}}_S(G,Y) \times_S \underline{\mathrm{Mor}}_S(G,Y)$ by $\alpha \mapsto (\alpha, \alpha)$. The right vertical morphism identifies with $\underline{\mathrm{Mor}}_S(id_G, \Delta) : \underline{\mathrm{Mor}}_S(G,Y) \to \underline{\mathrm{Mor}}_S(G, Y \times_S Y)$ where $\Delta : Y \to Y \times_S Y$ is the diagonal morphism and a closed embedding since Y is separated over S.
To show that $\underline{Y}^G \to \underline{Y}$ is a closed subfunctor, it suffices to show that $\underline{\mathrm{Mor}}_S(id_G, \Delta)$ is a closed embedding. This follows from:
Let $i : V \to W$ be a closed embedding of S-schemes and $U = \mathrm{Spec}_S \mathscr{A}$ be an affine S-scheme with $\mathscr{A}$ locally free on S. Then $\underline{\mathrm{Mor}}_S(U,V) \to \underline{\mathrm{Mor}}_S(U,W)$ induced by i is a closed subfunctor.
Proof: To show that this subfunctor is closed, one has to show that for any S-scheme R and any morphism $\mathrm{Mor}_S(\,\cdot\,, R) \to \underline{\mathrm{Mor}}_S(U,W)$ given by $id_R \mapsto (\alpha : U_R \to W_R)$ there is a closed subscheme $R' \subseteq R$ such that for $\beta \in \mathrm{Mor}_S(T,R)$: β factors through R' if and only if $\alpha_\beta : U_T \to W_T$ (base extension of α via β) factors through $V_T \subseteq W_T$.
Let $\psi : \mathscr{A}_R \to \mathscr{B}$ be the surjective homomorphism of $\mathcal{O}_R$-algebras corresponding to the closed embedding $\mathrm{Spec}_R \mathscr{B} = \alpha^{-1}(V_R) \to U_R = \mathrm{Spec}_R \mathscr{A}_R$. Then $\ker \psi = 0$ if and only if α factors through V_R.
The question is local on R and $\mathscr{A}_R$ is locally free, so we may assume $\mathscr{A} \cong \bigoplus_{j \in J} \mathcal{O}_R^{(j)}$ as an $\mathcal{O}_R$-module for some index set J. The closed subscheme $R' \subseteq R$, defined by the ideal sheaf generated by the $\ker(\mathcal{O}_R^{(j)} \to \mathscr{B})$, $j \in J$, has the desired properties. $\qquad\square$

The equivariant Quot functor can be considered as a subfunctor of the original Quot functor (canonical inclusion by forgetting the G-sheaf structures) and as such compared with the fixed point subfunctor.

Proposition 4.7. *The equivariant Quot functor coincides with the fixed point subfunctor.*

Proof. A T-valued point of $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ given by a subsheaf $\mathscr{H} \subseteq \mathscr{F}_T$ is a T-valued point of the fixed point subfunctor if and only if for all flat T-schemes T' and $g \in G(T')$ the isomorphism $g_* \lambda_g^{\mathscr{F}_{T'}} : \mathscr{F}_{T'} = g_* g^* \mathscr{F}_{T'} \to g_* \mathscr{F}_{T'}$ given by the G-sheaf structure of $\mathscr{F}$ restricts to an isomorphism $\mathscr{H}_{T'} \to g_* \mathscr{H}_{T'}$. By remark 3.37 this is equivalent to the statement that $\mathscr{H}$ is a G-subsheaf of $\mathscr{F}_T$ or equivalently that it defines a T-valued point of the equivariant Quot functor. $\qquad\square$

Corollary 4.8. *Let S be a noetherian scheme, G be an affine group scheme over S such that $G = \operatorname{Spec}_S \mathscr{A}$ with $\mathscr{A}$ locally free on S. Let X be a (quasi)projective G-scheme over S and $\mathscr{F}$ be a coherent G-sheaf on X. Then the equivariant Quot functor $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G,p}$ for a fixed Hilbert polynomial $p \in \mathbb{Q}[z]$ is represented by a (quasi)projective S-scheme $\operatorname{Quot}_{\mathscr{F}/X/S}^{G,p}$.*

Proof. Theorem 4.1 resp. its extension to the quasiprojective case, theorem 4.6 and proposition 4.7. □

4.2.2 The G-Hilbert scheme

In this subsection let S be a noetherian scheme over a field K. Let $f : X \to S$ be a (quasi) projective S-scheme and G be a group scheme over K. Assume that $G = \operatorname{Spec} A$ is affine with A cosemisimple. Let G operate on X over S, i.e. $G \times_K X \to X$ is an S-morphism. Let $\mathscr{F}$ be a coherent G-sheaf on X.

Remark 4.9. Here we have different base schemes for X and G. The results of the last subsection concerning representability are applicable, we may consider the S-morphism $G \times_K X \to X$ as an operation $G_S \times_S X \to X$ of the group scheme G_S and an equivariant sheaf on X either as a G-sheaf or as a G_S-sheaf. We write $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K}, \underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,p}$ for the corresponding Quot functors.

In the equivariant case the Quot functors for a constant Hilbert polynomial $n \in \mathbb{N}$ have a finer decomposition given by the isomorphism classes of n-dimensional representations of G over K. For $n \in \mathbb{N}$ and V an isomorphism class of n-dimensional representations of G over K one has a subfunctor $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,V} \subseteq \underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,n}$ given by (T a locally noetherian S-scheme)

$$\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,V}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \operatorname{supp} \mathscr{G} \textit{ finite over } T, \\ \textit{for } t \in T : H^0(X_t, \mathscr{G}_t) \cong V_{\kappa(t)} \textit{ as representations over } \kappa(t) \end{array} \right\}$$

where we have replaced "supp$\mathscr{G}$ *proper over* T, supp$\mathscr{G}_t$ 0-*dimensional*" by the equivalent condition "supp$\mathscr{G}$ *finite over* T" (here proper, quasifinite implies finite by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)]).

Remark 4.10. As in the case of Hilbert schemes of n points in subsection 4.1.1 $f_{T*}\mathscr{G}$ is a locally free sheaf on T and the canonical homomorphisms $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \to H^0(X_t, \mathscr{G}_t)$ of representations over $\kappa(t)$ (that can be constructed using the adjunction between inverse and direct image functors for G-sheaves, see remark 3.56) are isomorphisms. Thus one may rewrite the above functor as

$$\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,V}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \operatorname{supp} \mathscr{G} \textit{ finite over } T, \\ \textit{for } t \in T : \kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \cong V_{\kappa(t)} \textit{ as representations over } \kappa(t) \end{array} \right\}$$

Proposition 4.11. *For V an isomorphism class of n-dimensional representations of G over K the functor $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,V}$ is an open and closed subfunctor of $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,n}$.*

Proof. Let T be an S-scheme and $[\mathscr{F}_T \to \mathscr{G}]$ a quotient of G-sheaves defining a T-valued point of $\underline{\operatorname{Quot}}_{\mathscr{F}/X/S}^{G/K,n}$. The sheaf $f_{T*}\mathscr{G}$ is a locally free G-sheaf on T, as explained after proposition 3.68 it has a decomposition into isotypic components $f_{T*}\mathscr{G} = \bigoplus_i \mathscr{G}_i$ with $\mathscr{G}_i$ the components corresponding to the isomorphism classes V_i of simple representations of G over K.

The decomposition $f_{T*}\mathscr{G} = \bigoplus_i \mathscr{G}_i$ determines the decomposition $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} = \bigoplus_i (\kappa(t) \otimes_{\mathcal{O}_T} \mathscr{G}_i)$ into isotypic components, in particular the multiplicity of $(V_i)_{\kappa(t)}$ in $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G}$ is determined by the rank of $\mathscr{G}_i$, which is locally free as direct summand of the locally free sheaf $f_{T*}\mathscr{G}$. It follows, that these multiplicities are locally constant on T and therefore the condition $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \cong V_{\kappa(t)}$ is open and closed. $\qquad\square$

Corollary 4.12. *If X is (quasi)projective over S, then the functor $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}^{G/K,V}$ is represented by a (quasi)projective S-scheme $\mathrm{Quot}_{\mathscr{F}/X/S}^{G/K,V}$.* $\qquad\square$

For G finite over K of degree $|G|$ the G-Quot functor G-$\mathrm{Quot}_{\mathscr{F}/X/S}$ arises by taking for V the regular representation of G. In particular, the G-Hilbert functor is defined by

$$\underline{\text{G-Hilb}}_S\, X(T) := \left\{ \begin{array}{l} \textit{Quotient G-sheaves } \ [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \ \textit{ on } X_T, \\ \textit{Z finite flat over } T, \ \textit{ for } t \in T : H^0(Z_t, \mathcal{O}_{Z_t}) \textit{ isomorphic} \\ \textit{to the regular representation} \end{array} \right\} \qquad (4.3)$$

To summarise the arguments in this special case, $\underline{\mathrm{Hilb}}_S^{|G|} X$ is representable by a (quasi)projective S-scheme by theorem 4.1, $\underline{\text{G-Hilb}}_S X$ is an open and closed subfunctor of the equivariant Hilbert functor $\underline{\mathrm{Hilb}}_S^{G/K,|G|} X$ or, what amounts to the same by proposition 4.7, the fixed point subfunctor of $\underline{\mathrm{Hilb}}_S^{|G|} X$, which is a closed subfunctor of $\underline{\mathrm{Hilb}}_S^{|G|} X$ by theorem 4.6.

Corollary 4.13. *If X is (quasi)projective over S, then the functor $\underline{\text{G-Hilb}}_S X$ is represented by a (quasi)projective S-scheme G-$\mathrm{Hilb}_S X$.* $\qquad\square$

4.3 G-Hilb X and X/G

In the following let K be a field and $G = \operatorname{Spec} A$ a finite group scheme over K with A cosemisimple. Let S be a K-scheme and X an S-scheme with G-operation over S. The G-Hilbert functor for X is defined as in equation (4.3).

4.3.1 The morphism G-Hilb $X \to X/G$

For an extension field L of K an L-valued point of G-Hilb$_K X$, that is a finite closed subscheme $Z \subseteq X_L$ such that $H^0(Z, \mathcal{O}_Z)$ is isomorphic to the regular representation of G over L, is sometimes called a G-cluster. Since the regular representation only contains one copy of the trivial representation, the support of a G-cluster consists of only one G-orbit and thus defines a point of the quotient. We look for a morphism which contains this map as the map of points.

Lemma 4.14. *Let T be an S-scheme and $Z \subseteq X_T$ a closed subscheme defining an element of* $\underline{\text{G-Hilb}}_S X(T)$*. Then the projection $Z \to T$ is a geometric quotient of Z by G.*

Proof. Z is finite flat over T, $f_{T*}\mathcal{O}_Z$ is a locally free $\mathcal{O}_T$-algebra of finite rank and as a G-sheaf has the property that $f_{T*}\mathcal{O}_Z \otimes_{\mathcal{O}_T} \kappa(t)$ for $t \in T$ is isomorphic to the regular representation.
To show that $Z \to T$ is a geometric quotient one may take invariants over affine open subschemes of T. (Of course, the conditions (i)-(iv) of a geometric quotient also can be verified directly.) $\quad\square$

Theorem 4.15. *Let Y be a G-scheme over S with trivial G-operation and $\varphi : X \to Y$ an equivariant morphism. Then there is a unique morphism of functors $(S\text{-schemes})^{\circ} \to (sets)$*

$$\tau : \underline{\text{G-Hilb}}_S X \to \underline{Y}$$

such that for S-schemes T and $Z \in \underline{\text{G-Hilb}}_S X(T)$ with image $\tau(Z) \in Y(T)$ the diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ \tau(Z)\ } & Y
\end{array}
\tag{4.4}
$$

commutes.

Proof. By Lemma 4.14 the morphism $Z \to T$ for $Z \in \underline{\text{G-Hilb}}_S X(T)$ is a geometric and thus also categorical quotient of Z by G. The existence and uniqueness of a morphism $\tau(Z)$ such that diagram (4.4) commutes follow from its universal property.
The maps $\underline{\text{G-Hilb}}_S X(T) \to Y(T)$, $Z \mapsto \tau(Z)$ are functorial in T, that is $\tau(Z_{T'}) = \tau(Z) \circ \alpha$ for morphisms $\alpha : T' \to T$ of S-schemes (by uniqueness of $\tau(Z_{T'})$). Therefore they define a morphism of functors $\tau : \underline{\text{G-Hilb}}_S X \to \underline{Y}$. $\quad\square$

Corollary 4.16. *There is a unique morphism of functors $(S\text{-schemes})^{\circ} \to (sets)$*

$$\tau : \underline{\text{G-Hilb}}_S X \to \underline{X/G}$$

such that for S-schemes T and $Z \in \underline{\text{G-Hilb}}_S X(T)$ with image $\tau(Z) \in X/G(T)$ the diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ \tau(Z)\ } & X/G
\end{array}
\tag{4.5}
$$

commutes. $\quad\square$

Construction 4.17. We construct the morphism of the theorem in a more concrete way without using lemma 4.14 by defining a morphism of S-functors $\underline{\text{G-Hilb}}_S X \to \underline{\text{Hilb}}_S^1 Y \to \underline{Y}$.

Let $f : X \to S$, $g : Y \to S$ be S-schemes, $Y \to S$ separated. Let G operate on X over S and $\varphi : X \to Y$ be an equivariant morphism, Y with trivial G-operation. Similar to G-Hilbert functors for S-schemes without finiteness condition we define the functor $\underline{\text{Hilb}}_S^1 Y$ by

$$\underline{\text{Hilb}}_S^1 Y(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves}\ \ [0 \to \mathscr{I} \to \mathcal{O}_{Y_T} \to \mathcal{O}_Z \to 0]\ \ \textit{on}\ Y_T, \\ Z\ \textit{finite flat over}\ T,\ \textit{for}\ t \in T : h^0(Z_t, \mathcal{O}_{Z_t}) = 1 \end{array} \right\}$$

Let T be an S-scheme and $[\mathcal{O}_{X_T} \to \mathcal{O}_Z] \in \underline{\text{G-Hilb}}_S X(T)$. The restriction $\varphi_Z := (\varphi_T)|_Z : Z \to Y_T$ is a finite morphism. Because the G-operation on Y_T is trivial, the corresponding G-equivariant homomorphism $\mathcal{O}_{Y_T} \to \varphi_{Z*}\mathcal{O}_Z$ factors through $\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G$ (this corresponds to the universal property of the quotient $Z \to T$ used in the proof of the theorem).

The homomorphism $\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G$ is surjective because already $g_{T*}O_{Y_T} \to g_{T*}(\varphi_{Z*}\mathcal{O}_Z)^G$ is (the composition of the canonical homomorphisms $\mathcal{O}_T \to g_{T*}O_{Y_T} \to g_{T*}(\varphi_{Z*}\mathcal{O}_Z)^G = (f_{T*}\mathcal{O}_Z)^G$ is an isomorphism). Since $\mathcal{O}_Z$ is flat over T, $\varphi_{Z*}\mathcal{O}_Z$ is flat over T and the direct summand $(\varphi_{Z*}\mathcal{O}_Z)^G$ is flat as well. The representation $H^0(Y_t, (\varphi_{Z*}\mathcal{O}_Z)_t) \cong \kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathcal{O}_Z \cong H^0(X_t, (\mathcal{O}_Z)_t)$ over $\kappa(t)$ (see also remark 4.10) is isomorphic to the regular representation, therefore $h^0(Y_t, (\varphi_{Z*}\mathcal{O}_Z)_t^G) = 1$ for $t \in T$. Thus the quotient $[\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G]$ defines an element of $\underline{\text{Hilb}}_S^1 Y(T)$.

The maps $\underline{\text{G-Hilb}}_S X(T) \to \underline{\text{Hilb}}_S^1 Y(T)$, $[\mathcal{O}_{X_T} \to \mathcal{O}_Z] \mapsto [\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G]$ are functorial in T, i.e. for S-morphisms $T' \to T$ the quotients $[\mathcal{O}_{Y_{T'}} \to (\varphi_{Z*}\mathcal{O}_Z)_{T'}^G]$ and $[\mathcal{O}_{Y_{T'}} \to (\varphi_{Z_{T'}*}\mathcal{O}_{Z_{T'}})^G]$ coincide: The natural homomorphisms of G-equivariant $\mathcal{O}_{Y_{T'}}$-algebras $(\varphi_{Z*}\mathcal{O}_Z)_{T'} \to \varphi_{Z_{T'}*}\mathcal{O}_{Z_{T'}}$ arising from the diagrams

$$\begin{array}{ccc} Z_{T'} & \longrightarrow & Z \\ \varphi_{Z_{T'}} \downarrow & & \downarrow \varphi_Z \\ Y_{T'} & \longrightarrow & Y_T \end{array}$$

(see remark 3.56) are isomorphisms (φ_Z is finite). So these maps define a morphism $\underline{\text{G-Hilb}}_S X \to \underline{\text{Hilb}}_S^1 Y$.

Taking into account the canonical morphism $\underline{\text{Hilb}}_S^1 Y \to \underline{Y}$ that arises exactly as in proposition 4.4, one obtains a morphism $\underline{\text{G-Hilb}}_S X \to \underline{Y}$ such that the diagrams (4.4) commute: The element $[\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G] \in \underline{\text{Hilb}}_S^1 Y(T)$ corresponds to a closed subscheme $W \subseteq Y_T$ through which $\varphi_Z : Z \to Y_T$ factors. $\tau(Z)$ then is constructed using the inverse of the isomorphism $W \to T$:

$$\begin{array}{ccccc} Z \subseteq X_T & \longrightarrow & & X \\ \varphi_Z \downarrow & \downarrow \varphi_T & & \downarrow \varphi \\ W \subseteq Y_T & \longrightarrow & & Y \\ \nwarrow \ \ \swarrow & & \tau(Z) & \\ T & & & \end{array}$$

The uniqueness of a morphism $T \to Y$ making diagram (4.4) commute can also be seen via this construction.

Remark 4.18. A similar construction can be used in more general situations to form morphisms $\underline{\text{G-Hilb}}_S X \to \underline{\text{H-Hilb}}_S Y$ or at least morphisms defined on open subschemes.

4.3.2   Change of the base scheme of G-Hilbert functors

In this subsection we pursue the general idea of varying the base scheme for G-Hilbert functors. The base scheme always has trivial G-operation, a in some sense maximal possible base scheme, that is one with minimal fibers, for the G-Hilbert functor of a given G-scheme X is the quotient X/G. We show that $\underline{\text{G-Hilb}}_S X$ considered as a X/G-functor via the morphism τ of corollary 4.16 is isomorphic to $\underline{\text{G-Hilb}}_{X/G} X$.

We begin with some generalities on changing the base scheme of functors with respect to a morphism $\varphi : S' \to S$.

- base restriction: For a functor $F' : (S'\text{-schemes})^\circ \to (sets)$ define $_S F' : (S\text{-schemes})^\circ \to (sets)$ by taking disjoint unions

$$_S F'(T) := \bigsqcup \left\{ F'(\alpha) \mid \alpha \in \text{Mor}_S(T, S') \right\}$$

For an S'-scheme Y the corresponding construction is to consider Y as an S-scheme by composing the structure morphism $Y \to S'$ with $S' \to S$.

- base extension: For a functor $F : (S\text{-schemes})^\circ \to (sets)$ define $F_{S'} : (S'\text{-schemes})^\circ \to (sets)$ by

$$F_{S'}(\alpha : T \to S') = F(\varphi \circ \alpha)$$

which is the restriction of F to the category of S'-schemes. $F_{S'}$ also can be realised as the fibered product $F \times_{\underline{S}} \underline{S'}$ considered as S'-functor. For an S-scheme X this corresponds to the usual base extension $X_{S'} = X \times_S S'$.

- If in addition a morphism of S-functors $\psi : F \to \underline{S'}$ is given, then one can consider F as a functor on the category of S'-schemes via ψ, in other words let $F_{(S',\psi)} : (S'\text{-schemes})^\circ \to (sets)$ be given by

$$F_{(S',\psi)}(\alpha : T \to S') = \{\beta \in F(\varphi \circ \alpha) \mid \psi(\beta) = \alpha\} = \{\beta \in F_{S'}(\alpha) \mid \psi(\beta) = \alpha\}$$

For schemes this means to consider an S-scheme X as an S'-scheme via a given S-morphism $X \to S'$. Note that $F_{(S',\psi)}$ can be considered as subfunctor of $F_{S'}$ and that $_S(F_{(S',\psi)}) \cong F$.

Remark 4.19. (Base extension for G-Hilbert schemes).
(1) Let X be a G-scheme over S and S' an S-scheme. Then there is the isomorphism of S'-functors

$$(\underline{\text{G-Hilb}}_S X)_{S'} \cong \underline{\text{G-Hilb}}_{S'} X_{S'}$$

which derives from the natural isomorphisms $X \times_S T \cong X_{S'} \times_{S'} T$ for S'-schemes T.
(2) Sometimes, preferably in the case $S' = \text{Spec}\, K' \to S = \text{Spec}\, K$ with K, K' fields, it can be useful to involve the group scheme into the base change. Then

$$(\underline{\text{G-Hilb}}_K X)_{K'} \cong \underline{\text{G-Hilb}}_{K'} X_{K'} \cong \underline{G_{K'}\text{-Hilb}}_{K'} X_{K'}$$

Theorem 4.20. *Let Y be a G-scheme over S with trivial G-operation and $\varphi : X \to Y$ an equivariant morphism. Let $\tau : \underline{\text{G-Hilb}}_S X \to \underline{Y}$ be the morphism constructed in theorem 4.15. Then there is an isomorphism of functors $(Y\text{-schemes})^\circ \to (sets)$*

$$(\underline{\text{G-Hilb}}_S X)_{(Y,\tau)} \cong \underline{\text{G-Hilb}}_Y X$$

Proof. Both functors are subfunctors of $(\underline{\text{G-Hilb}}_S X)_Y : (Y\text{-schemes})^\circ \to (\text{sets})$:

$$(\underline{\text{G-Hilb}}_S X)_{(Y,\tau)}(\alpha : T \to Y) = \{Z \in (\underline{\text{G-Hilb}}_S X)_Y(\alpha : T \to Y) \mid \tau(Z) = \alpha\}$$

$$\underline{\text{G-Hilb}}_Y X(T) = \{Z \in (\underline{\text{G-Hilb}}_S X)_Y(T) \mid Z \hookrightarrow X \times_S T \text{ factors through } X \times_Y T\}$$

We show that they coincide. Let T be a Y-scheme and $Z \subseteq X \times_S T$ a closed subscheme defining an element of $(\underline{\text{G-Hilb}}_S X)_Y(T)$. Then there are the equivalences

$$
\begin{aligned}
Z \in \underline{\text{G-Hilb}}_Y X(T) \quad &\Longleftrightarrow \quad Z \hookrightarrow X \times_S T \text{ factors through } X \times_Y T \\
&\Longleftrightarrow \quad \text{diagram (4.4) commutes for } Z \text{ and the given morphism } T \to Y \\
&\Longleftrightarrow \quad Z \in (\underline{\text{G-Hilb}}_S X)_{(Y,\tau)}(T)
\end{aligned}
$$

where for the last one one uses uniqueness of a morphism $T \to Y$ making diagram (4.4) commute. $\qquad\square$

Corollary 4.21. *Let $\tau : \underline{\text{G-Hilb}}_S X \to X/G$ be the morphism of corollary 4.16. Then there is an isomorphism of functors $(X/G\text{-schemes})^\circ \to (\text{sets})$*

$$(\underline{\text{G-Hilb}}_S X)_{(X/G,\tau)} \cong \underline{\text{G-Hilb}}_{X/G} X$$

$\qquad\square$

Remark 4.22. We explicitly write down some direct consequences.
(1) The fact that there is an isomorphism $(\underline{\text{G-Hilb}}_S X)_{(X/G,\tau)} \cong \underline{\text{G-Hilb}}_{X/G} X$ of functors $(X/G\text{-schemes})^\circ \to (\text{sets})$ of course implies an isomorphism $\underline{\text{G-Hilb}}_S X \cong {}_S(\underline{\text{G-Hilb}}_{X/G} X)$ of functors $(S\text{-schemes})^\circ \to (\text{sets})$.
(2) If $\underline{\text{G-Hilb}}_{X/G} X$ is representable, then so is $\underline{\text{G-Hilb}}_S X$ and in this case the corresponding schemes $\text{G-Hilb}_{X/G} X$ and $\text{G-Hilb}_S X$ are isomorphic as X/G-schemes, i.e. there is the commutative diagram

$$
\begin{array}{ccc}
\text{G-Hilb}_{X/G} X & \xleftarrow{\quad \sim \quad} & \text{G-Hilb}_S X \\
& \searrow \quad \swarrow_{\tau} & \\
& X/G &
\end{array}
$$

(3) In the situation of (2), after base restriction by $X/G \to S$ there is the isomorphism of S-schemes ${}_S(\text{G-Hilb}_{X/G} X) \cong \text{G-Hilb}_S X$.

Remark 4.23. (Fibers of $\tau : \text{G-Hilb}_S X \to X/G$).
Assume that $\underline{\text{G-Hilb}}_{X/G} X$ is representable (then so is $\underline{\text{G-Hilb}}_S X$). For points $y \in X/G$ one may consider the fibers of the morphism $\tau : \text{G-Hilb}_S X \to X/G$ of corollary 4.16. Corollary 4.21 gives an isomorphism $\text{G-Hilb}_S X \cong {}_S(\text{G-Hilb}_{X/G} X)$ which identifies τ with the structure morphism of $\text{G-Hilb}_{X/G} X$. So the fibers of τ are the fibers $(\text{G-Hilb}_{X/G} X)_y$ of the X/G-scheme $\text{G-Hilb}_{X/G} X$, these are isomorphic to G-Hilbert schemes $\text{G-Hilb}_{\kappa(y)} X_y$ over $\kappa(y)$ of the G-schemes X_y which are the fibers of $X \to X/G$ over y.

Another application (result also contained in [Té04]):

Corollary 4.24. *Let X, Y be G-schemes over S, Y with trivial G-operation. Then*

$$\underline{\text{G-Hilb}}_S X \times_S Y \cong (\underline{\text{G-Hilb}}_S X) \times_S Y$$

Proof. The projection $X \times_S Y \to Y$ is G-equivariant, theorem 4.15 then constructs a morphism of S-functors $\tau : \underline{\text{G-Hilb}}_S X \times_S Y \to \underline{Y}$, by theorem 4.20 there is an isomorphism of Y-functors $(\underline{\text{G-Hilb}}_S X \times_S Y)_{(Y,\tau)} \cong \underline{\text{G-Hilb}}_Y X \times_S Y$ and $\underline{\text{G-Hilb}}_Y X \times_S Y \cong (\underline{\text{G-Hilb}}_S X) \times_S Y$ by the usual base extension. Restricting the base to S yields the result. $\qquad\square$

4.3.3 G-Hilbert schemes and G-Grassmannians

In this subsection we show in the case that the G-scheme X is affine over S, that there is a natural closed embedding of the G-Hilbert functor $\underline{\text{G-Hilb}}_S X$ into a G-Grassmannian functor, this implies that $\underline{\text{G-Hilb}}_S X$ is representable. This construction is useful for X finite over S, in which case G-Hilb$_S X$ is projective over S.

A special case of a G-Quot functor is the G-Grassmannian functor $\underline{\text{G-Grass}}_S(\mathscr{F}) = \underline{\text{G-Quot}}_{\mathscr{F}/S/S}$ for a quasicoherent G-sheaf $\mathscr{F}$ on S:

$$\underline{\text{G-Grass}}_S(\mathscr{F})(T) = \left\{ \begin{array}{l} \text{\textit{Quotient G-sheaves}} \ \ [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \ \ \text{\textit{on T,}} \ \mathscr{G} \ \text{\textit{locally free}} \\ \text{\textit{of finite rank with fibers isomorphic to the regular representation}} \end{array} \right\}$$

Let X be a G-scheme over S, assume that X is affine over S, $X = \text{Spec}_S \mathscr{B}$ for a quasicoherent G-sheaf of $\mathcal{O}_S$-algebras. One may rewrite the G-Hilbert functor for $X \to S$ as

$$\underline{\text{G-Hilb}}_S X(T) = \left\{ \begin{array}{l} \text{\textit{Quotient G-sheaves}} \ \ [0 \to \mathscr{I} \to \mathscr{B}_T \to \mathscr{C} \to 0] \\ \text{\textit{of }} \mathscr{B}_T\text{\textit{-modules on T,}} \ \mathscr{C} \ \text{\textit{locally free of finite rank}} \\ \text{\textit{with fibers isomorphic to the regular representation}} \end{array} \right\}$$

There is a morphism of S-functors $\underline{\text{G-Hilb}}_S X \to \underline{\text{G-Grass}}_S(\mathscr{B})$ consisting of injective maps

$$\underline{\text{G-Hilb}}_S X(T) \to \underline{\text{G-Grass}}_S(\mathscr{B})(T), \quad [\mathscr{B}_T \to \mathscr{C}] \mapsto [\mathscr{B}_T \to \mathscr{C}]$$

defined by forgetting the algebra structure of $\mathscr{B}$, this way $\underline{\text{G-Hilb}}_S X$ becomes a subfunctor of $\underline{\text{G-Grass}}_S(\mathscr{B})$. The essential point we will show is the closedness of the additional condition for $\mathscr{I} \subseteq \mathscr{B}_T$ to be an ideal of $\mathscr{B}_T$, consequence will be the following theorem.

Theorem 4.25. *Let $X = \text{Spec}_S \mathscr{B} \to S$ be an affine S-scheme with G-operation over S. Then the natural morphism of S-functors*

$$\underline{\text{G-Hilb}}_S X \to \underline{\text{G-Grass}}_S(\mathscr{B})$$

is a closed embedding.

Proof. We show that the canonical inclusion defined above is a closed embedding.

One has to show that for any S-scheme S' and any S'-valued point of the Grassmannian $[0 \to \mathscr{H} \overset{\varphi}{\to} \mathscr{B}_{S'} \to \mathscr{G} \to 0]$ there is a closed subscheme $Z \subseteq S'$ such that for every S-morphism $\alpha : T \to S'$: α factors through Z if and only if the quotient $[\mathscr{B}_T \to \alpha^*\mathscr{G}] \in \underline{\text{G-Grass}}_S(\mathscr{B})(T)$ determined by α comes from a quotient of the Hilbert functor.

For $\alpha : T \to S'$ the quotient $[\alpha^*\mathscr{H} \overset{\alpha^*\varphi}{\to} \mathscr{B}_T \to \alpha^*\mathscr{G} \to 0]$ of $\mathcal{O}_T$-modules comes from a quotient of $\mathscr{B}_T$-modules if and only if the image of $\alpha^*\mathscr{H}$ in $\mathscr{B}_T$ is a $\mathscr{B}_T$-submodule. Equivalently, the composition $\chi_T : \mathscr{B}_T \otimes_{\mathcal{O}_T} \alpha^*\mathscr{H} \to \mathscr{B}_T \otimes_{\mathcal{O}_T} (\alpha^*\varphi)(\alpha^*\mathscr{H}) \to \mathscr{B}_T \to \alpha^*\mathscr{G}$, where the arrow in the middle is defined using the multiplication map $\mathscr{B}_T \otimes_{\mathcal{O}_T} \mathscr{B}_T \to \mathscr{B}_T$ of the algebra $\mathscr{B}_T$, is zero.

To define Z, consider $\chi_{S'} : \mathscr{B}_{S'} \otimes_{\mathcal{O}_{S'}} \mathscr{H} \to \mathscr{G}$. Define Z to be the closed subscheme with ideal sheaf $\mathscr{I}$ that is the ideal of $\mathcal{O}_{S'}$ minimal with the property $\text{im}(\chi_{S'}) \subseteq \mathscr{I}\mathscr{G}$. Locally $\mathscr{I}$ can be described as follows: If $\mathscr{G}|_U \cong \bigoplus_j \mathcal{O}_U^{(j)}$ for an open $U \subseteq S'$, then $\mathscr{I}$ is the ideal sheaf generated by the images of the coordinate maps $\chi_U^{(j)} : (\mathscr{B}_{S'} \otimes_{\mathcal{O}_{S'}} \mathscr{H})|_U \to \mathcal{O}_U^{(j)}$.

This Z has the required property: One has to show that $\alpha : T \to S'$ factors through Z if and only if $(\alpha^*\varphi)(\alpha^*\mathscr{H}) \subseteq \mathscr{B}_T$ is a $\mathscr{B}_T$-submodule. χ_T identifies with $\alpha^*\chi_{S'}$. The question is local on S', assume that $\mathscr{G} \cong \bigoplus_j \mathcal{O}_{S'}^{(j)}$. Then there are the equivalences:

$$
\begin{aligned}
(\alpha^*\varphi)(\alpha^*\mathscr{H}) \text{ is a } \mathscr{B}_T\text{-submodule} \iff{}& \mathrm{im}(\chi_T) = 0 \ in \ \alpha^*\mathscr{G} \\
\iff{}& \mathrm{im}(\alpha^*\chi_{S'}) = 0 \ in \ \alpha^*\mathscr{G} \\
\iff{}& \forall\, j :\ \mathrm{im}(\alpha^*\chi_{S'}^{(j)}) = 0 \ in \ \mathcal{O}_T^{(j)} \\
\iff{}& \forall\, j :\ \mathrm{im}(\chi_{S'}^{(j)}) \subseteq \ker(\mathcal{O}_{S'} \to \alpha_*\mathcal{O}_T) \\
\iff{}& \mathscr{I} \subseteq \ker(\mathcal{O}_{S'} \to \alpha_*\mathcal{O}_T) \\
\iff{}& \alpha : T \to S \text{ factors through } Z
\end{aligned}
$$

$\square$

The Grassmannian functor $\underline{\mathrm{Grass}}_S^n(\mathscr{B})$ is representable (theorem 4.2) and so is the G-Grassmannian as a component of the equivariant Grassmannian $\underline{\mathrm{Grass}}_S^{G,n}(\mathscr{B})$ (same argument as in proposition 4.11), which is a closed subfunctor of $\underline{\mathrm{Grass}}_S^n(\mathscr{B})$ (proposition 4.7 and theorem 4.6). Further, if $f : X \to S$ is finite, then $\mathscr{B} = f_*\mathcal{O}_X$ is quasicoherent of finite type, the Grassmannian $\mathrm{Grass}_S^n(\mathscr{B})$ and thereby the G-Grassmannian $G\text{-Grass}_S(\mathscr{B})$ projective over S.

Corollary 4.26. *If $X \to S$ is finite, then the S-functor $\underline{G\text{-Hilb}}_S X$ is representable by a projective S-scheme.* $\square$

Remark 4.27. The same applies to the equivariant Hilbert functor and the Hilbert functor of n points, there are the closed embeddings

$$\underline{\mathrm{Hilb}}_S^n X \to \underline{\mathrm{Grass}}_S^n(\mathscr{B}), \quad \underline{\mathrm{Hilb}}_S^{G,n} X \to \underline{\mathrm{Grass}}_S^{G,n}(\mathscr{B})$$

and if $X \to S$ is finite, then the S-functors $\underline{\mathrm{Hilb}}_S^n X, \underline{\mathrm{Hilb}}_S^{G,n} X$ are representable by projective S-schemes.

4.3.4   Representability of G-Hilbert functors

For finite group schemes an affine quotient morphism of an algebraic K-scheme $X \to X/G$ is finite. This allows to use the closed embedding of $\underline{G\text{-Hilb}}_{X/G} X$ into a Grassmannian of theorem 4.25 to prove that the G-Hilbert functor $\underline{G\text{-Hilb}}_{X/G} X$ is represented by a scheme $G\text{-Hilb}_{X/G} X$ projective over X/G directly without using theorem 4.1 about representability of Quot and Hilbert functors. The behaviour of G-Hilbert functors under variation of the base scheme then implies that $\underline{G\text{-Hilb}}_K X$ is representable and the morphism $\tau : G\text{-Hilb}_K X \to X/G$ projective.

Theorem 4.28. *(Representability of $\underline{G\text{-Hilb}}_K X$).*
Let $G = \mathrm{Spec}\, A$ be a finite group scheme over a field K with A cosemisimple. Let X be a G-scheme algebraic over K and assume that a geometric quotient $\pi : X \to X/G$, π affine, of X by G exists. Then the G-Hilbert functor $\underline{G\text{-Hilb}}_K X$ is represented by an algebraic K-scheme $G\text{-Hilb}_K X$ and the morphism $\tau : G\text{-Hilb}_K X \to X/G$ of corollary 4.16 is projective.

Proof. Since the group scheme G is finite and X is algebraic, π is a finite morphism and X/G is algebraic. Thus corollary 4.26 applies to the G-Hilbert functor $\underline{G\text{-Hilb}}_{X/G} X$ and implies that $\underline{G\text{-Hilb}}_{X/G} X$ is represented by an algebraic K-scheme $G\text{-Hilb}_{X/G} X$ projective over X/G.
Corollary 4.16 constructs a morphism $\tau : \underline{G\text{-Hilb}}_K X \to X/G$, by corollary 4.21 there is the isomorphism of X/G-functors $(\underline{G\text{-Hilb}}_K X)_{(X/G,\tau)} \cong \underline{G\text{-Hilb}}_{X/G} X$. In particular there is an

isomorphism of K-functors $\underline{\text{G-Hilb}}_K X \cong {}_K(\underline{\text{G-Hilb}}_{X/G} X)$ that identifies τ with the structure morphism of $\underline{\text{G-Hilb}}_{X/G} X$.

It follows that $\underline{\text{G-Hilb}}_K X$ is represented by an algebraic K-scheme G-Hilb$_K X$ and that there is an isomorphism of K-schemes G-Hilb$_K X \cong$ G-Hilb$_{X/G} X$ that identifies the morphism $\tau :$ G-Hilb$_K X \to X/G$ with the structure morphism of G-Hilb$_{X/G} X$ which is projective. $\square$

4.3.5 Free operation

Let the finite group scheme G over K operate on an algebraic K-scheme X.

Proposition 4.29. *Let $f : X \to S$ be a finite flat morphism and let G operate on X over S such that the fibers $f_* \mathcal{O}_X \otimes_{\mathcal{O}_S} \kappa(s)$ for $s \in S$ are isomorphic to the regular representation. Then there is an isomorphism of S-functors $\underline{\text{G-Hilb}}_S X \cong \underline{S}$.*

Proof. For any S-scheme T X_T is finite flat over T with fibers isomorphic to the regular representation and the only closed subscheme of X_T with this property. Thus there are canonical bijections $\underline{\text{G-Hilb}}_S X(T) \leftrightarrow S(T)$ functorial in T. $\square$

Corollary 4.30. *If $X \to S$ is a G-torsor, then $\underline{\text{G-Hilb}}_S X \cong \underline{S}$.* $\square$

For the following corollaries assume that a geometric quotient $\pi : X \to X/G$, π affine, exists. Then by theorem 4.28 $\underline{\text{G-Hilb}}_K X$ is representable by an algebraic K-scheme G-Hilb$_K X$ and there is the projective morphism $\tau :$ G-Hilb$_K X \to X/G$.

Corollary 4.31. *If the operation of G on X is free, then the morphism $\tau :$ G-Hilb$_K X \to X/G$ is an isomorphism.*

Proof. The quotient morphism $\pi : X \to X/G$ is a G-torsor since the operation is free [Mu, AV, Ch. III.12, Thm. 1, p. 111,112], [Mu, GIT, Ch. 0, §4, Prop. 9, p. 16]. Then by corollary 4.30 there is an isomorphism of X/G-schemes G-Hilb$_{X/G} X \cong X/G$ and therefore $\tau :$ G-Hilb$_K X \to X/G$ an isomorphism. $\square$

Corollary 4.32. *Assume that X is an irreducible K-variety. Let G operate on X such that the operation is free on a dense open subscheme. Then there is a unique irreducible component W of $(\text{G-Hilb}_K X)_{\text{red}}$ such that $\tau|_W : W \to X/G$ is birational. In particular, if G-Hilb$_K X$ is reduced and irreducible then τ is birational.*

Proof. Let $U \subseteq X$ be a G-stable dense open subscheme on which the G-operation is free. Then U/G is open dense in X/G, the restriction $\tau|_{\tau^{-1}(U/G)} : \tau^{-1}(U/G) \to U/G$ is an isomorphism by corollary 4.31 and $W :=$ *closure of* $\tau^{-1}(U/G)$ *in* G-Hilb$_K X$ is the unique irreducible component of $(\text{G-Hilb}_K X)_{\text{red}}$ such that $\tau|_W : W \to X/G$ is birational. $\square$

Corollary 4.33. *Let G operate on X such that the operation is free on a dense open subscheme. Then, if the quotient morphism $\pi : X \to X/G$ is flat, $\tau :$ G-Hilb$_K X \to X/G$ is an isomorphism.*

Proof. The operation is free on an open dense subscheme and there the representations on the fibers of π are isomorphic to the regular representation. The isomorphism class of the representations on the fibers is locally constant, since $\pi_* \mathcal{O}_X$ is locally free and so are the direct summands corresponding to the isomorphism classes of simple representations of G over K – their rank determines the multiplicity of the corresponding simple representation in the representation on the fibers (as in the proof of proposition 4.11). Therefore the representation $\pi_* \mathcal{O}_X \otimes_{\mathcal{O}_{X/G}} \kappa(y)$ is isomorphic to the regular representation over $\kappa(y)$ for any $y \in X/G$. Then by proposition 4.29 G-Hilb$_{X/G} X \cong X/G$ as X/G-schemes and thus $\tau :$ G-Hilb$_K X \to X/G$ is an isomorphism. $\square$

4.4   Differential study of G-Hilbert schemes

In this section we carry out the differential study of Quot schemes [Gr61, Section 5] in the equivariant setting. The results are applied to determine the sheaf of relative differentials and relative tangent spaces of G-Hilbert schemes. In the last subsection we find relations between relative tangent spaces of G-Hilb$_K \mathbb{A}_K^n$ over $\mathbb{A}_K^n/G$ and the stratification described in subsection 1.2.2.

4.4.1   Differentials, derivations and extensions of morphisms

Let S be a scheme and Q an S-scheme.

Differentials and derivations. For a quasicoherent $\mathcal{O}_Q$-module $\mathscr{M}$ let $\mathscr{D}er_S(\mathcal{O}_Q, \mathscr{M})$ be the sheaf of S-derivations of $\mathcal{O}_Q$ in $\mathscr{M}$, that is locally for affine open $V = \operatorname{Spec} B \subseteq Q$ over $W = \operatorname{Spec} A \subseteq S$ the B-module $\operatorname{Der}_A(B, M)$, where $M = \mathscr{M}(V)$. The sheaf $\Omega_{Q/S}$ is characterised as the sheaf of universal differentials:

Proposition 4.34. [EGA, IV (4), (16.5.5)].
Let $\mathscr{M}$ be a quasicoherent $\mathcal{O}_Q$-module. Then there is the isomorphism

$$\mathscr{H}om_{\mathcal{O}_Q}(\Omega_{Q/S}, \mathscr{M}) \cong \mathscr{D}er_S(\mathcal{O}_Q, \mathscr{M})$$

of $\mathcal{O}_Q$-modules.   □

Derivations and extensions of morphisms. Let $\widetilde{T}$ be an S-scheme, $j : T \hookrightarrow \widetilde{T}$ a closed subscheme whose defining ideal $\mathscr{I}$ has the property $\mathscr{I}^2 = 0$ (then $\mathscr{I}$ is an $\mathcal{O}_T$-module and the underlying topological spaces of T and $\widetilde{T}$ coincide). Let $q : T \to Q$ be a morphism of S-schemes. One is interested in extensions $\widetilde{q}$ of q, that is morphisms $\widetilde{q} : \widetilde{T} \to Q$ satisfying $\widetilde{q} \circ j = q$.
For open $U \subseteq T$ put $\widetilde{U} := (U, \mathcal{O}_{\widetilde{T}}|_U)$ and define a sheaf of sets $\mathscr{E}$ by

$$\mathscr{E}(U) := \left\{ \widetilde{q} : \widetilde{U} \to Q \ \Big| \ \widetilde{q} \circ j|_U = q|_U \right\}$$

In the affine case $\operatorname{Spec} C = \widetilde{U} \subseteq \widetilde{T}$, $U = \operatorname{Spec} C/I$ defined in $\widetilde{U}$ by the ideal $I \subseteq C$ with $I^2 = 0$, $V = \operatorname{Spec} B \subseteq Q$ such that $q(U) \subseteq V$ over $W = \operatorname{Spec} A$, a morphism $\widetilde{q} : \widetilde{U} \to Q$ such that $\widetilde{q} \circ j|_U = q|_U$ corresponds to a homomorphism $B \to C$ over A, such that the composition $B \to C \to C/I$ corresponds to $q|_U^\# : \mathcal{O}_Q|_V \to q|_{U*}\mathcal{O}_U$. Given a $\widetilde{q}_0 : \widetilde{U} \to Q$ having this property, any other such $\widetilde{q}$ differs from it by a derivation with values in the ideal I, more precisely by an element of $\operatorname{Der}_A(B, {}_BI) \cong \operatorname{Hom}_B(\Omega_{B/A}, {}_BI) \cong {}_B\operatorname{Hom}_{C/I}(\Omega_{B/A} \otimes_B C/I, I)$. This way the $\mathcal{O}_T$-module $\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{I})$ considered as a sheaf of groups operates on the sheaf of sets $\mathscr{E}$, one is led to the following proposition.

Proposition 4.35. [EGA, IV (4), (16.5.17)].
The sheaf of sets $\mathscr{E}$ is a principal homogeneous sheaf (or pseudo-torsor) under the sheaf of groups $\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{I})$.   □

The term "formal principal homogeneous sheaf" or "pseudo-torsor" means, that for any open $U \subseteq T$ the set $\mathscr{E}(U)$ is either empty or a principal homogeneous set under the group $\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{I})(U)$ (see [Bour, Algebra, Ch. I, §5.6, Def. 7], [EGA, IV (4), (16.5.15)]).

4.4.2 Equivariant deformations

Let G be a group scheme over a scheme S, X a G-scheme over S and $\mathscr{F}$ a quasicoherent G-sheaf on X.

Let $\widetilde{T}$ be an S-scheme and $T \subseteq \widetilde{T}$ be a closed subscheme defined by a quasicoherent ideal sheaf $\mathscr{I}$ with the property $\mathscr{I}^2 = 0$. Let $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ be a quotient G-sheaf on X_T (in the sense of subsection 4.2.1) with $\mathscr{G}$ flat over T. By

$$
\mathscr{E}(U) := \left\{
\begin{array}{l}
\textit{Quotient } G\textit{-sheaves } [0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{U}} \to \widetilde{\mathscr{G}} \to 0] \textit{ on } X_{\widetilde{U}},\ \widetilde{\mathscr{G}} \textit{ flat over } \widetilde{U}, \\
[\mathscr{F}_U = \mathscr{F}_{\widetilde{U}} \otimes_{\mathcal{O}_{\widetilde{U}}} \mathcal{O}_U \to \widetilde{\mathscr{G}} \otimes_{\mathcal{O}_{\widetilde{U}}} \mathcal{O}_U] = [\mathscr{F}_U \to \mathscr{G}|_{X_U}]
\end{array}
\right\}
$$

for open $U \subseteq T$, where $\widetilde{U} := (U, \mathcal{O}_{\widetilde{T}}|_U)$, one has defined a sheaf of sets $\mathscr{E}$ on T. Elements of $\mathscr{E}(U)$ are also called flat deformations of the quotient $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ over $\widetilde{U}$. Define an $\mathcal{O}_T$-module $\mathscr{A}$ by

$$
\mathscr{A}(U) = \mathrm{Hom}^G_{X_U}(\mathscr{H}|_{X_U}, \mathscr{G}|_{X_U} \otimes_{\mathcal{O}_U} \mathscr{I}|_U)
$$

for open $U \subseteq T$.

Proposition 4.36. *There is a natural operation of the sheaf of groups $\mathscr{A}$ on the sheaf of sets $\mathscr{E}$ which makes $\mathscr{E}$ a formal principal homogeneous sheaf (or pseudo-torsor) under $\mathscr{A}$.*

The next lemma will be used in the proof of the proposition.

Lemma 4.37. *Let $\widetilde{A} \to A$ be a surjective homomorphism of rings, assume that its kernel $I \subseteq \widetilde{A}$ is nilpotent. Then for an $\widetilde{A}$-module $\widetilde{M}$ one has:*

$$
\begin{array}{ll}
\widetilde{M} \textit{ is a flat } \widetilde{A}\textit{-module} \iff & A \otimes_{\widetilde{A}} \widetilde{M} \textit{ is a flat } A\textit{-module and} \\
& \textit{the natural map } I \otimes_{\widetilde{A}} \widetilde{M} \to \widetilde{M} \textit{ is injective}
\end{array}
$$

Proof. [Ma, CA, Theorem 49, p. 146] or [Ma, CRT, Theorem 22.3], also [Bour, Commutative Algebra, Ch. III, §5.2]. $\square$

Proof of proposition 4.36. We subdivide the proof into two steps, the main one is the following statement:

Assume that there exists a flat deformation $[0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}} \to 0]$ of $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$. Then there is the exact sequence of G-sheaves on X_T

$$
0 \longrightarrow \mathscr{G} \otimes_{\mathcal{O}_T} \mathscr{I} \longrightarrow \mathscr{B} \longrightarrow \mathscr{H} \longrightarrow 0 \tag{4.6}
$$

where $\mathscr{B} = \ker(\mathscr{F}_{\widetilde{T}} \to \mathscr{G})/(\mathscr{H} \otimes_{\mathcal{O}_T} \mathscr{I})$, and its splittings are in bijection with the set of flat deformations of $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$.

Let $[0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}} \to 0]$ be a flat deformation of the given quotient $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$. Tensoring $0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}} \to 0$ with the exact sequence $0 \to \mathscr{I} \to \mathcal{O}_{\widetilde{T}} \to \mathcal{O}_T \to 0$

gives the commutative exact diagram (solid arrows)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \dashrightarrow & & \mathscr{H}' & \xrightarrow{h} & \mathscr{F}_T & \xrightarrow{g} & \mathscr{G} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle p} & & \uparrow & & \\
0 \longrightarrow & & \widetilde{\mathscr{H}} & \xrightarrow{\widetilde{h}} & \mathscr{F}_{\widetilde{T}} & \xrightarrow{\widetilde{g}} & \widetilde{\mathscr{G}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I} & \longrightarrow & \mathscr{F}_T\otimes_{\mathcal{O}_T}\mathscr{I} & \longrightarrow & \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where $\mathscr{H}' := \widetilde{\mathscr{H}}\otimes_{\mathcal{O}_{\widetilde{T}}}\mathcal{O}_T$ and $\widetilde{\mathscr{H}}\otimes_{\mathcal{O}_{\widetilde{T}}}\mathscr{I} \cong \mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I}$ etc. (use that $\mathscr{I}^2 = 0$). $\mathscr{H}' \to \mathscr{F}_T$ is injective, since $\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \cong \widetilde{\mathscr{G}}\otimes_{\mathcal{O}_{\widetilde{T}}}\mathscr{I} \to \widetilde{\mathscr{G}}$ is injective ($\widetilde{\mathscr{G}}$ is flat), so one may replace $\mathscr{H}'$ by $\mathscr{H}$. The homomorphisms $\mathscr{F}_T\otimes_{\mathcal{O}_T}\mathscr{I} \to \mathscr{F}_{\widetilde{T}} \xrightarrow{p} \mathscr{F}_T$ induce the exact sequence (4.6)

$$
0 \longrightarrow \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \longrightarrow \mathscr{B} \xrightarrow{\overline{p}} \mathscr{H} \longrightarrow 0
$$

where $\mathscr{B} = \ker(\mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}})/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I})$ is the middle homology of the complex $\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I} \longrightarrow \mathscr{F}_{\widetilde{T}} \longrightarrow \widetilde{\mathscr{G}}$. Here $\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \to \mathscr{B}$ is injective, since $\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \to \widetilde{\mathscr{G}}$ is.

$\widetilde{h}: \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}}$ induces a homomorphism $\widehat{h}: \mathscr{H} \to \mathscr{B} \subseteq \mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I})$, by construction $\ker(\mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I}) \to \widetilde{\mathscr{G}}) = \operatorname{im}(\widehat{h})$ and $\overline{p}\circ\widehat{h} = id_{\mathscr{H}}$. So $\widehat{h}$ defines a splitting of (4.6) which determines the kernel of $\mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}}$ and thus the deformation $[0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}} \to 0]$.

Reversely, given a splitting $\widehat{h}: \mathscr{H} \to \mathscr{B} \subseteq \mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I})$ of (4.6), define $\mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}}$ as $\mathscr{F}_{\widetilde{T}} \to \mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I})$ composed with the cokernel of $\widehat{h}$. Then $[\mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}}]$ restricted to a quotient over T is $[\mathscr{F}_T \to \mathscr{G}]$. The natural homomorphism $\widetilde{\mathscr{G}}\otimes_{\mathcal{O}_{\widetilde{T}}}\mathscr{I} \to \widetilde{\mathscr{G}}$ is the homomorphism $\widetilde{\mathscr{G}}\otimes_{\mathcal{O}_{\widetilde{T}}}\mathscr{I} \cong \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \to \widetilde{\mathscr{G}}$ induced by $\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \to \mathscr{B} \subseteq \mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I})$. That it is injective, can be seen by the exact, commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{F}_{\widetilde{T}}/(\mathscr{H}\otimes_{\mathcal{O}_T}\mathscr{I}) & \longrightarrow & \widetilde{\mathscr{G}} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{B} \cong \mathscr{H}\oplus(\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I}) & \longrightarrow & \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} & \longrightarrow & 0
\end{array}
$$

By Lemma 4.37 $\widetilde{\mathscr{G}}$ is flat.

Assume that there is a splitting of (4.6). *Then the set of splittings of* (4.6) *is a principal homogeneous set under the group* $\operatorname{Hom}^G_{X_T}(\mathscr{H}, \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I})$.

Relative to a given splitting

$$
0 \longrightarrow \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I} \longrightarrow (\mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I})\oplus\mathscr{H} \longrightarrow \mathscr{H} \longrightarrow 0
$$

of (4.6) any other splitting is uniquely determined by a homomorphism $\mathscr{H} \to \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I}$, so the set of splittings is principal homogeneous under $\operatorname{Hom}^G_{X_T}(\mathscr{H}, \mathscr{G}\otimes_{\mathcal{O}_T}\mathscr{I})$.

So far, for simplicity we have considered the case $U = T$, of course the same applies to the situation over arbitrary open $U \subseteq T$ and these constructions respect the restriction maps. $\square$

4.4.3 Application to G-Quot schemes

Let S be scheme over a field K and $G = \operatorname{Spec} A$ a finite group scheme over K with A cosemisimple. Let X be a G-scheme over S and $\mathscr{F}$ a coherent G-sheaf on X.

The equivariant Quot functor $\underline{\operatorname{Quot}}^G_{\mathscr{F}/X/S} : (S\text{-}schemes)^\circ \to (sets)$ had been defined as

$$\underline{\operatorname{Quot}}^G_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient G-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T \end{array} \right\}$$

Because the additional conditions defining the G-Quot functor are open and closed, the differential study of the G-Quot functor reduces to that of the equivariant Quot functor, so let $\underline{Q}$ be an open and closed component of $\underline{\operatorname{Quot}}^G_{\mathscr{F}/X/S}$ like the G-Quot functor. Sometimes we assume that $\underline{Q}$ is represented by an S-scheme Q. We will discuss a general situation that includes the following cases which will later be studied in detail:

(a) X a G-scheme over $S = \operatorname{Spec} K$ and one considers the G-Hilbert functor $\underline{G\text{-Hilb}}_K X$.

(b) Let X be a G-scheme over $\operatorname{Spec} K$ and assume that an affine geometric quotient $\pi : X \to X/G$ exists. One considers the relative G-Hilbert functor $\underline{G\text{-Hilb}}_{X/G} X$ over $S = X/G$.

We will apply the preceding results to state a general theorem that can be specialised to determine the sheaf of relative differentials and relative tangent spaces of Q over S.

Let T be an S-scheme and $\mathscr{M}$ a quasicoherent $\mathcal{O}_T$-module. Considering $\mathscr{M}$ as an ideal of square zero (write this as $\varepsilon\mathscr{M}$), one has a closed embedding of S-schemes $T \to \widetilde{T}$, where $\widetilde{T} = (T, \mathcal{O}_T \oplus \varepsilon\mathscr{M})$, the ideal sheaf defining T in $\widetilde{T}$ being $\varepsilon\mathscr{M}$. Here one has a projection $\widetilde{T} \to T$ such that the composition $T \to \widetilde{T} \to T$ is id_T.
Further let be given a morphism of S-schemes $q : T \to Q$, it corresponds to a quotient of G-sheaves $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ on X_T.

We want to determine the extensions of q to $\widetilde{T}$, i.e. morphisms $\widetilde{q} : \widetilde{T} \to Q$ whose restriction to T is q. These correspond to flat deformations of the quotient $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ on X_T to quotients $[0 \to \widetilde{\mathscr{H}} \to \mathscr{F}_{\widetilde{T}} \to \widetilde{\mathscr{G}} \to 0]$ on $X_{\widetilde{T}}$. The sets of these over open $U \subseteq T$ had been assembled to the sheaf $\mathscr{E}$ in subsection 4.4.1 resp. 4.4.2.

In the present case there is always a trivial deformation: One has $\mathcal{O}_{\widetilde{T}} = \mathcal{O}_T \oplus \varepsilon\mathscr{M}$ as an $\mathcal{O}_T$-module, therefore $\mathscr{F}_{\widetilde{T}} \cong \mathscr{F}_T \oplus \varepsilon(\mathscr{F}_T \otimes_{\mathcal{O}_T} \mathscr{M})$. The deformations correspond to splittings of the sequence $0 \to \mathscr{G} \otimes_{\mathcal{O}_T} \mathscr{M} \to \mathscr{B} \to \mathscr{H} \to 0$, here we have a given isomorphism $\mathscr{B} \cong \mathscr{H} \oplus \varepsilon(\mathscr{G} \otimes_{\mathcal{O}_T} \mathscr{M})$, so a natural zero-splitting. This corresponds to the zero-deformation $\mathscr{F}_{\widetilde{T}} = \mathscr{F}_T \oplus \varepsilon(\mathscr{F}_T \otimes_{\mathcal{O}_T} \mathscr{M}) \to \mathscr{G} \oplus \varepsilon(\mathscr{G} \otimes_{\mathcal{O}_T} \mathscr{M}) \to 0$ (it also arises by pulling back the quotient $[\mathscr{F}_T \to \mathscr{G}]$ on X_T to $X_{\widetilde{T}}$ via the projection $\widetilde{T} \to T$).

It follows, that for open $U \subseteq T$ the sets $\mathscr{E}(U)$ as defined in subsection 4.4.1 resp. 4.4.2 are not emtpy. Then by propositions 4.35 resp. 4.36 $\mathscr{E}$ is a torsor under $\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{M})$ resp. $\mathscr{A}$. Moreover, the natural zero-extension resp. zero-deformation gives identifications of $\mathscr{E}$ with the $\mathcal{O}_T$-modules $\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{M})$ resp. $\mathscr{A}$. This provides $\mathscr{E}$ with the structure of an $\mathcal{O}_T$-module (this $\mathcal{O}_T$-module structure on $\mathscr{E}$ can also be constructed more directly, see e.g. [EH, VI.1.3] for the case of $T = \operatorname{Spec} K$, K a field).

We arrive at the following theorem:

Theorem 4.38. *Let Q be as above, T be an S-scheme, $\mathscr{M}$ a quasicoherent $\mathcal{O}_T$-module and $\widetilde{T}$ the S-scheme $(T, \mathcal{O}_T \oplus \varepsilon\mathscr{M})$. Let $q : T \to Q$ be a morphism over S, $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ the corresponding quotient of G-sheaves on X_T and let $\mathscr{E}$ be the sheaf of sets on T of extensions of q to $\widetilde{T}$.*

Then $\mathscr{E}$ has the natural structure of an $\mathcal{O}_T$-module and there are isomorphisms of $\mathcal{O}_T$-modules

$$\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{M}) \cong \mathscr{E} \cong \mathscr{A}$$

where $\mathscr{A}$ is given by

$$\mathscr{A}(U) = \mathrm{Hom}_{X_U}^G(\mathscr{H}|_{X_U}, \mathscr{G}|_{X_U} \otimes_{\mathcal{O}_U} \mathscr{M}|_U))$$

for open $U \subseteq T$. □

This theorem allows to determine the sheaf of relative differentials and relative tangent spaces of Q over S.

The sheaf of differentials $\Omega_{Q/S}$. id_Q corresponds to the universal quotient $[0 \to \mathscr{H} \to \mathscr{F}_Q \to \mathscr{G} \to 0]$ of G-sheaves on X_Q. Applying theorem 4.38 with $T = Q$, $q = id_Q$ and $\widetilde{T} = (Q, \mathcal{O}_Q + \varepsilon\mathscr{M})$, $\mathscr{M}$ a quasicoherent $\mathcal{O}_T$-module, one obtains the following corollary, that characterises the sheaf of relative differentials $\Omega_{Q/S}$:

Corollary 4.39. *For quasicoherent O_T-modules $\mathscr{M}$ there is an isomorphism of $\mathcal{O}_Q$-modules*

$$\mathscr{H}om_{\mathcal{O}_Q}(\Omega_{Q/S}, \mathscr{M}) \cong \mathscr{A}$$

where $\mathscr{A}$ is given by

$$\mathscr{A}(U) = \mathrm{Hom}_{X_U}^G(\mathscr{H}|_{X_U}, \mathscr{G}|_{X_U} \otimes_{\mathcal{O}_U} \mathscr{M}|_U)$$

for open $U \subseteq Q$. □

Relative tangent spaces. Let L be an extension field of K and $q : \mathrm{Spec}\, L \to Q$. The morphism q corresponds to a quotient $[0 \to \mathscr{H} \to \mathscr{F}_L \to \mathscr{G} \to 0]$ of G-sheaves on X_L. We want to describe the relative tangent space of the S-scheme Q over S at q, i.e. the tangent space of the fiber Q_L over $\mathrm{Spec}\, L \xrightarrow{q} Q \to S$ at q. According to theorem 4.38 applied to $T = \mathrm{Spec}\, L$, $q : \mathrm{Spec}\, L \to Q$, $\mathscr{M} = L$, $\widetilde{T} = \mathrm{Spec}\, L[\varepsilon]$ one has the following corollary for the relative tangent space:

Corollary 4.40. *The relative tangent space of Q at $q : \mathrm{Spec}\, L \to Q$, L some extension field of K, is the L-vector space*

$$T_q Q = \mathrm{Hom}_{X_L}^G(\mathscr{H}, \mathscr{G})$$

□

4.4.4 Tangent spaces of G-Hilbert schemes

Let X be an algebraic K-scheme, $G = \operatorname{Spec} A$ a finite group scheme over K. Assume that G operates on X over K such that a geometric quotient $\pi : X \to X/G$, π affine, exists. Then by theorem 4.28 the functor $\underline{\text{G-Hilb}}_{X/G} X$ is represented by a scheme $\text{G-Hilb}_{X/G} X$ projective over X/G. Write $\text{G-Hilb}_K X$ for this scheme considered as a K-scheme, it represents the functor $\underline{\text{G-Hilb}}_K X$. As an application of theorem 4.38 resp. its corollaries one obtains:

Corollary 4.41. *(Tangent spaces of* $\text{G-Hilb}_K X$*).*
Let $h : \operatorname{Spec} L \to \text{G-Hilb}_K X$, L *an extension field of* K, *be a morphism of* K*-schemes corresponding to a quotient* $[0 \to \mathscr{I} \to \mathcal{O}_{X_L} \to \mathcal{O}_Z \to 0]$ *of* G*-sheaves on* X_L. *Then one has*

$$T_h \, \text{G-Hilb}_K X \cong \operatorname{Hom}_{X_L}^G(\mathscr{I}, \mathcal{O}_Z)$$

for the tangent space of $\text{G-Hilb}_K X$ *at* h. $\qquad\square$

Remark 4.42.
(1) There is the isomorphism

$$\operatorname{Hom}_{X_L}^G(\mathscr{I}, \mathcal{O}_Z) \cong \operatorname{Hom}_{X_L}^G(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_Z)$$

Thus the tangent space $T_q \, \text{G-Hilb}_K X$ coincides with the G-invariant global sections of the normal sheaf $\mathscr{H}om_{\mathcal{O}_{X_L}}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_Z)$.
(2) Using the exact sequence $0 \to \mathscr{I} \to \mathcal{O}_{X_L} \to \mathcal{O}_Z \to 0$ one derives the isomorphism

$$\operatorname{Hom}_{X_L}^G(\mathscr{I}, \mathcal{O}_Z) \cong \operatorname{Ext}_{X_L}^{G,1}(\mathcal{O}_Z, \mathcal{O}_Z)$$

The theorem also applies to relative G-Hilbert schemes, either directly or by means of the preceding corollary applied to the fibers over points of X/G.

Corollary 4.43. *(Relative tangent spaces of* $\text{G-Hilb}_{X/G} X$ *over* X/G*).*
Let $h : \operatorname{Spec} L \to \text{G-Hilb}_{X/G} X$, L *an extension field of* K, *be a morphism lying over a morphism of* K*-schemes* $h_0 : \operatorname{Spec} L \to X/G$. *Let* $[0 \to \mathscr{I} \to \mathcal{O}_{X_{h_0}} \to \mathcal{O}_Z \to 0]$ *be the quotient of* G*-sheaves corresponding to* h, *where* $X_{h_0} := \operatorname{Spec} L \times_{X/G} X$, *the fiber product formed by* h_0. *Then one has*

$$T_h \, \text{G-Hilb}_{X/G} X \cong \operatorname{Hom}_{X_{h_0}}^G(\mathscr{I}, \mathcal{O}_Z)$$

for the relative tangent space of $\text{G-Hilb}_{X/G} X$ *in* h. $\qquad\square$

Remark 4.44. Since the morphism $\tau : \text{G-Hilb}_K X \to X/G$ defined in corollary 4.16 identifies with the structure morphism of $\text{G-Hilb}_{X/G} X$ by corollary 4.21, corollary 4.43 describes as well relative tangent spaces of $\text{G-Hilb}_K X$ over X/G with respect to τ.

Example 4.45. Let $X = \mathbb{A}_K^n$, $G = \operatorname{Spec} A \subset \operatorname{GL}(n, K)$ a finite subgroup scheme with A cosemisimple. Then a geometric quotient $\pi : \mathbb{A}_K^n \to \mathbb{A}_K^n/G$ exists, the G-Hilbert functor is represented by a K-scheme $\text{G-Hilb}_K \mathbb{A}_K^n$, there is the projective morphism $\tau : \text{G-Hilb}_K \mathbb{A}_K^n \to \mathbb{A}_K^n/G$. Let $S := K[x_1, \dots, x_n]$, let $O \in \mathbb{A}_K^n$ be the origin, $\mathfrak{m} \subset S$ the corresponding maximal ideal, $\overline{O} := \pi(O) \in \mathbb{A}_K^n/G$ with corresponding maximal ideal $\mathfrak{n} \subset S^G$, let $\overline{S} := S/\mathfrak{n}S$. An L-valued point of the fiber $E := \tau^{-1}(\overline{O})$ corresponds to a G-cluster defined by an ideal $I \subset S_L$ such that $\mathfrak{n}_L \subseteq I$ or equivalently an ideal $\overline{I} \subset \overline{S}_L = S_L/\mathfrak{n}_L S_L$.
The tangent space of $\text{G-Hilb}_K \mathbb{A}_K^n$ (over K) at I is the L-vector space

$$T_I \, \text{G-Hilb}_K \mathbb{A}_K^n \cong \operatorname{Hom}_{S_L}^G(I, S_L/I)$$

The relative tangent space of $\text{G-Hilb}_K \mathbb{A}_K^n$ over $\mathbb{A}_K^n/G$ at I or equivalently the tangent space of the fiber $E \cong \text{G-Hilb}_K \pi^{-1}(\overline{O})$ over $\overline{O}$ at I is the L-vector space

$$T_I E \cong \operatorname{Hom}_{\overline{S}_L}^G(\overline{I}, \overline{S}_L/\overline{I})$$

4.4.5 Relative tangent spaces and stratification

In [ItNm96], [ItNm99] (see subsection 1.2.2) has been defined a certain stratification of the G-Hilbert scheme, that will at least partially provided with a geometric meaning in this subsection.

For simplicity let $K = \mathbb{C}$, consider the situation in example 4.45, that is $X = \mathbb{A}_{\mathbb{C}}^n$, $G \subset \mathrm{GL}(n, \mathbb{C})$ a finite subgroup. Using the notations introduced there, for an ideal $I \subset S$ with $\mathfrak{n} \subseteq I$ or equivalently $\overline{I} \subseteq \overline{S}$ defining a G-cluster and thus a $\mathbb{C}$-valued point of $E = \tau^{-1}(\overline{O})$ one can consider the representation

$$\overline{I}/\overline{\mathfrak{m}}\overline{I} \cong I/(\mathfrak{m}I + \mathfrak{n}S)$$

This representation has been used in [ItNm96], [ItNm99] in the case of finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ to give a natural construction for the bijection observed in [McK80] between isomorphism classes of nontrivial irreducible representations of G and irreducible components of the exceptional divisor of the minimal resolution G-Hilb$_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^2 \to \mathbb{A}_{\mathbb{C}}^2/G$. Subsequently it has been considered in the case of finite small subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$ as well, see for example [Is02].

The relative tangent space of G-Hilb$_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^n$ over $\mathbb{A}_{\mathbb{C}}^n/G$ for points $\overline{I} \in E$ is given as

$$\mathrm{Hom}_{\overline{S}}^G(\overline{I}, \overline{S}/\overline{I})$$

Since a homomorphism of $\overline{S}$-modules $\overline{I} \to \overline{S}/\overline{I}$ is determined by the images of the generators of $\overline{I}$, one has an injective homomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_{\overline{S}}^G(\overline{I}, \overline{S}/\overline{I}) \to \mathrm{Hom}_{\mathbb{C}}^G(\overline{I}/\overline{\mathfrak{m}}\overline{I}, \overline{S}/\overline{I}) \tag{4.7}$$

In the 2-dimensional case, looking at the explicit structure of the fiber $\overline{S}$ of $\mathbb{A}_{\mathbb{C}}^2 \to \mathbb{A}_{\mathbb{C}}^2/G$ over $\overline{O}$, one observes:

Observation 4.46. *The homomorphism* (4.7) *is an isomorphism for the cyclic groups* $G = \mu_r \subset \mathrm{SL}(2, \mathbb{C})$ *naturally operating on* $\mathbb{A}_{\mathbb{C}}^2$.

This relates the stratification discussed in [ItNm96], [ItNm99] to relative tangent spaces in the (A_n) cases. For the nonabelian finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ some further considerations will be necessary – tangent spaces do not suffice to describe the spaces $\mathrm{Hom}_{\mathbb{C}}^G(\overline{I}/\overline{\mathfrak{m}}\overline{I}, \overline{S}/\overline{I})$.

Part III

McKay correspondence over non algebraically closed fields

Chapter 5

Galois operation and irreducibility

This chapter is concerned with the behaviour of irreducibility with respect to Galois extensions of the ground field for components of schemes as well as for representations and written with regard to applications to the McKay correspondence in the next chapter. In general, after base extension $K \to L$ an irreducible component or an irreducible representation may decompose. If L is a Galois extension of K then there is an operation of the Galois group, we show that in both cases an object irreducible over K corresponds to a Galois orbit of irreducible objects over the extension field L.

In algebraic geometry a point or irreducible component corresponds to a prime ideal of a commutative K-algebra. In the representation theory of finite group schemes with cosemisimple Hopf algebra an isomorphism class of irreducible representations corresponds to a simple subcoalgebra or dually a minimal two-sided ideal in the group algebra. Thus, the main aim is to determine the effect of Galois extensions to commutative algebras that occur in algebraic geometry as well as for not necessarily commutative algebras and coalgebras that occur in representation theory.

First the behaviour of (co)semisimplicity with respect to extensions $K \to L$ of the base field is considered. Then we apply Galois descend for vector spaces to simple algebras resp. coalgebras to show that a minimal two-sided ideal resp. a simple subcoalgebra over K decomposes over L into a Galois orbit of minimal two-sided ideals resp. simple subcoalgebras.

For points and irreducible components of algebraic K-schemes the investigation of irreducibility with respect to Galois extensions reduces to that of semisimple commutative K-algebras and results for these can be applied. Further, we describe the Galois operation on functors of points, in particular on the G-Hilbert functor.

For representations and more generally for G-sheaves we introduce the notion of Galois conjugate representations resp. G-sheaves. We apply the results concerning Galois extensions of the ground field for cosemisimple coalgebras to isotypic decompositions of representations and G-sheaves.

5.1   Semisimple algebras and coalgebras and Galois extensions

Let K be a field and $K \to L$ a Galois extension, $\Gamma := \mathrm{Aut}_K(L)$.

5.1.1   Semisimplicity and Galois extensions

Let A be a K-algebra.

The radical $\mathrm{rad}\, A$ of a K-algebra A is the two-sided ideal defined to be the intersection of all maximal left ideals of A (or equivalently all maximal right ideals), we say that A is without radical if $\mathrm{rad}\, A = 0$ [Bour, Algébre, Ch. VIII, §6.3]. Remember the definition of semisimplicity [Bour, Algébre, Ch. VIII, §5.1].

Proposition 5.1. *A is semisimple if and only if it is artinian without radical.*

Proof. [Bour, Algébre, Ch. VIII, §6.4, Thm. 4, Cor. 2]. □

The radical of a K-algebra A has the following base change property, use that $K \to L$ is separable:

Proposition 5.2. *If* $\mathrm{rad}\, A = 0$, *then* $\mathrm{rad}\, A_L = 0$.

Proof. [Bour, Algébre, Ch. VIII, §7.6, Thm. 3, Cor. 3]. □

Proposition 5.3. *If A is semisimple and A_L artinian, then A_L is semisimple.*
If A_L is semisimple, then A is semisimple.

Proof. The first statement follows from propositions 5.1 and 5.2, the second from [Bour, Algébre, Ch. VIII, §7.6, Thm. 3, Cor. 4]. □

In particular for finite dimensional K-algebras A: *A is semisimple $\iff$ A_L is semisimple.*

Remember the definition of cosemisimplicity (definition 3.19). Let A be a finite dimensional K-algebra and $C = A^\vee$ the dual coalgebra. The notions "semisimple" and "cosemisimple" are dual, that is A is semisimple if and only if C is cosemisimple.

Corollary 5.4. *Let C be a finite dimensional coalgebra over K. Then C is cosemisimple if and only if C_L is cosemisimple.* □

Corollary 5.5. *For any finite group scheme $G = \mathrm{Spec}\, A$ over a field K of characteristic 0 the coalgebra A is cosemisimple.*

Proof. Group schemes over fields of characteristic 0 are reduced (see e.g. [Mu, AV, Ch. III.11, p. 101]), therefore for a suitable algebraic extension $K \to L$ the group scheme G_L is discrete. Then $(A_L)^\vee$ is isomorphic to a group algebra of a finite group and thus A_L is cosemisimple. By corollary 5.4 A is cosemisimple. □

5.1.2   Galois descent

Let V be a vector space over K, write $V_L := V \otimes_K L$ for its base extension to L. V_L has an operation of Γ given by $\gamma(v \otimes l) \mapsto v \otimes \gamma(l)$. We call an element $w \in V_L$ resp. an L-subspace $W \subseteq V_L$ K-rational, if $w = v \otimes 1$ for some $v \in V$ resp. $W = U_L$ for some K-subspace $U \subseteq V$.

Proposition 5.6. *Let V be a vector space over K. Then an element $w \in V_L$ resp. an L-subspace $W \subseteq V_L$ is K-rational, if and only if $\gamma(w) = w$ resp. $\gamma(W) \subseteq W$ for all $\gamma \in \Gamma$.*

Proof. [Bour, Algebra II, Ch. V, §10.4]. □

5.1.3 Application to algebras and coalgebras

In this subsection we use the proposition about Galois descent to describe the behaviour of decompositions of semisimple algebras resp. cosemisimple coalgebras into simple components with respect to extensions of the ground field.

A K-algebra A is called simple, if A is semisimple, $A \neq 0$ and A has no other two-sided ideals than $\{0\}$ and A (see [Bour, Algébre, Ch. VIII, §5.2]).

Proposition 5.7. *Let A be a simple K-algebra. Assume that A_L is semisimple, let $A_L = \bigoplus_{i=1}^{r} A_{L,i}$ be its decomposition into simple components. Then Γ permutes the simple summands $A_{L,i}$ and the operation on the set $\{A_{L,1}, \ldots, A_{L,r}\}$ is transitive.*

Proof. The $A_{L,i}$ are the minimal two-sided ideals of A_L. Since any $\gamma \in \Gamma$ is an automorphism of A_L as a K-algebra or ring, the $A_{L,i}$ are permuted by Γ.
Let $U = \sum_{\gamma \in \Gamma} \gamma A_{L,1}$ and V the sum over the remaining $A_{L,i}$. Then $A_L = U \oplus V$, U and V are Γ-stable and thus $U = U'_L$, $V = V'_L$ for K-subspaces $U', V' \subseteq F$ by proposition 5.6 since $K \to L$ is a Galois extension. It follows that $A = U' \oplus V'$ with U', V' two-sided ideals of A. Since A is simple, $V' = 0$, $U = A_L$ and the operation is transitive. $\square$

The situation for coalgebras is dual and proven completely analogously, note that simple coalgebras (see definition 3.10) are finite dimensional by proposition 3.18 and its base extensions are cosemisimple by corollary 5.4.

Proposition 5.8. *Let C be a simple coalgebra over K. Then C_L is cosemisimple, and if $C_L = \bigoplus_i C_{L,i}$ is its decomposition into simple components, then Γ transitively permutes the simple summands $C_{L,i}$.* $\square$

The following corollary will be applied to representations and G-sheaves. Since for these we use the formulation in terms of comodules this corollary is stated for coalgebras.

Corollary 5.9. *Let C be a cosemisimple coalgebra over K and $C \cong \bigoplus_j C_j$ its decomposition into simple subcoalgebras. Then C_L is cosemisimple, and if $C_L \cong \bigoplus_i C_{L,i}$ is the decomposition of C_L into simple subcoalgebras, then:*

(i) *The decomposition $C_L \cong \bigoplus_i C_{L,i}$ is a refinement of the decomposition $C_L \cong \bigoplus_j (C_j)_L$.*

(ii) *Γ transitively permutes the summands $C_{L,i}$ of $(C_j)_L$ for any j.*
Therefore $(C_j)_L = \sum_{\gamma \in \Gamma} \gamma C_{L,i}$, if $C_{L,i}$ is a summand of $(C_j)_L$.

Proof. C_L is cosemisimple: C decomposes into simple subcoalgebras, the base extensions of these are cosemisimple by proposition 5.8.
(i) is clear, Γ operates on the base extensions $(C_j)_L$ of the simple coalgebras C_j, then (ii) follows from proposition 5.8. $\square$

5.2   Galois operation on schemes

5.2.1   Irreducible components of schemes and Galois extensions

Let X be a K-scheme. For an extension field L of K the group $\Gamma = \mathrm{Aut}_K(L)$ operates on X_L by automorphisms of K-schemes such that the diagrams

$$
\begin{array}{ccc}
X_L & \xrightarrow{\ \ \gamma\ \ } & X_L \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,L & \xrightarrow{\ \ \gamma\ \ } & \mathrm{Spec}\,L
\end{array}
$$

commute. For simplicity we denote the morphisms $\mathrm{Spec}\,L \to \mathrm{Spec}\,L$, $X_L \to X_L$ coming from $\gamma : L \to L$ by γ as well.

A point of X may decompose over L, this way a point $x \in X$ corresponds to a set of points of X_L, the preimage of x with respect to the projection $X_L \to X$. In particular this applies to closed points and to irreducible components. These sets are known to be exactly the Γ-orbits.

Proposition 5.10. *Let X be an algebraic K-scheme and $K \to L$ be a Galois extension, $\Gamma := \mathrm{Aut}_K(L)$. Then points of X correspond to Γ-orbits of points of X_L, the Γ-orbits are finite.*

Proof. Taking fibers, the proposition reduces to the following statement:

Let F be the quotient field of a commutative integral K-algebra of finite type. Then $F_L = F \otimes_K L$ has only finitely many prime ideals and they are Γ-conjugate.

Proof. F_L is integral over F because this property is stable under base extension [Bour, Commutative Algebra, Ch. V, §1.1, Prop. 5]. It is clear that every prime ideal of F_L lies above the prime ideal (0) of F. There are no inclusions between the prime ideals of F_L [Bour, Commutative Algebra, Ch. V, §2.1, Proposition 1, Corollary 1]. Since every prime ideal of F_L is a maximal ideal and F_L is noetherian (a localisation of an L-algebra of finite type), F_L is artinian, it has only finitely many prime ideals $Q_1, \ldots, Q_r$.

F_L has trivial radical [Bour, Algébre, Ch. VIII, §7.3, Thm. 1, also §7.5 and §7.6, Cor. 3]. Being an artinian ring without radical, i.e. semisimple [Bour, Algébre, Ch. VIII, §6.4, Thm. 4, Cor. 2 and Prop. 9], F_L decomposes as a L-algebra into a direct sum

$$
F_L \cong \bigoplus_{i=1}^{r} F_{L,i}
$$

of fields $F_{L,i} \cong F_L/Q_i$ (this can easily be seen directly, however, it is part of the general theory of semisimple algebras developed in [Bour, Algébre, Ch. VIII] that contains the representation theory of finite groups schemes with cosemisimple Hopf algebra as another special case).

Γ operates on F_L, it permutes the Q_i and the simple components $F_{L,i}$ of F_L transitively by proposition 5.7. $\square$

For a closed subscheme $Z \subseteq X_L$ given by an ideal sheaf $\mathscr{I}$ the conjugate subscheme γZ is given by the ideal sheaf $\gamma_* \mathscr{I}$, there is the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \gamma_* \mathscr{I} & \longrightarrow & \mathcal{O}_{X_L} & \longrightarrow & \mathcal{O}_{\gamma Z} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \gamma_* \mathscr{I} & \longrightarrow & \gamma_* \mathcal{O}_{X_L} & \longrightarrow & \gamma_* \mathcal{O}_Z & \longrightarrow & 0
\end{array}
$$

where $\mathcal{O}_{X_L} \to \gamma_* \mathcal{O}_{X_L}$ is given by the morphism $\gamma : X_L \to X_L$. Let $X = \mathrm{Spec}\,A$ be affine, Z corresponding to an ideal $I \subseteq A_L$, then γZ corresponds to the ideal $\gamma^{-1} I \subseteq A_L$. In particular this applies to a point and the corresponding prime ideal (sheaf).

5.2.2 Galois operation on G-Hilbert schemes

Let Y be a K-scheme. For an extension field L of K the group $\Gamma = \mathrm{Aut}_K(L)$ operates on the functor of Y_L over K by automorphisms over K (not preserving morphisms over L):

$$\gamma: \quad Y_L(T) \quad \to \quad Y_L(T)$$
$$\alpha \quad \mapsto \quad \gamma \circ \alpha$$

For an L-scheme $f: T \to \mathrm{Spec}\, L$ and $\gamma \in \Gamma$ define the L-scheme $\gamma_* T$ to be the scheme T with structure morphism $\gamma \circ f$. For a morphism $\alpha: T' \to T$ of L-schemes let $\gamma_* \alpha$ be the same morphism α considered as an L-morphism $\gamma_* T' \to \gamma_* T$.

With this definition, for any L-scheme T, taking into account the L-scheme structure, one has for the subset $Y_L(T)_{L\text{-morph.}} \subseteq Y_L(T)$ of morphisms over L

$$\gamma: \quad Y_L(T)_{L\text{-morph.}} \quad \to \quad Y_L(\gamma_* T)_{L\text{-morph.}}$$
$$\alpha \quad \mapsto \quad \gamma \circ (\gamma_* \alpha)$$

Here $\gamma: \gamma_* Y_L \to Y_L$ is a morphism of L-schemes.

For a morphism $\alpha: Y_L \to Y'_L$ of L-schemes and $\gamma \in \Gamma$ define the conjugate morphism α^γ by $\alpha^\gamma := \gamma \circ (\gamma_* \alpha) \circ \gamma^{-1}$, which again is a morphism of L-schemes.

In the case that T is defined over K, that is $T = T'_L$ for some K-scheme T', one can describe the operation of Γ by an operation of Γ on the set $Y_L(T)_{L\text{-morph.}}$ of L-morphisms

$$\gamma: \quad Y_L(T)_{L\text{-morph.}} \quad \to \quad Y_L(T)_{L\text{-morph.}}$$
$$\alpha \quad \mapsto \quad \alpha^\gamma = \gamma \circ (\gamma_* \alpha) \circ \gamma^{-1}$$

Consider the case of G-Hilbert schemes: Let G be a finite group scheme over K, X be a G-scheme over K and assume that the G-Hilbert functor is represented by a K-scheme G-$\mathrm{Hilb}_K X$. There is a canonical isomorphism of L-schemes $(G\text{-}\mathrm{Hilb}_K X)_L \cong G_L\text{-}\mathrm{Hilb}_L X_L$ obtained by identifying $X \times_K T = X_L \times_L T$ for L-schemes T.

Proposition 5.11. *Let T be an L-scheme defined over K. Then, for a morphism $\alpha: T \to G_L\text{-}\mathrm{Hilb}_L X_L$ of L-schemes corresponding to a quotient $[0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0]$ and for $\gamma \in \Gamma$, the γ-conjugate morphism α^γ corresponds to the quotient $[0 \to \gamma_* \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_{\gamma Z} \to 0]$.*

Proof. For a morphism of L-schemes $\alpha: T \to G_L\text{-}\mathrm{Hilb}_L X_L \cong (G\text{-}\mathrm{Hilb}_K X) \times_K \mathrm{Spec}\, L$ consider the commutative diagram of L-morphisms

The morphism α is given by a quotient $[0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0]$ on $X_T = X_L \times_L T$. Under the identification $G_L\text{-}\mathrm{Hilb}_L X_L = (G\text{-}\mathrm{Hilb}_K X_K)_L$ the T-valued point α corresponds to a morphism $T \to G\text{-}\mathrm{Hilb}_K X$ of K-schemes, that is a quotient $[0 \to \mathscr{I} \to \mathcal{O}_{X \times_K T} \to \mathcal{O}_Z \to 0]$ on

$X \times_K T$, and the structure morphism $f : T \to \operatorname{Spec} L$. We have the correspondences

$$
\begin{aligned}
\alpha &\longleftrightarrow & \left\{ \begin{array}{l} [0 \to \mathscr{I} \to \mathcal{O}_{X \times_K T} \to \mathcal{O}_Z \to 0] \\ f : T \to \operatorname{Spec} L \end{array} \right. \\[2mm]
\gamma \circ (\gamma_* \alpha) &\longleftrightarrow & \left\{ \begin{array}{l} [0 \to \mathscr{I} \to \mathcal{O}_{X \times_K \gamma_* T} \to \mathcal{O}_Z \to 0] \\ \gamma \circ (\gamma_* f) : \gamma_* T \to \operatorname{Spec} L \end{array} \right. \\[2mm]
\alpha^\gamma = \gamma \circ (\gamma_* \alpha) \circ \gamma^{-1} &\longleftrightarrow & \left\{ \begin{array}{l} [0 \to \gamma^{-1*} \mathscr{I} \to \mathcal{O}_{X \times_K T} \to \gamma^{-1*} \mathcal{O}_Z \to 0] \\ f = \gamma \circ (\gamma_* f) \circ \gamma^{-1} : T \to \operatorname{Spec} L \end{array} \right. \\[2mm]
&\longleftrightarrow & \left\{ \begin{array}{l} [0 \to \gamma_* \mathscr{I} \to \mathcal{O}_{X \times_K T} \to \mathcal{O}_{\gamma Z} \to 0] \\ f = \gamma \circ (\gamma_* f) \circ \gamma^{-1} : T \to \operatorname{Spec} L \end{array} \right.
\end{aligned}
$$

Under the identification $(\text{G-Hilb}_K X)_L = \text{G}_L\text{-Hilb}_L X_L$ the last morphism corresponds to the quotient $[0 \to \gamma_* \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_{\gamma Z} \to 0]$ on $X_T = X_L \times_L T$. $\qquad\square$

In particular, in the case $X = \mathbb{A}^2_K$ the γ-conjugate of an L-valued point given by an ideal $I \subseteq L[x_1, x_2]$ or a G_L-cluster $Z \subset \mathbb{A}^2_L$ is given by the γ-conjugate ideal $\gamma^{-1} I \subset L[x_1, x_2]$ or the γ-conjugate G_L-cluster $\gamma Z \subset \mathbb{A}^2_L$.

Every point x of the L-scheme $\text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$ such that $\kappa(x) = L$ corresponds to a unique L-valued point $\alpha : \operatorname{Spec} L \to \text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$. The γ-conjugate point γx corresponds to the γ-conjugate L-valued point $\alpha^\gamma : \operatorname{Spec} L \to \text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$.

Corollary 5.12. *Let x be a closed point of $\text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$ such that $\kappa(x) = L$, $\alpha : \operatorname{Spec} L \to \text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$ the corresponding L-valued point given by an ideal $I \subset L[x_1, x_2]$. Then for $\gamma \in \Gamma$ the conjugate point γx corresponds to the γ-conjugate L-valued point $\alpha^\gamma : \operatorname{Spec} L \to \text{G}_L\text{-Hilb}_L \mathbb{A}^2_L$, which is given by the ideal $\gamma^{-1} I \subset L[x_1, x_2]$.* $\qquad\square$

5.3 Conjugate G-sheaves

Let K be a field, $K \to L$ a field extension and $\Gamma = \mathrm{Aut}_K(L)$. Let G be a group scheme over K and X a G-scheme over K.

5.3.1 Conjugate G-sheaves

The base extension X_L of X can equivalently be considered as a G-scheme or a G_L-scheme, these structures given by $(s_X)_L : G \times_K X_L = G_L \times_L X_L \to X_L$. Further, for an $\mathcal{O}_{X_L}$-module $\mathscr{F}$ one may equivalently consider G-sheaf or G_L-sheaf structures (for $G = \mathrm{Spec}\, A$ in the case of trivial operation this will make a difference with respect to decompositions into isotypic components corresponding to simple subcoalgebras of A resp. A_L).

Remark 5.13.
(1) For a G-sheaf $\mathscr{F}$ on X the sheaf $f^*\mathscr{F}$, where $f : X_L \to X$ is the projection, has a natural G-sheaf structure by proposition 3.51.
(2) For G-sheaves $\mathscr{F}, \mathscr{G}$ the tensor product $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ has a natural G-sheaf structure by proposition 3.46. If $\mathscr{F}, \mathscr{G}$ are quasicoherent and moreover $\mathscr{F}$ is finitely presented, then the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ has a natural G-sheaf structure by proposition 3.47.
(3) For G-sheaves $\mathscr{F}, \mathscr{G}$ on X there is the isomorphism $(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G})_L \cong \mathscr{F}_L \otimes_{\mathcal{O}_{X_L}} \mathscr{G}_L$ of G- or G_L-sheaves on X_L by proposition 3.59.
(4) Let $\mathscr{F}, \mathscr{G}$ be G-sheaves on X, assume that $\mathscr{F}$ is finitely presented. Then there is an isomorphism $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_L \cong \mathscr{H}om_{\mathcal{O}_{X_L}}(\mathscr{F}_L, \mathscr{G}_L)$ of G- or G_L-sheaves on X_L by proposition 3.58. In the case of trivial G-operation it follows $\mathscr{H}om^G_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_L \cong \mathscr{H}om^{G_L}_{\mathcal{O}_{X_L}}(\mathscr{F}_L, \mathscr{G}_L)$.

Again, for elements $\gamma \in \Gamma$ there are the automorphisms $\gamma : X_L \to X_L$ over K (see subsection 5.2.1). These are G-equivariant, since the group scheme operation on X_L is defined over K. For $\gamma \in \Gamma$ we introduce the notion of γ-conjugate G_L-sheaves.

Proposition - Definition 5.14. *Let $\mathscr{F}$ be a G_L-sheaf on X_L. For $\gamma \in \Gamma$ the $\mathcal{O}_{X_L}$-module $\gamma_*\mathscr{F}$ has a natural G_L-sheaf structure given by*

$$\begin{array}{ccc}
\gamma_* s^*_{X_L}\mathscr{F} & \xrightarrow{\ \gamma_*\lambda^{\mathscr{F}}\ } & \gamma_* p^*_{X_L}\mathscr{F} \\
\wr\uparrow & & \uparrow\wr \\
s^*_{X_L}\gamma_*\mathscr{F} & \xrightarrow{\ \lambda^{\gamma_*\mathscr{F}}\ } & p^*_{X_L}\gamma_*\mathscr{F}
\end{array} \tag{5.1}$$

This G_L-sheaf $\gamma_*\mathscr{F}$ is called the γ-conjugate G_L-sheaf of $\mathscr{F}$.
For a morphism of G_L-sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ the morphism $\gamma_*\varphi : \gamma_*\mathscr{F} \to \gamma_*\mathscr{G}$ is a morphism of G_L-sheaves between the sheaves $\gamma_*\mathscr{F}$ and $\gamma_*\mathscr{G}$ with γ-conjugate G_L-sheaf structures.

Proof. Equivalently work with G-sheaf structures, γ is an equivariant automorphism of the G-scheme X_L over K, the statements then follow from proposition 3.51. □

Remark 5.15. This way functors γ_* are defined, similarly one may define functors γ^*, then γ_* and $(\gamma^{-1})^*$ are isomorphic. The functors γ_*, γ^* are autoequivalences of categories like $\mathrm{Mod}^{G_L}(X_L)$, $\mathrm{Qcoh}^{G_L}(X_L)$, $\mathrm{Coh}^{G_L}(X_L)$. In the case of trivial operation they preserve trivial G-sheaf structures.

The functors γ_* commute with functors f^*_L, f_{L*} for equivariant morphisms f and with bifunctors like $\mathscr{H}om$ and $\otimes$:

Lemma 5.16. *There are the following natural isomorphisms of G_L-sheaves:*

(i) *For G_L-sheaves $\mathscr{F}, \mathscr{G}$ on X_L : $\gamma_*(\mathscr{F} \otimes_{\mathcal{O}_{X_L}} \mathscr{G}) \cong \gamma_*\mathscr{F} \otimes_{\mathcal{O}_{X_L}} \gamma_*\mathscr{G}$.*

(ii) *Let $f : Y \to X$ be an equivariant morphism of G-schemes over K and $\mathscr{F}$ a G_L-sheaf on X_L. Then $\gamma_* f_L^* \mathscr{F} \cong f_L^* \gamma_* \mathscr{F}$.*

(iii) *For quasicoherent G_L-sheaves $\mathscr{F}, \mathscr{G}$ on X_L with $\mathscr{F}$ finitely presented: $\gamma_* \mathscr{H}om_{\mathcal{O}_{X_L}}(\mathscr{F}, \mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_{X_L}}(\gamma_*\mathscr{F}, \gamma_*\mathscr{G})$. If the G-operation on X is trivial, it follows that $\gamma_*(\mathscr{H}om^{G_L}_{\mathcal{O}_{X_L}}(\mathscr{F}, \mathscr{G})) \cong \mathscr{H}om^{G_L}_{\mathcal{O}_{X_L}}(\gamma_*\mathscr{F}, \gamma_*\mathscr{G})$.*

Proof. Work with $(\gamma^{-1})^*$, then (i) follows from proposition 3.59, (ii) is clear and (iii) follows from proposition 3.58. □

Remark 5.17.
(1) If a sheaf $\mathscr{F}$ on X_L is rational over K, i.e. $\mathscr{F} = \mathscr{F}'_L$, $\mathscr{F}'$ an $\mathcal{O}_X$-module, then $\gamma_*\mathscr{F} \cong (id_X \times \gamma)_*(\mathscr{F}' \otimes_K L) \cong \mathscr{F}' \otimes_K (\gamma_*L)$ and there is the isomorphism of $\mathcal{O}_{X_L}$-modules $\gamma : \mathscr{F} \to \gamma_*\mathscr{F}$, the diagram

$$
\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\quad \gamma \quad} & \gamma_*\mathscr{F} \\
\| & & \| \\
\mathscr{F}' \otimes_K L & \xrightarrow{id_{\mathscr{F}'} \otimes \gamma} & \mathscr{F}' \otimes_K (\gamma_*L)
\end{array}
$$

commutes. If $\mathscr{F}$ has the structure of a G-sheaf, then $\mathscr{F}_L$ is a G_L-sheaf on X_L and the isomorphism of $\mathcal{O}_{X_L}$-modules $\gamma : \mathscr{F} \to \gamma_*\mathscr{F}$ is an isomorphism of G_L-sheaves: One has to show that the diagram

$$
\begin{array}{ccc}
s_{X_L}^* \gamma_*\mathscr{F} & \xrightarrow{\lambda^{\gamma_*\mathscr{F}}} & p_{X_L}^* \gamma_*\mathscr{F} \\
{\scriptstyle s_{X_L}^*\gamma} \big\uparrow & & \big\uparrow {\scriptstyle p_{X_L}^*\gamma} \\
s_{X_L}^* \mathscr{F} & \xrightarrow{\lambda^{\mathscr{F}}} & p_{X_L}^* \mathscr{F}
\end{array}
$$

commutes, but this is the commutative diagram

$$
\begin{array}{ccc}
(s_X^*\mathscr{F}') \otimes_K (\gamma_*L) & \xrightarrow{\lambda^{\mathscr{F}'} \otimes id_{\gamma_*L}} & (p_X^*\mathscr{F}') \otimes_K (\gamma_*L) \\
{\scriptstyle id \otimes \gamma} \big\uparrow & & \big\uparrow {\scriptstyle id \otimes \gamma} \\
(s_X^*\mathscr{F}') \otimes_K L & \xrightarrow{\lambda^{\mathscr{F}'} \otimes id_L} & (p_X^*\mathscr{F}') \otimes_K L
\end{array}
$$

(2) Taking the original structure on the target, Γ operates on K-rational $\mathscr{F}$ by maps (not L-linear) $\gamma : \mathscr{F} \to \mathscr{F}$.
(3) Consider images of subsheaves $\mathscr{G} \subseteq \mathscr{F}$ under the maps $\gamma : \mathscr{F} \to \mathscr{F}$ for K-rational $\mathscr{F}$. Taking the conjugate G_L-sheaf structure on the target, γ becomes an isomorphism of G_L-sheaves, it restricts to an isomorphism of G_L-sheaves $\gamma^{-1}\mathscr{G} \to \gamma_*\mathscr{G}$.

5.3.2   Conjugate comodules and representations

Let $G = \operatorname{Spec} A$ be an affine group scheme over K. Assume that G operates trivially on X. Then for an $\mathcal{O}_{X_L}$-module $\mathscr{F}$ a G_L-sheaf structure is equivalent to an A_L-comodule structure by proposition 3.63. We describe the effect of Galois conjugation to A_L-comodules.

Remark 5.18. For $\gamma \in \Gamma$ there are maps $\gamma : A_L \to A_L$. Taking the canonically defined conjugate Hopf algebra structure on the target, these maps become isomorphisms $\gamma : A_L \to \gamma_* A_L$ of Hopf algebras over L. They correspond to isomorphisms $\gamma : \gamma_* G_L \to G_L$ of group schemes over L (compare to the notation $\gamma_* T$ for schemes introduced in subsection 5.2.2).

Proposition 5.19. *Let $\mathscr{F}$ be a G_L-sheaf on X_L, X with trivial G-operation, the G_L-sheaf structure equivalent to an A_L-comodule structure $\varrho^{\mathscr{F}} : \mathscr{F} \to A_L \otimes_L \mathscr{F}$. Then for $\gamma \in \Gamma$ the G_L-sheaf structure of the γ-conjugate G_L-sheaf $\gamma_* \mathscr{F}$ is equivalent to the comodule structure $\varrho^{\gamma_* \mathscr{F}} : \gamma_* \mathscr{F} \to A_L \otimes_L \gamma_* \mathscr{F}$ determined by commutativity of the diagram*

$$
\begin{array}{ccc}
\gamma_* \mathscr{F} & \xrightarrow{\ \gamma_* \varrho^{\mathscr{F}}\ } & \gamma_* A_L \otimes_L \gamma_* \mathscr{F} \\
{\scriptstyle id} \big\uparrow & & \big\uparrow {\scriptstyle \gamma \otimes id} \\
\gamma_* \mathscr{F} & \xrightarrow[\ \varrho^{\gamma_* \mathscr{F}}\ ]{} & A_L \otimes_L \gamma_* \mathscr{F}
\end{array}
\tag{5.2}
$$

Proof. Remember the construction in the proof of proposition 3.63 that relates G_L-sheaf and A_L-comodule structures. Application of $p_{X_L *}$ to diagram (5.1) gives

$$
\begin{array}{ccc}
p_{X_L *} \gamma_* p_{X_L}^* \mathscr{F} & \xrightarrow{\ p_{X_L *} \gamma_* \lambda^{\mathscr{F}}\ } & p_{X_L *} \gamma_* p_{X_L}^* \mathscr{F} \\
{\scriptstyle \wr} \big\uparrow & & \big\uparrow {\scriptstyle \wr} \\
p_{X_L *} p_{X_L}^* \gamma_* \mathscr{F} & \xrightarrow[\ p_{X_L *} \lambda^{\gamma_* \mathscr{F}}\ ]{} & p_{X_L *} p_{X_L}^* \gamma_* \mathscr{F}
\end{array}
$$

which can be rewritten as

$$
\begin{array}{ccc}
\gamma_* A_L \otimes_L \gamma_* \mathscr{F} & \xrightarrow{\quad\quad} & \gamma_* A_L \otimes_L \gamma_* \mathscr{F} \\
{\scriptstyle \gamma \otimes id} \big\uparrow & & \big\uparrow {\scriptstyle \gamma \otimes id} \\
A_L \otimes_L \gamma_* \mathscr{F} & \xrightarrow{\quad\quad} & A_L \otimes_L \gamma_* \mathscr{F}
\end{array}
$$

Its restriction on the left side to $\gamma_* \mathscr{F}$ (or composition with the natural maps $\gamma_* \mathscr{F} \to \gamma_* A_L \otimes_L \gamma_* \mathscr{F}$ resp. $\gamma_* \mathscr{F} \to A_L \otimes_L \gamma_* \mathscr{F}$, see proof of proposition 3.63) is diagram (5.2). $\square$

In the special case of representations the definition of conjugate G-sheaves leads to the notion of a conjugate representation: Instead of a sheaf $\gamma_* \mathscr{F}$ one has an L-vector space $\gamma_* V$, the vector space structure given by $(l, v) \mapsto \gamma(l)v$ using the original structure. The choice of a K-structure $V = V_L'$ gives an isomorphism $\gamma : V \to \gamma_* V$ of L-vector spaces and leads to the diagram

$$
\begin{array}{ccc}
\gamma_* V & \xrightarrow{\ \gamma_* \varrho^V\ } & \gamma_* A_L \otimes_L \gamma_* V \\
{\scriptstyle id} \big\uparrow & & \big\uparrow {\scriptstyle \gamma \otimes id} \\
\gamma_* V & \xrightarrow{\ \varrho^{\gamma_* V}\ } & A_L \otimes_L \gamma_* V \\
{\scriptstyle \gamma} \big\uparrow & & \big\uparrow {\scriptstyle id \otimes \gamma} \\
V & \xrightarrow[\ (\varrho^V)^{\gamma}\ ]{} & A_L \otimes_L V
\end{array}
$$

for definition of the γ-conjugate A_L-comodule structure $(\varrho^V)^{\gamma}$ on V — this definition is made, such that $\gamma : (V, (\varrho^V)^{\gamma}) \to (\gamma_* V, \varrho^{\gamma_* V})$ is an isomorphism of A_L-comodules. We write V^{γ} for V with the conjugate A_L-comodule structure.

Remark 5.20. Let V' be an A-comodule over K and $V = V_L'$. Then as a special case of remark 5.17 there are maps $\gamma : V \to V$ resp. isomorphisms of A_L-comodules $\gamma : V \to \gamma_* V$. For any A_L-subcomodule $U \subseteq V$ these restrict to isomorphisms of A_L-comodules $\gamma^{-1} U \xrightarrow{\sim} \gamma_* U \cong U^{\gamma}$.

5.3.3   Decomposition into isotypic components and Galois extensions

Let $G = \operatorname{Spec} A$ be an affine group scheme over a field K, assume that A is cosemisimple. Let $K \to L$ be a Galois extension, $\Gamma = \operatorname{Aut}_K(L)$. Then by corollary 5.9 A_L is cosemisimple as well.

Remember the relations between the Galois operation on A_L given by maps $\gamma : A_L \to A_L$ resp. isomorphisms $\gamma : A_L \to \gamma_* A_L$ of Hopf algebras or of A_L-comodules (see remark 5.18 or 5.20) and the decompositions $A = \bigoplus_{j \in J} A_j$ and $A_L = \bigoplus_{i \in I} A_{L,i}$ into simple subcoalgebras described in corollary 5.9: Any $(A_j)_L$ decomposes over L into a Galois orbit $(A_j)_L = \sum_{\gamma \in \Gamma} \gamma A_{L,i}$ for some i. We relate this to conjugation of representations: The subcoalgebras $A_{L,i}$ are the isotypic components of A_L as a left-(or right-)comodule (proposition 3.26), let V_i be the isomorphism class of simple A_L-comodules corresponding to $A_{L,i}$. Define an operation of Γ on the index set I by $V_{\gamma(i)} = V_i^\gamma$.

Lemma 5.21. $\gamma^{-1} A_{L,i} = A_{L,\gamma(i)}$.

Proof. By remark 5.20 the isomorphism of A_L-comodules $\gamma : A_L \to \gamma_* A_L$ restricts to an isomorphism of A_L-comodules $\gamma^{-1} A_{L,i} \to \gamma_* A_{L,i}$, so $\gamma^{-1} A_{L,i}$ is isotypic of isomorphism class $\gamma_* V_i = V_i^\gamma = V_{\gamma(i)}$. □

As constructed in proposition 3.68, corollary 3.71, the decomposition of A into simple subcoalgebras $A = \bigoplus_j A_j$ gives decompositions of representations and more generally of G-sheaves on G-schemes with trivial G-operation into isotypic components corresponding to the A_j. After base extension one has decompositions of G_L-sheaves, we compare it with the decomposition coming from the decomposition of A_L into simple subcoalgebras.

Proposition 5.22. *Let X be a G-scheme with trivial operation, $\mathscr{F}$ a G-sheaf on X and let*

$$\mathscr{F} = \bigoplus_j \mathscr{F}_j, \qquad \mathscr{F}_L = \bigoplus_i \mathscr{F}_{L,i}$$

be the decompositions into isotypic components as a G-sheaf resp. G_L-sheaf. Then:

(i) *$\mathscr{F}_L = \bigoplus_i \mathscr{F}_{L,i}$ is a refinement of $\mathscr{F}_L = \bigoplus_j (\mathscr{F}_j)_L$.*

(ii) *The operation of Γ on $\mathscr{F}_L$ (see remark 5.17) permutes the isotypic components $\mathscr{F}_{L,i}$ of $\mathscr{F}_L$. It is $\gamma^{-1} \mathscr{F}_{L,i} = \mathscr{F}_{L,\gamma(i)}$, if $V_{\gamma(i)} = V_i^\gamma$.*

(iii) *$(\mathscr{F}_j)_L = \sum_{\gamma \in \Gamma} \gamma \mathscr{F}_{L,i}$, if $\mathscr{F}_{L,i}$ is a summand of $(\mathscr{F}_j)_L$.*

Proof. (i) follows from corollary 5.9.(i) and the construction of corollary 3.71, proposition 3.68. For (ii) consider the isomorphism of G_L-sheaves $\gamma : \mathscr{F}_L \to \gamma_* \mathscr{F}_L$ (remark 5.17), it restricts to an isomorphism $\gamma^{-1} \mathscr{F}_{L,i} \to \gamma_* \mathscr{F}_{L,i}$. $\mathscr{F}_{L,i}$ is an $A_{L,i}$-comodule, from proposition 5.19 derives the commutative diagram

$$
\begin{array}{ccc}
\gamma_* \mathscr{F}_{L,i} & \xrightarrow{\;\gamma_* \varrho^{\mathscr{F}_{L,i}}\;} & \gamma_* A_{L,i} \otimes_L \gamma_* \mathscr{F}_{L,i} \\
{\scriptstyle id} \big\uparrow & & \big\uparrow {\scriptstyle \gamma \otimes id} \\
\gamma_* \mathscr{F}_{L,i} & \xrightarrow{\;\varrho^{\gamma_* \mathscr{F}_{L,i}}\;} & \gamma^{-1} A_{L,i} \otimes_L \gamma_* \mathscr{F}_{L,i}
\end{array}
$$

So the A_L-comodule structure of $\gamma_* \mathscr{F}_{L,i}$ restricts to an $\gamma^{-1} A_{L,i}$-comodule structure and $\gamma^{-1} \mathscr{F}_{L,i}$ is the isotypic part corresponding to the simple subcoalgebra $\gamma^{-1} A_{L,i}$. By lemma 5.21 $\gamma^{-1} A_{L,i} = A_{L,\gamma(i)}$, thus $\gamma^{-1} \mathscr{F}_{L,i}$ is isotypic of isomorphism class V_i^γ.
(iii) follows from corollary 5.9.(ii). □

We now consider the case of representations.

Corollary 5.23. Γ *operates by* $V_i \mapsto V_i^\gamma$ *on the set* $\{V_i \mid i \in I\}$ *of isomorphism classes of irreducible representations of* G_L. *The subsets of* $\{V_i \mid i \in I\}$, *which occur by decomposing irreducible representations of* G *over* K *as representations over* L, *are exactly the* Γ-*orbits.* $\square$

Remark 5.24. For finite dimensional representations of a finite group scheme G over a field there is the usual notion of a character and as usual in characteristic 0 the character determines the isomorphism class [Bour, Algébre, Ch. VIII, §12.1, Prop. 3].

The relation to the representation theory of finite groups is as follows: Assume that A is finite dimensional, write $KG = A^\vee$ and $LG = A_L^\vee$ for the algebras dual to the coalgebras A and A_L. Dualising a finite dimensional A_L-comodule gives rise to an LG-module V or a homomorphism $LG \to \operatorname{End}_L(V)$. Here Γ operates on LG and, after choice of a K-structure $V = V_L'$, on $\operatorname{End}_L(V)$. Identify $\operatorname{End}_L(V)$ with a matrix algebra with respect to a K-rational basis of $V = V_L'$. If a γ-invariant element of LG is mapped to a matrix M, then the conjugate representation maps it to the matrix M^γ with conjugate entries.

In particular for a finite group scheme over K with all its closed points K-rational the character of the γ-conjugate representation is obtained from the character of the original representation by application of γ to the values of the character. Because in characteristic 0 a representation is characterised by its character, one has the following corollary:

Corollary 5.25. *Let* G *be a finite group, assume that the field* K *is of characteristic* 0. Γ *operates by* $\chi_i \mapsto \chi_i^\gamma$, *where* $\chi_i^\gamma(g) = \gamma(\chi_i(g))$ *for* $g \in G$, *on the set* $\{\chi_i \mid i \in I\}$ *of characters of irreducible representations of* G *over* L. *The subsets of* $\{\chi_i \mid i \in I\}$, *which occur by decomposing irreducible representations of* G *over* K *as representations over* L, *are exactly the* Γ-*orbits.* $\square$

Remark 5.26. For similar results in the representation theory of finite groups see e.g. [CR, Vol. I, §7B].

Chapter 6

McKay correspondence over non algebraically closed fields

In this chapter we establish a McKay correspondence for finite subgroup schemes $G \subset \mathrm{SL}(2, K)$ over not necessarily algebraically closed fields of characteristic 0 relating isomorphism classes of nontrivial irreducible representations and irreducible components of the exceptional divisor and moreover the representation graph and the intersection graph as in the original observation of McKay [McK80]. As already observed in [Li69], considering the rational double points over non algebraically closed fields one is lead to the remaining Dynkin diagrams of types (B_n), (C_n), (F_4), (G_2).

We use the McKay correspondence for finite groups over algebraically closed fields of characteristic 0. This situation arises for the operation of finite subgroup schemes of $\mathrm{SL}(2, K)$ after base extension to an algebraic closure, in addition there is an operation of the Galois group. In the last chapter it has been shown that concerning irreducibility both, components of schemes and representations, have the same behaviour with respect to Galois extensions, therefore any construction like the McKay correspondence relating such objects over the extension field that is equivariant with respect to the Galois operation determines a correspondence over K.

In the first section of this chapter we collect some data of the finite subgroup schemes of $\mathrm{SL}(2, K)$ and list possible representation graphs. In addition we investigate, under what conditions a finite subgroup of $\mathrm{SL}(2, C)$, C the algebraic closure of K, is realisable as a subgroup of $\mathrm{SL}(2, K)$.

The second section contains the theorems of McKay correspondence over non algebraically closed fields. We consider two constructions, the stratification of the G-Hilbert scheme and the tautological sheaves, originating from [ItNm96] and [GV83] respectively (in this thesis subsections 1.2.2 and 1.2.3), that are known to give a McKay correspondence over $\mathbb{C}$ and formulate them for non algebraically closed K.

In the last section we give some examples, we show that situations in which components irreducible over K split over the algebraic closure really do occur for any graph with nontrivial automorphism group.
Here we are primarily interested in the McKay correspondence, for which a detailed knowledge of the finite subgroup schemes $G \subset \mathrm{SL}(2, K)$ is not necessary. Though, it would be interesting to achieve a classification of finite subgroup schemes over fields that are not necessarily algebraically closed or of characteristic $p > 0$, which, as far as the author knows, has not been considered before, and to investigate the relationship between rational double points and quotients by finite group schemes.

6.1 The finite subgroup schemes of $\mathrm{SL}(2, K)$: representations and graphs

In this section K denotes a field of characteristic 0.

6.1.1 The finite subgroups of $\mathrm{SL}(2, C)$

By the well known classification any finite subgroup $G \subset \mathrm{SL}(2, C)$, C an algebraically closed field of characteristic 0, is isomorphic to one of the following groups:

$\mathbb{Z}/n\mathbb{Z}$ (cyclic group of order n), $n \geq 1$
BD_n (binary dihedral group of order $4n$), $n \geq 2$
BT (binary tetrahedral group)
BO (binary octahedral group)
BI (binary icosahedral group).

Presentations and the irreducible representations/character tables of these groups are listed in subsection 6.1.6.

6.1.2 Representation graphs

In the following definition we will introduce the (extended) representation graph as an in general directed graph. A loop is defined to be an edge emanating from and terminating at the same vertex. In addition we will attach a natural number called multiplicity to any vertex, and for homomorphisms of graphs in addition we will require, that for any vertex of the target its multiplicity is the sum of the multiplicities of its preimages.

Definition 6.1. *The extended representation graph* $\mathrm{Graph}(G, V)$ *associated to a finite subgroup scheme G of* $\mathrm{GL}(n, K)$*, V the given n-dimensional representation, is defined as the following directed graph:*

- vertices. A vertex of multiplicity n for each irreducible representation of G over K which decomposes over the algebraic closure of K into n irreducible representations.

- edges. Vertices V_i and V_j are connected by $\dim_K \mathrm{Hom}_K^G(V_i, V \otimes_K V_j)$ directed edges from V_i to V_j. In particular any vertex V_i has $\dim_K \mathrm{Hom}_K^G(V_i, V \otimes_K V_i)$ directed loops.

Define the representation graph to be the graph, which arises by leaving out the trivial representation and all edges emanating from or terminating at the trivial representation.

We say that a graph is undirected, if between any two different vertices the numbers of directed edges of both directions coincide and for any vertex the number of directed loops is even.
Then one can form a graph having only undirected edges by defining *(number of undirected edges between V_i and V_j) := (number of directed edges from V_i to V_j) = (number of directed edges from V_j to V_i)* for different vertices V_i, V_j and *(number of undirected loops of V_i) := $\frac{1}{2}$(number of directed loops of V_i)* for any vertex V_i.

Remark 6.2.
(1) For $G \subset \mathrm{SL}(2, K)$ the (extended) representation graph is undirected. There is the isomorphism $\mathrm{Hom}_K^G(V_i \otimes_K V, V_j) \cong \mathrm{Hom}_K^G(V_i, V \otimes_K V_j)$, which follows from the isomorphism $\mathrm{Hom}_K^G(V_i \otimes_K V, V_j) \cong \mathrm{Hom}_K^G(V_i, V^\vee \otimes_K V_j)$ and the fact that the 2-dimensional representation V given by inclusion $G \to \mathrm{SL}(2, K)$ is self-dual. Further, that the number of directed loops of any vertex is even, follows from the fact that over the algebraic closure C one has $\dim_C \mathrm{Hom}_C^G(U_i, V_C \otimes_C U_i) = 0$ for irreducible U_i over C.

(2) There is a definition of (extended) representation graph with another description of the edges: vertices V_i and V_j are connected by a_{ij} edges from V_i to V_j, where $V \otimes_K V_j = a_{ij} V_i \oplus$ *other summands*. The two definitions coincide over algebraically closed fields, always one has $a_{ij} \leq \dim_K \mathrm{Hom}_K^G(V_i, V \otimes_K V_j)$, inequality comes from the presence of nontrivial automorphisms.

Definition 6.3. *For a finite subgroup scheme* $G \subset \mathrm{SL}(2, K)$, V *the given* 2-*dimensional representation, define a* $\mathbb{Z}$-*bilinear form* $\langle \cdot, \cdot \rangle$ *on the representation ring of* G *by*

$$\langle V_i, V_j \rangle := \dim_K \mathrm{Hom}_K^G(V_i, V \otimes_K V_j) - 2 \dim_K \mathrm{Hom}_K^G(V_i, V_j)$$

Remark 6.4. The form $\langle \cdot, \cdot \rangle$ determines and is determined by the extended representation graph (the second equation follows from the fact, that $\dim_K \mathrm{Hom}_K^G(V_i, V_i) =$ *multiplicity of* V_i):

$\langle V_i, V_j \rangle = \langle V_j, V_i \rangle =$ *number of undirected edges between* V_i *and* V_j, *if* $V_i \not\cong V_j$
$\frac{1}{2} \langle V_i, V_i \rangle =$ *number of undirected loops of* V_i − *multiplicity of* V_i

6.1.3 Representation graphs and field extensions

Let $K \to L$ be a Galois extension, $\Gamma = \mathrm{Aut}_K(L)$ and let G be a finite subgroup scheme of $\mathrm{SL}(2, K)$.

An irreducible representation W of G over K decomposes as a representation of G_L over L into isotypic components $W = \bigoplus_i U_i$ which are Γ-conjugate by proposition 5.22. Every U_i decomposes into irreducible components $U_i = V_i^{\oplus m}$ (the same m for all i because of Γ-conjugacy). In the following we will write $m(W, L/K)$ for this number. It is related to the Schur index in the representation theory of finite groups (see e.g. [CR, Vol. II, §74]).

Proposition 6.5. *For finite subgroup schemes* G *of* $\mathrm{SL}(2, K)$ *it is* $m(W_j, L/K) = 1$ *for every irreducible representation* W_j *of* G. *It follows that* W_j *decomposes over* L *into a direct sum* $(W_j)_L \cong \bigoplus_i V_i$ *of* γ-*conjugate irreducible representations* V_i *of* G_L *nonisomorphic to each other.*

Proof. We may assume L algebraically closed. Further we may assume that G is not cyclic.
The natural 2-dimensional representation W given by inclusion $G \subset \mathrm{SL}(2, K)$ does satisfy $m(W, L/K) = 1$ because it is irreducible over L.
Following the discussion below without using this proposition one obtains the graphs in subsection 6.1.4 without multiplicities of vertices and edges but one knows which vertices over the algebraic closure may form a vertex over K and which vertices are connected. Argue that if an irreducible representation W_i satisfies $m(W_i, L/K) = 1$ then any irreducible W_j connected to W_i in the representation graph has to satisfy this property as well. $\square$

There is a morphism of graphs $\mathrm{Graph}(G_L, W_L) \to \mathrm{Graph}(G, W)$ (resp. of the nonextended graphs, the following applies to them as well): For W_j an irreducible representation of G the base extension $(W_j)_L$ is a sum $(W_j)_L = \bigoplus_i V_i$ of irreducible representations of G_L nonisomorphic to each other by proposition 6.5. The morphism $\mathrm{Graph}(G_L, W_L) \to \mathrm{Graph}(G, W)$ maps components of $(W_j)_L$ to W_j, thereby their multiplicities are added. Further, for irreducible representations $W_j, W_{j'}$ of G there is a bijection between the set of edges between W_j and $W_{j'}$ and the union of the sets of edges between the irreducible components of $(W_j)_L$ and $(W_{j'})_L$, again using proposition 6.5 $(W_j)_L$ and $(W_{j'})_L$ are sums $(W_j)_L = \bigoplus_i V_i$, $(W_{j'})_L = \bigoplus_{i'} V_{i'}$ of irreducible representations of G_L nonisomorphic to each other and one has

$$
\begin{aligned}
\dim_K \mathrm{Hom}_K^G(W_j \otimes_K W, W_{j'}) &= \dim_L(\mathrm{Hom}_K^G(W_j \otimes_K W, W_{j'}) \otimes_K L) \\
&= \dim_L \mathrm{Hom}_L^G((W_j)_L \otimes_L W_L, (W_{j'})_L) \\
&= \dim_L \mathrm{Hom}_L^G(\bigoplus_i V_i \otimes_L W_L, \bigoplus_{i'} V_{i'}) \\
&= \sum_{i,i'} \dim_L \mathrm{Hom}_L^G(V_i \otimes_L W_L, V_{i'})
\end{aligned}
$$

Γ operates on $\text{Graph}(G_L, W_L)$ by graph automorphisms: Irreducible representations are mapped to conjugate representations and equivariant homomorphisms to the conjugate homomorphisms. The vertices of $\text{Graph}(G, W)$ correspond to Γ-orbits of vertices of $\text{Graph}(G_L, W_L)$ by corollary 5.23.

Proposition 6.6. *The (extended) representation graph of G arises by identifying the elements of Γ-orbits of vertices of the (extended) representation graph of G_L, adding multiplicities. The edges between vertices W_j and $W_{j'}$ are in bijection with the edges between the isomorphism classes of irreducible components of $(W_j)_L$ and $(W_{j'})_L$.* $\qquad\square$

Remark 6.7. Taking $\frac{2}{-\langle V,V \rangle} V$ for the isomorphism classes of irreducible representations V as simple roots one can form the Dynkin diagram with respect to the form $-\langle \cdot, \cdot \rangle$ (see e.g. [Bour, Groupes et algèbres de Lie]). Between (extended) representation graphs and (extended) Dynkin diagrams there is the correspondence

(A_n)	$(A_2)'$	$(A_{2n+1})'$	$(A_{2n+2})'$	(D_n)	$(D_n)'$	$(D_4)''$	(E_6)	$(E_6)'$	(E_7)	(E_8)
(A_n)	$(C_1) = (A_1)$	(C_{n+1})	(C_{n+1})	(D_n)	(B_{n-1})	(G_2)	(E_6)	(F_4)	(E_7)	(E_8)

A long time ago, the occurence of the remaining Dynkin diagrams of types (B_n), (C_n), (F_4), (G_2) as resolution graphs had been observed in [Li69, p. 258] with a slightly different assignment of the non extended diagrams to the resolutions of these singularities, see also [Sl, Appendix I].

6.1.4   Representation graphs of the finite subgroup schemes of $\mathrm{SL}(2, K)$

We list the extended representation graphs $\text{Graph}(G, V)$ of the finite subgroups of $\mathrm{SL}(2, C)$ for C algebraically closed, their groups of automorphisms leaving the trivial representation fixed and the possible extended representation graphs for finite subgroup schemes over non algebraically closed K, which after suitable base extension become the graph $\text{Graph}(G, V)$. We use the symbol $\circ$ for the trivial representation.

- Cyclic groups

(A_{2n}), $n \geq 1$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $(A_{2n})'$
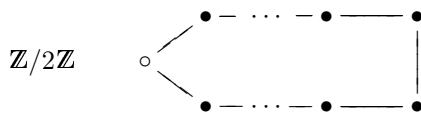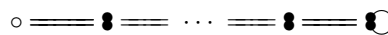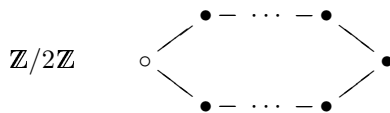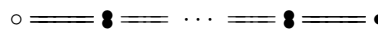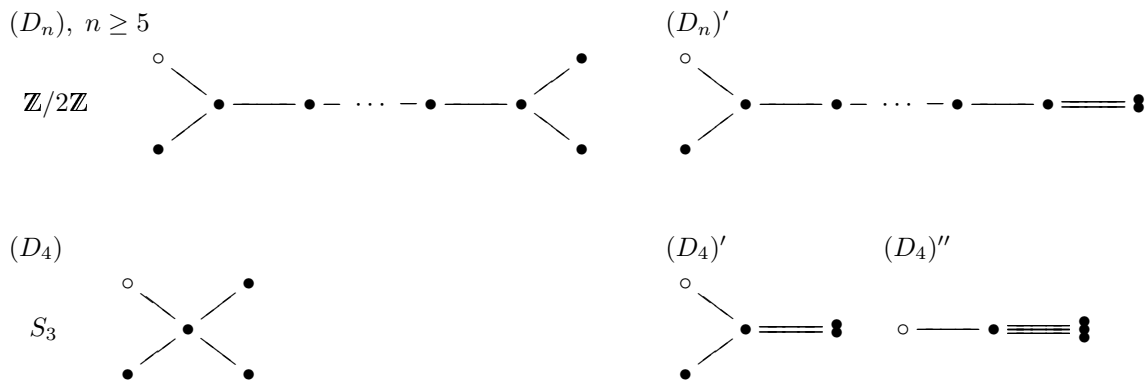
$\mathbb{Z}/2\mathbb{Z}$

(A_{2n+1}), $n \geq 1$ $\qquad\qquad\qquad\qquad\qquad$ $(A_{2n+1})'$

$\mathbb{Z}/2\mathbb{Z}$

(A_1)

$\{id\}$ $\quad \circ =\!=\!= \bullet$

- Binary dihedral groups

(D_n), $n \geq 5$ $(D_n)'$

$\mathbb{Z}/2\mathbb{Z}$

(D_4) $(D_4)'$ $(D_4)''$

S_3

- Binary tetrahedral group

(E_6) $(E_6)'$

$\mathbb{Z}/2\mathbb{Z}$

- Binary octahedral group

(E_7)

$\{id\}$

- Binary icosahedral group

(E_8)

$\{id\}$
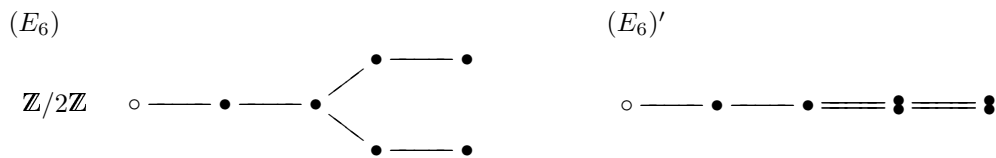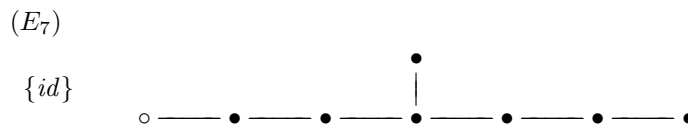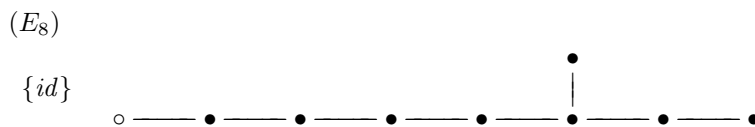
6.1.5 Finite subgroups of $\mathrm{SL}(2,K)$

Given a field K of characteristic 0, it is a natural question, which of the finite subgroups $G \subset \mathrm{SL}(2,C)$, C the algebraic closure of K, are realisable over the subfield K as subgroups (not just as subgroup schemes), that is there is an injective representation of the group G in $\mathrm{SL}(2,K)$.

For a finite subgroup G of $\mathrm{SL}(2,C)$ to occur as a subgroup of $\mathrm{SL}(2,K)$ it is necessary and sufficient that the given 2-dimensional representation in $\mathrm{SL}(2,C)$ is realisable over K. This is easy to show using the classification and the irreducible representations (see subsection 6.1.6)) of the individual groups. If a representation of a group G over C is realisable over K, necessarily its character has values in K. For the finite subgroups of $\mathrm{SL}(2,C)$ and the natural representation given by inclusion this means:

$\mathbb{Z}/n\mathbb{Z}$: $\xi + \xi^{-1} \in K$, $\xi \in C$ a primitive n-th root of unity.
BD_n: $\xi + \xi^{-1} \in K$, $\xi \in C$ a primitive $2n$-th root of unity.
BT: no condition.
BO: $\sqrt{2} \in K$.
BI: $\sqrt{5} \in K$.

To formulate sufficient conditions, we introduce the following notation (see [Se, CA, Part I, Chapter III, §1]):

Definition 6.8. *For a field K the Hilbert symbol $((\cdot,\cdot))_K$ is the map $K^* \times K^* \to \{-1,1\}$ defined by $((a,b))_K = 1$, if the equation $z^2 - ax^2 - by^2 = 0$ has a solution $(x,y,z) \in K^3 \setminus \{(0,0,0)\}$, and $((a,b))_K = -1$ otherwise.*

Remark 6.9. It is $((-1,b))_K = 1$ if and only if $x^2 - by^2 = -1$ has a solution $(x,y) \in K^2$.

Theorem 6.10. *Let G be a finite subgroup of $\mathrm{SL}(2,C)$ such that the values of the character of the natural representation given by inclusion are contained in K. Then:*

(i) *If $G \cong \mathbb{Z}/n\mathbb{Z}$, then G is realisable over K.*

(ii) *If $G \cong BD_n$: Let $\xi \in C$ be a primitive $2n$-th root of unity and $c := \frac{1}{2}(\xi + \xi^{-1})$. Then G is isomorphic to a subgroup of $\mathrm{SL}(2,K)$ if and only if $((-1, c^2 - 1))_K = 1$.*

(iii) *If $G \cong BT, BO$ or BI, then G is isomorphic to a subgroup of $\mathrm{SL}(2,K)$ if and only if $((-1,-1))_K = 1$.*

Proof. (i) For $n \geq 3$ let ξ be a primitive n-th root of unity and $c := \frac{1}{2}(\xi + \xi^{-1})$. By assumption $c \in K$. Then $\mathbb{Z}/n\mathbb{Z}$ is realisable over K, there is the representation

$$\mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(2,K), \qquad \overline{1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 2c \end{pmatrix}$$

(ii): Let $G = BD_n = \langle \sigma, \tau \mid \tau^2 = \sigma^n = (\tau\sigma)^2 \rangle$ (then the element $\tau^2 = \sigma^n = (\tau\sigma)^2$ has order 2) and let ξ be a primitive $2n$-th root of unity.
Then G is realisable as a subgroup of $\mathrm{SL}(2,K)$ if and only if the representation given by

$$\sigma \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.1}$$

is realisable over K.

The representation (6.1) *is realisable over K if and only if there is a 2×2-matrix M_τ over K having the properties*

$$\det(M_\tau) = 1, \ \ \mathrm{ord}(M_\tau) = 4, \ \ (M_\tau M_\sigma)^2 = -\mathbb{1}, \qquad \text{where } M_\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 2c \end{smallmatrix}\right), \ \ c = \tfrac{1}{2}(\xi + \xi^{-1}). \ \ (6.2)$$

If the representation (6.1) is realisable over K, then with respect to a suitable basis it maps $\sigma \mapsto M_\sigma$ and the image of τ is a matrix satisfying the properties (6.2).

On the other hand, if M_τ is a matrix having these properties, then $\sigma \mapsto M_\sigma$, $\tau \mapsto M_\tau$ is a representation of G in $\mathrm{SL}(2,K)$, which is easily seen to be isomorphic to the representation (6.1).

There is a 2×2-matrix M_τ over K having the properties (6.2) *if and only if the equation*

$$x^2 + y^2 - 2cxy + 1 = 0 \tag{6.3}$$

has a solution $(x,y) \in K^2$.

A matrix $M_\tau = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ satisfies the conditions (6.2) if and only if $(\alpha, \beta, \gamma, \delta) \in K^4$ is a solution of $\alpha\delta - \beta\gamma - 1 = 0$, $\quad \alpha + \delta = 0$, $\quad \beta + 2c\delta - \gamma = 0$. Such an element of K^4 exists if and only if there exists a solution $(x,y) \in K^2$ of equation (6.3).

The equation (6.3) *has a solution* $(x,y) \in K^2$ *if and only if* $(\!(-1, c^2 - 1)\!)_K = 1$.

We write the equation $x^2 + y^2 - 2cxy + 1 = 0$ as $(x,y) \left(\begin{smallmatrix} 1 & -c \\ -c & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = -1$. After diagonalisation $(x,y) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-c^2 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = -1$ or $x^2 + (1 - c^2)y^2 + 1 = 0$. This equation has a solution $(x,y) \in K^2$ if and only if $(\!(-1, c^2 - 1)\!)_K = 1$.

(iii) Let $G = BT, BO$ or BI, that is $G = \langle a, b \mid a^3 = b^k = (ab)^2 \rangle$ for $k \in \{3,4,5\}$. Let ξ be a primitive $2k$-th root of unity and $c = \frac{1}{2}(\xi + \xi^{-1})$. As in (ii), using the subgroup $\langle b \rangle$ instead of $\langle \sigma \rangle$, we obtain:

G is isomorphic to a subgroup of $\mathrm{SL}(2,K)$ if and only if there is a solution $(x,y) \in K^2$ of the equation

$$x^2 + y^2 - 2cxy - x + 2cy + 1 = 0 \tag{6.4}$$

Next we show:

Equation (6.4) *has a solution* $(x,y) \in K^2$ *if and only if* $(\!(-1, (2c)^2 - 3)\!)_K = 1$.

Equation (6.4) has a solution if and only if $(x,y,z) \left(\begin{smallmatrix} 1 & -c & -1/2 \\ -c & 1 & c \\ -1/2 & c & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = 0$ has a solution $(x,y,z) \in K^3$ with $z \neq 0$. The existence of a solution with $z \neq 0$ is equivalent to the existence of a solution $(x,y,z) \in K^3 \setminus \{(0,0,0)\}$ (if $(x,y,0)$ is a solution, then $(x,y,x-2cy)$ as well).

After diagonalisation: $(x,y,z) \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3-(2c)^2 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = 0$. The existence of a solution $(x,y,z) \in K^3 \setminus \{(0,0,0)\}$ for this equation is equivalent to $(\!(-1, (2c)^2 - 3)\!)_K = 1$.

For the individual groups we obtain:

BT: $c = \frac{1}{2}$, $(\!(-1,-2)\!)_K = 1$.

BO: $c = \frac{1}{\sqrt{2}}$, $(\!(-1,-1)\!)_K = 1$.

BI: $c = \frac{1}{4}(1 \pm \sqrt{5})$, $(\!(-1, \frac{1}{2}(-3 \pm \sqrt{5}))\!)_K = 1$.

Each of these conditions is equivalent to $(\!(-1,-1)\!)_K = 1$. For BI: $\frac{1}{2}(3 \pm \sqrt{5}) = (\frac{1}{2}(1 \pm \sqrt{5}))^2$. For BT one has maps between solutions (x,y) for $x^2 + y^2 = -1$ corresponding to $(\!(-1,-1)\!)_K$ and (x',y') for $x'^2 + 2y'^2 = -1$ corresponding to $(\!(-1,-2)\!)_K$ given by $x = \frac{x'+1}{2y'} \leftrightarrow x' = \frac{x+y}{x-y}$, $y = \frac{x'-1}{2y'} \leftrightarrow y' = \frac{1}{x-y}$ for $x \neq y$ resp. $y' \neq 0$ and by $(x,x) \mapsto (0,x)$, $(x',0) \leftrightarrow (x',0)$. $\qquad \square$

6.1.6 Finite subgroups of $\mathrm{SL}(2, C)$: Presentations and character tables

- Cyclic groups

The irreducible representations are $\varrho_j : \mathbb{Z}/n\mathbb{Z} \to C^*,\ \bar{i} \mapsto \xi^{ji}$ for $j \in \{0, \ldots, n-1\}$, where ξ is a primitive n-th root of unity.

- Binary dihedral groups: $BD_n = \langle \sigma, \tau \mid \tau^2 = \sigma^n = (\tau\sigma)^2 \rangle,\ -id := (\tau\sigma)^2$.

BD_n, n odd

	id	$-id$	σ^k	τ	$\tau\sigma$
1	1	1	1	1	1
1$'$	1	1	1	-1	-1
1$''$	1	-1	$(-1)^k$	i	$-i$
1$'''$	1	-1	$(-1)^k$	$-i$	i
2^j	2	$(-1)^j 2$	$\xi^{kj} + \xi^{-kj}$	0	0

BD_n, n even

	id	$-id$	σ^k	τ	$\tau\sigma$
1	1	1	1	1	1
1$'$	1	1	1	-1	-1
1$''$	1	1	$(-1)^k$	1	-1
1$'''$	1	1	$(-1)^k$	-1	1
2^j	2	$(-1)^j 2$	$\xi^{kj} + \xi^{-kj}$	0	0

In both cases: ξ a primitive $2n$-th root of unity and $j = 1, \ldots, n-1$.

- Binary tetrahedral group: $BT = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle,\ -id := (ab)^2$.

	id	$-id$	a	$-a$	b	$-b$	ab	
1	1	1	1	1	1	1	1	1
1$'$	1	1	ω	ω	ω^2	ω^2	1	$\mathbb{Z}/3\mathbb{Z}$
1$''$	1	1	ω^2	ω^2	ω	ω	1	$\mathbb{Z}/3\mathbb{Z}$
3	3	3	0	0	0	0	-1	A_4
2	2	-2	1	-1	1	-1	0	BT
2$'$	2	-2	ω	$-\omega$	ω^2	$-\omega^2$	0	BT
2$''$	2	-2	ω^2	$-\omega^2$	ω	$-\omega$	0	BT
	1	1	4	4	4	4	6	

ω a primitive 3rd root of unity.

- Binary octahedral group: $BO = \langle a, b \mid a^3 = b^4 = (ab)^2 \rangle,\ -id := (ab)^2$.

	id	$-id$	ab	a	$-a$	b	$-b$	b^2	
1	1	1	1	1	1	1	1	1	1
1$'$	1	1	-1	1	1	-1	-1	1	$\mathbb{Z}/2\mathbb{Z}$
2$'''$	2	2	0	-1	-1	0	0	2	S_3
3	3	3	1	0	0	-1	-1	-1	S_4
3$'$	3	3	-1	0	0	1	1	-1	S_4
2	2	-2	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0	BO
2$'$	2	-2	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$	0	BO
4	4	-4	0	-1	1	0	0	0	BO
	1	1	12	8	8	6	6	6	

- Binary icosahedral group: $BI = \langle a, b \mid a^3 = b^5 = (ab)^2 \rangle,\ -id := (ab)^2$.

	id	$-id$	a	$-a$	b	$-b$	b^2	$-b^2$	ab	
1	1	1	1	1	1	1	1	1	1	1
3	3	3	0	0	μ^+	μ^+	μ^-	μ^-	-1	A_5
3$'$	3	3	0	0	μ^-	μ^-	μ^+	μ^+	-1	A_5
4$'$	4	4	1	1	-1	-1	-1	-1	0	A_5
5	5	5	-1	-1	0	0	0	0	0	A_5
2	2	-2	1	-1	μ^+	$-\mu^+$	$-\mu^-$	μ^-	0	BI
2$'$	2	-2	1	-1	μ^-	$-\mu^-$	$-\mu^+$	μ^+	0	BI
4	4	-4	-1	1	1	-1	-1	1	0	BI
6	6	-6	0	0	-1	1	1	-1	0	BI
	1	1	20	20	12	12	12	12	30	

$\mu^+ := \frac{1}{2}(1 + \sqrt{5})$, $\mu^- := \frac{1}{2}(1 - \sqrt{5})$.

6.2 McKay correspondence for $G \subset \mathrm{SL}(2, K)$

Let G be a finite subgroup scheme of $\mathrm{SL}(2, K)$, K a field of characteristic 0, and C the algebraic closure of K. There is the geometric quotient $\pi : \mathbb{A}^2_K \to \mathbb{A}^2_K/G$ and the natural morphism $\tau : \text{G-Hilb}_K \mathbb{A}^2_K \to \mathbb{A}^2_K/G$, which is the minimal resolution of this quotient singularity.

6.2.1 Resolution of $\mathbb{A}^2_K/G$ by $\text{G-Hilb}_K \mathbb{A}^2_K$ and the intersection graph

The *G-Hilbert functor* $\underline{\text{G-Hilb}}_K \mathbb{A}^2_K : (K\text{-schemes})^\circ \to (sets)$

$$\underline{\text{G-Hilb}}_K \mathbb{A}^2_K(T) = \left\{ \begin{array}{l} \textit{Quotient G-sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{A}^2_T} \to \mathcal{O}_Z \to 0] \textit{ on } \mathbb{A}^2_T, \\ \textit{Z finite flat over } T, \textit{ the representation } H^0(\mathbb{A}^2_t, \mathcal{O}_{Z_t}) \textit{ for} \\ \textit{t} \in T \textit{ isomorphic to the regular representation} \end{array} \right\}$$

is representable by an algebraic K-scheme $\text{G-Hilb}_K \mathbb{A}^2_K$ (theorem 4.28) and the natural morphism $\tau : \text{G-Hilb}_K \mathbb{A}^2_K \to \mathbb{A}^2_K/G$ (corollary 4.16), as a map of points taking G-clusters to the corresponding orbits, is projective (theorem 4.28).

Proposition 6.11. *The G-Hilbert scheme $\text{G-Hilb}_K \mathbb{A}^2_K$ is irreducible and nonsingular. The morphism $\tau : \text{G-Hilb}_K \mathbb{A}^2_K \to \mathbb{A}^2_K/G$ is birational and the minimal resolution of $\mathbb{A}^2_K/G$.*

Proof. This is known for algebraically closed fields of characteristic 0 [ItNm96], [ItNm99], [BKR01]. From this the statements about irreducibility and nonsingularity for not necessarily algebraically closed K follow, use that $(\text{G-Hilb}_K \mathbb{A}^2_K)_C \cong G_C\text{-}\text{Hilb}_C \mathbb{A}^2_C$ (remark 4.19). The morphism $\tau : \text{G-Hilb}_K \mathbb{A}^2_K \to \mathbb{A}^2_K/G$ is known to be birational (corollary 4.32). The base extension $(\text{G-Hilb}_K \mathbb{A}^2_K)_C \to (\mathbb{A}^2_K/G)_C$ identifies with the natural morphism $G_C\text{-}\text{Hilb}_C \mathbb{A}^2_C \to \mathbb{A}^2_C/G_C$ (follows directly from the functorial definition of τ, see subsection 4.3.1). So the statement about minimality as well follows from the same statement for algebraically closed fields. $\square$

Define the exceptional divisor E by
$$E := \tau^{-1}(\overline{O})$$
where $\overline{O} = \pi(O)$, O the origin of $\mathbb{A}^2_K$. In general E is not reduced, denote by E_{red} the underlying reduced subscheme.

Definition 6.12. *The intersection graph of E_{red} is defined as follows:*
-vertices: A vertex of multiplicity n for each irreducible component $(E_{\text{red}})_i$ of E_{red} which decomposes over the algebraic closure of K into n irreducible components.
-edges: Different $(E_{\text{red}})_i$ and $(E_{\text{red}})_j$ are connected by $(E_{\text{red}})_i.(E_{\text{red}})_j$ undirected edges. $(E_{\text{red}})_i$ has $\frac{1}{2}(E_{\text{red}})_i.(E_{\text{red}})_i +$ multiplicity of $(E_{\text{red}})_i$ loops.

If K is algebraically closed, then the $(E_{\text{red}})_i$ are isomorphic to $\mathbb{P}^1_K$ and the self-intersection of each $(E_{\text{red}})_i$ is -2, because the resolution is crepant.

Let $K \to L$ be a Galois extension, $\Gamma = \text{Aut}_K(L)$. Γ operates on the intersection graph of $(E_{\text{red}})_L$ by graph automorphisms. The irreducible components $(E_{\text{red}})_i$ of E_{red} correspond to Γ-orbits of irreducible components $(E_{\text{red}})_{L,k}$ of $(E_{\text{red}})_L$ by proposition 5.10. For the intersection form one has
$$(E_{\text{red}})_i.(E_{\text{red}})_j = ((E_{\text{red}})_i)_L.((E_{\text{red}})_j)_L = \sum_{kl}(E_{\text{red}})_{L,k}.(E_{\text{red}})_{L,l}$$
where indices k and l run through the irreducible components of $((E_{\text{red}})_i)_L$ and $((E_{\text{red}})_j)_L$ respectively. Thus for the intersection graph there is a proposition similar to proposition 6.6 for representation graphs.

Proposition 6.13. *The intersection graph of E_{red} arises by identifying the elements of Γ-orbits of vertices of the intersection graph of $(E_{\mathrm{red}})_L$, adding multiplicities. The edges between vertices $(E_{\mathrm{red}})_i$ and $(E_{\mathrm{red}})_j$ are in bijection with the edges between the irreducible components of $((E_{\mathrm{red}})_i)_L$ and $((E_{\mathrm{red}})_j)_L$.* □

6.2.2   Irreducible components and irreducible representations

The basic statement of McKay correspondence is a bijection between the set of irreducible components of the exceptional divisor E and the set of isomorphism classes of nontrivial irreducible representations of the group scheme G.

Theorem 6.14. *There are bijections for intermediate fields $K \subseteq L \subseteq C$ between the set $\mathrm{Irr}(E_L)$ of irreducible components of E_L and the set $\mathrm{Irr}(G_L)$ of isomorphism classes of nontrivial irreducible representations of G_L having the property that for $K \subseteq L \subseteq L' \subseteq C$, if the bijection $\mathrm{Irr}(E_L) \to \mathrm{Irr}(G_L)$ for L maps $E_i \mapsto V_i$, then the bijection $\mathrm{Irr}(E_{L'}) \to \mathrm{Irr}(G_{L'})$ for L' maps irreducible components of $(E_i)_{L'}$ to irreducible components of $(V_i)_{L'}$.*

Proof. As described earlier, the Galois group $\Gamma = \mathrm{Aut}_L(C)$ of the Galois extension $L \to C$, operates on the sets $\mathrm{Irr}(G_C)$ and $\mathrm{Irr}(E_C)$. In both cases elements of $\mathrm{Irr}(G_L)$ and $\mathrm{Irr}(E_L)$ correspond to Γ-orbits of elements of $\mathrm{Irr}(G_C)$ and $\mathrm{Irr}(E_C)$ by corollary 5.23 and proposition 5.10 respectively. This way a given bijection between the sets $\mathrm{Irr}(G_C)$ and $\mathrm{Irr}(E_C)$ defines a bijection between $\mathrm{Irr}(G_L)$ and $\mathrm{Irr}(E_L)$ on condition that the bijection is equivariant with respect to the operations of Γ. Checking this for the bijection of McKay correspondence over the algebraically closed field C constructed via stratification or via tautological sheaves will give bijections over intermediate fields L having the property of the theorem. This will be done in the process of proving theorem 6.17 or theorem 6.20. □

Moreover, in the situation of the theorem the Galois group $\Gamma = \mathrm{Aut}_L(C)$ operates on the representation graph of G_C and on the intersection graph of $(E_{\mathrm{red}})_C$. Then in both cases the graphs over L arise by identifying the elements of Γ-orbits of vertices of the graphs over C by proposition 6.6 and 6.13. Therefore an isomorphism of the graphs over C, the bijection between the sets of vertices being Γ-equivariant, defines an isomorphism of the graphs over L.
For the algebraically closed field C this is the classical McKay correspondence for subgroups of $\mathrm{SL}(2, C)$ ([McK80], [GV83], [ItNm99]). The statement, that there is a bijection of edges between given vertices $(E_{\mathrm{red}})_{L,i} \leftrightarrow V_i$ and $(E_{\mathrm{red}})_{L,j} \leftrightarrow V_j$, can be formulated equivalently in terms of the intersection form as $(E_{\mathrm{red}})_{L,i}.(E_{\mathrm{red}})_{L,j} = \langle V_i, V_j \rangle$.

Theorem 6.15. *The bijections $E_i \leftrightarrow V_i$ of theorem 6.14 between irreducible components of E_L and isomorphism classes of nontrivial irreducible representations of G_L can be constructed such that $(E_{\mathrm{red}})_i.(E_{\mathrm{red}})_j = \langle V_i, V_j \rangle$ or equivalently that these bijections define isomorphisms of graphs between the intersection graph of $(E_{\mathrm{red}})_L$ and the representation graph of G_L.* □

We will consider two ways to construct bijections between nontrivial irreducible representations and irreducible components with the properties of theorem 6.14 and 6.15: A stratification of G-Hilb$_K \mathbb{A}^2_K$ ([ItNm96], [ItNm99], [Is02]) and the tautological sheaves on G-Hilb$_K \mathbb{A}^2_K$ ([GV83], [KaVa00], [Is02]).

6.2.3 Stratification

Let $S := K[x_1, x_2]$, let $O \in \mathbb{A}^2_K$ be the origin, $\mathfrak{m} \subset S$ the corresponding maximal ideal, $\overline{O} := \pi(O) \in \mathbb{A}^2_K/G$ with corresponding maximal ideal $\mathfrak{n} \subset S^G$, let $\overline{S} := S/\mathfrak{n}S$ with maximal ideal $\overline{\mathfrak{m}}$. An L-valued point of the fiber $E = \tau^{-1}(\overline{O})$ corresponds to a G-cluster defined by an ideal $I \subset S_L$ such that $\mathfrak{n}_L \subseteq I$ or equivalently an ideal $\overline{I} \subset \overline{S}_L = S_L/\mathfrak{n}_L S_L$. For such an ideal I define the representation $V(I)$ over L by

$$V(I) := \overline{I}/\overline{\mathfrak{m}}_L \overline{I}$$

Lemma 6.16. *For $\gamma \in \mathrm{Aut}_K(L)$: $V(\gamma^{-1}I) \cong V(I)^\gamma$.*

Proof. As an A_L-comodule $\overline{I} = \overline{I}_0 \oplus \overline{\mathfrak{m}}_L \overline{I}$, where $\overline{I}_0 \cong \overline{I}/\overline{\mathfrak{m}}_L \overline{I}$. Then $\gamma^{-1}\overline{I} = \gamma^{-1}\overline{I}_0 \oplus \overline{\mathfrak{m}}_L(\gamma^{-1}\overline{I})$ and $V(\gamma^{-1}I) = \gamma^{-1}\overline{I}/\overline{\mathfrak{m}}_L(\gamma^{-1}\overline{I}) \cong \gamma^{-1}\overline{I}_0 \cong \overline{I}_0^\gamma \cong V(I)^\gamma$ by remark 5.20 applied to $\overline{I}_0 \subseteq \overline{S}_L$. $\square$

Theorem 6.17. *There is a bijection $E_j \leftrightarrow V_j$ between the set $\mathrm{Irr}(E)$ of irreducible components of E and the set $\mathrm{Irr}(G)$ of isomorphism classes of nontrivial irreducible representations of G such that for any closed point $y \in E$: If $I \subset S_{\kappa(y)}$ is an ideal defining a $\kappa(y)$-valued point of the scheme $\{y\} \subset E$, then*

$$\mathrm{Hom}^G_{\kappa(y)}(V(I), (V_j)_{\kappa(y)}) \neq 0 \iff y \in E_j$$

and $V(I)$ is either irreducible or consists of two irreducible representations not isomorphic to each other. Applied to the situation after base extension $K \to L$, L an algebraic extension of K, one obtains bijections $\mathrm{Irr}(E_L) \leftrightarrow \mathrm{Irr}(G_L)$ having the properties of theorems 6.14 and 6.15.

Proof. In the case of algebraically closed K the theorem follows from [ItNm99] or [Is02].

In the general case denote by U_i the isomorphism classes of nontrivial irreducible representations of G_C over the algebraic closure C. Over C the theorem is valid, let $E_{C,i}$ be the component corresponding to the irreducible representation U_i of G_C.

We show that this bijection is equivariant with respect to the operations of $\Gamma = \mathrm{Aut}_K(C)$: Let $x \in E_{C,i}$ be a closed point such that $x \notin E_{C,i'}$ for $i' \neq i$. Then for the corresponding C-valued point $\alpha : \mathrm{Spec}\, C \to E_{C,i}$ given by an ideal $I \subset S_C$ one has $V(I) \cong U_i$. By corollary 5.12 the C-valued point corresponding to γx is α^γ given by the ideal $\gamma^{-1}I \subset S_C$. By lemma 6.16 $V(\gamma^{-1}I) \cong U_{\gamma(i)}$, where $U_{\gamma(i)} = U_i^\gamma$. Therefore $\gamma x \in E_{\gamma(i)}$ and $\gamma E_i = E_{\gamma(i)}$.

For an irreducible representation V_j of G define E_j to be the component of E, which decomposes over C into the irreducible components $E_{C,i}$ satisfying $U_i \subseteq (V_j)_C$. This method, applied to the situation after base extension $K \to L$, leads to bijections having the properties of theorems 6.14 and 6.15.

We show that this bijection is given by the condition in the theorem. Let y be a closed point of E and α a $\kappa(y)$-valued point of the scheme $\{y\}$ given by an ideal $I \subset S_{\kappa(y)}$. $K \to \kappa(y)$ is an algebraic extension, embed $\kappa(y)$ into C. After base extension $\kappa(y) \to C$ one has the C-valued point $\alpha_C : \mathrm{Spec}\, C \to \{y\}_C$ given by $I_C \subset S_C$. Then $V(I)_C \cong V(I_C)$ and I_C corresponds to a closed point $z \in \{y\}_C \subset E_C$. Therefore

$$
\begin{aligned}
y \in E_j &\iff z \in E_{C,i} \text{ for some } i \text{ satisfying } U_i \subseteq (V_j)_C \\
&\iff \mathrm{Hom}^G_C(V(I_C), U_i) \neq 0 \text{ for some } i \text{ satisfying } U_i \subseteq (V_j)_C \\
&\iff \mathrm{Hom}^G_{\kappa(y)}(V(I), (V_j)_{\kappa(y)}) \neq 0
\end{aligned}
$$

$\square$

6.2.4   Tautological sheaves

Let $0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{A}^2_Y} \to \mathcal{O}_Z \to 0$ be the universal quotient of $Y := \text{G-Hilb}_K \mathbb{A}^2_K$. The projection $p : Z \to Y$ is a finite flat morphism, $p_* \mathcal{O}_Z$ is a locally free G-sheaf on Y with fibers $p_* \mathcal{O}_Z \otimes_{\mathcal{O}_Y} \kappa(y)$ isomorphic to the regular representation over $\kappa(y)$.

Let $V_0, \ldots, V_s$ the isomorphism classes of irreducible representations of G, V_0 the trivial representation. The G-sheaf $\mathscr{G} := p_* \mathcal{O}_Z$ on Y decomposes into isotypic components (see corollary 3.71)

$$\mathscr{G} \cong \bigoplus_{j=0}^{s} \mathscr{G}_j$$

where $\mathscr{G}_j$ is the component for V_j. As in subsection 1.2.3 we define the tautological sheaves:

Definition 6.18. *For any isomorphism class V_j of irreducible representations of G over K define the sheaf $\mathscr{F}_j$ on $Y = \text{G-Hilb}_K \mathbb{A}^2_K$ by*

$$\mathscr{F}_j := \mathscr{H}om^G_{\mathcal{O}_Y}(V_j \otimes_K \mathcal{O}_Y, \mathscr{G}_j) = \mathscr{H}om^G_{\mathcal{O}_Y}(V_j \otimes_K \mathcal{O}_Y, \mathscr{G})$$

For a field extension $K \to L$ denote by $\mathscr{F}_{L,i}$ the sheaf $\mathscr{H}om^{G_L}_{\mathcal{O}_{Y_L}}(U_i \otimes_L \mathcal{O}_{Y_L}, \mathscr{G}_L)$ on Y_L, U_i an irreducible representation of G_L over L.

Remark 6.19.
(1) For $K = \mathbb{C}$ the sheaves $\mathscr{F}_j$ were studied in [GV83], [KaVa00] (see also subsection 1.2.3), they may be defined as well as $\mathscr{F}_j = \tau^* \mathscr{H}om^G_{\mathbb{A}^2_K/G}(V_j \otimes_K \mathcal{O}_{\mathbb{A}^2_K/G}, \pi_* \mathcal{O}_{\mathbb{A}^2_K})/(\mathcal{O}_Y\text{-}torsion)$ or $(p_* q^*(\mathcal{O}_{\mathbb{A}^2_K} \otimes_K V_j^\vee))^G$ using the canonical morphisms in the diagram

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\
Y & & \mathbb{A}^2_K \\
{\scriptstyle \tau}\searrow & & \swarrow{\scriptstyle \pi} \\
 & \mathbb{A}^2_K/G &
\end{array}
$$

(2) $\mathscr{F}_j$ is a locally free sheaf of rank $\dim_K V_j$.
(3) For each j there is the natural isomorphism of G-sheaves (see proposition 3.74)

$$\mathscr{F}_j \otimes_{\text{End}^G_K(V_j)} V_j \xrightarrow{\sim} \mathscr{G}_j$$

Let $K \to L$ be a Galois extension and $U_0, \ldots, U_r$ be the isomorphism classes of irreducible representations of G_L over L. Then a decomposition $(V_j)_L = \bigoplus_{i \in I_j} U_i$ over L of an irreducible representation V_j of G over K gives a decomposition of the corresponding tautological sheaf

$$
\begin{aligned}
(\mathscr{F}_j)_L &= \mathscr{H}om^G_{\mathcal{O}_Y}(V_j \otimes_K \mathcal{O}_Y, \mathscr{G})_L \\
&= \mathscr{H}om^{G_L}_{\mathcal{O}_{Y_L}}((V_j \otimes_K \mathcal{O}_Y)_L, \mathscr{G}_L) \\
&= \mathscr{H}om^{G_L}_{\mathcal{O}_{Y_L}}(\bigoplus_{i \in I_j} U_i \otimes_L \mathcal{O}_{Y_L}, \mathscr{G}_L) \\
&= \bigoplus_{i \in I_j} \mathscr{H}om^{G_L}_{\mathcal{O}_{Y_L}}(U_i \otimes_L \mathcal{O}_{Y_L}, \mathscr{G}_L) \\
&= \bigoplus_{i \in I_j} \mathscr{F}_{L,i}
\end{aligned}
$$

We have used the fact that the U_i occur with multiplicity 1 as it is the case for finite subgroup schemes of $\text{SL}(2, K)$, see proposition 6.5.

The tautological sheaves $\mathscr{F}_j$ can be used to establish a bijection between the set of irreducible components of E_{red} and the set of isomorphism classes of nontrivial irreducible representations of G by considering intersections $\mathscr{L}_j.(E_{\mathrm{red}})_{j'}$, i.e. the degrees of restrictions of the line bundles $\mathscr{L}_j := \bigwedge^{\mathrm{rk}\,\mathscr{F}_j} \mathscr{F}_j$ to the curves $(E_{\mathrm{red}})_{j'}$.

Theorem 6.20. *There is a bijection $E_j \leftrightarrow V_j$ between the set $\mathrm{Irr}(E)$ of irreducible components of E and the set $\mathrm{Irr}(G)$ of isomorphism classes of nontrivial irreducible representations of G such that*

$$\mathscr{L}_j.(E_{\mathrm{red}})_{j'} = \dim_K \mathrm{Hom}_K^G(V_j, V_{j'})$$

where $\mathscr{L}_j = \bigwedge^{\mathrm{rk}\,\mathscr{F}_j} \mathscr{F}_j$.
Applied to the situation after base extension $K \to L$, L an algebraic extension field of K, one obtains bijections $\mathrm{Irr}(E_L) \leftrightarrow \mathrm{Irr}(G_L)$ having the properties of theorems 6.14 and 6.15.

Proof. In the case of algebraically closed K the theorem follows from [GV83].

In the general case denote by $U_0, \ldots, U_r$ the isomorphism classes of irreducible representations of G_C over the algebraic closure C, U_0 the trivial one. Over C the theorem is valid, let $E_{C,i}$ be the component corresponding to the irreducible representation U_i of G_C, what means that $\mathscr{L}_{C,i}.(E_{\mathrm{red}})_{C,i'} = \delta_{ii'}$, where $\mathscr{L}_{C,i} = \bigwedge^{\mathrm{rk}\,\mathscr{F}_{C,i}} \mathscr{F}_{C,i}$.

To show that the bijection over C is equivariant with respect to the operations of $\Gamma = \mathrm{Aut}_K(C)$, one has to show that $\gamma_*\mathscr{L}_{C,i} \cong \mathscr{L}_{C,\gamma(i)}$, where $U_{\gamma(i)} = U_i^\gamma$. Then $\mathscr{L}_{C,i}.E_{C,i'} = \gamma_*\mathscr{L}_{C,i}.\gamma E_{C,i'} = \mathscr{L}_{C,\gamma(i)}.\gamma E_{C,i'}$ and therefore $\gamma E_{C,i'} = E_{C,\gamma(i')}$. It is $\gamma_*\mathscr{L}_{C,i} \cong \mathscr{L}_{C,\gamma(i)}$, because using corollary 5.16 and remark 5.17

$$
\begin{aligned}
\gamma_*\mathscr{F}_{C,i} &= \gamma_* \mathscr{H}om_{\mathcal{O}_{Y_C}}^{G_C}(U_i \otimes_C \mathcal{O}_{Y_C}, \mathscr{G}_C) \\
&\cong \mathscr{H}om_{\mathcal{O}_{Y_C}}^{G_C}(\gamma_*(U_i \otimes_C \mathcal{O}_{Y_C}), \gamma_*\mathscr{G}_C) \\
&\cong \mathscr{H}om_{\mathcal{O}_{Y_C}}^{G_C}(U_i^\gamma \otimes_C \mathcal{O}_{Y_C}, \mathscr{G}_C) \\
&= \mathscr{F}_{C,\gamma(i)}
\end{aligned}
$$

Since the bijection over C is equivariant with respect to the Γ-operations on $\mathrm{Irr}(G_C)$ and $\mathrm{Irr}(E_C)$, one can define a bijection $\mathrm{Irr}(G) \leftrightarrow \mathrm{Irr}(E)$: For $V_j \in \mathrm{Irr}(G)$ let E_j be the element of $\mathrm{Irr}(E)$ such that $(V_j)_C = \bigoplus_{i \in I_j} U_i$ and $(E_j)_C = \bigcup_{i \in I_j} E_{C,i}$ for the same subset $I_j \subseteq \{1, \ldots, r\}$. This method applied to the situation after base extension $K \to L$ leads to bijections having the properties of theorems 6.14 and 6.15.

We show that this bijection is given by the construction of the theorem. It is $(\mathscr{F}_j)_C = \bigoplus_{i \in I_j} \mathscr{F}_{C,i}$ and therefore

$$
\begin{aligned}
\mathscr{L}_j.(E_{\mathrm{red}})_{j'} &= (\mathscr{L}_j)_C.((E_{\mathrm{red}})_{j'})_C \\
&= \big(\textstyle\bigotimes_{i \in I_j} \mathscr{L}_{C,i}\big).\big(\textstyle\sum_{i' \in I_{j'}} (E_{\mathrm{red}})_{C,i'}\big) \\
&= \textstyle\sum_{i,i'} \mathscr{L}_{C,i}.(E_{\mathrm{red}})_{C,i'} \\
&= \textstyle\sum_{i,i'} \dim_C \mathrm{Hom}_C^{G_C}(U_i, U_{i'}) \\
&= \dim_C \mathrm{Hom}_C^{G_C}((V_j)_C, (V_{j'})_C) \\
&= \dim_K \mathrm{Hom}_K^G(V_j, V_{j'})
\end{aligned}
$$

$\square$

6.3   Examples

6.3.1   Abelian subgroup schemes and the toric resolution

Let C be a field of characteristic 0 containing a primitive n-th root of unity ξ. The cyclic group G of order n operates on $\mathbb{A}_C^2$ as a subgroup of the group $\mathrm{SL}(2, C)$ by the representation

$$G \to \mathrm{SL}(2, C), \qquad \overline{1} \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

The minimal resolution $\widetilde{X}$ of $X = \mathbb{A}_C^2/G$ can be constructed as a toric variety:
$\mathbb{A}_C^2 = \mathrm{Spec}\, C[x_1, x_2]$ is given by the cone $\sigma = \mathbb{Q}_{\geq 0}(1, 0) + \mathbb{Q}_{\geq 0}(0, 1) \subset \mathbb{Q}^2$ with respect to the lattice $N = \mathbb{Z}(1, 0) + \mathbb{Z}(0, 1) \subset \mathbb{Q}^2$. The inclusion $C[x_1^{\pm 1}, x_2^{\pm 1}]^G = C[x_1^{\pm n}, (x_1 x_2)^{\pm 1}] \subseteq C[x_1^{\pm 1}, x_2^{\pm 1}]$ corresponds to a sublattice $M' \subseteq M$ of $M = N^\vee$, $X = \mathbb{A}_C^2/G = \mathrm{Spec}\, C[x_1, x_2]^G = \mathrm{Spec}\, C[\sigma^\vee \cap M']$ is given by the cone σ with respect to the finer lattice $N' = \mathbb{Z}(1, 0) + \mathbb{Z}\frac{1}{n}(n - 1, 1) \subset \mathbb{Q}^2$ and the minimal resolution $\widetilde{X}$ by the fan with respect to N' consisting of the n maximal cones $\sigma_i = \mathbb{Q}_{\geq 0}(n - i, i) + \mathbb{Q}_{\geq 0}(n - i - 1, i + 1)$, $i \in \{0, \ldots, n - 1\}$. The exceptional divisors correspond to the additional 1-dimensional cones $\tau_i = \mathbb{Q}_{\geq 0}(n - i, i)$, $i \in \{1, n - 1\}$ and form the graph (A_{n-1}).
$\widetilde{X}$ is covered by the affine spaces

$$U_i = \mathrm{Spec}\, C[\sigma_i^\vee \cap M'] = \mathrm{Spec}\, C[s_i, t_i], \qquad i = 0, \ldots, n - 1$$

where

$$s_i = \frac{x_1^{i+1}}{x_2^{n-(i+1)}}, \quad t_i = \frac{x_2^{n-i}}{x_1^i}$$

The isomorphism from the toric resolution $\widetilde{X}$ to the G-Hilbert scheme corresponds to a quotient $0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{A}_{\widetilde{X}}^2} \to \mathcal{O}_Z \to 0$. It is given by the following flat families of G-clusters over the U_i (see [ItNm99, Thm. 12.3] for this and [Nm01] for the general case, in this thesis subsection 2.2.1 and example 2.27):

$$I_i = \left\langle x^{i+1} - s_i y^{n-(i+1)}, xy - s_i t_i, y^{n-i} - t_i x^i \right\rangle \subset C[s_i, t_i][x, y]$$

There are $n - 1$ 1-dimensional cones $\tau_i = \mathbb{Q}_{\geq 0}(n - i, i) = \sigma_{i-1} \cap \sigma_i$, $i \in \{1, \ldots, n - 1\}$ corresponding to the exceptional divisors $E_1, \ldots, E_{n-1}$. $E_i \cong \mathbb{P}_C^1$ is given by the equation $t_{i-1} = 0$ in the affine chart U_{i-1}, by $s_i = 0$ in U_i and is parametrised by $(a = x_1^i : b = x_2^{n-i})$.
The restriction of $\mathscr{I}$ to E_i is given by the restrictions of I_{i-1} and I_i to $\{t_{i-1} = 0\}$ and $\{s_i = 0\}$ respectively and thus by

$$I_{(a,b)} = \left\langle x^i - \tfrac{a}{b} y^{n-i}, xy, y^{n-(i-1)}, x^{i+1} \right\rangle \quad b \neq 0$$
$$I_{(a,b)} = \left\langle \tfrac{b}{a} x^i - y^{n-i}, xy, y^{n-(i-1)}, x^{i+1} \right\rangle \quad a \neq 0$$

The restriction of the summand $\mathscr{F}_j \otimes_C V_j$ of $p_* \mathcal{O}_Z$ corresponding to the irreducible representation V_j that occurs on the 1-dimensional subspace $\langle x_1^j \rangle_C$ or $\langle x_2^{n-j} \rangle_C \subset C[x_1, x_2]$ to E_i is isomorphic to

$$\begin{aligned} j \leq i - 1 : \quad & x^j \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i} \otimes_C V_j \\ j = i : \quad & (x^j \mathcal{O}_{E_i}(1) \oplus y^{n-j} \mathcal{O}_{E_i}(1))/\langle bx^j - ay^{n-j} \rangle \cong \mathcal{O}_{E_i}(1) \otimes_C V_j \\ j \geq i + 1 : \quad & y^{n-j} \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i} \otimes_C V_j \end{aligned}$$

It follows that the bijection between nontrivial irreducible representations of G and irreducible components of the exceptional divisor described in section 5.3 by a stratification and in section 5.4 by the tautological sheaves maps $V_i \longleftrightarrow E_i$.

Now we will consider the situation over a field that does not contain a primitive n-th root of unity. Let $K = \mathbb{Q}(\xi + \xi^{-1})$ and $C = \mathbb{Q}(\xi)$, ξ a primitive n-th root of unity. $K \to C$ is a Galois extension with Galois group $\Gamma := \mathrm{Aut}_K(C) = \{id, \gamma\}$, where $\gamma : \xi \mapsto \xi^{-1}$.

- Consider first the finite subgroup scheme $G = \mathrm{Spec}\, K[x]/\langle x^n - 1 \rangle \subset \mathrm{SL}(2, K)$ such that the map of C-valued points determined by the inclusion $G \to \mathrm{SL}(2, K)$ is the representation

$$G(C) \cong \mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(2, K)(C), \qquad \overline{1} \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

For $n \geq 3$ there exist closed points of this group scheme which are not K-rational. The closed points of G correspond to the sets of C-valued points $\{\overline{0}\}$, $\{\overline{1}, \overline{n-1}\}$, ..., these are the Γ-orbits in $G(C)$, Γ operating by $\gamma : \alpha \mapsto \alpha^\gamma$.

The coalgebra structure of $A = K[x]/\langle x^n - 1 \rangle$ is given by $x^i \mapsto x^i \otimes x^i$, A decomposes into simple subcoalgebras $A = \bigoplus_{i=0}^{n-1} Kx^i$, the irreducible representations are the characters $1, x, \ldots, x^{n-1}$. We assume that the representation corresponding to x occurs on $\langle x_1 \rangle_K \subset K[x_1, x_2]$.

Again one may construct the toric resolution, again the representation x^i corresponds to the divisor E_i and one has the graph (A_{n-1}).

- Secondly consider the finite subgroup scheme $G \subset \mathrm{SL}(2, K)$ with each of its closed points K-rational, the inclusion given by the representation

$$G(K) \cong \mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(2, K)(K), \qquad \overline{1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & \xi + \xi^{-1} \end{pmatrix}$$

After base extension $K \to C$ it is possible to construct the toric resolution: Choose a C-basis of $\langle x_1, x_2 \rangle_C$ such that the operation is diagonal, e.g. $x_1' = x_1 - \xi x_2$, $x_2' = x_1 - \xi^{-1} x_2$, and proceed as above.

It remains for $n \geq 3$ a nontrivial representation of Γ on the K-subspace $\langle x_1', x_2' \rangle_K \subset \langle x_1, x_2 \rangle_C$: With respect to the basis x_1', x_2' it is given by

$$\gamma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This way Γ operates on $C[x_1', x_2']$, it permutes the E_i and the U_i, one has $\gamma : C[s_i, t_i] \to C[s_{n-i-1}, t_{n-i-1}]$, $s_i \mapsto t_{n-i-1}$, $t_i \mapsto s_{n-i-1}$. This can be translated into an operation on the cone σ and the fan of the minimal resolution in $\mathbb{Q}^2$, γ interchanges the base vectors $(1, 0)$, $(0, 1)$ of $\mathbb{Q}^2$.

For $n \geq 3$ the Γ operates nontrivially on the set of irreducible representations of G_C over C by $V_i \mapsto V_i^\gamma = V_{n-i}$ (the operation on the closed points of G_C is trivial, the conjugate representation arises by application of γ to the entries of the matrix, here $\xi^i \mapsto \xi^{n-i}$) and on the set of irreducible components of the exceptional divisor by $\gamma : E_i \mapsto E_{n-i}$, one obtains the diagram $(A_{n-1})'$.

6.3.2 Finite subgroups of $\mathrm{SL}(2, K)$

In the case of subgroups $G \subset \mathrm{SL}(2, K)$ the representation graph can be read off from the table of characters of the group G over an algebraically closed field, since in this case representations are conjugate if and only if the values of their characters are. We have the following graphs for the finite subgroups of $\mathrm{SL}(2, K)$ (use theorem 6.10):

- Cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geq 1$: It is $\xi + \xi^{-1} \in K$, ξ a primitive n-th root of unity. (A_{n-1}) if $\xi \in K$, otherwise $(A_{n-1})'$.

- Binary dihedral group BD_n, $n \geq 2$: It is $c = \frac{1}{2}(\xi + \xi^{-1}) \in K$, ξ a primitive $2n$-th root of unity, and $((-1, c^2 - 1))_K = 1$.
(D_{n+2}) if n even or $\sqrt{-1} \in K$, otherwise $(D_{n+2})'$.

- Binary tetrahedral group BT: It is $((-1, -1))_K = 1$.
(E_6) if K contains a primitive 3rd root of unity, otherwise $(E_6)'$.

- Binary octahedral group BO: It is $((-1, -1))_K = 1$ and $\sqrt{2} \in K$.
(E_7)

- Binary icosahedral group BI: It is $((-1, -1))_K = 1$ and $\sqrt{5} \in K$.
(E_8)

Examples for the graphs $(A_n)'$, $(D_{2m+1})'$, $(E_6)'$:

$(A_n)'$:	$\mathbb{Z}/(n+1)\mathbb{Z}$ over $\mathbb{Q}(\xi + \xi^{-1})$, ξ a primitive $(n+1)$-th root of unity
$(D_{2m+1})'$:	BD_{2m-1} over $\mathbb{Q}(\xi)$, ξ a primitve $2(2m-1)$-th root of unity
$(E_6)'$:	BT over $\mathbb{Q}(\sqrt{-1})$

6.3.3   The graph $(D_{2m})'$

Let $n \geq 2$, ε a primitive $4n$-th root of unity and $\xi = \varepsilon^2$. Put $K = \mathbb{Q}(\varepsilon + \varepsilon^{-1})$, $C = \mathbb{Q}(\varepsilon)$ and $\Gamma = \mathrm{Aut}_K(C) = \{id, \gamma\}$.

One has the injective representation of $BD_n = \langle \sigma, \tau \mid \tau^2 = \sigma^n = (\tau\sigma)^2 \rangle$ in $\mathrm{SL}(2, C)$:

$$\sigma \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & 0 \end{pmatrix}$$

We will identify BD_n with its image in $\mathrm{SL}(2, C)$ and regard it as a subgroup scheme of $\mathrm{SL}(2, C)$.

Γ operates on $\mathrm{SL}(2, C)$, the K-automorphism $\gamma \in \Gamma$, $\gamma : \varepsilon \mapsto \varepsilon^{-1}$ of order 2 operates nontrivially on the closed points of BD_n:

$$\gamma : \quad \begin{aligned} \sigma &= \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} & \mapsto \quad \sigma^{-1} &= \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} \\ \tau &= \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon^{-1} & 0 \end{pmatrix} & \mapsto \quad \tau\sigma &= \begin{pmatrix} 0 & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix} \end{aligned}$$

The subgroup scheme $BD_n \subset \mathrm{SL}(2, C)$ is defined over K, let $G \subset \mathrm{SL}(2, K)$ such that $G_C = BD_n$. The closed points of G correspond to Γ-orbits of closed points of BD_n, they have the form $\{id\}$, $\{-id\}$, $\{\sigma^k, \sigma^{-k}\}$, $\{\tau\sigma^k, \tau\sigma^{-k+1}\}$.

γ operates on the characters of BD_n as follows:

n even			*n odd*		
$\mathbf{1}$	$\mapsto$	$\mathbf{1}$	$\mathbf{1}$	$\mapsto$	$\mathbf{1}$
$\mathbf{1}'$	$\mapsto$	$\mathbf{1}'$	$\mathbf{1}'$	$\mapsto$	$\mathbf{1}'$
$\mathbf{1}''$	$\mapsto$	$\mathbf{1}'''$	$\mathbf{1}''$	$\mapsto$	$\mathbf{1}''$
$\mathbf{1}'''$	$\mapsto$	$\mathbf{1}''$	$\mathbf{1}'''$	$\mapsto$	$\mathbf{1}'''$
$\mathbf{2}^j$	$\mapsto$	$\mathbf{2}^j$	$\mathbf{2}^j$	$\mapsto$	$\mathbf{2}^j$

One has the graph $(D_{n+2})'$ for n even and the graph (D_{n+2}) for n odd.

Bibliography

[Abe]   Eiichi Abe, *Hopf Algebras*, Cambridge University Press, 1980.

[AlKl]   A. Altman, S. Kleiman, *Introduction to Grothendieck Duality Theory*, Springer Lecture Notes 146, 1970.

[AlKl80]   A. Altman, S. Kleiman, *Compactifying the Picard Scheme*, Adv. in Math. 35 (1980), 50–112.

[Ar66]   M. Artin, *On isolated rational singularities of surfaces*, Am. J. Math. 88, 129–136 (1966).

[BH]   W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, 1993.

[BKR01]   T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories.* J. Amer. Math. Soc. 14 (2001), 535–554, `arXiv:math.AG/9908027`.

[Bl06a]   M. Blume, *McKay correspondence over non algebraically closed fields*, `arXiv:math.AG/0601550`.

[Bl06b]   M. Blume, *Construction of G-Hilbert schemes*, `arXiv:math.AG/0607577`.

[Bo06]   S. Boissiére, *On the McKay correspondences for the Hilbert scheme of points on the affine plane*, Math. Ann. 334 (2006), 419–438, `arXiv:math.AG/0410281`.

[BoSa05]   S. Boissiére, A. Sarti, *Contraction of excess fibres between the McKay correspondences in dimensions two and three*, `arXiv:math.AG/0504360`.

[Bour]   N. Bourbaki, *Éléments de mathématique / Elements of mathematics*, *Algebra II (Ch. 4-7), Algèbre Ch. 8, Commutative Algebra (Ch. 1-7), Groupes et algèbres de Lie (Ch. 4-6).*

[ChRo01]   Li Chiang, S. Roan, *Orbifolds and finite group representations*, Int. J. Math. Math. Sci. 26 (2001), no. 11, 649–669, `arXiv:math.AG/0007072`.

[ChRo03]   Li Chiang, S. Roan, *Crepant resolutions of $\mathbb{C}^n/A_1(n)$ and flops of n-folds for $n = 4, 5$*, Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), 27–41, Fields Inst. Commun. 38, Amer. Math. Soc., Providence, RI, 2003, `arXiv:math.AG/0208052`.

[ChRo04]   Li Chiang, S. Roan, *On hypersurface quotient singularities of dimension 4*, Int. J. Math. Math. Sci. 2004, no. 45-48, 2547–2581, `arXiv:math.AG/0011151`.

[CMT05]   A. Craw, D. Maclagan, R. Thomas, *Moduli of McKay quiver representations I: the coherent component*, `arXiv:math.AG/0505115`.

[CMT06]   A. Craw, D. Maclagan, R. Thomas, *Moduli of McKay quiver representations II: Groebner basis techniques*, `arXiv:math.AG/0611840`.

[Co76]   A. M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) 9 (1976), 379–436.

[CR]   C. Curtis, I. Reiner, *Methods of representation theory*, Volume I, II, Wiley-Interscience, 1981, 1987.

[Cr01]   A. Craw, *The McKay correspondence and representations of the McKay quiver*, Ph.D. thesis, University of Warwick, 2001.

[Cr05]   A. Craw, *An explicit construction of the McKay correspondence for A-Hilb $\mathbb{C}^3$*, J. Algebra 285, 682-705 (2005), `arXiv:math.AG/0010053`.

[CrRe02]   A. Craw, M. Reid, *How to calculate A-Hilb $\mathbb{C}^3$*, Soc. Math. France, Sémin. Congr. 6, 129-154 (2002), `arXiv:math.AG/9909085`.

[Da]   V. I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys 33:2, 97-154, (1978).

[DG]   M. Demazure, P. Gabriel, *Groupes Algébriques*, North-Holland Publishing Co., Amsterdam, 1970.

[EGA]   A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Publ. Math. IHES 4,8,11,17,20,24,28,32, 1960-1967.

[EGA1]   A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Springer, 1971.

[EH]   D. Eisenbud, J. Harris, *The Geometry of Schemes*, Springer GTM 197, 2000.

[FH]   W. Fulton, J. Harris, *Representation Theory*, Springer GTM 129, 1991.

[FL]   W. Fulton, S. Lang, *Riemann-Roch Algebra*, Springer Grundlehren 277, 1985.

[Fu]   W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies 131, Princeton University Press, 1993.

[GM]   S. I. Gelfand, Y. I. Manin, *Methods of Homological Algebra*, Springer 1997, 2003.

[GNS00]   Y. Gomi, I. Nakamura, K.-I. Shinoda, *Hilbert schemes of G-orbits in dimension three*, Asian J. Math. 4 (2000), 51–70.

[GNS04]   Y. Gomi, I. Nakamura, K.-I. Shinoda, *Coinvariant algebras of finite subgroups of* SL(3, $\mathbb{C}$), Can. J. Math. 56 (2004), 495–528.

[Gr61]   A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki 1960/61, No. 221.

[GV83]   G. Gonzalez-Sprinberg, J.-L. Verdier, *Construction géométrique de la correspondance de McKay*, Ann. Sci. École Norm. Sup. 16 (1983), 409–449.

[Ha, AG]   R. Hartshorne, *Algebraic Geometry*, Springer GTM 52, 1977.

[Ha, RD]   R. Hartshorne, *Residues and Duality*, Springer Lecture Notes 20, 1966.

[HL]   D. Huybrechts, M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Vieweg, Aspects of Mathematics Vol. E31, 1997.

[Is02]   A. Ishii, *On the McKay correspondence for a finite small subgroup of* $\mathrm{GL}(2, \mathbb{C})$, J. Reine Angew. Math. 549 (2002), 221–233.

[It01]   Y. Ito, *Special McKay correspondence*, `arXiv:math.AG/0111314`.

[ItNj00]   Y. Ito, H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Topology 39 (2000), 1155–1191, `arXiv:math.AG/9803120`.

[ItNm96]   Y. Ito, I. Nakamura, *McKay correspondence and Hilbert schemes*, Proc. Japan Acad. 72 Ser. A (1996), 135–138.

[ItNm99]   Y. Ito, I. Nakamura, *Hilbert schemes and simple singularities*, in *New trends in algebraic geometry* (Warwick 1996), 151–233, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[Ka]   C. Kassel, *Quantum Groups*, Springer GTM 155, 1995.

[KaVa00]   M. Kapranov, E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. 316 (2000), 565–576, `arXiv:math.AG/9812016`.

[Ke04]   O. Kedzierski, *Cohomology of the G-Hilbert scheme for* $\frac{1}{r}(1, 1, r-1)$, Serdica Math. J. 30 (2004), 293–302.

[Ko]   J. Kollár, *Rational Curves on Algebraic Varieties*, Springer, 1996.

[Li69]   J. Lipman, *Rational Singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. IHES 36 (1969), 195–279.

[Ma, CA]   H. Matsumura, *Commutative Algebra*, Benjamin, 1970.

[Ma, CRT]   H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics 8, Cambridge University Press, 1986.

[Man]   Y. I. Manin, *Lectures on the K-functor in Algebraic Geometry*, Russian Mathematical Surveys 24 (1969), 1–89.

[Mat]   K. Matsuki, *Introduction to the Mori Program*, Springer, 2002.

[McK80]   J. McKay, *Graphs, singularities and finite groups*, in *The Santa Cruz Conference on Finite Groups* (Santa Cruz 1979), Proc. Sympos. Pure Math. 37 (1980), 183–186.

[McL]   S. Mac Lane, *Categories for the Working Mathematician*, Springer GTM 5, 1971, 1978.

[Mi]   J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[Mu, AV]   D. Mumford, *Abelian Varieties*, Oxford University Press, 1974.

[Mu, CS]   D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, No. 59, Princeton University Press, 1974.

[Mu, GIT]   D. Mumford, *Geometric Invariant Theory*, Springer, 1965.

[Ni05]   N. Nitsure, *Construction of Hilbert and Quot schemes*, in *Fundamental Algebraic Geometry*, AMS Mathematical Surveys and Monographs Vol. 123, 2005, `arXiv:math.AG/0504590`.

[Nm01]   I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Alg. Geom. 10 (2001), 757–779.

[Oda]   T. Oda, *Convex Bodies and Algebraic Geometry*, Springer, 1988.

[Re80]   M. Reid, *Canonical 3-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, 273–310, (1980).

[Re87]   M. Reid, *Young Person's Guide to Canonical Singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math. 46, Amer. Math. Soc., 1987.

[Re97]   M. Reid, *McKay correspondence*, Proc. of algebraic geometry symposium (Kinosaki, Nov 1996), 14–41, `arXiv:alg-geom/9702016`.

[Re99]   M. Reid, *La correspondance de McKay*, Séminaire Bourbaki, Volume 1999/2000, Exposés 865-879, Société Mathématique de France, Astérisque 276, 53–72 (2002), `arXiv:math.AG/9911165`.

[Ro96]   S. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*, Topology 35, 489–508, 1996.

[Se, CA]   J.-P. Serre, *A Course in Arithmetic*, Springer GTM 7, 1973.

[Se, LR]   J.-P. Serre, *Linear Representations of Finite Groups*, Springer GTM 42, 1977.

[Se04]   M. Sebestean, *A smooth four-dimensional G-Hilbert scheme*, Serdica Math. J. 30 (2004), 283–292.

[Se05]   M. Sebestean, *Correspondance de McKay et équivalences dérivées*, Ph.D. thesis, Université Paris 7, 2005.

[SGA1]   A. Grothendieck et al., *Séminaire de Géométrie Algébrique 1960-61, Revêtements étales et Groupe Fondemental*, Springer Lecture Notes 224, 1971, `arXiv:math.AG/0206203`.

[SGA4]   A. Grothendieck et al., *Séminaire de Géométrie Algébrique 1963-64, Théorie des Topos et Cohomologie Étale de Schémas*, Springer Lecture Notes 269, 270, 305, 1972-73.

[SGA6]   A. Grothendieck et al., *Séminaire de Géométrie Algébrique 1966-67, Théorie des Intersections et Théorème de Riemann-Roch*, Springer Lecture Notes 225, 1971.

[Sl]   P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Springer Lecture Notes 815, 1980.

[Sw]   M. E. Sweedler, *Hopf Algebras*, Benjamin, 1969.

[Té04]   S. Térouanne, *Correspondance de McKay: variations en dimension trois*, Ph.D. thesis, Institut Fourier (Grenoble), 2004.

[Té06]   S. Térouanne, *Résolutions de singularités non-abéliennes en dimension trois*, C. R. Math. Acad. Sci. Paris 342 (2006), 329–332.

[Vi05]   A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, in *Fundamental Algebraic Geometry*, AMS Mathematical Surveys and Monographs Vol. 123, 2005, `arXiv:math.AG/0412512`.

[Vw77]   E. Viehweg, *Rational singularities of higher dimensional schemes*, Proc. Amer. Math. Soc. 63 (1977), 6–8.

[Wa]   W. Waterhouse, *Introduction to Affine Group Schemes*, Springer GTM 66, 1979.

Anhang A

Zusammenfassung in deutscher Sprache

Die Beobachtung von McKay, publiziert 1980, setzt exzeptionelle Kurven in der minimalen Auflösung von Quotientensingularitäten $\mathbb{A}_{\mathbb{C}}^2/G$ für endliche Untergruppen $G \subset \mathrm{SL}(2, \mathbb{C})$ in Beziehung zu der Darstellungstheorie der Gruppe G:

Beobachtung A.1. (*"Klassische McKay Korrespondenz"*, [McK80]).
Es gibt eine Bijektion zwischen der Menge der irreduziblen Komponenten des exzeptionellen Divisors E und der Menge der Isomorphieklassen nichttrivialer irreduzibler Darstellungen der Gruppe G und darüber hinaus einen Isomorphismus von Graphen zwischen dem Schnittgraphen von Komponenten von E_{red} und dem Darstellungsgraphen von G, wobei beides Graphen vom Typ ADE sind.

Dabei enthalten der Schnitt- bzw. Darstellungsgraph Informationen über die Konfiguration der exzeptionellen Kurven bzw. die Zerlegung von Tensorprodukten irreduzibler Darstellungen mit der gegebenen 2-dimensionalen Darstellung.

In der Folge wurden, um diese Beobachtung zu erklären, verschiedene Ansätze betrachtet, zudem durchlief dieses Thema zahlreiche Variationen und erfuhr beträchtliche Erweiterungen. Wir versuchen, die grundlegende Idee zu formulieren, ausführlichere Darstellungen dieses Forschungsgebiets findet man in [Re97], [Re99].

Sei G eine endliche Gruppe von Automorphismen einer glatten Varietät M über $\mathbb{C}$, etwa $M = \mathbb{A}_{\mathbb{C}}^n$ mit einer linearen Operation einer endlichen Untergruppe $G \subset \mathrm{SL}(n, \mathbb{C})$. Üblicherweise wird der Quotient M/G singulär sein und man betrachtet Singularitätenauflösungen $Y \to M/G$ mit gewissen Minimalitätseigenschaften (in Dimension 2 gibt eine bis auf Isomorphie eindeutige minimale Auflösung, in höheren Dimensionen hat man den Begriff einer krepanten Auflösung). Die McKay Korrespondenz im allgemeinen beschreibt die Auflösung Y mittels der Darstellungstheorie der Gruppe G, folgendes Prinzip wurde von Reid formuliert:

Prinzip A.2. ([Re99, Principle 1.1]). *Die Antwort auf jede wohlgestellte Frage über die Geometrie von Y ist die G-äquivariante Geometrie von M.*

Realisierungen dieses Prinzips sind ein Isomorphismus $K(Y) \cong K^G(M)$ zwischen der K-Theory von Y und der G-äquivarianten K-Theorie von M ("K-theoretische McKay Korrespondenz") oder eine Äquivalenz $D(Y) \simeq D^G(M)$ zwischen der derivierten Kategorie von Y und der G-äquivarianten derivierten Kategorie von M ("derivierte McKay Korrespondenz").

Eine Methode, Auflösungen von Quotientensingularitäten zu konstruieren, ist das G-Hilbertschema G-Hilb$_{\mathbb{C}}$ M. Es parametrisiert G-Cluster, dies sind G-stabile endliche abgeschlossene Unterschemata $Z \subseteq M$, deren Koordinatenring als Darstellung über $\mathbb{C}$ isomorph zu der regulären Darstellung von G ist. Ein freier G-Orbit ist ein G-Cluster, im Fall nichttrivialer Stabilisatorgruppe kann es jedoch viele nichtreduzierte G-Cluster mit demselben Orbit als Träger geben, zum Beipiel besteht für $G \subset \mathrm{SL}(2, \mathbb{C})$ der exzeptionelle Divisor $E \subset$ G-Hilb$_{\mathbb{C}}$ $\mathbb{A}^2_{\mathbb{C}}$ aus G-Clustern mit dem Ursprung als Träger.

Für $G \subset \mathrm{SL}(2, \mathbb{C})$ ist das G-Hilbertschema G-Hilb$_{\mathbb{C}}$ $\mathbb{A}^2_{\mathbb{C}}$ die minimale Auflösung der Quotientensingularität $\mathbb{A}^2_{\mathbb{C}}/G$. Ähnliche Aussagen gelten in Dimension 3, insbesondere ist für endliche Untergruppen $G \subset \mathrm{SL}(3, \mathbb{C})$ das G-Hilbertschema irreduzibel, nichtsingulär und G-Hilb$_{\mathbb{C}}$ $\mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}/G$ eine krepante Auflösung, all das gilt aber nicht mehr in höheren Dimensionen.

————————————

Die wesentlichen neuen Resultate dieser Doktorarbeit sind Verallgemeinerungen der McKay Korrespondenz sowie Erweiterungen und Verbesserungen bei der Konstruktion von G-Hilbertschemata. Diese Resultate sind enthalten in den Arbeiten [Bl06a], [Bl06b].

Zum ersten Mal betrachten wir McKay Korrespondenz über nicht notwendig algebraisch abgeschlossenen Grundkörpern und für endliche Gruppenschemata statt lediglich für endliche Gruppen. Sei $G \subset \mathrm{SL}(2, K)$ ein endliches Untergruppenschema über einem Körper K der Charakteristik 0. Über nicht algebraisch abgeschlossenem K kann es sowohl Darstellungen von G als auch Komponenten des exzeptionellen Divisors E geben, die irreduzibel über K sind, aber über dem algebraischen Abschluß zerfallen. Wir werden sehen, daß diese beiden Arten, bei Erweiterung des Grundkörpers zu zerfallen, miteinander in Beziehung stehen, und werden eine McKay Korrespondenz, die nichttriviale irreduzible Darstellungen mit exzeptionellen Primdivisoren in Beziehung setzt, für beliebige Körper K der Charakteristik 0 formulieren. Insbesondere ist die Schema-Struktur des Gruppenschemas G verknüpft mit der Schema-Struktur der exzeptionellen Faser E. Folgendes wird Konsequenz detaillierterer Theoreme in Kapitel 6 sein:

Theorem A.3. *Sei K ein Körper der Charakteristik 0 und $G \subset \mathrm{SL}(2, K)$ ein endliches Untergruppenschema. Dann gibt es eine Bijektion zwischen der Menge irreduzibler Komponenten des exzeptionellen Divisors E und der Menge der Isomorphieklassen nichttrivialer irreduzibler Darstellungen von G und darüber hinaus einen Isomorphismus zwischen dem Schnittgraphen von E_{red} und dem Darstellungsgraphen von G.*

Als Vorbereitung für Konstruktionen zur McKay Korrespondenz und zu G-Hilbertschemata muß einiges an Theorie G-äquivarianter Garben für Gruppenschemata G entwickelt werden, dies geschieht in Kapitel 3.

Mit dem Ziel, die McKay Korrespondenz zu verallgemeinern, verallgemeinern wir die Konstruktion von G-Hilbertschemata auf endliche Gruppenschemata. Das Arbeiten mit Gruppenschemata verlangt es, die Dinge in streng funktorieller Sprache zu formulieren, zudem werden Eigenschaften G-äquivarianter Garben für Gruppenschemata G gebraucht. In Kapitel 4 gelangen wir zu folgendem Theorem:

Theorem A.4. *Sei $G = \mathrm{Spec}\, A$ ein endliches Gruppenschema über einem Körper K mit A kohalbeinfach. Sei X ein G-Schema algebraisch über K, wir nehmen an, daß ein geometrischer Quotient $\pi : X \to X/G$, π affin, von X nach G existiert. Dann wird der G-Hilbertfunktor $\underline{\mathrm{G\text{-}Hilb}}_K X$ repräsentiert durch ein algebraisches K-Schema G-Hilb$_K X$ und der natürliche Morphismus $\tau : \mathrm{G\text{-}Hilb}_K X \to X/G$ ist projektiv.*

Neben der Verallgemeinerung auf endliche Gruppenschemata mit kohalbeinfacher Hopfalgebra über beliebigen Grundkörpern haben wir die folgenden Erweiterungen und Vereinfachungen bei der Konstruktion des G-Hilbertschemas: Wir führen relative G-Hilbertschemata zu einem Schema mit G-Operation über einem anderen Schema ein und variieren das Basisschema von G-Hilbertschemata. Dies erlaubt es, das G-Hilbertschema ohne Verwendung des Hilbertschemas von n Punkten zu konstruieren. Diese neue Konstruktion funktioniert unter natürlicheren Voraussetzungen, weiter liefert sie zusätzliche Informationen über den Morphismus vom G-Hilbertschema auf den Quotienten, dieser wird interpretiert als Strukturmorphismus eines relativen G-Hilbertschemas.

Als eine Anwendung ist es möglich, relative Tangentialräume des G-Hilbertschemas über dem Quotienten zu berechnen, diese stehen in Beziehung zu einer gewissen Stratifizierung des G-Hilbertschemas, die in Arbeiten zur McKay Korrespondenz betrachtet wird.

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Diese Doktorarbeit ist wie folgt gegliedert:

Teil I besteht aus zwei erklärenden Kapiteln. Im ersten Kapitel führen wir die Hauptthemen dieser Doktorarbeit, die McKay Korrespondenz und G-Hilbertschemata, ein. Wir erklären die Beobachtung von McKay, Konstruktionen wie die Stratifizierung des G-Hilbertschemas und die tautologischen Garben, wir diskutieren kurz Erweiterungen wie die K-theoretische und derivierte McKay Korrespondenz, aber wir beschränken uns auf den Fall endlicher Untergruppen $G \subset \mathrm{SL}(2, \mathbb{C})$. Das zweite Kapitel enthält allgemeine Theorie über Quotientensingularitäten, über die Beschreibung des G-Hilbertschemas für abelsche Gruppen G als torische Varietät und andere Themen sowie zahlreiche Beispiele für diese Theorie.

Teil II über G-Garben und G-Hilbertschemata bildet den technischen Kern dieser Doktorarbeit. In Kapitel 3 entwickeln wir die Theorie G-äquivarianter Garben für Gruppenschemata G, die in dieser Arbeit für die Konstruktionen zu G-Hilbertschemata und McKay Korrespondenz gebraucht wird. In Kapitel 4 besprechen, erweitern und vereinfachen wir die Konstruktion von G-Hilbertschemata. Wir konstruieren das G-Hilbertschema für endliche Gruppenschemata G und führen relative G-Hilbertschemata ein, wir finden eine Konstruktion für das G-Hilbertschema, die das Hilbertschema von n Punkten nicht benötigt und unter natürlicheren Voraussetzungen funktioniert. Zudem erhalten wir zusätzliche Informationen über den Morphismus vom G-Hilbertschema auf den Quotienten.

Teil III enthält Resultate über McKay Korrespondenz über nicht algebraisch abgeschlossenen Körpern. Kapitel 5 behandelt die Beziehungen, sowohl für Darstellungen als auch für Komponenten von Schemata, zwischen der Operation der Galoisgruppe und Irreduzibilität bei einer Galoiserweiterung des Grundkörpers, führt Galois-konjugierte G-Garben ein und beschreibt die Galoisoperation auf dem G-Hilbertschema. Im letzten Kapitel kommen wir zu der Situation des ersten Kapitels zurück und erweitern die klassische McKay Korrespondenz für endliche Gruppen $G \subset \mathrm{SL}(2, \mathbb{C})$ auf endliche Gruppenschemata $G \subset \mathrm{SL}(2, K)$ über Körpern K von Charakteristik 0, die nicht notwendig algebraisch abgeschlossen sind.

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Anhang B

Lebenslauf

20.8.1976	geboren in Hannover
1983-1996	Schulbesuch in Hannover:
	1983-1987 Grundschule Hägewiesen
	1987-1989 Orientierungsstufe Sahlkamp
	1989-1996 Gymnasium Herschelschule
1996	Abitur
1996-1997	Zivildienst
1997-2002	Studium der Mathematik und Physik
	an der Universität Hannover
10/2001-5/2002	Diplomarbeit "Darstellungen von Heisenberggruppen und
	Elliptische Kurven" betreut von Prof. Dr. Klaus Hulek
7/2002	Diplom Mathematik an der Universität Hannover
seit 10/2002	Wissenschaftlicher Angestellter am Mathematischen Institut
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