Formal Dialogue Semantics for Definitional Reasoning and Implications as Rules

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
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Tübingen
2012

Tag der mündlichen Qualifikation:	11. Juli 2012
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PREFACE

This doctoral dissertation has been written within the ESF research project "Dialogical Foundations of Semantics (DiFoS)", being part of the ESF-EUROCORES programme "LogICCC – Modelling Intelligent Interaction" (DFG grant Schr 275/15-1).

Most of the results presented here have also been presented by the author in the following talks:

- *Dialogues and Definitional Reasoning*. Universidade Federal de Goiás, Goiânia, Brazil, 22 February 2010.
- *Dialogues and Definitional Reasoning*. Department of Philosophy & TecMF, PUC-Rio, Rio de Janeiro, 10 March 2010.
- *Dialogues, End-Rules and Definitional Reasoning*. Cross-CRP workshop "Modelling Interaction, Dialog, Social Choice, and Vagueness (MIDiSoVa)" at the Institute for Logic, Language and Computation of the University of Amsterdam, 26–28 March 2010.
- *Implications as rules*. Cross-CRP workshop "Dialogues, Inference, and Proof – Logical and Empirical Perspectives (DIPLEAP)" at the Vienna University of Technology, 26–28 November 2010.
- *Dialogues, End-Rules and Definitional Reasoning*. "First Meeting on Logic in Centro-Oeste", at the Universidade Federal de Goiás, Goiânia, Brazil, 24 September 2010.
- *Implications as rules in dialogues*. Cross-CRP workshop "Proof and Dialogues (ProDi)", at the University of Tübingen, 25–27 February 2011.
- *Implications as Rules in Dialogical Semantics*. "LOGICA 2011", at the Hejnice Monastery, Czech Republic, 20–24 June, 2011.
- *Dialogues, implications as rules and definitional reasoning*. "LogICCC Final Conference", Berlin, 15–18 September 2011.

Some of the results are published in:

- Thomas Piecha, *Dialogues, End-Rules and Definitional Reasoning* (Abstract), ***First Meeting on Logic in Centro-Oeste, Universidade Federal de Goiás, Goiânia, Brazil, 24 September 2010***, Almeida & Clément Edições, 2010.
- Thomas Piecha and Peter Schroeder-Heister, *Implications as Rules in Dialogical Semantics*, ***The Logica Yearbook 2011*** (M. Peliš and V. Punčochář, editors), College Publications, London 2012 (forthcoming).

I would like to thank Peter Schroeder-Heister. The ideas on which this doctoral dissertation is based have been developed in collaboration with him.

Thomas Piecha
Tübingen, May 2012

Chapter 1

INTRODUCTION

Dialogues have first been proposed by Lorenzen [1960], [1961][1] as an alternative foundation for constructive or intuitionistic logic.[2] The general idea is that the logical constants are given an interpretation in certain game-theoretical terms. Dialogues are two-player games between a proponent and an opponent, where each of the two players can either attack claims made by the other player or defend their own claims. For example, an implication $A \to B$ is attacked by claiming A and defended by claiming B. This means that in order to have a winning strategy for $A \to B$, the proponent must be able to generate an argument for B depending on what the opponent can put forward in defense of A. The logical constant of implication has thus been given a certain game-theoretical or argumentative interpretation, and corresponding argumentative interpretations can be given for the other logical constants.[3] Different from standard constructive interpretations of the logical constants like the Brouwer–Heyting–Kolmogorov (BHK) interpretation[4], the attacker need not necessarily produce a full proof of A in the case of implications $A \to B$. Instead, the proponent may force the opponent to produce certain fragments of a proof of A which are sufficient to successfully defend B.[5] Starting from argumentative interpretations, formal dialogue semantics can be developed.

The two main goals of this dissertation are to provide dialogical foundations in the sense of formal dialogue semantics for definitional reasoning and for implications as rules.

In order to achieve the first main goal, we will first (in Chapter 2) introduce a new kind of dialogues for intuitionistic logic as a variant

[1]See also Lorenzen and Schwemmer [1973], Lorenzen and Lorenz [1978] and Lorenzen [1980], [1982], [1987].

[2]For a critical discussion of this approach see Hodges [2001] (cf. also Hodges [2009]) and the reply by Krabbe [2001]; see also Marion [2009].

[3]Cf. Felscher [2002]. See also Remark 2.1.3 below.

[4]See Heyting [1971].

[5]See also Piecha and Schroeder-Heister [2012].

of (standard) Lorenzen dialogues (respectively as a variant of Lorenzen dialogues as they have been presented by Felscher [1985], [2002]). The distinguishing feature of these new dialogues is that dialogues won by the proponent need not end with the assertion of an atomic formula (as it is the case in the standard dialogues) but may end with the assertion of a complex formula.[6] We will show (in Chapter 3) that the formulas justifiable or provable on the basis of these dialogues are exactly the formulas provable in sequent calculus for intuitionistic logic. Based on this result, we will then (in Chapter 4) extend these dialogues to definitional dialogues. They provide a means for definitional reasoning, that is, for reasoning about given definitions of atomic formulas whose defining conditions may be complex formulas. Such definitions have the form of (generalized) logic programs, and definitional reasoning is an extension of logic programming. Definitional dialogues are a formal semantics for this extension. A paradoxical definition is used as a test case for definitional reasoning, and the effects of the structural operation of contraction are considered.

The second main goal—that is, to provide a dialogical foundation for implications as rules—will be approached in Chapter 5. Different to BHK-like interpretations or to the dialogues considered so far, implications will be understood as being rules. Thus an implication $A \to B$ is understood as a rule which allows us to pass from A over to B. Such an understanding of implications as rules can be motivated from logic programming or natural deduction. In the latter, *modus ponens*

$$\frac{A \quad A \to B}{B}$$

can be read as the application of $A \to B$ as a rule, which is used to pass from A to B, and where the introduction

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$$

(where assumptions A can be discharged) of an implication $A \to B$ can be read as establishing a rule, namely by deriving its conclusion B from its premiss A. For sequent calculus, Schroeder-Heister [2011a], [2011b] has proposed an alternative left implication introduction rule

$$\frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

[6] To prevent any misunderstandings, we point out that atomic formulas do not contain any logical constant, whereas complex formulas contain at least one logical constant. That is, complex formulas are non-atomic formulas and do *not* comprise atomic formulas as a limit case.

which is motivated by a reading of implications as rules, and which replaces the standard left implication introduction rule

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}$$

We will carry the implications-as-rules approach over to dialogues. The general idea will be that any implication $A \to B$ claimed by the opponent is considered to be a rule in a sort of 'database'; this rule can later be used by the proponent to reduce the justification of its conclusion B to the justification of its premiss A. This is made possible by allowing the proponent to defend an attack on B by asserting A whenever $A \to B$ has been claimed by the opponent before. If no such claim has been made before, then the argument for B continues with an opponent attack on B, just as in the standard dialogues.[7] Special consideration will be given to the structural operation of cut in these new implications-as-rules dialogues, and an equivalence result will be shown for the corresponding sequent calculus. Finally, we will consider a combination of definitional dialogues with implications-as-rules dialogues.

Dialogues do also feature certain similarities with tableaux. We will point out these similarities—as well as the differences—at the end of Chapter 2 in Section 2.9.

All definitions of dialogues and sequent calculi that we will make use of can be looked up in Appendix A.

[7]See also Piecha and Schroeder-Heister [2012].

Chapter 2

DIALOGUES AND STRATEGIES FOR PROPOSITIONAL LOGIC

We define the concepts of argumentation form, dialogue and strategy, following the presentation of Felscher [1985], [2002] with slight deviations. After that, certain variants of dialogues are introduced. Of these, the newly introduced DI_c^p- and EI_c^p-dialogues will be of special importance: An equivalence result will be proved for EI_c^p-dialogues and a sequent calculus with complex initial sequents (see Chapter 3), and this result will be the basis of certain extensions to what we call 'definitional dialogues', introduced and analyzed in Chapter 4. We focus on dialogues for intuitionistic propositional logic.[8] Nonetheless, in order to highlight how dialogues for intuitionistic propositional logic differ from dialogues for classical propositional logic we will also consider the latter briefly. Finally, we will point out certain problems that arise when hypotheses (or assumptions) are allowed in dialogues.

2.1. Dialogues

We define our language, argumentation forms for logical constants and dialogues.

DEFINITION 2.1.1. The *language* consists of propositional *formulas* $A, B, C, \ldots$ that are constructed from *atomic formulas* (*atoms*) $a, b, c, \ldots$ with the *logical constants* $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication). Furthermore, $\vee$, $\wedge_1$ and $\wedge_2$ are used as *special symbols*. In addition, the letters P ('proponent') and O ('opponent') are used. An *expression* e is either a formula or a special symbol. For each expression e there is a *P-signed expression* $P\, e$ and an *O-signed expression* $O\, e$. A signed expression is called *assertion* if the expression is a formula; it is called *symbolic attack* if the expression is a special symbol. X and Y, where $X \neq Y$, are used as variables for P and O.

[8] Dialogues for intuitionistic first-order logic will be examined in Chapter 3.

DEFINITION 2.1.2. For each logical constant an *argumentation form* determines how a complex formula (having the respective constant in outermost position) that is asserted by X can be attacked by Y and how this attack can be defended (if possible) by X. The argumentation forms are as follows:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	(X chooses $i = 1$ or $i = 2$)
implication $\to$:	assertion:	$X\ A \to B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	

REMARK 2.1.3. By these argumentation forms the logical constants are given an *argumentative interpretation* (as Felscher [2002, p. 127] calls it) in the following sense:

(i) An argument on a conjunctive assertion made by X consists in Y choosing one conjunct of the assertion, and X continuing the argument with that chosen conjunct. In other words, the argumentative interpretation of conjunction is given by the reduction of the argument on a conjunctive assertion made by X to the argument on one of the conjuncts chosen by Y in the attack.

(ii) In an argument on a disjunctive assertion made by X, Y demands the continuation of the argument with any of the disjuncts. In other words, the argumentative interpretation of disjunction is given by the reduction of the argument on a disjunctive assertion made by X to the argument on one of the disjuncts chosen by X in the defense.

(iii) An argument on an implicative assertion made by X consists in Y stating the antecedent of the implication (whereby the antecedent functions as an assumption), and X continuing the argument with the succedent. Alternatively, X could continue with an attack on the assumed antecedent. In other words, the argumentative interpretation of implication is given by the reduction of the argument on an implicative assertion made by X to the argument on the succedent under the assumption of the antecedent.

(iv) An argument on a negative assertion $\neg A$ made by X consists in Y stating the assertion A, without X being able to continue the argument.

This argumentative interpretation of negation can be made clear by introducing the *falsum* $\bot$ as a constant which signifies absurdity (which is taken as a primitive notion). We can then define negation by implication and *falsum*: $\neg A := A \to \bot$. An argument on $\neg A$ is thus an argument on $A \to \bot$. However, X asserting $\bot$ would mean that Y could continue the argument with *any* assertion—assuming the principle of *ex falso quodlibet* to be applicable here. To avoid this, $\bot$ must not be asserted. Hence, an argument on $\neg A$ (i.e. on $A \to \bot$) can only continue with an argument on the assumption A, and cannot be reduced to an argument on $\bot$.[9]

DEFINITION 2.1.4. Let $\delta(n)$, for $n \geq 0$, be a signed expression and $\eta(n)$ a pair $[m, Z]$, for $0 \leq m < n$, where Z is either A (for 'attack') or D (for 'defense'), and where $\eta(0)$ is empty. Pairs $\langle\delta(n), \eta(n)\rangle$ are called *moves*.

A move $\langle\delta(n), \eta(n) = [m, A]\rangle$ is called *attack move*, and a move $\langle\delta(n), \eta(n) = [m, D]\rangle$ is called *defense move*.

REMARK 2.1.5. $\delta(n)$ is a function mapping natural numbers $n \geq 0$ to signed expressions $X\ e$, and $\eta(n)$ is a function mapping natural numbers $n \geq 0$ to pairs $[m, Z]$. The numbers in the domain of $\delta(n)$ (resp. in the domain of $\eta(n)$) are called *positions*.

When talking about a move $\langle\delta(n), \eta(n)\rangle$, we write $\langle\delta(n) = X\ e, \eta(n) = [m, Z]\rangle$ to express that $\delta(n)$ has the value $X\ e$ for position n, and that $\eta(n)$ has the value $[m, Z]$ for position n.[10]

For example, $\langle\delta(n) = P\ A, \eta(n) = [m, D]\rangle$ denotes a defense move which is made by the proponent P at position n by asserting the formula A; this defense move refers to a move made at position m. A concrete move like $\langle\delta(4) = P \wedge_1, \eta(4) = [3, A]\rangle$ will also be written as

$$4. \quad P \wedge_1 \quad [3, A]$$

This is an attack move with symbolic attack $P \wedge_1$; it is made at position 4 and refers to a move made at position 3.

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[9] This is similar to the treatment of negation in constructive semantics, respectively in the Brouwer–Heyting–Kolmogorov (BHK) interpretation of logical constants, as for example stated by Heyting [1971, p. 102]: "[...] $\neg\mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction." Where contradiction—or equivalently absurdity (here signified by $\bot$)—is usually considered to be a primitive notion.

[10] This deviates from the terminology in Felscher [1985], where the numbers n in the domain of $\delta(n)$ are called moves, places or positions; we call the numbers n positions as well, but what we call moves are not positions.

The notation $\langle\delta(n) = X\, e, \eta(n) = [m, Z]\rangle$ has the advantage that we can speak about a move $\langle X\, e, [m, Z]\rangle$ by including information about the position n at which this move is made.

Although moves are always pairs $\langle\delta(n), \eta(n)\rangle$, we will also refer to moves by giving only their $\delta(n)$-component, as long as it is clear from the context which move is meant, or if it is irrelevant whether the move is an attack or a defense, or if it is irrelevant to which position the move refers to. And instead of $\langle\delta(n) = X\, e, \eta(n)\rangle$ we will also speak of the move $X\, e$ made at position n. We will also speak simply about attacks and defenses in order to refer to attack moves and defense moves, respectively.

DEFINITION 2.1.6. A *dialogue* is a finite or infinite sequence of moves $\langle\delta(n), \eta(n)\rangle$ (for $n = 0, 1, 2, \dots$) satisfying the following conditions:

($D00$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

($D01$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

DEFINITION 2.1.7. An attack $\langle\delta(n), \eta(n) = [m, A]\rangle$ at position n on an assertion at position m is called *open at position* k for $n < k$ if there is no position n' such that $n < n' \leq k$ and $\langle\delta(n'), \eta(n') = [n, D]\rangle$, that is, if there is no defense at or before position k to an attack at position n.

REMARK 2.1.8. Since there is no defense to an attack $\langle\delta(n) = Y\, A$, $\eta(n) = [m, A]\rangle$ on $\delta(m) = X\, \neg A$ for $m < n$, the attack at position n is open at all positions k for $n < k$.

2.2. DI^p-dialogues and strategies

We define DI^p-dialogues and strategies. With regard to the literature on dialogical logic, DI^p-dialogues can be considered to be the standard dialogues for intuitionistic propositional logic. The following definition of DI^p-dialogues is based on the definition of dialogues.

DEFINITION 2.2.1. A DI^p*-dialogue* is a dialogue satisfying the following conditions (in addition to ($D00$), ($D01$) and ($D02$)):

($D10$) If, for an atomic formula a, $\delta(n) = P\, a$, then there is an m such that $m < n$ and $\delta(m) = O\, a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11$) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

($D12$) For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

($D13$) If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

A DI^p-dialogue beginning with $P\,A$ (i.e., $\delta(0) = P\,A$, where A is a complex formula) is called *DI^p-dialogue for the formula A*.

REMARK 2.2.2. The objects defined by the conditions ($D00$)–($D02$) alone are what Felscher [1985], [2002] calls 'dialogues', and the objects defined by adding ($D10$)–($D13$)—which we call 'DI^p-dialogues'—are called 'D-dialogues' by him.[11] Since here we are concerned with the objects defined by ($D00$)–($D02$) plus ($D10$)–($D13$), we simply speak of 'dialogues', omitting the specifier 'DI^p' as long as no confusion can arise.

REMARK 2.2.3. The conditions ($D00$)–($D13$) are also called 'frame rules' ('Rahmenregeln'), 'structural rules' or 'special rules of the game' ('spezielle Spielregeln') in the literature, and ($D10$) is sometimes called 'formal rule'. The argumentation forms are also called 'particle rules' ('Partikelregeln'), 'logical rules' or 'general rules of the game' ('allgemeine Spielregeln').[12]

We will stick to the notions 'dialogue condition(s)' (or just 'condition(s)') and 'argumentation form(s)'.

REMARK 2.2.4. Proponent P and opponent O are not interchangeable due to the asymmetries between P and O introduced in ($D10$) and ($D13$). For atomic formulas a, the proponent move $\langle \delta(n) = P\,a, \eta(n) = [m, Z]\rangle$ is possible only after an opponent move $\langle \delta(m) = O\,a, \eta(m) = [k, Z]\rangle$ for $k < m < n$, and O can attack a P-signed formula only once, whereas P can attack O-signed formulas repeatedly.

These asymmetries are introduced by dialogue conditions only. The argumentation forms themselves (as given in Definition 2.1.2) are symmetric with respect to the two players P and O. That is, they are independent of whether the assertion is made by the proponent P or by the opponent O; they are thus player-independent.[13]

[11] See Felscher [1985], [2002] for references on this kind of dialogues.

[12] For references cf. Krabbe [2006].

[13] Argumentation forms which are not player-independent in this sense will be considered in Section 3.9 (see Definition 3.9.7) and in Chapter 5.

REMARK 2.2.5. Condition $(D11)$ itself does not exclude the possibility of defending an already defended attack again at a position where exactly one attack is open (cf. Example 2.2.19).

DEFINITION 2.2.6. *P wins a dialogue for a formula A* if the dialogue is finite, begins with the move $P\,A$ and ends with a move of P such that O cannot make another move.

REMARK 2.2.7. A dialogue won by P ends with a move $\langle\delta(n) = P\,a, \eta(n) = [m, Z]\rangle$, where a is an atomic formula.

EXAMPLE 2.2.8. A dialogue for the formula $(a \vee b) \rightarrow \neg\neg(a \vee b)$ is the following:

0.	$P\,(a \vee b) \rightarrow \neg\neg(a \vee b)$	
1.	$O\,a \vee b$	$[0, A]$
2.	$P\,\vee$	$[1, A]$
3.	$O\,a$	$[2, D]$
4.	$P\,\neg\neg(a \vee b)$	$[1, D]$
5.	$O\,\neg(a \vee b)$	$[4, A]$
6.	$P\,a \vee b$	$[5, A]$
7.	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$

The dialogue starts with the assertion of the formula $(a \vee b) \rightarrow \neg\neg(a \vee b)$ by the proponent P in the initial move at position 0. This initial move is attacked ($\eta(1) = [0, A]$) by the opponent O with the assertion of the antecedent $a \vee b$ ($\delta(1) = O\,a \vee b$) of the implication asserted by P at position 0. The attack is thus made according to the argumentation form for implication.

At position 2, the proponent does not proceed according to the argumentation form for implication by defending O's attack move with the assertion of the succedent $\neg\neg(a \vee b)$ of the attacked implication. Instead, the proponent makes the symbolic attack $P\,\vee$ on O's assertion $a \vee b$. This move is thus made according to the argumentation form for disjunction. The attack is defended by O with the assertion of the left disjunct a (alternatively, O could also have chosen the right disjunct b). The moves at positions 1–3 are an instance of the argumentation form for disjunction.

As a is an atomic formula, it cannot be attacked. At position 4, the proponent defends O's attack $O\,a \vee b$ by asserting the succedent $\neg\neg(a \vee b)$ of the attacked implication $(a \vee b) \rightarrow \neg\neg(a \vee b)$. The moves at positions 0, 1 and 4 are an instance of the argumentation form for implication.

The opponent now attacks $P\,\neg\neg(a \vee b)$ at position 5 by asserting $O\,\neg(a \vee b)$ according to the argumentation form for negation. By this argumentation form there is no defense for the attack. But the proponent can attack $O\,\neg(a \vee b)$ with the assertion $P\; a \vee b$. The moves at positions 4 and 5 are an instance of the argumentation form for negation, and the moves at positions 5 and 6 are another instance of that argumentation form.

Next O attacks $P\; a \vee b$ with the symbolic attack $O \vee$ according to the argumentation form for disjunction at position 7. Finally, this attack is defended by P's assertion of the left disjunct a. The moves at positions 6–8 are made according to the argumentation form for disjunction. Note that P cannot defend here by asserting the right disjunct b: the opponent has not asserted the atomic formula b before, hence such a move is prohibited by condition $(D10)$.

The proponent's move at position 8 is the last one. The opponent cannot attack a, since it is an atomic formula. Each other P-signed formula has been attacked by O, thus no more attack moves can be made by O due to condition $(D13)$, as these would be repetitions of attacks already made. And since each proponent attack that can be defended according to an argumentation form has already been defended by O, no more defense moves are possible either, due to condition $(D12)$. The dialogue is finite, begins with the move $P\,(a \vee b) \to \neg\neg(a \vee b)$ and ends with a move of P such that O cannot make another move; the dialogue for the formula $(a \vee b) \to \neg\neg(a \vee b)$ is thus won by P.

We next introduce dialogue trees and define strategies. We explain first what we call a path.

DEFINITION 2.2.9. A *path* in a branch of a tree with root node n_0 is a sequence $n_0, n_1, \ldots, n_k$ of nodes for $k \geq 0$ where n_i and n_{i+1} are adjacent for $0 \leq i < k$.

DEFINITION 2.2.10. A *dialogue tree* is a tree whose branches contain as paths all possible dialogues for a given formula.

REMARK 2.2.11. For a given formula A there is exactly one dialogue tree, if we consider trees to be equal modulo swapping of branches.

DEFINITION 2.2.12. A *strategy* for a formula A is a subtree S of the dialogue tree for A such that S does not branch at even positions, S has as many nodes at odd positions as there are possible moves for O, and all branches of S are dialogues for A won by P.

REMARK 2.2.13. In more game-theoretic terms, the strategies defined here could also be called *winning strategies for the player P*, and a corresponding definition could be given of *winning strategies for the player O*. For the dialogical treatment of logic undertaken here, only the first notion is needed, however. We can thus simply speak of *strategies*.

REMARK 2.2.14. Strategies are finite for propositional formulas. All the branches in a strategy have finite length by definition, whereas dialogues that are not part of a strategy can be of infinite length (an example is given below in Remark 2.2.23). Dialogue trees are therefore infinite objects in general (cf. Example 2.2.18). As dialogue trees can be constructed breadth-first, of course, an existing strategy can always be found.

REMARK 2.2.15. Formulas can have no, exactly one or more than one strategy.

EXAMPLE 2.2.16. For the formula $(a \vee b) \to \neg\neg(a \vee b)$ there are the following three strategies, among others:

(i)

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\,a \vee b$			$[0, A]$
2.	$P\,\neg\neg(a \vee b)$			$[1, D]$
3.	$O\,\neg(a \vee b)$			$[2, A]$
4.	$P\,a \vee b$			$[3, A]$
5.	$O\,\vee$			$[4, A]$
6.	$P\,\vee$			$[1, A]$
7.	$O\,a$	$[6, D]$	$O\,b$	$[6, D]$
8.	$P\,a$	$[5, D]$	$P\,b$	$[5, D]$

(ii)

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\,a \vee b$			$[0, A]$
2.	$P\,\neg\neg(a \vee b)$			$[1, D]$
3.	$O\,\neg(a \vee b)$			$[2, A]$
4.	$P\,\vee$			$[1, A]$
5.	$O\,a$	$[4, D]$	$O\,b$	$[4, D]$
6.	$P\,a \vee b$	$[3, A]$	$P\,a \vee b$	$[3, A]$
7.	$O\,\vee$	$[6, A]$	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$	$P\,b$	$[7, D]$

(iii)

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$			
1.	$O\,a \vee b$			$[0, A]$
2.	$P\,\vee$			$[1, A]$
3.	$O\,a$	$[2, D]$	$O\,b$	$[2, D]$
4.	$P\,\neg\neg(a \vee b)$	$[1, D]$	$P\,\neg\neg(a \vee b)$	$[1, D]$
5.	$O\,\neg(a \vee b)$	$[4, A]$	$O\,\neg(a \vee b)$	$[4, A]$
6.	$P\,a \vee b$	$[5, A]$	$P\,a \vee b$	$[5, A]$
7.	$O\,\vee$	$[6, A]$	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$	$P\,b$	$[7, D]$

There are more strategies for this formula than the three shown here, because the proponent can repeatedly attack formulas asserted by the opponent. For example, in strategy (iii) the proponent could at position 4 (in the left as well as in the right dialogue) repeat the attack $P \vee$ on $O\, a \vee b$. The subtrees below these attacks (in both dialogues) would have the same form as the subtree below position 2 in strategy (iii).

EXAMPLE 2.2.17. There is exactly one strategy for the formula $a \to \neg\neg a$:

0.	$P\, a \to \neg\neg a$	
1.	$O\, a$	$[0, A]$
2.	$P\, \neg\neg a$	$[1, D]$
3.	$O\, \neg a$	$[2, A]$
4.	$P\, a$	$[1, A]$

EXAMPLE 2.2.18. There is no strategy for the formula $((a \to b) \to a) \to a$, an instance of Peirce's law. The dialogue tree has the form

0.	$P\, ((a \to b) \to a) \to a$		
1.	$O\, (a \to b) \to a$		$[0, A]$
2.	$P\, a \to b$		$[1, A]$
3.	$O\, a$ $[2, A]$		$O\, a$ $[2, D]$
4.	$\vdots$	$P\, a$ $[1, D]$	$P\, a \to b$ $[1, A]$
5.			$O\, a$ $[4, A]$
6.			$\vdots$

where at position 4 in the left dialogue and at position 6 in the right dialogue the proponent can only repeat the attack $P\, a \to b$ on $O\, (a \to b) \to a$. This attack can in turn be either attacked with $\langle \delta(5) = O\, a, \eta(5) = [4, A] \rangle$ or defended with $\langle \delta(5) = O\, a, \eta(5) = [4, D] \rangle$ (in the left branch), respectively with $\langle \delta(7) = O\, a, \eta(7) = [6, A] \rangle$ or $\langle \delta(7) = O\, a, \eta(7) = [6, D] \rangle$ (in the right branch). As the proponent can repeat the attack $P\, a \to b$ indefinitely while the opponent can always attack this attack with $O\, a$ or defend with $O\, a$, the dialogue tree for $((a \to b) \to a) \to a$ is infinite.

Not all dialogues beginning with the path $\langle P\, ((a \to b) \to a) \to a, \emptyset \rangle, \langle O\, (a \to b) \to a, [0, A] \rangle, \langle P\, a \to b, [1, A] \rangle, \langle O\, a, [2, A] \rangle$ are won by P. There is thus no strategy.

There would be a strategy, if condition $(D11)$ were dropped for P. Then P could win the left, respectively the right dialogue by defending the attack $O\, (a \to b) \to a$ made at position 1 (which is not the last open attack in the left and in the right dialogue) with the move $P\, a$ $[1, D]$ at position 4 in the left dialogue and at position 6 in the right dialogue.

This example also shows that dialogue trees which do not contain a strategy as a subtree can nevertheless contain dialogues which are won by the proponent P.

EXAMPLE 2.2.19. There is no strategy for the formula $a \vee \neg a$, an instance of *tertium non datur*. The only possible dialogue is

0.	$P\, a \vee \neg a$	
1.	$O \vee$	$[0, A]$
2.	$P\, \neg a$	$[1, D]$
3.	$O\, a$	$[2, A]$

and P does not win.

There would be a strategy, if condition $(D12)$ were dropped for P. Then P could defend the attack $O \vee$ a second time by stating a, thereby winning the dialogue. Condition $(D11)$ does not have to be dropped because there are not more than one open attacks at position 3 (there is exactly one open attack at position 3; the attack $O \vee$ is not open there since it has already been defended at position 2).

EXAMPLE 2.2.20. There is no strategy for the formula $\neg\neg a \to a$, an instance of double negation elimination. The only possible dialogue is

0.	$P\, \neg\neg a \to a$	
1.	$O\, \neg\neg a$	$[0, A]$
2.	$P\, \neg a$	$[1, A]$
3.	$O\, a$	$[2, A]$

and P does not win.

There would be a strategy, if condition $(D11)$ were dropped for P. Then P could defend the attack $O\, \neg\neg a$ by stating a at position 4, thereby winning the dialogue. Since the attack $O\, \neg\neg a$ has not been defended before, condition $(D12)$ would not be violated by P.

DEFINITION 2.2.21. A formula A is called *dialogue-provable* (or DI^p-*dialogue-provable*) if there is a strategy for A. Notation: $\vdash_{DI^p} A$.

REMARK 2.2.22. We speak of dialogue-*provable* formulas here, in accordance with Felscher [2002]. Contrasting Gentzen's calculi[14] with dialogues, Felscher [2002, p. 127] remarks:

> Gentzen's calculi of proofs are easily explained in that they represent the weakest consequence relation for which the provability interpretation is valid. The connection between dialogues and the argumentative interpretation of logical operations is [...] located on a different level: it is not the dialogues but the *strategies* for dialogues which will correspond to proofs. I thus formulate the *basic purpose* for the use of dialogues:
>
> (A_0) Logically provable assertions shall be those which, for *purely formal* reasons, can be upheld by a strategy covering every dialogue chosen by $[O]$.

[14]Sequent calculi will be considered below; cf. Chapter 3.

However, the fact that we speak of *provability* in the context of dialogues (thus following Felscher) should not be misunderstood in a way that would imply that dialogues cannot be seen as a (formal) semantics (as opposed to considering dialogues only as a proof system or calculus).

Of course, such a misunderstanding could only arise if one's notion of semantics is limited to truth-conditional semantics, as opposed to proof-theoretic semantics (like Brouwer–Heyting–Kolmogorov (BHK) semantics[15], or related justificationist, verificationist, pragmatist or falsificationist approaches in the tradition of Dummett and Prawitz[16]) where the notion of proof or closely related notions are of central importance.

As the meaning of the logical constants is in some sense given by the argumentation forms in terms of how assertions containing the logical constants can be used in an argumentation, dialogues might very well be seen as a semantics under the heading "meaning is use", and were indeed introduced for that purpose.

REMARK 2.2.23. (i) The dialogue tree for a dialogue-provable formula can contain dialogues not won by P. For example, the dialogue tree for the dialogue-provable formula $(a \vee b) \to \neg\neg(a \vee b)$ contains the following dialogue which is infinite and thus not won by P:

$$
\begin{array}{lll}
0. & P\,(a \vee b) \to \neg\neg(a \vee b) & \\
1. & O\,a \vee b & [0, A] \\
2. & P \vee & [1, A] \\
3. & O\,a & [2, D] \\
4. & P \vee & [1, A] \\
5. & O\,a & [4, D] \\
 & \vdots &
\end{array}
$$

(ii) The dialogue tree for a dialogue-provable formula can also contain finite dialogues which are not won by P because they end in a move made by O such that P cannot make another move. For example, the dialogue tree for $\neg a \vee (a \to a)$ contains the dialogue

$$
\begin{array}{lll}
0. & P\,\neg a \vee (a \to a) & \\
1. & O \vee & [0, A] \\
2. & P\,\neg a & [1, D] \\
3. & O\,a & [2, A]
\end{array}
$$

which is finite and not won by P. (The atomic formula a asserted by O in the last move cannot be attacked, and the attack $O \vee$ cannot be defended again due to condition $(D12)$.)

[15] Cf. Heyting [1971].

[16] See e.g. Dummett [1991], Prawitz [1971], [2006], [2007] and Schroeder-Heister [2006].

REMARK 2.2.24. The dialogue-provable formulas are exactly the formulas provable in intuitionistic logic. This has been shown (also for intuitionistic first-order logic) by Felscher [1985] by proving for Gentzen's sequent calculus *LJ*[17] (for intuitionistic first-order logic) that every (first-order) strategy can be transformed into a proof in *LJ*, and vice versa.[18]

2.3. Classical dialogues

Although we will only be concerned with intuitionistic logic, we point out here how dialogues for classical (propositional) logic relate to dialogues for intuitionistic (propositional) logic.

REMARK 2.3.1. If the conditions $(D11)$ and $(D12)$ are restricted to apply only to O (and no more to P), then the formulas provable on the basis of the thus modified dialogues are exactly the formulas provable in classical logic.

DEFINITION 2.3.2. A *classical dialogue* is a dialogue where the conditions $(D11)$ and $(D12)$ do hold for O but not for P, that is, where conditions $(D11)$ and $(D12)$ are replaced by the following conditions $(D11^+)$ and $(D12^+)$, respectively:

$(D11^+)$ If $\eta(p) = [n, D]$ for even n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by P, then only the last of them may be defended by O at position p.

$(D12^+)$ For every even m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by P may be defended by O at most once.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for dialogues are directly carried over to the corresponding notions for classical dialogues.

DEFINITION 2.3.3. A formula A is called *classically dialogue-provable* if there is a classical strategy for A. Notation: $\vdash_{DK^p} A$. (Classical strategies are called 'C-strategies' in Felscher [2002].)

REMARK 2.3.4. It was shown by Felscher [1986, p. 367] (see also Felscher [2002, p. 139]) that

$\vdash_{DK^p} A$ if and only if A is provable in classical propositional logic.

---

[17] See Gentzen [1935].

[18] Cf. also Barth and Krabbe [1982].

(Felscher uses Glivenko's theorem in order to show this. Since this theorem only holds for propositional logic, the result for classical first-order logic would have to be shown by other means.)

REMARK 2.3.5. That not only one but both conditions $(D11)$ and $(D12)$ have to be modified in order to get classical logic can be seen by considering Example 2.2.18 and Example 2.2.19. Neither of the two formulas considered there holds in intuitionistic logic, but both do hold in classical logic.

If only $(D11)$ is modified, then $((a \to b) \to a) \to a$ would be provable (cf. Example 2.2.18), whereas $a \vee \neg a$ would not be provable (cf. Example 2.2.19). And if only $(D12)$ is modified, then $a \vee \neg a$ but not $((a \to b) \to a) \to a$ would be provable. This is not connected to the fact that $(a \vee \neg a) \to (((a \to b) \to a) \to a)$ is a theorem in intuitionistic logic, whereas $(((a \to b) \to a) \to a) \to (a \vee \neg a)$ is not, although intuitionistic logic plus *tertium non datur* $A \vee \neg A$ (for any formula A), respectively plus Peirce's law $((A \to B) \to A) \to A$ (for any formulas A, B), is equivalent to classical logic.

The formal systems resulting from modifying either only condition $(D11)$ or only condition $(D12)$ may thus seem strange. However, the difference between schematic formulas containing propositional variables and formulas containing only specific propositions is essential here, since dialogues are only defined for the latter, and it is only the argumentation forms which are given for the former. The object

0.	$P\,A \to B$	
1.	$O\,A$	$[0, A]$
2.	$P\,B$	$[1, D]$

is therefore not a dialogue (or even a strategy), but can only be understood as a class of dialogues whose properties depend on the form of A and B. Note that the only possible dialogue for the (non-schematic) proposition $a \to b$ is

0.	$P\,a \to b$	
1.	$O\,a$	$[0, A]$

which is not won by P. The attack at position 1 cannot be defended with the move $P\,b$ due to $(D10)$. In the object above, the moves $\langle \delta(2) = P\,B, \eta(2) = [1, D] \rangle$ would only be allowed for complex formulas B or for atomic formulas A, B where $A \equiv B$. This, however, is not what one would like to express by the schematic use of propositional variables.

These observations concerning propositional variables versus specific propositions do apply not only to classical dialogues but also to the other kinds of dialogues treated here.

EXAMPLE 2.3.6. There is a classical strategy for the formula $a \vee \neg a$:

0.	$P\ a \vee \neg a$	
1.	$O \vee$	$[0, A]$
2.	$P \neg a$	$[1, D]$
3.	$O\ a$	$[2, A]$
4.	$P\ a$	$[1, D]$

The last move is possible due to the replacement of condition $(D12)$ by condition $(D12^{+})$. In the presence of $(D12)$ this move is not possible, and there is thus no DI^{p}-strategy for (any instance of) *tertium non datur* (cf. Example 2.2.19).

EXAMPLE 2.3.7. There is a classical strategy for the formula $\neg\neg a \to a$:

0.	$P \neg\neg a \to a$	
1.	$O \neg\neg a$	$[0, A]$
2.	$P \neg a$	$[1, A]$
3.	$O\ a$	$[2, A]$
4.	$P\ a$	$[1, D]$

The last move is possible due to the replacement of condition $(D11)$ by condition $(D11^{+})$. In the presence of $(D11)$ this move is not possible, and there is thus no DI^{p}-strategy for (any instance of) double negation elimination (cf. Example 2.2.20).

In the following we will not consider classical dialogues again. We consider only intuitionistic logic.[19]

2.4. Closure under substitution

We saw that dialogues cannot be used to prove (or validate) logical laws directly. Strategies (as based on the dialogues defined here) can only be presented for instances of logical laws. In order to show that a logical law holds, one has thus to show that there is a strategy for *each* instance of the law. That is, one has to prove closure under substitution—more precisely: closure under uniform substitution of arbitrary formulas for the atomic formulas in a formula—for strategies. This does not have to be done directly if strategies can be shown to be equivalent to derivations in a calculus that is known to be closed under substitution. Such an equivalence result will be given in Chapter 3.

[19] Besides classical and intuitionistic logic, many other logics have been considered from a dialogical perspective and have been given a dialogue semantics; cf. e.g. Blass [1992], [1997], Fermüller [2003], [2008], [2010], Fermüller and Ciabattoni [2003], Rahman [2012], Rahman and Rückert [2001] and Rückert [2007].

2.5. DI_c^p-dialogues

We introduce DI_c^p-dialogues as a variant of DI^p-dialogues. Their distinguishing feature with respect to strategies is that the DI_c^p-dialogues of a strategy need not end with a proponent move $P\ a$ asserting an atomic formula a as in DI^p-dialogues of a strategy. Instead, the DI_c^p-dialogues of a strategy can also end with a proponent move $P\ A$ asserting a complex formula A.

Regarding the definition of DI_c^p-dialogues, this is achieved by just adding one further condition to the definition of DI^p-dialogues.

DEFINITION 2.5.1. A DI_c^p-*dialogue* is a DI^p-dialogue with the additional condition

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

The full definition of a DI_c^p-*dialogue* is thus given by the following argumentation forms (as already given in Definition 2.1.2):

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	(X chooses $i = 1$ or $i = 2$)
implication $\rightarrow$:	assertion:	$X\ A \rightarrow B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	

together with the following conditions (where conditions ($D00$)–($D02$) and ($D10$)–($D13$) are as already given in Definitions 2.1.6 and 2.2.1):

($D00$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

($D01$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D10$) If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11$) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

($D12$) For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

($D13$) If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^p-dialogues are directly carried over to the corresponding notions for DI_c^p-dialogues.

DEFINITION 2.5.2. A formula A is called *DI_c^p-dialogue-provable* if there is a DI_c^p-strategy for A. Notation: $\vdash_{DI_c^p} A$.

REMARK 2.5.3. A DI_c^p-dialogue won by P ends with the assertion of a complex formula or with the assertion of an atomic formula. Whereas a DI^p-dialogue won by P can only end with the assertion of an atomic formula.

EXAMPLE 2.5.4. The following DI_c^p-dialogue is a DI_c^p-strategy for the formula $(a \vee b) \to \neg\neg(a \vee b)$:

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$	
1.	$O\,a \vee b$	$[0, A]$
2.	$P\,\neg\neg(a \vee b)$	$[1, D]$
3.	$O\,\neg(a \vee b)$	$[2, A]$
4.	$P\,a \vee b$	$[3, A]$

The opponent O cannot attack $a \vee b$, since neither of the two conditions (i) and (ii) of ($D14$) is satisfied: $a \vee b$ has already been asserted by O, and $a \vee b$ has not been attacked by P. The DI_c^p-dialogue is won by P, and it is a DI_c^p-strategy for $(a \vee b) \to \neg\neg(a \vee b)$.

Clearly, this dialogue cannot be a DI^p-strategy, since O could make another move in this case (cf. the DI^p-strategy (i) in Example 2.2.16).

REMARK 2.5.5. A motivation for condition ($D10$) is given in Felscher [2002, p. 129f.]. This condition is related to the use of the so-called *Ipse* *dixisti!-remark* as stipulated in Barth and Krabbe [1982]. The use of the *Ipse dixisti!-remark* is, however, not restricted to atomic formulas as in condition $(D10)$ (cf. Krabbe [2001, p. 45]). If the *Ipse dixisti!-remark* can only be made by the proponent P, then its effect is similar to what is obtained by the addition of our condition $(D14)$.

2.6. EI^p-dialogues

We define EI^p-dialogues as a restricted form of DI^p-dialogues. They differ from DI^p-dialogues only in that each opponent move must now refer to the immediately preceding proponent move. This restriction yields certain technical advantages, without changing the extension of the set of dialogue-provable formulas.

DEFINITION 2.6.1. An EI^p*-dialogue* is a DI^p-dialogue with the additional condition

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n-1$.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^p-dialogues are directly carried over to the corresponding notions for EI^p-dialogues.

REMARK 2.6.2. The EI^p-dialogues as they are defined here are exactly the E-dialogues of Felscher [1985], [1986], [2002] (references to their original formulation are given therein).

DEFINITION 2.6.3. A formula A is called EI^p*-dialogue-provable* if there is an EI^p-strategy for A. Notation: $\vdash_{EI^p} A$.

EXAMPLE 2.6.4. The following EI^p-dialogue tree is an EI^p-strategy (see Felscher [2002]):

0.	$P\,((a \to b) \to (a \to c)) \to (a \to (b \to c))$			
1.			$O\,(a \to b) \to (a \to c)$	$[0, A]$
2.			$P\,a \to (b \to c)$	$[1, D]$
3.			$O\,a$	$[2, A]$
4.			$P\,b \to c$	$[3, D]$
5.			$O\,b$	$[4, A]$
6.			$P\,a \to b$	$[1, A]$
7.	$O\,a$	$[6, A]$	$O\,a \to c$	$[6, D]$
8.	$P\,b$	$[7, D]$	$P\,a$	$[7, A]$

(cont'd on next page)

9.			$O\,c$	$[8, D]$
10.			$P\,c$	$[5, D]$

For comparison, we show an embedding of this EI^p-strategy into the following DI^p-strategy:

0.	$P\,((a \to b) \to (a \to c)) \to (a \to (b \to c))$			
1.	$O\,(a \to b) \to (a \to c)$			$[0, A]$
2.	$P\,a \to (b \to c)$			$[1, D]$
3.	$O\,a$			$[2, A]$
4.	$P\,b \to c$			$[3, D]$
5.	$O\,b$			$[4, A]$
6.	$P\,a \to b$			$[1, A]$
7.	$O\,a$	$[6, A]$	$O\,a \to c$	$[6, D]$
8.	$P\,b$	$[7, D]$	$P\,a$	$[7, A]$
9.	$O\,a \to c$	$[6, D]$	$O\,c$	$[8, D]$
10.	$P\,a$	$[9, A]$	$P\,c$	$[5, D]$
11.	$O\,c$	$[10, D]$	$O\,a$	$[6, A]$
12.	$P\,c$	$[5, D]$	$P\,b$	$[11, D]$

The DI^p-strategy differs from the EI^p-strategy only in having additional moves at positions 9–12 in the left dialogue and at positions 11 and 12 in the right dialogue.

REMARK 2.6.5. It has been shown by Felscher that there is a recursive algorithm by which every EI^p-strategy can be embedded into a DI^p-strategy, and that therefore the EI^p-dialogue-provable formulas are exactly the formulas provable in intuitionistic propositional logic (see Felscher [1985, p. 221] and Felscher [2002, p. 119]; these results hold not only for the propositional but also for the first-order case). As the DI^p-dialogue-provable formulas are also exactly the formulas provable in intuitionistic propositional logic, the following holds: $\vdash_{EI^p} A$ *if and only if* $\vdash_{DI^p} A$.

2.7. EI^p_c-dialogues

As in the case of DI^p-dialogues and their variant, the DI^p_c-dialogues, we can introduce EI^p_c-dialogues as a variant of EI^p-dialogues by adding the condition $(D14)$ to the definition of EI^p-dialogues.

DEFINITION 2.7.1. An EI^p_c-*dialogue* is an EI^p-dialogue with the additional condition

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

Again, the notions ‘dialogue won by P’, ‘dialogue tree’ and ‘strategy’ as defined for DI^p-dialogues are directly carried over to the corresponding notions for EI_c^p-dialogues.

REMARK 2.7.2. Condition (E) implies condition $(D13)$. Furthermore, condition (E) implies condition $(D11)$ for odd p and condition $(D12)$ for odd n (cf. Definition 2.2.1).

In the presence of condition (E), condition $(D13)$ can therefore be omitted, and conditions $(D11)$ and $(D12)$ can be restricted to conditions $(D11')$ and $(D12')$, respectively, as follows:

$(D11')$ If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

$(D12')$ For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

REMARK 2.7.3. For EI_c^p-dialogues, we will use conditions $(D11')$ and $(D12')$ instead of conditions $(D11)$ and $(D12)$, and we omit condition $(D13)$. The full definition of *EI_c^p-dialogues* is thus given by the argumentation forms (as given in Definition 2.1.2)

negation $\neg$:	assertion:	$X \neg A$	
	attack:	$Y A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X A_1 \wedge A_2$	
	attack:	$Y \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X A_i$	
disjunction $\vee$:	assertion:	$X A_1 \vee A_2$	
	attack:	$Y \vee$	
	defense:	$X A_i$	(X chooses $i = 1$ or $i = 2$)
implication $\rightarrow$:	assertion:	$X A \rightarrow B$	
	attack:	$Y A$	
	defense:	$X B$	

together with the following conditions:

$(D00)$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

$(D01)$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D10$) If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n - 1, Z]\rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

DEFINITION 2.7.4. A formula A is called *EI_c^p-dialogue-provable* if there is an EI_c^p-strategy for A. Notation: $\vdash_{EI_c^p} A$.

We also speak of *EI_c^p-provable* formulas or of *EI_c^p-provability*. (Likewise for the other kinds of dialogues.)

2.8. Hypothetical dialogues

The dialogues considered so far do not allow for the use of hypotheses or assumptions. This could be changed by the following definition, which leads to problems, however. These problems will be discussed below.[20]

DEFINITION 2.8.1. *Hypothetical dialogues* are dialogues where at positions $n < 0$ there can be moves of the form $\langle\delta(n) = O\,A, \eta(n) = \emptyset\rangle$. The formulas A stated in such moves are called *assumptions* or *hypotheses*. The proponent P may attack these formulas A as permitted by the conditions defining dialogues.

[20] Rahman and Tulenheimo [2009] have proposed what they call ‘intuitionistic dialogues with hypotheses’. They consider only implication-free fragments, however. Although this might be unproblematic in the case of classical logic, it is an undue restriction if intuitionistic logic is to be obtained; and it is not clear, whether the proposed approach can be properly generalized for the case including implication.

REMARK 2.8.2. This definition can be applied to any kind of dialogues that we have defined above. In what follows, we consider *hypothetical* DI_c^p*-dialogues* as an example.

DEFINITION 2.8.3. A formula A is called *dialogue-provable under assumptions* $A_1, \ldots, A_n$ if there is a strategy for A when the formulas $A_1, \ldots, A_n$ are given as assumptions. Notation: $A_1, \ldots, A_n \vdash_{DI_c^p} A$ (likewise for the other kinds of dialogues defined above).

EXAMPLE 2.8.4. The following is a strategy for $a \wedge b$ under the assumptions a and b, given at positions -2 and -1, respectively:

$$\begin{array}{rllcll}
-2. & & & O\,a & & \\
-1. & & & O\,b & & \\
0. & & & P\,a \wedge b & & \\
1. & O \wedge_1 & [0, A] & \multicolumn{1}{|l}{O \wedge_2} & [0, A] & \\
2. & P\,a & [1, D] & \multicolumn{1}{|l}{P\,b} & [1, D] &
\end{array}$$

The proponent P can assert the formulas a respectively b in the moves at position 2 since O has already asserted them before at positions -2 and -1, respectively. We have thus: $a, b \vdash_{DI_c^p} a \wedge b$.

EXAMPLE 2.8.5. Under the assumption a, the dialogue

$$\begin{array}{rll}
-1. & O\,a & \\
0. & P\,a \vee b & \\
1. & O \vee & [0, A] \\
2. & P\,a & [1, D]
\end{array}$$

is a strategy for $a \vee b$. The proponent P can assert the formula a in the move at position 2 without violating condition $(D10)$ since O has already asserted a before at position -1. We have thus: $a \vdash_{DI_c^p} a \vee b$.

REMARK 2.8.6. Whether the collection of assumptions is treated as multiset, set or list depends on the structural properties given by the conditions that define the respective kind of dialogues. That is, this depends on whether (and to what extent) structural operations like thinning, contraction and exchange are embedded in the respective dialogues.

We do not elaborate further on this here. Structural properties will be examined in Section 3.8 of Chapter 3 as well as in Chapter 4; see also Chapter 5 for a dialogical treatment of the structural operation of cut.

Felscher [2002, p. 143] remarks that the assumptions should be given as a list of O-signed formulas, followed (or preceded) by the P-signed formula which is the consequence in question. He then argues that "no general rule on how to proceed from this initial list can be stated as long as we want to keep the alternation between P and [O] during the progress of our dialogue" (*ibid.*), because the initial list $O\,a$, $P\,a \vee b$ in Example 2.8.5 "*must* be followed by an attack of [O]", whereas the initial list $O\,a \wedge b$, $P\,a$ "*must* be followed by an attack of P" (*ibid.*).

The latter could only be represented analogously to Example 2.8.5 if the proponent P were also allowed to assert an atomic formula in the move at position 0 (contrary to the dialogues considered by Felscher and to the dialogues considered here so far):

$$\begin{array}{rll} -1. & O\,a \wedge b & \\ 0. & P\,a & \\ 1. & P\,\wedge_1 & [-1, A] \\ 2. & O\,a & [1, D] \end{array}$$

Note, however, that even for the thus modified dialogues there would not be a strategy for the formula a under the assumption $a \wedge b$, although $a \wedge b \vdash_{DI_c^p} a$ should hold. Felscher remarks further that "[c]ertainly, regulations circumventing these difficulties may be formulated, but apparently only at the cost of a loss in intuitive appeal" (*ibid.*).

As we just saw, these difficulties do not disappear if we only allow as an additional regulation that P can assert an atomic formula in the move at position 0 (as it is done, for example, in the definitional dialogues treated in Chapter 4 below). Furthermore—as atomic formulas cannot be attacked in the dialogues considered so far—the dialogue

$$\begin{array}{rl} -1. & O\,a \wedge b \\ 0. & P\,c \end{array}$$

would be a strategy for c under the assumption $a \wedge b$, that is, $a \wedge b \vdash_{DI_c^p} c$ would hold, which should not be the case.

This problem could be avoided if the assumptions $A_1, \ldots, A_n$ stated at positions $p < 0$ were (re)stated by the opponent O at position 1 as a conjunction $A_1 \wedge \ldots \wedge A_n$ (i.e., if such a (re)statement were allowed as an additional move for O). For example, there would then be no strategy for the atomic formula c under the assumption $a \wedge b$ (i.e., for the initial move $P\,c$ at position 0 with move $O\,a \wedge b$ at position -1 repeated at position 1), the dialogue tree being

$$\begin{array}{rll|ll} -1. & & \multicolumn{3}{c}{O\,a \wedge b} \\ 0. & & \multicolumn{3}{c}{P\,c} \\ 1. & & \multicolumn{3}{c}{O\,a \wedge b} \\ 2. & P\,\wedge_1 & [1, A] & P\,\wedge_2 & [1, A] \\ 3. & O\,a & [2, D] & O\,b & [2, D] \end{array}$$

But there would then be no strategy for the formula a under the assumption $a \wedge b$ either:

$$
\begin{array}{rllll}
-1. & & O\, a \wedge b & & \\
0. & & P\, a & & \\
1. & & O\, a \wedge b & & \\
2. & P \wedge_1 & [1, A] & \vert\; P \wedge_2 & [1, A] \\
3. & O\, a & [2, D] & \vert\; O\, b & [2, D]
\end{array}
$$

Whether the conjunction of assumptions is restated at position 1 is inessential to the problem. The move made by O at position 1 could also be a vacuous move. The collection of assumptions is then simply a multiset, a set or a list of O-signed formulas which is not subject to the alternation of P- and O-moves beginning at position 0. The assumptions can be attacked by P according to the respective conditions defining dialogues, and the alternation would be kept for all moves at positions $n \geq 0$. In any case, the assertion by the opponent O of the formula stated by the proponent P at position 0 would have to be interpreted as a condition for P winning the dialogue.

However, this would not solve the problem in general, since there would be no strategy for $a \vee b$ under the assumption $a \wedge b$, for example. What is needed in addition is that P is allowed to state the formula stated at position 0 also in a defense move to the opponent's move at position 1. But this amounts to the reduction of hypothetical dialogues for formulas A under assumptions $A_1, \dots, A_n$ to dialogues for formulas $(A_1 \wedge \dots \wedge A_n) \to A$. The notion of assumption would thus be dependent on the treatment of conjunction and—more importantly—of implication. The resulting dialogues are then not hypothetical anymore, since no formula would be used as an assumption in the genuine sense.

2.9. Digression: dialogues and tableaux

In this digression, we want to put dialogues into a broader context. We describe tableaux as well as Kripke semantics for intuitionistic propositional logic, and we compare them with dialogues. Historically, tableaux have been developed after sequent calculi (cf. Chapter 3) and before dialogues. They are related to both, although with stronger similarities to dialogues. For example, in Felscher's [1985] proof of the equivalence result for sequent calculus derivations and strategies, tableaux[21] are used as an intermediate step. There it is shown that sequent calculus derivations as well as strategies can be transformed into these tableaux, and vice versa.

[21] These are special kinds of tableaux, which Felscher [1985] calls 'IC-protableaux', 'IC-tableaux' and 'irreducible IC-protableaux', respectively.

We first present so-called analytic tableaux for classical propositional logic. Tableaux for intuitionistic propositional logic are then given by imposing certain restrictions on analytic tableaux. After that we will compare Kripke semantics and tableaux for intuitionistic propositional logic. This will be done by reformulating tableaux in terms of the forcing relation used in Kripke semantics.

The purpose of this digression is to point out some similarities as well as differences between dialogues, tableaux and Kripke semantics for intuitionistic propositional logic.[22] We will see that closed tableaux are quite similar to strategies, although there are important differences between dialogues and tableaux as such.

2.9.1. Analytic tableaux. Before we can give the definition of analytic tableaux, we have to extend our language by two more signatures:

DEFINITION 2.9.1. We introduce additional signatures t and f. For formulas A, expressions of the form t A are called t-*signed formulas*, and expressions of the form f A are called f-*signed formulas*. We use $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$ as variables for t- or f-signed formulas.

DEFINITION 2.9.2. (i) An *analytic tableau*[23] for classical propositional logic is a tree of signed formulas which is generated by non-branching α-*rules* of the form $\begin{array}{c}\alpha\\ \alpha_1\\ \alpha_2\end{array}$ or $\begin{array}{c}\alpha\\ \alpha_1\end{array}$ and branching β-*rules* of the form $\begin{array}{c}\beta\\ \beta_1 \mid \beta_2\end{array}$:

$$\alpha\text{-rules}\left\{\begin{array}{ccccc} \mathrm{t}\,A\wedge B & \mathrm{f}\,A\vee B & \mathrm{f}\,A\to B & \mathrm{t}\,\neg A & \mathrm{f}\,\neg A\\ \mathrm{t}\,A & \mathrm{f}\,A & \mathrm{t}\,A & \mathrm{f}\,A & \mathrm{t}\,A\\ \mathrm{t}\,B & \mathrm{f}\,B & \mathrm{f}\,B & & \end{array}\right.$$

$$\beta\text{-rules}\left\{\begin{array}{ccc} \mathrm{f}\,A\wedge B & \mathrm{t}\,A\vee B & \mathrm{t}\,A\to B\\ \mathrm{f}\,A \mid \mathrm{f}\,B & \mathrm{t}\,A \mid \mathrm{t}\,B & \mathrm{f}\,A \mid \mathrm{t}\,B \end{array}\right.$$

(ii) An *analytic tableau for A* is an analytic tableau with root node f A.

(iii) A *closed branch* of an analytic tableau is a branch containing t a as well as f a for any atomic formula a. In this case we also say that the formula a occurs with *contradicting signatures*.

(iv) An *open branch* is a branch which is not closed, that is, a branch in which no atomic formula occurs with contradicting signatures.

[22] For more details and results on semantical completeness for intuitionistic logic we refer to Troelstra and van Dalen [1988b, chapter 13]. Cf. also Troelstra [1993].

[23] Cf. Smullyan [1995] or Smullyan [2009].

(v) An analytic tableau is called *closed* if all its branches are closed, and it is called *open* if it contains an open branch.

DEFINITION 2.9.3. A formula A is called *analytic tableau-provable* if there is a closed analytic tableau for A. Notation: $\vdash_{TA^p} A$.

REMARK 2.9.4. We consider tableaux where to each complex formula a rule may be applied repeatedly in each branch.

For analytic tableaux for classical propositional logic we could impose the restriction that to each complex formula a rule may be applied at most once. For classical first-order logic this restriction would have to be dropped.

In tableaux for intuitionistic logic we will have to allow for repeated rule applications to any complex formula already in the propositional case (cf. Example 2.9.12 below).

EXAMPLE 2.9.5. The following is a closed analytic tableau for $(a \vee b) \to \neg\neg(a \vee b)$:

$$
\begin{array}{lcc|cc}
0. & \multicolumn{4}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} \\
1. & \multicolumn{3}{c}{\mathrm{t}\,a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\,\neg\neg(a \vee b)} & (0) \\
3. & \mathrm{t}\,a & (1) & \mathrm{t}\,b & (1) \\
4. & \mathrm{t}\,\neg(a \vee b) & (2) & \mathrm{t}\,\neg(a \vee b) & (2) \\
5. & \mathrm{f}\,a \vee b & (4) & \mathrm{f}\,a \vee b & (4) \\
6. & \mathrm{f}\,a & (5) & \mathrm{f}\,a & (5) \\
7. & \underline{\mathrm{f}\,b} & (5) & \underline{\mathrm{f}\,b} & (5) \\
 & 3 \times 6 & & 3 \times 7 &
\end{array}
$$

REMARK 2.9.6. For our convenience, we use the following decorations:

(i) We put line numbers to the left of tableaux, and we use numbers (n) to the right of formulas to refer to the line number of the formula a rule has been applied to.

For example, in the above Example 2.9.5 the two nodes t a and t b in line 3 are marked with (1), since they were generated by an application of the β-rule for disjunction to the t-signed formula t $a \vee b$ in line 1.

(ii) The expression $n \times m$ indicates a closed branch, where the numbers n and m refer to the lines containing occurrences of an atomic formula with contradicting signatures in this branch.

These decorations are not part of tableaux as such.

REMARK 2.9.7. For the consequence relation $\vDash$ of classical propositional logic, the following holds: $\vDash A \iff \vdash_{TA^p} A$.[24]

[24] See Smullyan [1995].

2.9.2. Tableaux for intuitionistic logic. Tableaux for intuitionistic logic were first conceived by Beth [1955], [1956]. Here we consider two kinds of such tableaux which are similar to those presented in Fitting [1969], [1983]. Following Fitting [1983], we call the one kind of intuitionistic tableaux *Beth-tableaux* and the other kind *Gentzen-tableaux*.

Beth-tableaux. For the definition of Beth-tableaux we first need the following distinction concerning logical constants:

DEFINITION 2.9.8. We distinguish the *regular logical constants* $\wedge$ and $\vee$ from the *special logical constants* $\neg$ and $\rightarrow$. A *formula A* is *regular*, respectively *special*, if its outermost logical constant is regular, respectively special.[25]

DEFINITION 2.9.9. A *Beth-tableau* is an analytic tableau in which applications of α-rules for special formulas are possible only if all f-signed formulas are deleted from the branch where the α-rule for a special formula is applied. That is, all f-signed formulas (including α if f-signed) in the branch above $\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}$ in the case of implication, respectively above α_1 in the case of negation, have to be deleted. This includes the case where α is not immediately above $\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}$, respectively α_1. That is, if there are f-signed formulas between α and $\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}$, respectively α_1, then those formulas have to be deleted as well.

This deletion is called *intuitionistic branch modification*, and the corresponding rule is called *intuitionistic branch modification rule*, short: IBMR. Its application is indicated by an additional number d_b in a list $(n, d_0, d_1, \ldots, d_k)$ written to the right of the deleted formula. The number d denotes the line and b the branch (numbered 1 to k from left to right), respectively the path belonging to all branches (in which case $b = 0$), of the special formula where the IBMR has been called. (As in analytic tableaux, the number n in the list $(n, d_0, d_1, \ldots, d_k)$ refers to the line number of the formula a rule has been applied to.)

The deletion is restricted to the branch from which the IBMR has been called, that is, the deleted formulas cannot be used further only in that branch, whereas the deletion is irrelevant for all other branches. A deletion in the path belonging to all branches ($b = 0$) is relevant for all branches. Deletions noted in a branch will not be noted again at subsequent calls of the IBMR in that branch.

DEFINITION 2.9.10. A formula A is called *Beth-tableau-provable* if there is a closed Beth-tableau for A. Notation: $\vdash_{TB^p} A$.

---

[25] See Fitting [1983] for this distinction and its motivation.

EXAMPLE 2.9.11. We construct a Beth-tableau for the formula $a \vee \neg a$. Its development starts with the f-signed formula f $a \vee \neg a$ in line 0:

$$\begin{array}{lcl} 0. & \mathrm{f}\, a \vee \neg a & (–) \end{array}$$

As this formula is not the result of a rule application to a formula in a preceding line n, we note the placeholder '–' for n in the comment list: (–). In the next step, we apply the α-rule for disjunction to f $a \vee \neg a$. This is a regular formula, so the IBMR is not called. We get

$$\begin{array}{lcl} 0. & \mathrm{f}\, a \vee \neg a & (–) \\ 1. & \mathrm{f}\, a & (0) \\ 2. & \mathrm{f}\, \neg a & (0) \end{array}$$

Now we apply the α-rule for negation to the special formula f $\neg a$ in line 2 $(= d)$. This yields the formula t a in line 3 and calls the IBMR, deleting all f-signed formulas in this branch 0 $(= b)$. This deletion is noted by the addition of 2_0 $(= d_b)$ in the comments to the right of each deleted formula:

$$\begin{array}{lcl} 0. & \mathrm{f}\, a \vee \neg a & (–, 2_0) \\ 1. & \mathrm{f}\, a & (0, 2_0) \\ 2. & \mathrm{f}\, \neg a & (0, 2_0) \\ 3. & \underline{\mathrm{t}\, a} & (2) \\ & \circ & \end{array}$$

The tableau is open (indicated by $\circ$), since the only remaining signed formula is t a. All other signed formulas have been deleted, and there is thus no contradicting signed formula f a.

Note that no further (repeated) rule applications are possible here, since all complex formulas have been deleted in the last rule application when the IBMR was called. Repeated applications of the α-rule for disjunction to f $a \vee \neg a$ would have been possible only before that. However, no closed tableaux could result in this case. We can therefore argue that any Beth-tableau for the formula $a \vee \neg a$ is either open or infinite, but never closed. The formula $a \vee \neg a$ is thus not Beth-tableau-provable.

EXAMPLE 2.9.12. The following is a closed Beth-tableau for the formula $\neg\neg(a \vee \neg a)$, showing thus $\vdash_{TB^p} \neg\neg(a \vee \neg a)$:

$$\begin{array}{lcl} 0. & \mathrm{f}\, \neg\neg(a \vee \neg a) & (–, 0_0) \\ 1. & \mathrm{t}\, \neg(a \vee \neg a) & (0) \\ 2. & \mathrm{f}\, a \vee \neg a & (1, 4_0) \\ 3. & \mathrm{f}\, a & (2, 4_0) \\ 4. & \mathrm{f}\, \neg a & (2, 4_0) \\ 5. & \mathrm{t}\, a & (4) \end{array}$$

(cont'd on next page)

$$
\begin{array}{lcl}
6. & \mathrm{f}\, a \vee \neg a & (1) \\
7. & \mathrm{f}\, a & (6) \\
8. & \underline{\mathrm{f}\, \neg a} & (6) \\
 & 5 \times 7 &
\end{array}
$$

The formula in line 0 is a special formula. In the first step, the α-rule for f-signed negation was applied to it, calling the IBMR, which deletes the f-signed formula in line 0. Then the α-rule for t-signed negation was applied to the special formula in line 1, also calling the IBMR (however, no further modifications have to be done in this branch, since the only f-signed formula above has already been deleted; we do not extend the list besides the formula in line 0 to $(-, 0_0, 1_0)$ but keep $(-, 0_0)$). This yields the formula in line 2, and an application of the α-rule for disjunction to it yields $\mathrm{f}\, a$ and $\mathrm{f}\, \neg a$. The latter formula is special and the application of the α-rule for f-signed negation to it yields $\mathrm{t}\, a$ and deletes the formulas in lines 2–4.

The only remaining formulas are $\mathrm{t}\, \neg(a \vee \neg a)$ (in line 1) and $\mathrm{t}\, a$ (in line 5). The latter being atomic, only a (repeated) rule application to the former is possible. This yields line 6, and finally lines 7 and 8 by an α-rule application to the formula in line 6. The formulas $\mathrm{t}\, a$ (in line 5) and $\mathrm{f}\, a$ (in line 7) have contradicting signatures, hence the tableau is closed.

Note that there would not be a closed Beth-tableau if repeated rule applications were not allowed.

EXAMPLE 2.9.13. The following are three closed Beth-tableaux for the formula $(a \vee b) \to \neg\neg(a \vee b)$:

(i)

$$
\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\, a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\, \neg\neg(a \vee b)} & (0, 2_0) \\
3. & \multicolumn{3}{c}{\mathrm{t}\, \neg(a \vee b)} & (2) \\
4. & \multicolumn{3}{c}{\mathrm{f}\, a \vee b} & (3) \\
5. & \mathrm{t}\, a & (1) & \mathrm{t}\, b & (1) \\
6. & \mathrm{f}\, a & (4) & \mathrm{f}\, a & (4) \\
7. & \underline{\mathrm{f}\, b} & (4) & \underline{\mathrm{f}\, b} & (4) \\
 & 5 \times 6 & & 6 \times 7 &
\end{array}
$$

(ii)

$$
\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\, a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\, \neg\neg(a \vee b)} & (0, 2_0) \\
3. & \multicolumn{3}{c}{\mathrm{t}\, \neg(a \vee b)} & (2) \\
4. & \mathrm{t}\, a & (1) & \mathrm{t}\, b & (1) \\
5. & \mathrm{f}\, a \vee b & (3) & \mathrm{f}\, a \vee b & (3) \\
6. & \mathrm{f}\, a & (5) & \mathrm{f}\, a & (5)
\end{array}
$$

(cont'd on next page)

$$
\begin{array}{lcc|cc}
7. & \dfrac{\mathrm{f}\, b}{4 \times 6} & (5) & \dfrac{\mathrm{f}\, b}{4 \times 7} & (5)
\end{array}
$$

(iii)

$$
\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\, a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\, \neg\neg(a \vee b)} & (0, 2_1, 2_2) \\
3. & \mathrm{t}\, a & (1) & \mathrm{t}\, b & (1) \\
4. & \mathrm{t}\, \neg(a \vee b) & (2) & \mathrm{t}\, \neg(a \vee b) & (2) \\
5. & \mathrm{f}\, a \vee b & (4) & \mathrm{f}\, a \vee b & (4) \\
6. & \mathrm{f}\, a & (5) & \mathrm{f}\, a & (5) \\
7. & \dfrac{\mathrm{f}\, b}{3 \times 6} & (5) & \dfrac{\mathrm{f}\, b}{3 \times 7} & (5)
\end{array}
$$

They correspond to the strategies (i), (ii) and (iii), respectively, given in Example 2.2.16.

REMARK 2.9.14. As we have to allow for more than one rule application to any complex formula in a branch, branches can be infinite. Infinite branches in a tableau correspond to infinite dialogues. For example, consider the following Beth-tableau:

$$
\begin{array}{lcc|cc|cc|cc}
0. & \multicolumn{7}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{7}{c}{\mathrm{t}\, a \vee b} & (0) \\
2. & \multicolumn{7}{c}{\mathrm{f}\, \neg\neg(a \vee b)} & (0, 2_3, 2_4) \\
3. & \multicolumn{5}{c}{\mathrm{t}\, a} & (1) & \mathrm{t}\, b & (1) \\
4. & \multicolumn{3}{c}{\mathrm{t}\, a} & (1) & \mathrm{t}\, b & (1) & \mathrm{t}\, \neg(a \vee b) & (2) \\
5. & \mathrm{t}\, a & (1) & \mathrm{t}\, b & (1) & \mathrm{t}\, \neg(a \vee b) & (2) & \mathrm{f}\, a \vee b & (4) \\
6. & \vdots & & & & \mathrm{f}\, a \vee b & (5) & \mathrm{f}\, a & (5) \\
7. & & & & & \mathrm{f}\, a & (6) & \dfrac{\mathrm{f}\, b}{3 \times 7} & (6) \\
8. & & & & & \dfrac{\mathrm{f}\, b}{3 \times 7} & (7) & &
\end{array}
$$

The leftmost branch is infinite (and thus not closed) if repetitive rule applications are made on the signed formula in line 1. It corresponds to the infinite dialogue in Remark 2.2.23 (i).

Gentzen-tableaux. An alternative to Beth-tableaux are Gentzen-tableaux for intuitionistic logic. The latter are more similar to Gentzen's sequent calculus for intuitionistic logic than the former. For a detailed treatment (also extending Gentzen-tableaux to intuitionistic first-order logic) we refer to Fitting [1983].

DEFINITION 2.9.15. A *Gentzen-tableau* is an analytic tableau where in applications of α- and β-rules to f-signed formulas the IBMR has to be applied. To t-signed formulas the α- and β-rules can be applied as in analytic tableaux, that is, without any application of the IBMR.

DEFINITION 2.9.16. A formula A is called *Gentzen-tableau-provable* if there is a closed Gentzen-tableau for A. Notation: $\vdash_{TG^p} A$.

EXAMPLE 2.9.17. For comparison with the closed Beth-tableau shown in Example 2.9.12, we give a closed Gentzen-tableau for the same formula $\neg\neg(a \vee \neg a)$:

$$\begin{array}{rcl}
0. & \mathrm{f}\,\neg\neg(a \vee \neg a) & (-, 0_0) \\
1. & \mathrm{t}\,\neg(a \vee \neg a) & (0) \\
2. & \mathrm{f}\,a \vee \neg a & (1, 2_0) \\
3. & \mathrm{f}\,a & (2, 4_0) \\
4. & \mathrm{f}\,\neg a & (2, 4_0) \\
5. & \mathrm{t}\,a & (4) \\
6. & \mathrm{f}\,a \vee \neg a & (1, 6_0) \\
7. & \mathrm{f}\,a & (6) \\
8. & \underline{\mathrm{f}\,\neg a} & (6) \\
 & 5 \times 7 &
\end{array}$$

The formula $\neg\neg(a \vee \neg a)$ is thus Gentzen-tableau-provable.

REMARK 2.9.18. Gentzen-tableaux are more closely related to sequent calculi like LI^p (see Definition 3.1.1 in the next chapter or Appendix A.8) than Beth-tableaux are. With the exception of α-rule applications to disjunctive f-signed formulas $\mathrm{f}\,A_1 \vee A_2$, each branch in a Gentzen-tableau contains at most one f-signed formula. This corresponds to the fact that LI^p-sequents can have at most one formula in the succedent, where the f-signed formulas in Gentzen-tableaux are exactly the formulas in the succedents of LI^p-sequents. The exception with disjunctive f-signed formulas $\mathrm{f}\,A_1 \vee A_2$ corresponds to the fact that in LI^p the right introduction rule for disjunction $(\vdash \vee)$ can have the sequent $\Gamma \vdash A_1$ or the sequent $\Gamma \vdash A_2$ as premiss.

For example, compare the following LI^p-derivation (see also Remark 3.8.3) with the Gentzen-tableau shown in the above Example 2.9.17:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{Id}_a)\;\dfrac{}{a \vdash_{LI^p} a}}{a \vdash_{LI^p} a \vee \neg a}\,(\vdash \vee)}{a, \neg(a \vee \neg a) \vdash_{LI^p}}\,(\neg\vdash)}{\neg(a \vee \neg a) \vdash_{LI^p} \neg a}\,(\vdash \neg)}{\neg(a \vee \neg a) \vdash_{LI^p} a \vee \neg a}\,(\vdash \vee)}{\neg(a \vee \neg a), \neg(a \vee \neg a) \vdash_{LI^p}}\,(\neg\vdash)}{\neg(a \vee \neg a) \vdash_{LI^p}}\,(\mathrm{Contr})}{\vdash_{LI^p} \neg\neg(a \vee \neg a)}\,(\vdash \neg)$$

(The application of contraction (Contr) in this LI^p-derivation corresponds in the Gentzen-tableau to the twofold application of the α-rule for negation to the t-signed formula $\mathrm{t}\,\neg(a \vee \neg a)$ in line 1. This yields $\mathrm{f}\,a \vee \neg a$ in line 2 and again in line 6.)

Beth-tableaux are less closely related to the sequent calculus LI^p (or to similar sequent calculi with at most one formula in the succedent of sequents), since Beth-tableaux allow for more than one usable f-signed formula in general.

EXAMPLE 2.9.19. For comparison with the three closed Beth-tableaux given in Example 2.9.13, we show the following three corresponding closed Gentzen-tableaux for the formula $(a \vee b) \to \neg\neg(a \vee b)$:

(i)

$$\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\,a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\,\neg\neg(a \vee b)} & (0, 2_0) \\
3. & \multicolumn{3}{c}{\mathrm{t}\,\neg(a \vee b)} & (2) \\
4. & \multicolumn{3}{c}{\mathrm{f}\,a \vee b} & (3, 4_0) \\
5. & \mathrm{t}\,a & (1) & \mathrm{t}\,b & (1) \\
6. & \mathrm{f}\,a & (4) & \mathrm{f}\,a & (4) \\
7. & \underline{\mathrm{f}\,b} & (4) & \underline{\mathrm{f}\,b} & (4) \\
 & 5 \times 6 & & 5 \times 7 &
\end{array}$$

(ii)

$$\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\,a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\,\neg\neg(a \vee b)} & (0, 2_0) \\
3. & \multicolumn{3}{c}{\mathrm{t}\,\neg(a \vee b)} & (2) \\
4. & \mathrm{t}\,a & (1) & \mathrm{t}\,b & (1) \\
5. & \mathrm{f}\,a \vee b & (3, 5_0) & \mathrm{f}\,a \vee b & (3, 5_0) \\
6. & \mathrm{f}\,a & (5) & \mathrm{f}\,a & (5) \\
7. & \underline{\mathrm{f}\,b} & (5) & \underline{\mathrm{f}\,b} & (5) \\
 & 4 \times 6 & & 4 \times 7 &
\end{array}$$

(iii)

$$\begin{array}{lcc|cc}
0. & \multicolumn{3}{c}{\mathrm{f}\,(a \vee b) \to \neg\neg(a \vee b)} & (-, 0_0) \\
1. & \multicolumn{3}{c}{\mathrm{t}\,a \vee b} & (0) \\
2. & \multicolumn{3}{c}{\mathrm{f}\,\neg\neg(a \vee b)} & (0, 2_1, 2_2) \\
3. & \mathrm{t}\,a & (1) & \mathrm{t}\,b & (1) \\
4. & \mathrm{t}\,\neg(a \vee b) & (2) & \mathrm{t}\,\neg(a \vee b) & (2) \\
5. & \mathrm{f}\,a \vee b & (4, 5_1) & \mathrm{f}\,a \vee b & (4, 5_2) \\
6. & \mathrm{f}\,a & (5) & \mathrm{f}\,a & (5) \\
7. & \underline{\mathrm{f}\,b} & (5) & \underline{\mathrm{f}\,b} & (5) \\
 & 3 \times 6 & & 3 \times 7 &
\end{array}$$

These three closed Gentzen-tableaux correspond to the strategies (i), (ii) and (iii), respectively, as shown in Example 2.2.16.

REMARK 2.9.20. The Gentzen-tableaux in Examples 2.9.17 and 2.9.19 differ from the Beth-tableaux in Examples 2.9.12 and 2.9.13, respectively, only with respect to the signed formulas deleted by applications of the IBMR. The trees of signed formulas are the same, respectively.

REMARK 2.9.21. For Gentzen-tableaux, the *disjunction property*

$$\text{if} \vdash_{TG^p} A \vee B, \text{ then} \vdash_{TG^p} A \text{ or} \vdash_{TG^p} B$$

can be easily established as follows (cf. Fitting [1983, proposition 5.1, p. 463]): A closed Gentzen-tableaux for $A \vee B$ begins with

$$\begin{array}{lcl} 0. & \mathrm{f}\, A \vee B & (-, 0_0) \\ 1. & \mathrm{f}\, A & (0) \\ 2. & \mathrm{f}\, B & (0) \\ 3. & \vdots & \end{array}$$

leaving only f A and f B for the next rule application. In both cases, this rule application calls the IBMR, which applies to all formulas above line 3. Thus an application to f A will delete f B and an application to f B will delete f A. Therefore there has to be either a closed tableau for f A or a closed tableau for f B.

REMARK 2.9.22. It can be seen that for the disjunction property the use of the IBMR in Gentzen-tableaux has the same effect as condition $(D12)$ in dialogues.

If the proponent P has a strategy for a closed formula of the form $A \vee B$, then P can defend the attack $O \vee$ on this formula only by asserting either A or B. Because of condition $(D12)$, the proponent P cannot later defend again against this attack by asserting B, respectively A. There must thus be a strategy for A or for B.[26]

2.9.3. Kripke semantics and intuitionistic tableaux. We now compare Beth- and Gentzen-tableaux with Kripke semantics for intuitionistic propositional logic. Kripke semantics[27] can be motivated by considering weak counterexamples to classical logical laws.[28]

Weak counterexamples. Consider the following example of a classical, non-constructive proof:

EXAMPLE 2.9.23. The statement

"There exist two irrational numbers x and y such that x^y is rational"

[26] Cf. Rückert [2007, p. 23f.].

[27] See Kripke [1963], [1965]. Cf. also Moschovakis [2010], van Dalen [2001], [2002], [2008] or van Dalen and van Atten [2006], for example.

[28] Cf. Dummett [2000], Moschovakis [2010], Troelstra and van Dalen [1988a] or van Dalen [2008], for example.

can be easily proved as follows:

Suppose $\sqrt{2}^{\sqrt{2}}$ is rational. Then there exist two irrational numbers x and y such that x^y is rational. Suppose $\sqrt{2}^{\sqrt{2}}$ is irrational. Then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is rational. Using *tertium non datur* ($\sqrt{2}^{\sqrt{2}}$ is rational or not, i.e. irrational) the above statement can be inferred.

REMARK 2.9.24. (i) The proof in Example 2.9.23 is *not constructive*, since it does not produce two numbers x and y such that the number x^y is rational.

(ii) Example 2.9.23 is a *weak counterexample* for the *tertium non datur*. From a constructivist perspective, the *tertium non datur* $A \vee \neg A$ says that for any statement A we have a proof of A or a proof of $\neg A$ (i.e., a construction which transforms a hypothetical proof of A into a proof of $\perp$). This would enable us to decide for any statement A whether this statement holds or not. But an example like the still undecided validity of the statement "There exist infinitely many twin primes" shows that this is not the case.

(iii) The counterexample is *weak* because it does not refute the *tertium non datur*, that is, from the assumption of the *tertium non datur* there has not been derived a contradiction. It was merely shown that the *tertium non datur* is not a constructively acceptable logical principle.

(iv) Furthermore, it is impossible (from a constructivist perspective) to refute the *tertium non datur* by presenting a statement A for which $\neg(A \vee \neg A)$ holds, since $\neg\neg(A \vee \neg A)$ holds constructively for all statements A.

Kripke semantics. The principle of *tertium non datur* does not hold in intuitionistic logic in general. A motivation for Kripke semantics for intuitionistic logic can be given as follows:

MOTIVATION 2.9.25. Weak counterexamples for the *tertium non datur* proceed on the assumption of a yet undecided statement A. But it is not precluded that a proof of A is found later. This situation can be depicted as follows:

$$\begin{tikzcd} k_1 \bullet \; A \\ k_0 \bullet \arrow[u] \end{tikzcd}$$

The state k_0 represents our present knowledge about A: We do not know whether A holds or $\neg A$ holds. The state k_1 represents a later point in time in which a proof of A has been found, that is, where we know that A holds. Thus we cannot assert A in k_0. But we cannot assert $\neg A$ in k_0 either, because a later state k_1 in which A holds is not precluded. Therefore we also cannot assert $A \vee \neg A$ in k_0, since for this we would have to be able to either assert A or $\neg A$ in k_0.[29]

DEFINITION 2.9.26. A (propositional) *Kripke-model* $\mathscr{K}$ is a triple

$$\langle K, \leq, \Vdash \rangle$$

where the *Kripke-frame* $\langle K, \leq \rangle$ is a non-empty partially ordered set, and $\Vdash$ (read: *forces*) is a binary relation (the *forcing relation*) of elements k of K and (propositional) formulas, such that for atomic formulas a the following monotonicity condition holds:

$$\text{If } k \Vdash a \text{ and } k \leq k', \text{ then } k' \Vdash a.$$

The elements k of K are called *states* or *possible worlds*. ($k \Vdash a$ is read "k forces a" or "a holds in k".)

The relation $\Vdash$ is extended to complex formulas A, B by the following clauses:

(i) $k \Vdash A \wedge B := k \Vdash A$ and $k \Vdash B$,

(ii) $k \Vdash A \vee B := k \Vdash A$ or $k \Vdash B$,

(iii) $k \Vdash A \to B :=$ For all $k' \geq k$: if $k' \Vdash A$, then $k' \Vdash B$,

(iv) not $k \Vdash \bot$ (resp. $k \nVdash \bot$), that is, there is no element k of K such that $\bot$ holds in k.

REMARK 2.9.27. (i) It follows from the definition that $k \Vdash \neg A$ if and only if $\forall k' \geq k\ (k' \nVdash A)$.

(ii) Furthermore, it holds that $k \Vdash \neg\neg A$ if and only if $\forall k' \geq k\ \neg\forall k'' \geq k'\ (k'' \nVdash A)$, since then $\forall k' \geq k\ (k' \nVdash \neg A)$ and thus $k \Vdash \neg\neg A$. (Classically this is equivalent to $\forall k' \geq k\ \exists k'' \geq k'\ (k'' \Vdash A)$.)

LEMMA 2.9.28. *Every formula A fulfills the monotonicity condition, that is, for all $k, k' \in K$: If $k \Vdash A$ and $k \leq k'$, then $k' \Vdash A$.*

PROOF. By induction on formulas. Consider as an example the case $A \equiv B \to C$: Let $k \Vdash B \to C$ and $k \leq k'$. If $k' \leq k''$ and $k'' \Vdash B$, then also $k \leq k''$ and $k'' \Vdash B$. Since $k \Vdash B \to C$ holds, $k'' \Vdash C$ holds then as well. Thus for all $k'' \geq k'$: If $k'' \Vdash B$, then $k'' \Vdash C$, that is, $k' \Vdash B \to C$. ⊣

DEFINITION 2.9.29. (i) A formula A is *valid in k* in a Kripke-model $\mathscr{K}$ if and only if $k \Vdash A$.

[29] This common epistemic/temporal interpretation serves here only as a possible motivation for the mathematical structure to be defined next. We are here only interested in that structure. Cf. Troelstra and van Dalen [1988a] for what follows.

(ii) A formula A is *valid in* $\mathscr{K}$ if and only if $k \Vdash A$ for all $k \in K$ (Notation: $\mathscr{K} \Vdash A$).

(iii) For a set X of formulas, $X \Vdash A$ holds if and only if in each Kripke-model $\mathscr{K}$ in which all $B \in X$ are valid, A is valid as well, that is, if for all $B \in X$: If $\mathscr{K} \Vdash B$, then $\mathscr{K} \Vdash A$.

(iv) A formula A is *Kripke-valid* if and only if $X = \emptyset$, that is, if $\emptyset \Vdash A$ (short: $\Vdash A$).

REMARK 2.9.30. If k_0 is the smallest state in the partially ordered set K, then by monotonicity (Lemma 2.9.28) it holds that A is valid in $\mathscr{K}$ if and only if A is valid in k_0.

REMARK 2.9.31. It can be shown that the Kripke-valid formulas are exactly the formulas provable in sequent calculus for intuitionistic propositional logic.

Kripke-models can thus be used to show that a formula cannot be proved in sequent calculus for intuitionistic propositional logic, by showing that the formula is not Kripke-valid, that is, by presenting a Kripke-model which is a counterexample for the formula in question.

EXAMPLE 2.9.32. Using atomic formulas a and b, we give Kripke-counterexamples in diagrammatic form.

(i) $\neg\neg a \vee \neg a$ is not Kripke-valid. Let $K = \{k_0, k_1, k_2\}$, $k_0 \leq k_1, k_0 \leq k_2$ and $k_1 \Vdash a$. Then $\mathscr{K} = \langle K, \leq, \Vdash \rangle$ can be represented by the following diagram, where to the right of every state $k \in K$ only those atomic formulas a are noted for which $k \Vdash a$ holds, and where the arrows express the partial order on K as given by $\leq$:

$$
\begin{tikzcd}
k_1 \bullet\ a & & k_2 \bullet \\
& k_0 \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Due to $k_1 \Vdash a$ we have $k_0 \nVdash \neg a$, and since $k_2 \Vdash \neg a$ holds, $k_0 \nVdash \neg\neg a$ holds. Therefore $k_0 \nVdash \neg\neg a \vee \neg a$ holds, and the Kripke-model $\mathscr{K}$ is a Kripke-counterexample for $\neg\neg a \vee \neg a$, that is, $\neg\neg a \vee \neg a$ is not Kripke-valid.

(ii) $\neg\neg a \to a$ is not Kripke-valid. In the Kripke-model

$$
\begin{tikzcd}
k_1 \bullet\ a \\
k_0 \bullet \arrow[u]
\end{tikzcd}
$$

$k_0 \nVdash a$ holds, and due to $k_1 \Vdash a$ we have $k_0 \Vdash \neg\neg a$ (cf. Remark 2.9.27 (ii)). Therefore $k_0 \nVdash \neg\neg a \to a$ holds, that is, $\neg\neg a \to a$ is not Kripke-valid.

(iii) $(a \to b) \to (\neg a \vee b)$ is not Kripke-valid. In the Kripke-model

$$\begin{tikzcd}
& k_3 \bullet\ a, b & \\
k_1 \bullet\ b \arrow[ur] & & k_2 \bullet\ a, b \arrow[ul] \\
& k_0 \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$k_3 \Vdash a \to b$ holds, and $k_1 \Vdash a \to b$ holds due to $k_1 \Vdash b$ and $k_3 \Vdash b$. Therefore with $k_0 \nVdash a$ and $k_2 \Vdash b$ also $k_0 \Vdash a \to b$ holds. $k_0 \nVdash \neg a$ holds due to $k_2 \Vdash a$, and since $k_0 \nVdash b$ holds, $k_0 \nVdash \neg a \vee b$ holds. Hence $k_0 \nVdash (a \to b) \to (\neg a \vee b)$ holds, that is, $(a \to b) \to (\neg a \vee b)$ is not Kripke-valid.

REMARK 2.9.33. In a certain sense, the construction of intuitionistic tableaux for a formula A could be understood as the search for a countermodel for A. However, from an open tableau for A alone we cannot necessarily extract a countermodel in general, as we just might not have found a closed tableau for A yet (this is different from analytic tableaux for classical (propositional) logic, where an open tableau for a formula A excludes the existence of a closed tableau for A, and is hence sufficient for having a countermodel; see also Remark 2.9.37 below).

Comparison with Beth- and Gentzen-tableaux. A common feature of intuitionistic tableaux and Kripke semantics can be seen in the use of certain ordered states. In Kripke-models, this is given directly by the frames $\langle K, \leq \rangle$. In intuitionistic tableaux, collections of signed formulas can be interpreted as states, which are given an ordering by applications of the IBMR. For example, in the Beth-tableau for $\neg\neg(a \vee \neg a)$ (cf. Example 2.9.12) we can distinguish the following three states as ordered by applications of the IBMR:

$$k_0 = \left\{ \ \mathrm{f}\, \neg\neg(a \vee \neg a) \right. \qquad k_1 = \left\{ \begin{array}{l} \mathrm{t}\, \neg(a \vee \neg a) \\ \mathrm{f}\, a \vee \neg a \\ \mathrm{f}\, a \\ \mathrm{f}\, \neg a \end{array} \right. \qquad k_2 = \left\{ \begin{array}{l} \mathrm{t}\, a \\ \mathrm{f}\, a \vee \neg a \\ \mathrm{f}\, a \\ \mathrm{f}\, \neg a \end{array} \right.$$

(For Gentzen-tableaux a similar interpretation can be upheld, but the condition for applying the IBMR leads to different states. In the following, we concentrate on Beth-tableaux as point of departure.)

To make this connection between intuitionistic tableaux and Kripke semantics clearer, we first reformulate the tableau rules by using the forcing relation $\Vdash$, respectively the negated forcing relation $\nVdash$, with unsigned formulas:[30]

$$\alpha\text{-rules}\begin{cases} k \Vdash A \wedge B & k \nVdash A \vee B & k \nVdash A \to B & k \Vdash \neg A & k \nVdash \neg A \\ k \Vdash A & k \nVdash A & k' \Vdash A & k' \nVdash A & k' \Vdash A \\ k \Vdash B & k \nVdash B & k' \nVdash B & & \end{cases}$$

$$\beta\text{-rules}\begin{cases} k \nVdash A \wedge B & k \Vdash A \vee B & k \Vdash A \to B \\ k \nVdash A \mid k \nVdash B & k \Vdash A \mid k \Vdash B & k' \nVdash A \mid k' \Vdash B \end{cases}$$

where $k \leq k'$.

Instead of using t- or f-signed formulas, the formulas $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$ in the α- and β-rules are now of the form $k \Vdash A$ and $k \nVdash A$.

We do not use any kind of intuitionistic branch modification like the one effected by the IBMR. Instead, the following proviso has to be observed:

($*$) In the case of α-rule applications to $k \nVdash A \to B$ and $k \nVdash \neg A$, the state k' has to be new; that is, the k' in $\begin{matrix}\alpha_1\\ \alpha_2\end{matrix}$, respectively α_1, of such an α-rule application must not occur in any formula preceding α_1. In the case of $k \Vdash A \to B$ and $k \Vdash \neg A$, the state k' can be any state such that $k \leq k'$.

A branch is closed if it contains the formulas $k \Vdash A$ and $k \nVdash A$.

EXAMPLE 2.9.34. The following is such a reformulated closed tableau for the formula $\neg\neg(a \vee \neg a)$:

$$\begin{array}{lcc} 0. & k_0 \nVdash \neg\neg(a \vee \neg a) & \\ 1. & k_1 \Vdash \neg(a \vee \neg a) & (0) \\ 2. & k_1 \nVdash a \vee \neg a & (1) \\ 3. & k_1 \nVdash a & (2) \\ 4. & k_1 \nVdash \neg a & (2) \\ 5. & k_2 \Vdash a & (4) \\ 6. & k_2 \nVdash a \vee \neg a & (1) \\ 7. & k_2 \nVdash a & (6) \\ 8. & \underline{k_2 \nVdash \neg a} & (6) \\ & 5 \times 7 & \end{array}$$

(Note that $k_1 \nVdash a$ (in line 3) does not contradict $k_2 \Vdash a$ (in line 5), since $k_1 \neq k_2$.)

The introduction of new states k' in tableaux according to the proviso ($*$) ensures that the monotonicity condition for the corresponding Kripke-models is only fulfilled for formulas A which are forced in k (i.e., where $k \Vdash A$), and that it cannot be fulfilled for formulas A which are not forced in k (i.e., where $k \nVdash A$).

[30] A similar formulation with signed formulas can be found in Nerode [1990].

The same result is achieved by using instead a trivially reformulated version of the IBMR which now deletes formulas of the form $\nVdash A$. The tableau rules can then be reformulated—without making any reference to states—as follows:

$$\alpha\text{-rules}\left\{\begin{array}{ccccc} \Vdash A \wedge B & \nVdash A \vee B & \nVdash A \to B & \Vdash \neg A & \nVdash \neg A \\ \Vdash A & \nVdash A & \Vdash A & \nVdash A & \Vdash A \\ \Vdash B & \nVdash B & \nVdash B & & \end{array}\right.$$

$$\beta\text{-rules}\left\{\begin{array}{ccc} \nVdash A \wedge B & \Vdash A \vee B & \Vdash A \to B \\ \nVdash A \mid \nVdash B & \Vdash A \mid \Vdash B & \nVdash A \mid \Vdash B \end{array}\right.$$

where the symbols $\Vdash$, respectively $\nVdash$, have now simply replaced the signatures t, respectively f, in the rules for Beth- and Gentzen-tableaux.

EXAMPLE 2.9.35. The following is a reformulated closed Beth-tableau (with the trivially reformulated IBMR) for the formula $\neg\neg(a \vee \neg a)$:

$$\begin{array}{lcl} 0. & \nVdash \neg\neg(a \vee \neg a) & (-, 0_0) \\ 1. & \Vdash \neg(a \vee \neg a) & (0) \\ 2. & \nVdash a \vee \neg a & (1, 4_0) \\ 3. & \nVdash a & (2, 4_0) \\ 4. & \nVdash \neg a & (2, 4_0) \\ 5. & \Vdash a & (4) \\ 6. & \nVdash a \vee \neg a & (1) \\ 7. & \nVdash a & (6) \\ 8. & \underline{\nVdash \neg a} & (6) \\ & 5 \times 7 & \end{array}$$

REMARK 2.9.36. It can be shown that $\vdash_{TB^p} A \iff \vdash_{TG^p} A \iff \Vdash A$.[31]

REMARK 2.9.37. For open tableaux the following holds: Whereas in the case of analytic tableaux (i.e., in the case of tableaux for classical propositional logic) one can always conclude from an open tableau for A that A is not valid, this is not so for intuitionistic tableaux. Here Kripke-validity of a formula A corresponds to the existence of a closed tableau for A, but the existence of an open tableau for A does in general not correspond to A not being Kripke-valid.

2.9.4. Relations between tableaux and dialogues. We have presented analytic tableaux for classical propositional logic to begin with.[32] Based on analytic tableaux we have then given two kinds of tableaux for intuitionistic propositional logic: Beth-tableaux and Gentzen-tableaux.[33] These in turn

[31] See Fitting [1983], Troelstra and van Dalen [1988b] and the references given therein.
[32] Cf. Smullyan [1995].
[33] Cf. Beth [1955], [1956] and Fitting [1969], [1983].

have been compared with Kripke semantics for intuitionistic propositional logic.

We conclude with some observations concerning the relations between closed tableaux and strategies, respectively between dialogues and tableaux.

On the level of dialogues and tableaux, one can observe the following similarities:

(i) Both argumentation forms and tableau rules operate with signed formulas. In both cases there are two signatures.

(ii) Each 'application' of an argumentation form or application of a tableau rule (i.e., of an α- or β-rule) yields a formula (or formulas) of less complexity than the complexity of the formula the argumentation form or tableau rule has been applied to.

(iii) Neither dialogues nor tableaux have explicit rules for structural operations like thinning, contraction or cut (for cut our EI°-dialogues[34] will be an exception). The structural operations of thinning and contraction are only implicitly given in both cases.

(iv) For both dialogues and tableaux there is given a condition to terminate their development after finitely many moves or tableau rule applications, respectively. For dialogues this is the condition for P winning a dialogue; for tableaux this is the condition for closing a branch (viz. by observing that an atomic formula occurs with contradicting signatures in a branch).

Differences between dialogues and tableaux are:

(i) Dialogues are linear, since they are given as sequences of moves; whereas tableaux are binary trees due to the branching β-rules.

(ii) In tableaux all signed expressions are formulas, whereas dialogues have also signed expressions which are not formulas but are special symbols (like e.g. $\wedge_1$ and $\wedge_2$).

(iii) Excepting the case of negation, each application of an α- or β-rule yields two signed formulas in tableaux, whereas dialogues are developed one move at a time, such that defense moves to given attack moves need not necessarily be made.

(iv) In dialogues the possible moves are regulated by conditions like $(D10)$–$(D13)$ etc. No such conditions are necessary in the case of analytic tableaux, and in the case of Beth- and Gentzen-tableaux the only additional condition is the IBMR. Note, however, that the IBMR only operates on signed formulas which are already part of a tableau

[34]See Chapter 5.

as a result of applications of α- or β-rules. In dialogues, on the other hand, the conditions can prevent moves which would be in accordance with the argumentation forms as such.

Despite the similarities between dialogues and tableaux, transformations of dialogues into tableaux, and vice versa, are rendered quite complicated by these differences (see Felscher [1985]).

Although there are important differences between *dialogues* and *tableaux*, it can be observed that *closed tableaux* and *strategies* are quite similar in structure.

2.10. Summary

We have introduced the basic notions needed for a dialogical treatment of propositional intuitionistic (resp. classical) logic: argumentation forms, dialogues and strategies. We have then introduced variants of dialogues, namely DI^p-dialogues, classical dialogues, DI_c^p-, EI^p- and EI_c^p-dialogues as well as hypothetical dialogues. The distinguishing feature of DI_c^p-dialogues and of EI_c^p-dialogues is that these dialogues—when won by the proponent P—need not end with the assertion of an atomic formula but can end with the assertion of a complex formula; whereas DI^p- and EI^p-dialogues won by the proponent P can only end with the assertion of an atomic formula. The DI_c^p- and EI_c^p-dialogues will be of special importance in the following investigations. In a final digression, dialogues have been put into a broader context by considering relations between dialogues and tableaux.

Chapter 3

EQUIVALENCE RESULTS FOR STRATEGIES AND DERIVATIONS

We consider the sequent calculus LI_c^p as a generalization of the sequent calculus LI^p for intuitionistic propositional logic. As a main result we then prove the equivalence of LI_c^p-provability and EI_c^p-dialogue-provability by showing that EI_c^p-strategies and LI_c^p-derivations can be transformed into each other. This is done similarly to the proof for classical propositional logic given by Sørensen and Urzyczyn [2006], [2007].

The consideration of the sequent calculus LI_c^p and the introduction of EI_c^p-dialogues is motivated by our goal to extend dialogues to *definitional dialogues* (see Chapter 4). Definitional dialogues will allow for reasoning about definitions for atomic formulas. The defining conditions of these atomic formulas do not necessarily have to be given by atomic formulas only, but can be given by any complex formula. We therefore have to make sure that it is possible that dialogues in a strategy can not only end with P-moves asserting atomic formulas, but that they can also end with P-moves asserting complex formulas. In sequent calculus this corresponds to having initial sequents not only for atomic but also for complex formulas. Definitional extensions of sequent calculus have been developed and analyzed by Hallnäs and Schroeder-Heister [1990], [1991].[35]

The equivalence result for EI_c^p-dialogues to be proved in this chapter will thus serve two purposes. First, it establishes the fact that EI_c^p-dialogues validate exactly intuitionistic propositional logic. Second, it shows that dialogues can in principle be used as a basis for reasoning about definitions of the kind described. We can therefore provide dialogical foundations for definitional reasoning.

We will also explain how structural operations like thinning, contraction and exchange are incorporated in dialogues, and how this corresponds to the structural rules of sequent calculi. Contraction is particularly important in definitional reasoning, and more will be said about it in Chapter 4.

[35]See also Hallnäs [1991] and Schroeder-Heister [1992], [1993], [1994a].

Finally, we will introduce dialogues and sequent calculi for intuitionistic first-order logic. Of special interest are the sequent calculus LI_c and EI_c-dialogues. These extend the sequent calculus LI_c^p and EI_c^p-dialogues, respectively, from intuitionistic propositional logic to intuitionistic first-order logic. It will be shown that the equivalence result for LI_c^p-provability and EI_c^p-dialogue-provability can be generalized for the sequent calculus LI_c and EI_c-dialogues. As a consequence, dialogical foundations in the sense of formal dialogue semantics can be provided for definitional reasoning about definitions containing any first-order formulas as defining conditions of atomic formulas; this will be done in Chapter 4.

3.1. The sequent calculus LI^p

We give the sequent calculus LI^p for intuitionistic propositional logic. Note that the axiom (Id_a) is restricted to atomic formulas, that is, initial sequents $a \vdash a$ can only contain atomic formulas a. The sequent calculus LI^p is in this (and only in this) respect more closely related to the system **G3i** of Troelstra and Schwichtenberg [2000], for example, than to Gentzen's original calculus LJ (see Gentzen [1935]). In the latter, initial sequents are of the form $A \vdash A$, where A can be an arbitrary formula. Such initial sequents will be allowed in LI_c^p.

DEFINITION 3.1.1. The *sequent calculus LI^p for intuitionistic propositional logic* consists of the following rules, where Γ and Δ are finite multisets of formulas (the comma in antecedents of sequents stands for multiset union, and singletons are written without braces):

Axiom

$$(\mathrm{Id}_a)\ \frac{}{a \vdash a}\ \text{(where } a \text{ is atomic)}$$

Logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1,2) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad\qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1,2)$$

(cont'd on next page)

$$(\to\vdash)\ \frac{\Gamma\vdash A \qquad \Delta, B\vdash C}{\Gamma,\Delta, A\to B\vdash C} \qquad\qquad \frac{\Gamma, A\vdash B}{\Gamma\vdash A\to B}\ (\vdash\to)$$

Structural rules

$$(\text{Thin}\vdash)\ \frac{\Gamma\vdash C}{\Gamma, A\vdash C} \qquad\qquad \frac{\Gamma\vdash}{\Gamma\vdash A}\ (\vdash\text{Thin})$$

$$(\text{Contr})\ \frac{\Gamma, A, A\vdash C}{\Gamma, A\vdash C}$$

$$(\text{Cut})\ \frac{\Gamma\vdash A \qquad \Delta, A\vdash B}{\Gamma,\Delta\vdash B}$$

The rules $(\neg\vdash)$, $(\wedge\vdash)$, $(\vee\vdash)$ and $(\to\vdash)$ are called *left introduction rules*, and the rules $(\vdash\neg)$, $(\vdash\wedge)$, $(\vdash\vee)$ and $(\vdash\to)$ are called *right introduction rules*.

REMARK 3.1.2. *(Sequent calculus) derivations* are defined as usual.

EXAMPLE 3.1.3. Without using structural rules, there are the following three derivations in LI^p of the sequent $\vdash(a\vee b)\to\neg\neg(a\vee b)$:

(i)

$$\dfrac{\dfrac{(\vee\vdash)\ \dfrac{\dfrac{(\mathrm{Id}_a)\ \dfrac{}{a\vdash a}}{a\vdash a\vee b}\ (\vdash\vee) \qquad \dfrac{(\mathrm{Id}_a)\ \dfrac{}{b\vdash b}}{b\vdash a\vee b}\ (\vdash\vee)}{(\neg\vdash)\ \dfrac{a\vee b\vdash a\vee b}{a\vee b,\neg(a\vee b)\vdash}}}{a\vee b\vdash\neg\neg(a\vee b)}\ (\vdash\neg)}{\vdash(a\vee b)\to\neg\neg(a\vee b)}\ (\vdash\to)$$

(ii)

$$\dfrac{\dfrac{(\vee\vdash)\ \dfrac{(\neg\vdash)\ \dfrac{\dfrac{(\mathrm{Id}_a)\ \dfrac{}{a\vdash a}}{a\vdash a\vee b}\ (\vdash\vee)}{a,\neg(a\vee b)\vdash} \qquad (\neg\vdash)\ \dfrac{\dfrac{(\mathrm{Id}_a)\ \dfrac{}{b\vdash b}}{b\vdash a\vee b}\ (\vdash\vee)}{b,\neg(a\vee b)\vdash}}{a\vee b,\neg(a\vee b)\vdash}}{a\vee b\vdash\neg\neg(a\vee b)}\ (\vdash\neg)}{\vdash(a\vee b)\to\neg\neg(a\vee b)}\ (\vdash\to)$$

(iii)

$$\dfrac{(\vee\vdash)\ \dfrac{\dfrac{(\neg\vdash)\ \dfrac{\dfrac{(\mathrm{Id}_a)\ \dfrac{}{a\vdash a}}{a\vdash a\vee b}\ (\vdash\vee)}{a,\neg(a\vee b)\vdash}}{a\vdash\neg\neg(a\vee b)}\ (\vdash\neg) \qquad \dfrac{(\neg\vdash)\ \dfrac{\dfrac{(\mathrm{Id}_a)\ \dfrac{}{b\vdash b}}{b\vdash a\vee b}\ (\vdash\vee)}{b,\neg(a\vee b)\vdash}}{b\vdash\neg\neg(a\vee b)}\ (\vdash\neg)}{a\vee b\vdash\neg\neg(a\vee b)}}{\vdash(a\vee b)\to\neg\neg(a\vee b)}\ (\vdash\to)$$

These three derivations correspond to the strategies (i), (ii) and (iii), respectively, that have been given in Example 2.2.16.

REMARK 3.1.4. Except for the restriction of initial sequents $a \vdash a$ to atomic formulas a, the sequent calculus LI^p is almost exactly like (the propositional part of) Gentzen's calculus *LJ* without the structural rule for exchange ("Vertauschung"), that is, without the rule

$$(\mathrm{Exch})\,\frac{\Delta, A, B, \Gamma \vdash C}{\Delta, B, A, \Gamma \vdash C}$$

In LI^p, exchange is implicitly given by the fact that the antecedents of sequents are multisets of formulas instead of lists of formulas. (This will also be the case in all other sequent calculi to be introduced below.)

3.2. The sequent calculus LI_c^p

We introduce the sequent calculus LI_c^p for intuitionistic propositional logic. This calculus differs from LI^p only in that initial sequents given by the axiom are not restricted to atomic formulas anymore, but can now contain complex formulas as well.

DEFINITION 3.2.1. We define LI_c^p to be LI^p without the axiom being restricted to atomic formulas, that is, instead of (Id_a) we use the following axiom:

$$(\mathrm{Id})\,\frac{}{A \vdash A}\quad (A \text{ atomic or complex})$$

The *sequent calculus* LI_c^p *with atomic or complex initial sequents for intuitionistic propositional logic* is thus given by the following rules:

Axiom

$$(\mathrm{Id})\,\frac{}{A \vdash A}\quad (A \text{ atomic or complex})$$

Logical rules

$$(\neg\vdash)\,\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\,(\vdash\neg)$$

$$(\wedge\vdash)\,\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,(\vdash\wedge)$$

$$(\vee\vdash)\,\frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad\qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\,(\vdash\vee)\ (i = 1, 2)$$

(cont'd on next page)

$$(\to\vdash)\ \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad\qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash \mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

As in LI^p, Γ and Δ are finite multisets of formulas, the comma in antecedents of sequents stands for multiset union, and singletons are written without braces.

EXAMPLE 3.2.2. The following LI_c^p-derivation corresponds to the EI_c^p-strategy in Example 2.5.4:

$$\cfrac{\cfrac{(\neg\vdash)\ \cfrac{(\mathrm{Id})\ \cfrac{}{a \vee b \vdash a \vee b}}{a \vee b, \neg(a \vee b) \vdash}}{a \vee b \vdash \neg\neg(a \vee b)}\ (\vdash\neg)}{\vdash (a \vee b) \to \neg\neg(a \vee b)}\ (\vdash\to)$$

THEOREM 3.2.3. *The calculi LI^p and LI_c^p are equivalent.*

PROOF. The axiom (Id_a) is a special case of (Id). Hence, every sequent derivable in LI^p is derivable in LI_c^p. And every sequent derivable in LI_c^p is derivable in LI^p because (Id) is derivable in LI^p.

For example, consider the case of implication for atomic formulas a and b:

$$\cfrac{(\to\vdash)\ \cfrac{(\mathrm{Id}_a)\ \cfrac{}{a \vdash a} \qquad (\mathrm{Id}_a)\ \cfrac{}{b \vdash b}}{a, a \to b \vdash b}}{a \to b \vdash a \to b}\ (\vdash\to)$$

Similarly for the other connectives. Then one can show by induction that every sequent of the form $C \vdash C$ for complex C is derivable in LI^p. Thus (Id) is derivable in LI^p. $\dashv$

DEFINITION 3.2.4. A formula A is called *LI_c^p-provable*, if there is a derivation of the sequent $\vdash A$ in LI_c^p. Notation: $\vdash_{LI_c^p} A$. (Likewise for LI^p and all other sequent calculi to be introduced below.)

3.3. Situations

In order to show that EI_c^p-strategies and LI_c^p-derivations can be transformed into each other, we introduce so-called *situations*. A situation codifies certain information that is available at a given position in a dialogue as a sequent. This sequent can then be used to give a corresponding LI_c^p-sequent. Situations are also used in the opposite direction, that is, in constructing an EI_c^p-strategy from an LI_c^p-derivation. In both cases, situations act as an intermediate step.[36]

DEFINITION 3.3.1. Let d be a (possibly empty) finite EI_c^p-dialogue for A. The empty EI_c^p-dialogue contains no moves and is called d_ε. The *situation after d* is written $\Gamma \vdash C$ and is defined as follows:

(i) The situation after the empty EI_c^p-dialogue d_ε is $\vdash A$.

(ii) If $\Gamma \vdash C$ is the situation after d, then the situation after $d, \langle \delta(n), \eta(n) \rangle$ is

  (a) $\Gamma \vdash C$, if $\langle \delta(n), \eta(n) \rangle$ is a proponent move, that is, a proponent move does not change the situation,

  (b) $\Gamma, B \vdash C$, if $\langle \delta(n), \eta(n) \rangle = \langle O\, B, [n-1, D] \rangle$, that is, if $\langle \delta(n), \eta(n) \rangle$ is an opponent defense asserting B at position n,

  (c) as follows, if $\langle \delta(n), \eta(n) \rangle = \langle O\, e, [n-1, A] \rangle$ where $\delta(m) = P\, D$, that is, if $\langle \delta(n), \eta(n) \rangle$ is an opponent attack stating e at position n on an assertion D at position m:

$$\begin{array}{ll} \Gamma, A \vdash & \text{if } D = \neg A \text{ and } e = A \\ \Gamma \vdash A_1 & \text{if } D = A_1 \wedge A_2 \text{ and } e = \wedge_1 \\ \Gamma \vdash A_2 & \text{if } D = A_1 \wedge A_2 \text{ and } e = \wedge_2 \\ \Gamma \vdash A_i & \text{if } D = A_1 \vee A_2 \text{ and } e = \vee \\ \Gamma, A \vdash B & \text{if } D = A \to B \text{ and } e = A \end{array}$$

REMARK 3.3.2. In what follows, we will use '$\vdash_{LI_c^p}$' as the sequent symbol in LI_c^p-sequents (now written: $\Gamma \vdash_{LI_c^p} C$), in order to distinguish them from situations (written: $\Gamma \vdash C$).

REMARK 3.3.3. Γ is a set in the case of situations $\Gamma \vdash C$, whereas in sequents $\Gamma \vdash_{LI_c^p} C$ it is a multiset.

LEMMA 3.3.4. *Let $\Gamma \vdash C$ be the situation after the EI_c^p-dialogue*

$$d = \langle \delta(0), \eta(0) \rangle, \langle \delta(1), \eta(1) \rangle, \langle \delta(2), \eta(2) \rangle, \ldots, \langle \delta(m), \eta(m) \rangle$$

[36] Cf. the definition of 'position after' in Sørensen and Urzyczyn [2006], respectively the definition of 'situation after' in Sørensen and Urzyczyn [2007].

for A. Then

(i) *$B \in \Gamma$ if and only if $\delta(i) = O\,B$ for $1 < i \leq m$, that is, if the opponent move is not a symbolic attack but a move asserting B.*

(ii) *$B = C$ if and only if $B = A$, or if there is an opponent attack $\langle\delta(j) = O\,e$, $\eta(j) = [i, A]\rangle$ on some formula D for which the proponent move $\langle\delta(k) = P\,B, \eta(k) = [j, D]\rangle$ is a defense.*

(iii) *$\emptyset = C$ if and only if there is an opponent attack $\langle\delta(j) = O\,A, \eta(j) = [i, A]\rangle$ for which there is no proponent defense $\langle\delta(k) = P\,B, \eta(k) = [j, D]\rangle$, that is, if and only if O has attacked $\neg A$.*

PROOF. By induction on the number of moves of the EI_c^p-dialogue d. $\dashv$

REMARK 3.3.5. If $\Gamma \vdash C$ is the situation after an EI_c^p-dialogue $d \neq d_\varepsilon$ for A not ending in a proponent move, then in the move after d the proponent can either attack any formula in Γ or assert C in a defense, if such a move is not prohibited by the conditions $(D10)$–$(D14)$.

3.4. If $\vdash_{EI_c^p} A$, then $\vdash_{LI_c^p} A$

It has to be shown that $\vdash_{EI_c^p} A$ *if and only if* $\vdash_{LI_c^p} A$. We first prove the implication from left to right.

THEOREM 3.4.1. *If $\vdash_{EI_c^p} A$, then $\vdash_{LI_c^p} A$.*

PROOF. Let S be a strategy[37] for A and $d = \langle\delta(0), \eta(0)\rangle, \langle\delta(1), \eta(1)\rangle, \langle\delta(2), \eta(2)\rangle, \ldots, \langle\delta(n), \eta(n)\rangle$ an EI_c^p-dialogue in S not ending in a proponent move. We show by induction on the subtree below d in S that if $\Gamma \vdash C$ is the situation after d, then $\Gamma \vdash_{LI_c^p} C$ holds.

Since d is part of a strategy S, there is a proponent move $\langle\delta(n+1) = P\,e$, $\eta(n+1) = [j, Z]\rangle$. This move is either an attack or a defense.

First, assume the proponent move is an attack $\langle\delta(n+1) = P\,e, \eta(n+1) = [j, A]\rangle$ on $\langle\delta(j) = O\,D, \eta(j) = [i, Z]\rangle$. Then $D \in \Gamma$ by Lemma 3.3.4. Let $\Gamma' = \Gamma \setminus D$, that is, the set Γ' is Γ without D, and the corresponding multiset has thus no occurrence of D either. We consider each form of D, where D is a negation, conjunction, disjunction or implication:

(1) $D = \neg A$. Then the subtree below d depends on the form of A. We first consider the cases where the conditions of $(D14)$ are satisfied, that is, where A either has not been asserted by O in d or where A has been asserted by O but was attacked by P in d:

[37] In what follows, we consider EI_c^p-dialogues and EI_c^p-strategies. The prefix 'EI_c^p' is sometimes omitted for brevity.

(a) If A is an atom a, then the subtree below d is

$$n+1. \quad P\,a \;\; [j, A]$$

and $a \in \Gamma$ due to $(D10)$ and Lemma 3.3.4. Then the situation is $\Gamma', a \vdash C$, and $\Gamma \vdash_{LI_c^p} C$ is derivable by (Id), (Thin$\vdash$), ($\neg\vdash$) and ($\vdash$Thin):[38]

$$\dfrac{(\neg\vdash)\,\dfrac{(\text{Thin}\vdash)\,\dfrac{(\text{Id})\,\dfrac{}{a \vdash_{LI_c^p} a}}{\Gamma', a \vdash_{LI_c^p} a}}{\Gamma', a, \neg a \vdash_{LI_c^p}}}{\Gamma', a, \neg a \vdash_{LI_c^p} C}\,(\vdash\text{Thin})$$

(b) For $A = \neg E$ the subtree below d is

$$\begin{array}{lll} n+1. & P\,\neg E & [j, A] \\ n+2. & O\,E & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma', E \vdash$. Then $\Gamma', E \vdash_{LI_c^p}$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\vdash\neg$), ($\neg\vdash$) and ($\vdash$Thin):

$$\dfrac{(\neg\vdash)\,\dfrac{\dfrac{\Gamma', E \vdash_{LI_c^p}}{\Gamma' \vdash_{LI_c^p} \neg E}\,(\vdash\neg)}{\Gamma', \neg\neg E \vdash_{LI_c^p}}}{\Gamma', \neg\neg E \vdash_{LI_c^p} C}\,(\vdash\text{Thin})$$

(c) For $A = E_1 \wedge E_2$ the subtree below d is

$$\begin{array}{lc} n+1. & P\,E_1 \wedge E_2\;[j, A] \\ n+2. & O\,\wedge_1\;\;[n+1, A] \mid O\,\wedge_2\;\;[n+1, A] \end{array}$$

and the situations after position $n+2$ are $\Gamma' \vdash E_1$ and $\Gamma' \vdash E_2$. Then $\Gamma' \vdash_{LI_c^p} E_1$ and $\Gamma' \vdash_{LI_c^p} E_2$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\vdash\wedge$), (Contr), ($\neg\vdash$) and ($\vdash$Thin):

$$\dfrac{(\neg\vdash)\,\dfrac{(\text{Contr})\,\dfrac{\dfrac{\Gamma' \vdash_{LI_c^p} E_1 \qquad \Gamma' \vdash_{LI_c^p} E_2}{\Gamma', \Gamma' \vdash_{LI_c^p} E_1 \wedge E_2}\,(\vdash\wedge)}{\Gamma' \vdash_{LI_c^p} E_1 \wedge E_2}}{\Gamma', \neg(E_1 \wedge E_2) \vdash_{LI_c^p}}}{\Gamma', \neg(E_1 \wedge E_2) \vdash_{LI_c^p} C}\,(\vdash\text{Thin})$$

[38] Where double lines indicate single or multiple applications of a rule.

(d) For $A = E_1 \vee E_2$ the subtree below d is

$$\begin{array}{lll} n+1. & P\, E_1 \vee E_2 & [j, A] \\ n+2. & O \vee & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma' \vdash E_i$. Then $\Gamma' \vdash_{LI^p_c} E_i$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI^p_c} C$ is derivable by $(\vdash\vee)$, $(\neg\vdash)$ and $(\vdash \text{Thin})$:

$$\cfrac{(\neg\vdash)\ \cfrac{\cfrac{\Gamma' \vdash_{LI^p_c} E_i}{\Gamma' \vdash_{LI^p_c} E_1 \vee E_2}\ (\vdash\vee)}{\Gamma', \neg(E_1 \vee E_2) \vdash_{LI^p_c}}}{\Gamma', \neg(E_1 \vee E_2) \vdash_{LI^p_c} C}\ (\vdash \text{Thin})$$

(e) For $A = E \to F$ the subtree below d is

$$\begin{array}{lll} n+1. & P\, E \to F & [j, A] \\ n+2. & O\, E & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma', E \vdash F$. Then $\Gamma', E \vdash_{LI^p_c} F$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI^p_c} C$ is derivable by $(\vdash\to)$, $(\neg\vdash)$ and $(\vdash \text{Thin})$:

$$\cfrac{(\neg\vdash)\ \cfrac{\cfrac{\Gamma', E \vdash_{LI^p_c} F}{\Gamma' \vdash_{LI^p_c} E \to F}\ (\vdash\to)}{\Gamma', \neg(E \to F) \vdash_{LI^p_c}}}{\Gamma', \neg(E \to F) \vdash_{LI^p_c} C}\ (\vdash \text{Thin})$$

Now we consider the cases where the conditions (i) and (ii) of $(D14)$ are not satisfied, that is, where A has been asserted by O in d without having been attacked by P in d. If the conditions of $(D14)$ are not satisfied at position $n+1$, then moves of the form $\langle \delta(n+2) = O\, e, \eta(n+2) = [n+1, A] \rangle$ are not possible. That is, for formulas $\neg A$ the subtrees below d all have the form

$$n+1. \quad P\, A \ \ [j, A]$$

where only the move of P at position $n+1$ remains. We know at position $n+1$ that $A \in \Gamma$ because the conditions of $(D14)$ are not satisfied at position $n+1$, and A therefore must be asserted in d by O. Let $\Gamma'' = \Gamma' \setminus A$. Then $\Gamma'', A \vdash_{LI^p_c} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI^p_c} C$ is derivable by (Id), (Thin$\vdash$), ($\neg\vdash$) and ($\vdash$Thin):

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{LI_c^p} A}\,(\mathrm{Id})}{\Gamma'', A \vdash_{LI_c^p} A}\,(\mathrm{Thin}\vdash)}{\Gamma'', A, \neg A \vdash_{LI_c^p}}\,(\neg\vdash)}{\Gamma'', A, \neg A \vdash_{LI_c^p} C}\,(\vdash\mathrm{Thin})
$$

(2) $D = A_1 \wedge A_2$. Then the subtree below d is

$$
\begin{array}{lll} n+1. & P \wedge_1 & [j, A] \\ n+2. & O\, A_1 & [n+1, D] \end{array} \quad \text{respectively} \quad \begin{array}{lll} n+1. & P \wedge_2 & [j, A] \\ n+2. & O\, A_2 & [n+1, D] \end{array}
$$

and the situation after position $n+2$ is $\Gamma', A_1 \vdash C$ or $\Gamma', A_2 \vdash C$, respectively. Then $\Gamma', A_1 \vdash_{LI_c^p} C$ respectively $\Gamma', A_2 \vdash_{LI_c^p} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\wedge\vdash$):

$$
(\wedge\vdash)\ \dfrac{\Gamma', A_i \vdash_{LI_c^p} C}{\Gamma', A_1 \wedge A_2 \vdash_{LI_c^p} C}
$$

(3) $D = A_1 \vee A_2$. Then the subtree below d is

$$
\begin{array}{lc} n+1. & P \vee \ [j, A] \\ n+2. & O\, A_1\ \ [n+1, D] \mid O\, A_2\ \ [n+1, D] \end{array}
$$

and the situations after position $n+2$ are $\Gamma', A_1 \vdash C$ and $\Gamma', A_2 \vdash C$. Then $\Gamma', A_1 \vdash_{LI_c^p} C$ and $\Gamma', A_2 \vdash_{LI_c^p} C$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\vee\vdash$) and (Contr):

$$
\dfrac{\dfrac{\Gamma', A_1 \vdash_{LI_c^p} C \qquad \Gamma', A_2 \vdash_{LI_c^p} C}{\Gamma', \Gamma', A_1 \vee A_2 \vdash_{LI_c^p} C}\,(\vee\vdash)}{\Gamma', A_1 \vee A_2 \vdash_{LI_c^p} C}\,(\mathrm{Contr})
$$

(4) $D = A \to B$. Then the subtree below d depends on the form of A. We first consider the cases where the conditions of ($D14$) are satisfied, that is, where A either has not been asserted by O in d or where A has been asserted by O but was attacked by P in d:

(a) If A is an atom a, then the subtree below d is

$$
\begin{array}{lll} n+1. & P\, a & [j, A] \\ n+2. & O\, B & [n+1, D] \end{array}
$$

and $a \in \Gamma$ by ($D10$) and Lemma 3.3.4. Let $\Gamma'' = \Gamma' \setminus a$. Then the situation after position $n+2$ is $\Gamma'', B \vdash C$. Then $\Gamma'', B \vdash_{LI_c^p} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by (Id) and ($\to\vdash$):

$$\dfrac{\dfrac{}{a \vdash_{LI_c^p} a}\,(\mathrm{Id}) \qquad \Gamma'', B \vdash_{LI_c^p} C}{\Gamma'', a, a \to B \vdash_{LI_c^p} C}\,(\to\vdash)$$

(b) For $A = \neg E$ the subtree below d is

$$\begin{array}{ll} n+1. & P\,\neg E\ [j, A] \\ n+2. & O\,E\ \ [n+1, A] \mid O\,B\ \ [n+1, D] \end{array}$$

and the situations after position $n+2$ are $\Gamma', E \vdash$ and $\Gamma', B \vdash C$. Then $\Gamma', E \vdash_{LI_c^p}$ and $\Gamma', B \vdash_{LI_c^p} C$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\vdash\neg$), ($\to\vdash$) and (Contr):

$$\dfrac{\dfrac{\dfrac{\Gamma', E \vdash_{LI_c^p}}{\Gamma' \vdash_{LI_c^p} \neg E}\,(\vdash\neg) \qquad \Gamma', B \vdash_{LI_c^p} C}{\Gamma', \Gamma', \neg E \to B \vdash_{LI_c^p} C}\,(\to\vdash)}{\Gamma', \neg E \to B \vdash C}\,(\mathrm{Contr})$$

(c) For $A = E_1 \wedge E_2$ the subtree below d is

$$\begin{array}{ll} n+1. & P\,E_1 \wedge E_2\ [j, A] \\ n+2. & O\wedge_1\ \ [n+1, A] \mid O\wedge_2\ \ [n+1, A] \mid O\,B\ \ [n+1, D] \end{array}$$

and the situations after position $n+2$ are $\Gamma' \vdash E_1$, $\Gamma' \vdash E_2$ and $\Gamma', B \vdash C$. Then $\Gamma' \vdash_{LI_c^p} E_1$, $\Gamma' \vdash_{LI_c^p} E_2$ and $\Gamma', B \vdash_{LI_c^p} C$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by ($\vdash\wedge$), ($\to\vdash$) and (Contr):

$$\dfrac{\dfrac{\dfrac{\Gamma' \vdash_{LI_c^p} E_1 \qquad \Gamma' \vdash_{LI_c^p} E_2}{\Gamma', \Gamma' \vdash_{LI_c^p} E_1 \wedge E_2}\,(\vdash\wedge) \qquad \Gamma', B \vdash_{LI_c^p} C}{\Gamma', \Gamma', \Gamma', (E_1 \wedge E_2) \to B \vdash_{LI_c^p} C}\,(\to\vdash)}{\Gamma', (E_1 \wedge E_2) \to B \vdash_{LI_c^p} C}\,(\mathrm{Contr})$$

(d) For $A = E_1 \vee E_2$ the subtree below d is

$$\begin{array}{ll} n+1. & P\,E_1 \vee E_2\ [j, A] \\ n+2. & O\vee\ \ [n+1, A] \mid O\,B\ \ [n+1, D] \end{array}$$

and the situations after position $n+2$ are $\Gamma' \vdash E_i$ and $\Gamma', B \vdash C$. Then $\Gamma' \vdash_{LI_c^p} E_i$ and $\Gamma', B \vdash_{LI_c^p} C$ are derivable by the induction hypothesis, and

$\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash\vee)$, $(\to\vdash)$ and (Contr):

$$
(\text{Contr})\,\dfrac{(\to\vdash)\,\dfrac{\dfrac{\Gamma' \vdash_{LI_c^p} E_i}{\Gamma' \vdash_{LI_c^p} E_1 \vee E_2}\,(\vdash\vee) \qquad \Gamma', B \vdash_{LI_c^p} C}{\Gamma', \Gamma', (E_1 \vee E_2) \to B \vdash_{LI_c^p} C}}{\Gamma', (E_1 \vee E_2) \to B \vdash_{LI_c^p} C}
$$

(e) For $A = E \to F$ the subtree below d is

$$
\begin{array}{ll}
n+1. & \qquad P\, E \to F\ [j, A] \\
n+2. & O\, E\ \ [n+1, A] \mid O\, B\ \ [n+1, D]
\end{array}
$$

and the situations after position $n+2$ are $\Gamma', E \vdash F$ and $\Gamma', B \vdash C$. Then $\Gamma', E \vdash_{LI_c^p} F$ and $\Gamma', B \vdash_{LI_c^p} C$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash\to)$, $(\to\vdash)$ and (Contr):

$$
(\text{Contr})\,\dfrac{(\to\vdash)\,\dfrac{\dfrac{\Gamma', E \vdash_{LI_c^p} F}{\Gamma' \vdash_{LI_c^p} E \to F}\,(\vdash\to) \qquad \Gamma', B \vdash_{LI_c^p} C}{\Gamma', \Gamma', (E \to F) \to B \vdash_{LI_c^p} C}}{\Gamma', (E \to F) \to B \vdash_{LI_c^p} C}
$$

Now we consider the cases where the conditions of $(D14)$ are not satisfied, that is, where A has been asserted by the opponent O in d without having been attacked by the proponent P in d. If the conditions of $(D14)$ are not satisfied at position $n+1$, then moves of the form $\langle \delta(n+2) = O\, e, \eta(n+2) = [n+1, A] \rangle$ are not possible. That is, for formulas $A \to B$ the subtrees below d all have the form

$$
\begin{array}{ll}
n+1. & P\, A\ \ [j, A] \\
n+2. & O\, B\ \ [n+1, D]
\end{array}
$$

where only the defense move of O remains. Then $B \in \Gamma'$ after position $n+2$ by Lemma 3.3.4, and the situation is $\Gamma' \vdash C$. Furthermore, we know at position $n+1$ that $A \in \Gamma$ because the conditions of $(D14)$ are not satisfied at position $n+1$, and A therefore must be asserted in d by O. Let $\Gamma'' = \Gamma' \setminus B$. Then $\Gamma'', B \vdash_{LI_c^p} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by (Id) and $(\to\vdash)$:

$$
(\to\vdash)\,\dfrac{(\text{Id})\,\dfrac{}{A \vdash_{LI_c^p} A} \qquad \Gamma'', B \vdash_{LI_c^p} C}{\Gamma'', A, A \to B \vdash_{LI_c^p} C}
$$

Second, assume the proponent move is a defense $\langle \delta(n+1) = P\, E, \eta(n+1) = [j, D] \rangle$ to $\langle \delta(j) = O\, D, \eta(j) = [i, Z] \rangle$, or the initial move $\langle \delta(0) = P\, E, \eta(0) = \emptyset \rangle$. Then $E = C$ by Lemma 3.3.4, where E is an atom, a negation, conjunction, disjunction or implication. We first consider the cases where the conditions of $(D14)$ are satisfied, that is, where E either has not been asserted by O in d or where E has been asserted by O but was attacked by P in d:

(1) E is an atom. Then E must be asserted by O in d due to $(D10)$, and $E \in \Gamma$ by Lemma 3.3.4. Let $\Gamma' = \Gamma \setminus E$. Then $\Gamma \vdash_{LI_c^p} C$ is derivable by (Id) and (Thin$\vdash$):

$$\text{(Thin}\vdash) \dfrac{\text{(Id)} \dfrac{}{E \vdash_{LI_c^p} E}}{\Gamma', E \vdash_{LI_c^p} E}$$

(2) $E = \neg F$. Then the subtree below d is

$$\begin{array}{lll} n+1. & P\,\neg F & [j, D] \\ n+2. & O\,F & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma, F \vdash$. Then $\Gamma, F \vdash_{LI_c^p}$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash \neg)$:

$$\dfrac{\Gamma, F \vdash_{LI_c^p}}{\Gamma \vdash_{LI_c^p} \neg F} (\vdash \neg)$$

(3) $E = F_1 \wedge F_2$. Then the subtree below d is

$$\begin{array}{lc} n+1. & P\,F_1 \wedge F_2\ [j, D] \\ n+2. & O \wedge_1\ [n+1, A] \mid O \wedge_2\ [n+1, A] \end{array}$$

and the situations after position $n+2$ are $\Gamma \vdash F_1$ and $\Gamma \vdash F_2$. Then $\Gamma \vdash_{LI_c^p} F_1$ and $\Gamma \vdash_{LI_c^p} F_2$ are derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash \wedge)$ and (Contr):

$$\text{(Contr)} \dfrac{\dfrac{\Gamma \vdash_{LI_c^p} F_1 \qquad \Gamma \vdash_{LI_c^p} F_2}{\Gamma, \Gamma \vdash_{LI_c^p} F_1 \wedge F_2} (\vdash \wedge)}{\Gamma \vdash_{LI_c^p} F_1 \wedge F_2}$$

(4) For $E = F_1 \vee F_2$ the subtree below d is

$$\begin{array}{lll} n+1. & P\,F_1 \vee F_2 & [j, D] \\ n+2. & O\,\vee & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma \vdash F_i$. Then $\Gamma \vdash_{LI_c^p} F_i$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash \vee)$.

$$\dfrac{\Gamma \vdash_{LI_c^p} F_i}{\Gamma \vdash_{LI_c^p} F_1 \vee F_2} (\vdash \vee)$$

(5) $E = F \to G$. Then the subtree below d is

$$\begin{array}{lll} n+1. & P\,F \to G & [j, D] \\ n+2. & O\,F & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma, F \vdash G$. Then $\Gamma, F \vdash_{LI_c^p} G$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by $(\vdash\to)$:

$$\frac{\Gamma, F \vdash_{LI_c^p} G}{\Gamma \vdash_{LI_c^p} F \to G}\ (\vdash\to)$$

Now we consider the cases where the conditions of $(D14)$ are not satisfied, that is, where E has been asserted by O in d without having been attacked by P in d. If the conditions of $(D14)$ are not satisfied at position $n+1$, then moves of the form $\langle \delta(n+2) = O\,e, \eta(n+2) = [n+1, A]\rangle$ are not possible. That is, for formulas E the subtrees below d all have the form

$$n+1. \quad P\,E \ \ [j, D]$$

where only the move of P at position $n+1$ remains. We know at position $n+1$ that $E \in \Gamma$ because the conditions of $(D14)$ are not satisfied at position $n+1$, and E therefore must be asserted in d by O. In addition, $E = C$ by Lemma 3.3.4. Let $\Gamma'' = \Gamma' \setminus E$. Then $\Gamma'', E \vdash_{LI_c^p} E$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c^p} C$ is derivable by (Id) and (Thin$\vdash$):

$$\text{(Thin}\vdash) \frac{\text{(Id)}\ \dfrac{}{E \vdash_{LI_c^p} E}}{\Gamma'', E \vdash_{LI_c^p} E}$$

Thus for every situation $\Gamma \vdash C$ in an EI_c^p-strategy there is a corresponding sequent $\Gamma \vdash_{LI_c^p} C$ in an LI_c^p-derivation. $\dashv$

3.5. Possible situations

We define the notions of possible situation and substrategy. These will be used in the proof of the converse direction of the equivalence result, that is, for proving the implication from right (derivations) to left (strategies).

DEFINITION 3.5.1. Let $\Gamma \vdash C$ be the situation after d. Then a *possible situation after* d is a situation $\Gamma' \vdash C'$, where $\Gamma' \subseteq \Gamma$ and $C' = C$ or $C' = \emptyset$. Furthermore, if $\Gamma, b, \neg b \vdash$ for atomic b is the situation after d, then $\Gamma, b \vdash b$ is a possible situation after d.

REMARK 3.5.2. Instead of defining $\Gamma, b \vdash b$ to be a possible situation after d if $\Gamma, b, \neg b \vdash$ is the situation after d for atomic b, we could add the axiom

$$(\mathrm{Id}')\ \frac{}{b, \neg b \vdash_{LI_c^p}}\quad (b \text{ atomic})$$

to LI_c^p. (Id') being derivable in LI_c^p, this would yield an equivalent calculus.

DEFINITION 3.5.3. A *substrategy* for a possible situation $\Gamma \vdash C$ is a subtree s of a dialogue tree t comprising as root node a node at an even position in t with possible situation $\Gamma \vdash C$ and all descendents in t such that s does not branch at even positions, s has as many nodes at odd positions as there are possible moves for O, and all leaves are proponent moves such that O cannot make another move.

REMARK 3.5.4. Substrategies are different from strategies in that the root node of a substrategy does not have to be the root node of the dialogue tree as in strategies. Every substrategy is a subtree of a strategy.

3.6. If $\vdash_{LI_c^p} A$, then $\vdash_{EI_c^p} A$

We conclude the proof of $\vdash_{EI_c^p} A$ *if and only if* $\vdash_{LI_c^p} A$ by proving the implication from right to left.

THEOREM 3.6.1. *If* $\vdash_{LI_c^p} A$, *then* $\vdash_{EI_c^p} A$.

PROOF. Let d be an EI_c^p-dialogue for A not ending in a proponent move. We show by induction on the number of LI_c^p-inferences that if $\Gamma \vdash_{LI_c^p} C$, and $\Gamma \vdash C$ is a possible situation after d, then there is a subtree t below d such that $\begin{matrix} d \\ t \end{matrix}$ is a strategy for A.

The derivation can consist of the axiom, end with a logical rule or end with a structural rule.

First, assume the derivation consists of the axiom

$$(\mathrm{Id})\ \frac{}{B \vdash_{LI_c^p} B}$$

Then $B \vdash B$ is a possible situation after the EI_c^p-dialogue d for A. Then by Lemma 3.3.4 the formula B must have been asserted in an opponent move because $B = \Gamma$. The formula B can be atomic or complex.

(1) B is atomic. Since the formula A (for which $\delta(0) = P\,A$) cannot be atomic, there is a proponent defense asserting B by Lemma 3.3.4. Then

$$\begin{array}{ll} & d \\ n+1. & P\,B \quad [j, D] \end{array}$$

is a strategy since atoms cannot be attacked.

(2) B is complex. Since $B = \Gamma$ and $B = C$, the conditions of $(D14)$ cannot be satisfied. That is, B has been asserted by O in d but cannot have been attacked by P in d. Otherwise, there would be a possible attack $\langle \delta(n+2) = O\, e, \eta(n+2) = [n+1, A] \rangle$ after the proponent move $\langle \delta(n+1) = P\, B, \eta(n+1) = [j, Z] \rangle$, yielding only possible situations where $B \neq C$ or $B \neq \Gamma$ and $\Gamma \neq \emptyset$ or $C \neq \emptyset$ because all formulas B' resulting from the opponent attack are of lower complexity than B. By Lemma 3.3.4 we know that B has been asserted by O in d since $B = \Gamma$; and if B has not been attacked by P in d, then

$$\begin{array}{lll} & & d \\ n+1. & P\, B & [j, D] \end{array}$$

is a strategy since $(D14)$ prohibits an opponent attack on B.

Second, assume the derivation ends with a logical rule. We consider first the cases where the derivation ends with a right introduction rule. If C is the formula introduced in the succedent of an LI_c^p-sequent, then $\Gamma \vdash C$ is a possible situation after the EI_c^p-dialogue d for A, and by Lemma 3.3.4 either (a) $C \neq A$ and there is some proponent defense $\langle \delta(n+1) = P\, C$, $\eta(n+1) = [j, D] \rangle$, or (b) $C = A$. In the latter case, d must be the empty dialogue d_ε because A can only occur once and at position 0. We consider both cases for each right introduction rule:

(1) The derivation ends with

$$\frac{\Gamma, B \vdash_{LI_c^p}}{\Gamma \vdash_{LI_c^p} \neg B} (\vdash \neg)$$

Then $\Gamma \vdash \neg B$ is a possible situation after the EI_c^p-dialogue d for A.

(a) If $\neg B \neq A$, then there is some proponent defense $\langle \delta(n+1) = P\, \neg B$, $\eta(n+1) = [j, D] \rangle$, and the subtree t below d is

$$\begin{array}{lll} n+1. & P\, \neg B & [j, D] \\ n+2. & O\, B & [n+1, A] \end{array}$$

and $\Gamma, B \vdash$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash \neg)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\, \neg B & [j, D] \\ n+2. & O\, B & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma \vdash \neg B$.

(b) If $\neg B = A$, then $\Gamma = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle \delta(0) = P\, \neg B, \eta(0) = \emptyset \rangle$. This is followed by the move

$\langle\delta(1) = O\,B, \eta(1) = [1, A]\rangle$ at position 1, after which there is a substrategy s by the induction hypothesis. Then $\langle\delta(0) = P\,\neg B, \eta(0) = \emptyset\rangle$, $\langle\delta(1) = O\,B$, $\eta(1) = [1, A]\rangle$, s is a strategy for $\vdash \neg B$.

(2) The derivation ends with

$$\frac{\Gamma \vdash_{LI_c^p} A_1 \qquad \Delta \vdash_{LI_c^p} A_2}{\Gamma, \Delta \vdash_{LI_c^p} A_1 \wedge A_2}\ (\vdash \wedge)$$

Then $\Gamma, \Delta \vdash A_1 \wedge A_2$ is a possible situation after the EI_c^p-dialogue d for A.

(a) If $A_1 \wedge A_2 \neq A$, then there is some proponent defense $\langle\delta(n+1) = P\,A_1 \wedge A_2, \eta(n+1) = [j, D]\rangle$, and the subtree t below d is

$$\begin{array}{lc|c}
n+1. & \multicolumn{2}{c}{P\,A_1 \wedge A_2\ [j, D]} \\
n+2. & O \wedge_1\ [n+1, A] & O \wedge_2\ [n+1, A]
\end{array}$$

and $\Gamma \vdash A_1$ and $\Delta \vdash A_2$ are possible situations after positions $n+2$ corresponding to the premisses of $(\vdash \wedge)$. By the induction hypothesis there are substrategies s and s' below t. Then

$$\begin{array}{lc|c}
n+1. & \multicolumn{2}{c}{P\,A_1 \wedge A_2\ [j, D]} \\
n+2. & O \wedge_1\ [n+1, A] & O \wedge_2\ [n+1, A] \\
 & s & s'
\end{array}$$

is a substrategy for $\Gamma, \Delta \vdash A_1 \wedge A_2$.

(b) If $A_1 \wedge A_2 = A$, then $\Gamma, \Delta = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle\delta(0) = P\,A_1 \wedge A_2, \eta(0) = \emptyset\rangle$. This is followed by the moves $\langle\delta(1) = O \wedge_1, \eta(1) = [1, A]\rangle$ and $\langle\delta(1) = O \wedge_2, \eta(1) = [1, A]\rangle$ at position 1, after which there are substrategies s and s' by the induction hypothesis. Then

$$\begin{array}{lc|c}
0. & \multicolumn{2}{c}{P\,A_1 \wedge A_2} \\
1. & O \wedge_1\ [0, A] & O \wedge_2\ [0, A] \\
 & s & s'
\end{array}$$

is a strategy for $\vdash A_1 \wedge A_2$.

(3) The derivation ends with

$$\frac{\Gamma \vdash_{LI_c^p} A_i}{\Gamma \vdash_{LI_c^p} A_1 \vee A_2}\ (\vdash \vee)$$

Then $\Gamma \vdash A_1 \vee A_2$ is a possible situation after the EI_c^p-dialogue d for A.

(a) If $A_1 \vee A_2 \neq A$, then the subtree t below d is

$$\begin{array}{lll}
n+1. & P\,A_1 \vee A_2 & [j, D] \\
n+2. & O \vee & [n+1, A]
\end{array}$$

and $\Gamma \vdash A_i$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash\vee)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\,A_1 \vee A_2 & [j, D] \\ n+2. & O\,\vee & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma \vdash A_1 \vee A_2$.

(b) If $A_1 \vee A_2 = A$, then $\Gamma, \Delta = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle \delta(0) = P\,A_1 \vee A_2, \eta(0) = \emptyset \rangle$. This is followed by the move $\langle \delta(1) = O\,\vee, \eta(1) = [1, A] \rangle$ at position 1, after which there is a substrategy s by the induction hypothesis. Then $\langle \delta(0) = P\,A_1 \vee A_2$, $\eta(0) = \emptyset \rangle, \langle \delta(1) = O\,\vee, \eta(1) = [1, A] \rangle, s$ is a strategy for $\vdash A_1 \vee A_2$.

(4) The derivation ends with

$$\frac{\Gamma, B \vdash_{LI_c^p} C}{\Gamma \vdash_{LI_c^p} B \to C}\,(\vdash\to)$$

Then $\Gamma \vdash B \to C$ is a possible situation after the EI_c^p-dialogue d for A.

(a) If $B \to C \neq A$, then the subtree t below d is

$$\begin{array}{lll} n+1. & P\,B \to C & [j, D] \\ n+2. & O\,B & [n+1, A] \end{array}$$

and $\Gamma, B \vdash C$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash\to)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\,B \to C & [j, D] \\ n+2. & O\,B & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma \vdash B \to C$.

(b) If $B \to C = A$, then $\Gamma, \Delta = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle \delta(0) = P\,B \to C, \eta(0) = \emptyset \rangle$. This is followed by the move $\langle \delta(1) = O\,B, \eta(1) = [1, A] \rangle$ at position 1, after which there is a substrategy s by the induction hypothesis. Then $\langle \delta(0) = P\,B \to C$, $\eta(0) = \emptyset \rangle, \langle \delta(1) = O\,B, \eta(1) = [1, A] \rangle, s$ is a strategy for $\vdash B \to C$.

Now we consider the cases where the derivation ends with a left introduction rule.

(1) The derivation ends with

$$(\neg\vdash)\,\frac{\Gamma \vdash_{LI_c^p} B}{\Gamma, \neg B \vdash_{LI_c^p}}$$

Then $\Gamma, \neg B \vdash$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain an opponent move $\langle \delta(j) = O\,\neg B, \eta(j) = [i, Z]\rangle$. Then for complex B not introduced by (Id)

$$\begin{array}{ll} n+1. & P\,B \;\; [j, A] \\ n+2. & O\,e \;\; [n+1, A] \end{array}$$

is a subtree t below d. The possible situation after position $n+2$ depends on the form of B, where B can be a negation, conjunction, disjunction or implication. Then B was introduced by a right introduction rule (i.e., by $(\vdash\neg)$, $(\vdash\wedge)$, $(\vdash\vee)$ or $(\vdash\rightarrow)$). We first consider these cases for complex B.[39] After that we consider the case where the complex formula B has been introduced by (Id); in this case the move $\langle \delta(n+2) = O\,e, \eta(n+2) = [n+1, A]\rangle$ is impossible. Finally, the remaining case where B is atomic is treated.

(a) For $B = \neg E$ the derivation ends with

$$(\neg\vdash)\,\dfrac{\dfrac{\Gamma, E \vdash_{LI_c^p}}{\Gamma \vdash_{LI_c^p} \neg E}\,(\vdash\neg)}{\Gamma, \neg\neg E \vdash_{LI_c^p}}$$

The subtree t below d is then

$$\begin{array}{lll} n+1. & P\,\neg E & [j, A] \\ n+2. & O\,E & [n+1, A] \end{array}$$

and $\Gamma, E \vdash$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash\neg)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\,\neg E & [j, A] \\ n+2. & O\,E & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \neg\neg E \vdash$.

(b) For $B = E_1 \wedge E_2$ the derivation ends with

$$(\neg\vdash)\,\dfrac{\dfrac{\Gamma \vdash_{LI_c^p} E_1 \quad \Delta \vdash_{LI_c^p} E_2}{\Gamma, \Delta \vdash_{LI_c^p} E_1 \wedge E_2}\,(\vdash\wedge)}{\Gamma, \Delta, \neg(E_1 \wedge E_2) \vdash_{LI_c^p}}$$

The subtree t below d is then

$$\begin{array}{lc} n+1. & P\,E_1 \wedge E_2\; [j, A] \\ n+2. & O\,\wedge_1\;\; [n+1, A] \;\big|\; O\,\wedge_2\;\; [n+1, A] \end{array}$$

[39]Without loss of generality we assume that the respective right introduction rules are applied in the last step of the derivation of the premiss of $(\neg\vdash)$.

and $\Gamma \vdash E_1$, respectively $\Delta \vdash E_2$, are possible situations in the left, respectively right branch after position $n+2$ corresponding to the premisses of $(\vdash \wedge)$. By the induction hypothesis there are substrategies s and s' below the branches of t. Then

$$\begin{array}{lcc|c} n+1. & & P\, E_1 \wedge E_2\ [j, A] & \\ n+2. & O \wedge_1\ [n+1, A] & & O \wedge_2\ [n+1, A] \\ & s & & s' \end{array}$$

is a substrategy for $\Gamma, \Delta, \neg(E_1 \wedge E_2) \vdash$.

(c) For $B = E_1 \vee E_2$ the derivation ends with

$$(\neg\vdash)\ \dfrac{\dfrac{\Gamma \vdash_{LI_c^p} E_i}{\Gamma \vdash_{LI_c^p} E_1 \vee E_2}\ (\vdash \vee)}{\Gamma, \neg(E_1 \vee E_2) \vdash_{LI_c^p}}$$

The subtree t below d is then

$$\begin{array}{lll} n+1. & P\, E_1 \vee E_2 & [j, A] \\ n+2. & O \vee & [n+1, A] \end{array}$$

and $\Gamma \vdash E_i$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash \vee)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\, E_1 \vee E_2 & [j, A] \\ n+2. & O \vee & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \neg(E_1 \vee E_2) \vdash$.

(d) For $B = E \to F$ the derivation ends with

$$(\neg\vdash)\ \dfrac{\dfrac{\Gamma, E \vdash_{LI_c^p} F}{\Gamma \vdash_{LI_c^p} E \to F}\ (\vdash \to)}{\Gamma, \neg(E \to F) \vdash_{LI_c^p}}$$

The subtree t below d is then

$$\begin{array}{lll} n+1. & P\, E \to F & [j, A] \\ n+2. & O\, E & [n+1, A] \end{array}$$

and $\Gamma, E \vdash F$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash \to)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\, E \to F & [j, A] \\ n+2. & O\, E & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \neg(E \to F) \vdash$.

(e) If B is complex and was introduced by (Id), then the derivation ends with

$$\dfrac{\dfrac{\dfrac{}{B \vdash_{LI^p_c} B}\,(\mathrm{Id})}{\Gamma, B \vdash_{LI^p_c} B}\,(\mathrm{Thin}\vdash)}{\Gamma, B, \neg B \vdash_{LI^p_c}}\,(\neg\vdash)$$

Then $\Gamma, B, \neg B \vdash$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain the opponent moves $\langle \delta(j) = O\,B, \eta(j) = [i, Z]\rangle$ and $\langle \delta(l) = O\,\neg B, \eta(l) = [k, Z]\rangle$. Since B is complex, the conditions of $(D14)$ cannot be satisfied, as was shown in the treatment of (Id), case (2) on page 60. That is, d contains the opponent move $\langle \delta(j) = O\,B, \eta(j) = [i, Z]\rangle$, but no proponent attack $\langle \delta(m) = P\,e, \eta(m) = [j, A]\rangle$. Then the move $\langle \delta(n+2) = O\,e, \eta(n+2) = [n+1, A]\rangle$ is impossible, and

$$\begin{array}{ll} & d \\ n+1. & P\,B \;\; [j, A] \end{array}$$

is a strategy for $\Gamma, B, \neg B \vdash$.

(f) Finally, we consider the case where B is atomic. Then B must have been introduced by (Id), and the derivation ends with

$$\dfrac{\dfrac{\dfrac{}{B \vdash_{LI^p_c} B}\,(\mathrm{Id})}{\Gamma, B \vdash_{LI^p_c} B}\,(\mathrm{Thin}\vdash)}{\Gamma, B, \neg B \vdash_{LI^p_c}}\,(\neg\vdash)$$

Then $\Gamma, B, \neg B \vdash$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain the opponent moves $\langle \delta(j) = O\,\neg B, \eta(j) = [i, Z]\rangle$ and $\langle \delta(l) = O\,B, \eta(l) = [k, Z]\rangle$. Since B is atomic, the move $\langle \delta(n+2) = O\,e, \eta(n+2) = [n+1, A]\rangle$ is impossible, and

$$\begin{array}{ll} & d \\ n+1. & P\,B \;\; [j, A] \end{array}$$

is a strategy for $\Gamma, B, \neg B \vdash$.

(2) The derivation ends with

$$\dfrac{\Gamma, A_i \vdash_{LI^p_c} C}{\Gamma, A_1 \wedge A_2 \vdash_{LI^p_c} C}\,(\wedge\vdash)$$

Then $\Gamma, A_1 \wedge A_2 \vdash C$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain an opponent move $\langle \delta(j) = O\,A_1 \wedge A_2, \eta(j) = [i, Z]\rangle$. Then

$$\begin{array}{ll} n+1. & P\,\wedge_1 \;\; [j, A] \\ n+2. & O\,A_1 \;\; [n+1, D] \end{array} \quad \text{respectively} \quad \begin{array}{ll} n+1. & P\,\wedge_2 \;\; [j, A] \\ n+2. & O\,A_2 \;\; [n+1, D] \end{array}$$

is a subtree t_1, respectively t_2, below d with possible situations $\Gamma, A_1 \vdash C$ respectively $\Gamma, A_2 \vdash C$ after positions $n+2$ corresponding to the premiss of $(\wedge\vdash)$. By the induction hypothesis there are substrategies s_1 and s_2 below the subtrees t_1, respectively t_2, of t. Then

$$\begin{array}{lll} n+1. & P\wedge_1 & [j, A] \\ n+2. & O\,A_1 & [n+1, D] \\ & & s_1 \end{array} \quad \text{respectively} \quad \begin{array}{lll} n+1. & P\wedge_2 & [j, A] \\ n+2. & O\,A_2 & [n+1, D] \\ & & s_2 \end{array}$$

are substrategies for $\Gamma, A_1 \wedge A_2 \vdash C$.

(3) The derivation ends with

$$(\vee\vdash)\,\frac{\Gamma, B \vdash_{LI_c^p} D \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, B \vee C \vdash_{LI_c^p} D}$$

Then $\Gamma, \Delta, B \vee C \vdash D$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain an opponent move $\langle \delta(j) = O\,B \vee C, \eta(j) = [i, Z]\rangle$. Then

$$\begin{array}{lc} n+1. & P\vee\,[j, A] \\ n+2. & O\,B\ \ [n+1, D] \mid O\,C\ \ [n+1, D] \end{array}$$

is a subtree t below d with possible situations $\Gamma, B \vdash D$ (left branch) and $\Delta, C \vdash D$ (right branch) after positions $n+2$ corresponding to the left respectively right premiss of $(\vee\vdash)$. By the induction hypothesis there are substrategies s and s' below the left, respectively right branch of t. Then

$$\begin{array}{lc} n+1. & P\vee\,[j, A] \\ n+2. & O\,B\ \ [n+1, D] \mid O\,C\ \ [n+1, D] \\ & s \qquad\qquad\qquad s' \end{array}$$

is a substrategy for $\Gamma, \Delta, B \vee C \vdash D$.

(4) The derivation ends with

$$(\rightarrow\vdash)\,\frac{\Gamma \vdash_{LI_c^p} B \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, B \rightarrow C \vdash_{LI_c^p} D}$$

Then $\Gamma, \Delta, B \rightarrow C \vdash D$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain an opponent move $\langle \delta(j) = O\,B \rightarrow C, \eta(j) = [i, Z]\rangle$. Then for complex B not introduced by (Id)

$$\begin{array}{lc} n+1. & P\,B\,[j, A] \\ n+2. & O\,e\ \ [n+1, A] \mid O\,C\ \ [n+1, D] \end{array}$$

is a subtree t below d with possible situation $\Delta, C \vdash D$ in the right branch after position $n+2$ corresponding to the right premiss of $(\rightarrow\vdash)$. The possible situation in the left branch after position $n+2$ depends on the form of B, where B can be a negation, conjunction, disjunction or implication, and where B was introduced by a right introduction rule (i.e., by $(\vdash\neg)$,

$(\vdash\wedge)$, $(\vdash\vee)$ or $(\vdash\rightarrow)$). We first consider these cases for complex B.[40] After that we consider the case where the complex formula B has been introduced by (Id); in this case the move $\langle\delta(n+2) = O\, e, \eta(n+2) = [n+1, A]\rangle$ in the left branch of t is impossible. Finally, the remaining case where B is atomic is treated.

(a) For $B = \neg E$ the derivation ends with

$$(\rightarrow\vdash)\ \dfrac{\dfrac{\Gamma, E \vdash_{LI_c^p}}{\Gamma \vdash_{LI_c^p} \neg E}\,(\vdash\neg) \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, \neg E \rightarrow C \vdash_{LI_c^p} D}$$

The subtree t below d is then

$$\begin{array}{ll} n+1. & \qquad\qquad P\,\neg E\ [j, A] \\ n+2. & O\,E\ \ [n+1, A] \mid O\,C\ \ [n+1, D] \end{array}$$

and $\Gamma, E \vdash$ is a possible situation in the left branch after position $n+2$ corresponding to the premiss of $(\vdash\neg)$. By the induction hypothesis there are substrategies s and s' below the left, respectively right branch of t. Then

$$\begin{array}{llcl} n+1. & & P\,\neg E\ [j, A] & \\ n+2. & O\,E\ \ [n+1, A] & \Big| & O\,C\ \ [n+1, D] \\ & \qquad s & & \qquad s' \end{array}$$

is a substrategy for $\Gamma, \Delta, \neg E \rightarrow C \vdash D$.

(b) For $B = E_1 \wedge E_2$ the derivation ends with

$$(\rightarrow\vdash)\ \dfrac{\dfrac{\Gamma \vdash_{LI_c^p} E_1 \qquad \Gamma \vdash_{LI_c^p} E_2}{\Gamma \vdash_{LI_c^p} E_1 \wedge E_2}\,(\vdash\wedge) \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, (E_1 \wedge E_2) \rightarrow C \vdash_{LI_c^p} D}$$

The subtree t below d is then

$$\begin{array}{ll} n+1. & \qquad\qquad\qquad P\,E_1 \wedge E_2\ [j, A] \\ n+2. & O\,\wedge_1\ \ [n+1, A] \mid O\,\wedge_2\ \ [n+1, A] \mid O\,C\ \ [n+1, D] \end{array}$$

and $\Gamma \vdash E_1$ respectively $\Gamma \vdash E_2$ are possible situations in the left, respectively middle branch after position $n+2$ corresponding to the premisses of $(\vdash\wedge)$. By the induction hypothesis there are substrategies s, s' and s'' below the branches of t. Then

$$\begin{array}{llclcl} n+1. & & & P\,E_1 \wedge E_2\ [j, A] & & \\ n+2. & O\,\wedge_1\ \ [n+1, A] & \Big| & O\,\wedge_2\ \ [n+1, A] & \Big| & O\,C\ \ [n+1, D] \\ & \qquad s & & \qquad s' & & \qquad s'' \end{array}$$

is a substrategy for $\Gamma, \Delta, (E_1 \wedge E_2) \rightarrow C \vdash D$.

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[40] Without loss of generality we assume that the respective right introduction rules are applied in the last step of the derivation of the left premiss of $(\rightarrow\vdash)$.

(c) For $B = E_1 \vee E_2$ the derivation ends with

$$(\to\vdash)\,\dfrac{\dfrac{\Gamma \vdash_{LI_c^p} E_i}{\Gamma \vdash_{LI_c^p} E_1 \vee E_2}\,(\vdash\vee) \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, (E_1 \vee E_2) \to C \vdash_{LI_c^p} D}$$

The subtree t below d is then

$$\begin{array}{ll} n+1. & \qquad P\, E_1 \vee E_2\ [j, A] \\ n+2. & O \vee\ [n+1, A] \mid O\, C\ \ [n+1, D] \end{array}$$

and $\Gamma \vdash E_i$ is a possible situation in the left branch after position $n+2$ corresponding to the premiss of $(\vdash\vee)$. By the induction hypothesis there are substrategies s and s' below the branches of t. Then

$$\begin{array}{lcc} n+1. & \multicolumn{2}{c}{P\, E_1 \vee E_2\ [j, A]} \\ n+2. & O \vee\ [n+1, A] & \vline\ O\, C\ \ [n+1, D] \\ & s & \vline\ \quad s' \end{array}$$

is a substrategy for $\Gamma, \Delta, (E_1 \vee E_2) \to C \vdash D$.

(d) For $B = E \to F$ the derivation ends with

$$(\to\vdash)\,\dfrac{\dfrac{\Gamma, E \vdash_{LI_c^p} F}{\Gamma \vdash_{LI_c^p} E \to F}\,(\vdash\to) \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, (E \to F) \to C \vdash_{LI_c^p} D}$$

The subtree t below d is then

$$\begin{array}{ll} n+1. & \qquad P\, E \to F\ [j, A] \\ n+2. & O\, E\ \ [n+1, A] \mid O\, C\ \ [n+1, D] \end{array}$$

and $\Gamma, E \vdash F$ is a possible situation in the left branch after position $n+2$ corresponding to the premiss of $(\vdash\to)$. By the induction hypothesis there are substrategies s and s' below the branches of t. Then

$$\begin{array}{lcc} n+1. & \multicolumn{2}{c}{P\, E \to F\ [j, A]} \\ n+2. & O\, E\ \ [n+1, A] & \vline\ O\, C\ \ [n+1, D] \\ & s & \vline\ \quad s' \end{array}$$

is a substrategy for $\Gamma, \Delta, (E \to F) \to C \vdash D$.

(e) If B is complex and was introduced by (Id), then the derivation ends with

$$(\to\vdash)\,\dfrac{(\text{Thin}\vdash)\,\dfrac{(\text{Id})\,\dfrac{}{B \vdash_{LI_c^p} B}}{\Gamma, B \vdash_{LI_c^p} B} \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, B, B \to C \vdash_{LI_c^p} D}$$

Then $\Gamma, \Delta, B, B \to C \vdash D$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain the opponent moves $\langle \delta(j) = O\, B, \eta(j) = [i, Z] \rangle$

and $\langle \delta(l) = O\ B \to C, \eta(l) = [k, Z] \rangle$. Since B is complex, the conditions of $(D14)$ cannot be satisfied, as was shown in the treatment of (Id), case (2) on page 60. That is, d contains the opponent move $\langle \delta(j) = O\ B, \eta(j) = [i, Z] \rangle$, but no proponent attack $\langle \delta(m) = P\ e, \eta(m) = [j, A] \rangle$. Then the move $\langle \delta(n+2) = O\ e, \eta(n+2) = [n+1, A] \rangle$ is impossible, and

$$\begin{array}{lll} n+1. & P\ B & [j, A] \\ n+2. & O\ C & [n+1, D] \end{array}$$

is a subtree t below d for complex B introduced by (Id), with possible situation $\Delta, C \vdash D$ after position $n+2$ corresponding to the right premiss of $(\to\vdash)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\ B & [j, A] \\ n+2. & O\ C & [n+1, D] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \Delta, B, B \to C \vdash D$.

(f) Finally, we consider the case where B is atomic. Then B must have been introduced by (Id), and the derivation ends with

$$\dfrac{\dfrac{\dfrac{}{B \vdash_{LI_c^p} B}\,(\mathrm{Id})}{\Gamma, B \vdash_{LI_c^p} B}\,(\mathrm{Thin}\vdash) \qquad \Delta, C \vdash_{LI_c^p} D}{\Gamma, \Delta, B, B \to C \vdash_{LI_c^p} D}\,(\to\vdash)$$

Then $\Gamma, \Delta, B, B \to C \vdash D$ is a possible situation after d, and by Lemma 3.3.4 the dialogue d must contain the opponent moves $\langle \delta(j) = O\ B \to C, \eta(j) = [i, Z] \rangle$ and $\langle \delta(l) = O\ B, \eta(l) = [k, Z] \rangle$. Since B is atomic, the move $\langle \delta(n+2) = O\ e, \eta(n+2) = [n+1, A] \rangle$ is impossible, and

$$\begin{array}{lll} n+1. & P\ B & [j, A] \\ n+2. & O\ C & [n+1, D] \end{array}$$

is a subtree t below d for atomic B with possible situation $\Delta, C \vdash D$ after position $n+2$ corresponding to the right premiss of $(\to\vdash)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\ B & [j, A] \\ n+2. & O\ C & [n+1, D] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \Delta, B, B \to C \vdash D$.

Third, assume the derivation ends with a structural rule. Since (Cut) is eliminable in LI_c^p, we consider only $(\mathrm{Thin}\vdash)$, $(\vdash\mathrm{Thin})$ and (Contr):

(1) The derivation ends with

$$(\text{Thin}\vdash)\ \frac{\Gamma \vdash_{LI_c^p} C}{\Gamma, B \vdash_{LI_c^p} C}$$

Then $\Gamma, B \vdash C$ is a possible situation after d, and $\Gamma \vdash C$ is a possible situation after d as well, since $\Gamma \subseteq \Gamma \cup B$.

(2) The derivation ends with

$$\frac{\Gamma \vdash_{LI_c^p}}{\Gamma \vdash_{LI_c^p} B}\ (\vdash \text{Thin})$$

Then $\Gamma \vdash B$ is a possible situation after d, and $\Gamma \vdash$ is a possible situation after d as well, since $\Gamma \subseteq \Gamma$ and the succedent is empty.

(3) The derivation ends with

$$(\text{Contr})\ \frac{\Gamma, B, B \vdash_{LI_c^p} C}{\Gamma, B \vdash_{LI_c^p} C}$$

Then $\Gamma, B \vdash C$ is a possible situation after d, and $\Gamma, B, B \vdash C$ is a possible situation after d as well, since $\Gamma \cup B \cup B = \Gamma \cup B$ in the case of situations (cf. Remark 3.3.3).

Thus for every sequent $\Gamma \vdash_{LI_c^p} C$ in an LI_c^p-derivation there is a corresponding situation $\Gamma \vdash C$ in an EI_c^p-strategy. $\dashv$

3.7. EI_c^p-provability is equivalent to LI_c^p-provability

We collect the results of the preceding sections into the following corollaries:

COROLLARY 3.7.1. *By Theorem* 3.4.1 *and Theorem* 3.6.1 *each EI_c^p-strategy can be transformed into an LI_c^p-derivation and vice versa.*

COROLLARY 3.7.2. *By Theorem* 3.2.3 *and Corollary* 3.7.1 *the EI_c^p-dialogue-provable formulas are exactly the formulas provable in LI^p. That is, $\vdash_{EI_c^p} A$ if and only if $\vdash_{LI^p} A$.*

COROLLARY 3.7.3. *By Remark* 2.2.24 *the dialogue-provable formulas are exactly the formulas provable in LI^p. That is, $\vdash_{DI^p} A$ if and only if $\vdash_{LI^p} A$. Hence by Corollary* 3.7.2 *also $\vdash_{EI_c^p} A$ if and only if $\vdash_{DI^p} A$.*

REMARK 3.7.4. We have thus:

$$\vdash_{EI^p} A \overset{2.6.5}{\iff} \vdash_{DI^p} A \overset{2.2.24}{\iff} \vdash_{LI^p} A \overset{3.2.3}{\iff} \vdash_{LI_c^p} A \overset{3.7.1}{\iff} \vdash_{EI_c^p} A.$$

REMARK 3.7.5. Equivalence for DI_c^p-provability and LI_c^p-provability can be proved similarly to the proofs of Theorem 3.4.1 and Theorem 3.6.1 for EI_c^p-provability.

Compared with DI_c^p-dialogues, EI_c^p-dialogues have the technical advantage that the O-move made at position $n+2$ must be an attack on or a defense to the immediately preceding P-move made at position $n+1$ (due to condition (E)). For DI_c^p-dialogues this does not have to be the case. Situations would have to be defined a bit more general for O-moves $\langle \delta(n), \eta(n)\rangle = \langle O\,e, [m, Z]\rangle$, where $m < n$ instead of $m = n-1$ (cf. Definition 3.3.1). Consequently, the induction steps would have to be formulated in a slightly more general way, too.

3.8. Structural reasoning in EI_c^p-dialogues

We have used sequent calculi where structural operations like thinning, contraction and cut are explicitly given by the structural rules. This is not the case for the dialogues considered so far. Here the structural operations of thinning and contraction are only implicitly given by the conditions defining the dialogues. The structural operation of exchange is implicit both in the dialogues and sequent calculi considered so far. In the latter this is due to the fact that the antecedents of sequents are conceived as multisets (cf. Remark 3.1.4).[41]

We will now explain how structural reasoning in dialogues corresponds to applications of structural rules in sequent calculi. Since the dialogues that have as yet been dealt with do not contain an operation of cut, we will consider only thinning, contraction and exchange here. The operation of cut will be dealt with in detail in Chapter 5.

REMARK 3.8.1. In EI_c^p-dialogues the case where the proponent P does not use a formula B asserted in an opponent move $\langle \delta(j) = O\,B, \eta(j) = [i, Z]\rangle$—either by attacking it with a move $\langle \delta(k) = P\,e, \eta(k) = [j, A]\rangle$ or by asserting it in a move—corresponds to an application of (Thin$\vdash$) introducing B in LI_c^p-derivations.

Consider the following example, where $a \to (b \to a)$ is not provable without such an assertion of b by O or without the corresponding application of (Thin$\vdash$), respectively:

0.	$P\,a \to (b \to a)$	
1.	$O\,a$	$[0, A]$
2.	$P\,b \to a$	$[1, D]$
3.	$O\,b$	$[2, A]$
4.	$P\,a$	$[3, D]$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{LI_c^p} a}\,(\mathrm{Id})}{a, b \vdash_{LI_c^p} a}\,(\mathrm{Thin}\vdash)}{a \vdash_{LI_c^p} b \to a}\,(\vdash\to)}{\vdash_{LI_c^p} a \to (b \to a)}\,(\vdash\to)$$

[41] Instead of using the structural rules (Thin$\vdash$), ($\vdash$Thin) and (Contr), sequent calculi with implicit thinning and contraction can be formulated as well.

The formula b asserted by O at position 3 cannot be attacked by P; it corresponds to the formula introduced by (Thin$\vdash$).

REMARK 3.8.2. In EI_c^p-dialogues the case where the proponent P does not use a formula B by asserting it in a defense $\langle \delta(k) = P\, B, \eta(k) = [j, D] \rangle$ to an opponent attack $\langle \delta(j) = O\, e, \eta(j) = [i, A] \rangle$ corresponds to an application of ($\vdash$Thin) introducing B in LI_c^p-derivations.

Consider the following example, where $a \to (\neg a \to \neg b)$ is not provable if P would assert $\neg b$ in a defense or without the corresponding application of ($\vdash$Thin), respectively:

0.	$P\, a \to (\neg a \to \neg b)$	
1.	$O\, a$	$[0, A]$
2.	$P\, \neg a \to \neg b$	$[1, D]$
3.	$O\, \neg a$	$[2, A]$
4.	$P\, a$	$[3, A]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{LI_c^p} a}\,(\text{Id})}{a, \neg a \vdash_{LI_c^p}}\,(\neg\vdash)}{a, \neg a \vdash_{LI_c^p} \neg b}\,(\vdash \text{Thin})}{a \vdash_{LI_c^p} \neg a \to \neg b}\,(\vdash\to)}{\vdash_{LI_c^p} a \to (\neg a \to \neg b)}\,(\vdash\to)$$

To win the dialogue, the formula $\neg b$ must not be asserted by the proponent in a defense to the opponent attack at position 3; it corresponds to the formula introduced by ($\vdash$Thin). Similarly for $a \to (\neg a \to b)$, where P cannot make the defense move $\langle \delta(4) = P\, b, \eta(4) = [3, D] \rangle$ because b has not been asserted by O before.

REMARK 3.8.3. In EI_c^p-dialogues the twofold use made by the proponent P of a formula asserted by the opponent O corresponds to an application of the structural rule (Contr) in LI_c^p-derivations. The twofold use can consist either

(1) in the twofold attack of a formula by the proponent P,

(2) in the twofold assertion by the proponent P of a formula asserted by the opponent O before,

or

(3) in an attack of a formula A by the proponent P together with the assertion of A by P.

That is, the twofold use can be of the following forms:

$$(1)\quad \begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,e & [k, A] \\ & \vdots & \\ m. & P\,e & [k, A] \end{array} \qquad\qquad (2)\quad \begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,A & [i < l, Z] \\ & \vdots & \\ m. & P\,A & [j < m, Z] \end{array}$$

$$
(3)\quad
\begin{array}{lll}
k. & O\,A & [k-1, Z] \\
 & \vdots & \\
l. & P\,e & [k, A] \\
 & \vdots & \\
m. & P\,A & [i < m, Z]
\end{array}
\qquad \text{respectively} \qquad
\begin{array}{lll}
k. & O\,A & [k-1, Z] \\
 & \vdots & \\
l. & P\,A & [i < l, Z] \\
 & \vdots & \\
m. & P\,e & [k, A]
\end{array}
$$

Consider the following two examples in which the twofold use made by P of an assertion made by O is of the form (1). The formulas $\neg(a \wedge \neg a)$ respectively $\neg\neg(a \vee \neg a)$ are not provable without a twofold attack on $a \wedge \neg a$ respectively $\neg(a \vee \neg a)$ by P, or without the corresponding application of (Contr) in the LI_c^p-derivations, respectively:

(i)

0.	$P\,\neg(a \wedge \neg a)$	
1.	$O\,a \wedge \neg a$	$[0, A]$
2.	$P\,\wedge_1$	$[1, A]$
3.	$O\,a$	$[2, D]$
4.	$P\,\wedge_2$	$[1, A]$
5.	$O\,\neg a$	$[4, D]$
6.	$P\,a$	$[5, A]$

$$
\dfrac{\text{(Contr)}\ \dfrac{(\wedge\vdash)\ \dfrac{(\wedge\vdash)\ \dfrac{(\neg\vdash)\ \dfrac{(\mathrm{Id})\ \dfrac{}{a \vdash_{LI_c^p} a}}{a, \neg a \vdash_{LI_c^p}}}{a, a \wedge \neg a \vdash_{LI_c^p}}}{a \wedge \neg a, a \wedge \neg a \vdash_{LI_c^p}}}{a \wedge \neg a \vdash_{LI_c^p}}}{\vdash_{LI_c^p} \neg(a \wedge \neg a)}\ (\vdash\neg)
$$

The twofold attack at positions 2 and 4 corresponds to the contraction of $a \wedge \neg a, a \wedge \neg a$ to $a \wedge \neg a$.

(ii)

0.	$P\,\neg\neg(a \vee \neg a)$	
1.	$O\,\neg(a \vee \neg a)$	$[0, A]$
2.	$P\,a \vee \neg a$	$[1, A]$
3.	$O\,\vee$	$[2, A]$
4.	$P\,\neg a$	$[3, D]$
5.	$O\,a$	$[4, A]$
6.	$P\,a \vee \neg a$	$[1, A]$
7.	$O\,\vee$	$[6, A]$
8.	$P\,a$	$[7, D]$

$$
\dfrac{\text{(Contr)}\ \dfrac{(\neg\vdash)\ \dfrac{\dfrac{\dfrac{(\neg\vdash)\ \dfrac{\dfrac{(\mathrm{Id})\ \dfrac{}{a \vdash_{LI_c^p} a}}{a \vdash_{LI_c^p} a \vee \neg a}\ (\vdash\vee)}{a, \neg(a \vee \neg a) \vdash_{LI_c^p}}}{\neg(a \vee \neg a) \vdash_{LI_c^p} \neg a}\ (\vdash\neg)}{\neg(a \vee \neg a) \vdash_{LI_c^p} a \vee \neg a}\ (\vdash\vee)}{\neg(a \vee \neg a), \neg(a \vee \neg a) \vdash_{LI_c^p}}}{\neg(a \vee \neg a) \vdash_{LI_c^p}}}{\vdash_{LI_c^p} \neg\neg(a \vee \neg a)}\ (\vdash\neg)
$$

The twofold attack at positions 2 and 6 corresponds to the contraction of $\neg(a \vee \neg a), \neg(a \vee \neg a)$ to $\neg(a \vee \neg a)$.

An example where the twofold use of a formula is of the form (2) is in the following EI_c^p-strategy:[42]

0.	$P\,(a \to (a \to b)) \to (a \to b)$	
1.	$O\,a \to (a \to b)$	$[0, A]$

(cont'd on next page)

[42] Cf. Keiff [2011].

2.	$P\, a \to b$	$[1, D]$
3.	$O\, a$	$[2, A]$
4.	$P\, a$	$[1, A]$
5.	$O\, a \to b$	$[4, D]$
6.	$P\, a$	$[5, A]$
7.	$O\, b$	$[6, D]$
8.	$P\, b$	$[3, D]$

The proponent P uses the formula a asserted by O at position 3 twice: once at position 4, and then again at position 6. The corresponding LI_c^p-derivation with the corresponding application of (Contr) is:

$$
\dfrac{(\to\vdash)\,\dfrac{(\mathrm{Contr})\,\dfrac{(\to\vdash)\,\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash_{LI_c^p} a} \qquad (\to\vdash)\,\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash_{LI_c^p} a} \quad (\mathrm{Id})\,\dfrac{}{b \vdash_{LI_c^p} b}}{a, a \to b \vdash_{LI_c^p} b}}{a, a, a \to (a \to b) \vdash_{LI_c^p} b}}{a, a \to (a \to b) \vdash_{LI_c^p} b}}{a \to (a \to b) \vdash_{LI_c^p} a \to b}}{\vdash_{LI_c^p} (a \to (a \to b)) \to (a \to b)}\,(\to\vdash)
$$

Two examples of strategies in which contraction is applied according to form (3) can be found in Example 4.3.5; see also Remark 4.3.7.

THEOREM 3.8.4. *There are eliminable applications of the structural rules* (Thin$\vdash$), (Contr) *or* (Cut) *in* LI_c^p *that cannot be reflected in* EI_c^p*-dialogues.*

PROOF. There is exactly one EI_c^p-strategy S for $a \to a$, which by Corollary 3.7.1 has exactly one corresponding LI_c^p-derivation without applications of structural rules:

0.	$P\, a \to a$	
1.	$O\, a$	$[0, A]$
2.	$P\, a$	$[1, D]$

$$
\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash_{LI_c^p} a}}{\vdash_{LI_c^p} a \to a}\,(\vdash\to)
$$

However, there are infinitely many LI_c^p-derivations of $\vdash_{LI_c^p} a \to a$ using (eliminable applications of) the structural rules (Thin$\vdash$) and (Contr), or (Cut), but there is only the one strategy S which corresponds to all of them. $\dashv$

THEOREM 3.8.5. *There are eliminable applications of the structural rules* (Thin$\vdash$) *and* (Contr) *in* LI_c^p *that can be reflected in* EI_c^p*-dialogues.*

PROOF. Consider the following LI_c^p-derivation of $\vdash_{LI_c^p} \neg(a \wedge \neg a)$ containing eliminable applications of (Thin$\vdash$) and (Contr) together with its corresponding EI_c^p-strategy:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash_{LI_c^p} a}\,(\mathrm{Id})}{a, \neg a \vdash_{LI_c^p}}\,(\neg\vdash)}{a, \neg a, a \vdash_{LI_c^p}}\,(\mathrm{Thin}\vdash)}{a, a \wedge \neg a, a \vdash_{LI_c^p}}\,(\wedge\vdash)}{a, a \wedge \neg a, a \wedge \neg a \vdash_{LI_c^p}}\,(\wedge\vdash)}{a \wedge \neg a, a \wedge \neg a, a \wedge \neg a \vdash_{LI_c^p}}\,(\wedge\vdash)}{a \wedge \neg a, a \wedge \neg a \vdash_{LI_c^p}}\,(\mathrm{Contr})}{a \wedge \neg a \vdash_{LI_c^p}}\,(\mathrm{Contr})}{\vdash_{LI_c^p} \neg(a \wedge \neg a)}\,(\vdash\neg)
$$

0.	$P\,\neg(a \wedge \neg a)$	
1.	$O\,a \wedge \neg a$	$[0, A]$
2.	$P\,\wedge_1$	$[1, A]$
3.	$O\,a$	$[2, D]$
4.	$P\,\wedge_1$	$[1, A]$
5.	$O\,a$	$[4, D]$
6.	$P\,\wedge_2$	$[1, A]$
7.	$O\,\neg a$	$[6, D]$
8.	$P\,a$	$[7, A]$

The (eliminable) application of (Thin$\vdash$) is reflected in an additional assertion of a by O which is not attacked by P (the use of a by P in the move at position 8 demands only one assertion of a by O), and the (eliminable) application of (Contr) is reflected in the two twofold attacks at positions 2 and 6, respectively at positions 4 and 6. $\dashv$

REMARK 3.8.6. Exchange is implicit in LI_c^p because the antecedents in LI_c^p-sequents are multisets. In EI_c^p-dialogues exchange is incorporated due to the fact that for any two given assertions made by O for which there are possible attacks by P, the proponent P can attack either of them, that is, irrespective of their order in the dialogue.

As a consequence of the presence of exchange in EI_c^p-dialogues, multiple assertions of the same formula made by the opponent O cannot in general be distinguished with respect to their role in the dialogue. For example, in the dialogue for $\neg(a \wedge \neg a)$ presented in the proof of Theorem 3.8.5 there are two assertions of a made by the opponent O. Only one of them is necessary for the move at position 8, while the other one is the result of thinning. Which assertion corresponds to what cannot be determined because of implicit exchange.

DEFINITION 3.8.7. *Contraction-free EI_c^p-dialogues* are obtained from EI_c^p-dialogues by adding the following condition:

($D13^*$) For any move $\langle\delta(k) = O\,A, \eta(k) = [j, Z]\rangle$ there is at most one move of the form $\langle\delta(l) = P\,e, \eta(l) = [k, A]\rangle$ or $\langle\delta(l) = P\,A, \eta(l) = [i, Z]\rangle$, where $j < k < l$ and $i < l$. That is, each assertion of an O-signed formula may be used by P at most once.

Contraction-free EI_c^p-dialogues are thus defined by the conditions ($D00$), ($D01$), ($D02$), ($D10$), ($D11'$), ($D12'$), ($D13^*$), ($D14$) and (E), with the argumentation forms as given for dialogues.

The results and observations just made on structural reasoning in EI_c^p-dialogues apply directly to DI_c^p-dialogues. The only exception concerns contraction-free DI_c^p-dialogues, which have to be defined as follows:

DEFINITION 3.8.8. *Contraction-free* DI_c^p*-dialogues* are DI_c^p-dialogues with the additional dialogue condition $(D13^*)$. They are thus defined by the conditions $(D00)$, $(D01)$, $(D02)$, $(D10)$, $(D11)$, $(D12)$, $(D13)$, $(D13^*)$ and $(D14)$, with the argumentation forms as given for dialogues.

3.9. Dialogues for first-order logic

So far, only propositional logic has been treated. We point out that the results generalize to first-order logic, if the quantifiers are just seen as generalizations of disjunction (in the case of the existential quantifier $\exists$) and of conjunction (in the case of the universal quantifier $\forall$) to the infinite case. The interesting notions are then already given by considering the propositional case.[43] Nonetheless, we here compare some different treatments of the first-order quantifiers $\forall$ and $\exists$ in a dialogue setting.

We first extend our language to first-order:

DEFINITION 3.9.1. We extend our language to a *first-order language* by adding *variables* $x, y, \ldots$, *terms* t (where variables are terms), and the *logical constants* $\forall$ (universal quantifier) and $\exists$ (existential quantifier). The expression $\exists$ and terms t are also used as *special symbols*.

In addition to the usual definitions of *bound* variables and *free* occurrences of variables, we consider an occurrence of a variable in a symbolic attack to be free.

Now we extend the concepts of argumentation form, dialogue and strategy to first-order logic, following again the presentation of Felscher [1985], [2002] with slight deviations.

DEFINITION 3.9.2. We add argumentation forms for $\forall$ and $\exists$:

universal quantifier $\forall$:	assertion:	$X\ \forall x A(x)$	
	attack:	$Y\ t$	$(Y$ chooses the term $t)$
	defense:	$X\ A(x)[t/x]$	
existential quantifier $\exists$:	assertion:	$X\ \exists x A(x)$	
	attack:	$Y\ \exists$	
	defense:	$X\ A(x)[t/x]$	$(X$ chooses the term $t)$

[43] It might be observed that due to the constructive treatment of implication in intuitionistic logic, quantification enters the picture already on the propositional level.

where $[t/x]$ is the substitution of the term t for the variable x, and $A(x)[t/x]$ is the result of substituting t for all occurrences of x in A. This substitution instance is also written $A(t)$.

DEFINITION 3.9.3. *Dialogues* and *strategies* for first-order logic are defined as propositional dialogues and strategies extended by the argumentation forms for $\forall$ and $\exists$.

EXAMPLE 3.9.4. The first-order formula $\neg\neg\forall x a(x) \to \forall x \neg\neg a(x)$ has the following strategy:

0.			$P\,\neg\neg\forall x a(x) \to \forall x \neg\neg a(x)$				
1.			$O\,\neg\neg\forall x a(x)$	$[0, A]$			
2.			$P\,\forall x \neg\neg a(x)$	$[1, D]$			
3.	$O\,t_1$	$[2, A]$	$O\,t_2$	$[2, A]$	$O\,t_3$	$[2, A]$	...
4.	$P\,\neg\neg a(t_1)$	$[3, D]$	$P\,\neg\neg a(t_2)$	$[3, D]$	$P\,\neg\neg a(t_3)$	$[3, D]$	...
5.	$O\,\neg a(t_1)$	$[4, A]$	$O\,\neg a(t_2)$	$[4, A]$	$O\,\neg a(t_3)$	$[4, A]$	...
6.	$P\,\neg\forall x a(x)$	$[1, A]$	$P\,\neg\forall x a(x)$	$[1, A]$	$P\,\neg\forall x a(x)$	$[1, A]$	...
7.	$O\,\forall x a(x)$	$[6, A]$	$O\,\forall x a(x)$	$[6, A]$	$O\,\forall x a(x)$	$[6, A]$	...
8.	$P\,t_1$	$[7, A]$	$P\,t_2$	$[7, A]$	$P\,t_3$	$[7, A]$	...
9.	$O\,a(t_1)$	$[8, A]$	$O\,a(t_2)$	$[8, A]$	$O\,a(t_3)$	$[8, A]$	...
10.	$P\,a(t_1)$	$[5, A]$	$P\,a(t_2)$	$[5, A]$	$P\,a(t_3)$	$[5, A]$	...

As the domain of terms is denumerably infinite, the strategy consists of denumerably infinitely many dialogues (indicated by '...').

REMARK 3.9.5. Infinite strategies can be avoided by replacing them by their so-called *skeletons* (cf. Felscher [1985], [2002]).

A *skeleton* for a formula A is a subtree S of the dialogue tree for A such that S does not branch at even positions, all branches of S are dialogues for A won by P, and S has as many nodes at odd positions as there are possible moves for O, with the following exceptions: Only one node at odd positions n has to be considered if

(i) $\langle\delta(n) = O\,y, \eta(n) = [m, A]\rangle$ for $\langle\delta(m) = P\,\forall x A(x), \eta(m) = [l, Z]\rangle$ where the variable y is not occurring free in any expression $\delta(k)$ with $k < n$. That is, O is attacking $P\,\forall x A(x)$ according to the argumentation form for $\forall$ (choosing $t = y$).

(ii) $\langle\delta(n) = O\,A(x)[y/x], \eta(n) = [m, D]\rangle$ for $\langle\delta(m) = P\,\exists, \eta(m) = [l, A]\rangle$ where the variable y is not occurring free in any expression $\delta(k)$ with $k < n$. That is, O is defending an attack $P\,\exists$ on an assertion $O\,\exists x A(x)$ according to the argumentation form for $\exists$ (choosing $t = y$).

REMARK 3.9.6. Skeletons are an improvement compared to strategies if finite objects are preferred to infinite ones. However, from a technical point of view one might want to improve on skeletons by avoiding the following property (cf. Felscher [1985], [2002]). Consider the two skeletons:

0.	$P\,\exists x a(x) \to \exists x a(x)$		0.	$P\,\exists x a(x) \to \exists x a(x)$	
1.	$O\,\exists x a(x)$	$[0, A]$	1.	$O\,\exists x a(x)$	$[0, A]$
2.	$P\,\exists$	$[1, A]$	2.	$P\,\exists x a(x)$	$[1, D]$
3.	$O\,a(y)$	$[2, D]$	3.	$O\,\exists$	$[2, A]$
4.	$P\,\exists x a(x)$	$[1, D]$	4.	$P\,\exists$	$[1, A]$
5.	$O\,\exists$	$[4, A]$	5.	$O\,a(y)$	$[4, D]$
6.	$P\,a(y)$	$[5, D]$	6.	$P\,a(y)$	$[3, D]$

The skeleton on the left corresponds to a sequent calculus derivation of $\exists x a(x) \to \exists x a(x)$ where the right introduction rule for the existential quantifier has to be applied before the left introduction rule due to the eigenvariable condition in the latter.[44]

In the left skeleton the move $O\,\exists$ (at position 5) comes after the move $P\,\exists$ (at position 3). In the skeleton on the right this order is reversed, and the attack $O\,\exists$ at position 3 can be defended by $P\,a(y)$ only at position 6 after the assertion of $a(y)$ in the defense move $O\,a(y)$ at position 5. Felscher observes that "[t]here are no phenomena of an analogous type in, say, the sequent calculus".[45]

Skeletons of the type on the right can be avoided by using the following *formal argumentation forms* for $\forall$ and $\exists$ instead of the ones used so far. The set of dialogue-provable formulas is not changed by that.

DEFINITION 3.9.7. We define *formal argumentation forms* for the quantifiers $\forall$ and $\exists$ as follows:[46]

($P\,\forall$-form):	assertion:	$P\,\forall x A(x)$	
	attack:	$O\,y$	(with eigenvariable condition)
	defense:	$P\,A(x)[y/x]$	
($O\,\forall$-form):	assertion:	$O\,\forall x A(x)$	
	attack:	$P\,t$	
	defense:	$O\,A(x)[t/x]$	
($P\,\exists$-form):	assertion:	$P\,\exists x A(x)$	
	attack:	$O\,t$	
	defense:	$P\,A(x)[t/x]$	

[44] Cf. Definition 3.10.1 below.

[45] See Felscher [1985, p. 223], Felscher [2002, p. 141].

[46] Cf. Felscher [1985], [2002]. As Felscher [2002, p. 142] explains (where the opponent is denoted by 'Q'), "[t]he adjective *formal* [...] refers to the fact that, contrary to the intuitive understanding, in the attack $Q\,t$ the term t is stated already by Q; eigenvariables chosen at a later position then *must* respect these expressions $Q\,t$".

($O\,\exists$-form):	assertion:	$O\,\exists x A(x)$
	attack:	$P\,\exists$
	defense:	$O\,A(x)[y/x]$ (with eigenvariable condition)

where the *eigenvariable condition* is that y does not occur free in an expression (i.e., in an assertion or in a symbolic attack) before. That is, the move $\langle\delta(n) = O\,y, \eta(n) = [m, A]\rangle$, respectively the move $\langle\delta(n) = O\,A(x)[y/x]$, $\eta(n) = [m, D]\rangle$, in a dialogue is only possible if y does not occur free at positions $k < n$ in that dialogue.

DEFINITION 3.9.8. A dialogue constructed in accordance with the formal argumentation forms is called *formal dialogue*.

A *formal dialogue tree* is a tree whose branches contain as paths all possible formal dialogues for a given formula.

P wins a formal dialogue for a formula A if the formal dialogue is finite, begins with the move $P\,A$ and ends with a move of P such that O cannot make another move.

DEFINITION 3.9.9. A formal dialogue is called *DI-dialogue* if it satisfies the conditions $(D00)$–$(D02)$ and $(D10)$–$(D13)$ as given in Definitions 2.1.6 and 2.2.1.

DEFINITION 3.9.10. A *DI_c-dialogue* is a *DI*-dialogue that satisfies the additional condition $(D14)$ as given in Definition 2.5.1.

DEFINITION 3.9.11. A formal dialogue is called *EI-dialogue* if it satisfies the conditions $(D00)$–$(D02)$, $(D10)$–$(D13)$ and (E) as given in Definitions 2.1.6, 2.2.1 and 2.6.1.

DEFINITION 3.9.12. A formal dialogue is called *EI_c-dialogue* if it satisfies the following conditions:

$(D00)$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

$(D01)$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11')$ If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and

$\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

$(D12')$ For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

DEFINITION 3.9.13. A *formal strategy* for a formula A is a subtree S of the formal dialogue tree for A such that S does not branch at even positions, all branches of S are formal dialogues for A won by P, and S has as many nodes at odd positions as there are possible moves for O, with the following exceptions: Only one node at odd positions n has to be considered if

(i) $\langle \delta(n) = O\, A(x)[y/x], \eta(n) = [m, D] \rangle$ for $\langle \delta(m) = P\, \exists, \eta(m) = [l, A] \rangle$. That is, the opponent is defending an attack $P\, \exists$ according to the formal argumentation form ($O\, \exists$-form).

(ii) $\langle \delta(n) = O\, y, \eta(n) = [m, A] \rangle$. That is, the opponent makes an attack move $O\, y$ according to the formal argumentation form ($P\, \forall$-form).

(iii) $\langle \delta(n) = O\, t, \eta(n) = [m, A] \rangle$. That is, the opponent makes an attack move $O\, t$ according to the formal argumentation form ($P\, \exists$-form).

Formal strategies will also be called *DI-strategies*, *DI_c-strategies*, *EI-strategies* or *EI_c-strategies*, depending on the respective underlying dialogues.

EXAMPLE 3.9.14. The first-order formula $\neg\neg\forall x a(x) \to \forall x \neg\neg a(x)$ has the following formal strategy:

0.	$P\, \neg\neg\forall x a(x) \to \forall x \neg\neg a(x)$	
1.	$O\, \neg\neg\forall x a(x)$	$[0, A]$
2.	$P\, \forall x \neg\neg a(x)$	$[1, D]$
3.	$O\, y$	$[2, A]$
4.	$P\, \neg\neg a(y)$	$[3, D]$
5.	$O\, \neg a(y)$	$[4, A]$
6.	$P\, \neg\forall x a(x)$	$[1, A]$
7.	$O\, \forall x a(x)$	$[6, A]$
8.	$P\, y$	$[7, A]$

(cont'd on next page)

9.	$O\, a(y)$	$[8, D]$
10.	$P\, a(y)$	$[5, A]$

Contrary to the infinite strategy of Example 3.9.4, this formal strategy is finite.

REMARK 3.9.15. Formal strategies are finite objects. Whereas formal dialogue trees are infinite in general: the formal dialogues can be infinite branches, and there can be infinitely many formal dialogues as branches in a formal dialogue tree.

Dialogue trees for propositional formulas could also be represented by objects containing infinitely many branches, since an infinite dialogue[47] can be represented as a tree with infinitely many branches of unbounded length. In our definition of dialogue trees (see Definition 2.2.10) an infinite dialogue is always represented by a branch containing infinitely many paths. However, in the case of first-order formulas there must be infinitely many branches if a formal argumentation form for a quantifier is applied. These branches cannot be represented as one infinite dialogue.

REMARK 3.9.16. Every formal strategy can be transformed into a skeleton by replacing all attack moves $O\, t$ in applications of the formal argumentation form ($P\, \exists$-form) by moves $O\, \exists$.

For example, the formal strategy

0.	$P\, \exists x a(x) \to \exists x a(x)$	
1.	$O\, \exists x a(x)$	$[0, A]$
2.	$P\, \exists$	$[1, A]$
3.	$O\, a(y)$	$[2, D]$
4.	$P\, \exists x a(x)$	$[1, D]$
5.	$O\, y$	$[4, A]$
6.	$P\, a(y)$	$[5, D]$

is transformed into the skeleton

0.	$P\, \exists x a(x) \to \exists x a(x)$	
1.	$O\, \exists x a(x)$	$[0, A]$
2.	$P\, \exists$	$[1, A]$
3.	$O\, a(y)$	$[2, D]$
4.	$P\, \exists x a(x)$	$[1, D]$
5.	$O\, \exists$	$[4, A]$
6.	$P\, a(y)$	$[5, D]$

Eigenvariable conditions for a term t in a formal strategy will not be violated if t is removed in the transformation to a skeleton.

[47]See Remark 2.2.23 (i) for an example of an infinite dialogue for a propositional formula.

Furthermore, it can be shown that every skeleton can be transformed into a formal strategy. A transformation is presented in Felscher [1985], where this is proved for E-strategies and E-skeletons. Since these concepts are closely related to the concepts of formal strategy and skeleton as they are used here, the result can be carried over directly.

We have thus the following notions of first-order intuitionistic dialogue-provability:

DEFINITION 3.9.17. A formula A is called *DI-dialogue-provable* if there is a *DI*-strategy for A. Notation: $\vdash_{DI} A$.

DEFINITION 3.9.18. A formula A is called *DI_c-dialogue-provable* if there is a DI_c-strategy for A. Notation: $\vdash_{DI_c} A$.

DEFINITION 3.9.19. A formula A is called *EI-dialogue-provable* if there is an *EI*-strategy for A. Notation: $\vdash_{EI} A$.

DEFINITION 3.9.20. A formula A is called *EI_c-dialogue-provable* if there is an EI_c-strategy for A. Notation: $\vdash_{EI_c} A$.

3.10. Sequent calculi for first-order logic

We define the sequent calculi *LI* and LI_c for intuitionistic first-order logic by adding logical rules for the quantifiers $\forall$ and $\exists$ to the propositional calculi LI^p and LI_c^p, respectively.

DEFINITION 3.10.1. The *sequent calculus LI for intuitionistic first-order logic* is the propositional calculus LI^p with additional left and right introduction rules for the quantifiers $\forall$ and $\exists$.[48] We give the whole calculus *LI*, repeating the rules of LI^p:

Axiom

$$(\mathrm{Id}_a)\ \frac{}{a \vdash a}\ \text{(where } a \text{ is atomic)}$$

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

(cont'd on next page)

[48]Again, Γ and Δ are finite multisets of formulas. The comma in antecedents of sequents stands for multiset union, and singletons are written without braces.

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\to\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

$(y$ does not occur free in $\Gamma)$

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

$(y$ does not occur free in $\Gamma, C)$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash\mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

EXAMPLE 3.10.2. A derivation of the sequent $\vdash \exists x a(x) \to \exists x a(x)$ in *LI* is the following:

$$\dfrac{(\exists\vdash)\ \dfrac{\dfrac{(\mathrm{Id})\ \dfrac{}{a(y) \vdash a(y)}}{a(y) \vdash \exists x a(x)}\ (\vdash\exists)}{\exists x a(x) \vdash \exists x a(x)}}{\vdash \exists x a(x) \to \exists x a(x)}\ (\vdash\to)$$

The derivation corresponds to the skeleton in Remark 3.9.16.

DEFINITION 3.10.3. We define LI_c to be *LI* without the axiom being restricted to atomic formulas, that is, instead of (Id_a) we use the following axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\quad (A \text{ atomic or complex})$$

The *sequent calculus LI_c with atomic or complex initial sequents for intuitionistic first-order logic* is thus given by the following rules:

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\to\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

$(y$ does not occur free in $\Gamma)$

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

$(y$ does not occur free in $\Gamma, C)$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash\mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

As in *LI*, Γ and Δ are finite multisets of formulas, the comma in antecedents of sequents stands for multiset union, and singletons are written without braces.

REMARK 3.10.4. The sequent calculus LI_c for intuitionistic first-order logic is almost exactly like Gentzen's sequent calculus *LJ* (see Gentzen

[1935]). As in the propositional case (cf. Remark 3.1.4), we do not need a structural rule for exchange, since we treat the antecedents of sequents not as lists of formulas (as in *LJ*) but as multisets of formulas.

Theorem 3.10.5. *The calculi LI and LI_c are equivalent.*

Proof. See Theorem 3.2.3. (Id_a) is a special case of (Id), and (Id) is derivable in *LI*. ⊣

3.11. First-order equivalence results

In what follows, we assume without proof that the set of EI_c-provable formulas is a conservative extension of the set of EI_c^p-provable formulas. Under this assumption we can extend the equivalence result for EI_c^p-provability and LI_c^p-provability by just adding the cases for the first-order fragment to the proofs of Theorem 3.4.1 and Theorem 3.6.1, without having to modify the proofs for the propositional fragment. Furthermore, we have to consider only formal dialogues and formal strategies. This has the advantage that no infinite strategies have to be considered.[49] Moreover, for each formal argumentation form there is then a corresponding sequent calculus rule and vice versa (where the formal argumentation forms ($P\,\forall$-form), ($O\,\forall$-form), ($P\,\exists$-form) and ($O\,\exists$-form) correspond to the sequent calculus rules ($\vdash\forall$), ($\forall\vdash$), ($\vdash\exists$) and ($\exists\vdash$), respectively).

We extend the definition of situations to first-order formulas (cf. Definition 3.3.1):

Definition 3.11.1. Let d be a (possibly empty) finite EI_c-dialogue for A. The empty EI_c-dialogue contains no moves and is called d_ε. The *(first-order) situation after d* is written $\Gamma \vdash C$ and is defined as follows:

(i) The situation after the empty EI_c-dialogue d_ε is $\vdash A$.

(ii) If $\Gamma \vdash C$ is the situation after d, then the situation after $d, \langle\delta(n), \eta(n)\rangle$ is

  (a) $\Gamma \vdash C$, if $\langle\delta(n), \eta(n)\rangle$ is a proponent move, that is, a proponent move does not change the situation,

  (b) $\Gamma, B \vdash C$, if $\langle\delta(n), \eta(n)\rangle = \langle O\,B, [n-1, D]\rangle$, that is, if $\langle\delta(n), \eta(n)\rangle$ is an opponent defense asserting B at position n,

[49]This could also be achieved by considering skeletons. However, the step to formal argumentation forms has the additional advantage that no dialogues like the skeleton on the right in Remark 3.9.6 can occur; these would not have a corresponding sequent calculus derivation.

(c) as follows, if $\langle\delta(n), \eta(n)\rangle = \langle O\, e, [n-1, A]\rangle$ where $\delta(m) = P\, D$, that is, if $\langle\delta(n), \eta(n)\rangle$ is an opponent attack stating e at position n on an assertion D at position m:

$\Gamma, A \vdash$	if $D = \neg A$ and $e = A$
$\Gamma \vdash A_1$	if $D = A_1 \wedge A_2$ and $e = \wedge_1$
$\Gamma \vdash A_2$	if $D = A_1 \wedge A_2$ and $e = \wedge_2$
$\Gamma \vdash A_i$	if $D = A_1 \vee A_2$ and $e = \vee$
$\Gamma, A \vdash B$	if $D = A \to B$ and $e = A$
$\Gamma \vdash A(x)[y/x]$	if $D = \forall x A(x)$ and $e = y$
$\Gamma \vdash A(x)[t/x]$	if $D = \exists x A(x)$ and $e = t$

(This definition differs from the one for the propositional case only in the addition of the last two (types of) situations for the first-order formulas. The definition of *possible situation after d* (see Definition 3.5.1) remains the same for first-order situations.)

REMARK 3.11.2. (i) We use the sequent symbol '$\vdash_{LI_c}$' in LI_c-sequents (i.e. $\Gamma \vdash_{LI_c} C$ instead of $\Gamma \vdash C$) in what follows, in order to distinguish them from situations (written $\Gamma \vdash C$).

(ii) Γ is a set in situations $\Gamma \vdash C$, whereas in LI_c-sequents $\Gamma \vdash_{LI_c} C$ it is a multiset.

(iii) Lemma 3.3.4 holds without change also for first-order situations.

THEOREM 3.11.3. *There is an EI_c-strategy for a formula A if and only if there is a derivation of the sequent $\vdash A$ (resp. $\vdash_{LI_c} A$) in LI_c, that is, $\vdash_{EI_c} A$ if and only if $\vdash_{LI_c} A$.*

PROOF. We first show the direction from left to right, that is, we show that if $\vdash_{EI_c} A$, then $\vdash_{LI_c} A$.

Let S be an EI_c-strategy for A and let

$$d = \langle\delta(0), \eta(0)\rangle, \langle\delta(1), \eta(1)\rangle, \langle\delta(2), \eta(2)\rangle, \ldots, \langle\delta(n), \eta(n)\rangle$$

be an EI_c-dialogue in S not ending in a proponent move. We show by induction on the subtree below d in S that if $\Gamma \vdash C$ is the situation after d, then there is a derivation of the sequent $\Gamma \vdash_{LI_c} C$.

Since d is part of an EI_c-strategy S, there is a proponent move $\langle\delta(n+1) = P\, e, \eta(n+1) = [j, Z]\rangle$. This move is either an attack or a defense.

First, assume the proponent move is an attack $\langle\delta(n+1) = P\, e, \eta(n+1) = [j, A]\rangle$ on $\langle\delta(j) = O\, D, \eta(j) = [i, Z]\rangle$. Then $D \in \Gamma$ by Lemma 3.3.4 (cf. Remark 3.11.2 (iii)). Let $\Gamma' = \Gamma \setminus D$, that is, the set Γ' is Γ without D, and the corresponding multiset has thus no occurrence of D either.

We consider the two first-order cases $D = \forall x A(x)$ and $D = \exists x A(x)$:

(1) $D = \forall x A(x)$. Then the subtree below d is

$$\begin{array}{lll} n+1. & P\,t & [j, A] \\ n+2. & O\,A(x)[t/x] & [n+1, D] \end{array}$$

and the situation after position $n+2$ is $\Gamma', A(x)[t/x] \vdash C$. Then the sequent $\Gamma', A(t) \vdash_{LI_c} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c} C$ is derivable by $(\forall\vdash)$:

$$(\forall\vdash)\ \frac{\Gamma', A(t) \vdash_{LI_c} C}{\Gamma', \forall x A(x) \vdash_{LI_c} C}$$

(2) $D = \exists x A(x)$. Then the subtree below d is

$$\begin{array}{lll} n+1. & P\,\exists & [j, A] \\ n+2. & O\,A(x)[y/x] & [n+1, D] \end{array}$$

if the eigenvariable condition is satisfied, and the situation after position $n+2$ is $\Gamma', A(x)[y/x] \vdash C$. Then $\Gamma', A(y) \vdash_{LI_c} C$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c} C$ is derivable by $(\exists\vdash)$:

$$(\exists\vdash)\ \frac{\Gamma', A(y) \vdash_{LI_c} C}{\Gamma', \exists x A(x) \vdash_{LI_c} C}$$

Second, assume the proponent move is a defense $\langle \delta(n+1) = P\,E, \eta(n+1) = [j, D]\rangle$ to $\langle \delta(j) = O\,D, \eta(j) = [i, Z]\rangle$ or the initial move $\langle \delta(0) = P\,E, \eta(0) = \emptyset\rangle$. Then $E = C$ by Lemma 3.3.4 (cf. Remark 3.11.2 (iii)). We first consider the cases where the conditions (i) and (ii) of $(D14)$ are satisfied, that is, where E either has not been asserted by O in d or where E has been asserted by O but was attacked by P in d.

Again, we consider only the two first-order cases $E = \forall x A(x)$ and $E = \exists x A(x)$:

(1) $E = \forall x A(x)$. Then the subtree below d is

$$\begin{array}{lll} n+1. & P\,\forall x A(x) & [j, D] \\ n+2. & O\,y & [n+1, A] \end{array}$$

if the eigenvariable condition is satisfied, and the situation after position $n+2$ is $\Gamma \vdash A(x)[y/x]$. Then $\Gamma \vdash_{LI_c} A(y)$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c} C$ is derivable by $(\vdash\forall)$:

$$\frac{\Gamma \vdash_{LI_c} A(y)}{\Gamma \vdash_{LI_c} \forall x A(x)}\ (\vdash\forall)$$

(2) For $E = \exists x A(x)$ the subtree below d is

$$\begin{array}{lll} n+1. & P\,\exists x A(x) & [j, D] \\ n+2. & O\,t & [n+1, A] \end{array}$$

and the situation after position $n+2$ is $\Gamma \vdash A(x)[t/x]$. Then $\Gamma \vdash_{LI_c} A(t)$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c} C$ is derivable by $(\vdash \exists)$.

$$\frac{\Gamma \vdash_{LI_c} A(t)}{\Gamma \vdash_{LI_c} \exists x A(x)} (\vdash \exists)$$

If the conditions (i) and (ii) of $(D14)$ are not satisfied, that is, if E has been asserted by O in d without having been attacked by P in d, then moves of the form $\langle \delta(n+2) = O\, e, \eta(n+2) = [n+1, A]\rangle$ are not possible. Hence for formulas E the subtrees below d all have the form

$$n+1. \quad P\,E \quad [j, D]$$

where only the move of P at position $n+1$ remains. At position $n+1$ we know that $E \in \Gamma$ because the conditions of $(D14)$ are not satisfied at position $n+1$; therefore E must have been asserted by O in d. Furthermore, $E = C$ by Lemma 3.3.4 (cf. Remark 3.11.2 (iii)). Let $\Gamma'' = \Gamma' \setminus E$. Then $\Gamma'', E \vdash_{LI_c} E$ is derivable by the induction hypothesis, and $\Gamma \vdash_{LI_c} C$ is derivable by (Id) and (Thin$\vdash$):

$$(\text{Thin}\vdash)\,\frac{(\text{Id})\,\dfrac{}{E \vdash_{LI_c} E}}{\Gamma'', E \vdash_{LI_c} E}$$

Therefore, for every situation $\Gamma \vdash C$ in an EI_c-strategy there is a corresponding sequent $\Gamma \vdash_{LI_c} C$ in an LI_c-derivation.

Now we show the remaining direction from right to left, that is, we show that if $\vdash_{LI_c} A$, then $\vdash_{EI_c} A$.

Let d be an EI_c-dialogue for A not ending in a proponent move. We show by induction on the length of the LI_c-derivation that if $\Gamma \vdash_{LI_c} C$, and $\Gamma \vdash C$ is a possible situation after d, then there is a subtree t below d such that $\begin{smallmatrix} d \\ t \end{smallmatrix}$ is an EI_c-strategy for A.

The derivation can consist of the axiom, end with a logical rule or end with a structural rule. Here we only have to consider the four cases of the first-order logical rules, that is, where the derivation ends with an application of $(\forall\vdash)$, $(\vdash\forall)$, $(\exists\vdash)$ or $(\vdash\exists)$.

We consider first the cases where the derivation ends with a right introduction rule. If C is the formula introduced in the succedent of an LI_c^p-sequent, then $\Gamma \vdash C$ is a possible situation after the EI_c^p-dialogue d for A, and by Lemma 3.3.4 (cf. Remark 3.11.2 (iii)) either (a) $C \neq A$ and there is some proponent defense $\langle \delta(n+1) = P\,C, \eta(n+1) = [j, D]\rangle$, or (b) $C = A$. In the latter case, d must be the empty dialogue d_ε because A can only occur once and at position 0. We consider both cases for each of the two right introduction rules:

(1) The derivation ends with

$$\frac{\Gamma \vdash_{LI_c} B(y)}{\Gamma \vdash_{LI_c} \forall x B(x)} \ (\vdash \forall)$$

Then the eigenvariable condition is satisfied, and $\Gamma \vdash \forall x B(x)$ is a possible situation after the EI_c-dialogue d for A.

(a) If $\forall x B(x) \neq A$, then there is some proponent defense $\langle \delta(n+1) = P\,\forall x B(x), \eta(n+1) = [j, D] \rangle$, and the subtree t below d is

$$\begin{array}{lll} n+1. & P\,\forall x B(x) & [j, D] \\ n+2. & O\,y & [n+1, A] \end{array}$$

and $\Gamma \vdash B(x)[y/x]$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash \forall)$. By the induction hypothesis there is a substrategy s below t. Then

$$\begin{array}{lll} n+1. & P\,\forall x B(x) & [j, D] \\ n+2. & O\,y & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma \vdash \forall x B(x)$.

(b) If $\forall x B(x) = A$, then $\Gamma = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle \delta(0) = P\,\forall x B(x), \eta(0) = \emptyset \rangle$. This is followed by the move $\langle \delta(1) = O\,y, \eta(1) = [1, A] \rangle$, after which there is a substrategy s by the induction hypothesis. Then

$$\begin{array}{lll} 0. & P\,\forall x B(x) & \\ 1. & O\,y & [0, A] \\ & \quad s & \end{array}$$

is a strategy for $\vdash \forall x B(x)$.

(2) The derivation ends with

$$\frac{\Gamma \vdash_{LI_c} B(t)}{\Gamma \vdash_{LI_c} \exists x B(x)} \ (\vdash \exists)$$

Then $\Gamma \vdash \exists x B(x)$ is a possible situation after the EI_c-dialogue d for A.

(a) If $\exists x B(x) \neq A$, then the subtree t below d is

$$\begin{array}{lll} n+1. & P\,\exists x B(x) & [j, D] \\ n+2. & O\,t & [n+1, A] \end{array}$$

and $\Gamma \vdash B(x)[t/x]$ is a possible situation after position $n+2$ corresponding to the premiss of $(\vdash \exists)$. By the induction hypothesis there is a substrategy s

below t. Then

$$\begin{array}{lll} n+1. & P\,\exists x B(x) & [j, D] \\ n+2. & O\,t & [n+1, A] \\ & & s \end{array}$$

is a substrategy for $\Gamma \vdash \exists x B(x)$.

(b) If $\exists x B(x) = A$, then $\Gamma = \emptyset$, and the root of the subtree t below d_ε is now the move $\langle \delta(0) = P\,\exists x B(x), \eta(0) = \emptyset \rangle$. This is followed by the move $\langle \delta(1) = O\,t, \eta(1) = [1, A] \rangle$, after which there is a substrategy s by the induction hypothesis. Then $\langle \delta(0) = P\,\exists x B(x), \eta(0) = \emptyset \rangle$, $\langle \delta(1) = O\,\vee$, $\eta(1) = [1, A] \rangle, s$ is a strategy for $\vdash \exists x B(x)$.

Finally, we consider the cases where the derivation ends with a left introduction rule.

(1) The derivation ends with

$$(\forall\vdash)\,\frac{\Gamma, B(t) \vdash_{LI_c} C}{\Gamma, \forall x B(x) \vdash_{LI_c} C}$$

Then $\Gamma, \forall x B(x) \vdash C$ is a possible situation after the EI_c-dialogue d for A, and by Lemma 3.3.4 (cf. Remark 3.11.2 (iii)) this dialogue d must contain an opponent move $\langle \delta(j) = O\,\forall x B(x), \eta(j) = [i, Z] \rangle$. Then

$$\begin{array}{lll} n+1. & P\,t & [j, A] \\ n+2. & O\,B(t) & [n+1, D] \end{array}$$

is a subtree t below d with possible situation $\Gamma, B(t) \vdash C$ after position $n+2$ corresponding to the premiss of $(\forall\vdash)$. By the induction hypothesis there is a substrategy s below the subtree t. Then

$$\begin{array}{lll} n+1. & P\,t & [j, A] \\ n+2. & O\,B(t) & [n+1, D] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \forall x B(x) \vdash C$.

(2) The derivation ends with

$$(\exists\vdash)\,\frac{\Gamma, B(y) \vdash_{LI_c} C}{\Gamma, \exists x B(x) \vdash_{LI_c} C}$$

Then the eigenvariable condition is satisfied, and $\Gamma, \exists x B(x) \vdash C$ is a possible situation after the EI_c-dialogue d for A. By Lemma 3.3.4 (cf. Remark 3.11.2 (iii)) this dialogue d must contain an opponent move $\langle \delta(j) = O\,\exists x B(x), \eta(j) = [i, Z] \rangle$. Then

$$\begin{array}{lll} n+1. & P\,\exists & [j, A] \\ n+2. & O\,B(y) & [n+1, D] \end{array}$$

is a subtree t below d with possible situation $\Gamma, B(y) \vdash C$ after position $n+2$ corresponding to the premiss of $(\exists\vdash)$. By the induction hypothesis there is a substrategy s below the subtree t. Then

$$\begin{array}{lll} n+1. & P\,\exists & [j, A] \\ n+2. & O\,B(y) & [n+1, D] \\ & & s \end{array}$$

is a substrategy for $\Gamma, \exists x B(x) \vdash C$.

Hence, for every first-order sequent $\Gamma \vdash_{LI_c} C$ in an LI_c-derivation there is a corresponding situation $\Gamma \vdash C$ in an EI_c-strategy.

For each given LI_c-derivation proving a formula A we can thus construct an EI_c-strategy for A and vice versa. $\dashv$

REMARK 3.11.4. We have thus:

$$\vdash_{EI_c} A \overset{3.11.3}{\Longleftrightarrow} \vdash_{LI_c} A \overset{3.10.5}{\Longleftrightarrow} \vdash_{LI} A \overset{2.2.24}{\Longleftrightarrow} \vdash_{DI} A \overset{2.6.5}{\Longleftrightarrow} \vdash_{EI} A.$$

Here we have only made use of *formal* EI_c-, *DI*- and *EI*-dialogues, respectively -strategies. Corresponding equivalence results can also be established for skeletons and for strategies based on dialogues using the following argumentation forms for the quantifiers $\forall$ and $\exists$ (as given in Definition 3.9.2)

universal quantifier $\forall$:	assertion:	$X\ \forall x A(x)$	
	attack:	$Y\ t$	(Y chooses the term t)
	defense:	$X\ A(x)[t/x]$	
existential quantifier $\exists$:	assertion:	$X\ \exists x A(x)$	
	attack:	$Y\ \exists$	
	defense:	$X\ A(x)[t/x]$	(X chooses the term t)

instead of the formal argumentation forms, by using results of Felscher [1985] which he summarizes as follows (see Felscher [1985, p. 238]):

> For each of the following types of objects
>
> (1) D-dialogues
> (2) D-skeletons,
> (3) E-skeletons,
> (4) formal E-strategies,
>
> we have defined algorithms, transforming the objects of the one type into those of the adjoining types and vice versa.

(Where formal E-strategies are exactly our *EI*-strategies, D-dialogues are our DI^p-dialogues when extended to first-order formulas by adding the above two argumentation forms for $\forall$ and $\exists$; D- and E-skeletons are like our *DI*- and *EI*-strategies when replaced by skeletons.)

Compared with formal dialogues, dialogues using the above two argumentation forms for $\forall$ and $\exists$ have the advantage that all argumentation forms are independent of whether the assertion is made by the proponent P or by the opponent O. These argumentation forms are player-independent, or P/O-symmetric, in this sense.

3.12. Summary

We have introduced the sequent calculus LI_c^p and EI_c^p-dialogues. Compared with the sequent calculus LI^p and EI^p-dialogues, their distinguishing feature is that initial sequents for complex formulas are allowed and that dialogues in EI_c^p-strategies can end with assertions of complex formulas, respectively. We have then proved as a main result that LI_c^p-provability is equivalent to EI_c^p-dialogue-provability, by showing that every LI_c^p-derivation can be transformed into an EI_c^p-strategy and vice versa. Together with further results this establishes the equivalence of EI^p-, DI^p-, LI^p-, LI_c^p- and EI_c^p-provability. We have then considered structural reasoning in EI_c^p-dialogues and have introduced contraction-free EI_c^p-dialogues. Finally, the equivalence results have been generalized to first-order logic: EI_c-, LI_c-, LI-, DI- and EI-provability are equivalent. The main result is of special importance for the extension of dialogues to definitional dialogues; this will be the subject of the next chapter.

Chapter 4

DIALOGUES FOR DEFINITIONAL REASONING

The principles of definitional reflection and definitional closure have been introduced as sequent-style inferences by Hallnäs and Schroeder-Heister [1990], [1991] (see also Hallnäs [1991] and Schroeder-Heister [1993]). We introduce dialogues containing definitional reflection and definitional closure as an additional argumentation form of definitional reasoning. The resulting definitional dialogues will enable us to reason about definitions for atomic formulas. The considered definitions are clausal definitions, where the defining conditions are not restricted to atomic formulas but can be given by arbitrary (first-order) formulas. These definitions are thus a generalization of monotone inductive definitions[50] or, equivalently, of (implication-free) definite Horn clause programs as they are used in standard logic programming based on the resolution principle.[51]

For definite Horn clause programs the principle of definitional closure corresponds to the resolution principle. The principle of definitional reflection can be seen as the dual to the principle of definitional closure. It yields an extension of standard logic programming already for definite Horn clause programs. A further extension results from the use of generalized Horn clause programs, that is, from the use of clausal definitions for atomic formulas whose defining conditions can be arbitrary formulas. The principle of definitional closure allows us to infer an atomic formula from any of its defining conditions. The principle of definitional reflection says that whatever follows from each of the defining conditions of an atomic formula follows from that atomic formula alone. With the argumentation form of definitional reasoning, definitional dialogues will provide the corresponding dialogical means for these extensions.

The clausal definitions need not be wellfounded. This leads to paradoxes like Russell's, whose dialogical treatment will be considered as an example of definitional reasoning. The example shows that the structural

[50] Cf. Aczel [1977], Moschovakis [1974].

[51] For logic programming we refer to Lloyd [1993], Doets [1994], Apt [1997], Nienhuys-Cheng and de Wolf [1997] and Jäger and Stärk [1998].

operation of contraction can be critical in the presence of non-wellfounded clausal definitions: without further restrictions, there are then strategies for contradictory assertions. Certain restrictions concerning contraction in definitional dialogues are considered. Finally, an alternative approach (due to Kreuger [1994]) that does not restrict contraction is carried over to definitional dialogues.

Definitional dialogues will be introduced in two steps: We first give a preliminary definition (see Definition 4.2.1) of definitional dialogues, which is based on the definition of EI_c-dialogues. These (preliminary) definitional dialogues are then examined for a non-wellfounded definition in a setting with contraction. A comparison with derivations in the sequent calculus $LI_c(\mathcal{D})$—that is LI_c (which has contraction) extended by definitional closure and definitional reflection—shows that provability by strategies of preliminary definitional dialogues is not equivalent to provability in that sequent calculus. A slight modification has to be made to the preliminary definition to render the two notions of definitional dialogue provability and sequent calculus provability coextensive. This modification yields the final definition of definitional dialogues (see Definition 4.3.3).

4.1. Definitional reasoning

We introduce the argumentation form of definitional reasoning for clausal definitions. Clausal definitions are collections of definitional clauses, which are formulated over a first-order language.

DEFINITION 4.1.1. The *first-order language* is as given in Definition 3.9.1, where for *variables* $x, y, \dots,$ *(individual) constants* $k, l, m, \dots$ and *function symbols* $f, g, \dots$ we define *terms* as follows:

(i) Every variable is a term.

(ii) Every individual constant is a term.

(iii) If f is an n-ary function symbol and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ is also a term.

As before, we use $a, b, c, \dots$ as *relation symbols* (or *predicate symbols*). If a is an n-ary relation symbol and if $t_1, \dots, t_n$ are terms, then $a(t_1, \dots, t_n)$ is an *atomic formula* (*atom*). Complex formulas are defined as usual.

DEFINITION 4.1.2. A *definitional clause* is an expression of the form

$$a \Leftarrow B_1 \wedge \dots \wedge B_n$$

for $n \geq 0$, where a is atomic and the B_i in the *body* $B_1 \wedge \ldots \wedge B_n$ of the clause are the *defining conditions* for the *head* a.[52] The defining conditions B_i need not be atomic but can be any complex formula. Clauses with empty body are called *facts*; we indicate empty bodies with the symbol '$\top$' (*verum*).

EXAMPLE 4.1.3. (i) $a \Leftarrow (b \to c) \wedge d$ is a (propositional) definitional clause with head a and body $(b \to c) \wedge d$, containing the two defining conditions $b \to c$ and d. (This clause can also be read as a first-order clause in which all relation symbols have arity 0.)

(ii) $a(x, y) \Leftarrow \neg b(k, l, x)$ is a (quantifier-free) first-order definitional clause with the binary relation $a(x, y)$ in the head and having as defining condition the complex formula $\neg b(k, l, x)$.

(iii) $a(x, y) \Leftarrow \exists x b(k, l, f(x))$ is a first-order definitional clause, where the variable x in the functional term $f(x)$ occurring as the third argument of the ternary relation symbol b is bound by the existential quantifier.

DEFINITION 4.1.4. A finite set $\mathcal{D}$ of definitional clauses

$$\mathcal{D}\begin{cases} a \Leftarrow \Gamma_1 \\ \quad\vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

is a *(clausal) definition of the atom a*, where $\Gamma_i = B_1^i \wedge \ldots \wedge B_{n_i}^i$ is the body of the i-th clause (for $1 \leq i \leq k$). These clauses are the *defining clauses* of a with respect to definition $\mathcal{D}$.

The *set of defining conditions of a* will be represented by $\mathcal{D}(a)$, that is, $\mathcal{D}(a) = \{\Gamma_1, \ldots, \Gamma_k\}$.

REMARK 4.1.5. We write the bodies Γ_i of definitional clauses as conjunctions

$$B_1^i \wedge \ldots \wedge B_{n_i}^i$$

of the defining conditions $B_{l_i}^i$.

They could also be written as a list or set $B_1^i, \ldots, B_{n_i}^i$, where the comma functions as a 'structural conjunction'. The latter notation is more convenient in a sequent calculus setting. However, for dialogues we would first have to introduce a means to handle such lists or sets, whereas we can handle conjunctions directly via the argumentation form for $\wedge$. We will therefore use the former notation throughout.

[52] The symbol '$\Leftarrow$' is used exclusively to write definitional clauses and should not be confused with implication '$\to$'.

DEFINITION 4.1.6. A *definition* is any finite set of definitional clauses. Definitions $\mathcal{D}$ have thus the general form

$$\mathcal{D}\begin{cases} a_1 \Leftarrow \Gamma_1^1 \\ \quad\vdots \\ a_1 \Leftarrow \Gamma_{k_1}^1 \\ \quad\vdots \\ a_n \Leftarrow \Gamma_1^n \\ \quad\vdots \\ a_n \Leftarrow \Gamma_{k_n}^n \end{cases}$$

(In logic programming terms, definitions $\mathcal{D}$ are (a generalization of) logic programs where the bodies of program clauses can be arbitrary formulas.)

We can now define an argumentation form that will allow us to reason about such definitions.

DEFINITION 4.1.7. For each atom a defined by definitional clauses

$$a \Leftarrow B_1^i \wedge \ldots \wedge B_{n_i}^i$$

with defining conditions

$$\Gamma_i = B_1^i \wedge \ldots \wedge B_{n_i}^i \quad \text{(where } 1 \leq i \leq k\text{)}$$

the following argumentation form of *definitional reasoning* determines how an atom a that is stated by X can be attacked by Y and how this attack can be defended by X. We use '$\mathscr{D}$' as a special symbol to indicate the attack.

definitional reasoning:	assertion:	$X\ a$
	attack:	$Y\ \mathscr{D}$ (only if $a \neq \top$)
	defense:	$X\ \Gamma_i$ (X chooses $i = 1, \ldots, k$)

For the *verum* $\top$ we impose the following restriction: The move $X\ \top$ cannot be attacked with $Y\ \mathscr{D}$.

REMARK 4.1.8. We have defined the argumentation form of definitional reasoning in such a way that atoms—with the exception of the *verum* $\top$—can be attacked independently of whether there are definitional clauses having these atoms in their head or not. In other words, whenever a player asserts an atom, the other player may ask for its definition, regardless of whether one has been given or not. And we will not give any dialogue conditions which would prohibit attacks on undefined atoms just because they are undefined.

The restriction with respect to the *verum* $\top$ is necessary if $\top$ is treated as an atomic formula. Otherwise it would be attackable as well. This would be in conflict with its intended meaning, suggested by its use as an indicator of empty bodies of definitional clauses, that is, by standing for the empty conjunction. The meaning of the *verum* $\top$ is stipulated by the imposed restriction.

REMARK 4.1.9. The argumentation form of definitional reasoning is formulated for atoms a defined by definitional clauses

$$\begin{array}{l} a \Leftarrow B^1_1 \wedge \ldots \wedge B^1_{n_1} \\ \quad\vdots \\ a \Leftarrow B^k_1 \wedge \ldots \wedge B^k_{n_k} \end{array}$$

That is, in definitional reasoning the Γ_i chosen by X in a defense to an attack $Y\,\mathscr{D}$ on $X\,a$ must be the body of a clause with head a in the case of propositional clauses; bodies of definitional clauses not defining a cannot be chosen.

REMARK 4.1.10. In the case of first-order clauses one has to consider substitution instances of heads and bodies of clauses.

Let the substitution σ be a most general unifier[53] for the atom a and the head a' of at least one first-order clause. Then the body Γ_i of such a clause with head a' can be chosen in a defense $X\,\Gamma_i\sigma$ to an attack $Y\,\mathscr{D}$ on $X\,a$ since $a\sigma \equiv a'\sigma$.[54] That is, in order to defend such an attack, we first have to look for a most general unifier σ which unifies a with the head of a clause $a' \Leftarrow \Gamma_i$. If it exists,[55] we apply it to Γ_i, and the defense move is $X\,\Gamma_i\sigma$.

For example, if the first-order clause $a(t) \Leftarrow b(x)$ is given by definition, then an attack $Y\,\mathscr{D}$ on a move $X\,a(x)$ can be defended with the move $b(t)$. That is, the definitional reasoning for the given clause is of the form

$$\begin{array}{l} X\,a(x) \\ Y\,\mathscr{D} \\ X\,b(t) \end{array}$$

where the substitution $\sigma = [t/x]$ is here the most general unifier for the atom $a(x)$ and the head $a(t)$ of the definitional clause. Applying σ to the body $b(x)$ of the clause yields $b(t)$, which is asserted in the defense move.

Although the unification step is made only implicitly in the definitional reasoning, and substitutions are not extra marked down in dialogues, we are in general interested in the substitutions computed in the construction of strategies (see Example 4.2.9 below).

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[53]A substitution σ is a *unifier* of two atoms a and b if $a\sigma \equiv b\sigma$, that is, if $a\sigma$ and $b\sigma$ are syntactically identical. A substitution σ is a *most general unifier* of two atoms a and b if for all unifiers τ of a and b it holds that $\tau = \sigma\rho$ for a substitution ρ.

[54]We write $A\sigma$ to denote the result of the application of a substitution σ to a formula A.

[55]This is decidable by the unification algorithm; see e.g. Lloyd [1993, p. 24f.].

REMARK 4.1.11. The argumentation form of definitional reasoning is the dialogical equivalent to the principles of definitional closure and definitional reflection. Both principles are incorporated in the one argumentation form of definitional reasoning.

REMARK 4.1.12. As sequent-style inferences the principles of definitional closure and reflection are formulated as follows. We consider the set

$$\mathcal{D}\begin{cases} a \Leftarrow \Gamma_1 \\ \quad\vdots \\ a \Leftarrow \Gamma_k \end{cases}$$

of definitional clauses defining the atom a, where $\Gamma_i = B_1^i \wedge \ldots \wedge B_{n_i}^i$ is the body of the i-th clause defining a. The *principle of definitional closure* has the form of a right introduction rule for atoms a defined by $\mathcal{D}$:[56]

(definitional closure) $$\frac{\Delta \vdash \Gamma_i}{\Delta \vdash a}\,(\vdash \mathcal{D})$$

The *principle of definitional reflection*[57] has the form of a left introduction rule for atoms a defined by $\mathcal{D}$:

(definitional reflection) $$(\mathcal{D}\vdash)\,\frac{\Delta, \Gamma_1 \vdash C \quad \ldots \quad \Delta, \Gamma_k \vdash C}{\Delta, a \vdash C}$$

For definitional clauses in a first-order language and substitutions σ replacing variables by terms the principle of definitional closure is

(definitional closure) $$\frac{\Delta \vdash \Gamma_i\sigma}{\Delta \vdash a\sigma}\,(\vdash \mathcal{D})$$

and the principle of definitional reflection is

(definitional reflection) $$(\mathcal{D}\vdash)\,\frac{\{\Delta, \Gamma_i\sigma \vdash C \mid b \Leftarrow \Gamma_i \in \mathcal{D} \text{ and } a = b\sigma\}}{\Delta, a \vdash C}$$

where for the correct handling of variables by means of substitution the following proviso has to be observed:

> For any substitution σ replacing variables by terms, the application of definitional reflection is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

[56] The defining conditions Γ_i need not be written as a conjunction $B_1^i \wedge \ldots \wedge B_{n_i}^i$ but could be given as a list or set $B_1^i, \ldots, B_{n_i}^i$ (cf. Remark 4.1.5). The principle of definitional closure would then have to be given in the form

$$\frac{\Delta \vdash B_1^i \quad \ldots \quad \Delta \vdash B_{n_i}^i}{\Delta \vdash a}\,(\vdash \mathcal{D})$$

(to stay within our context of intuitionistic logic).

[57] See Hallnäs [1991] and Hallnäs and Schroeder-Heister [1990], [1991].

That is, the set $\mathcal{D}(a\sigma)$ of defining conditions of $a\sigma$ has to be a subset of the set $(\mathcal{D}(a))\sigma$ of defining conditions obtained by applying the substitution σ to the defining conditions of a.[58]

Formulated as sequent-style inferences, the principles of definitional closure $(\vdash \mathcal{D})$ and definitional reflection $(\mathcal{D}\vdash)$ can be added to the sequent calculus LI_c for intuitionistic first-order logic. For any given definition $\mathcal{D}$ we then get a logic $LI_c(\mathcal{D})$ over the definition $\mathcal{D}$.

DEFINITION 4.1.13. For any given definition $\mathcal{D}$, the *sequent calculus* $LI_c(\mathcal{D})$ *for intuitionistic first-order logic over* $\mathcal{D}$ is:

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1,2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1,2)$$

$$(\rightarrow\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\vdash\rightarrow)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

$(y$ does not occur free in $\Gamma)$

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

$(y$ does not occur free in $\Gamma, C)$

(cont'd on next page)

[58]This proviso is part of the formulation of definitional reflection proposed in Hallnäs and Schroeder-Heister [1990], [1991]. For other variants of definitional reflection see Schroeder-Heister [2007a].

Structural rules

$$(\text{Thin}\vdash)\,\frac{\Gamma\vdash C}{\Gamma, A\vdash C} \qquad\qquad \frac{\Gamma\vdash}{\Gamma\vdash A}\,(\vdash\text{Thin})$$

$$(\text{Contr})\,\frac{\Gamma, A, A\vdash C}{\Gamma, A\vdash C}$$

$$(\text{Cut})\,\frac{\Gamma\vdash A \qquad \Delta, A\vdash B}{\Gamma, \Delta\vdash B}$$

Definitional rules

$$\frac{\Delta\vdash B_1^i\sigma \quad \ldots \quad \Delta\vdash B_{n_i}^i\sigma}{\Delta\vdash a\sigma}\,(\vdash\mathcal{D})$$

$$(\mathcal{D}\vdash)\,\frac{\{\Delta, \Gamma_i\sigma\vdash C \mid b\Leftarrow\Gamma_i\in\mathcal{D} \text{ and } a = b\sigma\}}{\Delta, a\vdash C}$$
$$(\text{where } \mathcal{D}(a\sigma)\subseteq(\mathcal{D}(a))\sigma)$$

4.2. Definitional dialogues

The argumentation form of definitional reasoning combines the principle of definitional closure and the principle of definitional reflection into one. This corresponds to the fact that only one argumentation form is needed for each logical constant, whereas in sequent calculi two rules—one left and one right introduction rule—would be needed for each logical constant.

For dialogues, the difference between definitional closure and definitional reflection appears on the level of strategies. Here only one defense move $P\,\Gamma_i$ has to be given for an attack $O\,\mathscr{D}$, whereas all possible defense moves $O\,\Gamma_i$ have to be given for an attack $P\,\mathscr{D}$. In other words, in the first case only the defining conditions Γ_i of one clause defining the attacked atom have to be given, whereas in the second case the defining conditions Γ_i of each clause defining the attacked atom have to be given.

Thus definitional reasoning in dialogues corresponds to the principles of definitional closure and definitional reflection in sequent calculus as follows:

(i) Instances of the argumentation form of definitional reasoning in which the attack move is $O\,\mathscr{D}$ correspond to applications of definitional closure, and

(ii) instances of the argumentation form of definitional reasoning in which the attack move is $P\,\mathscr{D}$ correspond to applications of definitional reflection.

Next we will formulate a preliminary definition of definitional dialogues based on EI_c-dialogues. A comparison with certain derivations in $LI_c(\mathcal{D})$ will indicate that something is lacking in this preliminary definition, if a full correspondence of definitional strategies with $LI_c(\mathcal{D})$-derivations is to be achieved. A final definition of definitional dialogues will be given in Definition 4.3.3 after a discussion of a paradoxical definitional clause.

DEFINITION 4.2.1. *(Preliminary) definitional dialogues* are EI_c-dialogues where the following changes are made.

Conditions $(D00)$ and $(D01)$ are replaced by the following conditions $(D00')$ and $(D01')$, respectively, where the restriction of the expressions in $\delta(0)$ and $\delta(m)$ to complex formulas is discarded; that is, a (preliminary) definitional dialogue can start with the assertion of an atomic formula, and atomic formulas can be attacked:

$(D00')$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

$(D01')$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

Condition $(D02)$ remains without change:

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Condition $(D10)$ is omitted altogether, so that P can now assert atomic formulas without O having asserted them before. Conditions $(D11')$, $(D12')$, $(D14)$ and (E) remain without change:

$(D11')$ If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

$(D12')$ For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

The following proviso for applications of definitional reasoning in the presence of variables is added:

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(This is the same proviso as in the principle of definitional reflection as given above in Remark 4.1.12.)

Thus (preliminary) definitional dialogues are defined by the conditions ($D00'$), ($D01'$), ($D02$), ($D11'$), ($D12'$), ($D14$), (S) and (E), with the additional argumentation form of definitional reasoning.

Given en bloc, the definition of (preliminary) definitional dialogues is as follows:

Argumentation forms for (preliminary) definitional dialogues:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	(Y chooses $i = 1$ or 2)
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	(X chooses $i = 1$ or 2)
implication $\to$:	assertion:	$X\ A \to B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	
($P\,\forall$-form):	assertion:	$P\ \forall x A(x)$	
	attack:	$O\ y$	(y not free before)
	defense:	$P\ A(x)[y/x]$	
($O\,\forall$-form):	assertion:	$O\ \forall x A(x)$	
	attack:	$P\ t$	
	defense:	$O\ A(x)[t/x]$	
($P\,\exists$-form):	assertion:	$P\ \exists x A(x)$	
	attack:	$O\ t$	
	defense:	$P\ A(x)[t/x]$	
($O\,\exists$-form):	assertion:	$O\ \exists x A(x)$	
	attack:	$P\ \exists$	
	defense:	$O\ A(x)[y/x]$	(y not free before)

definitional reasoning:	assertion:	$X\ a$	
	attack:	$Y\ \mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\ \Gamma_i$	(X chooses $i = 1, \ldots, k$)

Conditions for (preliminary) definitional dialogues:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01'$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\ \mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

The notions '(formal) dialogue won by P', '(formal) dialogue tree' and '(formal) strategy' as defined for EI_c-dialogues are directly carried over to the corresponding notions for (preliminary) definitional dialogues.

REMARK 4.2.2. Instead of the formal argumentation forms for the quantifiers (as used in EI_c-dialogues), one could use the following player-independent argumentation forms (as given in Definition 3.9.7):

universal quantifier $\forall$:	assertion:	$X\ \forall x A(x)$	
	attack:	$Y\ t$	(Y chooses the term t)
	defense:	$X\ A(x)[t/x]$	

existential quantifier $\exists$:	assertion:	$X\ \exists x A(x)$
	attack:	$Y\ \exists$
	defense:	$X\ A(x)[t/x]$ (X chooses the term t)

REMARK 4.2.3. The omission of condition $(D10)$ is compensated by the fact that O can attack any atom asserted by P with a move $O\,\mathscr{D}$, if allowed by condition $(D14)$.

REMARK 4.2.4. The proviso (S) is not a restriction on definitions, but only a condition for the applicability of definitional reasoning in the case of proponent attacks.

In order to explain why the proviso has to be installed, we point out two important consequences of it:[59]

(i) The proponent P cannot attack an atom a with a move $P\,\mathscr{D}$ if definitional clauses with defining conditions Γ_i would have to be taken into account in the opponent's defense move(s) $O\,\Gamma_i$ that are not relevant for a. For example, for the definition

$$\mathcal{D}\begin{cases} a(t) \Leftarrow \top \\ a(x) \Leftarrow b \end{cases}$$

(where a is unary and b has arity 0) there would be a strategy for $a(x) \to b$ if the proviso is not respected:

0.	$P\,a(x) \to b$	
1.	$O\,a(x)$	$[0, A]$
2.	$P\,\mathscr{D}$	$[1, A]$
3.	$O\,b$	$[2, D]$
4.	$P\,b$	$[1, D]$

The move at position 2 violates the proviso, since for the substitution $\sigma = [t/x]$ we have $\mathcal{D}(a(x)\sigma) = \mathcal{D}(a(t)) = \{b\sigma, \top\} = \{b, \top\}$ and $(\mathcal{D}(a(x)))\sigma = \{b\sigma\} = \{b\}$; thus $\mathcal{D}(a(x)\sigma) \not\subseteq (\mathcal{D}(a(x)))\sigma$. As $a(t)$ can be obtained by the definition $\mathcal{D}$ while b cannot, there should not be a strategy for the implication $a(x) \to b$ for the given definition $\mathcal{D}$. This is effected by the proviso. The second clause of $\mathcal{D}$ is irrelevant for having a strategy for the substitution instance $a(t)$ of $a(x)$:

0.	$P\,a(t)$	
1.	$O\,\mathscr{D}$	$[0, A]$
2.	$P\,\top$	$[1, D]$

[59] For a more detailed discussion of the proviso in the context of definitional reflection see Hallnäs and Schroeder-Heister [1991], Schroeder-Heister [2007a] and de Campos Sanz and Piecha [2009b].

(ii) Another consequence of the proviso is that P cannot attack an atom a with a move $P\,\mathscr{D}$ if the defining conditions of a contain variables not occurring in a, that is, if the body of a clause contains variables which are not occurring in the head a. For example, for the definition

$$\mathcal{D}\begin{cases} c(t) \Leftarrow \top \\ a(t') \Leftarrow c(x) \end{cases}$$

(where a and c are both unary) there would be a strategy for $a(t') \to c(t')$ if the proviso is not respected:

0.	$P\,a(t') \to c(t')$	
1.	$O\,a(t')$	$[0, A]$
2.	$P\,\mathscr{D}$	$[1, A]$
3.	$O\,c(t')$	$[2, D]$
4.	$P\,c(t')$	$[1, D]$

The move at position 2 violates the proviso, since for the substitution $\sigma = [t'/x]$ we have $\mathcal{D}(a(t')\sigma) = \mathcal{D}(a(t')) = \{c(x)\}$ and $(\mathcal{D}(a(t')))\sigma = \{c(x)\sigma\} = \{c(t')\}$); thus $\mathcal{D}(a(t')\sigma) \not\subseteq (\mathcal{D}(a(t')))\sigma$. As $a(t')$ can be obtained by the definition $\mathcal{D}$ while $c(t')$ cannot, there should be no strategy in this case, too. This is achieved by respecting the proviso.

EXAMPLE 4.2.5. We consider the definition

$$\mathcal{D}_e\begin{cases} a \Leftarrow \top \\ d \Leftarrow \top \\ d \Leftarrow a \\ c \Leftarrow a \wedge d \end{cases}$$

With respect to $\mathcal{D}_e$, the following is a strategy for the atom c:

0.		$P\,c$		
1.		$O\,\mathscr{D}$		$[0, A]$
2.		$P\,a \wedge d$		$[1, D]$
3.	$O \wedge_1$	$[2, A]$	$O \wedge_2$	$[2, A]$
4.	$P\,a$	$[3, D]$	$P\,d$	$[3, D]$
5.	$O\,\mathscr{D}$	$[4, A]$	$O\,\mathscr{D}$	$[4, A]$
6.	$P\,\top$	$[5, D]$	$P\,a$	$[5, D]$
7.			$O\,\mathscr{D}$	$[6, A]$
8.			$P\,\top$	$[7, D]$

At position 0 the proponent P asserts the atom c. In definitional dialogues this is allowed by condition $(D00')$, whereas in standard dialogues with condition $(D00)$ only complex formulas can be asserted in initial moves at position 0. At position 1 this assertion is attacked by O according to the argumentation form of definitional reasoning. The proponent P defends this attack by asserting the defining conditions $a \wedge d$ of the attacked atom c, as given by the last clause of definition $\mathcal{D}_e$. The opponent O attacks $a \wedge d$ at position 3, and P defends at position 4 by asserting the atoms a and d, respectively. The proponent P can assert the atomic formulas a and d—without O having asserted them before—as there is no condition $(D10)$ in definitional dialogues, which would prohibit these moves. However, the opponent O can attack any atoms asserted by P (if not prohibited by condition $(D14)$), and does so with the move $O\,\mathscr{D}$ at position 5 in each of the two dialogues.

In the left dialogue, the proponent defends the opponent's attack on a by asserting $\top$ at position 6 (there are no defining conditions for the atom a; it is given as a fact by the first clause in $\mathcal{D}_e$). In the right dialogue, the proponent chooses to defend by asserting the defining condition a of d, as given in the third clause of $\mathcal{D}_e$. The right dialogue then proceeds as the left one. Alternatively, the proponent could have defended the opponent's attack by choosing to use the second clause of $\mathcal{D}_e$. This clause gives d as a fact, and the proponent's defense would thus be the *verum* $\top$. That is, the right dialogue would end with the move $P\,\top$ already at position 6.

Both dialogues in the above strategy end with the assertion of the *verum* $\top$. As there is no attack possible on $\top$, both dialogues are won by P. The strategy contains only such applications of definitional reasoning in which the opponent attacks atomic formulas with moves $O\,\mathscr{D}$; that is, only the principle of definitional closure is employed here.

EXAMPLE 4.2.6. An example where the principle of definitional reflection is used with respect to the definition $\mathcal{D}_e$ (just given in Example 4.2.5) is the following strategy for the formula $d \to a$:

$$
\begin{array}{llll}
0. & & P\, d \to a & \\
1. & & O\, d & [0, A] \\
2. & & P\, \mathscr{D} & [1, A] \\
\end{array}
$$
$$
\begin{array}{lll|ll}
3. & O\,\top & [2, D] & O\, a & [2, D] \\
4. & P\, a & [1, D] & P\, a & [1, D] \\
5. & O\,\mathscr{D} & [4, A] & & \\
6. & P\,\top & [5, D] & & \\
\end{array}
$$

The first application of definitional reasoning (comprising positions 1–3) is according to the principle of definitional reflection. Here the defining conditions of each of the definitional clauses for the attacked atom d have to be considered. As $\mathcal{D}_e$ contains two clauses for d, there are two defense moves (made at position 3) to be considered. In the left dialogue, the proponent can only defend the opponent's attack made at position 1 by asserting the atom a. The following attack by O, asking for defining conditions of a, is defended by P with $\top$ (using the first clause of $\mathcal{D}_e$, which is the only definitional clause for a). Here the principle of definitional closure has been employed. In the right dialogue, the proponent makes the same defense move at position 4 as in the right dialogue. Due to condition $(D14)$ the opponent cannot attack the atom a: O has asserted a before (at position 3) without P having attacked a.

The proponent could also make the move $P\,\mathscr{D}$ at position 4 in the right dialogue instead. The dialogue would then end thus:

$$\begin{array}{lll} & \vdots & \\ 3. & O\,a & [2, D] \\ 4. & P\,\mathscr{D} & [3, A] \\ 5. & O\,\top & [4, D] \\ 6. & P\,a & [1, D] \end{array}$$

This yields a strategy in which the principle of definitional reflection has been employed twice.

EXAMPLE 4.2.7. For the definition $\mathcal{D}_e$ (see Example 4.2.5) there is no strategy for the formula $\neg d$. The dialogue tree is

$$\begin{array}{lll|ll} 0. & & & P\,\neg d & \\ 1. & & & O\,d & [0, A] \\ 2. & & & P\,\mathscr{D} & [1, A] \\ 3. & O\,\top & [2, D] & O\,a & [2, D] \\ 4. & \vdots & & P\,\mathscr{D} & [3, A] \\ 5. & & & O\,\top & [4, D] \\ & & & \vdots & \end{array}$$

and contains only infinite dialogues.

In the left branch, the proponent can repeat the attack $P\,\mathscr{D}$ on $O\,d$. In the right branch, the proponent can repeat the attacks on $O\,d$ and $O\,a$. In any case, the opponent can always defend these attacks. No other moves are possible, since the *verum* $\top$ cannot be attacked.

EXAMPLE 4.2.8. We now consider the following (first-order) definition $\mathcal{D}$, in which the atoms $even(x)$ and $odd(x)$ are two predicates, and s is a unary function symbol (interpreted as the successor function on natural numbers):[60]

$$\mathcal{D}\begin{cases} even(0) \Leftarrow \top \\ even(s(x)) \Leftarrow odd(x) \\ odd(x) \Leftarrow \neg\, even(x) \end{cases}$$

[60] See Hallnäs and Schroeder-Heister [1991, p. 657].

Then for the given definition $\mathcal{D}$ the following definitional dialogue is a strategy for $\neg\, even(s(0))$:

0.	$P \neg\, even(s(0))$	
1.	$O\ even(s(0))$	$[0, A]$
2.	$P\, \mathscr{D}$	$[1, A]$
3.	$O\ odd(0)$	$[2, D]$
4.	$P\, \mathscr{D}$	$[3, A]$
5.	$O \neg\, even(0)$	$[4, D]$
6.	$P\ even(0)$	$[5, A]$
7.	$O\, \mathscr{D}$	$[6, A]$
8.	$P\, \top$	$[7, D]$

The applications of definitional reasoning comprising the moves at positions 1–3 and 3–5, respectively, are according to the principle of definitional reflection. The opponent's first defense move depends on the substitution $[0/x]$, which unifies the attacked atom $even(s(0))$ with the head $even(s(x))$ of clause 2 and yields the corresponding defining condition $odd(x)[0/x] = odd(0)$, asserted by O at position 3. The opponent's second defense move depends on the same substitution $[0/x]$; it unifies $odd(0)$ with the head $odd(x)$ of the third clause, allowing the opponent to defend with the defining condition $\neg\, even(x)[0/x] = \neg\, even(0)$ in the move at position 5. The moves at positions 6–8 are definitional reasoning by the principle of definitional closure. As $\top$ cannot be attacked, the dialogue ends with the proponent's move at position 8. By reasoning about the definition $\mathcal{D}$ we have thus shown $\neg\, even(s(0))$.

From a logic programming perspective this can be described as follows: The initial move expresses in a formal way a query[61] about the given definition (or program) $\mathcal{D}$ like "Does $\neg\, even(s(0))$ hold with respect to $\mathcal{D}$?". We then try to answer that query by searching for a strategy with respect to $\mathcal{D}$, that is, by employing definitional reasoning (in addition to purely logical reasoning). Finding a strategy means that the query has a positive answer.

EXAMPLE 4.2.9. It is not only of interest whether a query can be answered positively or not. We are also interested in the substitutions made if a positive answer is obtained, that is, if a strategy is found.

We consider the following definition[62]

$$\mathcal{D} \begin{cases} disease(k) \Leftarrow symptom(l) \\ symptom(l) \Leftarrow symptom(m) \wedge disease(n) \end{cases}$$

[61] For a precise exposition of (generalized) queries in a sequent calculus setting of logic programming see Hallnäs and Schroeder-Heister [1990].

[62] See Hallnäs and Schroeder-Heister [1991, p. 657].

where k, l, m, n are constants, and $disease(x)$ and $symptom(x)$ are two unary predicates (whose intended meaning is indicated by their names).

(i) To find out what could be a possible disease if a certain symptom l is observed, we can pose a query like $symptom(l) \to disease(x)$. Here implication '$\to$' is used to express a query for $disease(x)$ under the assumption (or hypothesis) $symptom(l)$. In a sequent calculus setting this can be made explicit by writing $symptom(l) \vdash disease(x)$. In our dialogue setting this would require an extension to hypothetical dialogues (cf. Section 2.8 and the problems indicated there). However, for the intuitionistic interpretation of implication used here, the representation of such queries as implications is adequate. We obtain the following strategy:

0.	$P\ symptom(l) \to disease(x)$	
1.	$O\ symptom(l)$	$[0, A]$
2.	$P\ disease(x)$	$[1, D]$
3.	$O\ \mathscr{D}$	$[2, A]$
4.	$P\ symptom(l)$	$[3, D]$

The atom $symptom(l)$ asserted by P in the last move cannot be attacked with a move $O\ \mathscr{D}$ due to condition $(D14)$: The opponent has already asserted the atom $symptom(l)$ at position 2, and P has not attacked it. In the definitional reasoning the substitution $[k/x]$ has been applied, where the first clause of $\mathcal{D}$ was used to defend the opponent's attack $O\ \mathscr{D}$ by asserting the defining condition $symptom(l)$ for $disease(k)$. The answer to our query is therefore that k is a possible disease for the observed symptom l.

(ii) To find out which symptom is an indicator for the disease n, we can pose the query $symptom(x) \to disease(n)$. The following is a strategy for this query:

0.	$P\ symptom(x) \to disease(n)$	
1.	$O\ symptom(x)$	$[0, A]$
2.	$P\ \mathscr{D}$	$[1, A]$
3.	$O\ symptom(m) \wedge disease(n)$	$[2, D]$
4.	$P\ \wedge_2$	$[3, A]$
5.	$O\ disease(n)$	$[4, D]$
6.	$P\ disease(n)$	$[1, D]$

The last move cannot be further attacked with $O\ \mathscr{D}$ due to condition $(D14)$. At position 2 the proponent attacks the opponent's assertion of the atom $symptom(x)$ by definitional reasoning with the move $P\ \mathscr{D}$. The opponent defends this attack by giving the defining conditions $symptom(m) \wedge disease(n)$ for $symptom(l)$ according to the second clause in $\mathcal{D}$. This move depends on the application of the substitution $[l/x]$ to $symptom(x)$, which yields the head of the second clause. In other words, the atom $symptom(x)$ asserted at position 1 has first been unified with the head $symptom(l)$ of the second clause, the unifier being the substitution $[l/x]$. This substitution $[l/x]$—which has been produced in the construction of the strategy—is what we are interested in here: it tells us that we have to test for symptom l in order to find disease n.

In (i) the query was answered by using definitional reasoning with attack move $O\,\mathscr{D}$, that is, by an application of the principle of definitional closure. In (ii) there is no such possibility; only definitional reasoning with attack move $P\,\mathscr{D}$ can be employed to answer the query, that is, only an application of the principle of definitional reflection allows us to extract the desired information from definition $\mathcal{D}$.

REMARK 4.2.10. Queries under atomic assumptions were represented as implications having these assumptions as antecedents. We note that this is in general not the same as putting the assumptions as additional definitional clauses (in the form of facts) into the given definition.

For the query in Example 4.2.9 (i) this would make no difference. Instead of the query $symptom(l) \to disease(x)$ we would use the query $disease(x)$ and add the antecedent $symptom(l)$ as a fact to the given definition $\mathcal{D}$, yielding the following definition $\mathcal{D}'$:

$$\mathcal{D}' \begin{cases} symptom(l) \Leftarrow \top \\ disease(k) \Leftarrow symptom(l) \\ symptom(l) \Leftarrow symptom(m) \wedge disease(n) \end{cases}$$

We then obtain the strategy

0.	P $disease(x)$	
1.	$O\,\mathscr{D}$	$[0, A]$
2.	P $symptom(l)$	$[1, D]$
3.	$O\,\mathscr{D}$	$[2, A]$
4.	$P\,\top$	$[3, D]$

where in the first application of definitional reasoning (positions 1–3) the substitution $[k/x]$ has been applied. That is, we get the same answer to our query as in Example 4.2.9 (i).

However, when we do likewise for the query $symptom(x) \to disease(n)$ of Example 4.2.9 (ii), we do not even obtain a strategy for the new query $disease(n)$ with respect to the extended definition

$$\mathcal{D}'' \begin{cases} symptom(x) \Leftarrow \top \\ disease(k) \Leftarrow symptom(l) \\ symptom(l) \Leftarrow symptom(m) \wedge disease(n) \end{cases}$$

where the clause *symptom*$(x) \Leftarrow \top$ has now been added to $\mathcal{D}$ (instead of the clause *symptom*$(l) \Leftarrow \top$ as in $\mathcal{D}'$).

The dialogue tree for *disease*(n) with respect to $\mathcal{D}''$ is

0.	P *disease*(n)	
1.	$O\,\mathscr{D}$	$[0, A]$

The proponent cannot answer the attack because *disease*(n) is not unifiable with *disease*(k) (i.e. the head of the second clause in $\mathcal{D}''$), and the substitution $[l/x]$ can therefore not be obtained.

REMARK 4.2.11. The argumentation form of definitional reasoning allows for any attacks on atoms independently of whether they are defined by a given definition or not. The only exception is the *verum* $\top$, which cannot be attacked at all (cf. Remark 4.1.8). And we have imposed no dialogue conditions that would disallow attacks on undefined atoms just because they are undefined. This has consequences concerning the notion of falsity and its relation to negation (see Hallnäs and Schroeder-Heister [1991, section 4], where it is also pointed out how this relates to the conception of negation as finite failure[63]).

Let the *falsum* $\bot$ stand for any atom which is not defined by a given definition $\mathcal{D}$, that is, which is not the head of any definitional clause in $\mathcal{D}$.[64] Then there exists a strategy for $\bot \to A$, for any formula A:

0.	$P \bot \to A$	
1.	$O \bot$	$[0, A]$
2.	$P\,\mathscr{D}$	$[1, A]$

This means that definitional reasoning[65] provides a principle of *ex falso quodlibet*, as long as some atom is undefined[66].

We also observe that for any undefined atom a there is a strategy for its negation $\neg a$:

0.	$P \neg a$	
1.	$O\, a$	$[0, A]$
2.	$P\,\mathscr{D}$	$[1, A]$

There is no strategy for a itself in this case (i.e., when a is undefined), the dialogue tree being

[63] See Clark [1978].

[64] Any undefined nullary predicate will do. In case of predicates of arbitrary arity all possible substitution instances have to be undefined as well.

[65] More precisely: the principle of definitional reflection, whose application obtains when the proponent attacks an atom with the move $P\,\mathscr{D}$, as it is the case here.

[66] Definitions being *finite* sets of definitional clauses, this is always the case for any given definition if no restriction is made on the language.

0.	$P\, a$	
1.	$O\, \mathscr{D}$	$[0, A]$

Note, however, that a given definition need not decide for a defined atom a whether there is either a strategy for a or a strategy for $\neg a$. The definitional clause $a \Leftarrow a$ is a definition of the atom a. The dialogue trees for the formulas a and $\neg a$ are

0.	$P\, a$			0.	$P\, \neg a$	
1.	$O\, \mathscr{D}$	$[0, A]$		1.	$O\, a$	$[0, A]$
2.	$P\, a$	$[1, D]$	and	2.	$P\, \mathscr{D}$	$[1, A]$
3.	$O\, \mathscr{D}$	$[2, A]$		3.	$O\, a$	$[2, D]$
	$\vdots$			4.	$P\, \mathscr{D}$	$[3, A]$
					$\vdots$	

respectively. There is thus neither a strategy for a nor for $\neg a$, although a is defined.

So far, only preliminary definitional dialogues have been dealt with. We will next discuss a certain paradoxical definitional clause and will then arrive at a final definition of definitional dialogues. The observations made for preliminary definitional dialogues remain valid for definitional dialogues in the sense of this final definition.

4.3. Definitional dialogues and contraction

In the following, we consider the definitional clause

$$a \Leftarrow \neg a$$

(using $\neg A := A \to \bot$ for all formulas A, this is just an abbreviation for $a \Leftarrow a \to \bot$ here). It is related to Curry's Paradox[67]—respectively to one of its special cases, namely Russell's Paradox—where for $t \in \{x \mid A\} \Leftarrow A[t/x]$ and $t = \{x \mid \neg(x \in x)\}$ with $A = \neg(x \in x)$ we have $t \in t \Leftarrow \neg(t \in t)$. The latter clause is of the form $a \Leftarrow \neg a$.

Another version of Curry's Paradox can be formulated in model-theoretic terms for languages containing a truth predicate '*True*'.[68] Consider the sentence C defined by $\mathit{True}(\ulcorner C \urcorner) \to \bot$, where $\ulcorner C \urcorner$ is a representation of C (e.g. its Gödel-number). Assume $\mathit{True}(\ulcorner C \urcorner)$; then C by $\mathit{True}(\ulcorner A \urcorner) \models A$;[69] and by definition of C also $\mathit{True}(\ulcorner C \urcorner) \to \bot$. Then (using the assumption

[67] See Curry [1942].

[68] Cf. Field [2008], for example.

[69] Where '$\models$' signifies the usual model-theoretic relation of logical consequence.

$True(\ulcorner C \urcorner))$ by *modus ponens* $\bot$. That is, $True(\ulcorner C \urcorner) \vDash \bot$. By using the import theorem

$$\text{If } A \vDash B, \text{ then } \vDash A \to B$$

one gets

(1) $$\vDash True(\ulcorner C \urcorner) \to \bot$$

and by the definition of C we have $\vDash C$. Using $A \vDash True(\ulcorner A \urcorner)$ gives

(2) $$\vDash True(\ulcorner C \urcorner)$$

and an application of *modus ponens* to (1) and (2) yields $\vDash \bot$. That is, absurdity (i.e., the *falsum* $\bot$) is valid for the given definition of C.

On the one hand, Field [2008, p. 282] considers the import theorem ("If $A \vDash B$, then $\vDash A \to B$")[70] to be problematic, since *conditional assertion* ($A \vDash B$) and *assertion of a conditional* ($\vDash A \to B$) have to be kept apart according to him.

Weir [2005], on the other hand, sees a problem in the use of transitivity for the logical consequence relation $\vDash$. The argument concluding $True(\ulcorner C \urcorner) \vDash \bot$ uses the assumption $True(\ulcorner C \urcorner)$ actually twice. Thus instead of $True(\ulcorner C \urcorner) \vDash \bot$ one can only conclude

(3) $$True(\ulcorner C \urcorner), True(\ulcorner C \urcorner) \vDash \bot$$

from the two premisses

$$True(\ulcorner C \urcorner) \vDash True(\ulcorner C \urcorner) \to \bot$$

and

$$True(\ulcorner C \urcorner), True(\ulcorner C \urcorner) \to \bot \vDash \bot.$$

This step makes use of transitivity for the relation $\vDash$, which is problematized in Weir [2005].[71]

Another problem can be seen in the fact that the assumption $True(\ulcorner C \urcorner)$ has to be used twice. Thus to infer $True(\ulcorner C \urcorner) \vDash \bot$ from (3) requires the contraction of $True(\ulcorner C \urcorner), True(\ulcorner C \urcorner)$ to $True(\ulcorner C \urcorner)$. In the following, we will concentrate on the role of contraction in Curry's Paradox (in the form of $a \Leftarrow \neg a$).

EXAMPLE 4.3.1. For the given definitional clause $a \Leftarrow \neg a$ there is neither a strategy for a nor for $\neg a$. The dialogue tree for a has the form

[70] This is called '$\to$-Introduction' by Field.

[71] Proof-theoretically, this step corresponds to the use of cut in corresponding derivations of $\bot$. It can be shown that cut is not eliminable from such derivations. By demanding the eliminability of cut in general, one can thus exclude these derivations. This has already been pointed out by Prawitz [1965, appendix B, §1] for Russell's Paradox in the context of natural deduction. Cf. also Ekman [1994].

$$
\begin{array}{llll|ll}
0. & & & & P\,a & \\
1. & & & & O\,\mathscr{D} & [0,A] \\
2. & & & & P\,\neg a & [1,D] \\
3. & & & & O\,a & [2,A] \\
4. & & & & P\,\mathscr{D} & [3,A] \\
5. & & & & O\,\neg a & [4,D] \\
6. & P\,a & [5,A] & & P\,\mathscr{D} & [3,A] \\
7. & \vdots & & & O\,\neg a & [6,D] \\
 & & & & \vdots &
\end{array}
$$

which contains only infinite branches consisting of iterations of the moves at positions 1–6 in the left branch and of iterations of the moves starting at position 6 in the right branch, with corresponding $\eta(n)$. And the dialogue tree for $\neg a$ has the form

$$
\begin{array}{lll|ll|ll|ll}
0. & & & & & & P\,\neg a & & \\
1. & & & & & & O\,a & & [0,A] \\
2. & & & & & & P\,\mathscr{D} & & [1,A] \\
3. & & & & & & O\,\neg a & & [2,D] \\
4. & & & & P\,a & & [3,A] & P\,\mathscr{D} & [1,A] \\
5. & & & & O\,\mathscr{D} & & [4,A] & O\,\neg a & [4,D] \\
6. & P\,\mathscr{D} & [1,A] & P\,a & [3,A] & P\,\neg a & [5,D] & \vdots & \\
 & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

which also contains only infinite branches. They consist of iterations of the moves at positions 3–5, 5, 1–5 and 4–5 in the first, second, third and fourth branch, respectively, with corresponding $\eta(n)$.

REMARK 4.3.2. However, a as well as $\neg a$ are provable for the given definitional clause $a \Leftarrow \neg a$ $(= \mathcal{D})$ in $LI_c(\mathcal{D})$:[72]

$$
\dfrac{\dfrac{(\mathrm{Contr})\,\dfrac{(\mathcal{D}\vdash)\,\dfrac{(\neg\vdash)\,\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash a}}{a, \neg a \vdash}}{a, a \vdash}}{a \vdash}}{\vdash \neg a}\,(\vdash \neg)}{\vdash a}\,(\vdash \mathcal{D})
$$

In order to establish an equivalence also between strategies of definitional dialogues and sequent calculus derivations using definitional closure and definitional reflection with contraction, the opponent O must not attack any atom asserted by O before, even if there has been an attack on that atom by the proponent P.

[72] Cf. Hallnäs and Schroeder-Heister [1991, p. 638], where contraction is used implicitly, and Schroeder-Heister [1992]; see also Schroeder-Heister [2003].

Definition 4.3.3. The following condition is added to the preliminary Definition 4.2.1 of definitional dialogues in order to prohibit attacks by O on atoms asserted by O before:

$(D15)$ If for an atom a there is a move $\langle \delta(l) = O\, a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\, \mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\, a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

And condition $(D14)$ has to be replaced by the following condition $(D14^*)$ which is $(D14)$ restricted to complex formulas:

$(D14^*)$ O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

Definitional dialogues are thus defined by the conditions $(D00')$, $(D01')$, $(D02)$, $(D11')$, $(D12')$, $(D14^*)$, $(D15)$, (S) and (E), with the argumentation forms as given in Definition 4.2.1. That is, the final definition of *definitional dialogues* is given by the following argumentation forms and dialogue conditions:

Argumentation forms for definitional dialogues:

negation $\neg$:	assertion:	$X\, \neg A$	
	attack:	$Y\, A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\, A_1 \wedge A_2$	
	attack:	$Y\, \wedge_i$	(Y chooses $i = 1$ or 2)
	defense:	$X\, A_i$	
disjunction $\vee$:	assertion:	$X\, A_1 \vee A_2$	
	attack:	$Y\, \vee$	
	defense:	$X\, A_i$	(X chooses $i = 1$ or 2)
implication $\to$:	assertion:	$X\, A \to B$	
	attack:	$Y\, A$	
	defense:	$X\, B$	
($P\,\forall$-form):	assertion:	$P\, \forall x A(x)$	
	attack:	$O\, y$	(y not free before)
	defense:	$P\, A(x)[y/x]$	
($O\,\forall$-form):	assertion:	$O\, \forall x A(x)$	
	attack:	$P\, t$	
	defense:	$O\, A(x)[t/x]$	
($P\,\exists$-form):	assertion:	$P\, \exists x A(x)$	
	attack:	$O\, t$	
	defense:	$P\, A(x)[t/x]$	

($O\,\exists$-form):	assertion:	$O\,\exists x A(x)$	
	attack:	$P\,\exists$	
	defense:	$O\,A(x)[y/x]$	(y not free before)
definitional reasoning:	assertion:	$X\,a$	
	attack:	$Y\,\mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\,\Gamma_i$	(X chooses $i = 1, \ldots, k$)

Conditions for definitional dialogues:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01'$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

($D14^*$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D15$) If for an atom a there is a move $\langle \delta(l) = O\,a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\,\mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

REMARK 4.3.4. The restriction of condition ($D14$) to complex formulas was not necessary in the treatment of EI_c-dialogues because attacks on atomic formulas are not possible there.

EXAMPLE 4.3.5. For the given definitional clause $a \Leftarrow \neg a$ there is a strategy for a as well as for $\neg a$ if condition $(D15)$ is respected:

$$
\begin{array}{lll}
0. & P\,a & \\
1. & O\,\mathscr{D} & [0, A] \\
2. & P\,\neg a & [1, D] \\
3. & O\,a & [2, A] \\
4. & P\,\mathscr{D} & [3, A] \\
5. & O\,\neg a & [4, D] \\
6. & P\,a & [5, A]
\end{array}
\qquad\qquad
\begin{array}{lll}
0. & P\,\neg a & \\
1. & O\,a & [0, A] \\
2. & P\,\mathscr{D} & [1, A] \\
3. & O\,\neg a & [2, D] \\
4. & P\,a & [3, A]
\end{array}
$$

REMARK 4.3.6. These two strategies correspond to the following two $LI_c(\mathcal{D})$-derivations for the given definitional clause $a \Leftarrow \neg a$, respectively:

$$
\dfrac{\dfrac{(\mathrm{Contr})\,\dfrac{(\mathcal{D}\vdash)\,\dfrac{(\neg\vdash)\,\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash a}}{a, \neg a \vdash}}{a, a \vdash}}{a \vdash}}{\vdash \neg a}\,(\vdash \neg)}{\vdash a}\,(\vdash \mathcal{D})
\qquad\qquad
\dfrac{(\mathrm{Contr})\,\dfrac{(\mathcal{D}\vdash)\,\dfrac{(\neg\vdash)\,\dfrac{(\mathrm{Id})\,\dfrac{}{a \vdash a}}{a, \neg a \vdash}}{a, a \vdash}}{a \vdash}}{\vdash \neg a}\,(\vdash \neg)
$$

(Cf. also Remark 4.3.2.)

REMARK 4.3.7. The existence of a strategy for a as well as for $\neg a$ in Example 4.3.5 depends on the fact that in the last move the proponent P can state the formula a (in the moves $\langle \delta(6) = P\,a, \eta(6) = [5, A] \rangle$ and $\langle \delta(4) = P\,a, \eta(4) = [3, A] \rangle$, respectively), which has been attacked by P with definitional reasoning before (in the moves $\langle \delta(4) = P\,\mathscr{D}, \eta(4) = [3, A] \rangle$ and $\langle \delta(2) = P\,\mathscr{D}, \eta(2) = [1, A] \rangle$, respectively).

That a is stated in the last move of a dialogue in a strategy means that a is used without reference to its definition, like an assumption introduced by the initial sequent $a \vdash a$ in the corresponding sequent calculus derivation.

However, here this move is possible only after having reflected on the definition of a by definitional reasoning; this corresponds to the introduction of the assumption a by definitional reflection in the sequent calculus derivation. Hence, the formula a has been used both with and without referring to its definition. This means that the differently used occurrences of the formula a have been contracted implicitly, corresponding to the explicit contraction by the application of (Contr) in the corresponding sequent calculus derivations shown above in Remark 4.3.6 (cf. also Remark 4.3.2).

In other words, the proponent P has not only made twofold use of the formula a (asserted by O at position 3) in the moves at positions 4 and 6 of the left dialogue, respectively in the moves at positions 2 and 4 of the right dialogue—that is, contractions of the form (3) as given in Remark 3.8.3—but the formula a has also been used in two different senses: once as an arbitrary assumption and once according to its given definition.

4.4. Definitional dialogues without contraction

Alternatively, we can consider definitional dialogues without contraction. Whereas in all our sequent calculi contraction is explicitly given by the rule (Contr),[73] this is not the case in dialogues; here contraction is only implicitly given as the twofold use by the proponent P of an assertion made by the opponent O (see Remark 3.8.3).

DEFINITION 4.4.1. *Contraction-free definitional dialogues* are definitional dialogues where the following condition is added:

($D13^*$) For any move $\langle\delta(k) = O\,A, \eta(k) = [j, Z]\rangle$ there is at most one move of the form $\langle\delta(l) = P\,e, \eta(l) = [k, A]\rangle$ or $\langle\delta(l) = P\,A, \eta(l) = [i, Z]\rangle$, where $j < k < l$ and $i < l$. That is, each assertion of an O-signed formula may be used by P at most once.

That is, *contraction-free definitional dialogues* are defined by the conditions $(D00')$, $(D01')$, $(D02)$, $(D11')$, $(D12')$, $(D13^*)$, $(D14^*)$, $(D15)$, (S) and (E), with the argumentation forms as given in Definition 4.2.1.

EXAMPLE 4.4.2. For the given definitional clause $a \Leftarrow \neg a$ the contraction-free dialogue trees for a and $\neg a$ consist of the dialogues

$$\begin{array}{lll} 0. & P\,a & \\ 1. & O\,\mathscr{D} & [0, A] \\ 2. & P\,\neg a & [1, D] \\ 3. & O\,a & [2, A] \\ 4. & P\,\mathscr{D} & [3, A] \\ 5. & O\,\neg a & [4, D] \end{array} \qquad \text{respectively} \qquad \begin{array}{lll} 0. & P\,\neg a & \\ 1. & O\,a & [0, A] \\ 2. & P\,\mathscr{D} & [1, A] \\ 3. & O\,\neg a & [2, D] \end{array}$$

There is thus neither a strategy for a nor for $\neg a$. This corresponds to the fact that neither the sequent $\vdash a$ nor the sequent $\vdash \neg a$ is derivable without (Contr) in the sequent calculus $LI_c(\mathcal{D})$ for the given definitional clause.

4.5. Definitional dialogues with restricted contraction

A comparison of contraction-free definitional dialogues with definitional dialogues that have contraction shows that contraction is critical in reasoning about paradoxical clauses like $a \Leftarrow \neg a$.[74] However, not having any contraction at all yields a logic which might be considered too weak;[75]

[73] Equivalent sequent calculi with implicit contraction can be given too, of course; cf. Hudelmaier [1992], [1993] or Dyckhoff [1992].

[74] That contraction can lead to Russell's Paradox has been pointed out by Fitch [1936]. Cf. also Fitch [1952] and Ackermann [1950].

[75] Cf. Ono and Komori [1985] for examples of logics without contraction. Cf. also Došen and Schroeder-Heister [1993].

for example, there would be no strategy for $\neg(a \wedge \neg a)$ in such a logic (cf. Remark 3.8.3). Instead of abandoning contraction completely, it would therefore be desirable to introduce a restricted version of contraction.[76]

In sequent calculus, one such restriction could be to disallow contraction of two occurrences of a formula where one occurrence contains a subformula that has been introduced by definitional reasoning (i.e., by an application of definitional reflection $(\mathcal{D}\vdash)$ or by an application of definitional closure $(\vdash\mathcal{D})$). This restriction can be implemented by restricting the antecedents of sequents to lists. Formula occurrences introduced by definitional reflection or by definitional closure can then be labeled in order to distinguish them from formulas introduced by other left or right introduction rules, and the restriction on contraction can be given by saying that only formula occurrences containing no labeled subformulas can be contracted. The resulting logic without definitional reflection or definitional closure is then the same as the one with the non-restricted contraction. A less restrictive version of contraction could still allow for contractions of two formula occurrences that have the same labeled subformula occurrences in common; for example, the list $a \wedge \neg \underline{a}, a \wedge \neg \underline{a}$ (where the atoms in the respective right conjuncts have been labeled) in the antecedent of a sequent could be contracted in this case. It might also be worthwhile to study more fine-grained restrictions on contraction, for example by differentiating between the form of formulas a labeled formula is subformula of.

It should be possible to implement all such restrictions also for definitional dialogues. However, this is more difficult than in sequent calculus, since in dialogues contraction is not given explicitly by an argumentation form that would correspond to a rule like (Contr), but is only implicitly given as a twofold use of a formula by the proponent P according to one of the three forms given in Remark 3.8.3.

4.6. Definitional dialogues and Kreuger's rule

So far, we have only considered restrictions on contraction in order to prevent strategies for contradictory assertions in the presence of non-wellfounded clausal definitions like $a \Leftarrow \neg a$. An alternative approach which keeps contraction unrestricted has been proposed by Kreuger [1994] for a sequent calculus setting with the principles of definitional closure $(\vdash\mathcal{D})$ and definitional reflection $(\mathcal{D}\vdash)$. He gave a condition which restricts initial sequents $a \vdash a$ to such atomic formulas a whose only definitional clause in

[76] See Schroeder-Heister [2004].

a given definition $\mathcal{D}$ is the clause $a \Leftarrow a$. In other words, the axiom

$$(\mathrm{Id}_a)\ \frac{}{a \vdash a}\quad \text{(where } a \text{ is atomic)}$$

may only be used if the given definition contains the definitional clause $a \Leftarrow a$ and no other clause defining a, that is, no other clause with head a.

Equivalently (see Schroeder-Heister [1994a]), one can make use of a set $\mathcal{U}$ of atoms not defined by $\mathcal{D}$, that is, for a given definition $\mathcal{D}$ we use the set

$$\mathcal{U} := \{a \mid (a \Leftarrow \Gamma) \notin \mathcal{D} \text{ for any atoms } a \text{ and defining conditions } \Gamma\}.$$

Instead of the axiom (Id_a) we then use an axiom $(\mathrm{Id}_a)_{\mathcal{U}}$ which is restricted to the atoms in $\mathcal{U}$:

$$(\mathrm{Id}_a)_{\mathcal{U}}\ \frac{}{a \vdash a}\quad \text{(where } a \in \mathcal{U} \text{ for atomic } a\text{)}$$

That is, a sequent $a \vdash a$ can only be introduced by $(\mathrm{Id}_a)_{\mathcal{U}}$ if a is undefined. In addition, we restrict the principle of definitional reflection to the introduction of atoms which are not elements of $\mathcal{U}$. That is, instead of $(\mathcal{D}\vdash)$ we use the following rule $(\mathcal{D}\vdash)_{\mathcal{U}}$:

$$(\mathcal{D}\vdash)_{\mathcal{U}}\ \frac{\{\Delta, \Gamma_i\sigma \vdash C \mid b \Leftarrow \Gamma_i \in \mathcal{D} \text{ and } a = b\sigma\}}{\Delta, a \vdash C}\quad \text{(where } a \notin \mathcal{U}\text{)}$$

(with the same proviso that $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$). No restrictions on contraction are made.

In other words, the introduction of an assumption a by $(\mathrm{Id}_a)_{\mathcal{U}}$ is only possible if a has not been specified by being defined in a given definition $\mathcal{D}$; and the introduction of an assumption a by $(\mathcal{D}\vdash)_{\mathcal{U}}$ is only possible if a has been specified in the sense of being defined in $\mathcal{D}$.[77]

In such a system neither the sequent $\vdash a$ nor the sequent $\vdash \neg a$ is derivable for the given definitional clause $a \Leftarrow \neg a$: As $a \notin \mathcal{U}$, the derivation cannot start with an application of $(\mathrm{Id}_a)_{\mathcal{U}}$ introducing $a \vdash a$ as initial sequent. Neither can the derivation start with an application of $(\mathcal{D}\vdash)_{\mathcal{U}}$: all atoms except a are in $\mathcal{U}$ and can thus not be introduced by $(\mathcal{D}\vdash)_{\mathcal{U}}$; and an introduction of a by $(\mathcal{D}\vdash)_{\mathcal{U}}$ depends on a derivation of the sequent $\neg a \vdash$, which is not derivable.

This shows that the effects of a paradoxical clause like $a \Leftarrow \neg a$ can also be precluded without making any restrictions on contraction. However, this is only achieved by imposing strong restrictions[78] on the introduction of (atomic) formulas as assumptions (i.e., restrictions on the introduction of atomic formulas in the antecedents of sequents) by restricting the axiom

[77] As a consequence, definitional reflection in the form of $(\mathcal{D}\vdash)_{\mathcal{U}}$ cannot provide a principle of *ex falso quodlibet* anymore (cf. Remark 4.2.11).

[78] These restrictions might be considered to be too strong (cf. Schroeder-Heister [1994a], [2004]).

to $(\mathrm{Id}_a)_{\mathcal{U}}$ and the principle of definitional reflection to $(\mathcal{D}\vdash)_{\mathcal{U}}$. Note that we cannot use (Id) restricted to $\mathcal{U}$, that is,

$$(\mathrm{Id})_{\mathcal{U}}\ \frac{}{A \vdash A}\ \ (\text{where } a \in \mathcal{U} \text{ for } A \equiv a)$$

here instead of $(\mathrm{Id}_a)_{\mathcal{U}}$, since the sequent $a \vdash a$ would then be derivable by

$$(\mathcal{D}\vdash)_{\mathcal{U}}\ \dfrac{\dfrac{(\mathrm{Id})_{\mathcal{U}}\ \overline{\neg a \vdash \neg a}}{\neg a \vdash a}\ (\vdash\mathcal{D})}{a \vdash a}$$

and consequently—since no restrictions on contraction have been made—the sequents $\vdash a$ and $\vdash \neg a$ would become derivable as well.

One cannot argue against the use of the axiom $(\mathrm{Id})_{\mathcal{U}}$ (instead of using $(\mathrm{Id}_a)_{\mathcal{U}}$) on the basis of a distinction between assumptions which are specified or not specified by a given definition, as a non-atomic formula cannot be specified by the definitions considered here anyway. One could argue,[79] however, that the axiom is used in the context of a sequent calculus which has a left introduction rule for each logical constant. Although a complex assumption cannot be specified by a definition, it can thus be viewed as being specified by the left introduction rule for its outermost logical constant. On the principle that only formulas (be they atomic or complex) which are not specified in any way—that is, neither by definition nor by left introduction rules—should be allowed to be introduced by the axiom, no application of $(\mathrm{Id})_{\mathcal{U}}$ for complex formulas A will be possible.

For example, considering again the definition given by the single definitional clause $a \Leftarrow \neg a$, the application

$$(\mathrm{Id})_{\mathcal{U}}\ \frac{}{\neg a \vdash \neg a}$$

is then not possible, since the antecedent in $\neg a \vdash \neg a$ would have to be introduced by the left introduction rule $(\neg\vdash)$ for negation from a sequent with succedent a, presupposing either $\vdash a$ (which we want to derive) or $a \vdash a$ (which cannot be the conclusion of $(\mathrm{Id})_{\mathcal{U}}$ since $a \notin \mathcal{U}$).

The use of $(\mathrm{Id}_a)_{\mathcal{U}}$ (instead of $(\mathrm{Id})_{\mathcal{U}}$) embodies the principle that only completely unspecified formulas should be introduced as assumptions by the axiom. And the use of $(\mathcal{D}\vdash)_{\mathcal{U}}$ (instead of $(\mathcal{D}\vdash)$) strengthens this principle in that it ensures that such unspecified formulas can only be introduced by applications of $(\mathrm{Id}_a)_{\mathcal{U}}$.

If we adhere to these principles also for definitional dialogues, then the following changes have to be made in their definition:

(i) We have to revert from $(D14^{*})$ to a condition like $(D10)$; this corresponds to using (Id_a) instead of (Id).

[79] Cf. Schroeder-Heister [2004].

(ii) On this condition we have to impose the restriction to atomic formulas $a \in \mathcal{U}$; this then corresponds to using $(\mathrm{Id}_a)_{\mathcal{U}}$ instead of (Id_a).

(iii) And we have to restrict applications of definitional reasoning in such a way that proponent attacks $P\,\mathscr{D}$ on atoms a are only possible if $a \notin \mathcal{U}$; this corresponds to using $(\mathcal{D}\vdash)_{\mathcal{U}}$ instead of $(\mathcal{D}\vdash)$.

Changes (i) and (ii) are achieved by replacing condition $(D14^*)$ in Definition 4.3.3 by the following condition:

$(D10^*)$ If, for an atomic formula $a \in \mathcal{U}$, $\delta(n) = P\,a$ for $n \neq 0$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula a, which has been asserted by O before, only if $a \in \mathcal{U}$.

REMARK 4.6.1. Condition $(D10^*)$ does not affect initial moves. Definitional dialogues can still start with the assertion of any atomic (or complex) formula made by the proponent. But once the opponent has asserted an atomic formula, this formula can afterwards only be asserted by the proponent if the formula is undefined.

Change (iii) is achieved by adding the following condition to Definition 4.3.3:

(K) For $\langle \delta(m) = O\,a, \eta(m) = [l, Z]\rangle$ and $a \in \mathcal{U}$ there is no attack $\langle \delta(n) = P\,\mathscr{D}, \eta(n) = [m, A]\rangle$ for $l < m < n$. That is, P may attack an atom a by definitional reasoning only if $a \notin \mathcal{U}$.

DEFINITION 4.6.2. *Kreuger-restricted definitional dialogues* are defined by the conditions $(D00')$, $(D01')$, $(D02)$, $(D10^*)$, $(D11')$, $(D12')$, $(D15)$, (S), (K) and (E), with the argumentation forms as given in Definition 4.3.3.[80]

EXAMPLE 4.6.3. For Kreuger-restricted definitional dialogues there is neither a strategy for a nor for $\neg a$ for the definition given by the single definitional clause $a \Leftarrow \neg a$. The dialogue trees for a and $\neg a$ consist of the dialogues

0.	$P\,a$			0.	$P\,\neg a$	
1.	$O\,\mathscr{D}$	$[0, A]$	respectively	1.	$O\,a$	$[0, A]$
2.	$P\,\neg a$	$[1, D]$		2.	$P\,\mathscr{D}$	$[1, A]$
3.	$O\,a$	$[2, A]$		3.	$O\,\neg a$	$[2, D]$
4.	$P\,\mathscr{D}$	$[3, A]$				
5.	$O\,\neg a$	$[4, D]$				

[80]The complete definition of Kreuger-restricted definitional dialogues is shown in Section A.21 of Appendix A.

As a is defined, we have $a \notin \mathcal{U}$. The moves $P\,\mathscr{D}$ attacking $O\,a$ are not prohibited by condition (K). But the proponent cannot attack the moves $O\,\neg a$ by asserting a, since this would violate condition $(D10^*)$. (Note that condition $(D10^*)$ does not restrict initial moves; hence P can assert the defined atom a in the initial move of the left dialogue.)

The use of Kreuger-restricted definitional dialogues has here therefore the same effect as the use of contraction-free definitional dialogues (cf. Example 4.4.2), although this effect is achieved by very different means.

4.7. Summary

We have introduced the argumentation form of definitional reasoning as a dialogical formulation of the principles of definitional closure and definitional reflection (first introduced as sequent-style inferences by Hallnäs and Schroeder-Heister [1990], [1991][81]). Our dialogical formulation combines both principles in the one (player-independent) argumentation form of definitional reasoning. We have then introduced definitional dialogues as an extension of EI_c-dialogues. Definitional dialogues enable us to reason about clausal definitions of atomic formulas whose defining conditions can be given not only by atoms but also by complex formulas. As an example of such a clausal definition we studied a paradoxical clause and the effects of contraction within a preliminary framework of definitional dialogues. A comparison with the handling of this clause in a sequent calculus setting allowed us to arrive at a final version of definitional dialogues. These definitional dialogues are a generalization of standard dialogues also with respect to the formulas assertable in the initial move: standard dialogues like *DI*- or *EI*-dialogues can only begin with the assertion of a complex formula, whereas definitional dialogues can also begin with the assertion of an atomic formula. Atomic formulas can be given a meaning by providing a definition for them. As the defining conditions in such a definition can be given by atomic or complex formulas, definitional reasoning can in general lead from assertions of atomic formulas to assertions of complex formulas. By introducing definitional dialogues on the basis of EI_c-dialogues—that is, on the basis of dialogues which in an EI_c-strategy can end with assertions of complex formulas—we have provided an adequate dialogical foundation in the sense of formal dialogue semantics for definitional reasoning. Finally, as an elaboration on the effects of contraction in the presence of a paradoxical clause, certain possible restrictions on contraction were suggested, and an alternative approach due to Kreuger was considered.

[81] See also Hallnäs [1991] and Schroeder-Heister [1993].

Chapter 5

DIALOGUES FOR IMPLICATIONS AS RULES

In Schroeder-Heister [2011a], [2011b][82] an alternative left introduction rule for implication in sequent calculus is introduced. It is motivated by the interpretation of implications as rules.

In this chapter, we develop a dialogical framework for implications as rules; we analyze this framework and compare it with its sequent calculus counterpart. We will also combine the dialogical approach to implications as rules with definitional reasoning, and we will indicate an extension to hypothetical reasoning with implications as hypotheses.

5.1. Implications as rules

Usually, constructive interpretations of implication are more or less directly given by the Brouwer–Heyting–Kolmogorov (BHK) interpretation[83], according to which a proof of an implication $A \to B$ consists of a construction transforming any given proof of A into a proof of B. The standard dialogical interpretation of implication is based on the same idea: An implication $A \to B$ is attacked by claiming A and defended by claiming B. In order to have a (winning) strategy for $A \to B$, the proponent must be able to produce a substrategy[84] for B from what the opponent uses in defending A. A difference to standard constructive interpretations is that the opponent need not necessarily give a full proof of A which is then transformed into a proof of B. Instead, the proponent may force the opponent to produce certain fragments of a proof of A that are sufficient to produce a substrategy for B.

A more elementary view of implication is based on the conception that an implication $A \to B$ is a rule which allows one to pass over from A to B. This view is particularly supported by the treatment of implication in natural

[82]Cf. also Schroeder-Heister [1984].
[83]Cf. Heyting [1971].
[84]See Definition 5.4.2.

deduction. There *modus ponens*

$$\frac{A \quad A \to B}{B}$$

can be read as the application of $A \to B$ as a rule, which is used to infer B from A, that is, *modus ponens* can be read as a schema of rule application. The introduction of an implication $A \to B$ by

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}$$

(where assumptions A can be discharged) can be read as establishing a rule, namely by deriving its conclusion B from its premiss A. Applications of logic such as logic programming or deductive databases support this perspective. Reading implications as rules motivates an alternative left implication introduction rule

$$(\to\vdash)^{\circ}\ \frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

in sequent calculus.[85] This schema expresses that by assuming the implication-as-rule $A \to B$ we are entitled to infer B from A. It replaces the standard left implication introduction rule

$$(\to\vdash)\ \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}$$

while the right implication introduction rule remains the standard one:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

When implications are read as rules, an elementary meaning is given to implication which is conceptually prior to the meaning of the other logical constants.

In the following, we carry the implications-as-rules approach over to dialogues.[86] Once an implication $A \to B$ has been claimed by the opponent, it is considered to be a rule in a sort of 'database', which later on can be used by the proponent in order to reduce the justification of its conclusion B to that of A. This is achieved by allowing the proponent to defend an attack on B by asserting A whenever $A \to B$ has been claimed by the opponent before. In case no such claim has been made before (i.e., if no applicable rule is available in the database), the argument for B continues as usual with an opponent attack on B (which must eventually be defended by the proponent), depending on the respective form of B.

[85] See Schroeder-Heister [2011a], [2011b].

[86] See also Piecha and Schroeder-Heister [2012].

For brevity, only the propositional case will be considered in what follows; the results can be generalized to the first-order case.

5.2. The sequent calculus LI°

We give the sequent calculus LI° for intuitionistic propositional logic, which contains the alternative left implication introduction rule

$$(\to\vdash)^{\circ}\ \frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

as a replacement for the standard left implication introduction rule

$$(\to\vdash)\ \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}$$

of LI_c^p. This is the only difference to LI_c^p.

REMARK 5.2.1. The standard left implication introduction rule $(\to\vdash)$ satisfies the following properties:

(i) It contains only one logical constant.[87]

(ii) The logical constant occurs only in the conclusion of the rule, and there exactly once.

(iii) Each formula which is an argument of the logical constant occurs exactly once in the conclusion of the rule.[88]

The alternative rule $(\to\vdash)^{\circ}$ does only satisfy the properties (i) and (ii), but not (iii).

DEFINITION 5.2.2. The *sequent calculus LI° for intuitionistic propositional logic* consists of the following rules:[89]

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\quad (A \text{ atomic or complex})$$

Logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

(cont'd on next page)

[87]The rule is considered as a schema. Thus, although Γ, Δ can contain complex formulas and A, B, C can be complex formulas, the rule as a schema contains only the one logical constant $\to$.

[88]Here these formulas are A and B.

[89]Again, Γ and Δ are finite multisets of formulas (the comma in antecedents of sequents stands for multiset union, and singletons are written without braces).

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i=1,2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i=1,2)$$

$$(\rightarrow\vdash)^\circ\ \frac{\Gamma \vdash A}{\Gamma, A \rightarrow B \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\vdash\rightarrow)$$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash \mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

REMARK 5.2.3. The sequent calculus LI° does not have the cut elimination property. The sequent $a, a \rightarrow (b \wedge c) \vdash b$ (for atomic and distinct formulas a, b, c) can only be derived by using (Cut):

$$(\mathrm{Cut})\ \frac{(\rightarrow\vdash)^\circ\ \dfrac{(\mathrm{Id})\ \dfrac{}{a \vdash a}}{a, a \rightarrow (b \wedge c) \vdash b \wedge c} \qquad (\wedge\vdash)\ \dfrac{(\mathrm{Id})\ \dfrac{}{b \vdash b}}{b \wedge c \vdash b}}{a, a \rightarrow (b \wedge c) \vdash b}$$

This is the only kind of derivation where (Cut) cannot be eliminated (see Schroeder-Heister [2011b]).

REMARK 5.2.4. Although LI° does not have the cut elimination property, it does have the *weak cut elimination property*. That is, every LI°-derivation containing an application of (Cut) can be transformed into an LI°-derivation of the form

$$(\mathrm{Cut})\ \frac{(\rightarrow\vdash)^\circ\ \dfrac{\vdots}{\Gamma \vdash A} \qquad \begin{matrix}\vdots \\ \Delta, A \vdash C\end{matrix}}{\Gamma, \Delta \vdash C}$$

where the left premiss of (Cut) is the conclusion of an application of $(\rightarrow\vdash)^\circ$. Furthermore, the right premiss of (Cut) can be assumed to be either the conclusion of a derivation of the above form, or it is the endsequent in a derivation such that the cut formula A is the result of an application of a left introduction rule in the last step. As a consequence of the weak cut elimination property, LI° has the subformula property. (See Schroeder-Heister [2011b] for these results.)

REMARK 5.2.5. Since $(\rightarrow\vdash)$ is a derivable rule in LI°:

$$\dfrac{(\rightarrow\vdash)^{\circ}\,\dfrac{\Gamma\vdash A}{\Gamma, A\rightarrow B\vdash B} \qquad \Delta, B\vdash C}{\Gamma, \Delta, A\rightarrow B\vdash C}\,(\mathrm{Cut})$$

and $(\rightarrow\vdash)^{\circ}$ is a derivable rule in LI^{p}_{c}:

$$(\rightarrow\vdash)\,\dfrac{\Gamma\vdash A \qquad (\mathrm{Id})\,\dfrac{}{B\vdash B}}{\Gamma, A\rightarrow B\vdash B}$$

(cf. Schroeder-Heister [2011b]), we have $\vdash_{LI^{\circ}} A$ if and only if $\vdash_{LI^{p}_{c}} A$.

5.3. EI°-dialogues

We now introduce dialogues for the implications-as-rules approach. The guiding idea is the following: When making an assertion A, the proponent P must be prepared to either defend A in the ‘standard’ way against an attack of the opponent O, or else make the assertion C for some C, for which O has already claimed $C \rightarrow A$, that is, for which the implication-as-rule $C \rightarrow A$ is sufficient to generate A. This is modeled by saying that every assertion made by P is symbolically questioned by O, following which P chooses which of the two ways described P is prepared to take. Contrary to the proponent P, the opponent O is not given a choice. The opponent’s non-implicational assertions are attacked and defended as usual, whereas the opponent’s implicational assertions are considered as providing rules which the proponent can use, but not question; so there are no attacks and defenses defined for them.

DEFINITION 5.3.1. For each logical constant we first define *argumentation forms* which determine how a complex formula (having the respective constant in outermost position) that has been asserted by the opponent O can be attacked (if possible) and how this attack can be defended (if possible):[90]

AF$(\neg\vdash)$:	assertion:	$O\,\neg A$	
	attack:	$P\,A$	
	defense:	*no defense*	
AF$(\wedge\vdash)$:	assertion:	$O\,A_1 \wedge A_2$	
	attack:	$P\,\wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\,A_i$	

[90] In Definition 2.1.2 argumentation forms were defined independently of whether the assertion is made by the proponent P or by the opponent O. This symmetry is not preserved here.

AF($\vee\vdash$):	assertion:	$O\ A_1 \vee A_2$	
	attack:	$P\ \vee$	
	defense:	$O\ A_i$	(O chooses $i = 1$ or $i = 2$)
AF($\to\vdash$)$^\circ$:	assertion:	$O\ A \to B$	
	attack:	*no attack*	
	defense:	*no defense*	

Except for AF($\to\vdash$)$^\circ$, these argumentation forms coincide with the standard ones (cf. Definition 2.1.2) in case of assertions made by the opponent O.[91]

We now extend our language by the two special symbols ? and $|\cdot|$. For assertions made by the proponent P there is a pair of argumentation forms for each logical constant (depicted below as trees having two branches which are separated by $|$). An assertion A made by the proponent P can be questioned by the opponent with the move O ? (such a move is only possible if the expression stated in the P-move is an assertion, that is, a formula; if it is not an assertion but a symbolic attack, then it cannot be questioned with the move O ?).

The proponent P can then answer this question either by allowing an attack on the assertion (this is indicated by the special symbol $|\cdot|$; see the argumentation forms on the left side of $|$ below), or by asserting any formula C for which O has asserted the implication $C \to A$ at an earlier position. We call this the *rule condition* (R):

(R) P may answer a question O ? on a formula A by choosing C provided O has asserted the formula $C \to A$ before.

The argumentation forms for assertions made by the proponent P are then defined as follows:

AF($\vdash\neg$):	assertion:	$P\ \neg A$	
	question:	O ?	
	choice:	$P\ \vert\neg A\vert$	$P\ C$ (R)
	attack:	$O\ A$	
	defense:	*no defense*	

AF($\vdash\wedge$):	assertion:	$P\ A_1 \wedge A_2$	
	question:	O ?	
	choice:	$P\ \vert A_1 \wedge A_2\vert$	$P\ C$ (R)
	attack:	$O\ \wedge_i$ $(i = 1$ or $2)$	
	defense:	$P\ A_i$	

[91] The argumentation form AF($\to\vdash$)$^\circ$ could also be omitted, to the same effect. However, we prefer to give the argumentation form AF($\to\vdash$)$^\circ$ in order to make it explicit that implications $A \to B$ asserted by O cannot be attacked. Being able to refer to AF($\to\vdash$)$^\circ$ will also help to state some results more clearly.

$$
\begin{array}{llll}
\mathrm{AF}(\vdash\vee): & \text{assertion:} & & P\,A_1\vee A_2\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|A_1\vee A_2| & \multicolumn{1}{|l}{P\,C \quad (R)}\\
& \text{attack:} & O\,\vee & \multicolumn{1}{|l}{}\\
& \text{defense:} & P\,A_i \quad (i=1 \text{ or } 2) & \multicolumn{1}{|l}{}
\end{array}
$$

$$
\begin{array}{llll}
\mathrm{AF}(\vdash\to): & \text{assertion:} & & P\,A\to B\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|A\to B| & \multicolumn{1}{|l}{P\,C \quad (R)}\\
& \text{attack:} & O\,A & \multicolumn{1}{|l}{}\\
& \text{defense:} & P\,B & \multicolumn{1}{|l}{}
\end{array}
$$

In the case of an attack $O \wedge_i$ according to the argumentation form AF($\vdash \wedge$) for conjunctive formulas asserted by P, the opponent O chooses $i = 1$ or $i = 2$, and in the case of a defense $P\,A_i$ to an attack $O \vee$ according to the argumentation form AF($\vdash \vee$) for disjunctive formulas asserted by P, the proponent P chooses $i = 1$ or $i = 2$. The argumentation forms on the left (i.e., the respective left branches) correspond to the argumentation forms given in Definition 2.1.2 for 'standard' dialogues (where the device of question and choice moves is not needed). The argumentation forms on the right (i.e., the respective right branches) reflect the implications-as-rules view.

For assertions of atomic formulas a made by the proponent P an argumentation form is given by the rule condition (R) itself:

$$
\begin{array}{lll}
\mathrm{AF}(R): & \text{assertion:} & P\,a\\
& \text{question:} & O\,?\\
& \text{choice:} & P\,C \quad \text{only if } O \text{ has asserted } C\to a \text{ before}
\end{array}
$$

In addition, we define an argumentation form AF(Cut) such that any expression e (i.e., question, symbolic attack or formula) stated by O can be followed by a move $P\,A$, and this move can then be followed by the opponent move $O\,A$:

$$
\begin{array}{lll}
\mathrm{AF(Cut)}: & \text{statement:} & O\,e\\
& \text{cut:} & P\,A\\
& \text{cut:} & O\,A
\end{array}
$$

The formula A is called *cut formula* in this argumentation form.

REMARK 5.3.2. The argumentation form AF(Cut) differs from the other argumentation forms in that the move $O\,e$ need not be an assertion (i.e., the statement of a formula) but can be the statement of any expression e (i.e., question, symbolic attack or formula).

Another difference is that the cut formula is completely independent of the expression e. Calling the P-move an attack and the subsequent O-move a defense as in the other argumentation forms would thus be inadequate. We therefore simply speak of *cut moves* in both cases.

The idea behind AF(Cut) is that at any (even) position the proponent P can introduce an arbitrary formula A as a lemma. The proponent P must then later be prepared both to defend this lemma A as an assertion and to defend the original claim (i.e., the assertion made in the initial move at position 0) given this lemma, that is, given the opponent's claim of A.

DEFINITION 5.3.3. We extend the definition of *moves* (see Definition 2.1.4) as follows:

As before, pairs $\langle\delta(n), \eta(n)\rangle$ are called *moves*, where $\delta(n)$, for $n \geq 0$, is again a signed expression, and $\eta(n)$ is again a pair $[m, Z]$, for $0 \leq m < n$, where Z is now either A (for 'attack'), D (for 'defense'), Q (for 'question'), C (for 'choice') or Cut. As before, $\eta(n) = [m, Z]$ is empty for $n = 0$, that is, $\eta(0) = \emptyset$. In addition, m in $\eta(n) = [m, Z]$ is empty for $Z = Cut$.

We have thus the following types of moves:

$$\begin{array}{rl} \textit{attack move} & \langle\delta(n) = X\, e, \eta(n) = [m, A]\rangle, \\ \textit{defense move} & \langle\delta(n) = X\, A, \eta(n) = [m, D]\rangle, \\ \textit{question move} & \langle\delta(n) = O\, ?, \eta(n) = [m, Q]\rangle, \\ \textit{choice move} & \langle\delta(n) = P\, |A|, \eta(n) = [m, C]\rangle, \\ & \langle\delta(n) = P\, A, \eta(n) = [m, C]\rangle, \\ \textit{cut move} & \langle\delta(n) = X\, A, \eta(n) = [Cut]\rangle. \end{array}$$

REMARK 5.3.4. A question move can only be made by O, and a choice move can only be made by P. The other types of moves are available for both the proponent P and the opponent O.

In a choice move, $\delta(n)$ can have the form $P\,|A|$ or $P\,A$. In the first case, P allows an attack on the formula A. In the second case, P asserts the formula A in accordance with the rule condition (R), that is, A is the antecedent of an implication asserted by O before.

Dialogues for the implications-as-rules approach can now be defined as follows.

DEFINITION 5.3.5. An *EI°-dialogue* is a sequence of moves $\langle\delta(n), \eta(n)\rangle$ $(n = 0, 1, 2, \dots)$ satisfying the following conditions:

$(D00')$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

$(D01^\circ)$ If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

$(D03^{\circ})$ If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark '?'.

$(D04^{\circ})$ If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

$(D05^{\circ})$ If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (i.e., the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(m) = [Cut]$ and $\delta(n) = O\,A$ for $\delta(m) = P\,A$ (where $m < n$).

$(D11')$ If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

$(D12')$ For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

$(D14')$ O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n - 1, Z]\rangle$. That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$, or it is a cut move with $\delta(n) = O\,A$ for $\delta(n - 1) = P\,A$.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for DI^{p}-dialogues are directly carried over to the corresponding notions for EI°-dialogues.

REMARK 5.3.6. *EI°-dialogues* are similar to EI_c^{p}-dialogues without condition $(D10)$ for the argumentation forms given in Definition 5.3.1 and satisfying the condition $(D14')$ instead of $(D14)$, where $(D14')$ differs from $(D14)$ only in that the latter is a condition for O attacking a formula C, whereas the former is a condition for O questioning a formula C.

Condition $(D00')$ is the same as for definitional dialogues (cf. Definition 4.3.3). It allows EI°-dialogues to start with the assertion of an atomic formula, contrary to the restriction to complex formulas as, for example, in EI^p_c-dialogues (cf. Definition 2.7.1). Condition $(D01^\circ)$ differs from condition $(D01)$ in EI^p_c-dialogues in that it allows for attacks by O on expressions of the form $|A|$ for complex formulas A. Condition $(D02)$ is as given in Definition 2.1.6 for dialogues. Conditions $(D03^\circ)$ and $(D04^\circ)$ have been added for the question and choice moves, respectively, and condition $(D05^\circ)$ has been added for the cut moves.

We recall that condition (E) implies condition $(D13)$. In addition, condition (E) implies condition $(D11)$ for odd p and condition $(D12)$ for odd n (cf. Definition 2.2.1 and Remark 2.7.2). The conditions $(D11)$ and $(D12)$ have thus been weakened here to the conditions $(D11')$ and $(D12')$, respectively. In the presence of (E), this weakening does not make any difference regarding the extension (i.e., the set specified by the concept) of EI°-dialogues because $(D11') + (D12') + (E)$ is neither more nor less restrictive than $(D11) + (D12) + (E)$.

REMARK 5.3.7. Dialogues for the implications-as-rules approach can also be defined on the basis of DI^p- or DI^p_c-dialogues (instead of EI^p_c-dialogues). This has been done in Piecha and Schroeder-Heister [2012], where the D-dialogues used there correspond to the DI^p-dialogues given here in Definition 2.2.1.

DEFINITION 5.3.8. A formula A is called *EI°-dialogue-provable* (short: *EI°-provable*) if there is an EI°-strategy for A. Notation: $\vdash_{EI^\circ} A$.

EXAMPLE 5.3.9. An EI°-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$ is the following:

$$\begin{array}{rll|ll}
0. & \multicolumn{3}{l}{P\, a \to ((a \to (b \wedge c)) \to b)} & \\
1. & \multicolumn{3}{c}{O\, ?} & [0, Q] \\
2. & \multicolumn{3}{c}{P\, |a \to ((a \to (b \wedge c)) \to b)|} & [1, C] \\
3. & \multicolumn{3}{c}{O\, a} & [2, A] \\
4. & \multicolumn{3}{c}{P\, (a \to (b \wedge c)) \to b} & [3, D] \\
5. & \multicolumn{3}{c}{O\, ?} & [4, Q] \\
6. & \multicolumn{3}{c}{P\, |(a \to (b \wedge c)) \to b|} & [5, C] \\
7. & \multicolumn{3}{c}{O\, a \to (b \wedge c)} & [6, A] \\
8. & \multicolumn{3}{c}{P\, b \wedge c} & [Cut] \\
9. & O\, ? & [8, Q] & O\, b \wedge c & [Cut] \\
10. & P\, a & [9, C] & P\, \wedge_1 & [9, A] \\
11. & & & O\, b & [10, D] \\
12. & & & P\, b & [7, D]
\end{array}$$

The moves at positions 0–4 and at positions 4–7 + 12 (in the right dialogue) are made according to the argumentation form AF($\vdash \to$). In the choice moves at positions 2 and 6 the proponent P can only choose $|a \to ((a \to (b \wedge c)) \to b)|$ and $|(a \to (b \wedge c)) \to b|$, respectively, since

O has not asserted any implications before which could be used as rules by choosing their antecedents. At position 7 the opponent asserts the implication $a \to (b \wedge c)$. The succedent $b \wedge c$ of this implication is asserted by P in the cut move at position 8; it is questioned at position 9 (in the left dialogue). In accordance with the rule condition (R), the proponent can now answer this question move by asserting in the choice move at position 10 (in the left dialogue) the antecedent a of the implication whose succedent has been questioned. The implication $a \to (b \wedge c)$ has thus been used as a rule. The opponent cannot question the formula a due to condition $(D14')$: O has already asserted a (in the attack move at position 3), and P has not attacked a (such an attack is not even possible, since a is atomic). At position 8 the proponent P could defend the attack $O\; a \to (b \wedge c)$ by asserting b, since assertions by P of atomic formulas not asserted by O before are not prohibited in EI°-dialogues (they would be prohibited by condition $(D10)$, for example in EI^{p}_{c}-dialogues). However, this could be questioned by O at position 9, and P would lose this dialogue as P can neither choose $|b|$ nor C (since there is no move $O\; C \to b$ for such a formula C) in the next move, so that only a cut move is possible (there is no attack for $O\; a \to (b \wedge c)$ (by definition of $\mathrm{AF}(\to\vdash)^{\circ}$), and for a being atomic there is no attack for the move $O\; a$ at position 3):

$$
\begin{array}{rll|ll}
0. & \multicolumn{3}{c}{P\; a \to ((a \to (b \wedge c)) \to b)} & \\
1. & \multicolumn{3}{c}{O\,?} & [0, Q] \\
2. & \multicolumn{3}{c}{P\; |a \to ((a \to (b \wedge c)) \to b)|} & [1, C] \\
3. & \multicolumn{3}{c}{O\; a} & [2, A] \\
4. & \multicolumn{3}{c}{P\; (a \to (b \wedge c)) \to b} & [3, D] \\
5. & \multicolumn{3}{c}{O\,?} & [4, Q] \\
6. & \multicolumn{3}{c}{P\; |(a \to (b \wedge c)) \to b|} & [5, C] \\
7. & \multicolumn{3}{c}{O\; a \to (b \wedge c)} & [6, A] \\
8. & \multicolumn{3}{c}{P\; b} & [7, D] \\
9. & \multicolumn{3}{c}{O\,?} & [8, Q] \\
10. & \multicolumn{3}{c}{P\; b \wedge c} & [Cut] \\
11. & O\,? & [10, Q] & O\; b \wedge c & [Cut] \\
12. & P\; a & [9, C] & P\; \wedge_1 & [9, A] \\
13. & & & O\; b & [10, D] \\
 & & & \;\vdots &
\end{array}
$$

In the right dialogue, the proponent P could continue by attacking $O\; b \wedge c$ again (with $P\; \wedge_1$ or $P\; \wedge_2$) or by making another cut move, but neither would lead to a strategy.

REMARK 5.3.10. For comparison with the EI°-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$ just given above in Example 5.3.9, we give an EI^{p}_{c}-strategy for the same formula:

0.	$P\ a \to ((a \to (b \wedge c)) \to b)$	
1.	$O\ a$	$[0, A]$
2.	$P\ (a \to (b \wedge c)) \to b$	$[1, D]$
3.	$O\ a \to (b \wedge c)$	$[2, A]$
4.	$P\ a$	$[3, A]$
5.	$O\ b \wedge c$	$[4, D]$
6.	$P \wedge_1$	$[5, A]$
7.	$O\ b$	$[6, D]$
8.	$P\ b$	$[3, D]$

Except for the attack move $P\ a$ made at position 4, each attack or defense move in this EI^p_c-strategy also occurs as an attack or defense move in the EI°-strategy.

Proponent attacks on implications asserted by the opponent—like the attack move $P\ a$ made at position 4—are not possible in EI°-dialogues. In the corresponding EI°-strategy, the proponent asserts the antecedent a of the implication $a \to (b \wedge c)$ (asserted by the opponent at position 7) in the choice move at position 10 (in the left dialogue) in accordance with the rule condition (R).

REMARK 5.3.11. The absence of condition $(D10)$ in the definition of EI°-dialogues is compensated for by allowing the opponent O to question assertions of atomic formulas made by the proponent P. In dialogues with $(D10)$ there is, for example, no strategy for the formula $a \to b$, since the dialogue

0.	$P\ a \to b$	
1.	$O\ a$	$[0, A]$

cannot be continued with the move $\langle \delta(2) = P\ b, \eta(2) = [1, D] \rangle$; this would only be possible if b were asserted by O before.

In EI°-dialogues (where $(D10)$ is absent) there is no strategy for $a \to b$ either. The EI°-dialogue begins with the moves

0.	$P\ a \to b$	
1.	$O\ ?$	$[0, Q]$
2.	$P\ \lvert a \to b \rvert$	$[1, C]$
3.	$O\ a$	$[2, A]$
4.	$P\ b$	$[3, D]$
5.	$O\ ?$	$[4, Q]$

where P can now assert b at position 4 without O having asserted it before. However, the opponent O can make a question move at position 5—in accordance with the argumentation form $\mathrm{AF}(R)$—, and the proponent P can now continue only with a cut move; this leads to an infinite dialogue.

The proponent P cannot make the choice move $\langle\delta(6) = P\,|b|, \eta(6) = [5, C]\rangle$ here, since there is no such argumentation form for atomic formulas. The only possible choice move would be one according to the argumentation form $\mathrm{AF}(R)$, that is, a move of the form $\langle\delta(6) = P\,C, \eta(6) = [5, C]\rangle$ for a formula $C \to a$ asserted by the opponent O before. But such a formula has not been asserted by O here.

REMARK 5.3.12. Due to condition $(D14')$, EI°-dialogues won by P need not end with the assertion of an atomic formula but can end with the assertion of a complex formula.

For example, the following dialogue is an EI°-strategy for the formula $(a \vee b) \to \neg\neg(a \vee b)$ (cf. Example 2.5.4):

0.	$P\,(a \vee b) \to \neg\neg(a \vee b)$	
1.	$O\,?$	$[0, Q]$
2.	$P\,\|(a \vee b) \to \neg\neg(a \vee b)\|$	$[1, C]$
3.	$O\,a \vee b$	$[2, A]$
4.	$P\,\neg\neg(a \vee b)$	$[3, D]$
5.	$O\,?$	$[4, Q]$
6.	$P\,\|\neg\neg(a \vee b)\|$	$[5, C]$
7.	$O\,\neg(a \vee b)$	$[6, A]$
8.	$P\,a \vee b$	$[7, A]$

The opponent O cannot question $a \vee b$, since neither of the two conditions (i) and (ii) of $(D14')$ is satisfied: $a \vee b$ has already been asserted by O at position 3, and $a \vee b$ has not been attacked by P.

EXAMPLE 5.3.13. The following dialogue is an EI°-strategy for the formula $(a \to b) \to ((b \to c) \to (a \to c))$:

0.	$P\,(a \to b) \to ((b \to c) \to (a \to c))$	
1.	$O\,?$	$[0, Q]$
2.	$P\,\|(a \to b) \to ((b \to c) \to (a \to c))\|$	$[1, C]$
3.	$O\,a \to b$	$[2, A]$
4.	$P\,(b \to c) \to (a \to c)$	$[3, D]$
5.	$O\,?$	$[4, Q]$
6.	$P\,\|(b \to c) \to (a \to c)\|$	$[5, C]$
7.	$O\,b \to c$	$[6, A]$
8.	$P\,a \to c$	$[7, D]$
9.	$O\,?$	$[8, Q]$
10.	$P\,\|a \to c\|$	$[9, C]$
11.	$O\,a$	$[10, A]$
12.	$P\,c$	$[11, D]$
13.	$O\,?$	$[12, Q]$

(cont'd on next page)

14.	$P\,b$	$[13, C]$
15.	$O\,?$	$[14, Q]$
16.	$P\,a$	$[15, C]$

At position 3, the opponent asserts the implication $a \to b$. The formula b—which occurs also as the succedent of this implication—is questioned at position 15. In accordance with the rule condition (R), the proponent asserts a—the antecedent of the implication—in the last move; the opponent cannot question this move due to condition $(D14')$. The implication $b \to c$ is asserted by O in the move at position 7. The opponent questions c at position 13, which enables P to answer according to the rule condition (R) with the choice move $P\,b$ at position 14. The implications $a \to b$ and $b \to c$ have thus been used as rules: the latter implication-as-rule allowed P to answer the question on c with b, and the former allowed P to answer the question on b with a.

For comparison, we consider the corresponding LI°-derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\to\vdash)^{\circ}\;\dfrac{(\to\vdash)^{\circ}\;\dfrac{(\mathrm{Id})\;\dfrac{}{a \vdash a}}{a, a \to b \vdash b}}{a, a \to b, b \to c \vdash c}}{a \to b, b \to c \vdash a \to c}\;(\vdash\to)}{a \to b \vdash (b \to c) \to (a \to c)}\;(\vdash\to)}{\vdash (a \to b) \to ((b \to c) \to (a \to c))}\;(\vdash\to)$$

The moves at positions 0–4, 4–8 and 8–12 correspond to the last, second to last and first application of $(\vdash\to)$, respectively. The moves at positions 12–14 and 14–16 correspond to the second and first application of $(\to\vdash)^{\circ}$, respectively.

REMARK 5.3.14. In Example 5.3.13 no cut moves where necessary for having a strategy for $(a \to b) \to ((b \to c) \to (a \to c))$, whereas in Example 5.3.9 it was shown that there is no strategy for $a \to ((a \to (b \wedge c)) \to b)$ without cut moves.

The implications-as-rules approach as such is independent of the presence of cut. However, cut moves have to be allowed if not only a fragment of intuitionistic (propositional) logic is to be captured. We will have a closer look on cut in the next section.

5.4. LI°-provability is equivalent to EI°-dialogue-provability

In order to show that EI°-dialogue-provability is equivalent to provability for the sequent calculus LI°—in the sense that there is an EI°-strategy for a formula A if and only if A is provable in LI°—we prove first that there is an EI°-strategy for a formula A if and only if there is an EI_c^p-strategy for A (see Theorem 5.4.6 below). The equivalence result (see Corollary 5.4.10 below) follows then with Corollary 3.7.3, Remark 2.2.24, Theorem 3.2.3 and Remark 5.2.5.

REMARK 5.4.1. The equivalence of the sequent calculi LI° and LI_c^p (and thus also of LI^p) could be established by just showing that $(\to\vdash)$ is a derivable rule in LI° and that $(\to\vdash)^{\circ}$ is a derivable rule in LI_c^p (see Remark 5.2.5). This cannot be done as easily for EI°- and EI_c^p-dialogues. Whereas the sequent calculi LI° and LI_c^p differ only with respect to the left implication introduction rule, the EI°- and EI_c^p-dialogues differ not only in the argumentation form for implication. Their difference is rather spread across all argumentation forms, and they differ in the conditions defining the dialogues as well: In EI_c^p-dialogues there is only one argumentation form for each logical constant, whereas in EI°-dialogues there are always two (although the left branches in AF($\vdash\neg$), AF($\vdash\wedge$), AF($\vdash\vee$) and AF($\vdash\to$) correspond to the respective argumentation forms of EI_c^p-dialogues, of course). The argumentation form AF$(\to\vdash)^{\circ}$ does not correspond to the argumentation form for implication of EI_c^p-dialogues. Contrary to EI_c^p-dialogues, EI°-dialogues can begin with the assertion of an atomic formula; and condition $(D10)$ is absent in EI°-dialogues.

DEFINITION 5.4.2. A *substrategy* is a subtree s of a dialogue tree t comprising as root node a node at an even position in t and all descendents in t such that s does not branch at even positions, s has as many nodes at odd positions as there are possible moves for O, and all leaves are proponent moves such that O cannot make another move.[92]

LEMMA 5.4.3. (i) *The weak cut elimination property holds for EI°-strategies. That is, every EI°-strategy containing cut moves made according to the argumentation form* AF(Cut) *can be transformed into an EI°-strategy of the form*

$$
\begin{array}{rll}
 & \qquad\vdots & \\
m. & O\ A \to B\ [m-1, Z] & \\
 & \qquad\vdots & \\
n. & \qquad P\ B\ [\mathit{Cut}] & \\
n+1. & O\ ?\ \ [n, Q] & O\ B\ \ [\mathit{Cut}] \\
n+2. & P\ A\ \ [n+1, C] & \quad s_2 \\
n+3. & O\ ?\ \ [n+2, Q] & \\
 & \qquad s_1 &
\end{array}
$$

[92]Substrategies have been defined for possible situations in Definition 3.5.3. Here we make use of substrategies as such.

where the O-move at position m is either an attack or a defense (i.e., either $Z = A$ or $Z = D$), and the move $\langle \delta(n+1) = O\,B, \eta(n+1) = [Cut] \rangle$ is the uppermost cut move made by O (i.e., there is no cut move at positions $k < n-1$). The O-move at position $n+3$ might not be possible due to $(D14')$. In this case the left dialogue ends with the P-move at position $n+2$.

(ii) Furthermore, the substrategy s_2 is either of the same form as the above EI°-strategy, or it depends on a sequence of moves made according to AF($\neg\vdash$), AF($\wedge\vdash$), AF($\vee\vdash$) *or* AF($\to\vdash$)$^{\circ}$.

PROOF. By induction on the complexity of cut formulas in EI°-strategies. ⊣

COROLLARY 5.4.4. *As a consequence of the weak cut elimination property, EI°-strategies have the subformula property.*[93]

LEMMA 5.4.5. *(i) EI°-strategies for formulas of the form*

$$A \to ((A \to (B \wedge C)) \to B)$$

containing a cut move where the cut formula is of the form $B \wedge C$ cannot be transformed into EI°-strategies (for the respective formula) containing no cut move. However, they can be transformed into EI_c^p-strategies (for the respective formula).

(ii) Every other EI°-strategy (for a given formula) containing a cut move can be transformed into an EI_c^p-strategy (for the given formula) as well.

PROOF. (i) By Lemma 5.4.3 an EI°-strategy (for a given formula) containing a cut move with cut formula $B \wedge C$ can be transformed into an EI°-strategy (for the given formula) of the form

$$
\begin{array}{rc}
0. & P\, A \to ((A \to (B \wedge C)) \to B) \\
 & \vdots \\
m. & O\, A \to (B \wedge C)\ [m-1, A] \\
 & \vdots \\
n. & P\, B \wedge C\ [Cut]
\end{array}
$$

$$
\begin{array}{rll|l}
n+1. & O\,? & [n, Q] & O\, B \wedge C\ \ [Cut] \\
n+2. & P\, A & [n+1, C] & \quad s_2 \\
n+3. & O\,? & [n+2, Q] & \\
 & & s_1 &
\end{array}
$$

where the uppermost cut move made by O is at position $n+1$. (If the O-move at position $n+3$ is not possible due to $(D14')$, then the left dialogue ends with the P-move at position $n+2$.)

---

[93]This is in full analogy to the results on the weak cut elimination property and the subformula property for LI°-derivations; see Remark 5.2.4.

By the argumentation given in Example 5.3.9 there can then be no EI°-strategy (for the given formula) that does not contain a cut move. However, for every EI°-strategy of the above form, the following is an EI_c^p-strategy:

$$
\begin{array}{rc}
0. & P\, A \to ((A \to (B \wedge C)) \to B) \\
 & \vdots \\
m. & O\, A \to (B \wedge C)\ [m-1, A] \\
 & \vdots \\
n. & P\, A\ [m, A] \\
n+1. & O\, e\ \ [n, A] \;\Big|\; O\, B \wedge C\ \ [n, D] \\
n+2. & s_1' \qquad\quad\ \Big|\qquad\quad s_2'
\end{array}
$$

(ii) Let S be an EI°-strategy containing a cut move with a cut formula $D = b, \neg B, B \vee C$ or $B \to C$. By Lemma 5.4.3 and Corollary 5.4.4 we can assume that S has the following form as shown in Lemma 5.4.3:

$$
\begin{array}{rl|l}
 & & \vdots \\
m. & & O\, A \to D\ [m-1, Z] \\
 & & \vdots \\
n. & & P\, D\ [Cut] \\
n+1. & O\, ?\ \ [n, Q] & O\, D\ \ [Cut] \\
n+2. & P\, A\ \ [n+1, C] & \quad s_2 \\
n+3. & O\, ?\ \ [n+2, Q] & \\
 & \quad s_1 &
\end{array}
$$

where in the O-move at position m either $Z = A$ or $Z = D$. If s_2 is of the same form as S, then the cut formula in s_2 has to be of lower complexity than the cut formula in the uppermost pair of cut moves. Otherwise, s_2 must depend on a sequence of moves which are made according to AF(R), AF$(\neg\vdash)$, AF$(\wedge\vdash)$, AF$(\vee\vdash)$ or AF$(\to\vdash)^\circ$, and S can be transformed into the following EI_c^p-strategy S':

$$
\begin{array}{rc}
 & \vdots \\
m. & O\, A \to D\ [m-1, Z] \\
 & \vdots \\
n. & P\, A\ [m, A] \\
n+1. & O\, e\ \ [n, A] \;\Big|\; O\, D\ \ [n, D] \\
n+2. & s_1' \qquad\quad \Big|\qquad\quad s_2'
\end{array}
$$

In case $D = b$ (i.e., when the cut formula D is atomic) the substrategy s_2 can only depend on moves made above position n, since P cannot attack atomic formulas. The same holds for s_2'.

It follows by induction on the complexity of the respective cut formulas that every EI°-strategy containing one or more cut moves can be transformed into an EI_c^p-strategy. (Since EI_c^p-dialogues are defined without an argumentation form like AF(Cut), these EI_c^p-strategies are necessarily free of cut moves.) $\dashv$

THEOREM 5.4.6. *There is an EI°-strategy for a formula A if and only if there is an EI_c^p-strategy for A, that is, $\vdash_{EI^\circ} A$ if and only if $\vdash_{EI_c^p} A$.*

PROOF. We first show the direction from right to left, that is, we show that if there is an EI_c^p-strategy for A, then there is an EI°-strategy for A.

Let S be an EI_c^p-strategy for A, and let $d = \langle\delta(0), \eta(0)\rangle, \langle\delta(1), \eta(1)\rangle, \langle\delta(2), \eta(2)\rangle, \dots, \langle\delta(n), \eta(n)\rangle$ be an EI_c^p-dialogue in S not ending in a proponent move. We show by induction on the subtree below d in S that there is then also an EI°-strategy S' for A.

Since d is part of an EI_c^p-strategy S, there is a proponent move $\langle\delta(n+1) = P\,e, \eta(n+1) = [j, Z]\rangle$. This move is either an attack or a defense.

First, assume the proponent move is an attack $\langle\delta(n+1) = P\,e, \eta(n+1) = [j, A]\rangle$ on a move $\langle\delta(j) = O\,D, \eta(j) = [j-1, Z]\rangle$. We consider all possible cases:

(1) $e = A$, where A is a formula such that condition (i) or (ii) in $(D14)$ is satisfied. Since the proponent move is an attack, the formula D is either $\neg A$ or $A \to B$, where A can be atomic or complex. The following subcases have to be considered:

(a) *$D = \neg A$ and A is an atomic formula a.* In this case, a cannot be a formula such that condition (i) or (ii) in $(D14)$ is satisfied. The subtree below d in S would consist in the single move

$$n+1. \quad P\,a \quad [j, A]$$

But this move is only possible if O has asserted a before, due to condition $(D10)$; condition (i) in $(D14)$ would not be satisfied. And P cannot have attacked a, since there is no attack for atomic formulas; condition (ii) in $(D14)$ would not be satisfied. See case (4) below.

(b) *$D = \neg A$ for complex A.* Again, there is no defense move for O. Thus the O-move has to be an attack according to the argumentation forms for EI_c^p-dialogues. That is, the subtree below d in S has the form

$$\begin{array}{lll} n+1. & P\,A & [j, A] \\ n+2. & O\,e & [n+1, A] \\ & \quad\mathcal{S} & \end{array}$$

where e depends on the form of A. Then the subtree below d' in S' has the form

$$\begin{array}{lll} n+1. & P\,A & [j, A] \\ n+2. & O\,? & [n+1, Q] \\ n+3. & P\,|A| & [n+2, C] \end{array}$$

(cont'd on next page)

$$\begin{array}{lll} n+4. & O\,e & [n+3, A] \\ & s' & \end{array}$$

(c) *$D = A \to B$ and A is an atomic formula a.* Conditions (i) and (ii) in $(D14)$ cannot be satisfied. See case (4) below.

(d) *$D = A \to B$ for complex A.* The subtree below d in S has the following form

$$\begin{array}{lll|l} n+1. & & P\,A\,[j, A] & \\ n+2. & O\,e & [n+1, A] & O\,B\ \ [n+1, D] \\ & & s_1 & \quad s_2 \end{array}$$

and the subtree below d' in S' has the form

$$\begin{array}{lll|l} n+1. & & & P\,B\,[Cut] \\ n+2. & O\,? & [n+1, Q] & O\,B\ \ [Cut] \\ n+3. & P\,A & [n+2, C] & \quad s_2' \\ n+4. & O\,? & [n+3, Q] & \\ n+5. & P\,|A| & [n+4, C] & \\ n+6. & O\,e & [n+5, A] & \\ & & s_1' & \end{array}$$

The last move in d' is the assertion of $A \to B$ by O. Since there is no attack move according to $\mathrm{AF}(\to\vdash)^{\circ}$, the only possible move for P at position $n+1$ is a cut move according to AF(Cut). This move can be either followed by the move $\langle \delta(n+2) = O\,B, \eta(n+2) = [Cut] \rangle$ (right dialogue) or questioned (left dialogue).

The right dialogue continues with the substrategy s_2' which corresponds to s_2 and exists by the induction hypothesis. Note that s_2 is a substrategy independently of the first move $\langle \delta(n+1) = P\,A, \eta(n+1) = [j, A] \rangle$ in the subtree below d in S, because—due to condition (E)—the substrategy s_2 cannot contain a move $\langle \delta(k) = O\,e, \eta(k) = [n+1, A] \rangle$ for $k > n+2$; that is, the proponent move $\langle \delta(n+1) = P\,A, \eta(n+1) = [j, A] \rangle$ is irrelevant in s_2.

In the left dialogue, P chooses to assert A in accordance with the rule condition (R), that is, P chooses to assert the antecedent of the implication $A \to B$ asserted by O in the last move of d'. This is then questioned by O in the move at position $n+4$, and is answered at position $n+5$ by P choosing to allow an attack on A. This attack is made at position $n+6$, and the dialogue continues with the substrategy s_1' which corresponds to s_1 and exists by the induction hypothesis.

This corresponds to the fact that the left implication introduction rule $(\to\vdash)$ of LI_c^p is a derivable rule in LI°, by an application of $(\to\vdash)^{\circ}$ and (Cut); compare Remark 5.2.5.

(2) $e = \wedge_i$. Then D has the form $A_1 \wedge A_2$ and O can only make a defense move. The subtree below d in S has thus the form

$$\begin{array}{lll} n+1. & P \wedge_i & [j, A] \\ n+2. & O\, A_i & [n+1, D] \\ & \quad s & \end{array}$$

and the subtree below d' in S' is

$$\begin{array}{lll} n+1. & P \wedge_i & [j, A] \\ n+2. & O\, A_i & [n+1, D] \\ & \quad s' & \end{array}$$

(3) $e = \vee$. Then D has the form $A_1 \vee A_2$ and O can only make a defense move. The subtree below d in S has thus the form

$$\begin{array}{lc|c} n+1. & \multicolumn{2}{c}{P \vee [j, A]} \\ n+2. & O\, A_1 \;\; [n+1, D] & O\, A_2 \;\; [n+1, D] \\ & s_1 & s_2 \end{array}$$

and the subtree below d' in S' is

$$\begin{array}{lc|c} n+1. & \multicolumn{2}{c}{P \vee [j, A]} \\ n+2. & O\, A_1 \;\; [n+1, D] & O\, A_2 \;\; [n+1, D] \\ & s'_1 & s'_2 \end{array}$$

(4) $e = A$, where A is a formula such that the conditions (i) and (ii) in $(D14)$ are not satisfied, that is, where A has been asserted by O in d without having been attacked by P in d.

In this case, moves of the form $\langle \delta(n+2) = O\, e, \eta(n+2) = [n+1, A] \rangle$ are not possible. That is, the subtree below d consists in a single move of the form

$$n+1. \quad P\, A \quad [j, A]$$

The conditions (i) and (ii) in $(D14')$ are then not satisfied too (they are the same as in $(D14)$), hence opponent moves of the form $\langle \delta(n+2) = O\, ?, \eta(n+2) = [n+1, Q] \rangle$ are not possible, and the subtree below the corresponding dialogue d' in S' consists in a single move $\langle \delta(n+1) = P\, A$, $\eta(n+1) = [k, A] \rangle$ (where $k > j$).

Second, assume the proponent move is a defense $\langle \delta(n+1) = P\, A$, $\eta(n+1) = [j, D] \rangle$ to $\langle \delta(j) = O\, e, \eta(j) = [j-1, A] \rangle$. The following cases have to be considered:

(1) A is a formula such that condition (i) or (ii) in $(D14)$ is satisfied, where A can be atomic or complex.

(a) *A is an atomic formula a*. The subtree below d in S would have to be given by the single move

$$n + 1. \quad P\,a \;\; [j, D]$$

Due to $(D10)$, this move would only be possible if O had asserted a before. But then condition (i) in $(D14)$ cannot be satisfied. Furthermore, P could not have attacked this assertion, since there is no attack on atomic formulas; thus condition (ii) in $(D14)$ cannot be satisfied either. Hence there is no strategy in this case.

(b) *A is a complex formula.* The O-move has to be an attack according to the argumentation forms for EI_c^p-dialogues. The subtree below d in S then has the form

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,e & [n+1, A] \\ & s & \end{array}$$

where e depends on the form of A. Then the subtree below d' in S' has the form

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,? & [n+1, Q] \\ n+3. & P\,|A| & [n+2, C] \\ n+4. & O\,e & [n+3, A] \\ & s' & \end{array}$$

(2) A is a formula such that the conditions (i) and (ii) in $(D14)$ are not satisfied, that is, where A has been asserted by O in d without having been attacked by P in d. As in case (4) above, moves of the form $\langle \delta(n+2) = O\,e, \eta(n+2) = [n+1, A]\rangle$ are not possible, and the subtree below d consists in a single move of the form

$$n + 1. \quad P\,A \;\; [j, D]$$

The conditions (i) and (ii) in $(D14')$ are then not satisfied too, hence moves of the form $\langle \delta(n+2) = O\,?, \eta(n+2) = [n+1, Q]\rangle$ are not possible, and the subtree below the corresponding dialogue d' in S' consists in a single move $\langle \delta(n+1) = P\,A, \eta(n+1) = [k, D]\rangle$ (where $k > j$).

This concludes the proof that if $\vdash_{EI_c^p} A$, then $\vdash_{EI^{\circ}} A$.

It remains to show the direction from left to right, that is, we show that if there is an EI°-strategy for A, then there is an EI_c^p-strategy for A.

First, assume the proponent move is an attack $\langle \delta(n+1) = P\,e, \eta(n+1) = [j, A]\rangle$ on an opponent move $\langle \delta(j) = O\,D, \eta(j) = [i, Z]\rangle$. We consider all possible cases:

(1) $e = A$. Then $D = \neg A$, where A is atomic or complex. (This is the only possibility if $e = A$, since there is no attack in case $D = A \to B$, that is, according to the argumentation form AF$(\to\vdash)^\circ$.)

(a) *A is an atomic formula a.* If condition (i) or (ii) in $(D14')$ is satisfied, then O can question a, and this is the only move possible. Note that due to the absence of condition $(D10)$ from the definition of EI°-dialogues—contrary to EI_c^p-dialogues—P can assert a without O having asserted a before; thus the move $\langle \delta(n+1) = P\, a, \eta(n+1) = [j, A]\rangle$ is possible in any case. The subtree below the EI°-dialogue d' in the EI°-strategy S' is

$$\begin{array}{rll} n+1. & P\,a & [j, A] \\ n+2. & O\,? & [n+1, Q] \\ & s' & \end{array}$$

If O has asserted a in d', then the subtree below the EI_c^p-dialogue d in the EI_c^p-strategy S is the single move

$$n+1. \quad P\,a \;\; [j, A]$$

because atomic formulas cannot be attacked, and there is no defense to an attack on $D = \neg a$.

In case O has not asserted a in d', we have to consider the possible P-moves in s' answering the question move at position $n+2$. The P-move can be made according to AF(R) or according to AF(Cut). We consider both cases:

The P-move is made according to AF(R). Then O must have asserted a formula $C \to a$ in d', and S' has the form

$$\begin{array}{rll} & \vdots & \\ m. & O\,C \to a & [m-1, Z] \\ & \vdots & \\ n-1. & P\,a & [j, A] \\ n. & O\,? & [n-1, Q] \\ n+1. & P\,C & [n, C] \\ n+2. & O\,? & [n+1, Q] \\ & s_1' & \end{array} \left.\vphantom{\begin{array}{c}\vdots\\ m\\ \vdots\end{array}}\right\} d'$$

If the O-move at position m is not a cut move, and if C is complex, then the EI_c^p-strategy S has the form

$$\begin{array}{rll} & \vdots & \\ m. & O\,C \to a & [m-1, Z] \\ & \vdots & \\ n+1. & P\,C & [m, A] \end{array} \left.\vphantom{\begin{array}{c}\vdots\\ m\\ \vdots\end{array}}\right\} d$$

(cont'd on next page)

$$\begin{array}{lll|ll} n+2. & O\,f & [n+1, A] & O\,a & [n+1, D] \\ n+3. & & s_1 & P\,a & [j, A] \end{array}$$

where the expression f is either of a symbolic attack or a formula, depending on the form of C. The opponent move $\langle\delta(n+2) = O\,f, \eta(n+2) = [n+1, A]\rangle$ is only possible in S if the formula C is complex. In case C is atomic, S has the form

$$\left.\begin{array}{lll} & \vdots & \\ m. & O\,C \to a & [m-1, Z] \\ & \vdots & \end{array}\right\} d$$
$$\begin{array}{lll} n+1. & P\,C & [m, A] \\ n+2. & O\,a & [n+1, D] \\ n+3. & P\,a & [j, A] \end{array}$$

in accordance with condition $(D10)$.

If the O-move at position m is a cut move, then a corresponding EI_c^p-strategy S can be given by Lemma 5.4.5.

The P-move is made according to AF(Cut). S' has the form

$$\begin{array}{lll} n+1. & P\,a & [j, A] \\ n+2. & O\,? & [n+1, Q] \\ n+3. & P\,B & [Cut] \\ n+4. & O\,B & [Cut] \\ & s_2' & \end{array}$$

with cut formula B. By Lemma 5.4.3 there must then be a move $\langle\delta(m) = O\,C \to B, \eta(m) = [m-1, Z]\rangle$ (where $m < n-2$) for $Z = A$ or D in S', that is, S' has the form

$$\begin{array}{lllll} & & \vdots & & \\ m. & & O\,C \to B & [m-1, Z] & \\ & & \vdots & & \\ n-2. & & P\,a & [j, A] & \\ n-1. & & O\,? & [n+1, Q] & \\ n. & & P\,B & [Cut] & \end{array}$$
$$\begin{array}{lll|ll} n+1. & O\,? & [n, Q] & O\,B & [Cut] \\ n+2. & P\,C & [n+1, C] & & s_2' \\ n+3. & O\,? & [n+2, Q] & & \\ & s_1' & & & \end{array}$$

By Lemma 5.4.5 there is then an EI_c^p-strategy S.

(b) *A is a complex formula.* This case is like the case in which the proponent move is a defense move; see page 149 (subcase (2)).

(2) $e = \wedge_i$. Then D has the form $A_1 \wedge A_2$ and O can only make a defense move. The subtree below d' in S' has thus the form

$$\begin{array}{ll} n+1. & P\wedge_i \;\; [j,A] \\ n+2. & O\,A_i \;\; [n+1,D] \\ & \quad s' \end{array}$$

and the subtree below d in S is

$$\begin{array}{ll} n+1. & P\wedge_i \;\; [j,A] \\ n+2. & O\,A_i \;\; [n+1,D] \\ & \quad s \end{array}$$

(3) $e = \vee$. Then D has the form $A_1 \vee A_2$ and O can only make a defense move. The subtree below d' in S' has thus the form

$$\begin{array}{lc|c} n+1. & \multicolumn{2}{c}{P\vee\;[j,A]} \\ n+2. & O\,A_1\;\;[n+1,D] & O\,A_2\;\;[n+1,D] \\ & s_1' & s_2' \end{array}$$

and the subtree below d in S is

$$\begin{array}{lc|c} n+1. & \multicolumn{2}{c}{P\vee\;[j,A]} \\ n+2. & O\,A_1\;\;[n+1,D] & O\,A_2\;\;[n+1,D] \\ & s_1 & s_2 \end{array}$$

Second, assume the proponent move is a defense $\langle \delta(n+1) = P\,A, \eta(n+1) = [j,D]\rangle$ to $\langle \delta(j) = O\,e, \eta(j) = [i,Z]\rangle$. The O-move at position $n+2$ can only be a question move $\langle \delta(n+2) = O\,?, \eta(n+2) = [n+1,Q]\rangle$. The EI°-strategy S' either depends on a choice move $\langle \delta(n+3) = P\,e, \eta(n+3) = [n+1,C]\rangle$ answering this question move or it does not. If it does not depend on such a choice move, then the subtree below d' in S' has the form

$$\begin{array}{ll} n+1. & P\,A \;\; [j,D] \\ n+2. & O\,? \;\; [n+1,Q] \\ & \quad s' \end{array}$$

and the subtree below the corresponding dialogue d in S is

$$\begin{array}{ll} n+1. & P\,A \;\; [j,D] \\ & \quad s \end{array}$$

If the EI°-strategy S' depends on a choice move, then the subtree below d' in S' has the form

$$\begin{array}{ll} n+1. & P\,A \;\; [j,D] \\ n+2. & O\,? \;\; [n+1,Q] \end{array}$$

(cont'd on next page)

$$\begin{array}{lll} n+3. & P\,e & [n+2, C] \\ & s' & \end{array}$$

where e is either $|A|$ (according to the left option in the respective argumentation forms AF($\vdash\neg$), AF($\vdash\wedge$), AF($\vdash\vee$) and AF($\vdash\rightarrow$)) or e is C for an implication $C \rightarrow A$ asserted by O before (according to the right option in the respective argumentation forms AF($\vdash\neg$), AF($\vdash\wedge$), AF($\vdash\vee$) and AF($\vdash\rightarrow$), respectively AF(R) for atomic A).

(1) $e = |A|$. Then the subtree below d' in S' has the form

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,? & [n+1, Q] \\ n+3. & P\,|A| & [n+2, C] \\ n+4. & O\,f & [n+3, A] \\ & s' & \end{array}$$

where the expression f in the attack move is either of a symbolic attack or a formula, depending on the form of A in each case. The subtree d in S has then the form

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,f & [n+2, A] \\ & s & \end{array}$$

(2) $e = C$ *for an implication* $C \rightarrow A$ *asserted by* O *before.* The dialogue d' contains a move $\langle \delta(m) = O\,C \rightarrow A, \eta(m) = [m-1, Z]\rangle$ (where $Z = A$ or D, $m < n-1$) and S' has the form

$$\begin{array}{lll} & \vdots & \\ m. & O\,C \rightarrow A & [m-1, Z] \\ & \vdots & \end{array} \Bigg\} d'$$

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,? & [n+1, Q] \\ n+3. & P\,C & [n+2, C] \\ n+4. & O\,? & [n+3, Q] \\ & s' & \end{array}$$

Then S has the form

$$\begin{array}{ll} & \vdots \\ m. & O\,C \rightarrow A\,[m-1, Z] \\ & \vdots \end{array} \Bigg\} d$$

$$\begin{array}{ll} n+1. & P\,C\,[m, A] \end{array}$$

(cont'd on next page)

$$\begin{array}{llll|ll} n+2. & O\,f & [n+1, A] & & O\,A & [n+1, D] \\ n+3. & & s & & P\,A & [j, D] \end{array}$$

where the expression f is either of a symbolic attack or a formula, depending on the form of C.[94] In the right dialogue, the opponent O cannot attack the last P-move at position $n+3$ due to condition $(D14)$: O has already asserted A in the move at position $n+2$, and P has not attacked A.

This corresponds to the fact that the rule $(\rightarrow\vdash)^{\circ}$ of LI° is a derivable rule in LI_c^p, by an application of $(\rightarrow\vdash)$ and (Id); compare Remark 5.2.5. Consider the following derivation $\mathfrak{D}_S$:

$$(\rightarrow\vdash)\,\dfrac{\begin{array}{c}\mathfrak{d}_s\\ \Gamma\vdash C\end{array} \qquad (\mathrm{Id})\,\dfrac{}{A\vdash A}}{\Gamma, C\rightarrow A\vdash A}$$

where $\mathfrak{d}_s$ shall be a subderivation corresponding to the substrategy s in S. The left dialogue in S then corresponds to the left branch in this derivation $\mathfrak{D}_S$, and the right dialogue corresponds to the right branch in $\mathfrak{D}_S$.

Third, assume the proponent move is a choice $\langle\delta(n+1) = P\,e, \eta(n+1) = [n, C]\rangle$ with respect to $\langle\delta(n) = O\,?, \eta(n) = [n-1, Q]\rangle$. The expression e in the choice move can either be $|A|$ for a formula A questioned by O or it can be C for an implication $C\rightarrow A$ asserted by O before.

(1) $e = |A|$. Condition (i) or (ii) in $(D14')$ is satisfied or not.

(a) If condition (i) or (ii) in $(D14')$ is satisfied, then O can make an attack move $\langle\delta(n+2) = O\,f, \eta(n) = [n+1, A]\rangle$, where the expression f is either of a symbolic attack or a formula, depending on the form of A in each case. The subtree below d' in S' has then the form

$$\begin{array}{lll} n+1. & P\,|A| & [n, C] \\ n+2. & O\,f & [n+1, A] \\ & s' & \end{array}$$

Condition (i) or (ii) in $(D14)$ is then also satisfied, and the subtree below d in S has the form

$$\begin{array}{lll} n+1. & P\,A & [j, D] \\ n+2. & O\,f & [n+2, A] \\ & s & \end{array}$$

(b) If conditions (i) and (ii) in $(D14')$ are not satisfied, then O has asserted A before, without having been attacked by P. Hence O cannot question A.

[94] We do not have to distinguish between atomic and complex formulas A in this case, since A is asserted by O before it is asserted by P. Hence the P-move at position $n+3$ cannot be in violation of $(D10)$.

The subtree below d' in S' then ends with the move at position $n+1$ as follows:

$$n+1. \quad P\,|A|\;\;[n, C]$$

Condition (i) or (ii) in $(D14)$ is then also satisfied, O cannot attack A, and the subtree below d in S ends with

$$n+1. \quad P\,A\;\;[j, D]$$

(2) $e = C$. The dialogue d' must contain a move $\langle \delta(m) = O\,C \to A$, $\eta(m) = [m-1, Z]\rangle$ (where $Z = A$ or D, $m < n-1$) and also a move $\langle \delta(n-1) = P\,A, \eta(n-1) = [j, Z]\rangle$ to which the question move made by O at position n refers. Thus S' has the form[95]

$$\begin{array}{rcl} & \vdots & \\ m. & O\,C \to A & [m-1, Z] \\ & \vdots & \\ n-1. & P\,A & [j, Z] \\ n. & O\,? & [n-1, Q] \\ n+1. & P\,C & [n, C] \\ n+2. & O\,? & [n+1, Q] \\ & s' & \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} d'$$

and for complex C the EI_c^p-strategy S has the form

$$\begin{array}{rl} & \qquad\vdots \\ m. & \qquad O\,C \to A\,[m-1, Z] \;\Big\}\, d \\ & \qquad\vdots \\ n+1. & \qquad P\,C\,[m, A] \\ n+2. & O\,f\;\;[n+1, A] \;\big|\; O\,A\;\;[n+1, D] \\ n+3. & \qquad s \qquad\;\;\; \big|\; P\,A\;\;[j, D] \end{array}$$

where the expression f is either of a symbolic attack or a formula, depending on the form of C. The formula A can be atomic; since A is asserted by O already at position $n+2$ in the right dialogue in S, condition $(D10)$ is not violated by the P-move at position $n+4$.

The move $\langle \delta(n+2) = O\,f, \eta(n+2) = [n+1, A]\rangle$ is only possible in S if C is complex. In case C is atomic, S has the form

$$\begin{array}{rcl} & \vdots & \\ m. & O\,C \to A & [m-1, Z] \\ & \vdots & \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} d$$

(cont'd on next page)

[95] Cf. the case where the proponent move is a defense move (i.e., the second case, subcase (2) on page 149).

$n+1.$	$P\ C$	$[m, A]$
$n+2.$	$O\ A$	$[n+1, D]$
$n+3.$	$P\ A$	$[j, D]$

Fourth, assume the proponent move is a cut $\langle \delta(n+1) = P\ A, \eta(n+1) = [Cut] \rangle$. By Lemma 5.4.5 there is then an EI^p_c-strategy for each such EI°-strategy. ⊣

We collect the equivalence results for EI°-dialogues:

COROLLARY 5.4.7. $\vdash_{EI^\circ} A$ *if and only if* $\vdash_{DI^p} A$, *by Corollary* 3.7.3 *and the just proved Theorem* 5.4.6.

COROLLARY 5.4.8. $\vdash_{LI^p} A$ *if and only if* $\vdash_{EI^\circ} A$, *by Corollary* 5.4.7 *and Remark* 2.2.24.

COROLLARY 5.4.9. $\vdash_{LI^p_c} A$ *if and only if* $\vdash_{EI^\circ} A$, *by Corollary* 5.4.8 *and Theorem* 3.2.3.

COROLLARY 5.4.10. $\vdash_{LI^\circ} A$ *if and only if* $\vdash_{EI^\circ} A$, *by Corollary* 5.4.9 *and Remark* 5.2.5.

REMARK 5.4.11. We therefore have

$$\vdash_{EI^\circ} A \overset{5.4.10}{\iff} \vdash_{LI^\circ} A \overset{5.2.5}{\iff} \vdash_{LI^p_c} A \overset{3.7.1}{\iff} \vdash_{EI^p_c} A,$$

and by using Remark 3.7.5 we also have $\vdash_{EI^\circ} A \iff \vdash_{DI^p_c} A$.

REMARK 5.4.12. A detailed discussion comparing the proof-theoretic approach toward implications-as-rules (using sequent calculus) with the dialogical approach developed here can be found in Piecha and Schroeder-Heister [2012]; see also the concluding Chapter 6 below.

5.5. Definitional dialogues for implications as rules

EI°-dialogues can be extended to definitional dialogues. We consider here only a possible extension by definitional dialogues *with contraction*, for the quantifier-free fragment. An important difference is the presence of cut moves in EI°-dialogues, given by the argumentation form AF(Cut) together with condition $(D05^\circ)$. For EI°-dialogues extended to definitional dialogues with contraction it can be shown that the addition of AF(Cut), together with $(D05^\circ)$, yields strategies for formulas that have no strategy otherwise. In other words, there are instances of AF(Cut) which are eliminable in EI°-dialogues but are not eliminable in EI°-dialogues extended to definitional dialogues with contraction. This is due to the fact that for definitional dialogues with contraction and the given definitional clause

$a \Leftarrow a \to \bot$[96] there is a strategy for $\bot$ if cut moves can be made according to the argumentation form AF(Cut), while this is not the case without AF(Cut).[97] Furthermore, the following properties hold:

(i) For any given formula A, let $\mathcal{D}_{\text{triv}}$ be the (special) definition containing for each atomic subformula a of A the trivial definitional clause $a \Leftarrow a$.[98] Then the formula A is EI°-dialogue-provable if and only if A is provable with (propositional) definitional dialogues for $\mathcal{D}_{\text{triv}}$ as the given (special) definition.

(ii) Definitional dialogues with contraction are defined with condition $(D14^{*})$. This condition is effectively the same as condition $(D14')$ in EI°-dialogues when considered for the respectively given argumentation forms (cf. Remark 5.3.6). They can thus be used together without causing any unwanted interferences.

(iii) Definitional dialogues as well as EI°-dialogues have been defined on the basis of EI_c^p-dialogues. Both kinds of dialogues incorporate conditions $(D11')$, $(D12')$ and (E) from EI_c^p-dialogues.

(iv) Both EI°-dialogues and definitional dialogues can begin with the assertion of an atomic formula at position 0 due to condition $(D00')$.

(v) Two (unproblematic) differences are given by condition $(D01')$ in definitional dialogues (instead of $(D01)$ in EI°-dialogues) and by the additional conditions $(D03^{\circ})$, $(D04^{\circ})$ and $(D05^{\circ})$ in EI°-dialogues. These differences are due to the different kinds of moves available for the respectively given argumentation forms.

(vi) Finally, conditions $(D15)$ and (S) in definitional dialogues are only relevant in applications of the argumentation form of definitional reasoning. They have no effect on applications of the argumentation forms of EI°-dialogues.

The following definition extends EI°-dialogues to definitional dialogues with contraction. We consider only the quantifier-free fragment here, and do therefore not give corresponding argumentation forms for the quantifiers

[96] Cf. Section 4.3, where we wrote $a \Leftarrow \neg a$ as an abbreviation for the definitional clause $a \Leftarrow a \to \bot$.

[97] It has been shown in Schroeder-Heister [1992] for the corresponding sequent calculus with definitional reflection that cut-elimination fails if implication can be used in the bodies of definitional clauses and contraction is present. Without one or the other, cut-elimination holds. That is, for implication-free definitions with or without contraction cut-elimination holds as well as for the contraction-free calculus for definitions that are implication-free or not.

[98] This special definition $\mathcal{D}_{\text{triv}}$ is not a definition in the sense of Definition 4.1.6, since it is not a finite set of definitional clauses.

$\forall$ and $\exists$. Note, however, that definitional clauses may contain formulas with free variables nonetheless.

DEFINITION 5.5.1. The *extension of EI°-dialogues to definitional dialogues with contraction* can be given as follows:
Argumentation forms:

AF($\neg\vdash$):	assertion:	$O\ \neg A$	
	attack:	$P\ A$	
	defense:	*no defense*	
AF($\wedge\vdash$):	assertion:	$O\ A_1 \wedge A_2$	
	attack:	$P\ \wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\ A_i$	
AF($\vee\vdash$):	assertion:	$O\ A_1 \vee A_2$	
	attack:	$P\ \vee$	
	defense:	$O\ A_i$	(O chooses $i = 1$ or $i = 2$)
AF($\to\vdash$)°:	assertion:	$O\ A \to B$	
	attack:	*no attack*	
	defense:	*no defense*	

AF($\vdash\neg$):	assertion:		$P\ \neg A$
	question:		$O\ ?$
	choice:	$P\ \lvert\neg A\rvert$	$\lvert\ P\ C \quad (R)$
	attack:	$O\ A$	$\lvert$
	defense:	*no defense*	$\lvert$
AF($\vdash\wedge$):	assertion:		$P\ A_1 \wedge A_2$
	question:		$O\ ?$
	choice:	$P\ \lvert A_1 \wedge A_2\rvert$	$\lvert\ P\ C \quad (R)$
	attack:	$O\ \wedge_i \quad (i = 1 \text{ or } 2)$	$\lvert$
	defense:	$P\ A_i$	$\lvert$
AF($\vdash\vee$):	assertion:		$P\ A_1 \vee A_2$
	question:		$O\ ?$
	choice:	$P\ \lvert A_1 \vee A_2\rvert$	$\lvert\ P\ C \quad (R)$
	attack:	$O\ \vee$	$\lvert$
	defense:	$P\ A_i \quad (i = 1 \text{ or } 2)$	$\lvert$
AF($\vdash\to$):	assertion:		$P\ A \to B$
	question:		$O\ ?$
	choice:	$P\ \lvert A \to B\rvert$	$\lvert\ P\ C \quad (R)$
	attack:	$O\ A$	$\lvert$
	defense:	$P\ B$	$\lvert$

AF(R):	assertion:	$P\ a$	
	question:	$O\ ?$	
	choice:	$P\ C$	only if O has asserted $C \to a$ before

AF(Cut):	statement:	$O\ e$
	cut:	$P\ A$
	cut:	$O\ A$

definitional reasoning:	assertion:	$X\ a$	
	attack:	$Y\ \mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\ \Gamma_i$	(X chooses $i = 1, \ldots, k$)

Conditions:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01°$) If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D03°$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark '?'.

($D04°$) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

($D05°$) If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (i.e., the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(m) = [Cut]$ and $\delta(n) = O\ A$ for $\delta(m) = P\ A$ (where $m < n$).

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd n there is at most one m such that $\eta(m) = [n, D]$. That is, an attack by O may be defended by P at most once.

($D14'$) O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D14^*$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D15$) If for an atom a there is a move $\langle\delta(l) = O\,a, \eta(l) = [k, Z]\rangle$, then there is no attack $\langle\delta(n) = O\,\mathscr{D}, \eta(n) = [m, A]\rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

(S) For any substitution σ of variables $x, y, \ldots$ by terms t, the application of definitional reasoning is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$. That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n-1$, or it is a cut move with $\delta(n) = O\,A$ for $\delta(n-1) = P\,A$.

Remark 5.5.2. By adding argumentation forms for the quantifiers $\forall$ and $\exists$ following the schema of the argumentation forms given above for the propositional case, EI°-dialogues (and their extension to definitional dialogues just given) can be extended to first-order logic.

5.6. Hypothetical EI°-dialogues

One further step would be to extend EI°-dialogues to *hypothetical EI°-dialogues*, that is, to EI°-dialogues which allow for a 'database' of implications assumed to be asserted by the opponent before any initial move is made at position 0.

Allowing for hypotheses can lead to problems; they have been pointed out in Section 2.8 about hypothetical dialogues. The situation would be different, however, for hypothetical EI°-dialogues. Whereas in the hypothetical dialogues considered in Section 2.8 the proponent can attack hypotheses, this is not possible in hypothetical EI°-dialogues. Here the proponent cannot attack any implications asserted by the opponent (regardless of whether they are asserted hypothetically or not), and the hypothetically asserted implications can only be used as rules by the proponent in choice moves which are made according to the rule condition (R). The problems concerning hypothetical dialogues that we have described in Section 2.8 would thus be avoided.

5.7. Summary

The proof-theoretic approach to implications as rules has been carried over to the dialogical setting in the form of EI°-dialogues. These dialogues differ in several respects from the dialogues considered before. One of the new features is the addition of an argumentation form for cut. It could be shown that EI°-provability is equivalent to LI°-provability. An adequate treatment of implications as rules is thus not only possible in the proof-theoretic approach, but can also be realized in the dialogical approach. Moreover, we have proposed an extension of EI°-dialogues to definitional EI°-dialogues with contraction, and we have finally sketched how EI°-dialogues could be used for hypothetical reasoning when implications are given as hypotheses. Implications as rules and their dialogical framework will be discussed further in the concluding Chapter 6.

Chapter 6

CONCLUSION

We have developed dialogical foundations in the sense of formal dialogue semantics for definitional reasoning and for implications as rules.

In comparison with the corresponding proof-theoretic approaches, certain complications can be observed in the dialogical approaches.

For example, modifications concerning structural operations like thinning or contraction can be implemented more easily in a proof-theoretic environment using sequent calculus. This is due to the fact that the structural operations are in sequent calculus explicitly given by structural rules like (Thin$\vdash$) and (Contr), whereas they are only implicitly given in dialogues. Of course, this need not be seen as a defect of dialogues but can rather be described as their strong point: argumentation forms are given only for the logical constants, and everything else is—in part implicitly—dealt with by the dialogue conditions.

It can also be observed that the EI°-dialogues for the interpretation of implications as rules is not as straightforward as the corresponding sequent calculus LI°, which can be read as the proof-theoretic semantics for implications as rules. The definitions of LI° and LI_c^p differ only in the left implication introduction rule, whereas the definition of EI°-dialogues differs quite a lot from the one for EI_c^p-dialogues.[99]

One of the main differences between standard dialogues (like e.g. EI^p-dialogues) and EI°-dialogues is that the argumentation forms in the latter are no longer symmetric with respect to proponent and opponent; that is, the player independence of the argumentation forms that obtains in the standard dialogues is given up in EI°-dialogues. Although proponent and opponent are not interchangeable in standard dialogues due to the dialogue conditions (cf. Remark 2.2.4), there is a perfect symmetry with respect to the argumentation forms. Just attacks and defenses are defined, not different ways of attacking and defending for proponent or opponent. If the idea of having player independent argumentation forms is considered to be essential

[99] See also Piecha and Schroeder-Heister [2012] for what follows.

in the dialogical paradigm, then giving it up may seem to amount to giving up the dialogical setting itself as a foundational approach. However, from the implications-as-rules point of view it could be argued that implication is different from the other logical constants, and that this difference requires an asymmetric treatment with respect to the argumentation forms.

As a consequence of this asymmetry in the treatment of implication there is another asymmetry: In *EI*°-dialogues the proponent can defend an assertion by means of the rule condition (R) independently of its logical form. This is not possible in standard dialogues where a defense of an assertion always depends on its logical form, and where formulas are always decomposed into subformulas according to their logical form.

But certain tenets within the dialogical tradition—such as the player independence of argumentation forms or the decomposition of formulas according to their logical form—do not have to be taken as being essential in dialogical approaches. Particularly not for implications as rules: Rules are *not* logical constants but belong to the general structural framework that underlies definitions of logical constants.[100] Given that the proponent has the dialogical role of claiming something to hold, and the opponent the role of providing the assumptions under which something is supposed to hold, the implication-as-rule $A \to B$ means for the proponent that B must be defended on the background A, whereas the opponent only grants with $A \to B$ the right to *use* this implication as a rule, without any propositional claim. This is exactly what is captured in our *EI*°-dialogues for implications-as-rules.

A crucial aspect here is the significance which is given to *modus ponens*

$$\frac{A \quad A \to B}{B}$$

in general, and in natural deduction in particular. For the implications-as-rules view, *modus ponens* is essential for the meaning of implication as it expresses the idea of *application*, which is the characteristic feature of a rule. As already mentioned in the introduction (see Chapter 1), in natural deduction *modus ponens* can be understood as the application of the implication $A \to B$ as a rule which allows us to infer B from A.

The sequent calculus *LI*° can be viewed as a system representing the idea of *modus ponens* at the sequent-calculus level via the left implication introduction rule

$$(\to\vdash)^\circ \; \frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

The standard interpretation of implication in the dialogical setting corresponds instead to the symmetric sequent calculus *LJ* (as well as to our

[100] Cf. Schroeder-Heister [2007a], de Campos Sanz and Piecha [2009a], [2009b] and Piecha and de Campos Sanz [2010].

related sequent calculi LI^p, LI^p_c, LI_c) with the left implication introduction rule

$$(\to\vdash)\,\frac{\Gamma\vdash A \quad B,\Delta\vdash C}{\Gamma, A\to B,\Delta\vdash C}$$

It is based on the 'implications-as-links' view, according to which an implication $A \to B$ introduced by means of $(\to\vdash)$ on the left side of the sequent symbol, links an occurrence of A on the right side of the left premiss with an occurrence of B on the left side of the right premiss of $(\to\vdash)$.[101] The standard dialogical approach favors sequent-style reasoning in the sense of $(\to\vdash)$. The idea of implications-as-rules fits very well into natural-deduction style reasoning with *modus ponens*. We have shown that the sequent-style rendering of this kind of reasoning via the sequent calculus LI° can be fully represented in the dialogical setting. That implications-as-rules received an asymmetric treatment in our dialogical representation is not a defect of the dialogical setting or of the modeled sequent calculus LI°; it are rather the differences between natural deduction and the symmetric sequent calculus which are reflected therein.

Another complication is introduced by the need of (a restricted version of) cut in order to achieve full intuitionistic logic. This need is present in both the proof-theoretic and the dialogical setting for implications-as-rules. In the dialogical setting, however, the handling of cut is difficult and by far not as plausible as in the proof-theoretic setting. For dialogues one has to model the claim of the cut formula—made by both the proponent and the opponent—according to the pattern of attack and defense as employed in the other argumentation forms, although with slight deviations. The addition of an argumentation form for cut might also be conceived as being alien to the dialogical approach as such, as it has always been considered as being cut-free per se (this is a legacy which dialogues share with tableaux; cf. Section 2.9). But such a view proves to be too narrow from the perspective of implications-as-rules if full intuitionistic logic is to be achieved.

Overall, this dissertation demonstrates that the dialogical framework is versatile enough to cope with approaches like definitional reasoning and implications-as-rules that have originally been developed in the realm of proof-theoretic semantics. Certain complications on the dialogical side have already been pointed out; one example were the structural operations, whose dialogical representation renders investigations of substructural logics more difficult than in a proof-theoretic setting. Leaving such questions of practicality aside, more general arguments are needed if one wants to give preference either to proofs or to dialogues as the appropriate foundational approach.

[101] See Schroeder-Heister [2011b].

Appendix A

DEFINITIONS OF DIALOGUES AND SEQUENT CALCULI

This appendix collects all definitions of dialogues and sequent calculi we have made use of.

A.1. Dialogues

Dialogues are defined by the following argumentation forms and conditions:

Argumentation forms:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	$(Y$ chooses $i = 1$ or $i = 2)$
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	$(X$ chooses $i = 1$ or $i = 2)$
implication $\to$:	assertion:	$X\ A \to B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	

Conditions:

$(D00)$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

$(D01)$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

A.2. DI^p-dialogues

We add the following conditions to the definition of dialogues:

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11)$ If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

A.3. Classical dialogues

We add the following conditions to the definition of dialogues:

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11^+)$ If $\eta(p) = [n, D]$ for even n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by P, then only the last of them may be defended by O at position p.

$(D12^+)$ For every even m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by P may be defended by O at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

A.4. DI_c^p-dialogues

We add the following conditions to the definition of dialogues:

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11)$ If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

A.5. EI^p-dialogues

We add the following conditions to the definition of dialogues:

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11)$ If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

Since condition (E) implies condition $(D13)$, and (E) furthermore implies $(D11)$ respectively $(D12)$ for odd p respectively for odd n, EI^p-dialogues can also be defined as follows:

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n - 1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

A.6. EI_c^p-dialogues

We add the following conditions to the definition of dialogues:

($D10$) If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n - 1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

A.7. Hypothetical dialogues

See Section 2.8.

A.8. The sequent calculus LI^p

The *sequent calculus LI^p for intuitionistic propositional logic* consists of the following rules, where Γ and Δ are finite multisets of formulas (the comma in antecedents of sequents stands for multiset union, and singletons are written without braces):

Axiom

$$(\mathrm{Id}_a)\,\frac{}{a \vdash a}\ \text{(where } a \text{ is atomic)}$$

Logical rules

$$(\neg\vdash)\,\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\,(\vdash\neg)$$

$$(\wedge\vdash)\,\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,(\vdash\wedge)$$

$$(\vee\vdash)\,\frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\,(\vdash\vee)\ (i = 1, 2)$$

$$(\rightarrow\vdash)\,\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\,(\vdash\rightarrow)$$

Structural rules

$$(\mathrm{Thin}\vdash)\,\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\,(\vdash\mathrm{Thin})$$

$$(\mathrm{Contr})\,\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\,\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

A.9. The sequent calculus LI_c^p

The *sequent calculus LI_c^p with atomic or complex initial sequents for intuitionistic propositional logic* is defined as LI^p but without the axiom being restricted to atomic formulas (as in LI^p, Γ and Δ are finite multisets of formulas, the comma in antecedents of sequents stands for multiset union, and singletons are written without braces):

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\rightarrow\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\vdash\rightarrow)$$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash \mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

A.10. Contraction-free EI_c^p-dialogues

Contraction-free EI_c^p-dialogues are obtained by adding the following condition to the definition of EI_c^p-dialogues:

($D13^*$) For any move $\langle \delta(k) = O\,A, \eta(k) = [j, Z]\rangle$ there is at most one move of the form $\langle \delta(l) = P\,e, \eta(l) = [k, A]\rangle$ or $\langle \delta(l) = P\,A, \eta(l) = [i, Z]\rangle$, where $j < k < l$ and $i < l$. That is, each assertion of an O-signed formula may be used by P at most once.

Contraction-free EI_c^p-dialogues are thus defined by the conditions ($D00$), ($D01$), ($D02$), ($D10$), ($D11'$), ($D12'$), ($D13^*$), ($D14$) and (E), with the argumentation forms as given for dialogues.

A.11. Contraction-free DI_c^p-dialogues

Contraction-free DI_c^p-dialogues are obtained by adding the following condition to the definition of DI_c^p-dialogues:

($D13^*$) For any move $\langle\delta(k) = O\,A, \eta(k) = [j, Z]\rangle$ there is at most one move of the form $\langle\delta(l) = P\,e, \eta(l) = [k, A]\rangle$ or $\langle\delta(l) = P\,A, \eta(l) = [i, Z]\rangle$, where $j < k < l$ and $i < l$. That is, each assertion of an O-signed formula may be used by P at most once.

Contraction-free DI_c^p-dialogues are thus defined by the conditions ($D00$), ($D01$), ($D02$), ($D10$), ($D11$), ($D12$), ($D13$), ($D13^*$) and ($D14$), with the argumentation forms as given for dialogues.

A.12. Dialogues for first-order logic

We add argumentation forms for the quantifiers $\forall$ and $\exists$:

universal quantifier $\forall$:	assertion:	$X\,\forall x A(x)$	
	attack:	$Y\,t$	(Y chooses the term t)
	defense:	$X\,A(x)[t/x]$	
existential quantifier $\exists$:	assertion:	$X\,\exists x A(x)$	
	attack:	$Y\,\exists$	
	defense:	$X\,A(x)[t/x]$	(X chooses the term t)

where $[t/x]$ is the substitution of the term t for the variable x, and $A(x)[t/x]$ is the result of substituting t for all occurrences of x in A. This substitution instance is also written $A(t)$.

Dialogues and *strategies* for first-order logic are defined as propositional dialogues and strategies extended by the argumentation forms for $\forall$ and $\exists$.

A.13. Formal dialogues (for first-order logic)

We define *formal argumentation forms* for the quantifiers $\forall$ and $\exists$ as follows:

($P\,\forall$-form):	assertion:	$P\,\forall x A(x)$	
	attack:	$O\,y$	(with eigenvariable condition)
	defense:	$P\,A(x)[y/x]$	
($O\,\forall$-form):	assertion:	$O\,\forall x A(x)$	
	attack:	$P\,t$	
	defense:	$O\,A(x)[t/x]$	

($P\,\exists$-form):	assertion:	$P\,\exists x A(x)$
	attack:	$O\,t$
	defense:	$P\,A(x)[t/x]$
($O\,\exists$-form):	assertion:	$O\,\exists x A(x)$
	attack:	$P\,\exists$
	defense:	$O\,A(x)[y/x]$ (with eigenvariable condition)

where the *eigenvariable condition* is that y does not occur free in an expression (i.e. in an assertion or in a symbolic attack) before. That is, the move $\langle \delta(n) = O\,y, \eta(n) = [m, A]\rangle$, respectively the move $\langle \delta(n) = O\,A(x)[y/x],$ $\eta(n) = [m, D]\rangle$, is only possible if y does not occur free at positions $k < n$.

A dialogue constructed in accordance with the formal argumentation forms is called *formal dialogue*. A *formal dialogue tree* is a tree whose branches contain as paths all possible formal dialogues for a given formula. *P wins a formal dialogue for a formula A* if the formal dialogue is finite, begins with the move $P\,A$ and ends with a move of P such that O cannot make another move.

A formal dialogue is called *DI-dialogue* if it satisfies the conditions $(D00)$–$(D02)$ and $(D10)$–$(D13)$ as given in Definitions 2.1.6 (see also Section A.1) and 2.2.1 (see also Section A.2).

A *DI_c-dialogue* is a *DI*-dialogue that satisfies the additional condition $(D14)$ as given in Definition 2.5.1 (see also Section A.4).

A formal dialogue is called *EI-dialogue* if it satisfies the conditions $(D00)$–$(D02)$, $(D10)$–$(D13)$ and (E) as given in Definitions 2.1.6, 2.2.1 (see also Sections A.1 and A.2) and 2.6.1 (see also Section A.5).

A formal dialogue is called *EI_c-dialogue* if it satisfies the following conditions:

$(D00)$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

$(D01)$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

$(D10)$ If, for an atomic formula a, $\delta(n) = P\,a$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14$) O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n - 1, Z]\rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

A *formal strategy* for a formula A is a subtree S of the formal dialogue tree for A such that S does not branch at even positions, all branches of S are formal dialogues for A won by P, and S has as many nodes at odd positions as there are possible moves for O, with the following exceptions: Only one node at odd positions n has to be considered if

(i) $\langle\delta(n) = O\, A(x)[y/x], \eta(n) = [m, D]\rangle$ for $\langle\delta(m) = P\,\exists, \eta(m) = [l, A]\rangle$. That is, O is defending an attack $P\,\exists$ according to the formal argumentation form ($O\,\exists$-form).

(ii) $\langle\delta(n) = O\, y, \eta(n) = [m, A]\rangle$. That is, O makes an attack move $O\, y$ according to the formal argumentation form ($P\,\forall$).

(iii) $\langle\delta(n) = O\, t, \eta(n) = [m, A]\rangle$. That is, O makes an attack move $O\, t$ according to the formal argumentation form ($P\,\exists$-form).

A.14. The sequent calculus *LI*

The *sequent calculus LI for intuitionistic first-order logic* is the propositional calculus LI^p with additional left and right introduction rules for the quantifiers $\forall$ and $\exists$. Γ and Δ are finite multisets of formulas. The comma in antecedents of sequents stands for multiset union, and singletons are written without braces. We give the whole calculus *LI*, repeating the rules of LI^p:

Axiom

$$(\mathrm{Id}_a)\ \frac{}{a \vdash a}\ \text{(where } a \text{ is atomic)}$$

(cont'd on next page)

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\to\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

$$(y \text{ does not occur free in } \Gamma)$$

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

$$(y \text{ does not occur free in } \Gamma, C)$$

Structural rules

$$(\text{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash\text{Thin})$$

$$(\text{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\text{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

A.15. The sequent calculus LI_c

The *sequent calculus LI_c with atomic or complex initial sequents for intuitionistic first-order logic* is defined as *LI* but without the axiom being restricted to atomic formulas (as in *LI*, Γ and Δ are finite multisets of formulas, the comma in antecedents of sequents stands for multiset union, and singletons are written without braces):

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\quad (A \text{ atomic or complex})$$

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1,2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1,2)$$

$$(\rightarrow\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\vdash\rightarrow)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

(y does not occur free in Γ)

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

(y does not occur free in Γ, C)

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash \mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

A.16. The sequent calculus $LI_c(\mathcal{D})$

For any given definition $\mathcal{D}$, the *sequent calculus* $LI_c(\mathcal{D})$ *for intuitionistic first-order logic over* $\mathcal{D}$ is:

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Propositional logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\to\vdash)\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\vdash\to)$$

First-order logical rules

$$(\forall\vdash)\ \frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x A(x) \vdash B} \qquad \frac{\Gamma \vdash A(y)}{\Gamma \vdash \forall x A(x)}\ (\vdash\forall)$$

(y does not occur free in Γ)

$$(\exists\vdash)\ \frac{\Gamma, A(y) \vdash C}{\Gamma, \exists x A(x) \vdash C} \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\vdash\exists)$$

(y does not occur free in Γ, C)

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash\mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

(cont'd on next page)

Definitional rules

$$\frac{\Delta \vdash B_1^i\sigma \quad \dots \quad \Delta \vdash B_{n_i}^i\sigma}{\Delta \vdash a\sigma}\ (\vdash \mathcal{D})$$

$$(\mathcal{D}\vdash)\ \frac{\{\Delta, \Gamma_i\sigma \vdash C \mid b \Leftarrow \Gamma_i \in \mathcal{D} \text{ and } a = b\sigma\}}{\Delta, a \vdash C}$$
$$\text{(where } \mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma)$$

A.17. Preliminary definitional dialogues

Preliminary definitional dialogues are EI_c-dialogues where the following changes are made.

Conditions $(D00)$ and $(D01)$ are replaced by the following conditions $(D00')$ and $(D01')$, respectively, where the restriction of the expressions in $\delta(0)$ and $\delta(m)$ to complex formulas is discarded; that is, a definitional dialogue can start with the assertion of an atomic formula, and atomic formulas can be attacked:

$(D00')$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

$(D01')$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

Condition $(D02)$ remains without change:

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Condition $(D10)$ is omitted altogether, so that P can now assert atomic formulas without O having asserted them before. Conditions $(D11')$, $(D12')$, $(D14)$ and (E) remain without change:

$(D11')$ If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

$(D12')$ For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n-1$.

The following proviso for applications of definitional reasoning in the presence of variables is added:

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

Thus (preliminary) definitional dialogues are defined by the conditions $(D00')$, $(D01')$, $(D02)$, $(D11')$, $(D12')$, $(D14)$, (S) and (E), with the additional argumentation form of definitional reasoning. Given en bloc:

Argumentation forms:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	$(Y$ chooses $i = 1$ or $2)$
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	$(X$ chooses $i = 1$ or $2)$
implication $\rightarrow$:	assertion:	$X\ A \rightarrow B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	
$(P\,\forall$-form):	assertion:	$P\ \forall x A(x)$	
	attack:	$O\ y$	$(y$ not free before)
	defense:	$P\ A(x)[y/x]$	
$(O\,\forall$-form):	assertion:	$O\ \forall x A(x)$	
	attack:	$P\ t$	
	defense:	$O\ A(x)[t/x]$	
$(P\,\exists$-form):	assertion:	$P\ \exists x A(x)$	
	attack:	$O\ t$	
	defense:	$P\ A(x)[t/x]$	
$(O\,\exists$-form):	assertion:	$O\ \exists x A(x)$	
	attack:	$P\ \exists$	
	defense:	$O\ A(x)[y/x]$	$(y$ not free before)

definitional reasoning:	assertion:	$X\,a$	
	attack:	$Y\,\mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\,\Gamma_i$	(X chooses $i = 1, \ldots, k$)

Conditions:

$(D00')$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

$(D01')$ If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

$(D02)$ If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

$(D11)$ If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

$(D14)$ O can attack a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

A.18. Definitional dialogues

Definitional dialogues are defined by adding to the preliminary definition of definitional dialogues the following condition:

$(D15)$ If for an atom a there is a move $\langle \delta(l) = O\,a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\,\mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

Furthermore, condition $(D14)$ is replaced by the following condition $(D14^*)$ which is $(D14)$ restricted to complex formulas:

($D14^*$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

Definitional dialogues are thus defined as follows:

Argumentation forms:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	(Y chooses $i = 1$ or 2)
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	(X chooses $i = 1$ or 2)
implication $\rightarrow$:	assertion:	$X\ A \rightarrow B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	
($P\ \forall$-form):	assertion:	$P\ \forall x A(x)$	
	attack:	$O\ y$	(y not free before)
	defense:	$P\ A(x)[y/x]$	
($O\ \forall$-form):	assertion:	$O\ \forall x A(x)$	
	attack:	$P\ t$	
	defense:	$O\ A(x)[t/x]$	
($P\ \exists$-form):	assertion:	$P\ \exists x A(x)$	
	attack:	$O\ t$	
	defense:	$P\ A(x)[t/x]$	
($O\ \exists$-form):	assertion:	$O\ \exists x A(x)$	
	attack:	$P\ \exists$	
	defense:	$O\ A(x)[y/x]$	(y not free before)
definitional reasoning:	assertion:	$X\ a$	
	attack:	$Y\ \mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\ \Gamma_i$	(X chooses $i = 1, \ldots, k$)

Conditions:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01'$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14^*$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D15$) If for an atom a there is a move $\langle\delta(l) = O\,a, \eta(l) = [k, Z]\rangle$, then there is no attack $\langle\delta(n) = O\,\mathscr{D}, \eta(n) = [m, A]\rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

A.19. Definitional dialogues without contraction

Contraction-free definitional dialogues are definitional dialogues where the following condition ($D13^*$) is added:

($D13^*$) For any move $\langle\delta(k) = O\,A, \eta(k) = [j, Z]\rangle$ there is at most one move of the form $\langle\delta(l) = P\,e, \eta(l) = [k, A]\rangle$ or $\langle\delta(l) = P\,A, \eta(l) = [i, Z]\rangle$, where $j < k < l$ and $i < l$. That is, each assertion of an O-signed formula may be used by P at most once.

Contraction-free definitional dialogues are thus defined by the conditions ($D00'$), ($D01'$), ($D02$), ($D11'$), ($D12'$), ($D13^*$), ($D14^*$), ($D15$), (S) and (E), with the argumentation forms as given for definitional dialogues.

A.20. Definitional dialogues with restricted contraction

See Section 4.5.

A.21. Kreuger-restricted definitional dialogues

For a set $\mathcal{U}$ of atoms a which are not defined by a given definition $\mathcal{D}$, that is, for $\mathcal{U} := \{a \mid (a \Leftarrow \Gamma) \notin \mathcal{D}$ for any atoms a and defining conditions $\Gamma\}$, the *Kreuger-restricted definitional dialogues* are defined by the following argumentation forms and conditions:

Argumentation forms:

negation $\neg$:	assertion:	$X\ \neg A$	
	attack:	$Y\ A$	
	defense:	*no defense*	
conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	$(Y$ chooses $i = 1$ or $2)$
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	$(X$ chooses $i = 1$ or $2)$
implication $\rightarrow$:	assertion:	$X\ A \rightarrow B$	
	attack:	$Y\ A$	
	defense:	$X\ B$	
$(P\ \forall$-form$)$:	assertion:	$P\ \forall x A(x)$	
	attack:	$O\ y$	$(y$ not free before$)$
	defense:	$P\ A(x)[y/x]$	
$(O\ \forall$-form$)$:	assertion:	$O\ \forall x A(x)$	
	attack:	$P\ t$	
	defense:	$O\ A(x)[t/x]$	
$(P\ \exists$-form$)$:	assertion:	$P\ \exists x A(x)$	
	attack:	$O\ t$	
	defense:	$P\ A(x)[t/x]$	
$(O\ \exists$-form$)$:	assertion:	$O\ \exists x A(x)$	
	attack:	$P\ \exists$	
	defense:	$O\ A(x)[y/x]$	$(y$ not free before$)$
definitional reasoning:	assertion:	$X\ a$	
	attack:	$Y\ \mathscr{D}$	(only if $a \neq \top$)
	defense:	$X\ \Gamma_i$	$(X$ chooses $i = 1, \ldots, k)$

Conditions:

$(D00')$ $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01'$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D10^*$) If, for an atomic formula $a \in \mathcal{U}$, $\delta(n) = P\,a$ for $n \neq 0$, then there is an m such that $m < n$ and $\delta(m) = O\,a$. That is, P may assert an atomic formula a, which has been asserted by O before, only if $a \in \mathcal{U}$.

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D15$) If for an atom a there is a move $\langle \delta(l) = O\,a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\,\mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(K) For $\langle \delta(m) = O\,a, \eta(m) = [l, Z] \rangle$ and $a \in \mathcal{U}$ there is no attack $\langle \delta(n) = P\,\mathscr{D}, \eta(n) = [m, A] \rangle$ for $l < m < n$. That is, P may attack an atom a by definitional reasoning only if $a \notin \mathcal{U}$.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n - 1, Z] \rangle$. That is, an opponent move made at position n is either an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$.

A.22. The sequent calculus LI°

The *sequent calculus LI° for intuitionistic propositional logic* consists of the following rules (where Γ and Δ are finite multisets of formulas; the comma in antecedents of sequents stands for multiset union, and singletons are written without braces):

Axiom

$$(\mathrm{Id})\ \frac{}{A \vdash A}\ (A \text{ atomic or complex})$$

Logical rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A}{\Gamma, \neg A \vdash} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ (\vdash\neg)$$

$$(\wedge\vdash)\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ (i = 1, 2) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vdash\vee)\ (i = 1, 2)$$

$$(\rightarrow\vdash)^\circ\ \frac{\Gamma \vdash A}{\Gamma, A \rightarrow B \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ (\vdash\rightarrow)$$

Structural rules

$$(\mathrm{Thin}\vdash)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ (\vdash\mathrm{Thin})$$

$$(\mathrm{Contr})\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\mathrm{Cut})\ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

A.23. *EI*°-dialogues

EI°-dialogues are defined by the following argumentation forms and conditions:

Argumentation forms for assertions made by O:

AF($\neg\vdash$):	assertion:	$O\ \neg A$	
	attack:	$P\ A$	
	defense:	*no defense*	
AF($\wedge\vdash$):	assertion:	$O\ A_1 \wedge A_2$	
	attack:	$P\ \wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\ A_i$	

AF($\vee\vdash$):	assertion:	$O\ A_1 \vee A_2$	
	attack:	$P \vee$	
	defense:	$O\ A_i$	(O chooses $i = 1$ or $i = 2$)

AF($\rightarrow\vdash$)°:	assertion:	$O\ A \rightarrow B$
	attack:	*no attack*
	defense:	*no defense*

The argumentation forms for assertions made by the proponent P are formulated with the following *rule condition*

(R) P may answer a question $O\,?$ on a formula D by choosing C only if O has asserted the formula $C \rightarrow D$ before.

AF($\vdash\neg$):	assertion:		$P\ \neg A$
	question:		$O\,?$
	choice:	$P\ \lvert\neg A\rvert$	$P\ C$ (R)
	attack:	$O\ A$	
	defense:	*no defense*	

AF($\vdash\wedge$):	assertion:		$P\ A_1 \wedge A_2$
	question:		$O\,?$
	choice:	$P\ \lvert A_1 \wedge A_2\rvert$	$P\ C$ (R)
	attack:	$O\ \wedge_i$ ($i = 1$ or 2)	
	defense:	$P\ A_i$	

AF($\vdash\vee$):	assertion:		$P\ A_1 \vee A_2$
	question:		$O\,?$
	choice:	$P\ \lvert A_1 \vee A_2\rvert$	$P\ C$ (R)
	attack:	$O \vee$	
	defense:	$P\ A_i$ ($i = 1$ or 2)	

AF($\vdash\rightarrow$):	assertion:		$P\ A \rightarrow B$
	question:		$O\,?$
	choice:	$P\ \lvert A \rightarrow B\rvert$	$P\ C$ (R)
	attack:	$O\ A$	
	defense:	$P\ B$	

For assertions of atomic formulas a made by the proponent P an argumentation form is given by the rule condition (R) itself:

AF(R):	assertion:	$P\ a$
	question:	$O\,?$
	choice:	$P\ C$ only if O has asserted $C \rightarrow a$ before

By the following argumentation form AF(Cut) any expression e (i.e., question, symbolic attack or formula) stated by O can be attacked with the move $P\,A$, and this attack can then be defended with the move $O\,A$:

AF(Cut): assertion: $O\,e$
attack: $P\,A$
defense: $O\,A$

The conditions for EI°-dialogues are the following:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01^{\circ}$) If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D03^{\circ}$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark '?'.

($D04^{\circ}$) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

($D05^{\circ}$) If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (i.e., the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(m) = [Cut]$ and $\delta(n) = O\,A$ for $\delta(m) = P\,A$ (where $m < n$).

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p-1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14'$) O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

(E) All moves $\langle\delta(n), \eta(n)\rangle$ for n odd are of the form $\langle\delta(n), \eta(n) = [n-1, Z]\rangle$. That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n-1$, or it is a cut move with $\delta(n) = O\,A$ for $\delta(n-1) = P\,A$.

A.24. EI°-dialogues extended to definitional dialogues with contraction

The *extension of EI°-dialogues to definitional dialogues with contraction* is given as follows:

Argumentation forms:

AF($\neg\vdash$):	assertion:	$O\,\neg A$	
	attack:	$P\,A$	
	defense:	*no defense*	
AF($\wedge\vdash$):	assertion:	$O\,A_1 \wedge A_2$	
	attack:	$P\,\wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\,A_i$	
AF($\vee\vdash$):	assertion:	$O\,A_1 \vee A_2$	
	attack:	$P\,\vee$	
	defense:	$O\,A_i$	(O chooses $i = 1$ or $i = 2$)
AF($\to\vdash$)$^\circ$:	assertion:	$O\,A \to B$	
	attack:	*no attack*	
	defense:	*no defense*	

AF($\vdash\neg$):	assertion:		$P\,\neg A$
	question:		$O\,?$
	choice:	$P\,\lvert\neg A\rvert$	$P\,C$ (R)
	attack:	$O\,A$	
	defense:	*no defense*	
AF($\vdash\wedge$):	assertion:		$P\,A_1 \wedge A_2$
	question:		$O\,?$
	choice:	$P\,\lvert A_1 \wedge A_2\rvert$	$P\,C$ (R)
	attack:	$O\,\wedge_i$ $(i = 1$ or $2)$	
	defense:	$P\,A_i$	

AF($\vdash\vee$): assertion: $P\, A_1 \vee A_2$
question: $O\,?$
choice: $P\,|A_1 \vee A_2|$ | $P\, C$ (R)
attack: $O\, \vee$
defense: $P\, A_i$ $(i = 1$ or $2)$

AF($\vdash\rightarrow$): assertion: $P\, A \rightarrow B$
question: $O\,?$
choice: $P\,|A \rightarrow B|$ | $P\, C$ (R)
attack: $O\, A$
defense: $P\, B$

AF(R): assertion: $P\, a$
question: $O\,?$
choice: $P\, C$ only if O has asserted $C \rightarrow a$ before

AF(Cut): statement: $O\, e$
cut: $P\, A$
cut: $O\, A$

definitional reasoning: assertion: $X\, a$
attack: $Y\, \mathscr{D}$ (only if $a \neq \top$)
defense: $X\, \Gamma_i$ (X chooses $i = 1, \ldots, k$)

Conditions:

($D00'$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01°$) If $\eta(n) = [m, A]$ for even n, then the expression in $\delta(m)$ is a complex formula. If $\eta(n) = [n-1, A]$ for odd n, then the expression in $\delta(n-1)$ is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D03°$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark '?'.

($D04^{\circ}$) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

($D05^{\circ}$) If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (i.e., the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(m) = [Cut]$ and $\delta(n) = O\,A$ for $\delta(m) = P\,A$ (where $m < n$).

($D11'$) If $\eta(p) = [n, D]$ for odd n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks by O, then only the last of them may be defended by P at position p.

($D12'$) For every odd m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack by O may be defended by P at most once.

($D14'$) O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D14^{*}$) O can attack a complex formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

($D15$) If for an atom a there is a move $\langle \delta(l) = O\,a, \eta(l) = [k, Z] \rangle$, then there is no attack $\langle \delta(n) = O\,\mathscr{D}, \eta(n) = [m, A] \rangle$ for $\delta(m) = P\,a$ with $k < l < m < n$. That is, O may attack an atom a by definitional reasoning only if it has not been asserted by O before.

(S) For any substitution σ replacing variables $x, y, \ldots$ by terms t, the application of definitional reasoning with attack move $P\,\mathscr{D}$ is restricted to the cases where $\mathcal{D}(a\sigma) \subseteq (\mathcal{D}(a))\sigma$.

(E) All moves $\langle \delta(n), \eta(n) \rangle$ for n odd are of the form $\langle \delta(n), \eta(n) = [n-1, Z] \rangle$. That is, an opponent move made at position n is either a question, an attack or a defense of the immediately preceding move made by the proponent at position $n - 1$, or it is a cut move with $\delta(n) = O\,A$ for $\delta(n-1) = P\,A$.

A.25. Hypothetical EI°-dialogues

See Section 5.6.

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