EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

University of Tübingen
Working Papers in
Economics and Finance

No. 73

A Policy Response to a Downside of the Integration of Economies: An Impossibility Theorem

by

Oded Stark

Faculty of Economics and Social Sciences
www.wiwi.uni-tuebingen.de

A policy response to a downside of the integration of economies: An impossibility theorem

by

Oded Stark

Universities of Bonn, Tuebingen, and Warsaw

Mailing Address: Oded Stark
ZEF, University of Bonn
Walter-Flex-Strasse 3
D-53113 Bonn
Germany

E-Mail Address: ostark@uni-bonn.de

May 2014

Abstract

Recent research shows that the merger of economies increases aggregate stress. This paper shows that there is no income distribution policy which will ensure that the wellbeing of the individuals belonging to merging economies does not fall below their pre-merger level.

Keywords: Merger of populations; Revision of social space; Aggregate relative deprivation; Societal stress; Policy response

JEL classification: D04; D63; F55; H53; P51

1. Introduction

In recent research we have studied the integration of economies, which we view as a merger of populations, and the consequent changes in social space and people's comparison sets (Stark, 2013). Specifically, we have looked at the merger of populations as a merger of income vectors; we measured social stress by aggregate relative deprivation; and we showed that (except in the special case in which the merged populations have identical income distributions) a merger increases aggregate relative deprivation. We referred to this result as "superadditivity." Given this increase, in the current paper we assess whether a budget-constrained policy-maker can reverse the increase by means of an income redistribution that retains individual levels of wellbeing at their pre-merger levels. We show that such a reversal is not feasible; there is not enough of a gain to be skimmed off to compensate for the loss. We refer to this result as a general impossibility theorem. The theorem reported in the present paper goes beyond the examples set out in Stark (2010) and, as such, the result presented here supports the notion that the preliminary and specific finding reported in Stark (2010) is robust and general.

When populations merge, the social horizons faced by the individuals who constitute the merged population change: people who were previously outside the individuals' social domain are brought in. The (thus far six) successive monetary mergers of European countries constitute an example: the replacement of diverse currencies by a common currency brings about an instantaneous change in the comparison environment, expanding the reference space of individuals in a given country to encompass individuals from the adjoining countries. Although, prior to the introduction of the euro as a common currency, individuals in specific European countries were certainly able to compare their incomes with the incomes of individuals in other European countries, the comparison was not immediate. It required effort to convert incomes denominated in different currencies and was presumably not often attempted. Upon currency unification, the comparison environment changes *instantaneously*, enabling, indeed inviting, an easy comparison with others' incomes. For example, workers who perform the same task and are employed by a manufacturer with plants located in different countries will be able to compare their earnings with each other directly, effortlessly, and routinely.

How can a policy be designed to mitigate the heightened social stress? Policy-makers need only look around to appreciate the speed and ferocity with which social stress can cascade into social unrest. Naturally, it will be desirable to enact policies to ensure that individuals' wellbeing does not fall below its pre-merger level. Drawing on the example of the merger of a one-person population with a two-person population, Stark (2010) demonstrated that such a policy, which is a staple of public finance (a Pareto neutralizing transfer from the gainers to the losers), cannot be implemented. In the present paper, we generalize this "impossibility result" to the merger of populations of any size: the loss always outweighs the gain. In combination, the superadditivity theorem in Stark (2013) and the present paper's impossibility theorem raise the specter of a dark side of the integration of economies that cannot be easily reversed.

In Section 2 we present measures of individual and aggregate relative deprivation, and we restate the superadditivity theorem: the aggregate relative deprivation of merged populations is larger than or equal to the sum of the pre-merger levels of the aggregate relative deprivation of the constituent populations. In Section 3 we study a policy response to the increase in post-merger discontent. We show that an income redistribution which seeks to retain the pre-merger levels of wellbeing cannot be implemented, a general impossibility result. Section 4 concludes.

2. A measure of deprivation and the superadditivity of aggregate relative deprivation (*ARD*) with respect to the merger of two populations

We measure the stress level of a population by adding the levels of stress experienced by the individuals who constitute the population. We refer to this sum as the aggregate relative deprivation (*ARD*) of the population. We measure the stress of an individual by the extra income units that others in the population have, we sum up these excesses, and we normalize by the size of the population. (A detailed exposition is in Stark, 2013).

For an ordered vector of incomes in population *P* of size n, $x=\left(x_1,...,x_n\right)$, where $x_1 \le x_2 \le ... \le x_n$, the relative deprivation of the *i*-th individual whose income is x_i, $i=1,2,...,n$, is defined as

$$RD\left(x_i,x\right) \equiv \frac{1}{n}\sum_{j=i+1}^{n}\left(x_j - x_i\right) \tag{1}$$

where it is understood that $RD\left(x_n,x\right)=0$. To ease the analysis that follows, an alternative representation of the relative deprivation measure is helpful. Let $F\left(x_i\right)$ be the fraction of those in population P whose incomes are smaller than or equal to x_i. The relative deprivation of an individual earning x_i in population P with an income vector $x=(x_1,...,x_n)$ is equal to the fraction of those whose incomes are higher than x_i times their mean excess income, namely,

$$RD(x_i,x)=[1-F(x_i)]\cdot E(x-x_i \mid x>x_i). \tag{2}$$

A proof is in Stark (2013).

The aggregate relative deprivation is, in turn, the sum of the individual levels of relative deprivation

$$ARD(x)=\sum_{i=1}^{n} RD(x_i,x)=\sum_{i=1}^{n}\frac{\sum_{j=i+1}^{n}(x_j-x_i)}{n}. \tag{3}$$

$ARD(x)$ is our index of the level of "stress" of population P.

We now consider two populations, P_1 and P_2, with ordered income vectors $x^1=(x_i^1)$ and $x^2=(x_i^2)$ of dimensions n_1 and n_2, respectively. The total population size is $n=n_1+n_2$. The ordered income vector of the merged population is denoted $x^1\circ x^2$, and is the n-dimensional income vector obtained by merging the two income vectors and ordering the resulting n components from lowest to highest.

The following theorem states that the difference $ARD(x^1\circ x^2)-ARD(x^1)-ARD(x^2)$ is non-negative: a merger increases aggregate relative deprivation or leaves it unchanged. Namely, if we conceptualize the merger of two income vectors as an addition operator, then *ARD* is a superadditive function of the income vectors. (A function H is superadditive if for all x, y it satisfies $H(x+y)-H(x)-H(y)\geq 0$.)

Superadditivity theorem. Let P_1 and P_2 be two populations with ordered income vectors x^1 and x^2, and let $x^1\circ x^2$ be the ordered vector of merged incomes. Then

$$ARD(x^1\circ x^2)-ARD(x^1)-ARD(x^2)\geq 0.$$

Proof. See Stark (2013).

Example 1: consider the merger of populations P_1 and P_2 with income vectors $x^1=(1,2)$ and $x^2=(3,4)$, respectively. The pre-merger levels of aggregate relative deprivation are $ARD\left(x^1\right)=1/2$ and $ARD\left(x^2\right)=1/2$. In the merged population with income vector $x^1 \circ x^2 = (1,2,3,4)$, we have that $ARD\left(x^1 \circ x^2\right)=5/2>1=ARD\left(x^1\right)+ARD\left(x^2\right)$. This example vividly illustrates further why a formal proof of the superadditivity result is needed. Even in the simple case in which the two populations do not overlap and a relatively poor, two-person population $x^1 = (1,2)$ merges with a relatively rich, two-person population $x^2 = (3,4)$, the overall relative deprivation effect cannot be pre-ascertained. In such a case, it is quite clear that upon integration members of the poorer population are subjected to more relative deprivation, whereas members of the richer population other than the richest are subjected to less relative deprivation. Because one constituent population experiences an increase of its *ARD* while another experiences a decrease, whether the *ARD* of the merged population is higher than the sum of the *ARD*s of the constituent populations cannot be determined without formal analysis. Put differently, in a setting in which others could only bring negative externalities, a smaller population will always experience less aggregate relative deprivation. But in a setting such as ours when others joining in can confer both negative externalities (of 3 and 4 upon 1 and 2) and positive externalities (of 1 and 2 upon 3), it is impossible to determine without proof whether the expansion of a population will entail a reduction in aggregate relative deprivation or an increase.

Because throughout we have kept incomes unchanged, the incomes of the members of a constituent population are not affected by its merger with another population: in our setting, a merger changes the social comparisons space that governs the sensing and calculation of relative income (relative deprivation), but it leaves absolute incomes intact. If we assume that individuals' wellbeing depends positively on absolute income and negatively on the relative deprivation experienced, a merger leads to a deterioration in the aggregate wellbeing of at least one of the merged populations.

We next ask how a government that is concerned about the wellbeing (utility) levels of individuals falling below their pre-merger levels and the consequent increase of the aggregate level of social stress will be able to respond in a cost-effective manner. Governments must be well aware that an increase in social stress could translate into social unrest, and there have been plenty of episodes, historical and current, to remind governments of the short distance between social stress and social protest. Clark and Senik (2010) reviewed data collected in 2006/7 as part of Wave 3 of the European Social Survey. Their analysis of a usable sample of around 19,000 observations for 18 countries reveals that income comparisons are acknowledged as at least somewhat important by a majority of Europeans; are mostly upward; and are associated with lower levels of happiness. When the merger of populations, in and by itself, exacerbates social stress on account of less favorable upward comparisons, governments will want to employ measures aimed at reversing the surge in stress.

3. Policy response to the post-merger increase in *ARD*

The unwanted effects of a merger on the wellbeing of populations and individuals call for the design and assessment of policies aimed at counteracting the increase in individuals' distress. We consider the viability of a self-contained, non-publicly-financed policy aimed at preserving the wellbeing of individuals at its pre-merger level. We find, though, that a policy that seems to be attractive may not be implementable.

We assume that the wellbeing of an individual is a function of his absolute income and of his relative deprivation, with the partial first derivatives being, respectively, positive and negative. Correspondingly, we define the preferences of the individuals in population P with an ordered income vector x as

$$u_i = u\left(x_i, x\right) = \alpha_i x_i - \left(1-\alpha_i\right) RD\left(x_i, x\right) \tag{4}$$

where $0<\alpha_i<1$, $i=1,2,...,n$. This form of the individuals' utility function, in which the coefficients sum up to one, is equivalent to a social planner "giving" to an individual 100 percent of weight that he can assign to income and relative deprivation in any way that he wants. Then, we can ascertain that each individual's preferences enter the maximization problem with equal "importance:" the sum of the coefficients is constant for all individuals.

The underlying idea of this policy response is to skim off income from those who reap a gain as a consequence of the merger, and to distribute that income to those who experience a loss, such that following the merger no individual will be worse off in terms of the utility measure defined in (4). There are several difficulties with such a scheme, however.

First, a necessary condition is that there has to be at least one gainer. Without a gainer, there will be no surplus to tap. But as is quite obvious, there may not be any as, for example, when population with income vector $x^1 = (1,2,3,4)$ joins population with income vector $x^2 = (5,5)$.

Second, for the policy to be applicable, the policy-maker would need to know the α_i 's. If each individual has his own distinct preference structure, the information required is colossal. Two possibilities then come to mind: that all the individuals share the same distaste for relative deprivation, or that they do not. We consider in detail the former possibility: $\alpha_i = \alpha \ \ \forall i, \ i = 1, 2, ..., n$.

That all the individuals share the same distaste for relative deprivation eases drastically the information requirements, allowing us to work with a single α. But then, even in the simplest configuration of incomes, impossibility strikes; Stark (2010) presents an example of this impossibility for the simple case of the merger of a one-person population with a two-person population. We next state and prove that what this simple case reveals generalizes to an impossibility that applies to the merger of any two populations with a uniform α.

Impossibility theorem. A wellbeing-preserving "tax and transfer" scheme administered upon the merger of two populations and a uniform distaste for relative deprivation, $1-\alpha$, $0 < \alpha < 1$, cannot be self-sustaining.

Proof. Let $\mathbf{R}_+^n$ be the space of *n*-dimensional vectors with non-negative components. We consider a population P characterized by an ordered income vector $x = (x_i) \in \mathbf{R}_+^n$, $x_1 \le x_2 \le ... \le x_n$, and a uniform distaste for relative deprivation represented by the parameter $1-\alpha$, where $0 < \alpha < 1$. We first prove two auxiliary propositions, and introduce some necessary notation.

Proposition 1. Let $u = (u_1, ..., u_n)$ denote the vector of ordered wellbeing levels defined according to

$$u_i = u(x_i, x) = \alpha x_i - (1-\alpha) RD(x_i, x), \ i = 1, \ldots, n. \tag{5}$$

Then, the ranking of the individuals by their incomes (from the smallest to the largest) is identical to the ranking of the individuals by their levels of wellbeing (from the smallest to the largest). In other words, the order of the individuals whose levels of wellbeing are components of the wellbeing vector *u* is identical to the order of the individuals whose incomes are components of the income vector *x* related to *u*,

$$x_j > x_k \Leftrightarrow u_j > u_k \text{ for all } k = 1, \ldots, n-1, \ j = k+1, \ldots, n.$$

Proof. Let $x_j > x_k$. Because any individual who has a higher income is less relatively deprived than an individual who has a lower income, that is, $RD\left(x_j, x\right) < RD\left(x_k, x\right)$, it follows immediately from (5) that $u_j > u_k$.

Furthermore, we seek to show that $u_j > u_k \Rightarrow x_j > x_k$. We do this indirectly by assuming $x_j \le x_k$. This implies

$$\alpha x_j \le \alpha x_k \quad \Rightarrow \quad \alpha x_j - (1-\alpha) RD\left(x_j, x\right) \le \alpha x_k - (1-\alpha) RD\left(x_k, x\right) \quad \Rightarrow \quad u_j \le u_k ,$$

which contradicts the assumption $u_j > u_k$. This concludes the proof of Proposition 1. □

Proposition 2. For a population characterized by the ordered income vector $x = \left(x_i\right) \in \mathbf{R}_+^n$, $x_1 \le x_2 \le ... \le x_n$, the population's aggregate income is given by

$$Y(x) = \frac{n}{\alpha} \sum_{i=1}^{n} \left[\frac{n-(i-1)}{(i-1)\alpha + n-(i-1)} - \frac{n-i}{i\alpha + n - i} \right] u_i \ . \tag{6}$$

Proof. From Proposition 1 we know that $x_j > x_k$ implies $u_j > u_k$ for all $k = 1, \ldots, n-1$, $j = k+1, \ldots, n$ where $u = \left(u_1, ..., u_n\right)$ denotes the vector of ordered levels of wellbeing corresponding to the ordered vector of income levels $x = \left(x_1, ..., x_n\right)$. In matrix parlance, the ordered vector of the individual levels of wellbeing defined by (5) can be represented as

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} - (1-\alpha) \begin{pmatrix} \frac{1}{n} \sum\limits_{k=2}^{n} \left(x_k - x_1\right) \\ \frac{1}{n} \sum\limits_{k=3}^{n} \left(x_k - x_2\right) \\ \vdots \\ \frac{1}{n} \sum\limits_{k=n-1}^{n} \left(x_k - x_{n-1}\right) \\ 0 \end{pmatrix} = Bx , \tag{7}$$

where

$$B=\begin{pmatrix} \alpha+\frac{n-1}{n}(1-\alpha) & -\frac{1}{n}(1-\alpha) & \cdots & -\frac{1}{n}(1-\alpha) & -\frac{1}{n}(1-\alpha) \\ 0 & \alpha+\frac{n-2}{n}(1-\alpha) & -\frac{1}{n}(1-\alpha) & \cdots & -\frac{1}{n}(1-\alpha) \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & 0 & \alpha+\frac{1}{n}(1-\alpha) & -\frac{1}{n}(1-\alpha) \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}.$$

Because for $0<\alpha<1$, $B=\left(b_{ij}\right)_{n\times n}$ is a full rank matrix, equation (7) is equivalent to $x=Cu$, where $C\equiv B^{-\mathbf{1}}=\left(c_{ij}\right)_{n\times n}$ with

$$c_{ij}=\begin{cases} \dfrac{n(1-\alpha)}{(j\alpha+n-j)\left((j-1)\alpha+n-(j-1)\right)} & \text{for } j>i \\ \dfrac{n}{j\alpha+n-j} & \text{for } i=j \\ 0 & \text{for } j<i. \end{cases}$$

The aggregate income of a population that is characterized by the ordered income vector $x=\left(x_i\right)\in\mathbf{R}_+^n$ is then given by

$$\begin{aligned} Y(x) &= \sum_{i=1}^n x_i=\sum_{i=1}^n\sum_{j=1}^n c_{ij}\,u_j=\sum_{j=1}^n\sum_{i=1}^n c_{ij}\,u_j=\sum_{j=1}^n\frac{n}{j\alpha+n-j}\left(1+\frac{(1-\alpha)(j-1)}{(j-1)\alpha+n-(j-1)}\right)u_j \\ &= \sum_{j=1}^n\frac{n}{j\alpha+n-j}\left(\frac{n}{(j-1)\alpha+n-(j-1)}\right)u_j \\ &= \frac{n}{\alpha}\sum_{j=1}^n\left[\frac{n-(j-1)}{(j-1)\alpha+n-(j-1)}-\frac{n-j}{j\alpha+n-j}\right]u_j \end{aligned}$$

which is equation (6). This concludes the proof of Proposition 2. □

We next consider two populations, denoted by P_1 and P_2, characterized by the ordered income vectors $x^1=\left(x_i^1\right)$ and $x^2=\left(x_i^2\right)$ of dimensions n_1 and n_2, respectively. These populations are merged, and we presuppose that some of the $n=n_1+n_2$ individuals thereby experience an increase in individual wellbeing, whereas others experience a decrease.

Before proceeding, we introduce some notation. If $u^1=\left(u_i^1\right)$, $i=1,\ldots,n_1$, and $u^2=\left(u_j^2\right)$, $j=1,\ldots,n_2$, are the ordered wellbeing vectors related to the income vectors

$x^1 = \left(x_i^1\right)$ and $x^2 = \left(x_j^2\right)$, respectively, that are merged, we can write the ordered merged vector of the pre-merged wellbeing levels as

$$u = u^1 \circ u^2 = \left(u_{\nu(1)}^{\varphi(1)}, u_{\nu(2)}^{\varphi(2)}, \dots, u_{\nu(n)}^{\varphi(n)}\right),$$

where $n = n_1 + n_2$, $\varphi(k)$ is 1 (or 2) if the k-th overall wellbeing level belongs to population P_1 (or P_2), and $\nu(k)$ is the rank (from smallest to largest) within $P_{\varphi(k)}$ of this k-th overall smallest wellbeing level.

Because $u_{\nu(1)}^{\varphi(1)}$ is the smallest wellbeing level in the merged population, we have that

$$\left(\varphi(1), \nu(1)\right) \equiv \begin{cases} (1,1) \text{ if } u_1^1 \le u_1^2 \\ (2,1) \text{ otherwise.} \end{cases}$$

In order to express the other $u_{\nu(k)}^{\varphi(k)}$ terms, we define

$$q_1(k) = 2k - \sum_{p=1}^{k} \varphi\left(p\right)$$

and

$$q_2(k) = \sum_{p=1}^{k} \varphi\left(p\right) - k,$$

which are, respectively, the number of wellbeing levels from P_1 and the number of wellbeing levels from P_2 among the first k wellbeing levels of the merged population. Therefore, $q_1(1) = 1$ and $q_2(1) = 0$ if $\varphi(1) = 1$, and $q_1(1) = 0$ and $q_2(1) = 1$ if $\varphi(1) = 2$. With this notation in place, the first $k-1$ wellbeing levels of the merged population are made up of $q_1(k-1)$ wellbeing levels from P_1, and of $q_2(k-1)$ wellbeing levels from P_2. Obviously, $k = q_1(k) + q_2(k)$. The $\nu(k)$ terms and $\varphi(k)$ terms are therefore defined recursively as

$$\left(\varphi(k), \nu(k)\right) \equiv \begin{cases} \left(1, q_1(k-1)+1\right) \text{ if } u_{q_1(k-1)+1}^1 \le u_{q_2(k-1)+1}^2 \\ \left(2, q_2(k-1)+1\right) \text{ otherwise.} \end{cases}$$

We are now ready to prove the theorem itself. To avoid any decline in individual wellbeing, we transfer income to losers of wellbeing, taking away income from the gainers of wellbeing in order to finance the transfers. The resulting post-policy ordered income vector of the merged population, denoted by $x^* = \left(x_1^*, \ldots, x_n^*\right)$, $x_1^* \le x_2^* \le \ldots \le x_n^*$, $n = n_1 + n_2$, characterizes a population in which - once the “tax and transfer” policy has been implemented

- no individual forgoes wellbeing due to the merger, that is, $u\left(x_k^*, x^*\right) = u_{\nu(k)}^{\varphi(k)}$ for every $k = 1, \ldots, n$. We show that no self-financed wellbeing-preserving redistribution of incomes can exist by demonstrating that it is not possible for the aggregate post-merger income, $Y\left(x^1\right) + Y\left(x^2\right) = \sum_{j=1}^{n_1} x_j^1 + \sum_{k=1}^{n_2} x_k^2$, to be equal to (or greater than) the post-wellbeing-preserving tax and transfer policy income, $Y\left(x^*\right) = \sum_{i=1}^{n} x_i^*$ (except in one degenerate income structure identified at the end of the proof).

Let

$$e\left(l, n\right) \equiv \frac{n-l}{l\alpha + n - l}, \; l = 0, 1, \ldots, n.$$

Using (6), we can express the aggregate level of income of the merged population after the wellbeing-preserving policy is implemented as[1]

$$Y\left(x^*\right) = \frac{n}{\alpha} \sum_{i=1}^{n} \left[e\left(i-1, n\right) - e\left(i, n\right)\right] u_{\nu(i)}^{\phi(i)}.$$

Then, the difference between $Y\left(x^*\right)$ and the sum of the pre-merger aggregate incomes of the constituent populations, is

$$\begin{aligned} Y\left(x^*\right) - Y\left(x^1\right) - Y\left(x^2\right) &= \frac{n}{\alpha} \sum_{i=1}^{n} \left[e\left(i-1, n\right) - e\left(i, n\right)\right] u_{\nu(i)}^{\phi(i)} \\ -\frac{n_1}{\alpha} \sum_{j=1}^{n_1} \left[e\left(j-1, n_1\right) - e\left(j, n_1\right)\right] u_j^1 &- \frac{n_2}{\alpha} \sum_{k=1}^{n_2} \left[e\left(k-1, n_2\right) - e\left(k, n_2\right)\right] u_k^2 = \sum_{i=1}^{n} f(i) u_{\nu(i)}^{\phi(i)}, \end{aligned} \tag{8}$$

where

$$f(k) \equiv \frac{n\left[e\left(k-1, n\right) - e\left(k, n\right)\right] - n_{\phi(k)}\left[e\left(\nu(k)-1, n_{\phi(k)}\right) - e\left(\nu(k), n_{\phi(k)}\right)\right]}{\alpha}.$$

We define the non-negative sequence $d(k)$, $k = 1, 2, \ldots, n$, as

$$d(1) \equiv u_{\nu(1)}^{\phi(1)} \text{ and } d(k) \equiv u_{\nu(k)}^{\phi(k)} - u_{\nu(k-1)}^{\phi(k-1)}, \; k = 2, 3, \ldots, n$$

[1] Note that following the merger but prior to implementing the wellbeing-preserving policy, equality of the ordering of the incomes and of the pre-merger levels of wellbeing may not be preserved amongst the overall population, but after the implementation of the policy the ordering of the revised incomes $x^* = \left(x_1^*, \ldots, x_n^*\right)$ must again observe the ordering of the levels of wellbeing, allowing us to resort to Proposition 2 for writing the aggregate level of income of the merged population after the wellbeing-preserving policy is implemented.

to have

$$u_{\nu(i)}^{\phi(i)} = \sum_{k=1}^{i} d(k) \geq 0\,,\; k = 1, 2, \ldots, n\,,$$

and we are able to rewrite (8) as

$$\begin{aligned} Y\left(x^*\right) - Y\left(x^1\right) - Y\left(x^2\right) &= \sum_{i=1}^{n} f(i) \sum_{k=1}^{i} d(k) = \sum_{k=1}^{n} d(k) \sum_{i=k}^{n} f(i) \\ &= d(1) \sum_{i=1}^{n} f(i) + \sum_{k=2}^{n} d(k) \left(\sum_{i=1}^{n} f(i) - \sum_{i=1}^{k-1} f(i) \right) \\ &= d(1) \sum_{i=1}^{n} f(i) + \sum_{k=2}^{n} d(k) \sum_{i=1}^{n} f(i) - \sum_{k=2}^{n} d(k) \sum_{i=1}^{k-1} f(i) \\ &= \sum_{i=1}^{n} f(i) \sum_{k=1}^{n} d(k) - \sum_{k=1}^{n-1} d(k+1) \sum_{i=1}^{k} f(i). \end{aligned} \tag{9}$$

For the partial sums, $\sum_{i=1}^{k} f(i)$, $k \leq n$, in (9) we get

$$\begin{aligned} \sum_{i=1}^{k} f(i) &= \frac{n}{\alpha} \sum_{i=1}^{k} \left[e\left(i-1, n\right) - e\left(i, n\right) \right] \\ &\quad - \frac{n_1}{\alpha} \sum_{i=1}^{q_1(k)} \left[e\left(i-1, n_1\right) - e\left(i, n_1\right) \right] - \frac{n_2}{\alpha} \sum_{i=1}^{q_2(k)} \left[e\left(i-1, n_2\right) - e\left(i, n_2\right) \right] \\ &= \frac{n}{\alpha} \left[e\left(0, n\right) - e\left(k, n\right) \right] - \frac{n_1}{\alpha} \left[e\left(0, n_1\right) - e\left(q_1(k), n_1\right) \right] - \frac{n_2}{\alpha} \left[e\left(0, n_2\right) - e\left(q_2(k), n_2\right) \right] \\ &= \frac{1}{\alpha} \left(n - \frac{n\left(n-k\right)}{n - \left(1-\alpha\right)k} - n_1 + \frac{n_1\left(n_1 - q_1(k)\right)}{n_1 - \left(1-\alpha\right) q_1(k)} - n_2 + \frac{n_2\left(n_2 - q_2(k)\right)}{n_2 - \left(1-\alpha\right) q_2(k)} \right) \\ &= \frac{\left(\alpha - 1\right)\left(n_2 q_1(k) - n_1 q_2(k)\right)^2}{\left(n - \left(1-\alpha\right)k\right)\left(n_1 - \left(1-\alpha\right) q_1(k)\right)\left(n_2 - \left(1-\alpha\right) q_2(k)\right)} \leq 0, \end{aligned} \tag{10}$$

where we used the facts that $k = q_1(k) + q_2(k)$ and that $n = n_1 + n_2$.

We note that, for $0 < \alpha < 1$, (10) is equal to zero only if $n_2 q_1(k) - n_1 q_2(k) = 0$, which is the case, for example, for $k = n$ because then, $q_1(k) = n_1$ and $q_2(k) = n_2$; and that (10) is strictly negative if $n_2 q_1(k) - n_1 q_2(k) \neq 0$. In particular, $\sum_{i=1}^{n} f(i) = 0$.

Using (10), we return to (9) and get

$$\begin{aligned} Y\left(x^*\right) - Y\left(x^1\right) - Y\left(x^2\right) &= \sum_{i=1}^{n} f(i) \sum_{k=1}^{n} d(k) - \sum_{k=1}^{n-1} d(k+1) \sum_{i=1}^{k} f(i) = -\sum_{k=1}^{n-1} d(k+1) \sum_{i=1}^{k} f(i) \\ &= (1-\alpha) \sum_{k=1}^{n-1} d(k+1) \frac{\left(n_2 q_1(k) - n_1 q_2(k)\right)^2}{\left(k\alpha + n - k\right)\left(q_1(k)\alpha + n_1 - q_1(k)\right)\left(q_2(k)\alpha + n_2 - q_2(k)\right)} \geq 0. \end{aligned} \tag{11}$$

The inequality in (11) is not strict only if for each $i = 2, 3, \ldots, n$ such that $d(i) = u_{\nu(i)}^{\phi(i)} - u_{\nu(i-1)}^{\phi(i-1)} > 0$ we have that $n_2 q_1(i-1) = n_1 q_2(i-1)$, in which case $Y\left(x^*\right)$ will be equal to $Y\left(x^1\right) + Y\left(x^2\right)$. This condition means that the two merged populations have exactly the same structure of incomes or, put differently, that apart from re-scaling, the structures are the same. Thus, if population P_1 consists of k_1 individuals with income y_1, k_2 individuals with income y_2 and, in general, for any i, of k_i individuals with income y_i, then population P_2 consists of rk_1 individuals with income y_1, rk_2 individuals with income y_2 and, in general, for any i, of rk_i individuals with income y_i, where $r \in \mathbb{Q}$ is such that $rk_i \in \mathbb{Z}$ for any i. In this case, no transfer is needed to compensate for increased stress because such an increase does not occur at all.

We thus conclude that as long as the just-described degenerate case does not apply, the aggregate post-(wellbeing-preserving) policy income, $Y\left(x^*\right)$, will always exceed the aggregate pre-policy income, $Y\left(x^1\right) + Y\left(x^2\right)$. This completes the proof of the theorem. □

In sum: a "tax and transfer" scheme cannot achieve its aim because there is not enough of a gain to placate the losers while still keeping the gainers at least as well off as prior to the merger. In a way, this impossibility theorem is akin to the aggregate relative deprivation superadditivity theorem: here as there, welfare takes a beating.

4. Conclusion

An increase in aggregate relative deprivation is a downside to the integration of economies. To aid a social planner who seeks cost-effectively to counter this negative effect, we analyzed a policy measure in which individual levels of wellbeing are not allowed to fall. We showed that implementation of a self-contained "tax and transfer" scheme aimed at retaining individuals' wellbeing at their pre-merger levels is not viable because there is not enough of a gain to placate the losers while still keeping the gainers at least as well off as prior to the merger. A governmental infusion of funds is needed, or the efficiency gains to be had from integration need to raise incomes sufficiently to facilitate an effective tax and transfer scheme. When the possibility of a merger is contemplated, an interesting question to address is whether the anticipated boost in productivity will suffice to pay for the cost of the policy discussed above.

Acknowledgements

I am indebted to Marcin Jakubek, Martyna Kobus, and Grzegorz Kosiorowski for thoughtful advice, and to an anonymous referee for perceptive comments.

References

Clark, A.E., Senik, C., 2010. Who compares to whom? The anatomy of income comparisons in Europe. Economic Journal 120(544), 573-594.

Stark, O., 2010. Looking at the integration of nations through the lens of the merger of populations: preliminary superadditivity and impossibility results. Swiss Journal of Economics and Statistics 146(4), 661-675.

Stark, O., 2013. Stressful integration. European Economic Review 63, 1-9.