Numerical Analysis of the Stochastic Navier-Stokes Equations

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Abstract

The main subject of this thesis is to analyse various discretisations schemes for the stochastic Navier-Stokes equations on bounded two and three dimensional domains.

The motivation for this numerical analysis is twofold: First this is an important model problem which combines algebraic constraints, non-Lipschitz nonlinearity, and stochastic forcing. The methods developed for this model problem may be applied to a wide range of nonlinear stochastic partial differential equations driven by a Wiener noise.

Second, these methods may be used in applications. Such a system has been introduced to better understand turbulence phenomena, well posedness of the deterministic problem and random fluctuations in hydrodynamic models. It may be used to model relevant physical phenomena, such as turbulence.

In the first part of this thesis, we address the finite element based approximation of weak martingale solutions in two and three dimensions, i.e., a system consisting of a filtered probability space, a Wiener process on it, and a solution to the equations. The discretisation is conceived such that all the elements of the system are constructed using continuous perturbations of the discrete iterates, and convergence without rates for subsequences of approximating solutions is proved. Moreover we show the same convergence properties for a scheme which uses general random variables to approximate the time increments of the stochastic forcing. In the two-dimensional case, thanks to a local monotonicity argument, the same scheme with Wiener process increments is shown to produce iterates that converge towards the unique strong solution.

In the second part of this thesis, we study the convergence properties of projection based splitting schemes applied to the unsteady stochastic Stokes equations. In this simplified setting, we observe that the Lagrange multiplier affects the convergence behavior of the scheme, due to its irregularity This motivates the introduction of a new discretisation scheme, which is stable under this irregularity. Finite element discretisations are also considered, and their convergence proved.

In the third part of the thesis, we consider implicit Euler based approximation schemes for the two-dimensional stochastic Navier-Stokes equations with periodic boundary conditions, and study convergence with rates. Due to the non-Lipschitz character of the nonlinearity, we prove convergence only on a set with probability arbitrarily close to one for the proposed schemes in a general setting. However, for additive noise we show convergence on the whole realisation space for the time discretisation. Finite element approximations for the corresponding time discretisations are considered, and convergence analysed.

All the parts are concluded with simulations to illustrate the convergence results, and compare the efficiency of the different discretisations.

Zusammenfassung

Die Zielsetzung dieser Arbeit ist die Analyse von Diskretisierungen der stochastischen Navier-Stokes Gleichungen auf beschränkten, zwei- und dreidimensionalen Gebieten.

Die Motivation für diese numerische Analysis ist zweifach: Zum einen sind die Gleichungen ein wichtiges Modellproblem, das algebraische Nebenbedingungen mit einer stochastischen Kraft und einer Nichtlinearität, die nicht Lipschitz ist, kombiniert. Die hier entwickelten Methoden können auf andere ähnliche nichtlineare stochastische Partielle Differentialgleichungen angewandt werden. Zum anderen sind diese Methoden für praktische Anwendungen wichtig. Die hier betrachteten Gleichungen dienen Dazu, turbulente Strömungen, Wohlgestelltheit des entsprechenden deterministischen Problems und zufällige Schwankungen in hydrodynamischen Modellen besser zu verstehen.

Im ersten Teil der Arbeit wird die auf finiten Elemente basierte Approximation schwacher Martingallösungen in drei Dimensionen betrachtet. Dies sind Systeme, die aus drei Bestandteilen bestehen: einem filtrierten Wahrscheinlichkeitsraum, einem Wienerprozess und einer Lösung der Gleichungen. Die Diskretisierung ist so konstruiert, dass alle Elemente des Lösungssystems unter Anwendung stetiger Störungen der diskreten Iterationen konstruiert werden können. Konvergenz ohne Raten wird hierfür gezeigt. Zusätzlich werden für dieses Schema mittels allgemeiner Zufallsvariablen, die die Zeitinkremente der stochastischen Kraft approximieren, ähnliche Konvergenzeigenschaften gezeigt. Für zweidimensionale Gebiete konvergiert dasselbe Schema gegen die eindeutige starke Lösung, was unter Verwendung lokaler Monotonie gezeigt wird.

Im zweiten Teil werden Konvergenzeigenschaften von Projektionsverfahren studiert, die auf die instationären stochastischen Stokes Gleichungen angewandt werden. In diesem vereinfachten Zusammenhang wird untersucht, welchen Einfluss die Irregularität des Lagrange Multiplikator auf die Konvergenz hat. Hierzu wird auch die Konvergenz der finite Elemente Diskretisierung gezeigt.

Im dritten Teil der Arbeit werden Schemata für die zweidimensionale Navier-Stokes Gleichungen mit periodischen Randbedingungen untersucht, die auf finiten Elementen und dem impliziten Eulerverfahren basieren, und eine Konvergenzanalyse mit Raten durchgeführt. Da die Nichtlinearität nicht Lipschitz ist, kann Konvergenz im Allgemeinen nur bis auf eine Menge mit beliebig kleiner Wahrscheinlichkeit gezeigt werden. Im Falle additiven Rauschens wird dennoch Konvergenz auf dem ganzen Realisierungsraum für die Zeitdiskretisierung gezeigt. Dazu wird auch die finite Elemente Approximation betrachtet, und die Konvergenzanalyse für verschiedene Fälle durchgeführt.

In den jeweils anschliessenden Simulationen wird das Konvergenzverhalten der vorgestellten Algorithmen gezeigt und deren Leistungen verglichen.

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Contents

Chapter 1

Introduction

1.1 Stochastic Navier-Stokes Equations

We consider the system of equations describing the motion of incompressible fluids subject to a random force in a bounded polygonal domain $D \subset \mathbb{R}^d$, $d = 2, 3$. Let $\mathfrak{P} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, i.e., a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, together with a filtration $\mathbb{F} = \{\mathcal{F}\}_{t\geq 0}$ satisfying the usual assumptions, and $\mathcal{F}_t \subset \mathcal{F}$ for all $t \geq 0$. We assume that $\mathbf{W} \equiv \{\mathbf{W}(t)\ ;\ t \in [0, T]\}$ is a Wiener process that takes values in a suitable Hilbert space. Then we look for a random velocity vector field $\mathbf{u} : D_T \times \Omega \to \mathbb{R}^d$ and a random pressure scalar field $\pi : D_T \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \dot{\mathbf{u}} - \nu\Delta\mathbf{u} + [\mathbf{u} \cdot \nabla]\mathbf{u} + \nabla\pi = \mathbf{f} + \mathbf{g}(\mathbf{u})\dot{\mathbf{W}} \quad & \text{in } D_T \times \Omega\,, \\ \operatorname{div} \mathbf{u} = 0 \quad & \text{in } D_T \times \Omega\,, \end{aligned} \tag{1.1.1}$$

where $D_T := (0, T) \times D$, and the dot above $\mathbf{u}$ and $\mathbf{W}$ indicates the distributional derivative with respect to the variable t. This equations are supplemented with the initial condition

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0 \quad \text{in } D\,, \tag{1.1.2}$$

and with homogeneous Dirichlet or periodic boundary conditions. Depending on the type of initial condition (random variable or deterministic function), we will specify in each situation studied later in which sense equality (1.1.2) has to be understood. Here, $\mathbf{f} : D_T \to \mathbb{R}^d$ is an external force, and $\mathbf{g}$ is a strongly continuous operator valued map with linear growth. Its additional properties will be specified later, depending on the kind of problem we are considering. The above equations are known as the stochastic Navier-Stokes equations (briefly SNSEs), and may also be given in differential notation

$$\begin{aligned} d\mathbf{u} + \big(- \nu\Delta\mathbf{u} + [\mathbf{u} \cdot \nabla]\mathbf{u} + \nabla\pi\big)\, dt &= \mathbf{f}\, dt + \mathbf{g}(\mathbf{u})\, d\mathbf{W}\,, \\ \operatorname{div} \mathbf{u}\, dt &= 0\,. \end{aligned}$$

Setting $\mathbf{g} \equiv 0$, we obtain the deterministic Navier-Stokes equations (NSEs), which are used to describe the motion of an incompressible fluid in the domain D subject to the external force $\mathbf{f}$. This model was introduced by the French physicist C.L.M.H. Navier in 1822 and by the British mathematician G.G. Stokes in 1845 independently (it is worth noticing that also Poisson in 1830 and Saint-Vénant in 1843 derived these equations). The first proofs of the well-posedness of the problem appeared in the works of Odqvist [105] and of Leray [94, 95], while Hopf was the first to give a probabilistic description of fluid flows, obtaining in [73] the equation for the characteristic functional of the statistical solution[1]. Since then, the problem has been widely studied from both, the physical and mathematical point of view, and despite these efforts, there are still many open question of mathematical and physical nature. Actually, the most important problem is the question about the uniqueness of a weak solution in three space dimensions and the existence of global smooth solutions. Other open questions regard e.g. the analysis of singularities that may arise, or the full understanding of turbulent phenomena. Beyond the analytical studies, the development of computational methods for the NSEs has become another focus of the highest priority for the application of the mathematical theory. In particular, one of the main challenges is the efficient simulation of flows in turbulent regimes, which is crucially connected with the full understanding of turbulent phenomena. In this context, the introduction of a stochastic forcing ($\mathbf{g} \neq 0$) in (1.1.1) is an extension of the original model, and is meant to be a complementatory model to the NSEs, which may represent a helpful tool in the statistical description of turbulence, thanks to some additional properties of the corresponding solutions. For example, the two-dimensional SNSEs have a unique invariant measure, hence exhibit ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data: see [47]. Despite continuing efforts of many scientists during the last three decades, such a property, which could lead to profound understanding of the nature of turbulence, has not yet been proved for the deterministic equations.

From a mathematical point of view, the introduction of the noise is intended to recover properties that are not valid for the corresponding deterministic problem. In this direction, an important tool arising from the stochastic model (1.1.1) is that of stationary solution, namely a solution that is a stationary process, and thus its joint distribution is independent of the time. In general, the concept of stationary solution is useful for the quantitative knowledge of the statistics of the stationary regime of a turbulent flow, see [44, Remark 3.4]. In the specific case of the SNSEs, it is possible to show the existence of stationary solutions $\mathbf{u}_{stat}$, see e.g. [46, Section 4] that satisfy an energy inequality in a mean sense. As a consequence of the stationarity, the expectation of the solution is independent of the time, hence there holds

$$\mathbb{E}\left[\int_D \left|\nabla \mathbf{u}_{stat}(t,\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right] < \infty$$

for all $t \geq 0$. This implies that

$$\int_D \left|\nabla \mathbf{u}_{stat}(t,\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} < \infty$$

for all $t \geq 0$ and for almost every realisation of the noise. A corresponding property for the three-dimensional deterministic problem, where it would imply uniqueness, does not hold.

Another fact that makes SNSEs interesting, is that in general a stochastically forced model may lead to uniqueness results which are not available for the deterministic case at the moment. To support this opinion, we recall that there exist examples of ordinary differential equations which lack of uniqueness, e.g.,

$$\mathrm{d}u(t) = \sqrt{|u(t)|}\,\mathrm{d}t\,, \quad u(0) = 0\,,$$

but the same equation perturbed by a real-valued Wiener process W

$$\mathrm{d}u(t) = \sqrt{|u(t)|}\mathrm{d}t + \mathrm{d}W(t)\,, \quad u(0) = 0\,,$$

[1]The functional is given by

$$\mathbf{\Phi}_t[\mathbf{y}] = \langle \exp[i(\mathbf{u}(t),\mathbf{y})]\rangle\,,$$

where $(\cdot,\cdot)$ denotes the $\mathbb{L}^2$ scalar product, $\mathbf{u}$ is the velocity vector field, and $\langle\cdot\rangle$ denotes the average over all possible initial data. This functional is a tool to conveniently describe the probability distribution of $\mathbf{u}$ and its correlation moments (the latter by means of variational derivatives of the functional).

has a unique solution for almost every realisation of the noise. A similar remarkable result, but more related to stochastic hydrodynamics is obtained in [4] for the dyadic model

$$\begin{aligned} \mathrm{d}X_n(t) &= \big(k_{n-1}X_{n-1}^2(t) - k_n X_n(t)X_{n+1}(t)\big)\,\mathrm{d}t \quad t \geq 0\,, \\ X_n(0) &= x_n\,, \end{aligned} \tag{1.1.3}$$

for $n \geq 1$ with $k_n > 0$, $X_0 \equiv x_0 = 0$ and $\{x_n\}_{n\geq 1}$ square summable. This model shares some properties with the stochastic Euler equations ($\nu = 0$ in (1.1.1)), like the quadratic nonlinearity, and the fact that the energy is (formally) constant, namely

$$\sum_{n=1}^{\infty} X_n^2(t) \equiv C > 0\,.$$

It is known, see [5], that for this model there are initial conditions such that there exist at least two solutions with continuous components on some interval [0,T]. Let $\{W_n\}_{n=1}^{\infty}$ be a family of independent i.i.d. real valued Wiener processes, and $\circ$ denotes the Stratonovich noise; see the (1.2.2) for its meaning. If stochastic perturbation of the form

$$\sigma k_{n-1}X_{n-1}(t) \circ \mathrm{d}W_{n-1}(t) - \sigma k_{n+1}X_{n+1}(t) \circ \mathrm{d}W_n(t) \quad \sigma \neq 0\,,$$

is added to problem (1.1.3), uniqueness in law of the solution on the space $C([0,T];\mathbb{R})^{\mathbb{N}}$ can be shown for every initial data.

It is worth noticing that in [10], the author derives a model for turbulence driven by a fast unidirectional flow. The most important properties for this model are the existence and uniqueness of an invariant measure, which is then used to prove the scaling laws conjectured by Kolmogorov in [82], proved under the assumption of additive noise as external body force.

Turning to a physical point of view, the idea to study NSEs using additional stochastic terms mainly originates from the modeling of small fluctuations. This idea appears in [91, Chapter 17], where the authors consider the classical balance laws for mass, energy and momentum forced by a random noise, to describe the fluctuations, in particular local stresses and temperature, which are not related to the gradient of the corresponding quantities. The authors then derive correlations for the random forcing by following the general theory of fluctuations; see [90, Chapter 12]. In a different framework, Kolmogorov conjectures the use of randomly forced equations for turbulent flows, see [83], and since then, lots of model involving the Navier-Stokes equations and a stochastic forcing have been introduced and studied. A first method to justify the noise in the SNSEs is to consider the randomly forced NSEs as Langevin equations as in the classical theory of noises; see [127]. In fact, the random force method in the Lagrangian description (when the motion of a system of fixed liquid particles is traced) of turbulence was proposed in [103] to statistically describe the motion of fluid particles in turbulent flows. More precisely, Langevin stochastic equations, because of the analogies that the author finds between turbulence and molecular statistics, see [93, 107, 108], are used to describe the (relative) motions $\{\mathbf{u}_i\}_{i\geq 1}$ of the fluid particles, i.e.,

$$\dot{\mathbf{u}}_i(t) = -\lambda \mathbf{u}_i(t) + \mathbf{z}_i(t)\,,$$

where $\mathbf{z}_i$ is the random force (gaussian white noise) and $\lambda > 0$. This method is then naturally generalised in [104] to the Euler description of turbulence, i.e. a description of the velocity field $\mathbf{u}(t, \mathbf{x})$, leading to the NSEs with a random forcing term like in (1.1.1).

A second way to justify the random forcing is the Markovian random coupling model introduced in [49], which is a slight modification of the stochastic model introduced by Kraichnan in [87] for the random oscillator. The main idea from [49] consists in considering, instead of a single set of Navier-Stokes equations, a family of N sets of equations for N turbulent flows $\{\mathbf{v}_j\}_{j=1}^N$ that are coupled through the nonlinear terms $\big[\mathbf{v}_j\cdot\nabla\big]\mathbf{v}_j$. It is assumed that the quadratic terms fluctuate around their means $\boldsymbol{\mu}_j = \mathbb{E}\left[\big[\mathbf{v}_j\cdot\nabla\big]\mathbf{v}_j\right]$ as follows

$$\big[\mathbf{v}_j\cdot\nabla\big]\mathbf{v}_j = \boldsymbol{\mu}_j + \frac{1}{N}\sum_{k,l=1}^{N}\gamma_{jkl}\big[\mathbf{v}_k\cdot\nabla\big]\mathbf{v}_l\,,$$

where the coupling coefficients $\{\gamma_{jkl}(t)\}_{j,k,l=1}^N$ are independent, identically distributed, centered gaussian white noises symmetric with respect to j, k and l. Thus we get the system

$$\begin{aligned}\mathrm{d}\mathbf{v}_j + \big(-\nu\Delta\mathbf{v}_j + \boldsymbol{\mu}_j + \nabla\pi_j\big)\,\mathrm{d}t + \frac{1}{N}\sum_{k,l=1}^{N}\big([\mathbf{v}_k\cdot\nabla]\mathbf{v}_l\big)\circ\mathrm{d}z_{jkl} &= \mathbf{Q}^{1/2}\mathrm{d}\mathbf{W}\,,\\ \operatorname{div}\mathbf{v}_j &= 0\,,\end{aligned}$$

supplied with some initial condition, with i.i.d. central white noises $\{\mathrm{d}z_{jkl}\}_{j,k,l=1}^N$ and a Wiener process $\mathbf{W}$. The random forcing $\mathbf{Q}^{1/2}\mathrm{d}\mathbf{W}$ is needed to ensure the existence of stationary solutions. Note that the system is "doubly" stochastic, in the sense that additionally to the stochastic forcing $\mathbf{Q}^{1/2}\mathrm{d}\mathbf{W}$ there are the random coupling coefficients $\{\mathrm{d}z_{jkl}(t)\}_{j,k,l=1}^N$. This model is the motivation for the first rigorous analytical study of the SNSEs by Bensoussan and Temam in [8]. This model is also studied in [34, Section 4], but from the point of view of the characteristic functional.

1.2 Derivation of the model and existence results

To end the discussion about the modeling aspects related to the equations studied in the present work, we consider a general model for turbulence rigorously derived in [99, 100] and reported here for the sake of completeness. Let $\mathfrak{P} := \big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$ be a stochastic basis. We assume that all the random fields are regular enough to justify the manipulations below, and we work with scalar valued Wiener processes, to simplify the mathematical setting. We assume that the fluid we are dealing with is newtonian. Similarly to the random force method introduced in [103], for a real valued Wiener process W_1 on $\mathfrak{P}$ we assume that the fluid particle motion is described by the Stratonovich stochastic equation

$$\mathrm{d}\boldsymbol{\eta}(t,\mathbf{x}) = \mathbf{u}(t,\boldsymbol{\eta}(t,\mathbf{x}))\,\mathrm{d}t + \boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\mathbf{x}))\circ\mathrm{d}W_1\,,\quad \boldsymbol{\eta}(0,\mathbf{x}) = \mathbf{x}\in\mathbb{R}^d\,, \tag{1.2.1}$$

where $\mathbf{u}:[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}^d$ is an unknown random field, $\boldsymbol{\sigma}:[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}^d$ is a given non-random function. The idea of decomposing the velocity into a slow and a fast oscillating part has often been used in the study of turbulent flows. The interest in flows of the form (1.2.1) originates from recent developments in modeling a turbulent flow by a generalised Gaussian random field with zero mean and a special covariance function, see [100, Introduction], following the seminal work of Kraichnan on turbulent transport [88], and then developed in [55, 51]. We assume that for every fixed $t\geq 0$ the map $\boldsymbol{\eta}(t,\cdot):\mathbb{R}^d\to\mathbb{R}^d$ is a diffeomorphism; see [116, Section 5.2] for the necessary assumptions. Here is

$$\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot))\circ\mathrm{d}W_1 = \frac{1}{2}[\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot))\cdot\nabla]\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot))\,\mathrm{d}t + \boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot))\,\mathrm{d}W_1\,. \tag{1.2.2}$$

Let W_2 be another real valued Wiener process on $\mathfrak{P}$, independent of W_1, and assume that the random field $\mathbf{u}$ has the following form

$$\mathrm{d}\mathbf{u}(t,\cdot) = \boldsymbol{\alpha}(t,\cdot)\,\mathrm{d}t + \boldsymbol{\beta}(t,\cdot)\,\mathrm{d}W_1 + \boldsymbol{\gamma}(t,\cdot)\,\mathrm{d}W_2\,, \tag{1.2.3}$$

where $\boldsymbol{\alpha} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, $\boldsymbol{\beta} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ and $\boldsymbol{\gamma} : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are unknown random fields adapted to the filtration generated by the processes W_1 and W_2. To have the incompressibility of the fluid characterised by (1.2.1), we need that $\boldsymbol{\eta}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is volume preserving for every $t \in [0,T]$. This holds if and only if

$$\operatorname{div} \mathbf{u}(t,\cdot) = \operatorname{div} \boldsymbol{\eta}(t,\cdot) = 0 \quad (t \geq 0)\,, \tag{1.2.4}$$

together with the assumption of Stratonovich noise in (1.2.1); see [100, Remark 1]. To see this it is necessary to derive an equation for the Jacobian of $\boldsymbol{\eta}$, and adapt the computations from [33, Section 1.1] to the stochastic setting. Now, to derive the equations for the velocity field $\mathbf{u}$ we apply Newtonian mechanics to the velocity field

$$\mathbf{U}(t,\cdot) = \mathbf{u}(t,\cdot)\,\mathrm{d}t + \boldsymbol{\sigma}(t,\cdot) \circ \mathrm{d}W_1\,,$$

in order to find the equations for the balance of momentum and mass. Consequently, because of the Newton second law together with (1.2.1), for the total force $\mathbf{F}(t,\boldsymbol{\eta}(t,\cdot))$ applied to the fluid particle with trajectory $\boldsymbol{\eta}(t,\cdot)$ there holds

$$\mathrm{d}\big(\mathrm{d}\boldsymbol{\eta}\big)(t,\cdot) = \mathbf{F}(t,\boldsymbol{\eta}(t,\cdot))\,, \tag{1.2.5}$$

where we assumed that the mass density is 1. This implies

$$\int_0^T \phi(t)\mathbf{F}(t,\boldsymbol{\eta}(t,\cdot))\,\mathrm{d}t = -\int_0^T \dot{\phi}(t)\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot)) \circ \mathrm{d}W_1(t) + \int_0^T \phi(t)\,\mathrm{d}\mathbf{u}(t,\boldsymbol{\eta}(t,\cdot))$$

for all smooth functions ϕ with compact support on $\mathbb{R}$. By the Itô-Wentzell formula [116, Theorem 1.4.9] together with (1.2.2)

$$\begin{aligned}
\mathrm{d}\mathbf{u}(t,\boldsymbol{\eta}(t,\cdot)) \;=\;& \Big([\mathbf{u}(t,\boldsymbol{\eta}(t,\cdot)) \cdot \nabla]\mathbf{u}(t,\boldsymbol{\eta}(t,\cdot)) + \sum_{i,j=1}^{d} \boldsymbol{\sigma}_i(t,\boldsymbol{\eta}(t,\cdot))\mathbf{u}_{x_i x_j}(t,\boldsymbol{\eta}(t,\cdot))\boldsymbol{\sigma}_j(t,\boldsymbol{\eta}(t,\cdot))] \\
&+\sum_{j=1}^{n} \big[[\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot)) \cdot \nabla]\sigma_j(t,\boldsymbol{\eta}(t,\cdot))\mathbf{u}_{x_j}(t,\boldsymbol{\eta}(t,\cdot))\big]\Big)\mathrm{d}t \\
&+\boldsymbol{\alpha}(t,\boldsymbol{\eta}(t,\cdot))\mathrm{d}t + \Big([\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot)) \cdot \nabla]\mathbf{u}(t,\boldsymbol{\eta}(t,\cdot)) + \boldsymbol{\beta}(t,\boldsymbol{\eta}(t,\cdot))\Big)\,\mathrm{d}W_1 \\
&+\boldsymbol{\gamma}(t,\boldsymbol{\eta}(t,\cdot))\,\mathrm{d}W_2 + [\boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,\cdot) \cdot \nabla]\boldsymbol{\beta}(t,\boldsymbol{\eta}(t,\cdot))\,\mathrm{d}t\,.
\end{aligned}$$

We assume that there exist given random fields $\mathbf{d}$, $\mathbf{f}$, $\mathbf{q}$ and $\mathbf{h}$, from $[0,T] \times \mathbb{R}^d \times \Omega$ to $\mathbb{R}^d$, such that

$$\int_0^T \phi(t)\mathbf{F}(t,\cdot)\,\mathrm{d}t = \; -\int_0^T \dot{\phi}(t)\mathbf{d}(t,\cdot) \circ \mathrm{d}W_1 + \int_0^T \phi(t)\big(\mathbf{f}(t,\cdot)\,\mathrm{d}t + \mathbf{q}(t,\cdot)\,\mathrm{d}W_1 + \mathbf{h}(t,\cdot)\,\mathrm{d}W_2\big)\,.$$

The random fields $\mathbf{d}$, $\mathbf{f}$, $\mathbf{q}$ and $\mathbf{h}$ will be specified below. We use the invertibility of $\boldsymbol{\eta}(t,\cdot)$, and matching the similar terms we arrive at

$$\begin{aligned}
\mathrm{d}\mathbf{u}(t,\cdot) \;=\;& \Big([\mathbf{u}(t,\cdot) \cdot \nabla]\mathbf{u}(t,\cdot) + \sum_{i,j=1}^{d} \sigma_i(t,\boldsymbol{\eta}(t,\cdot))\mathbf{u}_{x_i x_j}(t,\boldsymbol{\eta}(t,\cdot))\sigma_j(t,\boldsymbol{\eta}(t,\cdot)) \\
&+\sum_{j=1}^{n} \big[[\boldsymbol{\sigma}(t,\cdot) \cdot \nabla]\sigma_j(t,\cdot)\mathbf{u}_{x_j}(t,\cdot)\big] + [\boldsymbol{\sigma}(t,\cdot) \cdot \nabla]\boldsymbol{\beta}(t,\cdot)\Big)\mathrm{d}t \\
&+\boldsymbol{\alpha}(t,\cdot)\mathrm{d}t + \Big([\boldsymbol{\sigma}(t,\cdot) \cdot \nabla]\mathbf{u}(t,\cdot) + \boldsymbol{\beta}(t,\cdot)\Big)\,\mathrm{d}W_1 + \boldsymbol{\gamma}(t,\cdot)\,\mathrm{d}W_2\,.
\end{aligned}$$

First, we observe that for $A_{i,j}(t,\cdot) := \sigma_i(t,\cdot)\sigma_j(t,\cdot)$ we have

$$\sum_{i,j=1}^{d} \big(A_{i,j}(t,\cdot)\mathbf{u}_{x_i}(t,\cdot)\big)_{x_j} = \sum_{i,j=1}^{d} \sigma_i(t,\cdot)\mathbf{u}_{x_i x_j}(t,\cdot)\sigma_j(t,\cdot) + \sum_{j=1}^{n} \big[[\boldsymbol{\sigma}(t,\cdot)\cdot\nabla]\sigma^j(t,\cdot)\mathbf{u}_{x_j}(t,\cdot)\big]\,,$$

since $\mathbf{u}$ is divergence-free. After some algebraic manipulations, we get

$$\begin{aligned} \boldsymbol{\gamma}(t,\cdot) &= \mathbf{h}(t,\cdot)\,, \quad \boldsymbol{\beta}(t,\cdot) = -[\boldsymbol{\sigma}(t,\cdot)\cdot\nabla]\mathbf{u}(t,\cdot) + \mathbf{q}(t,\cdot)\,, \\ \boldsymbol{\alpha}(t,\cdot) &= [\boldsymbol{\sigma}(t,\cdot)\cdot\nabla]\boldsymbol{\beta}(t,\cdot) - [\mathbf{u}(t,\cdot)\cdot\nabla]\mathbf{u}(t,\cdot) + \operatorname{div}\big(\mathbf{A}(t,\cdot)\nabla\mathbf{u}(t,\cdot)\big) + \mathbf{f}(t,\cdot)\,, \end{aligned}$$

which results in the equations

$$\begin{aligned} \mathrm{d}\mathbf{u}(t,\cdot) =& \Big(\sum_{i,j=1}^{d} \big(A_{i,j}(t,\cdot)\mathbf{u}_{x_i}(t,\cdot)\big)_{x_j} - [\mathbf{u}(t,\cdot)\cdot\nabla]\mathbf{u}(t,\cdot) + [\boldsymbol{\sigma}(t,\cdot)\cdot\nabla]\mathbf{q}(t,\cdot)\Big)\,\mathrm{d}t \\ & + \mathbf{f}(t,\cdot)\,\mathrm{d}t + \Big(\mathbf{q}(t,\cdot) - [\boldsymbol{\sigma}(t,\cdot)\cdot\nabla]\mathbf{u}(t,\cdot)\Big)\,\mathrm{d}W_1 + \mathbf{h}(t,\cdot)\,\mathrm{d}W_2\,. \end{aligned} \tag{1.2.6}$$

Now, we precise the random fields $\mathbf{f}$, $\mathbf{q}$ and $\mathbf{h}$. We assume that the fluid is viscous, and following the derivation of the Navier-Stokes equations in [33, Section 1.3], we note that (1.2.1) corresponds to the velocity field $\mathbf{U}(t,\cdot)$. Consequently, by the usual assumptions on the stress tensor for Newtonian flows, see [33, Page 32], we obtain that the stress tensor has the form

$$\mathbf{T}(t,\cdot) := \frac{\nu}{2}\Big(\nabla\mathbf{U}(t,\cdot) + \big[\nabla\mathbf{U}(t,\cdot)\big]^T\Big)\,, \tag{1.2.7}$$

with the kinematic viscosity $\nu > 0$. Hence, for a surface S in the fluid, the total force exerted across the surface S per unit area at $\mathbf{x} \in S$ at time t is given by

$$-\pi(t,\mathbf{x})\,\mathbf{I}\,\mathbf{n}(\mathbf{x}) + \big[\mathbf{T}(t,\mathbf{x}) + \mathbf{G}(t,\mathbf{x})\big]\mathbf{n}(\mathbf{x})\,,$$

where $\mathbf{G}$ is the external body force, π is the pressure, $\mathbf{n}(\mathbf{x})$ is the outer normal at the point $\mathbf{x} \in \partial S$ and $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the identity matrix. By the divergence theorem we obtain that the total force is given by

$$-\nabla\pi(t,\cdot) + \operatorname{div}\big[\mathbf{T}(t,\cdot) + \mathbf{G}(t,\cdot)\big]\,.$$

We decompose the pressure π and the force $\operatorname{div}\mathbf{G}$ following the structure of $\mathrm{d}\mathbf{u}$ obtaining

$$\begin{aligned} \pi(t,\cdot) &= \pi^{\mathbf{f}}(t,\cdot)\,\mathrm{d}t + \pi^{\mathbf{q}}(t,\cdot)\,\mathrm{d}W_1 + \pi^{\mathbf{h}}(t,\cdot)\,\mathrm{d}W_2\,, \\ \operatorname{div}\mathbf{G}(t,\cdot) &= \tilde{\mathbf{f}}(t,\cdot)\,\mathrm{d}t + \tilde{\mathbf{q}}(t,\cdot)\,\mathrm{d}W_1 + \tilde{\mathbf{h}}(t,\cdot)\,\mathrm{d}W_2\,, \end{aligned}$$

for given random fields $\tilde{\mathbf{f}}$, $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{h}}$. Thus, noting the Stratonovich noise in (1.2.1), on any piece of fluid material we have that the force per unit volume is given by

$$\mathbf{f}(t,\cdot)\,\mathrm{d}t + \mathbf{q}(t,\cdot)\,\mathrm{d}W_1 + \mathbf{h}(t,\cdot)\,\mathrm{d}W_2$$

with

$$\begin{aligned} \mathbf{f} &= \nu\Delta\mathbf{u}(t,\cdot) + \frac{\nu^2}{2}\sum_{i=1}^{d}\Delta\sigma_i(t,\cdot)\Delta\boldsymbol{\sigma}_{x_i}(t,\cdot) - \nabla\pi^{\mathbf{f}}(t,\cdot) + \tilde{\mathbf{f}}(t,\cdot)\,, \\ \mathbf{q} &= \tilde{\mathbf{q}}(t,\cdot) - \nabla\pi^{\mathbf{q}}(t,\cdot) + \nu\Delta\boldsymbol{\sigma}(t,\cdot) \\ \mathbf{h} &= \tilde{\mathbf{h}}(t,\cdot) - \nabla\pi^{\mathbf{h}}(t,\cdot)\,. \end{aligned}$$

Putting all together, and dropping the arguments below, we arrive at the following Navier-Stokes equations

$$
\begin{aligned}
\mathrm{d}\mathbf{u} = \Big(-[\mathbf{u}\cdot\nabla]\mathbf{u} + \sum_{i,j=1}^{d} \big(A_{i,j}\mathbf{u}_{x_i}\big)_{x_j} + \nu\Delta\mathbf{u} + \frac{\nu^2}{2}\sum_{i=1}^{d}\Delta\sigma_i(t,\cdot)\Delta\boldsymbol{\sigma}_{x_i}(t,\cdot)\Big)\mathrm{d}t \\
+ \Big([\boldsymbol{\sigma}\cdot\nabla]\big(\tilde{\mathbf{q}} - \nabla\pi^{\mathbf{q}} + \nu\Delta\boldsymbol{\sigma}\big) - \nabla\pi^{\mathbf{f}} + \tilde{\mathbf{f}}\Big)\,\mathrm{d}t \\
+ \Big(\tilde{\mathbf{q}} + \nu\Delta\boldsymbol{\sigma} - \nabla\pi^{\mathbf{q}} - [\boldsymbol{\sigma}\cdot\nabla]\mathbf{u}\Big)\mathrm{d}W_1 + \Big(\tilde{\mathbf{h}} - \nabla\pi^{\mathbf{h}}\Big)\,\mathrm{d}W_2\,, \\
\operatorname{div}\mathbf{u} = 0\,.
\end{aligned}
\tag{1.2.8}
$$

As we see, this model is more general than equations (1.1.1), as it includes a noise which depends on the gradient of the velocity, a general diffusion term arising from the Lagrange formulation with Stratonovich noise, and some other (deterministic) forcing terms. It is clear, that if we set $\boldsymbol{\sigma}\equiv\boldsymbol{\beta}\equiv\boldsymbol{\gamma}\equiv\mathbf{0}$, we obtain the deterministic NSEs. Otherwise, by setting $\boldsymbol{\sigma}\equiv\boldsymbol{\beta}\equiv\mathbf{0}$ we obtain the following SNSEs

$$
\begin{aligned}
\mathrm{d}\mathbf{u} + \big(\nu\Delta\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla\pi^{\mathbf{f}}\big)\,\mathrm{d}t &= \tilde{\mathbf{f}} + \big(\tilde{\mathbf{h}} - \nabla\pi^{\mathbf{h}}\big)\mathrm{d}W_2\,, \\
\operatorname{div}\mathbf{u}\,\mathrm{d}t &= 0\,,
\end{aligned}
$$

which can be identified with (1.1.1), if we use an operator valued random field $\mathbf{h}$, an Hilbert space valued Wiener process W_2, and define $\pi = \pi^{\mathbf{f}}\,\mathrm{d}t + \pi^{\mathbf{h}}\,\mathrm{d}W_2$. If we are interested in an infinite dimensional forcing, we observe that we may have considered a field $\mathbf{u}$ of the form

$$
\mathrm{d}\mathbf{u}(t,\cdot) = \boldsymbol{\alpha}(t,\cdot)\,\mathrm{d}t + \sum_{i=1}^{\infty}\boldsymbol{\beta}_i(t,\cdot)\,\mathrm{d}W_1^i + \sum_{j=1}^{\infty}\boldsymbol{\gamma}_j(t,\cdot)\,\mathrm{d}W_2^j\,,
\tag{1.2.9}
$$

for suitable families of random fields $\{\boldsymbol{\beta}_i\}_{i=1}^{\infty}$, $\{\boldsymbol{\gamma}_i\}_{i=1}^{\infty}$ and independent families of scalar Wiener processes $\{W_j^i\}_{i=1}^{\infty}$, $j=1,2$. This would imply a more general model, leaving unchanged the considerations of the present section. A comparable derivation of the SNSEs is given in [17], where the authors consider equation (1.2.1) with a deterministic velocity field $\mathbf{u}$ and Itô noise. However, the SNSEs derived in that paper may be considered as a special case of the equations (1.2.8). Another derivation of the SNSEs model can be found in [86], where the SNSEs are obtained analysing a stochastic microscopic model underlying the two-dimensional NSEs for the vorticity of a viscous incompressible flows; see also [7, Sections 2.1 and 2.2] for a simplified derivation.

The first mathematically rigorous approach to the SNSEs is given in [8], where the authors study the SNSEs with constant diffusion coefficient $\mathbf{g}$ and one-dimensional Wiener process. There, a solution is defined as a random variable with values in a suitable Banach space, satisfying the (deterministic) NSEs for almost every realisation of the noise. Then, by means of a theorem on measurable sections, the existence is shown, and an energy inequality in the mean sense is also given, but the question of the uniqueness is left open, due to the interplay of nonlinearity and stochastics. Since then, this problem has been widely studied and several approaches have been proposed. The main problem in getting existence results for the SNSEs, is given by the difficulty, in most cases, to apply the tools that are used in the deterministic theory. As an example, in the deterministic case a compactness theorem is needed to show that there exists a strongly convergent subsequence in the set of solutions to the approximating problems. This argument requires uniform bounds for the time derivative of the approximating sequence, which are not available for the SNSEs due to the irregularity in time of the driving Wiener process. Thus, to prove existence, non-trivial estimates on the modulus of continuity of the approximating solutions, see e.g. [129] or the factorisation method for stochastic integrals, see e.g. [37], are needed. An important tool to get compactness in the stochastic case is given by the embedding theorems for fractional Sobolev spaces proved in [46, Section 2], see also Section 3.2.3. This theorems allow a treatment of the SNSEs that is similar to the one of the corresponding deterministic equations, and is easier than the estimation of the modulus of continuity of the approximating solutions.

Another question related to the SNSEs is the uniqueness of the solutions. In fact, the uniqueness is proved only for the two dimensional case in [120]. In the stochastic setting, again, it is not possible to apply the uniqueness proof from the deterministic theory, due to the the energy estimate, which holds only in a mean sense. As a consequence, to obtain uniqueness it is necessary to multiply the equation for the difference of two solutions with a positive integrating factor, which compensates the effects caused by the interplay of nonlinearity and stochastics. Even though the uniqueness of a solution to the three-dimensional problem is still an open problem, a striking result has been obtained in [36], where the existence of a measurable selection of solutions that depend continuously on the initial condition is proved. There is no equivalent result for the deterministic problem, and this represents an important step towards the proof of the uniqueness.

Let us cite some existence results that are methodologically connected with the present work. The existence of global weak martingale solutions of (1.1.1) which satisfy an energy inequality is shown in [46] in the case of a bounded domain and a noise which depends on the solution. For the proof a Faedo-Galerkin method is used together a compactness argument. In [101] the same is proved for the general model (1.2.8) in an unbounded domain using a regularisation procedure, and the result is extended to prove existence and uniqueness of a strong solution in the two-dimensional case. Since the domain is unbounded, the compactness argument has to be appropriately modified. Another existence proof, limited to the two-dimensional case can be found in [98], where a local monotonicity method is used to obtain existence only by weak convergence, and thus without any compactness theorem. The argument is then applied also in [97], where the same method is applied to the SNSEs with artificial compressibility.

Other existence proofs include [18, 7, 11, 22, 23, 24, 25, 26, 115, 119, 84, 129], and the methods used are, among others, semigroup theory, nonstandard analysis or the application of the Yamada-Watanabe theorem from [114].

1.3 Numerical Schemes for the SNSEs

Stochastic partial differential equations (SPDEs) are motivated, beyond stochastic hydrodynamics, by such phenomena as wave propagation in random media, nonlinear optics, phase separation models, neurophysiology and population biology. Thus, also the interest in the discretisation of such equations has grown in the last two decades, giving rise to a rich literature about the numerical analysis of SPDEs.

The first steps towards a suitable discretisation arise by using practical methods in an abstract setting. In fact, Rothe's method is considered in [57] for linear equations, in [77, 13] for equations with maximal monotone operators, and in [96] for an equation involving monotone operators, but in all papers it is used to show existence of a solution. To our knowledge, the numerical analysis of SPDES started with the paper [9], where a splitting method suggested by the Lie-Trotter product formulas for SPDEs is used. This method decomposes the SPDEs in a deterministic PDEs, and a stochastic equation not involving a differential operator, which are simpler for numerical computations, and are solved successively. Convergence (without rate) is proved, and an estimate of the approximation error is given for a particular case.

The global discretisation error is estimated first in [55], for a class of SPDEs on smooth domains with strongly monotone operator with positive eigenvalues in its drift, and scalar-valued

Wiener process. Using the spectral decomposition, a set of ordinary stochastic differential equations is recovered, and this is then discretised in time by a strong Itô-Taylor method, see [81]. The derived error estimate depends on the time-step and on the biggest eigenvalue considered in the spectral decomposition, and evidences a coupling between the time and the space discretisation parameters. This work was preceded by [65], where the one-dimensional heat equation with a nonlinear measurable term with linear growth, and an additive space-time white noise is studied. The authors consider only time discretistion of the corresponding solution by an implicit Rothe's method. Using the spectral decomposition of the Laplace operator on the unit interval, uniform Hölder continuity of the piecewise affine interpolation of the discrete solution is proved. This leads, using tightness of the laws of the corresponding discrete solutions, to the uniform convergence in probability of the solutions to the discretisation scheme. The result was then generalised in [66], where the authors consider a nonlinear Lipschitz continuous term in the drift and multiplicative noise with Lipschitz coefficient. Strong L^p-convergence with rates is obtained getting advantage of an estimate for the L^p-norm of the discrete solution, which is proved by Malliavin calculus. By relaxing the conditions on the nonlinear coefficients, convergence in probability is proved, extending the result of [65] to multiplicative noise. The space-time discretisation of the same problem is then considered in [60, 61], where a finite difference scheme for the space discretisation is considered, and both, implicit and explicit schemes are used. Under various assumptions on the nonlinear coefficients corresponding convergence types are proved. E.g., for Lipschitz continuous coefficients, strong L^p-convergence with rate is proved, while under the assumptions of measurability and local boundedness convergence in probability without rates holds. The proof relies on L^p-estimates for the Green function related to the Laplace operator, and on estimates for the corresponding convolutions.

An important generalisation of these results is given in [111], where the time discretisation of SPDEs under very general assumptions is considered. The equation is assumed to have a drift consisting of a linear operator, which generates an analytic semigroup, and a nonlinear term, together with a Lipschitz continuous diffusion term. First, the author considers a Lipschitz continuous nonlinearity to prove strong convergence in L^p with rates using tools from semigroup theory, and energy estimates for solutions. Then, the corresponding problem with a locally Lipschitz nonlinearity is studied. In this case only convergence in probability with rates can be shown. The proof relies on the Lipschitz truncation of the nonlinearity, in order to get advantage of the strong convergence. Then, the distance between the solution of the truncated problem and the original one is estimated using regularity properties of corresponding solutions. Despite the generality of the tools used in the proof, it is worth noticing that the methods used in the proof are specialised on the case of the one-dimensional Burgers equation, and thus if we want to apply the same tools, we need analogous properties for the solution of the problem we are studying. Another general discretisation procedure is given in [69], where general semilinear SPDEs with Lipschitz coeffcients are studied. The explicit Euler, the implicit Euler, and the Cranck-Nicholson scheme are investigated together with a spatial discretisation based on finite differences, spectral decomposition or wavelet approximation. Strong convergence with rates is proved under general assumptions on the related linear and nonlinear operators, obtaining a flexible discretisation which can be applied to a wide range of equations. This approach is then extended to SPDEs driven by a Banach space valued Wiener process in [68].

As regards the finite element method, we may cite the paper [1]. There, the discretisation of elliptic and parabolic SPDEs is studied by means of finite element and finite difference methods. Using the weak formulation of the problem, and the corresponding formulation obtained with the Green function, the authors show strong convergence with rates in L^2. A key tool in the proof is also an appropriate treatment of the Wiener process, whose approximation is taken to be piecewise constant in time and space on the discretisation intervals. Another important paper which studies the finite element discretisation is [132], where a numerical scheme based on piecewise affine finite elements and implicit Euler is studied for the stochastic heat equation with both, additive and multiplicative noise. The author derives new error estimates for the corresponding deterministic problem using semigroup theory, and deriving regularity estimates for the exact solution, optimal error estimates are proved. This result also includes the case where the equation is driven by an additive space-time white noise, and gives an error estimate for the error induced by the truncation in the series representation of the Wiener process.

Another series of papers which considers the discretisation of SPDEs is [62, 63, 64], where the authors study the convergence properties of discretisation schemes applied to SPDEs with strongly monotone coefficients. The considered noise is finite dimensional, but can be easily extended to an infinite dimensional one, if a Hilbert-Schmidt integrand is considered. In [62], implicit and explicit time discretisation schemes together with a general space discretisation are considered for equations with strongly monotone operators satisfying a polynomial growth condition, framework which includes, e.g., the p-Laplacian in the drift. Under some consistency assumptions which specify approximation properties for corresponding discrete operators and coefficients, the weak L^2-convergence without rates towards the unique solution is proved using variational methods. In [63] the time discretisation is applied to equations with monotone operators, but linear growth. Again, assuming appropriate approximation properties for the discrete versions of the corresponding operators and coefficients, strong convergence with rates is proved. The strong convergence with rates is then proved for space-time approximation in [64], for implicit and explicit schemes, and including space discretisations like finite differences, wavelets and finite elements.

Focusing on more general equations, which do not fit into the setting of equations with Lipschitz continuous coefficients or monotone operators, we can cite [80], where a finite element discretisation of the Allen-Cahn equation is considered, and the techniques from [1] are applied to the new problem setting to obtain numerical simulations. Another interesting example of discretisation is given by [38], where an implicit temporal discretisation is applied to the nonlinear Schrödinger equation driven by a Stratonovich noise. The authors consider a non-monotone nonlinearity with polynomial growth, and prove convergence in probability by a compactness argument, which is related to that used in Chapter 3. For the approximation of problems with coefficients which are not globally Lipschitz, we can also cite [78], where the order of convergence is estimated for every fixed realisation of the solution. The methods used in the proof base on the proof of strong convergence under the assumption of Lipschitz continuity, and then by applying a localisation technique for one sample path on a bounded set.

The list of the works which have studied numerical approximations of SPDEs is far from being exhaustive, but the works we cited here represent, to our opinion, some of the most important steps in the understanding of numerical methods for SPDEs.

Despite the growing interest about the theory of SNSEs and the discretisation of SPDEs, the question about numerical approximations of SNSEs is virtually untouched. For practical purposes, e.g. weather prediction or climatology, approximation of solutions over long time periods is crucial. It is believed that at large times the trajectories will typically stay on or near the attracting sets, and their distributions are characterised by the invariant measures. Therefore, the knowledge of the invariant measures and attractors as well as their numerical approximations is very often essential for practical issues. However only few works which are related to some type of approximation of (1.1.1) have been written, most of them not always in the framework of an implementable discretisation. In our opinion, the most important is [48], where a time discretisation of the SNSEs with additive noise is considered. Convergence in probability, in distribution and almost sure is proved for the implicit Euler method to approximate the solution of the SNSEs, splitted in an Ornstein-Uhlenbeck equation and a two-dimensional Navier-Stokes with stochastic coefficients. Then in [56] is considered the approximation of the noise by smooth functions, also called Wong-Zakai approximation, for the two-dimensional SNSEs, result which is extended to the three-dimensional case in [130]. In [12] the approximation of the SNSEs by an abstract Galerkin method is considered, and mean square convergence to the strong solution is proved applying properties of stopping times and convergence principles from functional analysis. A striking result has been obtained in [41], where convergence with rates for a weak approximation of the SNSEs is studied for the two-dimensional equations on the torus. The author uses a weighted space to obtain a generalised Feller condition, a spectral method for the vorticity equation, together with a Strang splitting for the corresponding semigroup, and a cubature method to approximate the stochastic integral. Weak convergence with rates is obtained under a coupling of the discretisation parameters and numerical experiments are provided.

Because of the motivations listed above, we think that a study of the discretisation of the SNSEs should be carried out. The objective of this work is twofold:

1. First, we provide a fully practical numerical discretisation of the equations (1.1.1), which can be used to study physically relevant phenomena, like turbulence. Our studies may provide tools for future studies of invariant measures to equations (1.1.1).

2. Second, for the discretisation of (1.1.1) we develop technical tools that may be applied to a wide range of nonlinear SPDEs with non-Lipschitz nonlinearity, or problems which possess only weak martingale solutions.

1. Construction of weak martingale solutions (Chapter 1). The first problem we address is the approximation of the weakest concept of solution, since, in the three-dimensional case, the only available concept of solution is that of weak martingale solution, which is a system consisting of a stochastic basis, a Wiener process and a solution, both defined on the stochastic basis. To construct a weak martingale solution we consider a pair of finite element spaces $(\mathbb{H}_h, L_h)$ satisfying the discrete Ladyzhenskaya-Babuska-Brezzi (LBB) condition

$$\sup_{\mathbf{\Phi}\in\mathbb{H}_h} \frac{(\operatorname{div}\mathbf{\Phi}, \Pi)}{\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}} \geq C\|\Pi\|_{L^2}\,, \tag{1.3.1}$$

with a constant $C > 0$ independent of the mesh size $h > 0$. Then we fix a stochastic basis $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a Wiener process $\mathbf{W}$ on it. Let $I_k := \{t_m\}_{m=0}^M$ be an equi-distant mesh of size $k > 0$ covering the time interval $[0, T]$, where $T \equiv t_M$. Then define $\mathbf{f}^m := \mathbf{f}(t_m)$, and $\Delta_m\mathbf{W} := \mathbf{W}(t_m) - \mathbf{W}(t_{m-1})$, for $m \geq 1$. We consider the following finite element scheme based on the implicit Euler method.

Algorithm 1.1. *Let $\mathbf{U}^0 \in \mathbb{H}_h$ be given. For every $m \in \{1, \ldots, M\}$ find an $\mathbb{H}_h \times L_h$-valued random variable $\big(\mathbf{U}^m, \Pi^m\big)$ such that for all $(\mathbf{\Phi}, \Lambda) \in \mathbb{H}_h \times L_h$,*

$$\begin{aligned} &\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \mathbf{\Phi}\big) + k\nu\big(\nabla\mathbf{U}^m, \nabla\mathbf{\Phi}\big) - k\big(\Pi^m, \operatorname{div}\mathbf{\Phi}\big) + k\big([\mathbf{U}^m\cdot\nabla]\mathbf{U}^m, \mathbf{\Phi}\big)\\ &\qquad + \frac{k}{2}\Big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \mathbf{\Phi}\Big) = k\big\langle\mathbf{f}^m, \mathbf{\Phi}\big\rangle + \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{\Phi}\big)\,,\\ &(\operatorname{div}\mathbf{U}^m, \Lambda) = 0\,. \end{aligned} \tag{1.3.2}$$

The existence of a sequence of random variables $\{\mathbf{U}^m\}_{m=1}^M$ is ensured by a variant of the Browuer fixed point theorem, while the existence of $\{\Pi^m\}_{m=1}^M$ follows by the discrete LBB condition. The measurability of $\mathbf{U}^m$ with respect to $\mathcal{F}_{t_m}$ follows by applying the methods from [38]. The next key tool are the following a priori estimates

$$
\begin{aligned}
&\mathbb{E}\Big[\max_{1\le m\le M}\frac{1}{2}\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 + k\nu\sum_{m=1}^{M}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2 + \frac{1}{2}\sum_{m=1}^{M}\|\mathbf{U}^m-\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\Big] \le C_T\,,\\
&\mathbb{E}\Big[\max_{1\le m\le M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^p} + k\nu\sum_{m=1}^{M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^{p-1}}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big] \le C_{T,p}\,,\\
&\mathbb{E}\Big[\Big(k\sum_{m=1}^{M}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)^{2^{p-1}}\Big] \le C_{T,p}\,.
\end{aligned}
\tag{1.3.3}
$$

Since the problem is nonlinear, we need a compactness argument in order to pass to the limit. This is accomplished in several steps. With the help of the a priori estimates, we prove the following bounds:

$$
\begin{aligned}
&\mathbb{E}\Big[k\sum_{m=0}^{M-\ell}t_\ell^{3/4}\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|_{\mathbb{L}^2}^2 + \|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}^2\Big] \le C_T t_\ell\,,\\
&\mathbb{E}\Big[k\sum_{m=0}^{M-\ell}\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}^p\Big] \le C_{T,p}t_\ell^{p/2}\,.
\end{aligned}
\tag{1.3.4}
$$

Thanks to the compact embeddings of Sobolev spaces of fractional order from [46, Section 2], the inequalities (1.3.4) imply compactness of the piecewise affine interpolation of the iterates $\{\mathbf{U}^m\}_{m=1}^M$ in the space

$$L^2(0,T;\mathbb{L}^2)\cap C([0,T],D(\mathbf{A}^{-\gamma})),\quad \gamma>1\,.$$

Now it is possible to apply the Skorokhod almost sure representation theorem, to extract an a.s. convergent subsequence of approximating solutions on a new stochastic basis $\mathfrak{P}'$. Using the energy estimates and the resulting uniform integrability, it is possible to show the strong convergence of the subsequence on $\mathfrak{P}'$, and thus the existence of a process $\mathbf{u}$ that solves the SNSEs.

The last step in the construction of the weak martingale solution is the construction of the Wiener process. This is accomplished using an almost sure representation theorem, which allows to “transfer” the old Wiener process on the new stochastic basis. The identification of the stochastic integral follows by using the related equation to express it by deterministic integrals involving a continuous perturbation of the discrete solution. Using the convergence properties of the approximating solutions is then possible to show that the limiting term corresponding to the stochastic integral is a square integrable martingale with quadratic variation process $\mathbf{R}$, where

$$\mathbf{R}_t = \int_0^t \mathbf{g}(\mathbf{u}(s))\mathbf{Q}\mathbf{g}^*(\mathbf{u}(s))\,\mathrm{d}s\,.$$

The identification of the stochastic integral then follows by a representation theorem for martingales.

We introduce then a second method to construct weak martingale solutions. The main difference with respect to the just enumerated procedure is the approximation of the Wiener process. In fact, we replace the brownian increments by general random variables, for instance bounded, whose first and second moments coincide with those of the Wiener process. Moreover, we need that the estimates for higher moments of the increments are consistent with the ones of the Wiener process. The method of construction is not very different as regards the construction of the process $\mathbf{u}$, solution to the SNSEs, but in the identification of the stochastic integral more work is needed. Since we approximate the (continuous) stochastic integral by a discrete time (discontinuous) process, to show that the limiting object corresponding to the discrete time process is a martingale with the desired quadratic variation, we can not use the techniques listed above. To accomplish this task we get advantage of a general theorem about the convergence of discrete time quadratic variation processes and corresponding martingales, which is proved in Appendix B.

2. Construction of strong solutions (Chapter 1).

For the two-dimensional Navier-Stokes equations, we prove that Algorithm 1.1 produces the approximation of a strong solution. To this end, we employ a local monotonicity argument developed in the papers [98, 97], where it is shown that

$$\Big(\mathbf{G}(\mathbf{u}) - \mathbf{G}(\mathbf{v}), \mathbf{u} - \mathbf{v}\Big) + \frac{27r^4}{2\nu^3}\|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2}^2 \geq \frac{\nu}{2}\|\nabla(\mathbf{u} - \mathbf{v})\|_{\mathbb{L}^2}^2 \tag{1.3.5}$$

for the nonlinear operator

$$\mathbf{G}: \ \mathbf{u} \mapsto -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla + \frac{1}{2}\mathrm{div}\,\mathbf{u})\mathbf{u}\,.$$

To get advantage of this result in the continuous setting, the solution of the Galerkin system corresponding to the SNSEs is multiplied with an appropriate process, in order to handle the second term on the left-hand side in (1.3.5) with Itô's formula, and use the Minty trick to identify the limits in the nonlinear terms. The principal advantage of this approach is that it prevents the need of strong convergence, and thus the use of compactness arguments.

First we consider the case of additive noise, in order to clarify the techniques. The local monotonicity argument needs the Itô lemma to be properly applied in the continuous setting, but in the discretisation we may only use piecewise constant approximations of the solution. Thus the Itô lemma may not be applied as in the continuous case. To prove the desired convergence properties, we apply the calculus rules for the discrete derivatives from Appendix A together with a Taylor expansion. Then we have to carefully handle all the residual terms, show that they vanish in the limit, and by the Minty trick we are able to identify the limit of the weakly convergent sequence.

This arguments are then applied to the case of multiplicative noise, for a Lipschitz diffusion coefficient, under a mild assumption on the Lipschitz constant.

3. Splitting schemes for the Stokes equations and rôle of Lagrange multiplier (Chapter 2).

In this chapter we investigate the approximation properties of efficient splitting scheme arising from the deterministic theory; see [31, 125]. Their application to stochastic problems may appear straightforward, but as we will explain, that is not the case. To better understand the differences between the stochastic and deterministic case, let us explain the rôle of the pressure in the context of SNSEs.

The pressure is defined as the Lagrange multiplier corresponding to the divergence-free constraint of the NSEs, which ensures that the resulting velocity field is incompressible. Despite its importance in applications, in the usual existence proofs of deterministic NSEs, the pressure is usually neglected by projecting the equations on the space of solenoidal vector fields using the Leray-projection. Thus, the concept of weak solution to the NSEs does not include the pressure explicitly, but it can be constructed as a distribution by means of the De Rahm theorem ; see e.g. [126]. However, using potential theoretical methods it is possible to show that for the pressure related to a weak solution the pressure possesses the regularity

$$\pi \in L^{5/3}(D_T)\,,$$

for bounded and unbounded domains $D \subset \mathbb{R}^3$; see [123, Theorem 2.2].

For SNSEs the situation is slightly more complicated. Consider the equations (1.1.1). We may notice, following to the derivation of the SNSE (1.2.8), that the pressure in (1.1.1) is written as

$$\pi(t,\cdot) = \pi^{\mathbf{f}}(t,\cdot)\,\mathrm{d}t + \pi^{\mathbf{g}}(t,\cdot)\,\mathrm{d}W_2\,, \tag{1.3.6}$$

where $\pi^{\mathbf{g}}$ results from the random forcing term. This suggests us that the pressure is highly irregular, since $\pi^{\mathbf{g}}\,\mathrm{d}W_2$ corresponds heuristically to the distributional derivative of the noise. This observation is then confirmed by the proof of the high roughness of the pressure given in [92], where it is proved that

$$\pi \in L^1\Big(W^{-1,\infty}\big(0,T;W^{1,2}(D)/\mathbb{R}\big)\Big)\,, \tag{1.3.7}$$

if the noise takes values in $\mathbb{L}^2$. This roughness may cause drawbacks (and in fact it has) in both, the theory and the computations of discrete solutions. Namely, in the deterministic case, see e.g. [112, 113], higher regularity is needed to show optimal approximation properties for splitting schemes. It is worth noticing, that for a solenoidal stochastic forcing, formal computations motivates the estimate

$$\mathbb{E}\left[\int_0^T \|\nabla\pi\|_{\mathbb{L}^2}^2\,\mathrm{d}t\right] \leq C\,.$$

Now that the subtle interplay of stochastics and Lagrange multiplier has been evidenced, we may explain its influence on the convergence results for splitting schemes. We analyse the Chorin scheme from [31] applied to the unsteady Stokes equations with homogeneous Dirichlet boundary conditions. This is a method which splits the computation of velocity and pressure in two subsequent iterations, and is based on the pressure stabilisation equation

$$\operatorname{div}\mathbf{u} - \varepsilon\,\Delta\pi = 0 \quad (\varepsilon > 0)\,,$$

to avoid the saddle point character of the problem. The algorithm is as follows:

Algorithm 1.2. 1. Let $1 \leq m \leq M$. For given $\mathbf{u}^{m-1} \in L^2(\Omega,\mathbb{V})$ and $\widetilde{\mathbf{u}}^{m-1} \in L^2\big(\Omega,\mathbb{W}_0^{1,2}(D)\big)$, find $\widetilde{\mathbf{u}}^m \in L^2\big(\Omega,\mathbb{W}_0^{1,2}(D)\big)$ such that

$$\big(\widetilde{\mathbf{u}}^m - \mathbf{u}^{m-1}\big) - k\,\Delta\widetilde{\mathbf{u}}^m = k\,\mathbf{f}^m + \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})\Delta\mathbf{W}_m \quad \text{in } D\,.$$

2. Compute $\mathbf{u}^m \in L^2\big(\Omega,\mathbb{V}\big)$, and $p^m \in L^2\big(\Omega,W^{1,2}(D)/\mathbb{R}\big)$,

$$\begin{aligned} \mathbf{u}^m - \widetilde{\mathbf{u}}^m + k\,\nabla p^{m+1} &= 0\,, \qquad \operatorname{div}\mathbf{u}^m = 0 \qquad \text{in } D\,,\\ \langle\mathbf{u}^m,\mathbf{n}\rangle = 0 &\qquad \text{on } \partial D\,. \end{aligned}$$

The last step may be reformulated as a Poisson problem for the approximating pressure, in order to gain computational efficiency. To understand the structure of error, we reformulate the scheme as a pressure stabilisation scheme with semi-explicit treatment of the pressure approximations

$$\begin{aligned} \big(\widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^{m-1}\big) - k\,\Delta\widetilde{\mathbf{u}}^m + k\,\nabla p^{m-1} &= k\,\mathbf{f}^m + \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})\Delta\mathbf{W}_m && \text{in } D\,,\\ \operatorname{div}\widetilde{\mathbf{u}}^m - k\,\Delta p^m &= 0 && \text{in } D\,,\\ \partial_{\mathbf{n}} p^m &= 0 && \text{on } \partial D\,, \end{aligned}$$

and $\widetilde{\mathbf{u}}^0 \equiv \mathbf{u}_0$ in D. This formulation may be seen as a perturbation of the implicit Euler scheme, and allows to identify all the sources of the error, which are the pressure stabilisation constraint $(k > 0)$ and the explicit treatment of the pressure approximations.

The error analysis uses the splitting of the error into every single contribution (pressure stabilisation, semi-explicit treatment of the pressure) by introducing auxiliary approximation schemes. Thus, by measuring the error between each subsequent scheme, we arrive at the final error estimate, which quantifies all the sources contributing to the error between the solution to the implicit Euler scheme for the unsteady stochastic Stokes equations, and the solution to Algorithm 1.2. Due to the pressure stabilisation, the error equation contains the pressure approximations, which then crucially affects the error estimates. In fact, to obtain the error estimates we need that corresponding pressure approximation satisfy

$$\mathbb{E}\left[k\sum_{i=1}^{M}\|\nabla p^m\|_{\mathbb{L}^2}^2\right] \leq C$$

uniformly in $k > 0$. This property is then transferred to the pressure iterates resulting from the auxiliary problems by the corresponding error equations. As one can maybe note, due to the decomposition (1.3.6) and the regularity property (1.3.7), the above bound is valid only if the noise takes values in a space of divergence-free functions, leading to optimal convergence only for such a stochastic perturbation. This result is then confirmed by the numerical computations, where Algorithm 1.2 is applied to equations with both, solenoidal and non-solenoidal noise, showing optimal convergence behavior for the former and suboptimal for the latter.

To obtain, independently of the type of noise, an optimal splitting-method from Algorithm 1.2 it is necessary to add an additional step which projects the noise increment onto a space of solenoidal functions, preventing the pressure field to be too rough. Then, the other steps of the algorithm remains unchanged, except for the calculation of the pressure approximation, where the resulting Lagrange multiplier eliminated by the first projection step is summed to the resulting pressure approximation. The resulting scheme is then called stochastic Chorin scheme, and shows optimal convergence properties again.

In addition to the proposed splitting schemes, we study their finite elements approximation by a general pair of approximation spaces. According to the deterministic theory, the finite element space need not to be LBB-stable, because of the pressure stabilisation, which prevents the divergence-free constraint. Thus, the choice of an unstable pairing is allowed, if the following coupling of time and space discretisation parameters holds:

$$k \geq Ch^2\,.$$

Even if we are not able to prove any error bound for the pressure, in the numerical experiments we observe that the expectation of the pressure error is subject to the phenomenon of boundary layers, due to the unphysical boundary conditions from the second step of Algorithm 1.2. The results for the expectation are similar to the corresponding ones for the deterministic problem.

The studies performed in this chapter shows that the application of deterministic methods to stochastic problem has to be accomplished carefully, due to the roughness of the noise, and its interplay with the Lagrange multiplier. This consideration becomes even more crucial in the study of the next problem.

4. Discretisation of the stochastic Navier-Stokes equations (Chapter 3).

In this last part, we consider algorithms from the first two parts, applied to the two-dimensional SNSEs with periodic boundary conditions and multiplicative noise, to obtain convergence with rates. The results from this chapter show that, as in the study of splitting methods, it is not possible just to apply the approximation methods from the deterministic theory, because of various effects caused by the subtle interplay of stochastic, nonlinearity and Lagrange multiplier. This problem is here amplified by the presence of the convection term, which, due to its non-Lipschitz character, is responsible for the resulting error estimates, that are valid only of a subset of the realisation set Ω. Objective of this part is the study of Algorithms 1.1 and 1.2 applied to the two-dimensional SNSEs with periodic boundary conditions.

We start with a fully implicit Euler based time discretisation. In order to perform the error analysis, we have to prove higher regularity properties of the exact solutions. This is possible thanks to the particular problem setting, for which there holds

$$\big([\mathbf{u}\cdot\nabla]\mathbf{u}, \mathbf{A}\mathbf{u}\big) = 0 \tag{1.3.8}$$

for divergence-free $\mathbf{u}$, where $\mathbf{A}$ is the Stokes operator. This property allows to get estimates pointwise in time for the higher moments of the norm of the solution gradient

$$\mathbb{E}\left[\sup_{0\le t\le T}\|\nabla\mathbf{u}(t)\|_{\mathbb{L}^2}^p\right] + \nu\,\mathbb{E}\left[\int_0^T \|\nabla\mathbf{u}(t)\|_{\mathbb{L}^2}^{p-2}\|\mathbf{A}\mathbf{u}(t)\|_{\mathbb{L}^2}^2\,\mathrm{d}t\right] \le C_{T,p}\,.$$

These estimates are then crucial to prove the Hölder continuity in time of the solution and of its gradient by semi-group theory, using the correspondence between mild and strong solutions in this framework:

$$\mathbb{E}\left[\big\|\mathbf{u}(t)-\mathbf{u}(s)\big\|_{\mathbb{V}}^{\tilde{p}}\right] \le C\,|t-s|^{\frac{\eta\tilde{p}}{2}} \qquad \forall\, 0<\eta<\frac{1}{2}, \quad p\in[2,\infty)\,.$$

Once that these regularity properties has been proved, we may prove a priori estimates that are the discrete counterpart of the estimates obtained for the exact solution. Moreover, we prove stability estimates for the pressure depending on the type of noise, which reads

$$\mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla\pi^m\|_{\mathbb{L}^2}^2\right] \le C_{t_M,2}k^{-1} \quad \text{for divergence-free noise,} \tag{1.3.9}$$

while for divergence-free noise, we obtain a bound uniformly in $k>0$. Then, subtracting the corresponding equations, testing with the error, using the Hölder continuity and the a priori estimates, we are able to prove an error bound on a set $\Omega_k\subset\Omega$ with the property that $\mathbb{P}[\Omega_k]\to 1$ for vanishing parameter $k>0$. This lack of a global estimate is due to the fact that the a priori estimates only hold in a mean sense, while, to bound properly the terms resulting from the non-Lipschitz nonlinearity, estimates independent of the stochastic parameter $\omega\in\Omega$ are needed. These are then obtained by means of the Markov inequality on a set of measure close to one, leading to the local error estimate. Unfortunately, the convergence behavior is affected by the Hölder exponent of the gradient, which is only $1/4$, leading to the half of the expected convergence order. We may increase the convergence order by considering weaker norm for the error. That is, we test the corresponding error equation with $\mathbf{A}^{-1}$ applied to the error. In this way, we do not have to consider the difference of the gradients of the discretised and of exact solution, which caused the suboptimal convergence, increasing the convergence rate to optimal. As a consequence, we get convergence in probability with rates for the corresponding approximations.

We then pass to the full discretisation based on the implicit Euler scheme together with a stable finite element pairing, studying the error between the time and the full discretisation in the case of general noise. First we derive an a priori estimate for the fully discrete solution. In this framework we only obtain estimates like those from Chapter 3, i.e., no pointwise in time stability for the gradient of the discrete solution is available because identity (1.3.8) does not hold any more. As a consequence, when we try to estimate the error induced by the convection term, we need the inverse estimates to control the gradient of the fully discrete solution pointwise in time on a set $\Omega_h \subset \Omega$ with probability close to one, leading to order limiting terms, arising from the stabilisation of the nonlinear term. We started the analysis with a spatial discretisation which does not deliver exactly divergence-free iterates. Thus, during the error analysis, the pressure approximation from the time discrete scheme does not vanish. This causes a dependence of the convergence rate on the estimate (1.3.9), which implies a coupling of the discretisation parameters to get convergence on a set $\Omega_h \subset \Omega$ with probability close to one.

A possibility to get better convergence behavior for general noise may be the choice of finite element pairings which deliver exactly divergence-free iterates. An example of this is the Scott-Vogelius element; see [121]. Because the iterates are pointwise divergence-free, the pressure vanishes preventing that the order of convergence is affected on the type of noise. Moreover the order of convergence is optimal (on Ω_h), since other order limiting terms caused by the stabilisation of the convection term are not necessary any more for the Scott-Vogelius element.

We observe that all the above convergence results hold on a set of probability close to one. In this last part, we show that for additive noise, it is possible to show strong convergence on the whole set Ω for the implicit time discretisation. The proof crucially depends on the exponential estimates for the gradient of the exact solution, which allow to prevent the use of the Markov inequality to obtain pathwise estimate on the set Ω_k.

Chapter 2

Stochastic integral in infinite dimensions

Objective of this chapter is to give an introduction to the stochastic integral in infinite dimensions, providing the essential tools that are necessary to understand the manipulations of the stochastic integrals in the discrete setting. The presentation of the topic is essentially based on the tools we use later. In fact we do not consider some of the more general aspects, like e.g. the localisation procedure for integrands from [37].

2.1 Nuclear and Hilbert-Schmidt operators

In this section we give a short introduction to Nuclear and Hilbert-Schmidt operators, which are crucial for the definition of the Wiener process. For more details we refer to [118, Section 3.7] and to [110, Appendix B]. For two vector spaces X and Y we denote by $\mathcal{L}(X,Y)$ the space of linear operators bounded with respect to the operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. We use the convention $\mathcal{L}(X) = \mathcal{L}(X,X)$. The space X' indicates the dual of the space X. Let $\big(H, (\cdot,\cdot)_H\big)$ and $\big(K, (\cdot,\cdot)_K\big)$ be two separable Hilbert spaces, with norms $|\cdot|_H$ and $|\cdot|_K$ respectively. The operator $A : K \to H$ is called nuclear if it can be represented in the form

$$x \mapsto Ax = \sum_{i=1}^{\infty} a_i (b_i, x)_K \,,$$

where $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ are sequences in H and K respectively, with the property

$$\sum_{i=1}^{\infty} |a_i|_H \cdot |b_i|_K < \infty \,.$$

The space of all nuclear operator nuclear operators from K to H is denoted by $\mathcal{I}_1(K,H)$, and endowed with the norm

$$\|A\|_{\mathcal{I}_1(K,H)} := \inf \left\{ \sum_{i=1}^{\infty} |a_i|_H \cdot |b_i|_K \;;\; Ax = \sum_{i=1}^{\infty} a_i (b_i, x)_K, \quad x \in K \right\} ,$$

it is a Banach space. An operator $B : K \to H$ is called Hilbert-Schmidt if

$$\sum_{i=1}^{\infty} |Be_i|_H < \infty \,,$$

where $\{e_i\}_{i\geq 1}$ is an orthonormal basis of K. The space of all Hilbert-Schmidt operators from K to H is denoted by $\mathcal{I}_2(K,H)$, and is a Hilbert space if endowed with the scalar product

$$\big(A,B\big)_{\mathcal{I}_2(K,H)} := \sum_{i=1}^{\infty} \big(Ae_i, Be_i\big)_H \quad \forall A,B \in \mathcal{I}_2(K,H)\,.$$

For an operator $C \in \mathcal{L}(H)$ we define the trace

$$\operatorname{Tr} C = \sum_{i=1}^{\infty} (Ce_i, e_i)_H\,,$$

if the series is convergent. All the above representations are independent of the choice of the orthonormal basis $\{e_i\}_{i\geq 1}$. An operator $D \in \mathcal{L}(H)$ is called positive, respectively nonnegative, if $(Dx,x)_H > 0$, respectively $(Dx,x)_H \geq 0$, for all $x \in H$.

We enumerate some known facts about nuclear and Hilbert-Schmidt operators. For (i), (ii) and (iii) we refer to [110, Remark B.0.9], [110, Proposition 2.3.4] and [110, Remark B.0.6] respectively. For (iv), we refer to [118, Chapter 3, Section 7.1, Corollary 2].

Proposition 2.1.1. (i) *If an operator $A \in \mathcal{L}(H)$ has finite trace, is self-adjoint and nonnegative, then it is nuclear.*

(ii) *If an operator $A \in \mathcal{L}(H)$ is nonnegative, symmetric and has finite trace, there exists a unique element $A^{1/2} \in \mathcal{L}(H)$ such that $A^{1/2} \circ A^{1/2} = A$. Moreover there holds $A^{1/2} \in \mathcal{I}_2(H)$ and $\|A^{1/2}\|^2_{\mathcal{I}_2(H)} = \operatorname{Tr} A$.*

(iii) *The composition of a Hilbert-Schmidt and a bounded linear operator is a Hilbert-Schmidt operator.*

(iv) *The composition of a nuclear and a bounded linear operator is a nuclear operator.*

(v) *For an operator $\mathbf{A} \in \mathcal{I}_2(H,K)$, there holds $\|\mathbf{A}\|_{\mathcal{I}_2(H,K)} = \|\mathbf{A}^*\|_{\mathcal{I}_2(K,H)}$, where $\mathbf{A}^*$ denotes the adjoint of $\mathbf{A}$.*

We use the conventions $\mathcal{I}_1(K) = \mathcal{I}_1(K,K)$ and $\mathcal{I}_2(K) = \mathcal{I}_2(K,K)$.

2.2 Construction of the stochastic integral

In this section, we define the Hilbert space valued Wiener process and construct the stochastic integral with respect to it. Let $T > 0$ and let $\mathfrak{P} := \big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ be a stochastic basis, with $\mathbb{F} = \big\{\mathcal{F}_t;\, t \in [0,T]\big\}$. Let $\big(\boldsymbol{\mathcal{K}}, (\cdot,\cdot)_{\boldsymbol{\mathcal{K}}}\big)$ be a Hilbert space. By $\mathcal{N}(\mathbf{m},\mathbf{Q})$ we denote the Gaussian measure on $\boldsymbol{\mathcal{K}}$ with expectation vector $\mathbf{m} \in \boldsymbol{\mathcal{K}}$ and covariance operator $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$. Analogously to the real-valued case, we have the following definition.

Definition 2.2.1. *A $\boldsymbol{\mathcal{K}}$-valued stochastic process $\mathbf{W} \equiv \{\mathbf{W}(t)\,;\, t \in [0,T]\}$ on $\mathfrak{P}$ is called a $\mathbf{Q}$-Wiener process, if*

1. $\mathbf{W}(0) = \mathbf{0}$,
2. $\mathbf{W}$ *as $\mathbb{P}$-a.s. trajectories,*
3. $\mathbf{W}$ *has independent increments, i.e. the random variables*

$$\mathbf{W}(t_n) - \mathbf{W}(t_{n-1}), \mathbf{W}(t_{n-1}) - \mathbf{W}(t_{n-2}), \ldots, \mathbf{W}(t_1) - \mathbf{W}(t_0)$$

are independent for any choice of the partition $0 \leq t_0 < t_1 < \ldots < t_n \leq T$, $n \in \mathbb{N}$.

4. the increments have the following laws

$$\mathbf{W}(t) - \mathbf{W}(s) \sim \mathcal{N}\big(0, (t-s)\mathbf{Q}\big) \quad \forall\, t, s \in [0,T].$$

Let $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ be self-adjoint and positive definite, and $\{\mathbf{e}_j\}_{j\geq 1}$ denotes an orthonormal basis of $\boldsymbol{\mathcal{K}}$ consisting of eigenfunctions of $\mathbf{Q}$. Then the $\mathbf{Q}$-Wiener process has the following representation

$$\mathbf{W}(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \beta^j(t)\mathbf{e}_j \qquad \forall\, t \in [0,T]\,, \tag{2.2.1}$$

where $\{q_j\}_{j\geq 1} \subset \mathbb{R}^+$ are the corresponding eigenvalues of $\mathbf{Q}$ and $\big\{\beta^j(t);\, t \in [0,T]\big\}_{j\in\mathbb{N}}$, is a sequence of independent $\mathbb{R}$-valued Brownian motions on $\mathfrak{P}$; see e.g. [37, Chapter 4]. This series converges in $L^2(\Omega; C([0,T];\boldsymbol{\mathcal{K}}))$. In particular, for any $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ there exists a $\mathbf{Q}$-Wiener process on $\boldsymbol{\mathcal{K}}$.

Remark 2.2.1. *The series* (2.2.1) *is convergent in the space* $L^2(\Omega; C([0,T];\boldsymbol{\mathcal{K}}))$ *because the embedding* $\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}) \hookrightarrow \boldsymbol{\mathcal{K}}$ *defines a Hilbert-Schmidt operator. If* $\mathbf{Q}$ *is not a trace-class operator, it is always possible possible to construct a Hilbert space* $\boldsymbol{\mathcal{J}}$ *such that the embedding* $\mathbf{R} : \boldsymbol{\mathcal{J}} \hookrightarrow \boldsymbol{\mathcal{K}}$ *is Hilbert-Schmidt; see [110, Remark 2.5.1]. Then, by defining* $\mathbf{Q} = \mathbf{R}\mathbf{R}^*$*, we may define a* $\mathbf{Q}$*-Wiener process which converges in the space* $L^2(\Omega; C([0,T];\boldsymbol{\mathcal{J}}))$.

Now we are in position to construct the integral with respect to the $\boldsymbol{\mathcal{K}}$-valued Wiener process, and to define the class of processes for with the stochastic integral exists. For $p \geq 1$ and $\boldsymbol{\mathcal{X}}$ being a Banach space, consider the space $M^p\big([0,T],\mathbb{F};\boldsymbol{\mathcal{X}}\big)$ of equivalence classes of all $\mathbb{F}$-progressively measurable processes $u : [0,T]\times\Omega \to \boldsymbol{\mathcal{X}}$ such that

$$\mathbb{E}\left[\int_0^T \|u(t)\|^p_{\boldsymbol{\mathcal{M}}}\,\mathrm{d}t\right] < \infty\,. \tag{2.2.2}$$

Let $\big(\boldsymbol{\mathcal{H}}, (\cdot,\cdot)_{\boldsymbol{\mathcal{H}}}\big)$ be a Hilbert space. For any process $\boldsymbol{\varphi} \in M^2\big([0,T],\mathbb{F};\mathcal{L}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{H}})\big)$ we define the stochastic integral $\big\{\int_0^t \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s);\, t\in[0,T]\big\}$ by the following three steps.

(i) First we consider a step process defined by

$$\boldsymbol{\varphi}(s) = \sum_{i=1}^{n} \mathbb{1}_{[t_m,t_{m+1})}(s)\boldsymbol{\varphi}_m, \tag{2.2.3}$$

where the random variables $\boldsymbol{\varphi}_m$ take only a finite number of values in $\mathcal{L}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{H}})$. Then define the integral as the continuous $\boldsymbol{\mathcal{H}}$-valued $\mathbb{F}$-martingale

$$E(\boldsymbol{\varphi}) := \int_0^t \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s) = \sum_{m=1}^{M} \boldsymbol{\varphi}_m\Big(\mathbf{W}(t\wedge t_m) - \mathbf{W}(t\wedge t_{m-1})\Big) \qquad \forall\, t\in[0,T]\,.$$

(ii) For any step process prove that there holds the Itô isometry

$$\mathbb{E}\left[\Big|\int_0^T \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s)\Big|^2_{\boldsymbol{\mathcal{H}}}\right] = \mathbb{E}\left[\int_0^T \|\boldsymbol{\varphi}(s)\circ\mathbf{Q}^{1/2}\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{H}})}\,\mathrm{d}s\right]\,. \tag{2.2.4}$$

(iii) Let $\boldsymbol{\mathcal{X}} := \mathcal{I}_2\big(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}), \boldsymbol{\mathcal{H}}\big)$. The processes of the form (2.2.3) are dense in the Banach space $M^2\big([0,T], \mathbb{F}; \boldsymbol{\mathcal{X}}\big)$. Then there exists a unique linear bounded operator

$$E : M^2\big([0,T], \mathbb{F}; \boldsymbol{\mathcal{X}}\big) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; \boldsymbol{\mathcal{H}}),$$

which is the extension to $M^2\big([0,T], \mathbb{F}; \boldsymbol{\mathcal{X}}\big)$ of the operator E, defined in (i) for all the step processes.

Remark 2.2.2. *In most of the paper we work with processes taking values in $\mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})$, which is less general case since $\mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}}) \subset \mathcal{I}_2\big((\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}), \boldsymbol{\mathcal{H}}\big)$, because of Proposition 2.1.1,* (iii).

Remark 2.2.3. *It is possible to enlarge the class of integrands by using a localisation procedure, which takes into account all progressively measurable processes $\boldsymbol{\varphi}$ such that*

$$\mathbb{P}\left[\int_0^T \|\boldsymbol{\varphi}(s) \circ \mathbf{Q}\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})}\, \mathrm{d}s < \infty\right] = 1\,.$$

Remark 2.2.4. *For the case of a $\mathbf{Q}$-Wiener process, where $\mathbf{Q}$ is not nuclear, e.g., $\mathbf{Q}$ is the identity, because of* (2.2.4)*, we need that $\boldsymbol{\varphi}$ takes values in the space of Hilbert-Schmidt operators.*

For a detailed construction of the stochastic integral, we may refer to [37, 110] for Hilbert space theory of stochastic integral, or to [16], for extensions to Banach space theory.

2.3 Properties of the stochastic integral

Here we list some properties of the stochastic integral that are use later in this work. The stochastic integral satisfies the Itô isometry (see [37, Proposition 4.5]), i.e., for each $\boldsymbol{\varphi} \in M^2\big([0,T]; \mathbb{F}; \mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})\big)$

$$\mathbb{E}\Big[\Big|\int_0^t \boldsymbol{\varphi}(s)\, \mathrm{d}\mathbf{W}(s)\Big|^2_{\boldsymbol{\mathcal{H}}}\Big] = \mathbb{E}\Big[\int_0^t \|\boldsymbol{\varphi}(s) \circ \mathbf{Q}^{1/2}\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})}\, \mathrm{d}s\Big] \quad \forall\, t \in [0,T]\,, \tag{2.3.1}$$

and the Burkholder-Davis-Gundy inequality, see [16, Theorem 2.4] which holds for $1 < r < \infty$:

$$\mathbb{E}\left[\sup_{0\le t\le T}\Big|\int_0^t \boldsymbol{\varphi}(s)\, \mathrm{d}\mathbf{W}(s)\Big|^r_{\boldsymbol{\mathcal{H}}}\right] \le C\mathbb{E}\left[\left(\int_0^t \|\boldsymbol{\varphi}(s) \circ \mathbf{Q}^{1/2}\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})}\, \mathrm{d}s\right)^{r/2}\right] \quad \forall\, t \in [0,T]\,. \tag{2.3.2}$$

and the following special due to Davis

$$\mathbb{E}\left[\sup_{0\le t\le T}\Big|\int_0^t \boldsymbol{\varphi}(s)\, \mathrm{d}\mathbf{W}(s)\Big|_{\boldsymbol{\mathcal{H}}}\right] \le 3\mathbb{E}\left[\left(\int_0^t \|\boldsymbol{\varphi}(s) \circ \mathbf{Q}^{1/2}\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})}\, \mathrm{d}s\right)^{1/2}\right] \quad \forall\, t \in [0,T]\,. \tag{2.3.3}$$

which can be found in [37, Theorem 3.14].

We recall that for a $\mathbf{Q}$-Wiener process $\mathbf{W}$, there holds the inequality

$$\mathbb{E}\left[|\mathbf{W}(t) - \mathbf{W}(s)|^{2n}_{\boldsymbol{\mathcal{K}}}\right] \le C_n (t-s)^n (\operatorname{Tr} \mathbf{Q})^n \quad \forall\, n \in \mathbb{N}, \tag{2.3.4}$$

where for $n = 1$ we have equality and $C_n = 1$; see for instance [75, Corollary 1.1].

An important property for the stochastic integral is the Itô formula.

Theorem 2.3.1. *Let* $\mathbf{W}$ *be a* $\boldsymbol{\mathcal{K}}$*-valued Wiener process with covariance operator* $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$*. Assume that* $\boldsymbol{\Phi} \in M^2\big([0,T], \mathbb{F}; \mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})\big)$*,* $\boldsymbol{\phi}$ *is a* $\boldsymbol{\mathcal{H}}$*-valued progressively measurable and Bochner integrable process. Define the process*

$$X(t) = X_0 + \int_0^t \boldsymbol{\phi}(s)\,\mathrm{d}s + \int_0^t \boldsymbol{\Phi}(s)\,\mathrm{d}\mathbf{W}\,,$$

where X_0 *is a* $\boldsymbol{\mathcal{H}}$*-valued* $\mathcal{F}_0$ *-measurable random variable. Let* $F : [0,T] \times \boldsymbol{\mathcal{H}} \to \mathbb{R}$ *be a function with derivatives* F_t*,* F_x*,* F_{xx} *that are uniformly continuous on bounded subsets of* $[0,T] \times \boldsymbol{\mathcal{H}}$*. Then there holds*

$$\begin{aligned} F(t, X(t)) =& F(0, X(0)) + \int_0^t \big(F_x(s, X(s)), \boldsymbol{\Phi}(s)\,\mathrm{d}\mathbf{W}\big)_{\boldsymbol{\mathcal{H}}} + \int_0^t F_t(s, X(s))\,\mathrm{d}s \\ &+ \int_0^t \Big\{ \big(F_x(s, X(s)), \boldsymbol{\phi}(s)\,\big)_{\boldsymbol{\mathcal{H}}} + \frac{1}{2}\mathrm{Tr}\left[F_{xx}(s, X(s))\boldsymbol{\Phi}\mathbf{Q}^{1/2}\big(\boldsymbol{\Phi}\mathbf{Q}^{1/2}\big)^*\right]\Big\}\,\mathrm{d}s\,. \end{aligned}$$

The Itô formula holds only for deterministic functions $\mathbf{F}$. For $\mathbf{F}$ that are stochastic processes, the Itô-Wentzell formula [116, Theorem 1.4.9] is a useful tool, which allows to consider $\mathbf{F}$ given by Itô processes.

Chapter 3

Construction of martingale and strong solutions

3.1 Introduction

We consider the system of equations describing the motion of an incompressible fluid subjected to a random force in a bounded polygonal domain $D \subset \mathbb{R}^d$, $d = 2, 3$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete, filtered probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t;\, t \in [0,T]\}$ satisfying the usual assumptions, and $\mathbf{W} \equiv \{\mathbf{W}(t);\, t \in [0,T]\}$ be a $\boldsymbol{\mathcal{K}}$-valued Wiener process adapted to $\mathbb{F}$, for a Hilbert space $\boldsymbol{\mathcal{K}}$. Then we look for a stochastic process $\mathbf{u} \equiv \{\mathbf{u}(t)\,;\, t \in [0,T]\}$ with values in $\mathbb{L}^2(D) = L^2(D;\mathbb{R}^d)$ such that

$$\dot{\mathbf{u}} - \nu\Delta\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla\pi = \mathbf{f} + \mathbf{g}(\mathbf{u})\dot{\mathbf{W}} \qquad \text{in } D_T \times \Omega\,, \tag{3.1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } D_T \times \Omega\,, \tag{3.1.2}$$

where $D_T := (0,T) \times D$, and the following initial and boundary conditions hold,

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{in } D\,, \qquad \text{and} \qquad \mathbf{u} = \mathbf{0} \quad \text{on } (0,T)\times\partial D\,. \tag{3.1.3}$$

Here, $\mathbf{u}\cdot\nabla = \sum_{i=1}^n u_i\partial_i$. Below, we will precise requirements for the mapping $\mathbf{g}$.

An existing result for the practical discretisation of problem (3.1.1)–(3.1.3), Flandoli and Tortorelli [48] consider a semi-discretisation in time of the stochastic 2D Navier-Stokes equations driven by an additive noise, where strong solutions exist. By splitting the problem in a linear SPDEs coupled with Navier-Stokes equations with stochastic coefficients, convergence in probability of the discrete solution is shown.

A paper which is related to this chapter from a methodological point of view is [38], where De Bouard and Debussche show convergence in probability towards the unique strong solution of a time-semidiscretisation of the stochastic Schrödinger equation with non-Lipschitz nonlinearity. Similarly to the present work, uniform bounds for higher moments of the discrete solution, and of its increments are used to establish compactness for the sequence of discrete solutions. The main difference between the present paper and [38] is that here one has to deal with weak martingale solutions, which requires to construct a filtered probability space together with an adapted Wiener process, and an adapted solution in the overall construction process.

A similar construction of a Wiener process and of a related σ-algebra for the weak martingale solution is considered in [3], where a discretisation of the stochastic Landau-Lifschitz-Gilbert equation with Stratonovich noise is considered. Important tools to accomplish this goal are the characterisation of the new probability space, which uses [128, Theorem 1.10.4 and Addendum 1.10.5], together with a theorem which allows to identify the limit of a sequence of quadratic variations of a martingales [20, Theorem C.2]. The method we employ can be considered a generalisation of that developed in [131] for the finite dimensional case.

The goals of this chapter are to consider implementable space-time discretisations of (3.1.1)–(3.1.3), and to show that corresponding iterates construct weak martingale solutions ($d=3$), and strong solutions ($d=2$). The related analysis addresses the interplay of space-time discretisation (see Algorithms 3.1 and 3.3), and its interaction with the discretisation of the noise (see Algorithm 3.2 below), where stability, control of increments of iterates, a compactness argument and some tools from stochastic analysis are needed. For this purpose, we study a finite element based space-time discretisation of problem (3.1.1)–(3.1.3). Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain D, where $h>0$ denotes the maximum mesh-size, and $\big(\mathbb{H}_h, L_h\big)$ denotes a corresponding Ladyzhenskaja-Babuska-Brezzi (LBB) stable pair of finite element spaces to approximate velocity and pressure fields, respectively. Let $I_k := \{t_m\}_{m=0}^M$ be an equi-distant partition of size $k>0$ covering the time interval $[0,T]$, where $T \equiv t_M$. Then, we consider the following discretisation of (3.1.1), where for $m \geq 1$, $\mathbf{f}^m := \mathbf{f}(t_m)$, and $\Delta_m \mathbf{W} := \mathbf{W}(t_m) - \mathbf{W}(t_{m-1})$.

Algorithm 3.1. *Let $\mathbf{U}^0 \in \mathbb{H}_h$ be given. For every $m \in \{1,\ldots,M\}$ find an $\mathbb{H}_h \times L_h$-valued random variable $\big(\mathbf{U}^m, \Pi^m\big)$ such that for all $(\boldsymbol{\Phi}, \Lambda) \in \mathbb{H}_h \times L_h$,*

$$\begin{aligned}&\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \boldsymbol{\Phi}\big) + k\nu\big(\nabla \mathbf{U}^m, \nabla \boldsymbol{\Phi}\big) - k\big(\Pi^m, \operatorname{div} \boldsymbol{\Phi}\big) + k\big([\mathbf{U}^m \cdot \nabla]\mathbf{U}^m, \boldsymbol{\Phi}\big)\\ &\quad + \frac{k}{2}\Big([\operatorname{div} \mathbf{U}^m]\mathbf{U}^m, \boldsymbol{\Phi}\Big) = k\langle \mathbf{f}^m, \boldsymbol{\Phi}\rangle + \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m \mathbf{W}, \boldsymbol{\Phi}\big)\,, \end{aligned} \tag{3.1.4}$$

$$(\operatorname{div} \mathbf{U}^m, \Lambda) = 0\,. \tag{3.1.5}$$

A general Faedo-Galerkin method, continuous in time, which uses divergence-free functions is used in [46] to construct weak martingale solutions of (3.1.1)–(3.1.3) as limit of approximate solutions of a related stochastic differential equation: a compactness argument that uses fractional Sobolev spaces, and the Skorokhod almost sure representation theorem, together with a martingale representation theorem to properly identify limits of both, the deterministic and the Itô integrals are used in [46]. The fact that divergence-free functions are used, and that only a space discretisation is considered is in contrast to practical implementations, where usually (3.1.2) is only approximatively satisfied.

Unfortunately, it is non-trivial to construct practicable numerical schemes where finite element functions are exactly divergence-free, which is why discretisations are preferred where solutions are only weakly divergence-free, i.e., (3.1.5) holds. General references in the deterministic setting include [15, 53]. Here, in general, sequences of space-time interpolants $\{\operatorname{div}\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ only converge weakly in L^2 to zero (as $k,h \to 0$). Hence a related consequence is the convergence to solutions of the limiting problem in larger spaces; see also Theorem 3.5.1.

In this work, we show convergence of the space-time discretisation (3.1.4)–(3.1.5) which consists of the following three steps:

- (i$_1$) For every $k,h>0$, show solvability by means of the Brouwer fixed point theorem, and derive useful a priori bounds for iterates $\{\mathbf{U}^m\}_{m=1}^M$, and increments of iterates (Section 3.3).
- (ii$_1$) Prove tightness of sequences of laws related to space-time continuous functions $\{\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ which are only discretely divergence-free, and use Prohorov's theorem together with the Skorokhod almost sure representation theorem to identify limits of related deterministic integrals (Section 3.4).
- (iii$_1$) In order to identify the limit of the stochastic part in (3.1.4)–(3.1.5), slightly perturb the sequence $\{\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ to $\{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}\}_{k,h}$: for corresponding filtrations $\{\mathbb{F}_{k,h}\}_{k,h}$, the resulting

stochastic integral term is a square-integrable $\mathbb{F}_{k,h}$-martingale, and a proper limit $(k, h \to 0)$ a square-integrable $\mathbb{F}'$-martingale which by martingale representation theorem is the Itô integral $\left\{\int_0^t \mathbf{g}\big(\mathbf{u}(s)\big)\,\mathrm{d}\mathbf{W}'(s);\ t \in [0,T]\right\}$; this part is accomplished in Section 3.5.1.

The practical construction of a weak martingale solution, since it is the only available solution concept for problem (3.1.1)–(3.1.3) in three space dimensions, is the first main contribution of this chapter. This requires to properly address the combined space-time discretisation effects interacting with the non-Lipschitz drift, and the nonlinear noise. In particular, to realise these steps for a space-time discretisation where iterates are only discretely divergence-free requires different tools than those used in e.g. [46]. In particular, this practical construction of a weak martingale solution for (3.1.1)–(3.1.3)

(i_2) has to account for the fact that involved velocity fields are only discretely divergence-free, which is the reason for the additional term that appears in (3.1.4), and a proper balancing of finite element spaces for both, velocity and pressure to obtain a well-posed discretisation ('discrete LBB condition'). This lack of incompressibility of iterates $\{\mathbf{U}^m\}_{m=1}^M$ requires different compactness arguments on larger spaces if e.g. compared to those in [46] for example; see Lemma 3.3.3.

(ii_2) requires relevant stability properties, including uniform control of higher moments of iterates, and increments of them (in $k, h > 0$), which motivates an implicit in time treatment of deterministic terms in (3.1.4), and an explicit in time treatment of the integrand in the stochastic part; see Lemma 3.3.1. The control of increments of the iterates in Lemma 3.3.3 in particular is then a crucial step which again benefits from the special space-time discretisation, and is a key property for the compactness argument in Lemma 3.4.3.

(iii_2) has to construct (a proper sequence of) filtrations $\mathbb{F}_{k,h}$ to validate the $\mathbb{F}_{k,h}$-martingale property of the related stochastic term, to eventually pass to the limit; see equation (3.5.2) in Section 3.5.1. Since the approximation is discrete in time, we use iterates from Algorithm 3.1 to construct time-continuous processes $\{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}\}_{k,h}$, which inherit relevant properties of those iterates. In the next step this construction allows the stochastic integral with respect to the given Wiener process $\mathbf{W}$ to be represented by a sum of deterministic integrals involving the continuous functions $\{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}\}_{k,h}$. Then, the convergence of a subsequence of discrete solutions obtained in Lemma 3.4.3 allows to pass to the limit in the deterministic integrals on a new probability space, see Proposition 3.4.1, leading to a martingale which may be identified as a stochastic integral with respect to a limiting Wiener process $\mathbf{W}'$ by a martingale representation theorem.

As will become clear from the following, these arguments differ significantly from existing works in the numerics of nonlinear SPDEs. We hope that the elaboration of the steps (i_2)–(iii_2) turns out useful for the practically relevant construction of weak martingale solutions for a broad range of nonlinear SPDEs; while (i_2)–(ii_2) are mainly based on tools from nonlinear numerical analysis, those developed within (iii_2) in Section 5 heavily draw from concepts of stochastic analysis in a discrete setting.

It is computationally advantageous to replace Gaussian increments of a Wiener process $\Delta_m \mathbf{W}$ in Algorithm 3.1 by simpler (for instance bounded and discrete) i.i.d. $\boldsymbol{\mathcal{K}}$-valued random variables $\boldsymbol{\xi}^m$, with appropriate moment conditions (SI_1)–(SI_3) to hold; see Section 3.5.2 for details. The following algorithm is then a simple modification of Algorithm 3.1.

Algorithm 3.2. *Let $\mathbf{U}^0 \in \mathbb{H}_h$ be given. For every $m \in \{1, \dots, M\}$ find an $\mathbb{H}_h \times L_h$-valued random variable $\big(\mathbf{U}^m, \Pi^m\big)$ such that for all $(\mathbf{\Phi}, \Lambda) \in \mathbb{H}_h \times L_h$,*

$$\begin{aligned} &\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \mathbf{\Phi}\big) + k\nu\big(\nabla \mathbf{U}^m, \nabla \mathbf{\Phi}\big) - k\big(\Pi^m, \operatorname{div} \mathbf{\Phi}\big) + k\big([\mathbf{U}^m \cdot \nabla]\mathbf{U}^m, \mathbf{\Phi}\big) \\ &\quad + \frac{k}{2}\Big([\operatorname{div} \mathbf{U}^m]\mathbf{U}^m, \mathbf{\Phi}\Big) = k\langle \mathbf{f}^m, \mathbf{\Phi}\rangle + \big(\mathbf{g}(\mathbf{U}^{m-1})\boldsymbol{\xi}^m, \mathbf{\Phi}\big)\,, \end{aligned} \tag{3.1.6}$$

$$(\operatorname{div} \mathbf{U}^m, \Lambda) = 0\,. \tag{3.1.7}$$

We introduce assumptions (SI_1)–(SI_3), see Section 3.5.2, on the random variables $\{\boldsymbol{\xi}^m\}_{m=1}^M$ to have measurability with respect to the time discrete filtration, to ensure that expectation and covariance coincide with those of the corresponding Brownian increments, and to have the right scaling for higher moments of $\boldsymbol{\xi}^m$ with respect to the time-step $k > 0$. It is thanks to these assumptions that it is possible to show existence and stability of corresponding iterates $\{\mathbf{U}^m\}_{m=1}^M$, but in this case we can not use the Burkholder-Davis-Gundy inequality to get higher moments, and the Doob's inequality is needed. Then the tightness of the corresponding piecewise affine interpolation of the resulting iterates follows exactly as in Step (ii_2); see Section 3.5.2. The main issue is then given by the construction of the stochastic integral. Now we can not take advantage of the perturbation $\{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}\}_{k,h}$ from (iii_2), since it is constructed by a stochastic integral. In this case we construct a time-discrete martingale $\{\mathbf{M}_{k,h}^m\}_{m=1}^M$, with corresponding filtration. By using a general theorem on the convergence of time discrete martingales, together with the properties of the discrete process $\{\mathbf{R}_{k,h}^m\}_{m=1}^M$, which has the rôle of the quadratic variation, we can identify the limit of $\{\mathbf{M}_{k,h}^m\}_{m=1}^M$ with the desired stochastic integral. This convergence result can be compared with the Donsker invariance principle. In fact, we can use any sequence of i.i.d. random variables which have the same expectation and covariance as the Brownian increments to construct a stochastic integral, since the convergence is independent of the distribution of the $\{\boldsymbol{\xi}^m\}_{m=1}^M$. The method we present can be seen as a generalisation of the method used in [134] in the case of SDEs.

Moreover, we can consider versions of Algorithms 3.1 and 3.2 where the convection term is treated in a semi-implicit way, see Algorithm 3.3, such that the solution of problem (3.1.1)–(3.1.3) is approximated by iterates solving linear problems only. For this scheme, the estimates for higher moments of the solution, and of its increments remain valid, leading to the same compactness properties as for the solution of Algorithm 3.1 or 3.2. This allows to conclude the same convergence properties as for Algorithms 3.1 and 3.2.

Our second main contribution is the convergence of the whole sequence of $\mathbb{H}_h$-valued iterates $\{\mathbf{U}^m\}_{m=0}^M$ from Algorithm 3.1 on a given probability space to the existing unique strong solutions of (3.1.1)–(3.1.3), for $d = 2$ and multiplicative noise. This method can be seen as a generalisation of the numerical methods presented in [62] to the 2D Navier-Stokes equations. Moreover, we extend arguments from [98] and [97] to the given fully discrete setting. In [98], the existence of strong solutions is shown by an abstract Faedo-Galerkin method. Here we use the implementable Algorithm 3.1 to construct strong solutions of (3.1.1)–(3.1.3). A crucial difference between the two approaches is the time discretisation: due to the fact that chain and product rules differ from their discrete counterparts, residual terms arise in the present setting which need to be shown to vanish for vanishing discretisation parameters. The main tool in this proof is a local monotonicity result for the operator

$$\mathbf{G}(\mathbf{u}) := -\nu\Delta\mathbf{u} + [\mathbf{u} \cdot \nabla]\mathbf{u} + \frac{1}{2}[\operatorname{div} \mathbf{u}]\mathbf{u},$$

taken from [97]. This property allows to identify the limit in the nonlinear terms without using strong convergence type results explicitly: the argument does not employ the Skorokhod theorem as well, and, as a consequence, we do not need to construct a new probability space together with a Wiener process on it.

This chapter is organized as follows. In Section 3.2 we collect necessary background material. In Section 3.3 we show stability properties of solutions from Algorithm 3.1, and construct weak martingale solutions of (3.1.1)–(3.1.3) from $\{\mathbf{U}^m\}_{m=0}^M$ in Sections 3.4 and 3.5.1. In Section 3.5.2, we generalize these results to Algorithm 3.2. In Section 3.6, we construct strong solutions of (3.1.1)–(3.1.3) as limits of solutions of Algorithm 3.1 for the case $d = 2$.

3.2 Preliminaries

3.2.1 General setting

Let us assume that D is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with polygonal or polyhedral boundary. Let $L^p(D)$ and $W^{m,p}(D)$ for $m \geq 0$, $1 \leq p \leq \infty$, denote the usual Lebesgue and Sobolev spaces, which are endowed with the standard norms $\|\cdot\|_{L^p}$ respectively $\|\cdot\|_{W^{m,p}}$ and, for $p = 2$, scalar products $(\cdot,\cdot)$ respectively $(\cdot,\cdot)_{W^{m,2}}$. By $W_0^{m,p}(D)$ we denote the closure in $W^{m,p}(D)$ of $C_0^\infty(D)$ of all smooth functions defined on D with compact support. Let $L_0^p(D)$ denote the subspace of functions from $L^p(D)$ with vanishing mean. The Lebesgue and the Sobolev spaces of vector-valued functions will be indicated with blackboard bold letters, e.g. $\mathbb{W}_0^{1,2}(D) = W_0^{1,2}(D, \mathbb{R}^d)$, for $d = 2, 3$. Since we work mainly with the domain D, we write usually $\mathbb{W}_0^{1,2}$ or $\mathbb{L}^2$. For a Banach space X, let $L^p\big(0, T; X\big)$, and $W^{m,p}\big(0, T, X\big)$ denote standard Lebesgue and Sobolev spaces of Bochner measurable X-valued functions. For the space $\mathbb{W}_0^{1,2}$ we denote by $\mathbb{W}^{-1,2}$ its dual, and by $\langle\cdot,\cdot\rangle$ the corresponding dual pairing. The following spaces play a fundamental rôle below.

$$
\begin{aligned}
\boldsymbol{\mathcal{V}} &= \big\{\mathbf{v} \in \mathbb{C}_0^\infty(D)\,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } D\big\}\,,\\
\mathbb{H} &= \big\{\mathbf{v} \in \mathbb{L}^2(D)\,;\, \operatorname{div}\mathbf{v} = 0 \text{ a.e. in } D,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ a.e. on } \partial D\big\}\,,\\
\mathbb{V} &= \big\{\mathbf{v} \in \mathbb{W}_0^{1,2}(D)\,;\, \operatorname{div}\mathbf{v} = 0 \text{ a.e. in } D\big\}\,,
\end{aligned}
$$

where '$\cdot$' denotes the standard scalar product in $\mathbb{R}^d$, and $\mathbf{n} : \partial D \to \mathbb{R}^d$ denotes the unit outer normal vector field.

We denote $D(\mathbf{A}) = \mathbb{W}^{2,2}(D) \cap \mathbb{V}$, and define the self-adjoint, inversely compact operator $\mathbf{A} : D(\mathbf{A}) \to \mathbb{H}$ via $\mathbf{A}\mathbf{u} := -\mathbf{P}_{\mathbb{H}}\Delta\mathbf{u}$, where $\mathbf{P}_{\mathbb{H}} : \mathbb{L}^2(D) \to \mathbb{H}$ denotes the Leray-projection. Below, we always suppose that the bounded domain $D \subset \mathbb{R}^d$ is such that the unique solution $\mathbf{w} \in \mathbb{V}$ of the stationary, incompressible Stokes problem $-\Delta\mathbf{w} + \nabla\pi = \mathbf{b}$ in $D \subset \mathbb{R}^d$ supplemented with Dirichlet boundary conditions belongs to $\mathbb{V} \cap \mathbb{W}^{2,2}(D)$ provided $\mathbf{b} \in \mathbb{L}^2(D)$, and satisfies $\|\mathbf{w}\|_{\mathbb{W}^{2,2}} \leq C\|\mathbf{b}\|_{\mathbb{L}^2}$.

We summarize the assumptions needed below for data $\mathbf{Q}$, $\mathbf{u}_0$, and $\mathbf{f}$; see [46] for similar assumptions.

(S$_1$) $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ is a symmetric, positive operator.

(S$_2$) $\mathbf{g} : \mathbb{L}^2 \to \mathcal{L}(\boldsymbol{\mathcal{K}}, \mathbb{L}^2)$ is (strongly) continuous with linear growth, i.e., there exists a constant $K_1 > 0$, such that

$$\|\mathbf{g}(\mathbf{v})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)} \leq K_1\|\mathbf{v}\|_{\mathbb{L}^2}.$$

(S$_3$) $\mathbf{u}_0 \in \mathbb{H}$, and $\mathbf{f} \in C\big([0,T]; \mathbb{W}^{-1,2}\big)$.

We recall the notion of a weak martingale solution of (3.1.1)–(3.1.3); see for instance [46].

Definition 3.2.1. *Let $T > 0$ be given, and* (S$_1$)–(S$_3$) *are valid. A weak martingale solution of* (3.1.1)–(3.1.3) *is a system* $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$, $\mathbf{u}$, $\mathbf{W}$, *where* $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ *is a stochastic basis,* $\mathbf{W}$ *is a* $\boldsymbol{\mathcal{K}}$*-valued* $\mathbb{F}$*-Wiener process with covariance operator* $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ *symmetric and positive, and*

$$\mathbf{u} \in L^2\Big(\Omega; C\big([0,T]; D(\mathbf{A}^{-\gamma})\big) \cap L^\infty\big(0,T;\mathbb{H}\big) \cap L^2\big(0,T;\mathbb{V}\big)\Big)$$

is an $\mathbb{F}$*-progressively measurable process, such that* $\gamma > 1$ *and* $\mathbb{P}$*-almost surely, for all* $t \in [0,T]$*, the following equality holds,*

$$\begin{aligned}\big(\mathbf{u}(t,\cdot), \mathbf{v}\big) + \int_0^t \big(\nabla \mathbf{u}(s,\cdot), \nabla \mathbf{v}\big)\,\mathrm{d}s - \int_0^t \Big([\mathbf{u}(s,\cdot)\cdot\nabla]\mathbf{v}, \mathbf{u}(s,\cdot)\Big)\,\mathrm{d}s \\ = \big(\mathbf{u}_0, \mathbf{v}\big) + \int_0^t \big(\mathbf{f}(s,\cdot), \mathbf{v}\big)\,\mathrm{d}s + \Big(\int_0^t \mathbf{g}\big(\mathbf{u}(s,\cdot)\big)\mathrm{d}\mathbf{W}(s,\cdot), \mathbf{v}\Big) \qquad \forall\, \mathbf{v} \in \boldsymbol{\mathcal{V}}\,.\end{aligned}$$

A weak martingale solution for (3.1.1)–(3.1.3) and $d = 3$ under the above assumptions is constructed in [46, Theorem 3.1] by an abstract Faedo-Galerkin method.

3.2.2 Space discretisation

For simplicity, throughout the work we assume that $\mathcal{T}_h$ is a quasi-uniform triangulation of the domain $D \subset \mathbb{R}^d$ ($d = 2, 3$) into triangles resp. tetrahedra of maximal diameter $h > 0$, and $\overline{D} = \bigcup_{K\in\mathcal{T}} \overline{K}$. Let $\mathbb{P}_i(K) \equiv [P_i(K)]^d$ be the space of polynomial vector fields on K with degree less or equal to i. We introduce finite element function spaces for fixed $i, j \in \mathbb{N}_0$,

$$\begin{aligned}\mathbb{H}_h &:= \big\{\mathbf{W} \in \mathbb{C}^0(\overline{D}) \cap \mathbb{W}_0^{1,2}(D)\,;\, \mathbf{W} \in \mathbb{P}_i(K) \quad \forall\, K \in \mathcal{T}_h\big\}\,, \\ L_h &:= \big\{\Pi \in L_0^2(D)\,;\, \Pi \in P_j(K) \quad \forall\, K \in \mathcal{T}_h\big\}\,.\end{aligned}$$

We assume that these spaces satisfy (for fixed i, j) the discrete LBB-condition, see e.g. [15, Section II.2.3],

$$\sup_{\mathbf{\Phi}\in\mathbb{H}_h} \frac{\big(\operatorname{div}\mathbf{\Phi}, \Pi\big)}{\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}} \geq C\|\Pi\|_{L^2} \quad \forall\, \Pi \in L_h\,, \tag{3.2.1}$$

for a constant $C > 0$ independent of the mesh size $h > 0$. Let

$$\mathbb{V}_h := \big\{\mathbf{\Phi} \in \mathbb{H}_h\,;\, (\operatorname{div}\mathbf{\Phi}, \Pi) = 0 \text{ for all } \Pi \in L_h\big\}\,.$$

Note that $\mathbb{V}_h$ is in general not contained in $\mathbb{H}$ or $\mathbb{V}$. We denote by $\mathbf{P}_h^0 : \mathbb{L}^2(D) \to \mathbb{H}_h$, and by $\mathbf{Q}_h^0 : \mathbb{L}^2(D) \to \mathbb{V}_h$ $\mathbb{L}^2$-orthogonal projections, defined by the identities

$$\big(\mathbf{z} - \mathbf{P}_h^0\mathbf{z}, \mathbf{\Phi}\big) = 0 \quad \forall\, \mathbf{\Phi} \in \mathbb{H}_h\,, \qquad \big(\mathbf{z} - \mathbf{Q}_h^0\mathbf{z}, \mathbf{\Phi}\big) = 0 \quad \forall\, \mathbf{\Phi} \in \mathbb{V}_h\,. \tag{3.2.2}$$

The following estimates are standard, see for instance [15, 53],

$$\|\mathbf{z} - \mathbf{Q}_h^0\mathbf{z}\|_{\mathbb{L}^2} + h\|\nabla\left(\mathbf{z} - \mathbf{Q}_h^0\mathbf{z}\right)\|_{\mathbb{L}^2} \leq Ch^2\|\mathbf{A}\mathbf{z}\|_{\mathbb{L}^2} \quad \forall\, \mathbf{z} \in \boldsymbol{\mathcal{V}} \cap \mathbb{W}^{2,2}(D)\,, \tag{3.2.3}$$

$$\|\mathbf{z} - \mathbf{Q}_h^0\mathbf{z}\|_{\mathbb{L}^2} \leq Ch\|\nabla\mathbf{z}\|_{\mathbb{L}^2} \quad \forall\, \mathbf{z} \in \boldsymbol{\mathcal{V}} \cap \mathbb{W}^{1,2}(D)\,. \tag{3.2.4}$$

We recall the inverse inequality [14, Lemma 4.5.3], which holds for quasi-uniform triangulations for every finite element function $\mathbf{v}_h \in \mathbb{H}_h$,

$$\|\mathbf{v}_h\|_{\mathbb{W}^{\ell,q_1}} \leq Ch^{m-\ell+d\min\{\frac{1}{q_1}-\frac{1}{q_2},0\}}\|\mathbf{v}_h\|_{\mathbb{W}^{m,q_2}} \qquad \forall\, 1 \leq q_1, q_2 \leq \infty\,, \quad 0 \leq m \leq \ell\,. \tag{3.2.5}$$

An important tool for proving the solvability and the energy estimates is the following identity

$$\big([\mathbf{\Phi}\cdot\nabla]\mathbf{\Phi}, \mathbf{\Phi}\big) + \frac{1}{2}\big([\operatorname{div}\mathbf{\Phi}]\mathbf{\Phi}, \mathbf{\Phi}\big) = 0 \quad \forall\, \mathbf{\Phi} \in \mathbb{H}_h\,. \tag{3.2.6}$$

3.2.3 Time discretisation

Let $I_k = \{t_m\}_{m=0}^M$ denote an equi-distant partition of $[0,T]$ of time-step size $k > 0$. To construct weak martingale solutions of the nonlinear problem (3.1.1)–(3.1.3) requires some strong convergence of constructed subsequences which for deterministic problems usually comes from the compactness results for instance of Aubin and Lions, and requires some estimates for the time derivatives of approximate solutions. This strategy does not extend to the stochastic case where approximate solutions are not differentiable; a key problem related to time-discretisation is to properly set up sequences of continuous processes $\{\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ related to the discrete ones $\{\mathbf{U}^m\}_{m=0}^M$ solving the iterative scheme, and to construct related filtrations $\{\mathbb{F}_{k,h}\}_{k,h}$ that they are adapted to. Another problem that has to be addressed in the present work is that approximating functions in general are not solenoidal. In the following, we use a method based on fractional Sobolev spaces, which are related to Nikolskii spaces. We refer to [122, Definition 1] for the following

Definition 3.2.2. *Let X be a Banach space, and $T > 0$.*
i) Fractional Sobolev spaces are defined for $0 < s < 1$, $1 \leq p < \infty$ by

$$W^{s,p}\big(0,T;X\big) = \big\{f \in L^p\big(0,T;X\big) : \|f\|_{W^{s,p}} < \infty\big\},$$

where

$$\|f\|_{W^{s,p}} = \Big(\int_0^T \int_0^T \Big(\frac{\|f(r)-f(t)\|_X}{|r-t|^s}\Big)^p \frac{\mathrm{d}r\mathrm{d}t}{|r-t|}\Big)^{\frac{1}{p}}.$$

ii) Hölder spaces are defined for $0 < s < 1$ by

$$\mathrm{Lip}^s\big([0,T];X\big) = \big\{f \in L^\infty\big(0,T;X\big) : \|f\|_{\mathrm{Lip}^s} < \infty\big\},$$

where

$$\|f\|_{\mathrm{Lip}^s} = \operatorname{ess\,sup}_{r,t\in[0,T]} \frac{\|f(r)-f(t)\|_X}{|r-t|^s}.$$

iii) Nikolskii spaces are defined for $0 < s < 1$, $1 \leq p < \infty$ by

$$N^{s,p}\big(0,T;X\big) = \big\{f \in L^p\big(0,T;X\big) : \|f\|_{N^{s,p}} < \infty\big\},$$

where

$$\|f\|_{N^{s,p}} = \sup_{\delta>0} \delta^{-s}\|f(\cdot+\delta)-f(\cdot)\|_{L^p(0,T-\delta;X)}.$$

The following properties of these spaces are known, see for instance [122].

(i) $W^{s,p} \subset N^{s,p}$,

(ii) $W^{s,p} \subset W^{r,p}$ and $N^{s,p} \subset N^{r,p}$, for $s \geq r$,

(iii) $W^{s,p}$ and $N^{s,p}$ are both embedded in $W^{r,p} \cap N^{r,p}$, provided $s > r$,

(iv) if $s > \frac{1}{p}$, then both, $W^{s,p}$ and $N^{s,p}$ are included in $\mathrm{Lip}^{s-\frac{1}{p}}$. In particular, they are included in the set of continuous functions,

(v) if $s - \frac{1}{p} \geq r - \frac{1}{q}$, then $W^{s,p} \subset W^{r,q}$ and $N^{s,p} \subset N^{r,q}$, provided $0 < r \leq s < 1$, and $1 \leq p \leq q < \infty$.

Let X be a Banach space, $k>0$ and let $I_k \equiv \{t_m\}_{m=0}^M$ be an equi-distant partition of $[0,T]$. Let us denote by $\mathcal{G}_k$ the set of functions belonging to $C\big([0,T];X\big)$ that are piecewise affine on subintervals $[t_m,t_{m+1}]$, $m=1,\cdots,M$.

The following criterion for the embedding of the space $\mathcal{G}_k$ into the spaces $N^{\alpha,p}(0,T;X)$ will be useful, see [3, Lemma 3.1] for a proof.

Lemma 3.2.1. *Assume that $0\le\alpha<1$ and $p\ge 1$. Assume that $f\in\mathcal{G}_k$ is such that for every $\ell\in\{1,\ldots,M\}$*

$$k\sum_{m=0}^{M-\ell}\big\|f(t_{m+\ell})-f(t_m)\big\|_E^p \le K^p t_\ell^{\alpha p}. \tag{3.2.7}$$

Then, there exists a constant $C>0$ not depending on f neither on $k>0$, such that

$$\|f\|_{N^{\alpha,p}(0,T;X)} \le CK\,. \tag{3.2.8}$$

3.2.4 Compactness results

The following compactness results will be needed below; see e.g. [46] for proofs and comments.

Lemma 3.2.2. *Assume that $X_0\subset X\subset X_1$ are Banach spaces, X_0 and X_1 being reflexive. Assume that the embedding $X_0\hookrightarrow X$ is compact, $q\in(1,\infty)$, and $\alpha\in(0,1)$. Then the embedding*

$$L^q\big(0,T;X_0\big)\cap W^{\alpha,q}\big(0,T;X_1\big)\hookrightarrow L^q\big(0,T;X\big)$$

is compact.

Lemma 3.2.3. *Assume that X_0, X are Banach spaces such that the embedding $X_0\hookrightarrow X$ is compact. Assume that $q\in(1,\infty)$ and $0<\alpha<\beta<1$. Then the embedding*

$$W^{\beta,q}\big(0,T;X_0\big)\hookrightarrow W^{\alpha,q}\big(0,T;X\big)$$

is compact.

The last compactness result relates Sobolev spaces to spaces of continuous functions.

Lemma 3.2.4. *Assume that X_0, X are Banach spaces such that the embedding $X_0\hookrightarrow X$ is compact, and some real numbers $\alpha\in(0,1)$, $q>1$ satisfy*

$$\alpha q>1\,.$$

Then the space $W^{\alpha,q}\big(0,T;X_0\big)$ is compactly embedded into $C\big([0,T];X\big)$.

3.3 Existence and stability of discretised solutions

Let $\mathfrak{P}=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a stochastic basis. Let us first observe that in view of the assumption (S$_1$) there exists a $\boldsymbol{\mathcal{K}}$-valued Wiener process $\mathbf{W}$ with covariance operator $\mathbf{Q}$. We fix this process, together with the filtration $\mathbb{F}$ generated by it, for the remainder of this section.

In a first step, we derive existence and stability properties for the solution to Algorithm 3.1. For this purpose, problem (3.1.4)–(3.1.5) is rewritten in the following form by using discretely solenoidal functions,

$$\begin{aligned}&(\mathbf{U}^m-\mathbf{U}^{m-1},\mathbf{\Phi})+k\nu(\nabla\mathbf{U}^m,\nabla\mathbf{\Phi})+k\big([\mathbf{U}^m\cdot\nabla]\mathbf{U}^m,\mathbf{\Phi}\big)\\ &+\frac{k}{2}\big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m,\mathbf{\Phi}\big)=k\big\langle\mathbf{f}^m,\mathbf{\Phi}\big\rangle+\big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{\Phi}\big)\qquad\forall\mathbf{\Phi}\in\mathbb{V}_h\,.\end{aligned}\tag{3.3.1}$$

Lemma 3.3.1. *Assume that $1 \leq p < \infty$ is a natural number and $\mathbf{U}^0 \in \mathbb{H}_h$ with $\|\mathbf{U}^0\|_{\mathbb{L}^2} \leq C$ for a constant independent of $h > 0$, and assume that $\mathbf{f} \in C\big(0,T;\mathbb{W}^{-1,2}\big)$. Then there exists a sequence of $\mathbb{H}_h \times L_h$-valued random variables $\{(\mathbf{U}^m, \Pi^m)\}_{m=1}^M$ which $\mathbb{P}$-almost surely solves Algorithm 3.1, and satisfies the following conditions for $T \equiv t_M$:*

(i) *for each $m \in \{0, \cdots, M\}$, the map $\mathbf{U}^m : \Omega \to \mathbb{H}_h$ is $\mathcal{F}_{t_m}$-measurable.*

(ii) $\mathbb{E}\Big[\max_{1\leq m\leq M} \|\mathbf{U}^m\|^2_{\mathbb{L}^2} + k\nu \sum_{m=1}^M \|\nabla \mathbf{U}^m\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{m=1}^M \|\mathbf{U}^m - \mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\Big] \leq C_T\,,$

(iii) $\mathbb{E}\Big[\max_{1\leq m\leq M} \|\mathbf{U}^m\|^{2^p}_{\mathbb{L}^2} + k\nu \sum_{m=1}^M \|\mathbf{U}^m\|^{2^p-1}_{\mathbb{L}^2}\|\nabla \mathbf{U}^m\|^2_{\mathbb{L}^2}\Big] \leq C_{T,p}\,,$

(iv) $\mathbb{E}\Big[\Big(k\sum_{m=1}^M \|\nabla \mathbf{U}^m\|^2_{\mathbb{L}^2}\Big)^{2^{p-1}}\Big] \leq C_{T,p}\,.$

Above, the constants $C_{T,p} \equiv C_{T,p}\big(\mathrm{Tr}\,\mathbf{Q}, \|\mathbf{U}^0\|_{L^2}, \|\mathbf{f}\|_{L^\infty(0,T;\mathbb{L}^2)}\big) > 0$, and $C_T \equiv C_{T,1}$ do not depend on $k, h > 0$.

Proof. **Step 1.** *Solvability.* We use Brouwer's fixed point theorem to show the existence of $\mathbb{V}_h$-valued random variables $\{\mathbf{U}^m(\omega)\}_{m=1}^M$ on Ω solving (3.3.1). We argue by induction: since $\mathbf{U}^0$ is given, and $|\Delta_m \mathbf{W}(\omega)|_{\mathcal{K}} < \infty$ for all $m \in \{1,\ldots,M\}$ $\mathbb{P}-a.s.$, we may then assume that $\mathbf{U}^1,\ldots,\mathbf{U}^{m-1}(\omega)$ are also given. For every $m \in \{1,\ldots,M\}$, consider the map $\boldsymbol{\mathcal{F}}^\omega_m : \mathbb{V}_h \to \mathbb{V}_h$, defined by

$$\begin{aligned}\big(\boldsymbol{\mathcal{F}}^\omega_m(\boldsymbol{\Phi}), \boldsymbol{\psi}\big) &= \big(\boldsymbol{\Phi} - \mathbf{U}^{m-1}(\omega), \boldsymbol{\psi}\big) + k\nu\big(\nabla\boldsymbol{\Phi}, \nabla\boldsymbol{\psi}\big) + k\big([\boldsymbol{\Phi}\cdot\nabla]\boldsymbol{\Phi}, \boldsymbol{\psi}\big) \\ &\quad + \frac{k}{2}\big([\mathrm{div}\,\boldsymbol{\Phi}]\boldsymbol{\Phi}, \boldsymbol{\psi}\big) - \Big(\mathbf{g}\big(\mathbf{U}^{m-1}(\omega)\big)\Delta_m\mathbf{W}(\omega), \boldsymbol{\psi}\Big) - \langle \mathbf{f}^m, \boldsymbol{\psi}\rangle \qquad \forall \boldsymbol{\psi} \in \mathbb{V}_h\,.\end{aligned}$$

Since $\mathbb{V}_h$, endowed with the $\mathbb{L}^2$ scalar product is a Hilbert space, $\boldsymbol{\mathcal{F}}^\omega_m$ is well-defined. Moreover, it can be easily shown that this mapping is continuous. Hence, on using identity (3.2.6) and Young's inequality, we have for all $\boldsymbol{\Phi} \in \mathbb{V}_h$

$$\begin{aligned}\big(\boldsymbol{\mathcal{F}}^\omega_m(\boldsymbol{\Phi}), \boldsymbol{\Phi}\big) &\geq \Big(\frac{1}{2}\|\boldsymbol{\Phi}\|^2_{\mathbb{L}^2} - \|\mathbf{U}^{m-1}(\omega)\|^2_{\mathbb{L}^2} - \frac{1}{2}\|\mathbf{g}\big(\mathbf{U}^{m-1}(\omega)\big)\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}(\omega)|^2_{\boldsymbol{\mathcal{K}}}\Big) \\ &\quad + k\nu\|\nabla\boldsymbol{\Phi}\|^2_{\mathbb{L}^2} - Ck\|\mathbf{f}^m\|^2_{\mathbb{W}^{-1,2}} - k\nu\|\nabla\boldsymbol{\Phi}\|^2_{\mathbb{L}^2}\,.\end{aligned} \tag{3.3.2}$$

By the inductive assumption and (S_2), there holds for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$R_m(\omega) := \|\mathbf{U}^{m-1}(\omega)\|^2_{\mathbb{L}^2}\Big(1 + \frac{K_1}{2}\big|\Delta_m\mathbf{W}(\omega)\big|^2_{\boldsymbol{\mathcal{K}}}\Big) + Ck\|\mathbf{f}^m\|^2_{\mathbb{W}^{-1,2}} < \infty\,.$$

Hence, we may conclude that, with $\mathcal{A}_m(\omega) \equiv \big\{\boldsymbol{\varphi} \in \mathbb{V}_h : \|\boldsymbol{\varphi}\|^2_{\mathbb{L}^2} \geq 2R_m(\omega)\big\}$ there holds

$$\big(\boldsymbol{\mathcal{F}}^\omega_m(\boldsymbol{\Phi}), \boldsymbol{\Phi}\big) \geq 0 \qquad \forall \boldsymbol{\Phi} \in \mathcal{A}_m(\omega).$$

By Brouwer's fixed point theorem, this implies the existence (but not uniqueness) of $\boldsymbol{\Phi}^*_\omega \in \mathbb{V}_h$, such that $\boldsymbol{\mathcal{F}}^\omega_m(\boldsymbol{\Phi}^*_\omega) = 0$, see for instance [53, Corollary 1.1, p. 279]. Hence $\mathbf{U}^m(\omega) \in \mathbb{V}_h$ exists $\mathbb{P}$-almost surely.

Since the discrete LBB-condition holds, there exists an L_h-valued pressure $\{\Pi^m\}_{m=1}^M$ such that (3.1.4) is valid.

Step 2. *Measurability. Proof of claim* (i). This can be done exactly as in [38], see also [3].

Step 3. *A priori energy estimates.* We will prove the three bounds from (ii) by first proving an auxiliary inequality to deduce the second and third part of the inequality in (ii). Then we prove the first part. We put $\mathbf{\Phi} = \mathbf{U}^m$ in equation (3.3.1) and by using the following fundamental identity

$$(\mathbf{a}, \mathbf{a}-\mathbf{b}) = \frac{1}{2}(\|\mathbf{a}\|_{\mathbb{L}^2}^2 - \|\mathbf{b}\|_{\mathbb{L}^2}^2) + \frac{1}{2}\|\mathbf{a}-\mathbf{b}\|_{\mathbb{L}^2}^2 \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{L}^2\,, \tag{3.3.3}$$

together with (3.2.6), we find out that

$$\begin{aligned} &\frac{1}{2}\Big(\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 + \|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\Big) + k\nu\|\nabla \mathbf{U}^m\|_{\mathbb{L}^2}^2 \\ &\quad = k\langle \mathbf{f}^m, \mathbf{U}^m\rangle + \Big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m \mathbf{W}, \mathbf{U}^m - \mathbf{U}^{m-1}\Big) + \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{U}^{m-1}\big)\,. \end{aligned} \tag{3.3.4}$$

Fix $m \in \{1, \ldots, M\}$. Note that in view of Step 2, the last term on the right-hand side of (3.3.4) vanishes when taking its expectation. By the Cauchy-Schwarz inequality, and taking the sum from $m = 1$ to $m = M$, after absorbing terms we obtain

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\Big[\|\mathbf{U}^M\|_{\mathbb{L}^2}^2 + \frac{1}{2}\sum_{m=1}^{M}\|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 + k\nu\sum_{m=1}^{M}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big] \\ &\quad \le \frac{1}{2}\|\mathbf{U}^0\|_{\mathbb{L}^2}^2 + Ck\sum_{m=1}^{M}\|\mathbf{f}^m\|_{\mathbb{W}^{-1,2}}^2 + \sum_{m=1}^{M}\mathbb{E}\big[\|\mathbf{g}\big(\mathbf{U}^{m-1}\big)\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2\big]\,. \end{aligned} \tag{3.3.5}$$

By the tower property of the conditional expectation, the independence of the increments of the Wiener process, and assumption (S_2) we find for the last term

$$\begin{aligned} &\mathbb{E}\Big[\|\mathbf{g}\big(\mathbf{U}^{m-1}\big)\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2\Big] = \mathbb{E}\Big[\mathbb{E}\big[\|\mathbf{g}\big(\mathbf{U}^{m-1}\big)\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2|\mathcal{F}_{t_{m-1}}\big]\Big] \\ &\quad = \mathbb{E}\Big[\|\mathbf{g}\big(\mathbf{U}^{m-1}\big)\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\mathbb{E}\big[|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2|\mathcal{F}_{t_{m-1}}\big]\Big] = k(\operatorname{Tr}\mathbf{Q})\,\mathbb{E}\big[\|\mathbf{g}\big(\mathbf{U}^{m-1}\big)\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\big] \\ &\quad \le (\operatorname{Tr}\mathbf{Q})\,K_1 k\,\mathbb{E}\big[\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\big]\,, \end{aligned} \tag{3.3.6}$$

where in the last equality we use (2.3.4) with $n = 1$. Hence we conclude by applying the discrete version of Gronwall's lemma (see e.g. [117, Lemma 1.4.2]) to inequality (3.3.5), together with (3.3.6), to obtain the following estimate

$$\max_{1\le m\le M}\mathbb{E}\Big[\|\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big] \le \Big(\|\mathbf{U}^0\|_{\mathbb{L}^2}^2 + Ck\sum_{m=1}^{M}\|\mathbf{f}^m\|_{\mathbb{W}^{-1,2}}^2\Big)\mathrm{e}^{CT} \le C_T. \tag{3.3.7}$$

In the next step, by using the estimate (3.3.7) together with (3.3.5), we obtain the second and third estimate in (ii). To prove the first inequality in (ii), we start from equality (3.3.4), and use the Cauchy-Schwarz's inequality on the first two terms of the right-hand side, sum from $m = 1$ to $m = i$, for a fixed natural number $i \ge 1$, take the maximum over $1 \le i \le M$, and apply the expectations. As a consequence, omitting the positive terms on the left-hand side, we find

$$\begin{aligned} \mathbb{E}\left[\max_{1\le i\le M}\|\mathbf{U}^i\|_{\mathbb{L}^2}^2\right] \le\ & \|\mathbf{U}^0\|_{\mathbb{L}^2} + Ck\sum_{m=1}^{M}\|\mathbf{f}^m\|_{\mathbb{W}^{-1,2}}^2 + k\mathbb{E}\left[\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}\|_{\mathbb{L}^2}^2\right] \\ & + 2\mathbb{E}\left[\max_{1\le i\le M}\sum_{m=1}^{i}\big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{U}^{m-1}\big)\right]\,, \end{aligned} \tag{3.3.8}$$

where we used the fact that for sums of positive terms, the maximum is reached for $i = M$. The first two terms are clearly bounded. To bound the third term we proceed like in getting (3.3.6) and use the auxiliary inequality (3.3.7), together with condition (S_2). It remains to bound the fourth term. This is accomplished with the help of the Burkholder-Davis-Gundy inequality, see [79, Theorem 3.3.28], and assumption (S_2), after treating the sum as the stochastic integral of a piecewise constant integrand:

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\leq i\leq M}\sum_{m=1}^{i}\left(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{U}^{m-1}\right)\right]\\
&\leq C\mathbb{E}\left[\left(k\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\right)^{1/2}\right]\\
&\leq \mathbb{E}\left[\max_{1\leq m\leq M}\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}\left(k\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\right)^{1/2}\right] \qquad (3.3.9)\\
&\leq \frac{1}{2}\|\mathbf{U}^0\|^2_{\mathbb{L}^2}+\frac{1}{2}\mathbb{E}\left[\max_{1\leq i\leq M}\|\mathbf{U}^i\|^2_{\mathbb{L}^2}+k\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\right]\\
&\leq \frac{1}{2}\|\mathbf{U}^0\|^2_{\mathbb{L}^2}+\frac{1}{2}\mathbb{E}\left[\max_{1\leq i\leq M}\|\mathbf{U}^i\|^2_{\mathbb{L}^2}+K_1^2k\sum_{m=1}^{M}\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\right].
\end{aligned}$$

In the third inequality, we change the index from m to i and shift it by one, taking in account the initial condition. Thus, we can absorb the first term on the left hand side (3.3.8); using the bound (3.3.7), we get then the first inequality in (ii).

Step 4. *Bounds for the higher moments of the velocity.* First we prove the assertion for $p = 2$. At the end we indicate how our argument can be extended to any natural number $p \geq 3$. Again we proceed like in Step 2, by proving an auxiliary bound analogous to (3.3.7), to deduce the second and then the first part of the inequality in (iii).

We begin by multiplying identity (3.3.4) by $\|\mathbf{U}^m\|^2_{\mathbb{L}^2}$, and consider the last two terms on the corresponding right-hand side:

$$I := \left(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{U}^m-\mathbf{U}^{m-1}\right)\|\mathbf{U}^m\|^2_{\mathbb{L}^2},\quad II := \left(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{U}^{m-1}\right)\|\mathbf{U}^m\|^2_{\mathbb{L}^2}.$$

By using the Cauchy-Schwarz's inequality, we infer that

$$\begin{aligned}
I &\leq \|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}|^2_{\boldsymbol{\mathcal{K}}}\|\mathbf{U}^m\|^2_{\mathbb{L}^2}+\frac{1}{4}\|\mathbf{U}^m-\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\,\|\mathbf{U}^m\|^2_{\mathbb{L}^2}\\
&= \|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}|^2_{\boldsymbol{\mathcal{K}}}\left(\left[\|\mathbf{U}^m\|^2_{\mathbb{L}^2}-\|^2\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\right]+\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\right)\\
&\quad +\frac{1}{4}\|\mathbf{U}^m-\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\,\|\mathbf{U}^m\|^2_{\mathbb{L}^2} \qquad (3.3.10)\\
&\leq \|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}|^2_{\boldsymbol{\mathcal{K}}}\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}+4\|\mathbf{g}(\mathbf{U}^{m-1})\|^4_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}|^4_{\boldsymbol{\mathcal{K}}}\\
&\quad +\frac{1}{16}\Big|\|\mathbf{U}^m\|^2_{\mathbb{L}^2}-\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\Big|^2+\frac{1}{4}\|\mathbf{U}^m-\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\,\|\mathbf{U}^m\|^2_{\mathbb{L}^2}\,,
\end{aligned}$$

and

$$\begin{aligned}
II &= \left(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{U}^{m-1}\right)\left(\|\mathbf{U}^m\|^2_{\mathbb{L}^2}-\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}+\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\right)\\
&\leq \left(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W},\mathbf{U}^{m-1}\right)\,\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}+4\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\Delta_m\mathbf{W}|^2_{\boldsymbol{\mathcal{K}}}\,\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\\
&\quad +\frac{1}{16}\Big|\|\mathbf{U}^m\|^2_{\mathbb{L}^2}-\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\Big|^2. \qquad (3.3.11)
\end{aligned}$$

Next, by using the identity (3.3.3), and the above estimates (3.3.10) and (3.3.11), we infer that there exists a constant $C>0$ which does not depend on $k,h>0$ such that

$$\begin{aligned}
&\frac{1}{4}\Big(\|\mathbf{U}^m\|_{\mathbb{L}^2}^4 - \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4 + \big(1-[\frac{1}{4}+\frac{1}{4}]\big)\Big|\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\Big|^2\Big) \\
&\qquad + \frac{1}{4}\|\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big(\|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 + k\nu\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big) \\
&\leq 4\big\|\mathbf{g}(\mathbf{U}^{m-1})\big\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^4|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^4 + 5\big\|\mathbf{g}(\mathbf{U}^{m-1})\big\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2\,\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 \\
&\qquad + Ck\|\mathbf{f}^m\|_{\mathbb{W}^{-1,2}}^4 + \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{U}^{m-1}\big)\,\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\,.
\end{aligned} \tag{3.3.12}$$

Then, proceeding as in (3.3.6), by using (2.3.4) for $n=2$, and the assumption (S_2), we get that

$$\begin{aligned}
\mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^4|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^4\Big] &\leq C(\operatorname{Tr}\mathbf{Q})^2k^2\,\mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^4\Big] \\
&\leq K_1^4C(\operatorname{Tr}\mathbf{Q})^2k^2\,\mathbb{E}\Big[\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\Big]\,.
\end{aligned}$$

Next, by (2.3.4), and the linear growth condition (S_2) for $\mathbf{g}$, we find that the second term in (3.3.12) can be estimated as follows,

$$\begin{aligned}
\mathbb{E}\Big[\big\|\mathbf{g}(\mathbf{U}^{m-1})\big\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2\,\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\Big] &\leq K_1^2C\mathbb{E}\Big[\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4|\Delta_m\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^2\Big] \\
&\leq K_1^2C(\operatorname{Tr}\mathbf{Q})k\mathbb{E}\Big[\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\Big]\,.
\end{aligned}$$

Since the fourth term on the right-hand side of (3.3.12) vanishes when taking expectation, by the discrete version of the Gronwall Lemma, proceeding as in Step 2, we obtain the second inequality in (ii) i.e.

$$\max_{1\leq m\leq M}\mathbb{E}\Big[\|\mathbf{U}^m\|_{\mathbb{L}^2}^4\Big] \leq C_{T,2}\,. \tag{3.3.13}$$

Next we use it to conclude

$$\begin{aligned}
&\max_{1\leq m\leq M}\mathbb{E}\Big[\|\mathbf{U}^m\|_{\mathbb{L}^2}^4\Big] + \frac{1}{2}\mathbb{E}\Big[\sum_{m=1}^{M}\Big|\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\Big|^2\Big] \\
&+\mathbb{E}\Big[\sum_{m=1}^{M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big(\|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 + k\nu\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)\Big] \leq C_{T,2}\,.
\end{aligned}$$

For the first inequality in (iii), we use a corresponding strategy as in (3.3.8) and (3.3.9).

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\leq i\leq M}\sum_{m=1}^{i}\big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{U}^{m-1}\big)\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\right] \\
&\leq C\mathbb{E}\left[\left(k\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^6\right)^{1/2}\right] \\
&\leq C\mathbb{E}\left[\max_{1\leq m\leq M}\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\left(k\sum_{m=1}^{M}\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\right)^{1/2}\right] \\
&\leq \frac{1}{2}\mathbb{E}\left[\max_{1\leq i\leq M}\|\mathbf{U}^i\|_{\mathbb{L}^2}^4\right] + C\|\mathbf{U}^0\|_{\mathbb{L}^2}^4 + C\mathbb{E}\left[K_1^2k\sum_{m=1}^{M}\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\right].
\end{aligned} \tag{3.3.14}$$

Thus, using (3.3.13), we may conclude the assertion for (iii) for $p = 2$.

We may now continue successively for $p \geq 3$, by multiplying (3.3.12) by $\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^{p-1}}$, to arrive at the following inequality

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{1\leq m\leq M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^p} + c_p\sum_{m=1}^{M}\Big|\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^{p-1}} - \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^{2^{p-1}}\Big|^2\Big] \\
&+\mathbb{E}\Big[\sum_{m=1}^{M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^{\xi_p}\Big(\|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 + k\nu\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)\Big] \leq C_{T,p}\,, \qquad (3.3.15)
\end{aligned}$$

for some constants $c_p \geq 0$ and $\xi_p = \sum_{\ell=1}^{p-1} 2^\ell = 2^p - 2$. We leave the details of the derivation of (3.3.15) to the reader.

Step 5. *A priori estimates for the gradient norm of the velocity.* Here we prove the inequality (iv). As in the previous step, we only consider in detail the case $p = 2$, and hints for the general case are given. We take the sum in (3.3.4) (with index i instead of m) from $i = 1$ to $i = m$, and afterwards square the resulting equality. This us leads to the following inequality

$$\begin{aligned}
&\frac{1}{C}\|\mathbf{U}^m\|_{\mathbb{L}^2}^4 \;+\; \Big(\frac{k\nu}{C}\sum_{i=1}^{m}\|\nabla\mathbf{U}^i\|_{\mathbb{L}^2}^2\Big)^2 \leq \|\mathbf{U}^0\|_{\mathbb{L}^2}^4 + \Big(k\sum_{i=1}^{m}\|\mathbf{f}^i\|_{\mathbb{L}^2}^2\Big)^2 \\
&+\;\Big(\sum_{i=1}^{m}\|\mathbf{g}(\mathbf{U}^{i-1})\Delta_i\mathbf{W}\|_{\mathbb{L}^2}^2\Big)^2 + \Big(\sum_{i=1}^{m}(\mathbf{g}(\mathbf{U}^{i-1})\Delta_i\mathbf{W}, \mathbf{U}^{i-1})\Big)^2. \qquad (3.3.16)
\end{aligned}$$

Next we consider only the expectation of the last two terms; the other terms can be bounded easily. In the first case, by the Cauchy-Schwarz's inequality for sums, and assumption (S_2), we have

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{i=1}^{m}\|\mathbf{g}(\mathbf{U}^{i-1})\Delta_i\mathbf{W}\|_{\mathbb{L}^2}^2\Big)^2\Big] &\leq m\sum_{i=1}^{m}\mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{i-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^4|\Delta_i\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^4\Big] \\
&\leq K_1^4 m\sum_{i=1}^{m}\mathbb{E}\Big[\|\mathbf{U}^{i-1}\|_{\mathbb{L}^2}^4|\Delta_i\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^4\Big] = \cdots .
\end{aligned}$$

Next, using the tower property, the inequality (2.3.4) for $n = 2$, and the already proved first part of assertion (iii) we get

$$\begin{aligned}
\cdots &= K_1^2 m\sum_{i=1}^{m}\mathbb{E}\Big[\|\mathbf{U}^{i-1}\|_{\mathbb{L}^2}^4\mathbb{E}\big[|\Delta_i\mathbf{W}|_{\boldsymbol{\mathcal{K}}}^4|\mathcal{F}_{t_{i-1}}\big]\Big] \\
&\leq C_2(\mathrm{Tr}\,\mathbf{Q})^2 mk^2\sum_{i=1}^{m}\mathbb{E}\big[\|\mathbf{U}^{i-1}\|_{\mathbb{L}^2}^4\big] \leq C_{T,2}\,.
\end{aligned}$$

In the last term of (3.3.16) we use the fact the discrete time sequence can be represented as a stochastic integral together with Burkholder-Davis-Gundy inequality, assumption (S_2), and first part of assertion (iii), to conclude that

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{m}(\mathbf{g}(\mathbf{U}^{i-1})\Delta_i\mathbf{W}, \mathbf{U}^{i-1})\Big)^2\Big] \leq C\mathbb{E}\Big[k\sum_{i=1}^{m}\|\mathbf{U}^{i-1}\|_{\mathbb{L}^2}^4\Big] \leq C_{T,2}\,. \qquad (3.3.17)$$

Putting things together in (3.3.16) shows inequality (iv) for $p = 2$.

For the case $p \geq 3$ we note that the following inequality

$$
\begin{aligned}
&\frac{1}{C_p}\|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^p} + \Big(\frac{k\nu}{C_p}\sum_{i=1}^m \|\nabla \mathbf{U}^i\|_{\mathbb{L}^2}^2\Big)^{2^{p-1}} \leq \|\mathbf{U}^0\|_{\mathbb{L}^2}^{2^p} + \Big(k\sum_{i=1}^m \|\mathbf{f}\|_{\mathbb{L}^2}^2\Big)^{2^{p-1}} \\
&+\Big(\sum_{i=1}^m \|\mathbf{g}(\mathbf{U}^{i-1})\Delta_i \mathbf{W}\|_{\mathbb{L}^2}^2\Big)^{2^{p-1}} + \Big(\sum_{i=1}^m (\mathbf{g}(\mathbf{U}^{i-1})\Delta_i \mathbf{W}, \mathbf{U}^{i-1})\Big)^{2^{p-1}}
\end{aligned}
\tag{3.3.18}
$$

holds.

We take expectation and bound each term independently. The first and second terms can easily be controlled. The third term on the right-hand side can be bounded using the inequality (2.3.4) together with the tower property, assumption (S_2), and inequality (iii). For the expectation of the last term in (3.3.18), we represent the discrete time sequence as a stochastic integral and use the Burkholder-Davis-Gundy inequality.

The proof of the Lemma is therefore complete. □

Remark 3.3.1. *To have $\|\mathbf{U}^0\|_{\mathbb{L}^2} \leq C$ independently of $h > 0$, we can choose for instance $\mathbf{U}^0 = \mathbf{Q}_h^0 \mathbf{u}_0$, because of the stability in $\mathbb{L}^2$ of the projection $\mathbf{Q}_h^0$; see [59, Lemma 3.1].*

In the deterministic case, a mesh constraint ensures the uniqueness property for iterates of Algorithm 5.1 by a contraction argument for restricted numerical parameters $F(k,h) > 0$. Here, stochastic effects only allow uniqueness of iterates $\{\mathbf{U}^m\} \subset L^2(\Omega;\mathbb{L}^2(D))$ of Algorithm 5.1 on a set Ω_ε, $\varepsilon = \varepsilon(k,h) > 0$, with probability close to one. The following lemma shows uniqueness on those sets $\Omega_\varepsilon \subset \Omega$ under restrictive conditions on numerical parameters, evidencing the subtle interplay of discretisation and stochastic effects in a general setup of data.

Lemma 3.3.2. *Let $d = 2, 3$, and $D \subset \mathbb{R}^d$ be a bounded polygonal domain. Suppose that the parameters $k, h, \varepsilon, \nu > 0$ satisfy the constraint*

$$\frac{Ck}{h^{2d-2}\varepsilon} < 1\,,$$

with a constant $C = C(d, D, \nu^{-1}) > 0$. Then iterates $\{\mathbf{U}^m\}_{m=1}^M$ of Algorithm 5.1 are unique on the set

$$\Omega_\varepsilon := \Big\{\omega \in \Omega \;\Big|\; \max_{1\leq m\leq M} \|\mathbf{U}^m(\omega)\|_{\mathbb{L}^2}^4 \leq \frac{1}{\varepsilon}\Big\}\,.$$

where $\mathbb{P}[\Omega_\varepsilon] \geq 1 - \mathbb{E}\big[\max_{1\leq m\leq M}\|\mathbf{U}^m\|_{\mathbb{L}^2}^4\big]\,\varepsilon$.

Proof. **a)** Let $d = 3$. Suppose that two sequences $\{\mathbf{U}^m\}_{m=1}^M$, $\{\mathbf{V}^m\}_{m=1}^M \subset L^2(\Omega;\mathbb{L}^2(D))$ solve Algorithm 5.1 for the initial condition $\mathbf{U}^0$. We subtract the corresponding equations to get for $\mathbf{\Phi} = \mathbf{Z}^m := \mathbf{U}^m - \mathbf{V}^m$ for $m \geq 1$

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\mathbf{Z}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{Z}^{m-1}\|_{\mathbb{L}^2}^2 + \|\mathbf{Z}^m - \mathbf{Z}^{m-1}\|_{\mathbb{L}^2}^2\Big) + k\nu\|\nabla \mathbf{Z}^m\|_{\mathbb{L}^2}^2 + k\big([\mathbf{U}^m\cdot\nabla]\mathbf{U}^m, \mathbf{Z}^m\big) \\
&-k\big([\mathbf{V}^m\cdot\nabla]\mathbf{V}^m, \mathbf{Z}^m\big) + \frac{k}{2}\big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \mathbf{Z}^m\big) - \frac{k}{2}\big([\operatorname{div}\mathbf{V}^m]\mathbf{V}^m, \mathbf{Z}^m\big) \\
&= \Big([\mathbf{g}(\mathbf{U}^{m-1}) - \mathbf{g}(\mathbf{V}^{m-1})]\Delta_m \mathbf{W}, \mathbf{Z}^m - \mathbf{Z}^{m-1}\Big) + \big([\mathbf{g}(\mathbf{U}^{m-1}) - \mathbf{g}(\mathbf{V}^{m-1})]\Delta_m \mathbf{W}, \mathbf{Z}^{m-1}\big)\,.
\end{aligned}
\tag{3.3.19}
$$

Taking into account the skew-symmetricity of the stabilised convective term, we obtain

$$
\begin{aligned}
&\big([\mathbf{U}^m\cdot\nabla]\mathbf{U}^m, \mathbf{Z}^m\big) - \big([\mathbf{V}^m\cdot\nabla]\mathbf{V}^m, \mathbf{Z}^m\big) + \frac{1}{2}\big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \mathbf{Z}^m\big) - \frac{1}{2}\big([\operatorname{div}\mathbf{V}^m]\mathbf{V}^m, \mathbf{Z}^m\big) \\
&= \big([\mathbf{Z}^m\cdot\nabla]\mathbf{U}^m, \mathbf{Z}^m\big) + \frac{1}{2}\big([\operatorname{div}\mathbf{Z}^m]\mathbf{U}^m, \mathbf{Z}^m\big)\,.
\end{aligned}
$$

Then, for the first nonlinear convective term we use Hölder's inequality together with interpolation of $\mathbb{L}^3$ between $\mathbb{L}^2$ and $\mathbb{W}^{1,2}$, and an inverse estimate,

$$\begin{aligned}\big([\mathbf{Z}^m\cdot\nabla]\mathbf{U}^m,\mathbf{Z}^m\big) &\le \|\mathbf{Z}^m\|_{\mathbb{L}^3}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}\|\mathbf{Z}^m\|_{\mathbb{L}^6}\le\frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu^3}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\nabla\mathbf{U}^m\|^4_{\mathbb{L}^2}\\ &\le \frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu^3}h^{-4}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\mathbf{U}^m\|^4_{\mathbb{L}^2}\,,\end{aligned}\tag{3.3.20}$$

and correspondingly

$$\big([\operatorname{div}\mathbf{Z}^m]\mathbf{U}^m,\mathbf{Z}^m\big) \le \frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu^3}h^{-4}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\mathbf{U}^m\|^4_{\mathbb{L}^2}\,.\tag{3.3.21}$$

Here $C=C(d,D)>0$ denotes a constant resulting from the Gagliardo-Nirenberg inequality, and from the inverse estimate. By Lemma 3.1 any finite moment of solutions of Algorithm 5.1 is bounded. Hence, we may define a set Ω_ε such that $\max_{1\le m\le M}\|\mathbf{U}^m(\omega)\|^4_{\mathbb{L}^2}\le\frac{1}{\varepsilon}$ for a.e. $\omega\in\Omega_\varepsilon$ and find by Chebyshev's inequality

$$\mathbb{P}[\Omega_\varepsilon]\ge 1-\varepsilon\mathbb{E}\left[\max_{1\le m\le M}\|\mathbf{U}^m\|^4_{\mathbb{L}^2}\right]\ge 1-C_{T,2}\varepsilon \qquad (\varepsilon>0)\,.$$

Thus, after multiplying (3.3.19) with the indicator function of Ω_ε, and taking expectation, we may proceed by induction to establish $\max_{1\le m\le M}\mathbb{E}\left[\|\mathbf{U}^m-\mathbf{V}^m\|^2_{\mathbb{L}^2}\right]=0$.

First step: $m=1$. Using the fact that $\mathbf{Z}^0\equiv\mathbf{0}$ together with (3.3.20) and (3.3.21), we obtain

$$\mathbb{E}\left[\left(\frac{1}{2}-C\frac{k}{h^4\varepsilon\nu^3}\right)\mathbb{1}_{\Omega_\varepsilon}\|\mathbf{Z}^1\|^2_{\mathbb{L}^2}+k\nu\mathbb{1}_{\Omega_\varepsilon}\|\nabla\mathbf{Z}^1\|^2_{\mathbb{L}^2}\right]\le 0\,,\tag{3.3.22}$$

provided $Ckh^{-4}\varepsilon^{-1}\nu^{-3}<1/2$.

Induction step: $m-1\to m$. Assuming $\max_{1\le i\le m-1}\mathbb{E}\left[\mathbb{1}_{\Omega_\varepsilon}\|\mathbf{Z}^i\|^2_{\mathbb{L}^2}\right]=0$, we want to prove

$$\max_{1\le i\le m}\mathbb{E}\left[\mathbb{1}_{\Omega_\varepsilon}\|\mathbf{Z}^i\|^2_{\mathbb{L}^2}\right]=0\,.\tag{3.3.23}$$

First we observe that because of the induction assumption, there holds $\mathbf{U}^{m-1}=\mathbf{V}^{m-1}$ $\mathbb{P}$-a.s. on Ω_ε, which implies that the term corresponding to the stochastic integral disappears. Then, because of (3.3.19) we have

$$\mathbb{E}\left[\frac{1}{2}\mathbb{1}_{\Omega_\varepsilon}\Big(\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}+k\nu\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}\Big)\right]\le C\frac{k}{h^4\nu^{-3}\varepsilon^{-1}}\mathbb{E}\left[\mathbb{1}_{\Omega_\varepsilon}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\right]\,.$$

This implies (3.3.23).

b) Let now $d=2$. The main difference with respect to the three-dimensional case is that now we interpolate $\mathbb{L}^4$ between $\mathbb{L}^2$ and $\mathbb{W}^{1,2}$, obtaining

$$\begin{aligned}\big([\mathbf{Z}^m\cdot\nabla]\mathbf{U}^m,\mathbf{Z}^m\big) &\le \|\mathbf{Z}^m\|^2_{\mathbb{L}^4}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}\le\frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\nabla\mathbf{U}^m\|^2_{\mathbb{L}^2}\\ &\le \frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu}h^{-2}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\mathbf{U}^m\|^2_{\mathbb{L}^2}\,,\end{aligned}$$

and correspondingly

$$\big([\operatorname{div}\mathbf{Z}^m]\mathbf{U}^m,\mathbf{Z}^m\big) \le \frac{\nu}{2}\|\nabla\mathbf{Z}^m\|^2_{\mathbb{L}^2}+\frac{C}{\nu^3}h^{-2}\|\mathbf{Z}^m\|^2_{\mathbb{L}^2}\|\mathbf{U}^m\|^4_{\mathbb{L}^2}\,.$$

Proceeding now as in past **a)** settles the assertion for $Ckh^{-2}\varepsilon^{-1}\nu^{-3}<1/2$. □

The next result quantifies the time variation of the iterates $\{\mathbf{U}^m\}_{m=0}^M$ from Algorithm 3.1. It will be useful later on to validate the related compactness properties of the iterates $\{\mathbf{U}^m\}_{m=0}^M$. Its proof evidences the interaction of noise, the incompressibility constraint, and the chosen space-time discretisation.

Lemma 3.3.3.(i) *There exists a constant $C_T > 0$ such that for every $\ell \in \{1, \dots, M\}$*

$$\mathbb{E}\Big[k \sum_{m=0}^{M-\ell} t_\ell^{3/4} \|\mathbf{U}^{m+\ell} - \mathbf{U}^m\|_{\mathbb{L}^2}^2 + \|\mathbf{U}^{m+\ell} - \mathbf{U}^m\|_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}^2\Big] \le C_T t_\ell \,. \tag{3.3.24}$$

(ii) *For every $p \ge 2$ there exists a constant $C_p > 0$ such that for every $\ell \in \{1, \dots, M\}$*

$$\mathbb{E}\Big[k \sum_{m=0}^{M-\ell} \|\mathbf{U}^{m+\ell} - \mathbf{U}^m\|_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}^p\Big] \le C_{T,p} t_\ell^{p/2} \,. \tag{3.3.25}$$

Proof. **Step 1.** *Proof of the first part of inequality (i).* **a)** First assume $m \ge 1$. Fix a natural number $\ell \in \{1, \dots, M\}$. We replace the index m by i in (3.3.1), and take the sum from $i = m+1$ to $i = m+\ell$. Then choosing $\mathbf{\Phi} = \mathbf{U}^{m+\ell} - \mathbf{U}^m$, and finally taking the sum over $m \in \{1, \dots, M-\ell\}$ leads us to the following identity

$$\begin{aligned} k \sum_{m=1}^{M-\ell} \|\mathbf{U}^{m+\ell} - \mathbf{U}^m\|_{\mathbb{L}^2}^2 &= -\nu k^2 \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \Big(\nabla \mathbf{U}^{m+i}, \nabla[\mathbf{U}^{m+\ell} - \mathbf{U}^m]\Big) \\ &- k^2 \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \Big([\mathbf{U}^{m+i} \cdot \nabla]\mathbf{U}^{m+i} + \frac{1}{2}[\operatorname{div} \mathbf{U}^{m+i}]\mathbf{U}^{m+i}, \mathbf{U}^{m+\ell} - \mathbf{U}^m\Big) \\ &+ k^2 \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \Big(\mathbf{f}^{m+i}, \mathbf{U}^{m+\ell} - \mathbf{U}^m\Big) \\ &+ k \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \Big(\mathbf{g}(\mathbf{U}^{m+i-1})\Delta_{m+i}\mathbf{W}, \mathbf{U}^{m+\ell} - \mathbf{U}^m\Big) \\ &=: I + II + III + IV \,. \end{aligned} \tag{3.3.26}$$

Our aim is to show that the expectation of each of these four terms can be estimated from above by $Ct_\ell^{1/4}$. We begin with the first term. By the Cauchy-Schwarz inequality, and Lemma 3.3.1, (ii), we have

$$\begin{aligned} \mathbb{E}[I] &\le \nu \, \mathbb{E}\Big[k \sum_{m=1}^{M-\ell} \|\nabla[\mathbf{U}^{m+\ell} - \mathbf{U}^m]\|_{\mathbb{L}^2} \Big(k \sum_{i=1}^{\ell} \|\nabla \mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big)\Big] \\ &\le 2\nu\sqrt{t_\ell}\, \mathbb{E}\Big[k \sum_{m=1}^{M} \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2} \Big(k \sum_{i=1}^{\ell} \|\nabla \mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\Big)^{1/2}\Big] \\ &\le 2\nu\sqrt{t_\ell}\, \mathbb{E}\Big[k \sum_{m=1}^{M} \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2} \Big(k \sum_{m=1}^{M} \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)^{1/2}\Big] \qquad (3.3.27) \\ &\le C_T \nu \sqrt{t_\ell}\sqrt{T}\, \mathbb{E}\Big[k \sum_{m=1}^{M} \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2}^2\Big] \le C_T t_\ell^{1/2} \le C T^{1/4} t_\ell^{1/4} \,. \end{aligned}$$

Since both terms in II are similar to each other, we only treat the first one. Since the Sobolev embedding $\mathbb{W}^{1,2} \subset \mathbb{L}^6$ is continuous, by using the Gagliardo-Nirenberg inequality for $d = 3$, the

Cauchy-Schwarz inequality for sums, the Hölder's inequality, and the second part of (iii) and (iv) in Lemma 3.3.1, and similar ideas as in the estimates of I, we infer that

$$\begin{aligned}
\mathbb{E}\big[II\big] &\leq k^2\mathbb{E}\Big[\sum_{m=1}^{M-\ell}\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|_{\mathbb{L}^6}\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^3}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big] \\
&\leq Ck\,\mathbb{E}\left[\sum_{m=1}^{M-\ell}\|\nabla[\mathbf{U}^{m+\ell}-\mathbf{U}^m]\|_{\mathbb{L}^2}k\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{1/2}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{3/2}\right] \\
&\leq Ct_\ell^{1/4}k\,\mathbb{E}\left[2\sum_{m=1}^{M}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}\Big(k\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{2/3}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\Big)^{3/4}\right] \\
&\leq Ct_\ell^{1/4}\left(\mathbb{E}\Big[\Big(k\sum_{m=1}^{M}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}\Big)^4\Big]\right)^{1/4}\mathbb{E}\left[k\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{2/3}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\right]^{3/4} \\
&\leq C_T t_\ell^{1/4}\,. \qquad (3.3.28)
\end{aligned}$$

The last estimate follows from the

$$\mathbb{E}\left[k\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{2/3}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\right] \leq C\mathbb{E}\left[k\sum_{i=1}^{\ell}\big(\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2+1\big)\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\right], \qquad (3.3.29)$$

We omit the straightforward estimates for $\mathbb{E}\big[III\big]$. For what concerns the last term in the equality (3.3.26), by the Young's inequality, the Itô isometry (2.3.1), and the assumption (S_2), we infer that

$$\mathbb{E}\left[IV\right] \leq C_T(\mathrm{Tr}\,\mathbf{Q})T^{3/4}t_\ell^{1/4}\Big(\max_{1\leq m\leq M}\mathbb{E}\big[\|\mathbf{U}^m\|_{\mathbb{L}^2}^2\big]\Big)+\frac{1}{4}k\sum_{m=1}^{M-\ell}\mathbb{E}\Big[\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big]\,.$$

We conclude by observing that the second term above can be absorbed into the left-hand side (3.3.26) and the first can be estimated with Lemma 3.3.1, (ii).

b) Consider the case $m \geq 0$. It is enough to consider $m = 0$ and $\ell = M-1$ in (3.3.26). Then we have

$$k\mathbb{E}\big[\|\mathbf{U}^\ell-\mathbf{U}^0\|_{\mathbb{L}^2}^2\big] \leq 2\big(k\mathbb{E}\big[\|\mathbf{U}^\ell-\mathbf{U}^1\|_{\mathbb{L}^2}^2\big]+k\mathbb{E}\big[\|\mathbf{U}^1-\mathbf{U}^0\|_{\mathbb{L}^2}^2\big]\big).$$

Because of Lemma 3.3.1, we have that $k\mathbb{E}\big[\|\mathbf{U}^1-\mathbf{U}^0\|_{\mathbb{L}^2}^2\big] \leq C_T k$. Thus, using **a)**, we have

$$k\mathbb{E}\big[\|\mathbf{U}^\ell-\mathbf{U}^1\|_{\mathbb{L}^2}^2\big]+k\mathbb{E}\big[\|\mathbf{U}^1-\mathbf{U}^0\|_{\mathbb{L}^2}^2\big] \leq C(t_{\ell-1}^{1/4}+k) \leq Ct_\ell^{1/4},$$

for k small enough. Thus the first part of (i) is proved.

Step 2. *Proof of the second part of inequality (i).* Let us fix a natural number $\ell \in \{1,\ldots,M\}$. Let us recall that there exist $C > 0$ such that the following inequality holds

$$\|\cdot\|_{(\mathbb{V}\cap\mathbb{W}^{2,2})'} \leq C\sup\Big\{(\cdot,\boldsymbol{\varphi}) : \boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}} \leq 1\Big\}.$$

Consider (3.3.1), with index i instead of m, and sum it up from $i = m+1$ to $i = m+\ell$. Then take the sum of it from $m = 0$ to $m = M-\ell$, and finally the norm, and the expectation of it. We then arrive at the following estimate

$$
\begin{aligned}
&\frac{k}{C}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|^2_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}\Big]\\
&\quad\leq\ k^3\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(\sum_{i=1}^{\ell}\nabla\mathbf{U}^{m+i},-\nabla[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}-\nabla\boldsymbol{\varphi}\Big)^2\Big]\\
&\quad+\ k^3\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(\sum_{i=1}^{\ell}[\mathbf{U}^{m+i}\cdot\nabla]\mathbf{U}^{m+i}+\frac{1}{2}[\operatorname{div}\mathbf{U}^{m+i}]\mathbf{U}^{m+i},[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\Big)^2\Big]\\
&\quad+\ k^3\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(\sum_{i=1}^{\ell}\mathbf{f}^{m+i},[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\Big)^2\Big]\\
&\quad+\ k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(\sum_{i=1}^{\ell}\mathbf{g}(\mathbf{U}^{m+i-1})\Delta_{m+i}\mathbf{W},[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\Big)^2\Big]\\
&\quad=:\ I+II+III+IV\,.
\end{aligned}\tag{3.3.30}
$$

We proceed separately with the terms $I,\dots,IV$.

By the second part of inequality (3.2.3), estimates (3.2.5), and the first part of (i) in Lemma 3.3.1, we get

$$
\begin{aligned}
I\ &\leq\ k^3\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(\sum_{i=1}^{\ell}\Big\{\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\|\nabla[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}\|_{\mathbb{L}^2}\Big\}+\big(\mathbf{U}^{m+i},-\mathbf{A}\boldsymbol{\varphi}\big)\Big)^2\Big]\\
&\leq\ k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(k\sum_{i=1}^{\ell}Ch^{-1}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}h\|\mathbf{A}\boldsymbol{\varphi}\|_{\mathbb{L}^2}+\big(\mathbf{U}^{m+i},\mathbf{A}\boldsymbol{\varphi}\big)\Big)^2\Big]\\
&\leq\ Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\Big(k\sum_{i=1}^{\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big)^2\Big]\leq Ct_\ell^2\,k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\max_{1\leq i\leq\ell}\|\mathbf{U}^{m+i}\|^2_{\mathbb{L}^2}\Big] \qquad (3.3.31)\\
&\leq\ C_Tt_\ell^2\leq C_TTt_\ell\,.
\end{aligned}
$$

We continue with term II in (3.3.30). Using integration by parts and the Cauchy-Schwarz inequality, we obtain the following estimate.

$$
\begin{aligned}
II\ &=\ -k\sum_{m=0}^{M-\ell}\mathbb{E}\Bigg[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\big[\mathbf{U}^{m+i}\cdot\nabla\big]\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\},\mathbf{U}^{m+i}\Big)\\
&\qquad -\frac{1}{2}\Big(\big[\operatorname{div}\mathbf{U}^{m+i}\big]\mathbf{U}^{m+i},\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\}\Big)\Big\}^2\Bigg] \qquad (3.3.32)\\
&\leq\ Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Bigg[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\big[\mathbf{U}^{m+i}\cdot\nabla\big]\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\},\mathbf{U}^{m+i}\Big)\Big\}^2\\
&\qquad+\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\big[\operatorname{div}\mathbf{U}^{m+i}\big]\mathbf{U}^{m+i},\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\}\Big)\Big\}^2\Bigg]=II_1+II_2.
\end{aligned}
$$

Then the continuous embedding $\mathbb{W}^{1,2} \subset \mathbb{L}^6$, (3.2.3) and (3.2.5), like in (3.3.31), yield

$$\begin{aligned} II_1 &= -k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\big[\mathbf{U}^{m+i}\cdot\nabla\big]\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\},\mathbf{U}^{m+i}\Big)\Big\}^2\Big] \\ &\leq k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\|\mathbf{U}^{m+i}\|_{\mathbb{L}^3}\|\nabla\big\{[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}\|_{\mathbb{L}^2}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big)\Big\}^2\Big] \\ &\quad +k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big\{\sum_{i=1}^{\ell}k\Big(\|\mathbf{U}^{m+i}\|_{\mathbb{L}^3}\|\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^6}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big)\Big\}^2\Big] \qquad (3.3.33)\\ &\leq Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\Big(\sum_{i=1}^{\ell}k\|\mathbf{U}^{m+i}\|_{\mathbb{L}^3}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\Big)^2\Big] \leq \dots \end{aligned}$$

Then the Gagliardo-Nirenberg inequality for $d = 3$, Lemma 3.3.1, (iv), and the first part of (iii) further lead us to

$$\begin{aligned} \dots &\leq Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\max_{1\leq i\leq\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^3\Big(k\sum_{i=1}^{\ell}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^{1/2}\Big)^2\Big] \\ &\leq Ct_\ell^{3/2}\,k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\max_{1\leq i\leq\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^3\Big(k\sum_{i=1}^{\ell}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\Big)^{1/2}\Big] \qquad (3.3.34)\\ &\leq Ct_\ell^{3/2}k\sum_{m=0}^{M-\ell}\Big(\mathbb{E}\Big[\max_{1\leq i\leq\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^4\Big]\Big)^{3/4}\Big(\mathbb{E}\Big[\Big(k\sum_{i=1}^{\ell}\|\nabla\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\Big)^2\Big]\Big)^{1/4} \leq CT^{1/2}t_\ell\,. \end{aligned}$$

The additional term II_2 is now the order-limiting term. Since in general $\mathbb{V}_h \not\subset \mathbb{V}$, the iterates $\big\{\operatorname{div}\mathbf{U}^m\big\}_{m=1}^M$ will be controlled by the second part of Lemma 3.3.1, (ii), for $\big\{\nabla\mathbf{U}^m\big\}_{m=1}^M$, such that

$$\begin{aligned} II_b &\leq \frac{Ck}{2}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\leq 1}}\Big(k\sum_{i=1}^{\ell}\big\|\operatorname{div}\mathbf{U}^{m+i}\big\|_{L^2}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}\big\|[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\big\|_{\mathbb{L}^\infty}\Big)^2\Big] \\ &\leq Ct_\ell k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\max_{1\leq i\leq\ell}\|\mathbf{U}^{m+i}\|_{\mathbb{L}^2}^2\Big(k\sum_{i=1}^{\ell}\|\operatorname{div}\mathbf{U}^{m+i}\|_{L^2}^2\Big)\Big] \leq Ct_\ell\,, \qquad (3.3.35) \end{aligned}$$

where the following bound is used for every $0 < \delta \leq 3$,

$$\begin{aligned} \|[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}\|_{\mathbb{L}^\infty} &\leq C\|\nabla[\mathbf{Q}_h^0-\boldsymbol{\mathcal{I}}_h]\boldsymbol{\varphi}\|_{\mathbb{L}^{3+\delta}} + C\|\nabla[\boldsymbol{\mathcal{I}}_h-\mathbf{Id}]\boldsymbol{\varphi}\|_{\mathbb{L}^{3+\delta}} \\ &\leq Ch^{-d\frac{1+\delta}{6+2\delta}}\|\nabla[(\mathbf{Q}_h^0-\mathbf{Id})+(\mathbf{Id}-\boldsymbol{\mathcal{I}}_h)]\boldsymbol{\varphi}\|_{\mathbb{L}^2} + C\|\boldsymbol{\varphi}\|_{\mathbb{W}^{2,2}} \\ &\leq Ch^{-d\frac{1+\delta}{6+2\delta}}\Big(h+h\Big)\|\boldsymbol{\varphi}\|_{\mathbb{W}^{2,2}} + C\|\boldsymbol{\varphi}\|_{\mathbb{W}^{2,2}} \leq C\,, \end{aligned}$$

by the Sobolev embedding, the approximation properties of the Lagrange interpolation, see Appendix C , inverse estimates (3.2.5), the interpolation estimate for the Lagrange interpolation (for instance [53, Lemma 4.4.4]), and (3.2.3).

A proper control of III is immediate. For IV, by assertion (S_2), the tower property, and first part of Lemma 3.3.1, (i),

$$IV \leq Ck\sum_{m=0}^{M}\sum_{i=1}^{\ell}k\mathbb{E}\Big[\big(1+\|\mathbf{U}^{\ell-1}\|_{\mathbb{L}^2}^2\big)\Big] \leq Ct_\ell\,.$$

The proof of (i) is thus concluded.

Step 3. *Proof of assertion (ii) by modifying the argument in Step 2.* Fix some $\ell \in \{1,\dots,M\}$, change the index in (3.3.1) from m to i, and sum up from $i=m$ to $i=m+\ell$. Choose $\mathbf{\Phi} = \mathbf{Q}_h^0\boldsymbol{\varphi} \in \mathbb{V}_h$ for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}$. Summation over $m \in \{0,\dots,M-\ell\}$, property $(3.2.2)_2$, and taking expectations then leads to

$$\begin{aligned}
&\frac{k}{C_p}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\|\mathbf{U}^{m+\ell}-\mathbf{U}^m\|^p_{(\mathbb{V}\cap\mathbb{W}^{2,2})'}\Big]\\
&\quad\le k^{p+1}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\le 1}}\Big(\sum_{i=1}^{\ell}\nabla\mathbf{U}^{m+i},-\nabla[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}-\nabla\boldsymbol{\varphi}\Big)^p\Big]\\
&\qquad+k^{p+1}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\le 1}}\Big(\sum_{i=1}^{\ell}[\mathbf{U}^{m+i}\cdot\nabla]\mathbf{U}^{m+i}+\frac{1}{2}[\operatorname{div}\mathbf{U}^{m+i}]\mathbf{U}^{m+i},[\mathbf{Q}_h^0-\mathbf{Id}]\boldsymbol{\varphi}+\boldsymbol{\varphi}\Big)^p\Big]\\
&\qquad+k^{p+1}\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\le 1}}\Big(\sum_{i=1}^{\ell}\mathbf{f}^{m+i},\boldsymbol{\varphi}\Big)^p\Big]\\
&\qquad+k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}},\\ \|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{2,2}}\le 1}}\Big(\sum_{i=1}^{\ell}\mathbf{g}\big(\mathbf{U}^{m+i-1}\big)\Delta_{m+i}\mathbf{W},\boldsymbol{\varphi}\Big)^p\Big] \qquad (3.3.36)\\
&\quad=: I+II+III+IV\,.
\end{aligned}$$

A simple adaptation of the arguments from Step 2 leads to

$$I+II+III \le C_T\big(t_\ell^p+t_\ell^{3p/4}+t_\ell^{p/2}\big)\le CT^{p/2}t_\ell^{p/2}\,.$$

For IV, by the Burkholder-Davis-Gundy inequality, assumption (S_2), and the first part of Lemma 3.3.1, (ii),

$$\begin{aligned}
IV &\le Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\Big\|\sum_{i=1}^{\ell}\mathbf{g}\big(\mathbf{U}^{m+i-1}\big)\Delta_{m+i}\mathbf{W}\Big\|^p_{\mathbb{L}^2}\Big]\\
&\le Ck\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\Big(k\sum_{i=1}^{\ell}\|\mathbf{g}(\mathbf{U}^{m+i-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\,\mathrm{d}s\Big)^{p/2}\Big]\\
&\le CK_1^p t_\ell^{p/2}\,\mathbb{E}\left[\max_{1\le m\le M}(1+\|\mathbf{U}^m\|^p_{\mathbb{L}^2})\right]\le Ct_\ell^{p/2}\,.
\end{aligned}$$

This concludes the proof of the assertion (ii) and also of the whole Lemma. □

We define the piecewise affine, globally continuous process

$$\boldsymbol{\mathcal{U}}_{k,h}(t,\mathbf{x}) := \frac{t-t_{m-1}}{k}\mathbf{U}^m(\mathbf{x})+\frac{t_m-t}{k}\mathbf{U}^{m-1}(\mathbf{x})\quad\forall\,(t,\mathbf{x})\in[t_{m-1},t_m)\times D\,, \qquad (3.3.37)$$

and

$$\boldsymbol{\mathcal{U}}^-_{k,h}(t,\mathbf{x}) := \mathbf{U}^{m-1}(\mathbf{x})\quad\forall\,(t,\mathbf{x})\in[t_{m-1},t_m)\times D\,, \qquad (3.3.38)$$

$$\boldsymbol{\mathcal{U}}^+_{k,h}(t,\mathbf{x}) := \mathbf{U}^{m}(\mathbf{x})\quad\forall\,(t,\mathbf{x})\in(t_{m-1},t_m]\times D\,. \qquad (3.3.39)$$

Note that

$$\boldsymbol{\mathcal{U}}_{k,h}(t,\cdot) - \boldsymbol{\mathcal{U}}^-_{k,h}(t,\cdot) = \frac{t-t_{m-1}}{k}\big(\mathbf{U}^m - \mathbf{U}^{m-1}\big) \qquad \forall\, t \in [t_{m-1}, t_m)\,. \tag{3.3.40}$$

Hence, the $C\big([0,T];\mathbb{V}_h\big)$-valued random variable $\boldsymbol{\mathcal{U}}_{k,h}$ satisfies for every $t \in [t_{m-1}, t_m)$,

$$\begin{aligned}
&\big(\boldsymbol{\mathcal{U}}_{k,h}(t) - \boldsymbol{\mathcal{U}}^-_{k,h}, \boldsymbol{\Phi}\big) + (t-t_{m-1})\Big\{\nu\big(\nabla \boldsymbol{\mathcal{U}}^+_{k,h}, \nabla\boldsymbol{\Phi}\big) + \big([\boldsymbol{\mathcal{U}}^+_{k,h}\cdot\nabla]\boldsymbol{\mathcal{U}}^+_{k,h}, \boldsymbol{\Phi}\big)\Big\} \\
&+\frac{1}{2}(t-t_{m-1})\Big([\mathrm{div}\boldsymbol{\mathcal{U}}^+_{k,h}]\boldsymbol{\mathcal{U}}^+_{k,h}, \boldsymbol{\Phi}\Big) = (t-t_{m-1})\big(\mathbf{f}(t_m), \boldsymbol{\Phi}\big) \\
&+\frac{t-t_{m-1}}{k}\Big[\int_{t_{m-1}}^t \big(\mathbf{g}(\mathbf{U}^-)\,\mathrm{d}\mathbf{W}(s), \boldsymbol{\Phi}\big) + \big(\mathbf{g}(\mathbf{U}^-)[\mathbf{W}(t_m) - \mathbf{W}(t)], \boldsymbol{\Phi}\big)\Big]\,.
\end{aligned} \tag{3.3.41}$$

3.4 Compactness properties of iterates

Lemma 3.3.3 controls fractional derivatives of the process $\boldsymbol{\mathcal{U}} \equiv \boldsymbol{\mathcal{U}}_{k,h} : D_T \to \mathbb{R}^d$. Since $N^{s_1,r} \subset W^{s_2,r}$ continuously if $s_1 > s_2$ by [122, Cor. 24], the following result immediately follows from Lemma 3.2.1.

Lemma 3.4.1. *Let $k, h > 0$, and $T \equiv t_M > 0$. There exists $C_T > 0$ such that the solution $\boldsymbol{\mathcal{U}}_{k,h}$ of (3.3.41) satisfies for every $\alpha \in (0, \frac{1}{8})$ the bound*

$$\mathbb{E}\Big[\|\boldsymbol{\mathcal{U}}_{k,h}\|^2_{W^{\alpha,2}(0,T;\mathbb{L}^2)}\Big] \leq C_T\,.$$

For integer $p \geq 2$, $\beta \in (0, \frac{1}{2})$, there holds

$$\mathbb{E}\Big[\|\boldsymbol{\mathcal{U}}_{k,h}\|^2_{W^{\beta,p}(0,T;D(\mathbf{A}^{-\gamma}))}\Big] \leq C_T \qquad \forall\, \gamma \geq 1\,.$$

The bounds in Lemma 3.3.1, and $\mathbb{E}\big[\|\boldsymbol{\mathcal{U}}_{k,h}\|^2_{L^2(0,T;\mathbb{W}^{1,2}_0)}\big] \leq C$ as well as Lemma 3.4.1 allow for the following compactness result that follows from

(i) Lemma 3.2.2, with $X_0 = \mathbb{W}^{1,2}_0$ and $X = X_1 = \mathbb{L}^2$, for $\alpha \in (0, \frac{1}{8})$, and $q = 2$,

(ii) Lemma 3.2.3, with $X_0 = \mathbb{L}^{6/5}$ and $X = \mathbb{W}^{-1,2}$, and $\beta \in (0, \frac{1}{8})$,

(iii) Lemma 3.2.4, with $X_0 = D(\mathbf{A}^{-\widetilde{\gamma}})$ and $X = D(\mathbf{A}^{-\gamma})$ for $1 \leq \widetilde{\gamma} < \gamma$, and $\alpha \in (0, \frac{1}{2})$, $q = p$.

Lemma 3.4.2. *Assume that $\alpha \in (0, \frac{1}{8})$ and $\mathbf{U}^0 \to \mathbf{u}_0$ in $\mathbb{L}^2$. Then*
i) The sequence of laws $\{\mathcal{L}(\boldsymbol{\mathcal{U}}_{k,h})\}_{k,h}$ is tight on $L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big)$.
ii) If $\gamma > 1$, and $p > 2$, the sequence is tight on $C\big([0,T];D(\mathbf{A}^{-\gamma})\big)$.

By Lemma 3.4.2, we can find a subsequence $\{\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ denoted in the same way as the full sequence, such that the laws $\big\{\mathcal{L}(\boldsymbol{\mathcal{U}}_{k,h})\big\}_{k,h}$ converge weakly to a certain probability measure μ on $L^2\big(0,T;\mathbb{L}^2\big)$. The following result is based on the Skorokhod theorem [76, p. 9], which allows to turn over to possibly another sequence $\{\boldsymbol{\mathcal{U}}'_{k,h}\}_{k,h}$ with improved convergence properties.

Proposition 3.4.1. *Let $\alpha \in (0, \frac{1}{8})$, $\gamma > 1$, and $\mathbf{U}^0 \to \mathbf{u}_0$ in $\mathbb{L}^2$ for $h \to 0$. There exists a filtered probability space $\mathfrak{P}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, and*
i) a sequence $\big\{\boldsymbol{\mathcal{U}}'_{k,h}\big\}_{k,h}$ such that for all indices k, h

$$\boldsymbol{\mathcal{U}}'_{k,h} : \Omega \to L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big)$$

is a measurable map and

$$\mathcal{L}(\boldsymbol{\mathcal{U}}_{k,h}) = \mathcal{L}'(\boldsymbol{\mathcal{U}}'_{k,h}) \qquad \text{on } L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big)\,.$$

ii) an $L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big)$*-valued random variable* $\boldsymbol{u}$ *defined on* $\mathfrak{P}'$ *such that*

$$\mathcal{L}'(\boldsymbol{u}) = \mu \qquad \text{on } L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big)\,,$$

and $\mathbb{P}'$*-almost surely*

$$\boldsymbol{\mathcal{U}}'_{k,h} \to \boldsymbol{u} \qquad \text{in } L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big) \qquad (k,h \to 0)\,.$$

iii) Let $p > 2$*, there exists a sequence* $\{\boldsymbol{\mathcal{U}}'_{k,h}\}_{k,h}$ *such that for all indices* k, h

$$\boldsymbol{\mathcal{U}}'_{k,h} : \Omega \to L^2\big(0,T;\mathbb{L}^2\big) \cap C\big(0,T;D(\mathbf{A}^{-\gamma})\big)$$

is a measurable map and

$$\mathcal{L}(\boldsymbol{\mathcal{U}}_{k,h}) = \mathcal{L}'(\boldsymbol{\mathcal{U}}'_{k,h}) \qquad \text{on } L^2\big(0,T;\mathbb{L}^2\big) \cap C\big(0,T;D(\mathbf{A}^{-\gamma})\big)\,.$$

iv) an $L^2\big(0,T;\mathbb{L}^2\big) \cap C\big(0,T;D(\mathbf{A}^{-\gamma})\big)$*-valued random variable* $\boldsymbol{u}$ *defined on* $\mathfrak{P}'$ *such that*

$$\mathcal{L}'(\boldsymbol{u}) = \mu \qquad \text{on } L^2\big(0,T;\mathbb{L}^2\big) \cap C\big(0,T;D(\mathbf{A}^{-\gamma})\big)\,,$$

for $p > 2$*, and* $\mathbb{P}'$*-almost surely*

$$\boldsymbol{\mathcal{U}}'_{k,h} \to \boldsymbol{u} \qquad \text{in } L^2\big(0,T;\mathbb{L}^2\big) \cap C\big(0,T;D(\mathbf{A}^{-\gamma})\big) \qquad (k,h \to 0)\,.$$

We resume all the convergence results in the following

Lemma 3.4.3. *Let* α *and* γ *be as in Proposition 3.4.1, and* $\mathbf{U}^0 \to \mathbf{u}_0$ *in* $\mathbb{L}^2$ *for* $h \to 0$*. Then there exists a filtered probability space* $\mathfrak{P}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$*, such that the following convergences hold for* $k, h \to 0$*:*

$$\begin{aligned}
\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+ &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{in } L^p\big(\Omega'; L^\infty(0,T;\mathbb{L}^2)\big)\,,\\
\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+ &\rightharpoonup \boldsymbol{u} && \text{in } L^2\big(\Omega'; L^2(0,T;\mathbb{W}^{1,2}_0)\big)\,,\\
\boldsymbol{\mathcal{U}}'_{k,h} &\to \boldsymbol{u} && \text{in } L^2\big(\Omega'; L^2(0,T;\mathbb{L}^2)\big)\,,\\
\operatorname{div}\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+,\ \operatorname{div}\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^- &\rightharpoonup 0 && \text{in } L^2\big(\Omega'; L^2(0,T;L^2)\big)\,,\\
\operatorname{div}\boldsymbol{\mathcal{U}}'_{k,h} &\rightharpoonup 0 && \text{in } L^2\big(\Omega'; L^2(0,T;L^2)\big)\,.
\end{aligned} \tag{3.4.1}$$

Proof. The assertions $(3.4.1)_1$ and $(3.4.1)_2$ follow from the fact that the sequence $\{\boldsymbol{\mathcal{U}}'_{k,h}\}_{k,h}$ of piecewise affine $\mathbb{H}_h$-valued processes and the original sequence $\{\boldsymbol{\mathcal{U}}_{k,h}\}_{k,h}$ satisfy the same estimates, since they have the same laws; see Lemmata 3.3.1 and 3.3.3.

Convergence $(3.4.1)_3$ follows by Lemma 3.3.1, (ii), Proposition 3.4.1, (3.3.40) and the uniform integrability given by Lemma 3.3.1, (ii).

The first part of property $(3.4.1)_4$ follows from $(3.4.1)_2$, the discrete divergence-free constraint (3.1.5), and approximation properties of the Lagrange interpolation $\mathcal{I}_h : C^\infty(D) \to L_h$ (see [14, Theorem 4.4.4]), such that for all $t \in [0,T]$ and all $\lambda \in C^\infty(D)$,

$$\begin{aligned}
\mathbb{E}'\Big[\int_0^T \big(\operatorname{div}\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+, \lambda\big)\,\mathrm{d}s\Big] &= \mathbb{E}'\Big[\int_0^T \big(\operatorname{div}\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+, [\lambda - \mathcal{I}_h\lambda] + \mathcal{I}_h\lambda\big)\,\mathrm{d}s\Big]\\
&= \mathbb{E}'\Big[\int_0^T \big(\operatorname{div}\big(\boldsymbol{\mathcal{U}}'_{k,h}\big)^+, \lambda - \mathcal{I}_h\lambda\big)\,\mathrm{d}s\Big] \to 0 \qquad (k,h \to 0)\,.
\end{aligned}$$

The proof of the second part of $(3.4.1)_4$ follows accordingly, using the strong convergence of the discrete initial data

$$\begin{aligned}\mathbb{E}'\Big[\int_0^T \big(\operatorname{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^-,\lambda\big)\,\mathrm{d}s\Big] &= \mathbb{E}'\Big[\int_{t_1}^T \big(\operatorname{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^-,\lambda-\mathcal{I}_h\lambda\big)\,\mathrm{d}s\Big] - k\mathbb{E}'\Big[\big(\mathbf{U}^0,\nabla\lambda\big)\Big]\\ &\to 0 \quad (k,h\to 0)\,.\end{aligned}$$

Finally, property $(3.4.1)_5$ follows exactly in the same way, since $\boldsymbol{\mathcal{U}}'$ is a linear combination of $\big(\boldsymbol{\mathcal{U}}'\big)^+$ and $\big(\boldsymbol{\mathcal{U}}'\big)^-$. □

We now may identify limits of the deterministic integral in Algorithm 3.1.

Lemma 3.4.4. *For $T>0$, let $\{\boldsymbol{\mathcal{U}}'_{k,h}\}_{k,h}$ be the sequence from Proposition 3.4.1. Then for every $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}}$ and every $t\in[0,T]$,*

$$\text{(i)}\qquad \lim_{k,h\to 0}\mathbb{E}'\left[\int_0^t \big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla\mathbf{Q}_h^0\boldsymbol{\varphi}\big)\,\mathrm{d}s\right] = \mathbb{E}'\left[\int_0^t \big(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}\big)\,\mathrm{d}s\right].$$

$$\begin{aligned}\text{(ii)}\qquad &\lim_{k,h\to 0}\mathbb{E}'\Big[\int_0^t \Big([(\boldsymbol{\mathcal{U}}'_{k,h})^+\cdot\nabla](\boldsymbol{\mathcal{U}}'_{k,h})^+ + \frac{1}{2}[\operatorname{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^+](\boldsymbol{\mathcal{U}}'_{k,h})^+,\mathbf{Q}_h^0\boldsymbol{\varphi}\Big)\,\mathrm{d}s\Big]\\ &= \mathbb{E}'\Big[\int_0^t \Big([\boldsymbol{u}\cdot\nabla]\boldsymbol{u},\boldsymbol{\varphi}\Big)\,\mathrm{d}s\Big].\end{aligned}$$

Proof. In addition to $(3.4.1)_2$ and $(3.4.1)_4$, we have the convergence

$$(\boldsymbol{\mathcal{U}}'_{k,h})^-,(\boldsymbol{\mathcal{U}}'_{k,h})^+\to\boldsymbol{u} \qquad \text{in } L^2\big(\Omega';L^2(0,T;\mathbb{L}^2)\big), \tag{3.4.2}$$

which is a consequence of the third part of (ii) from Lemma 3.3.1 together with (3.3.40), for the sequence on the new probability space $\mathfrak{P}'$.

In order to prove part (i) of the lemma, let us fix $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}}$. We note that

$$\begin{aligned}&\int_0^t \Big\{\Big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla\mathbf{Q}_h^0\boldsymbol{\varphi}\Big) - \big(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}\big)\Big\}\,\mathrm{d}s\\ &\quad = \int_0^t \Big\{\Big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla[\mathbf{Q}_h^0\boldsymbol{\varphi}-\boldsymbol{\varphi}]\Big) + \Big(\nabla[(\boldsymbol{\mathcal{U}}'_{k,h})^+-\boldsymbol{u}],\nabla\boldsymbol{\varphi}\Big)\Big\}\,\mathrm{d}s\,.\end{aligned}$$

Then, we infer that

$$\lim_{k,h\to 0}\mathbb{E}'\left[\int_0^t \Big\{\Big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla[\mathbf{Q}_h^0\boldsymbol{\varphi}-\boldsymbol{\varphi}]\Big) + \Big(\nabla[(\boldsymbol{\mathcal{U}}'_{k,h})^+-\boldsymbol{u}],\nabla\boldsymbol{\varphi}\Big)\Big\}\,\mathrm{d}s\right] = 0\,.$$

For the first term in the integral this follow by the uniform boundedness of $\{\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\}_{k,h}$ in $L^2\big(\Omega';L^2(0,T;\mathbb{L}^2)\big)$ and strong convergence for $\boldsymbol{\varphi}-\mathbf{Q}_h^0\boldsymbol{\varphi}$ in $\mathbb{V}$. For the second term, this is a consequence of the weak convergence of $\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+$ in $\mathbb{L}^2$.

To prove (ii), we integrate by parts in the leading term on the left-hand side, and use (3.4.2) for the first term, and (3.4.2) together with $(3.4.1)_2$ for the second:

$$\begin{aligned}&-\lim_{k,h\to 0}\mathbb{E}'\Big[\int_0^t \Big\{\Big([(\boldsymbol{\mathcal{U}}'_{k,h})^+\cdot\nabla]\mathbf{Q}_h^0\boldsymbol{\varphi},(\boldsymbol{\mathcal{U}}'_{k,h})^+\Big) + \frac{1}{2}\Big([\operatorname{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^+](\boldsymbol{\mathcal{U}}'_{k,h})^+,\mathbf{Q}_h^0\boldsymbol{\varphi}\Big)\Big\}\,\mathrm{d}s\Big]\\ &\qquad = -\mathbb{E}'\Big[\int_0^t \Big([\boldsymbol{u}\cdot\nabla]\boldsymbol{\varphi},\boldsymbol{u}\Big)\,\mathrm{d}s\Big].\end{aligned}$$

Integration by parts and using $\operatorname{div}\boldsymbol{u}=0$ Lebesgue a.e. in D_T, and $\mathbb{P}'$-almost surely then implies assertion (ii). This completes the proof. □

Due to the strong $\mathbb{W}^{1,2}$-convergence of $\boldsymbol{\varphi} - \mathbf{Q}_h^0\boldsymbol{\varphi}$ for $h \to 0$, Lemma 3.4.4 remains valid if $\mathbf{Q}_h^0\boldsymbol{\varphi}$ is replaced by $\boldsymbol{\varphi}$. Therefore

$$[(\boldsymbol{\mathcal{U}}'_{k,h})^+ \cdot \nabla](\boldsymbol{\mathcal{U}}'_{k,h})^+ + \frac{1}{2}[\mathrm{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^+](\boldsymbol{\mathcal{U}}'_{k,h})^+ \rightharpoonup [\boldsymbol{u} \cdot \nabla]\boldsymbol{u} \quad \text{in } L^2(\Omega; L^2(0,T; D(\mathbf{A}^{-\gamma}))) .$$

3.5 Construction of weak martingale solutions

3.5.1 Construction by exact increments

We now prove that the random process $\boldsymbol{u}$ constructed in Section 3.4, together with a probability space $\mathfrak{P}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, and a Wiener process to be constructed is a weak martingale solution of equations (3.1.1)–(3.1.3). For this aim, we follow an argument from [3, Section 6]. According to the modified Skorokhod theorem formulated and proved in [128, Theorem 1.10.4 and Addendum 1.10.5], the new probability space $\mathfrak{P}'$, together with a family of measurable maps $\phi_{k,h} : \Omega' \to \Omega$ can be constructed such that

$$\begin{aligned} \mathbb{P} &= \mathbb{P}' \circ \phi_{k,h} , \\ \boldsymbol{\mathcal{U}}'_{k,h} &= \boldsymbol{\mathcal{U}}_{k,h} \circ \phi_{k,h} . \end{aligned}$$

We can then define a $C(\mathbb{R}^+, \boldsymbol{\mathcal{K}})$-valued random variable

$$\mathbf{W}'_{k,h} := \mathbf{W} \circ \phi_{k,h} , \tag{3.5.1}$$

and using [132, Section 2] it can be shown that $\mathbf{W}'_{k,h}$ it is a $\mathbf{Q}$-Wiener process. In fact, for a real-valued Brownian motion β^i from (2.2.1), we define $(\beta^i)'_{k,h}(\omega') := \beta^i(\phi_{k,h}(\omega'))$ for all $\omega' \in \Omega'$, such that

$$\mathbb{P}'[(\beta^i)' \in D] = \mathbb{P}'[\phi_{k,h}^{-1}(\beta^i)^{-1}(D)] = \mathbb{P}[\beta^i \in D].$$

The claim then follows by noting the representation (2.2.1).

In Section 3.4, the $\mathbb{H}_h$-valued process $\boldsymbol{\mathcal{U}}_{k,h}$ is defined in a natural way via piecewise affine interpolation of iterates $\{\mathbf{U}^m\}_{m=0}^M$ from (3.1.4)–(3.1.5). However, in order to construct an appropriate Wiener process, we use another continuous $\mathbb{L}^2$-valued process $\widetilde{\boldsymbol{\mathcal{U}}}_{k,h} \equiv \{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}(t)\,;\, t \in [0,T]\}$ defined by

$$\begin{aligned} &\big(\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}(t) - \widetilde{\boldsymbol{\mathcal{U}}}_{k,h}(0), \boldsymbol{\Phi}\big) + \int_0^t \Big\{\nu\big(\nabla\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h}, \nabla\boldsymbol{\Phi}\big) + \Big([\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h} \cdot \nabla]\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h}, \boldsymbol{\Phi}\Big)\Big\}\,\mathrm{d}s \\ &\quad + \int_0^T \frac{1}{2}\Big([\mathrm{div}\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h}]\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h}, \boldsymbol{\Phi}\Big)\,\mathrm{d}s = \int_0^t \langle \mathbf{f}(s), \boldsymbol{\Phi}\rangle\,\mathrm{d}s + \int_0^t \big(\mathbf{g}(\widetilde{\boldsymbol{\mathcal{U}}}^-)\,\mathrm{d}\mathbf{W}(s), \boldsymbol{\Phi}\big) \qquad \forall \boldsymbol{\Phi} \in \mathbb{V}_h . \end{aligned} \tag{3.5.2}$$

Note that this process coincides with the process $\boldsymbol{\mathcal{U}}_{k,h}$ on the grid points $\{t_m\}_{m=0}^M$, i.e.

$$\widetilde{\boldsymbol{\mathcal{U}}}^+_{k,h}(t_m) = \mathbf{U}^m = \boldsymbol{\mathcal{U}}^+_{k,h}(t_m) , \quad m = 0, 1, \ldots, M .$$

Below, we will show the strong L^2-convergence of a subsequence of $\{\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}\}_{k,h}$ to $\boldsymbol{u}$ obtained from (3.4.1). Using the properties of the stochastic integral for piecewise constant processes, we compute

$$\begin{aligned} &\mathbf{Q}_h^0\Big(\int_{t_{m-1}}^t \mathbf{g}\big(\mathbf{U}^{m-1}\big)\,\mathrm{d}\mathbf{W}(s)\Big) - \mathbf{Q}_h^0\mathbf{g}\big(\mathbf{U}^{m-1}\big)\Big\{\frac{t - t_{m-1}}{k}\big(\mathbf{W}(t_m) - \mathbf{W}(t_{m-1})\big)\Big\} \\ &= \mathbf{Q}_h^0\mathbf{g}\big(\mathbf{U}^{m-1}\big)\Big[\frac{t_m - t}{k}\big(\mathbf{W}(t) - \mathbf{W}(t_{m-1})\big) - \frac{t - t_{m-1}}{k}\big(\mathbf{W}(t_m) - \mathbf{W}(t)\big)\Big] . \end{aligned}$$

By subtracting (3.5.2) from (3.3.41), setting $\mathbf{\Phi} = \widetilde{\boldsymbol{\mathcal{U}}}_{k,h}(t) - \boldsymbol{\mathcal{U}}_{k,h}(t)$, taking expectation, using the Cauchy-Schwarz inequality and the tower property, we get for $t \in [t_{m-1}, t_m)$

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\Big[\|\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}(t) - \boldsymbol{\mathcal{U}}_{k,h}(t)\|_{\mathbb{L}^2}^2\Big] \quad &\leq \quad \mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\Big|\frac{t_m - t}{k}\big(\mathbf{W}(t) - \mathbf{W}(t_{m-1})\big)\Big|_{\boldsymbol{\mathcal{K}}}^2\Big] \\
&\quad + \mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\Big|\frac{t - t_{m-1}}{k}\big(\mathbf{W}(t_m) - \mathbf{W}(t)\big)\Big|_{\boldsymbol{\mathcal{K}}}^2\Big] \\
&\leq \quad \mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\Big](\mathrm{Tr}\,\mathbf{Q})\Big(t_m - t + t - t_{m-1}\Big) \\
&\leq \quad Ck\Big(1 + \mathbb{E}\big[\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2\big]\Big)\,.
\end{aligned} \tag{3.5.3}
$$

We may define correspondingly the family of $\mathbb{L}^2$-valued processes $\{\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h}\}_{k,h}$ analogously to (3.5.2), with $\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h} : D_T \times \Omega' \to \mathbb{R}^d$. Then, by (3.5.1) and [3, Proposition 6.3],

$$
\begin{aligned}
&\Big(\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h}(t) - \widetilde{\boldsymbol{\mathcal{U}}}'_{k,h}(0), \mathbf{\Phi}\Big) + \int_0^t \Big\{\nu\Big(\nabla(\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^+, \nabla\mathbf{\Phi}\Big) + \Big([(\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^+ \cdot \nabla](\widetilde{\boldsymbol{\mathcal{U}}}_{k,h})^+, \mathbf{\Phi}\Big)\Big\}\,\mathrm{d}s \\
&+ \int_0^T \frac{1}{2}\Big([\mathrm{div}\,(\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^+](\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^+, \mathbf{\Phi}\Big)\,\mathrm{d}s = \int_0^t \langle \mathbf{f}(s), \mathbf{\Phi}\rangle\,\mathrm{d}s \\
&+ \int_0^t \big(\mathbf{g}\big((\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^-\big)\,\mathrm{d}\mathbf{W}'_{k,h}(s), \mathbf{\Phi}\big) \qquad \forall \mathbf{\Phi} \in \mathbb{V}_h \quad \forall t \in [0,T]\,.
\end{aligned} \tag{3.5.4}
$$

Using (3.5.4) we may now follow the argument in (3.5.3) to conclude

$$
\frac{1}{2}\mathbb{E}'\Big[\|\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h}(t) - \boldsymbol{\mathcal{U}}'_{k,h}(t)\|_{\mathbb{L}^2}^2\Big] \leq Ck\Big(1 + \mathbb{E}'\big[\|(\boldsymbol{\mathcal{U}}'_{k,h})^-\|_{\mathbb{L}^2}^2\big]\Big) \quad \forall t \in [0,T]\,.
$$

Thanks to this estimate and $(3.4.1)_3$, we obtain,

$$
\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h} \to \boldsymbol{u} \qquad \text{in } L^2\big(\Omega'; L^2(0,T;\mathbb{L}^2)\big) \quad \text{as } k,h \to 0. \tag{3.5.5}
$$

Let $\mathbb{F}_{k,h}$ resp. $\mathbb{F}'_{k,h}$ denote the natural filtration generated by the processes $\widetilde{\boldsymbol{\mathcal{U}}}_{k,h}$ resp. $\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h}$. We introduce the following $\mathbb{V}_h$-valued $\mathbb{F}_{k,h}$- resp. $\mathbb{F}'_{k,h}$-martingale defined on the filtered probability spaces $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ resp. $\big(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}'\big)$ via

$$
\begin{aligned}
\big(\mathbf{X}_{k,h}(t), \mathbf{\Phi}\big) \quad &:= \quad \int_0^t \Big(\mathbf{g}(\widetilde{\boldsymbol{\mathcal{U}}}^-_{k,h})\,\mathrm{d}\mathbf{W}(s), \mathbf{\Phi}\Big) \qquad \forall \mathbf{\Phi} \in \mathbb{V}_h\,, \quad t \geq 0 \\
\big(\mathbf{X}'_{k,h}(t), \mathbf{\Phi}\big) \quad &:= \quad \int_0^t \Big(\mathbf{g}\big((\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^-\big)\,\mathrm{d}\mathbf{W}'_{k,h}(s), \mathbf{\Phi}\Big) \qquad \forall \mathbf{\Phi} \in \mathbb{V}_h\,, \quad t \geq 0\,.
\end{aligned}
$$

In the following, we identify the limit of the quadratic variation of the $\mathbb{V}_h$-valued process $\mathbf{X}'_{k,h}$ for $k,h \to 0$. This will be accomplished by verifying all assumptions of [20, Theorem C.2], and taking into account Theorem B.0.2, which is used to represent the quadratic variation by dual pairing between $D(\mathbf{A}^{-\gamma})$ and $D(\mathbf{A}^{\gamma})$, instead by means of scalar product on $D(\mathbf{A}^{-\gamma})$. This choice is motivated by the fact that in the limiting equation we have test function from $D(\mathbf{A}^{\gamma})$. The identification of the stochastic integral is accomplished in three steps.

Step 1. We show the convergence of the quadratic variation process of $\mathbf{X}'_{k,h}$. Let $\mathbf{R}_{k,h} \equiv \{\mathbf{R}_{k,h}(t)\ ;\ t \in [0,T]\}$ be the quadratic variation process of the process $\mathbf{X}_{k,h}$. By [37, Section 3], thanks to (3.5.2) we have

$$
\mathbf{R}_{k,h}(t) := \int_0^t \mathbf{Q}_h^0\mathbf{g}(\widetilde{\boldsymbol{\mathcal{U}}}^-_{k,h})\mathbf{Q}^{1/2}\big[\mathbf{Q}_h^0\mathbf{g}(\widetilde{\boldsymbol{\mathcal{U}}}^-_{k,h})\mathbf{Q}^{1/2}\big]^*\,\mathrm{d}s\,. \tag{3.5.6}
$$

Since the laws of the processes $(\mathbf{W},\widetilde{\boldsymbol{\mathcal{U}}}_{k,h})$ and $(\mathbf{W}'_{k,h},\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})$ are equal due to (3.5.1), (3.5.2) and (3.5.4), we infer that the laws of $\mathbf{X}_{k,h}$ and $\mathbf{X}'_{k,h}$ also coincide. Therefore, in view of representations (3.5.2), (3.5.4) and [3, Lemma 6.3], the quadratic variation of the $\mathbb{F}'_{k,h}$-martingale $\mathbf{X}'_{k,h}$ is given by

$$\mathbf{R}'_{k,h}(t) := \int_0^t \mathbf{Q}_h^0 \mathbf{g}\big((\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^-\big)\mathbf{Q}^{1/2}\big[\mathbf{Q}_h^0\mathbf{g}\big((\widetilde{\boldsymbol{\mathcal{U}}}'_{k,h})^-\big)\mathbf{Q}^{1/2}\big]^*\,\mathrm{d}s \qquad \forall\, t\in[0,T]\,. \tag{3.5.7}$$

Let $\gamma>1$. We define as a natural candidate for the martingale part of the process $\boldsymbol{u}$ the following $D(\mathbf{A}^{-\gamma})$-valued process $\mathbf{X}'$ on $\mathfrak{P}'$, for all $t\in[0,T]$,

$$\begin{aligned}\big\langle \mathbf{X}'(t),\boldsymbol{\varphi}\big\rangle_{D(\mathbf{A}^{-\gamma})} &= \big(\boldsymbol{u}(t)-\mathbf{u}_0,\boldsymbol{\varphi}\big)\\ &+\int_0^t \Big\{\nu\big(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}\big)+\Big([\boldsymbol{u}\cdot\nabla]\boldsymbol{u},\boldsymbol{\varphi}\Big)-\langle \mathbf{f},\boldsymbol{\varphi}\rangle\Big\}\,\mathrm{d}s \qquad \forall\boldsymbol{\varphi}\in D(\mathbf{A}^{\gamma})\,,\end{aligned} \tag{3.5.8}$$

where $\big\langle\cdot,\cdot\big\rangle_{D(\mathbf{A}^{-\gamma})}$ denotes the dual pairing between $D(\mathbf{A}^{-\gamma})$ and $D(\mathbf{A}^{\gamma})$. We take the limit $k,h\to 0$ of the $\mathcal{I}_1(\mathbb{V}_h)$-valued quadratic variation process $\mathbf{R}'_{k,h}$. Because of (3.5.5), (3.2.3), assumption (S$_2$) and Theorem B.0.2, we find for every $0\le t\le T$,

$$\begin{aligned}\big\langle\widetilde{\mathbf{R}}'_{k,h}(t)\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\big\rangle_{D(\mathbf{A}^{-\gamma})} := \big(\mathbf{R}'_{k,h}(t)\mathbf{Q}_h^0\boldsymbol{\varphi}_1,\mathbf{Q}_h^0\boldsymbol{\varphi}_2\big) &\to \big\langle\widetilde{\mathbf{R}}(t)\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\big\rangle_{D(\mathbf{A}^{-\gamma})}\\ \forall\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\in\boldsymbol{\mathcal{V}} &\qquad (k,h\to 0)\,,\end{aligned} \tag{3.5.9}$$

where $\widetilde{\mathbf{R}}(t)=\mathbf{R}(t)\circ I: D(\mathbf{A}^{\gamma})\to D(\mathbf{A}^{-\gamma})$, $\gamma>1$, with the isometric isomorphism $I: D(\mathbf{A}^{\gamma})\to D(\mathbf{A}^{-\gamma})$; see Appendix B. Because of Theorem B.0.2, $\mathbf{R}$ is then defined by

$$\mathbf{R}(t) := \int_0^t \mathbf{g}(\boldsymbol{u})\mathbf{Q}^{1/2}\big[\mathbf{g}(\boldsymbol{u})\mathbf{Q}^{1/2}\big]^*\,\mathrm{d}s \qquad \forall\, t\in[0,T]\,. \tag{3.5.10}$$

Step 2. We have to show that the $D(\mathbf{A}^{-\gamma})$-valued process $\mathbf{X}'$ defined in (3.5.8) is a square integrable martingale with respect to the naturally augmented filtration $\overline{\mathbb{F}}$ generated by $\boldsymbol{u}$, with quadratic variation given by (3.5.10). In order to prove that $\mathbf{X}'$ is an $\overline{\mathbb{F}}$-martingale we first note that in view of [40, p. 75] it is enough to show that the process $\mathbf{X}'$ is an $\mathbb{F}$-martingale. For this aim let us choose $n\in\mathbb{N}$ and fix times $s,t\in[0,T]$ and $0\le s_1<\cdots<s_n\le s$, bounded and continuous functions $h_i: D(\mathbf{A}^{-\gamma})\to\mathbb{R}$, $i=1,\ldots,n$ and vectors $\boldsymbol{\varphi}\in D(\mathbf{A}^{\gamma})$, $i=1,\ldots,n$. We have to show the following equality

$$\mathbb{E}'\Big[\big\langle\mathbf{X}'(t)-\mathbf{X}'(s),\boldsymbol{\varphi}\big\rangle_{D(\mathbf{A}^{-\gamma})}\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]=0\,. \tag{3.5.11}$$

By the definition of $\mathbf{X}'$ we obtain the following

$$\begin{aligned}&\mathbb{E}'\Big[\big\langle\mathbf{X}'(t)-\mathbf{X}'(s),\boldsymbol{\varphi}\big\rangle_{D(\mathbf{A}^{-\gamma})}\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]\\ &=\mathbb{E}'\Big[\big(\boldsymbol{u}(t)-\boldsymbol{u}(s),\boldsymbol{\varphi}\big)\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]+\mathbb{E}'\Big[\Big(\int_s^t(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})\,\mathrm{d}s\Big)\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]\\ &\quad+\mathbb{E}'\Big[\Big(\int_s^t([\boldsymbol{u}\cdot\nabla]\boldsymbol{u},\boldsymbol{\varphi})\,\mathrm{d}s\Big)\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]-\mathbb{E}'\Big[\Big(\int_s^t\langle\mathbf{f},\boldsymbol{\varphi}\rangle\,\mathrm{d}s\Big)\prod_{i=1}^n h_i\big(\boldsymbol{u}(s_i)\big)\Big]\\ &=: I+II+III-IV\,.\end{aligned} \tag{3.5.12}$$

We only consider the terms I and III since the other two can be treated in a similar way. By the convergence property (3.5.5), we infer that I satisfies the following identity

$$I = \lim_{k,h\to 0} \mathbb{E}'\Big[\Big(\widetilde{\mathcal{U}}'_{k,h}(t) - \widetilde{\mathcal{U}}'_{k,h}(s), \mathbf{Q}_h^0\boldsymbol{\varphi}\Big)\prod_{i=1}^n h_i\big(\mathcal{U}'_{k,h}(s_i)\big)\Big].$$

By Lemma 3.4.4, (i) we infer that III satisfies the following identity

$$III = \lim_{k,h\to 0} \mathbb{E}'\Big[\Big(\int_s^t \Big\{[(\widetilde{\mathcal{U}}'_{k,h})^+\cdot\nabla](\widetilde{\mathcal{U}}'_{k,h})^+ + \frac{1}{2}\big[\mathrm{div}\,(\widetilde{\mathcal{U}}'_{k,h})^+\big](\widetilde{\mathcal{U}}'_{k,h})^+\Big\}\,\mathrm{d}s, \mathbf{Q}_h^0\boldsymbol{\varphi}\Big)\prod_{i=1}^n h_i\big(\mathcal{U}'_{k,h}(s_i)\big)\Big].$$

Therefore, by (3.5.2) and Lemma 3.4.4 (ii), the right-hand side of equality (3.5.12) is equal to 0.

Step 3. It remains to show that $\mathbf{R}$ is the quadratic variation process $\langle\langle \mathbf{X}'\rangle\rangle$ of $\mathbf{X}'$. According to [20, Theorem C.2] together with Theorem B.0.2, we need to show that for every $t \in [0,T]$ and some $r > 1$,

$$\sup_{k,h} \mathbb{E}'\Big[\|\mathbf{X}'_{k,h}(t)\|^{2r}_{D(\mathbf{A}^{-\gamma})}\Big] \;<\; \infty\,, \tag{3.5.13}$$

$$\sup_{k,h} \mathbb{E}'\Big[\|\widetilde{\mathbf{R}}'_{k,h}(t)\|^{r}_{\mathcal{L}\left(D(\mathbf{A}^{\gamma}),D(\mathbf{A}^{-\gamma})\right)}\Big] \;<\; \infty\,. \tag{3.5.14}$$

To show estimate (3.5.13), we use Lemma 3.3.1, $(\text{ii})_1$, and assumption (S_2), after reformulating (3.5.2) in a manner similar to (3.5.8). To verify the condition (3.5.14), we use the definition of $\widetilde{\mathcal{U}}'_{k,h}$ and Lemma 3.3.1, after noting (3.5.9). Now that we have shown that these two assumptions are satisfied, we obtain the following equation of the limiting process $\mathbf{X}'$ at all times $t \in [0,T]$,

$$\big\langle \mathbf{X}'(t), \boldsymbol{\varphi}\big\rangle_{D(\mathbf{A}^{-\gamma})} = \big(\boldsymbol{u}(t) - \mathbf{u}_0, \boldsymbol{\varphi}\big) + \int_0^T \Big\{\big(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi}\big) + \big([\boldsymbol{u}\cdot\nabla]\boldsymbol{u}, \boldsymbol{\varphi}\big) - \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\Big\}\,\mathrm{d}s \quad \forall \boldsymbol{\varphi} \in D(\mathbf{A}^{\gamma})\,. \tag{3.5.15}$$

Hence, the process $\mathbf{X}'$ is a square integrable $D(\mathbf{A}^{-\gamma})$-valued martingale with respect to the filtration $\overline{\mathbb{F}}$, with quadratic variation given by (3.5.10).

The remainder of the construction of a weak martingale solution for (3.1.1)–(3.1.3) is now standard, using the martingale representation theorem; see for instance [37, Section 8.4], or [46, Step 3 in the proof of Theorem 3.1].

We summarize the results proven in Sections 3.3 through 3.5.1.

Theorem 3.5.1. *Let $D \subset \mathbb{R}^d$, $d = 3$ be a polyhedral bounded domain, $T > 0$, $\mathbf{U}^0 \in \mathbb{H}_h$ such that $\|\mathbf{U}^0\|_{\mathbb{L}^2} \leq C$ uniformly for $h > 0$. Let $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ be a filtered probability space, and suppose* (S_1) *through* (S_3)*. For every finite $(k,h) > 0$, let $\mathcal{T}_h$ be a quasi-uniform triangulation of D, $I_k := \{t_m\}_{m=0}^M$ be an equi-distant partition covering $[0,T]$, and $\big(\mathbb{H}_h, L_h\big)$ a pair of finite element spaces that satisfies the discrete LBB condition. There exists $\big\{(\mathbf{U}^m, \Pi^m)\big\}_{m=1}^M$ which solves* (3.1.4)–(3.1.5)*, and satisfies Lemma 3.3.1.*

Let $\widetilde{\mathcal{U}}_{k,h} : D_T \to \mathbb{R}^d$ be the continuous process obtained from iterates $\{\mathbf{U}^m\}_{m=1}^M$ in (3.5.2) *for $k,h > 0$, and $\mathbf{U}^0 \to \mathbf{u}_0$ in $\mathbb{L}^2$ for $h \to 0$. Then, there exist a filtered probability space $\big(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}'\big)$, a convergent subsequence $\{\widetilde{\mathcal{U}}'_{k,h}\}_{k,h}$, and $\boldsymbol{u}$ such that for all $\alpha \in (0, \frac{1}{8})$,*

$$\widetilde{\mathcal{U}}'_{k,h} \to \boldsymbol{u} \qquad \text{in } L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big) \qquad (k,h\to 0)$$

$\mathbb{P}'$-almost surely, and an $\mathbb{F}'$-progressively measurable $\mathbf{Q}$-Wiener process $\mathbf{W}'$ such that the system $\boldsymbol{u}$, $\mathbf{W}'$, $\big(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}'\big)$ is a weak martingale solution of problem (3.1.1)–(3.1.3).

In addition, provided that we fix $\gamma > 1$ and $p > 2$ from Proposition 3.4.1 and Lemma 3.3.3, then, $\mathbb{P}'$-almost surely,

$$\widetilde{\mathcal{U}}'_{k,h} \to \boldsymbol{u} \qquad \text{in } L^2\big(0,T;\mathbb{L}^2\big) \cap C\big([0,T];D(\mathbf{A}^{-\gamma})\big) \qquad (k,h \to 0)\,.$$

Algorithm 3.1 amounts to solving nonlinear algebraic problems. Revisiting the above proofs shows that weak martingale solutions of (3.1.1)–(3.1.3) may also be constructed by iterates which successively solve linear algebraic problems.

Remark 3.5.1. *We may modify Algorithm 3.1 to the following linear scheme.*

Algorithm 3.3. *Let $\mathbf{U}^0 \in \mathbb{H}_h$ be given. For every $1 \le m \le M$ find $\mathbb{H}_h \times L_h$-valued $\big(\mathbf{U}^m, \Pi^m\big)$ such that for all $(\boldsymbol{\Phi}, \Lambda) \in \mathbb{H}_h \times L_h$,*

$$\begin{aligned}\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \boldsymbol{\Phi}\big) + k\nu\big(\nabla\mathbf{U}^m, \nabla\boldsymbol{\Phi}\big) - k\big(\Pi^m, \operatorname{div}\boldsymbol{\Phi}\big) + k\big([\mathbf{U}^{m-1}\cdot\nabla]\mathbf{U}^m, \boldsymbol{\Phi}\big)&\\ +\frac{k}{2}\Big([\operatorname{div}\mathbf{U}^{m-1}]\mathbf{U}^m, \boldsymbol{\Phi}\Big) = k\big\langle \mathbf{f}^m, \boldsymbol{\Phi}\big\rangle + \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \boldsymbol{\Phi}\big)\,,& \qquad (3.5.16)\end{aligned}$$

$$(\operatorname{div}\mathbf{U}^m, \Lambda) = 0\,. \qquad (3.5.17)$$

Accordingly, Lemmata 3.3.1 and 3.3.3, as well as Theorem 3.5.1 hold for iterates of Algorithm 3.3.

Remark 3.5.2. *For simplicity, we used a function $\mathbf{f}: D_T \to \mathbb{R}^d$ with*

$$\mathbf{f} \in C([0,T];\mathbb{L}^2)$$

but we can also assume that

$$\mathbf{f} \in L^p(0,T;\mathbb{W}^{-1,2})\,,$$

for the fixed p from Theorem 3.5.1, using a piecewise constant approximation like in [125, Section 3.4.2].

3.5.2 Construction by approximate increments

The goal of this section is to construct a weak solution to problem (3.1.1)–(3.1.3) when the increments of the Wiener process are replaced by general, not necessarily Gaussian, random variables. Choosing simpler, for instance bounded, random variables may also improve convergence of nonlinear algebraic solvers for Algorithm 3.2.

To this end let us assume that $I_k := \{t_m\}_{m=0}^M$ is an equi-distant partition of size $k > 0$ covering the time interval $[0,T]$, $\mathbb{F}_k = \big\{\mathcal{F}_{t_m} : t_m \in I_k\big\}$ is a (discrete time) filtration, $(\Omega, \mathcal{F}, \mathbb{F}_k, \mathbb{P})$ is a complete, filtered probability space, $\boldsymbol{\mathcal{K}}$ is a real separable Hilbert space, $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ be symmetric and positive, and $\{\boldsymbol{\xi}^m\}_{m=1}^M$ a sequence of $\boldsymbol{\mathcal{K}}$-valued i.i.d. random variables such that each satisfies

(SI_1) $\boldsymbol{\xi}^m$ is $\mathcal{F}_{t_m}$-measurable and independent of $\mathcal{F}_{t_{m-1}}$,

(SI_2) $\mathbb{E}\,[\boldsymbol{\xi}^m] = \mathbf{0}$, $\mathbb{E}\,[(\boldsymbol{\xi}^m, \mathbf{x})_{\boldsymbol{\mathcal{K}}}(\boldsymbol{\xi}^m, \mathbf{y})_{\boldsymbol{\mathcal{K}}}] = k(\mathbf{Q}\mathbf{x}, \mathbf{y})_{\boldsymbol{\mathcal{K}}} \qquad \forall\, \mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{K}}$,

(SI_3) $\mathbb{E}\left[|\boldsymbol{\xi}^m|_{\boldsymbol{\mathcal{K}}}^{2p}\right] \le Ck^p$, for all integer $p \ge 1$.

Note that the assumptions (SI_1) and (SI_2) are generalized versions of the assumptions needed to show the existence of the $\mathbb{R}$-valued Wiener process by a piecewise continuous interpolation of an appropriate random walk on $\mathbb{R}$. In the finite dimensional case, by the Donsker invariance principle, the limiting distribution of increments is Gaussian. In fact, the Donsker invariance principle is a direct consequence of the central limit theorem. Here we do not use it, and the only thing we need to get the right quadratic variation process is the condition on the first moment and on the covariance operator. In the present case, these conditions on the first and second moment allow us to show that, for $k, h \to 0$, the limiting process corresponding to the stochastic integral has the desired quadratic variation, which, by a martingale representation theorem, is enough to show the existence of an appropriate Wiener process.

We now give examples of $\mathbb{R}$-, and more general $\boldsymbol{\mathcal{K}}$-valued sequences of random variables $\{\boldsymbol{\xi}^m\}_{m=1}^M$ that satisfy (SI$_1$)–(SI$_3$).

Example 3.5.1. *1. Let us assume that $\boldsymbol{\mathcal{K}} = \mathbb{R}$ and $\{\widetilde{\xi}^m\}_{m=1}^M$ be a sequence of i.i.d. $\mathbb{R}$-valued random variables. We put $\xi^m = \sqrt{k}\widetilde{\xi}^m$. Below we list two admissible choices which satisfy the conditions* (SI$_1$)*–*(SI$_3$)*.*

(i) $\mathbb{P}[\widetilde{\xi}^m = \pm 1] = \frac{1}{2}$, *or*

(ii) $\mathbb{P}[\widetilde{\xi}^m = \pm\sqrt{3}] = \frac{1}{6}$, *and* $\mathbb{P}[\widetilde{\xi}^m = 0] = \frac{2}{3}$.

We refer to [102, Section 6.4] and [81, Section 9.7] for a further discussion of simplified schemes of SODEs.

2. Let $\boldsymbol{\mathcal{K}}$ be a real separable Hilbert space and $\{\mathbf{e}_j;\, j \geq 1\}$ is an orthonormal basis (ONB) of $\boldsymbol{\mathcal{K}}$. Then the conditions (SI$_1$)*–*(SI$_3$) *are satisfied for the following system of random variables*

$$\boldsymbol{\xi}^m = \sum_{j=1}^{\infty} \sqrt{q_j}\xi^{j,m}\mathbf{e}_j, \quad m \geq 1,$$

where $\{\{\xi^{j,m}\}_{j=1}^{\infty}\}_{m=1}^M$ are i.i.d. $\mathbb{R}$-valued random variables from part 1. of this example.

In what follows we will show that the iterates from Algorithm 3.2 inherit the stability properties of those from Algorithm 3.1. Once this result is obtained, we will be able to identify the limits of the deterministic integrals exactly as in Section 3.4. Finally, we will use a general theorem on the convergence of discrete time martingales to identify the limit of the term corresponding to the stochastic integral.

Existence and stability of solutions

Assume that $\mathbf{U}_0$ is a function in $\mathbb{H}_h$ with $\|\mathbf{U}^0\|_{\mathbb{L}^2} \leq C$. We note that the random walk defined through

$$\mathbf{S}_0 = 0, \quad \mathbf{S}_n = \sum_{i=1}^{n} \mathbf{g}(\mathbf{U}^{i-1})\boldsymbol{\xi}^i$$

is a discrete time $\boldsymbol{\mathcal{H}}$-valued $\mathbb{F}_k$-martingale. The following result holds.

Lemma 3.5.1. *Let $\mathbf{U}^0$ be a given function with $\|\mathbf{U}^0\|_{\mathbb{L}^2} \leq C$, and suppose that the assumptions* (S$_1$)–(S$_3$) *and* (SI$_1$)–(SI$_3$) *hold. Then all results of Lemmata 3.3.1 and 3.3.3 remain valid.*

Proof. We only sketch those parts of the proof which indicate the rôle played by the general random variables $\{\boldsymbol{\xi}^m\}_{m=1}^M$.

We begin with the proof of Lemma 3.3.1. We consider the term

$$\sum_{m=1}^{M} \mathbb{E}\left[\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\boldsymbol{\xi}^m|^2_{\boldsymbol{\mathcal{K}}}\right].$$

By the tower property, and assumptions (SI$_3$) and (S$_2$) we compute,

$$
\begin{aligned}
&\mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\boldsymbol{\xi}^m|^2_{\boldsymbol{\mathcal{K}}}\Big] = \mathbb{E}\Big[\mathbb{E}\big[\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}|\boldsymbol{\xi}^m|^2_{\boldsymbol{\mathcal{K}}}\big|\mathcal{F}_{t_{m-1}}\big]\Big] \\
(3.5.18)\qquad &= \mathbb{E}\Big[\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\mathbb{E}\big[|\boldsymbol{\xi}^m|^2_{\boldsymbol{\mathcal{K}}}\big|\mathcal{F}_{t_{m-1}}\big]\Big] \leq Ck\,\mathbb{E}\big[\|\mathbf{g}(\mathbf{U}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\big] \\
&\leq K_1^2 Ck\mathbb{E}\big[\|\mathbf{U}^{m-1}\|^2_{\mathbb{L}^2}\big]\,.
\end{aligned}
$$

As a consequence, the assertions (3.3.7) and (3.3.13) are satisfied. Consequently, the second and third part of (ii), and the second part of (iii), for $p = 2$, in Lemma 3.3.1 follow. We can prove (iv) from Lemma 3.3.1, and as a by-product we get the first part of (iii). This implies the first part of (ii) by interpolation of L^p-spaces. Taking the maximum, and using the Doob's inequality for $p = 2$, as well as the independence of the random variables $\{\boldsymbol{\xi}^i\}_{i=1}^M$ we compute

$$
\begin{aligned}
&\mathbb{E}\left[\Big(\max_{1\leq m\leq M}\sum_{i=1}^m\big(\mathbf{g}(\mathbf{U}^{i-1})\boldsymbol{\xi}^i,\mathbf{U}^{i-1}\big)\Big)^2\right] \leq C_2\mathbb{E}\left[\sum_{m=1}^M\big(\mathbf{g}(\mathbf{U}^{m-1})\boldsymbol{\xi}^m,\mathbf{U}^{m-1}\big)^2\right] \\
&\leq \delta\mathbb{E}\left[\max_{1\leq m\leq M}\|\mathbf{U}^m\|^4_{\mathbb{L}^2}\right] + \|\mathbf{U}^0\|^4_{\mathbb{L}^2} + C_2\mathbb{E}\left[\sum_{m=1}^M\|\mathbf{g}(\mathbf{U}^{m-1})\boldsymbol{\xi}^m\|^4_{\mathbb{L}^2}\right] \qquad (\delta > 0).
\end{aligned}
$$

The first term can be absorbed in the left-hand side, while last term can be bounded with the tower property for martingales. This shows assertion (iv) of Lemma 3.3.1, together with the first part of (iii) for $p = 2$. Then the first part of (ii) follows. This argument can be applied for all integers $p \geq 3$.

Now we will prove Lemma 3.3.3. The proofs of estimates of the $\mathbb{L}^2$-norm the time increments in Lemma 3.3.3 are straightforward by using the discrete martingale property of the random walk $\{\mathbf{S}_m\}_{m=1}^M$.

For integer $p > 2$, we use again the independence of the random variables $\{\boldsymbol{\xi}^m\}_{m=1}^M$ to compute

$$
\begin{aligned}
k\sum_{m=0}^{M-\ell}\mathbb{E}\Big[\Big(\sum_{i=1}^{\ell}\mathbf{g}(\mathbf{U}^{m+i-1})\boldsymbol{\xi}^{m+i},\mathbf{Q}_h^0\boldsymbol{\varphi}\Big)^p\Big] &= k\sum_{m=0}^{M-\ell}\mathbb{E}\left[\sum_{i=1}^{\ell}\big(\mathbf{g}(\mathbf{U}^{m+i-1})\boldsymbol{\xi}^{m+i},\mathbf{Q}_h^0\boldsymbol{\varphi}\big)^p\right] \\
&\leq Ct_\ell^{p/2},
\end{aligned}
$$

where in the last estimate we use the estimates from the first part of the proof. □

Compactness of the iterates and identification of the deterministic integrals

Recall the definitions of linear interpolation $\boldsymbol{\mathcal{U}}_{k,h}$ in the framework from current section. For $t \in [t_{m-1}, t_m)$ the following equality holds for all $\boldsymbol{\Phi} \in \mathbb{V}_h$,

$$
\begin{aligned}
(3.5.19)\ &\big(\boldsymbol{\mathcal{U}}_{k,h}(t) - \boldsymbol{\mathcal{U}}^-_{k,h},\boldsymbol{\Phi}\big) + (t-t_{m-1})\Big\{\big(\nabla\boldsymbol{\mathcal{U}}^+_{k,h},\nabla\boldsymbol{\Phi}\big) + \big([\boldsymbol{\mathcal{U}}^+_{k,h}\cdot\nabla]\boldsymbol{\mathcal{U}}^+_{k,h},\boldsymbol{\Phi}\big)\Big\} \\
&+\frac{1}{2}(t-t_{m-1})\Big([\mathrm{div}\boldsymbol{\mathcal{U}}^+_{k,h}]\boldsymbol{\mathcal{U}}^+_{k,h},\boldsymbol{\Phi}\Big) = (t-t_{m-1})\big(\mathbf{f}(t_m),\boldsymbol{\Phi}\big) + \frac{t-t_{m-1}}{k}\Big(\mathbf{g}(\mathbf{U}^{m-1})\boldsymbol{\xi}^m,\boldsymbol{\Phi}\Big)\,.
\end{aligned}
$$

An argument similar to that used in Section 3.4 allows us to formulate the following result.

Proposition 3.5.1. *Let α, γ, and $p > 2$ be as in Section 3.4. Let $\mathbf{U}^0 \in \mathbb{H}_h$ be such that $\mathbf{U}^0 \to \mathbf{u}_0$ in $\mathbb{L}^2$ for $h \to 0$. Then there exist a filtered probability space $\mathfrak{P}' = (\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, a subsequence $\big\{\boldsymbol{\mathcal{U}}'_{k,h}\big\}_{k,h}$ of solutions from (3.5.19) defined on $\mathfrak{P}'$, and a process $\boldsymbol{u}$ which satisfy all properties in Lemma 3.4.2, Proposition 3.4.1, Lemma 3.4.3, and Lemma 3.4.4.*

Identification of the stochastic integral

In this section let the elements of the time grid I_k be denoted by $\{t_n^k;\, 0 \leq n \leq M\}$ to avoid confusion in the identification of the integral.

In order to identify the stochastic integral we need to show that the assumptions of Theorem B.0.3 are satisfied and so conclude by means of a representation theorem. To this end, let $(\boldsymbol{\mathcal{U}}'_{k,h})^+$, respectively $\boldsymbol{u}$, play the rôle of $U^+_{k,h}$, respectively U in Theorem B.0.3, and let $V = D(\mathbf{A}^\gamma)$ $E = D(\mathbf{A}^{-\gamma}), \gamma > 1$. For the filtrations, we use the notations from Theorem B.0.3.

First we need a process which is a discrete version of the quadratic variation. Put $(\mathbf{U}^{i-1})' := (\boldsymbol{\mathcal{U}}')^-(t_i)$. Then we define the discrete-time process

$$(\mathbf{R}^m_{k,h})' := k \sum_{i=1}^{m} \Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2}\Big)\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2}\Big)^*,$$

together with the corresponding piecewise constant interpolation process $(\mathbf{R}'_{k,h})^+$. Denote $(\mathbf{M}^m_{k,h})' = \sum_{i=1}^m \mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\boldsymbol{\xi}^i$. We claim that the process ($\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{\mathcal{V}}$)

$$(N'_{k,h})^+(t^k_n) := \big((\mathbf{M}'_{k,h})^+(t^k_n), \boldsymbol{\phi}\big)\big((\mathbf{M}'_{k,h})^+(t^k_n), \boldsymbol{\psi}\big) - \big((\mathbf{R}'_{k,h})^+(t^k_n)\boldsymbol{\phi}, \boldsymbol{\psi}\big) \quad n = 0, 1, \ldots, M$$

is an $\mathbb{F}^k_n$-martingale. For this aim by using the independence of the random variable $\{\boldsymbol{\xi}^m\}_{m=1}^M$ we get, for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{\mathcal{V}}$,

$$\mathbb{E}\Bigg[\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\boldsymbol{\xi}^i, \boldsymbol{\phi}\Big)\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\boldsymbol{\xi}^j, \boldsymbol{\psi}\Big)$$
$$-k\sum_{i=1}^{m+1}\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2}(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2})^*\boldsymbol{\phi}, \boldsymbol{\psi}\Big)\Big|\mathcal{F}^{k,h}_{t^k_m}\Bigg]$$

$$= \mathbb{E}\Bigg[\sum_{i=1}^{m+1}\Big\{\Big(\boldsymbol{\xi}^i, \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^0_h\boldsymbol{\phi}\Big)\Big(\boldsymbol{\xi}^i, \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^0_h\boldsymbol{\psi}\Big)$$
$$-k\Big(\mathbf{Q}^0_h \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2}(\mathbf{Q}^0_h \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2})^*\boldsymbol{\phi}, \boldsymbol{\psi}\Big)\Big\}\Big|\mathcal{F}^{k,h}_{t^k_m}\Bigg] = \ldots$$

Using the fact that ξ^m is $\mathcal{F}_{t_m}$ measurable we get

$$\ldots = \sum_{i=1}^{m}\Big(\boldsymbol{\xi}^i, \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^0_h\boldsymbol{\phi}\Big)\Big(\boldsymbol{\xi}^i, \mathbf{g}^*\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^0_h\boldsymbol{\psi}\Big)$$
$$-k\sum_{i=1}^{m}\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2}(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^{i-1})'\big)\mathbf{Q}^{1/2})^*\boldsymbol{\phi}, \boldsymbol{\psi}\Big)$$
$$+\mathbb{E}\Bigg[\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^m)'\big)\boldsymbol{\xi}^{m+1}, \boldsymbol{\phi}\Big)\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^m)'\big)\boldsymbol{\xi}^{m+1}, \boldsymbol{\psi}\Big)$$
$$-k\Big(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}(\mathbf{Q}^0_h \mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2})^*\boldsymbol{\phi}, \boldsymbol{\psi}\Big)\Big|\mathcal{F}^{k,h}_{t^k_m}\Bigg].$$

Thus the claim is a consequence of the following train of identities.

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\boldsymbol{\xi}^{m+1},\boldsymbol{\phi}\Big)\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\boldsymbol{\xi}^{m+1},\boldsymbol{\psi}\Big)\\
&\quad -k\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big)^*\boldsymbol{\phi},\boldsymbol{\psi}\Big)\Big|\mathcal{F}^{k,h}_{t_m^k}\Big]\\
&=\mathbb{E}\Big[\Big(\boldsymbol{\xi}^{m+1},\mathbf{g}^*\big((\mathbf{U}^m)'\big)\mathbf{Q}_h^0\boldsymbol{\phi}\Big)\Big(\boldsymbol{\xi}^{m+1},\mathbf{g}^*\big((\mathbf{U}^m)'\big)\mathbf{Q}_h^0\boldsymbol{\psi}\Big)\\
&\quad -k\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big)^*\boldsymbol{\phi},\boldsymbol{\psi}\Big)\Big|\mathcal{F}^{k,h}_{t_m^k}\Big]\\
&=k\Big(\mathbf{Q}\mathbf{g}^*\big((\mathbf{U}^m)'\big)\mathbf{Q}_h^0\boldsymbol{\phi},\mathbf{g}^*\big((\mathbf{U}^m)'\big)\mathbf{Q}_h^0\boldsymbol{\psi}\Big)\\
&\quad -k\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big(\mathbf{Q}_h^0\mathbf{g}^*\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big)^*\boldsymbol{\phi},\boldsymbol{\psi}\Big)\\
&=k\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big)^*\boldsymbol{\phi},\boldsymbol{\psi}\Big)-\\
&\quad k\Big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big(\mathbf{Q}_h^0\mathbf{g}\big((\mathbf{U}^m)'\big)\mathbf{Q}^{1/2}\big)^*\boldsymbol{\phi},\boldsymbol{\psi}\Big)=0\,,
\end{aligned}$$

where in the second equality, we used that $\boldsymbol{\xi}^{m+1}$ is independent of $\mathcal{F}^k_{t_m^k}$, together with (SI_2), and $\mathbf{g}\big((\mathbf{U}^m)'\big)$ is $\mathcal{F}^k_{t_m^k}$-measurable.

We will now verify that the properties (B.0.9) and (B.0.10) hold. Firstly we observe that (B.0.9) follows by the strong convergence of $\boldsymbol{\mathcal{U}}'_{k,h}$ in the space $C\big(0,T;D(\mathbf{A}^{-\gamma})\big)$, $\gamma>1$.

In order to prove the latter, let us recall that, in view of equality (3.5.19) we have

$$\begin{aligned}
\big((\mathbf{M}^m_{k,h})',\boldsymbol{\Phi}\big) \;=\;& \Big((\boldsymbol{\mathcal{U}}'_{k,h})^+(t_m^k)-\boldsymbol{\mathcal{U}}_{k,h}(0),\boldsymbol{\Phi}\Big)\\
&+\int_0^{t_m^k}\nu\Big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla\boldsymbol{\Phi}\Big)+\Big([(\boldsymbol{\mathcal{U}}'_{k,h})^+\cdot\nabla](\boldsymbol{\mathcal{U}}'_{k,h})^+,\boldsymbol{\Phi}\Big)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^{t_m^k}\Big([\mathrm{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^+](\boldsymbol{\mathcal{U}}'_{k,h})^+,\boldsymbol{\Phi}\Big)\,\mathrm{d}s-\int_0^{t_m^k}\big(\mathbf{f}(s),\boldsymbol{\Phi}\big)\,\mathrm{d}s\qquad\forall\boldsymbol{\Phi}\in\mathbb{V}_h\,.
\end{aligned}\tag{3.5.20}$$

Noting an obvious identity $\int_0^t 1\,\mathrm{d}s=\int_0^{t_m^k}1\,\mathrm{d}s+\int_{t_m^k}^t 1\,\mathrm{d}s$ and using the strong convergence of $\boldsymbol{\mathcal{U}}'_{k,h}$ in the space $C\big(0,T;D(\mathbf{A}^{-\gamma})\big)$, $\gamma>1$, we conclude that the convergence (B.0.10) holds true.

Now we will verify whether the assumption (B.0.11) of Theorem B.0.3 is satisfied. For this aim let us fix a time $t\in[0,T]$ and a sequence $(t_m^k)_{k>0}$ such that $t_m^k\to t$. Since the function $\mathbf{g}$ is Lipschitz continuous, by the strong convergence of $\boldsymbol{\mathcal{U}}'_{k,h}$ in the space $L^2\big(\Omega;L^2(0,T;\mathbb{L}^2)\big)$, and by the fact that the projection $\mathbf{Q}_h^0$ converges strongly in $\mathbb{L}^2$ (i.e. $\mathbf{Q}_h^0\mathbf{x}\to\mathbf{x}$, for all $\mathbf{x}\in\mathbb{V}$) to the identity, we get

$$\Big\langle(\widetilde{\mathbf{R}}'_{k,h})^+(t_m^k)\boldsymbol{\phi},\boldsymbol{\psi}\Big\rangle_{D(\mathbf{A}^{-\gamma})}:=\Big((\mathbf{R}'_{k,h})^+(t_m^k)\boldsymbol{\phi},\boldsymbol{\psi}\Big)\to\Big\langle\widetilde{\mathbf{R}}(t)\boldsymbol{\phi},\boldsymbol{\psi}\Big\rangle_{D(\mathbf{A}^{-\gamma})}\quad\forall\boldsymbol{\phi},\boldsymbol{\psi}\in\boldsymbol{\mathcal{V}},$$

where $\widetilde{\mathbf{R}}=\mathbf{R}\circ I:D(\mathbf{A}^{\gamma})\to D(\mathbf{A}^{-\gamma})$, with the isometric isomorphism $I:D(\mathbf{A}^{\gamma})\to D(\mathbf{A}^{-\gamma})$ from Appendix B, and

$$\mathbf{R}(t):=\int_0^t\big(\mathbf{g}(\mathbf{u}(s))\mathbf{Q}^{1/2}\big)\big(\mathbf{g}(\mathbf{u}(s))\mathbf{Q}^{1/2}\big)^*\,\mathrm{d}s\,.$$

Note that it follows from the above formula that the process $\mathbf{R}$ is $\mathbb{F}$-progressively measurable.

The next assumption of Theorem B.0.3 to be checked is (B.0.12). We compute

$$\begin{aligned}\frac{1}{C_r}|\big((\mathbf{M}'_{k,h})^+,\mathbf{\Phi}\big)|^{2r} \leq\ & |\Big((\boldsymbol{\mathcal{U}}'_{k,h})^+(t_m^k),\mathbf{\Phi}\Big)|^{2r} + |\Big(\boldsymbol{\mathcal{U}}'_{k,h}(0),\mathbf{\Phi}\Big)|^{2r} \\ +\ & \Big|\int_0^{t_m^k}(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla\mathbf{\Phi})\,\mathrm{d}s\Big|^{2r} + \Big|\int_0^{t_m^k}\Big([(\boldsymbol{\mathcal{U}}'_{k,h})^+\cdot\nabla](\boldsymbol{\mathcal{U}}'_{k,h})^+,\mathbf{\Phi}\Big)\,\mathrm{d}s\Big|^{2r} \\ +\ & \Big|\frac{1}{2}\int_0^{t_m^k}\Big([\mathrm{div}\,(\boldsymbol{\mathcal{U}}'_{k,h})^+](\boldsymbol{\mathcal{U}}'_{k,h})^+,\mathbf{\Phi}\Big)\,\mathrm{d}s\Big|^{2r} + \Big|\int_0^{t_m^k}\big(\mathbf{f}(s),\mathbf{\Phi}\big)\,\mathrm{d}s\Big|^{2r} \quad \forall\mathbf{\Phi}\in\mathbb{V}_h\,.\end{aligned}$$

Thus, for $\mathbf{\Phi} = \mathbf{Q}_h^0\boldsymbol{\varphi}$ for all $\boldsymbol{\varphi}\in D(\mathbf{A}^\gamma)$, $\gamma > 1$ we have

$$\begin{aligned}\mathbb{E}\Big[\Big|\Big((\boldsymbol{\mathcal{U}}'_{k,h})^+(t_m^k),\mathbf{\Phi}\Big)\Big|^{2r}\Big] &\leq \mathbb{E}\Big[\|(\boldsymbol{\mathcal{U}}'_{k,h})^+(t_m^k)\|_{\mathbb{L}^2}^{2r}\Big]\|\mathbf{\Phi}\|_{\mathbb{L}^2}^{2r}\,, \\ \mathbb{E}\left[\Big|\int_0^{t_m^k}\Big(\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+,\nabla\mathbf{\Phi}\Big)\,\mathrm{d}s\Big|^{2r}\right] &\leq \mathbb{E}\left[\Big(\int_0^{t_m^k}\|\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}\,\mathrm{d}s\Big)^{2r}\right]\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}^{2r} \\ &\leq C\mathbb{E}\left[\Big(\int_0^{t_m^k}\|\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big)^{r}\right]\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}^{2r}\,,\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[\Big|\int_0^{t_m^k}\Big([(\boldsymbol{\mathcal{U}}'_{k,h})^+\cdot\nabla](\boldsymbol{\mathcal{U}}'_{k,h})^+,\mathbf{\Phi}\Big)\,\mathrm{d}s\Big|^{2r}\right] \\ &\quad\leq C\mathbb{E}\left[\Big(\int_0^{t_m^k}\|(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\|\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big)^{r}\right]\|\mathbf{\Phi}\|_{\mathbb{L}^\infty}^{2r} \\ &\quad\leq C\mathbb{E}\left[\Big(\sup_{t\in[0,T]}\|(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\int_0^{t_m^k}\|\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big)^{r}\right]\|\mathbf{\Phi}\|_{\mathbb{L}^\infty}^{2r} \\ &\quad\leq C\mathbb{E}\left[\Big(\sup_{t\in[0,T]}\|(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\Big)^{2r}\right]^{1/2}\mathbb{E}\left[\Big(\int_0^{t_m^k}\|\nabla(\boldsymbol{\mathcal{U}}'_{k,h})^+\|_{\mathbb{L}^2}^2\,\mathrm{d}s\Big)^{2r}\right]^{1/2}\|\mathbf{\Phi}\|_{\mathbb{L}^\infty}^{2r}\,.\end{aligned}$$

Thanks to Lemma 3.3.1, we can conclude that

$$\sup_{h,k>0}\mathbb{E}\left[\|(\mathbf{M}'_{k,h})^+(t)\|_{D(\mathbf{A}^{-\gamma})}^{2r}\right] < \infty, \quad \gamma > 1.$$

To prove (B.0.13), we note the sublinearity of $\mathbf{g}\in\mathcal{I}_2(\boldsymbol{\mathcal{K}},\mathbb{L}^2)$, together with the uniform boundedness of the projection $\mathbf{Q}_h^0 : \mathbb{L}^2\to\mathbb{V}_h$ and of $\mathbf{Q}\in\mathcal{I}_1(\boldsymbol{\mathcal{K}})$. Now we have that the limit $\mathbf{M}$ is a square integrable martingale with quadratic variation process

$$\mathbf{R}(t) := \int_0^t(\mathbf{g}(\mathbf{u}(s))\mathbf{Q}^{1/2})(\mathbf{g}(\mathbf{u}(s))\mathbf{Q}^{1/2})^*\,\mathrm{d}s.$$

The existence of the Wiener process follows from a standard martingale representation theorem as in Section 3.5.1. Hence we proved that both Theorem 3.5.1 and Remark 3.5.1 hold for Algorithm 3.2.

We can summarize the results proved in this section in the following result.

Theorem 3.5.2. *Let $D\subset\mathbb{R}^d$, where $d = 2$ or $d = 3$ be a polyhedral bounded domain, $T > 0$ and $\mathbf{U}^0\in L^p\big(\Omega,\mathbb{H}_h\big)$ for some $p\geq 2$. Let $\big(\Omega,\mathcal{F},\mathbb{P}\big)$ be a probability space, $\boldsymbol{\mathcal{K}}$ be a real separable*

Hilbert space and let $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ *be a positive and symmetric operator. Suppose* (S_2), (S_3), *and* (SI_1) *through* (SI_3). *For every finite* $(k,h) > 0$, *let* $\mathcal{T}_h$ *be a quasi-uniform triangulation of* D, *let* $I_k := \{t_m\}_{m=0}^M$ *be an equi-distant partition covering* $[0,T]$, *and* $(\mathbb{H}_h, L_h)$ *be a pair of finite element spaces that satisfies the discrete LBB condition. There exists* $\{(\mathbf{U}^m, \Pi^m)\}_{m=1}^M$ *which solves* (3.1.4)–(3.1.5), *and satisfies Lemma 3.3.1.*

Let $\boldsymbol{\mathcal{U}}_{k,h} : D_T \to \mathbb{R}^d$ *be the continuous process obtained from iterates* $\{\mathbf{U}^m\}_{m=1}^M$ *in* (3.5.19) *for* $k,h > 0$, *and* $\mathbf{U}^0 \to \mathbf{u}_0$ *in* $L^2(\Omega;\mathbb{L}^2)$ *for* $h \to 0$. *Then, there exist a filtered probability space* $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, *a subsequence* $\{\boldsymbol{\mathcal{U}}'_{k,h}\}_{k,h}$, *an* $\mathbb{F}'$*-progressively measurable process* $\boldsymbol{u}$, *and an* $\mathbb{F}'$*-progressively measurable* $\mathbf{Q}$*-Wiener process* $\mathbf{W}'$ *such that the following two conditions are satisfied.*

(1) For all $\alpha \in (0, \frac{1}{8})$, $\mathbb{P}'$*-almost surely,*

$$\lim_{k,h\to 0} \boldsymbol{\mathcal{U}}'_{k,h} = \boldsymbol{u} \quad \text{in } L^2\big(0,T;\mathbb{L}^2\big) \cap W^{\alpha,2}\big(0,T;\mathbb{W}^{-1,2}\big). \tag{3.5.21}$$

(2) The system $(\boldsymbol{u}, \mathbf{W}', \Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ *is a weak martingale solution of problem* (3.1.1)–(3.1.3).

In addition, provided that $\gamma > 1$ *and* $p > 2$, $\mathbb{P}'$*-almost surely,*

$$\lim_{k,h\to 0} \boldsymbol{\mathcal{U}}'_{k,h} = \boldsymbol{u} \text{ in } L^2\big(0,T;\mathbb{L}^2\big) \cap C\big([0,T]; D(\mathbf{A}^{-\gamma})\big). \tag{3.5.22}$$

3.6 Construction of strong solutions in the 2-D case

In this section we will show, using tools from papers [98, 97], that iterates from Algorithm 3.1 converge to a strong solution to problem (3.1.1)–(3.1.3) for $k,h \to 0$. The key tool in our analysis, as well as in these two papers, is a sort of local monotonicity property of the operator $-\nu\Delta + \mathbf{F}$ defined below in (3.6.4). This property is used to identify a strong solution and allows to avoid the use of a compactness methods. Our study in this section is restricted to the 2-dimensional case, but we impose weaker conditions on the initial random variable $\mathbf{u}_0$ and on $\mathbf{f}$.

We summarize the assumptions needed below for data $\mathbf{Q}$, $\mathbf{u}_0$, and $\mathbf{f}$.

(SII_1) $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$ is a symmetric, positive operator.

(SII_2) $\mathbf{u}_0 \in L^2(\Omega, \mathcal{F}_0; \mathbb{H})$, and $\mathbf{f} \in L^2\Big(\Omega; L^2\big([0,T]; \mathbb{W}^{-1,2}\big)\Big)$.

(SII_3) The mapping $\mathbf{g} : \mathbb{L}^2 \to \mathcal{I}_2\big(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}), \boldsymbol{\mathcal{H}}\big)$ satisfies the Lipschitz condition

$$\|\mathbf{g}(\mathbf{v}) - \mathbf{g}(\mathbf{u})\|_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\boldsymbol{\mathcal{H}})} \le K_2 \|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^2}\,,$$

with a constant $K_2 > 0$, such that

$$\frac{\nu}{2}\|\nabla(\mathbf{u}-\mathbf{v})\|_{\mathbb{L}^2}^2 - K_2^2\|\mathbf{u}-\mathbf{v}\|_{\mathbb{L}^2}^2 \ge 0\,. \tag{3.6.1}$$

Inequality (3.6.1) holds for all $K_2 \le \lambda_1\sqrt{\frac{\nu}{2}}$, where $\lambda_1 > 0$ is the smallest eigenvalue of $\mathbf{A}$.

Definition 3.6.1. *Let* $T > 0$, *and suppose* (SII_1)*–*(SII_3). *Let* $\mathbf{W}$ *be an* $\mathbb{F}$*-progressively measurable* $\mathbf{Q}$*-Wiener process on the probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. *A strong solution of* (3.1.1)–(3.1.3) *is an* $\mathbb{F}$*-progressively measurable stochastic process* $\mathbf{u}$ *such that*

$$\mathbf{u} \in L^2(\Omega; L^\infty(0,T;\mathbb{H})) \cap L^2\big(\Omega; L^2(0,T;\mathbb{V})\big)\,,$$

and the following equation holds for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\big(\mathbf{u}(t,\cdot),\mathbf{v}\big) &+ \nu\int_0^t \big(\nabla\mathbf{u}(s,\cdot),\nabla\mathbf{v}\big)\,\mathrm{d}s - \int_0^t \Big(\big[\mathbf{u}(s,\cdot)\cdot\nabla\big]\mathbf{v},\mathbf{u}(s,\cdot)\Big)\,\mathrm{d}s \\ &= \big(\mathbf{u}_0,\mathbf{v}\big) + \int_0^t \big(\mathbf{f}(s,\cdot),\mathbf{v}\big)\,\mathrm{d}s + \Big(\int_0^t \mathbf{g}(\big(\mathbf{u}(s,\cdot)\big)\mathrm{d}\mathbf{W}(s,\cdot),\mathbf{v}\Big) \quad \forall\,\mathbf{v}\in\boldsymbol{\mathcal{V}}\,. \end{aligned} \tag{3.6.2}$$

In the first part of this section we consider the additive noise to point out the key steps that are used in the construction of a strong solution. Then, we consider the multiplicative noise, where the computations are quite similar but more involved. For brevity, we write $\mathcal{I}_2$ to indicate $\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)$.

For a given $r > 0$, let us define

$$\mathbb{B}_r = \{\mathbf{v}\in\mathbb{W}_0^{1,2} \;:\; \|\mathbf{v}\|_{\mathbb{L}^4}\le r\}\,. \tag{3.6.3}$$

The following result from [97] is crucial.

Lemma 3.6.1. *Let us assume that $d = 2$. Then the nonlinear operator*

$$\mathbf{G}:\;\mathbf{u}\mapsto -\nu\Delta\mathbf{u} + \mathbf{F}\mathbf{u} := -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla + \frac{1}{2}\mathrm{div}\,\mathbf{u})\mathbf{u} \tag{3.6.4}$$

is monotone in the ball $\mathbb{B}_r$ i.e., the following inequality is satisfied

$$\Big(\mathbf{G}(\mathbf{u})-\mathbf{G}(\mathbf{v}),\mathbf{u}-\mathbf{v}\Big) + \frac{27r^4}{2\nu^3}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{L}^2} \ge \frac{\nu}{2}\|\nabla(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L}^2} \tag{3.6.5}$$

for all $\mathbf{u}\in\mathbb{W}_0^{1,2}$, $\mathbf{v}\in\mathbb{B}_r$. Moreover, the operator $\mathbf{G}$ is hemicontinuous.

3.6.1 Additive Noise

First we assume the following weakened form of (SII$_3$).

(SII$'_3$) $\mathbf{g}:\Omega\times[0,T]\to\mathcal{L}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{H}})$ belongs to $L^2\Big(\Omega;C\big([0,T];\mathcal{L}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{H}})\big)\Big)$.

Then we define $\mathbf{g}^-$ by

$$\mathbf{g}^-(t) := \mathbf{g}^{m-1}\equiv\mathbf{g}(t_{m-1})\quad\forall\,t\in[t_{m-1},t_m).$$

Assume that $T = t_M$. The process $\boldsymbol{\mathcal{U}}^+_{k,h}$ has been defined in formula (3.3.39), which is constructed by Algorithm 3.1. We recall that we will not use the Skorokhod theorem anymore. By Lemma 3.3.1, formula (ii), and (3.3.40) we can find a subsequence of the sequence (k,h), which for simplicity of notation will denoted in the same way as the old one, such that

$$\boldsymbol{\mathcal{U}}^+_{k,h}\rightharpoonup\mathbf{u}\quad\text{in}\quad L^2\big(\Omega;L^2(0,T;\mathbb{W}_0^{1,2})\big)\qquad(k,h\to 0)\,. \tag{3.6.6}$$

Hence, we define $\mathbf{G}_0$ as

$$\mathbb{E}\left[\int_0^T(\mathbf{G}_0,\boldsymbol{\varphi})\;\mathrm{d}s\right] = \lim_{k,h}\mathbb{E}\left[\int_0^T\Big(\mathbf{G}(\boldsymbol{\mathcal{U}}^+_{k,h}),\mathbf{Q}^0_h\boldsymbol{\varphi}\Big)\,\mathrm{d}s\right]\quad\forall\,\boldsymbol{\varphi}\in\boldsymbol{\mathcal{V}}\,. \tag{3.6.7}$$

Here

$$\Big(\mathbf{G}(\boldsymbol{\mathcal{U}}^+_{k,h}),\boldsymbol{\Phi}\Big) := \nu\Big(\nabla\boldsymbol{\mathcal{U}}^+_{k,h},\nabla\boldsymbol{\Phi}\Big) + \Big([\boldsymbol{\mathcal{U}}^+_{k,h}\cdot\nabla]\boldsymbol{\mathcal{U}}^+_{k,h} + \frac{1}{2}[\mathrm{div}\boldsymbol{\mathcal{U}}^+_{k,h}]\boldsymbol{\mathcal{U}}^+_{k,h},\boldsymbol{\Phi}\Big)\quad\forall\,\boldsymbol{\Phi}\in\mathbb{V}_h\,.$$

Since in our special case we obviously have, by (SII_3'), the convergence

$$\mathbf{g}^- \to \mathbf{g} \quad \text{in} \quad L^2\Big(\Omega; L^2(0,T;\mathcal{I}_2)\Big)\,, \tag{3.6.8}$$

we infer that the process $\mathbf{u}$ satisfies the following equation, for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}$, $\mathbb{P} - a.s.$, and for all $t \in [0,T]$,

$$\big(\mathbf{u}(t) - \mathbf{u}_0, \boldsymbol{\varphi}\big) + \int_0^t (\mathbf{G}_0(s), \boldsymbol{\varphi})\,\mathrm{d}s = \int_0^t (\mathbf{f}(s), \boldsymbol{\varphi})\,\mathrm{d}s + \int_0^t \Big(\mathbf{g}(s)\mathrm{d}\mathbf{W}(s), \boldsymbol{\varphi}\Big)\,. \tag{3.6.9}$$

Hence, in order to prove that the limiting process $\mathbf{u}$ is a strong solution of Problem 3.1.1, we have to prove that

$$\mathbb{E}\left[\int_0^T (\mathbf{G}_0(s), \boldsymbol{\varphi})\;\mathrm{d}s\right] = \mathbb{E}\Big[\int_0^T \Big\{\nu\Big(\nabla\mathbf{u}(s), \nabla\boldsymbol{\varphi}\Big) + \Big([\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s), \boldsymbol{\varphi}\Big)\Big\}\mathrm{d}s\Big]\,.$$

Let $\rho : [0,T] \to \mathbb{R}$ be a bounded monotonically decreasing function with $\rho(0) = 0$. Define a finite sequence $\{\rho^m\}_{m=0}^M$ by $\rho^m = \rho(t_m)$, $m \in \mathbb{N}$. Then multiplying the discrete energy inequality (3.3.4) by e^{ρ^m} leads us to the following inequality

$$\begin{aligned} \mathrm{e}^{\rho^m}\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 - \mathrm{e}^{\rho^m}\|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^2 \;\le\; & \mathrm{e}^{\rho^m}\left\{2k\left(-\mathbf{G}(\mathbf{U}^m), \mathbf{U}^m\right) + \|\mathbf{g}^{m-1}\Delta_m\mathbf{W}\|_{\mathbb{L}^2}^2\right\} \\ & + \mathrm{e}^{\rho^m}\left\{\left(\mathbf{U}^{m-1}, \mathbf{g}^{m-1}\Delta_m\mathbf{W}\right) + k\,\|\mathbf{f}^m\|_{\mathbb{L}^2}^2\right\}. \end{aligned} \tag{3.6.10}$$

We note that because of the Itô isometry (2.3.1) the following equality holds

$$\mathbb{E}\left[\sum_{m=1}^M \|\mathbf{g}^{m-1}\Delta_m\mathbf{W}\|_{\mathbb{L}^2}^2\right] = \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{g}^{m-1}\mathbf{Q}^{1/2}\|_{\mathcal{I}_2(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}^2\right] = \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{g}^{m-1}\|_{\mathcal{I}_2}^2\right].$$

By Lemma A.0.1 the discrete time derivative $d_t\rho^m := (\rho^m - \rho^{m-1})/k$, see (A.0.3), satisfies the following equality

$$\mathrm{e}^{\rho^+} d_t\|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2 = d_t\left(\mathrm{e}^{\rho^+}\|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2\right) - \|\boldsymbol{\mathcal{U}}_{k,h}^-\|_{\mathbb{L}^2}^2 d_t\mathrm{e}^{\rho^+}\,.$$

We infer, by taking the sum in (3.6.10) from $m = 1$ to $m = M$, then taking the expectation, using Lemma A.0.2, and identity (A.0.7), and finally using the tower property and Itô's isometry, we get the following inequality

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{\rho^+(T)}\|\boldsymbol{\mathcal{U}}_{k,h}^+(T)\|_{\mathbb{L}^2}^2\right] \;\le\; & \mathbb{E}\left[\|\boldsymbol{\mathcal{U}}_{k,h}^-(0)\|_{\mathbb{L}^2}^2 + \int_0^T\left[d_t\mathrm{e}^{\rho^+}\|\boldsymbol{\mathcal{U}}_{k,h}^-\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\right] \\ & + \mathbb{E}\left[\int_0^T\left[\mathrm{e}^{\rho^+}2\left(-\mathbf{G}(\boldsymbol{\mathcal{U}}_{k,h}^+), \boldsymbol{\mathcal{U}}_{k,h}^+\right)\right]\mathrm{d}s\right] \\ & + \mathbb{E}\left[\int_0^T \mathrm{e}^{\rho^+}\left(\|\mathbf{g}^-\|_{\mathcal{I}_2}^2 + \|\mathbf{f}^+\|_{\mathbb{L}^2}^2\right)\,\mathrm{d}s\right]. \end{aligned} \tag{3.6.11}$$

Applying first the identity (A.0.7) to the function ρ^+ we find a bounded function $g : [0,T] \to \mathbb{R}$ and a function $\xi : [0,T] \to \mathbb{R}$ such that $\xi(t) \in (\rho^-(t), \rho^+(t))$ and the inequality (3.6.11) becomes

$$\begin{aligned} & \mathbb{E}\left[\mathrm{e}^{\rho^+(T)}\|\boldsymbol{\mathcal{U}}_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\boldsymbol{\mathcal{U}}_{k,h}^-(0)\|_{\mathbb{L}^2}^2\right] + \int_0^T \mathrm{e}^{\rho^+}2\left(\mathbf{G}(\boldsymbol{\mathcal{U}}_{k,h}^+), \boldsymbol{\mathcal{U}}_{k,h}^+\right)\mathrm{d}s \\ & \le \mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{U}}_{k,h}^-\|_{\mathbb{L}^2}^2\left[\mathrm{e}^{\rho^+}g(s)d_t\rho^+ + \mathrm{e}^{\xi}\frac{(\rho^+ - \rho^-)^2}{2k}\right]\,\mathrm{d}s\right] \\ & + \mathbb{E}\left[\int_0^T \mathrm{e}^{\rho^+}\left(\|\mathbf{g}^-\|_{\mathcal{I}_2}^2 + \|\mathbf{f}^+\|_{\mathbb{L}^2}^2\right)\,\mathrm{d}s\right]. \end{aligned} \tag{3.6.12}$$

Next we choose a function ρ in such a way that for all $1 \leq m \leq M$

$$\rho^+(t) := -r^+(t) := -\frac{27}{\nu^3} k \sum_{i=0}^{m} \|\mathbf{Q}_h^0 \mathbf{v}(t_i)\|_{\mathbb{L}^4}^4 \qquad \forall t \in (t_{m-1}, t_m]\,, \tag{3.6.13}$$

for a fixed $\mathbf{v} \in C([0,T];\boldsymbol{\mathcal{V}})$. Then, thanks to Lemma 3.6.1, the following inequality holds,

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\Big(2\mathbf{G}(\boldsymbol{\mathcal{V}}^+) + d_t r^+ \boldsymbol{\mathcal{V}}^+, \boldsymbol{\mathcal{V}}^+ - \boldsymbol{\mathcal{U}}_{k,h}^+\Big)\,\mathrm{d}s\right] \\ &\geq \mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\Big(2\mathbf{G}(\boldsymbol{\mathcal{U}}_{k,h}^+) + d_t r^+ \boldsymbol{\mathcal{U}}_{k,h}^+, \boldsymbol{\mathcal{V}}^+ - \boldsymbol{\mathcal{U}}_{k,h}^+\Big)\,\mathrm{d}s\right], \end{aligned} \tag{3.6.14}$$

for all $\boldsymbol{\mathcal{U}}_{k,h}^+ \in L^2\Big(\Omega; L^2(0,T;\mathbb{W}_0^{1,2})\Big)$, and every $\boldsymbol{\mathcal{V}}^+ = \mathbf{Q}_h^0 \mathbf{v}^+ \in L^\infty([0,T];\mathbb{W}_0^{1,2})$. Note that the radius $r(t)$ of the time varying ball $B_{r(t)}$ is determined by the function $\boldsymbol{\mathcal{V}}^+$.

Now we try to pass to the limit in inequality (3.6.14). The terms which cause most problems are those involving the discrete derivative, because they do not fit exactly in the framework of (3.6.14). First note that because of Lemma 3.3.1, (i)$_3$

$$\lim_{k\to 0} \mathbb{E}\left[\|\boldsymbol{\mathcal{U}}_{k,h}^+ - \boldsymbol{\mathcal{U}}_{k,h}^-\|_{L^2(0,T;\mathbb{L}^2)}\right] = 0,$$

which implies the existence of an $L^2(\Omega; L^2(0,T))$-valued sequence $0 < \{\epsilon_k\}$ such that the convergence $\mathbb{E}\left[\|\epsilon_k\|_{L^2(0,T)}^2\right] \to 0$ for $k \to 0$ holds, and

$$\mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] = \mathbb{E}\left[\int_0^T \left[\|\boldsymbol{\mathcal{U}}_{k,h}^-\|_{\mathbb{L}^2}^2 + \epsilon_k\right]\mathrm{d}s\right].$$

Then we use the fact that $\boldsymbol{\mathcal{V}}^+ \to \mathbf{v}$ in $L^q(0,T;\mathbb{W}_0^{1,2})$ for $h \to 0$ by (3.2.3) to conclude

$$\begin{aligned} &\liminf_{k,h\to 0} \mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{U}}_{k,h}^-\|_{\mathbb{L}^2}^2 \left[\mathrm{e}^{-r^+} g(s) d_t r^+ + \mathrm{e}^{\xi}\frac{(r^+ - r^-)^2}{2k}\right]\mathrm{d}s\right] = \\ &\liminf_{k,h\to 0} \mathbb{E}\left[\int_0^T \Big(\|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2 - \epsilon_k\Big)\left[\mathrm{e}^{-r^+}(1 + g(s) - 1) d_t r^+ + \mathrm{e}^{\xi}\frac{(r^+ - r^-)^2}{2k}\right]\mathrm{d}s\right] \\ &= \liminf_{k,h\to 0} \mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2 \mathrm{e}^{-r^+} d_t r^+\,\mathrm{d}s\right] + \liminf_{k,h\to 0} \mathcal{I}_R\,, \end{aligned} \tag{3.6.15}$$

where

$$\begin{aligned} \mathcal{I}_R = &\,\mathbb{E}\left[\int_0^T -\epsilon_k \left[\mathrm{e}^{-r^+} g(s) d_t r^+ + \mathrm{e}^{\xi}\frac{(r^+ - r^-)^2}{k}\right]\mathrm{d}s\right] \\ &+ \mathbb{E}\left[\int_0^T \|\boldsymbol{\mathcal{U}}_{k,h}^+\|_{\mathbb{L}^2}^2 \left[\mathrm{e}^{-r^+}(g(s) - 1) d_t r^+ + \mathrm{e}^{\xi}\frac{(r^+ - r^-)^2}{k}\right]\mathrm{d}s\right]. \end{aligned} \tag{3.6.16}$$

All terms in (3.6.16) converge to 0 in $L^1(\Omega, L^1(0,T))$ by Lebesgue dominated convergence theorem, and so the residual term $\mathcal{I}_R$ converges to 0 for $k, h \to 0$. Thus we get

$$\begin{aligned}
&\liminf_{k,h\to 0} \mathbb{E}\left[-\int_0^T \left(\mathrm{e}^{-r^+} 2\mathbf{G}(\boldsymbol{\mathcal{U}}_{k,h}^+) + \left[\mathrm{e}^{-r^+} d_t r^+\right] \boldsymbol{\mathcal{U}}_{k,h}^+, \boldsymbol{\mathcal{U}}_{k,h}^+\right) \mathrm{d}s\right] \\
&\quad = \liminf_{k,h\to 0} \mathbb{E}\left[-\int_0^T \left(\mathrm{e}^{-r^+} 2\mathbf{G}(\boldsymbol{\mathcal{U}}_{k,h}^+), \boldsymbol{\mathcal{U}}_{k,h}^+\right) \mathrm{d}s\right] \\
&\qquad + \liminf_{k,h\to 0} \mathbb{E}\left[-\int_0^T \left(\left[\mathrm{e}^{-r^-} d_t r^+ + \mathrm{e}^{\xi} \frac{(r^+ - r^-)^2}{k}\right] \boldsymbol{\mathcal{U}}_{k,h}^-, \boldsymbol{\mathcal{U}}_{k,h}^-\right) \mathrm{d}s\right] \\
&\quad \geq \liminf_{k,h\to 0} \mathbb{E}\left[\mathrm{e}^{-r^+(T)} \|\boldsymbol{\mathcal{U}}_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\boldsymbol{\mathcal{U}}_{k,h}^+(0)\|_{\mathbb{L}^2}^2 - \int_0^T \mathrm{e}^{-r^+} \left(\|\mathbf{g}^-\|_{\mathcal{I}_2}^2 + \|\mathbf{f}^+\|_{\mathbb{L}^2}^2\right) \mathrm{d}s\right] \\
&\quad \geq \mathbb{E}\left[\mathrm{e}^{-r^+(T)} \|\mathbf{u}(T)\|_{\mathbb{L}^2}^2 - \|\mathbf{u}_0\|_{\mathbb{L}^2}^2 - \int_0^T \mathrm{e}^{-r} \left(\|\mathbf{g}\|_{\mathcal{I}_2}^2 + \|\mathbf{f}\|_{\mathbb{L}^2}^2\right) \mathrm{d}s\right] \\
&\quad = \mathbb{E}\left[-\int_0^T \mathrm{e}^{-r} \left(2\mathbf{G}_0 + \frac{27}{\nu^3} \|\mathbf{v}\|_{\mathbb{L}^4}^4 \mathbf{u}, \mathbf{u}\right) \mathrm{d}s\right],
\end{aligned}$$

where we used (3.6.15) in the first step, inequality (3.6.11) in the second step, lower semicontinuity of norms, strong convergence of the initial data, strong convergence of $\mathbf{g}^-$ and $\mathbf{f}^+$ from (3.6.8) in the third step, and Itô formula for the limit equation in the fourth step. Now we are allowed to take the limit in (3.6.14) to get

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-r}\Big(2\mathbf{G}(\mathbf{v}) + \partial_s r \mathbf{v}, \mathbf{v} - \mathbf{u}\Big)\, \mathrm{d}s\right] \geq \mathbb{E}\left[\int_0^T \mathrm{e}^{-r}\Big(2\mathbf{G}_0 + \partial_s r \mathbf{u}, \mathbf{v} - \mathbf{u}\Big)\, \mathrm{d}s\right]$$

where $r : [0,T] \to \mathbb{R}$ is defined by

$$\partial_s r(s) = \frac{27}{\nu^3} \|\mathbf{v}(\mathbf{s})\|_{\mathbb{L}^4}^4 \quad \mathbf{v} \in C([0,T], \boldsymbol{\mathcal{V}}).$$

By density, we get that the inequality holds for all $\mathbf{v} \in L^4(\Omega; L^\infty(0,T;\mathbb{H})) \cap L^2(\Omega; L^2(0,T;\mathbb{V}))$. Now the conclusion of the proof follows by a standard argument of monotone linear operators. Set $\mathbf{v} := \mathbf{u} + \lambda \mathbf{w}$, for $\mathbf{w} \in L^4(\Omega; L^\infty(0,T;\mathbb{H})) \cap L^2(\Omega; L^2(0,T;\mathbb{V}))$ and $\lambda > 0$, divide the inequality by λ, and use hemicontinuity of the operator $\mathbf{G}$ to let $\lambda \to 0$ and get

$$\mathbb{E}\left[\int_0^T \Big(\mathbf{G}(\mathbf{u}(s)) - \mathbf{G}_0(s), \mathbf{w}(s)\Big) \mathrm{d}s\right] \geq 0,$$

which implies $\mathbf{G}(\mathbf{u}(t)) = \mathbf{G}_0(t)$ with probability 1, a.e. in $[0,T] \times D$, since $\mathbf{w}$ is arbitrary.

3.6.2 Multiplicative noise

Now we assume that $\mathbf{g}$ is a nonlinear function of $\mathbf{u}$ satisfying (SII$_3$). In addition to the convergences (3.6.6) and (3.6.7), we have

$$\mathbf{g}(\boldsymbol{\mathcal{U}}_{k,h}^-) \rightharpoonup \mathbf{g}_0(t) \quad \text{in} \quad L^2(\Omega; L^2(0,T;\mathcal{I}_2)). \tag{3.6.17}$$

Then it follows that the limit $\mathbf{u}$ is solution of the following equation

$$\begin{aligned}
&(\mathbf{u}(t) - \mathbf{u}_0, \boldsymbol{\varphi}) + \int_0^t (\mathbf{G}_0(s), \boldsymbol{\varphi})\, \mathrm{d}s \\
&\quad = \int_0^t (\mathbf{f}(s), \boldsymbol{\varphi})\, \mathrm{d}s + \int_0^t (\mathbf{g}_0(s) \mathrm{d}\mathbf{W}(s), \boldsymbol{\varphi}) \quad \forall\, t \in [0,T],
\end{aligned} \tag{3.6.18}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}$. To prove that $\mathbf{u}$ is a strong solution, we need to identify the terms $\mathbf{G}_0(t)$ and $\mathbf{g}_0(t)$. A necessary condition for the uniqueness is a Lipschitz condition on the mapping $\mathbf{g} : \mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}) \to \boldsymbol{\mathcal{H}}$,

$$\|\mathbf{g}(\mathbf{v}) - \mathbf{g}(\mathbf{u})\|_{\mathcal{I}_2} \leq K_2 \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2} . \tag{3.6.19}$$

As we will see in the proof, we have to combine the Lipschitz condition with the condition given in (3.6.5). Summing these two inequalities we get

$$\begin{aligned} -\Big(\mathbf{G}(\mathbf{u}) - \mathbf{G}(\mathbf{v}), \mathbf{u} - \mathbf{v}\Big) - \frac{27r^4}{2\nu^3}\|\mathbf{u} - \mathbf{v}\|^2_{\mathbb{L}^2} + \|\mathbf{g}(\mathbf{v}) - \mathbf{g}(\mathbf{u})\|^2_{\mathcal{I}_2} \\ \leq -\frac{\nu}{2}\|\nabla(\mathbf{u} - \mathbf{v})\|^2_{\mathbb{L}^2} + K_2^2\|\mathbf{u} - \mathbf{v}\|^2_{\mathbb{L}^2} \quad \forall \mathbf{u} \in \mathbb{W}^{1,2}_0, \quad \forall \mathbf{v} \in \mathbb{B}^r . \end{aligned}$$

This implies (SII_3). Then, using Lemma 3.3.1, we have

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-r^+(T)}\|\boldsymbol{\mathcal{U}}^+_{k,h}(T)\|^2_{\mathbb{L}^2}\right] \leq\ & \mathbb{E}\left[\|\boldsymbol{\mathcal{U}}^-_{k,h}(0)\|^2_{\mathbb{L}^2}\right] \\ & +\mathbb{E}\left[\int_0^T \left[d_t\mathrm{e}^{-r^+}\|\boldsymbol{\mathcal{U}}^-_{k,h}\|^2_{\mathbb{L}^2} - \mathrm{e}^{-r^+}\Big(2\mathbf{G}(\boldsymbol{\mathcal{U}}^+_{k,h}),\boldsymbol{\mathcal{U}}^+_{k,h}\Big)\right] \mathrm{d}s\right] \\ & +\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\Big(\|\mathbf{g}(\boldsymbol{\mathcal{U}}^-_{k,h})\|^2_{\mathcal{I}_2} + (\mathbf{f},\boldsymbol{\mathcal{U}}^+_{k,h})\Big)\,\mathrm{d}s\right] . \end{aligned} \tag{3.6.20}$$

Then for every $\mathbf{v} \in C([0,T];\mathbb{V})$, define $\boldsymbol{\mathcal{V}}^+$ as in the computations for the additive noise case and conclude

$$\begin{aligned} &\mathbb{E}\left[\mathrm{e}^{\rho^+(T)}\|\boldsymbol{\mathcal{U}}^+_{k,h}(T)\|^2_{\mathbb{L}^2} - \|\boldsymbol{\mathcal{U}}^+_{k,h}(0)\|^2_{\mathbb{L}^2}\right] \\ &\leq \mathbb{E}\left[\int_0^T -\mathrm{e}^{-r^+}\Big(2\mathbf{G}(\boldsymbol{\mathcal{U}}^+_{k,h}) - \mathbf{G}(\boldsymbol{\mathcal{V}}^+),\boldsymbol{\mathcal{U}}^+_{k,h} - \boldsymbol{\mathcal{V}}^+\Big)\,\mathrm{d}s\right] \\ &\quad +\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\|\mathbf{g}(\boldsymbol{\mathcal{U}}^+_{k,h}) - \mathbf{g}(\boldsymbol{\mathcal{V}}^+)\|^2_{\mathcal{I}_2}\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \Big(d_t\mathrm{e}^{-r^+}\Big)\,\|\boldsymbol{\mathcal{U}}^+_{k,h} - \boldsymbol{\mathcal{V}}^+\|^2_{\mathbb{L}^2}\,\mathrm{d}s\right] \\ &\quad +\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\|\mathbf{g}(\boldsymbol{\mathcal{U}}^+_{k,h}) - \mathbf{g}(\boldsymbol{\mathcal{U}}^-_{k,h})\|^2_{\mathcal{I}_2}\,\mathrm{d}s\right] \\ &\quad +\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\Big(\Big(-2\mathbf{G}(\boldsymbol{\mathcal{V}}^+),\boldsymbol{\mathcal{U}}^+_{k,h}\Big) - 2\Big(\mathbf{G}(\boldsymbol{\mathcal{U}}^+_{k,h}) - \mathbf{G}(\boldsymbol{\mathcal{V}}^+),\boldsymbol{\mathcal{V}}^+\Big)\Big)\mathrm{d}s\right] \\ &\quad +\mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}\Big(2\Big(\mathbf{g}(\boldsymbol{\mathcal{U}}^+_{k,h}),\mathbf{g}(\boldsymbol{\mathcal{V}}^+)\Big)_{\mathcal{I}_2} + -\|\mathbf{g}(\boldsymbol{\mathcal{V}}^+)\|^2_{\mathcal{I}_2}\Big)\,\mathrm{d}s\right] \\ &\quad +\mathbb{E}\left[\int_0^T \Big(d_t\mathrm{e}^{-r^+}\Big)\Big(2\Big(\boldsymbol{\mathcal{U}}^-_{k,h},\boldsymbol{\mathcal{V}}^+\Big) - \|\boldsymbol{\mathcal{V}}^+\|^2_{\mathbb{L}^2}\Big)\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \mathrm{e}^{-r^+}(\mathbf{f},\boldsymbol{\mathcal{U}}^+_{k,h})\,\mathrm{d}s\right] . \end{aligned}$$

With Assumption (SII_3), using the representation (3.6.15) for the discrete derivative of the exponential function and condition (S_2), we can get rid of the first three terms on the right-hand side leaving only the rest term from (3.6.16). The fourth term does not cause any difficulty, since $\mathbf{g}$ is Lipschitz continuous and the difference $\boldsymbol{\mathcal{U}}^+_{k,h} - \boldsymbol{\mathcal{U}}^-_{k,h}$ converges strongly to 0. Then, by the same argument for the last term, to fit all the terms in the setting of (SII_3)', by using the lower semicontinuity of the norm, strong convergence of the initial data, strong convergence of

$\boldsymbol{\mathcal{V}}^+$, and part (iii) of Lemma 3.3.1, we get

$$\begin{aligned}
&\lim_{k,h\to 0} \mathbb{E}\left[e^{\rho^+(T)} \|\boldsymbol{\mathcal{U}}^+_{k,h}(T)\|^2_{\mathbb{L}^2} - \|\boldsymbol{\mathcal{U}}^-_{k,h}(0)\|^2_{\mathbb{L}^2} \right] \\
&\leq \mathbb{E}\left[-\int_0^T e^{-r} \left((2\mathbf{G}(\mathbf{v}(s)), \mathbf{u}(s)) - 2\left(\mathbf{G}_0 - \mathbf{G}(\mathbf{v}(s)), \mathbf{v}(s)\right) \right) \mathrm{d}s \right] \\
&+\mathbb{E}\left[\int_0^T e^{-r} \left(2\left(\mathbf{g}_0, \mathbf{g}(\mathbf{v})\right)_{\mathcal{I}_2} - \|\mathbf{g}(\mathbf{v})\|^2_{\mathcal{I}_2} \right) \mathrm{d}s \right] \\
&+\mathbb{E}\left[\int_0^T \left(\partial_t e^{-r}\right) \left(2\left(\mathbf{u}, \mathbf{v}\right) - \|\mathbf{v}\|^2_{\mathbb{L}^2} \right) \mathrm{d}s \right] + \mathbb{E}\left[\int_0^T e^{-r} \left(\mathbf{f}, \mathbf{u}\right) \mathrm{d}s \right].
\end{aligned}$$

Using (3.6.18) and Itô formula to substitute the left-hand side, and rearranging we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r} \left(2\mathbf{G}(\mathbf{v}) - 2\mathbf{G}_0 + \partial_s r\ [\mathbf{v} - \mathbf{u}], \mathbf{v} - \mathbf{u} \right) \right] \\
&\qquad - \mathbb{E}\left[\int_0^T e^{-r} \|\mathbf{g}(\mathbf{v}) - \mathbf{g}_0\|^2_{\mathcal{I}_2} \right] \geq 0 \quad \forall\, \mathbf{v} \in C([0,T]; \mathbb{V}).
\end{aligned}$$

Extending by the density argument to all $\mathbf{v} \in L^4(\Omega; L^\infty(0,T;\mathbf{H})) \cap L^2(\Omega; L^2(0,T;\mathbb{V}))$ and setting $\mathbf{v} := \mathbf{u}$, we get $\mathbf{g}(\mathbf{u}(t)) = \mathbf{g}_0(t)$ with probability 1, a.e. in $[0,T] \times D$. Then using the same argument as in the previous subsection, we can prove $\mathbf{G}(\mathbf{u}(t)) = \mathbf{G}_0(t)$ with probability 1, a.e. in $[0,T] \times D$, finishing the proof of the existence of a strong solution.

According to [119, 98], strong solutions of (3.1.1)–(3.1.3) for $d = 2$ are unique. The arguments used to show uniqueness are similar in both works. The authors consider the function $y(t)\big(\mathbf{u}(t) - \mathbf{v}(t)\big)$, for $\mathbf{u}$ and $\mathbf{v}$ two solutions and for some appropriate choice of $y(t)$, show a corresponding energy inequality, and with the properties of the convective term in $d = 2$, they prove that $\mathbf{u}$ and $\mathbf{v}$ coincide $\mathbb{P}$-almost surely for the same initial condition. As a consequence for our study, uniqueness implies that the whole sequence $\{\boldsymbol{\mathcal{U}}^+_{k,h}\}_{k,h}$ generated by the iterates of Algorithm 3.1 converges to the strong solution.

We summarize the results of this section in the following theorem, which uses the stronger assumption (SII_3') instead of (S_2), in particular

Theorem 3.6.1. *Let $D \subset \mathbb{R}^2$ be a polyhedral domain, and $T > 0$. Suppose (SII_1), (SII_2), and (SII_3'), and let $\mathbf{W}$ be a $\mathbf{Q}$-Wiener process on the filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$. For every finite $(k,h) > 0$, let $\mathcal{T}_h$ be a quasi-uniform triangulation of D, I_k be an equi-distant partition covering $[0,T]$, and $\big(\mathbb{H}_h, L_h\big)$ a pair of finite element spaces that satisfies the discrete LBB condition.*

Let $\boldsymbol{\mathcal{U}}^+_{k,h} : D_T \to \mathbb{R}^2$ be the piecewise constant process obtained from iterates $\{\mathbf{U}^m\}_{m=0}^M$ for $k,h > 0$, and let $\mathbf{U}^0 \to \mathbf{u}_0$ in $L^2\big(\Omega; \mathbb{L}^2\big)$ for $h \to 0$. Then the whole sequence $\big\{\boldsymbol{\mathcal{U}}^+_{k,h}\big\}_{k,h}$ converges

$$\boldsymbol{\mathcal{U}}^+_{k,h} \rightharpoonup \mathbf{u} \qquad \text{in } L^2\big(\Omega; L^2\big(0,T;\mathbb{W}^{1,2}_0\big)\big) \qquad (k,h \to 0)\,,$$

where $\mathbf{u} : D_T \times \Omega \to \mathbb{R}^2$ is a strong solution of problem (3.1.1)–(3.1.3) *in the sense of Definition 3.6.1.*

3.7 Computational experiments

In this section we give some examples of discretisations of the stochastic Navier-Stokes equations (5.1.1)–(5.1.3). To this end we consider the semi-implicit Euler scheme given by Algorithm 3.3

for the discretisation. For the spatial discretisation we choose the LBB-stable MINI element, which consists of continuous piecewise linear functions enriched with a 'bubble' part for the velocity, and continuous piecewise linear functions for the pressure; see e.g. [74].

All the simulations are performed for the case $d = 2$ on a cartesian mesh of the domain $D = (0,1)^2$ on the time interval $[0,1]$. For the noise we consider the representation $\mathbf{W} := (W_1, W_2)^T$, for two independent Wiener processes W_i, $i = 1,2$ with values in the space $W_0^{1,2}$. We then consider the following expansions; see (2.1).

(i) Exact Gaussian increments

$$\Delta_m W_i \approx \sum_{j,k=1}^{N} \sqrt{\lambda_{j,k}^i}(\Delta_m \beta_{j,k}^i)\mathbf{e}_{j,k}\,.$$

(ii) Approximated increments

$$\Delta_m W_i \approx \sqrt{k} \sum_{j,k=1}^{N} \sqrt{\lambda_{j,k}^i}\xi_{j,k}^{i,m}\mathbf{e}_{j,k}\,.$$

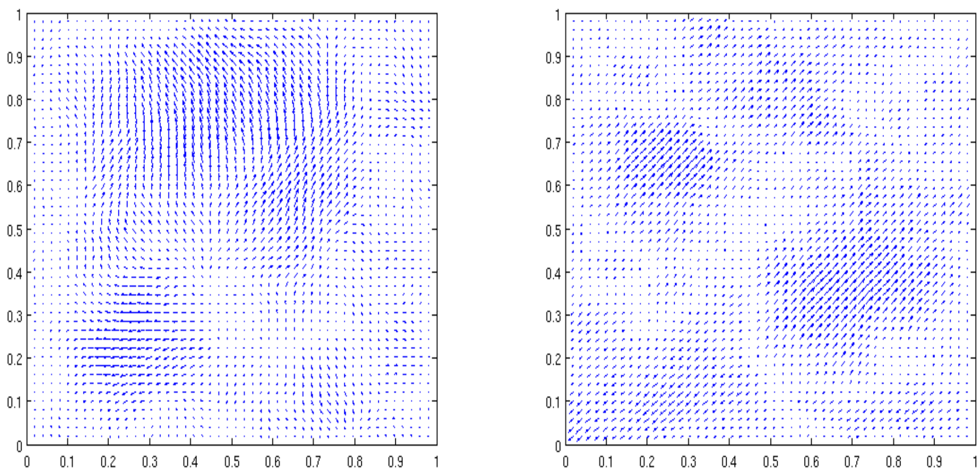

Figure 3.7.1: Wiener process for eigenvalues from point **a)**: Exact (left) and approximate increments, $M = 256$.

Here the elements of the basis are defined as the eigenvalues of the Laplace operator with homogeneous Dirichlet boundary conditions on the domain $(0,1)^2$, i.e.

$$\mathbf{e}_{j,k} := \sin(j\pi x)\,\sin(k\pi y) \quad x,y \in (0,1), \quad j,k \in \mathbb{N}\,.$$

In the case (i), the $\{\beta_{j,k}^i\}_{j,k=1}^{\infty}$, $i = 1,2$, are two independent families of independent $\mathbb{R}$-valued brownian motions, while the random variables $\{\{\xi_{j,k}^{i,m}\}_{m=1}^{\infty}\}_{j,k=1}^{\infty}$, $i = 1,2$, are independent and defined as

$$\mathbb{P}[\xi_{j,k}^{i,m} = 1] = \mathbb{P}[\xi_{j,k}^{i,m} = -1] = \frac{1}{2}\,.$$

For computational purposes, the above expansions are truncated, i.e. only a sum over a finite number N of terms is considered. This number N depends on the space discretisation parameter

h, in the sense that its behavior should be $O(h^{-1})$ in order to ensure convergence; see e.g. [132, Section 4] for the stochastic heat equation. To avoid the influence of the truncation error, we consider always finite dimensional covariance operator, which means that the sequences $\{\lambda^i_{j,k}\}_{j,k=1}^{\infty}$, $i = 1, 2$ are non zero for only a fixed finite number of terms.

We set $\mathbf{f} = (f_1, f_2)^T$ with $f_i := -\Delta u_i + \partial_{x_i}\pi$ for

$$\begin{aligned} u_1(t, x, y) &:= x^2(1-x)^2(2y - 6y^2 + 4y^3)\,, \\ u_2(t, x, y) &:= -y^2(1-y)^2(2x - 6x^2 + 4x^3)\,, \\ \pi(x, y, t) &:= (x^2 + y^2 - 2/3)(1 + t^2)\,, \end{aligned}$$

and $\mathbf{u}_0 = \mathbf{0}$.

3.7.1 Wiener process

Here we consider the discretisation of the noise for $M = 256$. We show plot of the increments of the noise for different choices of eigenvalues. We consider finite sums in (i) and (ii) above, setting 30 as the number of non-zero modes.

a) $\lambda_{k,j} = \frac{1}{(k+j)^2}$ for $jk \leq 30$,

b) $\lambda_{k,j} = 1$ for $jk \leq 30$,

In Figure 3.7.1 we can see the increments $\Delta_m \mathbf{W}$ and the random variables $\boldsymbol{\xi}_m$ for $M = 256$, plotted for $h = 1/50$ for part **a)**. For part **b)**, the increments are plotted in Figure 3.7.2, for $M = 256$ and $h = 1/100$. In this case, higher definition is needed to show the effects of the higher order modes. The non-decaying behavior of the Eigenvalues from **b)** causes the increments to be rougher, having a bigger $\mathbb{L}^2$-norm.

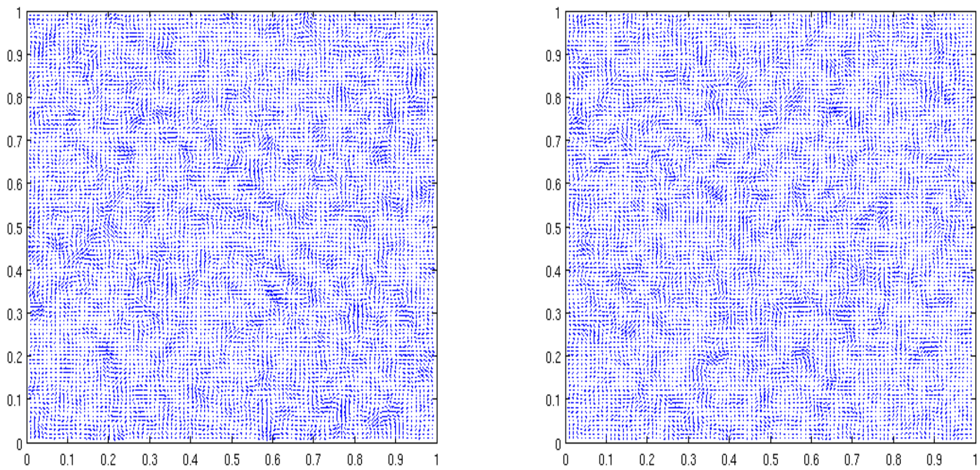

Figure 3.7.2: Wiener process with eigenvalues from point **b)**: Exact (left) and approximate increments, $M = 256$.

3.7.2 Brownian and discrete increments

Here we consider the discretisation of the solution corresponding to the forcing terms given in **a)** and **b)**. First we consider the Eigenvalues from **a)**. In Figure 3.7.3 are depicted snapshots for the pressure and velocity at times $t = 0.5$ and $t = 1$ in the case of Wiener increments, while in Figure 3.7.4 the same plot are given for discrete increments.

As regards the Eigenvalues from **b)**, in Figure 3.7.5 are depicted the snapshots at time $t = 1$ for both, velocity and pressure, computed with exact and approximated increments.

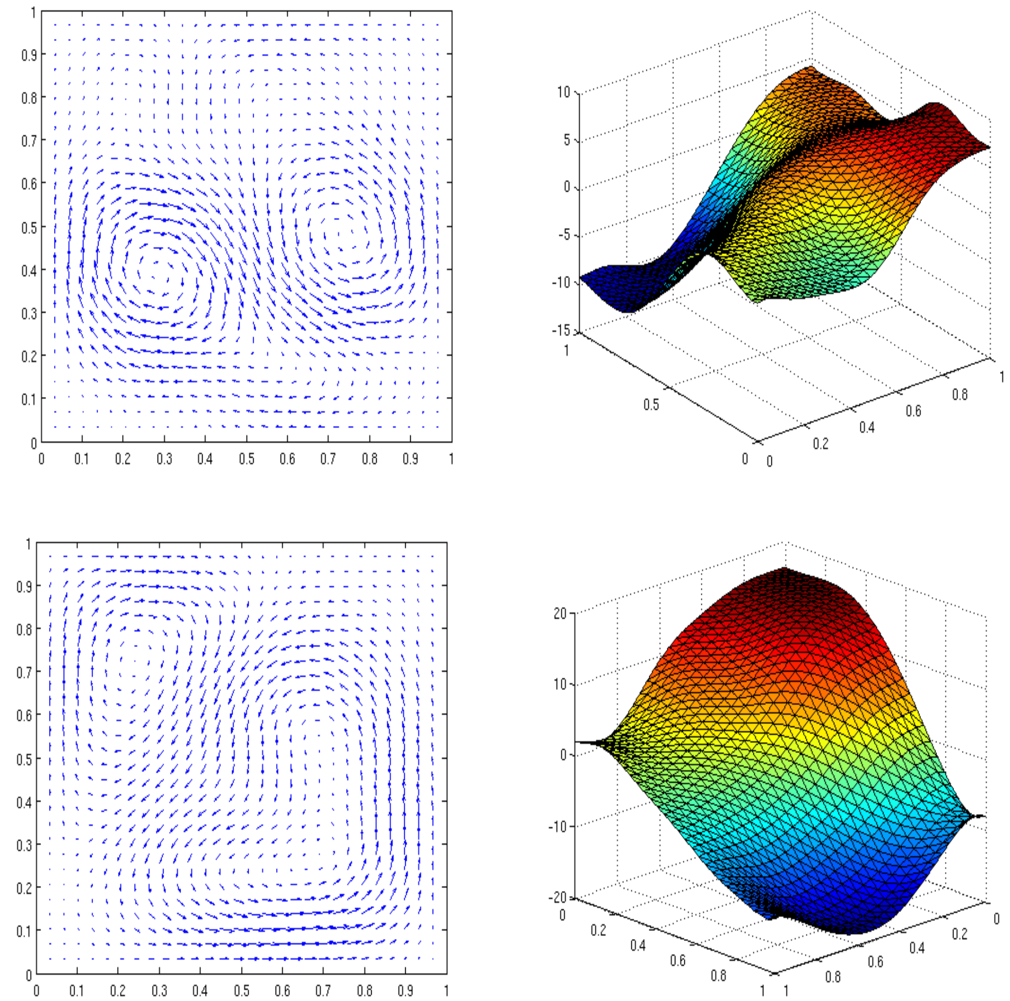

Figure 3.7.3: Velocity (left column) and pressure (right) for $t = 0.5$ (first row) and $t = 1$ for the eigenvalues from **a)**, computed with the exact increments.

From the computations, we observe how the regularity of the solution is influenced by the noise. In particular, and this will be crucial in the next chapters, we notice that while the velocity seems to keep some regularity even for nuclear noise, the pressure is particularly affected by the regularity of the noise. Moreover, the pressure computed by the approximated increments is smaller, due to the fact that in this case the increments are always bounded.

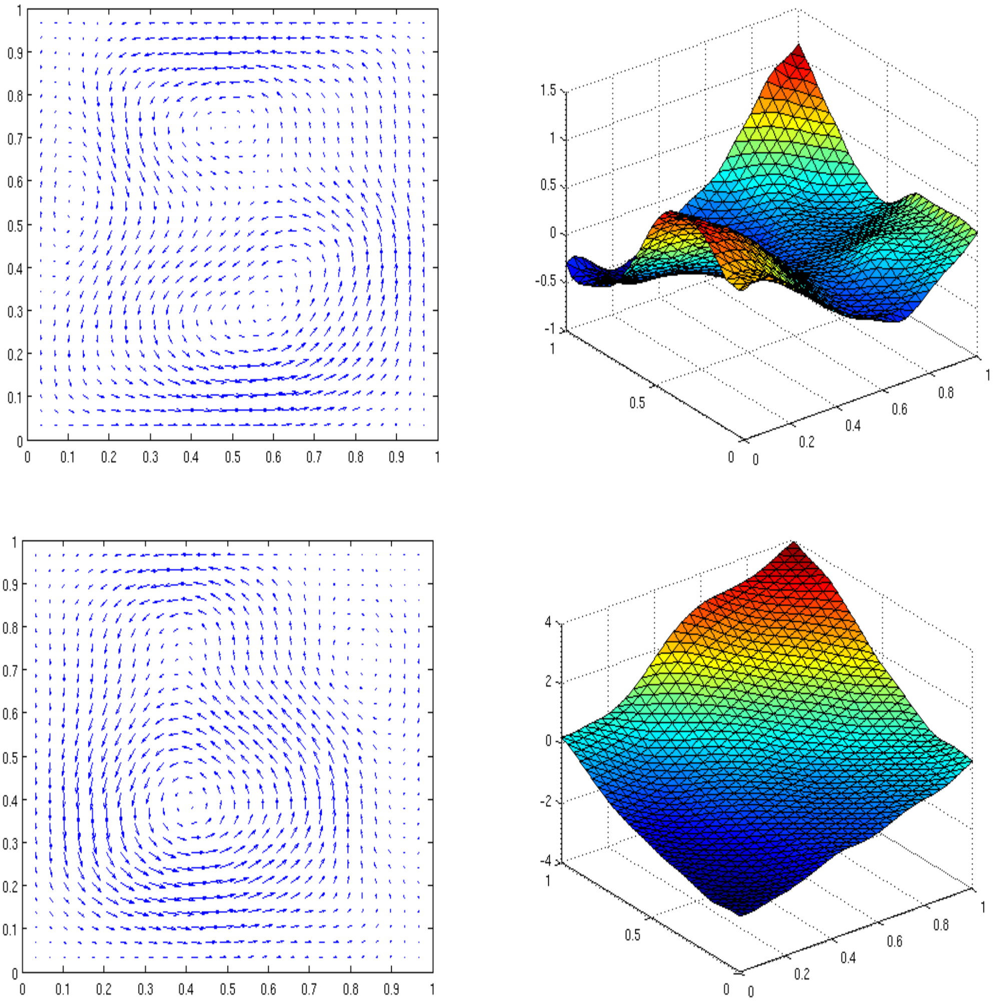

Figure 3.7.4: Velocity (left column) and pressure (right) for $t = 0.5$ (first row) and $t = 1$ for the Eigenvalues from **b)**, computed with the approximate increments.

3.7.3 Navier-Stokes-Coriolis equations

Let $\{\mathbf{b}_i\}_{i=1}^3$ be the canonical basis of $\mathbb{R}^3$, and $D = \mathbb{R}^2 \times (0, b)$, $b > 0$. We consider the following Navier-Stokes-Coriolis equations

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla\pi + \zeta(\mathbf{b}_3 \times \mathbf{u}) = \mathbf{0} \qquad \text{in } D_T \times \Omega\,, \tag{3.7.1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } D_T \times \Omega\,, \tag{3.7.2}$$

with boundary conditions for the third component given by

$$\mathbf{u}(t, x_1, x_2, 0) = \mathbf{0}, \quad \mathbf{u}(t, x_1, x_2, b) = u_b\mathbf{b}_3 \quad x_1, x_2 \in \mathbb{R},\ t > 0\,.$$

Here $u_b \in \mathbb{R}$ is a constant and $\zeta > 0$ denotes the speed of rotation. These equations are used to describe the motion of a fluid under the influence of the Coriolis force, which is represented by the last term on the left-hand side of (3.7.1). Beyond the practical importance of these equations (e.g. meteorology), an important consequence is that for ζ big enough, the system admits well posedness, evidencing the dissipative effect of the Coriolis force, which dominates the convection;

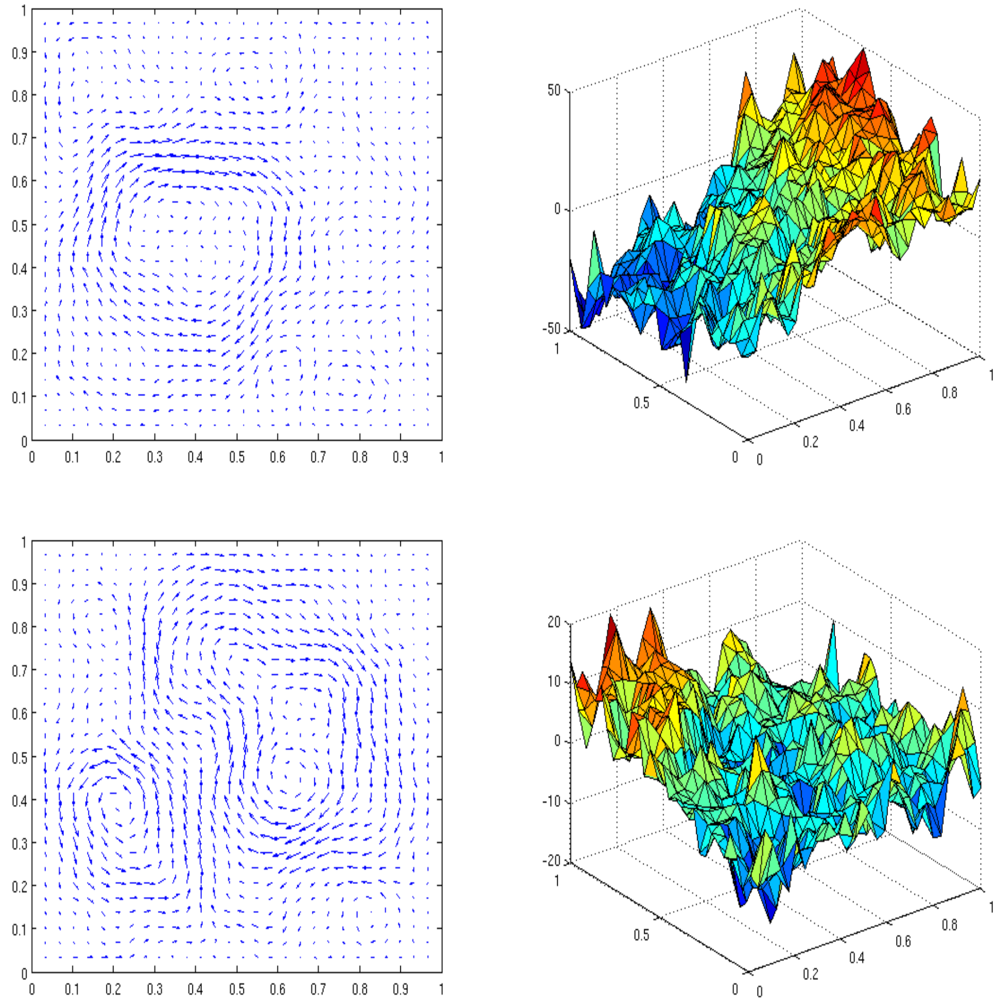

Figure 3.7.5: Velocity (left column) and pressure (right) for $t = 1$, computed with the Eigenvalues from Example **b)** computed with exact (first row) and approximate increments (second row).

see e.g. [85]. It is remarkable that this system admits an explicit stationary solution given by

$$\mathbf{u}_{stat}(x_1, x_2, x_3) = \tilde{u}_b \begin{pmatrix} 1 - \mathrm{e}^{-\frac{x_3}{\delta}} \cos\left(\frac{x_3}{\delta}\right) \\ \mathrm{e}^{-\frac{x_3}{\delta}} \sin\left(\frac{x_3}{\delta}\right) \\ 0 \end{pmatrix},$$

where $\delta = b/\pi k$, for $k \in \mathbb{Z}$, and

$$\tilde{u}_b := \begin{cases} u_b(1 - \mathrm{e}^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is even} \\ u_b(1 + \mathrm{e}^{-\frac{b}{\delta}})^{-1} & \text{if } k \text{ is odd} \end{cases}.$$

This solution is called Ekman spiral, and describes the theoretical displacement of current direction by the Coriolis effect[1]. Given this stationary solution, it is worth to study its stability. The stability under deterministic perturbations has been considered by many authors, and exponential convergence of solution towards the Ekman spiral has been proved. For stochastic

[1] Encyclopaedia Britannica.

perturbations, the problem has been considered in [72], where its stochastic analogue is introduced, and its stochastic stability is studied by means of stochastic stationary solutions. the stochastic problem reads.

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla\pi + \zeta(\mathbf{k}\times\mathbf{u}) = \varepsilon \mathrm{d}\mathbf{W} \qquad \text{in } D_T\times\Omega\,, \tag{3.7.3}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } D_T\times\Omega \tag{3.7.4}$$

with the same boundary conditions.

Objective of this example is to consider a two dimensional version of the problem, and simulate a practical example. Our setting is slightly different, since we consider the two dimensional problem with Dirichlet boundary conditions

$$\mathbf{u} = (1,0)^T \quad \text{for } \mathbf{x}\in E\,, \quad \mathbf{u} = \mathbf{0} \quad \text{for } \in E^c\,,$$

where $E = [0,1]\times\{1\}$, and non-solenoidal Wiener noise. In this setting, the Coriolis force reads

$$\zeta(\mathbf{k}\times\mathbf{u}) = \zeta\begin{pmatrix} u_2 \\ -u_1 \\ 0 \end{pmatrix}.$$

To understand the interplay between the Coriolis force and the noise, we consider the discretisation of these equations for $\zeta = 40$ and $\varepsilon = 0.1$ at a fixed time $t_M = 3$. The results are depicted in Figure 3.7.6.

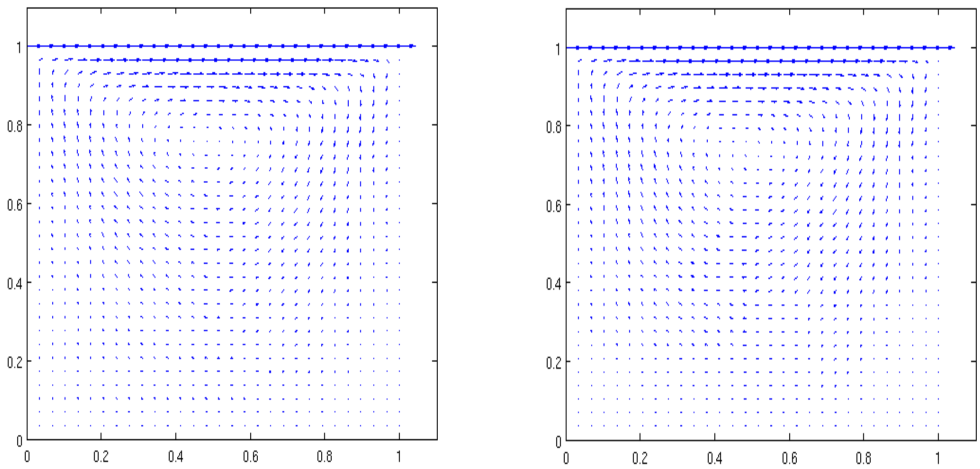

Figure 3.7.6: Velocities for $t_M = 3$, computed for the Navier-Stokes-Coriolis equations, with the Eigenvalues from Example **b)** computed with exact (left) and approximate increments (right).

We note that the solutions is stable in the sense that it fluctuates around the profile shown in Figure 3.7.6, for both, the exact and the approximate increments.

3.8 Summary and outlook

We analysed the qualitative convergence of different space-time discretisations, based on the (semi-) implicit Euler scheme with a discretely stable LBB-stable finite element pairing.

In three dimensions, the problem is caused by the need of compactness to handle the nonlinear term, by the weakly divergence-free finite element pairing, and by the construction of the stochastic integral. The main tools are suitable a priori bounds for the discrete solution and for corresponding increments. For the construction of the stochastic integral we consider two procedures: the computation by Gaussian increments, and the approximation by general random variables with consistent assumptions on the moments. To show that the limiting object is the desired stochastic integral, we make use of appropriated convergence theorems for continuous and time discrete martingales, together with a representation theorem.

In two dimensions, we address the construction of strong solutions by a local monotonicity method, which uses weak convergence only. The main problem is to adapt computations from continuous to a discrete setting. This is accomplished by using a priori estimates for a perturbed problem, together with a careful handling of the residual term arising from the discrete derivatives.

Further steps in the numerical analysis of the SNSEs are the quantitative statement on the convergence of approximating sequences, the application of splitting methods to increase the efficiency of the discretisation, and what are the properties of the Lagrange multiplier and how this affects the convergence.

Acknowledgment: Parts of this chapter were written when I visited the Newton Institute, Cambridge (UK) in the period January-May 2010 to participate on the program: *Stochastic Partial Differential Equations.*

Chapter 4

Splitting-based scheme for the Stokes equations

4.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, and $D \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polyhedral domain and define $D_T = (0, T) \times D$. We consider the d-dimensional stochastic Stokes equation

$$\begin{aligned} \dot{\mathbf{u}} - \nu\Delta\mathbf{u} + \nabla\pi &= \mathbf{f} + \mathbf{g}(\cdot, \mathbf{u})\dot{\mathbf{W}} && \text{in } D_T \times \Omega\,, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } D_T \times \Omega\,, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } D \times \Omega\,, \end{aligned} \tag{4.1.1}$$

together with the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial D_T := (0, T) \times \partial D$. Here, the velocity $\mathbf{u} = (u_1, .., u_d)$ and the pressure p are unknown random fields on D_T, and $\mathbf{W}$ is a $\boldsymbol{\mathcal{H}}$-valued, $\mathbb{F}$-adapted Wiener process, where $\boldsymbol{\mathcal{H}}$ is a separable Hilbert space. Finally, $\mathbf{g}(\cdot, \mathbf{u})$ be an appropriate operator to be specified later. The study of the stochastic incompressible Stokes system is e.g. motivated from modeling microfluids, where inertial effects are generally negligible, and microscopic fluctuations are relevant contributions to fluid flow dynamics; cf. [124, 44].

Strong solutions $\mathbf{u} \in L^2\big(\Omega; C([0, T]; \mathbb{H})\big) \cap L^2\big(\Omega_T; \mathbb{V}\big)$, where $\Omega_T := \Omega \times (0, T)$, of (4.1.1) for proper operators $\mathbf{g}$ are usually obtained by a Galerkin method which employs *divergence-free* approximates from finite-dimensional spaces $\mathbb{H}_n \subset \mathbb{H}$ $(n \geq 1)$ to remove the pressure from the problem. This strategy is different from a numerical setting, where the choice of the finite dimensional ansatz space for the pressure, as well as regularity properties of the pressure from (4.1.1) crucially determine both, stability and convergence behavior of the resulting scheme, see [53].

To properly handle the incompressibility constraint is a non-trivial issue, and is usually accomplished in a variational rather than a pointwise sense; as it is well-known for the corresponding deterministic problem, discretisation strategies based on implicit methods cause a significant computational effort due to the coupled computation of both, velocity and pressure iterates. Moreover, choices of stable finite element pairings are restricted by the LBB-constraint. As a consequence, splitting algorithms turn out to be a very promising alternative to reduce the complexity of actual computations by successively updating velocity and pressure iterates; we refer to [58] for a recent survey on this topic. It is evident that such a strategy is desirable to solve the stochastic partial differential equation (4.1.1), where a significant number of trajectories has to be computed to obtain statistically relevant results for quantities of interest. The goal of this paper is to show that the interplay of time-splitting strategies and the 'stochastic nature' of problem (4.1.1) is subtle, leading to a poor convergence behavior of known time-splitting schemes which perform well in the deterministic case. Computational experiments detail this assertion, which roots in the non-regular pressure process in (4.1.1). In a second step, an optimally convergent stochastic time-splitting scheme is constructed that distinguishes between approximations of the (non-regular) stochastic pressure, and the (more regular) deterministic pressure.

To illustrate the problematic issue to construct a proper time-splitting scheme of a stochastic equation, we start with Chorin's projection method [30, 32, 125], which is one of the first splitting schemes to solve the deterministic incompressible (Navier-)Stokes equation. Let $\mathbf{f}^m := \mathbf{f}(t_m, \cdot) \in L^2(\Omega, \mathbb{L}^2)$, suppose that $\mathbf{u}_0 \in L^2(\Omega, \mathbb{V})$ is given, and consider i.i.d. stochastic increments $\Delta \mathbf{W}_m := \mathbf{W}(t_m) - \mathbf{W}(t_{m-1})$, where $k = t_m - t_{m-1} > 0$ denotes the mesh-size of the equi-distant grid $I_k := \{t_m\}_{m=0}^M$ covering $[0, T]$.

Algorithm 4.1. 1. Let $1 \leq m \leq M$. For given $\mathbf{u}^{m-1} \in L^2(\Omega, \mathbb{V})$ and $\widetilde{\mathbf{u}}^{m-1} \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$, find $\widetilde{\mathbf{u}}^m \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$ such that

$$(4.1.2) \qquad \big(\widetilde{\mathbf{u}}^m - \mathbf{u}^{m-1}\big) - k\nu\, \Delta \widetilde{\mathbf{u}}^m = k\, \mathbf{f}^m + \mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1}) \Delta \mathbf{W}_m \quad \text{in } D \times \Omega\,.$$

2. Compute $\mathbf{u}^m \in L^2\big(\Omega, \mathbb{H}\big)$, and $\pi^m \in L^2\big(\Omega, W^{1,2}(D)/\mathbb{R}\big)$,

$$(4.1.3) \qquad \mathbf{u}^m - \widetilde{\mathbf{u}}^m + k\, \nabla \pi^m = 0\,, \qquad \operatorname{div} \mathbf{u}^m = 0 \qquad \text{in } D \times \Omega\,,$$

$$(4.1.4) \qquad \langle \mathbf{u}^m, \mathbf{n} \rangle = 0 \qquad \text{on } \partial D \times \Omega\,.$$

We start a discussion of the scheme which ignores the stochastic term for a moment: the latter step can be reformulated as a problem for the pressure function only,

$$(4.1.5) \qquad -\Delta \pi^m = -\frac{1}{k} \operatorname{div} \widetilde{\mathbf{u}}^m \quad \text{on } D \times \Omega\,, \qquad \partial_{\mathbf{n}} \pi^m = 0 \quad \text{on } \partial D \times \Omega\,.$$

Hence, each step consists of (4.1.2), (4.1.5), and the algebraic update (4.1.3) to obtain $\mathbf{u}^m \in \mathbb{H}$.

In order to understand error effects inherent to discretisation in time, and operator splitting in Chorin's scheme, we shift the index in $(4.1.3)_1$ back, and add the resulting equation to (4.1.2); together with (4.1.5), we then arrive at

$$\begin{aligned}
&(4.1.6)\big(\widetilde{\mathbf{u}}^m - \widetilde{\mathbf{u}}^{m-1}\big) - k\nu\, \Delta \widetilde{\mathbf{u}}^m + k\, \nabla \pi^{m-1} &&= k\, \mathbf{f}^m + \mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1}) \Delta \mathbf{W}_m && \text{in } D \times \Omega\,,\\
&(4.1.7) \qquad \operatorname{div} \widetilde{\mathbf{u}}^m - k\, \Delta \pi^m &&= 0 && \text{in } D \times \Omega\,,\\
&(4.1.8) \qquad \partial_{\mathbf{n}} \pi^m &&= 0 && \text{on } \partial D \times \Omega\,,
\end{aligned}$$

and $\widetilde{\mathbf{u}}^0 \equiv \mathbf{u}_0$ in $D \times \Omega$. We make the following observations: (i) iterates $\big\{\widetilde{\mathbf{u}}^m\big\}_{m=1}^M$ of Algorithm 4.1 are not divergence-free any more, but satisfy the 'quasi-compressibility equation' (4.1.7), with a penalisation parameter equal to k, (ii) iterates of the pressure satisfy a homogeneous Neumann boundary condition, which is in contrast to pressure $\pi : D_T \to \mathbb{R}$ from (4.1.1), and (iii) the pressure iterate in (4.1.6) is used in an explicit fashion, which rules out an immediate discrete energy law, where test functions $\mathbf{u}^m$ and π^m are used.

For the deterministic case, by assuming $D \subset \mathbb{R}^d$ to be a convex polyhedral domain, $\mathbf{u}_0 \in \mathbb{H} \cap \mathbb{W}^{2,2}(D)$, and $\mathbf{f} \in W^{2,\infty}\big(0, T; \mathbb{L}^2(D)\big)$, the following optimal estimates are proved in [112, Theorem 6.1],

$$(4.1.9) \qquad \max_{1 \leq m \leq M} \Big\{ \sqrt{\tau^m} \|\mathbf{u}(t_m, \cdot) - \widetilde{\mathbf{u}}^m\|_{\mathbb{L}^2} + \sqrt{k} \|\mathbf{u}(t_m, \cdot) - \widetilde{\mathbf{u}}^m\|_{\mathbb{W}^{1,2}} \Big\} \leq Ck\,,$$

where $\tau^m := \min\{1, t_m\}$. Its proof consists of three steps: first, optimal error estimates for the implicit Euler discretisation using solenoidal velocity fields are derived, where its derivation benefits from valid regularity properties of solutions $\mathbf{u} \in C\big([0,T]; \mathbb{V} \cap \mathbb{W}^{2,2}\big) \cap W^{2,2}\big(0,T; \mathbb{V}'\big)$; then, a modified version of (4.1.6)–(4.1.8) is studied with respect to both, convergence and stability properties, where the pressure iterate π^{m-1} in (4.1.6) is shifted to π^m; a key property here is the existing bound $\pi \in L^\infty\big(0,T; W^{1,2}/\mathbb{R}\big)$ for the deterministic evolutionary incompressible Stokes problem. We remark that this pressure-stabilisation method (with parameter $\varepsilon = k$) is of its own interest, since it allows for more choices of finite element pairings [15, 74], which are usually restricted by the discrete LBB condition. Finally, the third step accounts for the explicit treatment of the pressure in (4.1.6), which strongly benefits from the upper bound of $\int_0^T \tau(s)\|\nabla \pi_t(s)\|^2 \, \mathrm{d}s$ for the time derivative of the pressure from (4.1.1) in terms of the data $\mathbf{u}_0$, $\mathbf{f}$, and D_T, where $\tau(s) = \min\{1, s\}$.

The goal of the present work is to study convergence properties of $\mathbb{W}_0^{1,2}(D)$-valued iterates $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M$ from Algorithm 4.1 to approximate solutions of (4.1.1). The main difficulties which enter in the stochastic setting are due to *restricted regularity properties* (in time) of solutions $(\mathbf{u}, \pi)$ to (4.1.1), which are due to the driving stochastic term: for instance, the pressure which is constructed by Helmholtz decomposition after $\mathbf{u}$ is found, need not even be absolutely continuous with respect to time [92], see (4.2.6), but its regularity properties are crucial for the convergence analysis of our splitting method as detailed above. Hence, there is the question whether splitting effects inherent to Algorithm 4.1 will deteriorate convergence rates of computed iterates $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M$ — if compared to divergence-free velocity iterates $\{\mathbf{w}^m\}_{m=1}^M$, approximating $\{\mathbf{u}(t_m, \cdot)\}_{m=1}^M$, solving the coupled Euler-Maruyama time discretisation of (4.1.1),

$$
\begin{aligned}
&(4.1.10)\quad \big(\mathbf{w}^m - \mathbf{w}^{m-1}\big) - k\nu\, \Delta \mathbf{w}^m + k\nabla \sigma^m &&= k\mathbf{f}^m + \mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta \mathbf{W}_m \qquad &&\text{in } D \times \Omega\,,\\
&(4.1.11)\quad \operatorname{div} \mathbf{w}^m &&= 0 &&\text{in } D \times \Omega\,,
\end{aligned}
$$

where $\mathbf{w}^0 \equiv \mathbf{u}_0$ in $D \times \Omega$. Note that the pressure $\sigma^m : D \to \mathbb{R}$, which approximates $p(t_m, \cdot)$, will be eliminated from the convergence analysis where solenoidal test functions are used. As a consequence, the following rates of strong convergence of Euler iterates $\{\mathbf{w}^m\}_{m=1}^M$ are proved in [62],

(4.1.12)

$$
\max_{1 \le m \le M} \Big(\mathbb{E}\Big[\|\mathbf{u}(t_m, \cdot) - \mathbf{w}^m\|_{\mathbb{L}^2}^2\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[k \sum_{1 \le m \le M} \|\nabla\big(\mathbf{u}(t_m, \cdot) - \mathbf{w}^m\big)\|_{\mathbb{L}^2}^2\Big]\Big)^{1/2} \le C_T \sqrt{k}\,.
$$

In fact, [62, Theorem 3.1] gives the rate of convergence for a finite dimensional Wiener process. However, the proof can be modified in such a way, that the same result holds for the integral of a Hilbert-Schmidt operator valued process with respect to a cylindrical Wiener process. Since this will be the case (see Assumption (4.2.2) below), Theorem 3.1 of [62] is applicable.

The first main result in this paper is Theorem 4.3.1, which shows property (4.1.12) for iterates $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M$ from Algorithm 4.1 in the case of solenoidal noise. A discretisation in space using equal-order finite elements is studied in Section 4.4, and overall error estimates for related finite element iterates $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M$ are given in Theorem 4.4.1. Then, computational studies are provided in Section 4.5 which compare (anisotropic) convergence behavior of iterates from Algorithm 4.1 and (4.1.10)–(4.1.11) for different noise, and highlight that solenoidal noise is imperative to assure optimal convergence behavior of the splitting Algorithm 4.1, which in the case of general noise deteriorates to a poor convergence behavior. Those computational studies motivate the new time-splitting scheme (Algorithm 4.3) in Section 4.5.3, which distinguishes between approximate deterministic and stochastic pressure iterates. As a consequence, optimal rates of convergence for general noise is shown both, theoretically (see Theorem 4.5.1), and computationally.

The remainder of this chapter is organised as follows. Necessary background for the stochastic partial differential equation (4.1.1), and useful stability bounds for Euler iterates $\{\mathbf{w}^m\}_{m=1}^M$

solving (4.1.10)–(4.1.11) are provided in Section 2. In Section 4.3, we estimate the additional different perturbation effects due to the quasi-compressibility constraint (4.1.7), and the splitting character of Algorithm 4.1 due to the explicit treatment of the pressure in (4.1.6), which then leads to Theorem 4.3.1. In Section 4.4, a finite element discretisation of Algorithm 4.1 is proposed, where the study of the coupled error effects due to time discretisation, time splitting, and spatial discretisation leads to Theorem 4.4.1. Computational evidence to highlight failure of Chorin's method in the case of general noise is reported in Section 4.5, as well as the modified Algorithm 4.3 that performs optimally for general noise.

4.2 Preliminaries

4.2.1 The Problem

Let $\mathfrak{P} := \big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ be a stochastic basis where $\mathbb{F} = \{\mathcal{F}_t;\, t \geq 0\}$. For the definition of stochastic integral, we refer to Chapter 2. Let $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{H}}$, be two Hilbert spaces as in Chapter 2, and let the spaces $\mathbb{V}$ and $\mathbb{H}$ be defined as in Section 3.2.1.

Recall the Stokes operator $\mathbf{A} \equiv -\mathbf{P}_{\mathbb{H}}\Delta$, with domain $D(\mathbf{A}) = \mathbb{V} \cap \mathbb{W}^{2,2}(D)$. Here, $\mathbf{P}_{\mathbb{H}} : \mathbb{L}^2 \to \mathbb{H}$ denotes the (Leray) projection operator. For Lipschitz domains $D \subset \mathbb{R}^d$ and on $\mathbb{V} \cap \mathbb{W}^{2,2}(D)$, the operator norm $\|\cdot\|_{D(\mathbf{A})}$ is equivalent to the $\mathbb{W}^{2,2}(D)$-norm. Throughout the paper, let

$$\mathbf{u}_0 \in L^2\big(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V}\big), \qquad \text{and} \qquad \mathbf{f} \in L^2\big(\Omega \times (0,T); \mathbb{L}^2(D)\big)\,. \tag{4.2.1}$$

Suppose that $\mathbf{g} : \Omega \times (0,T) \times \mathbb{W}^{1,2}_0 \to \mathcal{I}_2\big(\boldsymbol{\mathcal{H}}; \mathbb{V}\big)$ is measurable Lipschitz, and sublinear; more precisely, there exists a constant $C_T > 0$ such that for $\boldsymbol{\mathcal{K}} = \mathbb{L}^2$ and $\boldsymbol{\mathcal{K}} = \mathbb{W}^{1,2}_0$,

$$\big\|\mathbf{g}(t,\mathbf{v}) - \mathbf{g}(t,\mathbf{w})\big\|_{\mathcal{I}_2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}})} \leq C_T \|\mathbf{v} - \mathbf{w}\|_{\boldsymbol{\mathcal{K}}} \qquad \forall\, t \in [0,T], \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbb{W}^{1,2}_0\,, \tag{4.2.2}$$

and for $\mathbf{v} \in \mathbb{W}^{1,2}_0$

$$\mathbf{g}(\cdot, \mathbf{v}) \in L^2\Big(\Omega; L^2\big(0,T; \mathcal{I}_2(\boldsymbol{\mathcal{H}}, \mathbb{W}^{1,2}_0)\big)\Big)\,. \tag{4.2.3}$$

Definition 4.2.1. *We call an adapted stochastic process a strong solution (in the stochastic sense) of* (4.1.1) *if* $\mathbf{u} \in L^2\big(\Omega; C([0,T]; \mathbb{H})\big) \cap M^2\big([0,T], \mathbb{F}; \mathbb{V}\big)$ *such that for all* $t \in [0,T]$ *and all* $\boldsymbol{\psi} \in \mathbb{V}$ *holds* $\mathbb{P}$*-a.s.,*

$$\big(\mathbf{u}(t), \boldsymbol{\psi}\big) + \nu \int_0^t \big(\nabla \mathbf{u}(s), \nabla \boldsymbol{\psi}\big)\,\mathrm{d}s = \big(\mathbf{u}_0, \boldsymbol{\psi}\big) + \int_0^t \big(\mathbf{f}(s), \boldsymbol{\psi}\big)\,\mathrm{d}s + \Big(\int_0^t \mathbf{g}(s,\mathbf{u})\mathrm{d}\mathbf{W}(s), \boldsymbol{\psi}\Big)\,.$$

The existence of a unique strong solution $\mathbf{u} \in L^2\big(\Omega; \mathscr{C}([0,T]; \mathbb{V})\big) \cap L^2\big(\Omega; L^2(0,T; D(\mathbf{A}))\big)$ which satisfies the energy equation

$$\begin{aligned} &\|\mathbf{u}(t)\|^2_{\mathbb{L}^2} + 2\nu \int_0^t \|\nabla \mathbf{u}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s = \|\mathbf{u}_0\|^2_{\mathbb{L}^2} + 2\int_0^t \big(\mathbf{f}(s), \mathbf{u}(s)\big)\,\mathrm{d}s \\ &\quad + 2\int_0^t \Big(\mathbf{g}\big(s,\mathbf{u}(s)\big)\mathrm{d}\mathbf{W}(s), \mathbf{u}(s)\Big) + \int_0^t \|\mathbf{g}\big(s,\mathbf{u}(s)\big)\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{H}};\mathbb{L}^2)}\,\mathrm{d}s \qquad \forall\, t \in [0,T] \end{aligned} \tag{4.2.4}$$

is well-known; see for instance [37, Theorem 6.19]. Moreover, standard arguments yield for $\mathbf{u}_0 \in L^2\big(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V}\big)$ that $\mathbf{u} \in L^2\big(\Omega; \mathscr{C}([0,T]; \mathbb{V})\big) \cap L^2\big(\Omega; L^2(0,T; D(\mathbf{A}))\big)$, and

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0 \leq t \leq T} \|\mathbf{u}(t)\|^2_{\mathbb{W}^{1,2}}\Big] + \nu \mathbb{E}\Big[\int_0^T \|\mathbf{A}\mathbf{u}(s)\|^2_{\mathbb{L}^2}\,\mathrm{d}s\Big] \leq C_T \Big\{1 + \mathbb{E}\big[\|\mathbf{u}_0\|^2_{\mathbb{W}^{1,2}}\big] + \\ &\quad + \mathbb{E}\Big[\int_0^T \|\mathbf{f}(s)\|^2_{\mathbb{W}^{-1,2}}\,\mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^T \|\mathbf{g}(s,\mathbf{0})\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{H}},\mathbb{L}^2)}\,\mathrm{d}s\Big]\Big\}\,, \end{aligned} \tag{4.2.5}$$

see e.g. [27, Thm. 4.4 & Sec. 5]. In contrast, the limited available analytical results about the pressure in (4.1.1) indicate very restricted smoothness: For $\mathbf{g} : \Omega \times [0,T] \times \mathbf{L}^2(D) \to \mathcal{L}_2(\boldsymbol{\mathcal{H}}, \mathbf{L}^2(D))$, there exists a unique (distributional) pressure (see [92, Theorem 4.1 and Remark 4.3])

$$\pi \in L^1\Big(\Omega, \mathbb{F}, \mathbb{P}; W^{-1,\infty}\big(0,T; W^{1,2}(D)/\mathbb{R}\big)\Big)\,, \tag{4.2.6}$$

such that $\mathbb{P}$-a.s.

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla\pi = \mathbf{f} + \mathbf{g}(\cdot,\mathbf{u})\dot{\mathbf{W}} \qquad \text{in } \big(\mathcal{D}'(D_T)\big)^d\,, \tag{4.2.7}$$

$$\int_D \pi \,\mathrm{d}\mathbf{x} = 0 \qquad \text{in } \mathcal{D}'(0,T)\,. \tag{4.2.8}$$

This result evidences a deregularising effect upon the pressure in (4.1.1) which is exerted by a general noise. This feedback effect of general noise onto the (lack of) regularity of the pressure may be avoided by analytical constructions using Leray projection, but causes severe deteriorations with respect to accuracy of well-known numerical schemes where accurate pressures are needed.

As will be shown in Lemma 4.2.1 below, pressure iterates of the (coupled) Euler-Maruyama scheme (4.1.10)–(4.1.11) are smoother for noise that is solenoidal, which is why we assume

$$\mathbf{g} : \Omega \times [0,T] \times \mathbb{W}_0^{1,2}(D) \to \mathcal{I}_2(\boldsymbol{\mathcal{H}}, \mathbb{V}) \tag{4.2.9}$$

in Sections 4.3 and 4.4. Conversely, computational experiments in Section 4.5.1 show that Chorin's projection method only performs optimally in the case of solenoidal noise. Since in our case $\mathbf{g} : \Omega \times (0,T) \times \mathbb{W}_0^{1,2} \to \mathcal{I}_2\big(\boldsymbol{\mathcal{H}}; \mathbb{V}\big)$ the following Lemma 4.2.1 motivates

$$\pi \in L^1\Big(\Omega, \mathbb{F}, \mathbb{P}; L^2(0,T; W^{1,2}(D)/\mathbb{R})\Big)\,, \tag{4.2.10}$$

which provides enough regularity of the pressure such that the splitting scheme performs optimally. However, we are not aware of a rigorous analytical motivation of this fact.

Remark 4.2.1. *A velocity field* $\mathbf{u}$ *that solves the stochastic incompressible (Navier-) Stokes equations is usually constructed by an ("inner approximation") Galerkin method that employs solenoidal test functions, and thus eliminates the pressure* p *from the problem in a first step; a pressure* p *is then later obtained by de Rham's theorem; see e.g. [11, 46, 27, 92]. A different strategy is to obtain solutions by perturbing the incompressibility constraint ('quasi-compressibility method') to avoid the saddle-point character of the problem, for example* ($\varepsilon > 0$)*:*

$$\begin{array}{llll}
\text{(i)} & \operatorname{div}\mathbf{u}^\varepsilon + \varepsilon\pi^\varepsilon = 0 \quad \textit{in } D_T\,, & & \\
\text{(ii)} & \operatorname{div}\mathbf{u}^\varepsilon - \varepsilon\Delta\pi^\varepsilon = 0 \quad \textit{in } D_T\,, & \partial_{\mathbf{n}}\pi^\varepsilon = 0 \quad \textit{on } \partial D_T\,, & \\
\text{(iii)} & \operatorname{div}\mathbf{u}^\varepsilon + \varepsilon\pi^\varepsilon_t = 0 \quad \textit{in } D_T\,, & \pi^\varepsilon(0) = \pi(0) \quad \textit{on } D\,, & \\
\text{(iv)} & \operatorname{div}\mathbf{u}^\varepsilon - \varepsilon\Delta\pi^\varepsilon_t = 0 \quad \textit{in } D_T\,, & \partial_{\mathbf{n}}\pi^\varepsilon = 0 \quad \textit{on } \partial D_T\,, & \pi^\varepsilon(0) = \pi(0) \quad \textit{on } D\,.
\end{array}$$

The penalty method (i) is used in [25], and the artificial compressibility method (iii) in [97] to construct solutions of the stochastic incompressible Navier-Stokes equations. The pressure stabilisation ansatz (ii) is related to Algorithm 4.1 where $\varepsilon = k$ *is chosen in* (4.1.5)*; the pressure correction method (iv) is used for numerical schemes as well; cf. [112] for further details.*

4.2.2 Euler scheme

The Euler scheme (4.1.10)–(4.1.11) is strongly consistent. Suppose that (4.2.1)–(4.2.3), and (4.2.9) are valid throughout the section. For every $m \geq 1$, there exists a solution $\mathbf{w}^m \in L^2\big(\Omega; \mathbb{V}\big)$ such that $\mathbf{w}^0 = \mathbf{u}_0$, and

$$\begin{aligned}&(\mathbf{w}^m - \mathbf{w}^{m-1}, \boldsymbol{\varphi}) + k\nu(\nabla \mathbf{w}^m, \nabla \boldsymbol{\varphi}) \\ &\qquad = k(\mathbf{f}^m, \boldsymbol{\varphi}) + \Big(\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta \mathbf{W}_m, \boldsymbol{\varphi}\Big) \qquad \forall \boldsymbol{\varphi} \in \mathbb{V}\,.\end{aligned} \tag{4.2.11}$$

Moreover, solutions satisfy the error estimate given in (4.1.12) and shown in [62].

Some bounds for solutions of (4.1.10)–(4.1.11) in strong norms will be useful later, where the first one mimics (4.2.4) on a discrete level.

Lemma 4.2.1. *Let* $\{\mathbf{w}^m\}_{m=1}^M \subset L^2\big(\Omega; \mathbb{V}\big)$ *be a solution of* (4.2.11), *and let* (4.2.1),(4.2.2),(4.2.3), (4.2.9) *be valid. Then*

(i)
$$\max_{1\leq m\leq M} \mathbb{E}\Big[\|\mathbf{w}^m\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\Big[\sum_{m=1}^M \|\mathbf{w}^m - \mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2\Big] + \nu\mathbb{E}\Big[k\sum_{m=1}^M \|\nabla \mathbf{w}^m\|_{\mathbb{L}^2}^2\Big] \leq C_T\Big\{\mathbb{E}\Big[\|\mathbf{u}_0\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\Big[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbb{L}^2}^2\Big]\Big\}\,,$$

(ii)
$$\max_{1\leq m\leq M} \mathbb{E}\Big[\|\nabla\mathbf{w}^m\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\Big[\sum_{m=1}^M \|\nabla[\mathbf{w}^m - \mathbf{w}^{m-1}]\|_{\mathbb{L}^2}^2\Big] + \nu\mathbb{E}\Big[k\sum_{m=1}^M \|\mathbf{A} \mathbf{w}^m\|_{\mathbb{L}^2}^2\Big] \leq C_T\Big\{\mathbb{E}\Big[\|\nabla\mathbf{u}_0\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\Big[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbb{L}^2}^2\Big]\Big\}\,,$$

(iii)
$$\mathbb{E}\Big[k\sum_{m=1}^M \|\nabla\sigma^m\|_{\mathbb{L}^2}^2\Big] \leq C_T\Big\{\mathbb{E}\Big[\|\nabla\mathbf{u}_0\|_{\mathbb{L}^2}^2\Big] + \mathbb{E}\Big[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbb{L}^2}^2\Big]\Big\}\,,$$

where $C_T \equiv C\big(\mathbf{u}_0, \mathbf{g}, \mathbf{f}, D, T\big) > 0$ *is a generic constant that does not depend on* k.

Proof. Assertion (i). Choose $\boldsymbol{\varphi} = \mathbf{w}^m$, and use the algebraic identity $2\big\langle \mathbf{a}-\mathbf{b}, \mathbf{a}\big\rangle = |\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2$ to obtain

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbf{w}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2 + \|\mathbf{w}^m - \mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2\Big) + \nu k\|\nabla \mathbf{w}^m\|_{\mathbb{L}^2}^2 = k\big(\mathbf{f}^m, \mathbf{w}^m\big) \\ &\quad + \Big(\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta \mathbf{W}_m, \mathbf{w}^m - \mathbf{w}^m\Big) + \Big(\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta \mathbf{W}_m, \mathbf{w}^{m-1}\Big)\,.\end{aligned} \tag{4.2.12}$$

Taking expectations puts the last term in (4.2.12) to zero. For the remaining stochastic term, we use Itô isometry, and (4.2.2), (4.2.3) to conclude that

$$\begin{aligned}&\mathbb{E}\Big[\Big|\Big(\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta \mathbf{W}_m, \mathbf{w}^m - \mathbf{w}^{m-1}\Big)\Big|\Big] \\ &\quad = k\mathbb{E}\Big[\|\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\|_{\mathcal{L}_2(\boldsymbol{\mathcal{H}}, \mathbb{L}^2)}^2\Big] + \frac{1}{4}\mathbb{E}\Big[\|\mathbf{w}^m - \mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2\Big] \\ &\quad \leq C_T k\Big(1 + \mathbb{E}\big[\|\mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2\big]\Big) + \frac{1}{4}\mathbb{E}\Big[\|\mathbf{w}^m - \mathbf{w}^{m-1}\|_{\mathbb{L}^2}^2\Big]\,.\end{aligned}$$

We now use the discrete version of Gronwall's lemma in (4.2.12) to obtain assertion (i).

Assertion (ii). Choose $\boldsymbol{\varphi} = \mathbf{A}\mathbf{w}^m$, and proceed as before. We use (4.2.9) and integrate by parts in the stochastic term to find

$$(4.2.13)\ \Big(\nabla\big[\mathbf{g}(t_m,\mathbf{w}^{m-1})\Delta\mathbf{W}_m\big],\nabla\mathbf{w}^{m-1}\Big) \quad + \quad \Big(\nabla\big[\mathbf{g}(t_m,\mathbf{w}^{m-1})\Delta\mathbf{W}_m\big],\nabla\big[\mathbf{w}^m-\mathbf{w}^{m-1}\big]\Big)\,.$$

After taking expectations, only the second term is non-zero; by Itô's isometry, (4.2.2), (4.2.3) an upper bound for it is

$$\begin{aligned}&\frac{1}{4}\mathbb{E}\Big[\|\nabla[\mathbf{w}^m-\mathbf{w}^{m-1}]\|^2_{\mathbb{L}^2}\Big] + Ck\mathbb{E}\Big[\|\mathbf{g}(t_{m-1},\mathbf{w}^{m-1})\|^2_{\mathcal{I}_2(\boldsymbol{\mathcal{H}},\mathbb{W}^{1,2})}\Big]\\ &\qquad\le \frac{1}{4}\mathbb{E}\Big[\|\nabla[\mathbf{w}^m-\mathbf{w}^{m-1}]\|^2_{\mathbb{L}^2}\Big] + C_Tk\Big(1+\mathbb{E}\big[\|\mathbf{w}^{m-1}\|^2_{\mathbb{W}^{1,2}}\big]\Big)\,.\end{aligned}$$

Putting things together, and using discrete Gronwall's inequality then leads to assertion (ii).

Assertion (iii). For every $m\ge 1$, consider (4.1.10) as identity on $L^2(\Omega,\mathbb{L}^2)$, which is justified from the previous step. Term-wise multiplication with $\nabla\sigma^m$ and integration in space then leads to

$$\frac{k}{2}\|\nabla\sigma^m\|^2_{\mathbb{L}^2} \le Ck\Big(\|\Delta\mathbf{w}^m\|^2_{\mathbb{L}^2} + \|\mathbf{f}^m\|^2_{\mathbb{L}^2}\Big)\,,$$

where we use (4.2.9). Assertion (ii) then validates the assertion. □

4.3 Perturbation effects: Quasi-Compressibility and Operator-Splitting

Solutions $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M \subset \mathbb{W}_0^{1,2}(D)$ of Algorithm 4.1 satisfy (4.1.6)–(4.1.8), which illustrates the different error effects due to time discretisation, quasi-incompressibility, and splitting character in the scheme. The main result of this section is the following

Theorem 4.3.1. *Let $T>0$, $D\subset\mathbb{R}^d$, $d=2,3$ be a bounded convex polyhedral domain, and* (4.2.1)–(4.2.3), (4.2.9) *be valid. Denote by $\mathbf{u}\in L^2\big(\Omega;C([0,T];\mathbb{V})\big)\cap L^2\big(\Omega;L^2(0,T;\mathbb{W}^{2,2})\big)$ the strong solution of* (4.1.1)*, and $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M\subset L^2\big(\Omega,\mathbb{W}_0^{1,2}(D)\big)$ solves Algorithm 4.1. There exists a constant $C\equiv C\big(\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{W}^{1,2}}],D_T\big)>0$, such that*

$$(4.3.1)\qquad \max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[k\sum_{m=1}^M\|\nabla\big(\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\big)\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \le C\sqrt{k}\,.$$

The proof is split into several steps: first, we study solutions

$$\big\{(\mathbf{v}^m,\rho^m)\big\}_{m=1}^M \subset L^2\big(\Omega;,\mathbb{W}_0^{1,2}(D)\big)\times L^2\big(\Omega,W^{1,2}(D)/\mathbb{R}\big)$$

of the auxiliary problem (note that, in contrast to (4.1.6) where the approximation of the pressure is given from the previous time-step, it is here computed by an implicit procedure)

$$(4.3.2)\qquad\begin{aligned}&\big(\mathbf{v}^m-\mathbf{v}^{m-1},\boldsymbol{\phi}\big)+k\nu\big(\nabla\mathbf{v}^m,\nabla\boldsymbol{\phi}\big)-k\big(\rho^m,\nabla\boldsymbol{\phi}\big)\\ &\qquad\qquad= \big(k\mathbf{f}^m,\boldsymbol{\phi}\big)+\big(\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})\Delta\mathbf{W}_m,\boldsymbol{\phi}\big) &&\forall\boldsymbol{\phi}\in\mathbb{W}_0^{1,2}(D)\quad \mathbb{P}-a.s.\,,\\ &\big(\operatorname{div}\mathbf{v}^m,\phi\big)+k\big(\nabla\rho^m,\nabla\phi\big)=0 &&\forall\,\phi\in W^{1,2}(D)\quad\mathbb{P}-a.s.\,,\\ &\qquad\qquad\partial_{\mathbf{n}}\rho^m=0 &&\text{on }\partial D\times\Omega\,,\end{aligned}$$

and $\mathbf{v}^0\equiv\mathbf{u}_0$ in $D\times\Omega$, both, with respect to convergence towards the solution of (4.1.10), and stability behavior. Then, we study convergence behavior for solutions of (4.3.2) to those of (4.1.6)–(4.1.8).

Proof. Step 1. The pressure stabilisation problem (4.3.2)*: rates of convergence.* We show the following convergence estimate for solutions $\{\mathbf{w}^m\}_{m=1}^M \subset L^2(\Omega, \mathbb{V})$ of (4.2.11), and $\{\mathbf{v}^m\}_{m=1}^M \subset L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$ of (4.3.2),

$$\max_{1\leq m\leq M}\Big(\mathbb{E}\Big[\|\mathbf{w}^m-\mathbf{v}^m\|_{\mathbb{L}^2}^2\Big]\Big)^{1/2}+\Big(\mathbb{E}\Big[k\sum_{m=1}^M\|\nabla\big(\mathbf{w}^m-\mathbf{v}^m\big)\|_{\mathbb{L}^2}^2\Big]\Big)^{1/2}$$
$$+\Big(\mathbb{E}\Big[k^2\sum_{m=1}^M\|\nabla\big(\sigma^m-\rho^m\big)\|_{\mathbb{L}^2}^2\Big]\Big)^{1/2}\leq C\sqrt{k}\,. \tag{4.3.3}$$

Let $\mathbf{e}^m := \mathbf{w}^m - \mathbf{v}^m \in L^2\big(\Omega, \mathbb{W}^{1,2}(D)\big)$, and $\chi^m := \sigma^m - \rho^m \in L^2\big(\Omega, W^{1,2}(D)/\mathbb{R}\big)$. Taking the difference of (4.1.10) and (4.3.2) then leads to

$$\begin{aligned}&\big(\mathbf{e}^m-\mathbf{e}^{m-1},\boldsymbol{\phi}\big)+k\nu\big(\nabla\mathbf{e}^m,\nabla\boldsymbol{\phi}\big)-k\big(\chi^m,\nabla\boldsymbol{\phi}\big)\\&\quad=\Big(\big(\mathbf{g}(t_{m-1},\mathbf{w}^{m-1})-\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})\big)\Delta\mathbf{W}_m,\boldsymbol{\phi}\Big)\qquad\forall\boldsymbol{\phi}\in\mathbb{W}_0^{1,2}(D)\quad\mathbb{P}-a.s.\,,\\&\big(\operatorname{div}\mathbf{v}^m,\phi\big)+k\big(\nabla\chi^m,\nabla\phi\big)=k\big(\nabla\sigma^m,\nabla\phi\big)\qquad\forall\,\phi\in W^{1,2}(D)\quad\mathbb{P}-a.s.\,,\\&\qquad\partial_{\mathbf{n}}\chi^m=\partial_{\mathbf{n}}\sigma^m\qquad\text{on }\partial D\,,\end{aligned}\tag{4.3.4}$$

and $\mathbf{e}^0 \equiv \mathbf{0}$ in $D \times \Omega$. By setting $\boldsymbol{\phi} := \mathbf{e}^m$, $\phi = \chi^m$ in (4.3.4), using the Lipschitz continuity of $\mathbf{g}$, adding both identities, and using Young's inequality, then leads to

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbf{e}^m\|_{\mathbb{L}^2}^2-\|\mathbf{e}^{m-1}\|_{\mathbb{L}^2}^2+\|\mathbf{e}^m-\mathbf{e}^{m-1}\|_{\mathbb{L}^2}^2\Big)+k\nu\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2+k^2\|\nabla\chi^m\|_{\mathbb{L}^2}^2\\&\quad\leq\Big([\mathbf{g}(t_{m-1},\mathbf{w}^{m-1})-\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})]\Delta\mathbf{W}_m,\mathbf{e}^{m-1}\Big)+\frac{1}{4}\|\mathbf{e}^m-\mathbf{e}^{m-1}\|_{\mathbb{L}^2}^2\\&\qquad+\|[\mathbf{g}(t_{m-1},\mathbf{w}^{m-1})-\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})]\Delta\mathbf{W}_m\|_{\mathbb{L}^2}^2+\frac{1}{4}k^2\|\nabla\chi^m\|_{\mathbb{L}^2}^2+k^2\|\nabla q^m\|_{\mathbb{L}^2}^2\,.\end{aligned}$$

The leading term on the right-hand side vanishes when we take its expectation. By Itô's isometry, and (4.2.2), there holds for the remaining stochastic integral term

$$\mathbb{E}\Big[\|[\mathbf{g}(t_{m-1},\mathbf{w}^{m-1})-\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})]\Delta\mathbf{W}_m\|_{\mathbb{L}^2}^2\Big]\leq Ck\Big(1+\mathbb{E}\big[\|\mathbf{e}^{m-1}\|_{\mathbb{L}^2}^2\big]\Big)\,.$$

We now take expectation term-wise, and sum over all steps $1 \leq m \leq m^* \leq M$; because of $\mathbb{E}\big[\|\mathbf{e}^0\|_{\mathbb{L}^2}^2\big] = 0$, Lemma 4.2.1, (iii), and the discrete version of Gronwall's inequality, after summation we arrive at

$$\frac{1}{2}\mathbb{E}\Big[\|\mathbf{e}^{m^*}\|_{\mathbb{L}^2}^2\Big]+\frac{1}{4}\mathbb{E}\Big[\sum_{m=1}^{m^*}\|\mathbf{e}^m-\mathbf{e}^{m-1}\|_{\mathbb{L}^2}^2\Big]+\nu\mathbb{E}\Big[k\sum_{m=1}^{m^*}\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2\Big]$$
$$+\frac{3}{4}\mathbb{E}\Big[k^2\sum_{m=1}^{m^*}\|\nabla\chi^m\|_{\mathbb{L}^2}^2\Big]\leq C_{t_{m^*}}\mathbb{E}\Big[k^2\sum_{m=1}^{m^*}\|\nabla q^m\|_{\mathbb{L}^2}^2\Big]\leq C_Tk\,. \tag{4.3.5}$$

Step 2. The pressure stabilisation problem (4.3.2)*: stability.* Proper bounds are needed for the pressure in (4.3.2) to validate optimal error estimates between solutions of (4.3.2) and (4.1.6)–(4.1.8) below. We show

$$\max_{1\leq m\leq M}\mathbb{E}\Big[\|\mathbf{v}^m\|_{\mathbb{W}^{1,2}}^2\Big]+\mathbb{E}\Big[k\sum_{m=1}^M\|\mathbf{v}^m\|_{\mathbb{W}^{2,2}}^2\Big]+\nu\mathbb{E}\Big[k\sum_{m=1}^M\|\nabla r^m\|_{\mathbb{L}^2}^2\Big]\leq C_T\,. \tag{4.3.6}$$

Hence, for solutions of problem (4.3.2) there hold the same estimates which are valid for solutions of (4.1.10)–(4.1.11) from Lemma 4.2.1.

Property $(4.3.6)_1$ follows from the term $(4.3.3)_2$, and Lemma 4.2.1, (ii), and property $(4.3.6)_3$ is a consequence of $(4.3.3)_3$, and Lemma 4.2.1, (iii). A formal derivation of $(4.3.6)_2$ uses $(4.3.2)_1$, which we multiply by $-\Delta \mathbf{v}^m$, and then integrate over D. After summing up over all $1 \leq m \leq M$, by taking expectations and absorbing terms we arrive at

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\|\nabla \mathbf{v}^M\|_{\mathbb{L}^2}^2\Big] + \frac{1}{2}\mathbb{E}\Big[\sum_{m=1}^{M} \|\nabla(\mathbf{v}^m - \mathbf{v}^{m-1})\|_{\mathbb{L}^2}^2\Big] + \frac{\nu}{4}\mathbb{E}\Big[k\sum_{m=1}^{M}\|\Delta \mathbf{v}^m\|_{\mathbb{L}^2}^2\Big] \\
&\leq \frac{1}{2}\mathbb{E}\big[\|\nabla \mathbf{v}^0\|_{\mathbb{L}^2}^2\big] + C\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla r^m\|_{\mathbb{L}^2}^2\Big] + C\mathbb{E}\Big[k\sum_{m=1}^{M}\|\mathbf{f}^m\|_{\mathbb{L}^2}^2\Big] \\
&\quad + k\mathbb{E}\Big[\sum_{m=1}^{M-1}\|\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})\|_{\mathcal{I}_2(\boldsymbol{\mathcal{H}},\mathbb{W}^{1,2})}^2\Big] + \frac{1}{4}\mathbb{E}\Big[\sum_{m=1}^{M}\|\nabla[\mathbf{v}^m - \mathbf{v}^{m-1}]\|_{\mathbb{L}^2}^2\Big],
\end{aligned} \tag{4.3.7}$$

where we use the fact that $\mathbb{E}\big[\sum_{m=1}^{M}(\nabla\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})\Delta\mathbf{W}_m, \nabla\mathbf{v}^{m-1})\big] = 0$, and Itô's isometry. By (4.2.2),(4.2.3), we have $\mathbb{E}\big[\|\mathbf{g}(t_{m-1},\mathbf{v}^{m-1})\|_{\mathcal{I}_2(\boldsymbol{\mathcal{H}},\mathbb{W}^{1,2})}\big] \leq C\big(1+\mathbb{E}[\|\mathbf{v}^{m-1}\|_{\mathbb{W}^{1,2}}]\big)$. The bounds $(4.3.6)_{1,3}$ then allow to conclude $(4.3.6)_2$ from (4.3.7).

Step 3. The splitting error: comparison of problems (4.3.2) *and* (4.1.6)–(4.1.8). We estimate the differences $\boldsymbol{\varepsilon}^m := \mathbf{v}^m - \widetilde{\mathbf{u}}^m \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$, and $\eta^m := \rho^m - \pi^m \in L^2\big(\Omega, W^{1,2}(D)/\mathbb{R}\big)$, which are determined by the following system of equations,

$$\begin{aligned}
\big(\boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}^{m-1}, \varphi\big) + k\nu\big(\nabla\boldsymbol{\varepsilon}^m, \nabla\boldsymbol{\phi}\big) + k\big(\nabla\eta^m, \boldsymbol{\phi}\big) &= \Big(\boldsymbol{\mathcal{E}}^m, \boldsymbol{\phi}\Big) \quad &&\forall\,\boldsymbol{\phi} \in \mathbb{W}_0^{1,2}(D) \quad \mathbb{P}-a.s., \\
\big(\operatorname{div}\boldsymbol{\varepsilon}^m, \phi\big) + k\big(\nabla\eta^m, \nabla\phi\big) &= 0 &&\forall\,\phi \in W^{1,2}(D) \quad \mathbb{P}-a.s., \\
\partial_{\mathbf{n}}\eta^m &= 0 &&\text{on } \partial D\,,
\end{aligned} \tag{4.3.8}$$

where $\boldsymbol{\varepsilon}_0 \equiv \mathbf{0}$, and

$$\boldsymbol{\mathcal{E}}^m := -k\nabla[\rho^m - \rho^{m-1}] + \Big(\mathbf{g}(t_{m-1},\mathbf{v}^{m-1}) - \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})\Big)\Delta\mathbf{W}_m\,. \tag{4.3.9}$$

Upon testing $(4.3.8)_1$ by $\boldsymbol{\varepsilon}^m$, and $(4.3.8)_2$ by η^m, adding both identities, using Young's inequality with $\delta_1 > 0$ and absorbing terms then yields

$$\begin{aligned}
&\frac{1}{2}\Big(\|\boldsymbol{\varepsilon}^m\|_{\mathbb{L}^2}^2 - \|\boldsymbol{\varepsilon}^{m-1}\|_{\mathbb{L}^2}^2 + \|\boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}^{m-1}\|_{\mathbb{L}^2}^2\Big) + k\nu\|\nabla\boldsymbol{\varepsilon}^m\|_{\mathbb{L}^2}^2 + k^2\big(\nabla\eta^m, \nabla\eta^{m-1}\big) \\
&\leq \Big([\mathbf{g}(t_{m-1},\mathbf{v}^{m-1}) - \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})]\Delta\mathbf{W}_m, \boldsymbol{\varepsilon}^{m-1}\Big) - k\Big(\nabla(\rho^m - \rho^{m-1}), \boldsymbol{\varepsilon}^m\Big) \\
&\quad + C_{\delta_1}\|[\mathbf{g}(t_{m-1},\mathbf{v}^{m-1}) - \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})]\Delta\mathbf{W}_m\|_{\mathbb{L}^2}^2 + \delta_1\|\boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}^{m-1}\|_{\mathbb{L}^2}^2\,.
\end{aligned} \tag{4.3.10}$$

Again, the expectation of the leading term on the right-hand side vanishes; Itô's isometry, and (4.2.2) yields to

$$\mathbb{E}\Big[\|[\mathbf{g}(t_{m-1},\mathbf{v}^{m-1}) - \mathbf{g}(t_{m-1},\widetilde{\mathbf{u}}^{m-1})]\Delta\mathbf{W}_m\|_{\mathbb{L}^2}^2\Big] \leq C_T k\Big(1 + \mathbb{E}[\|\boldsymbol{\varepsilon}^{m-1}\|_{\mathbb{L}^2}^2]\Big)\,.$$

There remains to deal with terms which contain pressures. We use $(4.3.8)_2$ and Young's inequality with $\delta_2 > 0$ to conclude that

$$\begin{aligned}
k^2\big(\nabla\eta^m, \nabla\eta^{m-1}\big) &= k^2\|\nabla\eta^m\|_{\mathbb{L}^2}^2 - k^2\Big(\nabla\eta^m, \nabla[\eta^m - \eta^{m-1}]\Big) \\
&= k^2\|\nabla\eta^m\|_{\mathbb{L}^2}^2 - k\big(\nabla\eta^m, \boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}^{m-1}\big) \\
&\geq \big(1 - \frac{1}{4\delta_2}\big)k^2\|\nabla\eta^m\|_{\mathbb{L}^2}^2 - \delta_2\|\boldsymbol{\varepsilon}^m - \boldsymbol{\varepsilon}^{m-1}\|_{\mathbb{L}^2}^2\,.
\end{aligned} \tag{4.3.11}$$

The remaining crucial term in (4.3.10) is bounded as follows,

$$k^2\Big(\nabla[\rho^m-\rho^{m-1}],\nabla\eta^m\Big)\le k^2\delta_3\|\nabla\eta^m\|^2_{\mathbb{L}^2}+\frac{k^2}{4\delta_3}\|\nabla[\rho^m-\rho^{m-1}]\|^2_{\mathbb{L}^2}\,.$$

where we used Young's inequality with $\delta_3>0$. To keep the corresponding terms in (4.3.12) nonnegative, we choose parameters $\delta_i>0$, $i=1,2,3$ such that

$$1-\frac{1}{4\delta_2}-\delta_3\ge 0\,,\qquad\text{and}\qquad \frac{1}{2}-\delta_1-\delta_2\ge 0\,,$$

Next, we sum over all $1\le m\le m^*\le M$ in (4.3.10), and take expectations. Then, by the discrete Gronwall inequality,

$$\begin{aligned}&\mathbb{E}\Big[\|\boldsymbol{\varepsilon}^{m^*}\|^2_{\mathbb{L}^2}\Big]+\Big(\frac{1}{2}-\delta_1-\delta_2\Big)\mathbb{E}\Big[\sum_{m=1}^{m^*}\|\boldsymbol{\varepsilon}^m-\boldsymbol{\varepsilon}^{m-1}\|^2_{\mathbb{L}^2}\Big]+\nu\mathbb{E}\Big[k\sum_{m=1}^{m^*}\|\nabla\boldsymbol{\varepsilon}^m\|^2_{\mathbb{L}^2}\Big]\\ &+k^2\Big(1-\frac{1}{4\delta_2}-\delta_3\Big)\mathbb{E}\Big[\sum_{m=1}^{m^*}\|\nabla\eta^m\|^2_{\mathbb{L}^2}\Big]\le C_T4\frac{k^2}{4\delta_3}\sum_{m=0}^{m^*}\|\nabla\rho^m\|^2_{\mathbb{L}^2}\le C_{t_T}k\,,\end{aligned}\tag{4.3.12}$$

where the last estimate uses $(4.3.6)_3$, and $r^0\equiv 0$ is a consequence of $(4.3.2)_2$.

By putting together results (4.1.12), (4.3.3), and (4.3.12) yields the error bound,

$$\begin{aligned}&\max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2}\\ &+\Big(\nu\mathbb{E}\Big[k\sum_{1\le m\le M}\|\nabla\big(\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\big)\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2}\le C\Big(\sqrt{k}+\sqrt{k}+\sqrt{k}\Big)\,,\end{aligned}\tag{4.3.13}$$

which proves Theorem 4.3.1. □

The following stability result for solutions of Algorithm 4.1 will be helpful in Section 4.4, where we consider an optimally convergent, practical finite element discretisation.

Lemma 4.3.1. *Let $T>0$, $D\subset\mathbb{R}^d$, $d=2,3$ be a convex polyhedral domain, and (4.2.1)–(4.2.3), (4.2.9) be valid. $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M\subset L^2\big(\Omega,\mathbb{W}^{1,2}_0(D)\big)$ be the solution of Algorithm 4.1. Then, all estimates given in Lemma 4.2.1 remain valid.*

Proof. We use (4.3.6) and (4.3.12) to validate bounds (i), (iii), and $(\text{ii})_{1,2}$ in Lemma 4.2.1 for $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M$. In order to (formally) verify $\mathbb{E}\big[k\sum_{m=1}^M\|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\big]\le C$, we multiply (4.1.6) by $-\Delta\widetilde{\mathbf{u}}^m$, integrate over D, and consider expectations. Similar arguments as above lead to

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\Big[\|\nabla\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}-\|\nabla\widetilde{\mathbf{u}}^{m-1}\|^2_{\mathbb{L}^2}+\frac{3}{4}\|\nabla[\widetilde{\mathbf{u}}^m-\widetilde{\mathbf{u}}^{m-1}]\|^2_{\mathbb{L}^2}\Big]+\frac{3k\nu}{4}\mathbb{E}\big[\|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\big]\\ &\le\mathbb{E}\big[k\|\nabla\pi^{m-1}\|^2_{\mathbb{L}^2}\big]+Ck\Big(1+\mathbb{E}\big[\|\widetilde{\mathbf{u}}^{m-1}\|^2_{\mathbb{W}^{1,2}}\big]\Big)+Ck\mathbb{E}\big[\|\mathbf{f}^m\|^2_{\mathbb{L}^2}\big]\,.\end{aligned}$$

We now sum up $1\le m\le M$, use the fact that $\nabla\pi^0=0$, and may use the available bound $\mathbb{E}\big[\sum_{m=1}^M\|\nabla\pi^m\|^2_{\mathbb{L}^2}\big]\le C$ to obtain

$$\mathbb{E}\Big[k\sum_{m=1}^M\|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\Big]\le C_T\,.$$

□

4.4 Finite element discretisation

Let $\mathcal{T}$ be a quasi-uniform triangulation of the polygonal or polyhedral bounded Lipschitz domain $D \subset \mathbb{R}^d$ into triangles or tetrahedra for $d = 2$ or $d = 3$, respectively. We define the lowest order finite element space

$$H_h = \left\{ \Phi \in \mathscr{C}(\overline{D}) : \Phi\big|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T} \right\},$$

where $\mathcal{P}_1(K)$ denotes the set of polynomials of degree less or equal than one if restricted to the element $K \in \mathcal{T}$. We introduce equal-order finite element function spaces

$$\mathbb{H}_h := \left[H_h\right]^d, \qquad \text{and} \qquad L_h := H_h \cap L^2_0(\Omega),$$

and $\mathbb{H}^0_h := \mathbb{H}_h \cap \mathbb{W}^{1,2}_0(D)$. We recall the $\mathbb{L}^2$-orthogonal projection $\mathbf{P}^0_h : \mathbb{L}^2 \to \mathbb{H}^0_h$, where

$$\big(\boldsymbol{\phi} - \mathbf{P}^0_h \boldsymbol{\phi}, \boldsymbol{\xi}\big) = 0 \qquad \forall \boldsymbol{\xi} \in \mathbb{H}^0_h,$$

for which holds

$$\|\boldsymbol{\phi} - \mathbf{P}^0_h \boldsymbol{\phi}\|_{\mathbb{L}^2} + h\|\nabla(\boldsymbol{\phi} - \mathbf{P}^0_h \boldsymbol{\phi})\|_{\mathbb{L}^2} \leq Ch^2 \|\boldsymbol{\phi}\|_{\mathbb{W}^{2,2}} \qquad \forall \boldsymbol{\phi} \in \mathbb{W}^{2,2}.$$

Accordingly, there holds for $P^1_h : W^{1,2}(D)/\mathbb{R} \to L_h$, where

$$\big(\nabla[\chi - P^1_h \chi], \nabla \eta\big) = 0 \qquad \forall \eta \in L_h$$

that

$$\|\chi - P^1_h \chi\|_{L^2} + h\|\nabla[\chi - P^1_h \chi]\|_{L^2} \leq Ch^2 \|\chi\|_{W^{2,2}} \qquad \forall \chi \in W^{1,2}/\mathbb{R} \cap W^{2,2}.$$

Below we use finite elements for a fully discrete version of Algorithm 4.1. Moreover, for simplicity we assume that $\mathbf{g}$ is independent of time.

Algorithm 4.2. 1. Let $m \geq 1$. For given $\mathbf{U}^{m-1} \in L^2(\Omega, \mathbb{H}_h)$, find $\widetilde{\mathbf{U}}^m \in L^2\big(\Omega, \mathbb{H}^0_h\big)$ such that

$$\begin{aligned} &\big(\widetilde{\mathbf{U}}^m - \mathbf{U}^{m-1}, \boldsymbol{\Psi}\big) + k\nu\big(\nabla \widetilde{\mathbf{U}}^m, \nabla \boldsymbol{\Psi}\big) \\ &\quad = k\big(\mathbf{f}^m, \boldsymbol{\Psi}\big) + \Big(\mathbf{g}(t_{m-1}, \widetilde{\mathbf{U}}^{m-1})\Delta \mathbf{W}_m, \boldsymbol{\psi}\Big) \qquad \forall \boldsymbol{\Psi} \in \mathbb{H}^0_h. \end{aligned} \tag{4.4.1}$$

2. For given $\widetilde{\mathbf{U}}^m \in L^2\big(\Omega, \mathbb{H}^0_h(D)\big)$, compute $\Pi^m \in L^2(\Omega, L_h)$ from

$$\begin{aligned} \big(\nabla \Pi^m, \nabla \chi\big) &= \frac{1}{k}\big(\widetilde{\mathbf{U}}^m, \nabla \chi\big) \qquad \forall \chi \in L_h \\ \partial_{\mathbf{n}} \Pi^m &= 0 \quad \text{on } \partial D. \end{aligned} \tag{4.4.2}$$

3. Update

$$\big(\mathbf{U}^m, \boldsymbol{\varphi}\big) = \big(\widetilde{\mathbf{U}}^m, \boldsymbol{\varphi}\big) - k\big(\nabla \Pi^m, \boldsymbol{\varphi}\big) \qquad \forall \boldsymbol{\varphi} \in \mathbb{H}_h.$$

The following result provides error estimates for the fully discrete scheme.

Theorem 4.4.1. *Let the assumptions in Lemma 4.3.1 hold. Let* $\{\widetilde{\mathbf{U}}^m\}_{m=1}^M \subset L^2(\Omega, \mathbb{H}^0_h)$ *be computed from Algorithm 4.2. Then*

$$\max_{1\leq m\leq M} \Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{U}}^m\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[k\sum_{m=1}^M \|\nabla[\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{U}}^m]\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \leq C\big(\sqrt{k} + h + \frac{h^2}{\sqrt{k}}\big).$$

Because of Theorem 4.3.1, it is sufficient to control the error between the solutions of Algorithms 4.1 and 4.2, for which Lemma 4.3.1 is relevant. In order to balance the coupling error $\mathcal{O}\big(\frac{h^2}{\sqrt{k}}\big)$ with the other two errors due to time discretisation, splitting, and spatial discretisation motivates a (non-critical) balancing $h \leq C\sqrt{k}$. We remark that this requirement is well-known in the deterministic setting, where stability of equal-order finite element pairings using the pressure stabilisation ansatz

$$\operatorname{div} \mathbf{u}^\varepsilon - \varepsilon \Delta \pi^\varepsilon = 0 \quad \text{in } D\,, \qquad \partial_{\mathbf{n}} \pi^\varepsilon = 0 \quad \text{on } \partial D$$

requires choices $\varepsilon \geq Ch^2$; cf. [74, 112]: since $\varepsilon = k$ in (4.4.2), the restriction $k \geq Ch^2$ then leads to a stable discretisation in space by equal-order finite element pairings.

Proof. For every $m \geq 1$, let

$$\big(\mathbf{E}^m, \Lambda^m\big) := \Big(\widetilde{\mathbf{u}}^m - \widetilde{\mathbf{U}}^m, \pi^m - \Pi^m\Big) \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D) \times W^{1,2}(D)\big)$$

be the solution of the following set of error equations, for all $\big(\boldsymbol{\Psi}, \chi\big) \in \mathbb{H}_h^0 \times H_h$,

$$\big(\mathbf{E}^m - \mathbf{E}^{m-1}, \boldsymbol{\Psi}\big) + k\nu\big(\nabla \mathbf{E}^m, \nabla \boldsymbol{\Psi}\big) + k\big(\nabla \Lambda^{m-1}, \boldsymbol{\Psi}\big) = \Big([\mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1}) - \mathbf{g}(t_{m-1}, \widetilde{\mathbf{U}}^{m-1})]\Delta \mathbf{W}_m, \boldsymbol{\Psi}\Big)\,, \tag{4.4.3}$$

$$\big(\operatorname{div} \mathbf{E}^m, \chi\big) + k\big(\nabla \Lambda^m, \nabla \chi\big) = 0\,. \tag{4.4.4}$$

The equations follow from the reformulation of Algorithm 4.1 in the form (4.1.6)–(4.1.8), and corresponding equations for (4.2). We may choose $\boldsymbol{\Psi} = \mathbf{P}_h^0 \mathbf{E}^m$ as test function in (4.4.3). For any $\delta_1 > 0$, we use Young's inequality to conclude

$$\begin{aligned} &\frac{1}{2}\Big(\|\mathbf{E}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2 + \|\mathbf{E}^m - \mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2\Big) + \frac{3k\nu}{4}\|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2 + k\big(\nabla \Lambda^m, \mathbf{P}_h^0 \mathbf{E}^m\big) \\ &\leq \Big([\mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1}) - \mathbf{g}(t_{m-1}, \widetilde{\mathbf{U}}^{m-1})]\Delta \mathbf{W}_m, \mathbf{P}_h^0 \mathbf{E}^m\Big) + k\|\nabla[\widetilde{\mathbf{u}}^m - \mathbf{P}_h^0 \widetilde{\mathbf{u}}^m]\|_{\mathbb{L}^2}^2 \\ &\quad + C_{\delta_1}\big\|[\mathbf{g}(\widetilde{\mathbf{u}}^{m-1}) - \mathbf{g}(\widetilde{\mathbf{U}}^{m-1})]\Delta \mathbf{W}_m\big\|_{\mathbb{L}^2}^2 + \delta_1\|\mathbf{E}^m - \mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2\,. \end{aligned} \tag{4.4.5}$$

A lower bound for the last term on the left-hand side is as follows ($\delta_2 > 0$),

$$\begin{aligned} k\big(\nabla \Lambda^{m-1}, \mathbf{P}_h^0 \mathbf{E}^m\big) \;\geq\; & k\big(\nabla P_h^1 \Lambda^{m-1}, \mathbf{E}^m\big) - \delta_2 k^2 \|\nabla \Lambda^{m-1}\|_{\mathbf{L}^2}^2 - \frac{1}{4\delta_2}\|\widetilde{\mathbf{u}}^m - \mathbf{P}_h^0 \widetilde{\mathbf{u}}^m\|_{\mathbb{L}^2}^2 \\ & - k\|\pi^{m-1} - P_h^1 \pi^{m-1}\|_{L^2}^2 - \frac{k}{4}\|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2\,. \end{aligned} \tag{4.4.6}$$

We use $\chi = P_h^1 \Lambda^{m-1}$ in (4.4.4) to conclude $k\big(\nabla P_h^1 \Lambda^{m-1}, \mathbf{E}^m\big) = k^2\big(\nabla P_h^1 \Lambda^{m-1}, \nabla \Lambda^m\big)$. We use properties of P_h^1 to conclude

$$\begin{aligned} k^2\big(\nabla P_h^1 \Lambda^{m-1}, \nabla \Lambda^m\big) &= k^2\big(\nabla P_h^1 \Lambda^m, \nabla \Lambda^m\big) - k^2\big(\nabla P_h^1[\Lambda^m - \Lambda^{m-1}], \nabla \Lambda^m\big) \\ &= k^2\|\nabla \Lambda^m\|^2 + k^2\big(\nabla[\pi^m - P_h^1 \pi^m], \nabla \Lambda^m\big) \\ &\quad - k^2\big(\nabla[\Lambda^m - \Lambda^{m-1}], \nabla P_h^1 \Lambda^m\big)\,. \end{aligned}$$

Because of (4.4.4) we may now conclude ($\delta_3, \delta_4 > 0$)

$$\begin{aligned} k^2\big(\nabla P_h^1 \Lambda^{m-1}, \nabla \Lambda^m\big) &= k^2\big(1 - \delta_3\big)\|\nabla \Lambda^m\|_{\mathbb{L}^2}^2 - C_{\delta_3} k^2 \|\nabla[\pi^m - P_h^1 \pi^m]\|_{\mathbb{L}^2}^2 \\ &\quad - k\big(\mathbf{E}^m - \mathbf{E}^{m-1}, \nabla P_h^1 \Lambda^m\big) \\ &\geq k^2\big(1 - \delta_3 - \delta_4\big)\|\nabla \Lambda^m\|_{\mathbb{L}^2}^2 - C_{\delta_3} k^2 \|\nabla \pi^m\|_{\mathbb{L}^2}^2 - \frac{1}{4\delta_4}\|\mathbf{E}^m - \mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2\,. \end{aligned}$$

Because of standard approximation results and Lemma 4.3.1, arising interpolation error terms in (4.4.5)–(4.4.6) may be controlled as follows,

$$
\begin{aligned}
&\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla(\widetilde{\mathbf{u}}^m-\mathbf{P}_h^0\widetilde{\mathbf{u}}^m)\|_{\mathbb{L}^2}^2\Big]\le Ch^2\mathbb{E}\Big[k\sum_{m=1}^{M}\|\Delta\widetilde{\mathbf{u}}^m\|_{\mathbb{L}^2}^2\Big]\le Ch^2\,,\\
&\mathbb{E}\Big[\sum_{m=1}^{M}\|\widetilde{\mathbf{u}}^m-\mathbf{P}_h^0\widetilde{\mathbf{u}}^m\|_{\mathbb{L}^2}^2\Big]\le Ch^4\mathbb{E}\Big[\sum_{m=1}^{M}\|\Delta\widetilde{\mathbf{u}}^m\|_{\mathbb{L}^2}^2\Big]\le C\frac{h^4}{k}\,,\\
&\mathbb{E}\Big[k\sum_{m=1}^{M}\|\pi^m-P_h^1\pi^m\|_{L^2}^2\Big]\le Ch^2\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla\pi^m\|_{\mathbb{L}^2}^2\Big]\le Ch^2\,,
\end{aligned}
\tag{4.4.7}
$$

where $(4.4.7)_2$ comes from (4.4.6), which involves a coupling of discretisation scales in space and time.

To keep the corresponding terms in (4.4.8) nonnegative, it is possible to choose $\delta_i > 0$, such that

$$
1-\delta_2-\delta_3-\delta_4>0\,,\qquad \text{and}\qquad \frac{1}{2}-\delta_1-\frac{1}{4\delta_4}\ge 0\,.
$$

Next, we sum over all $1 \le m \le m^* \le M$ in (4.3.10), and take expectations. Then, by the discrete Gronwall inequality, and (4.4.7),

$$
\begin{aligned}
&\mathbb{E}\Big[\|\mathbf{E}^{m^*}\|_{\mathbb{L}^2}^2\Big]+\Big(\frac{1}{2}-\delta_1-\frac{1}{4\delta_4}\Big)\mathbb{E}\Big[\sum_{m=1}^{m^*}\|\mathbf{E}^m-\mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2\Big]+\nu\mathbb{E}\Big[k\sum_{m=1}^{m^*}\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}^2\Big]\\
&+k^2\Big(1-\delta_2-\delta_3-\delta_4\Big)\mathbb{E}\Big[\sum_{m=1}^{m^*}\|\nabla\Lambda^m\|_{\mathbb{L}^2}^2\Big]\le C_{t_{m^*}}\Big(h^2+\frac{h^4}{k}\Big)\,.
\end{aligned}
\tag{4.4.8}
$$

This proves the theorem. □

4.5 Computational Experiments

In this section, we report on comparative computational studies for both, the Euler method (4.1.10)–(4.1.11), and the splitting Algorithm 4.2. For a stable discretisation in space, we use the LBB-stable MINI element; cf. [15, 74] for details. For the underlying domain $D = (0,1)^2 \subset \mathbb{R}^2$ and a deterministic applied forcing term $\mathbf{f}$, we consider the finite-dimensional Wiener process ($t \in [0,T]$)

$$
\mathbf{W}(t)=\sum_{j,k=1}^{N}\lambda_{j,k}\beta_{j,k}(t)\mathbf{e}_{j,k}\qquad (1\le N<\infty)\,,
$$

where $\lambda_{j,k} = \frac{1}{(j+k)^2}$, $\{\beta_{j,k}\}_{j,k=1}^N$ is a family of independent, real-valued Wiener processes on $\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$, as well as $\{\mathbf{e}_{j,k}\}_{j,k=1}^{\infty}$ are (for $\mathbf{x} := (x,y) \in \mathbb{R}^2$)

(i) non-solenoidal functions

$$
\mathbf{e}_{j,k}(x,y):=\Big(\sin(j\pi x)\sin(k\pi y),\sin(j\pi x)\sin(k\pi y)\Big)^\top,
$$

(ii) solenoidal functions

$$
\mathbf{e}_j(x,y):=\Big(-\cos(j\pi x-\frac{\pi}{2})\sin(j\pi y-\frac{\pi}{2}),\sin(j\pi x-\frac{\pi}{2})\cos(j\pi y-\frac{\pi}{2})\Big)^\top.
$$

Hence $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{K}} = \mathbb{W}_0^{1,2}$ for the basis from (i), and $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{K}} = \mathbb{W}_0^{1,2} \cap \mathbb{H}$ in the case (ii). Notice that in the analysis we use a Wiener process with an infinite number of terms, but all the results hold also for $\mathbf{W}$ defined as above. In this section we assume that equation (4.1.1) is driven by a finite dimensional noise.

In the experiments below we take $N = 4$, and address the following topics in the following Sections 4.5.1 and 4.5.2.

(A) How does non-solenoidal resp. solenoidal noise affect strong approximation properties of Algorithm 4.2? Is Theorem 4.3.1 sharp with respect to the restriction to solenoidal noise?

(B) Chorin's projection scheme in the deterministic setting is known to exhibit anisotropic error structures for the pressure, such as boundary layers of magnitude $\mathcal{O}(\sqrt{k}\,|\log k|)$, cf. e.g. [113]. What may be concluded accordingly in the stochastic setting for both, trajectories and expectations of pressure iterates?

It is evident that if compared to Euler's method the splitting scheme discussed here causes reduced computational effort, which in particular pays off in the present stochastic setting where a significant number of realisations has to be computed to obtain expectations.

For the experiments below we use $T = 1$, and compute on cartesian meshes of size $h = \frac{1}{50}$, for a number of realisations $N_p = 3000$, a minimum time discretisation parameter $k_0 = \frac{1}{4096}$, and a constant operator $\mathbf{g}$ in (4.1.1). To approximate strong errors $(1 \leq M^* \leq M)$

$$\Big(\mathbf{E}\big[\|\mathbf{U}_{k_0}^{M^*} - \mathbf{U}_{k_i}^{M^*}\|_{\mathbb{L}^2}^2\big]\Big)^{1/2} \approx \Big(\frac{1}{N_p}\sum_{\ell=1}^{N_p} \|\mathbf{U}_{k_0}^{M^*}(\omega_\ell) - \mathbf{U}_{k_i}^{M^*}(\omega_\ell)\|_{\mathbb{L}^2}^2\Big)^{1/2} \qquad (i \geq 1)\,,$$

we use $\mathbf{U}_{k_0}^{M^*} \approx \mathbf{u}(t_{M^*}, \cdot)$ as (approximate) solution to (4.1.1) which is computed for the smallest $k_0 \ll 1$, whereas $\{\mathbf{U}_{k_i}^{M^*}\}_{i \geq 1}$ are obtained from Algorithm 4.2 for $k_i = 2^i k_0$ with $i = 1, 2, 3, \cdots$.

4.5.1 Strong errors for different noise

We compare computed velocity iterates of both, the Euler scheme (4.1.10)–(4.1.11) and Algorithm 4.2 for both solenoidal and non-solenoidal noise. The theoretical study in the previous sections needed the uniform bound

$$\mathbb{E}\big[k \sum_{m=1}^{M} \|\nabla \sigma^m\|_{\mathbb{L}^2}^2\big] \leq C \tag{4.5.1}$$

for pressure iterates of (4.1.10)–(4.1.11); this property is shown in Lemma 4.2.1, (iii) in the case of solenoidal noise, and in Lemma 4.3.1 for pressure iterates of Algorithm 4.1 in this case. The computational results in Figure 4.5.1 evidence $\frac{1}{2}$ as convergence rate for velocity iterates from Algorithm 4.2, which is in accordance with Theorem 4.4.1. Figure 4.5.2 reports on corresponding results for applied non-solenoidal noise; we observe a reduction of the convergence rate for velocity iterates of Algorithm 4.2 by approximately 50%, while Euler iterates converge optimally. To further evidence this loss of accuracy for iterates of the splitting Algorithm 4.2 in the presence of non-solenoidal noise, our computations suggest

$$\Big(\mathbb{E}\big[k \sum_{m=1}^{M} \|\nabla \Pi^m\|_{\mathbb{L}^2}^2\big]\Big)^{1/2} \approx \frac{C}{\sqrt{k}}\,,$$

which is a bound that we obtain for $\{\sigma^m\}_{m=1}^{M}$ instead of item (iii) in Lemma 4.2.1 for applied non-solenoidal noise.

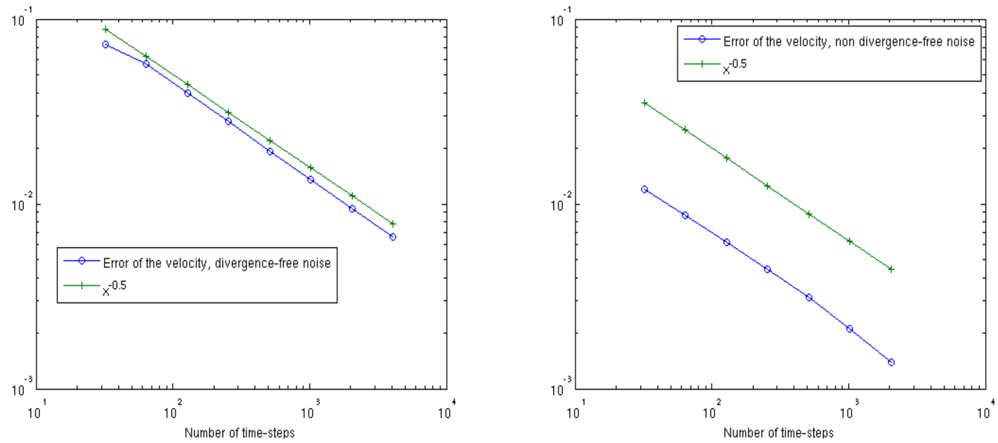

Figure 4.5.1: Solenoidal noise: Rates of convergence for velocity iterates of Algorithm 4.2 (left), and the Euler scheme (4.1.10)–(4.1.11) (right).

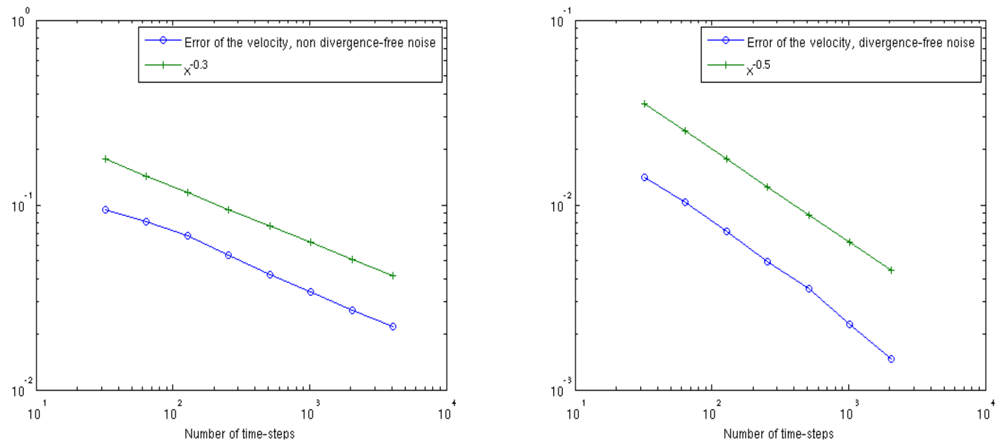

Figure 4.5.2: Non-solenoidal noise: Rates of convergence for velocity iterates of Algorithm 4.2 (left), and the Euler scheme (4.1.10)–(4.1.11) (right).

4.5.2 Approximation of pressures

The reformulation (4.1.6)–(4.1.8) of Algorithm 4.1 evidences error effects due to homogeneous boundary conditions, which are well-known in the deterministic setting to cause artificial boundary layers of thickness $\mathcal{O}(\sqrt{k}\,|\log k|)$; see e.g. [58, 112, 113] and the literature cited in these works. Hence, it is reasonable to ask if corresponding anisotropic errors for pressure iterates from Algorithm 4.1 occur in the stochastic setting as well. We remark that no results regarding (rates of) convergence of iterates $\{P^m\}_{m\geq 1}$ from Algorithm 4.1 have been obtained in the previous sections. The following results show error profiles for the pressure computed by Algorithm 4.2 both, pathwise and expectation-wise, computed for $h = 1/30$ and $k_0 = 1/512$. Again, we distinguish between computations for applied solenoidal and non-solenoidal noise.

Pressure error functions in the case of solenoidal noise for different time-step sizes are depicted in Figure 4.5.3 both, for a single path (first line), and expectations (second line); in both cases, we observe an anisotropic structure of the error, which is more pronounced for expectations, and clearly grows for increasing time-steps $k_i > 0$.

The influence of applied non-solenoidal noise on the accuracy of pressure iterates can be deduced from the plots in Figure 4.5.4: no local error structures are visible for a single realization;

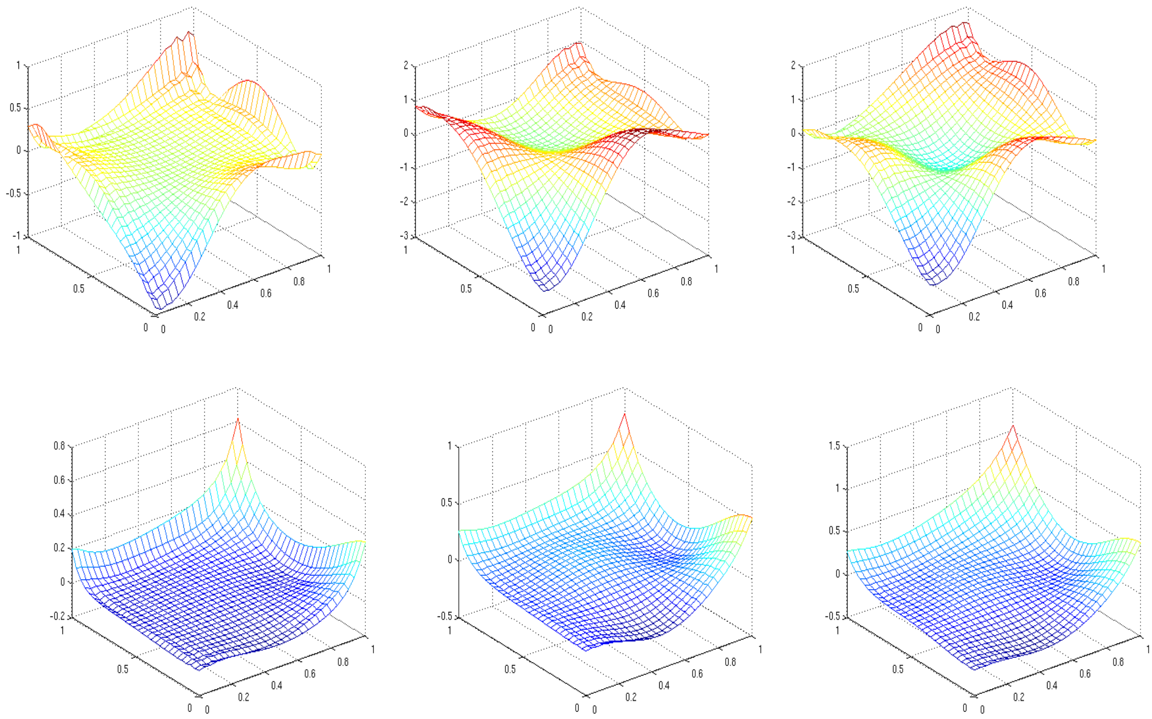

Figure 4.5.3: Solenoidal noise: Error of pressure from Algorithm 4.2 at $T = 1$ for $k_i = \frac{1}{512}, \frac{1}{256}, \frac{1}{128}$ for one realization (first line), and its expectation (seond line).

this is different from corresponding plots for expectations which still show marked boundary layers that increase for growing values $k_i > 0$.

The influence of applied non-solenoidal noise on the accuracy of pressure iterates can be deduced from the plot in Figure 4.5.5. There the function

$$k_i \mapsto k_i \mathbb{E}\left[\sum_{m=1}^{M} \|\nabla \Pi_{k_i}^m\|_{\mathbb{L}^2}^2\right]^{1/2}$$

is plotted, showing that the norm of the pressure divergence for small time-steps. Thsu, no local error structures are visible for a single realisation and for corresponding expectations.

4.5.3 Chorin scheme with stochastic pressure correction.

As has been shown so far, the proposed splitting Algorithm 4.1 shows optimal convergence behavior only in the case of solenoidal noise. Here we try to modify Algorithm 4.1, in order to validate optimal convergence behavior also in the case that the sequence of random variables $\big\{\mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1})\Delta \mathbf{W}_m\big\}_{m=1}^{M}$ approximates general noise. The scheme that we propose is the following:

Algorithm 4.3. Let $m \geq 0$.
1. For given $\widetilde{\mathbf{u}}^{m-1} \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$, find $\boldsymbol{\xi}^m \in \mathbb{H} \cap \mathbb{W}^{1,2}(D)$ such that

$$\begin{aligned} \boldsymbol{\xi}^m + \nabla s^m &= \frac{1}{\sqrt{k}}\mathbf{g}(t_{m-1}, \widetilde{\mathbf{u}}^{m-1})\Delta \mathbf{W}_m && \text{in } D,\\ \operatorname{div} \boldsymbol{\xi}^m &= 0 && \text{in } D, \qquad (4.5.2)\\ \langle \boldsymbol{\xi}^m, \mathbf{n}\rangle &= 0 && \text{on } \partial D\,. \end{aligned}$$

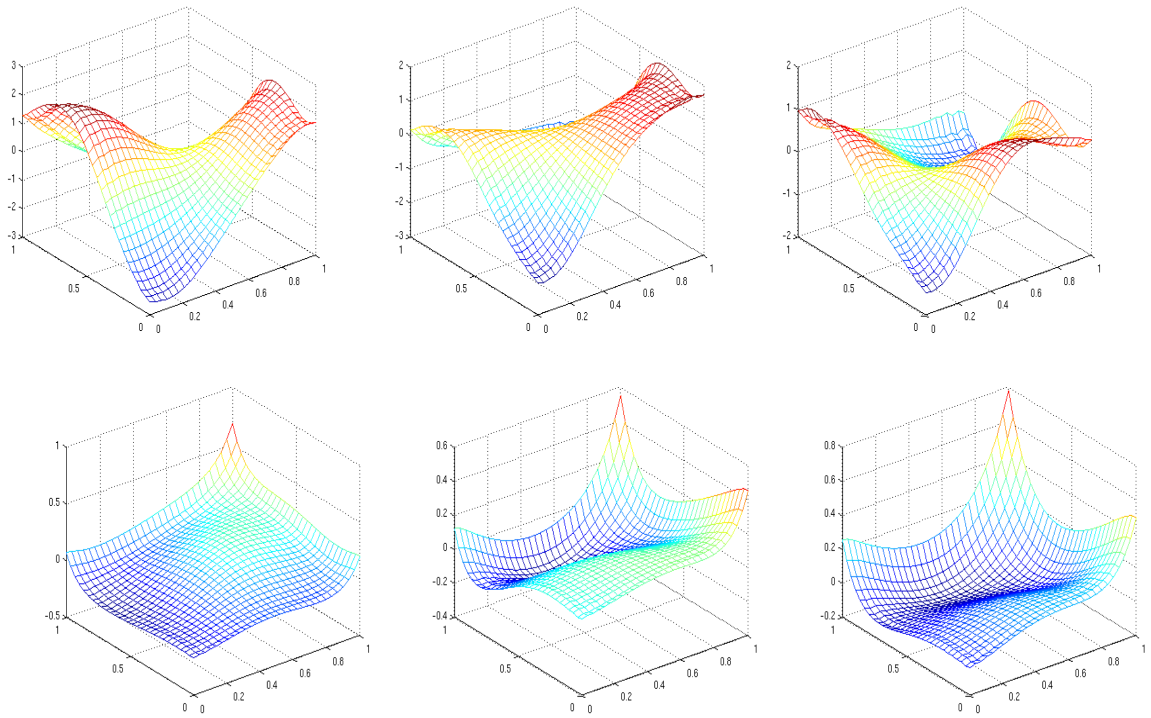

Figure 4.5.4: Non-solenoidal noise: Error of pressure from Algorithm 4.2 at $T = 1$ for $k_i = \frac{1}{512}, \frac{1}{256}, \frac{1}{128}$ for one realization (first line), and its expectation (seond line).

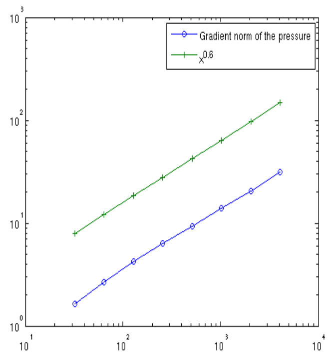

Figure 4.5.5: Non-solenoidal noise: gradient norm of the pressure plotted with respect to the time-step.

2. For given $\mathbf{u}^{m-1} \in L^2(\Omega, \mathbb{H})$, find $\widetilde{\mathbf{u}}^m \in L^2\big(\Omega, \mathbb{W}_0^{1,2}(D)\big)$ such that

$$\big(\widetilde{\mathbf{u}}^m - \mathbf{u}^{m-1}\big) - k\nu\Delta\widetilde{\mathbf{u}}^m = k\mathbf{f}^m + \sqrt{k}\boldsymbol{\xi}^m \quad \text{in } D\,. \tag{4.5.3}$$

3. Compute $\mathbf{u}^m \in L^2\big(\Omega, \mathbb{H}\big)$, and $\pi^m \in L^2\big(\Omega, W^{1,2}(D)/\mathbb{R}\big)$ from

$$\begin{aligned} \mathbf{u}^m - \widetilde{\mathbf{u}}^m + k\nabla\pi^m &= 0 \quad \text{in } D\,, \\ \operatorname{div}\mathbf{u}^m &= 0 \quad \text{in } D\,, \\ \langle \mathbf{u}^m, \mathbf{n}\rangle &= 0 \quad \text{on } \partial D\,. \end{aligned} \tag{4.5.4}$$

4. Compute the approximation of the pressure p via
$$r^m = \pi^m + \frac{1}{\sqrt{k}} s^m.$$

The underlying idea for this algorithm is to distinguish between deterministic and stochastic (forcing) integrals on the right-hand side of (4.1.1), which scale differently in a time discretisation scheme. Corresponding Helmholtz decompositions of both terms involve gradient functions, which are then referred to a *deterministic* and then *stochastic pressures.* It is by Step 1 that the gradient of the *stochastic pressure* $\{s^m\}_{m=1}^M$ in $(4.5.2)_1$ has no influence on computing velocity iterates in Steps 1 to 3, where only the deterministic pressure $\{\pi^m\}_{m=1}^M$ is involved. This argument is further detailed by the following formal computation for Euler iterates from (4.1.10)–(4.1.11):

$$\begin{aligned}\mathbf{w}^m - k\nu\Delta\mathbf{w}^m + k\nabla\pi^m &= \mathbf{w}^{m-1} + k\mathbf{f}^m + \mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta\mathbf{W}_m \\ &= \mathbf{w}^{m-1} + k\mathbf{f}^m + \mathbf{P}_{\mathbb{H}}\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta\mathbf{W}_m + \sqrt{k}\nabla s^{m-1}\,.\end{aligned}$$

So we get

$$\mathbf{w}^m - k\nu\Delta\mathbf{w}^m + k\nabla\pi^m = \mathbf{w}^{m-1} + k\mathbf{f}^m + \mathbf{P}_{\mathbb{H}}\mathbf{g}(t_{m-1}, \mathbf{w}^{m-1})\Delta\mathbf{W}_m\,,$$

where
$$\pi^m = \pi^m - \frac{1}{\sqrt{k}} s^m\,.$$

In fact, Algorithm 4.3 is Algorithm 4.1, which is applied to the same equation with projected noise. So, the proof of the convergence rate could follow directly from Theorem 4.3.1. Unfortunately, for $\mathbf{v} \in \mathbb{W}_0^{1,2}(D)$ the projection $\mathbf{P}_{\mathbb{H}}\mathbf{v} \in \mathbb{W}^{1,2}(D)$ is not an element of $\mathbb{W}_0^{1,2}(D)$. As a consequence, in formula (4.2.13) we obtain an additional boundary integral which is difficult to bound, and properties (ii) and (iii) of Lemma 4.2.1 are not clear to hold in this setting anymore. To avoid this problematic issue, we consider Problem 4.1.1 with space periodic boundary conditions on a set $Q = [0, L]^d$, $L > 0$. Let $\mathbb{H}^{per}$, $\mathbb{V}^{per}$ and $\mathbb{W}^{n,2}_{per}$ denote the space periodic analogues of the spaces $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{W}^{n,2}$. In this case optimal convergence of splitting Algorithm 4.3 also holds for general noise. We have the following

Theorem 4.5.1. *Let $T > 0$, $D \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded convex polyhedral domain, and* (4.2.1)–(4.2.3) *be valid. Denote by $\mathbf{u} \in L^2\big(\Omega; \mathscr{C}([0, T]; \mathbb{V}^{per})\big) \cap L^2\big(\Omega; L^2(0, T; \mathbb{W}^{2,2}_{per})\big)$ the strong solution of* (4.1.1)*, and $\{\widetilde{\mathbf{u}}^m\}_{m=1}^M \subset L^2\big(\Omega, \mathbb{W}^{1,2}_{per}(D)\big)$ solves Algorithm 4.3. There exists a constant $C \equiv C\big(\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{W}^{1,2}}], D_T\big) > 0$, such that*

$$\max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m, \cdot) - \widetilde{\mathbf{u}}^m\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[k\sum_{m=1}^M \|\nabla\big(\mathbf{u}(t_m, \cdot) - \widetilde{\mathbf{u}}^m\big)\|^2_{\mathbb{L}^2}\Big]\Big)^{1/2} \le C\sqrt{k}\,. \tag{4.5.5}$$

Again, note that condition (4.2.9) is not needed in this case to validate (4.5.5).

Here we give some numerical motivations for Algorithm 4.3 by considering the same setting as at the beginning of this section. Figure 4.5.6 shows error plots for different types of noise. We observe an improvement in the case of general noise to almost optimal order, which is rooted in the improved regularity of the deterministic pressure, which is exclusively needed for optimal convergence behavior of this time-splitting scheme,

$$\mathbb{E}\left[k\sum_{m=1}^M \|\nabla\pi^m\|^2_{\mathbb{L}^2}\right] \le C.$$

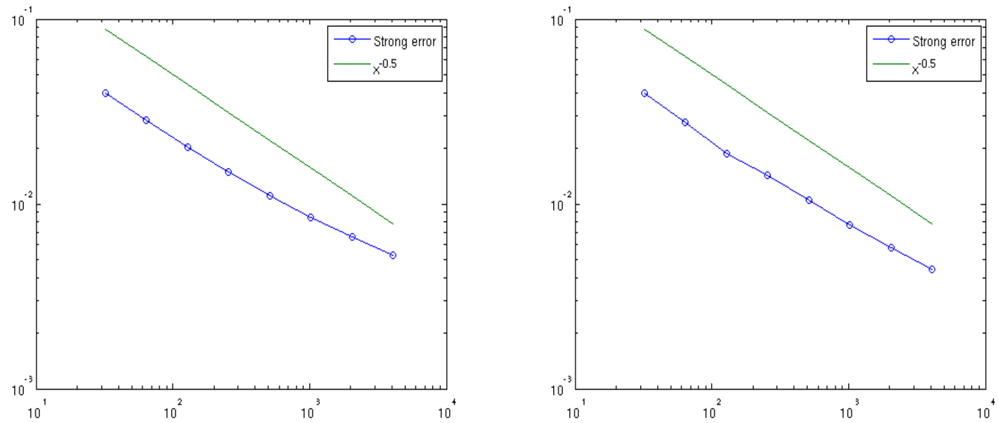

Figure 4.5.6: Solenoidal noise: Rates of convergence for velocity iterates of Algorithm 4.3 with non divergence-free noise (left), and solenoidal noise (right).

4.6 Summary and outlook

We analysed splitting-based schemes for the stochastic Stokes equations.

We proved error estimates for the stochastic version of the Chorin scheme, and theory evidences how this estimate depends on the regularity of the Lagrange multiplier, which is determined by the noise. For the proof we split the error in several parts, to control the error contributions from the pressure stabilisation and from the semi-implicit treatment of the pressure. To prove optimal error estimates, we have to assume that the noise takes values in the space of solenoidal vector fields.

Numerical experiments confirm that the irregularity of the pressure affects the convergence of the splitting scheme. Thus, we consider a generalised version of the splitting scheme, which includes a step where the noise is projected on the space of solenoidal vector fields. This scheme delivers optimal convergence independently of the type of noise.

Further steps in this direction may be the analysis of other splitting schemes, e.g. the Chorin-Penalty scheme from [113], or the application of such schemes to the SNSEs, and study the interaction of noise, Lagrange multiplier and convection term.

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Chapter 5

Strong rates of convergence

5.1 Introduction

We consider the system of equations describing the motion of incompressible Newtonian fluids subject to random forcing on the torus $D = (0, L)^2 \subset \mathbb{R}^2$. Let $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, with filtration $\mathbb{F} = \{\mathcal{F}_t \ ; \ t \in [0, T]\}$. We denote $D_T := (0, T) \times D$, for $0 < T < \infty$. Let $\mathbf{u}_0 : \Omega \times D \to \mathbb{R}^2$ be a given random variable. We seek a vector field $\mathbf{u} : \Omega \times D_T \to \mathbb{R}^2$, and a scalar field $\pi : \Omega \times D_T \to \mathbb{R}$ such that the stochastic Navier-Stokes equations

$$\dot{\mathbf{u}} - \nu \Delta \mathbf{u} + [\mathbf{u} \cdot \nabla]\mathbf{u} + \nabla \pi = \mathbf{g}(\mathbf{u})\dot{\mathbf{W}} \quad \text{in } D_T \times \Omega \,, \tag{5.1.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D_T \times \Omega \,, \tag{5.1.2}$$

are satisfied, together with initial and boundary conditions

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0 \quad \text{in } D \times \Omega \qquad \text{and} \qquad \mathbf{u}(t, \mathbf{x} + L\mathbf{b}_i) = \mathbf{u}(t, \mathbf{x}) \quad \text{on } (0, T) \times \partial D \times \Omega \tag{5.1.3}$$

for $i = 1, 2$. Here $\mathbf{u} \cdot \nabla = \sum_{i=1}^n u_i \partial_i$, $\{\mathbf{b}_j\}_{j=1}^2$ is the canonical basis of $\mathbb{R}^2$, and $\mathbf{g}$ is an operator-valued random field, while $\mathbf{W}$ is a Wiener process on $\mathfrak{P}$ that takes values in a Hilbert space to be specified later.

Stochastic Navier-Stokes equations are employed since long time as a complementatory model of the deterministic one to better understand the rôle of small perturbations or (thermodynamic) fluctuations, which are present in fluid flows, and to get further insight regarding possible non-uniqueness, and loss of initial regularity of solutions in the deterministic case (in 3D). In applications, the noise can be used to e.g. model structural vibrations in aeronautics, heating and industrial pollution in atmospheric dynamics, or the influence of ice or vulcans in some models of atmosphere-ocean evolution.

In this chapter, we show convergence with rates for space-time discretisations of the incompressible Navier-Stokes equations (5.1.1)–(5.1.3) using variational methods; we exploit improved spatial regularity of solutions of (5.1.1)–(5.1.3), and Hölder continuity in time of the velocity gradient to arrive at convergence in probability with rate 1/4 for the implicit Euler scheme in the $L^\infty(0, T; \mathbb{L}^2(D))$-norm, which improves to the rate 1/2 in the $L^2(0, T; \mathbb{L}^2(D))$-norm. In fact, Euler iterates $\{\mathbf{u}^m\}_{m=1}^M$ are only accounted for on a subset $\Omega_k \subset \Omega$, with $\mathbb{P}[\Omega \setminus \Omega_k] \to 0$ for $k \to 0$, where $k > 0$ is the (equi-distant) time step; our first result is then the estimate

$$\mathbb{E}\left[\mathbb{1}_{\Omega_k}\left(\max_{1 \le m \le M} \left\|\mathbf{u}(t_m) - \mathbf{u}^m\right\|_{\mathbb{L}^2}^2 + k^{1-\eta} \sum_{m=1}^M \left\|\mathbf{u}(t_m) - \mathbf{u}^m\right\|_{\mathbb{L}^2}^2\right)\right] \le C\, k^{\eta - \varepsilon} \quad (\varepsilon > 0)\,, \tag{5.1.4}$$

for all $\eta \in (0, 1/2)$; see Section 5.3. This bound reflects the interplay of the stochastic forcing and the non-Lipschitz nonlinearity: It is the set Ω_k which singles out trajectories whose approximation could then be handled by a discrete Gronwall argument. As a consequence of (5.1.4), convergence in probability with asymptotic rate 1/4, and 1/2 respectively follows; see Corollary 5.3.1:

$$\lim_{\widetilde{C}\to\infty}\lim_{k\to 0}\mathbb{P}\left[\max_{1\le m\le M}\big\|\mathbf{u}(t_m)-\mathbf{u}^m\big\|_{\mathbb{L}^2}^2+k^{1-\eta}\sum_{m=1}^{M}\big\|\mathbf{u}(t_m)-\mathbf{u}^m\big\|_{\mathbb{L}^2}^2\ge \widetilde{C}\,k^{\eta-\varepsilon}\right]=0\,, \tag{5.1.5}$$

for all $\varepsilon > 0$.

Since the proposed scheme is nonlinear, in a second step, we analyse the error of a semi-implicit treatment of the nonlinear drift. Estimates (5.1.4) and (5.1.5) remain valid for corresponding iterates $\{\mathbf{v}^m\}_{m=1}^M$ from the semi-implicit scheme, but the related subset Ω_k may not be constructed from $\{\mathbf{v}^m\}_{m=1}^M$ anymore: Ω_k is again defined in terms of the implicit Euler iterates $\{\mathbf{u}^m\}_{m=1}^M$ as in (5.1.4), for which $\mathbb{P}[\Omega_k]\to 1$ holds in case $k\to 0$.

The third goal of this paper is the finite element approximation of problem (5.1.1)–(5.1.3). Stable pairings of velocity and pressure ansatz spaces $(\mathbb{H}_h, L_h)$, each assembled from functions of vanishing mean, satisfy the discrete LBB-condition

$$\sup_{\mathbf{\Phi}\in\mathbb{H}_h}\frac{(\operatorname{div}\mathbf{\Phi},\Pi)}{\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}}\ge C\,\|\Pi\|_{L^2}\quad\forall\,\Pi\in L_h\,, \tag{5.1.6}$$

with a constant $C > 0$ independent of the mesh size $h > 0$. We may then define the space $\mathbb{V}_h\subset\mathbb{H}_h$ of discretely divergence-free velocity fields:

$$\mathbb{V}_h=\big\{\mathbf{\Phi}\in\mathbb{H}_h\,\big|\,(\operatorname{div}\mathbf{\Phi},\Lambda)=0\quad\forall\,\Lambda\in L_h\big\}\not\subseteq\mathbb{V}^{per}\,, \tag{5.1.7}$$

where $\mathbb{V}^{per}$ denotes the space of periodic divergence-free functions, see Section 5.2.1.

As it will turn out from the error analysis in Section 5.4, a main difficulty are the weak stability properties of solutions of the space-time discretisation (see Algorithm 5.3) in Lemma 5.4.1, as opposed to those for Algorithm 5.1 in Lemma 5.3.1 for the time discretisation. To compensate for this lack of stability, a perturbation argument in combination with a bootstrapping argument is employed to show optimal convergence in Theorem 5.4.1. This is in contrast to the deterministic version of (5.1.1)–(5.1.3), where a corresponding 'pathwise' version of Lemma 5.4.1 is enough to show optimal convergence of the related modification of Algorithm 5.3; cf. [71]. Moreover, as can be seen from Theorem 5.4.1, the explicit appearance of the pressure of Algorithm 5.1 in the error estimate illustrates the rôle of noise that determines the convergence behavior of a general discrete LBB-stable discretisation in Algorithm 5.3. To illustrate this problematic issue, we recall the following estimate for approximates $\mathbf{V}\in\mathbb{V}_h$ of the deterministic stationary incompressible Stokes equations with solutions $(\mathbf{v},\pi)\in(\mathbb{V}^{per}\cap\mathbb{W}^{2,2})\times W^{1,2}/\mathbb{R}$,

$$\|\nabla(\mathbf{v}-\mathbf{V})\|_{\mathbb{L}^2}\le C\left(\inf_{\mathbf{\Phi}\in\mathbb{V}_h}\|\nabla(\mathbf{v}-\mathbf{\Phi})\|_{\mathbb{L}^2}+\inf_{\Xi\in L_h}\sup_{\mathbf{0}\neq\mathbf{\Phi}\in\mathbb{V}_h}\frac{(\operatorname{div}\mathbf{\Phi},\pi-\Xi)}{\|\nabla\mathbf{\Phi}\|_{\mathbb{L}^2}}\right). \tag{5.1.8}$$

As a consequence, the finite element error for the velocity also depends on the best-approximation error of the pressure. Hence, it is because of the limited regularity of the pressure. As opposed to the deterministic case, the restricted regularity of the pressure, see [92],

$$\pi\in L^1\Big(\Omega;W^{-1,\infty}\big(0,T;\mathbb{W}^{1,2}(D)/\mathbb{R}\big)\Big) \tag{5.1.9}$$

that also the accuracy of velocity fields computed from general discrete LBB-stable mixed finite elements may be affected in the presence of a non-solenoidal noise; see Theorem 5.4.1.

In order to avoid this problematic issue, an alternative discretisation strategy that uses an inner approximation $\mathbb{V}_h\subset\mathbb{V}^{per}$ by finite element pairings may be preferred. With this ansatz, the second term on the right-hand side of (5.1.8) vanishes, leading to the best approximation error

$$\|\nabla(\mathbf{v}-\mathbf{V})\|_{\mathbb{L}^2} \le C \inf_{\mathbf{\Phi}\in\mathbb{V}_h} \|\nabla(\mathbf{v}-\mathbf{\Phi})\|_{\mathbb{L}^2}\,.$$

An example for a divergence-free finite element pairing is the Scott-Vogelius element from [121], which is recalled in Section 5.4. In this case, variational arguments similar to those given for the time discretisation may be applied to deduce optimal rates of convergence for a related space-time discretisation of (5.1.1)–(5.1.3),

$$\mathbb{E}\left[\mathbb{1}_{\Omega_k\cap\Omega_h} \max_{1\le m\le M} \left\|\mathbf{u}(t_m)-\mathbf{U}^m\right\|_{\mathbb{L}^2}^2\right] \le C\left(k^{\eta-\varepsilon}+h^{2-\varepsilon}\right) \qquad (\varepsilon>0)\,,$$

$$\mathbb{E}\left[\mathbb{1}_{\Omega_k\cap\Omega_h} k\sum_{m=1}^{M} \left\|\mathbf{u}(t_m)-\mathbf{U}^m\right\|_{\mathbb{L}^2}^2\right] \le C\left(k^{2\eta-\varepsilon}+h^{2-\varepsilon}\right) \qquad (\varepsilon>0)\,,$$

where $\{\mathbf{U}^m\}_{m=1}^M$ is the fully discrete $\mathbb{V}^{per}$-valued solution of Algorithm 5.3, and $\Omega_h\subset\Omega$ satisfies $\mathbb{P}[\Omega_h]\to 1$ for $h\to 0$.

The convergence analysis presented in this work may be simplified to cover the stochastic incompressible Stokes equations as well, where the regularity properties given by Lemmata 5.2.1 and 5.3.1 remain valid; by taking into account the improved Hölder continuity for the gradient of the solution of the Stokes equations, we obtain strong convergence with rate 1/2 in time, and 1 in space. It is due to the absence of a nonlinear drift that we may choose a set $\Omega_k\equiv\Omega_h\equiv\Omega$, and $\varepsilon=0$ in estimate (5.1.4); see Remarks 5.3.3 and 5.4.3 for further details.

Our hope is that the tools developed in this work to accurately discretise the stochastic Navier-Stokes equations may be applied to construct both, reliable and efficient space-time discretisations of more general constrained, nonlinear stochastic evolution equations. The remainder of this chapter is organised as follows. In Section 5.2 we collect background material, and necessary assumptions on data in (5.1.1)–(5.1.3). In Section 5.3 we show stability and convergence results for the solutions of the different time discretisation schemes, while in Section 5.4 we extend these results to the full discretisation in space and time.

5.2 Preliminaries

5.2.1 Function spaces

Let $\left(L^p(D),\|\cdot\|_{L^p}\right)$ and $\left(W^{k,p}(D),\|\cdot\|_{W^{k,p}}\right)$ denote the usual Lebesgue and Sobolev spaces endowed with their usual norms; spaces with blackboard letters (e.g., $\mathbb{W}^{k,p}(D):=[W^{k,p}(D)]^2$) represent the spaces of vector valued functions. We denote by $L^p_{per}(D)$ and $W^{k,p}_{per}(D)$ the Lebesgue and Sobolev space of functions that are periodic and have vanishing mean respectively. For a normed space X, $L^p(0,T;X)$ denotes the Bochner space of strongly measurable functions $\varphi:[0,T]\to X$ such that $\int_0^T \|\varphi\|_X^p\,\mathrm{d}t<\infty$, and X^* denotes the dual space of X. The inner product for functions in $\mathbb{L}^2(D)$ is defined as $(\mathbf{u},\mathbf{v}):=\sum_{i=1}^2\int_D u_i v_i\,\mathrm{d}\mathbf{x}$.

For the study of our problem we need the following spaces:

$$\mathbb{H}^{per} = \left\{\mathbf{u}\in\mathbb{L}^2_{per}(D)\mid \operatorname{div}\mathbf{u}=0 \text{ weakly in } D\right\},$$
$$\mathbb{V}^{per} = \left\{\mathbf{u}\in\mathbb{W}^{1,2}_{per}(D)\mid \operatorname{div}\mathbf{u}=0 \text{ weakly in } D\right\}.$$

Note that, since $\|\mathbf{v}\|_{\mathbb{V}}=\|\mathbf{v}\|_{\mathbb{V}^{per}}$ for all $\mathbf{v}\in\mathbb{V}^{per}$, we will use only $\|\cdot\|_{\mathbb{V}}$ in the following. We define the self-adjoint, inversely compact operator $\mathbf{A}:D(\mathbf{A})\to\mathbb{H}^{per}$ via $\mathbf{A}\mathbf{u}:=-\mathbf{P}^{per}_{\mathbb{H}}\Delta\mathbf{u}$, where $\mathbf{P}^{per}_{\mathbb{H}}:\mathbb{L}^2_{per}(D)\to\mathbb{H}^{per}$ denotes the Leray-projection, and define $D(\mathbf{A})=\mathbb{W}^{2,2}(D)\cap\mathbb{V}^{per}$. We recall that this projection is stable in $\mathbb{L}^2$ and $\mathbb{W}^{1,2}$; see [125, Remark 1.6]. The space of bounded linear operators between two vector spaces X and Y will be denoted by $\mathcal{L}(X,Y)$.

5.2.2 General assumptions, and spatial regularity of the solution

We summarise the assumptions needed below for data $\mathbf{W}$, $\mathbf{Q}$, $\mathbf{g}$, and $\mathbf{u}_0$. In the following we either choose $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$, or $\boldsymbol{\mathcal{H}} = \mathbb{V}^{per}$, to distinguish between non-solenoidal or solenoidal noise in (5.1.1)–(5.1.3). The following assumptions will be made.

(S_1) For $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$, let $\mathbf{W} = \{\mathbf{W}(t,\cdot)\ ;\ t \in [0,T]\}$ be a $\mathbf{Q}$-Wiener process with values in a separable Hilbert space $\boldsymbol{\mathcal{K}}$ defined on the stochastic basis $\mathfrak{P}$.

(S_2) Let the operator $\mathbf{g} : \mathbb{L}^2 \to \mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})$ be Lipschitz continuous with linear growth, i.e., there exist constants $K_1, K_2 > 0$ such that

$$\|\mathbf{g}(\mathbf{u})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)} \leq K_1 \|\mathbf{u}\|_{\mathbb{L}^2}\,, \qquad \|\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)} \leq K_2 \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2}\,.$$

(S_3) The operators $\mathbf{A} : D(\mathbf{A}) \to \mathbb{H}^{per}$, $\mathbf{g} : \mathbb{L}^2 \to \mathcal{L}(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})$, and $\mathbf{Q} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ satisfy the following estimates, with $K_3 \in (0, \nu/3)$, and $K_4 > 0$,

$$\mathrm{Tr}\Big(\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u})\Big) \leq K_3 \|\mathbf{u}\|^2_{\mathbb{V}}\,, \qquad \mathrm{Tr}\Big(\mathbf{A}\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u})\Big) \leq K_4 \|\mathbf{u}\|^2_{\mathbb{V}}\,.$$

(S_4) $\mathbf{u}_0 \in L^p\big(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V}^{per}\big)$ for some $p \geq 2$.

Remark 5.2.1. *Conditions (S_1), (S_2), and (S_4) are needed to prove existence of solutions; see e.g. [46, 27]. The second condition in (S_3) is needed to conclude Lemma 5.2.1, (i), for $p = 2$; see [25, Remark 4.1]. The first condition in (S_3) is needed to generalise this result to $p > 2$: admitting only K_3 sufficiently small allows to control the noise by the main part of the drift. The second condition in (S_3) implies the first, in case $K_3 = K_4/\lambda_1$, and $K_3 \in (0, 2\nu)$, where λ_1 denotes the smallest eigenvalue of $\mathbf{A}$.*

We give the definition of a strong solution, see for instance [27].

Definition 5.2.1. *A $\mathbb{V}^{per}$-valued adapted process $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbb{V})) \cap M^2([0,T], \mathbb{F}; \mathbb{V}^{per} \cap \mathbb{W}^{2,2})$ on $\mathfrak{P}$ is a strong solution to problem* (5.1.1)–(5.1.3) *if for every $t \in [0,T]$, and all $\boldsymbol{\phi} \in \mathbb{V}^{per}$, there holds*

$$\begin{aligned}\big(\mathbf{u}(t), \boldsymbol{\phi}\big) + \nu \int_0^t \big(\nabla \mathbf{u}(s), \nabla \boldsymbol{\phi}\big)\mathrm{d}s + \int_0^t \Big([\mathbf{u}(s) \cdot \nabla]\mathbf{u}(s), \boldsymbol{\phi}\Big)\mathrm{d}s \\ = \big(\mathbf{u}_0, \boldsymbol{\phi}\big) + \int_0^t \Big(\boldsymbol{\phi}, \mathbf{g}(\mathbf{u}(s))\mathrm{d}\mathbf{W}(s)\Big) \qquad \mathbb{P} - a.s.\end{aligned}$$

Higher moments with stronger norms are obtained for solutions in the following result; its proof requires $\nu > 0$ (see (S_3), first part) to be sufficiently large, in order to control the effects given by the random forcing term. This is a slight generalisation of corresponding results in [46, 27]. A similar result, but for additive noise is proved in [42, Lemma A.1].

Lemma 5.2.1. *Assume (S_1)–(S_4), and $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$. Then solutions of* (5.1.1)–(5.1.3) *satisfy for $2 \leq p \leq 8$*

$$\text{(i)} \qquad \sup_{0\leq t\leq T} \mathbb{E}\left[\|\mathbf{u}(t)\|^p_{\mathbb{V}}\right] + \nu\, \mathbb{E}\left[\int_0^T \|\mathbf{u}(t)\|^{p-2}_{\mathbb{V}} \|\mathbf{A}\mathbf{u}(t)\|^2_{\mathbb{L}^2}\,\mathrm{d}t\right] \leq C_{T,p}\,,$$

$$\text{(ii)} \qquad \mathbb{E}\left[\sup_{0\leq t\leq T} \|\mathbf{u}(t)\|^p_{\mathbb{V}}\right] \leq C_{T,p}\,,$$

where $C_{T,p} = C_{T,p}\Big(\mathrm{Tr}\,\mathbf{Q}, \mathbb{E}\left[\|\mathbf{u}_0\|^p_{\mathbb{V}}\right]\Big) > 0$.

Proof. Use Itô's formula for the process given by $\{\|\mathbf{u}(t)\|_{\mathbb{V}}^p\ ;\ t \geq 0\}$. We compute the Fréchet derivatives of the norm,

$$D\|\mathbf{u}\|_{\mathbb{V}}^p = p\|\mathbf{u}\|_{\mathbb{V}}^{p-2}\mathbf{A}\mathbf{u}\,, \qquad D^2\|\mathbf{u}\|_{\mathbb{V}}^p = p(p-2)\|\mathbf{u}\|_{\mathbb{V}}^{p-4}\mathbf{A}\mathbf{u}\otimes\mathbf{A}\mathbf{u} + p\|\mathbf{u}\|_{\mathbb{V}}^{p-2}\mathbf{A}\,,$$

such that

$$\begin{aligned}\mathrm{Tr}\left(D^2\|\mathbf{u}\|_{\mathbb{V}}^p\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u})\right) &\leq p(p-2)\|\mathbf{u}\|_{\mathbb{V}}^{p-4}\|\mathbf{A}\mathbf{u}\|_{\mathbb{L}^2}^2\,\mathrm{Tr}\Big(\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u})\Big)\\ &+p\|\mathbf{u}(s)\|_{\mathbb{V}}^{p-2}\mathrm{Tr}\left(\mathbf{A}\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u})\right).\end{aligned}$$

To obtain inequality (i), we use Itô's formula and assumption (S_3) to get

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{u}(t)\|_{\mathbb{V}}^p + p\nu\int_0^t \|\mathbf{u}(s)\|_{\mathbb{V}}^{p-2}\|\mathbf{A}\mathbf{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\\ &\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^p + \frac{p}{2}\int_0^t \|\mathbf{u}(s)\|_{\mathbb{V}}^{p-2}\mathrm{Tr}(\mathbf{A}\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u}))\,\mathrm{d}s\right]\\ &\quad+\frac{p(p-2)}{2}\mathbb{E}\left[\int_0^t \|\mathbf{u}\|_{\mathbb{V}}^{p-4}\|\mathbf{A}\mathbf{u}\|_{\mathbb{L}^2}^2\,\mathrm{Tr}(\mathbf{g}(\mathbf{u})\mathbf{Q}\mathbf{g}^*(\mathbf{u}))\,\mathrm{d}s\right]\\ &\leq \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^p\right] + K_4\frac{p}{2}\int_0^t \mathbb{E}\left[\|\mathbf{u}(s)\|_{\mathbb{V}}^p\right]\,\mathrm{d}s + \mathbb{E}\left[K_3\frac{p(p-2)}{2}\int_0^t \|\mathbf{u}\|_{\mathbb{V}}^{p-2}\|\mathbf{A}\mathbf{u}\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right].\end{aligned}$$

In particular, after absorbing the the last term of the right-hand side to the left, we obtain the term

$$\mathbb{E}\left[p\left(\nu - \frac{(p-2)}{2}K_3\right)\int_0^t \|\mathbf{u}\|_{\mathbb{V}}^{p-2}\|\mathbf{A}\mathbf{u}\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]$$

which is positive thanks to assumption (S_3), since we have $\nu - 3K_3 > 0$.

For assertion (ii), after the Itô formula we use the Burkholder-Davis-Gundy inequality together with assumption (S_3), following [44, Appendix 1]. □

5.2.3 Hölder continuity of the solution.

Let $\{e^{-t\mathbf{A}}\ ;\ t \geq 0\}$ denote the analytic contraction semigroup in $\mathbb{H}^{p,per} := \{\mathbf{u} \in \mathbb{L}_{per}^p\ ;\ \mathrm{div}\,\mathbf{u} = 0 \text{ weakly}\}$ for $1 < p < \infty$, which is generated by the Stokes operator $\mathbf{A}$ in the case of periodic boundary conditions. There holds $\mathbb{H}^{per} = \mathbb{H}^{per,2}$.

Lemma 5.2.2. *Let $a > 0$, $b \in [0,1]$, and $1 < p < \infty$. There exist constants $C_{a,p}, C_{b,p} > 0$, such that for all $t > 0$ there holds*

(i) $\left\|\mathbf{A}^a e^{-t\mathbf{A}}\right\|_{\mathcal{L}(\mathbb{H}^{per,p})} \leq C_{a,p}\,t^{-a}$,

(ii) $\left\|\mathbf{A}^{-b}\big(\mathbf{Id}_{\mathbb{H}^{per,p}} - e^{-t\mathbf{A}}\big)\right\|_{\mathcal{L}(\mathbb{H}^{per,p})} \leq C_{b,p}\,t^b$.

Proof. Estimate (i) is a consequence of the analyticity of the semigroup. Once we have (i), we can proceed as in the proof of [109, Theorem 2.6.13] to conclude (ii). □

We may now study Hölder continuity properties of strong solutions of (5.1.1)–(5.1.3) in $\mathbb{L}^p$-norms. For this purpose we use tools from semigroup theory (see, e.g. [111, Appendix]).

Lemma 5.2.3. *Suppose (S_1)–(S_4), and $\boldsymbol{\mathcal{H}} = \mathbb{W}_{per}^{1,2}$. For the solution of problem* (5.1.1)–(5.1.3), *with $\mathbf{u}_0 \in L^{2\widetilde{p}}(\Omega;\mathbb{V})$, $2 \leq \widetilde{p} < \infty$, there holds*

$$\text{(i)} \qquad \mathbb{E}\left[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbb{L}^4}^{\widetilde{p}}\right] \leq C\,|t-s|^{\eta\widetilde{p}} \qquad \forall\, 0 < \eta < \frac{1}{2}\,,$$

$$\text{(ii)} \qquad \mathbb{E}\left[\big\|\mathbf{u}(t) - \mathbf{u}(s)\big\|_{\mathbb{V}}^{\widetilde{p}}\right] \leq C\,|t-s|^{\frac{\eta\widetilde{p}}{2}} \qquad \forall\, 0 < \eta < \frac{1}{2}\,.$$

Proof. Due to the regularity estimate given in Lemma 5.2.1, and to [110, Proposition F.0.5, (i)], we have the following representation for the strong solution of (5.1.1)–(5.1.3),

$$\mathbf{u}(t) \;=\; \mathrm{e}^{-t\mathbf{A}}\mathbf{u}_0 + \int_0^t \mathrm{e}^{-(t-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}\mathbf{g}(\mathbf{u})\,\mathrm{d}\mathbf{W}(s)\,.$$

For $t_2 < t_1$, there holds

$$\begin{aligned}
&\mathbf{u}(t_1) - \mathbf{u}(t_2) = \big(\mathrm{e}^{-t_1\mathbf{A}} - \mathrm{e}^{-t_2\mathbf{A}}\big)\mathbf{u}_0 \\
&\quad + \Big(\int_0^{t_1} \mathrm{e}^{-(t_1-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s - \int_0^{t_2} \mathrm{e}^{-(t_2-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s\Big) \\
&\quad + \Big(\int_0^{t_1} \mathrm{e}^{-(t_1-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}\mathbf{g}(\mathbf{u})\,\mathrm{d}\mathbf{W}(s) - \int_0^{t_2} \mathrm{e}^{-(t_2-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}\mathbf{g}(\mathbf{u})\,\mathrm{d}\mathbf{W}(s)\Big) \\
&=: I + II + III\,.
\end{aligned} \tag{5.2.1}$$

First step: proof of (i). We only bound term II in (5.2.1). The control of terms I and III follows from standard estimates; see [111, Proof of Proposition 3.4]. From [54, Lemma 2.2] we have

$$\|\mathbf{A}^{-1/2}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}\cdot\nabla]\mathbf{u}\|_{\mathbb{L}^4} \le C\,\|\mathbf{A}^{1/8}\mathbf{u}\|^2_{\mathbb{L}^4}\,.$$

Using the Sobolev embeddings

$$\mathbb{W}^{1,2}(D) \subset \mathbb{W}^{3/4,2}(D) \subset \mathbb{W}^{1/4,4}(D)\,,$$

we then obtain

$$\|\mathbf{A}^{-1/2}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}\cdot\nabla]\mathbf{u}\|_{\mathbb{L}^4} \le C\,\|\mathbf{A}^{1/2}\mathbf{u}\|^2_{\mathbb{L}^2}\,. \tag{5.2.2}$$

We rearrange terms as follows,

$$\begin{aligned}
II &= \int_0^{t_1} \mathrm{e}^{-(t_1-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s - \int_0^{t_2} \mathrm{e}^{-(t_2-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s \\
&= \int_0^{t_2} \Big(\mathrm{e}^{-(t_1-s)\mathbf{A}} - \mathrm{e}^{-(t_2-s)\mathbf{A}}\Big)\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s \\
&\qquad + \int_{t_2}^{t_1} \mathrm{e}^{-(t_1-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s)\,\mathrm{d}s \\
&= II_a + II_b\,.
\end{aligned}$$

For II_a we have by (5.2.2), Lemma 5.2.2, (i) and (ii),

$$\begin{aligned}
\|II_a\|_{\mathbb{L}^4} &\le \int_0^{t_2} \Big\|\mathbf{A}^{1/2}\Big(\mathrm{e}^{-(t_1-s)\mathbf{A}} - \mathrm{e}^{-(t_2-s)\mathbf{A}}\Big)\Big\|_{\mathcal{L}(\mathbb{H}^{per,4})} \|\mathbf{A}^{-1/2}\mathbf{P}_{\mathbb{H}}[\mathbf{u}\cdot\nabla]\mathbf{u}\|_{\mathbb{L}^4}\,\mathrm{d}s \\
&\le C\int_0^{t_2} \Big\|\mathbf{A}^{1/2}\Big(\mathrm{e}^{-(t_1-s)\mathbf{A}} - \mathrm{e}^{-(t_2-s)\mathbf{A}}\Big)\Big\|_{\mathcal{L}(\mathbb{H}^{per,4})} \|\mathbf{A}^{1/2}\mathbf{u}\|^2_{\mathbb{L}^2}\,\mathrm{d}s \\
&\le C\int_0^{t_2} \Big\|\mathbf{A}^{1/2+\eta}\mathrm{e}^{-(t_2-s)\mathbf{A}}\Big\|_{\mathcal{L}(\mathbb{H}^{per,4})} \Big\|\mathbf{A}^{-\eta}\Big(\mathrm{e}^{-(t_1-t_2)\mathbf{A}} - \mathbf{Id}\Big)\Big\|_{\mathcal{L}(\mathbb{H}^{per,4})} \|\mathbf{A}^{1/2}\mathbf{u}\|^2_{\mathbb{L}^2}\,\mathrm{d}s \\
&\le C\,|t_1-t_2|^{\eta}\int_0^{t_2} \frac{\|\mathbf{A}^{1/2}\mathbf{u}(s)\|^2_{\mathbb{L}^2}}{|t_2-s|^{1/2+\eta}}\,\mathrm{d}s\,.
\end{aligned}$$

Then, for any $\widetilde{p} \geq 1$, and $0 < \eta < \frac{1}{2}$,

$$\begin{aligned}\mathbb{E}\left[\|II_a\|_{\mathbb{L}^4}^{\widetilde{p}}\right]^{1/\widetilde{p}} &\leq C\,|t_1 - t_2|^{\eta}\,\mathbb{E}\left[\left(\int_0^{t_2} \frac{1}{|t_2 - s|^{1/2+\eta}} \|\mathbf{A}^{1/2}\mathbf{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^{\widetilde{p}}\right]^{1/\widetilde{p}} \\ &\leq C\,|t_1 - t_2|^{\eta}\left(\sup_{0\leq t\leq T} \mathbb{E}\left[\|\mathbf{A}^{1/2}\mathbf{u}(t)\|_{\mathbb{L}^2}^{2\widetilde{p}}\right]^{1/\widetilde{p}}\right).\end{aligned}$$

For the second term II_b, by using the same methods, we have

$$\mathbb{E}\left[\|II_b\|_{\mathbb{L}^4}^{\widetilde{p}}\right]^{1/\widetilde{p}} \leq C\,|t_1 - t_2|^{\eta}\left(\sup_{0\leq t\leq T} \mathbb{E}\left[\|\mathbf{A}^{1/2}\mathbf{u}(t)\|_{\mathbb{L}^2}^{2\widetilde{p}}\right]^{1/\widetilde{p}}\right).$$

Second step: proof of (ii). From [54, Lemma 2.2] we have

$$\|\mathbf{A}^{-1/4}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}\cdot\nabla]\mathbf{u}\|_{\mathbb{L}^2} \leq C\,\|\mathbf{A}^{1/2}\mathbf{u}\|_{\mathbb{L}^2}^2\,. \tag{5.2.3}$$

Then, following the computations in the first step, with $\mathbb{H}^{per,4}$ replaced by $\mathbb{H}^{per}$, and using (5.2.3), we have

$$\begin{aligned}\|II_a\|_{\mathbb{L}^2} &\leq \int_0^{t_2} \left\|\mathbf{A}^{3/4}\left(\mathrm{e}^{-(t_1-s)\mathbf{A}} - \mathrm{e}^{-(t_2-s)\mathbf{A}}\right)\right\|_{\mathcal{L}(\mathbb{H}^{per})} \|\mathbf{A}^{-1/4}\mathbf{P}_{\mathbb{H}^{per}}[\mathbf{u}\cdot\nabla]\mathbf{u}\|_{\mathbb{L}^2}\,\mathrm{d}s \\ &\leq C\,|t_1 - t_2|^{\eta/2}\int_0^{t_2} \frac{\|\mathbf{A}^{1/2}\mathbf{u}(s)\|_{\mathbb{L}^2}^2}{|t_2 - s|^{3/4+\eta/2}}\,\mathrm{d}s\,.\end{aligned}$$

This estimate leads to

$$\mathbb{E}\left[\|II_a\|_{\mathbb{L}^2}^{\widetilde{p}}\right]^{1/\widetilde{p}} \leq C\,|t_1 - t_2|^{\eta/2}\left(\sup_{0\leq t\leq T} \mathbb{E}\left[\|\mathbf{A}^{1/2}\mathbf{u}(t)\|_{\mathbb{L}^2}^{2\widetilde{p}}\right]^{1/\widetilde{p}}\right),$$

and, in the same way we obtain

$$\mathbb{E}\left[\|II_b\|_{\mathbb{L}^2}^{\widetilde{p}}\right]^{1/\widetilde{p}} \leq C\,|t_1 - t_2|^{\eta/2}\left(\sup_{0\leq t\leq T} \mathbb{E}\left[\|\mathbf{A}^{1/2}\mathbf{u}(t)\|_{\mathbb{L}^2}^{2\widetilde{p}}\right]^{1/\widetilde{p}}\right).$$

□

5.2.4 Discretisation in space

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain $D \subset \mathbb{R}^2$, using triangles of maximal diameter $h > 0$, and $\overline{D} = \bigcup_{K\in\mathcal{T}} \overline{K}$. Let $\mathbb{P}_i(K) \equiv [P_i(K)]^2$ be the space of polynomial vector fields on K of degree less or equal to i. We introduce finite element function spaces ($i, j \in \mathbb{N}_0$)

$$\begin{aligned}\mathbb{H}_h &:= \left\{\mathbf{U} \in \mathbb{C}^0(\overline{D}) \cap \mathbb{W}^{1,2}(D) : \mathbf{U} \in \mathbb{P}_i(K) \quad \forall\, K \in \mathcal{T}_h\right\}, \\ L_h &:= \left\{\Pi \in L_0^2(D) : \Pi \in P_j(K) \quad \forall\, K \in \mathcal{T}_h\right\},\end{aligned}$$

which satisfy the discrete LBB-condition (5.1.6), and the space $\mathbb{V}_h$ is defined in (5.1.7). We denote by $\mathbf{Q}_h^0 : \mathbb{L}^2 \to \mathbb{V}_h$ the $\mathbb{L}^2$-orthogonal projection,

$$\left(\mathbf{z} - \mathbf{Q}_h^0\mathbf{z}, \mathbf{\Phi}\right) = 0 \quad \forall\,\mathbf{\Phi} \in \mathbb{V}_h\,. \tag{5.2.4}$$

The following estimates are standard, see for instance [70],

$$\|\mathbf{z}-\mathbf{Q}_h^0\mathbf{z}\|_{\mathbb{L}^2}+h\,\|\nabla\left(\mathbf{z}-\mathbf{Q}_h^0\mathbf{z}\right)\|_{\mathbb{L}^2} \leq Ch^2\,\|\mathbf{A}\mathbf{z}\|_{\mathbb{L}^2} \quad \forall\,\mathbf{z}\in\mathbb{V}^{per}\cap\mathbb{W}^{2,2}(D)\,, \tag{5.2.5}$$

$$\|\mathbf{z}-\mathbf{Q}_h^0\mathbf{z}\|_{\mathbb{L}^2} \leq Ch\,\|\nabla\mathbf{z}\|_{\mathbb{L}^2} \quad \forall\,\mathbf{z}\in\mathbb{V}^{per}\cap\mathbb{W}^{1,2}(D)\,. \tag{5.2.6}$$

We need the following stability estimates from [59, Lemma 3.1],

$$\|\mathbf{Q}_h^0\mathbf{v}\|_{\mathbb{L}^2} \leq C\,\|\mathbf{v}\|_{\mathbb{L}^2} \quad \forall\,\mathbf{v}\in\mathbb{L}^2\,, \tag{5.2.7}$$

$$\|\nabla\mathbf{Q}_h^0\mathbf{v}\|_{\mathbb{L}^2} \leq C\,\|\nabla\mathbf{v}\|_{\mathbb{L}^2} \quad \forall\,\mathbf{v}\in\mathbb{V}^{per}\,. \tag{5.2.8}$$

We recall the inverse inequality, which holds for all $\mathbf{\Phi}\in\mathbb{H}_h$,

$$\|\mathbf{\Phi}\|_{\mathbb{W}^{\ell,q_1}} \leq C\,h^{m-\ell+d\min\{\frac{1}{q_1}-\frac{1}{q_2},0\}}\,\|\mathbf{\Phi}\|_{\mathbb{W}^{m,q_2}} \qquad \forall\,1\leq q_1,q_2\leq\infty\,,\quad 0\leq m\leq\ell\,.$$

5.2.5 Convergence in probability

We recall the definition of convergence in probability with rates from [111, Definition 2.7].

Definition 5.2.2. *Let X be a Banach space, and $I_k=\{t_m\}_{m=1}^M$ an equi-distant mesh covering $[0,T]$, where $k=t_m-t_{m-1}$ for all $m=1,\dots,M$ is the time-step. Let $\{u^m\}_{m=1}^M\subset L^2(\Omega;X)$ be a sequence of X-valued random variables, and u be a stochastic process in $L^2(\Omega;C([0,T];X))$. We say that $\{u^m\}_{m=1}^M$ converges in probability with rate $\alpha>0$ towards $\{u(t_m)\}_{m=1}^M$, if*

$$\lim_{C\to+\infty}\lim_{k\to 0}\mathbb{P}\left[\max_{1\leq m\leq M}\|u^m-u(t_m)\|_X\geq C\,k^\alpha\right]=0\,.$$

For a comparison of the various types of convergence, we refer to [111, Lemma 2.8].

5.3 Convergence with rates of time discretisation schemes

In this section, we consider different time discretisation schemes on the equi-distant mesh $I_k:=\{t_m\}_{m=1}^M$ covering $[0,T]$ with mesh-size $k=T/M>0$, where $t_0=0$, and $t_M=T$.

Algorithm 5.1. *Let $\mathbf{u}^0:=\mathbf{u}_0$ be a given $\mathbb{V}^{per}$-valued random variable. Find for every $m\in\{1,\dots,M\}$ a tuple of random variables $(\mathbf{u}^m,\pi^m)$ with values in $\mathbb{V}^{per}\times L^2_{per}$, such that $\mathbb{P}$-almost surely*

$$\begin{aligned}(\mathbf{u}^m-\mathbf{u}^{m-1},\boldsymbol{\phi})+k\nu\,(\nabla\mathbf{u}^m,\nabla\boldsymbol{\phi})+&\\ k\big([\mathbf{u}^m\cdot\nabla]\mathbf{u}^m,\boldsymbol{\phi}\big)-k(\pi^m,\operatorname{div}\boldsymbol{\phi}) &= \big(\mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W},\boldsymbol{\phi}\big)\,,\end{aligned} \tag{5.3.1}$$

$$(\operatorname{div}\mathbf{u}^m,\psi) = 0\,, \tag{5.3.2}$$

for all $\boldsymbol{\phi}\in\mathbb{W}^{1,2}_{per}$ and $\psi\in L^2_{per}$.

Here $\Delta_m\mathbf{W}=\mathbf{W}(t_m)-\mathbf{W}(t_{m-1})\sim\mathcal{N}(0,k\mathbf{Q})$, with uniform mesh size $k:=t_{m+1}-t_m$. The existence of pressure iterates follows by de Rham theory ([125, p. 10]).

To analyse the error behavior of Algorithm 5.1, we reformulate equations (5.3.1) and (5.3.2) in the following form:

$$\begin{aligned}(\mathbf{u}^m-\mathbf{u}^{m-1},\boldsymbol{\phi})+\nu k(\nabla\mathbf{u}^m,\nabla\boldsymbol{\phi})&\\ +k\Big([\mathbf{u}^m\cdot\nabla]\mathbf{u}^m,\boldsymbol{\phi}\Big) &= \big(\mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W},\boldsymbol{\phi}\big) \quad \forall\,\boldsymbol{\phi}\in\mathbb{V}^{per},\quad \mathbb{P}-a.s.\end{aligned} \tag{5.3.3}$$

The stability of solutions $\{\mathbf{u}^m\}_{m=1}^M\subset L^2(\Omega;\mathbb{V}^{per})$ is studied in the following lemma.

Lemma 5.3.1. *Let* $\mathbf{u}^0 \in L^{2^q}\big(\Omega; \mathbb{V}^{per}\big)$ *for an integer* $1 \leq q < \infty$ *be given, such that* $\mathbb{E}\big[\|\mathbf{u}^0\|_{\mathbb{V}}^{2^q}\big] \leq C$*. Assume that (S*$_1$*)–(S*$_4$*) are valid, with* $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$*. Then* $\{\mathbf{u}^m\}_{m=1}^M \subset L^2(\Omega; \mathbb{V}^{per})$ *from Algorithm 5.1 satisfies*

$$\text{(i)} \qquad \mathbb{E}\Big[\max_{1\leq m\leq M} \|\mathbf{u}^m\|_{\mathbb{V}}^{2^q} + 2\nu k \sum_{m=1}^M \|\mathbf{u}^m\|_{\mathbb{V}}^{2^q-2}\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2\Big] \leq C_{t_M,q}\,,$$

$$\text{(ii)} \qquad \mathbb{E}\left[\sum_{m=1}^M \|\mathbf{u}^m - \mathbf{u}^{m-1}\|_{\mathbb{L}^2}^2\|\mathbf{u}^m\|_{\mathbb{L}^2}^2 + \sum_{m=1}^M \|\nabla(\mathbf{u}^m - \mathbf{u}^{m-1})\|_{\mathbb{L}^2}^2\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2\right] \leq C_{t_M,2}\,,$$

$$\text{(iii)} \qquad \mathbb{E}\left[\left(\sum_{m=1}^M \|\mathbf{u}^m - \mathbf{u}^{m-1}\|_{\mathbb{V}}^2\right)^4 + \left(k\sum_{m=1}^M \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{V}}^2\right)^4\right] \leq C_{t_M,3}\,,$$

where $C_{t_M,q} \equiv C_{t_M,q}\big(\operatorname{Tr}\mathbf{Q}, \mathbb{E}[\|\mathbf{u}^0\|_{\mathbb{V}}^{2^q}]\big) > 0$ *does not depend on* $k, h > 0$*. Moreover, the iterates* $\mathbf{u}^m$ *are* $\mathcal{F}_{t_m}$*-measurable.*

Proof. The measurability, assertion (i), the second part of (ii), and (iii) may be shown as in the proof of Lemma 3.3.1, using $\boldsymbol{\phi} = \mathbf{A}\mathbf{u}^m$ in (5.3.3). The first part of (ii) may be found in Lemma 3.3.1. □

In case of a semi-implicit treatment of the nonlinearity, we do not get the improved space regularity, since in general

$$\Big([\mathbf{u}^{m-1}\cdot\nabla]\mathbf{u}^m, \mathbf{A}\mathbf{u}^m\Big) \neq 0\,. \tag{5.3.4}$$

The next lemma, which is essential for the study of the splitting scheme and the finite element discretisation, shows that to obtain uniform bounds for the pressure iterates crucially depends on the noise used in Algorithm 5.1.

Lemma 5.3.2. *Let* $2 \leq q < \infty$*. Consider the iterates* $\{\mathbf{u}^m, \pi^m\}_{m=1}^M$ *from Algorithm 5.1.*

(i) *For* $\boldsymbol{\mathcal{H}} = \mathbb{V}^{per}$ *in (S*$_2$*), there holds*

$$\mathbb{E}\left[k\sum_{m=1}^M \|\nabla\pi^m\|_{\mathbb{L}^2}^2\right] \leq C_{t_M,2}\,. \tag{5.3.5}$$

(ii) *For* $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$ *in (S*$_2$*), there holds*

$$\mathbb{E}\left[k\sum_{m=1}^M \|\nabla\pi^m\|_{\mathbb{L}^2}^2\right] \leq C_{t_M,2}k^{-1}\,. \tag{5.3.6}$$

Proof. (i) Let $\boldsymbol{\mathcal{H}} = \mathbb{V}^{per}$. We formally multiply the strong form of (5.3.1) with $\nabla\pi^m$. Then we use $\big(\mathbf{g}^{m-1}\Delta_m\mathbf{W}, \nabla\pi^m\big) = 0$ to conclude

$$\begin{aligned} k\,\|\nabla\pi^m\|_{\mathbb{L}^2}^2 &\leq k\,\|\Delta\mathbf{u}^m\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2} + k\,\|[\mathbf{u}^m\cdot\nabla]\mathbf{u}^m\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2} \\ &\leq Ck\Big(\|\Delta\mathbf{u}^m\|_{\mathbb{L}^2}^2 + \|[\mathbf{u}^m\cdot\nabla]\mathbf{u}^m\|_{\mathbb{L}^2}^2\Big) + \frac{k}{2}\,\|\nabla\pi^m\|_{\mathbb{L}^2}^2\,. \end{aligned} \tag{5.3.7}$$

Since

$$\|[\mathbf{u}^m\cdot\nabla]\mathbf{u}^m\|_{\mathbb{L}^2}^2 \leq C\,\|\mathbf{u}^m\|_{\mathbb{L}^2}\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2\|\Delta\mathbf{u}^m\|_{\mathbb{L}^2}\,,$$

we may conclude the assertion from Lemma 5.3.1, (i).

(ii) Let $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$. We proceed accordingly to obtain

$$\begin{aligned} k\|\nabla\pi^m\|^2_{\mathbb{L}^2} &\leq k\,\|\Delta\mathbf{u}^m\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2} \\ &\quad +k\,\|[\mathbf{u}^m\cdot\nabla]\mathbf{u}^m\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2} + \|\mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W}\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2}\,. \end{aligned} \tag{5.3.8}$$

We only have to estimate the last term on the right-hand side. We use Assumption (S_2), and the tower property for martingales to compute

$$\begin{aligned} \mathbb{E}\left[\|\mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W}\|_{\mathbb{L}^2}\|\nabla\pi^m\|_{\mathbb{L}^2}\right] &\leq \frac{C}{k}\mathbb{E}\left[\|\mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W}\|^2_{\mathbb{L}^2}\right] + \frac{k}{8}\,\mathbb{E}\left[\|\nabla\pi^m\|^2_{\mathbb{L}^2}\right] \\ &\leq C\,\mathbb{E}\left[\|\mathbf{u}^{m-1}\|^2_{\mathbb{L}^2}\right] + \frac{k}{8}\,\mathbb{E}\left[\|\nabla\pi^m\|^2_{\mathbb{L}^2}\right]. \end{aligned}$$

By Lemma 5.3.1, (i), we then infer

$$\mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla\pi^m\|^2_{\mathbb{L}^2}\right] \leq \frac{C_{t_M}}{k}\,.$$

□

The remainder of this section is as follows: Rates of convergence for iterates of Algorithm 5.1 are established in Section 5.3.1 below. Then we consider Algorithm 5.2 which uses a semi-implicit discretisation of the convection term. Here, we use the stability of the fully implicit scheme to show corresponding error estimates.

5.3.1 Fully implicit time discretisation

Let $\mathbf{u} \equiv \{\mathbf{u}(t)\ ;\ t\in[0,T]\}$ be the strong solution to equations (5.1.1)–(5.1.3). Define the error $\mathbf{e}^m = \mathbf{u}(t_m) - \mathbf{u}^m$. We subtract the equation (5.3.3) from the equation in Definition 5.2.1, (iii), to get

$$\begin{aligned} &(\mathbf{e}^m - \mathbf{e}^{m-1},\boldsymbol{\phi}) + \nu\int_{t_{m-1}}^{t_m}\big(\nabla\big(\mathbf{u}(s)-\mathbf{u}^m\big),\nabla\boldsymbol{\phi}\big)\,\mathrm{d}s + \int_{t_{m-1}}^{t_m}\Big(\big[\mathbf{u}(s)\cdot\nabla\big]\mathbf{u}(s),\boldsymbol{\phi}\Big)\,\mathrm{d}s \\ &-\int_{t_{m-1}}^{t_m}\Big([\mathbf{u}^m\cdot\nabla]\mathbf{u}^m,\boldsymbol{\phi}\Big)\,\mathrm{d}s = \Big(\int_{t_{m-1}}^{t_m}\mathbf{g}(\mathbf{u})\,\mathrm{d}\mathbf{W} - \mathbf{g}(\mathbf{u}^{m-1})\Delta_m\mathbf{W},\boldsymbol{\phi}\Big) \quad \forall\boldsymbol{\phi}\in\mathbb{V}^{per}\,. \end{aligned} \tag{5.3.9}$$

In this section we prove the first main result, Theorem 5.3.1, in three steps: in the first we perform an error analysis on the whole set Ω. In the second step we introduce sample subsets with asymptotic probability one of Ω, in order to handle the stochastic integral and the terms arising from the nonlinearity. In the last step we apply a Gronwall argument on the sample subsets to obtain an error bound

Step 1. *A preparatory error analysis on* Ω. Set $\boldsymbol{\phi} = \mathbf{e}^m$ and consider the diffusion term

$$\Big(\nabla\big(\mathbf{u}(s)-\mathbf{u}^m\big),\nabla\mathbf{e}^m\Big) = \|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2} + \Big(\nabla\big(\mathbf{u}(s)-\mathbf{u}(t_m)\big),\nabla\mathbf{e}^m\Big)\,.$$

For the second term, thanks to Lemma 5.2.3, (ii), we compute

$$\begin{aligned} &\int_{t_{m-1}}^{t_m}\mathbb{E}\Big[-\Big(\nabla\big(\mathbf{u}(s)-\mathbf{u}(t_m)\big),\nabla\mathbf{e}^m\Big)\Big]\,\mathrm{d}s \\ &\leq \int_{t_{m-1}}^{t_m}\mathbb{E}\Big[C_{\delta_1}\big\|\nabla\big(\mathbf{u}(s)-\mathbf{u}(t_m)\big)\big\|^2_{\mathbb{L}^2} + \delta_1\|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2}\Big]\,\mathrm{d}s \\ &\leq C_{\delta_1}k^{1+\eta} + \delta_1 k\,\mathbb{E}\left[\|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2}\right] \quad (\delta_1>0)\,. \end{aligned} \tag{5.3.10}$$

Then, for the convection term, on using the skew-symmetricity, there holds

$$
\begin{aligned}
&\int_{t_{m-1}}^{t_m} \Big([\mathbf{u}^m \cdot \nabla]\mathbf{u}^m - \big[\mathbf{u}(s)\cdot\nabla]\mathbf{u}(s), \mathbf{e}^m \Big)\,\mathrm{d}s \\
&\qquad = \int_{t_{m-1}}^{t_m} \Big\{ \Big([\mathbf{e}^m \cdot \nabla]\mathbf{u}^m, \mathbf{e}^m \Big) - \Big([(\mathbf{u}(t_m) - \mathbf{u}(s))\cdot\nabla]\mathbf{e}^m, \mathbf{u}(t_m) \Big) \Big\}\,\mathrm{d}s \\
&\qquad\quad - \int_{t_{m-1}}^{t_m} \Big([\mathbf{u}(s)\cdot\nabla]\mathbf{e}^m, \mathbf{u}(t_m) - \mathbf{u}(s) \Big)\,\mathrm{d}s = \int_{t_{m-1}}^{t_m} \big(I + II + III \big)\,\mathrm{d}s\,.
\end{aligned} \tag{5.3.11}
$$

Term I can be bounded as follows,

$$
\mathbb{E}\,[I] \le C_{\delta_2}\,\mathbb{E}\left[\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2 \|\mathbf{e}^m\|_{\mathbb{L}^2}^2\right] + \delta_2\,\mathbb{E}\left[\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2\right] \quad (\delta_2 > 0)\,. \tag{5.3.12}
$$

For term II, we find

$$
\begin{aligned}
II &\le \|\mathbf{u}(t_m) - \mathbf{u}(s)\|_{\mathbb{L}^4} \|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2} \|\mathbf{e}^m\|_{\mathbb{L}^4} \\
&\le C\,\|\mathbf{u}(t_m) - \mathbf{u}(s)\|_{\mathbb{L}^4} \|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2} \|\mathbf{e}^m\|_{\mathbb{L}^2}^{1/2} \|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^{1/2} \\
&\le C\,\|\mathbf{u}(t_m) - \mathbf{u}(s)\|_{\mathbb{L}^4}^2 \|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2 + C_{\delta_3}\,\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 + \delta_3\,\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2 \quad (\delta_3 > 0)\,.
\end{aligned} \tag{5.3.13}
$$

Then, using Lemmata 5.2.3, (i) and 5.2.1, (i), for the first term leads to

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_{m-1}}^{t_m} \|\mathbf{u}(t_m) - \mathbf{u}(s)\|_{\mathbb{L}^4}^2 \|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] \\
&\le \int_{t_{m-1}}^{t_m} \mathbb{E}\left[\|\mathbf{u}(t_m) - \mathbf{u}(s)\|_{\mathbb{L}^4}^4\right]^{1/2} \mathbb{E}\left[\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^4\right]^{1/2}\,\mathrm{d}s \\
&\le C\,k^{1+2\eta} \Big(\mathbb{E}\left[\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^4\right]^{1/2} \Big) \le C\,k^{1+2\eta}\,.
\end{aligned} \tag{5.3.14}
$$

For the last inequality, we use Lemma 5.2.3, (i) and the boundedness of the eighth moment of the initial data. Accordingly, we may conclude that

$$
\mathbb{E}\left[\int_{t_{m-1}}^{t_m} III\,\mathrm{d}s\right] \le C_{\delta_3}\,k^{1+2\eta} + \delta_3\,k\,\mathbb{E}\left[\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2\right] \quad (\delta_4 > 0)\,. \tag{5.3.15}
$$

To bound the stochastic integral in (5.3.9), we divide it into two terms.

$$
\begin{aligned}
&\mathbb{E}\left[\max_{1\le n\le M} \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \Big(\big[\mathbf{g}\big(\mathbf{u}(s)\big) - \mathbf{g}(\mathbf{u}^{m-1})\big]\,\mathrm{d}\mathbf{W}(s), \mathbf{e}^m \Big)\right] \\
&\quad = \mathbb{E}\left[\max_{1\le n\le M} \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \Big(\big[\mathbf{g}\big(\mathbf{u}(s)\big) - \mathbf{g}(\mathbf{u}^{m-1})\big]\,\mathrm{d}\mathbf{W}(s), \mathbf{e}^{m-1} \Big)\right] \\
&\qquad + \mathbb{E}\left[\max_{1\le n\le M} \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \Big(\big[\mathbf{g}\big(\mathbf{u}(s)\big) - \mathbf{g}(\mathbf{u}^{m-1})\big]\,\mathrm{d}\mathbf{W}(s), \mathbf{e}^m - \mathbf{e}^{m-1} \Big)\right] = IV + V\,.
\end{aligned}
$$

For IV, we use the Burkholder-Davis-Gundy inequality, condition (S_2), Young's inequality, and Lemma 5.2.3, (i), to get

$$\begin{aligned}
IV &\leq C\,\mathbb{E}\left[\left(\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\|^2_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)}\|\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\mathrm{d}s\right)^{1/2}\right]\\
&\leq C\,\mathbb{E}\left[\max_{1\leq m\leq M}\|\mathbf{e}^{m-1}\|_{\mathbb{L}^2}\left(\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\|^2_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)}\mathrm{d}s\right)^{1/2}\right] \qquad (5.3.16)\\
&\leq C\,\mathbb{E}\Bigg[\max_{1\leq m\leq M}\|\mathbf{e}^{m}\|_{\mathbb{L}^2}\Bigg(\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\Big[\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}\big(\mathbf{u}(t_{m-1})\big)\|^2_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)}\\
&\qquad\qquad +K_2\,\|\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\Big]\mathrm{d}s\Bigg)^{1/2}\Bigg]\\
&\leq \delta_4\,\mathbb{E}\left[\max_{1\leq m\leq M}\|\mathbf{e}^{m}\|^2_{\mathbb{L}^2}\right]+C_{\delta_4}\,k\sum_{m=1}^{M}\Big(k^{2\eta}+\mathbb{E}\left[\|\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\right]\Big)\,,
\end{aligned}$$

while for the second we have ($\delta_5 > 0$)

$$\begin{aligned}
V &\leq \mathbb{E}\left[\sum_{m=1}^{M}\left(C_{\delta_5}\left\|\int_{t_{m-1}}^{t_m}\Big(\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\Big)\,\mathrm{d}\mathbf{W}(s)\right\|^2_{\mathbb{L}^2}+\delta_5\,\|\mathbf{e}^m-\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\right)\right]\\
&= \mathbb{E}\left[\sum_{m=1}^{M}\left(C_{\delta_5}\int_{t_{m-1}}^{t_m}\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\|^2_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)}\,\mathrm{d}s+\delta_5\,\|\mathbf{e}^m-\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\mathrm{d}s\right)\right] \qquad (5.3.17)\\
&\leq C_{\delta_5}\,k\sum_{m=1}^{M}\Big(k^{2\eta}+\mathbb{E}\big[\|\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\big]\Big)+\delta_5\,\mathbb{E}\left[\sum_{m=1}^{M}\|\mathbf{e}^m-\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}\right].
\end{aligned}$$

Here, we use Itô's isometry for the equality, and for the subsequent inequality the estimate

$$\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\|^2_{\mathcal{I}_2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{K}}),\mathbb{L}^2)}\leq\Big(\mathrm{Tr}\,\mathbf{Q}\Big)\,\|\mathbf{g}\big(\mathbf{u}(s)\big)-\mathbf{g}(\mathbf{u}^{m-1})\|^2_{\mathcal{L}(\boldsymbol{\mathcal{K}},\mathbb{L}^2)}\,,$$

condition (S_2), and Lemma 5.2.3, (i).

Step 2. *Introduction of sample subsets.* After these preparations, we are ready for the main argument of the error analysis which deals with the inherent nonlinear effects. By summing up (5.3.9) with $\boldsymbol{\phi}=\mathbf{e}^m$ from $m=1$ to M, and using the identity $(\mathbf{a},\mathbf{a}-\mathbf{b})=\frac{1}{2}\big(\|\mathbf{a}\|^2-\|\mathbf{b}\|^2+\|\mathbf{a}-\mathbf{b}\|^2\big)$, as well as considering the estimates (5.3.12), (5.3.13) (without expectation), and absorbing on the left-hand side corresponding terms, we obtain

$$\begin{aligned}
&\max_{1\leq n\leq m}\left[\|\mathbf{e}^n\|^2_{\mathbb{L}^2}+\frac{1}{2}\sum_{\ell=1}^{n}\|\mathbf{e}^\ell-\mathbf{e}^{\ell-1}\|^2_{\mathbb{L}^2}+\frac{\nu}{2}k\sum_{\ell=1}^{n}\|\nabla\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\right]\leq C\,k\sum_{\ell=1}^{m}(\|\nabla\mathbf{u}^\ell\|^2_{\mathbb{L}^2}+1)\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\\
&\qquad +C\,k\sum_{\ell=1}^{m}\Big[\|\nabla\Big(\mathbf{u}(t_\ell)-\mathbf{u}(t_{\ell-1})\Big)\|^2_{\mathbb{L}^2}+\|\mathbf{u}(t_\ell)-\mathbf{u}(t_{\ell-1})\|^2_{\mathbb{L}^4}\|\nabla\mathbf{u}(t_\ell)\|^2_{\mathbb{L}^2}\Big]+M_m \qquad (5.3.18)
\end{aligned}$$

$\mathbb{P}$-almost surely, where

$$M_m=\max_{1\leq n\leq m}\sum_{\ell=1}^{n}\Big(\int_{t_{\ell-1}}^{t_\ell}\mathbf{g}\big(\mathbf{u}(s)\big)\,\mathrm{d}\mathbf{W}(s)-\mathbf{g}(\mathbf{u}^{\ell-1})\Delta_\ell\mathbf{W},\mathbf{e}^\ell\Big)\,.$$

Here, the discrete Gronwall inequality may not be used since the application of Lemma 4.2.1, (i) only refers to expectations of $\{\|\nabla \mathbf{u}^\ell\|^2_{\mathbb{L}^2}\}_{\ell=1}^M$.

To overcome this problem, we consider a subset

$$\widetilde{\Omega}^1_{\kappa,m} = \Big\{\omega \in \Omega \ \Big| \ \max_{1\le \ell\le m} \|\nabla \mathbf{u}^\ell\|^2_{\mathbb{L}^2} \le \kappa\Big\} \subset \Omega \qquad (\kappa > 0)\,. \tag{5.3.19}$$

Thus, Markov's inequality yields that

$$\mathbb{P}\left[\max_{1\le\ell\le M} \|\nabla \mathbf{u}^\ell\|^2_{\mathbb{L}^2} \le \kappa\right] \ge 1 - \frac{\mathbb{E}\left[\max_{1\le\ell\le M}\|\nabla \mathbf{u}^\ell\|^2_{\mathbb{L}^2}\right]}{\kappa} \quad \forall\, \kappa > 0\,, \tag{5.3.20}$$

which is close to one thanks to Lemma 4.2.1, $(\mathrm{i})_1$.

Now consider the error inequality (5.3.18) for some $1 \le \ell \le M$, multiply it by $\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}}$, sum over the index from $\ell = 1$ to n, take the maximum between 1 and $m \le M$, and then the expectation. The choice of this indicator function is necessary such that the term corresponding to the stochastic integral is a martingale, which allows the use of the Burkholder-Davis-Gundy inequality. So we obtain for the first two terms on the left-hand side of (5.3.18)

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\le n\le m} \sum_{\ell=1}^n \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}} (\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2} - \|\mathbf{e}^{\ell-1}\|^2_{\mathbb{L}^2})\right]\\
&= \mathbb{E}\left[\max_{1\le n\le m}\left(\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|^2_{\mathbb{L}^2} - \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,0}}\|\mathbf{e}^0\|^2_{\mathbb{L}^2} + \sum_{\ell=2}^n (\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-2}} - \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}})\|\mathbf{e}^{\ell-1}\|^2_{\mathbb{L}^2}\right)\right]\\
&\ge \mathbb{E}\left[\max_{1\le n\le m} \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|^2_{\mathbb{L}^2}\right]\,,
\end{aligned} \tag{5.3.21}$$

where we use the fact that the sum in the second line is positive because $\widetilde{\Omega}^1_{\kappa,m} \subset \widetilde{\Omega}^1_{\kappa,m-1}$, and that $\mathbf{e}^0 = \mathbf{0}$ $\mathbb{P}$-almost surely. The next term to be considered corresponds to IV from (5.3.16). Recall that now we have the stochastic integral

$$\widetilde{M}_m = \max_{1\le n\le m} \sum_{\ell=1}^n \Big(\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \mathbf{g}\big(\mathbf{u}(s)\big)\, \mathrm{d}\mathbf{W}(s) - \mathbf{g}(\mathbf{u}^{\ell-1})\Delta_\ell \mathbf{W}, \mathbf{e}^\ell\Big)\,.$$

The bound corresponding to (5.3.16) now reads

$$\begin{aligned}
\widetilde{IV} \ &\le\ C\,\mathbb{E}\left[\left(\sum_{\ell=1}^m \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\mathbf{g}\big(\mathbf{u}(s)\big) - \mathbf{g}(\mathbf{u}^{\ell-1})\|^2_{\mathcal{L}_2(\boldsymbol{\mathcal{K}}_0,\mathbb{L}^2)} \|\mathbf{e}^{\ell-1}\|^2_{\mathbb{L}^2} \mathrm{d}s\right)^{1/2}\right]\\
&\le\ C\,\mathbb{E}\left[\max_{1\le n\le m} \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^{n-1}\|_{\mathbb{L}^2}\left(\sum_{\ell=1}^m \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\mathbf{g}\big(\mathbf{u}(s)\big) - \mathbf{g}(\mathbf{u}^{\ell-1})\|^2_{\mathcal{L}_2(\boldsymbol{\mathcal{K}}_0,\mathbb{L}^2)} \mathrm{d}s\right)^{1/2}\right]\\
&\le\ \delta_4\,\mathbb{E}\left[\max_{1\le n\le m} \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|^2_{\mathbb{L}^2}\right] + C_{\delta_4}\, k \sum_{\ell=1}^m \Big(k^{2\eta} + \mathbb{E}\Big[\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\Big]\Big) \qquad (\delta_4 > 0)\,,
\end{aligned}$$

where we use the Burkholder-Davis-Gundy inequality, and, in the last inequality, the fact that $\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n}}\|\mathbf{e}^n\|_{\mathbb{L}^2} \le \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|_{\mathbb{L}^2}$ for every $1 \le n \le M$. The term corresponding to V may be dealt with as before.

We split the first term on the right-hand side of (5.3.18), in order to get advantage of the set defined in (5.3.23) below. We obtain

$$C\,k \sum_{\ell=1}^m \|\nabla \mathbf{u}^\ell\|^2_{\mathbb{L}^2}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2} \le Ck\sum_{\ell=1}^m \|\nabla(\mathbf{u}^\ell - \mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2} + Ck\sum_{\ell=1}^m \|\nabla \mathbf{u}^{\ell-1}\|^2_{\mathbb{L}^2}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\,.$$

Then using estimates (5.3.21), (5.3.12), (5.3.10), (5.3.14), (5.3.15), and the bound for $\widetilde{IV}$, we conclude

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\leq n\leq m}\left(\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|^2_{\mathbb{L}^2}+\frac{1}{2}\sum_{\ell=1}^n \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}}\|\mathbf{e}^\ell-\mathbf{e}^{\ell-1}\|^2_{\mathbb{L}^2}+\frac{\nu}{2}k\sum_{\ell=1}^n \mathbb{1}_{\widetilde{\Omega}^1_{\kappa,\ell-1}}\|\nabla\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\right)\right]\\
(5.3.22)\ &\leq C\,\kappa\,k\sum_{\ell=1}^m \mathbb{E}\left[\max_{1\leq n\leq \ell}\mathbb{1}_{\widetilde{\Omega}^1_{\kappa,n-1}}\|\mathbf{e}^n\|^2_{\mathbb{L}^2}\right]+C\big(k^\eta+k^{2\eta}\big)\\
&\quad +Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\right].
\end{aligned}$$

Recall that the order-limiting term Ck^η is a consequence of (5.3.10), which itself follows from the Hölder regularity properties in Lemma 5.2.3. We compute

$$Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\mathbf{e}^\ell\|^2_{\mathbb{L}^2}\right]\leq Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}(\|\mathbf{u}^\ell\|^2_{\mathbb{L}^2}+\|\mathbf{u}(t_\ell)\|^2_{\mathbb{L}^2})\right].$$

To control the first term, we use Poincaré inequality to obtain

$$Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\mathbf{u}^\ell\|^2_{\mathbb{L}^2}\right]\leq Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\nabla\mathbf{u}^\ell\|^2_{\mathbb{L}^2}\right]\leq Ck\,,$$

thanks to Lemma 4.2.1, (ii). For the second, we use Hölder's inequality and Lemma 4.2.1, (iii)

$$\begin{aligned}
&Ck\sum_{\ell=1}^m \mathbb{E}\left[\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\|\mathbf{u}(t_\ell)\|^2_{\mathbb{L}^2}\right]\\
&\leq Ck\mathbb{E}\left[\max_{1\leq\ell\leq m}\|\mathbf{u}(t_\ell)\|^4_{\mathbb{L}^2}\right]^{1/2}\mathbb{E}\left[\left(\sum_{\ell=1}^m\|\nabla(\mathbf{u}^\ell-\mathbf{u}^{\ell-1})\|^2_{\mathbb{L}^2}\right)^2\right]^{1/2}\leq Ck\,.
\end{aligned}$$

The first term is bounded because of Lemma 5.3.1, (i), and the last term can be bounded with Lemma 5.3.1, (iii).

Step 3. *Gronwall argument.* Consider the following subset and inclusion,

$$(5.3.23)\qquad \widetilde{\Omega}^1_\kappa=\left\{\omega\in\Omega\ \Big|\ \max_{1\leq m\leq M}\|\nabla\mathbf{u}^m\|^2_{\mathbb{L}^2}\leq\kappa\right\}\subset\widetilde{\Omega}^1_{\kappa,m}\qquad(\kappa>0,\ m\leq M)\,.$$

The discrete Gronwall inequality then leads to

$$\mathbb{E}\Big[\mathbb{1}_{\widetilde{\Omega}^1_\kappa}\max_{1\leq n\leq M}\left(\|\mathbf{e}^n\|^2_{\mathbb{L}^2}+\frac{1}{2}\sum_{m=1}^n\|\mathbf{e}^m-\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}+\frac{\nu}{2}k\sum_{m=1}^n\|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2}\right)\Big]\leq C\mathrm{e}^{Ct_M\kappa}\,k^\eta\,,$$

provided that $C\kappa k<1$. With the constant $C>0$ from (5.3.22), we set

$$\kappa:=C^{-1}\log k^{-\varepsilon}\qquad(\varepsilon>0)\,,$$

and define $\Omega^1_k:=\widetilde{\Omega}^1_{\kappa(k)}$. Then

$$\mathbb{E}\left[\mathbb{1}_{\Omega^1_k}\max_{1\leq n\leq M}\left(\|\mathbf{e}^n\|^2_{\mathbb{L}^2}+\frac{1}{2}\sum_{m=1}^n\|\mathbf{e}^m-\mathbf{e}^{m-1}\|^2_{\mathbb{L}^2}+\frac{\nu}{2}k\sum_{m=1}^n\|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2}\right)\right]\leq C\,k^{\eta-\varepsilon}\qquad(\varepsilon>0)\,.$$

For the computable set $\Omega_k^1 \equiv \Omega_k^1\big(\{\mathbf{u}^m\}_m\big)$ we have, thanks to Lemmata 5.2.1 and 5.3.1, and for $k < 1$,

$$\mathbb{P}\left[\Omega_k^1\right] \;\geq\; 1 - \frac{\mathbb{E}\left[\max_{1\leq m\leq M} \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2}^2\right]}{\log k^{-\varepsilon}} := 1 + \frac{1}{\widetilde{\varepsilon}\log k}\,,$$

for $\widetilde{\varepsilon} = \varepsilon/\mathbb{E}\left[\max_{1\leq m\leq M}\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2\right]$. We can now state our first main result.

Theorem 5.3.1. *Let $D = (0, L)^2$, $T > 0$, and $\mathfrak{P} := \big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$ be a filtered probability space. Assume that (S_1) through (S_4) hold, with $\boldsymbol{\mathcal{H}} = \mathbb{W}^{1,2}_{per}$. Let $\mathbf{u}_0 \in L^8(\Omega; \mathbb{V}^{per})$ be a given random variable, $\mathbf{W}$ be an $\mathbb{F}$-progressively measurable $\mathbf{Q}$-Wiener process on $\mathfrak{P}$ with $\mathbf{Q} \in \mathcal{I}_1(\boldsymbol{\mathcal{K}})$, and $\mathbf{u} \in L^8\big(\Omega; C([0, T]; \mathbb{V}^{per})\big)$ be the strong solution of* (5.1.1)–(5.1.3)*. Let $I_k = \{t_m\}_{m=0}^M$ be an equidistant mesh covering $[0, T]$, for $k \leq k_0(T, \operatorname{Tr}\mathbf{Q}, \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^8\right])$ sufficiently small, and $\big\{\mathbf{u}^m\big\}_{m=1}^M \subset \mathbb{L}^8(\Omega; \mathbb{V}^{per})$ be iterates from Algorithm 5.1. Then, for every $\varepsilon > 0$, the computable set*

$$\Omega_k^1 \;=\; \Big\{\omega\in\Omega \;\Big|\; \max_{1\leq m\leq M}\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2 \leq \log k^{-\varepsilon}\Big\}\,,$$

satisfies

$$\mathbb{P}\left[\Omega_k^1\right] \geq 1 + \frac{1}{\varepsilon\log k}\,,$$

and iterates $\{\mathbf{u}^m\}_{m=1}^M$ of Algorithm 5.1 satisfy

$$\text{(i)}\qquad \mathbb{E}\left[\mathbb{1}_{\Omega_k^1}\left(\max_{1\leq m\leq M}\|\mathbf{u}(t_m)-\mathbf{u}^m\|_{\mathbb{L}^2}^2 + k\sum_{m=1}^M\|\mathbf{u}(t_m)-\mathbf{u}^m\|_{\mathbb{V}}^2\right)\right] \leq C\,k^{\eta-\varepsilon} \qquad \Big(\eta\in(0,\frac{1}{2})\Big)\,.$$

Suppose in addition that the following assumption holds:

(S_2)′ There exists a constant $K_2' > 0$ such that

$$\|\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v})\|_{\mathcal{L}(\boldsymbol{\mathcal{K}},(\mathbb{V}^{per})')} \leq K_2'\,\|\mathbf{u}-\mathbf{v}\|_{(\mathbb{V}^{per})'} \qquad \forall\,\mathbf{u},\mathbf{v}\in\mathbb{L}^2\,.$$

Then there exists a set

$$\Omega_k^2 \;=\; \Big\{\omega\in\Omega \;\Big|\; \max_{1\leq m\leq M}\Big(\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^4 + \|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^4\Big) \leq \log k^{-\varepsilon}\Big\}\,,$$

such that

$$\mathbb{P}\left[\Omega_k^2\right] \geq 1 + \frac{1}{\varepsilon\log k}\,,$$

and the iterates $\{\mathbf{u}^m\}_{m=1}^M$ of Algorithm 5.1 satisfy the estimate

$$\text{(ii)}\qquad \mathbb{E}\left[\mathbb{1}_{\Omega_k^2}\left(\max_{1\leq m\leq M}\|\mathbf{u}(t_m)-\mathbf{u}^m\|_{(\mathbb{V}^{per})'}^2 + k\sum_{m=1}^M\|\mathbf{u}(t_m)-\mathbf{u}^m\|_{\mathbb{L}^2}^2\right)\right] \leq C\,k^{2\eta-\varepsilon}\,.$$

Recall that $\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^8\right] \leq C$ is needed to validate the bounds (5.3.14), and (5.3.15), which control the nonlinear drift.

Proof. Estimate (i) is shown above. The order-limiting term in (5.3.22) is (5.3.10), which is bounded in Lemma 5.2.3, (ii). To prove estimate (ii) we use $\boldsymbol{\phi} = \mathbf{A}^{-1}\mathbf{e}^m$ in (5.3.9) instead, such that the term corresponding to (5.3.10) has no spatial derivatives anymore; Lemma 5.2.3,

(i) then improves the convergence behavior of the term (5.3.10). For the nonlinear convection term, we compute

$$\Big([\mathbf{e}^m\cdot\nabla]\mathbf{u}^m, \mathbf{A}^{-1}\mathbf{e}^m\Big) = -\Big([\mathbf{e}^m\cdot\nabla]\mathbf{A}^{-1}\mathbf{e}^m, \mathbf{u}^m\Big) \le \|\mathbf{e}^m\|_{\mathbb{L}^2}^2\|\nabla\mathbf{A}^{-1}\mathbf{e}^m\|_{\mathbb{L}^4}^2\|\mathbf{u}^m\|_{\mathbb{L}^4}^2\,,$$

where the second term on the right-hand side can be bounded by the Gagliardo-Nirenberg inequality. We remark that the set $\Omega_k^2 \equiv \Omega_k^2\big(\{\mathbf{u}^m\}_{m=1}^M, \mathbf{u}\big)$ also depends on the strong solution of (5.1.1)–(5.1.3), since in general

$$\Big([\mathbf{u}(t_m)\cdot\nabla]\mathbf{A}^{-1}\mathbf{e}^m, \mathbf{e}^m\Big) \neq 0\,.$$

Condition $(\mathrm{S}_2)'$ is necessary to bound the stochastic integral in (5.3.9) as in (5.3.16), (5.3.17). Again, we have to control the differences

$$I + II := Ck\mathbb{E}\left[\sum_{\ell=1}^m \|\mathbf{u}^\ell - \mathbf{u}^{\ell-1}\|_{\mathbb{L}^4}^4\|\mathbf{e}^\ell\|_{\mathbb{V}'}^2\right] + Ck\mathbb{E}\left[\sum_{\ell=1}^m \|\mathbf{u}(t_\ell) - \mathbf{u}(t_{\ell-1})\|_{\mathbb{L}^4}^4\|\mathbf{e}^\ell\|_{\mathbb{V}'}^2\right]\,.$$

To control the first difference we bound the norm of the solution by the norm of its gradient, and proceed as below (5.3.22). For the second, we use the same argument, taking into account Lemma 5.2.3. □

Remark 5.3.1. *To avoid the condition $k \le k_0(T, \operatorname{Tr}\mathbf{Q}, \mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^8\right])$, we may proceed as follow:*

$$k\sum_{m=1}^M (\|\nabla\mathbf{u}^{m-1}\|_{\mathbb{L}^2}^2 + 1)\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 = k\sum_{m=1}^{M-1}(\|\nabla\mathbf{u}^{m-1}\|_{\mathbb{L}^2}^2 + 1)\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 + k(\|\nabla\mathbf{u}^{M-1}\|_{\mathbb{L}^2}^2 + 1)\|\mathbf{e}^M\|_{\mathbb{L}^2}^2\,.$$

The expectation of the last term can be bounded by Ck thanks to Lemma 5.3.1. This allows the use of the explicit version of the Gronwall inequality, preventing any condition on the smallness of $k > 0$.

A consequence of this theorem is the convergence in probability of the scheme.

Corollary 5.3.1. *Under the assumptions of Theorem 5.3.1, iterates $\{\mathbf{u}^m\}_{m=1}^M$ of Algorithm 5.1 converge in probability with order $\alpha_1 < \frac{1}{2}(\eta - \varepsilon)$, respectively $\alpha_2 < \eta - \varepsilon$ for all $\varepsilon > 0$, i.e., there holds*

$$\lim_{\widetilde{C}\to\infty}\lim_{k\to 0}\mathbb{P}\Big[\max_{1\le m\le M}\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbb{L}^2} + \Big(k\sum_{m=1}^M\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbb{V}}^2\Big)^{1/2} \ge \widetilde{C}\,k^{\alpha_1}\Big] = 0\,,$$

$$\lim_{\widetilde{C}\to\infty}\lim_{k\to 0}\mathbb{P}\Big[\max_{1\le m\le M}\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{(\mathbb{V}^{per})'} + \Big(k\sum_{m=1}^M\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbb{L}^2}^2\Big)^{1/2} \ge \widetilde{C}\,k^{\alpha_2}\Big] = 0\,.$$

Proof. We estimate

$$\begin{aligned}\mathbb{P}\Big[\max_{1\le m\le M}\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 \ge \widetilde{C}k^\alpha\Big] &\le \mathbb{P}\Big[\Big\{\max_{1\le m\le M}\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 \ge \widetilde{C}k^\alpha\Big\}\cap\Omega_k^1\Big] + \mathbb{P}\left[\Omega\setminus\Omega_k^1\right]\\ &\le \frac{Ck^{\eta-\varepsilon}}{\widetilde{C}k^\alpha} - \frac{1}{\varepsilon\log k} \le \frac{C}{\widetilde{C}} - \frac{1}{\varepsilon\log k}\,.\end{aligned}$$

Passing to the limit for $k \to 0$ then implies

$$\lim_{k\to 0}\mathbb{P}\Big[\max_{1\le m\le M}\|\mathbf{e}^m\|_{\mathbb{L}^2}^2 \ge \widetilde{C}\,k^\alpha\Big] \le \frac{C}{\widetilde{C}}\,. \tag{5.3.24}$$

Eventually then passing to the limit for $\widetilde{C}\to\infty$ proves the assertion. The other terms can be handled in exactly the same way. □

Remark 5.3.2. *Next to strong convergence, and convergence in probability, the weak convergence is a relevant concept to study the approximation of related laws. Weak convergence rates for discretisations are proved in [39] for a problem with Lipschitz nonlinear drift. In [39], weak convergence is proved by semigroup theory and Malliavin calculus, leading to asymptotic weak convergence order* $1/2$ *for the equation*

$$u_t + Au + f(u) = g(u)\,\dot{W}\,, \tag{5.3.25}$$

with Lipschitz nonlinearity f*, and linear self-adjoint, positive operator* A*. In the present paper, as a direct consequence of Theorem 5.3.1 together with the mean value theorem, we may conclude the weak convergence of Euler iterates from Algorithm 5.1 with order* $\frac{1}{2}\eta - \varepsilon$ *on the set* Ω_k^1*, i.e.*

$$\mathbb{E}\left[\mathbb{1}_{\Omega_k^1} \max_{1\le m\le M} |\phi(\mathbf{u}(t_m)) - \phi(\mathbf{u}^m)|\right] \le C\,k^{\frac{1}{2}\eta-\varepsilon} \quad (\varepsilon > 0)\,,$$

for all $\phi \in C_b^1(\mathbb{W}^{1,2}_{per}, \mathbb{R})$*, where the constant depends on the problem data and on the function* ϕ*.*

Remark 5.3.3. *The convergence analysis performed in this section can be applied successfully to the stochastic incompressible Stokes equations*

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla\pi &= \mathbf{g}(\mathbf{u})\dot{\mathbf{W}} && \text{in } \Omega \times D_T \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega \times D_T \end{aligned}$$

in the periodic case, as well as in case of more general domains D *with Dirichlet boundary conditions. By standard arguments, the Hölder exponent related to* $\{\nabla\mathbf{u}(t)\ ;\ t \in [0,T]\}$ *is twice as large if compared to (ii) from Lemma 5.2.3, which then implies for Euler iterates strong convergence of order* η*,*

$$\mathbb{E}\left[\max_{1\le m\le M} \|\mathbf{u}^m - \mathbf{u}(t_m)\|^2_{\mathbb{L}^2} + k\sum_{m=1}^{M} \|\mathbf{u}^m - \mathbf{u}(t_m)\|^2_{\mathbb{V}}\right] \le C\,k^{2\eta}\,. \tag{5.3.26}$$

The absence of a nonlinear drift leads to $\Omega_k^1 \equiv \Omega$*, and* $\varepsilon = 0$ *in Theorem 5.3.1, since Gronwall's inequality may now be used directly; see* (5.3.18) *and* (5.3.22)*.*

Remark 5.3.4. (i) *The main problem in proving the error estimate* (i) *from Theorem 5.3.1 is the error induced by the nonlinearity* $([\mathbf{u}\cdot\nabla]\mathbf{u}, \boldsymbol{\phi})$*. Its proof is accomplished by using the Markov inequality* (5.3.20) *to control the stochastic effects of the nonlinearity on a subset* Ω_k^1 *of* Ω*, together with a discrete Gronwall argument. This results in an estimate which depends exponentially on the parameters arising from the Markov inequality, thus evidencing the subtle interplay of non-Lipschitz nonlinearity and stochastics. As a consequence, the measure of the set* $\Omega_k^1 \subset \Omega$ *converges logarithmically to one, i.e.*

$$\mathbb{P}[\Omega_k^1] \ge 1 + \frac{1}{\varepsilon \log k} \quad (\varepsilon > 0)\,.$$

(ii) *A complementatory strategy is proposed in [111], where a discretisation of the problem with truncated nonlinearity in the* $1D$ *Burgers equation leads to optimal strong convergence rates for the truncated solution on the whole set* Ω*; this auxiliary result may then be used to conclude rates of convergence in probability of the discretised, truncated problem towards the original one. In this work we leave the original problem* (5.1.1)–(5.1.3) *unaffected, and use a truncated sample set* Ω_k^1 *to account for nonlinear effects and obtain rates of convergence in probability for the stochastic Navier-Stokes equations* (5.1.1)–(5.1.3)*.*

5.3.2 Semi-implicit time semi-discretisation

We consider a discretisation with linearised drift.

Algorithm 5.2. *Let* $\mathbf{v}^0 := \mathbf{u}_0$. *Find for every* $m \in \{1, \ldots, M\}$ *a pair of random variables* $(\mathbf{v}^m, \rho^m)$ *with values in* $\mathbb{H}^{per} \times L^2_{per}$, *such that* $\mathbb{P}$*-almost surely*

$$(\mathbf{v}^m - \mathbf{v}^{m-1}, \boldsymbol{\phi}) + k\nu\,(\nabla \mathbf{v}^m, \nabla \boldsymbol{\phi})$$

$$+k\Big([\mathbf{v}^{m-1}\cdot\nabla]\mathbf{v}^m, \boldsymbol{\phi}\Big) - k(\rho^m, \operatorname{div}\boldsymbol{\phi}) = \Big(\mathbf{g}(\mathbf{v}^{m-1})\Delta_m \mathbf{W}, \boldsymbol{\phi}\Big) \qquad \forall\, \boldsymbol{\phi} \in \mathbb{W}^{1,2}_{per}\,, \tag{5.3.27}$$

$$(\operatorname{div}\mathbf{v}^m, \psi) = 0 \qquad \forall\, \psi \in L^2_{per}\,. \tag{5.3.28}$$

Iterates $\{\mathbf{v}^m\}_{m=1}^M$ resulting from Algorithm 5.2 do not satisfy Lemma 5.3.1 because of (5.3.4). As a consequence, we employ a perturbation analysis below to again benefit from the strong stability properties of Euler iterates $\{\mathbf{u}^m\}_{m=1}^M$ from Algorithm 5.1, and show that iterates $\{\mathbf{v}^m\}_{m=1}^M$ from Algorithm 5.2 inherit the same convergence properties from Theorem 5.3.1.

We rewrite Algorithm 5.2 by using divergence-free test functions and compare its solution to that of (5.3.3) by subtracting the two equations and choosing $\boldsymbol{\phi} = \overline{\mathbf{e}}^m := \mathbf{u}^m - \mathbf{v}^m$. The term governing the error is the convection term, which, by skew-symmetricity property, may be restated as follows,

$$\begin{aligned}&\Big([\mathbf{v}^{m-1}\cdot\nabla]\mathbf{v}^m - \big[\mathbf{u}^m\cdot\nabla]\mathbf{u}^m, \overline{\mathbf{e}}^m\Big)\\ &\quad = \Big([\overline{\mathbf{e}}^{m-1}\cdot\nabla]\mathbf{v}^m, \overline{\mathbf{e}}^m\Big) - \Big([(\mathbf{u}^m - \mathbf{u}^{m-1})\cdot\nabla]\overline{\mathbf{e}}^m, \mathbf{u}^m\Big) = I + II\,.\end{aligned}$$

The term I is estimated as follows,

$$\begin{aligned}&\Big([\overline{\mathbf{e}}^{m-1}\cdot\nabla]\mathbf{v}^m, \overline{\mathbf{e}}^m\Big) = \Big([\overline{\mathbf{e}}^{m-1}\cdot\nabla]\mathbf{u}^m, \overline{\mathbf{e}}^m\Big)\\ &\quad \le C\|\nabla\mathbf{u}^m\|^2_{\mathbb{L}^2}\big(\|\overline{\mathbf{e}}^{m-1}\|^2_{\mathbb{L}^2} + \|\overline{\mathbf{e}}^m\|^2_{\mathbb{L}^2}\big) + \frac{\nu}{8}\big(\|\nabla\overline{\mathbf{e}}^{m-1}\|^2_{\mathbb{L}^2} + \|\nabla\overline{\mathbf{e}}^m\|^2_{\mathbb{L}^2}\big)\,,\end{aligned} \tag{5.3.29}$$

where we use again the skew-symmetricity property of the trilinear form to base a corresponding bound on the result for $\|\nabla\mathbf{u}^m\|^2_{\mathbb{L}^2}$ in Lemma 5.3.1, (i), instead of $\|\nabla\mathbf{v}^m\|^2_{\mathbb{L}^2}$. From term II, we conclude

$$II \;\le\; C\Big(\|\mathbf{u}^m - \mathbf{u}^{m-1}\|^2_{\mathbb{L}^2}\|\mathbf{u}^m\|^2_{\mathbb{L}^2} + \|\nabla(\mathbf{u}^m - \mathbf{u}^{m-1})\|^2_{\mathbb{L}^2}\|\nabla\mathbf{u}^m\|^2_{\mathbb{L}^2}\Big) + \frac{\nu}{8}\|\nabla\overline{\mathbf{e}}^m\|^2_{\mathbb{L}^2}\,. \tag{5.3.30}$$

The expectation of the sum of the first two terms in (5.3.30), if multiplied by k, may be bounded by Ck; see Lemma 5.3.1, (ii). We may now employ the set $\Omega^1_k \equiv \Omega^1_k\big(\{\mathbf{u}^m\}_{m=1}^M\big)$ from above, and proceed as in the error analysis of the previous section to conclude the following result.

Theorem 5.3.2. *Let* $\{\mathbf{u}^m\}_{m=1}^M$ *be the solution of Algorithm 5.1, and* $\{\mathbf{v}_m\}_{m=1}^M$ *be the solution of Algorithm 5.2. Then, under the assumptions of Theorem 5.3.1, there holds*

$$\mathbb{E}\left[\mathbb{1}_{\Omega^1_k}\left(\max_{1\le m\le M}\|\mathbf{u}^m - \mathbf{v}^m\|^2_{\mathbb{L}^2} + k\sum_{m=1}^M\|\mathbf{u}^m - \mathbf{v}^m\|^2_{\mathbb{V}^{per}}\right)\right] \le C\,k^{1-\varepsilon}\,,$$

where the set Ω^1_k *is defined in Theorem 5.3.1.*

Theorem 5.3.2 compares iterates $\{\mathbf{u}^m\}_{m=1}^M$ and $\{\mathbf{v}^m\}_{m=1}^M$, and thus avoids to use the Hölder regularity property in Lemma 5.2.3, (ii) for strong solutions $\{\nabla\mathbf{u}(t)\,;\; t\in[0,T]\}$. By combining Theorem 5.3.1, (ii), and Theorem 5.3.2, on using the fact that $\Omega^2_k \subset \Omega^1_k$, we may conclude that for every $\eta \in (0, 1/2)$

$$\mathbb{E}\left[\mathbb{1}_{\Omega^2_k}\left(\max_{1\le m\le M}\|\mathbf{u}(t_m) - \mathbf{v}^m\|^2_{(\mathbb{V}^{per})'} + k^{1-\eta}\sum_{m=1}^M\|\mathbf{u}(t_m) - \mathbf{v}^m\|^2_{\mathbb{L}^2}\right)\right] \le C\,k^{\eta-\varepsilon}\,,$$

where Ω^2_k is defined in Theorem 5.3.1, (ii).

5.3.3 Additive noise: global convergence.

In this section we generalise the assertion (i) of Theorem 5.3.1. Using the results from [67], we are able to control the nonlinear effects, which cause the constant in the error estimate to be exponentially dependent on the pathwise regularity of the solution. To apply the desired stability, see [67, Lemma 4.10], we consider the following framework: Let $\mathbf{R} \in \mathcal{I}_2(\boldsymbol{\mathcal{K}})$ be a Hilbert-Schmidt operator and let $\mathbf{W}$ be a cylindrical Wiener process on $\mathbb{H}$. Throughout this section we consider the equations

$$\dot{\mathbf{u}} - \nu\Delta\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla\pi = \mathbf{R}\dot{\mathbf{W}} \quad \text{in } D_T \times \Omega\,, \tag{5.3.31}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } D_T \times \Omega\,, \tag{5.3.32}$$

together with the boundary conditions given in (5.1.3) and a deterministic initial value $\mathbf{u}_0 \in \mathbb{V}^{per}$.

We first begin with some technical Lemmata, and provide their proof in order to make clear the interplay of the various problem data: covariance operator, viscosity parameter and time interval. The first result we give, can be found, together with its proof, in [67, Lemma 4.10, Part 1]. Let $\mathbf{w} := \operatorname{curl}\mathbf{u}$. Then, $\mathbf{w}$ satisfies the Navier-Stokes equations in the vorticity formulation; see [67]. For the vorticity $\mathbf{w}$ there holds an exponential estimate.

Lemma 5.3.3. *There exist constants $C > 0$ and η_0, such that for every $t > 0$ and every $\eta \in (0, \eta_0]$ there holds the bound*

$$\mathbb{E}\left[\exp\left(\eta \sup_{t\geq s}\|\mathbf{w}(t)\|^2_{\mathbb{L}^2}\right)\right] \leq \exp\left(\eta e^{-\nu s}\|\mathbf{w}_0\|^2_{\mathbb{L}^2}\right). \tag{5.3.33}$$

The constant η_0 can be chosen as the constant $\alpha/2$ such that there holds

$$\nu\|\mathbf{w}\|^2_{\mathbb{L}^2} \geq \frac{\alpha}{2}\|\mathbf{Q}^*\mathbf{w}\|^2_{\mathbb{L}^2}\,,$$

evidencing the interaction of noise and viscosity.
Thanks to [67, Formula A.3]

$$\|\nabla\mathbf{u}\|_{\mathbb{L}^2} \leq \|\operatorname{curl}\mathbf{u}\|_{\mathbb{L}^2} \quad \forall\,\mathbf{u} \in \mathbb{V}^{per}\,,$$

from (5.3.33) we can conclude

$$\mathbb{E}\left[\exp\left(\eta \sup_{t\geq s}\|\nabla\mathbf{u}(t)\|^2_{\mathbb{L}^2}\right)\right] \leq \exp\left(\eta e^{-\nu t}\|\operatorname{curl}\mathbf{u}_0\|^2_{\mathbb{L}^2}\right). \tag{5.3.34}$$

Consider the $\mathbb{P}$-a.s. inequality (5.3.18)

$$\begin{aligned}
&\max_{1\leq m\leq M}\|\mathbf{e}^m\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{m=1}^{M}\|\mathbf{e}^m - \mathbf{e}^{m-1}\|^2_{\mathbb{L}^2} + \frac{\nu}{2}k\sum_{m=1}^{M}\|\nabla\mathbf{e}^m\|^2_{\mathbb{L}^2}\\
&\leq C\,k\sum_{m=1}^{M-1}(\|\nabla\mathbf{u}(t_m)\|^2_{\mathbb{L}^2} + 1)\|\mathbf{e}^m\|^2_{\mathbb{L}^2} + k(\|\nabla\mathbf{u}(t_M)\|^2_{\mathbb{L}^2} + 1)\|\mathbf{e}^M\|^2_{\mathbb{L}^2}\\
&\quad + C\,k\sum_{m=1}^{M}\Big[\|\nabla\big(\mathbf{u}(t_m) - \mathbf{u}(t_{m-1})\big)\|^2_{\mathbb{L}^2} + \|\mathbf{u}(t_m) - \mathbf{u}(t_{m-1})\|^2_{\mathbb{L}^4}\|\nabla\mathbf{u}(t_m)\|^2_{\mathbb{L}^2}\Big] + I_M\,.
\end{aligned} \tag{5.3.35}$$

Now, since we have additive noise, we can subtract the corresponding noises pathwise, because there is no stochastic integral. We may now use the Gronwall inequality before taking expectation, since $k\sum_{m=1}^{M}\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2$ is $\mathbb{P}$-a.s. finite. We then use the Hölder inequality to obtain

$$\mathbb{E}\left[\max_{1\le m\le M}\|\mathbf{e}^m\|_{\mathbb{L}^2}^2+\frac{\nu}{2}k\sum_{m=1}^{M}\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2\right]\le\mathbb{E}\left[\exp\Big(Ct_{M-1}\sup_{1\le m\le M}(\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2+1)\Big)E_M\right]$$
$$\le\mathbb{E}\left[\exp\Big(2Ct_{M-1}\sup_{1\le m\le M}(\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2+1)\Big)\right]^{1/2}\mathbb{E}\left[E_M^2\right]^{1/2},$$

where

$$E_M:=I+II+III:=k(\|\nabla\mathbf{u}(t_M)\|_{\mathbb{L}^2}^2+1)\|\mathbf{e}^M\|_{\mathbb{L}^2}^2$$
$$+C\,k\sum_{m=1}^{M}\Big[\|\nabla\big(\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\big)\|_{\mathbb{L}^2}^2+\|\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\|_{\mathbb{L}^4}^2\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2\Big].$$

We estimate each term separately. For the first there holds

$$\mathbb{E}\left[I^2\right]\le Ck^2\mathbb{E}\left[(\|\nabla\mathbf{u}(t_M)\|_{\mathbb{L}^2}^2+1)^2+\|\mathbf{e}^M\|_{\mathbb{L}^2}^4\right],$$

while for II we use the Hölder inequality for sums to get

$$\mathbb{E}\left[\left(k\sum_{m=1}^{M}\|\nabla\big(\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\big)\|_{\mathbb{L}^2}^2\right)^2\right]\le\mathbb{E}\left[Ct_Mk\sum_{m=1}^{M}\|\nabla\big(\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\big)\|_{\mathbb{L}^2}^4\right]\le Ck^{2\eta}.$$

For III we use

$$\mathbb{E}\left[\left(k\sum_{m=1}^{M}\|\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\|_{\mathbb{L}^4}^2\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2\right)^2\right]$$
$$\le\mathbb{E}\left[t_M\Big(\sup_{1\le m\le M}\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^4\Big)k\sum_{m=1}^{M}\|\mathbf{u}(t_m)-\mathbf{u}(t_{m-1})\|_{\mathbb{L}^4}^4\right]\le Ck^{4\eta}.$$

Putting the inequalities together and using the triangle inequality we have

$$\mathbb{E}\left[\max_{1\le m\le M}\|\mathbf{e}^m\|_{\mathbb{L}^2}^2+\frac{\nu}{2}k\sum_{m=1}^{M}\|\nabla\mathbf{e}^m\|_{\mathbb{L}^2}^2\right]$$
$$\le C\mathbb{E}\left[\exp\Big(2Ct_{M-1}\sup_{1\le m\le M}(\|\nabla\mathbf{u}(t_m)\|_{\mathbb{L}^2}^2+1)\Big)\right]^{1/2}\big(k+k^{\eta}+k^{2\eta}\big).\tag{5.3.36}$$

This can be resumed in the following

Theorem 5.3.3. *Let $D=(0,L)^2$, $T>0$, and $\mathfrak{P}:=\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$ be a filtered probability space. Assume that*

(i) $\mathbf{R}:\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}$ *is a Hilbert-Schmidt operator,*

(ii) $\mathbf{u}_0\in\mathbb{V}^{per}$.

Define $\mathbf{Q} := \mathbf{R}\mathbf{R}^*$ *and let* $\mathbf{W}$ *be an* $\mathbb{F}$*-progressively measurable* $\mathbf{Q}$*-Wiener process on* $\mathfrak{P}$*, and* $\mathbf{u} \in L^p\big(\Omega; C([0,T]; \mathbb{V}^{per})\big)$ *be the strong solution of* (5.1.1)–(5.1.3) *for all* $p \geq 2$*. Let* $I_k = \{t_m\}_{m=0}^M$ *be an equi-distant mesh covering* $[0,T]$*, for* $k \leq k_0(T, \operatorname{Tr}\mathbf{Q}, \|\mathbf{u}_0\|_{\mathbb{V}})$ *sufficiently small, and* $\big\{\mathbf{u}^m\big\}_{m=1}^M \subset \mathbb{L}^p(\Omega; \mathbb{V}^{per})$ *be iterates from Algorithm 5.1. Then there holds*

$$\text{(i)} \qquad \mathbb{E}\left[\left(\max_{1\leq m\leq M} \|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbb{L}^2}^2 + k\sum_{m=1}^M \|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbb{V}}^2\right)\right] \leq C\, k^\eta \qquad \big(\eta \in (0, \frac{1}{2})\big)\,,$$

under the assumption $2Ct_{M-1} \leq \eta_0$*; see* (5.3.34) *and* (5.3.36).

The result from this section shows that if exponential estimates for the velocity are available, it is possible to show convergence on the whole realisation set Ω. However, this result is limited to problems with small data, i.e., a covariance operator with small trace, a big viscosity, see below (5.3.33), and a small time interval $(0, T)$, evidencing the subtle interplay of stochastics and problem parameters.

5.4 Space-time discretisation

The goal in this section is to study the interplay of nonlinear effects, stochasticity, and general discrete LBB-stable finite element discretisations of problem (5.1.1)–(5.1.3). As will be shown below, since velocity approximates typically are only discretely rather than exactly divergence-free, the weak stability properties of the related Lagrange multiplier again crucially affect convergence properties of such discretisations, while spatial discretisations that lead to exactly divergence-free velocity approximates circumvent those problems and may be shown to be optimally convergent.

In order to illustrate the interaction of a general space discretisation with the type of the driving noise, we apply a general mixed method $\big(\mathbb{H}_h, L_h\big)$ to Algorithm 5.1 that satisfies the discrete LBB-constraint (5.1.6).

Algorithm 5.3. *Let* $\mathbf{U}^0$ *be a given* $\mathbb{H}_h$*-valued random variable. Find for every* $m \in \{1, \dots, M\}$ *a tuple of random variables* $\big(\mathbf{U}^m, \Pi^m\big)$ *with values in* $\mathbf{H}_h \times L_h$*, such that the following equations hold for all* $(\mathbf{\Phi}, \Lambda) \in \mathbf{H}_h \times L_h$ *and* $\mathbb{P}$*-almost surely,*

$$\begin{aligned}
&\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \mathbf{\Phi}\big) + k\nu\big(\nabla\mathbf{U}^m, \nabla\mathbf{\Phi}\big) + k\Big([\mathbf{U}^m \cdot \nabla]\mathbf{U}^m, \mathbf{\Phi}\Big) \\
&\qquad + \frac{k}{2}\Big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \mathbf{\Phi}\Big) - k\big(\Pi^m, \operatorname{div}\mathbf{\Phi}\big) = \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \mathbf{\Phi}\big)\,, && (5.4.1)\\
&\big(\operatorname{div}\mathbf{U}^m, \Lambda\big) = 0\,. && (5.4.2)
\end{aligned}$$

The additional term $\frac{k}{2}\Big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \mathbf{\Phi}\Big)$ is used to control nonlinear effects in the presence of discretely divergence-free velocity iterates, and thus allows for the following stability properties of the scheme; see [19, Lemma 3.1] for a proof.

Lemma 5.4.1. *Let* $1 \leq q < \infty$*, and* $\mathbf{U}^0 \in L^{2^q}\big(\Omega; \mathbb{H}_h\big)$ *be given, such that* $\mathbb{E}\big[\|\mathbf{U}^0\|_{\mathbb{L}^2}^{2^q}\big] \leq C$*. Suppose (S1), (S2), (S4). Then there exists* $\{(\mathbf{U}^m, \Pi^m)\}_{m=1}^M \subset L^2(\Omega; \mathbb{H}_h \times L_h)$ *which* $\mathbb{P}$*-almost surely solves Algorithm 5.3, and satisfies,*

$$\mathbb{E}\left[\max_{1\leq m\leq M} \|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^q} + 2\nu k\sum_{m=1}^M \|\mathbf{U}^m\|_{\mathbb{L}^2}^{2^q-2}\|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\right] \leq C_{t_M,q}\,, \tag{5.4.3}$$

where $C_{t_M,q} \equiv C_{t_M,q}\big(\operatorname{Tr}\mathbf{Q}, \mathbb{E}[\|\mathbf{U}^0\|_{\mathbb{L}^2}^{2^q}]\big) > 0$ *does not depend on* $k, h > 0$.

These stability results for iterates $\{\mathbf{U}^m\}_{m=1}^M$ of Algorithm 5.3 are weaker if compared to the ones in Lemma 5.3.1 for iterates $\{\mathbf{u}^m\}_{m=1}^M$ from Algorithm 5.1, where the test function $\boldsymbol{\phi} = \mathbf{A}\mathbf{u}^m$ in (5.3.1) allowed to control the nonlinear term in Algorithm 5.1. This strategy is not successful any more for Algorithm 5.3, and leads to further difficulties in the related error analysis below.

Remark 5.4.1. *A finite element analysis for the deterministic version of (5.1.1)–(5.1.3) via a deterministic version of Algorithm 5.3 is given in [71], which bases on stability properties for iterates similar to those in Lemma 5.4.1, and uses a discrete Gronwall argument to control nonlinear effects. This strategy is not straightforwardly applicable to problem* (5.1.1)–(5.1.3) *and Algorithm 5.3, where only expectations of corresponding terms are controlled, and iterates* $\{\mathbf{U}^m\}_{m=1}^M$ *satisfy the weaker stability property* (5.4.3)*; as a consequence, an 'iterative perturbation argument' is employed below to successively compensate for the lack of stronger stability bounds than those available in Lemma 5.4.1.*

Remark 5.4.2. *The convergence analysis in Section 5.3.1 for iterates* $\{\mathbf{u}_m\}_{m=1}^M$ *of Algorithm 5.1 relies on the strong stability properties in Lemma 5.3.1,* (i)*; iterates* $\{\mathbf{v}^m\}_{m=1}^M$ *of Algorithm 5.2 in Section 5.3.2 miss these strong stability properties, and only satisfy the discrete energy law corresponding to* (5.4.3)*. However, the fact that velocity iterates* $\{\mathbf{v}^m\}_{m=1}^M$ *are exactly divergence-free, together with estimate* (ii) *in Lemma 5.3.1 allows to efficiently control nonlinear effects, and leads to the optimal error estimates stated in Theorem 5.3.2. This property of the nonlinear term is not inherited by a general space discretisation in Algorithm 5.3, which is the reason for a different convergence analysis below.*

We start with an analysis of a general mixed discrete LBB-stable finite element discretisation, and show that the control of errors is crucially affected by the type of noise: an 'iterative perturbation argument' will be needed to compensate for the weak stability properties in Lemma 5.4.1. Moreover, the error estimate crucially depends on stability properties of the pressure in Algorithm 5.1, and thus causes a restrictive coupling of the space and time discretisation parameters for general noise.

To avoid this drawback, we analyse a spatial discretisation which delivers exactly divergence-free iterates, such as the Scott-Vogelius finite element pairing from [121]. The pressure term arising in the computations then disappears from the error analysis, leading to error estimates which are not affected by the type of noise used in (5.1.1)–(5.1.3).

Another advantage which goes along with pointwise divergence-free elements is that the stabilisation term $\frac{k}{2}\big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m,\boldsymbol{\Phi}\big)$ for the convection term in (5.4.1) vanishes. This term is subtle in the subsequent error analysis, and initiates the 'iterative perturbation argument' for general discretely LBB-stable elements; exactly divergence-free elements avoid this one, and structural stability properties of the nonlinearity again allow a direct argument in the error analysis.

We reformulate (5.4.1) using discretely divergence-free test functions,

$$
\begin{aligned}
&(\mathbf{U}^m - \mathbf{U}^{m-1}, \boldsymbol{\Phi}) + \nu k\,(\nabla\mathbf{U}^m, \nabla\boldsymbol{\Phi}) + k\big([\mathbf{U}^m\cdot\nabla]\mathbf{U}^m, \boldsymbol{\Phi}\big)\\
&\quad + \frac{k}{2}\big([\operatorname{div}\mathbf{U}^m]\mathbf{U}^m, \boldsymbol{\Phi}\big) = (\mathbf{g}(\mathbf{U}^{m-1})\Delta_m\mathbf{W}, \boldsymbol{\Phi}) \quad \forall\,\boldsymbol{\Phi}\in\mathbb{V}_h\,.
\end{aligned} \tag{5.4.4}
$$

For the sake of brevity, we introduce the following notation

$$
\widetilde{b}(\mathbf{U},\mathbf{V},\mathbf{W}) := \Big([\mathbf{U}\cdot\nabla]\mathbf{V},\mathbf{W}\Big) + \frac{1}{2}\Big([\operatorname{div}\mathbf{U}]\mathbf{V},\mathbf{W}\Big) \quad \forall\,\mathbf{U},\mathbf{V},\mathbf{W}\in\mathbf{H}_h\,. \tag{5.4.5}
$$

We bound the error between the semi-discrete and the fully discrete problem. By taking into account equation (5.3.3), the equation for $\mathbf{E}^m := \mathbf{u}^m - \mathbf{U}^m$ is

$$\begin{aligned}&(\mathbf{E}^m - \mathbf{E}^{m-1}, \mathbf{\Phi}) + \nu k\,(\nabla \mathbf{E}^m, \nabla \mathbf{\Phi}) + k\widetilde{b}\,(\mathbf{u}^m, \mathbf{u}^m, \mathbf{\Phi})\\ &-k\widetilde{b}\,(\mathbf{U}^m, \mathbf{U}^m, \mathbf{\Phi}) - k\,(\pi^m, \operatorname{div} \mathbf{\Phi}) = \Big((\mathbf{g}(\mathbf{u}^{m-1}) - \mathbf{g}(\mathbf{U}^{m-1}))\Delta_m \mathbf{W}, \mathbf{\Phi}\Big) \qquad \forall \mathbf{\Phi} \in \mathbb{H}_h\,.\end{aligned}$$

To obtain an error estimate we set

$$\mathbf{\Phi} = \mathbf{Q}_h^0 \mathbf{E}^m = \mathbf{E}^m - \big(\mathbf{u}^m - \mathbf{Q}_h^0 \mathbf{u}^m\big)$$

as test function, leading to

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbf{Q}_0^h \mathbf{E}^m\|_{\mathbb{L}^2}^2 - \|\mathbf{Q}_0^h \mathbf{E}^{m-1}\|_{\mathbb{L}^2}^2 + \|\mathbf{Q}_0^h(\mathbf{E}^m - \mathbf{E}^{m-1})\|_{\mathbb{L}^2}^2\Big) + \nu k \|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2\\ &\quad + k\widetilde{b}\,\big(\mathbf{u}^m, \mathbf{u}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big) - k\widetilde{b}\,\big(\mathbf{U}^m, \mathbf{U}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big)\\ &\quad = k\Big(\nabla \mathbf{E}^m, \nabla(\mathbf{u}^m - \mathbf{Q}_h^0 \mathbf{u}^m)\Big)\\ &\qquad + k\big(\pi^m, \operatorname{div} \mathbf{Q}_h^0 \mathbf{E}^m\big) + \Big((\mathbf{g}(\mathbf{u}^{m-1}) - \mathbf{g}(\mathbf{U}^{m-1}))\Delta_m \mathbf{W}, \mathbf{Q}_h^0 \mathbf{E}^m\Big)\,.\end{aligned} \tag{5.4.6}$$

Step 1. *Preliminary analysis on* Ω. The first term on the right-hand side is bounded with the help of (5.2.5),

$$k\Big(\nabla \mathbf{E}^m, \nabla(\mathbf{u}^m - \mathbf{Q}_h^0 \mathbf{u}^m)\Big) \leq \frac{\nu}{4} k\,\|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2 + C\,kh^2\,\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2\,. \tag{5.4.7}$$

By the multi-linearity of the form $\widetilde{b}$, there holds

$$\begin{aligned}&\widetilde{b}\,\big(\mathbf{u}^m, \mathbf{u}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big) - \widetilde{b}\,\big(\mathbf{U}^m, \mathbf{U}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big)\\ &\quad = \widetilde{b}\,\big(\mathbf{E}^m, \mathbf{u}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big) + \widetilde{b}\,\big(\mathbf{U}^m, \mathbf{E}^m, \mathbf{Q}_h^0 \mathbf{E}^m\big) \qquad =: I + II\,.\end{aligned}$$

For term I, we use (5.4.5), the Gagliardo-Nirenberg inequality, and Young's inequality to conclude

$$\begin{aligned}I &\leq \|\mathbf{E}^m\|_{\mathbb{L}^4} \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2} \|\mathbf{Q}_h^0 \mathbf{E}^m\|_{\mathbb{L}^4} + \frac{1}{2}\,\|\operatorname{div} \mathbf{E}^m\|_{\mathbb{L}^2} \|\mathbf{u}^m\|_{\mathbb{L}^4} \|\mathbf{Q}_h^0 \mathbf{E}^m\|_{\mathbb{L}^4}\\ &\leq C\,\|\mathbf{E}^m\|_{\mathbb{L}^2}^2 \Big(1 + \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2}^2 + \|\mathbf{u}^m\|_{\mathbb{L}^2}^2 \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2}^2\Big) + \frac{\nu}{8}\,\|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2\,.\end{aligned}$$

For term II, we employ the skew-symmetricity property of $\widetilde{b}$ to conclude

$$\begin{aligned}II &= \widetilde{b}\Big(\mathbf{U}^m, \mathbf{E}^m, \mathbf{Q}_h^0 \mathbf{E}^m - \mathbf{E}^m\Big)\\ &\leq \|\mathbf{U}^m\|_{\mathbb{L}^4} \|\nabla \mathbf{E}^m\|_{\mathbb{L}^2} \|\mathbf{u}^m - \mathbf{Q}_h^0 \mathbf{u}^m\|_{\mathbb{L}^4} + \frac{1}{2}\,\|\nabla \mathbf{U}^m\|_{\mathbb{L}^2} \|\mathbf{E}^m\|_{\mathbb{L}^4} \|\mathbf{u}^m - \mathbf{Q}_h^0 \mathbf{u}^m\|_{\mathbb{L}^4}\\ &:= II_a + II_b\,.\end{aligned}$$

By Gagliardo-Nirenberg inequality, and (5.2.5), (5.2.6), we find for the first part

$$II_a \leq \frac{\nu}{8} \|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2 + C\,h^2\,\|\mathbf{U}^m\|_{\mathbb{L}^2} \Big(\|\mathbf{U}^m\|_{\mathbb{L}^2} + \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2}\Big) \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2} \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}\,. \tag{5.4.8}$$

We use Young's inequality to resume

$$II_a \leq \frac{\nu}{8} \|\nabla \mathbf{E}^m\|_{\mathbb{L}^2}^2 + Ch^2 \Big(\|\mathbf{U}^m\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^m\|_{\mathbb{L}^2}^2 \|\nabla \mathbf{U}^m\|_{\mathbb{L}^2}^2 + \|\nabla \mathbf{u}^m\|_{\mathbb{L}^2}^2 \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2\Big)\,.$$

It remains to bound II_b. Since Lemma 5.4.1 only provides pointwise bounds for iterates rather than related gradients, we use the inverse inequality

$$\|\nabla \mathbf{U}^m\|_{\mathbb{L}^2} \leq C\, h^{-\beta_0}\, \|\mathbf{U}^m\|_{\mathbb{L}^2} \qquad \text{for } \beta_0 = 1\,, \tag{5.4.9}$$

the bound $\|\mathbf{u}^m - \mathbf{Q}_h^0\mathbf{u}^m\|_{\mathbb{L}^4} \leq C\, h^{3/2}\, \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}$, the inverse estimate (5.4.9), and Young's inequality to verify the upper bound

$$\begin{aligned} II_b &\leq C\, h^{\frac{3}{2}-\beta_0}\, \|\mathbf{U}^m\|_{\mathbb{L}^2}\|\mathbf{E}^m\|_{\mathbb{L}^2}^{1/2}\Big(\|\mathbf{E}^m\|_{\mathbb{L}^2} + \|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}\Big)^{1/2}\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2} \\ &\leq C\left(\|\mathbf{U}^m\|_{\mathbb{L}^2}^2 + \|\mathbf{U}^m\|_{\mathbb{L}^2}^4\right)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2 + \frac{\nu}{8}\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}^2 + C\, h^{3-2\beta_0}\, \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2\,. \end{aligned} \tag{5.4.10}$$

Putting things together then yields the estimate

$$\begin{aligned} I + II &\leq C\Big(\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^m\|_{\mathbb{L}^2}^4\Big)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2 + C\, h^{3-2\beta_0}\, \|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2 + \frac{\nu}{4}\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}^2 \\ &\quad + C\, h^2\left(\|\mathbf{U}^m\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^m\|_{\mathbb{L}^2}^2 + \|\nabla\mathbf{U}^m\|_{\mathbb{L}^2}^2\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^2\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}^2\right) \qquad \text{for } \beta_0 = 1\,. \end{aligned} \tag{5.4.11}$$

For the term involving the pressure in (5.4.6), we use (5.4.2), and the L^2-orthogonal projection $P_h^0 : L^2_{per}(D) \to L_h$,

$$\begin{aligned} k\left(\pi^m, \operatorname{div}\mathbf{Q}_h^0\mathbf{E}^m\right) &= k\left(\pi^m - P_h^0\pi^m, \operatorname{div}\mathbf{Q}_h^0\mathbf{E}^m\right) \\ &\leq C\, kh^2\, \|\nabla\pi^m\|_{\mathbb{L}^2}^2 + \frac{\nu}{4}k\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}^2\,, \end{aligned} \tag{5.4.12}$$

where the first term can be controlled by Lemma 5.3.2. To control the stochastic integral term in (5.4.6), we proceed like in (5.3.16), (5.3.17), after using the $\mathbb{L}^2$-stability of the Leray-projection.

Step 2. *Introduction of sample subsets.* Set

$$\Theta_{m-1} = \max_{1\leq\ell\leq m-1} C\left(\|\nabla\mathbf{u}^\ell\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^\ell\|_{\mathbb{L}^2}^4\right),$$

and consider the following estimates

$$\begin{aligned} &k\sum_{m=1}^{M}\mathbb{E}\left[\Big(\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^m\|_{\mathbb{L}^2}^4\Big)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2\right] \\ &\leq Ck\sum_{m=1}^{M}\mathbb{E}\left[\Big(\|\nabla\mathbf{u}^m - \mathbf{u}^{m-1}\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^m - \mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\Big)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2\right] \\ &\quad + Ck\sum_{m=1}^{M}\mathbb{E}\left[\Big(\|\mathbf{u}^{m-1}\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\Big)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2\right] \end{aligned} \tag{5.4.13}$$

The expectation of the first sum on the right-hand side of (5.4.13) can be bounded by Ck as in the proof of Theorem 5.3.1. Using Θ_{m-1} we can bound the second sum as follows

$$k\sum_{m=1}^{M}\mathbb{E}\left[\mathbb{1}_{\{\Theta_{m-1}\leq\kappa\}}\Big(\|\nabla\mathbf{u}^{m-1}\|_{\mathbb{L}^2}^4 + \|\mathbf{U}^{m-1}\|_{\mathbb{L}^2}^4\Big)\|\mathbf{E}^m\|_{\mathbb{L}^2}^2\right] \leq k\sum_{m=1}^{M}\kappa\,\mathbb{E}\left[\mathbb{1}_{\{\Theta_{m-1}\leq\kappa\}}\|\mathbf{E}^m\|_{\mathbb{L}^2}^2\right].$$

To control the initial condition we assume that

$$\mathbb{E}\left[\|\mathbf{U}^0 - \mathbf{u}^0\|_{\mathbb{L}^2}^2\right] \leq C\, h^2\,.$$

Thus, after multiplying the error equality (5.4.6) (with index ℓ) with the indicator function of the set ($\varepsilon > 0$)

$$\Omega^3_{h,\ell-1} := \left\{\omega \in \Omega \,\middle|\, \Theta_{\ell-1} \leq \kappa := \log h^{-\varepsilon}\right\}\,, \tag{5.4.14}$$

which depends on both, $\{\mathbf{u}^m\}_{m=1}^{\ell-1}$ and $\{\mathbf{U}^m\}_{m=1}^{\ell-1}$, and summing up from $\ell = 1$ to m in (5.4.6), we get, thanks to estimates (5.4.11) through (5.4.13), and Lemma 4.2.1, (i), as well as Lemma 5.4.1, the estimate ($\varepsilon > 0$)

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\leq n\leq m}\left(\mathbb{1}_{\Omega^3_{h,n-1}}\|\mathbf{Q}_h^0\mathbf{E}^n\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{\ell=1}^{n}\mathbb{1}_{\Omega^3_{h,\ell-1}}\|\mathbf{Q}_h^0(\mathbf{E}^\ell - \mathbf{E}^{\ell-1})\|^2_{\mathbb{L}^2}\right)\right]\\
&+\mathbb{E}\left[\max_{1\leq n\leq m}\left(Ck\sum_{\ell=1}^{n}\mathbb{1}_{\Omega^3_{h,\ell-1}}\|\nabla\mathbf{E}^\ell\|^2_{\mathbb{L}^2}\right)\right] \leq C\,\log\frac{1}{h^\varepsilon}\,k\sum_{\ell=1}^{m}\mathbb{E}\Big[\max_{1\leq n\leq \ell}\mathbb{1}_{\Omega^3_{h,n-1}}\|\mathbf{E}^n\|^2_{\mathbb{L}^2}\Big]\\
&+C\,\left(k + h^2 + h^{3-2\beta_0} + h^2\,\mathbb{E}\Big[k\sum_{\ell=1}^{m}\|\nabla\pi^\ell\|^2_{\mathbb{L}^2}\Big]\right)\,.
\end{aligned} \tag{5.4.15}$$

We observe the error Ck, which is caused by the first sum on the right-hand side of (5.4.13) together with Lemma 4.2.1, (ii). The last term on the right-hand side may be bounded by Lemma 5.3.2. Then, defining the set

$$\Omega^3_h := \left\{\omega \in \Omega \,\middle|\, \Theta \leq \log h^{-\varepsilon}\right\} \subset \Omega^3_{h,\ell} \qquad (1 \leq \ell \leq M)\,,$$

for $\Theta = C\max_{1\leq m\leq M}\left(\|\nabla\mathbf{u}^m\|^4_{\mathbb{L}^2} + \|\mathbf{U}^m\|^4_{\mathbb{L}^2}\right)$, which satisfies $\mathbb{P}\left[\Omega^3_h\right] \to 1$ for $h \to 0$ by Lemma 5.4.1 and Markov's inequality. Using the implicit version of the discrete Gronwall inequality, which requires the non-restrictive mesh-constraint

$$h^{-\varepsilon} < \exp\Big(\frac{1}{Ck}\Big) \tag{5.4.16}$$

to cope with the leading term on the right-hand side in (5.4.15), and (5.4.14), lead to

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{\Omega^3_h}\max_{1\leq n\leq M}\Big(\|\mathbf{E}^n\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{m=1}^{n}\|(\mathbf{E}^m - \mathbf{E}^{m-1})\|^2_{\mathbb{L}^2} + k\sum_{m=1}^{n}\|\nabla\mathbf{E}^m\|^2_{\mathbb{L}^2}\Big)\Big]\\
&\qquad\leq C\Big(h^{3-2\beta_0-\varepsilon} + h^{2-\varepsilon} + kh^{-\varepsilon} + h^{2-\varepsilon}\,\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla\pi^m\|^2_{\mathbb{L}^2}\Big]\Big)\\
&\qquad\leq C\Big(h^{3-2\beta_0-\varepsilon} + kh^{-\varepsilon} + h^{2-\varepsilon}\,\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla\pi^m\|^2_{\mathbb{L}^2}\Big] + Ch^2\Big)
\end{aligned} \tag{5.4.17}$$

where $C \equiv C\Big(C_{t_M,2}; t_M\Big) > 0$ is from Lemma 4.2.1. Here we used the bound $\|\mathbf{u}^m - \mathbf{Q}_h^0\mathbf{u}^m\|^2_{\mathbb{L}^2} \leq Ch^2\|\nabla\mathbf{u}^m\|^2_{\mathbb{L}^2}$, together with Lemma 5.4.1.

Step 3. *Bootstrapping argument.* The error estimate (5.4.17) is dominated by the error $Ch^{3-2\beta_0-\varepsilon}$ that results from term II_b on in (5.4.10). To improve (5.4.17) we consider the set

$$\Omega^E_{h,m-1} = \Omega^3_{h,m-1} \cap \left\{\max_{1\leq\ell\leq m-1}\|\mathbf{Q}_h^0\mathbf{E}^\ell\|^2_{\mathbb{L}^2} \leq Ch^{3-2\beta_0-2\varepsilon} + kh^{-2\varepsilon} + C_{\nabla\pi}h^{2-2\varepsilon}\right\} \subset \Omega\,, \tag{5.4.18}$$

where $C_{\nabla\pi} = \mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla\pi^m\|^2_{\mathbb{L}^2}\Big]$. The probability of this set can be estimated by Markov's inequality

$$\mathbb{P}[\Omega^3_{h,m-1} \setminus \Omega^E_{h,m-1}] \leq \frac{\mathbb{E}\left[\mathbb{1}_{\Omega^3_{h,m-1}}\max_{1\leq\ell\leq m-1}\|\mathbf{Q}_h^0\mathbf{E}^\ell\|^2_{\mathbb{L}^2}\right]}{Ch^{1-2\varepsilon} + kh^{-2\varepsilon} + C_{\nabla\pi}h^{2-2\varepsilon}} \leq Ch^\varepsilon\,,$$

where in the last inequality we use (5.4.17). Thus, on the set $\Omega^E_{h,m-1}$, using inverse inequalities and (5.4.17) we compute

$$\begin{aligned}\|\nabla\mathbf{U}^{m-1}\|_{\mathbb{L}^2} &\leq C\Big(\|\nabla\mathbf{u}^{m-1}\|_{\mathbb{L}^2}+\|\nabla\mathbf{Q}_h^0\mathbf{E}^{m-1}\|_{\mathbb{L}^2}\Big)\leq C\|\nabla\mathbf{u}^{m-1}\|_{\mathbb{L}^2}+Ch^{-1}\|\mathbf{Q}_h^0\mathbf{E}^{m-1}\|_{\mathbb{L}^2}\\ &\leq C\|\nabla\mathbf{u}^{m-1}\|^2_{\mathbb{L}^2}+Ch^{-\beta_0}\big(h^{1/2-\varepsilon}+\sqrt{k}h^{-\varepsilon}+h^{1-\varepsilon}\big)\,. \qquad (5.4.19)\end{aligned}$$

This leads to the following improvement on the set $\Omega^E_{h,m-1}$ of the corresponding term from (5.4.10)

$$\begin{aligned}II_b &\leq C\,h^{\frac{3}{2}-\beta_0}\big(h^{1/2-\varepsilon}+\sqrt{k}h^{-\varepsilon}+h^{1-\varepsilon}\big)\,\|\mathbf{E}^m\|^{1/2}_{\mathbb{L}^2}\Big(\|\mathbf{E}^m\|_{\mathbb{L}^2}+\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}\Big)^{1/2}\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2}\\ &\quad+\|\nabla\mathbf{u}^m\|_{\mathbb{L}^2}\|\mathbf{E}^m\|^{1/2}_{\mathbb{L}^2}\Big(\|\mathbf{E}^m\|_{\mathbb{L}^2}+\|\nabla\mathbf{E}^m\|_{\mathbb{L}^2}\Big)^{1/2}\|\mathbf{A}\mathbf{u}^m\|_{\mathbb{L}^2} \qquad (5.4.20)\\ &\leq C\Big(\|\mathbf{U}^m\|^2_{\mathbb{L}^2}+\|\mathbf{U}^m\|^4_{\mathbb{L}^2}+\|\nabla\mathbf{u}^m\|^4_{\mathbb{L}^2}\Big)\|\mathbf{E}^m\|^2_{\mathbb{L}^2} \qquad (5.4.21)\\ &\quad+\frac{\nu}{8}\|\nabla\mathbf{E}^m\|^2_{\mathbb{L}^2}+C\,\Big(h^{4-2\beta_0}+kh^{3-2\beta_0-2\varepsilon}+h^{5-2\beta_0-2\varepsilon}+h^3\Big)\,\|\mathbf{A}\mathbf{u}^m\|^2_{\mathbb{L}^2}\,.\end{aligned}$$

because of (5.4.17).

We may now state the following result about the convergence of a general space-time discretisation (see Algorithm 3.3) of (5.1.1)–(5.1.3) with general noise, which asymptotically justifies optimal first order of convergence for the space discretisation, and evidences the critical influence of the pressure on the overall convergence behavior. We may proceed as in (5.3.21) with the sets $\Omega^3_{h,n}$ and defining a set

$$\Omega^E_h=\Big\{\max_{1\leq m\leq M}\|\mathbf{Q}^0_h\mathbf{E}^m\|^2_{\mathbb{L}^2}\leq Ch^{1-2\varepsilon}+kh^{-2\varepsilon}+C_{\nabla\pi}h^{2-2\varepsilon}\Big\}\cap\Omega^3_h\quad(1\leq m\leq M)\,.$$

Theorem 5.4.1. *Let $D=(0,L)^2$, $T>0$, and $\mathfrak{P}:=\big(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}\big)$ be a filtered probability space. Assume that (S_1) through (S_4) hold, with $\boldsymbol{\mathcal{H}}=\mathbb{W}^{1,2}_{per}$. Let $\mathbf{u}_0\in L^8(\Omega;\mathbb{V})$ be given. Let $\mathbf{W}$ be an $\mathbb{F}$-adapted measurable $\mathbf{Q}$-Wiener process on $\mathfrak{P}$ with $\mathbf{Q}\in\widehat{\mathcal{L}}_1(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{K}})$. For $k,h>0$, let $\mathcal{T}_h$ be a quasi-uniform triangulation of D, and $I_k=\{t_m\}_{m=0}^M$ be a uniform partition covering $[0,T]$, and assume (5.4.16). Let $\big(\mathbb{H}_h,L_h\big)$ be a pair of finite element spaces that satisfies the discrete LBB-condition (5.1.6). Assume that $\mathbf{U}^0$ is a $\mathbb{H}_h$-valued random variable with $\mathbb{E}\big[\|\mathbf{U}^0\|^8_{\mathbb{L}^2}\big]\leq C$, such that $\mathbb{E}\big[\|\mathbf{U}^0-\mathbf{u}^0\|^2_{\mathbb{L}^2}\big]\leq C\,h^2$. Let $\{\mathbf{u}^m\}_{m=1}^M$ be the solution given by Algorithm 5.1, and $\big\{(\mathbf{U}^m,\Pi^m)\big\}_{m=1}^M$ solves Algorithm 3.3. Then, the set Ω^3_h which is defined in* (5.4.14) *satisfies*

$$\Omega^3_h\subset\Omega,\quad\mathbb{P}[\Omega^3_h]\geq 1+\frac{C}{\varepsilon\log h}\quad\forall\,\varepsilon>0\,.$$

Moreover there exists a set $\Omega^E_h\subset\Omega$, and $\varepsilon>0$ such that

$$\mathbb{P}[\Omega^3_h\setminus\Omega^E_h]\leq Ch^{\varepsilon}\,,$$

and the following estimate holds

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_{\Omega^3_h\cap\Omega^E_h}\left(\max_{1\leq m\leq M}\|\mathbf{u}^m-\mathbf{U}^m\|^2_{\mathbb{L}^2}+k\sum_{m=1}^M\Big\|\nabla\big(\mathbf{u}^m-\mathbf{U}^m\big)\Big\|^2_{\mathbb{L}^2}\right)\right]\\ &\qquad\leq C(h^{2-3\varepsilon}+kh^{1-3\varepsilon}+kh^{-\varepsilon})+Ch^{2-3\varepsilon}\,\mathbb{E}\Big[k\sum_{m=1}^M\|\nabla\pi^m\|^2_{\mathbb{L}^2}\Big]\,. \qquad (5.4.22)\end{aligned}$$

The overall convergence behavior of iterates $\{\mathbf{U}^m\}_{m=1}^M$ from Algorithm 5.3 is then controlled by Theorems 5.3.1 and 5.4.1.

Corollary 5.4.1. *Let $\mathbf{u}_0 \in L^8\big(\Omega;\mathbb{V}\big)$. Under the same assumptions as stated in Theorems 5.3.1 and 5.4.1, for $\eta \in (0,\frac{1}{2})$ there hold*

$$\text{(i)} \qquad \mathbb{E}\left[\mathbb{1}_{\Omega_k^1\cap\Omega_h^3\cap\Omega_h^E}\Big(\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)\right] \le C\left(k^{\eta-\varepsilon}+C(h^{2-3\varepsilon}+kh^{-\varepsilon})+Ch^{2-3\varepsilon}\,\mathbb{E}\left[k\sum_{m=1}^M\|\nabla\pi^m\|_{\mathbb{L}^2}^2\right]\right),$$

$$\text{(ii)} \qquad \mathbb{E}\left[\mathbb{1}_{\Omega_k^2\cap\Omega_h^3\cap\Omega_h^E}\Big(k\sum_{m=1}^M\|\mathbf{u}(t_m)-\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)\right] \le C\left(k^{2\eta-\varepsilon}+C(h^{2-3\varepsilon}+kh^{-\varepsilon})+Ch^{2-3\varepsilon}\,\mathbb{E}\left[k\sum_{m=1}^M\|\nabla\pi^m\|_{\mathbb{L}^2}^2\right]\right).$$

Here, $\mathbf{u}_0 \in L^8\big(\Omega;\mathbb{V}\big)$ is needed, since estimates (5.3.14) and (5.3.15) are involved, which require higher moments of the gradient of the strong solution of (5.1.1)–(5.1.3); see Lemma 5.2.3. To derive rates of convergence in probability for corresponding errors now follows as in Corollary 5.3.1.

Corollary 5.4.2. *Under the same assumptions as stated in Theorems 5.3.1 and 5.4.1, if we assume $k = O(h)$ Algorithm 5.3 is convergent in probability with order α_1 (resp. α_2) in time (see Corollary 5.3.1), and order $\beta < 1$ in space both, with respect to the $L^\infty(0,T;\mathbb{L}^2(D))$-norm and the $\Big(k\sum_{m=1}^M\|\cdot\|_{\mathbb{W}^{1,2}}^2\Big)^{1/2}$-norm,*

$$\text{(i)} \lim_{\widetilde{C}\to\infty}\ \lim_{k,h\to 0}\ \mathbb{P}\left[\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{U}^m\|_{\mathbb{L}^2} \ge \widetilde{C}\left(k^{\alpha_1}+h^\beta+\sqrt{C_{\nabla\pi}}h^\beta\right)\right],$$

$$\text{(ii)} \lim_{\widetilde{C}\to\infty}\ \lim_{k,h\to 0}\ \mathbb{P}\left[\Big(k\sum_{m=1}^M\|\mathbf{u}(t_m)-\mathbf{U}^m\|_{\mathbb{L}^2}^2\Big)^{\frac{1}{2}} \ge \widetilde{C}\left(k^{\alpha_2}+h^\beta+\sqrt{C_{\nabla\pi}}h^\beta\right)\right].$$

Remark 5.4.3. *The time discretisation of the non-stationary stochastic Stokes equations is discussed in Remark 5.3.3. By Lemma 5.3.2, Remark 5.3.3 and Theorem 5.4.1, a simplified argumentation leads to*

$$\mathbb{E}\left[\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{U}^m\|_{\mathbb{L}^2}^2+k\sum_{m=1}^M\Big\|\nabla\big(\mathbf{u}(t_m)-\mathbf{U}^m\big)\Big\|_{\mathbb{L}^2}^2\right] \le C\left(k+h^2+h^2\,\mathbb{E}\Big[k\sum_{m=1}^M\|\nabla\pi^m\|_{\mathbb{L}^2}^2\Big]\right).$$

A severe restriction of Theorem 5.4.1 is the coupling of discretisation parameters evidenced in (5.4.22) for general noise; this choice may be avoided for mixed methods $(\mathbb{H}_h, L_h)$ where $\mathbb{V}_h \subset \mathbb{V}^{per}$. A well-known pairing here is the Scott-Vogelius mixed element [121, 133], which uses continuous polynomials of degree $i \ge 1$ to approximate the velocity, and discontinuous polynomials of degree $i-1$ for the pressure. A relevant property is then $\operatorname{div}\mathbf{V} \in L_h$ for $\mathbf{V} \in \mathbb{H}_h$, which can be used to prove that the condition $(\operatorname{div}\mathbf{V},\chi) = 0$ for all $\chi \in L_h$ implies $\operatorname{div}\mathbf{V} = 0$

pointwise in D. As a consequence, the following terms vanish (see (5.4.6)) in the error analysis for an Algorithm 5.3 that is based on the Scott-Vogelius mixed element,

$$\big(\pi^m, \operatorname{div} \mathbf{Q}_h^0 \mathbf{E}^m\big) = 0\,, \qquad \text{and} \qquad \Big([\operatorname{div} \mathbf{U}^m]\mathbf{U}^m, \mathbf{E}^m\Big) = 0\,. \tag{5.4.23}$$

Moreover, the convection term $k\big([\mathbf{U}^m \cdot \nabla]\mathbf{U}^m, \mathbf{\Phi}\big)$ in (5.4.1) can be handled as in the corresponding error analysis around (5.3.29), and allows to avoid the 'bootstrapping argument' from above to verify the following result.

Theorem 5.4.2. *Suppose that the same assumptions as in Theorem 5.4.1 are valid. Moreover assume that the finite element pairing is chosen such that* $\mathbb{V}_h \subset \mathbb{V}^{per}$. *Then for the set*

$$\Omega_h^4 := \big\{\omega \in \Omega : \max_{1 \le m \le M} \|\nabla \mathbf{u}_m\|_{\mathbb{L}^2}^2 \le \log\, h^{-\varepsilon}\big\} \qquad (\varepsilon > 0)$$

there holds the estimate

$$\mathbb{E}\left[\mathbb{1}_{\Omega_h^4}\left(\max_{1\le m\le M} \|\mathbf{u}^m - \mathbf{U}^m\|_{\mathbb{L}^2}^2 + k\sum_{m=1}^M \Big\|\nabla\big(\mathbf{u}^m - \mathbf{U}^m\big)\Big\|_{\mathbb{L}^2}^2\right)\right] \le C\, h^{2-\varepsilon}\,.$$

By evidence, this estimate does not involve the pressure of Algorithm 5.1 any more, which suggests a more robust discretisation in Algorithm 5.3 for general noise.

Remark 5.4.4. *If we consider the result from Section 5.3.3, we note that the application of the space discretisation to get convergence on the whole set* Ω *is not possible, since we need exponential estimates to control the time-discrete iterations. This remains an unsolved question, leaving open the question of global convergence for the full discretisation in the case of additive noise.*

5.5 Computational experiments

In this section, we report on computational experiments which show the the convergence behavior of the proposed algorithms and compare the convergence results of Euler and Chorin based schemes. The space discretisation is accomplished by the stable MINI element; cf. [15, 74] for details.

We consider the finite-dimensional Wiener process ($t \in [0,T]$)

$$\mathbf{W}(t) = \sum_{j,k=1}^N \lambda_{j,k}\beta_{j,k}(t)\mathbf{e}_{j,k} \qquad (1 \le N < \infty)\,,$$

where $\lambda_{j,k} \equiv 20$, $\{\beta_{j,k}\}_{j,k=1}^N$ is a family of independent, real-valued Wiener processes on $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$, as well as $\{\mathbf{e}_{j,k}\}_{j,k=1}^\infty$ are functions specified below. For the initial conditioni we choose $\mathbf{u}_0 = \mathbf{0}$.

For the discretisation we consider the following fully practical scheme based on the Euler scheme:

$$\begin{aligned}
&\big(\mathbf{U}^m - \mathbf{U}^{m-1}, \mathbf{\Phi}\big) + k\nu\big(\nabla \mathbf{U}^m, \nabla\mathbf{\Phi}\big) + k\Big([\mathbf{U}^{m-1}\cdot\nabla]\mathbf{U}^m, \mathbf{\Phi}\Big) \\
&\quad + \frac{k}{2}\Big([\operatorname{div} \mathbf{U}^{m-1}]\mathbf{U}^m, \mathbf{\Phi}\Big) - k\big(\Pi^m, \operatorname{div}\mathbf{\Phi}\big) = \big(\mathbf{g}(\mathbf{U}^{m-1})\Delta_m \mathbf{W}, \mathbf{\Phi}\big)\,, \\
&(\operatorname{div} \mathbf{U}^m, \Lambda) = 0\,.
\end{aligned}$$

We remark that a simple adaptation of the previous calculation leads to the same error estimates as for the fully implicit scheme. In all the simulations the number of realisations is fixed to $N_p = 500$, which is a good compromise in order to get convergence without too big fluctuations, and a reasonable computation time (recall the two nonlinear terms, for which we have to assemble the corresponding finite element representation at every time-step).

5.5.1 Error for periodic boundary conditions

We consider the domain $D = (0, \pi)^2$ and periodic boundary conditions. We consider the eigenfunctions

$$\mathbf{e_s} = \begin{cases} \frac{1}{\sqrt{2\pi}}\mathbf{s}^\perp \sin(\mathbf{s}\cdot\mathbf{x}) & \mathbf{s} \in \mathbb{Z}^2_+, \\ \frac{1}{\sqrt{2\pi}}\mathbf{s}^\perp \cos(\mathbf{s}\cdot\mathbf{x}) & \mathbf{s} \in -\mathbb{Z}^2_+, \end{cases}$$

where $s^\perp = (-s_2, s_1)^T$, for $s_1, s_2 \in \mathbb{R}$, and

$$\mathbb{Z}^2_+ = \{(s_1, s_2) \,|\, s_1 > 0 \text{ or } (s_1 = 0, s_2 > 0)\}\,.$$

We consider a Wiener noise only for $\mathbf{s} \in \{(1,1), (2,2), (3,3), (4,4)\}$ in order to prevent the influence of the error caused by the truncation of the series representation of the Wiener process. The computations are performed on the domain $D_T = (0, 2\pi)^2 \times (0, 1/4)$, We consider time steps $k \in \frac{1}{1024}, \frac{1}{512}, \frac{1}{256}, \frac{1}{128}, \frac{1}{64}$, where the first time-step is used to compute the reference solution, ans plays the rôle of the exact solution for the computation of the error. The plot is shown in Figure 5.5.1 for the semi-implicit Euler scheme. First we note that the convergence rate is almost 1/2, which shows, together with Theorem 5.3.1, that the Hölder regularity is probably better than what we proved.

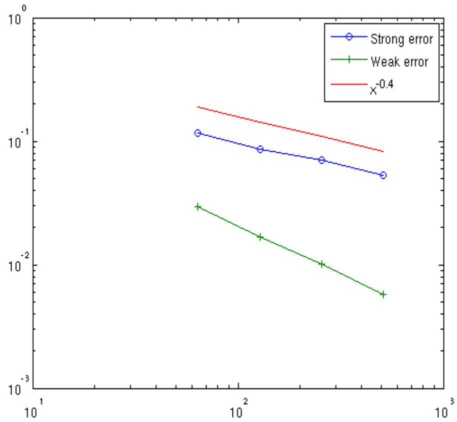

Figure 5.5.1: Error for periodic boundary conditions and periodic divergence-free noise.

This numerical example let us suppose that the results given in Theorems 5.3.1 and 5.4.1 may not be optimal in general, while results like Theorem 5.3.3 may be expected. We consider several numbers of realisations $N_p \leq 500$, and we do not note the exponential dependence of the error constant described in Theorem 5.4.1. On the contrary, the fluctuation of the error are less pronounced for a higher number of realisations. We also consider the weak error for the Euler scheme and it is clearly better than what is proved Remark 5.3.2.

5.5.2 Error for Dirichlet boundary conditions

The strong error analysis for the schemes we proposed strongly relies on the regularity given by Lemma 5.2.1, which is valid only for periodic boundary conditions. Goal of this example it to analyse the error behavior for homogeneous Dirichlet boundary conditions, and give a practical motivation for the numerical analysis of the corresponding problem. We consider now two different cases. First the eigenbasis from Section 5.5.1, and in addition an eigenbasis of the Laplace operator on the domain $D = (0, 1)^2$ with homogeneous boundary conditions, i.e.

$$\mathbf{e}_{j,k}(x, y) := \Big(\sin(j\pi x)\sin(k\pi y), \sin(j\pi x)\sin(k\pi y)\Big)^\top,$$

for $j, k \in \{1, 2, 3, 4\}$ again to avoid the influence of the truncation error. The plot of the

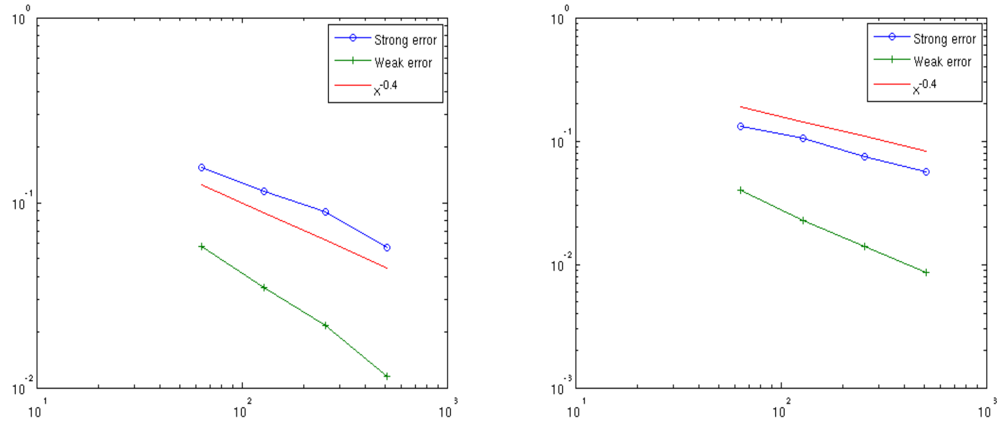

Figure 5.5.2: Error for Dirichlet boundary conditions. Divergence-free noise (left) and non divergence-free noise (right).

error is shown in Figure 5.5.2, and shows that the scheme converges (almost) optimally also for Dirichlet boundary conditions, and both, solenoidal and non-solenoidal noise, evidencing that the coupling of the parameters caused by the irregular pressure seems not to hold, in contrast to our theoretical studies. This result may be interpreted as a consequence of the fact that the rôle of the Lagrange multiplier is not well understood in both, theory and numerical analysis. Again, we notice that the error is not affected by the number of realisations.

5.5.3 Dependence on the Reynolds number

Here, we study the dependence of the solutions on the Reynolds number. In the same setting as in Section 5.5.2, we consider the discretisation for $Re = \in \{1, 10, 100\}$, an $\nu = \frac{1}{Re}$, and divergence-free noise. According to the theory, proof of Theorems 5.3.1 and 5.4.1, the error depends linearly on $\frac{1}{\nu} = Re$, and we expect that the convergence rate is not affected by the viscosity of the fluids, although the magnitude of the error suffers from it.

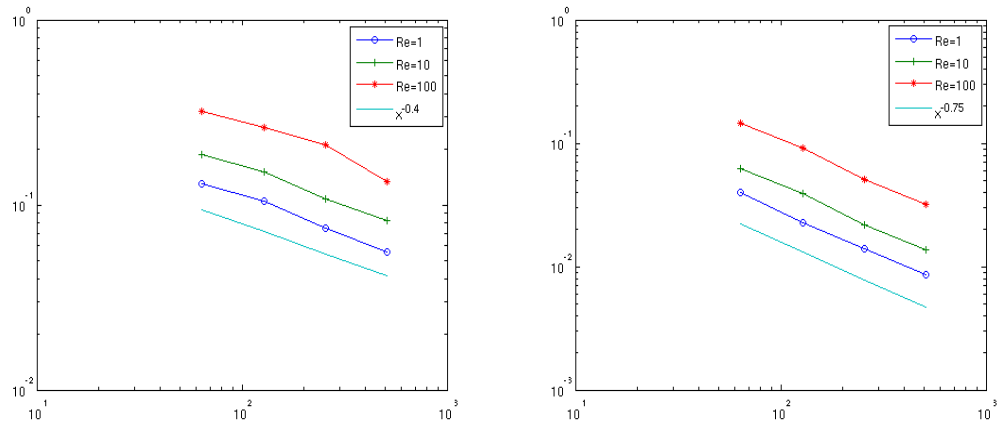

Figure 5.5.3: Error for Dirichlet boundary conditions and different viscosities. Strong (left) and weak convergence (right).

The results are depicted in Figure 5.5.3, showing comparable results for strong and weak error.

5.6 Summary and outlook

We analysed various space-time discretisations of the 2D stochastic Navier-Stokes equations (5.1.1)–(5.1.3), including the (semi-)implicit Euler scheme, and discretely LBB-stable mixed finite elements. The main problems are caused by the subtle interplay of nonlinearity, algebraic constraint, stochasticity, and space-time discretisation. To handle these problems, we perform the error analysis on various subsets of Ω to use a Gronwall-type argument, and to derive error bounds with rates; see e.g. Theorems 5.3.1 and 5.4.1. The main tools are bounds to control higher moments of the solutions of (5.1.1)–(5.1.3) (Lemma 5.2.1), the Hölder regularity for $\nabla\mathbf{u}$ (Lemma 5.2.3), a perturbation argument (Theorem 5.4.1), and exactly divergence-free elements (Theorem 5.4.2). All results in this work are obtained for periodic boundary conditions; it would be interesting to generalise them to boundary value problems, and to the three-dimensional case.

Piecewise solenoidal mixed finite elements of non-conforming type have been developed in [2] to e.g. reduce the computational effort of the Scott-Vogelius finite elements. The interaction of the non-conformity, nonlinearity, and stochasticity in a corresponding space discretisation of (5.1.1)–(5.1.3) remains an open problem. Another open question is the weak convergence behavior of the schemes proposed in this work, and the concomitant clarification of the interaction with the Lagrange multiplier, nonlinearity, and stochasticity in this framework.

The numerical experiments indicate that the bounds from Theorems 5.3.1 and 5.4.1 are not optimal. In particular, the subtle interplay between nonlinearity and stochastic forcing that arises from the proofs, appears not to be a source of problems for the convergence behavior of the analysed discretisation schemes.

Acknowledgment: I warmly thank Dr. Philipp Dörsek, who pointed to my attention the exponential estimates from [67], making possible the results from Section 5.3.3.

Appendices

Appendix A

Discrete derivatives

In this section we derive some rules for the computations with discrete derivatives. Let I_k be a given mesh of the interval $[0,T]$ of size $k>0$. Given a finite sequences $\{f^m\}_{m=0}^M$ we define two piecewise constant functions $f^+ : [0,T] \to \mathbb{R}$ and $f^-+ : [0,T] \to \mathbb{R}$ as

$$f^+(t) := f^m \quad \forall\, t \in [t_{m-1}, t_m)\,, \tag{A.0.1}$$

$$f^-(t) := f^{m-1} \quad \forall\, t \in (t_{m-1}, t_m]\,. \tag{A.0.2}$$

with $f^+(T) := f^M$, and $f^-(0) := f^0$.

Let us define the so called discrete time derivative of the sequence $\{f^m\}_{m=0}^M$ by

$$d_t f^m := \frac{f^m - f^{m-1}}{k} \qquad (1 \le m \le M)\,. \tag{A.0.3}$$

Then we have

$$\begin{aligned} d_t(f^m g^m) &= \frac{1}{k}\left(f^m g^m - f^{m-1} g^{m-1}\right) \\ &= \frac{1}{k}\left(f^m g^m - f^m g^{m-1} + f^m g^{m-1} - f^{m-1} g^{m-1}\right) \\ &= f^m d_t g^m + g^{m-1} d_t f^m. \end{aligned} \tag{A.0.4}$$

Let us also define the operator d_t acting on f^+ by

$$d_t f^+ = \frac{f^+ - f^-}{k}. \tag{A.0.5}$$

Then, defining piecewise constant functions $g^+, g^- : [0,T] \to \mathbb{R}$ corresponding to a sequence $(g^m)_{m=0}^M$ as above, we have the following discrete product rule.

Lemma A.0.1. *In the above framework the following identity holds*

$$d_t(f^+ g^+) = f^+ d_t g^+ + g^- d_t f^+\,. \tag{A.0.6}$$

Another useful tool is the following

Lemma A.0.2. *We have*

$$\int_0^T d_t f^+ \,\mathrm{dt} = f^+(T) - f^-(0).$$

Proof of Lemma A.0.1. $\int_0^T d_t f^+ \,\mathrm{dt} = k \sum_{i=1}^M \frac{f^i - f^{i-1}}{k} = f^M - f^0.$ □

A consequence of this lemma is the following identity

$$\int_0^T f^+ d_t g^+(t)\,\mathrm{d}t = (f^+g^+)(T) - (f^-g^-)(0) - \int_0^T (d_t f^+(t))g^-(t)\,\mathrm{d}t. \tag{A.0.7}$$

We derive now a discrete version of the chain rule for a function $\mathrm{e}^{f^+} : [0,T] \to \mathbb{R}$. Applying Taylor's formula to functions $\phi(x) = \mathrm{e}^x$, we get

$$\frac{\mathrm{e}^b - \mathrm{e}^a}{b-a} = \mathrm{e}^a + \mathrm{e}^\xi \frac{b-a}{2}$$

for some $\xi \in (a,b)$. Putting $a = f^-$ and $b = f^+$, we infer that there exists $\xi \in (f^-, f^+)$ such that

$$\begin{aligned} d_t \mathrm{e}^{f^+} &= \mathrm{e}^{f^-} d_t f^+ + \mathrm{e}^\xi \frac{(f^+ - f^-)^2}{2k} \\ &= \mathrm{e}^{f^- - f^+} \mathrm{e}^{f^+} d_t f^+ + \mathrm{e}^\xi \frac{(f^+ - f^-)^2}{2k}. \end{aligned} \tag{A.0.8}$$

We end this chapter with the discrete version of the Gronwall inequality; see [71, Lemma 5.1]

Lemma A.0.3. *Let k, B, and a_j, b_j, c_j, d_j be nonnegative numbers for $j \geq 0$, such that*

$$a_n + k\sum_{j=0}^{M} b_j \leq k\sum_{j=0}^{M} d_j a_j + k\sum_{j=0}^{M} c_j + B \quad \text{for } M \geq 0\,.$$

If $k\gamma_j < 1$ for all $j \geq 0$, there holds

$$a_n + k\sum_{j=0}^{M} b_j \leq \left(k\sum_{j=0}^{M} c_j + B\right) \exp\left(k\sum_{m=0}^{M} \sigma_j d_j\right)$$

for $\sigma_j = \frac{1}{1-kd_j}$.

We may also consider the following explicit version: there holds

$$a_m \leq \Big(B + k\sum_{j=1}^{m-1} c_j\Big)\mathrm{e}^{k\sum_{j=1}^{l-1} d_j}$$

provided

$$a_m \leq B + k\sum_{j=1}^{m-1}(d_j a_j + c_j)$$

for any $m \geq 1$.

Appendix B

Quadratic variation

We propose a generalization of [20, Theorem C.2]. In this section let the elements of the time grid I_k be denoted by t_m^k to avoid confusion in the proof of Theorem B.0.1. For two Hilbert spaces $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{H}}$, let $\mathcal{I}_1(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{H}})$ be the space of nuclear operators from $\boldsymbol{\mathcal{K}}$ to $\boldsymbol{\mathcal{H}}$.

Theorem B.0.1. *Assume that E and $\big(H, (\cdot, \cdot)\big)$ are a separable metric, and a Hilbert space respectively. Assume that $\mathcal{B} := (\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $h, k > 0$ and $I_k := \{t_m^k\}_{m=0}^M$. Suppose that for every pair (h, k) we are given two discrete time processes $\big\{U_{k,h}^m\big\}_{m=0}^M$ and $\big\{M_{k,h}^m\big\}_{m=0}^M$, such that*

$$U_{k,h}^m : \Omega \to E \qquad \text{and} \qquad M_{k,h}^m : \Omega \to H\,.$$

Given these processes, we define the following piecewise constant interpolation processes

$$U_{k,h}^+ : [0, T] \times \Omega \to E \qquad \text{and} \qquad M_{k,h}^+ : [0, T] \times \Omega \to H$$

as in the previous sections.
We assume also that

$$U : [0, T] \times \Omega \to E \qquad \text{and} \qquad M : [0, T] \times \Omega \to H$$

are stochastic processes such that for every $t \in [0, T]$, $\mathbb{P}$-almost surely, as $k \to 0$ and $t_m^k \to t$,

$$U_{k,h}^+(t_m^k) \to U(t) \ \text{ in } E, \tag{B.0.1}$$

$$M_{k,h}^+(t_m^k) \to M(t) \ \text{ in } H. \tag{B.0.2}$$

We denote by $\mathbb{F}^{k,h} = \big\{\mathcal{F}_{t_m^k}^{k,h};\, m = 1, \ldots, M\big\}$ the filtration on the probability space $\mathcal{B}$ generated by the process $\big\{U_{k,h}^+(t_m^k)\big\}_{m=0}^M$. Similarly, we denote by $\mathbb{F}$ the filtration on the probability space $\mathcal{B}$ generated by the process U. Finally, we denote by $\overline{\mathbb{F}}$ the augmentation of the filtration $\mathbb{F}$. For each h, k, assume that $R_{k,h}$ is an operator-valued process defined on I_k such that the process

$$\Big\{M_{k,h}^+(t_m^k) \otimes M_{k,h}^+(t_m^k) - R_{k,h}^+(t_m^k);\, m = 1, \ldots, M\Big\}$$

is an $\mathbb{F}^{k,h}$-martingale. Assume that R is an $\infty_1(H)$-valued $\overline{\mathbb{F}}$-progressively measurable process such that for every $t \in [0, T]$, $\mathbb{P}$-almost surely

$$(R_{k,h}^+(t_m^k)x, y) \to (R(t)x, y) \quad \forall x, y \in H, \tag{B.0.3}$$

for $k \to 0$, and $t^k_m \to t$. Assume also that for some $r > 1$ and for every $t \in [0,T]$,

$$\sup_{h,k>0} \mathbb{E}\left[\|M^+_{k,h}(t)\|^{2r}_E\right] < \infty, \tag{B.0.4}$$

$$\sup_{h,k>0} \mathbb{E}\left[\|R^+_{k,h}(t)\|^{r}_{\mathcal{L}(E)}\right] < \infty. \tag{B.0.5}$$

Then R is equal to $\langle\langle M\rangle\rangle$, the quadratic variation process of the $\mathbb{F}$-martingale M.

Proof. In view of the Doob-Meyer Theorem it is enough to prove that for all $x, y \in H$ the process $N = (N(t))_{t\geq 0}$, $t \in [0,T]$ defined by

$$N(t) = (M(t),x)(M(t),y) - (R(t)x,y), \quad t \in [0,T]$$

is an $\overline{\mathbb{F}}$-martingale. To show this, notice that by [40, p. 75] it is enough to show the martingale property with respect to $\mathbb{F}$.

Let us fix $x, y \in H$, and $t_1, t_2 \in [0,T]$ such that $t_2 \leq t_1$. We have to show that for any choice of times $0 \leq s_1 < s_2 < \ldots < s_n \leq t_2$, where $n \in \mathbb{N}$, and any bounded and continuous functions $h_i : E \to \mathbb{R}$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, the following equality holds

$$\mathbb{E}\left[\Big(N(t_1) - N(t_2)\Big)\prod_{i=1}^N h_i(U(s_i))\right] = 0.$$

By assumption, the process $\{N_{k,h}(t^k_n)\}^M_{n=1}$, defined by

$$N_{k,h}(t^k_n) := (M_{k,h}(t^k_n),x)(M_{k,h}(t^k_n),y) - (R_{k,h}(t^k_n)x,y)$$

is an $\mathbb{F}^{k,h}$-martingale. Let us fix a partition $0 \leq s_1 < s_2 < \ldots < s_n \leq t_2$ and let us choose a corresponding sequence of partitions $0 \leq s^k_1 < s^k_2 < \ldots < s^k_n \leq t^k_2$ with $s^k_i \nearrow s_i$ and $t^k_i \nearrow t_i$ for all indices i. Thus, for every sequence satisfying the properties given above, the following equality

$$\mathbb{E}\left[\Big(N_{k,h}(t^k_1) - N_{k,h}(t^k_2)\Big)\prod_{i=1}^N h_i(U_{k,h}(s^k_i))\right] = 0. \tag{B.0.6}$$

holds.

Now, in view of assumptions (B.0.3), (B.0.4) and (B.0.5), since the functions h_i are bounded, the process on the left-hand side of equality (B.0.6) is uniformly integrable. Using the almost sure pointwise convergence of

$$\Big(N_{k,h}(t^k_1) - N_{k,h}(t^k_2)\Big)\prod_{i=1}^N h_i(U_{k,h}(s^k_i)),$$

ensured by assumptions (B.0.1) and (B.0.2), and a well-known result from [106, Appendix C], we can conclude

$$0 = \lim_{h,k\to 0} \mathbb{E}\left[\Big(N^+_{k,h}(t^k_1) - N_{k,h}(t^k_2)\Big)\prod_{i=1}^N h_i(U_{k,h}(s^k_i))\right] = \mathbb{E}\left[\Big(N(t_1) - N(t_2)\Big)\prod_{i=1}^N h_i(U(s_i))\right].$$

□

Now we give a characterisation of the quadratic variation of a martingale with values in a dual of a Hilbert space.

Let V be a Hilbert space, with dual denoted by V'. Let $V \subset H \cong H' \subset V'$ be a Gelfand triple. We denote the scalar products of V and H respectively by $(\cdot,\cdot)_V$ and $(\cdot,\cdot)$. The induced norms are denoted by $\|\cdot\|$ and $|\cdot|$ respectively. By $\langle\cdot,\cdot\rangle$ we denote the dual pairing between V and V'. By the Lax-Milgram lemma we have an isomorphism $I : V \to V'$ defined by $I^{-1}(f) = u$, where for $f \in V'$, u is the solution of

$$(u,v)_V = \langle f, v\rangle = \langle I(u), v\rangle \quad \forall v \in V\,. \tag{B.0.7}$$

Due to the definition of I, it is easy to see that

$$\|I(u)\|_{V'} = \|u\|,$$

where $\|u\|_{V'} = \sup\{|\langle u,v\rangle|\ ;\ \|v\| \le 1\}$, for all $u \in V'$, establishing the isometry property of the operator I. Thus we may define the scalar product on V' by

$$(u,v)_{V'} := \big(I^{-1}(u), I^{-1}(v)\big)_V \quad \forall\, u,v \in V'\,.$$

Defining a norm on V' by means of the scalar product, i.e., $|||f|||_{V'} := \sqrt{\langle f, f\rangle_{V'}}$, there holds

$$|||f|||_{V'} = \|f\|_{V'} \qquad \forall\, f \in V'\,.$$

Let $M \equiv \{M_t\ ;\ t \in [0,T]\}$ be a square-integrable V'-valued martingale. A positive process $R \equiv \{R(t)\ ;\ t \in [0,T]\}$ is the quadratic variation of M if and only if the process

$$(M,a)_{V'}(M,b)_{V'} - (Ra,b)_{V'} \quad \forall\, a,b \in V',$$

is a martingale; see [37, p. 81]. We give a representation of the quadratic variation R by means of dual pairing instead of scalar product on V', by means of the isometric isomorphism I. We begin with a lemma which can be easily proved.

Lemma B.0.4. *There holds*

$$\langle f, x\rangle = \big(f, I(x)\big)_{V'} \qquad \forall\, f \in V', \quad \forall\, x \in V.$$

Let us define an operator $\widetilde{R} := R \circ I : V \to V'$. Then there exists a martingale $N \equiv \{N_t\ ;\ t \in [0,T]\}$ such that for all $t \in [0,T]$

$$\begin{aligned} \langle \widetilde{R}(t)u, v\rangle &= \big\langle R(t)I(u), v\big\rangle = \Big(R(t)I(u), I(v)\Big)_{V'} \\ &= \big(M_t, I(u)\big)_{V'}\big(M_t, I(v)\big)_{V'} + N_t \qquad \forall\, u,v \in V\,. \end{aligned} \tag{B.0.8}$$

Since I is invertible, we have the following

Theorem B.0.2. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. If $V \subset H \cong H' \subset V'$ is a Gelfand triple, and M is a V'-valued $\mathbb{F}$-martingale, then there exists an increasing $\mathcal{I}_1(V,V')$-valued process $\widetilde{R}$ such that for all $u,v \in V$ the process*

$$\langle M, u\rangle\langle M, v\rangle - \langle \widetilde{R}u, v\rangle$$

is a martingale. Moreover $\widetilde{R} = R \circ I$, where R is the quadratic variation of M and $I : V \to V'$ is the isometric isomorphism given in (B.0.7).

Now combining Theorems B.0.1 and B.0.2, we have a new version of B.0.1, where it is enough to prove the assumptions for the process $\widetilde{R}$.

Theorem B.0.3. *Assume that $\big(V,(\cdot,\cdot)\big)$ and E are respectively a Hilbert and a separable metric space, and suppose that we are given an Hilbert space H such that $V \subset H \cong H' \subset V'$ is a Gelfand triple. We denote by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V'. Assume that $\mathcal{B} := (\Omega,\mathcal{F},\mathbb{P})$ is a probability space. Let $h,k>0$, and $I_k := \{t^k_m\}_{m=0}^M$. Suppose that for every pair (h,k) we are given two discrete time processes $\big\{U^m_{k,h}\big\}_{m=0}^M$ and $\big\{M^m_{k,h}\big\}_{m=0}^M$, such that*

$$U^m_{k,h} : \Omega \to E \qquad \text{and} \qquad M^m_{k,h} : \Omega \to V' \,.$$

Given these processes we define the following piecewise constant interpolation processes

$$U^+_{k,h} : [0,T]\times\Omega \to E \qquad \text{and} \qquad M^+_{k,h} : [0,T]\times\Omega \to V'$$

as in the previous sections.
We assume also that

$$U : [0,T]\times\Omega \to E \qquad \text{and} \qquad M : [0,T]\times\Omega \to V'$$

are stochastic processes such that for every $t\in[0,T]$, $\mathbb{P}$-almost surely, as $k\to 0$ and $t^k_m \to t$,

$$U^+_{k,h}(t^k_m) \to U(t) \ \text{ in } E, \tag{B.0.9}$$

$$M^+_{k,h}(t^k_m) \to M(t) \ \text{ in } V'. \tag{B.0.10}$$

We denote by $\mathbb{F}^{k,h} = \big\{\mathcal{F}^{k,h}_{t^k_m};\, m=1,\dots,M\big\}$ the filtration on the probability space $\mathcal{B}$ generated by the process $\big\{U^+_{k,h}(t^k_m)\big\}_{m=0}^M$. Similarly, we denote by $\mathbb{F}$ the filtration on the probability space $\mathcal{B}$ generated by the process U. Finally, we denote by $\overline{\mathbb{F}}$ the augmentation of the filtration $\mathbb{F}$. For each h,k, assume that $R_{k,h}$ is an operator-valued process defined on I_k such that the process

$$\Big\{\big\langle M^+_{k,h}(t^k_m),u\big\rangle\big\langle M^+_{k,h}(t^k_m),v\big\rangle - \big\langle \widetilde{R}^+_{k,h}(t^k_m)u,v\big\rangle\,;\, m=1,\dots,M\Big\}$$

is an $\mathbb{F}^{k,h}$-martingale for all $u,v\in V$. Assume that $\widetilde{R}$ is an $\mathcal{I}_1(V,V')$-valued $\overline{\mathbb{F}}$-progressively measurable process such that for every $t\in[0,T]$, $\mathbb{P}$-almost surely

$$\langle \widetilde{R}^+_{k,h}(t^k_m)x,y\rangle \to \langle \widetilde{R}(t)x,y\rangle \quad \forall x,y\in V, \tag{B.0.11}$$

for $k\to 0$, and $t^k_m\to t$. Assume also that for some $r>1$ and for every $t\in[0,T]$,

$$\sup_{h,k>0} \mathbb{E}\Big[\|M^+_{k,h}(t)\|^{2r}_{V'}\Big] < \infty, \tag{B.0.12}$$

$$\sup_{h,k>0} \mathbb{E}\Big[\|\widetilde{R}^+_{k,h}(t)\|^{r}_{\mathcal{L}(V,V')}\Big] < \infty. \tag{B.0.13}$$

Define $R := \widetilde{R}\circ I$, where I is the isometric isomorphism given in (B.0.7). *Then R is equal to $\langle\langle M\rangle\rangle$, the quadratic variation process of the $\mathbb{F}$-martingale M.*

Appendix C

Finite elements

Here we recall some basic concepts from the finite element theory. Most of them can be found in [14].

Definition C.0.1 (Definition 3.3.11, [14]). *A triangulation of a polygonal domain D is a subdivision consisting of triangles having the property that no vertex of any triangle lies in the interior of an edge of another triangle.*

The definition can be extended to a subdivision of tetrahedra analogously

Definition C.0.2 (Definition 4.4.13, [14]). *Let D be a given domain and let $\{\mathcal{T}_h\}_{0<h\leq 1}$, be a family of subdivision such that*

$$\max\{diam(T)\,|\,T\in\mathcal{T}_h\}\leq Ch\,diam(D)\,.$$

The family is said to be quasi-uniform if the exists $\rho>0$ such that

$$\min\{diam(B_T)\,|\,T\in\mathcal{T}_h\}\geq\rho\,h\,diam(D)$$

for all $h\in(0,1]$, where B_T is the largest ball contained in T such that T is star-shaped with respect to B_T. The family is said to be non-degenerate if there exists $\sigma>0$ such that for all $T\in\mathcal{T}_h$ and for all $h\in(0,1]$,

$$diam(B_T)\geq\sigma\,diam(T)\,.$$

If a family is quasi-uniform, then is is non-degenerate, but not conversely.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of a bounded polygonal domain or polyhedral domain $D\subset\mathbb{R}^d$, $d-2,3$ into triangles of tetrahedra of maximal diameter $h>0$. Let $\mathcal{N}=\{N_1,\ldots,N_k\}$ be the functional corresponding to the evaluation at the vertices $\{\mathbf{x}_\ell\}_{\ell=1}^k$ of $\mathcal{T}_h$, i.e.,

$$N_\ell(u)=u(\mathbf{x}_\ell)\quad u\in C(\overline{D}),\ \ell\in\{1,\ldots,\ell\}\,,$$

and let

$$\{\boldsymbol{\phi}_\ell\,|\,\ell=1,\ldots,k\}$$

be the nodal basis for $\mathcal{T}_h$. Then the corresponding Lagrange interpolator is defined by

$$\mathcal{I}u:=\sum_{\ell=1}^{k}N(u)\boldsymbol{\phi}_\ell\quad\forall\,u\in C^0(\overline{D})\,.$$

Then, for a triangle T of maximal diameter $h\in(0,1]$ there holds the following approximation property, see [14, Theorem 4.4.4]

$$|u-\mathcal{I}u|_{W^{i,p}(T)}\leq C(m,d)h^{m-i}|u|_{W^{m,p}(T)}\quad\text{for } m-\frac{d}{p}>0\,.$$

for $0 \leq i \leq m$. Moreover, the following inverse estimate holds

$$\|v\|_{W^{i,p}} \leq C(l,m,p,q)h^{m-i+\frac{d}{p}-\frac{d}{q}}\|v\|_{W^{m,q}}$$

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