Algorithms for Mori Dream Spaces

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Introduction

The central objects of this thesis are *Mori dream spaces* as introduced by Y. Hu and S. Keel [60], i.e., algebraic varieties such that the cone of effective divisors has a polyhedral *Mori chamber decomposition*. Well-known examples are toric varieties, spherical varieties or smooth Fano varieties. Hu and Keel have shown that the Mori dream spaces X are characterized by finite generation of the *Cox ring* $\mathcal{R}(X)$. There are strong relations between the geometry of X and its Cox ring. In fact, a Mori dream space X can be retrieved from the spectrum $\overline{X}$ of $\mathcal{R}(X)$ as a quotient $X = \widehat{X} /\!\!/ H$ of an open subset $\widehat{X} \subseteq \overline{X}$ by a quasitorus H with the class group $\mathrm{Cl}(X)$ as its characters. F. Berchtold and J. Hausen [19, 51] proposed an explicit description of Mori dream spaces X in terms of *bunched rings*, i.e., pairs (R, Φ) consisting of a factorially K-graded ring R given by generators and relations and a collection (bunch) Φ of overlapping polyhedral cones in $K \otimes \mathbb{Q}$. Then R determines $\overline{X} = \operatorname{Spec} R$, the K-grading gives rise to the H-action on $\overline{X}$ and the open subset $\widehat{X} \subseteq \overline{X}$ is constructed from Φ by geometric invariant theory. Building on the approach via bunched rings [19, 51, 5, 18, 6], our main focus lies on the development, implementation and application of explicit algorithms for general Mori dream spaces.

We now give a summary of the results of this thesis. A first series of results is a toolkit for basic computations with (not only projective) Mori dream spaces. We present algorithms for

- basics on finitely generated abelian groups and algebras graded by them,
- Picard group, local class groups, the cones of effective, movable or semiample divisor classes,
- canonical toric ambient variety, stratification, irrelevant ideal,
- tests for being quasismooth, smooth, ($\mathbb{Q}$-) factorial, complete, (quasi-) projective, singularities (in terms of strata),
- for complete intersection Cox rings: intersection numbers, graph of exceptional divisors, anticanonical divisor class, test for ($\mathbb{Q}$-) Gorenstein and Fano properties, Gorenstein index,
- for varieties with the action of a torus of codimension one: resolution of singularities, test for being almost-homogeneous, roots of the automorphism group.

We have implemented these algorithms in a software package, called *MDSpackage*, see [54, 55]. This extends packages for toric varieties, for example [20, 47, 66, 69, 99].

As an application of our algorithms, we study *del Pezzo surfaces*, i.e., Fano surfaces, with a non-trivial $\mathbb{K}^*$-action. Recall that V. Alexeev and V. Nikulin [2] classified the log-terminal del Pezzos surfaces of Picard number one and Gorenstein index at most two. More is known for del Pezzos surfaces with torus action. A. Kasprzyk, M. Kreuzer and B. Nill [68] classified the toric del Pezzo surfaces with Gorenstein-index $n \leq 16$ and at most log-terminal singularities. H. Süß [94, Ch. 6] listed the non-toric, log-terminal del Pezzo $\mathbb{K}^*$-surfaces with $n \leq 3$ and Picard number $\varrho(X) \leq 2$ and E. Huggenberger [61, Ch. 5] classified the non-toric, log-terminal cases with $n = 1$. Using our algorithms, we classify the (not necessarily log-terminal) non-toric del Pezzo $\mathbb{K}^*$-surfaces with $n \le 6$, Picard number two and hypersurface Cox ring that are *combinatorially minimal*, i.e., those without contractible curves.

Theorem. *The following table lists the non-toric, combinatorially minimal del Pezzo $\mathbb{K}^*$-surfaces of Picard number two with hypersurface Cox ring and Gorenstein index $n \le 6$.*

Cox ring $\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$	n	*degree*
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	32	2	2
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$	256	4	1
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/6\mathbb{Z}$	864	6	$\frac{2}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^2\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	108	3	$\frac{4}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\mathbb{Z}^2$	9	3	$\frac{8}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	72	3	$\frac{4}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^2\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}$	500	5	$\frac{4}{5}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^5\rangle$	$\mathbb{Z}^2$	25	5	$\frac{8}{5}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^5\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	200	5	$\frac{4}{5}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$	576	6	$\frac{2}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^4 + T_3T_4^4 + T_5^4\rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	432	6	$\frac{2}{3}$
$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2 + T_3^2T_4 + T_5^2\rangle$	$\mathbb{Z}^2$	4	1	4

The explicit $\mathrm{Cl}(X)$*-grading of the Cox ring* $\mathcal{R}(X)$ *and further geometric properties of the surfaces are listed in Theorem 2.5.1.*

As a first advanced algorithm we show how to compute the Mori chamber decomposition of a given Mori dream space. More generally, we provide a method to compute the *GIT-fan* of torus-actions on affine varieties. The GIT-fan is a polyhedral fan parameterizing quotients from D. Mumford's [83] geometric invariant theory (GIT). For quotients associated to ample bundles the fan structure has been described by I. Dolgachev, Y. Hu and M. Thaddeus [35, 97]. F. Berchtold and J. Hausen [18] then provided an explicit construction of the GIT-fan of an affine variety. Building on the latter, Algorithm 3.2.9 computes the maximal GIT-cones by traversing a spanning tree of the implicitly given dual graph of the GIT-fan. We have published our algorithm in [71].

A next series of advanced algorithms concerns the impact of modifications on Cox rings. More precisely, given a modification $X_2 \to X_1$ of projective varieties where one of the Cox rings $\mathcal{R}(X_i)$ is known, we present methods to compute the other Cox ring in terms of generators and relations. The methods and results have been published jointly with J. Hausen and A. Laface in [57]. The case of a contraction $X_2 \to X_1$ of a Mori dream space X_2 can be answered purely theoretically, see Proposition 4.2.3. The case of a blow up $X_2 \to X_1$ of a Mori dream space X_1 is more delicate; we may even lose finite generation of $\mathcal{R}(X_2)$. We develop algorithms to

- verify finite generation of $\mathcal{R}(X_2)$,
- verify a guess of generators for $\mathcal{R}(X_2)$,
- produce a guess of generators for $\mathcal{R}(X_2)$,
- determine the ideal of relations.

Our starting point are toric ambient modifications as developed by J. Hausen in [51]. More precisely, given a Mori dream space X_1, there is a canonical embedding $X_1 \subseteq Z_1$ into a toric variety Z_1. Each toric modification $\pi\colon Z_2 \to Z_1$ induces a modification $X_2 \to X_1$ of the embedded variety where $X_2 \subseteq Z_2$ is the proper transform under π. Hausen has given a list of geometric criteria under which we can describe the Cox ring of X_2. A key step is to reduce these criteria to a series of primality tests, see Theorem 4.1.3 and the joint paper [10] with H. Bäker and J. Hausen. As a first consequence, we obtain algorithms for items two and four, i.e., to verify a guess of generators and to determine the ideal of relations.

As an application, we explicitly compute the Cox rings of the Gorenstein log-terminal del Pezzo surfaces of Picard number one without $\mathbb{K}^*$-action. See [2] for a classification of these surfaces in terms of singularity types. The Cox rings of the non-toric cases with $\mathbb{K}^*$-action have been determined by J. Hausen and H. Süß [59], the toric ones are well-known, see, e.g., R. Koelmann [72]. Our result completes the list. The idea is to present the surfaces X as $\mathbb{P}_2 \leftarrow X' \rightarrow X$ where enough information on the Cox ring of X' is known by the work of B. Hassett, Y. Tschinkel, U. Derenthal, M. Artebani, A. Garbagnati and A. Laface [49, 33, 4].

Theorem. *The following table lists the Cox rings of the Gorenstein log-terminal del Pezzo surfaces X of Picard number one that do not admit a non-trivial $\mathbb{K}^*$-action.*

$\mathrm{S}(X)$	$\sharp$ *gen.s of* $\mathcal{R}(X)$	$\sharp$ *relations*	$\mathrm{Cl}(X)$
$2A_4$	6	5	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
D_8	4	1	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
D_5A_3	5	2	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
D_62A_1	5	2	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
E_6A_2	4	1	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
E_7A_1	4	1	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
E_8	4	1	$\mathbb{Z}$
A_7	4	1	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
A_8	4	1	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
A_7A_1	5	2	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$A_5A_2A_1$	7	9	$\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$
$2A_3A_1$	9	20	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$4A_2$	10	27	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

Moreover, an explicit description of each Cox ring in terms of generators and relations is listed in Theorem 4.4.1.

A more elaborate algorithm addresses items one and three in the enumeration on page 2, i.e., given a blow up $X_2 \rightarrow X_1$ of a Mori dream space X_1 our Algorithm 4.5.12 provides a systematic guess for generators of $\mathcal{R}(X_2)$ and verifies its result. The idea is to show that $\mathcal{R}(X_2)$ is isomorphic to a certain saturated Rees algebra. Generators for $\mathcal{R}(X_2)$ can then be obtained by a traversal of the components of the Rees algebra. This is a complete answer to the problem as it terminates if and only if $\mathcal{R}(X_2)$ is finitely generated. Furthermore, we present an algorithm that verifies finite generation of $\mathcal{R}(X_2)$ for the case of infeasible computation.

As an application of our algorithms, we consider certain blow ups of the projective space $\mathbb{P}_3$. A.-M. Castravet and J. Tevelev [24] provided generators of the Cox ring of blow ups of $\mathbb{P}_n$ at points that lie on a rational normal curve. Relations have been determined by B. Sturmfels and Z. Xu [93]. Moreover, in [92] Sturmfels and M. Velasco computed for $n \leq 8$ the Cox rings of blow ups of $\mathbb{P}_n$ at $n+3$ points in general position. Applying a specialized version of our algorithm, we explicitly determine the Cox rings of blow ups of $\mathbb{P}_3$ in six points in *edge-special* position, i.e., four points are general and at least one point lies in two hyperplanes spanned by the others.

Theorem. *Let $x_1, \ldots, x_4 \in \mathbb{P}_3$ be the standard toric fixed points. The following table lists the $\mathbb{Z}^7$-graded Cox rings of the blow up of $\mathbb{P}_3$ in the following typical edge-special configurations $x_1, \ldots, x_6$.*

x_5	x_6	$\sharp$ *gen.s of* $\mathcal{R}(X)$	$\sharp$ *relations*
$[1,1,0,0]$	$[0,1,1,1]$	16	15

$[2,1,0,0]$	$[1,1,0,1]$	15	10
$[1,0,0,1]$	$[0,1,0,1]$	13	5
$[1,0,0,1]$	$[0,1,1,0]$	12	2
$[2,1,0,0]$	$[1,2,0,0]$	12	2

Moreover, an explicit description of each Cox ring in terms of generators and relations is listed in Theorem 4.6.6.

As the major result of this thesis, we determine the Cox rings of the smooth rational surfaces X with Picard number $\varrho(X) \leq 6$; for $\varrho(X) = 6$, we restrict ourselves to the non-$\mathbb{K}^*$-cases. The corollary that each smooth rational surface with $\varrho(X) \leq 6$ is a Mori dream space was also observed by Testa, Várilly-Alvarado and Velasco in [95]. So far, Cox rings of smooth rational surfaces have only been determined systematically for the special class of *(weak) del Pezzo surfaces*, i.e., blow ups of $\mathbb{P}_2$ in points in (almost) general position; see the work of V. Batyrev, U. Derenthal, O. Popov, M. Stillman, D. Testa and M. Velasco [13, 33, 32, 89].

Our approach makes use of the fact that each smooth rational surface X can be obtained as a blow up of the projective plane $\mathbb{P}_2$ in up to five points or as a blow up of the a-th Hirzebruch surface $\mathbb{F}_a$ in up to four points where $a \in \mathbb{Z}_{\geq 0}$. This enables us to use our methods for Cox ring computations of blow ups of Mori dream spaces: the blow ups of $\mathbb{P}_2$ can be handled in a purely computational way whereas the blow ups of $\mathbb{F}_a$ require a theoretical treatment. Here, we use our methods in a formal way to deal with the parameter $a \in \mathbb{Z}_{\geq 0}$. We give a complete classification of all surfaces X with $\varrho(X) \leq 5$ and list the generators and relations of the Cox rings of each class. For Picard number six we explicitly determine the Cox rings of the surfaces that do not admit a non-trivial $\mathbb{K}^*$-action; the remaining surfaces are known to be Mori dream spaces and their Cox rings can be obtained by combinatorial methods, see [59, 61].

Theorem. *Each smooth rational surface X with Picard number $\varrho(X) \leq 6$ is a Mori dream space. Moreover, the following statements hold.*

(i) *If $\varrho(X) \leq 5$ holds, then X is isomorphic to $\mathbb{P}_2$ or exactly one of the surfaces listed in Propositions 5.2.4, 5.2.5, 5.2.8 or Theorem 5.3.1. There, the Cox ring $\mathcal{R}(X)$ is listed explicitly in terms of generators and relations.*

(ii) *If $\varrho(X) = 6$ holds, then X admits a non-trivial $\mathbb{K}^*$-action or is isomorphic to exactly one of the following surfaces where $a \in \mathbb{Z}_{\geq 3}$.*

Cox ring $\mathcal{R}(X)$	*degree matrix*
$\mathbb{K}[T_1,\ldots,T_{10}]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 - T_6T_7T_8T_{10}$, $T_1T_2^2T_3T_4T_5 - T_6^2T_7 - T_9T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & 0 & 5 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
$\mathbb{K}[T_1,\ldots,T_{10}]/I$ *with I generated by* $T_3T_5T_8 - T_2T_6 - T_9T_{10}$, $T_1T_5 + T_7T_8 - T_2T_6^2T_4T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 3 & -2 \end{bmatrix}$

$\mathbb{K}[T_1, \dots, T_{11}]/I$

with I generated by

$T_3^2T_4T_5^2T_8 - T_2T_7 - T_{11}T_{10}$,
$T_2^2T_4T_6^2T_{11} - T_5T_9 + T_8T_{10}$,
$T_1T_5 + T_7T_8 - T_2T_4T_6^2T_{11}^2$,
$T_3^2T_4T_5T_8^2 + T_1T_2 - T_9T_{11}$,
$T_3^2T_4^2T_5T_8T_2T_6^2T_{11} - T_7T_9$
$-T_1T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 3 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{10}]/I$

with I generated by

$T_1T_5 + T_7T_8 - T_2T_4T_6T_{10}$,
$T_3T_5T_7T_8^2 - T_2^2T_4 - T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{13}]/I$

with I generated by

$T_1T_{11} - T_4T_3T_9 - T_8T_{12}$,
$T_1T_7 - T_2T_8 + T_3T_9T_{13}$,
$T_2T_6 + T_7T_{10} - T_3T_5T_{13}$,
$T_1T_6 + T_8T_{10} - T_3T_4T_{13}$,
$T_2T_{11} - \lambda T_5T_3T_9 - T_7T_{12}$,
$(\lambda - 1)T_1T_5 - T_{10}T_9 - T_{12}T_{13}$,
$(\lambda - 1)T_5T_8 + T_6T_9 - T_{11}T_{13}$,
$T_{10}T_{11} - (\lambda - 1)T_4T_3T_5 + T_6T_{12}$,
$(\lambda - 1)T_4T_7 + \lambda T_6T_9 - T_{11}T_{13}$,
$(\lambda - 1)T_2T_4 - \lambda T_{10}T_9 - T_{12}T_{13}$,

where $\lambda \in \mathbb{K}^* \setminus \{1\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{16}]/I$

with I generated by

$T_6T_{12} + \lambda T_7T_{14} - T_8T_{13}$,
$T_5T_{12} - \mu T_7T_{15} - T_9T_{13}$,
$T_4T_{13} - \lambda T_5T_{14} - \mu T_6T_{15}$,
$T_4T_{12} - \mu T_8T_{15} - \lambda T_9T_{14}$,
$T_3T_{11} + T_7T_{14} - T_8T_{13}$,
$T_1T_{13} - T_2T_{14} - T_3T_{15}$,
$T_1T_{11} - T_8T_{15} - T_9T_{14}$,
$T_2T_{11} - T_7T_{15} - T_9T_{13}$,
$(\lambda - \mu)T_3T_5 + \mu T_7T_{10} - T_{13}T_{16}$,
$(-\lambda + 1)T_5T_{14} + (-\mu + 1)T_6T_{15} + T_{10}T_{11}$,
$(\lambda - 1)T_5T_8 + (-\mu + 1)T_6T_9 - T_{11}T_{16}$,
$(\lambda - 1)T_4T_7 + (\lambda - \mu)T_6T_9 - T_{11}T_{16}$,
$(\mu - 1)T_3T_4 - \mu T_8T_{10} + T_{14}T_{16}$,
$(-\lambda + 1)T_2T_{14} + (-\mu + 1)T_3T_{15} + T_{10}T_{12}$,
$(\lambda\mu - \mu)T_2T_8 + (-\lambda\mu + \lambda)T_3T_9 - T_{12}T_{16}$,
$(\lambda - \mu)T_2T_6 + \lambda T_7T_{10} - T_{13}T_{16}$,
$(\lambda - 1)T_2T_4 - \lambda T_9T_{10} - T_{15}T_{16}$,
$(\lambda\mu - \mu)T_1T_7 + (\lambda - \mu)T_3T_9 - T_{12}T_{16}$,
$(\mu - 1)T_1T_6 - T_8T_{10} + T_{14}T_{16}$,
$(\lambda - 1)T_1T_5 - T_9T_{10} - T_{15}T_{16}$

where $\lambda \neq \mu \in \mathbb{K}^* \setminus \{1\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{11}]/I$

with I generated by

$T_6T_2T_4 + T_5T_9 - T_8T_{10}$,
$T_3T_4T_8 - T_1T_6 - T_9T_{11}$,
$T_3T_4T_5 + T_6T_7 - T_{11}T_{10}$,
$T_1T_5 + T_7T_8 - T_2T_4T_{11}$,
$T_3T_4^2T_2 - T_7T_9 - T_1T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{10}]/I$

with I generated by

$T_3T_5T_8 - T_2T_6 - T_9T_{10}$,
$T_1T_5 + T_7T_8 - T_2T_4T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

$\mathbb{K}[T_1, \dots, T_{10}]/I$

with I generated by

$T_1T_5T_{10} - T_2T_6 - T_7T_8$,
$T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5$
$-T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -a+1 & a & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -a+2 & a-1 & -1 & 1 \end{bmatrix}$$

$\mathbb{K}[T_1, \ldots, T_{11}]/I$
with I generated by ††
$T_7T_8 - T_2^aT_4T_6^{a-1}T_{11}^a + T_3T_5$,
$T_9T_{11} - T_1^aT_4T_5^aT_8^{a-1} - \kappa T_6T_7$,
$T_{10}T_{11} - T_1^aT_4T_5^{a-1}T_8^a + \kappa T_3T_6$,
$-\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9 - T_5T_{10}$,
$T_1^aT_2^aT_4^2T_5^{a-1}T_6^{a-1}T_8^{a-1}T_{11}^{a-1} - T_3T_9 - T_7T_{10}$
where $\kappa \in \mathbb{K}^*$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & a-1 & 0 & 1 & 2a-3 & 2a-2 & -a+2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & a-1 & 1 & 0 & 2a-2 & 2a-3 & -a+2 \\ 0 & 0 & 0 & 0 & 0 & a & -1 & 1 & 2a-2 & 2a-1 & -a+1 \end{bmatrix}$$

$\mathbb{K}[T_1, \ldots, T_{10}]/I$
with I generated by
$T_2^aT_4 - T_3T_5T_6^2T_{10} - T_7T_8$,
$T_1T_2^{a-1}T_4T_8 - T_3T_6 - T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2a-1 & -a+1 & -a & a \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 3 & -2 \end{bmatrix}$$

$\mathbb{K}[T_1, \ldots, T_{10}]/I$
with I generated by
$T_2^aT_4 - T_3T_6T_{10}T_5 - T_7T_8$,
$T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5$
$-T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3a-1 & -2a+1 & -a & a \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$\mathbb{K}[T_1, \ldots, T_9]/I$
with I generated by
$T_1T_2^{2a-1}T_4^2 - T_3^2T_5$
$-T_1T_2^{a-1}T_3T_4T_5T_6T_7T_9$
$+T_7T_8T_9^2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 2a-1 & 0 & 0 & 4a-3 & -a+1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$\mathbb{K}[T_1, \ldots, T_9]/I$
with I generated by
$T_2^aT_4 - T_3T_5T_6^2$
$-T_7T_9T_1T_2^{a-1}T_4T_5T_6$
$+T_7T_9^2T_8$

$$\begin{bmatrix} 1 & 1 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In this table, the first 12 classes do not admit a non-trivial $\mathbb{K}^*$-action. In the ††-case, the listed ring is the Cox ring for $a \le 15$ whereas for $a > 15$ the Cox ring is finitely generated and given by the equivariant normalization of $\mathbb{K}[T_1, \ldots, T_{11}]/(I : (T_1 \cdots T_{11})^\infty)$.

This thesis is divided into five chapters and an appendix. We now give a brief overview.

In the first chapter, for the convenience of the reader, we recall the fundamental theory of Cox rings, GIT, bunched rings and Mori dream spaces. Also, we recapitulate basics on surfaces, of modifications and of complexity-one T-varieties. This chapter is mainly taken from [5, 32, 61].

In Chapter 2, based on [19, 51, 5], we present an algorithmic toolkit for explicit computations with Mori dream spaces. We introduce the required data types and provide basic algorithms for finitely generated abelian groups, algebras that are graded by a finitely generated abelian group, for general Mori dream spaces as well as for the special class of complexity-one varieties; see Sections 1, 2, 3 and 4. As an application, we classify in Section 5 the combinatorially minimal $\mathbb{K}^*$-surfaces of Picard number two of Gorenstein index at most six whose Cox ring has a single relation.

Using the construction [19], we develop in Chapter 3 an algorithm to compute the GIT-fan for affine varieties $V(\mathfrak{a}) \subseteq \mathbb{K}^r$ with torus action. As a first step, we need to determine the torus orbits of $\mathbb{K}^r$ meeting $V(\mathfrak{a})$, the so-called $\mathfrak{a}$-*faces* or $\mathfrak{F}$-*faces* of the positive orthant $\mathbb{Q}^r_{\ge 0}$, see Section 1. We then show how to compute GIT-cones and the GIT-fan in Section 2. A special case is the Mori chamber decomposition. In Section 3 of Chapter 3 we give a direct algorithm to determine the $(H,2)$-*maximal subsets* [51], certain more general open subsets admitting a good quotient.

Given a modification $X_2 \to X_1$ of projective varieties where one of the Cox rings $\mathcal{R}(X_i)$ is known, we show in Chapter 4 how to determine the other one. As a first step, we translate in Sections 1 and 2 the methods of toric ambient modifications developed in [51] to a computable version. We also present a theoretical solution to compute the Cox ring of a contraction. In Section 3, we then are in position to present algorithms to verify a guess of generators and to determine relations. Based on [2], we apply these algorithms in Section 4 to compute the Cox rings of the Gorenstein, log-terminal del Pezzo surfaces of Picard number one that do not admit a non-trivial $\mathbb{K}^*$-action. In Section 5, we develop an algorithm to explicitly compute the Cox ring $\mathcal{R}(X_2)$ of a blow up $X_2 \to X_1$ of a Mori dream space X_1 along an irreducible subvariety inside the smooth locus. Moreover, for the case of infeasible computation, we present an algorithm that verifies finite generation of $\mathcal{R}(X_2)$. Specializing to blow ups $X_2 \to \mathbb{P}_n$ of point configurations in a projective space $\mathbb{P}_n$, in Section 6, we provide an algorithm that verifies that $\mathcal{R}(X_2)$ is generated by transforms of hyperplanes in $\mathbb{P}_n$. We explore some relations to the underlying combinatorial structures. Using our algorithm, we determine the Cox rings of blow ups of $\mathbb{P}_3$ in point configurations consisting of six distinct points in edge-special position.

In Chapter 5, we explicitly compute the Cox rings of the smooth rational surfaces X with Picard number $\varrho(X) \le 6$. Our classification is complete for $\varrho(X) \le 5$ and contains the non-$\mathbb{K}^*$-cases for $\varrho(X) = 6$. As X is obtained as a sequence of blow ups of $\mathbb{P}_2$ or the Hirzebruch surface $\mathbb{F}_a$, we can use the techniques developed in Chapter 4. We first classify the point configurations on $\mathbb{P}_2$ and $\mathbb{F}_a$ that we need to consider to obtain X as a blow up, see Section 1. Afterwards, in Sections 2, 3 and 4 we iteratively blow up a point of a surface of lower Picard number $\varrho(X)$ until we arrive at $\varrho(X) = 6$. In each step, we remove redundancies.

In Appendix A, we describe an implementation of the algorithms developed throughout this thesis. It is aimed towards usability and computations of up to medium examples. The appendix serves as a manual.

Throughout this thesis we made extensive use of the software systems `gfan`, `Macaulay2`, `magma`, `Maple`, `polymake` and `Singular` for computer algebra or polyhedral computations, see [63, 45, 23, 79, 43, 31].

CHAPTER 1

Preliminaries

We recall basic notions from algebraic geometry and thereby fix our notation for the subsequent chapters. This chapter is a summary of the sources referenced at the beginning of each section and does not contain results by the author. Our primary reference is the book by Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen and Antonio Laface [5].

Basics on geometric invariant theory for affine quasitorus actions can be found in the first section. In Section 2, we introduce Cox rings and toric varieties. We recall the theory of bunched rings and the correspondence to Mori dream spaces in Section 3. Section 4 deals with fundamental surface geometry and modifications, e.g., blow ups. In the final section, i.e., Section 5, we recall the construction and basic properties of $\mathbb{K}^*$-surfaces and complexity-one T-varieties.

Throughout this document, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a *variety*, we always mean a separated prevariety over $\mathbb{K}$.

1. GIT and good quotients

We recall basics on good quotients, the correspondence between graded affine algebras and affine varieties with quasitorus action as well as the GIT-fan. This section is a summary of mainly Sections III.1 and I.2 of [5]; see also [7, 18].

An *affine algebraic group* is an affine variety G together with a group structure such that the group operations are morphisms. A variety X with the action $G \times X \to X$ of an affine algebraic group G is called a *G-variety*. Given an affine algebraic group G, denote its group of *characters*, i.e., homomorphisms $G \to \mathbb{K}^*$ of algebraic groups, by $\mathbb{X}(G)$.

Definition 1.1.1. A *quasitorus* is an affine algebraic group H with its algebra of regular functions $\Gamma(H, \mathcal{O})$ generated by the characters $\chi \in \mathbb{X}(H)$. A connected quasitorus is a *torus*.

Each quasitorus is isomorphic to a product of a torus and a finite abelian group. The *standard torus* is the torus $\mathbb{T}^n := (\mathbb{K}^*)^n$. Homomorphic images of tori are again tori. Note that homomorphisms of tori correspond to integral matrices.

Remark 1.1.2. There are exact functors between finitely generated abelian groups and quasitori that are essentially inverse to each other; the assignments are

$$K \ \mapsto \ \operatorname{Spec} \mathbb{K}[K], \qquad \psi \ \mapsto \ \operatorname{Spec} \mathbb{K}[\psi], \qquad \mathbb{X}(H) \ \leftarrow\!\shortmid \ H, \qquad \varphi^* \ \leftarrow\!\shortmid \ \varphi.$$

We now briefly recall the correspondence between affine varieties with quasitorus action and affine algebras that are graded by a finitely generated abelian group. Given such an affine variety X with the action of a quasitorus H, we obtain a $\mathbb{X}(H)$-graded algebra

$$\Gamma(X, \mathcal{O}) \ = \bigoplus_{\chi \in \mathbb{X}(H)} \Gamma(X, \mathcal{O})_\chi, \qquad \Gamma(X, \mathcal{O})_\chi \ := \ \{f \in \Gamma(X, \mathcal{O});\ f(h \cdot x) = \chi(h) f(x)\}.$$

Vice versa, consider a finitely generated abelian group K and a K-graded, affine $\mathbb{K}$-algebra R. Set $X := \operatorname{Spec} R$ and choose K-homogeneous generators $f_1, \ldots, f_r$ for R. We have an embedding

$$X \;\to\; \mathbb{K}^r, \qquad x \;\mapsto\; (f_1(x), \ldots, f_r(x))$$

and $X \subseteq \mathbb{K}^r$ is invariant under the *diagonal action* of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ on $\mathbb{K}^r$ with comorphism

$$R \;\to\; \mathbb{K}[K] \otimes_{\mathbb{K}} R, \qquad R_w \ni f \;\mapsto\; \chi^w \otimes f.$$

Moreover, a *morphism* of affine H_i-varieties X_1 and X_2 with quasitori H_i is a pair $(\varphi, \widetilde{\varphi})$ consisting of a morphism of varieties $\varphi\colon X_1 \to X_2$ and a homomorphism of algebraic groups $\widetilde{\varphi}\colon H_1 \to H_2$ such that

$$\varphi(h \cdot x) \;=\; \widetilde{\varphi}(h) \cdot \varphi(x) \qquad \text{for all } \; x \in X_1, \; h \in H_1.$$

Proposition 1.1.3. *We have contravariant, exact functors that are essentially inverse to each other between the categories of affine $\mathbb{K}$-algebras that are graded by a finitely generated abelian group and affine varieties with the action of a quasitorus*

$$\begin{array}{rcl} R & \mapsto & \operatorname{Spec} R, \\ \left(\psi, \widetilde{\psi}\right) & \mapsto & \left(\operatorname{Spec} \psi,\, \operatorname{Spec} \mathbb{K}\big[\widetilde{\psi}\big]\right), \end{array} \qquad \begin{array}{rcl} \Gamma(X, \mathcal{O}) & \leftarrow\!\shortmid & X, \\ (\varphi^*,\, \widetilde{\varphi}^*) & \leftarrow\!\shortmid & (\varphi,\, \widetilde{\varphi}). \end{array}$$

By a *reductive algebraic group* we mean an affine algebraic group G such that every rational representation of G splits into irreducible ones. Examples of reductive groups include $\mathrm{SL}(n, \mathbb{Z})$, $\mathrm{GL}(n, \mathbb{Z})$, all finite groups and quasitori.

Consider a reductive algebraic group G and a G-variety X. A morphism $\varphi\colon X \to Y$ is *G-invariant* if $\varphi(x) = \varphi(g \cdot x)$ for all $x \in X$ and $g \in G$. We call φ *affine* if preimages of open affine subsets are again affine. The *ring of invariants* is the algebra

$$\mathcal{O}(X)^G \;:=\; \{f \in \Gamma(X, \mathcal{O});\; f(g \cdot x) \;=\; f(x) \text{ for each } x \in X,\; g \in G\}.$$

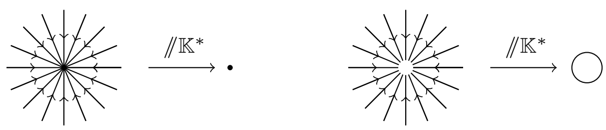
Definition 1.1.4. A *good quotient* for the G-action on X is an affine, G-invariant morphism $\pi\colon X \to Y$ such that $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism.

The quotient space Y of a good quotient $X \to Y$ for the G-action on X is unique up to isomorphism; we denote it by $X /\!\!/ G$. Note that good quotients need not exist. However, for an affine G-variety X with reductive group G, by a theorem of David Hilbert, the ring of invariants $\mathcal{O}(X)^G$ is finitely generated. Then the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$ yields the good quotient

$$X \;\to\; X /\!\!/ G \;=\; \operatorname{Spec} \mathcal{O}(X)^G.$$

If X is not affine, good quotients can be obtained by gluing together quotients of an affine covering.

Example 1.1.5. Define $G := \mathbb{K}^*$ and $X := \mathbb{K}^2$ as well as $U := \mathbb{K}^2 \setminus \{0\}$. The varieties X and U are G-varieties with the G-action given by $t \cdot (x, y) := (tx, ty)$. The quotient space $X /\!\!/ G$ is isomorphic to a point and $U /\!\!/ G$ is isomorphic to $\mathbb{P}_1$.



The following proposition subsumes some basic properties of good quotients. A subset $U \subseteq X$ of a G-variety X is *G-invariant* if $G \cdot U \subseteq U$.

Proposition 1.1.6. *Let X be a G-variety with a reductive group G and let $p\colon X \to Y$ be a good quotient for the G-action.*

(i) *The image of a closed, G-invariant subset is again closed.*
(ii) *The images of two closed, G-invariant, disjoint subsets are again disjoint.*
(iii) *For each point $y \in Y$, the fiber $p^{-1}(y)$ contains a closed G-orbit.*

A task of GIT is to construct *good G-sets* of a G-variety X, i.e., open subsets $U \subseteq X$ that admit a good quotient $U \to U /\!\!/ G$. We will concentrate on the affine case, i.e., assume $X = \operatorname{Spec} R$ with an affine, K-graded algebra R and the action of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$. Let $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$. The *weight cone* and the *orbit cone* of a point $x \in X$ are the convex polyhedral cones

$$\begin{array}{rcl} \vartheta_X & := & \operatorname{cone}(w \in K;\ R_w \neq \{0\}) \quad \subseteq \quad K_{\mathbb{Q}}, \\ \vartheta_x & := & \operatorname{cone}(w \in K;\ f(x) \neq 0 \text{ for some } f \in R_w) \quad \subseteq \quad K_{\mathbb{Q}}. \end{array}$$

It turns out that there are only finitely many orbit cones. Given a vector $w \in K_{\mathbb{Q}}$, we assign to w a *set of semistable points*, i.e., the open, H-invariant subset

$$X^{\mathrm{ss}}(w) \ := \ \{x \in X;\ f(x) \neq 0 \text{ with some } f \in R_{nw} \text{ and } n \in \mathbb{Z}_{\geq 1}\} \ \subseteq \ X.$$

Proposition 1.1.7. *In the above setting, $X^{\mathrm{ss}}(w) \neq \emptyset$ holds if and only if $w \in \vartheta_X$. In this case, the H-action on $X^{\mathrm{ss}}(w)$ admits a good quotient and $X^{\mathrm{ss}}(w) /\!\!/ H$ is projective over $X /\!\!/ H$. Moreover, given $w_1, w_2 \in \vartheta_X$ with $X^{\mathrm{ss}}(w_1) \subseteq X^{\mathrm{ss}}(w_2)$, we have a commutative diagram*

$$\begin{tikzcd} X^{\mathrm{ss}}(w_1) \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{\mathrm{ss}}(w_2) \arrow[d] \\ X^{\mathrm{ss}}(w_1) /\!\!/ H \arrow[r, "\varphi_{w_2}^{w_1}"'] & X^{\mathrm{ss}}(w_2) /\!\!/ H \end{tikzcd}$$

with $\varphi_{w_2}^{w_1}$ projective and surjective. Furthermore, given a third vector $w_3 \in \vartheta_X$ with $X^{\mathrm{ss}}(w_2) \subseteq X^{\mathrm{ss}}(w_3)$, we have $\varphi_{w_3}^{w_1} = \varphi_{w_3}^{w_2} \circ \varphi_{w_2}^{w_1}$.

A *quasifan* in a rational vector space $N_{\mathbb{Q}}$ is a finite collection Σ of polyhedral, convex cones in $N_{\mathbb{Q}}$ such that for each $\sigma \in \Sigma$, also all faces $\tau \preceq \sigma$ are elements of Σ and given $\sigma, \sigma' \in \Sigma$ the cone $\sigma \cap \sigma'$ is a face in both σ and σ'. A *fan* is a quasifan consisting of pointed cones. We write $\Sigma \subseteq N_{\mathbb{Q}}$ if the cones of Σ lie in $N_{\mathbb{Q}}$.

Definition 1.1.8. Given $w \in \vartheta_X$, the corresponding *GIT-cone* or *GIT-chamber* is the nonempty polyhedral cone

$$\lambda(w) \ := \ \bigcap_{\substack{x \in X, \\ w \in \vartheta_x}} \vartheta_x \ \subseteq \ K_{\mathbb{Q}}.$$

We call the collection $\Lambda(X, H) := \{\lambda(w);\ w \in \vartheta_X\}$ of all GIT-cones the *GIT-fan* of the H-action on X.

The term "fan" is justified by the following theorem which also relates GIT-cones to sets of semistable points. In particular, the number of GIT-cones is finite.

Theorem 1.1.9. *In the above setting, the GIT-fan $\Lambda(X, H)$ is a pure quasifan in $K_{\mathbb{Q}}$ with support ϑ_X. Furthermore, given $w_1, w_2 \in \vartheta_X$, we have*

$$\begin{array}{rclcrcl} \lambda(w_1) & \subseteq & \lambda(w_2) & \iff & X^{\mathrm{ss}}(w_1) & \supseteq & X^{\mathrm{ss}}(w_2), \\ \lambda(w_1) & = & \lambda(w_2) & \iff & X^{\mathrm{ss}}(w_1) & = & X^{\mathrm{ss}}(w_2). \end{array}$$

It then makes sense to define $X^{\mathrm{ss}}(\lambda) := X^{\mathrm{ss}}(w)$ with any element w in the relative interior of the given GIT-cone λ. Theorem 1.1.9 tells us that the structure of $\Lambda(X, H)$ reflects the variation of GIT-quotients; they form the *GIT-system* given by the quotients $Y(\lambda) := X^{\mathrm{ss}}(\lambda) /\!\!/ H$ and morphisms $\varphi^{\lambda_i}_{\lambda_j}$ of Proposition 1.1.7.

$$
\begin{tikzcd}[column sep=small]
 & Y(\lambda_i) \arrow[dl] \arrow[dr, "\varphi^{\lambda_i}_{\lambda_i\cap\lambda_j}"'] & & Y(\lambda_j) \arrow[dl, "\varphi^{\lambda_i}_{\lambda_j\cap\lambda_i}"] \arrow[dr] & \\
\cdots & & Y(\lambda_i \cap \lambda_j) & & \cdots
\end{tikzcd}
$$

If X is *H-factorial*, i.e., it is irreducible, normal and each H-invariant Weil divisor is the divisor of a rational homogeneous function, the GIT-fan gives a correspondence to good H-sets. A subset U_0 of a good H-set $U \subseteq X$ is *H-saturated* if $U_0 = p^{-1}(p(U_0))$ with the good quotient $p\colon U \to U /\!\!/ H$. By a *qp-maximal subset* of X we mean a good H-set $U \subseteq X$ with quasiprojective quotient space and U is maximal with respect to H-saturated inclusion among the good H-sets with quasiprojective quotient space.

Theorem 1.1.10. *In the above setting, assume that X is H-factorial. We have mutually inverse order-reversing bijections*

$$
\begin{array}{ccc}
\Lambda(X,H) & \longleftrightarrow & \{\text{ qp-maximal subsets of } X\ \} \\
\lambda & \mapsto & X^{\mathrm{ss}}(\lambda), \\
\bigcap\limits_{x\in U} \vartheta_x & \mapsfrom & U.
\end{array}
$$

Example 1.1.11. In Example 1.1.5, X and U are good $\mathbb{K}^*$-sets: the GIT-fan $\Lambda(X,\mathbb{K}^*)$ consists of cone(1) and cone(0). They correspond to the good $\mathbb{K}^*$-sets

$$
\begin{array}{ccccl}
X & = & X^{\mathrm{ss}}(0) & = & \{x \in X;\ f(x) \neq 0 \text{ for a } f \in \mathbb{K}[T_1,T_2]_0\}, \\
U & = & X^{\mathrm{ss}}(1) & = & \{x \in X;\ f(x) \neq 0 \text{ for a } f \in \mathbb{K}[T_1,T_2]_{n\cdot 1},\ n > 0\}.
\end{array}
$$

2. Cox rings

We recall the basic theory of Cox rings and toric varieties. This section summarizes parts of [5], mainly Chapter I, and [51, 61].

Consider an irreducible variety X that is *normal*, i.e., every local ring $\mathcal{O}_{X,x}$ integral and integrally closed in its quotient field. A *prime divisor* on X is an irreducible hypersurface $D \subseteq X$. The group generated by the prime divisors is the free abelian group $\mathrm{WDiv}(X)$, its elements are called *Weil divisors*. Let $\mathrm{ord}_D(f)$ be the *order* of vanishing of a rational function $f \in \mathbb{K}(X)^*$ along a prime divisor D. The *principal divisor* of f is

$$\mathrm{div}(f) \ := \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D \quad \in \quad \mathrm{WDiv}(X).$$

Note that $f \mapsto \mathrm{div}(f)$ is a homomorphism $\mathbb{K}(X)^* \to \mathrm{WDiv}(X)$ with the subgroup of principal divisors $\mathrm{PDiv}(X) \leq \mathrm{WDiv}(X)$ as its image. The *divisor class group* is the factor group

$$\mathrm{Cl}(X) \ := \ \mathrm{WDiv}(X) \,/\, \mathrm{PDiv}(X).$$

A Weil divisor is a *Cartier divisor* if it is locally principal; we write $\mathrm{CDiv}(X) \leq \mathrm{WDiv}(X)$ for the group of all Cartier divisors. Moreover, a Weil divisor $D = a_1D_1 + \ldots + a_nD_n \in \mathrm{WDiv}(X)$ is *effective* if all a_i are non-negative; we then write $D \geq 0$. Given an open subset $U \subseteq X$, the *restriction* of a Weil divisor $D \in \mathrm{WDiv}(X)$ is the Weil divisor $D_{|U} \in \mathrm{WDiv}(U)$ where $D_{|U} := D \cap U$ if D intersects U non-trivially and $D_{|U} := 0$ otherwise.

Construction 1.2.1. To a Weil divisor $D \in \mathrm{WDiv}(X)$ we assign the sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules where the sections over open subsets $U \subseteq X$ are

$$\Gamma(U, \mathcal{O}_X(D)) \;:=\; \left\{ f \in \mathbb{K}(X)^*;\; (\mathrm{div}(f) + D)_{|U} \geq 0 \right\} \;\cup\; \{0\}.$$

Note that given $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$, we have $f_1 f_2 \in \Gamma(U, \mathcal{O}_X(D_1 + D_2))$. To a subgroup $K \leq \mathrm{WDiv}(X)$ we associate the *sheaf of divisorial algebras*

$$\mathcal{S} \;:=\; \bigoplus_{D \in K} \mathcal{S}_D, \qquad \mathcal{S}_D \;:=\; \mathcal{O}_X(D).$$

The multiplication in $\mathcal{S}$ is defined by multiplying elements in the field of rational functions $\mathbb{K}(X)$.

Construction 1.2.2 (Cox ring)**.** Let X be an irreducible normal variety with finitely generated class group $\mathrm{Cl}(X)$ and $\mathbb{K}^* = \Gamma(X, \mathcal{O}^*)$, e.g., X is complete. Fix a subgroup $K \leq \mathrm{WDiv}(X)$ such that the homomorphism $c\colon K \to \mathrm{Cl}(X)$ mapping $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is surjective. Set $K^0 := \ker(c)$. Choose a group homomorphism

$$\chi\colon K^0 \;\to\; \mathbb{K}(X)^* \qquad \text{with} \qquad \mathrm{div}\,(\chi(E)) \;=\; E, \qquad E \in K^0.$$

Let $\mathcal{S}$ be the sheaf of divisorial algebras associated to K as in Construction 1.2.1. Consider the sheaf $\mathcal{I}$ of radical ideals that is locally defined by $1 - \chi(E)$ where E runs through K^0. On open subsets $U \subseteq X$, this means we have an ideal

$$\begin{aligned}\Gamma(U, \mathcal{I}) \;&=\; \left\{ f \in \Gamma(U, \mathcal{S});\ \text{locally } f = \sum_{E \in K^0} h_E(1 - \chi(E)) \text{ with } h_E \in \Gamma(U, \mathcal{S}) \right\} \\ &=\; \left\langle 1 - \chi(E_i);\ 1 \leq i \leq s \right\rangle_{\Gamma(U, \mathcal{S})}\end{aligned}$$

where $E_1, \ldots, E_s$ is a basis for K^0. Note that $1 \in \Gamma(U, \mathcal{S}_0)$ whereas $E \in \Gamma(U, \mathcal{S}_{-E})$. The *Cox sheaf* is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} \;=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} \;:=\; \pi\left(\bigoplus_{D' \in c^{-1}([D])} \mathcal{O}_X(D') \right)$$

and the projection $\pi\colon \mathcal{S} \to \mathcal{R}$. It is a quasicoherent sheaf of $\mathrm{Cl}(X)$-graded, reduced $\mathcal{O}_X$-algebras. The *Cox ring* of X is the ring of global sections of the Cox sheaf

$$\mathcal{R}(X) \;:=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma\left(X, \mathcal{R}_{[D]}\right) \;\cong\; \Gamma(X, \mathcal{S}) \,/\, \Gamma(X, \mathcal{I}).$$

One can show that the construction of the Cox ring in 1.2.2 does not depend on the choices made.

Remark 1.2.3. If in Construction 1.2.2 the class group $\mathrm{Cl}(X)$ is free, one can directly define $\mathcal{R}_{[D]} := \mathcal{O}_X(D)$.

In the situation of Construction 1.2.2, let $R := \mathcal{R}(X)$ be the Cox ring of X. A non-zero homogeneous element $f \in R \setminus R^*$ is $\mathrm{Cl}(X)$-*prime* if $f \mid gh$ with homogeneous elements g, h implies $f \mid g$ or $f \mid h$. We say that R is $\mathrm{Cl}(X)$-*factorial* if every homogeneous non-zero element $f \in R \setminus R^*$ is a product of $\mathrm{Cl}(X)$-primes.

Theorem 1.2.4. *In the above setting, the Cox ring $\mathcal{R}(X)$ is a $\mathrm{Cl}(X)$-factorial ring. If $\mathrm{Cl}(X)$ is free, then $\mathcal{R}(X)$ is a UFD.*

We turn to the geometric counterpart of the Cox sheaf. Let X be as before but assume additionally that $\mathcal{R}(X)$ is finitely generated. Then the Cox sheaf $\mathcal{R}$ is locally of finite type.

Construction 1.2.5. Let the setting be as above. Taking the *relative spectrum* we obtain an irreducible, normal variety $\widehat{X} := \operatorname{Spec}_X \mathcal{R}$ that is contained in $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$. The affine variety $\overline{X}$, called the *total coordinate space* of X, is invariant with respect to the action of the *characteristic quasitorus* $H_X := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$. Then the embedding $\widehat{X} \subseteq \overline{X}$ is H_X-equivariant and X can be retrieved as a quotient

$$\begin{tikzcd} \operatorname{Spec}_X \mathcal{R} \arrow[r, equal] & \widehat{X} \arrow[d, "/\!\!/ H_X"] & \subseteq & \overline{X} \arrow[r, equal] & \operatorname{Spec} \mathcal{R}(X) \\ & X & & & \end{tikzcd}$$

of the quasiaffine, good H_X-set $\widehat{X} \subseteq \overline{X}$ by H_X. The good quotient $p \colon \widehat{X} \to X$ is called the *characteristic space* of X.

In Construction 1.2.5, we call $z = (z_1, \dots, z_r) \in \widehat{X} \subseteq \mathbb{K}^r$ *Cox coordinates* for the point $x := p(z) \in X$. We write $x = [z]$ or $x = [z_1, \dots, z_r]$. Note that Cox coordinates are not unique, see 5.1.1.

Example 1.2.6. In Construction 1.2.5, $X = \mathbb{P}_2$ arises as a $\mathbb{K}^*$-quotient of the open subset $\widehat{X} = \mathbb{K}^2 \setminus \{0\}$ of $\overline{X} = \mathbb{K}^3$. Cox coordinates are the usual homogeneous coordinates.

A *toric variety* is an irreducible, normal variety Z with a basepoint $z_0 \in Z$ and the action $\mathbb{T}_Z \times Z \to Z$ of the torus $\mathbb{T}_Z$ such that the map $\mathbb{T}_Z \to Z$ defined by $t \mapsto t \cdot z_0$ is an open embedding. We then speak of the *dense torus* $\mathbb{T}_Z$ *of* Z. We briefly recall the connection to *lattice fans*, i.e., pairs (N, Σ) with a lattice N and a fan $\Sigma \subseteq N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Given a lattice fan (N, Σ) and a cone $\sigma \in \Sigma$, let $\sigma^\vee \subseteq N^*_{\mathbb{Q}} := N^* \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual cone. We obtain an affine toric variety

$$Z_\sigma \;:=\; \operatorname{Spec} \mathbb{K}\,[\sigma^\vee \cap N^*] \;=\; \operatorname{Spec} \bigoplus_{u \in \sigma^\vee \cap N^*} \mathbb{K} \cdot \chi^u$$

with dense torus $\mathbb{T}_N := \operatorname{Spec} \mathbb{K}[N^*]$. Gluing together the affine toric varieties Z_σ with $\sigma \in \Sigma$ produces the toric variety Z_Σ. For the other direction, it turns out that a toric variety Z can be covered by finitely many invariant open, affine toric subvarieties $Z_1, \dots, Z_s$. The cones $\sigma_{Z_1}, \dots, \sigma_{Z_s}$ of *convergent one-parameter subgroups* $\mathbb{K}^* \to Z_i$ of the Z_i then form a lattice fan $(\Sigma_Z, \Lambda(\mathbb{T}_Z))$ where $\Lambda(\mathbb{T}_Z)$ is the cone of convergent one-parameter subgroups of the dense torus $\mathbb{T}_Z$. Moreover, given a toric variety Z with fan Σ, note that to each cone $\sigma \in \Sigma$ one can assign a *distinguished point* $z(\sigma) \in Z$ such that the orbits $\mathbb{T}_Z \cdot z(\sigma')$ correspond bijectively to the cones $\sigma' \in \Sigma$.

A *morphism* of toric varieties Z and Z' is a pair $(\varphi, \widetilde{\varphi})$ consisting of a morphism of varieties $\varphi \colon Z \to Z'$ and a morphism of tori $\widetilde{\varphi} \colon \mathbb{T}_Z \to \mathbb{T}_{Z'}$ such that φ maps the basepoint of Z to the basepoint of Z' and $\varphi(t \cdot z) = \widetilde{\varphi}(t) \cdot \varphi(z)$ holds for all $t \in \mathbb{T}_Z$, $z \in Z$. Moreover, a *map of lattice fans* (Σ, N) and (Ω, M) is a lattice homomorphism $F \colon N \to M$ such that for each cone $\sigma \in \Sigma$ there is a cone $\omega \in \Omega$ such that $F(\sigma) \subseteq \omega$. To each such F one can assign a toric morphism $(\varphi_F, \widetilde{\varphi}_F) \colon Z_\Sigma \to Z_\Omega$, see [5, Ch. 2] or [42, 28] for details.

Proposition 1.2.7. *We have covariant functors that are essentially inverse to each other between the categories of lattice fans and toric varieties given by*

$$\begin{array}{rcl@{\qquad\qquad}rcl} (\Sigma, N) & \mapsto & (Z_\Sigma, \mathbb{T}_N, z_0), & (\Sigma_Z, \Lambda(\mathbb{T}_Z)) & \mapsfrom & (Z, \mathbb{T}_Z, z_0), \\ F & \mapsto & (\varphi_F, \widetilde{\varphi}_F)\,, & \widetilde{\varphi}_* & \mapsfrom & (\varphi, \widetilde{\varphi})\,. \end{array}$$

Remark 1.2.8. An irreducible, normal variety is toric if and only if its Cox ring is isomorphic to a polynomial ring.

3. Bunched rings and Mori dream spaces

We recall the basic theory of Mori dream spaces and bunched rings developed by F. Berchtold and J. Hausen in [19, 51]. This section is taken from [5], mainly Chapter III.

Let R be an integral affine $\mathbb{K}$-algebra that is graded by a finitely generated abelian group K, i.e.,

$$R \;=\; \bigoplus_{w\in K} R_w.$$

Similar to Section 2, a homogeneous element $f \in R\setminus\{0\}$ is called *K-prime* if $f \notin R^*$ and $f \mid gh$ with homogeneous g, h implies $f \mid g$ or $f \mid h$. We say that R is *factorially K-graded* or is *K-factorial* if each homogeneous element $f \in R \setminus \{0\}$ with $f \notin R^*$ is a product of K-prime elements.

If R is factorially K-graded, one can choose a system $\mathfrak{F} = (f_1, \dots, f_r)$ of K-prime pairwise non-associated generators. We then encode the K-grading of R in a *degree map*, i.e., a homomorphism $Q\colon \mathbb{Z}^r \to K$ of finitely generated abelian groups mapping the canonical basis vector $e_i \in \mathbb{Z}^r$ to $\deg(f_i) \in K$. Denote the positive orthant by $\gamma := \mathbb{Q}^r_{\geq 0}$.

Definition 1.3.1. An *$\mathfrak{F}$-face* is a face $\gamma_0 \preceq \gamma$ such that the product $\prod_{e_i\in\gamma_0} f_i$ is not an element of the radical of $\langle f_j;\, e_j \notin \gamma_0\rangle \subseteq R$.

Remark 1.3.2. The set of orbit cones $\Omega_{\overline{X}}$ of $\overline{X} := \operatorname{Spec} R$ as defined in Section 1 equals the collection of all images $Q(\gamma_0)$ such that $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face.

The $\mathfrak{F}$-faces store the algebraic information of R, see Chapter 3 for the computational aspects. We now turn to combinatorial data in $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$. The grading is *almost free* if $Q(\gamma_0 \cap \mathbb{Z}^r)$ generates the abelian group K for every facet $\gamma_0 \preceq \gamma$.

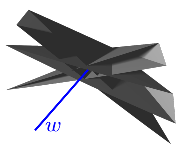
Definition 1.3.3.
(i) Let Ω be the set of *orbit cones*, i.e., the set of all cones $Q(\gamma_0) \subseteq K_{\mathbb{Q}}$ such that γ_0 is an $\mathfrak{F}$-face. An *$\mathfrak{F}$-bunch* is a non-empty collection $\Phi \subseteq \Omega$ such that
- each two $\vartheta_1, \vartheta_2 \in \Phi$ overlap, i.e., $\vartheta_1^\circ \cap \vartheta_2^\circ \neq \emptyset$.
- given $\vartheta_2 \in \Omega$ and $\vartheta_1 \in \Phi$ with $\vartheta_1^\circ \subseteq \vartheta_2^\circ$ then also $\vartheta_2 \in \Phi$.

(ii) An $\mathfrak{F}$-bunch Φ is *true* if $Q(\gamma_0) \in \Phi$ for each facet $\gamma_0 \preceq \gamma$.

(iii) An $\mathfrak{F}$-bunch Φ is *maximal* if no further projected $\mathfrak{F}$-face $Q(\gamma_0)$ can be added to Φ.

Example 1.3.4. Denote by $\vartheta := \operatorname{cone}(\deg(f_1), \dots, \deg(f_r)) \subseteq K_{\mathbb{Q}}$ the *weight cone* of R. Each vector $w \in \vartheta$ defines an $\mathfrak{F}$-bunch

$$\Phi(w) \;:=\; \{\vartheta_0 \in \Omega;\; w \,\in\, \vartheta_0^\circ\}\,.$$



We call the K-grading of R *pointed* if $R_0 = \mathbb{K}$ and the weight cone ϑ is pointed. The following notion will be of central interest in Chapter 2.

Definition 1.3.5. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$ where R is an almost freely factorially K-graded, integral, normal, affine $\mathbb{K}$-algebra with $\mathbb{K}^*$ as its group of homogeneous units. Moreover, $\mathfrak{F}$ is a system of pairwise non-associated K-prime generators for R and Φ is a true $\mathfrak{F}$-bunch in $K_{\mathbb{Q}}$.

To each bunched ring $(R, \mathfrak{F}, \Phi)$ we implicitly assign the degree map $Q\colon \mathbb{Z}^r \to K$ and the positive orthant $\gamma = \mathbb{Q}^r_{\geq 0}$. The relevant algebraic data of a bunched ring is stored in the collection of *relevant $\mathfrak{F}$-faces* and the *covering collection* given by

$$\begin{aligned} \mathrm{rlv}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \gamma_0 \text{ is an } \mathfrak{F}\text{-face and } Q(\gamma_0) \in \Phi\}, \\ \mathrm{cov}(\Phi) &:= \{\gamma_0 \in \mathrm{rlv}(\Phi);\ \gamma_0 \text{ minimal}\}. \end{aligned}$$

Recall that a variety X is called an *A_2-variety* if for each two points $x, x' \in X$ there is an affine, open neighborhood $U \subseteq X$ such that $x, x' \in U$. A normal variety is A_2 if and only if there is a closed embedding into a toric variety. The following construction is essential.

Construction 1.3.6. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. Write $\mathfrak{F} = (f_1, \ldots, f_r)$. Consider the action of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ on the affine variety $\overline{X} := \operatorname{Spec} R$. We assign to each $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ the open affine variety

$$\overline{X}_{\gamma_0} := \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \qquad \text{with } u = (u_1, \ldots, u_r) \in \gamma_0^\circ.$$

This is independent of the choice of $u \in \gamma_0^\circ$. We then obtain an open H-invariant subset

$$\widehat{X}(R, \mathfrak{F}, \Phi) := \widehat{X} := \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X}_{\gamma_0} = \bigcup_{\gamma_0 \in \mathrm{cov}(\Phi)} \overline{X}_{\gamma_0} \subseteq \overline{X}.$$

The H-action on $\widehat{X}$ admits a good quotient p. This means we have an irreducible normal A_2-variety $X(R, \mathfrak{F}, \Phi) := X := \widehat{X} /\!\!/ H$ with

$$\operatorname{Spec} R = \overline{X} \supseteq \widehat{X} \xrightarrow{\quad p \quad} X$$

Then X is of dimension $\dim(R) - \dim(K_{\mathbb{Q}})$. Moreover, the Cox ring of X is isomorphic to R, we have $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and there is an isomorphism

$$\mathrm{Cl}(X) \to K, \qquad [D_i] \mapsto \deg(f_i) \qquad \text{where } D_i := p\left(V(\widehat{X};\, f_i)\right)$$

and the $D_i \subseteq X$ are prime divisors. Furthermore, the affine open subsets $\overline{X}_{\gamma_0} \subseteq \widehat{X}$ with $\gamma_0 \in \mathrm{rlv}(\Phi)$ are H-saturated and we have an affine cover

$$X = \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} X_{\gamma_0}, \qquad X_{\gamma_0} := p\left(\overline{X}_{\gamma_0}\right) \subseteq X.$$

We call an embedding of varieties $\iota \colon X \to X'$ *big* if the codimension of $X' \setminus \iota(X)$ is at least two. Moreover, an A_2-variety X is called *A_2-maximal* if for each big open embedding $\iota \colon X \to X'$ with an A_2-variety X', we have $\iota(X) = X'$. Projective varieties are A_2-maximal.

Theorem 1.3.7. *Consider an irreducible, normal A_2-variety with finitely generated class group* $\mathrm{Cl}(X)$ *and finitely generated Cox ring* $R := \mathcal{R}(X)$. *Suppose* $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$. *Fix a system $\mathfrak{F}$ of pairwise non-associated* $\mathrm{Cl}(X)$*-prime generators for* $\mathcal{R}(X)$.

(i) *There is a maximal $\mathfrak{F}$-bunch Φ and a big open embedding $X \to X(R, \mathfrak{F}, \Phi)$.*
(ii) *If, in (i), X is A_2-maximal, then $X \cong X(R, \mathfrak{F}, \Phi)$.*

Definition 1.3.8. A *Mori dream space* is an irreducible, complete, normal variety X with finitely generated class group $\mathrm{Cl}(X)$ and finitely generated Cox ring $\mathcal{R}(X)$.

We will use the term Mori dream space in a slightly more general setting in Chapter 2. An $\mathfrak{F}$-bunch Φ of cones in $K_{\mathbb{Q}}$ is called *projective* if $\Phi = \Phi(w)$ for a vector $w \in K_{\mathbb{Q}}$ as in Example 1.3.4. The following corollary states the correspondence to bunched rings.

Corollary 1.3.9. *For each Mori dream space X there is a bunched ring $(R, \mathfrak{F}, \Phi)$ such that $X \cong X(R, \mathfrak{F}, \Phi)$. Moreover, there is a bijection*

$$\left\{ \begin{array}{c} \textit{projective Mori} \\ \textit{dream spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{bunched rings with a} \\ \textit{projective bunch of cones} \end{array} \right\}.$$

Observe that the GIT-fan presents the possible choices for the ample class w defining the projective bunch $\Phi(w)$. The *moving cone* of R is the polyhedral cone

$$\mathrm{Mov}(R) \quad := \quad \bigcap_{i=1}^{r} \mathrm{cone}(f_j;\ j \neq i) \quad \subseteq \quad K_{\mathbb{Q}}.$$

Remark 1.3.10. Fix a finitely generated abelian group K and a factorially K-graded, integral, normal, affine $\mathbb{K}$-algebra R with K-prime generators $f_1, \ldots, f_r$ and $\mathbb{K}^*$ as its homogeneous units. Let $H := \mathrm{Spec}\,\mathbb{K}[K]$ act on $\overline{X} := \mathrm{Spec}\,R$. Consider a projective Mori dream space Y with Cox ring $\mathcal{R}(Y) \cong R$ and class group $\mathrm{Cl}(Y) \cong K$. Then

$$Y \;\cong\; X(R, \mathfrak{F}, \Phi(w)) \qquad \text{for some} \qquad \lambda \in \Lambda(\overline{X}, H), \quad w \in \lambda^\circ \;\subseteq\; \mathrm{Mov}(R)^\circ.$$

Let X be an irreducible normal variety and Z a toric variety with dense torus $\mathbb{T}_Z$, basepoint $z_0 \in Z$ and invariant prime divisors $D_Z^1, \ldots, D_Z^r$. Write $D_Z^i = \overline{\mathbb{T}_Z \cdot z_i}$ with $z_i \in Z$. Let $\varphi\colon X \to Z$ be a morphism. Assume $\varphi^{-1}\left(D_Z^i\right) \subseteq X$ are pairwise different irreducible hypersurfaces for each $1 \leq i \leq r$. Define

$$Z' \quad := \quad \mathbb{T}_Z \cdot z_0 \;\cup\; \ldots \;\cup\; \mathbb{T}_Z \cdot z_r \quad \subseteq \quad Z.$$

Then the codimension of $X \setminus \varphi^{-1}(Z')$ in X is at least two and there is a canonical pullback homomorphism

$$\begin{tikzcd} \mathrm{CDiv}^{\mathbb{T}_Z}(Z') \arrow[r, "\varphi^*"] \arrow[d, equal] & \mathrm{CDiv}(\varphi^{-1}(Z')) \arrow[d, phantom, "\cap"] \\ \mathrm{WDiv}^{\mathbb{T}_Z}(Z) \arrow[r] \arrow[d] & \mathrm{WDiv}(X) \arrow[d] \\ \mathrm{Cl}(Z) \arrow[r] & \mathrm{Cl}(X) \end{tikzcd}$$

where $\mathrm{WDiv}^{\mathbb{T}_Z}(Z')$ and $\mathrm{CDiv}^{\mathbb{T}_Z}(Z')$ denote the respective $\mathbb{T}_Z$-invariant divisors. Since principal divisors are mapped to principal divisors, φ^* induces the map in the lower row which we denote again by $\varphi^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$.

Definition 1.3.11. Let X be an irreducible, normal variety and Z a toric variety with dense torus $\mathbb{T}_Z$ and invariant prime divisors $D_Z^1, \ldots, D_Z^r$. A closed embedding $\iota\colon X \to Z$ is a *neat embedding* if

$$\iota^{-1}\left(D_Z^1\right), \qquad \ldots, \qquad \iota^{-1}\left(D_Z^r\right) \quad \subseteq \quad X$$

are pairwise different irreducible hypersurfaces and the homomorphism $\iota^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism.

Construction 1.3.12 (Canonical toric ambient variety)**.** Consider a bunched ring $(R, \mathfrak{F}, \Phi)$ with degree map $Q\colon \mathbb{Z}^r \to K$ and positive orthant $\gamma = \mathbb{Q}^r_{\geq 0}$. Setting $M := \ker(Q)$ we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & \mathbb{Z}^r \arrow[r, "P"] & N \end{tikzcd}$$

$$\begin{tikzcd} 0 & K \arrow[l] & \mathbb{Z}^r \arrow[l, "Q"'] & M \arrow[l, "P^*"'] & 0 \arrow[l] \end{tikzcd}$$

For each face $\gamma_0 \preceq \gamma$, let $\gamma_0^* := \gamma_0^\perp \cap \delta$ be the dual face where $\delta := \gamma^\vee$. Consider the collection of faces Θ and the fans $\widehat{\Sigma} \subseteq \mathbb{Q}^r$ and $\Sigma \subseteq N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ given by

$$\begin{aligned}
\Theta &:= \{\gamma_0 \preceq \gamma;\ \text{there is } \gamma_1 \in \mathrm{rlv}(\Phi) \text{ with } \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ\}, \\
\widehat{\Sigma} &:= \{\delta_0 \preceq \delta;\ \ \text{there is } \gamma_0 \in \Theta \text{ with } \delta_0 \preceq \gamma_0^*\}, \\
\Sigma &:= \{P(\gamma_0^*);\ \gamma_0 \in \Theta\}.
\end{aligned}$$

As in Construction 1.3.6, let $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Consider the action of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} := \operatorname{Spec} R$. Let $\overline{Z} := \mathbb{K}^r$ be the toric variety corresponding to the cone δ. The generators $\mathfrak{F} = (f_1, \ldots, f_r)$ for R provide a closed embedding

$$\bar{\imath}\colon \overline{X} \ \to\ \overline{Z}, \qquad z \ \mapsto\ (f_1(z), \ldots, f_r(z))$$

which is H-equivariant if we install the diagonal H-action on $\overline{Z}$ given by the characters $\chi^{w_1}, \ldots, \chi^{w_r}$ with $w_i := \deg(f_i) \in K$. We have a diagram

$$\begin{tikzcd}
\overline{X} \arrow[rrrr, bend left, "\bar{\imath}"] & \supseteq & \widehat{X} \arrow[r, "\hat{\imath}"] \arrow[d, "/\!\!/ H"'] & \widehat{Z} \arrow[d, "/\!\!/ H"] & \subseteq \ \overline{Z} \\
& & X \arrow[r, "\imath"] & Z &
\end{tikzcd}$$

where $\widehat{Z}$ and Z are the toric varieties corresponding to $\widehat{\Sigma}$ and Σ respectively, $\hat{\imath}$ is the restriction of $\bar{\imath}$, the induced map of quotients $\imath$ is a neat embedding and the toric morphism $\widehat{Z} \to Z$, the *Cox construction*, corresponds to the matrix P.

Definition 1.3.13. In the setting of Construction 1.3.12, we call $\imath\colon X \to Z$ the *canonical toric embedding* and Z the *canonical toric ambient variety* of X.

We now give a short survey of the basic geometry of varieties arising from bunched rings. A first step is the decomposition into strata.

Construction 1.3.14. Let the situation be as in Construction 1.3.6. To an $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ we assign the locally closed set

$$\overline{X}(\gamma_0) \ := \ \left\{z \in \overline{X};\ f_i(z) \neq 0 \ \Leftrightarrow\ e_i \in \gamma_0 \text{ for each } 1 \leq i \leq r\right\} \ \subseteq\ \overline{X}.$$

Their union gives a disjoint covering of $\overline{X}$. We obtain a disjoint decomposition into locally closed *strata* as

$$X \ = \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} X(\gamma_0), \qquad X(\gamma_0) \ := \ p\left(\overline{X}(\gamma_0)\right) \ = \ X_{\gamma_0} \setminus \bigcup_{\substack{\gamma_1 \in \mathrm{rlv}(\Phi), \\ \gamma_0 \prec \gamma_1}} X_{\gamma_1}.$$

Let X be normal and irreducible. Given $x \in X$, the *local class group* $\mathrm{Cl}(X, x)$ is the factor group of $\mathrm{WDiv}(X)$ by the group of all divisors that are principal in a neighborhood of x. The *Picard group* $\mathrm{Pic}(X)$ is the group $\mathrm{CDiv}(X)/\mathrm{PDiv}(X)$ of Cartier divisors modulo principal divisors. Moreover, a point $x \in X$ is *factorial* if near x each Weil divisor is principal. Similarly, $x \in X$ is $\mathbb{Q}$-*factorial* if near x, for each Weil divisor $D \in \mathrm{WDiv}(X)$, there is $n \in \mathbb{Z}_{\geq 1}$ such that nD is principal. If each point is $\mathbb{Q}$-factorial, the variety X is $\mathbb{Q}$-*factorial*.

Proposition 1.3.15. *Consider the setting of Construction 1.3.6. and let $Q\colon \mathbb{Z}^r \to K$ be the degree matrix. For each $\gamma_0 \in \mathrm{rlv}(\Phi)$ and each point $x \in X(\gamma_0)$ we have a* *diagram*

$$\begin{tikzcd} \mathrm{Cl}(X) \arrow[r, "\cong"] \arrow[d] & K \arrow[d] \\ \mathrm{Cl}(X,x) \arrow[r, "\cong"] & K \,/\, Q(\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^r) \end{tikzcd}$$

It is independent of the choice of $x \in X(\gamma_0)$. The following claims hold.

(i) *The point x is factorial if and only if $Q(\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^r)$ equals K.*
(ii) *The point x is $\mathbb{Q}$-factorial if and only if $Q(\gamma_0)$ is of full dimension.*
(iii) *The point x is smooth if and only if it is factorial and there is a smooth point $z \in p^{-1}(x) \subseteq \widehat{X}$.*

Moreover, $\mathrm{Pic}(X)$ *is isomorphic to the Picard group of the canonical toric ambient variety Z and it is free if Z has a toric fixed point. Within $K \cong \mathrm{Cl}(X)$ it is given by*

$$\mathrm{Pic}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q\left(\mathrm{lin}(\gamma_0) \,\cap\, \mathbb{Z}^r\right).$$

The *effective cone* is the convex polyhedral cone $\mathrm{Eff}(X)$ in $\mathrm{Cl}(X)_{\mathbb{Q}} := \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the divisor classes of effective divisors. The *stable base locus* of a divisor $D \in \mathrm{WDiv}(X)$ is

$$\bigcap_{n \in \mathbb{Z}_{\geq 1}} \;\; \bigcap_{f \in \Gamma(X, \mathcal{O}(nD))} \mathrm{Supp}\left(\mathrm{div}_{nD}(f)\right).$$

We call a divisor $D \in \mathrm{WDiv}(X)$ *movable* if its stable base locus is of codimension at least two. The *moving cone* $\mathrm{Mov}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ is the convex polyhedral cone consisting of all movable divisor classes. Furthermore, a divisor $D \in \mathrm{WDiv}(X)$ with empty stable base locus is called *semiample*. It is *ample* if there is a covering of X by affine sets $X \setminus \mathrm{Supp}(\mathrm{div}_{nD}(f))$ with $n \in \mathbb{Z}_{\geq 1}$. The convex cones $\mathrm{SAmple}(X)$ and $\mathrm{Ample}(X) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ consist of all semiample or ample divisor classes, respectively.

Proposition 1.3.16. *Consider the situation of Construction 1.3.6 with degree map $Q\colon \mathbb{Z}^r \to K$ and $\gamma = \mathbb{Q}^r_{\geq 0}$. Within $K_{\mathbb{Q}} = \mathrm{Cl}(X)_{\mathbb{Q}}$ we have the cones*

$$\begin{aligned} \mathrm{Eff}(X) &= Q(\gamma), & \mathrm{SAmple}(X) &= \bigcap_{\tau \in \Phi} \tau, \\ \mathrm{Mov}(X) &= \bigcap_{\substack{\gamma_0 \preceq \gamma \\ \text{facet}}} Q(\gamma_0), & \mathrm{Ample}(X) &= \bigcap_{\tau \in \Phi} \tau^{\circ}. \end{aligned}$$

We now treat the case of a variety $X = X(R, \mathfrak{F}, \Phi)$ arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ with $\mathfrak{F} = (f_1, \ldots, f_r)$ and grading group K where R is a *complete intersection*, i.e., there are $d := r - \dim(R)$ polynomials $g_1, \ldots, g_d$ that are K-homogeneous and generate the kernel of

$$\mathbb{K}[T_1, \ldots, T_r] \;\to\; R, \qquad T_i \;\mapsto\; f_i.$$

We write $u_1, \ldots, u_d \in K$ for the degrees of $g_1, \ldots, g_d$ and $w_1, \ldots, w_r \in K$ for the degrees of $f_1, \ldots, f_r \in \mathbb{K}[T_1, \ldots, T_r]$. Recall that a variety X is *Gorenstein* if its anticanonical divisor is Cartier. If some positive multiple of its anticanonical divisor is Cartier, X is called $\mathbb{Q}$-*Gorenstein*. The variety X is *Fano* if it is irreducible, normal, projective and its anticanonical divisor is ample.

Proposition 1.3.17. *In the situation of Construction 1.3.6, assume $(R, \mathfrak{F}, \Phi)$ is a complete intersection bunched ring. Within $K = \mathrm{Cl}(X)$, the anticanonical divisor* *class of X is*

$$-w_X^{\mathrm{can}} \;=\; \sum_{i=1}^{r} w_i \;-\; \sum_{i=1}^{d} u_i \;\in\; K.$$

Moreover, the following properties hold.

(i) *X is Gorenstein if and only if $-w_X^{\mathrm{can}} \in \mathrm{Pic}(X)$.*
(ii) *X is $\mathbb{Q}$-Gorenstein if and only if $-w_X^{\mathrm{can}} \in \mathrm{lin}(\tau)$ for each $\tau \in \Phi$.*
(iii) *X is Fano if and only if $-w_X^{\mathrm{can}} \in \mathrm{Ample}(X)$.*

In the setting of Proposition 1.3.17, we say that the variety X is *n-Gorenstein* if $n \in \mathbb{Z}_{\geq 1}$ is minimal with $-nw_X^{\mathrm{can}} \in \mathrm{Pic}(X)$.

4. Modifications and surfaces

We recapitulate some basic geometry of *surfaces*, i.e., two-dimensional irreducible varieties. Also, the notions of contractions and blow ups of a variety are being recalled. This section is a summary of [5], mainly Chapter V, the thesis of U. Derenthal [33], mostly Chapter 1, and Chapters II.7 and V of R. Hartshorne's book [48]. Compare also [51, Sec. 6] and Beauville's book [14, Ch. II].

Consider a smooth, projective surface X and two distinct *curves* D_1, D_2 on X, i.e., irreducible subvarieties of dimension one. In other words, $D_1, D_2 \in \mathrm{WDiv}(X) = \mathrm{CaDiv}(X)$ are prime divisors with coefficient one. Their *intersection number* $D_1 \cdot D_2$ is the sum over all *intersection multiplicities* m_x, that is

$$D_1 \cdot D_2 \;:=\; \sum_{x \in D_1 \cap D_2} m_x, \qquad m_x \;:=\; \dim_{\mathbb{K}} \left(\mathcal{O}_{X,x} / \langle f_x^{(1)}, f_x^{(2)} \rangle \right)$$

where $f_x^{(i)} \in \mathcal{O}_{X,x}$ is a germ of a generator for the ideal $I(D_i)$ near x. Note that if D_1 and D_2 intersect transversally and are smooth, $D_1 \cdot D_2$ is the number of intersection points. Taking intersection numbers extends to a symmetric $\mathbb{Z}$-valued bilinear form on $\mathrm{WDiv}(X)$ that only depends on the classes of the involved divisors. This means we have an *intersection product*

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \;\to\; \mathbb{Z}, \qquad ([D_1], [D_2]) \;\mapsto\; D_1 \cdot D_2.$$

Given $D \in \mathrm{WDiv}(X)$, its *self-intersection number* is $D^2 := D \cdot D$. We call a curve $D \subseteq X$ *negative* or *non-negative* if $D^2 < 0$ or $D^2 \geq 0$, respectively. A *$(-k)$-curve* is an irreducible curve with $C^2 = -k$ that is isomorphic to $\mathbb{P}_1$.

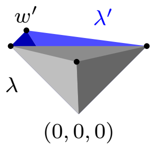
We turn to *modifications*, i.e., proper birational morphisms. Let X be a projective Mori dream space and $D \subseteq X$ a prime divisor. By a *contraction* of D, we mean a morphism $\pi\colon X \to X'$ mapping D to a point such that the restriction $\pi\colon X \setminus D \to X' \setminus \pi(D)$ is an isomorphism. For surfaces, the contractible divisors are exactly the negative, rational curves.

Theorem 1.4.1 (Castelnuovo criterion). *Let X be a projective, smooth surface and C an irreducible curve on X. Then C is a (-1)-curve if and only if there is a contraction $\pi\colon X \to X'$ of C with a smooth, projective surface X'.*

For general Mori dream spaces, we use the following remark. Let R be a K-graded $\mathbb{K}$-algebra as in Construction 1.3.6 with pairwise non-associated K-prime generators $f_1, \ldots, f_r$ and weight cone $\vartheta \subseteq K_{\mathbb{Q}}$. We say $w_i := \deg(f_i) \in K$ is *extremal* if $\vartheta \neq \mathrm{cone}(w_j;\, j \neq i)$. Note that this definition differs from the one in [5].

Remark 1.4.2. Let $X = X(R, \mathfrak{F}, \Phi)$ be a projective, $\mathbb{Q}$-factorial variety corresponding to a bunched ring as in Construction 1.3.6. Assume $\Phi = \Phi(w)$ holds for a full-dimensional GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ with $w \in \lambda^\circ$. Let $D_Z \subseteq Z$ be a prime divisor and $D_X := D_Z \cap X$ the corresponding prime divisor on X. Writing $w' := [D_Z] = [D_X] \in K$ for their classes, the following are equivalent.

(i) The vector $w' \in K$ is extremal and there is a full-dimensional cone $\lambda' \in \Lambda(\overline{X}, H)$ with $w' \in \lambda'$ such that $\lambda' \cap \lambda$ is of codimension one.
(ii) There is a contraction $X \to X'$ of D_X with a projective $\mathbb{Q}$-factorial variety X'.



We come to the blow up of a variety in a subvariety. For the case of the blow up of a surface in a point x, this means replacing x by a curve isomorphic to $\mathbb{P}_1$.

Construction 1.4.3 (Blow up)**.** Let $\iota\colon C \to X$ be a closed embedding of varieties. Then the ideal sheaf $\mathcal{I}_C$ on C, i.e., the kernel of $\iota^*\colon \mathcal{O}_X \to \iota_*\mathcal{O}_C$, is coherent and we have a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras

$$\mathcal{S} \ := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathcal{I}_C^d, \qquad \Gamma\left(U, \mathcal{I}_C^d\right) \ := \begin{cases} \Gamma(U, \mathcal{I}_C)^d, & d > 0, \\ \Gamma(U, \mathcal{O}_X), & d = 0. \end{cases}$$

It is possible to glue together the varieties $\operatorname{Proj}\Gamma(U, \mathcal{S})$ where $U \subseteq X$ is open and affine. We obtain a variety $X' := \operatorname{Proj}\mathcal{S}$ and a morphism $\pi\colon X' \to X$ such that $\pi^{-1}(U) \cong \operatorname{Proj}\Gamma(U, \mathcal{S})$ for all open, affine subsets $U \subseteq X$. We call π the *blow up* of X along the *center* C.

Example 1.4.4. In Construction 1.4.3, we blow up $X := \mathbb{K}^n$ at the origin, i.e., at $C = V(I) \subseteq X$ with $I = \langle T_1, \ldots, T_n \rangle$ and $\mathcal{S} = \mathcal{O}_X(X) \oplus I \oplus I^2 \oplus \ldots$. We have an epimorphism

$$\psi\colon \mathcal{O}_X(X)[S_1, \ldots, S_n] \ \to \ \mathcal{S}, \qquad S_i \ \mapsto \ T_i \ \in \ \mathcal{S}_1 \ = \ I.$$

Taking the Proj, we obtain an embedding of the blow up X' of X along C into $\operatorname{Proj}\mathcal{O}_X(X)[S_1, \ldots, S_n] \cong \mathbb{K}^n \times \mathbb{P}_{n-1}$. The homogeneous generators $T_iS_j - T_jS_i$ of $\ker(\psi)$ then give a description

$$X' \ \cong \ \{(x, y) \in \mathbb{K}^n \times \mathbb{P}_{n-1};\ x_iy_j \ = \ x_jy_i \text{ for all } i \neq j\} \ \subseteq \ \mathbb{K}^n \ \times \ \mathbb{P}_{n-1}.$$

Given a morphism $\varphi\colon X \to Y$ of varieties and the ideal sheaf $\mathcal{I}_Y$ on Y, the *inverse image ideal sheaf* $\varphi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_X$ on X is the following: viewing φ as a map of topological spaces, we have the preimage sheaf $\varphi^{-1}\mathcal{I}_Y$ in the sheaf of rings $\varphi^{-1}\mathcal{O}_Y$ on X. We define $\varphi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_X$ as the image of $\varphi^{-1}\mathcal{I}_Y$ under $\varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Proposition 1.4.5. *Let $\pi\colon X' \to X$ be the blow up of a variety X along a subvariety $C \subseteq X$ as in Construction 1.4.3.*

(i) *The morphism π is birational proper and surjective.*
(ii) *The restriction $\pi\colon \pi^{-1}(U) \to U$, where $U := X \setminus C$, is an isomorphism.*
(iii) *If X is projective, then X' is projective.*

For a morphism $\varphi\colon W \to X$ of varieties, let $\pi'\colon W' \to W$ be the blow up of W at the inverse image ideal sheaf $\varphi^{-1}\mathcal{I}_C \cdot \mathcal{O}_W$. Then there is a unique morphism φ' such that

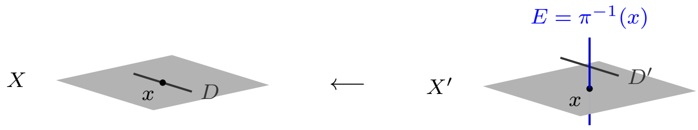
$$\begin{tikzcd} W' \arrow[r, "\varphi'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ W \arrow[r, "\varphi"] & X \end{tikzcd}$$

is commutative. Moreover, if $\varphi\colon W \to X$ is an embedding, also $\varphi'\colon W' \to X'$ is an embedding.

Given a blow up $\pi\colon X' \to X$ along $C \subseteq X$ as in Construction 1.4.3, we frequently also call the variety X' the *blow up of X along C*.

Definition 1.4.6. In Proposition 1.4.5, assume $W \subseteq X$. The *proper transform* of $W \subseteq X$ under $\pi\colon X' \to X$ is the subvariety $W' \subseteq X'$. The subvariety $E \subseteq X'$ defined by the inverse image ideal sheaf $\pi^{-1}\mathcal{I}_C \cdot \mathcal{O}_{X'}$ is the *exceptional divisor* of the blow up $\pi\colon X' \to X$.

Throughout this document we will mainly work with the blow up $\pi\colon X' \to X$ of a smooth projective surface X at a point $x \in X$. Then the proper transform of a prime divisor $D \in \mathrm{WDiv}(X)$ is the closure $D' := \overline{\pi^{-1}(D \setminus \{x\})}$ in X' and the exceptional divisor is the preimage $\pi^{-1}(x) \subseteq X'$; it is isomorphic to $\mathbb{P}_1$.



Remark 1.4.7 (Toric blow up)**.** Let Z be a toric variety with defining fan Σ. Then the blow up of Z at a toric fixed point $z_\sigma \in Z$ is the toric variety Z' with its fan Σ' obtained by the barycentric subdivision $\Sigma' \to \Sigma$ of the cone $\sigma \in \Sigma$.

We define the *Picard number* $\varrho(X) \in \mathbb{Z}_{\geq 0}$ of a surface X as the rank of the Picard group $\mathrm{Pic}(X)$. Note that for $\mathbb{Q}$-factorial X the Picard number equals the rank of the class group. Moreover, given a point x on a smooth surface X and a principal divisor $D = V(f) \in \mathrm{WDiv}(X)$, we write $\mu(x, D) \in \mathbb{Z}_{\geq 0}$ for the *multiplicity* of x in D; this is the maximal integer $r \in \mathbb{Z}_{\geq 0}$ such that $f \in \mathfrak{m}_x^r$ with the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. If $x \in D$, then $\mu(x, D) \geq 1$ and equality holds if D is smooth.

Proposition 1.4.8. *Let $\pi\colon X' \to X$ be the blow up of a smooth surface X at a point $x \in X$.*

(i) *The surface X' is smooth. If X is projective, then so is X'.*
(ii) *The class group* $\mathrm{Cl}(X')$ *is isomorphic to* $\mathrm{Cl}(X) \oplus \mathbb{Z}$ *and the Picard group* $\mathrm{Pic}(X')$ *is isomorphic to* $\mathrm{Pic}(X) \oplus \mathbb{Z}$*. In particular, the Picard number increases by one, i.e., $\varrho(X') = \varrho(X) + 1$.*
(iii) *The self intersection number of the exceptional divisor $E \subseteq X'$ is $E^2 = -1$ and E does not intersect the proper transforms of prime divisors* $D \in \mathrm{WDiv}(X)$ *with $x \notin D$.*
(iv) *Let* $D \in \mathrm{WDiv}(X)$ *be a prime divisor. Then the proper transform* $D' \in \mathrm{WDiv}(X')$ *of D has self-intersection number $(D')^2 = D^2 - \mu(x, D)^2$. In particular, if D is smooth and contains x then $(D')^2 = D^2 - 1$.*

Given surfaces $X_1, \ldots, X_n$, we call a sequence $(x_1, \ldots, x_n)$ of points $x_i \in X_i$ *infinitely near* if for all $2 \leq i \leq n$ the surface X_i is the blow up of X_{i-1} in x_{i-1} and $x_i \in X_i$ projects to x_{i-1} under $X_i \to X_{i-1}$. The union of the proper transforms of the n exceptional divisors is also called the *exceptional divisor over x_1*.

A *resolution (of singularities)* of a normal, projective variety X is a proper morphism $\pi\colon X' \to X$ with a smooth, projective surface X' such that the restriction

$$\pi^{-1}(U) \ \to \ U, \qquad U \ := \ X \setminus X^{\mathrm{sing}}$$

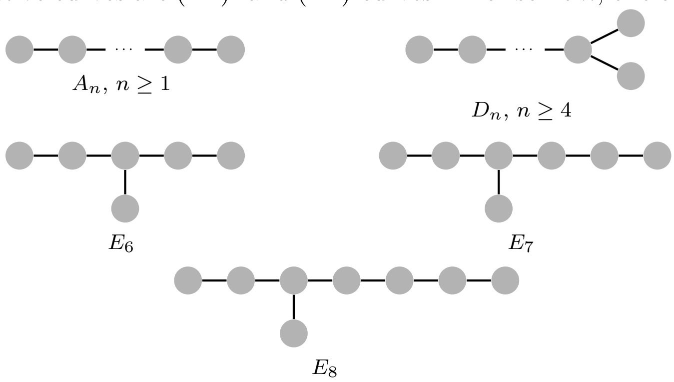
is an isomorphism. If X is a surface, there is a unique *minimal* resolution $X' \to X$, i.e., each other resolution $X'' \to X$ factors through $X' \to X$.

Construction 1.4.9 (Graph of exceptional curves)**.** Let X be a smooth, projective surface. Consider the colored, undirected, simple graph $G_X = (V, E)$ where V is the set of negative curves of X and the edges E are defined by

$$(D_1, D_2) \in E \qquad :\Leftrightarrow \qquad D_1 \cdot D_2 > 0.$$

The color function $V \to \mathbb{Z}$ is given by $D \mapsto D^2$. We call G_X the *graph of exceptional curves* or the *exceptional graph*.

Note that isomorphic surfaces X_1, X_2 have isomorphic graphs G_{X_1}, G_{X_2}. In particular, the number of $(-k)$-curves on X_1 and on X_2 must coincide for each $k \in \mathbb{Z}_{>0}$. Consider a singular point x on a normal surface X with the exceptional divisor E over x and the exceptional graph $G_{X'}$ of the minimal resolution $X' \to X$. Assume the only negative curves are (-1)- and (-2)-curves. If for some n, one of the graphs



is isomorphic to the subgraph $G_{X',-2} \subseteq G_{X'}$ of (-2)-curves occurring within E, then x is called an *ADE-singularity* of the type indicated below the fitting graph.

The next remark recalls the fact that the Cox ring of a Mori dream surface X already contains all information about X.

Remark 1.4.10. Let X be a projective Mori dream surface. Then there is exactly one full-dimensional GIT-cone $\lambda \in \Lambda(\overline{X}, H_X)$ with $\lambda \subseteq \operatorname{Mov}(X)^\circ$. In particular, in Construction 1.3.6 the bunch Φ can be obtained from $\mathcal{R}(X)$, i.e., is redundant.

5. Complexity-one T-varieties and $\mathbb{K}^*$-surfaces

We recall the theory of rational varieties with a torus action of codimension one, so-called complexity-one varieties. This class of varieties can be handled purely in terms of matrices. We put special emphasis on the surface case. This section is a summary of [5], mainly Chapters V.4 and III.4, and E. Huggenberger's thesis [61]. Compare also [53, 86]. We will work in the notation of [61]; the main difference to [5] is the ordering of the slopes in Construction 1.5.2.

Definition 1.5.1. A *complexity-one (T-) variety* is a rational, $\mathbb{Q}$-factorial, complete, normal variety X with an effective action of a torus T with $\dim(T) = \dim(X) - 1$. A $\mathbb{K}^*$*-surface* is a complexity-one T-surface.

We will first treat the important special case of $\mathbb{K}^*$-surfaces. Higher dimensional complexity-one T-varieties will be constructed at the end of this section. The defining data of a variety of complexity one, i.e., the bunched ring as in Section 3, is encoded in a *P-matrix*.

Construction 1.5.2 (P-matrix)**.** Let $r \in \mathbb{Z}_{\geq 1}$ and $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$ be positive integers. For each $0 \leq i \leq r$, consider tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$ and $d_i :=$

$(d_{i1}, \dots, d_{in_i}) \in \mathbb{Z}^{n_i}$ such that

$$\frac{d_{i1}}{l_{i1}} \ < \ \dots \ < \ \frac{d_{in_i}}{l_{in_i}}, \qquad\qquad \gcd(l_{ij}, d_{ij}) \ = \ 1 \ \text{ for all } i, j.$$

Set $n := n_0 + \dots + n_r$. We have an integral $r \times n$ block matrix L and an integral $1 \times n$ block matrix d

$$L \ := \ \begin{bmatrix} -l_0 & l_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -l_0 & 0 & \cdots & l_r \end{bmatrix}, \qquad d \ := \ \begin{bmatrix} d_0 & \cdots & d_r \end{bmatrix}.$$

We define four types of P-*matrices*, namely the following integral $(r+1) \times (n+m)$ matrices where $m \in \{0, 1, 2\}$ counts the number of additional columns

$$\text{(ee)} \quad P \ = \ \begin{bmatrix} L \\ d \end{bmatrix}, \qquad\qquad \text{(pp)} \quad P \ = \ \begin{bmatrix} L & 0 & 0 \\ d & 1 & -1 \end{bmatrix},$$

$$\text{(pe)} \quad P \ = \ \begin{bmatrix} L & 0 \\ d & 1 \end{bmatrix}, \qquad\qquad \text{(ep)} \quad P \ = \ \begin{bmatrix} L & 0 \\ d & -1 \end{bmatrix}.$$

We require the columns of P to be pairwise different and primitive and they must generate $\mathbb{Q}^{r+1}$ as a cone. We denote by v_{ij}, where $0 \le i \le r$ and $1 \le j \le n_i$, the first n columns of P and by v_k, where $1 \le k \le m$, the last m columns of P. Accordingly, we write e_{ij}, e_k for the canonical basis vectors of $\mathbb{Q}^{n+m}$.

Construction 1.5.3. Given $r \in \mathbb{Z}_{\ge 1}$, consider a $2 \times (r+1)$ matrix $A = [a_0, \dots, a_r]$ over $\mathbb{K}$ such that each two columns a_i, a_j are linearly independent for $i \ne j$. Let P be a $(r+1) \times (n+m)$ matrix as in Construction 1.5.2. Define

$$g_I \ := \ \det\left(\begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix}\right), \qquad T_j^{l_j} \ := \ T_{j1}^{l_{j1}} \cdots T_{jn_j}^{l_{jn_j}}, \qquad I \in \mathfrak{I}$$

where $\mathfrak{I}$ is the set of all triples $I = (i, i+1, i+2)$ with $0 \le i \le r-2$. We obtain a $\mathbb{K}$-algebra $R(P, A)$ that is a complete intersection ring

$$R(P, A) \ := \ \mathbb{K}\left[T_{ij}, \, S_k; \ 0 \le i \le r, \ 1 \le j \le n_i, \ 1 \le k \le m\right] \, / \, \langle g_I; \ I \in \mathfrak{I} \rangle.$$

Consider the projection $Q\colon \mathbb{Z}^{n+m} \to K$ with $K := \mathbb{Z}^{n+m}/\mathrm{Im}(P^*)$. We install a K-grading on $R(P, A)$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ Q(e_k).$$

Then the variables T_{ij}, S_k form a system of K-prime generators of $R(P, A)$ and the grading is K-factorial and almost free.

Theorem 1.5.4. *Each ring $R(P, A)$ is the Cox ring of a $\mathbb{Q}$-factorial, projective $\mathbb{K}^*$-surface $X(P, A)$. The surface $X(P, A)$ is determined by the matrices P and A up to isomorphism. Furthermore, each rational, normal, complete $\mathbb{K}^*$-surface is isomorphic to $X(P, A)$ for suitable matrices A and P.*

Corollary 1.5.5. *Let X be a surface with Cox ring $\mathbb{K}[T_1, \dots, T_n]/\langle c_1 T^{\nu_1} + c_2 T^{\nu_2} + c_3 T^{\nu_3}\rangle$ where $c_i \in \mathbb{K}^*$ and $\nu_i \in \mathbb{Z}^n_{\ge 0}$ are such that the T^{ν_i} are pairwise coprime. Then X admits a non-trivial $\mathbb{K}^*$-action.*

Remark 1.5.6. In Construction 1.5.3, the ideal $\langle g_I; \ I \in \mathfrak{I} \rangle$ is prime. In particular, each ideal $I \subseteq \mathbb{K}[T_1, \dots, T_r]$ generated by two polynomials

$$T^{\nu_1} + T^{\nu_2} + T^{\nu_3}, \qquad \lambda T^{\nu_2} + T^{\nu_3} + T^{\nu_4}, \qquad \lambda \in \mathbb{K}^* \setminus \{1\},$$

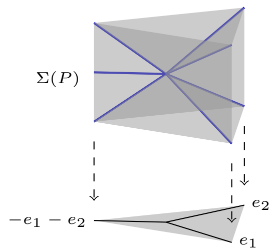
with exponent vectors $\nu_i \in \mathbb{Z}^r_{\ge 0}$ and pairwise coprime terms is a prime ideal.

We now construct the canonical toric ambient variety of a $\mathbb{K}^*$-surface explicitly, compare Construction 1.3.12.

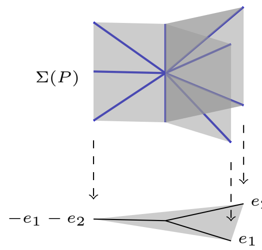
Construction 1.5.7. Let P be a P-matrix as in Construction 1.5.2; write v_{ij} for its first n columns and $v_1 = (0,\dots,0,1)$ and $v_2 = (0,\dots,0,-1)$ for its m possibly remaining columns. Define in $\mathbb{Z}^{n+m}$ the cones

$$
\begin{aligned}
\sigma^- &:= \operatorname{cone}(v_{01},\dots,v_{r1}), \\
\sigma^+ &:= \operatorname{cone}(v_{0n_0},\dots,v_{rn_r}), \\
\tau_{ij} &:= \operatorname{cone}(v_{ij},v_{ij+1}), && 0 \le i \le r,\ 1 \le j \le n_i - 1, \\
\tau_i^- &:= \operatorname{cone}(v_{i1},v_2), && 0 \le i \le r, \\
\tau_i^+ &:= \operatorname{cone}(v_{in_i},v_1), && 0 \le i \le r.
\end{aligned}
$$

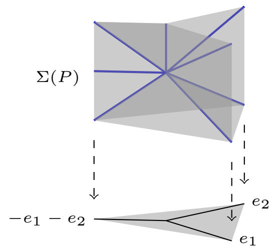
Depending on the type (ee), (pp), (pe) or (ep) of the P-matrix, we have a fan $\Sigma(P)$ in $\mathbb{Z}^{r+1}$ with the following maximal cones



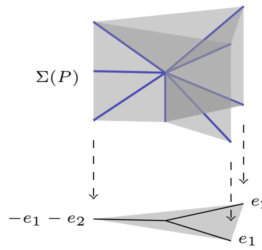
(ee): σ^+, all τ_{ij} and σ^-,



(pp): all τ_i^+, all τ_i^-, all τ_{ij},



(pe): all τ_i^+, all τ_{ij} and σ^-,



(ep): all τ_i^-, all τ_{ij} and σ^+.

Note that the drawings show the case $r = 2$. Moreover, $\Sigma(P)$ is the fan of the canonical toric ambient variety $Z(P)$ of $X(P,A)$ of Construction 1.3.12 independent of A. The $\mathbb{K}^*$-action on $X(P,A)$ arises from the one-parameter subgroup $t \mapsto (1,\dots,1,t)$ of the dense torus $\mathbb{T}^{r+1}$ of $Z(P)$.

Consider a P-matrix P as in Construction 1.5.2 with columns v_{ij} and v_1, v_2 if needed. A *block* of P is a $(r+1) \times n_i$ submatrix with columns $v_{i1},\dots,v_{in_i}$ for some $0 \le i \le r$ or, if present, the submatrix with all m occurring vectors v_1, v_2 as its columns. We call $U \in \mathrm{GL}(n+m,\mathbb{Z})$ *admissible* if the matrix $P \cdot U$ arises from P by switching columns within a block or by interchanging whole blocks. Similarly, $S \in \mathrm{GL}(r+1,\mathbb{Z})$ is *admissible* if the matrix $S \cdot P$ arises from P by a sequence of the operations

- add ± 1-multiples of the upper r rows of P to the last row,
- add ± 1-multiples of the first r rows to one of the first r rows in order to achieve a block matrix shape as in Construction 1.5.2,
- multiply the last row by ± 1.

Proposition 1.5.8. *Consider $\mathbb{K}^*$-surfaces $X_1 = X(P_1,A_1)$ and $X_2 = X(P_2,A_2)$ with P_i and A_i as in Construction 1.5.3. Then $X_1 \cong X_2$ if and only if*

$$
A_2 \;=\; B \cdot A_1 \cdot D \qquad \textit{and} \qquad P_2 \;=\; S \cdot P_1 \cdot U
$$

for a matrix $B \in \mathrm{GL}(2,\mathbb{K})$*, a diagonal matrix* $D \in \mathrm{GL}(r+1,\mathbb{K})$ *and admissible matrices* $S \in \mathrm{GL}(r+1,\mathbb{Z})$ *and* $U \in \mathrm{GL}(n+m,\mathbb{Z})$*.*

Note that in Proposition 1.5.8 the condition on S can be directly seen by an inspection of the g_I. The next remark specializes to the case $r = 2$.

Remark 1.5.9. If in Proposition 1.5.8 for both surfaces X_i we have $r = 2$, then the condition on the A_i can be dropped, i.e., X_1 is isomorphic to X_2 if and only if $P_2 = S \cdot P_1 \cdot U$ with admissible matrices $S \in \mathrm{GL}(r+1,\mathbb{Z})$ and $U \in \mathrm{GL}(n+m,\mathbb{Z})$.

We turn to the properties of the $\mathbb{K}^*$-action on a normal, projective $\mathbb{K}^*$-surface $X = X(P,A)$. Given $x \in X$, the orbit map $\mathbb{K}^* \to X$ with $(t,x) \mapsto t \cdot x$ can be extended to a morphism $\overline{\mu}_x \colon \mathbb{P}_1 \to X$. Define the *limit points*

$$\lim_{t\to 0} t\cdot x \ := \ \overline{\mu}_x(0), \qquad \lim_{t\to\infty} t\cdot x \ := \ \overline{\mu}_x(\infty).$$

Then, the image of $\overline{\mu}_x$ is the closure of $\mathbb{K}^* \cdot x$ and the limit points are distinct fixed points. Each fixed point is either *elliptic*, i.e., it is isolated and lies in the closure of infinitely many $\mathbb{K}^*$-orbits, *parabolic*, i.e., it belongs to a fixed point curve or it is *hyperbolic* which means it is isolated and lies in the closure of exactly two $\mathbb{K}^*$-orbits. We call $F^- \subseteq X$ the *source* and $F^+ \subseteq X$ the *sink* if there is a non-empty, open subset $U \subseteq X$ such that

$$\lim_{t\to 0} t\cdot x \ \in \ F^-, \qquad \lim_{t\to\infty} t\cdot x \ \in \ F^+ \qquad \text{for all } x \in U.$$

There always is exactly one source and sink and they are either single elliptic fixed points or curves of parabolic fixed points. Any fixed point outside of F^+ and F^- is hyperbolic.

Proposition 1.5.10. *Let* $X = X(P,A)$ *be as in Construction 1.5.7. Depending on the type of* X*, the following assertions hold.*

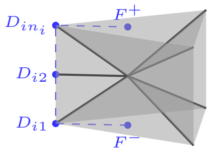
- (ee) *Both sink* F^- *and source* F^+ *consist of an elliptic fixed point.*
- (pp) *Both sink* F^- *and source* F^+ *are smooth rational curves consisting of parabolic fixed points.*
- (pe) *The source* F^- *is a smooth rational curve consisting of parabolic fixed points whereas the sink* F^+ *consists of an elliptic fixed point.*
- (ep) *The source* F^- *consists of an elliptic fixed point whereas the sink* F^+ *is a smooth rational curve consisting of parabolic fixed points.*

Let $D_{ij} \subseteq X$ *and* $D_k \subseteq X$ *be the toric divisors corresponding to the variables* T_{ij} *and* S_k *of* $R(P,A)$*. Then* D_{ij} *and* D_k *are rational curves. Let* $Z(P)$ *be the toric variety with fan* $\Sigma(P)$ *of Construction 1.5.7.*

- (i) *The toric orbit corresponding to the ray* $\mathbb{Q}_{\ge 0} \cdot v_{ij} \in \Sigma(P)$ *cuts out a non-trivial* $\mathbb{K}^*$*-orbit* $B_{ij} \subseteq D_{ij}$*.*
- (ii) *The toric orbit corresponding to a cone* $\tau_{ij} \in \Sigma(P)$ *cuts out a hyperbolic fixed point* $x_{ij} \in X$*.*
- (iii) *For each* $0 \le i \le r$*, the divisors* D_{ij} *form a chain of rational curves connecting the source* F^- *with the sink* F^+ *in the following sense: picking points* $b_{ij} \in B_{ij}$*, we have*

$$\lim_{t\to\infty} t\cdot b_{ij} \ = \ x_{ij+1} \ = \ \lim_{t\to 0} t\cdot b_{ij+1}$$
$$\lim_{t\to 0} t\cdot b_{i1} \ \in \ F^-, \qquad \lim_{t\to\infty} t\cdot b_{in_i} \ \in \ F^+.$$

- (iv) *Let* $x \in X$ *with* $x \notin F^+ \cup F^-$ *and* $x \notin D_{ij}$ *for all* i,j*. Then the limit points for* $t \to 0$ *and* $t \to \infty$ *are elements of* F^- *and* F^+ *respectively.*



$\Sigma(P)$, case (ee)

We now rephrase Constructions 1.5.2 and 1.5.3 in order to cope with higher-dimensional complexity-one T-varieties.

Construction 1.5.11. Choose $r \in \mathbb{Z}_{\geq 1}$, integers $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$ and $m, s \in \mathbb{Z}_{\geq 0}$ such that $0 < s < n + m - r$ where $n := n_0 + \ldots + n_r$. For each $0 \leq i \leq r$, pick a tuple $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$. Additionally, fix a $2 \times (r+1)$ matrix A over $\mathbb{K}$ with pairwise linearly independent columns as well as integral $s \times n$ and $s \times m$ matrices d and d' with the following property: in the $(r+s) \times (n+m)$ block matrix

$$P \ := \ \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix}, \qquad L \ := \ \begin{bmatrix} -l_0 & l_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -l_0 & 0 & \cdots & l_r \end{bmatrix}$$

the columns are pairwise different, primitive and the cone generated by them equals $\mathbb{Q}^{r+s}$.

Construction 1.5.12. Let P and A be as in Construction 1.5.11 and consider the ring $R := R(P, A)$ obtained from Construction 1.5.3. The system $\mathfrak{F} = (T_{ij}, S_k)$ consists of homogeneous K-prime generators for R. Given any true $\mathfrak{F}$-bunch Φ in $K_{\mathbb{Q}}$, we obtain a bunched ring $(R, \mathfrak{F}, \Phi)$. Construction 1.3.6 delivers $\overline{X}(P, A) :=$ Spec(R) and

$$\widehat{X}(P, A, \Phi) \ := \ \widehat{X}(R, \mathfrak{F}, \Phi), \qquad X(P, A, \Phi) \ := \ X(R, \mathfrak{F}, \Phi).$$

Define $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ where $K_0 := \mathbb{Z}^{n+m}/\mathrm{Im}(P_0^*)$ and $P_0 := (L, 0)$ consists of the first r rows of P. Then H_0 leaves $\widehat{X}(P, A, \Phi)$ invariant and there is an induced effective action of the torus $T := H_0/H = \mathrm{Spec}(\mathbb{Z}^s)$ on X.

Theorem 1.5.13. *In Construction 1.5.12, $X(P, A, \Phi)$ is an irreducible normal A_2-variety of dimension $s + 1$ with $\Gamma(X, \mathcal{O}) = \mathbb{K}$ and the torus $T := H_0/H$ acts effectively with maximal orbit dimension $\dim(X) - 1$ on X. In turn, each A_2-variety with $\Gamma(X, \mathcal{O}) = \mathbb{K}$ and torus action of complexity one arises from Construction 1.5.12.*

See, e.g., Example 2.4.7 for a complexity one T-variety X that arises as $X = X(P, A, \Phi)$ as in Theorem 1.5.13.

CHAPTER 2

Basic algorithms for Mori dream spaces

In this chapter, we present basic algorithms and data types needed to compute with Mori dream spaces. We use the correspondence between Mori dream spaces and bunched rings explained in Section 3 of Chapter 1.

Section 1 deals with gradings by a finitely generated abelian group, e.g., the $\mathrm{Cl}(X)$-grading of a Cox ring $\mathcal{R}(X)$. To this end, we introduce a data type for finitely generated abelian groups and their homomorphisms and show how to perform basic operations thereon. Afterwards, in Section 2, we show how to compute with rings that come with a grading by a finitely generated abelian group; this will be used to store the Cox ring of a Mori dream space. Section 3 then represents a Mori dream space X by its $\mathrm{Cl}(X)$-graded Cox ring and a bunch of cones. Several algorithms are given to explore the properties and geometry of X. Finally, Section 4 is concerned with algorithms that work in the more specialized setting of complexity one T-varieties.

This section uses results of [5, 17, 6], see the explicit references below. All presented algorithms have been implemented in the *MDSpackage* [54, 55], see Appendix A. Throughout this chapter, we will mainly use the following example.

Example 2.0.14. Consider the finitely generated abelian group $K := \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ and the factorially K-graded ring

$$R \ := \ \mathbb{K}[T_1, \ldots, T_8]/\langle f_1 \rangle, \qquad f_1 \ := \ T_1T_6 + T_2T_5 + T_3T_4 + T_7T_8$$

where the K-grading is encoded in the *degree matrix* having $\deg(T_1), \ldots, \deg(T_8)$ as its columns

$$Q \ = \ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}.$$

Choose in $K \otimes \mathbb{Q} = \mathbb{Q}^3$ the vector $w := (0, 0, 2)$. This defines a projective $\mathfrak{F}$-bunch $\Phi = \Phi(w)$ in R, see Example 1.3.4. By Construction 1.3.6, these data determine a Mori dream space $X = X(R, \mathfrak{F}, \Phi)$.

1. Finitely generated abelian groups and homomorphisms

In this section, we treat basic algorithms for finitely generated abelian groups and their homomorphisms. We will use them to work with gradings in subsequent sections. Some of these algorithms have also been implemented in [16] together with B. Bechtold, R. Birkner, L. Kastner, O. Motsak and A.-L. Winz. We assume that the reader is familiar with the basic algorithms on lattices as, e.g., used in [20]; compare the textbooks [75, 82]. We present the following algorithms:

- *Compare groups:* isomorphism test (Algorithm 2.1.4), test for containment (Algorithm 2.1.6), test for equality (Algorithm 2.1.8).
- *Construct new groups:* factor groups (Algorithm 2.1.9), product groups (Algorithm 2.1.10), free part (Algorithm 2.1.11), intersection (Algorithm 2.1.13).

- *Image, kernel:* image (Algorithm 2.1.17), preimage (Algorithm 2.1.18), kernel (Algorithm 2.1.19), test for being injective (Algorithm 2.1.20), test for being surjective (Algorithm 2.1.21), complete an exact sequence (Algorithm 2.1.22).
- *Gale duality:* compute the degree map Q out of P (Algorithm 2.1.24), compute P out of Q (Algorithm 2.1.25), compute the projection Q^0 onto the free part out of Q (Algorithm 2.1.26).
- *Homomorphisms as degree maps, etc.:* gradiator (Algorithm 2.1.29), test for being homogeneous (Algorithm 2.1.31), test for being almost free (Algorithm 2.1.32), section (Algorithm 2.1.34).

Remark 2.1.1. Let K be a finitely generated abelian group with elementary divisors $a_1, \ldots, a_k \in \mathbb{Z}$. Recall, e.g., from [76, Sec. I.10], that there is a sublattice $L \leq \mathbb{Z}^r$ such that

$$K \quad \cong \quad \mathbb{Z}^r/L \quad \cong \quad \mathbb{Z}^d \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/a_i\mathbb{Z}$$

with $d \in \mathbb{Z}_{\geq 0}$. Consider now a subgroup $H \leq K$. Then there is a sublattice $U \leq \mathbb{Z}^r$ such that

$$H \quad \cong \quad U\,/\,(U \cap L) \quad \cong \quad (U + L)\,/\,L \quad \leq \quad \mathbb{Z}^r/L.$$

Definition 2.1.2. Let a finitely generated abelian group H be given with a description $H \cong (U+L)/L$ as in Remark 2.1.1. We call the pair (U, L) in this setting an AG, an abbreviation for *abelian group*.

We do not differentiate between an AG and the underlying finitely generated abelian group.

Example 2.1.3. In Example 2.0.14, the grading group $K = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ is given as an AG $K = (\mathbb{Z}^4, \mathrm{lin}_{\mathbb{Z}}\,((0,0,0,2)))$.

Writing generators into the columns of a matrix, we may consider sublattices of $\mathbb{Z}^r$ as integral matrices. Given an integral $d \times r$ matrix A, write $\mathrm{lin}_{\mathbb{Z}}(A)$ for the sublattice of $\mathbb{Z}^r$ generated by its columns and $\mu_A \colon \mathbb{Z}^r \to \mathbb{Z}^d$ for the multiplication map $z \mapsto A \cdot z$. Elements of an AG (U, L) are usually given as elements of $\mathrm{lin}_{\mathbb{Z}}(U) \leq \mathbb{Z}^r$. The name of each algorithm is given in parentheses beside its number.

Algorithm 2.1.4 (AGareisom)**.** *Input:* AGs $G_1 = (U_1, L_1)$ and $G_2 = (U_2, L_2)$.

- Compute lattice bases for $\mu_{U_i}^{-1}(L_i)$ and write their elements in the columns of $d \times n_i$ matrices M_i.
- Compute Smith normal forms $S_i = (s^i_{kl})_{k,l}$ of M_i. Denote by r_i the number of zero-rows of S_i. Return *false* if $r_1 \neq r_2$.
- Return *false* if the two sets $\{|s^i_{jj}|;\ 1 \leq j \leq \min(n_i, d)\}$ are different. Otherwise, return *true*.

Output: *true* if G_1 is isomorphic to G_2 and *false* otherwise.

Proof. The abelian groups are isomorphic if their decompositions $\mathbb{Z}^{r_i} \oplus \bigoplus_j \mathbb{Z}/a_{ij}\mathbb{Z}$ as $\mathbb{Z}$-modules coincide. To show that the algorithm computes these descriptions, let $\mathrm{pr}_{U_i} \colon U_i \to U_i/U_i \cap L_i$ be the canonical projection. Then $\mathrm{lin}_{\mathbb{Z}}(M_i)$ is the kernel of $\pi_{U_i} = \mathrm{pr}_{U_i} \circ \mu_{U_i}$ where

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[rrr, "\pi_{U_i}"'] \arrow[dr] & & & U_i/L_i \cap U_i \\
& \mathbb{Z}^r/\mathrm{lin}_{\mathbb{Z}}(M_i) \arrow[r, "\cong"] & \mathbb{Z}^r/\mathrm{lin}_{\mathbb{Z}}(S_i) \arrow[ur, "\cong"] &
\end{tikzcd}$$

□

Remark 2.1.5. In Algorithm 2.1.4, the preimage can be computed as follows. Consider integral matrices A and B of sizes $n \times r$ and $n \times r'$. Compute a lattice basis L for the integral kernel of the concatenated matrix $[A, -B]$. A lattice basis of the projection of L onto the first r components delivers a basis for $\mu_A^{-1}(\mathrm{lin}_{\mathbb{Z}}(B))$; see [20].

Algorithm 2.1.6 (AGcontains)**.** *Input:* an AG (U_1, L_1) and either an AG (U_2, L_2) or a vector $w \in \mathbb{Z}^{r_1}$. In the latter case, define $U_2 := \mathrm{lin}_{\mathbb{Z}}(w)$ and $L_2 := L_1$.

- Return *false* if $L_1 \neq L_2$ or $U_1 + L_1 \not\subseteq U_2 + L_2$. Return *true* otherwise.

Output: *true* if (U_1, L_1) is a subgroup of (U_2, L_2) or if (U_1, L_1) contains $w + L_1$, respectively. Returns *false* if this is not the case.

Remark 2.1.7. In Algorithm 2.1.6, we can check containment of a vector $v \in \mathbb{Z}^r$ in a sublattice $L \leq \mathbb{Z}^r$ by the following steps. Compute a lattice basis K for the integral kernel of the enlarged matrix $[L, v]$. Then $v \in L$ if and only if the elements of the last row of K (considered as a matrix) are coprime; see [20].

Algorithm 2.1.8 (AGareequal)**.** *Input:* either two AGs $G_1 = (U_1, L_1)$ and $G_2 = (U_2, L_2)$ or two vectors $w, w' \in \mathbb{Z}^r$ and an AG $G = (U, L)$.

- Using Algorithm 2.1.6, in the first input case, return *true* if $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$. In the second input case, return *true* if $w - w' \in L$. Otherwise, return *false*.

Output: depending on the input, *true* if $G_1 = G_2$ or $w = w'$ as elements of G. Otherwise, *false* is returned.

Algorithm 2.1.9 (AGfactgrp)**.** *Input:* AGs $G_1 = (U_1, L_1)$ and $G_2 = (U_2, L_2)$ where $G_2 \leq G_1$ is a subgroup.

- Return the AG $(U_1, U_2 + L_2)$.

Output: an AG describing the factor group G_1/G_2.

Proof. Since $G_2 \leq G_1$ we have $L_1 = L_2$ and $U_2 \subseteq U_1$. The second isomorphism theorem yields the claim

$$\begin{aligned} G_1/G_2 &= ((U_1 + L_1)/L_1) \Big/ ((U_2 + L_2)/L_2) \\ &\cong (U_1 + L_2) \Big/ (U_2 + L_2) \\ &= (U_1 + (U_2 + L_2)) \Big/ (U_2 + L_2). \end{aligned}$$ □

Algorithm 2.1.10 (AGprodgrp)**.** *Input:* AGs $G_1 = (U_1, L_1)$ and $G_2 = (U_2, L_2)$.

- Consider U_i, L_i as matrices with generators for the respective lattices as their columns. Return the AG (U, L) where, for zero-matrices of fitting sizes, U and L are generated by the columns of

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \qquad \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

Output: an AG describing the product group $G_1 \times G_2$.

Given a finitely generated abelian group K, we denote by $K^{\mathrm{tor}} \leq K$ the subgroup of torsion elements, i.e., elements $w \in K$ such that $kw = 0$ for an integer $k \in \mathbb{Z}$. The *free part* is the factor group $K^0 := K/K^{\mathrm{tor}}$. Moreover, recall that the *saturation* of a lattice $L \leq \mathbb{Z}^r$ is the sublattice $L^{\mathrm{sat}} \leq \mathbb{Z}^r$ consisting of all $v \in \mathbb{Z}^r$ such that $kv \in L$ for a $k \in \mathbb{Z} \setminus \{0\}$.

Algorithm 2.1.11 (AGfreepart)**.** *Input:* an AG $K = (U, L)$.

- Return the AG (U, L^{sat}).

Output: an AG describing the free part K/K^{tor}.

Proof. The fact that the AG (U, L^{sat}) is the free part can be seen by using the second isomorphism theorem in

$$\begin{tikzcd} K = U/U \cap L \arrow[r, "v+U\cap L \mapsto v+U\cap L^{\mathrm{sat}}"] \arrow[d] & U/U \cap L^{\mathrm{sat}} \\ K/K^{\mathrm{tor}} \arrow[r, equal] & \left(U/U \cap L\right) \Big/ \left((U \cap L^{\mathrm{sat}})/U \cap L\right) \arrow[u, "\cong"'] \end{tikzcd}$$

□

Example 2.1.12. Consider the AGs $G := (\mathbb{Z}^4, \{0_{\mathbb{Z}^4}\})$ and $H := (L, \{0_{\mathbb{Z}^4}\})$ with $L := \mathrm{lin}_{\mathbb{Z}}((0,0,0,2))$. By Algorithm 2.1.6, we have $H \le G$. Moreover, by 2.1.9, the factor group G/H equals the grading group $K = (\mathbb{Z}^4, \mathrm{lin}_{\mathbb{Z}}((0,0,0,2)))$ of Example 2.1.3. Using 2.1.11, we see that the free part of K is $K^0 = \mathbb{Z}^3$.

Algorithm 2.1.13 (AGintersect)**.** *Input:* AGs $G_1 = (U_1, L)$ and $G_2 = (U_2, L)$.

- Return the AG $((U_1 + L) \cap (U_2 + L), L)$.

Output: an AG describing the intersection $G_1 \cap G_2$.

Proof. The intersection $G_1 \cap G_2$ is determined by intersecting the respective lattices as summarized by the following diagram; upward arrows stand for inclusion and downward arrows for projection.

$$\begin{tikzcd} U_1 + L \arrow[d] & & U_2 + L \arrow[d] \\ G_1 & (U_1 + L) \cap (U_2 + L) \arrow[ul] \arrow[ur] \arrow[d] & G_2 \\ & G_1 \cap G_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

□

Remark 2.1.14. In Algorithm 2.1.13 the intersection $L_1 \cap L_2$ of two sublattices $L_1, L_2 \le \mathbb{Z}^r$ can be computed as follows. Write lattice bases for L_i into the columns of $r \times n_i$ matrices A_i. Compute a lattice basis K for the kernel of the matrix $[A_1, -A_2]$. Denote by B the projection of K onto the first n_1 coordinates. A Hermite normal form of $A_1 B$ then has generators for $L_1 \cap L_2$ as its columns; see [20].

We turn to homomorphisms of finitely generated abelian groups. A typical example are degree matrices of Cox rings.

Definition 2.1.15. Consider a homomorphism $\varphi\colon G_1 \to G_2$ of finitely generated abelian groups. We encode φ in an *AGH*, i.e., a triple (G_1, G_2, A) where $G_i = (U_i, L_i)$ are AGs and A is an integral matrix satisfying

$$\mu_A(U_1 + L_1) \subseteq U_2 + L_2, \qquad \mu_A(L_1) \subseteq L_2.$$

This means we have a diagram

$$\begin{tikzcd}
\mathbb{Z}^s \arrow[d, equal] & & \mathbb{Z}^r \arrow[ll, "\mu_A"'] \\
\mathbb{Z}^s \arrow[d] & & U_1 + L_1 \arrow[ll, "\mu_A"'] \arrow[u] \arrow[d] \\
(U_2+L_2)/L_2 & & (U_1+L_1)/L_1 \arrow[ll, "\varphi"']
\end{tikzcd}$$

Example 2.1.16. In the situation of Examples 2.0.14 and 2.1.3, consider the degree map $Q\colon \mathbb{Z}^8 \to K$ with $Q(e_i) = \deg(T_i)$ and grading group $K = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$. We encode Q as an AGH (F, K, A) with AGs F, K and a 4×8 matrix A given by

$$F = \left(\mathbb{Z}^8, \{0_{\mathbb{Z}^8}\}\right), \quad K = \left(\mathbb{Z}^4, \mathbb{Z}\cdot\begin{pmatrix}0\\0\\0\\2\end{pmatrix}\right), \quad A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Algorithm 2.1.17 (AGHim)**.** *Input:* an AGH $\varphi = (G_1, G_2, A)$ with $G_i = (U_i, L_i)$ and an AG $H_1 = (U_1', L_1)$ with $H_1 \le G_1$.

- Compute the lattice M generated by the image $\mu_A(U_1' + L_1)$.
- Return the AG (M, L_2).

Output: an AG describing the image $\varphi(H_1)$ as a subgroup of G_2.

Proof. Let $\pi_i\colon \mathbb{Z}^{r_i} \to \mathbb{Z}^{r_i}/L_i$ be the canonical projections. We are in the situation of the diagram

$$\begin{tikzcd}
U_2 + L_2 \arrow[d, "\pi_2"] & U_1 + L_1 \arrow[l, "\mu_A"'] \arrow[d, "\pi_1"] & \geq & U_1' + L_1 \arrow[d, "\pi_1"] \\
G_2 & G_1 \arrow[l, "\varphi"'] & \geq & H_1
\end{tikzcd}$$

The correctness of the algorithm follows from the observation

$$\varphi(H_1) \;=\; \{\varphi(\pi_1(v));\, v \in U_1' + L_1\} \;=\; \{\pi_2(\mu_A(v));\, v \in U_1' + L_1\}. \qquad \square$$

Algorithm 2.1.18 (AGHpreim)**.** *Input:* an AGH $\varphi = (G_1, G_2, A)$ with $G_i = (U_i, L_i)$ and an AG $H_2 = (U_2', L_2)$ with $H_2 \le G_2$.

- Compute the sublattice $M := \mu_A^{-1}(U_2' + L_2)$ of $\mathbb{Z}^{r_1}$.
- Determine the sublattice $M' := M \cap (U_1 + L_1)$ of $\mathbb{Z}^{r_1}$.

Output: the AG (M', L_1) describing the preimage $\varphi^{-1}(H_2)$ as a subgroup of G_1.

Proof. Let $\pi_i\colon \mathbb{Z}^{r_i} \to \mathbb{Z}^{r_i}/L_i$ be the canonical projections. We are in the situation of the diagram

$$\begin{tikzcd}
U_2' + L_2 \arrow[d, "\pi_2"] & \leq & U_2 + L_2 \arrow[d, "\pi_2"] & U_1 + L_1 \arrow[l, "\mu_A"'] \arrow[d, "\pi_1"] \\
H_2 & \leq & G_2 & G_1 \arrow[l, "\varphi"']
\end{tikzcd}$$

The correctness of the algorithm follows from the observation

$$\begin{aligned}
\varphi^{-1}(H_2) &= \pi_1\left(\{v \in U_1 + L_1;\, \mu_A(v) \in U_2' + L_2\}\right) \\
&= \pi_1\left(\mu_A^{-1}(U_2' + L_2) \cap (U_1 + L_1)\right)
\end{aligned} \qquad \square$$

Algorithm 2.1.19 (AGHker)**.** *Input:* an AGH $\varphi = (G_1, G_2, A)$ with $G_i = (U_i, L_i)$.

- Call Algorithm 2.1.18 with input φ and the AG $(\{0\}, L_2)$.

Output: an AG describing $\varphi^{-1}(0)$ as a subgroup of G_1.

As a consequence of the previous algorithms, we can test a homomorphism of finitely generated abelian groups for being injective or surjective.

Algorithm 2.1.20 (AGHisinj)**.** *Input:* an AGH $\varphi = (G_1, G_2, A)$.

- Use Algorithm 2.1.19 to compute $M := \ker(\varphi)$.
- Return *true* if M equals the trivial subgroup of G_1 and return *false* otherwise.

Output: *true* if φ is injective and *false* otherwise.

Algorithm 2.1.21 (AGHissurj)**.** *Input:* an AGH $\varphi = (G_1, G_2, A)$.

- Use Algorithm 2.1.17 to compute $N := \mathrm{Im}(\varphi)$.
- Return *true* if, by Algorithm 2.1.8, $N = G_2$. Return *false* otherwise.

Output: *true* if φ is surjective and *false* otherwise.

Algorithm 2.1.22 (AGHcompleteseq)**.** *Input:* a surjective or injective AGH $\varphi = (G, G', A)$ where $G = (U, L)$ and $G' = (U', L')$ with sublattices $U,\ L \le \mathbb{Z}^r$ and U', $L' \le \mathbb{Z}^{r'}$.

- If, by Algorithm 2.1.21, φ is surjective
  - use Algorithm 2.1.19 to compute the AG $M := \ker(\varphi) \le G$,
  - return the AGH (M, G, E_r) where E_r is the $r \times r$-unit matrix.
- If, by Algorithm 2.1.20, φ is injective
  - compute the AG $K := G/G'$ with Algorithm 2.1.9,
  - return the AGH $(G', K, E_{r'})$ where $E_{r'}$ is the $r' \times r'$-unit matrix.

Output: if φ is surjective, an AGH $\iota = (M, G', B)$ is returned such that we have an exact sequence

$$0 \longleftarrow G' \xleftarrow[\varphi]{} G \xleftarrow[\iota]{} M \longleftarrow 0\ .$$

If φ is injective, an AGH $\pi = (G', K, B)$ is returned such that we have an exact sequence

$$0 \longleftarrow K \xleftarrow[\pi]{} G' \xleftarrow[\varphi]{} G \longleftarrow 0\ .$$

Example 2.1.23. We continue Example 2.1.16. Applying Algorithm 2.1.22 to the surjective AGH $\pi := Q = (\mathbb{Z}^8, K, A)$, we obtain an injective AGH $\varphi := (G, \mathbb{Z}^8, E_8)$, where E_8 is the 8×8 unit matrix and $G \le \mathbb{Z}^8$ is isomorphic to $\mathbb{Z}^5$ with

$$G = (\mathrm{lin}_{\mathbb{Z}} B,\ \{0_{\mathbb{Z}^8}\}), \qquad B \ := \ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 4 \\ 1 & 0 & 0 & 4 & 4 \\ -1 & -1 & -1 & -2 & -2 \\ -1 & -1 & -2 & -5 & -6 \end{bmatrix}.$$

Consider an $n \times r$ matrix P of full rank, and a surjective homomorphism $Q\colon \mathbb{Z}^r \to K$ of finitely generated abelian groups fitting into the diagram of dual exact sequences of $\mathbb{Z}$-modules

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & \mathbb{Z}^r \arrow[r, "P"] & \mathbb{Z}^n & \\ 0 & K \arrow[l] & \mathbb{Z}^r \arrow[l, "Q"'] & \mathbb{Z}^n \arrow[l, "P^*"'] & 0 \arrow[l] \end{tikzcd}$$

We then call Q and P *Gale dual*. If K is free, Q is given by an integral matrix which we call a *Gale dual matrix* for P. Note that Gale dual homomorphisms or matrices are not unique.

Algorithm 2.1.24 (AGHP2Q)**.** *Input:* an integral $n \times r$ matrix P of rank n.

- Compute a Smith normal form $S = V \cdot P^* \cdot W$ of the transpose P^* with invertible integral matrices V, W. Write $S = (S_{ij})_{i,j}$.
- Let $v_{i_1}, \dots, v_{i_d}$ be the rows of V with $i_k > \operatorname{rank}(S)$. Let $v_{j_1}, \dots, v_{j_s}$ be the rows of V with $1 < |S_{j_1 j_1}| < \dots < |S_{j_s j_s}|$. Define B as the matrix with rows $v_{i_1}, \dots, v_{i_d}, v_{j_1}, \dots, v_{j_s}$.
- Define the AG $K = (\mathbb{Z}^{r-l}, S')$ where S' is obtained from S by removing the l rows i of S with $S_{ii} = 1$. Then K is isomorphic to $\mathbb{Z}^d \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/|S_{j_i j_i}|\mathbb{Z}$.

Output: the AGH $(\mathbb{Z}^r, K, B)$. It represents a Gale dual homomorphism $Q\colon \mathbb{Z}^r \to \mathbb{Z}^r/\operatorname{Im}(P^*)$ of P.

Proof. Assume that in S there are l entries of absolute value one. Let $\operatorname{pr}\colon \mathbb{Z}^r \to \mathbb{Z}^{r-l}$ be the projection onto the other coordinates. By the diagram

$$\begin{tikzcd}
 & \mathbb{Z}^r/\operatorname{Im}(\mu_{P^*}) \arrow[ddd, bend left=80, "\cong"] & \\
\mathbb{Z}^{r-l} \arrow[d, equal] & \mathbb{Z}^r \arrow[l, "\mu_B"'] \arrow[u] \arrow[d, "\cong"', "\mu_V"] & \mathbb{Z}^n \arrow[l, "\mu_{P^*}"'] \\
\mathbb{Z}^{r-l} \arrow[d, "\pi"] & \mathbb{Z}^r \arrow[l, "\mathrm{pr}"] \arrow[d] & \mathbb{Z}^n \arrow[l, "\mu_S"] \arrow[u, "\mu_W", "\cong"'] \\
\mathbb{Z}^{r-l}/\operatorname{Im}(\mathrm{pr} \circ \mu_S) & \mathbb{Z}^r/\operatorname{Im}(\mu_S) \arrow[l, "\cong"] &
\end{tikzcd}$$

the group K is equal to $\mathbb{Z}^{r-l}/\operatorname{Im}(\operatorname{pr} \circ \mu_S)$ up to row permutations and, hence, also to $\mathbb{Z}^r/\operatorname{Im}(\mu_{P^*})$. By choice of B, the AGH $Q = (\mathbb{Z}^r, K, B)$ is surjective and $\pi \circ \mu_B \circ \mu_{P^*} = 0$ since for each $a \in \mathbb{Z}^n$ we have

$$\pi \circ \mu_B \circ \mu_{P^*}(a) \;=\; \pi \circ \operatorname{pr} \circ \mu_S \circ \mu_{W^{-1}}(a) \;=\; \overline{0} \;\in\; K. \qquad \square$$

We will see an application of Algorithm 2.1.24 in Example 2.1.30.

Algorithm 2.1.25 (AGHQ2P)**.** *Input:* a surjective AGH $Q = (\mathbb{Z}^r, K, A)$.

- Use Algorithm 2.1.19 to compute $\ker(Q) \le \mathbb{Z}^r$ as an AG $(U, 0)$.
- Return the transpose U^*.

Output: an integral $r \times n$ matrix P that is a Gale dual matrix for Q.

Given a surjective AGH $Q = (\mathbb{Z}^r, K, A)$ we can compute the free part $K^0 = K/K^{\mathrm{tor}}$, compare Algorithm 2.1.11. Then there is an integral, surjective matrix Q^0 fitting into the diagram

$$\begin{tikzcd}
K \arrow[d] & \mathbb{Z}^r \arrow[l, "Q"'] \arrow[dl, "Q^0"] \\
K^0 = K/K^{\mathrm{tor}} &
\end{tikzcd}$$

Algorithm 2.1.26 (AGHQ2Q0)**.** *Input:* a surjective AGH $Q = (\mathbb{Z}^r, K, A)$ where $K = (U, L)$.

- Compute a matrix B satisfying $\ker(B) = L^{\mathrm{sat}}$; see Remark 2.1.27.

- Define $Q^0 := B \cdot A$.

Output: the integral matrix Q^0. Considered as a map, we have $\mathrm{pr} \circ Q = Q^0$ with the projection $\mathrm{pr}\colon K \to K^0 = K/K^{\mathrm{tor}}$.

Proof. As an AG, we have $K^0 = (U, L^{\mathrm{sat}})$, see Algorithm 2.1.11. Then the canonical projection $\mathbb{Z}^r \to \mathbb{Z}^r/L^{\mathrm{sat}}$ is given by μ_B. The algorithm is correct by the diagram

$$\begin{tikzcd}
U+L \arrow[d] \arrow[dd, bend right, "\mu_B"'] & \\
U+L/L \arrow[d] & \mathbb{Z}^r \arrow[ul, "\mu_A"'] \arrow[l, "Q"'] \arrow[dl, "Q^0"] \\
U+L^{\mathrm{sat}}/L^{\mathrm{sat}} &
\end{tikzcd}$$

□

Remark 2.1.27. In Algorithm 2.1.26, B can be computed by first computing an invertible matrix U such that UL is in Hermite normal form. Removing the first $\mathrm{rank}(L)$ rows of U yields a matrix B with $\ker(B) = L^{\mathrm{sat}}$; see [20].

Example 2.1.28. Consider the AGH Q of Example 2.1.16. An application of Algorithms 2.1.25 and 2.1.26 yields the matrices

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 & -2 & -5 \\ 0 & 0 & 0 & 0 & 4 & 4 & -2 & -6 \end{bmatrix}, \qquad Q^0 = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Recall from [5, Con. III.2.4.2] that the *gradiator* of a list of polynomials $f_1, \ldots, f_s \in \mathbb{K}[T_1, \ldots, T_r]$ is a homomorphism $\mathbb{Z}^r \to K$ of finitely generated abelian groups such that $H := \operatorname{Spec} \mathbb{K}[K]$ is the maximal quasitorus in $\mathbb{T}^r$ leaving $V(f_1, \ldots, f_s) \subseteq \mathbb{K}^r$ invariant.

Algorithm 2.1.29 (AGHgradiator)**.** See [5, Constr. III.2.4.2]. *Input:* a list of polynomials $f_1, \ldots, f_s \in \mathbb{K}[T_1, \ldots, T_r]$. Write $f_i = a_{i,1}T^{\nu_{i,1}} + \ldots + a_{i,n_i}T^{\nu_{i,n_i}}$ with $a_{i,j} \in \mathbb{K}^*$.

- For $1 \leq i \leq s$ let P_i be the $(n_i - 1) \times r$ matrix with rows $\nu_{i,k} - \nu_{i,1}$ where $2 \leq k \leq n_i$. Let P be the vertical concatenation of $P_1, \ldots, P_s$.
- Use Algorithm 2.1.24 to compute an AGH $Q\colon \mathbb{Z}^r \to \mathbb{Z}^r/\mathrm{Im}(P^*)$.

Output: the pair (Q, P) where $Q\colon \mathbb{Z}^r \to \mathbb{Z}^r/\mathrm{Im}(P^*)$ is the gradiator and P is a Gale dual matrix.

Example 2.1.30. We continue Example 2.0.14. Applying Algorithm 2.1.29 to $f_1 \in \mathbb{K}[T_1, \ldots, T_8]$, we obtain a pair (Q', P') with an AGH $Q' = (\mathbb{Z}^8, \mathbb{Z}^5, A')$ where

$$P' = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Note that, given an AGH $Q = (\mathbb{Z}^r, K, A)$ with $K = (U, L)$, the degree of a homogeneous polynomial $f \in \mathbb{K}[T_1, \ldots, T_r]$ with respect to the grading $\deg(T_i) = Q(e_i)$ is $\deg(f) = A \cdot \nu + L \in K$ where T^ν is any non-zero monomial in f.

Algorithm 2.1.31 (AGHishomog)**.** *Input:* an AGH $Q = (\mathbb{Z}^r, K, A)$ and polynomials $f_1, \ldots, f_s \in \mathbb{K}[T_1, \ldots, T_r]$.

- For each $i = 1, \ldots, s$ do
  - if there are two monomials T^ν, T^μ of f_i such that, by Algorithm 2.1.8, $\deg(T^\nu) \neq \deg(T^\mu)$ in K, then return *false*.

- Return *true*.

Output: *true* if all f_i are homogeneous with respect to the K-grading $\deg(T_i) = Q(e_i)$. Returns *false* otherwise.

Algorithm 2.1.32 (AGHisalmostfree)**.** *Input:* an AGH $Q = (\mathbb{Z}^r, K, A)$.

- For each facet $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ do
  - If, by Algorithm 2.1.8, the subgroup $\langle Q(e_i);\, e_i \in \gamma_0)\rangle \leq K$ is different from K, then return *false*.
- Return *true*.

Output: *true* if the K-grading of $\mathbb{K}[T_1, \dots, T_r]$ given by $\deg(T_i) = Q(e_i)$ is almost free. Returns *false* otherwise.

Example 2.1.33. In Example 2.0.14 and 2.1.16, applying Algorithms 2.1.21, 2.1.32, and 2.1.31, we see that Q is surjective, the grading defined by $\deg(T_i) = Q(e_i)$ is almost free and $f_1 \in \mathbb{K}[T_1, \dots, T_8]$ is homogeneous of degree $(0, 0, 2, \overline{1}) \in K$. Moreover, Q has the AG G of Example 2.1.23 as its kernel.

Algorithm 2.1.34 (AGHsection)**.** *Input:* a surjective AGH $\varphi = (G_1, G_2, A)$ with AGs $G_i = (U_i, L_i)$.

- By a Hermite normal form computation, determine an integral matrix S such that $A \cdot S$ is the unit matrix.
- Check if $\psi = (G_2, G_1, S)$ defines an AGH, i.e., use Algorithm 2.1.6 to check whether $\mu_S(U_2 + L_2)$ is a subset of $U_1 + L_1$ and $\mu_S(L_2) \subseteq L_1$.
- Return ψ if the checks were positive. Return *false* otherwise.

Output: if no section $G_2 \to G_1$ for φ was found, *false* is returned. Otherwise, an AGH (G_2, G_1, S) representing such a section is returned.

2. Graded rings

Using the correspondence 1.3.7, a Mori dream space is determined by its graded Cox ring and a bunch of cones. In this section, we encode the algebraic data of the Cox ring in a data type. We present the following algorithms on graded rings.

- *Grading:* integral points (Algorithm 2.2.2), homogeneous component (Algorithm 2.2.3) and its dimension (Algorithm 2.2.5).
- *Tropical algorithms:* tropical variety for one equation (Algorithm 2.2.7), containment in the tropical variety (Algorithm 2.2.8).
- *Primality check, modifying polynomials:* check a variable for being K-prime (Algorithm 2.2.10), pullback and pushforward of a Laurent polynomial (Algorithms 2.2.12 and 2.2.13), closure (Algorithm 2.2.14).

Let $Q\colon \mathbb{Z}^r \to K$ be a surjective AGH. Consider an integral, normal, affine $\mathbb{K}$-algebra $R := \mathbb{K}[T_1, \dots, T_r]/I$ with an ideal $I \subseteq \mathbb{K}[T_1, \dots, T_r]$ that is homogeneous with respect to the K-grading

$$\deg(T_1) \;:=\; Q(e_1), \qquad \dots, \qquad \deg(T_r) \;:=\; Q(e_r).$$

Assume R is factorially K-graded, has $\mathbb{K}^*$ as its homogeneous units and the grading is almost free. Let $G \subseteq \mathbb{K}[T_1, \dots, T_r]$ be a set of K-prime generators for I and consider the matrix Q^0 fitting into

$$\begin{tikzcd} K \arrow[d] & \mathbb{Z}^r \arrow[l, "Q"'] \arrow[dl, "Q^0"] \\ K^0 & \end{tikzcd}$$

with the free part $K^0 = K/K^{\mathrm{tor}}$. Assume $\mathfrak{F} = (\overline{T}_1, \dots, \overline{T}_r)$ is a system of pairwise non-associated K-prime generators for R. Fix a Gale dual matrix P for Q and store the $\mathfrak{F}$-faces in a list $F_{\mathfrak{F}}$.

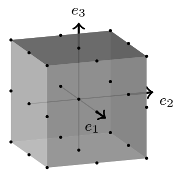
Definition 2.2.1. In the above setting, we encode the graded ring R in the tuple $(G, Q, Q^0, P, F_{\mathfrak{F}})$ and speak of a *GR*.

We do not differentiate between a GR and the underlying ring R. Note that G, P, Q^0, $F_{\mathfrak{F}}$ are all computable from a presentation $R = \mathbb{K}[T_1, \dots, T_r]/I$, see Algorithms 2.1.25, 2.1.26; we will postpone the computational aspects of $\mathfrak{F}$-faces until Chapter 3. We implicitly assign the respective positive orthant $\gamma := \mathbb{Q}^r_{\geq 0}$ to a GR. Some of the following algorithms need the list of *lattice points* $B \cap \mathbb{Z}^r$ or *interior points* $B^\circ \cap \mathbb{Z}^r$ of a polytope $B \subseteq \mathbb{Q}^r$. The following is an ad-hoc method using a bounding box; see, e.g., [11, 29] for advanced algorithms.

Algorithm 2.2.2 (intpoints)**.** *Input:* a polytope $B \subseteq \mathbb{Q}^r$.

- Compute for each $1 \leq i \leq r$ bounds ${b_i}^\bullet$ and ${b_i}_\bullet \in \mathbb{Z}^r$ such that for each vertex v of B, we have ${b_i}_\bullet \leq v_i \leq {b_i}^\bullet$.
- Set $\mathcal{L} := \emptyset$.
- For each $v \in \mathbb{Z}^r$ such that ${b_i}_\bullet \leq v_i \leq {b_i}^\bullet$ for all i do
  - if $v \in B$, then insert v into $\mathcal{L}$.

Output: the collection $\mathcal{L}$ of lattice points $B \cap \mathbb{Z}^r$.



For a K-graded ring $R = \bigoplus_{w \in K} R_w$, the $\mathbb{K}$-vector space R_w is called the *homogeneous component* or the *graded component* of R of degree w. Note that, by the decomposition $K = K^0 \oplus K^{\mathrm{tor}}$, we can decompose each $w \in K$ uniquely into $w = w^0 + w^{\mathrm{t}}$ with $w^0 \in K^0$ and $w^{\mathrm{t}} \in K^{\mathrm{tor}}$.

Algorithm 2.2.3 (GRgradedcomp)**.** *Input:* a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and $w \in K$. Assume that the grading is pointed. Decompose $w = w^0 + w^{\mathrm{t}}$ with $w^0 \in K^0$ and $w^{\mathrm{t}} \in K^{\mathrm{tor}}$.

- Let $\mathcal{W} := \emptyset$.
- For each $f \in G$ do
  - Let $\Delta := w^0 - \deg(f)^0 \in K^0$.
  - Use Algorithm 2.2.2 to compute the lattice points $M_\Delta := B_\Delta \cap \mathbb{Z}^r$ of the fiber polytope $B_\Delta := (Q^0)^{-1}(\Delta) \cap \gamma$.
  - For each $\nu \in M_\Delta$, insert the polynomial $T^\nu \cdot f$ into $\mathcal{W}$ if $\deg(T^\nu \cdot f) = w \in K$.
- Use Algorithm 2.2.2 to compute the lattice points $M_{w^0} := B_{w^0} \cap \mathbb{Z}^r$ of the fiber polytope $B_{w^0} := (Q^0)^{-1}(w^0) \cap \gamma$. Store the elements $\nu \in M_{w^0}$ with $\deg(T^\nu) = w \in K$ as ordered list $(\nu_1, \dots, \nu_k)$.
- Given $g \in \mathcal{W}$, let $v_g \in \mathbb{K}^k$ be the image of g under $T^{\nu_i} \mapsto e_i$. Let A be the matrix with the v_g as its columns where g runs through $\mathcal{W}$ in a fixed order.
- Compute a Smith normal form $S = U \cdot A \cdot V$ with integral, invertible matrices U, V. Write $s := \mathrm{rank}(S)$.
- Return the list $(\sum_{j=1}^k (b_i)_j T^{\nu_j};\ 1 \leq i \leq s)$ where $b_i := U^{-1} \cdot e_i$.

Output: a basis for $\langle G \rangle_w$ considered as a $\mathbb{K}$-vector space.

Lemma 2.2.4. *Consider a pointed grading $\bigoplus_{w\in K} \mathbb{K}[T_1,\dots,T_r]_w$ of the polynomial ring by a finitely generated abelian group K. Let $I \subseteq \mathbb{K}[T_1,\dots,T_r]$ be an ideal with K-homogeneous generators $g_1,\dots,g_s$. Then*

$$I_w \;=\; \mathrm{lin}_{\mathbb{K}}\left(T^\nu g_i;\ 1\le i\le s,\, \deg(T^\nu)+\deg(g_i)=w\right) \qquad \textit{for each } w\in K.$$

Proof. Given $f\in I_w$, write $f = h_1g_1+\ldots+h_sg_s$ with polynomials $h_1,\dots,h_s\in \mathbb{K}[T_1,\dots,T_r]$. By the direct sum decomposition, we may write each h_i uniquely as $h_i=\sum_k h_{i,k}$ with $h_{i,k}\in\mathbb{K}[T_1,\dots,T_r]_k$. Since f equals its degree-w part f_w, we have

$$f \;=\; \left(\sum_{i=1}^s h_ig_i\right)_w \;=\; \left(\sum_{i=1}^s\left(\sum_k h_{i,k}\right)g_i\right)_w \;=\; \sum_{i=1}^s h_{i,w-w_i}g_i,$$

where in the last step, we defined $w_i := \deg(g_i)$. The other inclusion is obvious. □

Proof of Algorithm 2.2.3. Note that since Q^0 is pointed, B_Δ and B_{w^0} are polytopes. Observe that the vector space $\mathbb{K}[T_1,\dots,T_r]_w$ has the $\mathbb{K}$-basis $(T^{\nu_1},\dots,T^{\nu_k})$ and $\mathrm{lin}_{\mathbb{K}}(\mathcal{W})=\langle G\rangle_w$ as a $\mathbb{K}$-vector space by Lemma 2.2.4. We now show that the remaining steps of the algorithm compute a basis for $\langle G\rangle_w$. Consider the isomorphism of $\mathbb{K}$-vector spaces

$$\varphi\colon \mathrm{lin}_{\mathbb{K}}(T^{\nu_1},\dots,T^{\nu_k}) \;\to\; \mathbb{K}^k, \qquad T^{\nu_i}\;\mapsto\; e_i.$$

That is $\varphi(g)=v_g$ for each polynomial $g\in\mathbb{K}[T_1,\dots,T_r]_w$. Moreover, by construction, $(b_1,\dots,b_s)$ is a basis for $\varphi(\mathrm{lin}_{\mathbb{K}}(\mathcal{W}))$. The last step in the algorithm applies the inverse map for φ, i.e., maps $b\in\mathbb{Q}^k$ to $\sum b_iT^{\nu_i}$. We summarize the situation by a diagram where rightward arrows are inclusions and the remaining arrows are isomorphisms.

$$\begin{tikzcd}
\langle G\rangle_w \arrow[r, equal] \arrow[dr] \arrow[ddr] & \mathrm{lin}_{\mathbb{K}}(\mathcal{W}) \arrow[r] \arrow[d, "\varphi"] & \mathrm{lin}_{\mathbb{K}}\left(T^{\nu_1},\dots,T^{\nu_k}\right) \arrow[r, equal] \arrow[d, "\varphi"'] & \mathbb{K}[T]_w \arrow[dl] \arrow[ddl] \\
 & \mathrm{lin}_{\mathbb{K}}(b_1,\dots,b_s) \arrow[r] \arrow[d, "\mu_U"] & \mathbb{K}^k \arrow[d, "\mu_U"'] & \\
 & \mathrm{lin}_{\mathbb{K}}(e_1,\dots,e_s) \arrow[r] & \mathbb{K}^k &
\end{tikzcd}$$

□

Algorithm 2.2.5 (GRgradedcompdim)**.** *Input:* a GR $R=(G,Q,Q^0,P,F_{\mathfrak{F}})$ and a vector $w\in Q^0(\gamma)$. Assume that the grading is pointed. Decompose $w=w^0+w^{\mathrm{t}}$ with $w^0\in K^0$ and $w^{\mathrm{t}}\in K^{\mathrm{tor}}$.

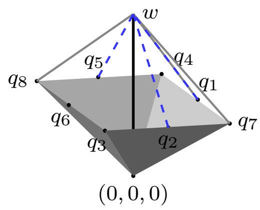
- Use Algorithm 2.2.2 to compute the set $M_{w^0} := B_{w^0}\cap\mathbb{Z}^r$ with the fiber polytope $B_{w^0} := (Q^0)^{-1}(w^0)\cap\gamma$. Denote by $n\in\mathbb{Z}_{\ge 0}$ the number of elements $\nu\in M_{w^0}$ with $\deg(T^\nu)=w\in K$.
- Compute a basis $\mathcal{B}$ for $\langle G\rangle_w$ with Algorithm 2.2.3. Write $d := n-|\mathcal{B}|$.

Output: the dimension $d\in\mathbb{Z}_{\ge 0}$ of the graded component $\mathbb{K}[T_1,\dots,T_r]_w/\langle G\rangle_w$.

Example 2.2.6. Consider in the setting of Example 2.0.14 the graded ring $R = (\{f_1\},Q,Q^0,P,F_{\mathfrak{F}})$ where Q is as in Example 2.1.16 and P and Q^0 are as in Example 2.1.28. We apply Algorithm 2.2.3 to R and $w := (1,0,3,\overline{0})\in K$. We have

$$\mathbb{K}[T_1,\dots,T_8]_w \;=\; \mathrm{lin}_{\mathbb{Q}}\big(T_1T_7T_8,\, T_1T_3T_4,\, T_1T_2T_5,\, T_1^2T_6\big),$$

where basis elements correspond to a positive combinations of w over the columns q_i of Q^0; e.g., for $T_1T_7T_8$ and $T_1T_2T_5$, we have:



Moreover, Algorithm 2.2.3 returns the basis (T_1f_1) for $\langle f_1\rangle_w$. As obtained with Algorithm 2.2.5, we have

$$\begin{aligned} & \dim\left(\mathbb{K}[T_1,\ldots,T_8]_w/\langle f_1\rangle_w\right) \\ = \ & \dim\left(\mathbb{K}[T_1,\ldots,T_8]_w/\mathrm{lin}_{\mathbb{Q}}(T_1^2T_6+T_1T_2T_5+T_1T_3T_4+T_1T_7T_8)\right) \\ = \ & 3. \end{aligned}$$

Recall, e.g., from [78, 22], that given an affine variety $\overline{X} := V(I) \subseteq \mathbb{K}^r$ with a monomial-free ideal $I \subseteq \mathbb{K}[T_1,\ldots,T_r]$, one assigns to I or to $\overline{X}$ the *tropical variety*

$$\mathrm{trop}(I) \ := \ \mathrm{trop}\left(\overline{X}\right) \ := \ \bigcap_{f\in I} \mathrm{trop}(f) \ \subseteq \ \mathbb{Q}^r,$$

where $\mathrm{trop}(f)$ is the support of the codimension one skeleton of the normal fan over the Newton polytope of f. There exists a fan $\Upsilon \subseteq \mathbb{Q}^r$ with support $|\Upsilon| = \mathrm{trop}(I)$; see [22] for its computation. This fan is a projectable fan, see [91, Prop. 2.8], so for the case of a Mori dream space $X = X(R,\mathfrak{F},\Phi)$ as in Construction 1.3.6, we may define $\mathrm{trop}(X) := P(\mathrm{trop}(\overline{X}))$ where $\overline{X}$ is the total coordinate space of X. For a single equation, we remark the following.

Algorithm 2.2.7 (GRtrop)**.** *Input:* a GR $R = (G,Q,Q^0,P,F_{\mathfrak{F}})$ where $P\colon \mathbb{Z}^r \to N$ and G contains a single polynomial f.

- Let Σ be the normal fan over the Newton polytope of f.
- Let $\widehat{\Upsilon}$ be the $(\dim(\Sigma)-1)$-skeleton of Σ and $\Upsilon := P(\widehat{\Upsilon})$.

Output: a fan Υ in $N_{\mathbb{Q}}$ with the tropical variety $\mathrm{trop}(\langle G\rangle)$ as its support.

Containment of a vector in the tropical variety can be done without computing the whole fan structure. For this, recall from [22, 41] that given a monomial-free ideal $I \subseteq \mathbb{K}[T_1,\ldots,T_r]$, the *Gröbner cone* of a vector $w \in \mathbb{Q}^r$ is the convex cone

$$\mathcal{C}(w) \ := \ \mathrm{cone}\left(w' \in \mathbb{Q}^r;\ \mathrm{in}_w(I) \ = \ \mathrm{in}_{w'}(I)\right) \ \subseteq \ \mathbb{Q}^r$$

where $\mathrm{in}_w(I)$ denotes the ideal $\langle \mathrm{in}_w(f);\ f \in I\rangle \subseteq \mathbb{K}[T_1,\ldots,T_r]$. Here, the initial form $\mathrm{in}_w(f)$ consists of all terms αT^ν of f that are maximal with respect to $\nu \mapsto \langle w,\nu\rangle$. Moreover, we have a description of the tropical variety

$$\mathrm{trop}(I) \ = \ \{w \in \mathbb{Q}^r;\ \mathrm{in}_w(I) \text{ is monomial-free}\} \ \subseteq \ \mathbb{Q}^r.$$

If I is homogeneous, the *Gröbner fan* is the collection of all Gröbner cones $\{\mathcal{C}(w);\ w \in \mathbb{Q}^r\}$. It turns out to be a convex, polyhedral, complete fan in $\mathbb{Q}^r$. The tropical variety $\mathrm{trop}(I)$ then is the support of the subfan Υ of the Gröbner fan of I that consists of all Gröbner cones $\mathcal{C}(w)$ such that $\mathrm{in}_w(I)$ is monomial-free.

Fix a monomial ordering $>$ on $\mathbb{K}[T_1,\ldots,T_r]$. Recall that for a vector $v \in \mathbb{Q}^r_{\geq 0}$, we obtain another monomial ordering $>_v$ on $\mathbb{K}[T_1,\ldots,T_r]$ given by

$$T^\nu \ >_v \ T^\mu \quad :\Leftrightarrow \quad \langle v,\nu\rangle > \langle v,\mu\rangle \quad \text{or} \quad \left[\langle v,\nu\rangle \ = \ \langle v,\mu\rangle \text{ and } T^\nu > T^\mu\right].$$

Algorithm 2.2.8 (GRtropcontains)**.** *Input:* a GR $R = (G,Q,Q^0,P,F_{\mathfrak{F}})$ and a vector $v \in N$ or $f \in \mathbb{Z}^r$ where $P\colon \mathbb{Z}^r \to N$. Assume that the cone ω over the columns $q_1,\ldots,q_r$ of Q^0 is pointed and no q_i is the zero-vector.

- If $v \in N$ was given, then choose $f \in \mathbb{Z}^r$ such that $P(f) = v$.

- Choose a linear form $u \in (\omega^\vee)^\circ$. Let $f^+ \in \mathbb{Z}^r_{>0}$ be the vector with components $(f^+)_i := u(q_i)$.
- Determine $a \in \mathbb{Z}_{\geq 0}$ such that $f' := f + af^+$ is an element of $\mathbb{Z}^r_{>0}$ and compute a Gröbner basis $\mathcal{G}_{>_{f'}}$ for $\langle G \rangle$ with respect to the monomial ordering $>_{f'}$.
- Consider the ideal $\mathfrak{a} := \langle \mathrm{in}_{f'}(g);\, g \in \mathcal{G}_{>_{f'}} \rangle \subseteq \mathbb{K}[T_1, \ldots, T_r]$.
- Return *false* if the radical membership test $T_1 \cdots T_r \in \sqrt{\mathfrak{a}}$ succeeds and return *true* otherwise.

Output: returns *true* if $f \in \mathrm{trop}(\langle G \rangle)$ or $v \in P(\mathrm{trop}(\langle G \rangle))$, respectively. Returns *false* otherwise.

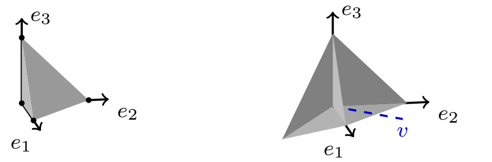
Proof. In the case of a given vector $v \in N$, since $P^{-1}(P(\mathrm{trop}(\langle G \rangle))) = \mathrm{trop}(\langle G \rangle)$, it suffices to choose any $f \in \mathbb{Z}^r$ such that $P(f) = v$. By the definition of the tropical variety, we have

$$\begin{aligned} f \in \mathrm{trop}(\langle G \rangle) \quad &\Leftrightarrow \quad T^\nu \notin \mathrm{in}_f(\langle G \rangle) \quad \text{for all } \nu \in \mathbb{Z}^r_{\geq 0} \\ &\Leftrightarrow \quad T_1 \cdots T_r \notin \sqrt{\mathrm{in}_f(\langle G \rangle)}. \end{aligned}$$

Note that since ω is pointed, $u_{|\omega \setminus \{0\}} > 0$. Hence, we can coarsify the grading as claimed, i.e., we find the vector $f' \in \mathbb{Z}^r_{>0}$. By [90, Prop. 1.12] and its proof, for each $g \in \langle G \rangle$ we have $\mathrm{in}_f(g) = \mathrm{in}_{f'}(g)$. Since f' is an element of the Gröbner cone of $\langle G \rangle$ with respect to the ordering $<_{f'}$, by [41, Cor. 2.14], we conclude

$$\mathrm{in}_f(\langle G \rangle) \;=\; \mathrm{in}_{f'}(\langle G \rangle) \;=\; \left\langle \mathrm{in}_{f'}(g);\, g \in \mathcal{G}_{>_{f'}} \right\rangle. \qquad \square$$

Example 2.2.9. Let Q' and P' be as in Example 2.1.30 and compute $(Q')^0$ with Algorithm 2.1.26. Consider the GR $R' = (\{f_1\}, Q', (Q^0)', P', F_{\mathfrak{F}})$ where f_1 is as in Example 2.0.14. Algorithm 2.2.7 then returns a two-dimensional pure fan $P'(\Upsilon)$ in $\mathbb{Q}^3$ with support $|P'(\Upsilon)| = \mathrm{trop}(f_1)$.



It is given as the codimension one skeleton of the normal fan over the standard-simplex. Moreover, using Algorithm 2.2.7, we verify that the vector $v := (1, 1, 0) \in \mathbb{Q}^3$ is contained in $\mathrm{trop}(f_1)$.

In the definition of bunched rings 1.3.5, we required the variables to define K-prime elements. We check this using the following direct method which was already published together with J. Hausen and A. Laface in [57, Alg. 4.2]. See [46, B.7] and [21, 25] for the computational background on how to compute the prime components $\mathfrak{c}_i$ and number fields $\mathbb{Q}(\alpha_i)$.

Algorithm 2.2.10 (GRisKprime)**.** *Input:* a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and an index $1 \leq k \leq r$ where we consider $Q\colon \mathbb{Z}^r \to K$ as a matrix and assume that the grading group is of shape $K = \mathbb{Z}^s \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_l\mathbb{Z}$. We further require that $G = \{f_1, \ldots, f_s\}$ is contained in $\mathbb{Q}[T_1, \ldots, T_r]$.

- If $\langle f_1, \ldots, f_s, T_k \rangle$ is not a radical ideal, return *false*.
- Compute a decomposition $\langle f_1, \ldots, f_s, T_k \rangle = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_m$ with prime ideals $\mathfrak{c}_i$ and number fields $\mathbb{Q}(\alpha_i)$ such that $\mathfrak{c}_i$ is defined over $\mathbb{Q}(\alpha_i)$.

- Denote by $q_1, \dots, q_l$ the last l rows of Q and by ζ_{a_i} the primitive a_i-th root of unity. Consider for any $b \in \mathbb{Z}^l_{\geq 0}$ the map
$$\varphi_b\colon\ \mathbb{Q}(\alpha_i)[T_1, \dots, T_r] \to \mathbb{L}[T_1, \dots, T_r], \qquad T_j \mapsto \zeta_{a_1}^{b_1 q_{1j}} \cdots \zeta_{a_l}^{b_l q_{lj}} T_j,$$
where $\mathbb{L} := \mathbb{Q}(\alpha_1, \dots, \alpha_m, \zeta_{a_1}, \dots, \zeta_{a_l})$. If for each two $i \neq j$ there is $b \in \mathbb{Z}^l_{\geq 0}$ such that $\varphi_b(\mathfrak{c}_i) = \mathfrak{c}_j$ in $\mathbb{L}$, then return *true*. Return *false* otherwise.

Output: *true* if T_k is K-prime in $\overline{\mathbb{Q}}[T_1, \dots, T_r]/\langle f_1, \dots, f_s\rangle$ and *false* otherwise.

Proof. Let $I := \langle G \rangle \subseteq \mathbb{K}[T_1, \dots, T_r]$. Consider the action of $H := \operatorname{Spec} \mathbb{K}[K]$ on $Y := V(I) \subseteq \mathbb{K}^r$. By [51, Prop. 3.2], T_k is K-prime in $R = \mathbb{K}[T_1, \dots, T_r]/I$ if and only if the divisor of T_k in Y is H-prime in the sense that it has only coefficients 0 or 1 and the prime divisors with coefficient 1 are transitively permuted by H. $\square$

Example 2.2.11. Consider the following factorially K-graded ring R where $K := \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$; we will encounter R later as the Cox ring of the surface with singularity type D_5A_3 in Theorem 4.4.1.
$$R \ = \ \mathbb{K}[T_1, \dots, T_5]/I, \qquad I = \langle T_1T_3 - T_4^2 - T_5^2,\ T_1T_2 - T_3^2 + T_4T_5 \rangle,$$
$$\deg(T_1) \ = \ (1, \overline{2}), \quad \deg(T_2) \ = \ (1, \overline{2}), \quad \deg(T_3) \ = \ (1, \overline{0}),$$
$$\deg(T_4) \ = \ (1, \overline{3}), \quad \deg(T_5) \ = \ (1, \overline{1}).$$
We show that the variable T_1 defines a K-prime element in R albeit it is not prime. The ideal $I + \langle T_1 \rangle \subseteq \mathbb{K}[T_1, \dots, T_5]$ has the two prime components
$$\mathfrak{c}_1 \ = \ \langle T_1,\, T_4 - T_5J,\, T_3^2 - T_4T_5 \rangle, \qquad \mathfrak{c}_2 \ = \langle T_1,\, T_4 + T_5J,\, T_3^2 - T_4T_5 \rangle$$
defined in $\mathbb{Q}(J)[T_1, \dots, T_5]$ with the imaginary unit $J \in \mathbb{C}$. In the notation of Algorithm 2.2.10, we have $a_1 = 4$, $\zeta_{a_1} = J$ and $\mathbb{L} = \mathbb{Q}(J)$.
$$\begin{tikzcd}
 & \overline{\mathbb{Q}} & \\
 & \mathbb{L} = \mathbb{Q}(J) \arrow[u, dash] \arrow[dl, equal] \arrow[dr, equal] & \\
\mathbb{Q}(\alpha_1) & & \mathbb{Q}(\alpha_2) \\
 & \mathbb{Q} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$
For $b := 1 \in \mathbb{Z}_{\geq 0}$, the torsion part of the degrees $\deg(T_i)$ defines the map
$$\varphi_b\colon\ \mathbb{Q}(J)[T_1, \dots, T_5] \ \to \ \mathbb{L}[T_1, \dots, T_5],$$
$$T_1 \mapsto -T_1, \quad T_2 \mapsto -T_2, \quad T_3 \mapsto T_3, \quad T_4 \mapsto -JT_4, \quad T_5 \mapsto JT_5.$$
Then, by the reasonsing of Algorithm 2.2.10, T_1 is K-prime since φ_b permutes the ideals $\mathfrak{c}_i \subseteq \mathbb{Q}(J)[T_1, \dots, T_5]$ transitively: their images are
$$\langle T_1,\, -JT_4 + T_5,\, T_3^2 - T_4T_5 \rangle \ = \ \mathfrak{c}_2, \quad \langle T_1,\, -JT_4 - T_5,\, T_3^2 - T_4T_5 \rangle \ = \ \mathfrak{c}_1.$$

We come to basic modifications of polynomials; compare [57, 44] and Section 3 of Chapter 4. Consider a homomorphism $p\colon \mathbb{T}^r \to \mathbb{T}^n$ of tori. Given a Laurent polynomial $g \in \mathbb{K}[S_1^{\pm 1}, \dots, S_n^{\pm 1}]$, its $\star$-*pull back* is a polynomial $p^\star g \in \mathbb{K}[T_1, \dots, T_r]$ such that $p^\star g$ and $p^* g$ coincide in $\mathbb{K}[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ up to units and the monomials of $p^\star g$ are coprime. The $\star$-pull back always exists and is unique up to constants.

Algorithm 2.2.12 (pull)**.** *Input:* a polynomial $f \in \mathbb{K}[S_1, \ldots, S_n]$ and an integral $n \times r$ matrix P of full rank.

- Let g' be the image of f under the homomorphism
$$\mathbb{K}[S_1, \ldots, S_n] \to \mathbb{K}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}], \qquad S_i \mapsto T^{P_{i*}}.$$
- Choose $\mu \in \mathbb{Z}^r_{\geq 0}$ such that $g := T^\mu g'$ is an element of $\mathbb{K}[T_1, \ldots, T_r]$.
- While there is $1 \leq i \leq r$ such that $T_i \mid g$, replace g by gT_i^{-1}.

Output: $g \in \mathbb{K}[T_1, \ldots, T_r]$ such that $g = p^\star g \in \mathbb{K}[T_1, \ldots, T_r]$ is the star pull back with the morphism of tori $p\colon \mathbb{T}^r \to \mathbb{T}^n$ corresponding to P.

Consider a homomorphism $p\colon \mathbb{T}^r \to \mathbb{T}^n$ of tori with kernel $H \subseteq \mathbb{T}^r$. Given an H-homogeneous polynomial $h \in \mathbb{K}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$, its $\star$*-push forward* is a polynomial $p_\star h \in \mathbb{K}[S_1, \ldots, S_n]$ such that $p^* p_\star h$ and h coincide in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$ up to units and the monomials of $p_\star g$ are coprime. The $\star$-push forward always exists and is unique up to constants.

Algorithm 2.2.13 (push)**.** *Input:* a polynomial $h \in \mathbb{K}[T_1, \ldots, T_r]$ and an integral $n \times r$ matrix P of full rank such that h is H-homogeneous where $H \subseteq \mathbb{T}^r$ is the kernel of the morphism of tori $p\colon \mathbb{T}^r \to \mathbb{T}^n$ corresponding to P.

- Compute a Smith normal form $D = U \cdot P \cdot V$ with integral invertible matrices U, V. Let $\varphi_U\colon \mathbb{T}^n \to \mathbb{T}^n$, $\varphi_D\colon \mathbb{T}^r \to \mathbb{T}^n$ and $\varphi_V\colon \mathbb{T}^r \to \mathbb{T}^r$ be the corresponding maps of tori.
- Use Algorithm 2.2.12 to compute $g := \varphi_V^\star h \in \mathbb{K}[T_1, \ldots, T_r]$.
- Write $D = [D', 0]$ as a block matrix where D' is a diagonal matrix with diagonal entries $d_1, \ldots, d_n \in \mathbb{Z} \setminus \{0\}$. Let $g' \in \mathbb{K}[S_1, \ldots, S_n]$ be the image of g under the map
$$\mathbb{K}[T_1^{d_1}, \ldots, T_n^{d_n}, T_{n+1}, \ldots, T_r] \to \mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}],$$
$$T_i^{d_i} \mapsto S_i \quad \text{for } 1 \leq i \leq n, \qquad T_i \mapsto 0 \quad \text{else}.$$
- Use Algorithm 2.2.12 to compute $f := \varphi_U^\star g' \in \mathbb{K}[S_1, \ldots, S_n]$.

Output: the $\star$-push forward $f = p_\star h \in \mathbb{K}[S_1, \ldots, S_n]$.

Proof. Note that D is indeed of the claimed form since P is of full rank. As V is invertible, we have $(\varphi_{V^{-1}})_\star h = \varphi_V^\star h$. The same argument holds for U. The claim follows from the decomposition

$$p_\star h \ = \ (\varphi_{U^{-1}})_\star (\varphi_D)_\star (\varphi_{V^{-1}})_\star h \qquad \begin{tikzcd} \mathbb{T}^r \arrow[r, "\varphi_{V^{-1}}", "\cong"'] \arrow[d, "p"'] & \mathbb{T}^r \arrow[d, "\varphi_D"] \\ \mathbb{T}^n & \mathbb{T}^n \arrow[l, "\cong", "\varphi_{U^{-1}}"'] \end{tikzcd}$$

and the fact that the $\star$-push forward under D is $(\varphi_D)_\star g = g'$ since $\varphi_D^\star g' = g$. Observe that (the map used to obtain) g' is well-defined. Since h is $\mathbb{Z}^r/\mathrm{Im}(P^*)$-homogeneous the pull back $\varphi_V^\star h$ is $\mathbb{Z}^r/\mathrm{Im}(D^*)$-homogeneous. This means that for each monomial T^ν in $\varphi_V^\star h$ we have $d_i \mid \nu_i$ for all $1 \leq i \leq n$. $\square$

Recall that the *saturation* of an ideal $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$ with respect to a polynomial $f \in \mathbb{K}[T_1, \ldots, T_r]$ is the ideal

$$I : f^\infty \ := \ \left\{g \in \mathbb{K}[T_1, \ldots, T_r];\ f^k g \in I \text{ for some } k \in \mathbb{Z}_{\geq 1}\right\} \ \subseteq \ \mathbb{K}[T_1, \ldots, T_r].$$

Algorithm 2.2.14 (closure)**.** Compare [64, pp. 23–24]. *Input:* a list of generators $f_1, \ldots, f_s$ for an ideal $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$.

- Compute generators $g_1, \ldots, g_n \in \mathbb{K}[T_1, \ldots, T_r]$ for $I : (T_1 \cdots T_r)^\infty$.

Output: polynomials $g_1, \dots, g_m \in \mathbb{K}[T_1, \dots, T_r]$ such that $\overline{V(\mathbb{T}^r; I)} \subseteq \mathbb{K}^r$ is given by $V(\mathbb{K}^r; g_1, \dots, g_s)$.

Proof. The associated primes of $J := I : (T_1 \cdots T_r)^\infty$ are the vanishing ideals $\mathfrak{p}_i \subseteq \mathbb{K}[T_1, \dots, T_r]$ of the irreducible components of $X := V(J) \subseteq \mathbb{K}^r$, see [46, Thm. 4.1.5]. We have $\mathfrak{p}_i = \mathfrak{p}_i : (T_1 \cdots T_r)^\infty$ for each i, see [64, Lem. 2.5.8]. In particular, no component of X is contained in a coordinate hyperplane. This shows that the closure in $\mathbb{K}^r$ is $\overline{V(J) \cap \mathbb{T}^r} = V(J)$. We now show $V(J) \cap \mathbb{T}^r = V(I) \cap \mathbb{T}^r$. By construction, $V(J) \subseteq V(I)$. For the reverse containment, consider $x \in V(I) \cap \mathbb{T}^r$ and $f \in J$. Then $(f \cdot (T_1 \cdots T_r)^n)(x) = 0$ for some $n \in \mathbb{Z}_{\geq 0}$, i.e., $f(x) = 0$. ☐

3. Mori dream spaces

We provide first algorithms to explore the properties and geometry of a Mori dream space. This section contains material from [5] as indicated near the respective items below. Here is an overview of the algorithms of this section:

- *Fundamental algorithms:* dimension (Algorithm 2.3.4), covering collection (Algorithm 2.3.6), relevant $\mathfrak{F}$-faces (Algorithm 2.3.5), toric ambient variety and completions (Algorithm 2.3.9), irrelevant ideal (Algorithm 2.3.11).
- *Cones of divisor classes:* effective cone (Algorithm 2.3.13), moving cone (Algorithm 2.3.14), semiample cone (Algorithm 2.3.15).
- *Groups:* class group (Algorithm 2.3.17), local class groups (Algorithm 2.3.18), Picard group (Algorithm 2.3.20), Picard index (Algorithm 2.3.21).
- *Singularities, properties:* test for being quasismooth and smooth (Algorithms 2.3.23 and 2.3.24), singularities (Algorithm 2.3.25), exceptional graph (Algorithm 2.3.27), test for being ($\mathbb{Q}$-) factorial (Algorithms 2.3.30 and 2.3.31), test for being complete (Algorithm 2.3.33), test for being (quasi-) projective (Algorithms 2.3.35 and 2.3.36), strata (Algorithm 2.3.39).
- *Complete intersection Cox rings:* anticanonical divisor class (Algorithm 2.3.41), test for being ($\mathbb{Q}$-) Gorenstein (Algorithms 2.3.43 and 2.3.44), Gorenstein index (Algorithm 2.3.45), test for being Fano (Algorithm 2.3.46), intersection numbers (Algorithm 2.3.48).

We work with Mori dream spaces by using the correspondence to bunched rings $(R, \mathfrak{F}, \Phi)$, compare Construction 1.3.6 and Corollary 1.3.9. To this end, we directly encode the true $\mathfrak{F}$-bunch Φ in a data type BUN and then define the central data type MDS. Let $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ be a GR. As before, we implicitly assign to R the grading group K, its free part K^0 and the positive orthant $\gamma = \mathbb{Q}^r_{\geq 0}$.

Definition 2.3.1. A *BUN* in R is a finite set $\Phi := \{\vartheta_1, \dots, \vartheta_s\}$ of polyhedral cones $\vartheta_i \subseteq K^0_{\mathbb{Q}}$ of the form $\vartheta_i = Q^0(\gamma_i)$ with an element $\gamma_i \in F_{\mathfrak{F}}$ such that

(i) for all i, j, we have $\vartheta_i^\circ \cap \vartheta_j^\circ \neq \emptyset$,
(ii) if $\vartheta_i^\circ \subseteq (Q^0(\gamma_j))^\circ$ with $\gamma_j \in F_{\mathfrak{F}}$, then $Q^0(\gamma_j)$ belongs to Φ,
(iii) for each facet $\gamma_0 \preceq \gamma$, the image $Q^0(\gamma_0)$ is an element of Φ.

Definition 2.3.2. Let R be a GR and Φ a BUN in R. We call the pair (R, Φ) a *MDS*.

We do not differentiate between a Mori dream space $X = X(R, \mathfrak{F}, \Phi)$ as in Construction 1.3.6 and its description as an MDS. For instance, we also write $\overline{X}$ for the total coordinate space of (the underlying Mori dream space of) an MDS X.

Example 2.3.3. Consider the GR R of Example 2.2.6. Then the Mori dream space X of Example 2.0.14 is given by the MDS $(R, \Phi(w))$ where $w := (0, 0, 2) \in K^0_{\mathbb{Q}}$.

Next, we treat essential algorithms like dimension, relevant $\mathfrak{F}$-faces, covering collection and the canonical toric ambient variety.

Algorithm 2.3.4 (MDSdim)**.** See [5, Thm. III.2.1.4]. *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute the dimension d_G of the ring R.
- Let $d := d_G - d^0$ where d^0 is the dimension of the rowspace of Q^0.

Output: the dimension $d = \dim(X)$.

Algorithm 2.3.5 (MDSrlv)**.** See [5, Con. III.2.1.3]. *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute the set L of all $\gamma_0 \in F_{\mathfrak{F}}$ such that $Q^0(\gamma_0) \in \Phi$.

Output: the set $L = \mathrm{rlv}(\Phi)$ of relevant $\mathfrak{F}$-faces.

Algorithm 2.3.6 (MDScov)**.** Compare [5, Con. III.2.1.3]. *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute $L := \mathrm{rlv}(\Phi)$ with Algorithm 2.3.5.
- Set $L^{\min} := \emptyset$.
- For each $\gamma_0 \in L$ do
  - insert γ_0 into $L^{\min}$ if there is no $\gamma_1 \in L$ such that $\gamma_1 \subsetneq \gamma_0$.

Output: the set $L^{\min} = \mathrm{cov}(\Phi)$ of all minimal relevant $\mathfrak{F}$-faces.

Example 2.3.7. The MDS $X = (R, \Phi)$ of Example 2.3.3 is of dimension $\dim(R) - 3 = 4$. Among the 83 relevant $\mathfrak{F}$-faces $\mathrm{rlv}(\Phi)$ returned by Algorithm 2.3.5, there are 14 minimal ones, i.e.,

$$\begin{aligned}\mathrm{cov}(\Phi) \;=\; \{&\{1,6,7,8\},\{2,5,7,8\},\{3,4,7,8\},\{1,2,5,6\},\{1,3,4,6\},\{2,3,4,5\},\\ &\{2,4,8\},\{1,3,8\},\{1,2,8\},\{5,6,7\},\{4,6,7\},\{3,5,7\},\{2,4,6\},\{1,3,5\}\}\end{aligned}$$

where we used Algorithm 2.3.6 and identify a subset $J \subseteq \{1, \dots, 8\}$ with the face $\mathrm{cone}(e_i;\ i \in J) \preceq \gamma = \mathbb{Q}^8_{\geq 0}$.

Algorithm 2.3.8 (MDSpointex)**.** *Input:* an MDS $X = (R, \Phi)$ and a vector $z \in \mathbb{K}^r$ where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- If $z \notin \overline{X} = V(G) \subseteq \mathbb{K}^r$, then return *false*.
- Compute $C := \mathrm{rlv}(\Phi)$ with Algorithm 2.3.5.
- For each $\gamma_0 \in C$ do
  - if $z_i \neq 0$ for all $e_i \in \gamma_0$, then return *true*.
- Return *false*.

Output: *true* if $[z] \in X$, i.e., $z \in \widehat{X} \subseteq \mathbb{K}^r$, and *false* otherwise.

Proof. The correctness directly follows from [5, Constr. III.3.1.1] where

$$\widehat{X} \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X} \setminus V\left(\prod_{e_i \in \gamma_0} T_i\right)$$

□

Given fans $\Sigma_1, \Sigma_2 \subseteq \mathbb{Q}^r$, denote by $\Sigma_1 \sqcap \Sigma_2$ their *coarsest common refinement*, i.e., the set of cone-wise intersections $\{\sigma_1 \cap \sigma_2;\ \sigma_i \in \Sigma_i\}$. The *Gelfand Kapranov Zelevinsky decomposition* of a matrix $Q^0 = [q_1, \dots, q_r]$ is the coarsest common refinement

$$\mathrm{GKZ}\left(Q^0\right) \;:=\; \prod \Lambda,$$

where Λ runs through all normal fans having their rays among the $\mathbb{Q}_{\geq 0} \cdot q_i$ and with support $|\Lambda| = \mathrm{cone}(q_1, \ldots, q_r)$; see Chapter 3 for the computational aspects.

Given an MDS X, the following algorithm computes the canonical toric ambient variety $Z_\Sigma \supseteq X$ as defined in Construction 1.3.12. Here, by a *completion* we mean a toric variety Z' with properties (i)-(iii) of [5, Prop. III.2.5.4]. That is, in the setting of 1.3.12, we have a toric characteristic space $p_{Z'} \colon \widehat{Z}' \to Z'$ and a neat, closed embedding $\iota \colon X \to Z'$ with $\widehat{X} = \bar{\iota}^{-1}(\widehat{Z}')$ such that $X_{\gamma_0} = \iota^{-1}(Z'_{P(\gamma_0^*)})$ for each $\gamma_0 \in \mathrm{rlv}(\Phi)$. Moreover

$$D_X^i \;=\; \iota^*(D_{Z'}^i) \quad \text{where} \quad D_X^i \;:=\; p_X\left(V(\widehat{X};\, T_i)\right), \quad D_{Z'}^i \;:=\; p_{Z'}\left(V(\widehat{Z}';\, T_i)\right).$$

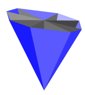
Algorithm 2.3.9 (MDSambtorvar)**.** See [5, Prop. III.2.5.4, Con. III.2.5.7] and Construction 1.3.12. *Input:* an MDS $X = (R, \Phi)$ where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$, $P \colon \mathbb{Z}^r \to N$ and $\gamma = \mathbb{Q}^r_{\geq 0}$. *Option:* `completions` is available if X is projective.

- If `completions` was asked
  - set $\mathcal{F} := \emptyset$,
  - compute $\lambda := \mathrm{SAmple}(X)$ with Algorithm 2.3.15,
  - compute the fan $\mathrm{GKZ}(Q^0)$ in $K^0_{\mathbb{Q}}$. This can be done using Algorithm 3.2.9 with input $\langle 0 \rangle$ and Q^0.
  - For each $\eta \in \Lambda(Q^0)$ such that $\eta^\circ \subseteq \lambda^\circ$ do
    - insert the fan $\Sigma(\eta) \subseteq N_{\mathbb{Q}}$ into $\mathcal{F}$ where
    $$\Sigma(\eta) \;=\; \left\{P(\gamma_0^*);\, \gamma_0 \preceq \gamma \text{ and } \eta^\circ \subseteq Q^0(\gamma_0)^\circ\right\}.$$
  - Return $\mathcal{F}$.
- Compute $C := \mathrm{cov}(\Phi)$ with Algorithm 2.3.6.
- Return the fan $\Sigma \subseteq N_{\mathbb{Q}}$ with maximal cones $\{P(\gamma_0^*);\ \gamma_0 \in C\}$.

Output: the fan $\Sigma \subseteq N_{\mathbb{Q}}$ of the canonical toric ambient variety $Z_\Sigma \supseteq X$. If `completions` was given and X is projective, a list $\mathcal{F}$ of complete fans $\Sigma_1, \ldots, \Sigma_k \subseteq N_{\mathbb{Q}}$ is returned such that $Z_{\Sigma_i} \supseteq X$ is a completion of Z_Σ.

Example 2.3.10. In Example 2.3.3, by Algorithm 2.3.9, the fan $\Sigma \subseteq \mathbb{Q}^5$ of the canonical toric ambient variety Z_Σ of X has eight five-dimensional and six four-dimensional maximal cones. Its rays are generated by the columns of the matrix P of Example 2.1.28. Moreover, the algorithm finds 17 completions since

$$\left|\left\{\eta \in \mathrm{GKZ}\left(Q^0\right);\, \eta^\circ \subseteq \mathrm{SAmple}(X)^\circ\right\}\right| \;=\; 17.$$



The *irrelevant ideal* of an MDS X is the ideal of the closed variety $\overline{X} \setminus \widehat{X}$. Let Σ be a fan in $\mathbb{Q}^n$ and $v_1, \ldots, v_r$ generators for the rays of Σ. As in [5, Prop. III.1.3.4], for each maximal cone $\sigma \in \Sigma$ we define

$$\nu \colon \Sigma \to \{0,1\}^r, \qquad \nu(\sigma)_i \;:=\; \begin{cases} 1, & v_i \notin \sigma, \\ 0, & v_i \in \sigma. \end{cases}$$

Algorithm 2.3.11 (MDSirrel)**.** See [5, Prop. III.1.3.4]. *Input:* an MDS $X = (R, \Phi)$ where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Use Algorithm 2.3.9 to compute the canonical toric ambient variety Z_Σ.
- Compute the ideal $J := \langle T^{\nu(\sigma)};\, \sigma \in \Sigma^{\max}\rangle + \langle G \rangle \subseteq \mathbb{K}[T_1, \ldots, T_r]$.

Output: a list of generators for the vanishing ideal J of $\overline{X} \setminus \widehat{X}$.

Example 2.3.12. Consider the MDS X defined in Example 2.3.3. By Algorithm 2.3.11, we have

$$\begin{aligned}\overline{X} \setminus \widehat{X} \quad = \quad & V\big(\mathbb{K}^8;\ T_3T_5T_7,\ T_4T_6T_7,\ T_5T_6T_7,\ T_1T_2T_8,\ T_1T_3T_8,\ T_2T_4T_8,\\ & T_1T_3T_5,\ T_2T_4T_6,\ T_1T_6T_7T_8,\ T_3T_4T_7T_8,\ T_2T_5T_7T_8,\ T_2T_3T_4T_5,\\ & T_1T_3T_4T_6,\ T_1T_2T_5T_6,\ T_1T_6 + T_2T_5 + T_3T_4 + T_7T_8\big).\end{aligned}$$

We come to algorithms that compute cones of divisor classes, i.e., the cones of effective, semiample, movable divisor classes.

Algorithm 2.3.13 (MDSeff)**.** See [5, Prop. III.3.2.9] and 1.3.16. *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ where Q^0 has the columns $q_1, \dots, q_r$.

- Compute $\omega := \mathrm{cone}(q_1, \dots, q_r)$.

Output: the cone $\omega = \mathrm{Eff}(X)$ in the vector space $K_{\mathbb{Q}}$.

Algorithm 2.3.14 (MDSmov)**.** See [5, Prop. III.3.2.9] and 1.3.16. *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ where Q^0 has the columns $q_1, \dots, q_r$.

- Compute the cone

$$\tau \ := \ \bigcap_{i=1}^{r} \mathrm{cone}\,(q_j;\ j \neq i) \ \subseteq \ K_{\mathbb{Q}}.$$

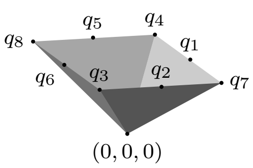
Output: the cone $\tau = \mathrm{Mov}(X)$ in the vector space $K_{\mathbb{Q}}$.

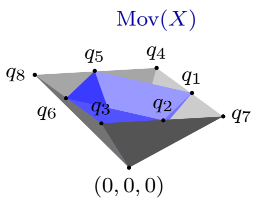
Algorithm 2.3.15 (MDSsample)**.** See [5, Prop. III.3.2.9] and 1.3.16. *Input:* an MDS $X = (R, \Phi)$.

- Compute the cone $\tau := \bigcap_{\tau \in \Phi} \tau \subseteq K_{\mathbb{Q}}$.

Output: the semiample cone $\tau = \mathrm{SAmple}(X)$ in the vector space $K_{\mathbb{Q}}$.

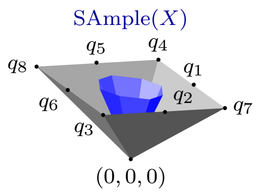
Example 2.3.16. Consider the MDS X of Example 2.3.3. By Algorithms 2.3.13 and 2.3.14, the effective cone of X is $Q^0(\mathbb{Q}^8_{\geq 0})$ and the moving cone of X is $\mathrm{cone}(q_1, q_5, q_6, q_2)$, i.e.,





Moreover, using Algorithm 2.3.15, we see that the semiample cone of X equals the GIT cone $\lambda((0, 0, 2))$, i.e.

$$\begin{aligned}\mathrm{SAmple}(X) \quad = \quad & \mathrm{cone}((2, 1, 3), (1, 1, 2), (-2, -1, 3),\\ & (-1, -1, 2), (-1, 0, 2), (1, 0, 2),\\ & (0, -1, 3), (0, 1, 3)).\end{aligned}$$



We turn to groups associated to an MDS. Set $\gamma_0 \preceq \gamma = \mathbb{Q}^r_{\geq 0}$. Given a canonical basis vector $e_i \in \mathbb{Z}^r$, set $\mathbb{Z}^{\chi_{\gamma_0}(e_i)} := \mathbb{Z}$ if $e_i \in \gamma_0$ and $\{0\}$ otherwise. Define the subgroup

$$H_{\gamma_0} \ := \ \mathrm{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^r \ = \ \bigoplus_{i=1}^{r} \mathbb{Z}^{\chi_{\gamma_0}(e_i)} \ \leq \ \mathbb{Z}^r. \tag{1}$$

Algorithm 2.3.17 (MDSclassgrp)**.** See [5, Thm. III.2.1.4] and 1.3.6. *Input:* an MDS $X = (R, \Phi)$ with degree map $Q\colon \mathbb{Z}^r \to K$.

- Return K.

Output: the AG K representing the class group $\mathrm{Cl}(X)$.

Algorithm 2.3.18 (MDSlocclassgrp)**.** See [5, Prop. III.3.1.5] and 1.3.15. *Input:* an MDS $X = (R, \Phi)$ and Cox coordinates $z \in \mathbb{K}^r$ for a point $x \in X$. Let $Q\colon \mathbb{Z}^r \to K$ be the degree map and $\gamma = \mathbb{Q}^r_{\geq 0}$ the positive orthant.

- Let $\gamma_0 := \mathrm{cone}(e_i;\, z_i \neq 0) \preceq \gamma$. Then $\gamma_0 \in \mathrm{rlv}(\Phi)$ and $x \in X(\gamma_0)$.
- Compute $G_x := K/Q(H_{\gamma_0})$ using (1) and Algorithms 2.1.9 and 2.1.17.

Output: the AG G_x representing the local class group $\mathrm{Cl}(X, x)$.

Example 2.3.19. Consider the MDS $X = (R, \Phi)$ of Example 2.3.3. Then $\gamma_0 := \mathrm{cone}(e_3, e_5, e_7) \preceq \mathbb{Q}^8_{\geq 0}$ is an element of $\mathrm{rlv}(\Phi)$, see Algorithm 2.3.5. By Algorithm 2.3.18, each $x \in X(\gamma_0)$ has the local class group

$$\mathrm{Cl}(X, x) \;=\; \left(\mathbb{Z}^4, L\right) \;\leq\; \mathrm{Cl}(X), \qquad L \;:=\; \mathrm{lin}_{\mathbb{Z}}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\right)$$

given as an AG. In particular, the local class group $\mathrm{Cl}(X, x)$ is isomorphic to the finite group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Algorithm 2.3.20 (MDSpic)**.** See [5, Cor. III.3.1.6] and 1.3.15. *Input:* an MDS $X = (R, \Phi)$ with degree map $Q\colon \mathbb{Z}^r \to K$.

- Define $G := K$ and compute $C := \mathrm{cov}(\Phi)$ using Algorithm 2.3.6.
- For each $\gamma_0 \in C$ do
  - use Algorithms 2.1.13 and 2.1.17 to redefine G as $G \cap Q(H_{\gamma_0})$ where $H_{\gamma_0} \leq \mathbb{Z}^r$ is as in (1).

Output: the Picard group $G = \mathrm{Pic}(X)$ as a subgroup of $\mathrm{Cl}(X)$.

Algorithm 2.3.21 (MDSpicind)**.** *Input:* a $\mathbb{Q}$-factorial MDS $X = (R, \Phi)$ with degree map $Q\colon \mathbb{Z}^r \to K$.

- Use Algorithms 2.3.20 and 2.1.9 to compute the factor group $H := K/\mathrm{Pic}(X)$ and its isomorphism type $H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_s\mathbb{Z}$.
- Return $a_1 \cdots a_s$.

Output: the index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \in \mathbb{Z}_{\geq 1}$.

Example 2.3.22. Continuing Example 2.3.3, we compute the Picard group of X with Algorithm 2.3.20 as the AG

$$\mathrm{Pic}(X) \;=\; \left(\mathrm{lin}_{\mathbb{Z}}\begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 12 & 12 & 24 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},\ \mathrm{lin}_{\mathbb{Z}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}\right) \;\leq\; \mathrm{Cl}(X).$$

Note that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}^3$. The Picard index is $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 6912$ by Algorithm 2.3.21.

We come to singularities and algorithms that determine further properties of X. An MDS X is called *quasismooth* if the open subset $\widehat{X} \subseteq \overline{X}$ is smooth. The following algorithm makes use of the computation of $\mathfrak{a}$-*faces*, i.e., faces $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $V(\mathbb{T}^r_{\gamma_0}; \mathfrak{a}) \neq \emptyset$ where $\mathbb{T}^r_{\gamma_0} \subseteq \mathbb{K}^r$ is the collection of all $z \in \mathbb{K}^r$ such that $z_i \neq 0 \Leftrightarrow e_i \in \gamma_0$. We will treat $\mathfrak{a}$-faces and their computation in Section 1 of Chapter 3.

Algorithm 2.3.23 (MDSisquasismooth)**.** *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$. Write $G = \{f_1, \ldots, f_s\}$ with $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

- Compute the Jacobian matrix, i.e., the $s \times r$ matrix $J := (\partial f_i / \partial T_j)_{i,j}$ over the polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$.
- Let $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ be the ideal generated by G and all $(r-d) \times (r-d)$ minors of J where d is the dimension of R.
- Compute $C := \operatorname{rlv}(\Phi)$ with Algorithm 2.3.5.
- For each $\gamma_0 \in C$ do
  - if, by Algorithm 3.1.2, γ_0 is an $\mathfrak{a}$-face, then return *false*.
- Return *true*.

Output: *true* if $\widehat{X}$ is smooth and *false* otherwise.

Proof. Note that $V(\mathfrak{a}) \subseteq \overline{X}$ equals the singular locus $\overline{X}^{\mathrm{sing}}$. Thus, the algorithm tests whether $\overline{X}^{\mathrm{sing}} \cap \widetilde{X} \neq \emptyset$ with the constructible set

$$\widetilde{X} \ := \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X}(\gamma_0) \ = \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X} \cap \mathbb{T}^r_{\gamma_0} \ \subseteq \ \widehat{X}$$

from [5, Con. III.3.1.1]. We now show that $\widetilde{X} \cap \overline{X}^{\mathrm{sing}}$ is empty if and only if $\widehat{X} \cap \overline{X}^{\mathrm{sing}}$ is empty. The reverse implication is obvious. For the direct one, suppose there were a point $x_1 \in \widehat{X} \cap \overline{X}^{\mathrm{sing}}$. Consider now the good quotient $p\colon \widehat{X} \to X$ by the characteristic quasitorus H. According to Proposition 1.1.6, the fiber $p^{-1}(p(x_1))$ contains a closed orbit

$$H \cdot x_0 \ \subseteq \ \widehat{X} \qquad \text{with} \qquad x_0 \ \in \ p^{-1}(p(x_1)).$$

Observe that $(H \cdot x_0) \cap (\overline{H \cdot x_1})$ is non-empty. Since the images $p(H \cdot x_0) = \{p(x_1)\}$ and $p(\overline{H \cdot x_1})$ are not disjoint, by Proposition 1.1.6, neither are $(H \cdot x_0)$ and $\overline{H \cdot x_1}$. Using the H-invariance of $\widehat{X}^{\mathrm{sing}} \subseteq \overline{X}$, we have $\overline{H \cdot x_1} \subseteq \widehat{X}^{\mathrm{sing}}$. Since $\widetilde{X}$ equals the union of all closed H-orbits of $\widehat{X}$ we conclude

$$\emptyset \ \neq \ (H \cdot x_0) \ \cap \ \left(\overline{H \cdot x_1}\right) \ \subseteq \ \widetilde{X} \cap \overline{X}^{\mathrm{sing}}. \qquad \square$$

Algorithm 2.3.24 (MDSissmooth)**.** See [5, Cor. III.3.1.12]. *Input:* an MDS X.

- Compute the fan Σ of the canonical toric ambient variety $Z_\Sigma \supseteq X$ with Algorithm 2.3.9.
- If Σ is not regular or $\widehat{X}$ is not smooth by Algorithm 2.3.23, then return *false*. Return *true* otherwise.

Output: *true* if X is smooth and *false* otherwise.

Algorithm 2.3.25 (MDSsing)**.** *Input:* an MDS $X = (R, \Phi)$ where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ with $G = \{f_1, \ldots, f_s\} \subseteq \mathbb{K}[T_1, \ldots, T_r]$.

- Compute the $s \times r$ Jacobian matrix $J := (\partial f_i / \partial T_j)_{i,j}$.
- Let $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ be the ideal generated by $f_1, \ldots, f_s$ and all $(r-d) \times (r-d)$ minors of J where $d \in \mathbb{Z}_{\geq 0}$ is the dimension of R.
- Compute $C := \operatorname{rlv}(\Phi)$ with Algorithm 2.3.5.
- Store all $\gamma_0 \in C$ which, by Algorithm 3.1.2, are $\mathfrak{a}$-faces and with $Q(\gamma_0 \cap \mathbb{Z}^r) = K$ in a list $\mathcal{F}$.

Output: the pair $(\mathfrak{a}, \mathcal{F})$. Then $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ is the vanishing ideal of $\overline{X}^{\mathrm{sing}} \subseteq \mathbb{K}^r$ and $\mathcal{F}$ is the list of all relevant $\mathfrak{F}$-faces $\gamma_1, \ldots, \gamma_l \preceq \mathbb{Q}^r_{\geq 0}$ such that $X(\gamma_i)$ is singular.

Example 2.3.26. Consider the MDS $X = (R, \Phi)$ of Example 2.3.3. By Algorithm 2.3.24, X is singular and Algorithm 2.3.25 provides us with $(\mathfrak{a}, \mathcal{F})$ where

$$\overline{X}^{\mathrm{sing}} \ = \ V\left(\mathbb{K}^8;\, \mathfrak{a}\right) \ = \ V\left(\mathbb{K}^8;\, T_1, \ldots, T_8\right) \ = \ \{0\}$$

Moreover, identifying a subset $J \subseteq \{1, \ldots, 8\}$ with the face $\operatorname{cone}(e_i;\, i \in J) \preceq \gamma = \mathbb{Q}^8_{\geq 0}$, the list $\mathcal{F}$ consists of the relevant $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ with singular stratum $X(\gamma_0)$.

It is

$$\begin{aligned}\mathcal{F} \;=\; \big(&\{3,5,7\},\ \{1,3,5,7\},\ \{2,3,4,5,7\},\ \{4,6,7\},\ \{1,3,4,6,7\},\ \{5,6,7\},\ \{3,5,6,7\},\\ &\{1,6,7,8\},\ \{1,3,6,7,8\},\ \{1,4,6,7,8\},\ \{3,4,6,7,8\},\ \{1,3,4,6,7,8\},\ \{3,4,7,8\},\\ &\{1,3,4,7,8\},\ \{2,3,4,7,8\},\ \{2,5,7,8\},\ \{2,3,5,7,8\},\ \{2,4,5,7,8\},\ \{3,4,5,7,8\},\\ &\{2,3,4,5,7,8\},\ \{1,2,8\},\ \{1,3,8\},\ \{2,4,8\},\ \{1,2,4,8\},\ \{2,3,4,5,8\},\ \{2,4,6,8\},\\ &\quad\{1,3,4,6,8\},\ \{1,3,5\},\ \{2,3,4,5\},\ \{2,4,6\},\ \{1,3,4,6\},\ \{1,2,5,6\}\big).\end{aligned}$$

We come to the graph of exceptional curves constructed in 1.4.9. We will use it primarily as an invariant. Note that Algorithm 2.3.48 can be used to compute the self-intersection numbers of its vertices.

Algorithm 2.3.27 (MDSintersgraph)**.** *Input:* a smooth, projective MDS $X = (R, \Phi)$ of dimension two with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Determine the extremal rays $\mathbb{Q}_{\geq 0} \cdot q_{i_1}, \ldots, \mathbb{Q}_{\geq 0} \cdot q_{i_k}$ of the effective cone $\operatorname{cone}(q_1, \ldots, q_r)$ where $q_1, \ldots, q_r$ are the columns of Q^0.
- Initialize $V := \{D_{i_1}, \ldots, D_{i_k}\}$ where $D_{i_j} := V(X; T_{i_j})$ and set $E := \emptyset$.
- Compute $C := \operatorname{rlv}(\Phi)$ with Algorithm 2.3.5.
- For each two distinct $i, j \in \{i_1, \ldots, i_k\}$ do
  - if $\operatorname{cone}(e_k;\ k \notin \{i,j\}) \in C$, then insert (D_i, D_j) into E.

Output: the graph of exceptional curves $G_X = (V, E)$ of X.

Proof. The negative curves $D_{i_1}, \ldots, D_{i_k} \subseteq X$ correspond to the extremal rays of the effective cone $\operatorname{Eff}(X)$, see [5, Ex. V.1] or [51, Prop. 6.7]. By basic properties [73, p. 96] of the good quotient p, we have

$$\begin{aligned} D_i \cap D_j \;\neq\; \emptyset \quad &\Leftrightarrow \quad p\Big(V(\widehat{X};\, T_i,\, T_j)\Big) \neq \emptyset \\ &\Leftrightarrow \quad \operatorname{cone}(e_k;\ k \notin \{i,j\}) \;\in\; \operatorname{rlv}(\Phi). \end{aligned}$$ ☐

Remark 2.3.28. Algorithm 2.3.48 can also be carried out for projective, $\mathbb{Q}$-factorial Mori dream spaces. It then computes the graph of contractible divisors in the sense of Remark 1.4.2.

Remark 2.3.29. Algorithm 2.3.48 can be used in conjunction with Algorithm 2.3.27 to compute the ADE-singularity type of the minimal resolution $X' \to X$ of a Mori dream surface X.

Algorithm 2.3.30 (MDSisfact)**.** Compare [5, Cor. III.1.4.5] and 1.3.15. *Input:* an MDS $X = (R, \Phi)$ with degree map $Q\colon \mathbb{Z}^r \to K$ and orthant $\gamma = \mathbb{Q}^r_{\geq 0}$. *Optional input:* Cox coordinates $z \in \mathbb{K}^r$ for a point $x \in X$.

- If $z \in \mathbb{K}^r$ was given, then set $C := \{\gamma_0\}$ with $\gamma_0 := \operatorname{cone}(e_i;\ z_i \neq 0) \preceq \gamma$. Otherwise, compute $C := \operatorname{rlv}(\Phi)$ using Algorithm 2.3.5.
- For each $\gamma_0 \in C$ do
  - if by Algorithms 2.1.8 and 2.1.17 the image $Q(H_{\gamma_0})$ with H_{γ_0} as in (1) does not generate K as a group, then return *false*.
- Return *true*.

Output: *true* if X is factorial and *false* otherwise. For the case of a given point $x \in X$, *true* is returned if x is factorial and *false* otherwise.

Proof. Let $\gamma_0 \in \operatorname{rlv}(\Phi)$. By [5, Cor. III.1.4.5], a point $x \in X(\gamma_0)$ is factorial if and only if $Q(\operatorname{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^r) = Q(H_{\gamma_0})$ generates K as a group. Use $X = \bigcup X(\gamma_0)$ where the union runs through all relevant $\mathfrak{F}$-faces. ☐

Algorithm 2.3.31 (MDSisQfact)**.** See [5, Cor. 3.1.9] and 1.3.15. *Input:* an MDS $X = (R, \Phi)$.

- If there is $\vartheta \in \Phi$ with $\dim(\vartheta) \neq \dim(K_{\mathbb{Q}})$, then return *false*. Return *true* otherwise.

Output: *true* if X is $\mathbb{Q}$-factorial and *false* otherwise.

Example 2.3.32. By applications of Algorithms 2.3.30 and 2.3.31, the MDS $X = (R, \Phi)$ of Example 2.3.3 is $\mathbb{Q}$-factorial but not factorial.

We defined an MDS to encode varieties arising from bunched rings and, hence, needs in general be neither projective nor complete. The following algorithm uses a result of J. Tevelev [96] to check for completeness.

Algorithm 2.3.33 (MDSiscomplete)**.** *Input:* a projective MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$.

- Compute a fan $\Upsilon \subseteq \mathbb{Q}^n$ with support $|\Upsilon| = \operatorname{trop}(X)$. If $|G| = 1$, Algorithm 2.2.7 can be used.
- Compute a complete fan Ω in $\mathbb{Q}^n$ having Υ as a subfan. If $G = \{g\}$, one can define Ω as the normal fan over the Newton polytope of $p_\star\, g$, compare Algorithm 2.2.13.
- Use Algorithm 2.3.9 to compute the fans Σ and $\overline{\Sigma}$ of the canonical ambient toric variety Z_Σ and a completion $Z_{\overline{\Sigma}}$ of Z_Σ.
- Compute the coarsest common refinements $\Sigma' := \Sigma \sqcap \Omega$ and $\Upsilon' := \overline{\Sigma} \sqcap \Upsilon$.
- For each maximal cone $\tau' \in \Upsilon'$ do
  - if $\tau' \not\subseteq \sigma'$ for each maximal cone $\sigma' \in \Sigma'$, then return *false*.
- Return *true*.

Output: *true* if X is complete and *false* otherwise.

Proof. First, note that in the case of $G = \{g\}$, Υ is the codimension-one skeleton of Ω and, therefore, Υ is a subfan of Ω. By [96, Prop. 2.3], X is complete if and only if the support $|\Sigma|$ contains the tropical variety $\operatorname{trop}(X) = |\Upsilon|$. Note that we have $|\Upsilon'| = \operatorname{trop}(X)$ and $|\Sigma| = |\Sigma'|$. Hence, X is complete if and only if in the following diagram of fans the dashed arrow is an inclusion.

$$\begin{tikzcd}
\Sigma & \subseteq & \overline{\Sigma} \\
\Sigma \sqcap \Omega \arrow[u] & \subseteq & \overline{\Sigma} \sqcap \Omega \arrow[u] \\
 & & \overline{\Sigma} \sqcap \Upsilon \arrow[u, phantom, sloped, "\subseteq"] \arrow[ull, dashed, bend left]
\end{tikzcd}$$

Since Υ is a subfan of Ω, each maximal cone τ' of $\Upsilon' = \Upsilon \sqcap \overline{\Sigma} \sqcap \Omega$ either is contained in Σ' or $\tau^\circ \cap |\Sigma'| = \emptyset$. This completes the proof. $\square$

Remark 2.3.34. In Algorithm 2.3.33, the fan Υ with $|\Upsilon| = \operatorname{trop}(X)$ can be computed using [63]. See [88, 41] for how to find Ω.

Algorithm 2.3.35 (MDSisquasiproj)**.** Compare [5, Cor. 1.4.5]. *Input:* an MDS $X = (R, \Phi)$. Write $\Phi = \{\vartheta_1, \dots, \vartheta_s\}$.

- Define $\tau := \vartheta_1$.
- For each $i = 2, 3, \dots, s$ do
  - if $\tau^\circ \cap \vartheta_i^\circ = \emptyset$, then return *false*. Otherwise, redefine τ as $\tau \cap \vartheta_i$.
- Return *true*.

Output: *true* if X is quasiprojective and *false* otherwise.

Proof. By [5, Cor. 1.4.5], X is quasiprojective if and only if $\vartheta_1^\circ \cap \ldots \cap \vartheta_s^\circ$ is non-empty. Clearly, if $\vartheta_i^\circ \cap \vartheta_j^\circ = \emptyset$ for some i, j, then X is not quasiprojective. Otherwise, by [87], the non-empty intersection $\vartheta_1^\circ \cap \ldots \cap \vartheta_s^\circ$ equals $(\vartheta_1 \cap \ldots \cap \vartheta_s)^\circ$. ☐

Algorithm 2.3.36 (MDSisproj)**.** *Input:* an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$. Assume that Q^0 has no zero-columns.

- Return *true* if, by Algorithms 2.3.35 and 2.3.13, X is quasiprojective and $\mathrm{Eff}(X)$ is pointed. Return *false* otherwise.

Output: *true* if X is projective and *false* otherwise.

Lemma 2.3.37. *Consider a surjective $k \times r$ matrix Q without zero-columns and the positive orthant $\gamma := \mathbb{Q}^r_{\geq 0}$. Then $Q(\gamma)$ is pointed if and only if* $\ker(Q) \cap \gamma = \{0\}$*.*

Proof. If $Q(\gamma_0)$ is pointed, clearly $Q(x) = 0$ with $x \in \gamma$ implies $x = 0$. On the other hand, if $Q(\gamma)$ contains $\mathrm{lin}_{\mathbb{Q}}(w)$ with a non-zero $w \in Q(\gamma)$, then there are $x, y \in \gamma \backslash \{0\}$ such that $Q(x) = w$ and $Q(y) = -w$. Thus, $0 \neq x + y \in \gamma \cap \ker(Q)$. ☐

Proof of Algorithm 2.3.36. Since X is quasiprojective, there is $w \in \mathrm{Mov}(X)^\circ$ such that $X = \overline{X}^{\mathrm{ss}}(w) /\!\!/ H_X$, see [5]. By [5, Prop. III.1.2.2] or Proposition 1.1.7, X is projective over $\overline{X} /\!\!/ H_X = \mathrm{Spec}\, R_0$. Set $\gamma := \mathbb{Q}^r_{\geq 0}$. We now prove that $Q^0(\gamma)$ is pointed if and only if $R_0 = \mathbb{K}$. Since X is an MDS, the classes of $T_1, \ldots, T_r$ are pairwise non-associated K-prime generators for R. By [52, Rem. 1.25], R_0 is generated by the classes of products T^ν where $\nu \in \mathbb{Z}^r_{\geq 0}$ runs through the elements of a Hilbert basis U for $\ker(Q^0) \cap \gamma$. By Lemma 2.3.37, $U = \{0\}$ is equivalent to $Q^0(\gamma)$ being pointed. This shows that X is projective if and only if $Q^0(\gamma)$ is pointed. ☐

Example 2.3.38. By Algorithm 2.3.36, the MDS X of Example 2.3.3 is projective.

As in Section 1 of Chapter 3, given $f \in \mathbb{K}[T_1, \ldots, T_r]$ and $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$, let $f^{\gamma_0} \in \mathbb{K}[T_1, \ldots, T_r]$ be the polynomial obtained from f by substitution of $T_i = 0$ for all $e_i \notin \gamma_0$. Define the $\mathbb{T}^r$-orbit $\mathbb{T}^r_{\gamma_0}$ as the set of elements $z \in \mathbb{K}^r$ such that $z_i = 0 \Leftrightarrow e_i \notin \gamma_0$.

Algorithm 2.3.39 (MDSstrat)**.** Compare Construction 1.3.14. *Input:* an MDS $X = (R, \Phi)$ where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute the collection $G_{\gamma_0} := \{f_1^{\gamma_0}, \ldots, f_s^{\gamma_0}\}$.
- Use Algorithm 2.2.13 to compute all $h_i := p_\star f_i^{\gamma_0} \in \mathbb{K}[S_1, \ldots, S_n]$ where $p \colon \mathbb{T}^r \to \mathbb{T}^n$ is the homomorphism of tori corresponding to P.

Output: $h_1, \ldots, h_s \in \mathbb{K}[S_1, \ldots, S_n]$. Then the stratum $X(\gamma_0)$ is given by the zero set $V(h_1, \ldots, h_s) \subseteq \mathbb{T}^n$ where the h_i are considered as elements of $\mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}]$.

Proof. Using standard properties of good quotients [73, p. 96], by [5, Con. III.3.1.1], we have

$$p\left(\overline{X}(\gamma_0)\right) \;=\; p\left(\overline{X} \cap \mathbb{T}^r_{\gamma_0}\right) \;=\; p\left(V\left(\mathbb{T}^r_{\gamma_0};\, f_1^{\gamma_0}, \ldots, f_s^{\gamma_0}\right)\right) \;=\; V\left(\mathbb{T}^n;\, h_1, \ldots, h_s\right). \quad ☐$$

Example 2.3.40. Let X be as in Example 2.3.3. For the following relevant $\mathfrak{F}$-face $\gamma_0 \preceq \gamma = \mathbb{Q}^8_{\geq 0}$ Algorithm 2.3.39 delivers

$$X(\gamma_0) \;=\; V\left(\mathbb{T}^5;\; T_1T_5 + T_3T_4\right), \qquad \gamma_0 := \mathrm{cone}(e_i;\, i \in \{1,3,4,6,7\}) \;\preceq\; \gamma.$$

We turn to algorithms on a *complete intersection* MDS $X = (R, \Phi)$, i.e., R is a complete intersection ring. Recall that this means that the kernel of $\mathbb{K}[T_1, \ldots, T_r] \to R$ with $T_i \mapsto f_i$ is generated by $r - \dim(R)$ polynomials.

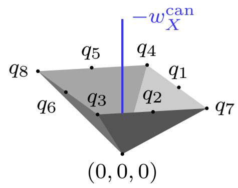
Algorithm 2.3.41 (MDSantican)**.** See [5, Thm. III.3.3.2] and 1.3.17. *Input:* a complete intersection MDS $X = (R, \Phi)$. Write $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ with $G = \{g_1, \ldots, g_d\}$. Let the degree map be given by the AGH $Q = (\mathbb{Z}^r, K, A)$ with an AG $K = (U, L)$.

- Define $w := \sum_{i=1}^r A_{*i} - \sum_{j=1}^d A \cdot \nu_j \in U + L$ where T^{ν_j} is a non-vanishing monomial in g_j.

Output: the vector $w \in U + L$. It represents the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$.

Example 2.3.42. Consider the complete intersection MDS X of Example 2.3.3. Using Algorithm 2.3.41, we obtain $-w_X^{\mathrm{can}} = (0, 0, 6, \bar{1}) \in K$ since

$$\sum_{i=1}^r A_{*i} - \sum_{j=1}^d A \cdot \nu_j = (0,0,8,4) - (0,0,2,1) = (0,0,6,3).$$



Algorithm 2.3.43 (MDSisQgorenstein)**.** See [5, Cor. III.3.3.3] and 1.3.17. *Input:* a complete intersection MDS $X = (R, \Phi)$.

- Compute the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$; see Algorithm 2.3.41.
- Determine the cone $\tau := \bigcap_{\vartheta \in \Phi} \mathrm{lin}_{\mathbb{Q}}(\vartheta)$ in $K_{\mathbb{Q}}$.
- Let $\pi\colon K \to K^0$ be the canonical projection. Return *true* if $\pi(-w_X^{\mathrm{can}}) \in \tau$ and *false* otherwise.

Output: *true* if X is $\mathbb{Q}$-Gorenstein and *false* otherwise.

Algorithm 2.3.44 (MDSisgorenstein)**.** See [5, Cor. III.3.3.3] and 1.3.17. *Input:* a complete intersection MDS X.

- Compute the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$; see Algorithm 2.3.41.
- Use Algorithm 2.3.20 to compute $G := \mathrm{Pic}(X) \leq K$.
- Return *true* if $-w_X^{\mathrm{can}} \in G$ and *false* otherwise.

Output: *true* if X is Gorenstein and *false* otherwise.

For a complete intersection $\mathbb{Q}$-Gorenstein MDS X, the *Gorenstein index* is the smallest integer $n \in \mathbb{Z}_{>0}$ such that $n \cdot (-w_X^{\mathrm{can}}) \in \mathrm{Pic}(X)$ where $-w_X^{\mathrm{can}}$ is the class of the anticanonical divisor of X.

Algorithm 2.3.45 (MDSgorensteinind)**.** *Input:* a complete intersection MDS X that is $\mathbb{Q}$-Gorenstein.

- Compute the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$; see Algorithm 2.3.41.
- Determine the Picard group $G := \mathrm{Pic}(X) \leq K$ with Algorithm 2.3.20.
- For $n = 1, 2, \ldots$ do
  - test with Algorithm 2.1.6, whether $n \cdot (-w_X^{\mathrm{can}}) \in G$. Return n if this is the case.

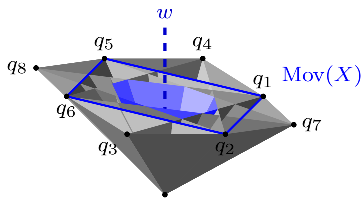
Output: the Gorenstein index $n \in \mathbb{Z}_{>0}$ of X.

Algorithm 2.3.46 (MDSisfano)**.** See [5, Cor. III.3.3.3] and 1.3.17. *Input:* a complete intersection MDS $X = (R, \Phi)$.

- Compute the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$; see Algorithm 2.3.41.
- For each $\vartheta \in \Phi$ do
  - return *false* if $-w_X^{\mathrm{can}} \notin \vartheta^\circ$.
- Return *true*.

Output: *true* if X is Fano and *false* otherwise.

Example 2.3.47. By Algorithms 2.3.46, 2.3.43 and 2.3.45, the MDS $X = (R, \Phi)$ with $\Phi = \Phi(w)$ of Example 2.3.3 is a Fano variety that is $\mathbb{Q}$-Gorenstein with Gorenstein index 4. Note that, by Examples 2.3.16 and 2.3.42, we would have lost the Fano property had we chosen $w \in \mathbb{Q}^3$ in a different GIT-cone within $\mathrm{Mov}(X)$ (the blue region).



For the surface case, we introduced intersection numbers in Section 4 of Chapter 1. We shortly recall from [5, Con. II.1.2.8] the construction for an n-dimensional complete toric variety Z with a lattice fan (Σ, N). Assume Σ is simplicial. Consider pairwise different, invariant prime divisors $D_1, \ldots, D_n$ on Z that correspond to rays $\mathbb{Q}_{\geq 0} \cdot v_1, \ldots, \mathbb{Q}_{\geq 0} \cdot v_n \in \Sigma$ with primitive vectors $v_i \in N$. Their intersection number is

$$D_1 \cdots D_n \;:=\; v_1 \cdots v_n \;:=\; \begin{cases} [N \cap \mathrm{lin}_{\mathbb{Q}}(\sigma) : \mathrm{lin}_{\mathbb{Z}}(v_1, \ldots, v_n)]^{-1}, & \sigma \in \Sigma, \\ 0, & \sigma \notin \Sigma \end{cases}$$

with $\sigma := \mathrm{cone}(v_1, \ldots, v_n)$. If $X = X(R, \mathfrak{F}, \Phi)$ is a projective Mori dream space with complete intersection Cox ring R and canonical toric ambient variety Z, it inherits intersection theory. Given invariant prime divisors $D_X^1, \ldots, D_X^n$ on X with $D_X^i = D_i \cap X$, by [5, Con. III.3.3.4], their intersection number is the toric intersection number

$$D_X^1 \cdots D_X^n \;=\; v_1 \cdots v_n \cdot u_1 \cdots u_d$$

where $u_1, \ldots, u_d$ are the degrees of the generators of the kernel of the map $T_i \mapsto f_i$ with $\mathfrak{F} = (f_1, \ldots, f_r)$; see [5, Con. III.3.3.4] for details. We state the next algorithm for the case of one equation. It is also able to produce self intersection numbers.

Algorithm 2.3.48 (MDSintersno)**.** Compare [5, Con. III.3.3.4]. *Input:* a quasiprojective MDS $X = (R, \Phi(w_0))$ and elements w, $w' \in K$ where $R = (\{g\}, Q, Q^0, P, F_{\mathfrak{F}})$. Let $q_1, \ldots, q_r \in K$ be the degrees of the variables $T_1, \ldots, T_r$ of R.

- Compute in $K_{\mathbb{Q}}$ the full-dimensional GIT-cone $\lambda(w_0) = \bigcap_{\vartheta \in \Phi} \vartheta$.
- Choose a random $w_0' \in \lambda(w_0)^\circ$ until the cone $\eta \in \mathrm{GKZ}(Q^0)$ with $w_0' \in \eta^\circ$ is of full dimension. See Algorithm 3.2.8 for how to compute η.
- Let $N = (n_{ij})_{i,j}$ be the $r \times r$ zero-matrix.
- For each two distinct $1 \leq i, j \leq r$ do
  - choose a non-zero monomial T^ν in g such that $\nu_i = \nu_j = 0$,
  - redefine $n_{ij} := \sum_{k=1}^r \nu_k (q_i \cdot q_j \cdot q_k)$ where we use Algorithm 2.1.9 and the $\mathbb{Z}$-module representation as in Algorithm 2.1.4 to calculate

    $$q_i \cdot q_j \cdot q_k \;=\; \begin{cases} [K : K_{ijk}]^{-1}, & \eta \subseteq \mathrm{cone}\left(q_l;\, l \notin \{i,j,k\}\right), \\ 0, & \text{else} \end{cases}$$

    with the subgroup $K_{i,j,k} := \langle q_l;\, l \notin \{i,j,k\} \rangle$ of K for $1 \leq k \leq r$.
- For each $1 \leq i \leq r$ do
  - compute a point $a' \in Q_i^{-1}(q_i) \subseteq \mathbb{Q}^{r-1}$ where Q_i is the matrix obtained from Q by removing the i-th column.

  - Adding another 0-entry, we have $a := (a'_1, \dots, a'_{i-1}, 0, a'_i, \dots, a'_{r-1})$ in $\mathbb{Q}^r$. Redefine n_{ii} as the sum $\sum_{j=1}^{r} a_j n_{ij}$.
- Compute α_i and $\beta_j \in \mathbb{Q}_{\geq 0}$ such that $w = \sum_i \alpha_i q_i$ and $w' = \sum_j \beta_j q_j$.
- Return $\sum_{i,j} \alpha_i \beta_j n_{ij} \in \mathbb{Q}$.

Output: the intersection number $D \cdot D' \in \mathbb{Q}$ of divisors D, D' on X with $[D] = w$ and $[D'] = w' \in K$.

Proof. This implements [5, Con. III.3.3.4]. Note that in line two, we have $\eta^\circ \subseteq \lambda(w_0)^\circ$ and in the first line of the first loop, the monomial T^ν exists since, otherwise, the codimension of $V(X; T_i, T_j)$ in X would be one. By construction and bilinearity of the intersection form, after the second loop, $n_{ij} = q_i \cdot q_j \cdot \deg(g)$ for all i, j. Then

$$w \cdot w' \cdot \deg(g) \quad = \quad \left(\sum_i \alpha_i q_i\right) \cdot \left(\sum_j \beta_j q_j\right) \cdot \deg(g) \quad = \quad \sum_{i,j} \alpha_i \beta_j \, n_{ij}. \qquad \square$$

4. Complexity-one T-varieties

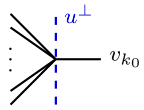
This section describes algorithms for the special class of Mori dream spaces of complexity-one T-varieties; see Section 5 of Chapter 1 for the background.

The algorithms concerning automorphisms have been developed by I. Arzhantsev, J. Hausen, E. Huggenberger and A. Liendo in [6]. The anticanonical complex and the related algorithms have been developed by B. Bechtold, J. Hausen and E. Huggenberger in [17]. See [5, 61] for the resolution of singularities for complexity-one T-varieties. Our Algorithm 2.4.8 works in a slightly more general setting. Here is an overview:

- *Automorphisms:* vertical and horizontal Demazure P-roots (Algorithms 2.4.1 and 2.4.2), roots of $\mathrm{Aut}(X)^0$ (Algorithm 2.4.6).
- *Singularities:* resolution of singularities (Algorithm 2.4.8).
- *Anticanonical complex:* anticanonical polytope (Algorithm 2.4.13), anticanonical complex (Algorithm 2.4.14), test for being (ϵ-log-) terminal (Algorithms 2.4.15 and 2.4.16).

Let $X = X(P, A, \Phi)$ be a complexity-one T-variety with matrices P, A and $\mathfrak{F}$-bunch Φ as in Construction 1.5.12. In order to compute the roots of the unit-component $\mathrm{Aut}(X)^0$, we first show how to calculate Demazure P-roots. Let $v_{ij}, v_k \in \mathbb{Z}^{r+s}$ be the columns of P and denote by M the dual lattice of $\mathbb{Z}^{r+s}$. Recall from [6, Def. 5.2] that a *vertical Demazure P-root* is a pair $(u, k_0) \in M \times \{1, \dots, m\}$ such that

$$\langle u, v_{ij} \rangle \;\geq\; 0 \quad \text{for all } i, j, \qquad \langle u, v_{k_0} \rangle \;=\; -1, \qquad \langle u, v_k \rangle \;\geq\; 0 \quad \text{for all } k \neq k_0.$$



A *horizontal Demazure P-root* is a tuple (u, i_0, i_1, C) in the following sense. We have $u \in M$, distinct indices $i_0, i_1 \in \{0, \dots, r\}$ and $C = (c_i)_i \in \prod_{i=0}^{r} \{1, \dots, n_i\}$ is such that $l_{ic_i} = 1$ for all $i \notin \{i_0, i_1\}$. Moreover, the scalar product $\langle u, v_k \rangle$ is at least zero for all k and

$$\langle u, v_{ic_i} \rangle \;=\; \begin{cases} 0, & i \notin \{i_0, i_1\}, \\ -1, & i = i_1, \end{cases} \qquad \langle u, v_{ij} \rangle \;\geq\; \begin{cases} l_{ij}, & i \notin \{i_0, i_1\}, \quad j \neq c_i, \\ 0, & i \in \{i_0, i_1\}, \quad j \neq c_i, \\ 0, & i = i_0, \qquad\quad\; j = c_i. \end{cases}$$

Given a horizontal or vertical Demazure P-root κ of the form $\kappa = (u, k_0)$ or $\kappa = (u, i_0, i_1, C)$, the last s coordinates of $u \in M$ give the *P-root* $\alpha_\kappa \in \mathbb{Z}^s$.

Algorithm 2.4.1 (MDSvdemazure)**.** See [6, Remark. 5.4]. *Input:* an MDS $X = (R, \Phi)$ where $X = X(P, A, \Phi)$ is a complexity-one variety with a $r \times (n+m)$ matrix P and a $2 \times (r+1)$ matrix A as in Construction 1.5.11.

- In the sense of 1.5.11, read out the blocks $l_i \in \mathbb{Z}_{\geq 0}^{n_i}$ of P where $0 \leq i \leq r$. Let $v_1, \ldots, v_{n+m}$ be the columns of P.
- Initialize $\mathcal{V} := \emptyset$.
- For each $k_0 \in \{1, \ldots, m\}$ do
  - define $\zeta \in \mathbb{Z}^{n+m}$ by setting $\zeta_{n+k_0} := -1$ and $\zeta_i := 0$ otherwise.
  - Compute in $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ the affine subspace and polytope
  $$\begin{aligned} \eta(k_0) &:= \{u \in M_{\mathbb{Q}};\ \langle u, v_{k_0} \rangle = -1\}, \\ B(k_0) &:= \left\{u \in \eta(k_0);\ P^* \cdot u \geq \zeta\right\}. \end{aligned}$$
  - Use Algorithm 2.2.2 to compute the set $U := B(k_0) \cap \mathbb{Z}^r$ of lattice points. Redefine $\mathcal{V}$ as the set $\mathcal{V} \cup \{(u, k_0);\ u \in U\}$.

Output: the set $\mathcal{V}$ of all vertical Demazure P-roots $(u, k_0) \in M \times \{1, \ldots, m\}$.

Algorithm 2.4.2 (MDShdemazure)**.** See [6, Remark. 5.4]. *Input:* an MDS $X = (R, \Phi)$ where $X = X(P, A, \Phi)$ is a complexity-one variety with a $r \times (n+m)$ matrix P and a $2 \times (r+1)$ matrix A as in Construction 1.5.11.

- In the sense of 1.5.11, read out the blocks $l_i \in \mathbb{Z}_{\geq 0}^{n_i}$ of P where $0 \leq i \leq r$. Write $n_i' := n_0 + \ldots + n_{i-1}$ and let $v_1, \ldots, v_{n+m}$ be the columns of P.
- Initialize $\mathcal{H} := \emptyset$.
- For each two distinct $0 \leq i_0, i_1 \leq r$ do
  - for each $C = (c_0, \ldots, c_r) \in \prod_{i=0}^{r} \{1, \ldots, n_i\}$ such that we have $l_{ic_i} = 1$ for all $i \in \{0, \ldots, r\} \setminus \{i_0, i_1\}$ do
    - let $\zeta \in \mathbb{Z}^{n+m}$ with $\zeta_{n+i} := 0$ for all $1 \leq i \leq m$. For its remaining components ζ_k we write $k = n_i' + j$ with $0 \leq i \leq r$ and $1 \leq j \leq n_i$ and define
    $$\zeta_k := \begin{cases} -1, & i = i_1 \quad \text{and } j = c_{i_1}, \\ l_{ij}, & i \notin \{i_0, i_1\} \quad \text{and } j \neq c_i, \\ 0, & \text{otherwise.} \end{cases}$$
    - Compute in $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ the affine subspace $\eta(i_0, i_1, C)$ which is given by
    $$\left\{u \in M_{\mathbb{Q}};\ \langle u, v_{n_i'+c_i} \rangle = 0 \text{ for } i \notin \{i_0, i_1\},\ \langle u, v_{n_{i_1}+c_{i_1}} \rangle = -1\right\}.$$
    - Compute the polytope $B(i_0, i_1, C) \subseteq M_{\mathbb{Q}}$ given by
    $$B(i_0, i_1, C) = \left\{u \in \eta(i_0, i_1, C);\ P^* \cdot u \geq \zeta\right\}.$$
    - Use Algorithm 2.2.2 to compute the set $U := B(i_0, i_1, C) \cap \mathbb{Z}^r$ of lattice points. Redefine $\mathcal{H}$ as $\mathcal{H} \cup \{(u, i_0, i_1, C);\ u \in U\}$.

Output: the set $\mathcal{H}$ of all horizontal Demazure P-roots (u, i_0, i_1, C).

Remark 2.4.3. Algorithms 2.4.2 and 2.4.1 depend only on the ring $R(P, A)$.

Remark 2.4.4. Algorithm 2.4.2 can be used to test a normal complete complexity-one T-variety for being *almost homogeneous*, i.e., its automorphism group acts with an open orbit. By [6, Thm. 6.1], this is equivalent to the existence of a horizontal Demazure P-root.

Example 2.4.5. We computationally verify [6, Ex. 5.3]. Consider the complexity-one variety $X = X(P, A, \Phi)$ where Φ is any $\mathfrak{F}$-bunch and

$$P = \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

are as in Construction 1.5.2. Since $m = 0$, Algorithm 2.4.1 returns the empty set. However, Algorithm 2.4.2 finds the single horizontal Demazure P-root

$$(u, 1, 2, C), \qquad u = (-1, -2, 3) \in M_{\mathbb{Q}} = \mathbb{Q}^3, \qquad C = (1, 1, 1).$$

Recall from [62, 6] that for a connected, semisimple linear algebraic group G there is a *root system* $\Phi_G \subseteq \mathbb{X}_{\mathbb{R}}(T)$ with respect to a given maximal torus $T \subseteq G$. Note that G is determined uniquely up to coverings by its root system. The following algorithm describes the roots of the unit component $\mathrm{Aut}(X)^0$ for a normal, complete complexity-one MDS $X = X(P, A, \Phi)$. By [6], for the semisimple part of $\mathrm{Aut}(X)$, only (sums of) the following root systems may occur

$$\begin{aligned} A_n &:= \{e_i - e_j;\ 1 \le i, j \le n+1,\ i \ne j\} \subseteq \mathbb{R}^{n+1}, \\ B_2 &:= \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\} \subseteq \mathbb{R}^2. \end{aligned}$$

Algorithm 2.4.6 (MDSautroots)**.** See [6, Thm. 5.5]. *Input:* an MDS $X = (R, \Phi)$, where $X = X(P, A, \Phi)$ is a complexity-one variety with P, A as in Construction 1.5.11 of size $r \times (n+m)$ and $2 \times (r+1)$.

- Compute the sets $\mathcal{V} \subseteq \mathbb{Z}^{r+s}$ and $\mathcal{H} \subseteq \mathbb{Z}^{r+s}$ of vertical and horizontal Demazure P-roots with Algorithms 2.4.1 and 2.4.2.
- Determine the set $A := \{\pi(u);\ u \in \mathcal{V} \cup \mathcal{H}\}$ where $\pi\colon \mathbb{Z}^{r+s} \to \mathbb{Z}^s$ is the projection onto the $\mathbb{Z}^s$-part.

Output: the set A of roots of the unit component $\mathrm{Aut}(X)^0$.

Example 2.4.7. Consider the complexity-one T-variety X arising from the data P, A, $\Phi = \Phi(1)$ as in Construction 1.5.12 where

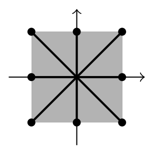
$$P = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Then X is three-dimensional and its Cox ring and degree map $Q\colon \mathbb{Z}^5 \to K := \mathbb{Z}^5/\mathrm{Im}(P^*) = \mathbb{Z}$ are

$$R = \mathbb{K}[T_1, \ldots, T_5]/\langle T_4T_5 + T_3^2 + T_1T_2\rangle, \qquad Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We compute the roots of $\mathrm{Aut}(X)^0$ using Algorithm 2.4.6. As predicted in [6, Thm. 7.2], we obtain the root system

$$\begin{aligned} B_2 = \ &\{(1, -1), (1, 1), (-1, -1), (-1, 1), \\ &(0, -1), (0, 1), (1, 0), (-1, 0)\} \subseteq \mathbb{Z}^2. \end{aligned}$$



We turn to resolutions of singularities; compare Section 4 of Chapter 1. The following algorithm has been developed for complexity-one T-varieties, see the book by I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface [5, Thm. III.4.4.9] and E. Huggenberger's thesis [61, Ch. 3]. For more general Mori dream spaces our algorithm computes a candidate for a resolution and tries to verify it. The algorithm uses the *weak tropical resolution* defined in [10].

Algorithm 2.4.8 (MDSresolvesing)**.** Compare [5, Thm. III.4.4.9] and [61, Ch. 3]. *Input:* a projective, $\mathbb{Q}$-factorial MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$. Write $G = \{f_1, \ldots, f_s\}$ and assume P is of size $n \times r$. *Options:* `verify`; `minimal` if X is a surface.

- Compute the fan $\overline{\Sigma}$ of a completion $Z_{\overline{\Sigma}}$ of the canonical toric ambient variety with Algorithm 2.3.9 and a fan $\Upsilon \subseteq \mathbb{Q}^n$ with support $\mathrm{trop}(X)$. If $s = 1$, Algorithm 2.2.7 can be used.

- Compute the coarsest common refinement $\Sigma' := \overline{\Sigma} \sqcap \Upsilon$ and subdivide its singular cones until the resulting fan $(\Sigma')^{\mathrm{reg}}$ is regular. Write primitive generators for its rays into the columns of a $n \times r'$ matrix $P' = [P, B]$.
- Let $p\colon \mathbb{T}^r \to \mathbb{T}^n$ and $p'\colon \mathbb{T}^{r'} \to \mathbb{T}^n$ be the homomorphisms of tori corresponding to P and P'. Use Algorithms 2.2.13 and 2.2.12 to compute the ideal $I \subseteq \mathbb{K}[T_1, \ldots, T_{r'}]$ generated by $(p')^\star p_\star f_1, \ldots, (p')^\star p_\star f_s$.
- Compute a set G' of generators for the ideal $I : (T_1 \cdots T_{r'})^\infty$.
- Use Algorithms 2.1.24 and 2.1.26 to compute the AGH $Q'\colon \mathbb{Z}^{r'} \to K'$ where $K' := \mathbb{Z}^{r'}/\mathrm{lin}_{\mathbb{Z}}(P')^*$ and the matrix $(Q')^0\colon \mathbb{Z}^{r'} \to (K')^0$. Define

$$\Phi' \;:=\; \left\{ (Q')^0\,(\delta_0^*);\ \delta_0 \preceq \delta \text{ and } P'(\delta_0) \in \Sigma' \right\}, \qquad \delta \;:=\; \mathbb{Q}^{r'}_{\geq 0}.$$

- Create the MDS $X' = (R', \Phi')$ with $R' = (G', Q', (Q^0)', P', F')$ where F' is the set of all $\delta_0 \preceq \delta$ such that $\delta_0^\circ \cap \tau \neq \emptyset$ for a maximal cone $\tau \in P^{-1}(\Upsilon)$.
- If `verify` was requested, then
  - use Algorithm 2.2.10 to check whether all variables $T_1, \ldots, T_{r'}$ define K'-primes in R'.
  - Check if $\dim(\langle G' \rangle) - \dim(\langle G' \rangle + \langle T_i, T_j \rangle) \geq 2$ for all $i \neq j$.
  - If $s = 1$, then check if $\mathrm{codim}_{\overline{X}'}((\overline{X}')^{\mathrm{sing}}) \geq 2$ with Algorithm 2.3.25. Otherwise, check if R' is normal.
  - Check whether X' is smooth with Algorithm 2.3.24.
- If `minimal` was requested and X is a surface, then redefine X' as the result of Algorithm 2.4.9 with input r and X'.

Output: $X' = (R', \Phi')$. If all `verify`-checks were successful or if X is a complexity-one variety as in Construction 1.5.11 then X' is a smooth MDS and $X' \to X$ a resolution of singularities. The resolution is minimal if `minimal` was requested and X is a surface.

Algorithm 2.4.9 (minimize)**.** *Input:* $r \in \mathbb{Z}_{\geq 0}$ and a two-dimensional smooth projective MDS $X' = (R', \Phi')$ that arises from a two-dimensional MDS X as in Algorithm 2.4.8.

- If R' is a complete intersection, then
  - use Algorithm 2.3.48 to determine the indices $i_1, \ldots, i_k$ such that $V(X'; T_{i_j})$ is a (-1)-curve.
- If R' is not a complete intersection, then
  - Compute $\vartheta := \mathrm{cone}(q_1, \ldots, q_{r'})$ where $q_1, \ldots, q_{r'}$ are the columns of the matrix $(Q^0)'$ representing the free part of the grading of R'.
  - Let $i_1, \ldots, i_k$ be such that the q_{i_j} are exactly the extremal vectors of ϑ with $i_j > r$ in the sense of Remark 1.4.2.
- Return X' if $k = 0$.
- For each $i \in \{i_1, \ldots, i_k\}$ do
  - compute the contraction $X' \to X_i$ of $V(X'; T_i)$ by an application of Algorithm 2.4.10.
  - If, by Algorithm 2.3.24, X_i is smooth, then return the result of the recursive call to Algorithm 2.4.9 with input r and X_i.

Output: a smooth MDS X_1 such that $X_1 \to X$ is a minimal resolution.

Proof. By Theorem 1.4.1, exactly the (-1)-curves can be contracted smoothly. Since X' is a surface the contractible divisors correspond to the extremal rays of $\vartheta = \mathrm{Eff}(X')$, see Remark 1.4.2. As the minimal resolution of a surface is unique, it suffices to consider any smooth contraction $X' \to X_{i_j}$.

$$\begin{tikzcd}
 & & X' \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
X_{j_1} \arrow[drr] & X_{j_2} \arrow[dr] & \cdots & X_{j_m-1} \arrow[dl] & X_{j_m} \arrow[dll] \\
 & & X & &
\end{tikzcd}$$

By choice of $i_j > r$, $X_{i_j} \to X$ is a resolution. Hence, the algorithm computes the minimal resolution in finitely many steps. □

Algorithm 2.4.10 (MDScontract)**.** *Input:* a projective, $\mathbb{Q}$-factorial, two-dimensional MDS $X = (R, \Phi)$ as in Algorithm 2.4.9 and a contractible curve $V(X; T_k)$. Write $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ with $G = \{f_1, \dots, f_s\}$.

- Let P' be the matrix obtained by deleting the k-th column of P. Compute $Q' \colon \mathbb{Z}^r \to \mathbb{Z}^r/\mathrm{Im}((P')^*)$ and the matrix $(Q^0)'$ using Algorithms 2.1.24 and 2.1.26.
- Set $G' := \{f_1', \dots, f_s'\}$ with the image f_i' of f_i under the map
$$\mathbb{K}[T_1, \dots, T_r] \ \to \ \mathbb{K}[T_1, \dots, T_{r-1}], \qquad T_i \ \mapsto \ \begin{cases} T_{i-1}, & i > k, \\ 1, & i = k, \\ T_i, & i < k. \end{cases}$$
- Compute the list $(F_{\mathfrak{F}})'$ of $\langle G' \rangle$-faces with Algorithm 3.1.2 and define $R' := (G', Q', (Q^0)', P', (F_{\mathfrak{F}})')$.
- Choose $w \in \mathrm{Mov}(R')^\circ$ and define the MDS $X' = (R', \Phi'(w))$, see Algorithm 2.3.14 and Example 1.3.4.

Output: the MDS X'. Then $X \to X'$ is the contraction of the divisor $V(X; T_k)$.

Proof. We will prove a similar statement in Algorithm 4.3.7 of Chapter 4. □

Proof of Algorithm 2.4.8. For complexity-one varieties, this is [5, Thm. III.4.4.9], see also [61, Ch. 3]. Hence, we only need to show that if the `verify` option was given and all tests succeed, then X' is a smooth MDS. In particular, $X' \to X$ then is an equivariant desingularization of X. Consider $\overline{X}' = V(G') \subseteq \mathbb{K}^{r'}$ and $H' = \mathrm{Spec}\, \mathbb{K}[K']$. In the case of $s = 1$, we required the open subset $U := \overline{X}' \backslash (\overline{X}')^{\mathrm{sing}}$ to be of codimension at least two in $\overline{X}'$. Thus, the ring R' is normal by Serre's criterion [73, 6.2], see Lemma 5.4.3. Moreover, the grading
$$Q'(e_1) \ = \ \deg(T_1), \qquad \dots, \qquad Q'(e_k) \ = \ \deg(T_k)$$
is almost free since, by construction, the columns of P' generate the whole space as a cone, are pairwise different and primitive. Furthermore, the codimension test ensures that the variables are pairwise non-associated. By Theorem 4.2.6, R' is the Cox ring of X'. We conclude that X' is a smooth MDS. □

Remark 2.4.11. In Algorithm 2.4.8, for higher-dimensional X, it is not clear how to obtain a "minimal" resolution. However, one may use [51, Thm. 6.2] which states that X arises from a *combinatorially minimimal* MDS X_0 by a finite sequence of small birationial maps and contractions.

In the following example, we apply Algorithm 2.4.8 to resolve the singularities of a surface X that does not admit a non-trivial $\mathbb{K}^*$-action. We will encounter X in Theorem 4.4.1 as the surface with singularity type E_6A_2.

Example 2.4.12. Consider the Mori dream surface X with class group $\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and the following Cox ring and degree matrix

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_1, \ldots, T_4]/\langle -T_1T_4^2 + T_2^3 + T_2T_3T_4 + T_3^3 \rangle, \qquad Q = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} \end{smallmatrix}\right].$$

By Algorithm 2.3.24, X is singular. Using Algorithm 2.4.8 with options `verify` and `minimal`, we obtain a minimal resolution $X' \to X$ with a smooth MDS X'. Its Cox ring is

$$\mathcal{R}\,(X') \;=\; \mathbb{K}[T_1, \ldots, T_{12}]\,/\,\langle g \rangle,$$
$$g := -T_1T_4^2T_5 + T_2^3T_{12}T_7^2T_8 + T_2T_3T_{11}T_4T_5T_6T_7T_8T_{10} + T_3^3T_9T_{11}^2T_{10}.$$

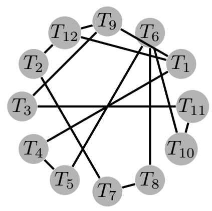
Note that $\mathcal{R}(X')$ is as predicted in [33, p. 40, type E_6A_2]. The class group of X' is $\mathbb{Z}^9$ and the degree matrix is

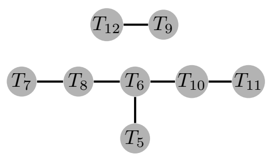
$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}.$$

Since X' is a complete intersection we can compute the self intersection numbers of all $D_i := V(X'; T_i)$ with Algorithm 2.3.48; they are

D_1^2	D_2^2	D_3^2	D_4^2	D_5^2	D_6^2	D_7^2	D_8^2	D_9^2	D_{10}^2	D_{11}^2	D_{12}^2
-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2

Observe that no further (-1)-curves D_i with $i > 4$ exist since we provided the option `minimal`. Combined with Algorithm 2.3.27, we obtain the exceptional graph $G_{X'}$ and the subgraph of (-2)-curves which indeed has E_6A_2 singularity type:



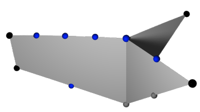


We come to algorithms concerning the so-called anticanonical complex introduced by B. Bechtold, J. Hausen and E. Huggenberger in [17]. Let $X = X(P, A, \Phi)$ be a complexity-one T-variety as in Construction 1.5.11 with P-matrix P of size $n \times r$. Its Cox ring is $R(P, A) = \mathbb{K}[T_1, \ldots, T_r]/I$ with an ideal $I = \langle f_1, \ldots, f_s \rangle$ and assume the $\mathrm{Cl}(X)$-grading is pointed; compare Construction 1.5.3. Let Q^0 be the degree matrix of the free part of the grading. In [17], the *anticanonical polytope* of X was defined as the polytope

$$A_X \;:=\; B_X^\vee, \qquad B_X \;:=\; (P^*)^{-1}\left(B_w + \sum_{i=1}^{s} \Delta(f_i) - (1, \ldots, 1)\right) \;\subseteq\; \mathbb{Q}^n,$$

with the Newton polytopes $\Delta(f_i) \subseteq \mathbb{Q}^r$ and the fiber-polytope $B_w := (Q^0)^{-1}(w) \cap \mathbb{Q}^r_{\geq 0}$ of the free part $w \in K^0$ of the anticanonical divisor class. Let Z_Σ be the canonical ambient toric variety of X as in Construction 1.5.7 and Υ a fan in $\mathbb{Q}^n$ with support $\mathrm{trop}(X)$. The *anticanonical complex* is the polyhedral complex

$$\mathcal{A}_X \;:=\; \mathrm{faces}(A_X) \sqcap \Upsilon \sqcap \Sigma \;\subseteq\; \mathbb{Q}^n,$$



where $\Upsilon \sqcap \Sigma$ is the coarsest common refinement and $\mathrm{faces}(A_X) \sqcap \Upsilon$ denotes the cone-wise intersection with the polytope A_X. The computation is a direct consequence of the definition.

Algorithm 2.4.13 (MDSanticanpoly)**.** See [17]. *Input:* an MDS $X = (R, \Phi)$ of complexity one. Let $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ with $G = \{f_1, \ldots, f_s\}$ and $P \colon \mathbb{Z}^r \to \mathbb{Z}^n$.

- Compute the free part $w := (-w_X^{\mathrm{can}})^0 \in K^0$ of the anticanonical divisor class with Algorithm 2.3.41. Determine $B_w := (Q^0)^{-1}(w) \cap \mathbb{Q}^r_{\geq 0}$.
- Compute the Minkowski sum $B := B_w + \Delta(f_1) + \ldots + \Delta(f_s) \subseteq \mathbb{Q}^r$ and the image $B_X := (P^*)^{-1}(B - (1, \ldots, 1))$ in $\mathbb{Q}^n$. Let $A_X := B_X^\vee$ be the dual.

Output: the anticanonical polytope $A_X \subseteq \mathbb{Q}^n$.

Algorithm 2.4.14 (MDSanticancomp)**.** See [17]. *Input:* an MDS $X = (R, \Phi)$ of complexity one with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute the anticanonical polytope $A_X \subseteq \mathbb{Q}^n$ with Algorithm 2.4.13.
- Compute a fan $\Upsilon \subseteq \mathbb{Q}^n$ with support $\mathrm{trop}(X)$, compare Algorithm 2.2.7.
- Compute the fan $\Sigma \subseteq \mathbb{Q}^n$ of the canonical toric ambient variety with Algorithm 2.3.9 and the coarsest common refinement $\Sigma' := \Upsilon \sqcap \Sigma$.
- Let $\mathcal{A}_X$ be the polyhedral complex with maximal cells $\sigma' \cap A_X$ where $\sigma' \in \Sigma'$ runs through the maximal cones of Σ'.

Output: the anticanonical complex $\mathcal{A}_X$ of X.

Consider a normal $\mathbb{Q}$-factorial variety X with Cartier canonical divisor D_X^{can}. Let $\varphi \colon X' \to X$ be a resolution of singularities. Recall, e.g., from [61], that X is *terminal* if $a_i > 0$ for all i in

$$D_{X'}^{\mathrm{can}} \;=\; \varphi^*\left(D_X^{\mathrm{can}}\right) \;+\; \sum_i a_i E_i$$

with the exceptional divisors E_i of φ. We say that X is *log-terminal* if $a_i > 1$ for all i. Similarly, given $0 < \varepsilon < 1$, we call X *ε-log-terminal* if we have $a_i > -1 + \varepsilon$ for all i.

Algorithm 2.4.15 (MDSisepslogterminal)**.** See [17]. *Input:* an MDS $X = (R, \Phi)$ of complexity one with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and a rational number $0 < \varepsilon < 1$.

- Compute the anticanonical complex $\mathcal{A}_X$ of X using Algorithm 2.4.14.
- For each maximal cell $C \in \mathcal{A}_X$ do
  - Let C_ε be the scaled cell $\varepsilon \cdot C$. Return *false* if C_ε is unbounded.
  - Compute the set of lattice points $U := C_\varepsilon \cap \mathbb{Z}^r$ with Algorithm 2.2.2.
  - Let $V \subseteq \mathbb{Q}^n$ consist of the zero-vector, the columns of P and the vertices of $\mathcal{A}_X$. Return *false* if $U \setminus V \neq \emptyset$.
- Return *true*.

Output: *true* if X is ε-log-terminal and *false* otherwise.

Algorithm 2.4.16 (MDSisterminal)**.** See [17]. *Input:* an MDS X of complexity one.

- Perform the same steps as in Algorithm 2.4.15 for $\varepsilon = 1$.

Output: *true* if X is terminal and *false* otherwise.

Note that we can also test a $\mathbb{K}^*$-surface for being log-terminal by an inspection of its P-matrix, see Remark 2.5.6.

Example 2.4.17. By an application of Algorithm 2.4.16, the complexity-one T-variety defined in Example 2.4.7 is terminal whereas the $\mathbb{K}^*$-surface of Example 2.4.5 is not. However, the latter is ε-log-terminal by Algorithm 2.4.15 for $\varepsilon := 1/2$.

5. Application: Combinatorially minimal $\mathbb{K}^*$-surfaces

In this section, we classify the non-toric, combinatorially minimal *del Pezzo*, i.e., Fano, $\mathbb{K}^*$-surfaces of Picard number two up to Gorenstein-index six. We apply the algorithms developed in the previous sections to study the resulting surfaces. This continues work of E. Huggenberger on the Gorenstein-case, see [61, Sec. 5.3].

We write d_X for the self-intersection number of the anticanonical divisor of the given surface X and denote the Picard index by $b := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. We say that X has *hypersurface Cox ring* if the spectrum $\overline{X}$ over the Cox ring is a hypersurface. Moreover, recall from [51, Sec. 6] that a variety X is *combinatorially minimal* if $\mathrm{Mov}(X) = \mathrm{Eff}(X)$. In the sense of Remark 1.4.2, this means that no further contraction is possible.

Theorem 2.5.1. *Up to isomorphism, there are only finitely many non-toric, combinatorially minimal del Pezzo $\mathbb{K}^*$-surfaces of Picard number two with hypersurface Cox ring and with Gorenstein index $n \in \mathbb{Z}_{>0}$. The following table is a classification of all surfaces with $n \leq 6$. No two shown surfaces are isomorphic.*

	Cox ring $\mathcal{R}(X)$	*degree matrix*	$\mathrm{Cl}(X)$	b	n	d_X
(1)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & -2 & -1 & 0 & -1 \\ \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	32	2	2
(2)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & -2 & -1 & 0 & -1 \\ \overline{1} & \overline{3} & \overline{2} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$	256	4	1
(3)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2\rangle$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & -2 & -1 & 0 & -1 \\ \overline{1} & \overline{5} & \overline{3} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/6\mathbb{Z}$	864	6	$\frac{2}{3}$
(4)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^2\rangle$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & -1 & -2 & 0 & -1 \\ \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	108	3	$\frac{4}{3}$
(5)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\begin{bmatrix} 3 & 0 & 0 & 1 & 1 \\ 0 & -1 & -3 & 0 & -1 \end{bmatrix}$	$\mathbb{Z}^2$	9	3	$\frac{8}{3}$
(6)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\begin{bmatrix} 3 & 0 & 0 & 1 & 1 \\ 0 & -1 & -3 & 0 & -1 \\ \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	72	3	$\frac{4}{3}$
(7)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^2\rangle$	$\begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & -1 & -2 & 0 & -1 \\ \overline{3} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}$	500	5	$\frac{4}{5}$
(8)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^5\rangle$	$\begin{bmatrix} 5 & 0 & 0 & 1 & 1 \\ 0 & -1 & -5 & 0 & -1 \end{bmatrix}$	$\mathbb{Z}^2$	25	5	$\frac{8}{5}$
(9)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^5\rangle$	$\begin{bmatrix} 5 & 0 & 0 & 1 & 1 \\ 0 & -1 & -5 & 0 & -1 \\ \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	200	5	$\frac{4}{5}$
(10)	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^3\rangle$	$\begin{bmatrix} 3 & 0 & 0 & 1 & 1 \\ 0 & -1 & -3 & 0 & -1 \\ \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$	576	6	$\frac{2}{3}$

(11)	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3T_4^4+T_5^4\rangle$	$\begin{bmatrix} 4 & 0 & 0 & 1 & 1 \\ \underline{0} & \underline{-1} & \underline{-4} & \underline{0} & \underline{-1} \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	432	6	$\frac{2}{3}$
(12)	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle$	$\begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & -2 & -1 & 0 & -1 \end{bmatrix}$	$\mathbb{Z}^2$	4	1	4

All surfaces are singular and exactly the surfaces of cases (5), (8) *and* (12) *are almost-homogeneous. The log-terminal surfaces are* (1), (2), (3), (4), (7) *and* (12). *Moreover, in each case* (i), *the following* P*-matrices can be chosen such that the respective* $\mathbb{K}^*$*-surface* X *satisfies* $X = X(P_i, A)$ *as in Construction 1.5.2:*

$$P_1 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 1 & -3 & 1 & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 2 & -5 & 1 & 1 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 2 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix},$$

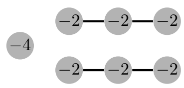
$$P_5 = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{bmatrix}, \qquad P_6 = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & -1 & 0 & 2 & 1 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 2 \\ -1 & 3 & -2 & 1 & 1 \end{bmatrix}, \qquad P_8 = \begin{bmatrix} -1 & -5 & 1 & 5 & 0 \\ -1 & -5 & 0 & 0 & 5 \\ -1 & -4 & 0 & 1 & 4 \end{bmatrix},$$

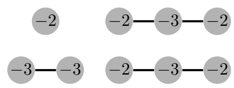
$$P_9 = \begin{bmatrix} -1 & -5 & 1 & 5 & 0 \\ -1 & -5 & 0 & 0 & 5 \\ -1 & -3 & 0 & 2 & 3 \end{bmatrix}, \qquad P_{10} = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & 1 & -1 & 1 & 2 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} -1 & -4 & 1 & 4 & 0 \\ -1 & -4 & 0 & 0 & 4 \\ -1 & -1 & 0 & 3 & 1 \end{bmatrix}, \qquad P_{12} = \begin{bmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

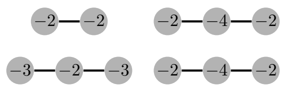
Furthermore, for the respective minimal resolution of singularities $X' \to X$, *the subgraphs of* $(-k)$*-curves,* $k \in \mathbb{Z}_{\geq 2}$, *of the graphs* $G_{X'}$ *of exceptional curves and the corresponding self-intersection numbers are*



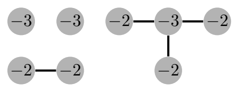
(1)



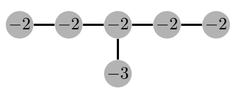
(2)



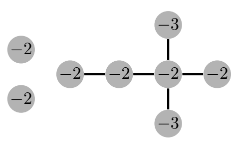
(3)



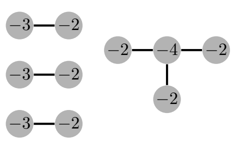
(4)



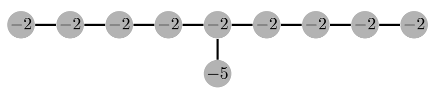
(5)



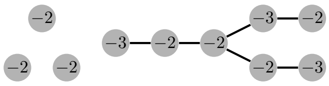
(6)



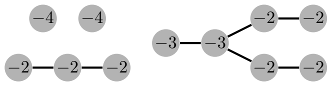
(7)



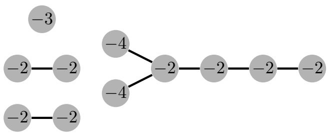
(8)



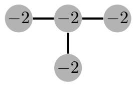
(9)



(10)



(11)



(12)

Remark 2.5.2. In Theorem 2.5.1, case (12) coincides with the Gorenstein surface with ADE-singularity type D_4 found in [61, Thm. 5.26]. Surfaces (1), (4) and (12) also appear in [94, Satz 6.13].

Lemma 2.5.3. *Let $w_1, \ldots, w_4 \in \mathbb{Z}^2$ be such that $\mathbb{Q}_{\geq 0} \cdot w_1 = \mathbb{Q}_{\geq 0} \cdot w_2$, we have $\mathbb{Q}_{\geq 0} \cdot w_3 = \mathbb{Q}_{\geq 0} \cdot w_4$ and $\mathbb{Q}_{\geq 0} \cdot w_1 \neq \mathbb{Q}_{\geq 0} \cdot w_3$. If $\mathrm{lin}_{\mathbb{Z}}(w_1, \ldots, w_4) = \mathbb{Z}^2$, then there is $S \in \mathrm{GL}(2, \mathbb{Z})$ such that*

$$S \cdot [w_1, \ldots, w_4] \ = \ \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \end{bmatrix} \qquad \text{with } \ a, b, c, d \ \in \ \mathbb{Z}.$$

Proof. After computing a Hermite normal form, we may assume that there are integers $a, b, c, d \in \mathbb{Z}$ satisfying

$$w_1 \ = \ (a, 0), \qquad w_2 \ = \ (b, 0), \qquad w_3 \ = \ c \cdot v, \qquad w_4 \ = \ d \cdot v$$

with a vector $v = (v_1, v_2) \in \mathbb{Z}^2$. Since the w_i generate $\mathbb{Z}^2$, we must have $v_2 = \pm 1$; we may assume $v_2 = 1$. Adding $-v_1$ times the last row to the first one yields

$$S \cdot \begin{bmatrix} a & b & c \cdot v_1 & d \cdot v_1 \\ 0 & 0 & c & d \end{bmatrix} \ = \ \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \end{bmatrix} \qquad \text{with } S \in \mathrm{GL}(2, \mathbb{Z}). \qquad \square$$

Lemma 2.5.4. *Let X be a combinatorially minimal $\mathbb{K}^*$-surface with hypersurface Cox ring and $\mathrm{rank}(\mathrm{Cl}(X)) = 2$. Then $X = X(P, A)$ with integral matrices*

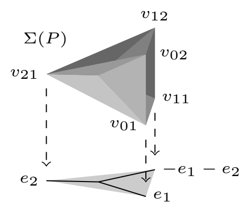
$$P \ = \ \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ -\frac{d_{12} l_{01}}{l_{12}} - \frac{d_{21} l_{01}}{l_{21}} & -\frac{d_{11} l_{02}}{l_{11}} - \frac{d_{21} l_{02}}{l_{21}} & d_{11} & d_{12} & d_{21} \end{bmatrix}, \qquad A \ = \ \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

as in Construction 1.5.2, satisfying $\gcd(l_{ij_i}, d_{ij_i}) = 1$ *for all* $0 \leq i \leq 2$ *and* $1 \leq j_i \leq n_i$ *and we have*

$$0 \ \leq \ d_{12} \ < \ l_{12}, \qquad 0 \ \leq \ d_{21} \ < \ l_{21}, \qquad \frac{d_{12}}{l_{12}} \ > \ \frac{d_{11}}{l_{11}}.$$

Furthermore, the fan $\Sigma(P)$ of the canonical toric ambient variety of X in Construction 1.5.7 is of type (ee) with exactly two three-dimensional cones

$$\sigma^+ \ := \ \mathrm{cone}(v_{02}, v_{12}, v_{21}),$$
$$\sigma^- \ := \ \mathrm{cone}(v_{01}, v_{11}, v_{21}).$$



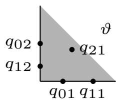
Moreover, the integral 2×5 matrix Q^0 representing the free part of the degree map $\mathbb{Z}^5 \to \mathrm{Cl}(X)$ is of shape

$$Q^0 \ = \ \begin{bmatrix} a & 0 & 0 & \frac{l_{01} a}{l_{12}} & \frac{l_{01} a}{l_{21}} \\ 0 & b & \frac{l_{02} b}{l_{11}} & 0 & \frac{l_{02} b}{l_{21}} \end{bmatrix}, \qquad a, b \ \in \ \mathbb{Z}_{>0}.$$

Proof. Since X is a surface and $\mathrm{rank}(\mathrm{Cl}(X)) = 2$, we have $\dim(\mathcal{R}(X)) = 4$. Together with the fact that the ideal of relations of $\mathcal{R}(X)$ is principal, the parameters in Construction 1.5.2 are $r = 2$, $n = 5$ and $m = 0$. This means $X = X(P, A)$ arises from matrices P and A where

$$P \;=\; \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{12} & d_{21} \end{bmatrix}, \qquad A \;=\; \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Let the 2×5 matrix $Q^0 = [q_{01}, \ldots, q_{21}]$ represent the projection $\mathbb{Z}^5 \to \mathbb{Z}^2$ onto the free part of $\mathrm{Cl}(X)$. Since X is combinatorially minimal, the columns $q_{01}, \ldots, q_{21}$ generate a pointed, two-dimensional cone $\vartheta \subseteq \mathbb{Q}^2$ where each of the two extremal rays of ϑ contains exactly two q_{ij} and $q_{21} \in \vartheta^\circ$ by homogenity of the defining equation $T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} T_{12}^{l_{12}} + T_{21}^{l_{21}}$ of the Cox ring. Moreover, by Lemma 2.5.3, there are $a, b, c, d, e, f \in \mathbb{Z}$ such that



$$Q^0 \;=\; \begin{bmatrix} a & 0 & 0 & d & e \\ 0 & b & c & 0 & f \end{bmatrix} \;=\; \begin{bmatrix} a & 0 & 0 & \frac{l_{01}a}{l_{12}} & \frac{l_{01}a}{l_{21}} \\ 0 & b & \frac{l_{02}b}{l_{11}} & 0 & \frac{l_{02}b}{l_{21}} \end{bmatrix},$$

where the last equality and the fact that all fractions are integers was obtained from $P \cdot (Q^0)^t = 0$. Multiplying the rows by ± 1, we may assume $a, b \in \mathbb{Z}_{>0}$. Using again $P \cdot (Q^0)^t = 0$, we have the additional conditions

$$ad_{01} \;=\; -\frac{ad_{12}l_{01}}{l_{12}} - \frac{ad_{21}l_{01}}{l_{21}}, \qquad bd_{02} \;=\; -\frac{bd_{11}l_{02}}{l_{11}} - \frac{bd_{21}l_{02}}{l_{21}}.$$

Division by a or b respectively gives the desired shape of P. The conditions $0 \leq d_{12} < l_{12}$ and $0 \leq d_{21} < l_{21}$ come from according row operations in P (before fixing the d_{0i}). Note that P is in the normal form of Construction 1.5.2 if we fix the ordering of the slopes of one block:

$$\frac{d_{02}}{l_{02}} > \frac{d_{01}}{l_{01}} \qquad \Leftrightarrow \qquad \frac{d_{12}}{l_{12}} > \frac{d_{11}}{l_{11}}. \qquad \square$$

Lemma 2.5.5. *Let $X = X(P, A)$ be as in Lemma 2.5.4. Then X is Fano if and only if the following inequalities hold*

$$1 \;<\; \frac{1}{l_{21}} + \frac{1}{l_{12}} + \frac{1}{l_{01}}, \qquad 1 \;<\; \frac{1}{l_{21}} + \frac{1}{l_{11}} + \frac{1}{l_{02}}.$$

Proof. Let Q^0 be the 2×5 matrix from Lemma 2.5.5 with columns q_{ij}. Also by Lemma 2.5.5 and Construction 1.5.3, the Cox ring of X is

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}]/\langle g \rangle, \qquad g \;:=\; T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} T_{12}^{l_{12}} + T_{21}^{l_{21}}.$$

By Proposition 1.3.17, X is Fano if and only if the free part $(-w_X^{\mathrm{can}})^0 \in \mathrm{Cl}(X)^0 = \mathbb{Z}^2$ of the anticanonical divisor class of X is an element of the ample cone, i.e.

$$\begin{aligned} (-w_X^{\mathrm{can}})^0 \;&=\; \left(\sum_{i,j} q_{ij} \right) - \deg(g) \\ &=\; \begin{pmatrix} a\left(\frac{l_{01}}{l_{21}} + \frac{l_{01}}{l_{12}} + 1 - l_{01}\right) \\ b\left(\frac{l_{02}}{l_{21}} + \frac{l_{02}}{l_{11}} + 1 - l_{02}\right) \end{pmatrix} \\ &\in\; \mathrm{cone}(e_1, e_2)^\circ. \end{aligned}$$

This means $(-w_X^{\mathrm{can}})_i^0 > 0$ for both $i = 1, 2$. Since $a, b \in \mathbb{Z}_{>0}$, division by l_{01} or $l_{02} \in \mathbb{Z}_{>0}$ respectively, gives the assertion. $\square$

The parameters occurring in the inequalities in Lemma 2.5.5 are *platonic triples*, i.e., triples $(a, b, c) \in \mathbb{Z}_{>0}^3$ such that $a^{-1} + b^{-1} + c^{-1} > 1$. Up to permutation, the only possible combinations are (compare [61, Ex. 3.20])

$$(1, x, y), \quad (2, 2, z), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5), \quad x, y \in \mathbb{Z}_{\geq 1}, \quad z \in \mathbb{Z}_{\geq 2}.$$

Remark 2.5.6. The platonic triples occurring in Lemma 2.5.5 are similar to the ones occurring in [61, Prop. 3.19, Ex. 3.20]: X being log-terminal is equivalent to

$$1 \; < \; \frac{1}{l_{02}} + \frac{1}{l_{12}} + \frac{1}{l_{21}}, \qquad 1 \; < \; \frac{1}{l_{01}} + \frac{1}{l_{11}} + \frac{1}{l_{21}}.$$

Lemma 2.5.7. *Let $X = X(P, A)$ be as in Lemma 2.5.4 and assume X is Fano. Then each possible choice of parameters l_{ij} must simultaneously satisfy cases A_k, B_l for some $k, l \in \{1, \ldots, 20\}$ where*

case	l_{01}	l_{12}	l_{21}
A_1	1	≥ 1	≥ 2
A_2	≥ 2	1	≥ 2
A_3	2	2	≥ 2
A_4	2	≥ 3	2
A_5	≥ 3	2	2
A_6	2	3	3
A_7	3	2	3
A_8	3	3	2
A_9	2	3	4
A_{10}	2	4	3
A_{11}	3	2	4
A_{12}	4	2	3
A_{13}	3	4	2
A_{14}	4	3	2
A_{15}	2	3	5
A_{16}	2	5	3
A_{17}	3	2	5
A_{18}	5	2	3
A_{19}	3	5	2
A_{20}	5	3	2

case	l_{02}	l_{11}	l_{21}
B_1	1	≥ 1	≥ 2
B_2	≥ 2	1	≥ 2
B_3	2	2	≥ 2
B_4	2	≥ 3	2
B_5	≥ 3	2	2
B_6	2	3	3
B_7	3	2	3
B_8	3	3	2
B_9	2	3	4
B_{10}	2	4	3
B_{11}	3	2	4
B_{12}	4	2	3
B_{13}	3	4	2
B_{14}	4	3	2
B_{15}	2	3	5
B_{16}	2	5	3
B_{17}	3	2	5
B_{18}	5	2	3
B_{19}	3	5	2
B_{20}	5	3	2

Proof. This is a direct application of Lemma 2.5.5 and the aforementioned possible choices for platonic triples. □

Lemma 2.5.8. *See* [61, Prop. 5.2]. *Let $X = X(P, A)$ be as in Lemma 2.5.4. Then X is n-Gorenstein if and only if $n \in \mathbb{Z}_{>0}$ is minimal such that all of the following divisibility constraints, called (a) to (j), are satisfied*

$$\begin{aligned}
l_{11}l_{21}d_{01} + l_{01}l_{21}d_{11} + l_{01}l_{11}d_{21} \quad &\Big| \quad n\left(l_{01}l_{11}l_{21} - l_{11}l_{21} - l_{01}l_{21} - l_{01}l_{11}\right),\\
l_{12}l_{21}d_{02} + l_{02}l_{21}d_{12} + l_{02}l_{12}d_{21} \quad &\Big| \quad n\left(l_{02}l_{12}l_{21} - l_{12}l_{21} - l_{02}l_{21} - l_{02}l_{12}\right),\\
l_{11}l_{21}d_{01} + l_{01}l_{21}d_{11} + l_{01}l_{11}d_{21} \quad &\Big| \quad n\left(l_{01}l_{21}d_{11} + l_{21}(d_{01} - d_{11}) + l_{01}(d_{21} - d_{11})\right),\\
l_{12}l_{21}d_{02} + l_{02}l_{21}d_{12} + l_{02}l_{12}d_{21} \quad &\Big| \quad n\left(l_{02}l_{21}d_{12} + l_{21}(d_{02} - d_{12}) + l_{02}(d_{21} - d_{12})\right),\\
l_{11}l_{21}d_{01} + l_{01}l_{21}d_{11} + l_{01}l_{11}d_{21} \quad &\Big| \quad n\left(l_{01}l_{11}d_{21} + l_{11}(d_{01} - d_{21}) + l_{01}(d_{11} - d_{21})\right),\\
l_{12}l_{21}d_{02} + l_{02}l_{21}d_{12} + l_{02}l_{12}d_{21} \quad &\Big| \quad n\left(l_{02}l_{12}d_{21} + l_{12}(d_{02} - d_{21}) + l_{02}(d_{12} - d_{21})\right),\\
l_{01}d_{02} - l_{02}d_{01} \quad &\Big| \quad n\left(d_{02} - d_{01}\right),\\
l_{01}d_{02} - l_{02}d_{01} \quad &\Big| \quad n\left(l_{01} - l_{02}\right),\\
l_{11}d_{12} - l_{12}d_{11} \quad &\Big| \quad n\left(d_{12} - d_{11}\right),\\
l_{11}d_{12} - l_{12}d_{11} \quad &\Big| \quad n\left(l_{11} - l_{12}\right).
\end{aligned}$$

Lemma 2.5.9. *Let $X = X(P, A)$ be as in Lemma 2.5.4. Set $q_1 := \gcd(l_{21}l_{12}, l_{01}l_{21})$ and $q_2 := \gcd(l_{21}l_{11}, l_{02}l_{21})$. If $q_1 \mid l_{12}l_{01}$ and $q_2 \mid l_{02}l_{11}$, then Q^0 can be chosen as* *the integral matrix*

$$Q^0 \;=\; \begin{bmatrix} \frac{l_{21}l_{12}}{q_1} & 0 & 0 & \frac{l_{01}l_{21}}{q_1} & \frac{l_{12}l_{01}}{q_1} \\ 0 & \frac{l_{21}l_{11}}{q_2} & \frac{l_{02}l_{21}}{q_2} & 0 & \frac{l_{02}l_{11}}{q_2} \end{bmatrix}.$$

Proof. Consider the sublattice $L \le \mathbb{Z}^5$ spanned by the two vectors $u_1, u_2 \in \mathbb{Z}^5$ which are given by

$$u_1 \;:=\; (l_{21}l_{12},\, 0,\, 0,\, l_{01}l_{21},\, l_{12}l_{01})\,, \qquad u_2 \;:=\; (0,\, l_{21}l_{11},\, l_{02}l_{21},\, 0,\, l_{02}l_{11})\,.$$

Since $L \le \ker(P)$ and $\mathrm{rank}(L) = 2 = \mathrm{rank}(\ker(P))$, the saturated lattice L^{sat} equals $\ker(P)$. Swapping coordinates, we have

$$\langle S_1 \cdot u_1,\, S_1 \cdot u_2\rangle \;=\; \left\langle \begin{pmatrix} l_{21}l_{12} \\ l_{01}l_{21} \\ 0 \\ 0 \\ l_{12}l_{01} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ l_{02}l_{21} \\ l_{21}l_{11} \\ l_{02}l_{11} \end{pmatrix} \right\rangle \;=:\; \left\langle \begin{pmatrix} v \\ 0 \\ l_{12}l_{01} \end{pmatrix}, \begin{pmatrix} 0 \\ v' \\ l_{02}l_{11} \end{pmatrix} \right\rangle$$

with $S_1 \in \mathrm{GL}(5,\mathbb{Z})$ and $v, v' \in \mathbb{Z}^2$. By basic algebra, given a primitive vector $w \in \mathbb{Z}^2$ and $d \in \mathbb{Z}$, there is $S \in \mathrm{GL}(2,\mathbb{Z})$ such that $S(d \cdot w) = (d, 0)$. Applying this to the primitive vectors vq_1^{-1} and $v'q_2^{-1}$, we have $S_2 \cdot v = (q_1, 0)$ and $S_2' \cdot v' = (q_2, 0)$ with matrices $S_2, S_2' \in \mathrm{GL}(2,\mathbb{Z})$. Then

$$S \cdot L \;=\; \langle Su_1,\, Su_2\rangle \;=\; \left\langle \begin{pmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ l_{12}l_{01} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ q_2 \\ 0 \\ l_{02}l_{11} \end{pmatrix} \right\rangle, \qquad S \;:=\; \left[\begin{array}{cc|c} S_2 & 0 & \\ 0 & S_2' & \\ \hline & & 1 \end{array}\right] \cdot S_1$$

with $S \in \mathrm{GL}(5,\mathbb{Z})$. Since $q_1 \mid l_{12}l_{01}$ and $q_2 \mid l_{02}l_{11}$, we have a sublattice $L' := \langle q_1^{-1}Su_1,\, q_2^{-1}Su_2\rangle \le \mathbb{Z}^5$. Since $L' = (L')^{\mathrm{sat}}$ and $L' \subseteq (S \cdot L)^{\mathrm{sat}}$ and both L' and $S \cdot L$ are of rank two, we have $L' = (S \cdot L)^{\mathrm{sat}}$. Then the rows of Q^0, i.e., generators for $\ker(P)$, are obtained by

$$\ker(P) \;=\; L^{\mathrm{sat}} \;=\; S^{-1}(L') \;=\; \left\langle q_1^{-1}u_1,\, q_2^{-1}u_2\right\rangle. \qquad \square$$

Lemma 2.5.10 (A_1B_1)**.** *Consider in Lemma 2.5.7 case A_1B_1, i.e., $l_{01} = 1 = l_{02}$, and $l_{11} \ge 1$, $l_{12} \ge 1$, $l_{21} \ge 2$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$2 \;\le\; l_{21} \;=\; l_{11} \;=\; l_{12} \;\le\; n, \qquad -n \;<\; -d_{11} \;<\; 2n.$$

Proof. Note that by Lemma 2.5.4, we require d_{01} and d_{02} to be integers of the following form

$$d_{01} \;=\; -\frac{d_{12}}{l_{12}} - \frac{d_{21}}{l_{21}} \;\in\; \mathbb{Z}, \qquad d_{02} \;=\; -\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} \;\in\; \mathbb{Z}.$$

Since also $l_{12}d_{01} \in \mathbb{Z}$ and $l_{11}d_{02} \in \mathbb{Z}$, this means $l_{21} \mid d_{21}l_{12}$ and $l_{21} \mid d_{21}l_{11}$. As $\gcd(l_{21}, d_{21}) = 1$, we have $l_{21} \mid l_{12}$ and $l_{21} \mid l_{11}$. Lemma 2.5.9 then allows us to choose the free part Q^0 of the grading as

$$Q^0 \;=\; \begin{bmatrix} l_{12} & 0 & 0 & 1 & \frac{l_{12}}{l_{21}} \\ 0 & l_{11} & 1 & 0 & \frac{l_{11}}{l_{21}} \end{bmatrix}.$$

Since the grading must be almost free, each four columns must generate $\mathbb{Z}^2$ as a lattice, i.e., both l_{11} and l_{11}/l_{21} as well as l_{12} and l_{12}/l_{21} have to be coprime. We obtain $l_{21} = l_{11} = l_{12}$. Now, condition (i) in Lemma 2.5.8 provides us with $l_{21} \mid n$ and, thus, $2 \le l_{21} = l_{11} = l_{12} \le n$. Moreover, by condition (a) of Lemma 2.5.8, we have $d_{11} - d_{12} \mid 2n$. Together with the condition $d_{12} > d_{11}$ obtained by the requirements on the slopes in Lemma 2.5.4, this implies

$$-n \;\le\; -l_{12} \;<\; -d_{12} \;<\; -d_{11} \;\le\; 2n - d_{12} \;<\; 2n. \qquad \square$$

Lemma 2.5.11 (A_1B_2)**.** *Consider in Lemma 2.5.7 case A_1B_2, i.e., $l_{01} = 1$, $l_{02} \geq 2$, $l_{11} = 1$, $l_{12} \geq 1$, $l_{21} \geq 2$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$2 \ \leq \ l_{21} \ = \ l_{02} \ = \ l_{12} \ \leq \ n, \qquad -1 \ < \ -d_{11} \ < \ \frac{3n}{2}.$$

Proof. Note that by Lemma 2.5.4, we require d_{01} and d_{02} to be integers of the following form

$$d_{01} \ = \ -\frac{d_{12}}{l_{12}} - \frac{d_{21}}{l_{21}} \ \in \ \mathbb{Z}, \qquad d_{02} \ = \ -d_{11}l_{02} - \frac{d_{21}l_{02}}{l_{21}} \ \in \ \mathbb{Z}.$$

For $d_{02} \in \mathbb{Z}$, this means $l_{21} \mid d_{21}l_{02}$ and $l_{21} \mid l_{02}$ follows from $\gcd(d_{21}, l_{21}) = 1$. Since also $l_{12}d_{01} \in \mathbb{Z}$ holds, we obtain $l_{21} \mid d_{21}l_{12}$. Using again $\gcd(d_{21}, l_{21}) = 1$, we arrive at $l_{21} \mid l_{12}$. Lemma 2.5.9 then allows us to choose the free part Q^0 of the grading as

$$Q^0 \ = \ \begin{bmatrix} l_{12} & 0 & 0 & 1 & \frac{l_{12}}{l_{21}} \\ 0 & 1 & l_{02} & 0 & \frac{l_{02}}{l_{21}} \end{bmatrix}.$$

Since the grading must be almost free, each four columns must generate $\mathbb{Z}^2$ as a lattice, i.e., both l_{02} and l_{02}/l_{21} as well as l_{12} and l_{12}/l_{21} have to be coprime. We obtain $l_{21} = l_{02} = l_{12}$. Now, condition (i) in Lemma 2.5.8 provides us with $l_{21} \mid n$ and, thus, $2 \leq l_{21} = l_{11} = l_{12} \leq n$. The requirements on the slopes in Lemma 2.5.4 and $0 \leq d_{21} < l_{21}$ gives us

$$d_{11} \ < \ \frac{d_{12}}{l_{12}} \ < \ 1.$$

Moreover, by condition (a) of Lemma 2.5.8, $-d_{11}l_{21} + d_{12}$ divides $n(l_{21}+1)$. Together, by division of $l_{21} = l_{12}$, we conclude

$$\begin{aligned} 0 \ \leq \ -d_{11} + \frac{d_{12}}{l_{21}} \ &\leq \ \frac{n(l_{21}+1)}{l_{21}} \ \leq \ \frac{3n}{2} \\ \Rightarrow \quad -1 \ < \ -\frac{d_{12}}{l_{12}} \ \leq \ -d_{11} \ &\leq \ \frac{3n}{2} - \frac{d_{12}}{l_{12}} \ < \ \frac{3n}{2}. \end{aligned}$$

□

Several of the cases of Lemma 2.5.7 can be directly left out by the following observation.

Lemma 2.5.12. *In the situation of Lemma 2.5.7, each case A_iB_j in which at least one of the following conditions is satisfied, is not possible.*

$l_{21} \nmid l_{01}l_{12}$,	$l_{12} \nmid l_{01}l_{21}$,
$l_{21} \nmid l_{02}l_{11}$,	$l_{11} \nmid l_{02}l_{21}$,
$l_{21} = l_{01}$ *and* $l_{12} \nmid l_{01}$,	$l_{12} = l_{01}$ *and* $l_{21} \nmid l_{01}$,
$l_{21} = l_{02}$ *and* $l_{11} \nmid l_{02}$,	$l_{11} = l_{02}$ *and* $l_{21} \nmid l_{02}$,
$l_{01} \geq 2$ *and* $l_{12} = l_{21}$ *and* $l_{01} \nmid l_{21}$,	$l_{02} \geq 2$ *and* $l_{11} = l_{21}$ *and* $l_{02} \nmid l_{21}$.

In particular, this rules out all cases A_iB_j with $i \geq 4$ or $j \geq 4$. Furthermore, in the cases A_3B_j and A_iB_3, we have $l_{21} = 2$, in the cases A_1B_j, we have $l_{12} \geq 2$ and in the cases A_iB_1, we have $l_{11} \geq 2$.

Proof. We enumerate the listed conditions row-wise from left to right and speak of conditions (i) to (x). In Lemma 2.5.4, we showed that the integers d_{01} and d_{02} are

$$d_{01} \ = \ -\frac{d_{12}l_{01}}{l_{12}} - \frac{d_{21}l_{01}}{l_{21}} \ \in \ \mathbb{Z}, \qquad d_{02} \ = \ -\frac{d_{11}l_{02}}{l_{11}} - \frac{d_{21}l_{02}}{l_{21}} \ \in \ \mathbb{Z}.$$

In particular, $l_{12}d_{01}$ and $l_{21}d_{01}$ as well as $l_{21}d_{02}$ and $l_{11}d_{02}$ are integers. Since $\gcd(l_{ij}, d_{ij}) = 1$ for all $0 \leq i \leq 2$ and $1 \leq j \leq n_i$, this is equivalent to

$$l_{21} \mid l_{01}l_{12}, \qquad l_{12} \mid l_{01}l_{21}, \qquad l_{21} \mid l_{02}l_{11}, \qquad l_{11} \mid l_{02}l_{21},$$

respectively. This proves cases (i) to (iv). In the next four cases, i.e., (v) to $(viii)$, ordered as in the claim, this becomes

$$\begin{array}{rclcrcl} d_{01} & = & -\frac{d_{12}l_{21}}{l_{12}} - d_{21} \ \in \ \mathbb{Z}, & \qquad & d_{01} & = & -d_{12} - \frac{d_{21}l_{01}}{l_{21}} \ \in \ \mathbb{Z}, \\ d_{02} & = & -\frac{d_{11}l_{02}}{l_{11}} - d_{21} \ \in \ \mathbb{Z}, & \qquad & d_{02} & = & -d_{11} - \frac{d_{21}l_{02}}{l_{21}} \ \in \ \mathbb{Z}. \end{array}$$

Since we require the d_{0i} to be integers, all occurring fractions must be integers, i.e., , enumerated from left to right, we have respectively

$$l_{12} \mid d_{12}l_{21}, \qquad l_{21} \mid d_{21}l_{01}, \qquad l_{11} \mid d_{11}l_{02}, \qquad l_{21} \mid d_{21}l_{02}.$$

Since l_{ij} and d_{ij} are coprime for all $0 \le i \le 2$ and $1 \le j \le n_i$, all d_{ij} can be removed from the above divisibility constraints, i.e., the respective claim follows from

$$l_{12} \mid l_{21}, \qquad l_{21} \mid l_{01}, \qquad l_{11} \mid l_{02}, \qquad l_{21} \mid l_{02}.$$

In the last two cases, we have respectively

$$d_{01} \ = \ \frac{l_{01}(-d_{12} - d_{21})}{l_{21}} \ \in \ \mathbb{Z}, \qquad d_{02} \ = \ \frac{l_{02}(-d_{11} - d_{21})}{l_{21}} \ \in \ \mathbb{Z}.$$

Since d_{01} and l_{01} as well as d_{02} and l_{02} are coprime, we must have $l_{01} \mid l_{21}$ and $l_{02} \mid l_{21}$ respectively. This completes the first part of the proof. The remaining assertions are direct consequences. □

Lemma 2.5.13 (A_1B_3)**.** *Consider in Lemma 2.5.7 case A_1B_3, i.e., $l_{01} = 1$, $l_{02} = 2$, $l_{11} = 2$, $l_{12} \ge 1$, $l_{21} \ge 2$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$1 \ \le \ l_{12} \ \le \ 2, \qquad l_{21} \ = \ 2, \qquad 0 \ > \ d_{11} \ \ge \ -n-1.$$

Proof. We have $l_{21} = 2$ by Lemma 2.5.12. Since $0 \le d_{21} < l_{21}$ and $\gcd(l_{21}, d_{21}) = 1$, we have $d_{21} = 1$. Moreover, using Lemma 2.5.4, the requirement

$$2d_{01} \ = \ -\frac{2d_{12}}{l_{12}} - 1 \ \in \ \mathbb{Z}$$

gives $l_{12} \mid 2d_{12}$ which in turn delivers $l_{12} \mid 2$ since $\gcd(l_{12}, d_{12}) = 1$. In particular $1 \le l_{12} \le 2$. Condition (h) in Lemma 2.5.8 supplies us with bounds

$$\frac{2d_{12}}{l_{12}} + 1 - d_{11} - 2 \,\Big|\, n \qquad \Rightarrow \qquad -d_{11} \ \le \ n + 1 - \frac{2d_{12}}{l_{12}} \ \le \ n+1.$$ □

Lemma 2.5.14 (A_2B_1)**.** *Consider in Lemma 2.5.7 case A_2B_1, i.e., $l_{01} \ge 2$, $l_{02} = l_{12} = 1$, $l_{21} \ge 2$ and $l_{11} \ge 1$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$2 \ \le \ l_{01} \ = \ l_{11} \ = \ l_{21} \ < \ 4, \qquad d_{12} \ = \ 0, \qquad 0 \ < \ -d_{11} \ \le \ 4n.$$

Proof. Note that by Lemma 2.5.4, we have $0 \le d_{12} < l_{12} = 1$, i.e., $d_{12} = 0$. we require d_{01} and d_{02} to be integers of the following form

$$d_{01} \ = \ -\frac{d_{21}l_{01}}{l_{21}} \ \in \ \mathbb{Z}, \qquad d_{02} \ = \ -\frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}} \ \in \ \mathbb{Z}.$$

For $d_{01} \in \mathbb{Z}$, this means $l_{21} \mid d_{21}l_{01}$ and $l_{21} \mid l_{01}$ follows from $\gcd(d_{21}, l_{21}) = 1$. Since also $l_{11}d_{01} \in \mathbb{Z}$, we obtain $l_{21} \mid d_{21}l_{11}$. Using again $\gcd(d_{21}, l_{21}) = 1$, we arrive at $l_{21} \mid l_{11}$. Lemma 2.5.9 then allows us to choose the free part Q^0 of the grading as

$$Q^0 \ = \ \begin{bmatrix} 1 & 0 & 0 & l_{01} & \frac{l_{01}}{l_{21}} \\ 0 & l_{11} & 1 & 0 & \frac{l_{11}}{l_{21}} \end{bmatrix}.$$

Since the grading must be almost free, each four columns must generate $\mathbb{Z}^2$ as a lattice, i.e., both l_{01} and l_{01}/l_{21} as well as l_{11} and l_{11}/l_{21} have to be coprime. We obtain $l_{21} = l_{01} = l_{11}$. By the requirements on the slopes in Lemma 2.5.4, we have $d_{11} < 0$. Now, conditions (a) and (b) in Lemma 2.5.8 provide us with

$$-d_{11} \mid n(l_{21} - 3), \qquad -d_{11} \mid n(l_{21} + 1).$$

Then $-d_{11}$ also divides the difference of the right hand sides, i.e., $-d_{11} \mid 4n$. In particular, $0 < -d_{11} \leq 4n$. Applying condition (j) of Lemma 2.5.8, i.e., $-d_{11} \mid nl_{21} - n$, we conclude

$$\begin{aligned} 0 \ < \ -d_{11} \ \leq \ nl_{21} - n \ \ &\Rightarrow \ \ -4n \ \leq \ d_{11} \ < \ -nl_{21} \ \leq \ d_{11} - n \\ &\Rightarrow \ \ 4 \ > \ l_{21} \ \geq \ -\frac{d_{11}}{n} + 1 \ > \ 1. \end{aligned}$$

□

Lemma 2.5.15 (A_2B_2)**.** *Consider in Lemma 2.5.7 case A_2B_2, i.e., $l_{01}, l_{02}, l_{21}, \geq 2$ and $l_{11} = l_{12} = 1$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$2 \ \leq \ l_{01} \ = \ l_{02} \ = \ l_{21} \ \leq \ 2n, \qquad 0 \ > \ d_{11} \ \geq \ -n.$$

Proof. Note that by Lemma 2.5.4, we have $0 \leq d_{12} < l_{12} = 1$, i.e., $d_{12} = 0$, and the condition on the slopes gives $0 > d_{11}$. Also, both l_{02} and d_{02} as well as l_{01} and d_{01} must be coprime. Using the description of d_{0i} obtained in Lemma 2.5.4, we have

$$d_{01} \ = \ -\frac{d_{21}l_{01}}{l_{21}} \ \in \ \mathbb{Z}, \qquad d_{02} \ = \ l_{02}(-d_{11}) - \frac{d_{21}l_{02}}{l_{21}} \ \in \ \mathbb{Z}.$$

Since $d_{02} \in \mathbb{Z}$, we must have $l_{21} \mid d_{21}l_{02}$ and the condition $\gcd(d_{21}, l_{21}) = 1$ then delivers $l_{21} \mid l_{02}$. Similarly, $l_{21} \mid d_{21}l_{01}$ implies $l_{21} \mid l_{01}$. Moreover, assume $l_{02} \nmid l_{21}$ or $l_{01} \nmid l_{21}$. Then there is a prime number $p \in \mathbb{Z}_{\geq 2}$ or a prime number $p' \in \mathbb{Z}_{\geq 2}$ such that

$$p \mid l_{02} \quad \text{and} \quad p \, \Bigg| \, \frac{d_{21}l_{02}}{l_{21}} \qquad\qquad \text{or} \qquad\qquad p' \mid l_{01} \quad \text{and} \quad p' \, \Bigg| \, \frac{d_{21}l_{01}}{l_{21}}$$

which means $\gcd(l_{02}, d_{02}) \geq p > 1$ or $\gcd(l_{01}, d_{01}) \geq p' > 1$, a contradiction. We obtain $l_{02} \mid l_{21} \mid l_{02}$ and $l_{01} \mid l_{21} \mid l_{01}$ and therefore $l_{01} = l_{21} = l_{02}$. Moreover, since X is n-Gorenstein, we have

$$2 \ \leq \ (-d_{11})l_{01}l_{21} \ = \ (-d_{11})l_{01}^2 \ \leq \ n(l_{21} + l_{01}) \ = \ 2nl_{01}$$

by condition (a) of Lemma 2.5.8. In particular, we obtain $l_{01} = l_{02} = l_{21} \leq 2n$ and $0 < -d_{11} \leq n$. We conclude that all free parameters in Lemma 2.5.4 are bounded. □

Lemma 2.5.16 (A_2B_3)**.** *Consider in Lemma 2.5.7 case A_2B_3, i.e., $l_{01} \geq 2$, $l_{02} = 2$, $l_{11} = 2$, $l_{12} = 1$ and $l_{21} \geq 2$. If X is n-Gorenstein, then there are only finitely many possibilities for the matrix P of Lemma 2.5.4. More precisely, we obtain the following bounds*

$$l_{01} \ = \ 2, \qquad l_{21} \ = \ 2, \qquad 0 \ > \ d_{11} \ \geq -n, \qquad d_{12} \ = \ 0, \qquad d_{21} \ = \ 1.$$

Proof. Note that by Lemma 2.5.4, we have $0 \leq d_{12} < l_{12} = 1$, i.e., $d_{12} = 0$, and the condition on the slopes gives $0 > d_{11}$. Lemma 2.5.12 provides us with $l_{21} = 2$. Thus, by Lemma 2.5.4, we have $d_{21} = 1$. Moreover, using again Lemma 2.5.4, we have

$$\gcd\left(l_{01}, d_{01}\right) \ = \ 1, \qquad d_{01} \ = \ -\frac{d_{21}l_{01}}{l_{21}} \ \in \ \mathbb{Z}.$$

Therefore, $2 = l_{21} \mid d_{21}l_{01}$ implies $2 \mid l_{01}$. Since $l_{01} \geq 2$, this means $2 = l_{01}$. Moreover, since X is n-Gorenstein, we have $0 < -d_{11} \mid n$ by condition (j) of Lemma 2.5.8. In particular, $0 > d_{11} \geq -n$. □

Lemma 2.5.17. *In the situation of Lemma 2.5.7, cases A_3B_1, A_3B_2 and A_3B_3 are not possible.*

Proof. By Lemma 2.5.12, in all three cases, we have $l_{21} = 2$ and, using Lemma 2.5.4, $d_{01} = -2$. This is a contradiction to $\gcd(d_{01}, l_{01}) = 1$. □

Proof of Theorem 2.5.1. By Theorem 1.5.4, each $\mathbb{K}^*$-surface X can be obtained as $X = X(P, A)$ where the matrices P and A are as in Lemma 2.5.4; compare Construction 1.5.2. Since X is a del Pezzo surface, Lemma 2.5.7 provides us with a finite list of cases A_iB_j where $i, j \in \{1, \dots, 20\}$. Lemma 2.5.12 then reduces this list to the cases A_iB_j with $i, j \in \{1, 2, 3\}$.

Using the bounds obtained in Lemma 2.5.10 and neglecting matrices not satisfying the requirements of Lemma 2.5.4, in case A_1B_1, the only possible P-matrices are

$$P_1 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 1 & -3 & 1 & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 2 & -5 & 1 & 1 \end{bmatrix}.$$

In a similar manner, in case A_1B_2, using the bounds given in Lemma 2.5.11, only the following P-matrices are possible:

$$P_4 = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 2 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}, \qquad P_5 = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & -1 & 0 & 2 & 1 \end{bmatrix}, \qquad P_7 = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 2 \\ -1 & 3 & -2 & 1 & 1 \end{bmatrix},$$

$$P_8 = \begin{bmatrix} -1 & -5 & 1 & 5 & 0 \\ -1 & -5 & 0 & 0 & 5 \\ -1 & -4 & 0 & 1 & 4 \end{bmatrix}, \qquad P_9 = \begin{bmatrix} -1 & -5 & 1 & 5 & 0 \\ -1 & -5 & 0 & 0 & 5 \\ -1 & -3 & 0 & 2 & 3 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} -1 & -3 & 1 & 3 & 0 \\ -1 & -3 & 0 & 0 & 3 \\ -1 & 1 & -1 & 1 & 2 \end{bmatrix}, \qquad P_{11} = \begin{bmatrix} -1 & -4 & 1 & 4 & 0 \\ -1 & -4 & 0 & 0 & 4 \\ -1 & -1 & 0 & 3 & 1 \end{bmatrix}.$$

Using again Lemma 2.5.4 in conjunction with Lemma 2.5.13, no valid P-matrix can be obtained in case A_1B_3. Moreover, making use of Lemmas 2.5.14 and 2.5.4, case A_2B_1 provides us with

$$P_{12} = \begin{bmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}, \qquad P_{13} = \begin{bmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 1 & -3 & 0 & 1 \end{bmatrix},$$

$$P_{14} = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 3 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}, \qquad P_{15} = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 3 \\ -2 & 0 & -2 & 0 & 2 \end{bmatrix},$$

$$P_{16} = \begin{bmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 2 & -5 & 0 & 1 \end{bmatrix}, \qquad P_{17} = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 3 \\ -1 & 1 & -4 & 0 & 1 \end{bmatrix}.$$

Combining Lemma 2.5.4 with Lemma 2.5.15 or 2.5.16, we see that case A_2B_3 does not yield valid P-matrices whereas we obtain from case A_2B_2 the P-matrices

$$P_{18} = \begin{bmatrix} -2 & -2 & 1 & 1 & 0 \\ -2 & -2 & 0 & 0 & 2 \\ -1 & 1 & -1 & 0 & 1 \end{bmatrix}, \qquad P_{19} = \begin{bmatrix} -2 & -2 & 1 & 1 & 0 \\ -2 & -2 & 0 & 0 & 2 \\ -1 & 3 & -2 & 0 & 1 \end{bmatrix},$$

$$P_{20} = \begin{bmatrix} -2 & -2 & 1 & 1 & 0 \\ -2 & -2 & 0 & 0 & 2 \\ -1 & 5 & -3 & 0 & 1 \end{bmatrix}.$$

By Lemma 2.5.17, the cases A_3B_j do not produce any valid P-matrix. We now remove isomorphic $\mathbb{K}^*$-surfaces. In our case, Proposition 1.5.8 tells us that $\mathbb{K}^*$-surfaces X_1 and X_2 with P-matrices P_1 and P_2 are isomorphic if and only if $P_2 =$

SP_1U with admissible matrices $S \in \mathrm{GL}(3,\mathbb{Z})$ and $U \in \mathrm{GL}(5,\mathbb{Z})$. The two invertible matrices

$$S \ := \ \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad U \ := \ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

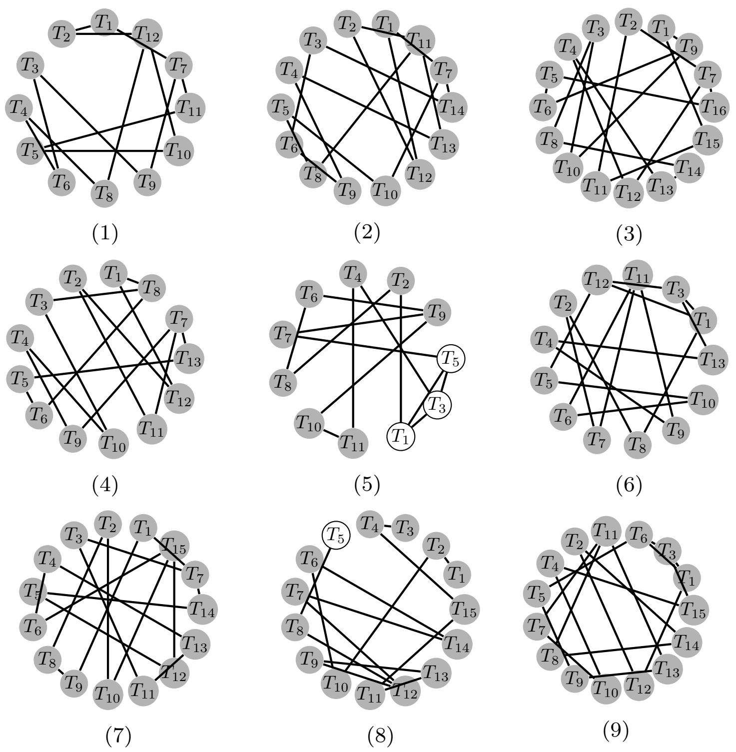
are admissible for all P-matrices P_i with $i > 12$. Hence, each $\mathbb{K}^*$-surface $X(P_i, A)$ with $i > 12$ is isomorphic to a $\mathbb{K}^*$-surface $X(P_j, A)$ with $1 \le j \le 12$ since

$$\begin{gathered} P_{13} \ = \ SP_4U, \qquad P_{14} \ = \ SP_5U, \qquad P_{15} \ = \ SP_6U, \\ P_{16} \ = \ SP_7U, \qquad P_{17} \ = \ SP_{10}U, \qquad P_{18} \ = \ SP_1U, \\ P_{19} \ = \ SP_2U, \qquad P_{20} \ = \ SP_3U. \end{gathered}$$

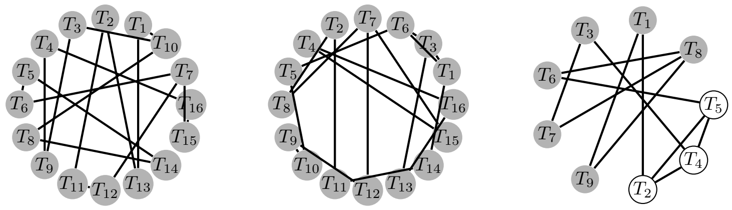
The remaining data shown in the first table of the theorem are applications of Algorithms 2.1.24, 2.3.21, 2.3.45, 2.3.24, 2.3.48 and 2.3.41. We come to the property of being almost-homogeneous. An application of Algorithm 2.4.2 shows that only in cases (5), (8) and (12) there are horizontal Demazure P-roots, namely

$$\begin{gathered} ((-1,-2,3),2,1,(1,1,1))), \qquad ((0,-4,5),2,0,(1,1,1))), \\ ((0,1,-2),2,0,(2,2,1))). \end{gathered}$$

By [6, Thm. 6.1], the existence of a horizontal Demazure P-root is equivalent to the surface being almost homogeneous. The statement about the log-terminal property is Remark 2.5.6. For the graphs of exceptional curves, we used Algorithm 2.4.8 to determine the minimal resolution $X' \to X$ of each surface X of the previous table. Algorithm 2.3.27 then delivers the respective graphs of exceptional curves $G_{X'}$; gray vertices stand for negative curves, white ones for nonnegative curves $V(X'; T_i)$.



(1) (2) (3) (4) (5) (6) (7) (8) (9)

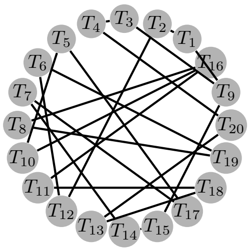


(10) (11) (12)

The subgraph of $(-k)$-curves, $k \in \mathbb{Z}_{\geq 2}$, then yields the graphs listed in the theorem. The needed self-intersection numbers are obtained using Algorithm 2.3.48; we write $d_i := V(X'; T_i)^2$.

case	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}
(1)	-1	-1	-1	-1	-1	-4	-2	-2	-2	-2	-2	-2				
(2)	-1	-1	-1	-1	-1	-3	-3	-2	-3	-2	-3	-2	-2	-2		
(3)	-1	-1	-1	-1	-1	-2	-4	-3	-4	-2	-2	-2	-3	-2	-2	-2
(4)	-1	-1	-1	-1	-1	-2	-3	-2	-2	-3	-2	-3	-2			
(5)	≥ 0	-1	≥ 0	-1	≥ 0	-2	-3	-2	-2	-2	-2					
(6)	-1	-1	-1	-1	-1	-2	-3	-2	-3	-2	-2	-2	-2			
(7)	-1	-1	-1	-1	-1	-2	-3	-3	-2	-2	-2	-2	-3	-2	-4	
(8)	-4	-1	-4	-1	≥ 0	-2	-2	-5	-2	-2	-2	-2	-2	-2	-2	
(9)	-1	-1	-1	-1	-1	-2	-2	-2	-2	-3	-2	-2	-3	-3	-2	
(10)	-1	-1	-1	-1	-1	-3	-3	-2	-4	-2	-2	-2	-4	-2	-2	-2
(11)	-1	-1	-1	-1	-1	-3	-2	-4	-2	-2	-2	-2	-2	-2	-4	-2
(12)	-1	≥ 0	-1	≥ 0	≥ 0	-2	-2	-2	-2							

For case (8), we computed a not necessarily minimal resolution $X' \to X$. The graph $G_{X'}$ and the self-intersection numbers $d_i := V(X'; T_i)^2$ are



d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
-5	-1	-5	-1	-1	-2	-2	-2	-1	-5	-2

d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}
-2	-2	-2	-2	-2	-2	-2	-2	-2

We apply the steps of Algorithm 2.4.8 with $r = 5$ formally to obtain the graph of exceptional curves of the minimal resolution. Since $d_9 = -1$, we contract $V(X'; T_9)$, where we write again X' for the contracted surface. This means we remove the corresponding vertex and edges and increase the self-intersection number of $V(X'; T_1)$, $V(X'; T_3)$ and $V(X'; T_{15})$ by one. Iterating this procedure, we contract $V(X'; T_i)$ with $i = 9, 15, 14, 7, 17$ and obtain the graph and intersection numbers shown above by shifting the indices of the remaining variables accordingly. □

Remark 2.5.18. Using the same methods, Theorem 2.5.1 can easily be expanded to higher Gorenstein index.

CHAPTER 3

Computing the Mori chamber decomposition

Given an action of a connected reductive linear algebraic group H on an algebraic variety X, Mumford [83] constructed good H-sets $X^{\mathrm{ss}}(\mathcal{L}) \subseteq X$ which depend on the choice of an ample, H-linearized line bundle $\mathcal{L}$ on X. In general, there are several distinct quotients and this *variation of GIT-quotients* is described by the GIT-fan. See the work of Dolgachev, Hu [35] and Thaddeus [97] for ample bundles on a projective variety and Arzhantsev, Berchtold, Hausen [7, 18] for the affine case. Based on [18], we provide in this chapter an algorithm to compute the GIT-fan of torus actions on affine varieties. In Mumford's sense [83], it describes the possible linearizations of the trivial bundle. Note that the torus-case is essential as more general group actions can be reduced to it [7]. An import special case is the Mori chamber decomposition of a Mori dream space.

The structure of this chapter is as follows. In Section 1, we present algorithms to compute $\mathfrak{F}$-faces, i.e., faces corresponding to torus orbits that meet an affine variety. Not only is this the basis for the computation of the GIT-fan but it is also essential for computations with Mori dream spaces, see Chapter 2. Section 2 is concerned with the computation of GIT-cones, the GIT-fan and the Mori chamber decomposition. The correspondence of the GIT-fan to qp-maximal good H-sets has been widened to the class of $(H,2)$-*maximal subsets* in [51, 5]. In Section 3, we recall the correspondence and present a direct algorithm for their computation.

Most parts of Sections 1 and 2 (as well as part of this introduction) have been published in the author's paper *Computing the GIT-fan*, see [71]. The algorithms of this chapter have been implemented in `Maple/convex` [70, 54] and also in joint work with J. Böhm and Y. Ren in `Singular` [31].

1. Computing $\mathfrak{F}$-faces

Let X be a Mori dream space with Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_r]/\mathfrak{a}$ where $\mathfrak{a} \subseteq \mathbb{K}[T_1,\dots,T_r]$ is an ideal. Then the $\mathfrak{F}$-faces of X in the sense of Section 3 of Chapter 1 are precisely the $\mathfrak{a}$-*faces* to be defined in Definition 3.1.1. In this section, we treat their computational aspects. Most of this section has appeared in [71, Sec. 2 and 3].

We will work with the following description of the toric orbits of $\mathbb{K}^r$ in terms of faces of the orthant $\gamma := \mathbb{Q}^r_{\geq 0}$: the standard torus $\mathbb{T}^r := (\mathbb{K}^*)^r$ acts via

$$\mathbb{T}^r \times \mathbb{K}^r \to \mathbb{K}^r, \qquad t \cdot x \ = \ (t_1x_1,\dots,t_rx_r).$$

Given a face $\gamma_0 \preceq \gamma$, define the *reduction* of an r-tuple z of, e.g., numbers along γ_0 as

$$z_{\gamma_0} \ := \ (z_1',\dots,z_r'), \qquad z_i' \ := \ \begin{cases} z_i, & e_i \in \gamma_0,\\ 0, & e_i \notin \gamma_0,\end{cases}$$

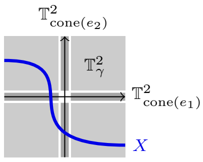
where $e_1,\dots,e_r \in \mathbb{Q}^r$ denote the canonical basis vectors. Then, one has a bijection

$$\{\text{ faces of } \gamma \ \} \ \leftrightarrow \ \{\ \mathbb{T}^r\text{-orbits}\ \}, \qquad \gamma_0 \ \mapsto \ \mathbb{T}^r_{\gamma_0} \ := \ \{t_{\gamma_0};\ t \in \mathbb{T}^r\}\ .$$

Note that in the notation of [42], $\mathbb{T}^r_{\gamma_0}$ is the $\mathbb{T}^r$-orbit through the distinguished point corresponding to the dual face $\gamma_0^* := \gamma^\perp \cap \gamma^\vee \preceq \gamma^\vee$.

Definition 3.1.1. Let $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ be an ideal. A face γ_0 of the positive orthant γ is an *$\mathfrak{a}$-face* if $V(\mathbb{T}^r_{\gamma_0}; \mathfrak{a}) \neq \emptyset$.

Let $X \subseteq \mathbb{K}^r$ be the zero set of an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$. Determining the torus orbits of $\mathbb{K}^r$ intersecting X means calculating the $\mathfrak{a}$-faces $\gamma_0 \preceq \gamma$.



Given a face $\gamma_0 \preceq \gamma$ and a polynomial $f \in \mathbb{K}[T_1, \ldots, T_r]$, we write $f_{\gamma_0} := f(T_{\gamma_0}) \in \mathbb{K}[T_{\gamma_0}]$ where $T := (T_1, \ldots, T_r)$, i.e., we replace each T_i with zero if $e_i \notin \gamma_0$. Set $\mathfrak{a}_{\gamma_0} := \langle f_{\gamma_0};\ f \in \mathfrak{a} \rangle \subseteq \mathbb{K}[T_{\gamma_0}]$. A direct $\mathfrak{a}$-face test is the following, based on a radical membership problem. This leads to a Gröbner based way to decide whether a given $\gamma_0 \preceq \gamma$ is an $\mathfrak{a}$-face.

Algorithm 3.1.2 ($\mathfrak{a}$-face verification I)**.** *Input:* a face $\gamma_0 \preceq \gamma$ and an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$.

- Return *false* if $\prod_{e_i \in \gamma_0} T_i \in \sqrt{\mathfrak{a}_{\gamma_0}}$ and return *true* otherwise.

Output: *true* if γ_0 is an $\mathfrak{a}$-face and *false* otherwise.

Remark 3.1.3. The radical membership test in Algorithm 3.1.2 can be replaced by a saturation: a face $\gamma_0 \preceq \gamma$ is an $\mathfrak{a}$-face if and only if $1 \notin \mathfrak{a}_{\gamma_0} : (T_1 \cdots T_r)^\infty$.

The main aim of this section is to speed up this direct approach by dividing out all possible torus symmetry. This is done in Algorithm 3.1.6. Further possible improvements are discussed at the end of the section.

First, consider any torus $\mathbb{T}$ and a monomial-free ideal $\mathfrak{c} \subseteq \mathcal{O}(\mathbb{T})$. Let $H \subseteq \mathbb{T}$ be the maximal subtorus leaving $V(\mathbb{T}; \mathfrak{c})$ invariant; compare Algorithm 2.1.29 and [5, Con. III.2.4.2]. Denote by $\pi\colon \mathbb{T} \to \mathbb{T}/H$ the quotient map. Note that $\mathbb{T}/H$ is again a torus. To describe π explicitly, we use the correspondence between integral matrices and homomorphisms of algebraic tori: every $n \times k$ matrix A defines a homomorphism $\alpha\colon \mathbb{T}^k \to \mathbb{T}^n$ by sending $t \in \mathbb{T}^k$ to $(t^{A_{1*}}, \ldots, t^{A_{n*}}) \in \mathbb{T}^n$ where the A_{i*} are the rows of A.

Remark 3.1.4. Let $\mathbb{T} = \mathbb{T}^k$ and $\mathbb{T}/H = \mathbb{T}^n$. The map $\pi\colon \mathbb{T}^k \to \mathbb{T}^n$ is given by any $n \times k$ matrix P of full rank satisfying

$$\ker(P) \quad = \quad \bigcap_{g \in \mathfrak{c}} \ker(P_g),$$

where to $g = a_0 T^{\nu_0} + \ldots + a_m T^{\nu_m} \in \mathfrak{c}$ we assign the $m \times k$ matrix P_g with rows $\nu_1 - \nu_0, \ldots, \nu_m - \nu_0$.

Remark 3.1.5. Let $\mathbb{T} = \mathbb{T}^k$. Fix a generating set $G := (g_1, \ldots, g_l)$ of the ideal $\mathfrak{c} \subseteq \mathbb{K}[T_1^{\pm 1}, \ldots, T_k^{\pm 1}]$. Let P_G be the stack matrix, i.e., the vertical concatenation, of $P_{g_1}, \ldots, P_{g_l}$. Compute the Hermite normal form $D = U \cdot P_G$ with an invertible integral matrix U. Choose P as the matrix consisting of the upper non-zero rows of D. Then P describes $\pi\colon \mathbb{T} \to \mathbb{T}/H$.

Proof. Clearly, P is of full rank. Since the exponent vectors of each $g \in \mathfrak{c}$ are linear combinations of the exponent vectors of $g_1, \ldots, g_l$, we have

$$\ker(P) \;=\; \ker(P_G) \;=\; \bigcap_{i=1}^{l} \ker(P_{g_i}) \;=\; \bigcap_{g \in \mathfrak{c}} \ker(P_g).$$

□

A *push forward* of $g \in \mathfrak{c}$ under π is a $h \in \mathcal{O}(\mathbb{T}/H)$ satisfying $\pi^* h = T^\mu g$ for some monomial T^μ. Suitably scaling push forwards by a monomial, we obtain the $\star$-push forward of Algorithm 2.2.13 on page 43. We define

$$\pi_\star\, \mathfrak{c} \;:=\; \langle \pi_\star\, g;\; g \in \mathfrak{c} \rangle \;\subseteq\; \mathcal{O}(\mathbb{T}/H)\,.$$

We now specialize to the case of $\mathfrak{a}$-face-verification. Given $\gamma_0 \preceq \gamma$, let $H(\gamma_0) \subseteq \mathbb{T}^r_{\gamma_0}$ be the maximal subgroup leaving $V(\mathbb{T}^r_{\gamma_0};\, \mathfrak{a}_{\gamma_0})$ invariant. Our approach reduces the dimension of the problem by using

$$V(\mathbb{T}^r_{\gamma_0};\, \mathfrak{a}) \neq \emptyset \quad \Leftrightarrow \quad V\left(\mathbb{T}^r_{\gamma_0}/H(\gamma_0);\, \pi_\star\, \mathfrak{a}_{\gamma_0}\right) \neq \emptyset\,.$$

Algorithm 3.1.6 ($\mathfrak{a}$-face verification II)**.** *Input:* an ideal $\mathfrak{a} = \langle f_1, \ldots, f_s \rangle$ in $\mathbb{K}[T_1, \ldots, T_r]$ and a face $\gamma_0 \preceq \gamma$. Set $g_i := (f_i)_{\gamma_0}$ and $G := (g_1, \ldots, g_s)$.

- Compute with Remark 3.1.5 a matrix P representing $\pi\colon \mathbb{T}^r \to \mathbb{T}^r_{\gamma_0}/H(\gamma_0)$.
- Apply Algorithm 2.2.13 to P to obtain $\pi_\star G := (\pi_\star g_1, \ldots, \pi_\star g_s)$.
- Test whether $T_1 \cdots T_n \in \sqrt{\langle \pi_\star G \rangle} \subseteq \mathbb{K}[T_1, \ldots, T_n]$. Return *false* if the test was successful. Return *true* otherwise.

Output: *true* if γ_0 is an $\mathfrak{a}$-face and *false* otherwise.

Proof. The map π is a good quotient for the $H(\gamma_0)$-action on $\mathbb{T}^r_{\gamma_0}$. Consequently, we have

$$\pi\left(\bigcap_{i=1}^{s} V\left(\mathbb{T}^r_{\gamma_0};\; g_i\right)\right) \;=\; \bigcap_{i=1}^{s} \pi\left(V\left(\mathbb{T}^r_{\gamma_0};\, g_i\right)\right) \;=\; V\left(\mathbb{T}^n;\; \pi_\star g_1, \ldots, \pi_\star g_s\right)$$

by standard properties of good quotients [73, p. 96]. This shows that $V(\mathbb{T}^r_{\gamma_0};\; \mathfrak{a}_{\gamma_0}) \neq \emptyset$ if and only if $V(\mathbb{T}^n;\; \pi_\star G)) \neq \emptyset$. □

Remark 3.1.7. If the total number of terms occurring among the generators is small as compared to the number of variables in the sense that $P = P_G$ in the first line of Algorithm 3.1.6, then we might speed up the algorithm using linear algebra as follows. Each term $\pi_\star g_i$ is linear by construction. Solve the linear system of equations $\pi_\star G = 0$. Then γ_0 is an $\mathfrak{a}$-face if and only if there is a solution in $\mathbb{T}^n$.

Remark 3.1.8. The efficiency of Algorithm 3.1.6 depends on the algorithms used for both Gröbner bases and Smith normal forms. An implementation using the respective built in functions of `Maple` gave the following timings.

	Algorithm 3.1.2	Algorithm 3.1.6 with 3.1.9(ii)
$\mathfrak{a}$-faces of $\mathfrak{a}_{2,5}$	21 s	10 s
$\mathfrak{a}$-faces of $\mathfrak{a}_{2,6}$	16 min	76 s
$\mathfrak{a}$-faces of $\mathfrak{a}_{2,7}$	$>$ 3 days	24.8 h
$\mathfrak{a}$-faces of $\mathfrak{a}_{2,3,3}$	4.03 h	44.1 min

Here, $\mathfrak{a}_{2,n}$ stands for the respective Plücker ideal and $\mathfrak{a}_{2,3,3}$ denotes the defining ideal of the Cox ring of the space $X(2,3,3)$ of complete rank two collineations [58, Thm. 1].

Let us briefly recall the connection to tropical geometry, compare [22, 78]. As seen in Section 2 of Chapter 2, we assign to a monomial-free ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ its *tropical variety*

$$\mathrm{trop}(\mathfrak{a}) \;:=\; \bigcap_{f\in\mathfrak{a}} \mathrm{trop}(f) \;\subseteq\; \mathbb{Q}^r$$

where $\mathrm{trop}(f)$ is the support of the codimension one skeleton of the normal fan of the Newton polytope of f. By [96],

$$(2) \qquad \gamma_0 \preceq \gamma \text{ is an } \mathfrak{a}\text{-face} \qquad \Leftrightarrow \qquad \mathrm{trop}(\mathfrak{a}) \cap (\gamma_0^*)^\circ \;\neq\; \emptyset\,.$$

Fixing a fan structure on $\mathrm{trop}(\mathfrak{a})$, this can be turned into a computable criterion. Note however that $\mathrm{trop}(\mathfrak{a})$ usually carries more information than needed to determine the $\mathfrak{a}$-faces and is in general harder to compute (see [22] for an algorithm).

Remark 3.1.9. To compute all $\mathfrak{a}$-faces, the number of calls to Algorithm 3.1.6 can be reduced by any of the following ideas.

(i) The *tropical prevariety* of a generating set $(f_1,\ldots,f_s)$ of $\mathfrak{a}$ is the coarsest common refinement $\prod_i \mathrm{trop}(f_i)$. Then each face $\gamma_0 \preceq \gamma$ whose dual face γ_0^* does not satisfy equation (2) with respect to $\prod_i \mathrm{trop}(f_i)$ is not an $\mathfrak{a}$-face.

(ii) A face $\gamma_0 \preceq \gamma$ is not an $\mathfrak{a}$-face if and only if there is $f \in \mathfrak{a}$ such that exactly one vertex of the Newton polytope of f lies in γ_0. Choosing any subset of $\mathfrak{a}$, we may identify some faces $\gamma_0 \preceq \gamma$ that are no $\mathfrak{a}$-faces.

(iii) Filter faces using the *Veronese embedding*: Let $\gamma_0 \preceq \gamma$ be such that there are (classically) homogeneous generators $g_1,\ldots,g_s$ of $\mathfrak{a}_{\gamma_0}$ of degree $d \in \mathbb{Z}_{\geq 0}$. The images of the g_i under

$$\mathbb{K}[T_{\gamma_0}] \to \mathbb{K}[S_\mu;\ \mu_1+\ldots+\mu_r = d]\ , \qquad T^\mu \mapsto S_\mu$$

give a linear system of equations with coefficient matrix A. If a Gauss-Jordan normal form of A contains a row with exactly one non-zero entry, γ_0 is no $\mathfrak{a}$-face. Adding redundant generators to $\mathfrak{a}_{\gamma_0}$ refines this procedure.

(iv) Let $\sigma \in S_r$ be a permutation of (the indices of) the variables $T_1,\ldots,T_r$ such that for each $f \in \mathfrak{a}$ there is $g_f \in \mathfrak{a}$ with $f \circ \sigma = g_f$. Then

$$\gamma_0 \preceq \gamma \ \mathfrak{a}\text{-face} \qquad \Leftrightarrow \qquad \sigma(\gamma_0) \;:=\; \mathrm{cone}(e_{\sigma(i)};\ e_i \in \gamma_0)\ \mathfrak{a}\text{-face}\ .$$

Proof. We prove statements (ii) and (iv). For (ii), we directly generalize the proof of [19, Prop. 9.3]: Define $\nu(f,\gamma_0)$ to be the number of vertices of the Newton polytope of f that lie in the given face $\gamma_0 \preceq \gamma$. Then γ_0 is an $\mathfrak{a}$-face if and only if there is $x \in \mathbb{K}^r$ such that

$$x_i \,\neq\, 0 \;\Leftrightarrow\; e_i \,\in\, \gamma_0\ , \qquad f(x) \;=\; f_{\gamma_0}(x) \;=\; 0 \quad \text{for all } f \in \mathfrak{a}.$$

Each polynomial f_{γ_0} is a sum of $\nu(f,\gamma_0)$ monomials. Clearly, if there is $f \in \mathfrak{a}$ such that $\nu(f,\gamma_0) = 1$ no such x can exists, i.e., γ_0 is not an $\mathfrak{a}$-face. Conversely, assume γ_0 is not an $\mathfrak{a}$-face. By definition

$$\prod_{e_i\in\gamma_0} T_i^n \;=\; \sum_j (h_j)_{\gamma_0}(f_j)_{\gamma_0} \;=\; \left(\sum_j h_j f_j\right)_{\gamma_0} \;=:\; f_{\gamma_0} \;\in\; \mathfrak{a}_{\gamma_0}$$

with a $n \in \mathbb{Z}_{>0}$ and polynomials $h_j \in \mathbb{K}[T_1,\ldots,T_r]$. Since $f \in \mathfrak{a}$ and $\nu(f,\gamma_0) = 1$ the proof is complete. For (iv), we only prove one direction. Assume $\gamma_0 \preceq \gamma$ is an $\mathfrak{a}$-face. Then there is $x \in \mathbb{K}^r$ with $f_i(x) = 0$ for all $f \in \mathfrak{a}$ and $x_i \neq 0$ if and only if $e_i \in \gamma_0$. Choose $y := (x_{\sigma(1)},\ldots,x_{\sigma(r)}) \in \mathbb{K}^r$. Then $y_i \neq 0$ if and only if $e_i \in \sigma(\gamma_0)$ and, by assumption, we have

$$f(y) \;=\; f\circ\sigma(x) \;=\; g_f(x) \;=\; 0 \qquad \text{for all } \ f \in \mathfrak{a}. \qquad \square$$

In the fourth statement of Remark 3.1.9, a special case are permutations $\sigma \in S_r$ that leave a fixed generating set of $\mathfrak{a}$ invariant. A naive approach for their detection is to test all $r!$ permutations for this property. Instead, we sketch an algorithm that uses graph theory.

Algorithm 3.1.10 (Generator symmetries)**.** *Input:* $f_1, \dots, f_s \in \mathbb{K}[T_1, \dots, T_r]$. Let $\nu_1^{(i)}, \dots, \nu_{m_i}^{(i)} \in \mathbb{Z}^r_{\geq 0}$ and $c_1^{(i)}, \dots, c_{m_i}^{(i)} \in \mathbb{K}^*$ be such that the $c_j^{(i)} T^{\nu_j^{(i)}}$ are the non-zero monomials of f_i.

- Let $G = (V, E)$ be the simple, directed, finite, bipartite graph with its set of vertices V and its set of edges E defined by

$$
\begin{aligned}
V \ &:= \ \{1, T_1, \dots, T_r\} \ \cup \ \bigcup_{i=1}^{s} \ \left\{\nu_1^{(i)}, \dots, \nu_{m_i}^{(i)}\right\}, \\
\left(\nu_j^{(i)}, T_k\right) \in E \quad &:\Leftrightarrow \quad \left(\nu_j^{(i)}\right)_k > 0, \\
\left(\nu_j^{(i)}, 1\right) \in E \quad &:\Leftrightarrow \quad \nu_j^{(i)} = (0, \dots, 0), \\
\left(\nu_j^{(i)}, \nu_k^{(i)}\right) \in E \qquad & \text{for all } i \neq j.
\end{aligned}
$$

- Assign to $\nu_j^{(i)} \in V$ the color $c_j^{(i)} \in \mathbb{K}^*$, to edges of type $(\bullet, T_k)$ the k-th entry of $\nu_j^{(i)}$, to edges of type $(\bullet, 1)$ the color $0 \in \mathbb{K}$ and to edges of the third type, we assign an unused color $c_i \in \mathbb{K}^*$.
- Compute the automorphism group $\mathrm{Aut}(G) \leq S_n$ with $n := |V|$, e.g., using [67, 81].
- Assume $T_1, \dots, T_r$ are the first r vertices in V. Return the set $\{\sigma \in S_r;\ (\sigma, \omega) \in \mathrm{Aut}(G)\}$.

Output: permutations $\sigma \in S_r$ such that, for all i, we have $f_i \circ \sigma = f_{\tau(i)}$ with a bijection $\tau \in \mathrm{Sym}(\{1, \dots, s\})$.

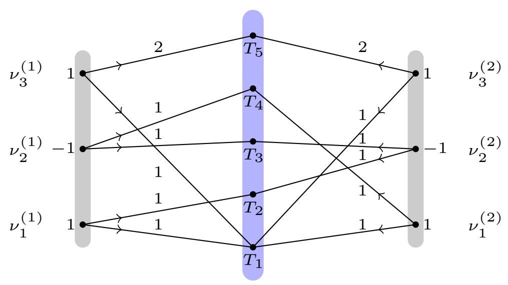
Proof. Note that we have $\mathrm{Aut}(G) \subseteq S_r \times S_{n-r}$ where n is the number of vertices. Each $(\sigma, \tau) \in \mathrm{Aut}(G)$ respects colors of edges and vertices. By construction, this means that each permutation $\sigma(T)$ of $T = (T_1, \dots, T_r)$ respects the coefficients and monomials of each f_i, i.e., induces a permutation of $f_1, \dots, f_s$. $\square$

Note that in order to use [67, 81] one first must translate G to an unweighted colored graph G' with $\mathrm{Aut}(G') \cong \mathrm{Aut}(G)$ as explained in, e.g., [80]. We sketch the construction in the following example.

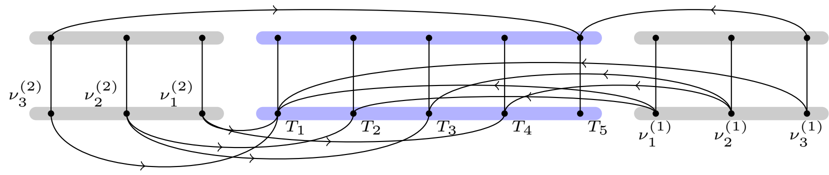
Example 3.1.11. Consider the ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \dots, T_5]$ that is generated by the two polynomials

$$
f_1 \ := \ T_1T_2 - T_3T_4 + T_1T_5^2, \qquad f_2 \ := \ T_1T_4 - T_2T_3 + T_1T_5^2
$$

where we order the exponents $\nu_j^{(i)}$ from left to right. Applying the first and second step of Algorithm 3.1.10, we obtain the colored, weighted, bipartite, directed, simple graph $G = (V, E)$



where we left out the edges of type $(\nu_j^{(i)}, \nu_k^{(i)})$ and the colors are drawn next to the vertices and edges (the shaded areas are only for highlighting purposes). We now transform G into a directed, colored, unweighted graph G' as explained in [80]. The removal of the weights (i.e., 1 and 2) of edges is achieved by adding a "layer" of vertices for each occurring weight.



Edges within the shaded areas and the colors of the vertices are not drawn. Assume $T_1, \ldots, T_5$ are the first five vertices. A direct inspection of G' shows that there is an automorphism (we show only the relevant part σ)

$$(\sigma, \tau) \;:=\; \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \ldots \\ 1 & 4 & 3 & 2 & 5 & \ldots \end{bmatrix} \;\in\; \mathrm{Aut}(G'),$$

which, interpreted as element of S_5, interchanges the variables T_2 and T_5. Then $f_1 \circ \sigma = f_2$. In particular, by Remark 3.1.9, $\gamma_0 \preceq \mathbb{Q}^5_{\geq 0}$ is an $\mathfrak{a}$-face if and only if $\sigma(\gamma_0)$ is an $\mathfrak{a}$-face.

2. Computing the GIT-fan

In this section, we develop algorithms for the computation of GIT-cones and the GIT-fan. First, we recall the necessary concepts from [18, 5]. Aspects of efficiency are discussed at the end of this section. Most of this section has been published in [71, Sec. 2 and 4].

Remark 3.2.1. Let G be a connected reductive algebraic group and X an irreducible, factorial, affine G-variety. Similar to the case of a quasitorus, one can define GIT-cones in $M_{\mathbb{Q}} := \mathbb{X}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the collection of GIT-cones $\Lambda(X, G)$ is a fan that is in bijection with the qp-maximal good G-sets of X, see Arzhantsev and Hausen [7, Sec. 3, Thm. 3.2]. Consider the good quotient

$$\pi \colon X \to Y, \qquad Y \;:=\; X /\!\!/ G^s$$

by the maximal connected semisimple subgroup $G^s \subseteq G$. Then $T := G/G^s$ is a torus, we may identify $\mathbb{X}(G) = \mathbb{X}(T)$ and $X^{\mathrm{ss}}(w) = \pi^{-1}(Y^{\mathrm{ss}}(w))$ holds for all $w \in M_{\mathbb{Q}}$, see [7, Lem. 3.3]. Moreover, there are methods to treat not necessarily affine G-varieties using torus actions on an affine variety, compare [7, Sec. 7].

Thus, by Remark 3.2.1, the case of torus actions on affine varieties is of special interest. Consider an affine variety $X \subseteq \mathbb{K}^r$ where we assume that its defining ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ is monomial-free and homogeneous with respect to a $\mathbb{Z}^k$-grading

$$q_i \;:=\; \deg(T_i) \;\in\; \mathbb{Z}^k, \qquad 1 \leq i \leq r\,.$$

Then the corresponding action of the torus $H = \mathbb{T}^k$ on $\mathbb{K}^r$ leaves the zero set $X = V(\mathbb{K}^r; \mathfrak{a}) \subseteq \mathbb{K}^r$ invariant. Let Q be the $k \times r$ matrix with columns $q_1, \ldots, q_r$. We assume that the cone $Q(\gamma) \subseteq \mathbb{Q}^k$ is of dimension k where $\gamma := \mathbb{Q}^r_{\geq 0}$. A *projected $\mathfrak{a}$-face* is a cone $Q(\gamma_0)$ with $\gamma_0 \preceq \gamma$ an $\mathfrak{a}$-face. These are exactly the *orbit cones* in the sense of Section 1 in Chapter 1. Write $\Omega_{\mathfrak{a}}$ for the set of all projected $\mathfrak{a}$-faces.

Definition 3.2.2. The *GIT-chamber* of a vector $w \in Q(\gamma) = \mathrm{cone}(q_1, \ldots, q_r) \subseteq \mathbb{Q}^k$ is the convex, polyhedral cone

$$\lambda(w) \;:=\; \bigcap_{\substack{\vartheta \in \Omega_{\mathfrak{a}},\\ w \in \vartheta}} \vartheta \;\subseteq\; \mathbb{Q}^k.$$

The *GIT-fan* of the H-action on $X = V(\mathbb{K}^r; \mathfrak{a})$ is the set $\Lambda(\mathfrak{a}, Q) = \{\lambda(w);\, w \in Q(\gamma)\}$ of all GIT-chambers.

As the name suggests, $\Lambda(\mathfrak{a}, Q)$ is indeed a (quasi-)fan in $\mathbb{Q}^k$ with $Q(\gamma)$ as its support, see [5, Thm. III.1.2.8]. However, note that the cones of the GIT-fan need not be pointed in general, compare [5, Ex. III.1.2.12]. The set of j-dimensional cones of $\Lambda(\mathfrak{a}, Q)$ will be denoted by $\Lambda(\mathfrak{a}, Q)^{(j)}$.

We turn to the computation of GIT-chambers. Let $\Omega := \{Q(\gamma_0);\, \gamma_0 \preceq \gamma\}$ be the set of projected faces of γ and let $\Omega^{(j)} \subseteq \Omega$ be the subset of j-dimensional cones. Similarly, $\Omega_{\mathfrak{a}}^{(j)} \subseteq \Omega_{\mathfrak{a}}$ is the subset of j-dimensional projected $\mathfrak{a}$-faces. We have

$$\Omega_{\mathfrak{a}}^{(k)} \;\subseteq\; \Omega_0^{(k)} \;:=\; \left\{\vartheta \in \Omega^{(k)};\ \text{all facets of } \vartheta \text{ are in } \Omega_{\mathfrak{a}}^{(k-1)}\right\} \;\subseteq\; \Omega^{(k)}$$

where the first containment is due to the fact that faces of projected $\mathfrak{a}$-faces are again projected $\mathfrak{a}$-faces, see [18, Cor. 2.4]. Given a vector w in the relative interior $Q(\gamma)^\circ$, set $\Omega^{(k)}(w)$ for the collection of all $\vartheta \in \Omega^{(k)}$ that contain w. The next algorithm determines the associated GIT-chamber $\lambda = \lambda(w)$.

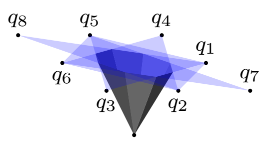
Remark 3.2.3.
(i) The set $\Omega^{(j)}$ is computed directly by taking cones over suitable subsets of $\{q_1, \ldots, q_r\}$.
(ii) The computation of $\Omega^{(j)}(w)$ can be sped up via *point location* similar to [77], i.e., we only consider cones $\vartheta \in \Omega^{(k)}$ with at least one generator lying on the same side as w of a random hyperplane subdividing $Q(\gamma)$.
(iii) For an efficient computation of $\Omega_{\mathfrak{a}}^{(j)}$, one reduces the amount of $\mathfrak{a}$-face tests as follows. Check for any $\vartheta \in \Omega^{(j)}$ if some $\gamma_0 \preceq \gamma$ with $Q(\gamma_0) = \vartheta$ is an $\mathfrak{a}$-face. As soon as such a face has been found, all other faces projecting to ϑ may be ignored in subsequent tests.

Algorithm 3.2.4 (GIT-chamber I). *Input:* a vector $w \in Q(\gamma)^\circ$ and $\Omega^{(k)}(w)$ as well as $\Omega_{\mathfrak{a}}^{(k-1)}$.

- Let $\lambda := \mathbb{Q}^k$.
- For each $\vartheta \in \Omega^{(k)}(w)$ do
  - if $\vartheta \not\supseteq \lambda$ and all facets of ϑ are in $\Omega_{\mathfrak{a}}^{(k-1)}$ then redefine λ as $\lambda \cap \vartheta$.

Output: the GIT-chamber $\lambda = \lambda(w)$ in $\mathbb{Q}^k$.

Example 3.2.5. The semiample cone computed in Example 2.3.16 is the GIT-chamber $\lambda(w)$ with $w = (0, 0, 1)$. It can be computed using only four orbit cones:



Lemma 3.2.6. *Let $\Sigma \subseteq \mathbb{Q}^k$ be a pure k-dimensional fan with convex support $|\Sigma|$ and let $\tau \in \Sigma$ be such that $\tau \cap |\Sigma|^\circ \neq \emptyset$. Then τ is the intersection over all $\sigma \in \Sigma^{(k)}$ satisfying $\tau \preceq \sigma$.*

Proof. Choose $\sigma \in \Sigma^{(k)}$ such that $\tau \preceq \sigma$. By [38, Thm. 1.11], we can write $\tau = \eta_1 \cap \ldots \cap \eta_m$ with facets $\eta_i \preceq \sigma$. Since $\tau^\circ \subseteq |\Sigma|^\circ$, also $\eta_i^\circ \subseteq |\Sigma|^\circ$ for all i. By convexity of $|\Sigma|$, for all i, there are cones $\sigma_{\eta_i}, \sigma'_{\eta_i} \in \Sigma^{(k)}$ such that $\sigma_{\eta_i} \cap \sigma'_{\eta_i} = \eta_i$. Note that $\tau \preceq \sigma_{\eta_i}$ and $\tau \preceq \sigma'_{\eta_i}$. Therefore

$$\tau \;=\; \bigcap_{i=1}^{m} (\sigma_{\eta_i} \cap \sigma'_{\eta_i}) \;\supseteq\; \bigcap_{\substack{\sigma \in \Sigma^{(k)},\\ \tau \preceq \sigma}} \sigma \;\supseteq\; \tau.$$

$\square$

Lemma 3.2.7. *Let $\lambda \in \Lambda(\mathfrak{a}, Q)^{(k)}$ and $\vartheta_0 \in \Omega_0^{(k)}$. If $\vartheta_0^\circ \cap \lambda^\circ \neq \emptyset$ then $\lambda \subseteq \vartheta_0$.*

Proof. Suppose $\lambda \not\subseteq \vartheta_0$. Choosing any $w \in \lambda^\circ \setminus \vartheta_0$ and $v \in \vartheta_0^\circ \cap \lambda^\circ$, the cone $\mathrm{cone}(v, w) \cap (\vartheta_0 \setminus \vartheta_0^\circ)$ lies on some facet $\eta_0 \preceq \vartheta_0$. By construction, $\eta_0^\circ \cap \lambda^\circ \neq \emptyset$. Since $\eta_0 \in \Omega_{\mathfrak{a}}^{(k-1)}$ holds, λ is not a GIT-chamber; a contradiction. $\square$

Proof of Algorithm 3.2.4. The algorithm terminates with a cone $\lambda \subseteq \mathbb{Q}^k$ containing the given $w \in Q(\gamma)^\circ$ and our task is to show that $\lambda = \lambda(w)$ holds. For this we establish

$$\lambda \;=\; \bigcap_{w \in \vartheta \in \Omega_0^{(k)}} \vartheta \;=\; \bigcap_{w \in \vartheta \in \Omega_{\mathfrak{a}}^{(k)}} \vartheta \;=\; \lambda(w).$$

The first equality is due to the algorithm. The third one follows from Lemma 3.2.6. Moreover, in the middle one, the inclusion "$\subseteq$" follows from $\Omega_0^{(k)} \supseteq \Omega_{\mathfrak{a}}^{(k)}$. Thus we are left with verifying "$\supseteq$" of the middle equality.

First suppose that $\lambda(w)$ is of full dimension. Then, for any $\vartheta_0 \in \Omega_0^{(k)}$ with $w \in \vartheta_0$, we obtain $\vartheta_0^\circ \cap \lambda(w)^\circ \neq \emptyset$, because $w \in \lambda(w)^\circ$ holds. Lemma 3.2.7 shows $\lambda(w) \subseteq \vartheta_0$. Thus, we obtain $\lambda \supseteq \lambda(w)$. The case of $\dim(\lambda(w)) < k$ then follows from the observation that $\lambda(w)$ is the intersection over all fulldimensional chambers $\lambda(w')$ with $w \in \lambda(w')$, see Lemma 3.2.6. $\square$

Working with $(k-1)$-dimensional projected $\mathfrak{a}$-faces in Algorithm 3.2.4 simplifies the necessary $\mathfrak{a}$-face tests compared to the following naive variant of the algorithm using k-dimensional ones.

Algorithm 3.2.8 (GIT-chamber II)**.** *Input:* a vector $w \in Q(\gamma)^\circ$ and $\Omega^{(k)}(w)$.

- Set $\lambda := \mathbb{Q}^k$.
- For each $\vartheta \in \Omega^{(k)}(w)$ do
  - if $\vartheta \not\supseteq \lambda$ and there is an $\mathfrak{a}$-face $\gamma_0 \preceq \gamma$ with $Q(\gamma_0) = \vartheta$ then redefine λ as $\lambda \cap \vartheta$.

Output: the GIT-chamber $\lambda = \lambda(w)$ in $\mathbb{Q}^k$.

The naive variant 3.2.8, in contrast, involves fewer convex geometric operations as 3.2.4 and thus can be more efficient if the latter ones are limiting the computation. See Remark 3.2.15 for a more concrete comparison of complexity aspects.

We turn to the GIT-fan. Given a full-dimensional cone $\lambda \subseteq \mathbb{Q}^k$, we denote by $\mathrm{innerfacets}(\lambda)$ the set of all facets of λ that intersect the relative interior $Q(\gamma)^\circ$. Moreover, for two sets A, B, we shortly write $A \ominus B$ for $(A \cup B) \setminus (A \cap B)$. The following algorithm computes the set of maximal cones of the GIT-fan $\Lambda(\mathfrak{a}, Q)$.

Algorithm 3.2.9 (GIT-fan)**.** *Input:* an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ and an integral matrix $Q = [q_1, \ldots, q_r]$ such that $\mathfrak{a}$ does not contain a monomial and is homogeneous with respect to the grading $q_i =: \deg(T_i)$.

- Initialize $\Lambda := \{\lambda_0\}$ with a random full-dimensional GIT-chamber λ_0 using Algorithm 3.2.4 or 3.2.8.
- $\mathcal{F} := \mathrm{innerfacets}(\lambda_0)$
- While there is $\eta \in \mathcal{F}$, do
  - use Algorithm 3.2.4 or 3.2.8 to compute the full-dimensional GIT-chamber $\lambda' \notin \Lambda$ with $\eta \preceq \lambda'$.
  - Redefine $\Lambda := \Lambda \cup \{\lambda'\}$ and $\mathcal{F} := \mathcal{F} \ominus \mathrm{innerfacets}(\lambda')$.

Output: the collection Λ of maximal cones of the GIT-fan $\Lambda(\mathfrak{a}, Q)$.

Remark 3.2.10. (i) In the first line of Algorithm 3.2.9, λ_0 can be found by successively testing whether $\lambda(w_0)$ is of full dimension where $w_0 =$

$a_1q_1 + \ldots + a_rq_r$ with random $a_i \in \mathbb{Q}_{\geq 0}$. Alternatively, if $\lambda(w_0)$ is low-dimensional, one may iteratively redefine w_0 as $w_0+\varepsilon\cdot v$ where $v \in \lambda(w_0)^\perp$ is a normal vector for some supporting hyperplane, $\varepsilon > 0$.

(ii) In the first line in the loop in Algorithm 3.2.9, let $\lambda \in \Lambda$ be the already found GIT-chamber with facet η. Then $\lambda' = \lambda(w')$ can be calculated with Algorithm 3.2.4 where $w' := w(\eta) - \varepsilon \cdot v$ for some $w(\eta) \in \eta^\circ$ and $v \in \lambda^\vee \cap \eta^\perp$ with a suitably small $\varepsilon > 0$. One possibly must reduce ε until $\lambda(w') \cap \lambda = \eta$.

Proof of Algorithm 3.2.9. Write $|\Lambda|$ for the union over all $\lambda \in \Lambda$ and $|\mathcal{F}|$ for the union over all $\eta \in \mathcal{F}$. Then, in each passage of the loop, a full-dimensional chamber of $\Lambda(\mathfrak{a}, Q)$ is added to Λ and, after adapting, $|\mathcal{F}| \cap Q(\gamma)^\circ$ is the boundary of $|\Lambda| \cap Q(\gamma)^\circ$ with respect to $Q(\gamma)^\circ$. The set $\mathcal{F}$ is empty if and only if $|\Lambda|$ equals $Q(\gamma)$. This shows that the algorithm terminates with the collection of maximal cones of $\Lambda(\mathfrak{a}, Q)$ as output. □

We can directly use Algorithm 3.2.9 to compute the *Mori chamber decomposition* of a Mori dream space X, i.e., the GIT-fan of the action of the torus $\operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)^0]$ on $\overline{X}$ where $\mathrm{Cl}(X)^0$ is the free part of the class group. The following algorithm is in the notation of Section 2.

Algorithm 3.2.11 (MDSchambers)**.** *Input:* an MDS $X = (R, \Phi)$ with a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Return the output of Algorithm 3.2.9 called with parameters $\langle G \rangle$ and Q^0.

Output: the Mori chamber decomposition of X.

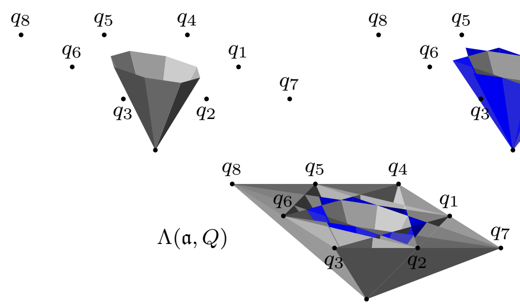
Example 3.2.12. We continue Example 2.0.14, i.e., we have a variety X with the $\mathrm{Cl}(X) = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$-graded Cox ring

$$\mathcal{R}(X) \quad = \quad \mathbb{K}[T_1, \ldots, T_r]/\langle f_1 \rangle, \qquad f_1 \ := \ T_1T_6 + T_2T_5 + T_3T_4 + T_7T_8$$

where the free parts of the degrees of the generators are given by the columns of the matrix

$$Q \ := \ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then Algorithm 3.2.11 returns the pure, three-dimensional fan $\Lambda(\mathfrak{a}, Q)$ with 37 maximal cones by the following steps:



Note that Algorithm 3.2.9 traverses a spanning tree of the (implicitly given) dual graph of $\Lambda(\mathfrak{a}, Q)$ with the maximal cones as its vertices; two vertices are connected by an edge if they share a common facet. Another traversal method for implicitly given graphs is reverse search by Avis and Fukuda [9]. By the following observation, it also can be applied to our problem.

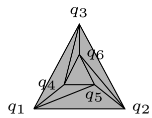
Proposition 3.2.13. *The GIT-fan $\Lambda(\mathfrak{a}, Q)$ is the normal fan of a polyhedron. If $\mathbb{Q}^k_{\geq 0} \subseteq Q(\gamma)$ then $\Lambda(\mathfrak{a}, Q)^{(k)}$ can be enumerated using reverse search.*

Proof. The first statement is [3, Cor. 10.4]. The second claim follows from the first one and [41, Sec. 3]. □

If we allow $\mathfrak{a}$ to contain monomials, the collection $\Lambda(\mathfrak{a}, Q)$ is not necessarily a quasi-fan and even if it is, Proposition 3.2.13 may not be valid any more.

Example 3.2.14. Consider the monomial ideal $\mathfrak{a} := \langle T_1T_6,\ T_2T_4,\ T_3T_5\rangle$ in the ring $\mathbb{K}[T_1, \ldots, T_6]$ and the $\mathbb{Z}^3$-grading with degree matrix

$$Q \ := \ [q_1, \ldots, q_6] \ := \ \begin{bmatrix} 2 & 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 1 & 3 \end{bmatrix}$$



Though $\mathfrak{a}$ contains monomials, using Algorithm 3.2.9 with input $\mathfrak{a}$ and Q returns the fan drawn on the right. It is not the normal fan of a polyhedron, see [30].

Remark 3.2.15. We compare the usage of Algorithm 3.2.4 (in 3.2.9) to that of 3.2.8. As a test, we compute the GIT-fans of the maximal torus action on the (affine cones over the) Grassmannians $G(2,5)$ and $G(2,6)$, using our `Maple/convex` implementation [70]. The following table lists the total number of $\mathfrak{a}$-face tests and the total number of cones ϑ entering the fourth line of Algorithms 3.2.4 and 3.2.8

	Algorithm 3.2.9 with 3.2.4		Algorithm 3.2.9 with 3.2.8	
	$\sharp$ $\mathfrak{a}$-face-tests	$\sharp$ cones ϑ	$\sharp$ $\mathfrak{a}$-face-tests	$\sharp$ cones ϑ
$G(2,5)$	300	21	469	20
$G(2,6)$	6574	50	21012	52

Note that in Algorithm 3.2.4, the $\mathfrak{a}$-face tests concern faces of lower dimension than in Algorithm 3.2.8 and thus are even faster.

Remark 3.2.16. (i) Intermediate storage of occurring cones and their intersections in Algorithms 3.2.4 and 3.2.8 saves time in further computations.
(ii) The traversal of the GIT-fan can take advantage of symmetries: Assume we know a subgroup $G \le S_r$ keeping the ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ invariant and each element of G induces a lattice isomorphism of $\mathbb{Z}^k$. Then in each step of Algorithm 3.2.9, we insert instead of λ' the orbit $G \cdot \lambda'$ into Λ and adjust $\mathcal{F}$ accordingly. See [65, Ch. 3.1] for a more thorough study of the traversal of symmetric fans.

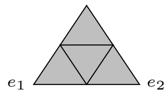
To finish this section, we consider torus actions on the affine cone over the Grassmannian $G(2,n)$ induced by a diagonal action on the Plücker coordinate space $\mathbb{K}^r$, where $r = \binom{n}{2}$. Such actions will be encoded by assigning to the variable T_i the i-th column q_i of a matrix $Q = [q_1, \ldots, q_r]$. Moreover, we write $\mathfrak{a}_{2,n} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ for the Plücker ideal.

We compute both, the GIT-fan of the torus action on $V(\mathbb{K}^r; \mathfrak{a}_{2,n})$ as well as the GIT fan of the ambient space $\mathbb{K}^r$. The latter coincides with the so-called *Gelfand Kapranov Zelevinsky decomposition* $\mathrm{GKZ}(Q)$, i.e., the coarsest common refinement of all normal fans having their rays among the cones over the columns of Q and with support $\mathrm{cone}(q_1, \ldots, q_r)$. In general, the Gelfand Kapranov Zelevinsky decomposition is a refinement of the GIT-fan. See [28] for a toric background.

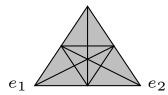
Below, the drawings show (projections of) the intersections of the respective fans with the standard simplex.

Example 3.2.17. (i) For $n = 4$, the ideal $\mathfrak{a}_{2,4} = \langle T_1T_6 - T_2T_5 + T_3T_4\rangle \subseteq \mathbb{K}[T_1, \ldots, T_6]$ is homogeneous with respect to

$$Q \ := \ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$



$\Lambda(\mathfrak{a}_{2,4}, Q)$

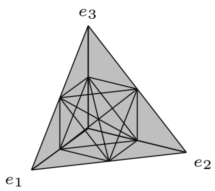


GKZ(Q)

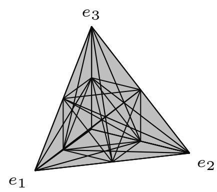
Using Algorithm 3.2.9, we obtain the four maximal GIT-chambers of $\Lambda(\mathfrak{a}_{2,4}, Q)$. The finer fan $\mathrm{GKZ}(Q)$ has twelve maximal cones.

(ii) For $n = 5$, the ideal $\mathfrak{a}_{2,5} \subseteq \mathbb{K}[T_1, \ldots, T_{10}]$ is homogeneous with respect to

$$Q \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$



$\Lambda(\mathfrak{a}_{2,5}, Q)$



GKZ(Q)

By Algorithm 3.2.9, there are twelve four-dimensional cones in $\Lambda(\mathfrak{a}_{2,5}, Q)$ whereas $\mathrm{GKZ}(Q)$ contains 336 such cones.

(iii) For $n = 6$, the ideal $\mathfrak{a}_{2,6} \subseteq \mathbb{K}[T_1, \ldots, T_{15}]$ is homogeneous with respect to

$$Q \ = \ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Using Algorithm 3.2.9, we obtain the 81 five-dimensional cones of the GIT-fan $\Lambda(\mathfrak{a}_{2,6}, Q)$. The fan $\mathrm{GKZ}(Q)$ has 61920 such cones.

3. Generalization to $(H, 2)$-maximal sets

We present a direct algorithm to characterize more general good H-sets defined in [51, 5]. First, we recall the required notions and correspondences from [51, 5] and then treat the computational aspects.

Let K be a finitely generated abelian group and consider an affine, irreducible, normal variety $X := \operatorname{Spec} A$ with an integral, normal, K-graded, affine algebra A. Then the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ acts on X. Write Ω_X for the set of orbit cones. To a collection $\Phi \subseteq \Omega_X$ we assign the a subset $U(\Phi) \subseteq X$ and to an H-invariant subset $U \subseteq X$ we assign a collection of orbit cones $\Phi(U) \subseteq \Omega_X$ where

$$\begin{aligned} U(\Phi) \ &:= \ \{x \in X;\ \text{there is } \vartheta \in \Phi \text{ with } \vartheta \preceq \vartheta_x\} \ \subseteq \ X, \\ \Phi(U) \ &:= \ \{\vartheta_x;\ x \in U \text{ and } H \cdot x \text{ closed in } U\} \qquad \subseteq \ \Omega_X. \end{aligned}$$

Recall that given a good H-set $U \subseteq X$ and an open subset $U' \subseteq U$, the inclusion $U' \subseteq U$ is *H-saturated* if $p^{-1}(p(U')) = U'$ with the good quotient $p\colon U \to U /\!\!/ H$.

Definition 3.3.1. We call a good H-set $U \subseteq X$ a *$(H, 2)$-maximal subset* if the quotient space $U /\!\!/ H$ is an A_2-variety such that U is maximal with respect to H-saturated inclusion amidst all good H-sets with an A_2-variety as quotient space.

Theorem 3.3.2. *See* [5, Thm. III.1.4.4]. *In the above situation, assume that X is H-factorial. We have mutually inverse, order-reversing bijections*

$$\left\{ \begin{array}{c} \textit{maximal bunches of} \\ \textit{orbit cones in } \Omega_X \end{array} \right\} \ \longleftrightarrow \ \{\, (H,2)\textit{-maximal subsets of } X \,\}$$

$$\begin{array}{ccc} \Phi & \mapsto & U(\Phi), \\ \Phi(U) & \leftarrow\!\shortmid & U. \end{array}$$

Under these maps, the qp-maximal subsets of X correspond to the sets of semistable points $U(\Phi(w)) = X^{\mathrm{ss}}(w)$ where $w \in \lambda^\circ$ with $\lambda \in \Lambda(X, H)$.

In particular, if we can compute all maximal bunches of orbit cones in Ω_X, we also obtain (the information stored in) the GIT-fan. We now turn to their computation in the setting of Chapter 2, i.e., consider an MDS $X = (R, \Phi)$ with $\Phi = \{\vartheta_1, \dots, \vartheta_s\}$ and GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$. Let Ω be the set of all orbit cones.

Definition 3.3.3. The *overlapping graph* is the finite, undirected, simple graph $G_\Omega = (V, E)$ with vertex set $V = \Omega$ and set of edges $E \subseteq V \times V$ given by

$$(\vartheta_1, \vartheta_2) \in E \qquad :\Leftrightarrow \qquad \vartheta_1^\circ \cap \vartheta_2^\circ \neq \emptyset.$$

Recall that a *clique* of a finite, directed, simple graph $G = (V, E)$ is a subset $C \subseteq V$ such that its induced subgraph is complete. A clique is *maximal* if it is maximal with respect to containment.

Algorithm 3.3.4 (GRH2max)**.** *Input:* a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

- Compute the collection $\Omega = \{\vartheta_1, \dots, \vartheta_m\}$ of orbit cones, i.e., the set of all $Q^0(\gamma_0)$ where $\gamma_0 \in F_{\mathfrak{F}}$.
- Calculate the overlapping graph $G_\Omega = (\Omega, E)$ by checking for each two orbit cones ϑ_i, $\vartheta_j \in \Omega$ whether $\vartheta_i^\circ \cap \vartheta_j^\circ \neq \emptyset$.
- Determine the maximal cliques $C_1, \dots, C_m$ of G as subgraphs $C_i = (\Phi_i, E_i)$. Interpret the $\Phi_i \subseteq \Omega$ as BUNs.

For the third step, we use the following algorithm *MaxCliques*(G, C', A', H') from Kreher and Stinson's book [74] with input $G := G_\Omega = (V, E)$ and $C', A', H' \subseteq V$ where we initialize $C' := A' := H' := \emptyset$. We assume the vertices of V are ordered with respect to a relation $>$.

- Set $R := \emptyset$, define $V_+ := \{v \in V;\, v > \max(x;\, x \in C')\}$ and set
$$H(C) \;:=\; \begin{cases} V, & C' = \emptyset, \\ A' \cap H' \cap V_+, & C' \neq \emptyset. \end{cases}$$
- If $H(C) = \emptyset$, then C' is a maximal clique and we redefine $R := \{C'\}$.
- For each $v \in H(C)$ do
  - define $C := C' \cup \{v\}$. Let $\mathrm{Adj}(v) \subseteq V$ be the neighbors of v in G.
  - Insert into R the result returned by the recursive call to *MaxCliques* with input G, C, $\mathrm{Adj}(v)$, and $H(C)$.
- Return R.

Output: the list $(\Phi_1, \dots, \Phi_m)$ of all maximal BUNs of orbit cones. They correspond to the $(H, 2)$-maximal subsets of $\operatorname{Spec} R$.

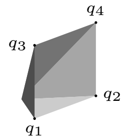
Proof. The correspondence to $(H, 2)$-maximal subsets of X is Theorem 3.3.2. The correctness of the algorithm *MaxCliques* is as in [74]: each maximal clique $C \subseteq V$ is obtained by enlarging a smaller clique $C' \subseteq C$ iteratively by an element $v \in \bigcap_{c \in C'} \mathrm{Adj}(c)$. To avoid repeated rediscoveries of the same clique, we may restrict to elements

$$v \;\in\; V_+ \;\cap\; \mathrm{Adj}(c^\bullet) \;\cap\; H(C' \setminus \{c^\bullet\}), \qquad c^\bullet \;:=\; \max(c;\, c \in C').$$

Thus, in each call, C' is a clique that is maximal if and only if $H(C) = \emptyset$. Since in the first step, the algorithm starts with all subsets of V of cardinality one, this completes the proof. $\square$

Example 3.3.5. In the situation of Algorithm 3.3.4, assume we are given a GR $R = (\emptyset, Q, Q^0, P, F_{\mathfrak{F}})$ with AGH $Q = (\mathbb{Z}^4, \mathbb{Z}^3, Q^0)$ where

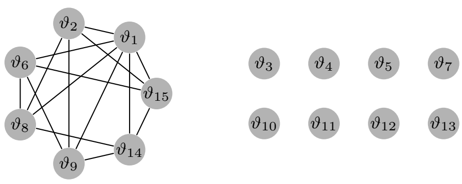
$$Q^0 \;:=\; [q_1, \ldots, q_4] \;:=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$



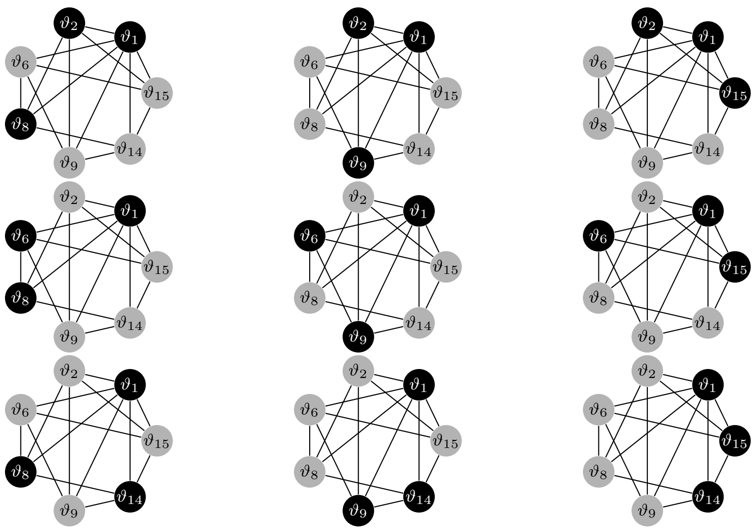
Then there are precisely fifteen orbit cones $\Omega = \{\vartheta_1, \ldots, \vartheta_{15}\}$ which we order by $\vartheta_i > \vartheta_j$ if $i > j$. They are

$$\begin{aligned}
\vartheta_1 &= \mathrm{cone}(q_1, q_2, q_3, q_4), & \vartheta_2 &= \mathrm{cone}(q_1, q_2, q_3),\\
\vartheta_3 &= \mathrm{cone}(q_4), & \vartheta_4 &= \mathrm{cone}(q_3),\\
\vartheta_5 &= \mathrm{cone}(q_1), & \vartheta_6 &= \mathrm{cone}(q_2, q_3, q_4),\\
\vartheta_7 &= \mathrm{cone}(q_3, q_4), & \vartheta_8 &= \mathrm{cone}(q_1, q_3, q_4),\\
\vartheta_9 &= \mathrm{cone}(q_1, q_4), & \vartheta_{10} &= \mathrm{cone}(q_2, q_4),\\
\vartheta_{11} &= \mathrm{cone}(q_2), & \vartheta_{12} &= \mathrm{cone}(q_1, q_2),\\
\vartheta_{13} &= \mathrm{cone}(q_1, q_3), & \vartheta_{14} &= \mathrm{cone}(q_2, q_3),\\
\vartheta_{15} &= \mathrm{cone}(q_1, q_2, q_3).
\end{aligned}$$

The overlapping graph G_Ω is the following undirected, simple graph with nine components



Besides the maximal cliques $\{\vartheta_i\}$ with $i \in \{3, 4, 5, 7, 10, 11, 12, 13\}$, Algorithm 3.3.4 detects the following nine 3-cliques (highlighted in black; isolated vertices are not drawn); in particular, there are seventeen $(H, 2)$-maximal subsets of Spec R.



Remark 3.3.6. In the situation of Algorithm 3.3.4, set $s := |\Omega|$. By [74, Sec. 4.3.1], the asymptotic worst case running time of the subroutine *MaxCliques* is $O(s \cdot n)$ where n is the number of (not necessarily maximal) cliques in G_Ω. Its average running time is $O(s^{\log_2(s)+1})$.

CHAPTER 4

Modifications of Mori dream spaces

This chapter is about the computation of Cox rings of modified Mori dream spaces. More precisely, given a modification $X_2 \to X_1$ of projective varieties, e.g., a sequence of blow ups, where one of the Cox rings $\mathcal{R}(X_i)$ is known, we provide computational methods to obtain information about the other Cox ring. To this end, we develop algorithms concerning the tasks

- verifying finite generation,
- producing a guess of generators,
- verifying a guess of generators,
- computing relations between generators.

Amongst others, we devise a technique to provide and verify a systematic guess for generators of the Cox ring of a blow up $X_2 \to X_1$ of a Mori dream space X_1; it terminates if and only if $\mathcal{R}(X_2)$ is finitely generated.

Section 1 develops the algebraic tools needed to relate the Cox rings $\mathcal{R}(X_1)$ and $\mathcal{R}(X_2)$. In Section 2 we adjust the methods of toric ambient modifications to our setting. Here, we also treat the contraction problem. We develop and present an algorithmic framework for modifications of Mori dream spaces in Section 3. This will enable us to compute explicit examples. As a first application, we compute in Section 4 the Cox rings of all Gorenstein log del Pezzo surfaces of Picard number one that do not admit a non-trivial $\mathbb{K}^*$-action. Section 5 concerns the third item and presents an automated approach to compute the Cox rings of blow ups of Mori dream spaces along a subvariety in the smooth locus. The last section, Section 6, treats the special case where the Cox ring is generated by proper transforms of hyperplanes. As an application, we determine the Cox rings of blow ups of $\mathbb{P}_3$ in certain special point configurations.

The first section has already been published in the paper *On Chow quotients of torus actions* [10] jointly with Hendrik Bäker and Jürgen Hausen. The remaining sections have been published in the paper *Computing Cox rings* [57] together with Jürgen Hausen and Antonio Laface. In an ongoing project with U. Derenthal, J. Hausen, A. Heim and A. Laface we have implemented the algorithms in the software system Singular [31] and plan to use it to compute Cox rings of cubic surfaces and smooth Fano threefolds [34].

1. Modifications and H-factoriality

In this section, we provide a general machinery to study the effect of modifications on the Cox ring. Similar to [51], we use toric embeddings. In contrast to the geometric criteria given there, our approach here is purely algebraic, based on results of Bechtold [15]. The crux of the matter is a construction of factorially graded rings out of given ones. This section has been published in [10, Sec. 3] together with H. Bäker and J. Hausen.

Let us recall from Sections 2 and 3 of Chapter 1 the necessary algebraic concepts. Let K be a finitely generated abelian group and R a finitely generated integral K-graded $\mathbb{K}$-algebra. A homogeneous nonzero nonunit $f \in R$ is called *K-prime* if $f \mid gh$ with homogeneous $g, h \in R$ always implies $f \mid g$ or $f \mid h$. The algebra R is called *factorially K-graded* if every homogeneous nonzero nonunit $f \in R$ is a product of K-primes.

We enter the construction of factorially graded rings. Consider a grading of the polynomial ring $\mathbb{K}[T_1, \ldots, T_{r_1}]$ by a finitely generated abelian group K_1 such that the variables T_i are homogeneous. Then we have a pair of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{k_1} \arrow[r, "Q_1^*"] & \mathbb{Z}^{r_1} \arrow[r, "P_1"] & \mathbb{Z}^n & \\ 0 & K_1 \arrow[l] & \mathbb{Z}^{r_1} \arrow[l, "Q_1"] & \mathbb{Z}^n \arrow[l, "P_1^*"] & 0 \arrow[l] \end{tikzcd}$$

where $Q_1 \colon \mathbb{Z}^{r_1} \to K_1$ is the degree map sending the i-th canonical basis vector e_i to $\deg(T_i) \in K_1$. We enlarge P_1 to an $n \times r_2$ matrix P_2 by concatenating further $r_2 - r_1$ columns. This gives a new pair of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{k_2} \arrow[r, "Q_2^*"] & \mathbb{Z}^{r_2} \arrow[r, "P_2"] & \mathbb{Z}^n & \\ 0 & K_2 \arrow[l] & \mathbb{Z}^{r_2} \arrow[l, "Q_2"] & \mathbb{Z}^n \arrow[l, "P_2^*"] & 0 \arrow[l] \end{tikzcd}$$

Construction 4.1.1. Given a K_1-homogeneous ideal $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$, we transfer it to a K_2-homogeneous ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ by taking extensions and contractions according to the scheme

$$\begin{tikzcd} \mathbb{K}[T_1, \ldots, T_{r_2}] \arrow[d, "\imath_2"'] & & \mathbb{K}[T_1, \ldots, T_{r_1}] \arrow[d, "\imath_1"] \\ \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] & \mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}] \arrow[l, "\pi_2^*"] \arrow[r, "\pi_1^*"'] & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}] \end{tikzcd}$$

where $\imath_1, \imath_2$ are the canonical embeddings and π_i^* are the homomorphisms of group algebras defined by $P_i^* \colon \mathbb{Z}^n \to \mathbb{Z}^{r_i}$.

Remark 4.1.2. From a geometric point of view, the passage from the Laurent polynomial ring to the polynomial ring corresponds to taking the closure. Back on the algebraic side this means saturating the ideal, see Algorithm 2.2.14.

Now, let $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$ be a K_1-homogeneous ideal and $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ the transferred K_2-homogeneous ideal. Our result relates factoriality properties of the algebras $R_1 := \mathbb{K}[T_1, \ldots, T_{r_1}]/I_1$ and $R_2 := \mathbb{K}[T_1, \ldots, T_{r_2}]/I_2$ to each other.

Theorem 4.1.3. *Assume R_1, R_2 are integral, $T_1, \ldots, T_{r_1}$ define K_1-primes in R_1 and $T_1, \ldots, T_{r_2}$ define K_2-primes in R_2. Then the following statements are equivalent.*

(i) *The algebra R_1 is factorially K_1-graded.*
(ii) *The algebra R_2 is factorially K_2-graded.*

Proof. First, observe that the homomorphisms π_j^* embed $\mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}]$ as the degree zero part of the respective K_j-grading and fit into a commutative diagram

$$
\begin{tikzcd}
I_2 \arrow[d] & \subseteq & \mathbb{K}[T_1, \ldots, T_{r_2}] \arrow[d, "\imath_2"'] & & \mathbb{K}[T_1, \ldots, T_{r_1}] \arrow[d, "\imath_1"] & \supseteq & I_1 \arrow[d] \\
I_2' & \subseteq & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] \arrow[rr, "\psi\colon T_i \mapsto \begin{cases} T_i & 1\le i\le r_1 \\ 1 & r_1+1\le i\le r_2 \end{cases}"] & & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}] & \supseteq & I_1' \\
& & I_2'' \arrow[u] \arrow[ull] & \mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}] \arrow[ul, "\pi_2^*"'] \arrow[ur, "\pi_1^*"] & I_1'' \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

The factor ring R_1' of the extension $I_1' := \langle \imath_1(I_1) \rangle$ is obtained from R_1 by the localization with respect to K_1-primes $T_1, \ldots, T_{r_1}$:

$$R_1' \ := \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]/I_1' \ \cong \ (R_1)_{T_1 \cdots T_{r_1}}.$$

The ideal I_1'' is the degree zero part of I_1'. Thus, its factor algebra is the degree zero part of R_1':

$$R_1'' \ := \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]_0/I_1'' \ \cong \ (R_1')_0.$$

Note that $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]$ and hence R_1' admit units in every degree. Thus, [15, Thm. 1.1] yields that R_1 is factorially K_1-graded if and only if R_1'' is a UFD.

The homomorphism ψ restricts to an isomorphism ψ_0 of the respective degree zero parts. Thus, the shifted ideal $I_2'' := \psi_0^{-1}(I_1'')$ defines an algebra R_2'' isomorphic to R_1'':

$$R_2'' \ := \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]_0/I_2'' \ \cong \ R_1''.$$

The ideal $I_2' := \langle \pi_2^*((\pi_1^*)^{-1}(I_1')) \rangle$ has I_2'' as its degree zero part and $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]$ admits units in every degree. The associated K_2-graded algebra

$$R_2' \ := \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]/I_2' \ \cong \ (R_2)_{T_1 \cdots T_{r_2}}$$

is the localization of R_2 by the K_2-primes $T_1, \ldots, T_{r_2}$. Again by [15, Thm. 1.1] we obtain that $R_2'' \cong R_1''$ is a UFD if and only if R_2 is factorially K_2-graded. This proves the assertion. $\square$

The following observation is intended for practical purposes; it reduces, for example, the number of necessary primality tests.

Proposition 4.1.4. *Assume that R_1 is integral and the canonical map $K_2 \to K_1$ admits a section (e.g., K_1 is free).*

(i) *Let $T_1, \ldots, T_{r_1}$ define K_1-primes in R_1 and $T_{r_1+1}, \ldots, T_{r_2}$ define K_2-primes in R_2. If no T_j with $j \geq r_1 + 1$ divides a T_i with $i \leq r_1$, then also $T_1, \ldots, T_{r_1}$ define K_2-primes in R_2.*

(ii) *The ring R_2 is integral. Moreover, if R_1 is normal and $T_{r_1+1}, \ldots, T_{r_2}$ define primes in R_2 (e.g., they are K_2-prime and K_2 is free), then R_2 is normal.*

Proof. The exact sequences involving the grading groups K_1 and K_2 fit into a commutative diagram where the upwards sequences are exact and $\mathbb{Z}^{r_2 - r_1} \to K_2'$ is an isomorphism:

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & K_1 \arrow[l] \arrow[u] & \mathbb{Z}^{r_1} \arrow[l, "Q_1"'] \arrow[u] & \mathbb{Z}^n \arrow[l, "P_1^*"'] \arrow[u] & 0 \arrow[l] \\
0 & K_2 \arrow[l] \arrow[u] & \mathbb{Z}^{r_2} \arrow[l, "Q_2"] \arrow[u] & \mathbb{Z}^n \arrow[l, "P_2^*"] \arrow[u] & 0 \arrow[l] \\
 & K_2' \arrow[u] & \mathbb{Z}^{r_2-r_1} \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

Moreover, denoting by $K_1' \subseteq K_2$ the image of the section $K_1 \to K_2$, there is a splitting $K_2 = K_2' \oplus K_1'$. As $K_2' \subseteq K_2$ is the subgroup generated by the degrees of $T_{r_1+1}, \dots, T_{r_2}$, we obtain a commutative diagram

$$
\begin{tikzcd}
\mathbb{K}[T_1, \dots, T_{r_2}] \arrow[d, "\imath_2"] & \\
\mathbb{K}[T_1, \dots, T_{r_1}, T_{r_1+1}^{\pm 1}, \dots, T_{r_2}^{\pm 1}] \arrow[r, "\psi\colon T_i \mapsto \begin{cases} T_i & 1 \le i \le r_1 \\ 1 & r_1+1 \le i \le r_2 \end{cases}"] & \mathbb{K}[T_1, \dots, T_{r_1}] \\
\mathbb{K}[T_1, \dots, T_{r_1}, T_{r_1+1}^{\pm 1}, \dots, T_{r_2}^{\pm 1}]_0 \arrow[u, "\mu"] \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

where the map μ denotes the embedding of the degree zero part with respect to the K_2'-grading. By the splitting $K_2 = K_2' \oplus K_1'$, the image of μ is precisely the Veronese subalgebra associated to the subgroup $K_1' \subseteq K_2$. For the factor rings R_2 and R_1 by the ideals I_2 and I_1, the above diagram leads to the following situation

$$
\begin{tikzcd}
R_2 \arrow[d, "\imath_2"] & \\
(R_2)_{T_{r_1+1} \cdots T_{r_2}} \arrow[r, "\psi"] & R_1 \\
\left((R_2)_{T_{r_1+1} \cdots T_{r_2}}\right)_0 \arrow[u, "\mu"] \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

To prove (i), consider a variable T_i with $1 \le i \le r_1$. We have to show that T_i defines a K_2-prime element in R_2. By the above diagram, T_i defines a K_1'-prime element in $((R_2)_{T_{r_1+1} \cdots T_{r_2}})_0$, the Veronese subalgebra of R_2 defined by $K_1' \subseteq K_2$. Since every K_2-homogeneous element of $(R_2)_{T_{r_1+1} \cdots T_{r_2}}$ can be shifted by a homogeneous unit into $((R_2)_{T_{r_1+1} \cdots T_{r_2}})_0$, we see that T_i defines a K_2-prime in $(R_2)_{T_{r_1+1} \cdots T_{r_2}}$, see [5, Lem. III.4.1.9]. By assumption, $T_{r_1+1}, \dots, T_{r_2}$ define K_2-primes in R_2 and are all coprime to T_i. It follows from [5, Lem. III.4.1.7] that T_i defines a K_2-prime in R_2.

We turn to assertion (ii). As just observed, the degree zero part $((R_2)_{T_{r_1+1} \cdots T_{r_2}})_0$ of the K_2'-grading is isomorphic to R_1 and thus integral (normal if R_1 is so). Moreover, the K_2'-grading is free in the sense that the associated torus $\operatorname{Spec} \mathbb{K}[K_2']$ acts freely on $\operatorname{Spec} (R_2)_{T_{r_1+1} \cdots T_{r_2}}$. It follows that $(R_2)_{T_{r_1+1} \cdots T_{r_2}}$ is integral (normal if R_1 is so). Construction 4.1.1 gives that R_2 is integral. Moreover, if $T_{r_1+1}, \dots, T_{r_2}$ define primes in R_2, we can conclude that R_2 is normal, see [5, Lem. IV.1.2.7]. $\square$

Let us apply the results to Cox rings, see Section 2 in Chapter 1 for the basic theory. In the setting fixed at the beginning of the section, we assume additionally that the columns of P_2 are pairwise different primitive vectors in $\mathbb{Z}^n$ and those of P_1 generate $\mathbb{Q}^n$ as a convex cone. Suppose we have toric Cox constructions $\pi_i\colon \widehat{Z}_i \to Z_i$ where $\widehat{Z}_i \subseteq \mathbb{K}^{r_i}$ are open toric subvarieties and π_i are toric morphisms defined by P_i, see [27]. Then the canonical map $Z_2 \to Z_1$ is a toric modification. Consider the ideal I_1 as discussed before and the geometric data

$$\overline{X}_1 \ := \ V(I_1) \subseteq \mathbb{K}^{r_1}, \qquad \widehat{X}_1 \ := \ \overline{X}_1 \cap \widehat{Z}_1, \qquad X_1 \ := \ \pi_1\left(\widehat{X}_1\right) \ \subseteq \ Z_1.$$

Assume that R_1 is factorially K_1-graded and $T_1, \ldots, T_{r_1}$ define pairwise non-associated K_1-prime elements in R_1. Then R_1 is the Cox ring of X_1, see [5]. Our statement concerns the Cox ring of the proper transform $X_2 \subseteq Z_2$ of $X_1 \subseteq Z_1$ with respect to $Z_2 \to Z_1$.

Corollary 4.1.5. *In the above setting, assume that R_2 is normal and the variables $T_1, \ldots, T_{r_2}$ define pairwise non-associated K_2-prime elements in R_2. Then the K_2-graded ring R_2 is the Cox ring of X_2.*

Proof. According to Theorem 4.1.3, the ring R_2 is factorially K_2-graded. Moreover, with the toric Cox construction $\pi_2\colon \widehat{Z}_2 \to Z_2$, we obtain that R_2 is the algebra of functions of the closure $\widehat{X}_2 \subseteq \widehat{Z}_2$ of $\pi_2^{-1}(X_2 \cap \mathbb{T}^{r_2})$. Thus, [5] yields that R_2 is the Cox ring of X_2. ☐

Example 4.1.6. We start with the UFD $R_1 = \mathbb{K}[T_1, \ldots, T_8]/I_1$ where the ideal I_1 is defined as

$$I_1 \ = \ \langle T_1T_2 + T_3T_4 + T_5T_6 + T_7T_8 \rangle.$$

Then I_1 is homogeneous with respect to the standard $K_1 = \mathbb{Z}$-grading given by $Q_1 = [1, \ldots, 1]$. Then $P_1 = [e_0, e_1, \ldots, e_7]$ is Gale dual to Q_1 where $e_0 = -e_1 - \ldots - e_7$ and the $e_i \in \mathbb{Z}^7$ are the canonical basis vectors. Concatenation of $e_1 + e_3$ yields a matrix P_2. Applying Construction 4.1.1, we obtain $R_2 = \mathbb{K}[T_1, \ldots, T_9]/I_2$ where

$$I_2 \ = \ \langle T_1T_2T_9 + T_3T_4T_9 + T_5T_6 + T_7T_8 \rangle$$

together with a $K_2 = \mathbb{Z}^2$-grading. As predicted by Theorem 4.1.3, R_2 is again a UFD.

2. Toric ambient modifications

Using the tools of Section 1, we upgrade the technique of toric ambient modifications developed in [51]. It will serve as foundation for the algorithmic framework for modifications of Mori dream spaces presented in Section 3. This section has been published together with J. Hausen and A. Laface, in [57, Sec. 2].

Recall from Section 2 in Chapter 1 that a *Mori dream space* is a normal projective variety with finitely generated Cox ring $\mathcal{R}(X)$ and class group $\mathrm{Cl}(X)$. The *characteristic quasitorus* $H := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ acts on $\widehat{X} := \operatorname{Spec}_X \mathcal{R}$ and the canonical map, the *characteristic space* over X, $p\colon \widehat{X} \to X$ is a good quotient for this action. One has the *total coordinate space* $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ and a canonical H-equivariant open embedding $\widehat{X} \subseteq \overline{X}$.

For Mori dream spaces X, we obtain canonical embeddings into toric varieties Z relating the geometry of X to that of its ambient variety. Let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise non-associated $\mathrm{Cl}(X)$-prime generators of the Cox ring $\mathcal{R}(X)$.

This gives rise to a commutative diagram

$$
\begin{array}{ccc}
\overline{X} & \subseteq & \mathbb{K}^r \\
\uparrow & & \uparrow \\
\widehat{X} & \subseteq & \widehat{Z} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
X & \subseteq & Z
\end{array}
$$

where the embedding $\overline{X} \subseteq \mathbb{K}^r$ of the total coordinate space is concretely given by $\overline{x} \mapsto (f_1(\overline{x}), \dots, f_r(\overline{x}))$, we have $\widehat{X} = \overline{X} \cap \widehat{Z}$ and $p\colon \widehat{Z} \to Z$ is the toric characteristic space; compare Construction 1.2.5 and [27]. The induced embedding $X \subseteq Z$ of quotients is as wanted, see [5, Sec. III.2.5] for details.

Definition 4.2.1. In the above setting, we call $X \subseteq Z$ a *canonically embedded Mori dream space (CEMDS).*

Remark 4.2.2. For a projective toric variety Z with Cox ring $\mathbb{K}[T_1, \dots, T_r]$, let $Q\colon \mathbb{Z}^r \to K := \mathrm{Cl}(Z)$ denote the degree map sending the i-th canonical basis vector to the degree of the i-th variable T_i and $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$ the Gale dual, i.e., P is dual to the inclusion $\ker(Q) \subseteq \mathbb{Z}^r$. If $w \in \mathrm{Cl}(Z)$ is an ample class, then the fans $\widehat{\Sigma}$ of $\widehat{Z}$ and Σ of Z are

$$
\widehat{\Sigma} \ := \ \{\widehat{\sigma} \preceq \mathbb{Q}^r_{\ge 0};\ w \in Q(\widehat{\sigma}^{\perp} \cap \mathbb{Q}^r_{\ge 0})\}, \qquad \Sigma^{\max} \ = \ \{P(\widehat{\sigma});\ \widehat{\sigma} \in \widehat{\Sigma}^{\max}\},
$$

where we write $\preceq$ for the face relation of cones and regard Q and P as maps of the corresponding rational vector spaces. If $X \subseteq Z$ is a CEMDS, then the ample class $w \in \mathrm{Cl}(Z) = \mathrm{Cl}(X)$ is also an ample class for X. Note that a different choice of the ample class $w' \in \mathrm{Cl}(X)$ may lead to another CEMDS $X \subseteq Z'$ according to the fact that the Mori chamber decomposition of Z refines the one of X.

We now consider modifications $\pi\colon X_2 \to X_1$ of normal projective varieties. A first general statement describes the Cox ring of X_1 in terms of the Cox ring of X_2.

Proposition 4.2.3. *Let $\pi\colon X_2 \to X_1$ be a proper birational morphism of normal projective varieties. Let $C \subseteq X_1$ be the center of the modification. Set $K_i := \mathrm{Cl}(X_i)$ and $R_i := \mathcal{R}(X_i)$ and identify $U := X_2 \setminus \pi^{-1}(C)$ with $X_1 \setminus C$. Then we have canonical surjective push forward maps*

$$
\begin{aligned}
\pi_*\colon K_2 \to K_1, \qquad & [D] \mapsto [\pi_* D], \\
\pi_*\colon R_2 \to R_1, \qquad & (R_2)_{[D]} \ \ni \ f \ \mapsto \ f_{|U} \ \in \ (R_1)_{[\pi_* D]}.
\end{aligned}
$$

Now suppose that $\mathcal{R}(X_2)$ is finitely generated, let $E_1, \dots, E_l \subseteq X_2$ denote the exceptional prime divisors and $f_1, \dots, f_l \in \mathcal{R}(X_2)$ the corresponding canonical sections. Then we have a commutative diagram

$$
\begin{tikzcd}
R_2 \arrow[rr, "\pi_*"] \arrow[dr, "\lambda"'] & & R_1 \\
& R_2/\langle f_i - 1;\ 1 \le i \le l\rangle \arrow[ur, "\psi"', "\cong"] &
\end{tikzcd}
$$

of morphisms of graded algebras where λ is the canonical projection with the projection $K_2 \to K_2/\langle \deg(f_i);\, 1 \le i \le l\rangle$ as accompanying homomorphism and the induced map ψ is an isomorphism.

Lemma 4.2.4. *See* [57, Lem. 2.3]. *Let R be a K_2-graded domain, $f \in R_w$ with w of infinite order in K_2 and consider the downgrading of R given by $K_2 \to K_1 := K_2/\langle w\rangle$. Then $f - 1$ is K_1-prime.*

Proof of Proposition 4.2.3. Let $x_i \in X_i$ be smooth points with $\pi(x_2) = x_1$ such that x_2 is not contained in any of the exceptional divisors. Consider the divisorial sheaf $\mathcal{S}^{x_i}$ on X_i associated to the subgroup of divisors avoiding the point x_i, see [5, Constr. 4.2.3]. Then we have canonical morphisms of graded rings

$$\Gamma(X_2, \mathcal{S}^{x_2}) \ \to\ \Gamma(U_2, \mathcal{S}^{x_2}) \ \to\ \Gamma(X_1, \mathcal{S}^{x_1}),$$

where $U_2 \subseteq X_2$ is the open subset obtained by removing the exceptional divisors of $\pi\colon X_2 \to X_1$ and the accompanying homomorphisms of the grading groups are the respective push forwards of Weil divisors. The homomorphisms are compatible with the relations of the Cox sheaves $\mathcal{R}^{x_i}$, see again [5, Constr. 4.2.3], and thus induce canonical morphisms of graded rings

$$\Gamma(X_2, \mathcal{R}^{x_2}) \ \to\ \Gamma(U_2, \mathcal{R}^{x_2}) \ \to\ \Gamma(X_1, \mathcal{R}^{x_1}).$$

This establishes the surjection $\pi_*\colon R_2 \to R_1$ with the canonical push forward of divisor class groups as accompanying homomorphism. Clearly, the canonical sections f_i of the exceptional divisors are sent to $1 \in R_1$.

We show that the induced map ψ is an isomorphism. As we may proceed by induction on l, it suffices to treat the case $l = 1$. Lemma 4.2.4 tells us that $f_1 - 1$ is K_1-prime. From [51, Prop. 3.2] we infer that $\langle f_1 - 1 \rangle$ is a radical ideal in R_2. Since $\mathrm{Spec}(\psi)$ is a closed embedding of varieties of the same dimension and equivariant with respect to the action of the quasitorus $\mathrm{Spec}\,\mathbb{K}[K_1]$, the assertion follows. □

As an immediate consequence, we obtain that X_1 is a Mori dream space provided X_2 is one; see also [85]. The converse question is in general delicate. The classical counterexample arises from the projective plane $X_1 = \mathbb{P}_2$: for suitably general points $x_1, \ldots, x_9 \in \mathbb{P}_2$, the blow up X_1 at the first eight ones is a Mori dream surface and the blow up X_2 of X_1 at x_9 is not.

We now upgrade the technique of toric ambient modifications developed in [51] and Section 1 according to our computational purposes. In the following setting, $\widehat{X}_i \to X_i$ needs (a priori) not be a characteristic space and $\overline{X}_i$ not a total coordinate space.

Setting 4.2.5. Let $\pi\colon Z_2 \to Z_1$ be a toric modification, i.e., Z_1, Z_2 are complete toric varieties and π is a proper birational toric morphism. Moreover, let $X_i \subseteq Z_i$ be closed subvarieties, both intersecting the big n-torus $\mathbb{T}^n \subseteq Z_i$, such that $\pi(X_2) = X_1$ holds. Then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{K}^{r_2} & \supseteq & \widehat{Z}_2 \arrow[d, "p_2"'] & \supseteq & \widehat{X}_2 \arrow[d, "p_2"'] & & \widehat{X}_1 \arrow[d, "p_1"] & \subseteq & \widehat{Z}_1 \arrow[d, "p_1"] & \subseteq & \mathbb{K}^{r_1} \\
& & Z_2 \arrow[rrrrrr, bend right, "\pi"'] & \supseteq & X_2 \arrow[rr] & & X_1 & \subseteq & Z_1 & &
\end{tikzcd}$$

where the downwards maps $p_i\colon \widehat{Z}_i \to Z_i$ are toric characteristic spaces and $\widehat{X}_i \subseteq \widehat{Z}_i$ are the closures of the inverse image $p_i^{-1}(X_i \cap \mathbb{T}^n)$. Let $I_i \subseteq \mathbb{K}[T_1, \ldots, T_{r_i}]$ be the vanishing ideal of the closure $\overline{X}_i \subseteq \mathbb{K}^{r_i}$ of $\widehat{X}_i \subseteq \widehat{Z}_i$ and set $R_i := \mathbb{K}[T_1, \ldots, T_{r_i}]/I_i$. Note that R_i is graded by $K_i := \mathrm{Cl}(Z_i)$.

Theorem 4.2.6. *Consider Setting 4.2.5.*

(i) *If $X_1 \subseteq Z_1$ is a CEMDS, the ring R_2 is normal and $T_1, \ldots, T_{r_2}$ define pairwise non-associated K_2-primes in R_2, then $X_2 \subseteq Z_2$ is a CEMDS. In particular, K_2 is the divisor class group of X_2 and R_2 is the Cox ring of X_2.*

(ii) *If $X_2 \subseteq Z_2$ is a CEMDS, then $X_1 \subseteq Z_1$ is a CEMDS. In particular, K_1 is the divisor class group of X_1 and R_1 is the Cox ring of X_1.*

Proof. First consider the lattice homomorphisms $P_i\colon \mathbb{Z}^{r_i} \to \mathbb{Z}^n$ associated to the toric morphisms $p_i\colon \widehat{Z}_i \to Z_i$. Viewing the P_i as matrices, we may assume that $P_2 = [P_1, B]$ with a matrix B of size $n \times (r_2 - r_1)$. We have a commutative diagram of lattice homomorphisms and the corresponding diagram of homomorphisms of tori:

(3)
$$
\begin{tikzcd}
 & \mathbb{Z}^{r_2} \arrow[dl, "{\substack{e_i \mapsto e_i \\ e_j \mapsto m_j e_j}}"'] \arrow[dr, "{[E_{r_1}, A]}"] & \\
\mathbb{Z}^{r_2} \arrow[d, "{P_2 = [P_1, B]}"'] & & \mathbb{Z}^{r_1} \arrow[d, "P_1"] \\
\mathbb{Z}^n \arrow[rr, "E_n"'] & & \mathbb{Z}^n
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathbb{T}^{r_2} \arrow[dl, "\mu"'] \arrow[dr, "\alpha"] & \\
\mathbb{T}^{r_2} \arrow[d, "p_2"'] & & \mathbb{T}^{r_1} \arrow[d, "p_1"] \\
\mathbb{T}^n \arrow[rr, "\mathrm{id}"'] & & \mathbb{T}^n
\end{tikzcd}
$$

where in the left diagram, the e_i are the first r_1, the e_j the last $r_2 - r_1$ canonical basis vectors of $\mathbb{Z}^{r_2}$, the m_j are positive integers and E_n, E_{r_1} denote the unit matrices of size n, r_1 respectively and A is an integral $r_1 \times (r_2 - r_1)$ matrix.

We prove (i). We first show that R_2 is integral. By construction, it suffices to show that $p_2^{-1}(X_1 \cap \mathbb{T}^n)$ is irreducible, compare Lemma 4.3.6. By assumption, $\overline{X}_1 \cap \mathbb{T}^{r_1}$ is irreducible. Since the Smith normal form of $[E_{r_1}, A]$ is simply $[E_{r_1}, 0]$, under α, preimages of irreducible subsets are again irreducible. This means $\alpha^{-1}(\overline{X}_1 \cap \mathbb{T}^{r_1})$ is irreducible. We conclude that $\overline{X}_2 \cap \mathbb{T}^{r_2} = \mu(\alpha^{-1}(\overline{X}_1 \cap \mathbb{T}^{r_1}))$ is irreducible. Moreover, since X_2 is complete and the K_2-grading of R_2 has a pointed weight cone, we obtain that R_2 has only constant units. Thus, Theorem 4.1.3 yields that R_2 is factorially K_2-graded. Since the T_i are pairwise non-associated K_2-primes and R_2 is normal, we conclude that R_2 is the Cox ring of X_2 and $X_2 \subseteq Z_2$ is a CEMDS.

We turn to (ii). Observe that for every $f \in I_2$, the Laurent polynomials $\mu^*(f)$ and $\alpha^*(f(t_1, \ldots, t_{r_1}, 1, \ldots, 1))$ differ by a monomial factor. We conclude

$$
\begin{aligned}
\mathbb{K}\left[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}\right] \cdot I_2 &= \langle \alpha^*(f(t_1, \ldots, t_{r_1}, 1, \ldots, 1));\ f \in I_2 \rangle \\
&\subseteq \mathbb{K}\left[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}\right].
\end{aligned}
$$

Now, Proposition 4.2.3 tells us that R_1 is the Cox ring of X_1. Since $T_1, \ldots, T_{r_1}$ define also in X_1 pairwise different prime divisors, we conclude that $X_1 \subseteq Z_1$ is a CEMDS. □

Remark 4.2.7. The verification of normality as well as the primality tests needed for Theorem 4.2.6 are computationally involved. Proposition 4.1.4 considerably reduces the effort in many cases.

As a consequence of Theorem 4.2.6, we obtain that the modifications preserving finite generation are exactly those arising from toric modifications as discussed. More precisely, let $Z_2 \to Z_1$ be a toric modification mapping $X_2 \subseteq Z_2$ onto $X_1 \subseteq Z_1$. We call $Z_2 \to Z_1$ a *good* toric ambient modification, if it is as in Theorem 4.2.6 (i).

Corollary 4.2.8. *Let $X_2 \to X_1$ be a birational morphism of normal, projective $\mathbb{Q}$-factorial varieties such that the Cox ring $\mathcal{R}(X_1)$ is finitely generated. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X_2)$ is finitely generated.*
(ii) *The morphism $X_2 \to X_1$ arises from a good toric ambient modification.*

Proof. The implication "(ii)⇒(i)" is Theorem 4.2.6. For the reverse direction, set $K_i := \mathrm{Cl}(X_i)$ and $R_i := \mathcal{R}(X_i)$. Let $f_1, \ldots, f_{r_2}$ be pairwise nonassociated K_2-prime generators for R_2. According to Proposition 4.2.3, we may assume, after suitably numbering, that $f_1, \ldots, f_{r_1}$ define generators of R_1, where $r_1 \le r_2$. Now take an ample class $w_1 \in K_1$. Then the pullback $w_2' \in K_2$ of w_1 under $X_2 \to X_1$ is semiample on X_2. Choose $w_2 \in K_2$ such that w_2 is ample on X_2 and the toric ambient variety Z_2 of X_2 defined by w_2 has an ample cone containing w_2' in its closure. Then, with the sets of semistable points $\widehat{Z}_2, \widehat{Z}_2' \subseteq \mathbb{K}^{r_2}$ defined by w_2, w_2' respectively and $\widehat{Z}_1 \subseteq \mathbb{K}^{r_1}$ the one defined by w_1. By [51, Lem. 6.7], we obtain morphisms

$$Z_2 \ = \ \widehat{Z}_2 /\!\!/ H_2 \ \to \ \widehat{Z}_2' /\!\!/ H_2 \ \cong \ \widehat{Z}_1 /\!\!/ H_2 \ = \ Z_1,$$

where $H_i := \operatorname{Spec} \mathbb{K}[K_i]$ denotes the characteristic quasitorus of Z_i; observe that $\widehat{Z}_2' \to \widehat{Z}_2' /\!\!/ H_2$ is in general not a toric characteristic space. Thus, we arrive at Setting 4.2.5 and $Z_2 \to Z_1$ is the desired good toric ambient modification inducing the morphism $X_2 \to X_1$. □

For a flexible use of Theorem 4.2.6 we will have to adjust given embeddings of a Mori dream space, e.g., bring general points of a CEMDS into a more special position, or remove linear relations from a redundant presentation of the Cox ring. The formal framework is the following.

Setting 4.2.9. Let Z_1 be a projective toric variety with toric characteristic space $p_1\colon \widehat{Z}_1 \to Z_1$ and ample class $w \in K_1 := \mathrm{Cl}(Z_1)$. Consider K_1-homogeneous polynomials $h_1, \ldots, h_l \in \mathbb{K}[T_1, \ldots, T_{r_1}]$ and, with $r_1' := r_1 + l$, the (in general non-toric) embedding

$$\bar\imath\colon \mathbb{K}^{r_1} \ \to \ \mathbb{K}^{r_1'}, \qquad\qquad (z_1, \ldots, z_{r_1}) \ \mapsto \ (z_1, \ldots, z_{r_1}, h_1(z), \ldots, h_l(z)).$$

Note that $\mathbb{K}[T_1, \ldots, T_{r_1'}]$ is graded by $K_1' := K_1$ via attaching to $T_1, \ldots, T_{r_1}$ their former K_1-degrees and to T_{r_1+i} the degree of h_i. The class $w \in K_1'$ defines a toric variety Z_1' and a toric characteristic space $p_1\colon \widehat{Z}_1' \to Z_1'$. Any closed subvariety $X_1 \subseteq Z_1$ and its image $X_1' := \imath(X_1)$ lead to a commutative diagram

$$\begin{tikzcd}
\mathbb{K}^{r_1} \arrow[rrrrrrrrr, bend left, "\bar\imath"] & \supseteq & \widehat{Z}_1 \arrow[d, "p_1"'] & \supseteq & \widehat{X}_1 \arrow[d, "p_1"'] & \widehat{X}_1' \arrow[d, "p_1'"] & \subseteq & \widehat{Z}_1' \arrow[d, "p_1'"] & \subseteq & \mathbb{K}^{r_1'} \\
 & & Z_1 \arrow[rrrrr, bend right, "\imath"'] & \supseteq & X_1 \arrow[r] & X_1' & \subseteq & Z_1' & &
\end{tikzcd}$$

where $\widehat{X}_1 \subseteq \widehat{Z}_1$ and $\widehat{X}_1' \subseteq \widehat{Z}_1'$ are the closures of the inverse image $p_1^{-1}(X_1 \cap \mathbb{T}^n)$ and $(p_1')^{-1}(X_1' \cap \mathbb{T}^{n'})$. Denote by I_1 and I_1' the respective vanishing ideals of the closures $\overline{X}_1 \subseteq \mathbb{K}^{r_1}$ of $\widehat{X}_1 \subseteq \widehat{Z}_1$ and $\overline{X}_1' \subseteq \mathbb{K}^{r_1'}$ of $\widehat{X}_1' \subseteq \widehat{Z}_1'$. Set $R_1 := \mathbb{K}[T_1, \ldots, T_{r_1}]/I_1$ and define $R_1' := \mathbb{K}[T_1, \ldots, T_{r_1'}]/I_1'$.

Remark 4.2.10. In Setting 4.2.5, the cone over the columns of the degree matrix Q_2 is pointed if the cone over the columns of Q_1 was pointed. Similarly, in Setting 4.2.9, the cone over the columns of Q_1' is pointed if the cone over the columns of Q_1 is pointed.

Proposition 4.2.11. *Consider Setting 4.2.9.*

(i) *If $X_1 \subseteq Z_1$ is a CEMDS and $T_1, \ldots, T_{r_1}, h_1, \ldots, h_l$ define pairwise non-associated K_1-primes in R_1 then $X_1' \subseteq Z_1'$ is a CEMDS.*

(ii) *If R_1' is normal, the localization $(R_1')_{T_1\cdots T_{r_1}}$ is factorially K_1'-graded and $T_1,\dots,T_{r_1}$ define pairwise non-associated K_1-primes in R_1 such that K_1 is generated by any r_1-1 of their degrees, then $X_1 \subseteq Z_1$ is a CEMDS.*

(iii) *If $X_1' \subseteq Z_1'$ is a CEMDS, then $X_1 \subseteq Z_1$ is a CEMDS.*

Proof. First, observe that the ideal I_1' equals $I_1 + \langle T_{r_1+1} - h_1, \dots, T_{r_1'} - h_l\rangle$. Consequently, we have a canonical graded isomorphism $R_1' \to R_1$ sending T_{r_1+i} to h_i. Assertion (i) follows directly.

We prove (ii). Since $(R_1')_{T_1\cdots T_{r_1}}$ is factorially K_1'-graded, we obtain that $(R_1)_{T_1\cdots T_{r_1}}$ is factorially K_1-graded. As $T_1,\dots,T_{r_1}$ define K_1-primes in R_1, we can apply [15, Thm. 1.2] to see that R_1 is factorially K_1-graded. Since $T_1,\dots,T_{r_1}$ are pairwise non-associated we conclude that $X_1 \subseteq Z_1$ is a CEMDS.

We turn to (iii). Note that by construction, $T_1,\dots,T_{r_1}$ also define K_1-primes in R_1. According to (ii), we only have to show that any r_1-1 of the degrees of $T_1,\dots,T_{r_1}$ generate K_1. For this, it suffices to show that each $\deg(T_j)$ for $j = r_1+1,\dots,r_1+l$ is a linear combination of any r_1-1 of the first r_1 degrees. Since $T_1,\dots,T_{r_1}$ generate R_1 and T_j is not a multiple of any T_i, we see that for any $i=1,\dots,r_1$, there is a monomial in h_j not depending on T_i. The assertion follows. □

3. Computing modifications of Mori dream spaces

Based on Section 2, we provide a general algorithmic framework for computations with modifications of Mori dream spaces. This section has been published together with J. Hausen and A. Laface in [57, Sec. 3].

In order to encode a canonically embedded Mori dream space $X_i \subseteq Z_i$ and its Cox ring R_i, we use the triple (P_i, Σ_i, G_i) where P_i and Σ_i are as in Remark 4.2.2 and $G_i = (g_1,\dots,g_s)$ is a system of generators of the defining ideal I_i of the Cox ring R_i. We call such a triple (P_i,Σ_i,G_i) as well a CEMDS.

Remark 4.3.1. In the sense of Chapter 2, each CEMDS (P,Σ,G) corresponds to a MDS (R,Φ) where the GR $R = (G,Q,Q^0,P,F_{\mathfrak{F}})$ is computed using Algorithms 2.1.24 and 2.1.26. The BUN Φ is obtained from Σ by Gale duality

$$\Phi \quad = \quad \{Q(\gamma_0);\ \gamma_0 \preceq \gamma \ \mathfrak{F}\text{-face},\ P(\gamma_0^*) \in \Sigma\}, \qquad \gamma \ := \ \mathbb{Q}^r_{\geq 0}.$$

In particular, given a CEMDS (P_i,Σ_i,G_i), the degree map $Q_i\colon \mathbb{Z}^{r_i} \to K_i$ and $\overline{X}_i$ as well as $p_i\colon \widehat{X}_i \to X_i$ are directly computable. Then Q_i and P_i are Gale dual to each other, i.e., Q_i is surjective and P_i is the dual of the inclusion $\ker(Q_i) \subseteq \mathbb{Z}^{r_i}$. The following two algorithms implement Proposition 4.2.11.

Algorithm 4.3.2 (StretchCEMDS)**.** *Input:* a CEMDS (P_1,Σ_1,G_1) and a list $(f_1,\dots,f_l)$ of polynomials $f_i \in \mathbb{K}[T_1,\dots,T_{r_1}]$ defining pairwise non-associated K_1-primes in R_1.

- Compute the Gale dual $Q_1\colon \mathbb{Z}^{r_1} \to K_1$ of P_1 with Algorithm 2.1.24.
- Let $Q_1'\colon \mathbb{Z}^{r_1+l} \to K_1$ be the extension of Q_1 by the degrees of $f_1,\dots,f_l$.
- Using Algorithm 2.1.25, compute the Gale dual $P_1'\colon \mathbb{Z}^{r_1+l} \to \mathbb{Z}^{n'}$ of Q_1' and the fan Σ' in $\mathbb{Z}^{n'}$ defined by P_1' and the ample class $w \in K_1' = K_1$ of Z_1.
- Set $G_1' := (g_1,\dots,g_s, T_{r_1+1}-f_1,\dots,T_{r_2}-f_l)$ where $G_1 = (g_1,\dots,g_s)$.

Output: the CEMDS (P_1',Σ_1',G_1').

The input of the second algorithm is more generally an *embedded space* $X_1 \subseteq Z_1$ that means just a closed normal subvariety intersecting the big torus. In particular, we do not care for the moment if R_1 is the Cox ring of X_1. We encode $X_1 \subseteq Z_1$ as well by a triple (P_1, Σ_1, G_1) and name it for short an ES. For notational reasons we write (P_1', Σ_1', G_1') for the input.

Algorithm 4.3.3 (CompressCEMDS)**.** *Input:* an ES (P_1', Σ_1', G_1') such that R_1' is normal, the localization $(R_1')_{T_1 \cdots T_{r_1}}$ is factorially K_1'-graded and the last l relations in G_1' are fake, i.e., of the form $f_i = T_i - h_i$ with h_i not depending on T_i. *Option:* `verify`.

- Successively substitute $T_i = h_i$ in G_1'. Set $G_1 := (f_1, \ldots, f_{r_1})$ where $G_1' = (f_1, \ldots, f_{r_1'})$ and $r_1 := r_1' - l$.
- Set $K_1 := K_1'$ and let $Q_1 \colon \mathbb{Z}^{r_1} \to K_1$ be the map sending e_i to $\deg(T_i)$ for $1 \le i \le r_1$.
- Compute a Gale dual $P_1 \colon \mathbb{Z}^{r_1} \to \mathbb{Z}^n$ of Q_1 with Algorithm 2.1.25 and the fan Σ_1 in $\mathbb{Z}^n$ defined by P_1 and the ample class $w \in K_1 = K_1'$ of Z_1'.
- If `verify` was asked then
  - check if any $r_1 - 1$ of the degrees of $T_1, \ldots, T_{r_1}$ generate K_1; see Algorithm 2.1.32.
  - check if $\dim(I_1) - \dim(I_1 + \langle T_i, T_j \rangle) \ge 2$ for all $i \ne j$,
  - check if $T_1, \ldots, T_{r_1}$ define K_1-primes in R_1; see Algorithm 2.2.10.

Output: the ES (P_1, Σ_1, G_1). If (P_1', Σ_1', G_1') is a CEMDS or all verifications were positive, then (P_1, Σ_1, G_1) is a CEMDS. In particular, then R_1 is the Cox ring of the corresponding subvariety $X_1 \subseteq Z_1$.

Remark 4.3.4. In Algorithm 4.3.3, observe that it is no restriction to assume that each fake relation $T_i - h \in \langle G_1' \rangle$ already satisfies $T_i - h \in G'$. Write $T_i - h = h_1 f_1 + \ldots + h_{r_1} f_{r_1}$ with $h_j \in \mathbb{K}[T_1, \ldots, T_r]$. Comparing the degrees of both sides with respect to a suitable monomial ordering, we obtain $T_i - h = f_j$ for some j.

We turn to the algorithmic version of Theorem 4.2.6. We will work with the *saturation* of an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ with respect to an ideal $\mathfrak{b} \subseteq \mathbb{K}[T_1, \ldots, T_r]$; recall that this is the ideal

$$\mathfrak{a} : \mathfrak{b}^\infty \ := \ \{g \in \mathbb{K}[T_1, \ldots, T_r];\ g\, \mathfrak{b}^k \subseteq \mathfrak{a} \text{ for some } k \in \mathbb{Z}_{\ge 0}\} \ \subseteq \ \mathbb{K}[T_1, \ldots, T_r].$$

In case of a principal ideal $\mathfrak{b} = \langle f \rangle$, we write $\mathfrak{a} : f^\infty$ instead of $\mathfrak{a} : \mathfrak{b}^\infty$. We say that an ideal $\mathfrak{a} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ is *f-saturated* if $\mathfrak{a} = \mathfrak{a} : f^\infty$ holds. We will only consider saturations with respect to $f = T_1 \cdots T_r \in \mathbb{K}[T_1, \ldots, T_r]$; we refer to [90, Chap. 12] for the computational aspect. Let us recall the basic properties, see also [64].

Lemma 4.3.5. *Consider $\mathbb{K}[T, U^{\pm 1}]$ with tupels of variables $T = (T_1, \ldots, T_{r_1})$ and $U = (U_1, \ldots, U_{r_2 - r_1})$. For $f := U_1 \cdots U_{r_2 - r_1} \in \mathbb{K}[U]$, one has mutually inverse bijections*

$$\begin{array}{ccc} \left\{ \text{ ideals in } \mathbb{K}[T, U^{\pm 1}] \ \right\} & \longleftrightarrow & \left\{ \ f\text{-saturated ideals in } \mathbb{K}[T, U] \ \right\} \\ \mathfrak{a} & \mapsto & \mathfrak{a} \cap \mathbb{K}[T, U] \\ \langle \mathfrak{b} \rangle_{\mathbb{K}[T, U^{\pm 1}]} & \leftarrow\!\shortmid & \mathfrak{b}. \end{array}$$

Under these maps, the prime ideals of $\mathbb{K}[T, U^{\pm 1}]$ correspond to the f-saturated prime ideals of $\mathbb{K}[T, U]$.

For transferring polynomials from $\mathbb{K}[T_1, \ldots, T_{r_1}]$ to $\mathbb{K}[T_1, \ldots, T_{r_2}]$ and vice versa, recall from Chapter 2 the following operations; compare also [44]. Consider a homomorphism $\pi \colon \mathbb{T}^n \to \mathbb{T}^m$ of tori and its kernel $H \subseteq \mathbb{T}^n$.

- By a *$\star$-pull back* of $g \in \mathbb{K}[S_1^{\pm 1}, \ldots, S_m^{\pm 1}]$ we mean a polynomial $\pi^\star g \in \mathbb{K}[T_1, \ldots, T_n]$ with coprime monomials such that $\pi^* g$ and $\pi^\star g$ are associated in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

- By a $\star$-*push forward* of an H-homogeneous $h \in \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ we mean a polynomial $\pi_\star h \in \mathbb{K}[S_1, \ldots, S_m]$ with coprime monomials such that h and $\pi^*\pi_\star h$ are associated in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Note that $\star$-pull backs and $\star$-push forwards always exist and are unique up to constants. The $\star$-pull back $\pi^\star g$ of a Laurent polynomial is its usual pull back $\pi^* g$ scaled with a suitable monomial. See Algorithm 2.2.13 on how to compute the $\star$-push forward.

Lemma 4.3.6. *Consider a monomial epimorphism $\pi\colon \mathbb{K}^{n_1} \times \mathbb{T}^{n_2} \to \mathbb{K}^m$. Write $T = (T_1, \ldots, T_{n_1})$ and $U = (U_1, \ldots, U_{n_2})$.*

(i) *If $\mathfrak{a} \subseteq \mathbb{K}[T, U^{\pm 1}]$ is a prime ideal, then $\langle \pi_\star \mathfrak{a} \rangle \subseteq \mathbb{K}[S^{\pm 1}]$ is a prime ideal.*
(ii) *If $\mathfrak{b} \subseteq \mathbb{K}[S^{\pm 1}]$ is a radical ideal, then $\langle \pi^\star \mathfrak{b} \rangle \subseteq \mathbb{K}[T, U^{\pm 1}]$ is a radical ideal.*

Proof. The first statement follows from $\langle \pi_\star \mathfrak{a} \rangle = (\pi^*)^{-1}(\mathfrak{a})$. To prove (ii), let $f \in \sqrt{\langle \pi^\star \mathfrak{b} \rangle}$. Since $\sqrt{\langle \pi^\star \mathfrak{b} \rangle} = I(\pi^{-1}(V(\mathfrak{b})))$ is invariant under $H := \ker(\pi|_{\mathbb{T}^{n_1+n_2}})$, we may assume that f is H-homogeneous, i.e., $f(h \cdot z) = \chi(h) f(z)$ holds with some character $\chi \in \mathbb{X}(H)$. Choose $\eta \in \mathbb{X}(\mathbb{T}^{n_1+n_2})$ with $\chi = \eta_{|H}$. Then $\eta^{-1} f$ is H-invariant and thus belongs to $\pi^*(I(V(\mathfrak{b}))$. Hilbert's Nullstellensatz and the assumption give $\pi^*(I(V(\mathfrak{b})) = \pi^*(\mathfrak{b})$. We conclude $f \in \langle \pi^\star \mathfrak{b} \rangle$. $\square$

We are ready for the first algorithm, treating the contraction problem. We enter a *weak CEMDS* (P_2, Σ_2, G_2) in the sense that G_2 provides generators for the extension of I_2 to $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]$ and a toric contraction $Z_2 \to Z_1$, encoded by (P_1, Σ_1), and obtain a CEMDS $X_1 \subseteq Z_1$.

Algorithm 4.3.7 (ContractCEMDS)**.** *Input:* a weak CEMDS (P_2, Σ_2, G_2) and a pair (P_1, Σ_1) where $P_2 = [P_1, B]$ and Σ_1 is a coarsening of Σ_2 removing the rays through the columns of B.

- For $G_2 = (g_1, \ldots, g_s)$, set $h_i := g_i(T_1, \ldots, T_{r_1}, 1, \ldots, 1) \in \mathbb{K}[T_1, \ldots, T_{r_1}]$.
- Compute a system of generators G_1' for $I_1' := \langle h_1, \ldots, h_s \rangle : (T_1 \cdots T_{r_1})^\infty$.
- Set $(P_1', \Sigma_1', G_1') := (P_1, \Sigma_1, G_1')$ and reorder the variables such that the last l relations of G_1' are as in Algorithm 4.3.3.
- Apply Algorithm 4.3.3 to (P_1', Σ_1', G_1') and write (P_1, Σ_1, G_1) for the output.

Output: (P_1, Σ_1, G_1). This is a CEMDS. In particular, R_1 is the Cox ring of the image $X_1 \subseteq Z_1$ of $X_2 \subseteq Z_2$ under $Z_2 \to Z_1$.

Proof. First we claim that in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]$, the ideal generated by $h_1, \ldots, h_s$ coincides with the ideal generated by $p_1^\star (p_2)_\star g_1, \ldots, p_1^\star (p_2)_\star g_r$. To see this, consider $p_i\colon \mathbb{T}^{r_i} \to \mathbb{T}^n$ and let $S_1, \ldots, S_n$ be the variables on $\mathbb{T}^n$. Then the claim follows from $(P_2)_{ij} = (P_1)_{ij}$ for $j \le r_1$ and

$$p_2^*(S_i) \;=\; T_1^{(P_2)_{i1}} \cdots T_{r_2}^{(P_2)_{ir_2}}, \qquad\qquad p_1^*(S_i) \;=\; T_1^{(P_1)_{i1}} \cdots T_{r_1}^{(P_1)_{ir_1}}.$$

As a consequence of the claim, we may apply Lemma 4.3.6 and obtain that G_1' defines a radical ideal in $\mathbb{K}[T_1^{\pm}, \ldots, T_{r_1}^{\pm}]$. Moreover, from Theorem 4.2.6 we infer that $\widehat{X}_1'$, defined as in Setting 4.2.5, is irreducible. Since G_1' has $\widehat{X}_1' \cap \mathbb{T}^{r_1}$ as its zero set, it defines a prime ideal in $\mathbb{K}[T_1^{\pm}, \ldots, T_{r_1}^{\pm}]$. Lemma 4.3.5 then shows that $I_1' \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$ is a prime ideal. Using Theorem 4.2.6 again, we see that (P_1', Σ_1', G_1') as defined in the third step of the algorithm is a CEMDS. Thus, we may enter Algorithm 4.3.3 and end up with a CEMDS. $\square$

We turn to the modification problem. Given a Mori dream space X_1 with Cox ring R_1 and a modification $X_2 \to X_1$, we want to know if X_2 is a Mori dream space, and if so, we ask for the Cox ring R_2 of X_2. Our algorithm verifies a guess of prospective generators for R_2 and, if successful, computes the relations. In practice, the generators are added via Algorithm 4.3.2.

Algorithm 4.3.8 (ModifyCEMDS)**.** *Input:* a weak CEMDS (P_1, Σ_1, G_1), a pair (P_2, Σ_2) with a matrix $P_2 = [P_1, B]$ and a fan Σ_2 having the columns of P_2 as its primitive generators and refining Σ_1. *Options:* `verify`.

- Compute $G_2' := (h_1, \dots, h_s)$ with $h_i = p_2^\star (p_1)_\star (g_i)$ and $G_1 = (g_1, \dots, g_s)$.
- Compute a list of generators G_2 for $I_2 := \langle h_1, \dots, h_s \rangle : (T_{r_1+1} \cdots T_{r_2})^\infty$.
- If `verify` was asked
  - compute a Gale dual $Q_2 \colon \mathbb{Z}^{r_2} \to K_2$ of P_2 with Algorithm 2.1.24,
  - check if $\dim(I_2) - \dim(I_2 + \langle T_i, T_j \rangle) \geq 2$ for all $i \neq j$,
  - check if $T_1, \dots, T_{r_2}$ define K_2-primes in R_2; see Algorithm 2.2.10.
  - check if R_2 is normal, e.g., using Proposition 4.1.4.

Output: (P_2, Σ_2, G_2), if the `verify`-checks were all positive, this is a CEMDS. In particular, R_2 then is the Cox ring of the strict transform $X_2 \subseteq Z_2$ of $X_1 \subseteq Z_1$ with respect to $Z_2 \to Z_1$.

Proof. We write shortly $\mathbb{K}[T, U^{\pm 1}]$ with the tuples $T = (T_1, \dots, T_{r_1})$ and $U = (U_1, \dots, U_{r_2 - r_1})$ of variables. Lemma 4.3.6 ensures that G_2 generates a radical ideal in $\mathbb{K}[T, U^{\pm}]$. To see that the zero set $V(G_2) \subseteq \mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1}$ is irreducible, consider the situation of equation (3) in the proof of Theorem 4.2.6. There, in the right hand side diagram, we may lift the homomorphisms of tori to

$$\begin{tikzcd}
& \mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1} \arrow[dl, "\mu"'] \arrow[dr, "\alpha"] & \\
\mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1} \arrow[d, "p_2"'] & & \mathbb{K}^{r_1} \arrow[d, "p_1"] \\
\mathbb{T}^n \arrow[rr, "\mathrm{id}"'] & & \mathbb{T}^n
\end{tikzcd}$$

Observe that we have an isomorphism $\varphi = \alpha \times \mathrm{id}$ given by

$$\mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1} \to \mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1}, \qquad (z, z') \ \mapsto \ \left(z_1 (z')^{A_{1*}}, \dots, z_{r_1} (z')^{A_{r_1 *}}, z' \right).$$

Since $\overline{X}_1$ is irreducible, so is $\varphi^{-1}(\overline{X}_1 \times \mathbb{T}^{r_2 - r_1}) = \alpha^{-1}(\overline{X}_1)$. Hence, the image $\mu(\alpha^{-1}(\overline{X}_1)) = \overline{X}_2 \cap \mathbb{K}^{r_1} \times \mathbb{T}^{r_2 - r_1} = V(G_2)$ is irreducible as well. Moreover, Lemma 4.3.5 implies that G_2 generates a prime ideal in $\mathbb{K}[T, U]$. If the `verify`-checks were all positive, then Theorem 4.2.6 tells us that (P_2, Σ_2, G_2) is a CEMDS. $\square$

Remark 4.3.9. If the canonical map $K_2 \to K_1$ admits a section, e.g., if K_1 is free, then, in the verification step of Algorithm 4.3.8, it suffices to check the variables $T_{r_1+1}, \dots, T_{r_2}$ for being K_2-prime in R_2, see Proposition 4.1.4.

4. Application: Gorenstein log del Pezzo surfaces

As an application of the algorithms developed in Section 3, mainly Algorithm 4.3.7, we compute Cox rings of Gorenstein log-terminal del Pezzo surfaces X of Picard number one that do not admit a non-trivial $\mathbb{K}^*$-action. This section has been published with J. Hausen and A. Laface in [57, Sec. 4].

Del Pezzo means that the anticanonical divisor $-\mathcal{K}_X$ is ample and the condition "Gorenstein log-terminal" implies that X has at most ADE-singularities. The idea is to present each such surface X, classified by Alekseev and Nikulin [2], as $\mathbb{P}_2 \leftarrow \widetilde{X} \to X$ with smooth $\widetilde{X}$ and information about $\mathcal{R}(\widetilde{X})$ is known from Hasset, Tschinkel, Derenthal, Artebani, Garbagnati and Laface [49, 33, 4]. The Cox rings admitting a non-trivial $\mathbb{K}^*$-action have been computed in [59], the toric ones in, e.g., [72].

In [2, Theorem 8.3], the Gorenstein log-terminal del Pezzo surfaces X of Picard number one have been classified according to the singularity type, i.e., the configuration $\mathrm{S}(X)$ of singularities. Besides $\mathbb{P}_2$, there are four toric Gorenstein log-terminal surfaces del Pezzo surfaces X of Picard number one, namely the singularity types A_1, A_1A_2, $2A_1A_3$ and $3A_2$. Moreover, there are thirteen (deformation types of) $\mathbb{K}^*$-surfaces; they represent the singularity types A_4, D_5, E_6, A_12A_3, $3A_1D_4$, A_1D_6, A_2A_5, E_7, A_1E_7, A_2E_6, E_8, $2D_4$ and their Cox rings have been determined in [59, Theorem 5.6].

We now compute the Cox rings of the remaining ones using Algorithm 4.3.2 and the knowledge of generators of their resolutions [33, 4]; note that the relations for Cox rings of the resolutions is still not known in all cases. In the sequel, we will write a Cox ring as a quotient $\mathbb{K}[T_1,\ldots,T_r]/I$ and specify generators for the ideal I. As before, the $\mathrm{Cl}(X)$-grading is encoded by a degree matrix, i.e., a matrix with $\deg(T_1),\ldots,\deg(T_r) \in \mathrm{Cl}(X)$ as columns.

Theorem 4.4.1. *The following table lists the Cox rings of the Gorenstein log-terminal del Pezzo surfaces X of Picard number one that do not allow a non-trivial $\mathbb{K}^*$-action.*

$\mathrm{S}(X)$	*Cox ring $\mathcal{R}(X)$*	$\mathrm{Cl}(X)$ *and degree matrix*
$2A_4$	$\mathbb{K}[T_1,\ldots,T_6]/I$ *with I generated by* $-T_2T_5+T_3T_4+T_6^2,\ -T_2T_4+T_3^2+T_5T_6,$ $T_1T_6-T_3T_5+T_4^2,\ T_1T_3-T_4T_6+T_5^2,$ $T_1T_2-T_3T_6+T_4T_5$	$\mathbb{Z}\oplus\mathbb{Z}/5\mathbb{Z}$ $\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{1} \end{array}\right]$
D_8	$\mathbb{K}[T_1,\ldots,T_4]/I$ *with I generated by* $T_1^2-T_4^2T_2T_3+T_4^4+T_3^4$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ $\left[\begin{array}{cccc} 2 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{array}\right]$
D_5A_3	$\mathbb{K}[T_1,\ldots,T_5]/I$ *with I generated by* $T_1T_3-T_4^2-T_5^2,\ T_1T_2-T_3^2+T_4T_5$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ $\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{2} & \bar{0} & \bar{3} & \bar{1} \end{array}\right]$
D_62A_1	$\mathbb{K}[T_1,\ldots,T_5]/I$ *with I generated by* $T_5T_2-T_5^2+T_3^2+T_4^2,\ -T_2^2+T_5T_2+T_1^2-T_4^2$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ $\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{array}\right]$
E_6A_2	$\mathbb{K}[T_1,\ldots,T_4]/I$ *with I generated by* $-T_1T_4^2+T_2^3+T_2T_3T_4+T_3^3$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ $\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{1} \end{array}\right]$
E_7A_1	$\mathbb{K}[T_1,\ldots,T_4]/I$ *with I generated by* $-T_1T_3^3-T_2^2+T_2T_3T_4+T_4^4$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ $\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{array}\right]$
E_8	$\mathbb{K}[T_1,\ldots,T_4]/I$ *with I generated by* $T_1^3+T_1^2T_4^2+T_2^2-T_3T_4^5$	$\mathbb{Z}$ $\left[\begin{array}{cccc} 2 & 3 & 1 & 1 \end{array}\right]$

A_7	$\mathbb{K}[T_1,\dots,T_4]/I$ *with* I *generated by* $T_1^2-T_4T_2T_3+T_4^4+T_3^4$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ $\begin{bmatrix} 2 & 2 & 1 & 1 \\ \overline{1} & \overline{1} & \overline{1} & \overline{0} \end{bmatrix}$
A_8	$\mathbb{K}[T_1,\dots,T_4]/I$ *with* I *generated by* $-T_1T_2T_3+T_2^3+T_3^3+T_4^3$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ \overline{1} & \overline{2} & \overline{0} & \overline{1} \end{bmatrix}$
A_7A_1	$\mathbb{K}[T_1,\dots,T_5]/I$ *with* I *generated by* $-T_2T_3+T_4^2-T_5^2,\ T_1^2-T_3^2+T_4T_5$	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \overline{2} & \overline{2} & \overline{0} & \overline{3} & \overline{1} \end{bmatrix}$
$A_5A_2A_1$	$\mathbb{K}[T_1,\dots,T_7]/I$ *with* I *generated by* $T_5^2+T_6^2-T_7T_1,\ T_4T_5+T_6T_1-T_2T_6-T_7^2,$ $-T_3T_6-T_5T_7+T_1T_4,\ T_3^2-T_6T_1+T_7^2,$ $T_1T_5-T_2T_5-T_4T_6+T_7T_3,$ $T_3T_4-T_6^2+T_7T_1-T_2T_7,$ *The class group and degree matrix are* $\mathbb{Z}\oplus\mathbb{Z}/6\mathbb{Z}$	$T_1T_3-T_2T_3+T_6T_5-T_7T_4,$ $T_1T_2-T_2^2-T_4^2+T_3T_5,$ $T_1^2-T_2^2-T_4^2+2T_3T_5-T_7T_6$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \overline{2} & \overline{2} & \overline{3} & \overline{5} & \overline{1} & \overline{4} & \overline{0} \end{bmatrix}$
$2A_3A_1$	$\mathbb{K}[T_1,\dots,T_9]/I$ *with* I *generated by* $-\frac{1}{2}T_4^2+T_5^2+\frac{1}{2}T_7T_9,$ $-\frac{1}{2}T_3T_8-\frac{1}{2}T_4T_5+T_6^2,$ $-\frac{1}{2}T_3T_4+T_5T_8+\frac{1}{2}T_6T_9,$ $T_2T_6-T_7T_9-4T_8^2,$ $T_5T_2-2T_3T_7+T_8T_9,$ $-\frac{1}{4}T_2T_4+\frac{1}{4}T_3T_9+T_7T_8,$ $T_1T_7+T_2T_7-4T_3T_4+2T_6T_9,$ $T_1T_6-2T_4^2+T_7T_9,$ $\frac{1}{2}T_1T_6-\frac{1}{2}T_2T_6+T_3^2-T_4^2,$ $T_1T_8-2T_4T_7+T_5T_9,$ *The class group and degree matrix are* $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	$T_1^2-16T_4T_5+8T_7^2-T_9^2,$ $T_2^2-16T_3T_8+8T_7^2-T_9^2,$ $T_2T_3-T_4T_9+4T_5T_7-8T_6T_8,$ $T_1T_2-8T_7^2-T_9^2,$ $T_1T_5+2T_3T_7-4T_4T_6+T_8T_9,$ $T_1T_3-T_4T_9-4T_5T_7,$ $-\frac{1}{8}T_4T_1+\frac{1}{8}T_2T_4+T_5T_6-T_7T_8,$ $-\frac{1}{16}T_9T_1+\frac{1}{16}T_2T_9-T_4T_8+T_6T_7,$ $-\frac{1}{8}T_9T_1+\frac{1}{8}T_2T_9+T_3T_5-T_4T_8,$ $\frac{1}{4}T_1T_8-\frac{1}{4}T_2T_8+T_6T_3-T_4T_7,$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \overline{1} & \overline{1} & \overline{0} & \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \\ \overline{3} & \overline{3} & \overline{2} & \overline{0} & \overline{2} & \overline{1} & \overline{3} & \overline{0} & \overline{1} \end{bmatrix}$
$4A_2$	$\mathbb{K}[T_1,\dots,T_{10}]/I$ *with* I *generated by* $3T_3T_6+3T_4T_7\zeta+(-3\zeta-3)T_5T_8,$ $(\zeta-1)T_2T_8+3T_3^2+(-\zeta-2)T_6T_9,$ $3T_2T_7\zeta+3T_6T_{10}+(-3\zeta-3)T_8T_9,$ $(-\zeta+1)T_2T_5+(\zeta-1)T_4T_9+3T_6T_8,$ $-\zeta T_1T_{10}+T_2T_{10}\zeta+3T_4T_7-3T_5T_8,$ $(\zeta+1)T_1T_{10}-T_2T_{10}\zeta+3T_5T_8-T_9^2,$ $-T_1T_9\zeta-T_2T_9+3T_3T_7+(\zeta+1)T_{10}^2,$ $-T_1T_9\zeta+T_2T_9\zeta+3T_3T_7-3T_8T_4,$ $(-\zeta+1)T_1T_8+(\zeta-1)T_2T_8+3T_3^2-3T_4T_5,$ $(\zeta+2)T_1T_7+(-\zeta-2)T_2T_7+3T_4T_3-3T_5^2,$ $T_2T_1+(-\zeta-1)T_2^2+3T_8T_3+T_9T_{10}\zeta,$ $-\zeta T_1T_2+3T_4T_6+T_9T_{10}\zeta,$ $T_1^2+(-\zeta-1)T_1T_2+3T_5T_7+T_9T_{10}\zeta$ *The class group and degree matrix are* $\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$(-\zeta-1)T_1T_9+(\zeta+1)T_2T_9-3T_8T_4+3T_5T_6,$ $(-2\zeta-1)T_1T_8+(\zeta-1)T_2T_8+3T_3^2+(\zeta-1)T_7T_{10},$ $-3T_1T_6+(3\zeta+3)T_2T_6+(-3\zeta-3)T_7T_9+3T_8T_{10},$ $(-\zeta-2)T_1T_7+(2\zeta+1)T_2T_7+3T_5^2+(-\zeta-2)T_8T_9,$ $(-\zeta-2)T_1T_6+(2\zeta+1)T_2T_6+3T_3T_5+(-\zeta-2)T_7T_9,$ $(2\zeta+1)T_1T_6+(-2\zeta-1)T_2T_6-3T_3T_5+3T_4^2,$ $(-\zeta+1)T_1T_5+(-\zeta-2)T_2T_5+(2\zeta+1)T_3T_{10}+3T_7^2,$ $(-\zeta+1)T_1T_5+(\zeta-1)T_2T_5-3T_6T_8+3T_7^2,$ $-3T_1T_4+(3\zeta+3)T_2T_4+(-3\zeta-3)T_3T_9+3T_5T_{10},$ $(-2\zeta-1)T_1T_4+(2\zeta+1)T_5T_{10}+3T_6^2,$ $(\zeta+2)T_1T_4+(-2\zeta-1)T_2T_4+(\zeta-1)T_3T_9+3T_7T_8,$ $(-\zeta+1)T_1T_3+(\zeta-1)T_5T_9+3T_6T_7,$ $3\zeta T_1T_3+3T_4T_{10}+(-3\zeta-3)T_5T_9,$ $(\zeta+2)T_1T_3+(-2\zeta-1)T_2T_3+(\zeta-1)T_5T_9+3T_8^2,$ *where* ζ *is a primitive third root of unity.* $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \overline{2} & \overline{2} & \overline{1} & \overline{0} & \overline{2} & \overline{1} & \overline{2} & \overline{0} & \overline{1} & \overline{0} \\ \overline{1} & \overline{1} & \overline{2} & \overline{2} & \overline{2} & \overline{0} & \overline{0} & \overline{0} & \overline{1} & \overline{1} \end{bmatrix}.$

Proof. According to [2, Thm. 8.3], the Gorenstein log del Pezzo surfaces X with $\varrho(X) = 1$ have ADE-singularity types

$$
\begin{gathered}
A_7,\quad A_8,\quad A_7A_1,\quad A_5A_2A_1,\quad 2A_4,\quad D_8,\quad D_5A_3,\quad D_62A_1,\quad 2A_32A_1,\quad 4A_2,\\
D_43A_1,\quad 2A_3A_1,\quad A_4,\quad A_5A_1,\quad A_5A_2,\quad D_5,\quad D_6A_1,\quad E_6A_2,\quad E_6,\quad E_7A_1,\\
E_7,\quad E_8,\quad 2D_4,\quad A_1,\quad A_2A_1,\quad A_32A_1,\quad 3A_2.
\end{gathered}
\tag{4}
$$

For each singularity type, up to isomorphism, there is exactly one such surface except for the cases E_6A_2, E_7A_1 and E_8 where exactly two isomorphism classes occur, and case $2D_4$ where there are infinitely many classes. As noted in the introduction, the singularity types shown in the last two rows of (4) are toric or are $\mathbb{K}^*$-surfaces; this includes all $2D_4$ cases.

Each of the remaining surfaces X, is obtained by contracting curves of a smooth surface X_2 arising as a blow up of $\mathbb{P}_2$ with generators for the Cox ring known by [33, 4]. A direct application of Algorithms 4.3.8 and 4.3.7 is not always feasible. However, we have enough information to present the blow ups of $\mathbb{P}_2$ as a weak CEMDS. As an example, we treat the D_5A_3-case. By [4], with $X_2 := X_{141}$, additional generators for $\mathcal{R}(X_2)$ correspond in $\mathcal{R}(\mathbb{P}_2)$ to

$$f_1 \;:=\; T_1 - T_2, \qquad\qquad f_2 \;:=\; T_1T_2 - T_2^2 + T_1T_3.$$

Using Algorithm 4.3.2 with input the CEMDS $\mathbb{P}_2$ and (f_1, f_2), we obtain a CEMDS X_1. Again by [4, Sec. 6], we know the degree matrix Q_2 of X_2. Write $Q_2 = [D, C]$ with submatrices D and C consisting of the first r_1 and the last $r_2 - r_1$ columns respectively. We compute a Gale dual matrix P_2 of the form $P_2 = [P_1, B]$ by solving $CB^t = -DP_1^t$. Let $p_1\colon \mathbb{T}^5 \to \mathbb{T}^4$ and $p_2\colon \mathbb{T}^{14} \to \mathbb{T}^4$ be the maps of tori corresponding to P_1 and P_2. Instead of using Algorithm 4.3.8, we directly produce the equations G_2' for X_2 on the torus:

$$
\begin{aligned}
p_2^\star(p_1)_\star\, f_1 &= T_1T_6T_7T_8T_{14} - T_2T_{10}T_{11}^2 - T_3T_{12}T_{13}^2,\\
p_2^\star(p_1)_\star\, f_2 &= T_1T_4T_{14}^2 + T_2T_3T_9T_{11}T_{13} - T_5T_6^2T_7.
\end{aligned}
$$

Note that by [4], the variables define pairwise non-associated $\mathrm{Cl}(X_2)$-prime generators for $\mathcal{R}(X_2)$. This makes X_2 a weak CEMDS with data (P_2, Σ_2, G_2') where Σ_2 is the stellar subdivision of the fan Σ_1 of the CEMDS X_1 at the columns of B. We now use Algorithm 4.3.7 to contract on X_2 the curves corresponding to the variables T_i with $i \in \{2, 3, 5, 7, 8, 9, 10, 12, 14\}$. The resulting ring is the one listed in the table of the theorem.

Observe that the given surfaces do not admit a non-trivial $\mathbb{K}^*$-action. As noted before, we only have to treat cases E_6A_2, E_7A_1 and E_8. Here, using e.g., Algorithm 2.4.8, one computes the minimal resolution $X' \to X$ and compares $\mathcal{R}(X')$ with the Cox ring given in [33, Sect. 3]. If the rings are isomorphic, again by [33, Sect. 3], X does not admit a non-trivial $\mathbb{K}^*$-action. See Example 2.4.12 for the E_6A_2 case. □

Remark 4.4.2. Note that the resolutions of the surfaces with singularity type E_6A_2, E_7A_1 and E_8 listed in Theorem 4.4.1 have a hypersurface as Cox ring; they have been computed in [33, Sect. 3, Table 9]. Moreover, these surfaces admit small degenerations into $\mathbb{K}^*$-surfaces. In fact, multiplying the monomials $T_2T_3T_4$ and $T_1^2T_4^2$ in the respective Cox rings with a parameter $\alpha \in \mathbb{K}$ gives rise to a flat family of Cox rings over $\mathbb{K}$. The induced flat family of surfaces over $\mathbb{K}$ has a $\mathbb{K}^*$-surface as zero fiber, compare also the corresponding Cox rings listed in [59, Theorem 5.6].

5. The lattice ideal method

We consider the blow up X_2 of a Mori dream space X_1 with known Cox ring and develop a method for the systematic guess and verification of generators for the new Cox ring $\mathcal{R}(X_2)$. A description of $\mathcal{R}(X_2)$ as saturated Rees algebra is used. As examples, we compute the Cox ring of Cayley's cubic surface and the Cox ring of the blow up of a weighted projective space in its base point. This section has been published in [57, Sec. 5] together with J. Hausen and A. Laface.

Let X_1 be a Mori dream space and $\pi\colon X_2 \to X_1$ the blow up of an irreducible subvariety $C \subseteq X_1$ contained in the smooth locus of X_1. As before, write $K_i := \mathrm{Cl}(X_i)$ for the divisor class groups and $R_i := \mathcal{R}(X_i)$ for the Cox rings. Then we have the canonical pullback maps

$$\begin{aligned} \pi^*\colon K_1 &\to K_2, & [D] &\mapsto [\pi^* D], \\ \pi^*\colon R_1 &\to R_2, & (R_1)_{[D]} \ni f &\mapsto \pi^* f \in (R_2)_{[\pi^* D]}. \end{aligned}$$

Moreover, identifying $U := X_2 \setminus \pi^{-1}(C)$ with $X_1 \setminus C$, we obtain canonical push forward maps

$$\begin{aligned} \pi_*\colon K_2 &\to K_1, & [D] &\mapsto [\pi_* D], \\ \pi_*\colon R_2 &\to R_1, & (R_2)_{[D]} \ni f &\mapsto f_{|U} \in (R_1)_{[\pi_* D]}. \end{aligned}$$

Let $J \subseteq R_1$ be the irrelevant ideal, i.e., the vanishing ideal of $\overline{X}_1 \setminus \widehat{X}_1$, and $I \subseteq R_1$ the vanishing ideal of $p_1^{-1}(C) \subseteq \overline{X}_1$ where $p_1\colon \widehat{X}_1 \to X_1$ is the characteristic space. We define the *saturated Rees algebra* to be the subalgebra

$$R_1[I]^{\mathrm{sat}} := \bigoplus_{d \in \mathbb{Z}} \left(I^{-d} : J^\infty\right) t^d \subseteq R_1[t^{\pm 1}], \qquad \text{where } I^k := R_1 \text{ for } k \le 0.$$

Note that this indeed makes $R_1[I]^{\mathrm{sat}}$ a graded algebra. For all $n, m \in \mathbb{Z}$ we have containment of $(I^n : J^\infty)(I^m : J^\infty)$ in $I^{n+m} : J^\infty$.

Remark 4.5.1. The usual Rees algebra $R_1[I] = \bigoplus_{d\in\mathbb{Z}} I^{-d}t^d$ is a subalgebra of the saturated Rees algebra $R_1[I]^{\mathrm{sat}}$. In the above situation, $I \subseteq R_1$ is a K_1-prime ideal. Since K_1-prime ideals are saturated with respect to K_1-homogeneous ideals, we have $I : J^\infty = I$. Consequently, $R_1[I]^{\mathrm{sat}}$ equals $R_1[I]$ if and only if $R_1[I]^{\mathrm{sat}}$ is generated in the $\mathbb{Z}$-degrees 0 and ± 1. In this case, $R_1[I]^{\mathrm{sat}}$ is finitely generated because $R_1[I]$ is so.

Note that the saturated Rees algebra $R_1[I]^{\mathrm{sat}}$ is naturally graded by $K_1 \times \mathbb{Z}$ as R_1 is K_1-graded and the ideals I, J are homogeneous. Let $E = \pi^{-1}(C)$ denote the exceptional divisor. Then we have a splitting $K_2 = \pi^* K_1 \times \mathbb{Z} \cdot [E] \cong K_1 \times \mathbb{Z}$; compare Proposition 1.4.8.

Proposition 4.5.2. *See* [57, Prop. 5.2]. *In the above situation, we have the following mutually inverse isomorphisms of graded algebras*

$$\begin{aligned} R_2 &\longleftrightarrow R_1[I]^{\mathrm{sat}}, \\ (R_2)_{[\pi^*D]+d[E]} \ni f &\mapsto \pi_* f \cdot t^d \in \left(R_1[I]^{\mathrm{sat}}\right)_{([D],d)}, \\ (R_2)_{[\pi^*D]+d[E]} \ni \pi^* f \cdot 1_E^d &\mapsfrom f \cdot t^d \in \left(R_1[I]^{\mathrm{sat}}\right)_{([D],d)}. \end{aligned}$$

For the computation of the Cox ring R_2, we work in the notation of Setting 4.2.5; in particular X_1 comes as a CEMDS $X_1 \subseteq Z_1$. As before, $C \subsetneq X_1$ is an irreducible subvariety contained in the smooth locus of X_1 and $\widehat{C} \subseteq \widehat{X}_1$ denotes its inverse image with respect to $p_1\colon \widehat{X}_1 \to X_1$. The idea is to stretch the given embedding $X_1 \subseteq Z_1$ by suitable generators of the vanishing ideal $I \subseteq R_1$ of $\widehat{C} \subseteq \overline{X}_1$ and then perform an ambient modification.

Algorithm 4.5.3 (BlowUpCEMDS)**.** *Input:* a CEMDS (P_1, Σ_1, G_1), a K_1-prime ideal $I = \langle f_1, \ldots, f_l \rangle \subseteq R_1$ with pairwise non-associated K_1-primes $f_i \in R_1$ defining an irreducible subvariety $C \subsetneq X_1$ inside the smooth locus and coprime positive integers $d_1, \ldots, d_l$ with $f_i \in I^{d_i} : J^\infty$.

- Compute the stretched CEMDS (P_1', Σ_1', G_1') by applying Algorithm 4.3.2 to (P_1, Σ_1, G_1) and $(f_1, \ldots, f_l)$.
- Define a multiplicity vector $v \in \mathbb{Z}^{r_1+l}$ by $v_i := 0$ for $1 \leq i \leq r_1$ and $v_i := d_{i-r_1}$ for $r_1 + 1 \leq i \leq r_1 + l$.
- Determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ of the fan Σ_1' along the ray through $P_1' \cdot v$. Write $P_2 := [P_1', P_1' \cdot v]$.
- Compute (P_2, Σ_2, G_2) by applying Algorithm 4.3.8 to (P_1', Σ_1', G_1') and the pair (P_2, Σ_2).
- Let T^ν be the product over all T_i with $C \not\subseteq D_i$ where $D_i \subseteq X_1$ is the divisor corresponding to T_i. Test whether $\dim(I_2 + \langle T_{r_2} \rangle) > \dim(I_2 + \langle T_{r_2}, T^\nu \rangle)$.
- Set $(P_2', \Sigma_2', G_2') := (P_2, \Sigma_2, G_2)$. Eliminate all fake relations by applying Algorithm 4.3.3. Call the output (P_2, Σ_2, G_2).

Output: (P_2, Σ_2, G_2). If the verification in the next to last step was positive, then (P_2, Σ_2, G_2) is a CEMDS describing the blow up X_2 of X_1 along C. In particular then the K_2-graded algebra R_2 is the Cox ring of X_2.

Lemma 4.5.4. *Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ be ideals and $f \in \mathbb{K}[T_1, \ldots, T_r]$ a polynomial. Then taking the saturation commutes with taking the localization, i.e.,*

$$\mathfrak{a}_f : (\mathfrak{b}_f)^\infty \;=\; (\mathfrak{a} : \mathfrak{b}^\infty)_f \;\subseteq\; \mathbb{K}[T_1, \ldots, T_r]_f.$$

Proof. Compare also [8, Cor. 3.15]. Consider $g \in (\mathfrak{a} : \mathfrak{b}^\infty)_f$. Then $gf^{-k} \in \mathfrak{a} : \mathfrak{b}^s$ for some $k \in \mathbb{Z}_{\geq 0}$ and an integer $s \geq 1$. This means $g\mathfrak{b}^s \subseteq \mathfrak{a}$. Since localizing commutes with taking products we obtain $g(\mathfrak{b}_f)^s \subseteq \mathfrak{a}_f$, see [8, Prop. 3.11].

For the other inclusion, let $g \in \mathfrak{a}_f : (\mathfrak{b}_f)^\infty$. As before, $g(\mathfrak{b}^s)_f \subseteq \mathfrak{a}_f$ for an $s \in \mathbb{Z}_{\geq 1}$. In particular, $g\mathfrak{b}^s \subseteq \mathfrak{a}_f$. Write $\mathfrak{b}^s = \langle b_1, \ldots, b_l \rangle$. Then there are $a_i \in \mathfrak{a}$ and $m_i \in \mathbb{Z}$ such that $gb_i = a_i f^{m_i} \in \mathfrak{a}_f$ which means $gf^{-m_i} b_i \in \mathfrak{a}$. We obtain $gf^{k'}\mathfrak{b}^s \subseteq \mathfrak{a}$ for suitable $k' \in \mathbb{Z}$. We arrive at $g \in (\mathfrak{a} : \mathfrak{b}^\infty)_f$. $\square$

Lemma 4.5.5. *In the situation of Algorithm 4.5.3, consider a monomial $T^\nu \in J \subseteq R_1$. Then the localization $(R_1[I]^{\mathrm{sat}})_{T^\nu}$ is isomorphic to $(R_1[I])_{T^\nu}$.*

Proof. Set $f := T^\nu$. Since the T_i with $1 \leq i \leq r_1$ are of $\mathbb{Z}$-degree zero, using Lemma 4.5.4, we obtain

$$\begin{aligned} (R_1[I]^{\mathrm{sat}})_f &= \bigoplus_{k<0} \left(I^{-k} : J^\infty\right)_f t^k \;\oplus\; \bigoplus_{k \geq 0} (R_1)_f t^k \\ &= \bigoplus_{k<0} \left(\left(I^{-k}\right)_f : R_1\right) t^k \;\oplus\; \bigoplus_{k \geq 0} (R_1)_f t^k \\ &= \bigoplus_{k<0} \left(I^{-k}\right)_f t^k \;\oplus\; \bigoplus_{k \geq 0} (R_1)_f t^k \\ &= (R_1[I])_f. \end{aligned}$$

$\square$

Lemma 4.5.6. *In the situation of Algorithm 4.5.3, let $R_1[I] \subseteq R_2 \subseteq R_1[I]^{\mathrm{sat}}$ be an inclusion of $K_2 = K_1 \times \mathbb{Z}$-graded algebras. Let T^ν be the product over all T_i such that $C \not\subseteq V(X_1; T_i)$. Then $R_2 = R_1[I]^{\mathrm{sat}}$ holds if*

$$\dim\left(I_2 + \langle T_{r_2} \rangle_{R_2}\right) \;>\; \dim\left(I_2 + \langle T_{r_2},\, T^\nu \rangle_{R_2}\right).$$

Proof. Denote by $(R_2)_d$ the degree d part of R_2 with respect to the natural $\mathbb{Z}$-grading. Assume that the inclusion $R_2 \subseteq R_1[I]^{\mathrm{sat}}$ is strict and let $n \in \mathbb{Z}_{\geq 1}$ be minimal such that there is $ft^{-n} \in R_1[I]^{\mathrm{sat}}_{-n} \setminus (R_2)_{-n}$. Observe that $n > 1$ by assumption. Moreover $ft^{-n+1} \in R_2$ since $f \in I^n \colon J^\infty \subseteq I^{n-1} \colon J^\infty$ and $(R_2)_{-n+1} = (R_1[I]^{\mathrm{sat}})_{-n+1}$. Therefore

$$ft^{-n+1} \ \in \ \langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap R_2, \qquad ft^{-n+1} \ \notin \ \langle t \rangle_{R_2}$$

so that the ideal $\langle t \rangle_{R_2}$ is strictly contained in $\langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap R_2$. Note that T^ν is an element of the irrelevant ideal J since

$$T^\nu \ \in \ J \qquad \Leftrightarrow \qquad \bigcap_{\nu_i = 0} D_i \ \neq \ \emptyset \qquad \Leftrightarrow \qquad \bigcap_{C \subseteq D_i} D_i \ \neq \ \emptyset.$$

Moreover, localizing by T^ν, Lemma 4.5.5 delivers

$$\left(\langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap R_2\right)_{T^\nu} \ = \ \langle t \rangle_{R_1[I]^{\mathrm{sat}}_{T^\nu}} \cap (R_2)_{T^\nu} \ = \ \langle t \rangle_{R_1[I]_{T^\nu}}.$$

In particular, both ideals are of the same dimension in R_2 and $\langle t \rangle_{R_2}$ equals the K_2-prime ideal $\langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap R_2$ in $(R_2)_{T^\nu}$, i.e., t is K_2-prime in $(R_2)_{T^\nu}$. Observe that t is K_2-prime in R_2. Given K_2-homogeneous elements $h_1, h_2 \in R_2$ with $t \mid h_1 h_2$, considered as elements of $(R_2)_{T^\nu}$, we have $t \mid h_1$ or $t \mid h_2$, i.e.,

$$h_1 (T^\nu)^{k_1} \ = \ t\alpha_1 \quad \text{or} \quad h_2 (T^\nu)^{k_2} \ = \ t\alpha_2, \qquad \alpha_i \in R_2,\ k_i \in \mathbb{Z}_{\geq 0}.$$

Since by the dimension requirement we know that t and T^ν are coprime in R_2, we obtain $t \mid h_1$ or $t \mid h_2$. Therefore, $\langle t \rangle_{R_2} = \langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap R_2$ in R_2, a contradiction. □

Proof of Algorithm 4.5.3. Consider the K_2-graded ring $R_2 = \mathbb{K}[T_1, \ldots, T_{r_2}]/I_2$ associated to the output (P_2, Σ_2, G_2). Assume all verifications were positive. The first step is to show that R_2 is normal; then (P_2, Σ_2, G_2) is a CEMDS and R_2 is the Cox ring of the output variety X_2. In a second step we show that X_2 equals the blow up of X_1 along C.

Consider the output (P_2, Σ_2, G_2) of the fourth item, i.e., the situation before entering the last step. The variables $T_{r_1+1}, \ldots, T_{r_2-1}$ correspond to $f_1, \ldots, f_l$ and T_{r_2} to the exceptional divisor. Observe that we have a canonical K_2-graded homomorphism $R_2 \to R_1[I]^{\mathrm{sat}}$ induced by

$$\mathbb{K}[T_1, \ldots, T_{r_2}] \ \to \ R_1[I]^{\mathrm{sat}}, \qquad T_i \ \mapsto \begin{cases} T_i, & 1 \leq i \leq r_1, \\ f_{i-r_1} t^{-v_i}, & r_1 < i < r_2, \\ t, & i = r_2. \end{cases}$$

Indeed, because C is contained in the smooth locus of X_1, the cone generated by the last l columns of P_1' is regular and, because in addition $d_1, \ldots, d_l$ are coprime, the vector $P_1' \cdot v$ is primitive. Thus, the ideal I_2 of X_2 is the saturation with respect to T_{r_2} of

$$I_1 + \langle T_i T_{r_2}^{v_i} - f_{i-r_1};\ r_1 < i < r_2 \rangle \ \subseteq \ \mathbb{K}[T_1, \ldots, T_{r_2}].$$

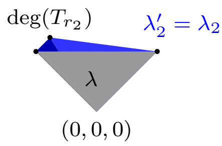
Consequently, the above assignment induces a homomorphism $R_2 \to R_1[I]^{\mathrm{sat}}$. This homomorphism yields an isomorphism of the K_2-graded localizations

$$(R_2)_{T_{r_2}} \ = \ \bigoplus_{d \in \mathbb{Z}} R_1 T_{r_2}^d \ \cong \ \bigoplus_{d \in \mathbb{Z}} R_1 t^d \ = \ \left(R_1[I]^{\mathrm{sat}}\right)_t$$

and hence is in particular injective. As the image $\varphi(R_2)$ contains generators

$$\varphi(T_{r_2}) \ = \ t, \qquad \varphi\left(T_{r_1+i} T_{r_2}^{v_i - 1}\right) \ = \ f_i t^{-1}, \qquad 1 \leq i \leq l$$

for the Rees algebra $R_1[I]$, we obtain $R_1[I] \subseteq \varphi(R_2) \subseteq R_1[I]^{\mathrm{sat}}$. In fact, by the dimension check in the last step and the definition of φ we may apply Lemma 4.5.6 which delivers $\varphi(R_2) = R_1[I]^{\mathrm{sat}}$. By Proposition 4.5.2, $R_1[I]^{\mathrm{sat}} \cong R_2$ is the Cox ring of the blow up X_2' of X_1 at C. In particular, $R_2 \cong R_2'$ is normal and we may apply Algorithm 4.3.3. Note that there is no need to use the `verify` option as the variables $T_1, \ldots, T_{r_2} \in R_2$ are K_2-prime and the generators surviving the elimination process are K_2-prime as well. As for any Cox ring, the K_2-grading is almost free.



We show that $X_2 \cong X_2'$ holds. Let $\lambda \subseteq \mathrm{Mov}(R_2)$ be the chamber representing X_1. Then λ is of codimension one in $\mathbb{Q} \otimes K_2$ and lies on the boundary of $\mathrm{Mov}(R_2)$. Since there are the contraction morphisms $X_2 \to X_1$ and $X_2' \to X_1$, the chambers λ_2, λ_2' corresponding to X_2, X_2' both have λ as a face. We conclude $\lambda_2 = \lambda_2'$ and thus $X_2 \cong X_2'$. □

An important special input case for Algorithm 4.5.9 is the blow up of a smooth point. The point $x_1 \in X_1$ can be given in Cox coordinates, i.e., as a point $z \in \widehat{X}_1 \subseteq \mathbb{K}^{r_1}$ with $x_1 = p_1(z)$.

Definition 4.5.7. Let P be an $s \times r$ integer matrix and $z \in \mathbb{K}^r$. Let $i_1, \ldots, i_k$ be the indices with $z_{i_j} \neq 0$ and $\nu_1, \ldots, \nu_s \in \mathbb{Z}^r$ a lattice basis for $\mathrm{im}(P^*) \cap \mathrm{lin}(e_{i_1}, \ldots, e_{i_k})$. Then the *associated ideal* to P and z is the saturation

$$\begin{aligned} I(P,z) \quad &:= \quad \left\langle z^{-\nu_1^+} T^{\nu_1^+} - z^{-\nu_1^-} T^{\nu_1^-}, \ldots, z^{-\nu_s^+} T^{\nu_s^+} - z^{-\nu_s^-} T^{\nu_s^-} \right\rangle : (T_1 \cdots T_r)^\infty \\ &\subseteq \quad \mathbb{K}[T_1, \ldots, T_r], \end{aligned}$$

where $\nu_i = \nu_i^+ - \nu_i^-$ is the unique decomposition with nonnegative vectors $\nu_i^+, \nu_i^- \in \mathbb{Z}^r$ and we write $T^\alpha = T_1^{\alpha_1} \cdots T_r^{\alpha_r}$ and $z^\alpha = z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ for any vector $\alpha \in \mathbb{Z}^r$.

Note that the ideal $I(P_1, z_1) + \langle T_j;\ j \neq i_1, \ldots, i_k \rangle$ describes the closure of the orbit through z_1 of the quasitorus $H = \mathrm{Spec}(\mathbb{K}[K_1])$ acting on $\mathbb{K}^{r_1}$ via the grading $\deg(T_i) = Q_1(e_i)$ with the projection $Q_1 \colon \mathbb{Z}^{r_1} \to K_1 := \mathbb{Z}^{r_1}/\mathrm{im}(P_1^*)$.

Remark 4.5.8. The ideal $I(P,z)$ is a so called lattice ideal. In particular it is generated by binomials, see [82].

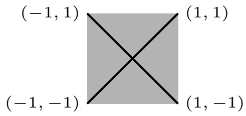
Algorithm 4.5.9 (BlowUpCEMDSpoint)**.** *Input:* a CEMDS $X_1 = (P_1, \Sigma_1, G_1)$ and a smooth point $x \in X_1$ given in Cox coordinates $z \in \mathbb{K}^{r_1}$.

- Compute a list $(f_1, \ldots, f_l)$ of pairwise non-associated K_1-prime generators for for $I(P_1, z) + \langle T_j;\ z_j = 0 \rangle \subseteq R_1$ and choose $d_i \in \mathbb{Z}_{\geq 1}$ such that $f_i \in I^{d_i} : J^\infty$.
- Call Algorithm 4.5.3 with input X_1, $(f_1, \ldots, f_l)$ and $(d_1, \ldots, d_l)$. Denote the result by (P_2, Σ_2, G_2).

Output: (P_2, Σ_2, G_2). If the verification was positive, then (P_2, Σ_2, G_2) is a CEMDS describing the blow up X_2 of X_1 in x. In particular then the K_2-graded algebra R_2 is the Cox ring of X_2.

We will see in Algorithm 4.5.12 how to choose the f_i more systematically. In the following example, we compute the Cox ring of *Cayley's nodal cubic surface*. We have published it in [56].

Example 4.5.10 (Cayley's cubic)**.** Let X_1 be the toric surface of Picard number two coming with four singularities of type A_1. So, X_1 arises from the fan Σ_1 in $\mathbb{Z}^2$ as indicated below



We determine the Cox ring of the blow up X_2 of X_1 at the unit element of the big torus $\mathbb{T}^2 \subseteq X_1$. Using Algorithm 4.5.9, all d_i are 1 and we obtain the Cox ring of X_2 as $R_2 = \mathbb{K}[T_1, \ldots, T_9]/I_2$ graded by $K_2 = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$ where generators of I_2 and the degree matrix Q_2 are

$$\begin{array}{l} T_5^2 - T_7^2 + T_6T_8,\ T_2T_5 - T_3T_6 + T_4T_7, \\ T_2^2 - T_4^2 - T_6T_9,\ T_3T_5 - T_1T_7 + T_2T_8, \\ T_4T_5 - T_1T_6 + T_2T_7,\ T_1T_5 - T_3T_7 + T_4T_8, \\ T_2T_3 - T_1T_4 - T_5T_9,\ T_1T_2 - T_3T_4 - T_7T_9, \\ T_1^2 - T_3^2 - T_8T_9 \end{array} \qquad \left[\begin{array}{rrrrrrrrr} -1 & 1 & -1 & 1 & 0 & 2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{array}\right]$$

Consider $w := (0, 1, 1, \bar{1}) \in K_2$ which is in fact the anticanonical class in $K_2 = \mathrm{Cl}(X_2)$. Then the homogeneous component $(R_2)_w$ is of dimension 4 and it is generated by the classes

$$z_0 \ := \ T_6T_8T_9, \qquad z_1 \ := \ T_4T_5T_6, \qquad z_2 \ := \ T_4T_7T_9, \qquad z_3 \ := \ T_5T_7T_8,$$

compare Algorithm 2.2.5. The rational map $X_2 \to \mathbb{P}_3$, given in Cox coordinates by $z \mapsto (z_0, z_1, z_2, z_3)$, is a closed embedding. We see that the image in $\mathbb{P}_3$ is Cayley's cubic surface as we have in $(R_2)_{3w}$ the relation

$$z_0z_1z_2 + z_0z_1z_3 + z_0z_2z_3 + z_1z_2z_3 \quad = \quad 0.$$

We now give an example where the Cox ring computation with Algorithm 4.5.3 fails depending on the input multiplicities $d_i \in \mathbb{Z}_{\geq 1}$. We will also see that the weighted ambient toric blow up induces a blow up of the embedded varieties. This serves also an example for the proof of the algorithm.

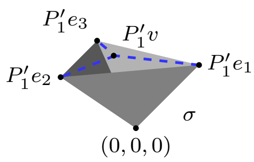
Example 4.5.11 (to the proof of Algorithm 4.5.3)**.** Consider the smooth two-dimensional CEMDS X_1 with $\mathbb{Z}^5$-graded Cox ring, degree matrix and irrelevant ideal

$$\begin{array}{rcl} R_1 & = & \mathbb{K}[T_1, \ldots, T_8] \,/\, \langle f \rangle, \\ f & := & T_3^3T_4^2T_5 - T_1^2T_2 - T_7T_8, \end{array} \qquad Q_1 \ := \ \left[\begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -3 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right],$$

$$J \ = \ \left\langle \begin{array}{c} T_1T_2T_3T_4T_7,\, T_1T_3T_4T_5T_7,\, T_1T_2T_5T_6T_8, \\ T_2T_4T_5T_6T_8,\, T_1T_3T_4T_7T_8,\, T_2T_5T_6T_7T_8, \\ T_1T_2T_3T_6,\, T_3T_4T_5T_6,\, T_3T_6T_7T_8 \end{array} \right\rangle \ \subseteq \ R_1.$$

We want to blow up X_1 in the point $x \in X_1$ having $z := (0, 1, 0, 1, 1, 1, 0, 1) \in \widehat{X}_1 \subseteq \mathbb{K}^8$ as Cox coordinates. Set $I := \langle T_1, T_3, T_7 \rangle \subseteq R_1$. By a computation, we have prime elements $f_i \in R_1$ with multiplicities $d_i \in \mathbb{Z}_{>0}$ where

$$\begin{array}{l} f_1 \ := \ T_1, \quad f_2 \ := \ T_3, \quad f_3 \ := \ T_7 \ \in \ R_1, \\ \quad d_1 \ = \ 1, \quad d_2 \ = \ 1, \quad d_3 \ = \ 2. \end{array}$$



Assume that we called Algorithm 4.5.3 instead with input X_1, I and lists (f_1, f_2, f_3) (d_1, d_2, d_3) where $d_1 = d_2 = d_3 = 1$. First, the call to Algorithm 4.3.2 provides the embedding

$$\bar{\imath}\colon \mathbb{K}^8 \ \to \ \mathbb{K}^{11}, \qquad\qquad z \ \mapsto \ (z,\ f_1(z),\ f_2(z),\ f_3(z))$$

and the stretched CEMDS X_1'. On X_1', we want to blow up the point $\iota(x)$ with Cox coordinates $(z, 0, 0, 0) \in \mathbb{K}^{11}$. The steps of 4.5.3 and its proof then are as follows.

Applying steps three and four, we obtain a modified ES X_2 with a $\mathbb{Z}^6$-graded ring

$$R_2 \;=\; \mathbb{K}[T_1, \ldots, T_{12}] \,/\, I_2, \qquad I_2 \;=\; \left\langle \begin{array}{c} T_9T_{12} - T_1,\; T_{10}T_{12} - T_3,\; T_{11}T_{12} - T_7, \\ T_{10}^3T_{12}^2T_4^2T_5 - T_9^2T_2T_{12} - T_8T_{11} \end{array} \right\rangle.$$

Recall from the proof of Algorithm 4.5.3 the monomorphism $\varphi\colon R_2 \to R_1[I]^{\mathrm{sat}}$ induced by

$$\mathbb{K}[T_1, \ldots, T_{12}] \;\to\; R_1[I]^{\mathrm{sat}}, \qquad T_i \;\mapsto\; \begin{cases} T_i, & 1 \le i \le 8, \\ f_{i-r_1}t^{-v_i}, & 8 < i < 12, \\ t, & i = 12. \end{cases}$$

As predicted, observe that we indeed obtain an inclusion $R_1[I] \subseteq \varphi(R_2) \subseteq R_1[I]^{\mathrm{sat}}$ of algebras. By [98, Prop. 7.9, Prop. 7.10], the Rees algebra $R_1[I]$ and $\varphi(R_2)$ even coincide:

$$\begin{aligned} R_1[I] \;&\cong\; \mathbb{K}[T_1,\, \ldots,\, T_8,\, U_1,\, U_2,\, U_3,\, t] \,/\, \langle g,\, U_1t - T_1,\, U_2t - T_3,\, U_3t - T_7 \rangle, \\ \varphi(R_2) \;&\cong\; \mathbb{K}[T_1,\, \ldots,\, T_8,\, U_1,\, U_2,\, U_3,\, t] \,/\, \langle g,\, U_1t - T_1,\, U_2t - T_3,\, U_3t - T_7 \rangle, \\ g \;&:=\; U_2^3T_4^2T_5t^2 - U_1^2T_2t - U_3T_8, \end{aligned}$$

Note that the inclusion $\varphi(R_2) \subsetneq R_1[I]^{\mathrm{sat}}$ is proper as, by a direct computation, we have $T_7 \in (I^2 : J^\infty) \setminus I^2$. This means T_7t^{-2} is an element of $(R_1[I]^{\mathrm{sat}})_{-2} \setminus \varphi(R_2)_{-2}$. Then T_7t^{-1} is an element of $\langle t \rangle_{R_1[I]^{\mathrm{sat}}} \cap \varphi(R_2)$ with $T_7t^{-1} \notin \langle t \rangle_{\varphi(R_2)}$. As predicted, the irrelevant ideal $J \subseteq R_1$ contains the product $T^\nu = T_2T_4T_5T_6T_8$ over all T_i with $x \notin V(X_1; T_i)$. Passing to localizations, according to Lemma 4.5.5, we have the equality $R_1[I]_{T^\nu} = \varphi(R_2)_{T^\nu} = R_1[I]^{\mathrm{sat}}_{T^\nu}$ of localized algebras. For instance, we now have

$$T_7 \;=\; T_8^{-1}\left(T_3^3T_4^2T_5 - T_1^2T_2\right) \quad \in \quad \left(I^2 : J^\infty\right)_{T^\nu} \;=\; \left(I^2\right)_{T^\nu}.$$

Then $\langle t \rangle_{R_1[I]^{\mathrm{sat}}_{T^\nu}} \cap \varphi(R_2)_{T^\nu}$ equals $\langle t \rangle_{\varphi(R_2)_{T^\nu}}$ which implies that t is a prime element in $\varphi(R_2)_{T^\nu}$. Consequently, t and T^ν are coprime in $R_1[I]^{\mathrm{sat}}$ but not in $\varphi(R_2)$ as

$$\dim\left(I_2 + \langle T_{12},\, T_8 \rangle_{R_2}\right) \;=\; \dim\left(I_2 + \langle T_{12} \rangle_{R_2}\right).$$

Thus, the verification step in Algorithm 4.5.9 with input (f_1, f_2, f_3) and $(1,1,1)$ fails. Had we chosen $d_3 = 2$, we would have obtained $U_2^3T_4^2T_5t - U_1^2T_2 - U_3T_8$ instead of g, the codimension test is successful and $\varphi(R_2) = R_1[I]^{\mathrm{sat}}$. This means the Cox ring of the blow up is the $\mathbb{Z}^6$-graded ring

$$\mathcal{R}(X_2) \;=\; \mathbb{K}[T_1, \ldots, T_9] \,/\, \langle T_3^3T_4^2T_5T_9 - T_1^2T_2 - T_7T_8 \rangle.$$

Albeit implied by the proof of Algorithm 4.5.9, we now show directly that the weighted ambient toric blow up induces a blow up of X_1 in x. Let $P_1 = [v_1, \ldots, v_8]$ be a Gale dual matrix of Q_1. Then $\mathbb{Q}_{\ge 0} \cdot v_1, \ldots, \mathbb{Q}_{\ge 0} \cdot v_8$ are the rays of the fan Σ_1 of the canonical toric ambient variety Z_1 of X_1. Choose coordinates

$$S_3 \;:=\; \frac{T_7}{T_2^3T_4^2T_5^4T_6^6T_8^5}, \qquad S_2 \;:=\; \frac{T_3}{T_2T_5T_6^2T_8^2}, \qquad S_1 \;:=\; \frac{T_1}{T_2T_4T_5^2T_6^3T_8^3}$$

for the affine chart $(Z_1)_\sigma$ with the smooth cone $\sigma := \mathrm{cone}(v_1, v_3, v_7) \in \Sigma_1$. Then

$$\mathcal{O}((Z_1)_\sigma \cap X_1) \;=\; \mathbb{K}[S_1, S_2, S_3] \,/\, \langle h \rangle, \qquad h \;:=\; (p_1)_\star f \;=\; S_2^3 - S_1^2 - S_3.$$

Since $d_1 = d_2 = 1$ and $d_3 = 2$, the blow up of X_1 in x is induced by a weighted blow $Z_2 \to Z_1$ of the toric ambient variety. By [84, Ex. 2], we have to blow up the sheaf of ideals $\mathcal{J}$ in $\mathcal{O}((Z_1)_\sigma \cap X_1)$ generated by

$$\langle S_1^2,\, S_2^2,\, S_3 \rangle \quad = \quad \langle S_1^2,\, S_2^2,\, S_2^3 - S_1^2 \rangle \quad = \quad \langle S_1^2,\, S_2^2 \rangle$$

where we used the special shape of h. Taking the Proj, by Proposition 1.4.5, we see that this induces the usual blow up of X_1 in x, i.e., $\mathrm{Proj}\,\mathcal{I} = \mathrm{Proj}\,\mathcal{J}$ where $\mathcal{I}$ is the sheaf of ideals in $\mathcal{O}((Z_1)_\sigma \cap X_1)$ generated by $\langle S_1, S_2 \rangle$. Note that we blew up a bigger orbit in Z_1 than anticipated; still, the insertion of the respective ray now cuts out $x \in X_1$.

The following algorithm produces a systematic guess for the generators and their multiplicities $d_i \in \mathbb{Z}_{\geq 1}$ of the Cox ring of a blow up of a Mori dream space.

Algorithm 4.5.12 (BlowUpCEMDS2)**.** *Input:* a CEMDS (P_1, Σ_1, G_1), a K_1-prime ideal I defining an irreducible subvariety $C \subseteq X_1$ inside the smooth locus.

- Let F and D be empty lists.
- For each $k = 1, 2, \ldots \in \mathbb{Z}_{\geq 1}$ do
  - compute a set G_k of generators for $A_k := I^k : J^\infty \subseteq R_1$. Let $f_{k1}, \ldots, f_{kl_i}$ be a maximal subset of pairwise non-associated elements of G_k with
$$f_{kj} \;\notin\; A_1 A_{k-1} + \ldots + A_{\lfloor \frac{k}{2} \rfloor} A_{\lceil \frac{k}{2} \rceil} \qquad \text{if} \;\; k > 1.$$
  - Determine integers $d_{k1}, \ldots, d_{kl_i} \in \mathbb{Z}_{\geq k}$ such that $f_{kj} \in A_{d_{ki}} \setminus A_{d_{ki}+1}$.
  - Add the elements of $f_{k1}, \ldots, f_{kl_i}$ to F that are not associated to any other element of F. Add the respective integers among $d_{k1}, \ldots, d_{kl_i}$ to D.
  - Run Algorithm 4.5.9 with input (P_1, Σ_1, G_1), F and D.
  - If Algorithm 4.5.9 terminated with (P_2, Σ_2, G_2) and positive verification, return (P_2, Σ_2, G_2).

Output (if provided): the algorithm terminates if and only if X_2 is a Mori dream space. In this case, the CEMDS (P_2, Σ_2, G_2) describes the blow up X_2 of X_1 along C. In particular, then the K_2-graded algebra R_2 is the Cox ring of X_2.

Proof. Note that each f_{ki} is a K_1-prime element. Otherwise, $f_{ki} = f_1 f_2$ with K_1-homogeneous elements $f_i \in R_1$. As I is K_1-prime, f_1 or f_2 lies in $A_{k'}$ with $k' < k$, i.e., $f_{ki} \in A_{k'}$. This contradicts the choice of f_{ki}.

By Proposition 4.5.2, the Cox ring R_2 of the blow up is isomorphic to the saturated Rees algebra $R_1[I]^{\mathrm{sat}}$. After the k-th step, $(F, T_1, \ldots, T_{r_1}, t)$ are generators for a subalgebra $B_k \subseteq R_1[I]^{\mathrm{sat}}$ such that
$$\mathbb{K}\left[\{t\} \cup R_1 \cup A_1 t^{-1} \cup \ldots \cup A_k t^{-k}\right] \;\subseteq\; B_k \;\subseteq\; \bigoplus_{k \in \mathbb{Z}} A_k t^{-k} \;=\; R_1[I]^{\mathrm{sat}}.$$
If the algorithm stops, by the correctness of Algorithm 4.5.9, the output then is a CEMDS describing the blow X_2 with Cox ring R_2. Vice versa, if X_2 has finitely generated Cox ring, there is $k_0 \geq 1$ with $R_1[I]^{\mathrm{sat}} = B_{k_0}$. Then Algorithm 4.5.9 is called with K_1-prime non-associated generators for $\mathcal{R}(X_2) \cong R_1[I]^{\mathrm{sat}}$ and thus terminates with positive verification. $\square$

Remark 4.5.13. Steps similar to the ones performed in Algorithm 4.5.12 can be used to determine generators and relations of each graded algebra $A = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_k$ if all A_k are finitely generated A_0-modules.

We now consider an example where an extra generator (found with Algorithm 4.5.12) with $d_i > 1$ is needed. Recall that given $a, b, c \in \mathbb{Z}_{\geq 1}$ with $\gcd(a, b, c) = 1$, the *weighted projective space* $\mathbb{P}(a, b, c)$ is the complete toric surface with the $\mathrm{Cl}(X) = \mathbb{Z}$-grading of $\mathcal{R}(\mathbb{P}(a, b, c)) = \mathbb{K}[T_1, \ldots, T_3]$ defined by the degree matrix
$$Q \;=\; \left[\begin{array}{ccc} a & b & c \end{array}\right].$$
Example 4.5.14 (Blow up of a weighted projective space)**.** We want to compute the Cox ring of the blow up of $X_1 := \mathbb{P}(3, 4, 5)$ at the general point with Cox coordinates $z_1 := (1, 1, 1) \in \mathbb{K}^3$. The lattice ideal of z_1 with respect to P_1 is
$$I(P_1, z_1) \;=\; \langle T_2^2 - T_1 T_3,\; T_1^2 T_2 - T_3^2,\; T_1^3 - T_2 T_3 \rangle, \qquad P_1 \;:=\; \left[\begin{array}{ccc} 1 & -2 & 1 \\ -2 & -1 & 2 \end{array}\right].$$

An application of Algorithm 4.5.9 with the three generators (f_1, f_2, f_3) of $I := I(P_1, z_1)$ and $d_i := 1$ for all i is unsuccessful: the dimension check fails. However, adding the additional generator

$$f_4 \quad := \quad T_1^5 - 3T_1^2T_2T_3 + T_1T_2^3 + T_3^3 \quad \in \quad I^2 : J^\infty$$

with $d_4 := 2$ to the input, Algorithm 4.5.9 returns the $\mathrm{Cl}(X_2) = \mathbb{Z}^2$-graded Cox ring $R_2 = \mathcal{R}(X_2)$ of the blow up X_2 of X_1 in $[z_1]$. All verifications are positive. The ring is given as $R_2 = \mathbb{K}[T_1, \ldots, T_8]/I_2$ with generators for I_2 and the degree matrix being

$$\begin{array}{ll} -T_1T_7 + T_4T_5 + T_6^2, & T_1T_4^2 - T_2T_7 + T_5T_6, \\ -T_1T_4T_6 - T_3T_7 + T_5^2, & -T_1T_5 + T_2T_6 + T_3T_4, \\ T_2^2 - T_1T_3 - T_4T_8, & T_1^3 - T_2T_3 - T_6T_8, \\ T_1^2T_4 - T_2T_5 + T_3T_6, & T_1^2T_6 + T_1T_2T_4 - T_3T_5 - T_7T_8, \\ T_1^2T_2 - T_3^2 - T_5T_8 & \end{array} \qquad \left[\begin{array}{rrrrrrrr} 3 & 4 & 5 & -1 & 1 & 0 & -3 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & -1 \end{array}\right].$$

In Algorithm 4.5.3, the saturation computation may become infeasible. In this case, the following variant can be used to, at least, obtain finite generation.

Algorithm 4.5.15 (Finite generation). *Input:* a CEMDS (P_1, Σ_1, G_1), a K_1-prime ideal $I = \langle f_1, \ldots, f_l \rangle \subseteq R_1$ with pairwise non-associated K_1-primes f_i defining an irreducible subvariety $C \subseteq X_1$ inside the smooth locus and coprime positive integers $d_1, \ldots, d_l$ with $f_i \in I^{d_i} : J^\infty$.

- Compute the stretched CEMDS (P_1', Σ_1', G_1') by applying Algorithm 4.3.2 to (P_1, Σ_1, G_1) and $(f_1, \ldots, f_l)$.
- Define a multiplicity vector $v \in \mathbb{Z}^{r_1+l}$ by $v_i := 0$ if $1 \le i \le r_1$ and $v_i := d_{i-r_1}$ for $r_1 + 1 \le i \le r_1 + l$.
- Determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ of the fan Σ_1' along the ray through $P_1' \cdot v$. Set $P_2 := [P_1', P_1' \cdot v]$.
- Use Algorithms 2.2.12 and 2.2.13 to compute $G_2' := (h_1, \ldots, h_s)$ where $h_i = p_2^\star (p_1')_\star (g_i)$ and $G_1' = (g_1, \ldots, g_s)$.
- Choose a system of generators G_2 of an ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ with $\langle G_2' \rangle : (T_1 \cdots T_{r_2})^\infty \supseteq I_2 \supseteq \langle G_2' \rangle$.
- Check if $\dim(I_2) - \dim(I_2 + \langle T_i, T_j \rangle) \ge 2$ for all $i \ne j$.
- Check if T_{r_2} is prime in $\mathbb{K}[T_j^{\pm 1};\ j \ne r_2][T_{r_2}]/I_2$.

Output: (P_2, Σ_2, G_2). The ES (P_2, Σ_2, G_2) describes the blow up X_2 of X_1 along C. If all verifications in the last steps were positive, the Cox ring $\mathcal{R}(X_2)$ is finitely generated and is given by the H_2-equivariant normalization of $\mathbb{K}[T_1, \ldots, T_{r_2}]/I_2 : (T_1 \cdots T_{r_2})^\infty$.

Proof. By the last verification, the exceptional divisor $D_{r_2} \subseteq X_2$ inherits a local defining equation from the toric ambient variety Z_2. Thus, the ambient modification is neat in the sense of [51, Def. 5.4]. By [51, Prop. 5.5], $X_2 \subseteq Z_2$ is a neat embedding. In turn, the dimension checks enable us to use [51, Cor. 2.7]. This completes the proof. $\square$

We will make frequent use of Algorithm 4.5.9 both directly as well as formally in Chapter 5. To close this section, we now use the Cox ring computation 4.5.9 to present the Rees algebras associated to certain binomial ideals in terms of generators and relations; we retrieve [98, Prop. 7.10].

Definition 4.5.16. Consider a binomial ideal $I = \langle T^{\nu_i^+} - T^{\nu_i^-};\ i = 1, \ldots, s \rangle$ in the polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$ and let P be the $s \times r$ matrix with the rows $\nu_i^+ - \nu_i^-$. Then $\mathbb{K}[T_1, \ldots, T_r]$ is graded by the *associated abelian group* $K_I := \mathbb{Z}^r / \mathrm{Im}(P^*)$ via $\deg(T_j) := e_j + \mathrm{Im}(P^*)$. We say that I is *general*, if the following properties hold:

(i) $\deg(T_1), \ldots, \deg(T_r)$ generate a pointed cone in $K_{\mathbb{Q}}$,

(ii) any $r-1$ of the $\deg(T_j)$ generate K_I as a group,
(iii) the ideal I is K_I-prime,
(iv) every T_j defines a K_I-prime in $\mathbb{K}[T_1, \dots, T_r]/I$.

Corollary 4.5.17. *Compare* [98, Prop. 7.10]. *Let I be a general binomial ideal in $R := \mathbb{K}[T_1, \dots, T_r]$ with K_I-prime generators $f_1, \dots, f_s$ and consider the ideal*

$$I' \;:=\; \langle SU_i - f_i;\, i = 1, \dots, s\rangle : S^\infty \;\subseteq\; \mathbb{K}[S, U_1, \dots, U_s, T_1, \dots, T_r] \;=:\; R'.$$

If S defines a K-prime element in R'/I', not associated to any of the U_k, T_j, then the Rees algebra $R[I]$ is isomorphic to the factor algebra R'/I'.

Proof. Let X be any projective toric variety having the K-graded polynomial ring $\mathbb{K}[T_1, \dots, T_r]$ as its Cox ring. Then, according to Algorithm 4.5.9, the ring R'/I' is the Cox ring of the blow up of X at the base point and thus, by Proposition 4.5.2, a saturated Rees algebra. Since R'/I' is generated in the Rees degrees $0, \pm 1$, it is the usual rees algebra. □

6. Linear generation

In this section, we consider blow ups $X_2 \to \mathbb{P}_n$ of the projective space $\mathbb{P}_n$ in k distinct points where the Cox ring $\mathcal{R}(X_2)$ is generated by proper transforms of hyperplanes. Certain relations to the underlying incidence structures are discussed. As an application, we consider certain blow ups of six special points on $\mathbb{P}_3$. Most of this section has been published in [57, Sec. 7] in joint work with J. Hausen and A. Laface

We consider the blow up X of a projective space $\mathbb{P}_n$ at k distinct points $x_1, \dots, x_k$ where $k > n+1$. Our focus is on special configurations in the sense that the Cox ring of X is generated by the exceptional divisors and the proper transforms of hyperplanes. We assume that $x_1, \dots, x_{n+1}$ are the standard toric fixed points, i.e., we have

$$x_1 \;=\; [1, 0, \dots, 0], \qquad \dots, \qquad x_{n+1} \;=\; [0, \dots, 0, 1].$$

Now, write $\mathcal{P} := \{x_1, \dots, x_k\}$ and let $\mathcal{L}$ denote the set of all hyperplanes $\ell \subseteq \mathbb{P}_n$ containing n (or more) points of $\mathcal{P}$. For every $\ell \in \mathcal{L}$, we fix a linear form $f_\ell \in \mathbb{K}[T_1, \dots, T_{n+1}]$ with $\ell = V(f_\ell)$. Note that the f_ℓ are homogeneous elements of degree one in the Cox ring of $\mathbb{P}_n$.

The idea is now to take all T_ℓ where $\ell \in \mathcal{L}$, as prospective generators of the Cox ring of the blow up X and then to compute the Cox ring using Algorithms 4.3.2, 4.3.8 and 4.3.3. Here comes the algorithmic formulation.

Algorithm 4.6.1 (LinearBlowUp)**.** *Input:* a collection $x_1, \dots, x_k \in \mathbb{P}_n$ of pairwise distinct points.

- Set $X_1 := \mathbb{P}_n$, let Σ_1 be the fan of $\mathbb{P}_n$ and P_1 the matrix with columns $e_0, \dots, e_n$ where $e_0 = -(e_1 + \dots + e_n)$.
- Compute the set $\mathcal{L}$ of all hyperplanes through any n points of $x_1, \dots, x_k$, let $(f_\ell;\ \ell \in \mathcal{L}')$ be the collection of the f_ℓ different from all T_i.
- Compute the stretched CEMDS (P_1', Σ_1', G_1') by applying Algorithm 4.3.2 to (P_1, Σ_1, G_1) and $(f_\ell;\ \ell \in \mathcal{L}')$.
- Determine the Cox coordinates $z_i' \in \mathbb{K}^{r_1'}$ of the points $x_i' \in X_1'$ corresponding to $x_i \in X_1$.
- Let Σ_2 be the barycentric subdivision of Σ_1' at the cones σ_i', corresponding to the toric orbits containing $x_i' = p_1'(z_i')$. Write primitive generators for the rays of Σ_2 into a matrix $P_2 = [P_1, B]$.
- Compute (P_2, Σ_2, G_2) by applying Algorithm 4.3.8 to (P_1', Σ_1', G_1') and the pair (P_2, Σ_2)

- Set $(P_2', \Sigma_2', G_2') := (P_2, \Sigma_2, G_2)$. Eliminate all fake relations by applying Algorithm 4.3.3 with option `verify`. Call the ouput (P_2, Σ_2, G_2).

Output: (P_2, Σ_2, G_2). If the verifications in the last step were positive, this is a CEMDS describing the blow up of $\mathbb{P}_n$ at the points $x_1, \ldots, x_k$. In particular, the K_2-graded algebra R_2 is the Cox ring of X_2.

Besides for the proof of Algorithm 4.6.1, the following lemma will primarily be used in Chapter 5; we have published it in [57, Lem. 6.3].

Lemma 4.6.2. *Consider Setting 4.2.5. Assume that $X_1 \subseteq Z_1$ is a CEMDS, $Z_2 \to Z_1$ arises from a barycentric subdivision of a regular cone $\sigma \in \Sigma_1$ and $X_2 \to X_1$ has as center a point $x \in X_1 \cap \mathbb{T}^n \cdot z_\sigma$. Let f be the product over all T_i where $P_1(e_i) \notin \sigma$, and choose $z \in \mathbb{K}^{r_1}$ with $p_1(z) = x$. Then $X_2 \to X_1$ is the blow up at x provided we have*

$$\langle T_i;\, z_i = 0\rangle_f + I(P_1, z)_f \;=\; \langle T_i;\, e_i \in \widehat{\sigma}\rangle_f + I(\overline{X}_1)_f \;\subseteq\; \mathbb{K}[T_1, \ldots, T_{r_1}]_f.$$

Proof. Compare also [39, Lem. 14.9]. Let $Z_{1,\sigma} \subseteq Z_1$ be the affine chart given by σ and set $X_{1,\sigma} := X_1 \cap Z_{1,\sigma}$. In order to see that the toric blow up $Z_2 \to Z_1$ induces a blow up $X_2 \to X_1$, we have to show

$$\mathfrak{m}_x \;=\; I(\mathbb{T}^n \cdot z_\sigma) + I(X_{1,\sigma}) \;\subseteq\; \Gamma(Z_{1,\sigma}, \mathcal{O}).$$

Consider the quotient map $p_1 \colon \widehat{Z}_1 \to Z_1$. Then we have $p_1^{-1}(Z_{1,\sigma}) = \mathbb{K}_f^{r_1}$ and, since σ is regular, $\Gamma(p_1^{-1}(Z_{1,\sigma}), \mathcal{O})$ admits units in every K_1-degree. This implies

$$p_1^*(\mathfrak{m}_x) \;=\; \langle T_i;\, z_i = 0\rangle_f + I(P_1, z)_f,$$
$$p_1^*(I(\mathbb{T}^n \cdot z_\sigma)) \;=\; \langle T_i;\, e_i \in \widehat{\sigma}\rangle_f, \qquad p_1^*(I(X_{1,\sigma})) \;=\; I(\overline{X_1})_f.$$

Consequently, the assumption together with injectivity of the pullback map p_1^* give the assertion. ☐

Lemma 4.6.3. *In Algorithm 4.6.1, for each x_i the barycentric subdivision of σ_i' induces a blow up of X_1' in x_i'.*

Proof. Observe that since any point x_i can be mapped to a toric fixed point by an automorphism of $\mathbb{P}_n$, there are n hyperplanes in $\mathcal{L}$ with intersection $\{x_i\} \subseteq \mathbb{P}_n$ for all i. Furthermore, note that is suffices to consider $\mathcal{L}'$ instead of $\mathcal{L}$ since the T_i are already present in $\mathcal{R}(\mathbb{P}_n)$.

In Algorithm 4.6.1, write $G_1' = \{T_{n+1+j} - f_j;\, 1 \le j \le s\}$. Since the following left hand side is H_1'-invariant and contains z_i', we have

$$\overline{X}_1' \cap V\left(T_j;\, e_j \in \widehat{\sigma}_i'\right) \;\supseteq\; V\left(I(P_1', z_i') + \langle T_j;\; (z_i')_j = 0\rangle\right) \;=\; \overline{H_1' \cdot z_i'}. \tag{5}$$

Assume that (5) is an equality. Taking ideals, this then implies that $I(\overline{X}_1') + \langle T_j;\, e_j \in \widehat{\sigma}_i'\rangle$ is equal to $I(P_1', z_i')$ as the ideals are linear and thus radical; the claim then follows from Lemma 4.6.2 since σ_i' is smooth. The remainder of this proof is concerned with showing that (5) is an equality. It suffices to compare their dimensions. Write the coefficients of the (linear) generators of $\mathfrak{b}_i := I(\overline{X_1'}) + \langle T_j;\, e_j \in \widehat{\sigma}_i'\rangle$ into a matrix $A_{\mathfrak{b}_i}$. Then

$$V\big(\mathbb{K}^{r_1'};\, \mathfrak{b}_i\big) \;=\; \Big\{x \in \mathbb{K}^{r_1'};\; A_{\mathfrak{b}_i} \cdot x \;=\; 0\Big\}, \quad \dim\Big(V\big(\mathbb{K}^{r_1'};\, \mathfrak{b}_i\big)\Big) \;=\; r_1' - \operatorname{rank}(A_{\mathfrak{b}_i}).$$

Changing coordinates, we may assume that $x_i = [0, \ldots, 0, 1] \in \mathbb{P}_n$, i.e., it is cut out by the n coordinate hyperplanes $V(T_1), \ldots, V(T_n) \subseteq \mathbb{P}_n$. Starting with the rows corresponding to these $V(T_j) \subseteq \mathbb{P}_n$ followed by the rows corresponding to the elements of G_1', the matrix $A_{\mathfrak{b}_i}$ is of shape

$$A_{\mathfrak{b}_i} \;=\; \left[\begin{array}{c|c} E_n \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & 0 \\ \hline \bullet & \bullet \\ \hline \bullet & E_s \end{array}\right]$$

where 0 stands for a zero matrix and $\bullet$ for an arbitrary matrix of fitting size. Then $\mathrm{rank}(A_{\mathfrak{b}_i}) = n+s$ and $r_1' = n+1+s$ shows that $V\big(\mathbb{K}^{r_1'};\ \mathfrak{b}_i\big)$ is one-dimensional. Let $n_0 \in \mathbb{Z}_{\geq 0}$ be the number of vanishing coordinates of z_i'. Then $B = (e_1, \ldots, e_m, e_0)$, where $m := r_1' - n_0 - 1$ and $e_0 := (-1, \ldots, -1)$, is a lattice basis for $\mathrm{Im}((P_1')^*) \cap \mathrm{lin}(e_j;\ (z_i')_j \neq 0)$ as in Definition 4.5.7. Consequently, the lattice ideal $I(P_1', z_i')$ is linear and we conclude

$$\dim\left(V\left(I(P_1', z_i') + \langle T_j;\ (z_i')_j = 0\rangle\right)\right) \;=\; r_1' - (m + n_0) \;=\; 1. \qquad \square$$

Proof of Algorithm 4.6.1. By Lemma 4.6.3, the modification $X_2 \to X_1'$ is the blow up at the points $x_1', \ldots, x_k'$. It remains to show that the input ring R_2' of the last step is normal; this is necessary for Algorithm 4.3.3. We only treat the case $k = 1$. Consider the stretched ring R_1' obtained from the third step and the ring R_2 obtained after the sixth step

$$R_1' \;=\; \mathbb{K}[T_1, \ldots, T_{r_1'}]/\langle G_1'\rangle, \qquad R_2 \;=\; \mathbb{K}[T_1, \ldots, T_{r_1'}, T_{r_2}]/\langle G_2\rangle$$

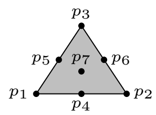
where T_{r_2} corresponds to the exceptional divisor. We assume that of the $r_1' - r_1$ new equations $T_i - f_i$ in G_1' the last l will result in fake relations in G_2. Localizing and passing to degree zero, we are in the situation

$$\begin{tikzcd} (R_1)_{T_1\cdots T_{r_1}} \arrow[r] & (R_1)_{T_1\cdots T_{r_1} f_1 \cdots f_{r_1'-l}} & (R_2)_{T_1\cdots T_{r_1'-l} T_{r_2}} \\ \big((R_1)_{T_1\cdots T_{r_1}}\big)_0 \arrow[r] & \Big((R_1)_{T_1\cdots T_{r_1} f_1\cdots f_{r_1'-l}}\Big)_0 \arrow[u] \arrow[r, equal] & \Big((R_2)_{T_1\cdots T_{r_1'-l} T_{r_2}}\Big)_0 \arrow[u] \end{tikzcd}$$

The upper left ring is K_1-factorial by assumption. By [15, Thm. 1.1] the middle ring in the lower row is a UFD and the ring on the upper right is K_2-factorial. Thus, R_2 is K_2-factorial. Since K_1 is free, also K_2 is, so R_2 is a UFD. In particular, R_2 is normal and we may apply Algorithm 4.3.3. $\square$

Example 4.6.4. Let X be the blow up of $\mathbb{P}_2$ in the seven points

$$x_1 := [1,0,0], \quad x_2 := [0,1,0], \quad x_3 := [0,0,1],$$
$$x_4 := [1,1,0], \quad x_5 := [1,0,-1], \quad x_6 := [0,1,1],$$
$$x_7 := [1,1,1].$$



Write S_i for the variables corresponding to x_i and let $T_1, \ldots, T_9$ correspond to the nine lines in $\mathcal{L}$. Algorithm 4.6.1 provides us with the Cox ring of X. It is given as the factor ring $\mathbb{K}[T_1, \ldots, T_9, S_1, \ldots, S_7]/I$ where I is generated by

$$\begin{array}{rr}
2T_8S_4S_6 - T_5S_2 + T_9S_7, & 2T_1S_3S_6 + T_5S_5 - T_6S_7, \\
2T_4S_1S_6 + T_6S_2 - T_9S_5, & -T_1S_2S_6 + T_2S_1S_5 - T_7S_4S_7, \\
2T_7S_3S_4 + T_6S_2 + T_9S_5, & -T_2S_5S_3 + T_3S_4S_2 - T_4S_7S_6, \\
2T_3S_1S_4 + T_5S_5 + T_6S_7, & T_1S_2S_3 + T_8S_4S_5 + T_4S_1S_7, \\
2T_2S_1S_3 + T_5S_2 + T_9S_7, & T_2T_6S_3 - T_3T_9S_4 - T_4T_5S_6, \\
T_3S_1S_2 + T_8S_5S_6 - T_7S_3S_7, & T_3T_9S_1 - T_5T_7S_3 - T_6T_8S_6, \\
T_2T_6S_1 - T_5T_7S_4 + T_1T_9S_6, & T_4T_5S_1 + T_1T_9S_3 + T_6T_8S_4, \\
T_3T_7S_4^2 + T_1T_4S_6^2 + T_2T_6S_5, & T_2T_7S_3^2 + T_4T_8S_6^2 + T_3T_9S_2, \\
T_1T_2S_3^2 + T_3T_8S_4^2 - T_4T_5S_7, & T_1T_3S_2^2 + T_2T_8S_5^2 + T_4T_7S_7^2, \\
T_3T_4S_1^2 + T_1T_7S_3^2 - T_6T_8S_5, & T_2T_4S_1^2 + T_7T_8S_4^2 - T_1T_9S_2,
\end{array}$$

$$T_2T_3S_1^2 + T_1T_8S_6^2 + T_5T_7S_7, \quad T_4T_5^2T_7 + T_2T_6^2T_8 + T_1T_3T_9^2$$

and the $\mathbb{Z}^8$-grading is given by the degree matrix

$$\left[\begin{array}{rrrrrrrrrrrrrrrr} 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Before eliminating fake relations, the ideal of the intersection of $\overline{X}_2$ with the ambient big torus $\mathbb{T}^{r_2}$ admits the following description in terms of finite geometries. In our setting, similar to [12], we call the pair $\mathfrak{L} := (\mathcal{P}, \mathcal{L})$ a *finite linear space* and the finite sets $\mathcal{P}$ and $\mathcal{L}$ *points* and *lines*, respectively. Note that each two points $p, p' \in \mathcal{P}$ lie on a common line $\ell \in \mathcal{L}$ and, for $n \geq 2$, there are three points not lying on a common line. An element $\ell \in \mathcal{L}$ is an *m-line* if it contains exactly m points. We call $\mathfrak{L}$ an *(m-)design* if each line is a m-line. Moreover, $\mathfrak{L}$ is a *near-pencil* if there are distinct lines $\ell_1, \ldots, \ell_{n-1} \in \mathcal{L}$ such that all points except one are contained in $\bigcap \ell_i$. An *m-arc* is a subset of $\mathcal{P}$ within which no three points lie on a common line.

Proposition 4.6.5. *At the end of the fifth step in Algorithm 4.6.1, the Cox ring of Z_2 is the polynomial ring $\mathbb{K}[T_\ell, S_p]$ with indices $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$. Consider the homomorphism*

$$\beta\colon \mathbb{K}[T_\ell;\, \ell \in \mathcal{L}] \ \to\ \mathbb{K}[T_\ell, S_p;\, \ell \in \mathcal{L},\, p \in \mathcal{P}], \qquad T_\ell \ \mapsto\ T_\ell \cdot \prod_{p \in \ell} S_p.$$

Then the extension of the ideal $I_2 \subseteq \mathbb{K}[T_\ell, S_p]$ to the Laurent polynomial ring $\mathbb{K}[T_\ell^{\pm}, S_p^{\pm}]$ is generated by $\beta(T_\ell - f_\ell)$ where $\ell \in \mathcal{L}'$. Moreover, properties of the linear space $\mathfrak{L} = (\mathcal{P}, \mathcal{L})$ lead to properties of I_2 as listed in the following table. The $\dagger$-cases require $n = 2$.

$\mathfrak{L}$	*property of R_2 or X_2*
design	*I_2 is classically homogeneous*
near pencil	*X_2 admits a non-trivial $\mathbb{K}^*$-action*
contains m-arc $\dagger$	*X_2 contains at least $\binom{m}{2} + k$ many (-1)-curves*
complete graph $\dagger$	*X_2 is smooth*
contains an m-line $\dagger$	*X_2 contains a $(1-m)$-curve*

Proof. Denote the elements of $\mathcal{L}'$ by ℓ_i. The first statement is directly seen by providing a weak B-lifting $[E_{r_1'}, A]$ in the sense of [10], i.e., we have a matrix A fitting into

$$\begin{tikzcd} \mathbb{Z}^{r_2} \arrow[r, "{[E_{r_1'}, A]}"] \arrow[d, "{P_2 = [P_1, B]}"'] & \mathbb{Z}^{r_1'} \arrow[d, "P_1"] \\ \mathbb{Z}^n \arrow[r, "\mathrm{id}"] & \mathbb{Z}^n \end{tikzcd} \qquad A \ =\ (a_{ij})_{ij}, \qquad a_{ij} \ :=\ \begin{cases} 1, & x_j \in \ell_i, \\ 0, & \text{else} \end{cases}$$

and the corresponding morphism $\alpha\colon \mathbb{T}^{r_2} \to \mathbb{T}^{r_1'}$ satisfies $\alpha^*(g) = p_2^*(p_1)_*(g)$ in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]$ for Laurent polynomials $g \in \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1'}^{\pm 1}]$.

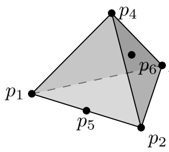
We come to the claims listed in the table. For the first one, note that all $\beta(T_\ell - f_\ell)$ are homogeneous and I_2, as a saturation, is obtained by a Gröbner basis computation. Gröbner bases of homogeneous ideals are again homogeneous, see for example [1, Ex. 1.8.3]. The second statement is due to the fact that, up to monomial factors, the ideal generated by the $\beta(T_\ell - f_\ell)$ is already of the shape of Construction 1.5.3. Since it is prime, it equals its saturation and therefore I_2; hence, X_2 admits a non-trivial $\mathbb{K}^*$-action by Theorem 1.5.13.

For the remaining three claims, as each $\ell \in \mathcal{L}$ has self-intersection number $\ell^2 = 1$, blowing up m different points on ℓ, the proper transform $\hat{\ell}$ has self-intersection number $\hat{\ell}^2 = 1 - m$, see Proposition 1.4.8. $\square$

We now treat blow ups of $\mathbb{P}_3$ in six distinct points $x_1, \ldots, x_6$. As before, we assume that $x_1, \ldots, x_4$ are the standard toric fixed points. We call the point configuration *edge-special* if at least one point of $\{x_5, x_6\}$ is contained in two different hyperplanes spanned by the other points.

Theorem 4.6.6. *Let X be the blow up of $\mathbb{P}_3$ at distinct points $x_1, \ldots, x_6$ not contained in a hyperplane. Then X is a Mori dream space. Moreover, for the following typical edge-special configurations, we obtain:*

(i) *For $x_5 := [1, 1, 0, 0]$, $x_6 := [0, 1, 1, 1]$, the Cox ring of X is $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_{16}]/I$ where I is generated by*

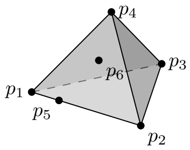


$$
\begin{array}{ll}
2T_4T_{13} - 2T_5T_{16} - 2T_3T_{14}, & T_4T_{12}T_{15} - T_2T_{14} - T_6T_{16}, \\
T_5T_{12}T_{15} - T_6T_{13} + T_7T_{14}, & T_3T_{12}T_{15} - T_2T_{13} - T_7T_{16}, \\
T_5T_{11}T_{12} - T_9T_{13} + T_{10}T_{14}, & T_4T_{11}T_{12} - T_8T_{14} - T_9T_{16}, \\
T_3T_{11}T_{12} - T_8T_{13} - T_{10}T_{16}, & T_1T_{12}T_{13} + T_7T_{11} - T_{10}T_{15}, \\
T_1T_{12}T_{14} + T_6T_{11} - T_9T_{15}, & T_1T_{12}T_{16} - T_2T_{11} + T_8T_{15}, \\
T_5T_8 - T_3T_9 + T_4T_{10}, & T_2T_5 - T_3T_6 + T_4T_7, \\
T_1T_5T_{12}^2 + T_7T_9 - T_6T_{10}, & T_1T_3T_{12}^2 + T_7T_8 - T_2T_{10}, \\
T_1T_4T_{12}^2 + T_6T_8 - T_2T_9 &
\end{array}
$$

with the $\mathbb{Z}^7$-grading given by the degree matrix

$$
\left[\begin{array}{rrrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].
$$

(ii) *For $x_5 := [2, 1, 0, 0]$, $x_6 := [1, 1, 0, 1]$, the Cox ring of X is $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_{15}]/I$ where I is generated by*

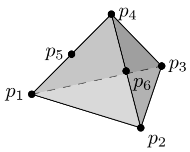


$$
\begin{array}{ll}
T_1T_{11} + T_7T_{14} + 2T_8T_{15}, & T_2T_{10} + T_7T_{14} + T_8T_{15}, \\
T_4T_{11}T_{14} - T_2T_{13} - T_5T_{15}, & T_4T_{10}T_{14} - T_1T_{13} - T_6T_{15}, \\
T_4T_{10}T_{11} + T_7T_{13} - T_9T_{15}, & T_6T_{11} - 2T_8T_{13} - T_9T_{14}, \\
T_5T_{10} - T_8T_{13} - T_9T_{14}, & 2T_4T_8T_{10} + T_6T_7 + T_1T_9, \\
T_4T_8T_{11} + T_5T_7 + T_2T_9, & T_4T_8T_{14} + T_1T_5 - T_2T_6
\end{array}
$$

with the $\mathbb{Z}^7$-grading given by the degree matrix

$$
\left[\begin{array}{rrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].
$$

(iii) *For $x_5 := [1, 0, 0, 1]$, $x_6 := [0, 1, 0, 1]$, the Cox ring of X is $\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_{13}]/I$ where I is generated by*

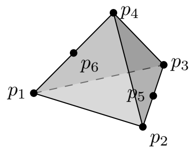


$$
\begin{array}{ll}
T_2T_8T_{11} - T_6T_9 + T_7T_{13}, & T_2T_{11}T_{12} - T_4T_9 + T_5T_{13}, \\
T_1T_9T_{11} - T_5T_8 + T_7T_{12}, & T_1T_{11}T_{13} - T_4T_8 + T_6T_{12}, \\
T_1T_2T_{11}^2 - T_5T_6 + T_4T_7 &
\end{array}
$$

with the $\mathbb{Z}^7$-grading given by the degree matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iv) *For $x_5 := [1,0,0,1]$, $x_6 := [0,1,1,0]$, the Cox ring of X is $\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_{12}]/I$ where I is generated by*

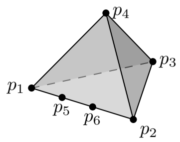


$$T_3T_8 - T_5T_{12} - T_2T_9,$$
$$T_4T_7 - T_6T_{11} - T_1T_{10}$$

with the $\mathbb{Z}^7$-grading given by the degree matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(v) *For $x_5 := [2,1,0,0]$, $x_6 := [1,2,0,0]$, the Cox ring of X is $\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_{12}]/I$ where I is generated by*



$$3T_2T_7 + 2T_5T_{11} + T_6T_{12},$$
$$3T_1T_8 + T_5T_{11} + 2T_6T_{12}$$

with the $\mathbb{Z}^7$-grading given by the degree matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. See [57, Thm. 7.5] for the proof of the first statement. The second part of the theorem is an application of Algorithm 4.6.1. ☐

Remark 4.6.7. Algorithm 4.6.1 usually is faster than the lattice ideal approach of Algorithm 4.5.9. For instance, some cases of Theorem 4.6.6 were only feasible using Algorithm 4.6.1.

CHAPTER 5

Smooth rational surfaces

In this chapter, we prove that each smooth rational surface of Picard number at most six is a Mori dream space. In terms of Cox rings, we present a complete classification for the case of Picard number at most five and a classification for the surfaces that do not admit a non-trivial $\mathbb{K}^*$-action for Picard number six. All Cox rings are listed explicitly in terms of generators and relations.

Using the fact that each smooth rational surface can be obtained as a blow up of the projective plane $\mathbb{P}_2$ or the Hirzebruch surface $\mathbb{F}_a$, we proceed by the following steps. In Section 1, we classify the needed point configurations on $\mathbb{P}_2$ and $\mathbb{F}_a$. Afterwards, we use Algorithm 4.5.9 to determine the Cox rings of the (possibly iterated) blow ups of these configurations. For blow ups of $\mathbb{F}_a$, we apply Algorithm 4.3.8 in a formal way.

In Sections 2 and 3, we classify the Cox rings of all families of smooth rational surfaces X of Picard number $\varrho(X) \leq 5$. Eliminating isomorphic surfaces, it turns out that besides $\overline{M}_{0,5}$ there are only surfaces with a non-trivial $\mathbb{K}^*$-action. In Section 4, we obtain the smooth rational surfaces of Picard number six as blow ups of the surfaces from the previous step. Here, we classify the Cox rings of the surfaces without a non-trivial $\mathbb{K}^*$-action. The result of Section 4 (and the proof of one of the cases) has been published in the paper *Computing Cox rings* together with J. Hausen and A. Laface [57, Sec. 6].

1. Point configurations on $\mathbb{P}_2$ and $\mathbb{F}_a$

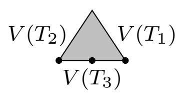
In this section we classify the point configurations on the projective plane and the Hirzebruch surface which we need to blow up in order to obtain the smooth rational surfaces of Picard number at most six in Sections 2, 3 and 4.

Recall from [5, Rem. III.2.5.5] that we can identify points on a Mori dream space by their *Cox coordinates*. This generalizes homogeneous coordinates on $\mathbb{P}_n$.

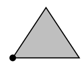
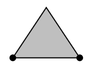
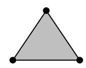
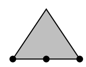
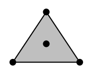
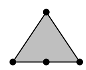
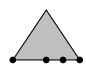
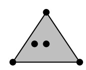
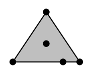
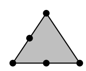
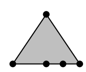
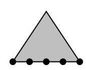
Notation 5.1.1. Let X be a $\mathbb{Q}$-factorial Mori dream space with characteristic space $p\colon \widehat{X} \to X$ and characteristic quasitorus H. For any $z \in \widehat{X} \subseteq \mathbb{K}^r$ we write $[z] := p(z) \in X$. Note that $[z] = [z']$ if and only if $z' \in H \cdot z$. Furthermore, given an ideal $I \subseteq \mathbb{K}[T_1, \dots, T_r]$ that is generated by $\mathrm{Cl}(X)$-homogeneous polynomials $f_1, \dots, f_n$, we write

$$V(X; f_1, \dots, f_n) \ := \ V(X; I) \ := \ p\left(V(\widehat{X}; I)\right) \ \subseteq \ X.$$

To symbolize point configurations, we draw $\mathbb{P}_2$ as the big torus (gray) together with its boundary divisors $V(\mathbb{P}_2; T_i)$ (the black bordering lines). Points are identified by their position on the torus or boundary divisors. For instance, the following picture symbolizes $\mathbb{P}_2$ with the points $[1,0,0]$, $[1,1,0]$ and $[0,1,0]$.



Proposition 5.1.2. *Each configuration of at most five distinct points on $\mathbb{P}_2$ can be moved by an automorphism of $\mathbb{P}_2$ to one of the following configurations. Occurring parameters are distinct elements of $\mathbb{K}^* \setminus \{1\}$.*

configuration	*points*	
$\mathbb{P}_2(\star)$	$\{[1,0,0]\}$	
$\mathbb{P}_2(\star\star)$	$\{[1,0,0],[0,1,0]\}$	
$\mathbb{P}_2(\star\star\star\ i)$	$\{[1,0,0],[0,1,0],[0,0,1]\}$	
$\mathbb{P}_2(\star\star\star\ ii)$	$\{[1,0,0],[0,1,0],[1,1,0]\}$	
$\mathbb{P}_2(\star\star\star\star\ i)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1]\}$	
$\mathbb{P}_2(\star\star\star\star\ ii)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,0]\}$	
$\mathbb{P}_2(\star\star\star\star\ iii)$	$\{[1,0,0],[0,1,0],[1,1,0],[1,\lambda,0]\}$	
$\mathbb{P}_2(\star\star\star\star\star\ i)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\lambda,\mu]\}$	
$\mathbb{P}_2(\star\star\star\star\star\ ii)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,\lambda,0]\}$	
$\mathbb{P}_2(\star\star\star\star\star\ iii)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1]\}$	
$\mathbb{P}_2(\star\star\star\star\star\ iv)$	$\{[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,\lambda,0]\}$	
$\mathbb{P}_2(\star\star\star\star\star\ v)$	$\{[1,0,0],[0,1,0],[1,1,0],[1,\lambda,0],[1,\mu,0]\}$	

Remark 5.1.3. Given triples (p_1,p_2,p_3) and (q_1,q_2,q_3) of non-collinear points in $\mathbb{P}_2$, there is exactly one projective linear transformation mapping the p_i to the q_i.

Proof of Proposition 5.1.2. Exemplarily, we treat the case of five distinct points $p_1,\ldots,p_5 \in \mathbb{P}_2$. *Case 1:* There is no line in $\mathbb{P}_2$ containing three of the points. By Remark 5.1.3, we can move p_1, p_2 and p_3 to the standard toric fixed points $[1,0,0]$,

$[0,1,0]$ and $[0,0,1] \in \mathbb{P}_2$. Applying a suitable torus element, we additionally achieve that p_4 is as claimed in $\mathbb{P}_2(\star\star\star\star\star\, i)$.

Case 2: There is a line in $\mathbb{P}_2$ containing exactly three of the points. *Subcase A:* No other line in $\mathbb{P}_2$ contains three points. We may assume that the line contains p_1, p_2 and p_4. After a projective transformation, we have

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1], \quad p_4 = [1,\lambda,0], \quad p_5 = [1,\mu,\nu].$$

Scaling p_5 with a torus element, we arrive at the configuration $\mathbb{P}_2(\star\star\star\star\star\, ii)$. *Subcase B:* There is an additional line in $\mathbb{P}_2$ containing three of the points. We may assume that p_1, p_2 and p_4 are elements of one line and p_1, p_3 and p_5 lie on the other line. After a projective transformation, we have

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1], \quad p_4 = [1,\lambda,0], \quad p_5 = [1,0,\mu].$$

Scaling p_4 and p_5 with a torus element leads to the configuration $\mathbb{P}_2(\star\star\star\star\star\, iii)$.

Case 4: There is a line in $\mathbb{P}_2$ containing exactly four of the points. We may assume that the line contains all points except p_3. After a projective transformation, we have

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1], \quad p_4 = [1,\lambda,0], \quad p_5 = [1,\mu,0].$$

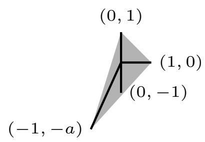
Scaling p_5 with a torus element, we arrive at the configuration $\mathbb{P}_2(\star\star\star\star\star\, iv)$.

Case 5: There is a line in $\mathbb{P}_2$ containing all points. After a projective transformation, we have

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [1,\lambda,0], \quad p_4 = [1,\mu,0], \quad p_5 = [1,\nu,0].$$

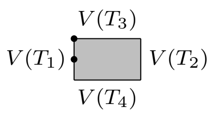
Scaling p_5 with a torus element, we arrive at the configuration $\mathbb{P}_2(\star\star\star\star\star\, v)$. $\square$

Recall that, given $a \in \mathbb{Z}_{\geq 0}$, the *a-th Hirzebruch surface* is the complete toric surface $\mathbb{F}_a$ corresponding to the following complete fan with its rays generated by the columns of P



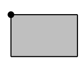
$$P := \begin{bmatrix} -1 & 1 & 0 & 0 \\ -a & 0 & 1 & -1 \end{bmatrix}, \qquad Q := \begin{bmatrix} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

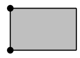
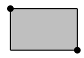
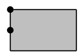
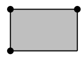
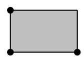
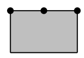
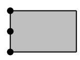
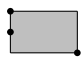
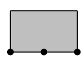
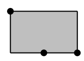
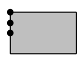
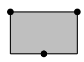
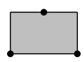
Then $\mathcal{R}(\mathbb{F}_a) = \mathbb{K}[T_1, \ldots, T_4]$ is graded by $\mathrm{Cl}(\mathbb{F}_a) = \mathbb{Z}^2$ via $\deg(T_i) = Q(e_i)$. Similar to $\mathbb{P}_2$, we draw $\mathbb{F}_a$ as the big torus (gray) together with its boundary divisors $V(\mathbb{F}_a; T_i)$ (the black bordering lines). Points are given in Cox coordinates as in 5.1.1 and are drawn according to their position on the torus or boundary divisors. For instance, the points $[0,1,0,1] \in \mathbb{F}_a$ and $[0,1,1,1] \in \mathbb{F}_a$ are drawn as follows.

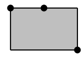
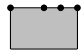
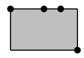
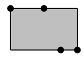
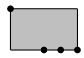
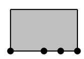
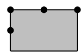
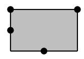
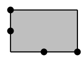
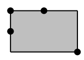
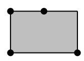
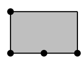
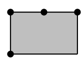
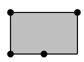
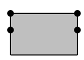
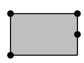


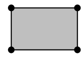
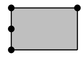
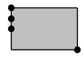
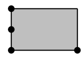
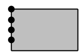
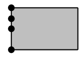
Note that the self-intersection numbers of the $V(\mathbb{F}_a; T_i)$ are a for $i = 3$, $-a$ for $i = 4$ and zero for $i \in \{1,2\}$, see [28, Thm. 10.4.4].

Proposition 5.1.4. *Let $a \geq 2$. Each configuration of at most $\min(a+1, 4)$ distinct points on $\mathbb{F}_a$ can be moved by an automorphism of $\mathbb{F}_a$ to one of the following configurations. Occurring parameters are distinct elements of $\mathbb{K}^* \setminus \{1\}$.*

configuration	*points*	
$\mathbb{F}_a(\star\, i)$	$\{[0,1,0,1]\}$	

$\mathbb{F}_a(\star\ ii)$	$\{[1,0,0,1]\}$	
$\mathbb{F}_a(\star\star\ i)$	$\{[0,1,0,1],[1,0,0,1]\}$	
$\mathbb{F}_a(\star\star\ ii)$	$\{[0,1,0,1],[0,1,1,0]\}$	
$\mathbb{F}_a(\star\star\ iii)$	$\{[0,1,0,1],[1,0,1,0]\}$	
$\mathbb{F}_a(\star\star\ iv)$	$\{[0,1,1,0],[1,0,1,0]\}$	
$\mathbb{F}_a(\star\star\ v)$	$\{[0,1,0,1],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\ i)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\ ii)$	$\{[0,1,0,1],[0,1,1,0],[1,0,1,0]\}$	
$\mathbb{F}_a(\star\star\star\ iii)$	$\{[0,1,0,1],[1,0,0,1],[1,1,0,1]\}$	
$\mathbb{F}_a(\star\star\star\ iv)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\ v)$	$\{[0,1,0,1],[0,1,1,0],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\ vi)$	$\{[0,1,0,1],[1,0,1,0],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\ vii)$	$\{[0,1,1,0],[1,0,1,0],[1,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\ viii)$	$\{[0,1,0,1],[1,0,1,0],[1,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\ ix)$	$\{[0,1,0,1],[0,1,1,1],[0,1,1,\lambda]\}$	
$\mathbb{F}_a(\star\star\star\ x)$	$\{[0,1,0,1],[1,0,0,1],[1,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\ xi)$	$\{[0,1,1,0],[1,0,1,0],[1,1,0,1]\}$	

$\mathbb{F}_a(\star\star\star\ xii)$	$\{[0,1,0,1],[1,0,1,0],[1,1,0,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ i)$	$\{[0,1,0,1],[1,0,0,1],[1,1,0,1],[1,\lambda,0,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ ii)$	$\{[0,1,0,1],[1,0,1,0],[1,1,0,1],[1,\lambda,0,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ iii)$	$\{[0,1,0,1],[1,0,1,0],[1,1,0,1],[1,\lambda,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ iv)$	$\{[0,1,0,1],[1,0,1,0],[1,1,1,0],[\lambda,\lambda,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ v)$	$\{[0,1,1,0],[1,0,1,0],[1,1,1,0],[1,\lambda,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ vi)$	$\{[0,1,0,1],[1,0,0,1],[1,1,0,1],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ vii)$	$\{[0,1,0,1],[1,0,0,1],[1,1,1,0],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ viii)$	$\{[0,1,0,1],[1,0,1,0],[1,1,1,0],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ ix)$	$\{[0,1,0,1],[1,0,1,0],[1,1,0,1],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ x)$	$\{[0,1,0,1],[0,1,1,0],[1,0,1,0],[1,1,0,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ xi)$	$\{[0,1,0,1],[1,0,1,0],[0,1,1,0],[1,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ xii)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,0],[1,1,0,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ xiii)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,0],[1,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ \ xiv)$ $\kappa \in \mathbb{K}^*$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,1],[1,0,1,\kappa]\}$	
$\mathbb{F}_a(\star\star\star\star\ xv)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,0],[1,0,1,1]\}$	

$\mathbb{F}_a(\star\star\star\star\ xvi)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,0],[1,0,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ xvii)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,1],[0,1,1,\lambda]\}$	
$\mathbb{F}_a(\star\star\star\star\ xviii)$	$\{[0,1,0,1],[1,0,0,1],[0,1,1,1],[0,1,1,0]\}$	
$\mathbb{F}_a(\star\star\star\star\ xix)$	$\{[0,1,0,1],[1,0,1,0],[0,1,1,1],[0,1,1,\lambda]\}$	
$\mathbb{F}_a(\star\star\star\star\ xx)$	$\{[0,1,0,1],[0,1,1,0],[1,0,1,0],[0,1,1,1]\}$	
$\mathbb{F}_a(\star\star\star\star\ xxi)$	$\{[0,1,0,1],[0,1,1,1],[0,1,1,\lambda],[0,1,1,\mu]\}$	
$\mathbb{F}_a(\star\star\star\star\ xxii)$	$\{[0,1,0,1],[0,1,1,0],[0,1,1,1],[0,1,1,\lambda]\}$	

Lemma 5.1.5. *Let X_1, X_2 be Mori dream surfaces. Then the following statements are equivalent.*

(i) *X_1 and X_2 are isomorphic.*
(ii) *$\mathcal{R}(X_1)$ and $\mathcal{R}(X_2)$ are isomorphic as $\mathrm{Cl}(X_i)$-graded algebras.*
(iii) *The affine H_{X_i}-varieties $\overline{X}_1$ and $\overline{X}_2$ are isomorphic.*

In particular, each H_i-equivariant automorphism of $\overline{X}_i$ is an H_i-equivariant automorphism of $\widehat{X}_i$.

Proof. The equivalence between the last two statements is Proposition 1.1.3 or [5, Thm. I.2.2.4]. We only need to show that (iii) implies (i). Let $(\varphi,\widetilde{\varphi})$ be an isomorphism between the affine H_i-varieties $\overline{X}_1$ and $\overline{X}_2$. Then both $\widehat{X}_2$ and $\varphi(\widehat{X}_1)$ are open, H_2-invariant subsets of $\overline{X}_2$ such that the complement in $\overline{X}_2$ is of codimension at least two, the good quotient by H_2 exists and is projective. Since X_2 is a surface, we obtain $\varphi(\widehat{X}_1)=\widehat{X}_2$. $\square$

Lemma 5.1.6. *Let H be the characteristic torus of $\mathbb{F}_a$.*

(i) *Given $A\in\mathrm{GL}(2,\mathbb{K})$, we have an H-equivariant automorphism of $\widehat{\mathbb{F}}_a\subseteq\mathbb{K}^4$ given by*

$$\varphi_A\colon \widehat{\mathbb{F}}_a\to\widehat{\mathbb{F}}_a,\quad (z_1,\dots,z_4)\mapsto(y_1,y_2,z_3,z_4),\quad y:=A\cdot(z_1,z_2).$$

(ii) *For each $t:=(t_a,t_{a-1},\dots,t_0)\in\mathbb{K}^{a+1}$, we have an H-equivariant automorphism*

$$\varphi_t\colon \widehat{\mathbb{F}}_a\to\widehat{\mathbb{F}}_a,\qquad z\mapsto\left(z_1,\,z_2,\,z_3+z_4\cdot\sum_{k=0}^{a}t_kz_1^kz_2^{a-k},\,z_4\right).$$

Proof. By Lemma 5.1.5, it suffices to show that φ_A and φ_t are H-equivariant automorphisms of $\overline{\mathbb{F}}_a$. Both φ_A and φ_t are H-equivariant since φ_A^* and φ_t^* are $\mathrm{Cl}(\mathbb{F}_a)$-graded; compare [6, Cor. 2.3]. The inverse map for (i) is given by $w\mapsto$

(x_1, x_2, w_3, w_4) with $x := A^{-1}(w_1, w_2)$. In (ii), given a vector $w \in \widehat{\mathbb{F}}_a$, the inverse map assigns $w_3 - (t_a w_1^a + \ldots + t_0 w_2^a)$ to w_3 and is the identity on the other entries. $\square$

Proof of Proposition 5.1.4. We treat exemplarily the case of four points and $a \geq 3$. Let $p_1, \ldots, p_4 \in \widehat{\mathbb{F}}_a$ be Cox coordinates for the four points on $\mathbb{F}_a$. Consider the 4×4 matrix $B := [p_1, \ldots, p_4]$ where we may assume that $\mathrm{rank}([p_1, p_2]) = \mathrm{rank}([p_i, p_j])$ for all i, j. In the case of $\mathrm{rank}([p_1, p_2]) = 2$ and of $\mathrm{rank}([p_1, p_2]) = 1$, respectively, choose $A \in \mathrm{GL}(2, \mathbb{K})$ such that

$$\left[\begin{array}{c|c} A & \\ \hline & E_2 \end{array}\right] \cdot B \;=\; \begin{bmatrix} 0 & 1 & \bullet & \bullet \\ 1 & 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & \bullet & \bullet \\ 1 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix},$$

where E_2 is the 2×2 unit matrix and $\bullet$ stands for an element of $\mathbb{K}$. By Lemma 5.1.6, the map corresponding to the above matrix multiplication is an automorphism. Write $q_1, \ldots, q_4$ for the columns of the resulting matrix. It suffices to treat the case $a = 3$ and $\mathrm{rank}([q_1, q_2]) = 2$, i.e., we have $q_{11} = q_{22} = 0$ and $q_{12} = q_{21} = 1$. We now look for $t = (t_3, \ldots, t_0) \in \mathbb{K}^4$ such that the automorphism φ_t defined in Lemma 5.1.6(ii) moves as many q_i to $V(\mathbb{F}_a; T_3)$ as possible. Given $z \in \mathbb{K}^4$, we have

$$\varphi_t(z) \,\in\, V\left(\widehat{\mathbb{F}}_a; T_3\right) \qquad \Leftrightarrow \qquad -z_3 \;=\; t_3 z_1^3 z_4 + t_2 z_1^2 z_2 z_4 + t_1 z_1 z_2^2 z_4 + t_0 z_2^3 z_4$$

by definition of φ_t. Thus, in the last equation, substituting q_i for z we look for solutions of the linear system of equations

$$A \cdot t \;=\; b, \qquad (A, b) \;:=\; \left[\begin{array}{cccc|c} 0 & 0 & 0 & q_{14} & -q_{13} \\ q_{24} & 0 & 0 & 0 & -q_{23} \\ q_{34} q_{31}^3 & q_{34} q_{31}^2 q_{32} & q_{34} q_{31} q_{32}^2 & q_{34} q_{32}^3 & -q_{33} \\ q_{44} q_{41}^3 & q_{44} q_{41}^2 q_{42} & q_{44} q_{41} q_{42}^2 & q_{44} q_{42}^3 & -q_{43} \end{array}\right].$$

We say that two points q_i and q_j *lie in a fiber* if $q_{i1} q_{j2} = q_{i2} q_{j1}$. If all $q_{i4} \neq 0$, replacing q_{i3} by $q_{i3} q_{i4}^{-1}$, we may assume that $q_{i4} = 1$ for all i. Then the rank of A equals the number of different fibers the points lie in:

$$\mathrm{rank}(A) = 2 + \mathrm{rank}\left(\begin{bmatrix} q_{31}^2 q_{32} & q_{31} q_{32}^2 \\ q_{41}^2 q_{42} & q_{41} q_{42}^2 \end{bmatrix}\right) = \begin{cases} 4, & \text{exactly 4 fibers with one point,} \\ 3, & \text{exactly 1 fiber with two points,} \\ 2, & \text{exactly 2 fibers with two points.} \end{cases}$$

In particular, the system $At = b$ is solvable if there are four different fibers containing exactly one point and we can move $\mathrm{rank}(A)$ many points to $V(\mathbb{F}_a; T_3)$ by an automorphism φ_t. To achieve the shown coordinates, one can move general points in $V(\mathbb{F}_a; T_3)$ and $V(\mathbb{F}_a; T_4)$ simultaneously or in all fibers simultaneously either by the automorphism φ_A of Lemma 5.1.6 or using a $\mathbb{K}^*$-action. We treat exemplarily configuration $\mathbb{F}_a(\star\star\star\star\ xiv)$. Here, we are in the case $\mathrm{rank}(A) = 2$, i.e., after the aforementioned transformation, the points are

$$q_1 \;=\; [0, 1, 0, 1], \qquad q_2 \;=\; [1, 0, 0, 1], \qquad q_3 \;=\; [0, b_2, b_3, b_4], \qquad q_4 \;=\; [c_1, 0, c_2, c_3],$$

with $b_i, c_j \in \mathbb{K}^*$. Note that there are $\mathbb{K}^*$-actions

$$\begin{aligned} \mathbb{K}^* \times \widehat{\mathbb{F}}_a \;&\to\; \widehat{\mathbb{F}}_a, \qquad (t, z) \;\mapsto\; (tz_1, tz_2, z_3, t^{-a} z_4), \\ \mathbb{K}^* \times \widehat{\mathbb{F}}_a \;&\to\; \widehat{\mathbb{F}}_a, \qquad (t, z) \;\mapsto\; (z_1, z_2, tz_3, tz_4). \end{aligned}$$

In particular, we see that $q_1 = [0, k_2, 0, k_4]$ and $q_2 = [l_1, 0, 0, l_4]$ for any $k_i, l_j \in \mathbb{K}^*$. Thus, scaling the last components of the listed Cox coordinates for q_3 and the first component of the Cox coordinates for q_4, we obtain the points $\{q_1, q_2, q_3', q_4'\}$ as shown in the table where

$$q_3' \;:=\; [0, 1, 1, 1], \qquad q_4' \;:=\; \left[1, 0, \frac{c_3}{b_3}, \frac{c_4}{b_4}\right] \;=\; [1, 0, 1, \kappa], \qquad \kappa \;:=\; \frac{b_3 c_4}{b_4 c_3} \;\in\; \mathbb{K}^*. \square$$

2. Smooth rational surfaces with $\varrho(X) \leq 4$

Building on the classification of point configurations in Section 1, we classify the smooth rational surfaces of Picard number at most four in terms of their Cox rings. The idea is a stepwise application of Algorithm 4.3.8 in the following formal way.

Remark 5.2.1 (Formal blow up)**.** Consider a smooth, two-dimensional CEMDS $X_1 = (G_1, P_1, \Sigma_1)$ with free class group K_1. Let X_2 be the blow up of X_1 in a point $x \in X_1$ with Cox coordinates $z \in \mathbb{K}^{r_1}$. We guess and verify a candidate for $\mathcal{R}(X_2)$ by the following steps.

- Choose prime elements $f_1, \ldots, f_l \in I(P_1, z) + \langle T_j;\, z_j = 0\rangle$ in R_1.
- Compute the stretched CEMDS X_1' by the steps of Algorithm 4.3.2 with input $f_1, \ldots, f_l$.
- Formally apply Algorithm 4.3.8 with option `verify`:
  - compute the ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ and show that it is saturated with respect to T_{r_2},
  - show that $\langle T_{r_2}\rangle + I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ is a prime ideal,
  - show that the codimension of $\overline{X}_2 \cap V(T_{r_2}, T_i)$ is at least two for all $i < r_2$ and that T_i is not associated to T_j for $i \neq j$.
- Prove that the performed modification was a blow up.
- Remove redundant generators with Algorithm 4.3.3 if necessary.

For details and correctness of the steps we refer to the proofs of the respective algorithms in Chapter 4, Theorem 4.2.6 and Proposition 4.1.4. In the following, we will only cite Theorem 4.2.6 when referring to the correctness of this procedure.

Notation 5.2.2. Denote by the prefix Bl the blow up of a point configuration. For instance, Bl $\mathbb{P}_2(\star\star\star\, ii)$ is the blow up of $\mathbb{P}_2$ in the configuration $\mathbb{P}_2(\star\star\star\, ii)$ defined in Proposition 5.1.2. Iterated blow ups are indicated by exponents and consecutive numbers; for example Bl $\mathbb{P}_2(\star^3\, ii)$ stands for (the second occurrence of) a blow up of $\mathbb{P}_2$ in three infinitely near points.

Remark 5.2.3. Each smooth rational surface X can be obtained as a blow up of $\mathbb{P}_2$ or as a blow up of the Hirzebruch surface $\mathbb{F}_a$ where $a \in \mathbb{Z}_{\geq 0}$; see [14, Thm. V.10]. Note that $\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{F}_1 = \mathrm{Bl}\, \mathbb{P}_2(\star)$. Moreover, each blow up of $\mathbb{P}_1 \times \mathbb{P}_1$ is isomorphic to a blow up of $\mathbb{F}_1$. In particular, it is a blow up of $\mathbb{P}_2$ in two points.

Proposition 5.2.4. *Let X be a smooth rational surface with Picard number $\varrho(X) = 2$. Then X is isomorphic to exactly one of the following.*

X	Cox ring $\mathcal{R}(X)$	degree matrix
Bl $\mathbb{P}_2(\star)$	$\mathbb{K}[T_1, \ldots, T_4]$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$
$\mathbb{F}_a$, $a \neq 1$	$\mathbb{K}[T_1, \ldots, T_4]$	$\begin{bmatrix} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Proof. By Remark 5.2.3, the surface X can either be obtained as a blow up of $\mathbb{P}_2$ in one point or we have $X = \mathbb{F}_a$ with $a \in \mathbb{Z}_{\geq 0}$. By Proposition 5.1.2, $\mathbb{P}_2(\star)$ is the only configuration we need to consider. Clearly, $\mathbb{F}_1 = \mathrm{Bl}\, \mathbb{P}_2(\star)$ and $\mathbb{F}_a$ is not isomorphic to the toric variety Bl $\mathbb{P}_2(\star)$ for $a \neq 1$. $\square$

Proposition 5.2.5. *Let X be a smooth rational surface with Picard number $\varrho(X) = 3$. Then X is isomorphic to exactly one of the following.*

X	*Cox ring* $\mathcal{R}(X)$	*degree matrix*
$\mathrm{Bl}\,\mathbb{P}_2(\star^2)$	$\mathbb{K}[T_1,\dots,T_5]$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star\star)$	$\mathbb{K}[T_1,\dots,T_5]$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star\ i)$ $a \ge 3$	$\mathbb{K}[T_1,\dots,T_5]$	$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -a+1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

The following lemma helps us to identify Gale dual matrices also for the case of formal parameters.

Lemma 5.2.6. *Let P be an integral $n \times r$ matrix with* $\operatorname{rank} P = n$ *and $r > n$. Assume $\mathbb{Z}^r/\mathrm{Im}(P^*)$ is free. Then an integral $(r-n) \times r$ matrix Q is a Gale dual matrix of P if all of the following conditions hold.*

(i) *We have* $\operatorname{rank}(Q) = r - n$.
(ii) *Each row of Q is an element of* $\ker(P)$.
(iii) *For each $1 \le k \le n$, the $k \times k$-minors of Q are coprime.*

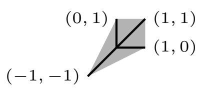
Proof. By (ii), the lattice $L \le \mathbb{Z}^r$ spanned by the rows of Q is contained in $\ker(P)$. Due to (i), the saturated lattice satisfies $L^{\mathrm{sat}} = \ker(P)$. The third condition means that the elementary divisors of L are all equal to one, i.e., $L^{\mathrm{sat}} = L$. □

Remark 5.2.7. Let X be a smooth rational surface. By [14], each negative curve C on X is an exceptional divisor and therefore smooth. In particular, the proper transform of C under the blow up $X' \to X$ of X in a point $x \in C$ is a (C^2-1)-curve, see Proposition 1.4.8.

Proof of Proposition 5.2.5. By Remark 5.2.3, each smooth rational surface X with Picard number $\varrho(X) = 3$ can be obtained as a blow up of $\mathbb{P}_2$ in two points or as the blow up of $\mathbb{F}_a$ in one point where $a \in \mathbb{Z}_{\ge 2}$. The configurations we need to consider are listed in Propositions 5.1.2 and 5.1.4.

We now compute the Cox rings of the listed surfaces. The variety $\mathrm{Bl}\,\mathbb{P}_2(\star^2)$ is a blow up of $Z' := \mathrm{Bl}\,\mathbb{P}_2(\star)$ in a point in the exceptional divisor. As a toric variety, Z' is given by a fan Σ' with its rays generated by the columns of

$$P := \begin{bmatrix} -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix},$$



The exceptional divisor of Z' is $V(Z'; T_4)$ which consists of the toric orbits through the points

$$[1,0,1,0], \qquad [1,1,0,0], \qquad [1,1,1,0] \quad \in \quad Z'.$$

Note that we can move both the third point and the second point to the first point by using the respective equivariant automorphisms

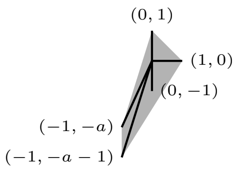
$$z \ \mapsto\ (z_1,\, z_2 - z_3,\, z_3,\, z_4), \qquad z \ \mapsto\ (z_1,\, z_3,\, z_2,\, z_4)$$

of $\widehat{Z}'$, compare Lemma 5.1.5. Thus, $\mathrm{Bl}\,\mathbb{P}_2(\star^2)$ is the toric variety with its fan given by the stellar subdivision of Σ' at $(2,1) \in \mathbb{Z}^2$.

The surfaces Bl $\mathbb{F}_a(\star\ i)$ and Bl $\mathbb{F}_a(\star\ ii)$ are obtained from $\mathbb{F}_a$ by stellar subdivision of the fan of $\mathbb{F}_a$ at $v \in \mathbb{Z}^2$ where $v = (-1, -a+1)$ or $v = (-1, -a-1)$, respectively. By Lemma 5.2.6, the degree matrices of the Cox rings $\mathbb{K}[T_1, \ldots, T_5]$ of Bl $\mathbb{F}_a(\star\ i)$ and Bl $\mathbb{F}_a(\star\ ii)$ are Gale dual matrices of

$$[P, v], \qquad P := \begin{bmatrix} -1 & 1 & 0 & 0 \\ -a & 0 & 1 & -1 \end{bmatrix}.$$

Note that Bl $\mathbb{F}_{a+1}(\star\ i)$ is isomorphic to Bl $\mathbb{F}_a(\star\ ii)$ for each $a \geq 2$: as toric varieties both surfaces share the same fan



Hence, we may remove Bl $\mathbb{F}_a(\star\ ii)$ from the list. Observe that we also may omit Bl $\mathbb{F}_2(\star\ i)$. Let Z be the blow up of $\mathbb{P}_2$ in the fixed point $[0,0,1]$. Blowing up the fixed point $[0,1,1,0] \in Z$, we obtain the toric variety Bl $\mathbb{F}_2(\star\ i)$. Therefore, Bl $\mathbb{F}_2(\star\ i)$ is isomorphic to Bl $\mathbb{P}_2(\star^2)$.

To show that the remaining listed surfaces X are pairwise non-isomorphic, we compare the self-intersection numbers of negative curves. These come from toric divisors, i.e., are of the form $V(X; T_i)$. See also Remark 5.2.7.

X	$V(T_1)^2$	$V(T_2)^2$	$V(T_3)^2$	$V(T_4)^2$	$V(T_5)^2$
Bl $\mathbb{P}_2(\star^2)$	≥ 0	-1	≥ 0	-2	-1
Bl $\mathbb{P}_2(\star\star)$	≥ 0	≥ 0	-1	-1	-1
Bl $\mathbb{F}_a(\star\ i)$, $a \geq 3$	-1	≥ 0	≥ 0	$-a$	-1

$\square$

Proposition 5.2.8. *Let X be a smooth rational surface with Picard number $\varrho(X) = 4$. Then X is isomorphic to exactly one of the following.*

X	*Cox ring* $\mathcal{R}(X)$	*degree matrix*
Bl $\mathbb{P}_2(\star^3\ i)$	$\mathbb{K}[T_1, \ldots, T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^3\ ii)$	$\mathbb{K}[T_1, \ldots, T_7]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 - T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^2\star\ i)$	$\mathbb{K}[T_1, \ldots, T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^2\star\ ii)$	$\mathbb{K}[T_1, \ldots, T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star\star\star\ i)$	$\mathbb{K}[T_1, \ldots, T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

Bl $\mathbb{P}_2(\star\star\star\ ii)$	$\mathbb{K}[T_1,\dots,T_7]/I$ *with I generated by* $T_2T_4 - T_1T_5 - T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star\star\ \ i)$ $a \ge 3$	$\mathbb{K}[T_1,\dots,T_6]$	$\begin{bmatrix} 1 & 0 & 0 & a-3 & -2 & -1 \\ 0 & 1 & 0 & a-3 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & a-2 & -1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star\star\ \ v)$ $a \ge 3$	$\mathbb{K}[T_1,\dots,T_7]/I$ *with I generated by* $T_6T_7 - T_2^aT_4 + T_3T_5$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & a & 2a-1 & -a+1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2\ \ i)$ $a \ge 3$	$\mathbb{K}[T_1,\dots,T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -a+2 & a-1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2\ \ ii)$ $a \ge 3$	$\mathbb{K}[T_1,\dots,T_6]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2a-1 & -a+1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$

In particular, the cases Bl $\mathbb{P}_2(\star^3\ ii)$, Bl $\mathbb{P}_2(\star\star\star\ ii)$ *and* Bl $\mathbb{F}_a(\star\star\ v)$ *are non-toric* $\mathbb{K}^*$*-surfaces. The remaining surfaces are toric.*

For the proof of Proposition 5.2.8, to avoid redundancies, we must be able to test surfaces for being isomorphic. We give a solution for toric and non-toric $\mathbb{K}^*$-surfaces. Afterwards, we provide easy to check conditions on whether an ambient modification was a blow up and on when a point in the total coordinate space is relevant.

Notation 5.2.9. Let A and B be $n \times r$ matrices. We write $A \risingdotseq B$ if A equals B up to permutation of the columns.

Remark 5.2.10. Let Z and Z' be complete toric surfaces. Write primitive generators for the rays of their fans into the columns of matrices P_Z and $P_{Z'}$. Then

$$Z_1 \ \cong\ Z_2 \qquad \Leftrightarrow \qquad A \cdot P_Z \ \risingdotseq\ P_{Z'} \qquad \text{for some } A \in \mathrm{GL}(2,\mathbb{Z}).$$

Algorithm 5.2.11 (Toric surface isomorphism test)**.** *Input:* $2 \times r$ matrices $P_Z = [p_1,\dots,p_r]$ and $P_{Z'} = [p'_1,\dots,p'_r]$ as in Remark 5.2.10 such that both the $p_1,\dots,p_r$ and $p'_1,\dots,p'_r$ contain a lattice basis for $\mathbb{Z}^2$.

- Choose a lattice basis (p_{i_1}, p_{i_2}) for $\mathbb{Z}^2$.
- If there is an ordered lattice basis (p'_{j_1}, p'_{j_2}) for $\mathbb{Z}^2$ such that $A_{j_1j_2} \cdot P_Z \risingdotseq P_{Z'}$ with the invertible matrix $A_{j_1j_2} := [p'_{j_1}, p'_{j_2}] \cdot [p_{i_1}, p_{i_2}]^{-1}$, then return *true*. Otherwise, return *false*.

Output: *true* if Z is isomorphic to Z' and *false* otherwise.

Proof. Each $A \in \mathrm{GL}(2,\mathbb{Z})$ maps a lattice basis to a lattice basis, i.e., $A \cdot [p_{i_1}, p_{i_2}] \risingdotseq [p'_{j_1}, p'_{j_2}]$ which means $A \risingdotseq A_{j_1,j_2}$. Thus, the algorithm runs through all possibilities of Remark 5.2.10. □

Algorithm 5.2.12 (Classify toric blow ups)**.** *Input:* the fan Σ_0 of a complete toric surface Z_0 and an integer $s \in \mathbb{Z}_{\ge 1}$.

- Initialize lists $L_0 := (\Sigma_0)$ and $L_i := \emptyset$ for $i \in \{1, \dots, s\}$.
- For each $i = 0, 1, \dots, s-1$ do
  - for each $\Sigma \in L_i$ do
    - for each maximal cone $\sigma \in \Sigma$ do
      - perform the barycentric subdivision $\Sigma' \to \Sigma$ of Σ at σ.
      - Use Algorithm 5.2.11 to test whether there is $\Sigma'' \in L_{i+1}$ such that the toric variety $Z_{\Sigma''}$ is isomorphic to $Z_{\Sigma'}$. Insert Σ' into L_{i+1} if this is not the case.

Output: L_s. This is a list of the fans of all complete toric surfaces that can be obtained from Z_0 by s blow up steps. Of these surfaces, no two are isomorphic.

We now turn to methods for testing whether two $\mathbb{K}^*$-surfaces are isomorphic. We use the fact that, in the sense of Proposition 1.5.8, *admissible* operations preserve the shape of the blocks of the P-matrices corresponding to the surfaces; compare Construction 1.5.2.

Remark 5.2.13. Consider $\mathbb{K}^*$-surfaces $X = X(P, A)$ and $X' = X(P', A')$ as in Construction 1.5.2. Assume that the Cox rings are of shape

$$\begin{aligned} \mathcal{R}(X) &= \mathbb{K}[T_1, \dots, T_{n+m}] \,/\, \langle g \rangle, & g &:= c_0 T^{l_0} + c_1 T^{l_1} + c_2 T^{l_2}, \\ \mathcal{R}(X') &= \mathbb{K}[T_1, \dots, T_{n+m}] \,/\, \langle g' \rangle, & g' &:= c'_0 T^{l'_0} + c'_1 T^{l'_1} + c'_2 T^{l'_2} \end{aligned}$$

with $c_i, c'_i \in \mathbb{K}^*$ and integral vectors $l_i \in \mathbb{Z}^{n_i}_{\geq 0}$, $l'_i \in \mathbb{Z}^{n'_i}_{\geq 0}$. By Remark 1.5.9, X and X' are isomorphic if and only if $P' = S \cdot P \cdot U$ with admissible matrices S, U. In particular, up to permutation, the sets of exponent vectors of g and of g' coincide:

$$\{\sigma_0(l_0), \, \sigma_1(l_1), \, \sigma_2(l_2)\} \;=\; \{l'_0, \, l'_1, \, l'_2\} \qquad \text{for some } \sigma_i \in \mathrm{Sym}(n_i).$$

Algorithm 5.2.14 ($\mathbb{K}^*$-surface isomorphism test)**.** *Input:* $\mathbb{K}^*$-surfaces X_1, X_2 as in Remark 5.2.13 with $\mathrm{Cl}(X_i) = \mathbb{Z}^k$ for both i. Let Q_i be the $k \times (n+m)$ degree matrices of $\mathcal{R}(X_i)$

- Return *false* if the criterion of Remark 5.2.13 fails.
- As in Algorithm 5.2.11, compute all matrices $A \in \mathrm{GL}(k, \mathbb{Z})$ such that $A \cdot Q_1 \cdot U_A = Q_2$ with a permutation matrix $U_A \in \mathrm{GL}(n+m, \mathbb{Z})$.
- Return *true* if one of the matrices U_A is admissible and *false* otherwise.

Output: *true* if $X_1 \cong X_2$ and *false* otherwise.

Proof. Let $X_i = X(P_i, A_i)$ be as in Construction 1.5.2. Then Q_i is a Gale dual matrix for P_i. Assume X_1 is isomorphic to X_2. By Remark 1.5.9, there is an admissible matrix $S \in \mathrm{GL}(3, \mathbb{Z})$ and an admissible permutation matrix $U \in \mathrm{GL}(n+m, \mathbb{Z})$ such that $P_2 = S \cdot P_1 \cdot U$. Then $Q'_2 := Q_1 (U^{-1})^t = Q_1 U$ satisfies $P_2 (Q'_2)^t = 0$. Hence, both Q_2 and Q'_2 have a basis for $\ker(P_2)$ as their rows, i.e.

$$Q_2 \;=\; A \cdot Q'_2 \;=\; A \cdot Q_1 \cdot U \qquad \text{for some} \qquad A \in \mathrm{GL}(k, \mathbb{Z}).$$

Similarly, for the reverse implication, assume that $Q_2 = A' \cdot Q_1 \cdot U'$ with $A' \in \mathrm{GL}(k, \mathbb{Z})$ and an admissible permutation matrix $U' \in \mathrm{GL}(n+m, \mathbb{Z})$. Then $P'_2 := P_1 \cdot ((U')^{-1})^t$ satisfies $P'_2 \cdot Q_2^t = 0$, i.e.

$$P_2 \;=\; S' \cdot P'_2 \qquad \text{for some} \qquad S' \in \mathrm{GL}(n+m-k, \mathbb{Z}).$$

Since U' is admissible, both P'_2 and $P_2 = S' \cdot P'_2$ are in block shape as in Construction 1.5.2. Therefore, the only possible row operations performed by multiplying S' from the left is adding multiples of the upper r rows to the lower ones, i.e., S' is admissible. $\square$

Lemma 5.2.15. *Consider Setting 4.2.5. Assume that $X_1 \subseteq Z_1$ is a Mori dream space embedded in its canonical toric ambient variety, $Z_2 \to Z_1$ arises from a* *barycentric subdivision of a regular cone $\sigma \in \Sigma_1$ and $X_2 \to X_1$ has as center a point $x \in X_1 \cap (\mathbb{T}^n \cdot z(\sigma))$ with Cox coordinates $z \in \mathbb{K}^r$. Then $X_2 \to X_1$ is the blow up at x provided the following conditions are fulfilled.*

(i) *The grading group K_1 is free.*
(ii) *The ideal $I(\overline{X}_1) + \langle T_i;\, z_i = 0\rangle \subseteq \mathbb{K}[T_1, \dots, T_{r_1}]$ is prime.*
(iii) *The torus $H_1 := \operatorname{Spec} \mathbb{K}[K_1]$ and $\overline{X}_1 \cap V(T_i;\, z_i = 0)$ are of the same dimension.*

Proof. Write $J_1 := I(\overline{X}_1) + \langle T_i;\, z_i = 0\rangle$ and $Y := V(\mathbb{K}^r;\, J_1)$. Since $z \in Y$, also the orbit closure $\overline{H_1 \cdot z}$ is contained in Y. It can be described as

$$\overline{H_1 \cdot z} \;=\; V(\mathbb{K}^r;\, J_2), \qquad J_2 \;:=\; I(P_1, z) + \langle T_i;\, z_i = 0\rangle.$$

By conditions (ii) and (iii), Y is irreducible and of the same dimension as $V(\mathbb{K}^r;\, J_2)$, which shows $Y = V(\mathbb{K}^r;\, J_2)$. Again by (ii), we conclude $J_1 = \sqrt{J_2}$. Since the grading group is free, by [82, Thm. 7.4], $I(P_1, z)$ is a prime ideal in the ring $\mathbb{K}[T_j;\, z_j \neq 0]$. The integral domain $\mathbb{K}[T_1, \dots, T_r]/J_2$ is isomorphic to $\mathbb{K}[T_j;\, z_j \neq 0]/I(P_1, z)$, so we arrive at $J_1 = J_2$. An application of Lemma 4.6.2 concludes the proof. $\square$

Lemma 5.2.16. *Consider Setting 4.2.9.*

(i) *Consider $z \in \overline{X}_1 \subseteq \mathbb{K}^{r_1}$. If $z_i = 0$ for exactly one $1 \le i \le r_1$, then $z \in \widehat{X}_1$.*
(ii) *If X_1 is a surface, then $\widehat{X}'_1 = \bar{\imath}(\widehat{X}_1)$.*

Proof. Let $\gamma := \mathbb{Q}^{r_1}_{\ge 0}$. Given a face $\gamma_0 \preceq \gamma$, write $T_{\gamma_0} \in \mathbb{K}[T_1, \dots, T_r]$ for the product of all T_i with $e_i \in \gamma_0$. By the construction of bunched rings 1.3.6, each facet $\gamma_0 \preceq \gamma$ is a relevant $\mathfrak{F}$-face and $\widehat{X}_1$ equals the union of all $\overline{X}_1 \setminus V(\overline{X}_1;\, T_{\gamma_0})$ where $\gamma_0 \preceq \gamma$ runs through the relevant $\mathfrak{F}$-faces. Choosing $\gamma_0 := \operatorname{cone}(e_i;\, z_i \neq 0)$ proves the first assertion. For (ii), consider the diagram

$$\begin{tikzcd}
\overline{X}_1 \arrow[d, "\bar{\imath}"', "\cong"] & \supseteq & \widehat{X}_1 \arrow[r, "/\!\!/ H_1"] \arrow[d] & X_1 \arrow[d, "\cong"] \\
\overline{X}'_1 & \supseteq & \widehat{X}'_1 \arrow[r, "/\!\!/ H_1"] & X'_1
\end{tikzcd}$$

with the characteristic quasitorus H_1 of X_1. Since X_1 is a surface, $\widehat{X}_1 \subseteq \overline{X}_1$ is the only open subset with its complement in $\overline{X}_1$ of codimension at least two that admits a quasiprojective quotient by H_1. Thus, as $\bar{\imath}\colon \overline{X}_1 \to \overline{X}'_1$ is an isomorphism, so is $\imath\colon X_1 \to X'_1$ and we obtain $\bar{\imath}(\widehat{X}_1) = \widehat{X}'_1$. $\square$

Lemma 5.2.17. *Let $f \in \mathbb{K}[T_1, \dots, T_r]$ be such that $T_j \nmid f$ for all j. Assume there is a variable T_i dividing exactly one non-zero monomial T^μ of f and $T_i^2 \nmid T^\mu$. Then f is prime.*

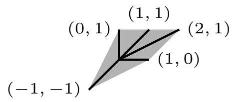
Proof. By choice of f, each factorization must be of the form $f = (g_1 + T_i g_2)h$ with $g_i,\, h \in \mathbb{K}[T_j;\, j \neq i]$. By assumption, h is not a monomial. Since we allow only one term to depend on T_i we conclude $h \in \mathbb{K}^*$, i.e., f is irreducible. $\square$

Proof of Proposition 5.2.8. By Remark 5.2.3, the surface X can be obtained as a blow up of $\mathbb{P}_2$ in three points or as the blow up of $\mathbb{F}_a$ in two points where $a \in \mathbb{Z}_{\ge 2}$. Propositions 5.1.2 and 5.1.4 list the configurations we need to consider. Moreover, X is a blow up of one of the surfaces classified in Proposition 5.2.5. See Remark 5.2.1 for the steps.

(I) Surfaces of type $\mathrm{Bl}\,\mathbb{P}_2(\star^3)$. The varieties of the form $\mathrm{Bl}\,\mathbb{P}_2(\star^3)$ are a blow up of $Z_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^2)$ in a point in the exceptional divisor of the first or of the second

blow up. As a toric variety, Z_1 is given by a fan Σ_1 with its rays generated by the columns of

$$P_1 := \begin{bmatrix} -1 & 1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix},$$



The exceptional divisors of Z_1 are $V(Z_1;\, T_4)$ and $V(Z_1;\, T_5)$. The union $V(Z_1;\, T_4)\cup V(Z_1;\, T_5)$ consists of the toric orbits through the points

$$\begin{array}{llll} q_1 := [1,0,1,1,0], & q_2 := [1,1,1,1,0], & q_3 := [1,1,1,0,0], & \\ q_4 := [1,1,1,0,1], & q_5 := [1,1,0,0,1] & \in Z_1. & \end{array}$$

The equivariant automorphism $z \mapsto (z_1, z_2, z_3 - z_2 z_5, z_4, z_5)$ of $\widehat{Z}_1$ maps q_4 to q_5; compare Lemma 5.1.5. Thus, if we blow up Z_1 in q_1, q_3 or in one of the points q_4, q_5 it is given by insertion of the respective rays

$$\mathbb{Q}_{\geq 0} \cdot (3,1), \qquad \mathbb{Q}_{\geq 0} \cdot (3,2), \qquad \mathbb{Q}_{\geq 0} \cdot (1,2)$$

into Σ_1 by means of a stellar subdivision. This covers the cases Bl $\mathbb{P}_2(\star^3\ i)$, Bl $\mathbb{P}_2(\star^3\ iii)$ and Bl $\mathbb{P}_2(\star^3\ iv)$. We now treat the case Bl $\mathbb{P}_2(\star^3\ ii)$ which is the blow up of Z_1 in q_2. This is done by the steps explained in Remark 5.2.1. Note that the point exists by Lemma 5.2.16. In the situation of Setting 4.2.9, we have $Z_1 = X_1$ and $K_1 = \mathbb{Z}^3$. Consider the embedding

$$\bar{\imath}\colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad x \mapsto (x, h_1(x)), \qquad h_1 := T_3^2 T_4 - T_1 T_2 \in \mathbb{K}[T_1,\dots,T_5].$$

Then $I_1' = \langle T_6 - h_1\rangle$ is the vanishing ideal of $\overline{X}_1'$. The $\mathbb{Z}^3$-grading on $\mathbb{K}[T_1,\dots,T_6]$ is given by the following degree matrix Q_1'. The columns of P_1', a Gale dual matrix of Q_1', generate the rays of the fan Σ_1' of the toric ambient variety Z_1' where

$$Q_1' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}, \qquad P_1' := \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 1 & 0 & -1 \end{bmatrix}.$$

We want to blow up $\imath(q_2) = [1,1,1,1,0,0] \in X_1'$. Note that this point exists by Lemma 5.2.16. Consider now Setting 4.2.5 with the toric modification $\pi\colon Z_2 \to Z_1'$ given by insertion of the ray through the sum $v := (-2,0,-1)$ of the fifth and sixth column of P_1' into Σ_1'. Thus, the fan of the toric variety Z_2 has its rays generated by the columns of $P_2 = [P_1', v]$. We obtain the ideal $I_2 = \langle g\rangle \subseteq \mathbb{K}[T_1,\dots,T_7]$ of the closure of the inverse image $\overline{X}_2$ by modifying the generator of I_1:

$$g := p_2^\star\,(p_1)_\star\,(T_6 - h_1) = T_6T_7 - T_3^2T_4 + T_1T_2 \in \mathbb{K}[T_1,\dots,T_7]$$

where p_i is the morphism of tori corresponding to P_i. Observe that g is already saturated with respect to T_7. All variables T_i define pairwise non-associated prime elements in $R_2 = \mathbb{K}[T_1,\dots,T_7]/\langle g\rangle$ by Lemma 5.2.17. Moreover, $T_7 \nmid T_i$ for all $i<7$ as

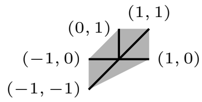
$$\dim\big(V(\mathbb{K}^7;\, T_7, T_i, g)\big) = 4 \qquad \text{for all } i < 7.$$

Using Theorem 4.2.6(i) with Proposition 4.1.4, R_2 is the Cox ring of the performed modification. Its degree matrix is a Gale dual matrix of P_2 as listed in the table. We now show that the modification was the blow up of $\imath(q_2) \in X_1'$. In $\mathbb{K}[T_1,\dots,T_6]$, the ideal $\langle T_5, T_6\rangle + I_1'$ is prime by Lemma 5.2.17, its zero set is three-dimensional and the latter contains $\bar{\imath}((1,1,1,1,0))$. By Lemma 5.2.15, the modification was the claimed blow up.

(II) Surfaces of type Bl $\mathbb{P}_2(\star^2\star)$. The cases Bl $\mathbb{P}_2(\star^2\star\ i)$ and Bl $\mathbb{P}_2(\star^2\star\ ii)$ are blow ups of $Z_1 :=$ Bl $\mathbb{P}_2(\star\star)$ in a point in one of the two exceptional divisors. As a toric

variety, Z_1 is given by a fan with its rays generated by the columns of

$$P_1 \;:=\; \begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & 0 \end{bmatrix},$$



Of the exceptional divisors $V(Z_1; T_4)$ and $V(Z_1; T_5)$ we consider the first one. It consists of the toric orbits through the points

$$q_1 \;:=\; [1,0,1,0,1], \qquad q_2 \;:=\; [1,1,1,0,1], \qquad q_3 \;:=\; [1,1,0,0,1] \;\in\; Z_1.$$

The automorphism $z \mapsto (z_1, z_2 - z_3 z_5, z_3, z_4, z_5)$ of $\widehat{Z}_1$ maps q_2 to q_1; compare Lemma 5.1.5. Thus, any blow up of Z_1 in one of the points q_i is isomorphic to the toric blow up obtained by insertion of the ray $\mathbb{Q}_{\ge 0} \cdot (2,1)$ or $\mathbb{Q}_{\ge 0} \cdot (1,2)$ into the fan of Z_1.

(III) Surfaces of type Bl $\mathbb{P}_2(\star\star\star)$. Surfaces of the form Bl $\mathbb{P}_2(\star\star\star)$ are again blow ups of $Z_1 :=$ Bl $\mathbb{P}_2(\star\star)$. By Proposition 5.1.2, we need to consider the following configurations.

If $X =$ Bl $\mathbb{P}_2(\star\star\star\ i)$ we have to perform the toric blow up of the fixed point $[0,0,1,1,1] \in Z_1$. This is done by the stellar subdivision of the fan of Z_1 at $(0,-1)$. For $X =$ Bl $\mathbb{P}_2(\star\star\star\ ii)$ we blow up the point $q := [1,1,0,1,1] \in Z_1$. Note that it exists by Lemma 5.2.16. The following steps are the same as in case Bl $\mathbb{P}_2(\star^3\ ii)$; compare Remark 5.2.1. We choose the embedding

$$\bar\imath\colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad x \mapsto (x, h_1(x)), \qquad h_1 \;:=\; T_2T_4 - T_1T_5 \;\in\; \mathbb{K}[T_1, \ldots, T_5]$$

and obtain a CEMDS X_1'. Its degree matrix Q_1' and the matrix P_1' of generators for the rays of the fan of the toric ambient variety Z_1' of X_1' are

$$Q_1' \;:=\; \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}, \qquad P_1' \;:=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

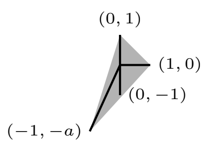
On X_1', to blow up $\iota(q) = [1,1,0,1,1,0]$, we perform the stellar subdivision of the fan of Z_1' at $v := (-1,-1,0)$. Set $P_2 := [P_1', v]$. The ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_7]$ of $\overline{X}_2$ is generated by

$$g \;:=\; p_2^\star\,(p_1)_\star\,(T_6 - h_1) \;=\; T_6T_7 - T_2T_4 + T_1T_5 \;\in\; \mathbb{K}[T_1, \ldots, T_7].$$

By the same arguments as in case Bl $\mathbb{P}_2(\star^3\ ii)$, using the simplifications of Proposition 4.1.4, we check that the requirements for Theorem 4.2.6(i) are fulfilled. Thus, $R_2 = \mathbb{K}[T_1, \ldots, T_7]/I_2$ is the Cox ring of the performed modification with the degree matrix as listed in the table. In $\mathbb{K}[T_1, \ldots, T_6]$, the ideal $\langle T_3, T_6, h_1 \rangle$ is prime by Lemma 5.2.17, its zero set is three-dimensional and it contains $(1,1,0,1,1,0)$. Thus, Lemma 5.2.15 can be applied.

(IV) Surfaces of type Bl $\mathbb{F}_a(\star\star)$. Recall that the fan Σ_a of the toric variety $\mathbb{F}_a$ has its rays generated by the columns of

$$P_1 \;:=\; \begin{bmatrix} -1 & 1 & 0 & 0 \\ -a & 0 & 1 & -1 \end{bmatrix},$$

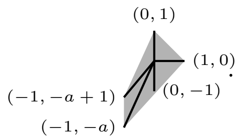


The first four cases Bl $\mathbb{F}_a(\star\star\ i)$ to Bl $\mathbb{F}_a(\star\star\ iv)$ are blow ups of $\mathbb{F}_a$ in the toric fixed points listed in Proposition 5.1.4 where $a \ge 2$. Each of these toric blow ups is a stellar subdivision $\Sigma_2 \to \mathbb{F}_a$ at two vectors $v_1,\ v_2 \in \mathbb{Z}^2$. The fan Σ_2 has its rays generated by the columns of $P_2 := [P_1, v_1, v_2]$. The choices for v_1 and v_2 are as follows. The degree matrices of the Cox rings are obtained as Gale dual matrices of P_2, compare Lemma 5.2.6.

X	v_1	v_2
Bl $\mathbb{F}_a(\star\star\ i)$	$(-1, -a+1)$	$(1, 1)$
Bl $\mathbb{F}_a(\star\star\ ii)$	$(-1, -a+1)$	$(-1, -a-1)$
Bl $\mathbb{F}_a(\star\star\ iii)$	$(-1, -a+1)$	$(1, -1)$
Bl $\mathbb{F}_a(\star\star\ iv)$	$(-1, -a-1)$	$(1, -1)$

We come to $X = \mathrm{Bl}\ \mathbb{F}_a(\star\star\ v)$. Recall from Proposition 5.2.5 that the toric variety $Z_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\ i)$ is the blow up of $\mathbb{F}_a$ in the fixed point $[0,1,0,1]$, i.e., the toric variety with the rays of its fan Σ_1 generated by the columns of

$$P_1 \ := \ \left[\begin{array}{ccccc} -1 & 1 & 0 & 0 & -1 \\ -a & 0 & 1 & -1 & -a+1 \end{array}\right],$$



Then X is the blow up of Z_1 in $q := [0,1,1,1,1]$. Note that q exists by Lemma 5.2.16 and projects to $[0,1,1,1] \in \mathbb{F}_a$ under the first blow up $z \mapsto (z_1z_5, z_2, z_3z_5, z_4)$. The subsequent steps are as in case Bl $\mathbb{P}_2(\star^3\ ii)$, compare Remark 5.2.1. Using the embedding

$$\bar{\imath}\colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad x \mapsto (x, h_1(x)), \qquad h_1 := T_2^aT_4 - T_3T_5 \in \mathbb{K}[T_1, \ldots, T_5],$$

we obtain a CEMDS X_1'. The degree matrix Q_1' and the matrix P_1' whose columns are generators for the rays of the fan Σ_1' of the toric ambient variety Z_1' of X_1' are

$$Q_1' \ := \ \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -a+1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right], \qquad P_1' \ := \ \left[\begin{array}{cccccc} 1 & a-1 & 0 & 1 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right].$$

On X_1', for the blow up of the point $\iota(q) = [0,1,1,1,1,0]$, we perform the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (0,-1,-1)$. Write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_7]$ of $\overline{X}_2$ is generated by

$$g \ := \ p_2^{\star}\,(p_1)_{\star}\,(T_6 - h_1) \ = \ T_6T_7 - T_2^aT_4 + T_3T_5 \ \in \ \mathbb{K}[T_1, \ldots, T_7].$$

By the same methods as before, we verify the requirements for Theorem 4.2.6 and Proposition 4.1.4. Thus, $R_2 = \mathbb{K}[T_1, \ldots, T_7]/I_2$ is the Cox ring of the performed modification with the degree matrix listed in the table. By Lemma 5.2.15, we did perform a blow up as the ideal $\langle T_3, T_6, h_1 \rangle$ in $\mathbb{K}[T_1, \ldots, T_6]$ is prime by Lemma 5.2.17 and its zero set contains $(0,1,1,1,1,0)$ while being three-dimensional.

(V) Surfaces of type Bl $\mathbb{F}_a(\star^2)$. Define Z_1 as in the previous case, i.e., Z_1 is the blow up of $\mathbb{F}_a$ in the fixed point $[0,1,0,1] \in \mathbb{F}_a$. We want to blow up a point in the exceptional divisor $V(Z_1; T_5)$. It consists of the toric orbits through the points

$$q_1 \ := \ [1,1,0,1,0], \qquad q_2 \ := \ [1,1,1,1,0], \qquad q_3 \ := \ [0,1,1,1,0] \ \in \ Z_1.$$

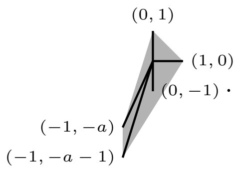
The automorphism $z \mapsto (z_1, z_2, z_3 - z_1z_2^{a-1}z_4, z_4, z_5)$ of $\widehat{Z}_1$ maps q_2 to q_1; compare Lemma 5.1.5. Thus, the remaining cases are the toric blow ups of Z_1 in q_1 and q_3 and are carried out by insertion of the respective rays

$$\mathbb{Q}_{\geq 0} \cdot v, \qquad v \ := \ (-1, -a+2) \qquad \text{or} \qquad v \ := \ (-2, -2a+1)$$

into Σ_1. The degree matrix of the Cox ring $\mathbb{K}[T_1, \ldots, T_6]$ of X is a Gale dual matrix of the enlarged matrix $[P_1, v]$, compare Lemma 5.2.6. This covers the cases Bl $\mathbb{F}_a(\star^2\ i)$ and Bl $\mathbb{F}_a(\star^2\ ii)$.

We come to blow ups of the toric variety Bl $\mathbb{F}_a(\star\ ii)$ in a point in the exceptional divisor. Recall that $Z_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\ ii)$ has the rays of its fan Σ_1 generated by the columns of

$$P_1 \ := \ \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ -a & 0 & 1 & -1 & -a-1 \end{bmatrix},$$



The exceptional divisor of Z_1 is $V(Z_1; T_5)$ and consists of the toric orbits through the points

$$q_1 \ := \ [0,1,1,1,0], \qquad q_2 \ := \ [1,1,1,1,0], \qquad q_3 \ := \ [1,1,1,0,0] \ \in \ Z_1.$$

The blow ups of the toric fixed points q_1 and q_3 are carried out by stellar subdivisions of Σ_1 at the rays

$$\mathbb{Q}_{\ge 0} \cdot v, \qquad v \ := \ (-2,-2a-1) \qquad \text{or} \qquad v \ := \ (-1,-a-2)$$

respectively. The degree matrix of the Cox ring $\mathbb{K}[T_1,\dots,T_6]$ of X is a Gale dual matrix of the enlarged matrix $[P_1, v]$, see Lemma 5.2.6. This covers the cases $\mathrm{Bl}\,\mathbb{F}_a(\star^2\ iii)$ and $\mathrm{Bl}\,\mathbb{F}_a(\star^2\ iv)$.

For $X = \mathrm{Bl}\,\mathbb{F}_a(\star^2\ v)$, we blow up Z_1 in q_2. Note that q_2 exists by Lemma 5.2.16. The steps are analogous to previous cases, e.g., $\mathrm{Bl}\,\mathbb{P}_2(\star^3\ ii)$. Choose the embedding

$$\bar{\imath}\colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad x \mapsto (x, h_1(x)), \qquad h_1 \ := \ T_2^{a+1}T_4 - T_1T_3 \ \in \ \mathbb{K}[T_1,\dots,T_5].$$

We obtain a CEMDS X_1', a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \ := \ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -a-1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad P_1' \ := \ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & a+1 & -1 & -a-1 & -a \end{bmatrix}.$$

On X_1', for the blow up of $\iota(q_2) = [1,1,1,1,0,0]$, we perform the stellar subdivision of Σ_1' at the vector $v := (-1,-2,-2a-1)$ in $\mathbb{Z}^3$. Set $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_7]$ of $\overline{X}_2$ is generated by

$$p_2^\star\,(p_1)_\star\,(T_6 - h_1) \ = \ T_6T_7 - T_2^{a+1}T_4 + T_1T_3 \ \in \ \mathbb{K}[T_1,\dots,T_7].$$

As in previous cases, one directly verifies the requirements for Theorem 4.2.6. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1,\dots,T_7]/I_2$. Its degree matrix is a Gale dual matrix of P_2, i.e.,

$$Q_2 \ = \ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -a-1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

By Lemma 5.2.15, the modification was a blow up as the ideal $\langle T_5, T_6, h_1\rangle$ in $\mathbb{K}[T_1,\dots,T_6]$ is prime by Lemma 5.2.17 and its zero set contains $(1,1,1,1,0,0)$ while being three-dimensional.

Isomorphisms: We now show that the surfaces listed in the proposition are pairwise non-isomorphic. As seen in the proof of Proposition 5.2.5, $\mathrm{Bl}\,\mathbb{F}_2(\star\ i)$ is isomorphic to $\mathrm{Bl}\,\mathbb{P}_2(\star^2)$. Therefore, we have $a \ge 3$ in the cases

$$\mathrm{Bl}\,\mathbb{F}_a(\star\star\ i), \quad \mathrm{Bl}\,\mathbb{F}_a(\star\star\ ii), \quad \mathrm{Bl}\,\mathbb{F}_a(\star\star\ iii), \quad \mathrm{Bl}\,\mathbb{F}_a(\star\star\ v),$$
$$\mathrm{Bl}\,\mathbb{F}_a(\star^2\ i), \qquad \mathrm{Bl}\,\mathbb{F}_a(\star^2\ ii).$$

We first treat the toric and then the non-toric cases. We compare the self-intersection numbers of negative curves. These curves are of the form $V(X; T_i)$, compare also 5.2.7.

a	X	$V(T_1)^2$	$V(T_2)^2$	$V(T_3)^2$	$V(T_4)^2$	$V(T_5)^2$	$V(T_6)^2$
	$\mathrm{Bl}\,\mathbb{P}_2(\star^3\ i)$	≥ 0	-2	≥ 0	-2	-2	-1
	$\mathrm{Bl}\,\mathbb{P}_2(\star^3\ iii)$	≥ 0	-1	≥ 0	-3	-2	-1
	$\mathrm{Bl}\,\mathbb{P}_2(\star^3\ iv)$	≥ 0	-1	-1	-3	-1	-1

	Bl $\mathbb{P}_2(\star^2 \star\ i)$	≥ 0	-1	-1	-2	-1	-1
	Bl $\mathbb{P}_2(\star^2 \star\ ii)$	≥ 0	≥ 0	-2	-2	-1	-1
	Bl $\mathbb{P}_2(\star\star\star\ i)$	-1	-1	-1	-1	-1	-1
≥ 3	Bl $\mathbb{F}_a(\star\star\ i)$	-1	-1	≥ 0	$-a$	-1	-1
≥ 3	Bl $\mathbb{F}_a(\star\star\ ii)$	-2	≥ 0	≥ 0	$-a-1$	-1	-1
≥ 3	Bl $\mathbb{F}_a(\star\star\ iii)$	-1	-1	≥ 0	$-a-1$	-1	-1
≥ 2	Bl $\mathbb{F}_a(\star\star\ iv)$	-1	-1	≥ 0	$-a-2$	-1	-1
≥ 3	Bl $\mathbb{F}_a(\star^2\ i)$	-1	≥ 0	≥ 0	$-a$	-2	-1
≥ 3	Bl $\mathbb{F}_a(\star^2\ ii)$	-2	≥ 0	≥ 0	$-a$	-2	-1
≥ 2	Bl $\mathbb{F}_a(\star^2\ iii)$	-2	≥ 0	≥ 0	$-a-1$	-2	-1
≥ 2	Bl $\mathbb{F}_a(\star^2\ iv)$	-1	≥ 0	≥ 0	$-a-2$	-2	-1

An inspection of the table shows that the listed surfaces are pairwise non-isomorphic except for the following. Using Algorithm 5.2.11 and Remark 5.2.10, we give isomorphisms

$$\begin{array}{rclrcl}
\mathrm{Bl}\ \mathbb{F}_{a+2}(\star^2\ i) & \to & \mathrm{Bl}\ \mathbb{F}_a(\star^2\ iv), & \mathrm{Bl}\ \mathbb{F}_{a+1}(\star\star\ i) & \to & \mathrm{Bl}\ \mathbb{F}_a(\star\star\ iii), \\
\mathrm{Bl}\ \mathbb{P}_2(\star^3\ iii) & \to & \mathrm{Bl}\ \mathbb{F}_3(\star^2\ i), & \mathrm{Bl}\ \mathbb{P}_2(\star^3\ iv) & \to & \mathrm{Bl}\ \mathbb{F}_3(\star\star\ i), \\
\mathrm{Bl}\ \mathbb{F}_{a+1}(\star\star\ iii) & \to & \mathrm{Bl}\ \mathbb{F}_a(\star\star\ iv), & \mathrm{Bl}\ \mathbb{F}_{a+1}(\star^2\ ii) & \to & \mathrm{Bl}\ \mathbb{F}_a(\star^2\ iii),
\end{array}$$

$$\mathrm{Bl}\ \mathbb{F}_{a+1}(\star^2\ i) \to \mathrm{Bl}\ \mathbb{F}_a(\star\star\ ii).$$

For each surface, we write primitive generators for the rays of its corresponding fan into a matrix as in Remark 5.2.10. In the first and the two last cases, the respective matrices already are the same up to column permutations. For the second to fifth isomorphism, in the notation of 5.2.9, the surfaces are isomorphic by Remark 5.2.10 since

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -a-1 & 0 & 1 & -1 & -a & 1 \end{bmatrix} \doteqdot \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -a & 0 & 1 & -1 & -a+1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \doteqdot \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 \\ -3 & 0 & 1 & -1 & -2 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \doteqdot \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -3 & 0 & 1 & -1 & -2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -a-1 & 0 & 1 & -1 & -a & -1 \end{bmatrix} \doteqdot \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -a & 0 & 1 & -1 & -a-1 & -1 \end{bmatrix}.$$

We come to isomorphisms between the $\mathbb{K}^*$-surfaces. This comprises the cases Bl $\mathbb{P}_2(\star^3\ ii)$, Bl $\mathbb{P}_2(\star\star\star\ ii)$, Bl $\mathbb{F}_a(\star\star\ v)$ and Bl $\mathbb{F}_a(\star^2\ v)$. Note that all these surfaces are non-toric since their total coordinate spaces have singularities. By Remark 5.2.13, no isomorphisms are possible except between $X_1 :=$ Bl $\mathbb{F}_{a+1}(\star\star\ v)$ and $X_2 :=$ Bl $\mathbb{F}_a(\star^2\ v)$. The degree matrices Q_i of $\mathcal{R}(X_i)$ coincide up to column permutations after applying the matrix $A \in \mathrm{GL}(4,\mathbb{Z})$:

$$\begin{aligned}
A \cdot Q_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -a-1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & a+1 & 2a+1 & -a \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & -a-1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
&= Q_2 \cdot U,
\end{aligned}$$

where $U \in \mathrm{GL}(7,\mathbb{Z})$ is the permutation matrix exchanging the first and fifth column. It is admissible in the sense of Proposition 1.5.8. By Algorithm 5.2.14, X_1 is isomorphic to X_2. □

3. Smooth rational surfaces with $\varrho(X) = 5$

In this section, we classify the smooth rational surfaces of Picard number five up to isomorphism and present their Cox rings explicitly. Each such surface can be obtained as a blow up of one of the smooth rational surfaces of Picard number four listed in Proposition 5.2.8. Whereas the Cox rings of blow ups of $\mathbb{P}_2$ can be calculated in a purely computational manner with Algorithm 4.5.9, for blow ups of $\mathbb{F}_a$, we apply the algorithm in a formal way as explained in Remark 5.2.1.

Theorem 5.3.1. *Let X be a smooth rational surface with Picard number $\varrho(X) = 5$. Then X is isomorphic to exactly one of the following surfaces.*

X	*Cox ring* $\mathcal{R}(X)$	*degree matrix*
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ iv)$ $\lambda \in \mathbb{K}^* \setminus \{1\}$	$\mathbb{K}[T_1,\ldots,T_9]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 - T_6T_7$, $(\lambda-1)T_3^2T_4 - T_8T_9 - \lambda T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ v)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 - T_6T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ vi)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 + T_6T_8^2T_7$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ ix)$	$\mathbb{K}[T_1,\ldots,T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 5 & -3 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xiii)$	$\mathbb{K}[T_1,\ldots,T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xiv)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ *with I generated by* $T_3^3T_4^2T_5 - T_1^2T_2 - T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -3 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xv)$	$\mathbb{K}[T_1,\ldots,T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iii)$	$\mathbb{K}[T_1,\ldots,T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 3 & -1 & -2 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iv)$	$\mathbb{K}[T_1,\ldots,T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix}$

Bl $\mathbb{P}_2(\star^2 \star \star\ i)$	$\mathbb{K}[T_1, \dots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^2 \star \star\ ii)$	$\mathbb{K}[T_1, \dots, T_8]/I$ *with* I *generated by* $T_2T_4T_6^2 - T_1T_5 - T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^2 \star \star\ iv)$	$\mathbb{K}[T_1, \dots, T_8]/I$ *with* I *generated by* $T_2T_4T_6 - T_1T_5 - T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & -2 \end{bmatrix}$
Bl $\mathbb{P}_2(\star \star \star\star\ i)$	$\mathbb{K}[T_1, \dots, T_{10}]/I$ *with* I *generated by* $T_4T_7 - T_5T_8 + T_6T_9$, $T_1T_7 - T_2T_8 + T_3T_9$, $T_3T_5 - T_2T_6 - T_7T_{10}$, $T_3T_4 - T_1T_6 - T_8T_{10}$, $T_2T_4 - T_1T_5 - T_{10}T_9$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star \star \star\star\ ii)$	$\mathbb{K}[T_1, \dots, T_8]/I$ *with* I *generated by* $T_2T_4 - T_1T_5 - T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star \star \star\star\ iii)$ $\lambda \in \mathbb{K}^* \setminus \{1\}$	$\mathbb{K}[T_1, \dots, T_9]/I$ *with* I *generated by* $T_2T_4 - T_1T_5 - T_6T_7$, $T_8T_9 - (\lambda - 1)T_2T_4 + \lambda T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star \star \star\ iii)$ $a \geq 3$	$\mathbb{K}[T_1, \dots, T_8]/I$ *with* I *generated by* $T_7T_8 - T_1T_5 + T_2T_6$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -a+2 & a-1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -a+3 & a-2 \end{bmatrix}$
Bl $\mathbb{F}_a(\star \star \star\ iv)$ $a \geq 3$	$\mathbb{K}[T_1, \dots, T_8]/I$ *with* I *generated by* $T_7T_8 - T_2^aT_4T_6^{a-1} + T_3T_5$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & a-2 & 1 & 2a-4 & -a+3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & a-1 & 1 & 2a-3 & -a+2 \end{bmatrix}$
Bl $\mathbb{F}_a(\star \star \star\ ix)$ $a \geq 3$ $\lambda \in \mathbb{K}^* \setminus \{1\}$	$\mathbb{K}[T_1, \dots, T_9]/I$ *with* I *generated by* $T_6T_7 - T_2^aT_4 + T_3T_5$, $T_8T_9 - (\lambda - 1)T_2^aT_4 + \lambda T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & a & 0 & a & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2 \star\ i)$ $a \geq 3$	$\mathbb{K}[T_1, \dots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & a-5 & -a+3 & -1 \\ 0 & 1 & 0 & 0 & a-4 & -a+3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & a-4 & -a+3 & -1 \\ 0 & 0 & 0 & 0 & a-3 & -a+2 & -1 \end{bmatrix}$

Bl $\mathbb{F}_a(\star^2 \star$ *iv*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_8]/I$ *with* I *generated by* $T_2^a T_4 - T_3 T_5 T_6^2 - T_7 T_8$	$\begin{bmatrix} 1 & 1 & 0 & -a & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2 \star$ *v*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 2a-4 & -a+3 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2a-3 & -a+2 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2 \star$ *viii*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_8]/I$ *with* I *generated by* $T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & a & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *i*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 & 0 & -a+3 & a-2 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *iii*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 2a-3 & -a+2 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *iv*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_8]/I$ *with* I *generated by* $T_1 T_2^{a-1} T_4 - T_3 T_6 - T_7 T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 0 & 0 & 0 & a-1 & 2a-3 & -a+2 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *v*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -a+2 & a-1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *viii*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 & -3a+2 & 2a-1 \\ 0 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *ix*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_8]/I$ *with* I *generated by* $T_1 T_2^{2a-1} T_4^2 - T_3^2 T_5 - T_7 T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 2a-1 & 0 & 3a-2 & -a+1 \\ 0 & 0 & 1 & 0 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^3$ *x*) $a \geq 3$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

In particular, each smooth rational surface of Picard number five either admits a non-trivial $\mathbb{K}^*$*-action or is isomorphic to* $\overline{M}_{0,5}$.

Remark 5.3.2. The $\mathbb{K}^*$-surfaces occurring in Theorem 5.3.1 are all embedded equivariantly into their canonical toric ambient varieties since the relations in the Cox rings are of trinomial shape as in Construction 1.5.3.

Lemma 5.3.3. *In $\mathbb{K}[T_1, \ldots, T_r]$, consider binomials $f_i := c_i^+ T^{\nu_i^+} - c_i^- T^{\nu_i^-}$ with $c_i^\pm \in \mathbb{K}^*$ where $1 \le i \le s$. Let A be the $s \times r$ matrix with rows $\nu_i^+ - \nu_i^-$. Then the dimension of $V(\mathbb{T}^r;\, f_1, \ldots, f_s)$ is $r - \operatorname{rank}(A)$.*

Proof. See [50, Satz 2.1.13]. Compute a Smith normal form $D = U \cdot A \cdot V$ with invertible matrices U, V and denote by φ_A, φ_D, φ_U the corresponding morphisms of tori. Then there is a finite abelian group Γ such that

$$\begin{aligned} V(\mathbb{T}^r;\, f_1, \ldots, f_s) &= \varphi_A^{-1}\left(-\frac{c_1^-}{c_1^+}, \ldots, -\frac{c_s^-}{c_s^+}\right) \\ &\cong \varphi_D^{-1}\left(\varphi_U\left(-\frac{c_1^-}{c_1^+}, \ldots, -\frac{c_s^-}{c_s^+}\right)\right) \\ &\cong \mathbb{T}^{r - \operatorname{rank} A} \times \Gamma. \end{aligned}$$

□

When checking a variable for being prime (e.g., with Algorithm 2.2.10) we often encounter ideals of the form $I = I_0 + I'$ with a binomial ideal I_0. The following observation may then simplify the computation. We have published a similar version in [56, Lem. 4.3].

Lemma 5.3.4 (Binomial trick)**.** *Consider an ideal $I = I_0 + I' \subseteq \mathbb{K}[T_1, \ldots, T_r]$ with a prime binomial ideal*

$$I_0 \;=\; \langle T^{\nu_1^+} - T^{\nu_1^-}, \ldots, T^{\nu_n^+} - T^{\nu_n^-} \rangle \qquad \text{where } \; \nu_i^+,\, \nu_i^- \in \mathbb{Z}_{\ge 0}^r.$$

Let B be the integral $n \times r$ matrix with rows $\nu_i^+ - \nu_i^-$ and A an integral $r \times s$ matrix the columns of which generate $\ker(B)$. We have a homomorphism

$$\psi_A \colon \mathbb{K}[T_1, \ldots, T_r] \;\to\; \mathbb{K}\left[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}\right], \qquad T^\nu \;\mapsto\; Y^{A^t \cdot \nu}.$$

Then $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$ is prime if one of the following conditions is fulfilled.

- (i) *$\langle \psi_A(I') \rangle$ is a prime ideal in the Veronese subalgebra $R \subseteq \mathbb{K}[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$ given by the monoid $S \subseteq \mathbb{Z}^s$ generated by the rows of A.*
- (ii) *$\ker(B) \cap \mathbb{Q}_{>0}^r \neq \emptyset$ and $\langle \psi_A(I') \rangle$ is a prime ideal in $\mathbb{K}[Y_1, \ldots, Y_s]$.*

Proof. Since I_0 is a prime binomial ideal, we have $I_0 = \ker(\psi_A)$. Let $s_1, \ldots, s_r$ be the rows of A. Then the image of ψ_A is the subalgebra

$$R \;=\; \mathbb{K}[Y^{s_1}, \ldots, Y^{s_r}] \;\subseteq\; \mathbb{K}[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}].$$

Thus, the map $\psi_A \colon \mathbb{K}[T_1, \ldots, T_r] \to R$ is surjective. As $I_0 = \ker(\psi_A)$, by basic algebra, we obtain $I = \psi_A^{-1}(\langle \psi_A(I') \rangle)$. The first assertion follows. For (ii), choose $w \in \ker(B) \cap \mathbb{Q}_{>0}^r$. We may replace each generator w' of $\ker(B)$ that does not lie in $\mathbb{Q}_{>0}^r$ with $w'' := bw + w'$ where $b \in \mathbb{Z}$ is large enough such that $w'' \in \ker(B) \cap \mathbb{Q}_{>0}^r$. In the above setting, the image R of ψ_A is contained in $\mathbb{K}[Y_1, \ldots, Y_s]$, i.e., $\langle \psi_A(I') \rangle$ is prime in R. The claim follows from (i). □

In order to classify blow ups of $\mathbb{P}_2$ and $\mathbb{F}_a$, in the proofs of Theorems 5.3.1 and 5.4.1, we have to choose points for the next blow up step. The possible choices of these points frequently is reflected in parameters in their Cox coordinates. With regard to $\mathbb{K}^*$-surfaces, parameters only occur when blowing up points on fixed point curves.

Remark 5.3.5. Let X be a $\mathbb{K}^*$-surface that is embedded equivariantly into its canonical toric ambient variety. Consider points $x, x' \in X$ belonging to exactly one of the divisors $V(X;\, T_i)$. By Proposition 1.5.10, we have

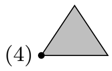
$$x \;=\; \lambda \cdot x' \quad \text{with} \quad \lambda \in \mathbb{K}^* \quad \Leftrightarrow \quad V(X;\, T_i) \;\text{ is not a fixed point curve.}$$

Remark 5.3.6. Computations involving Gröbner bases, e.g., in Algorithm 4.5.9, can also be carried out for equations depending on parameters $\lambda_1, \ldots, \lambda_s \in \mathbb{K}$

by changing the base field to $\mathbb{Q}(\lambda_1, \ldots, \lambda_s)$ and regard the λ_i as transcendental elements, see [46, p. 34].

Proof of Theorem 5.3.1. By Remark 5.2.3, the surface X can be obtained as a blow up of $\mathbb{P}_2$ in four points or as the blow up of $\mathbb{F}_a$ in three points where $a \in \mathbb{Z}_{\geq 2}$. This proof is structured as follows. We compute the resulting Cox rings grouped into originating point configurations; they are listed in Propositions 5.1.2 and 5.1.4. Each surface of Picard number five arises as a blow up of a surface of Picard number four as classified in Proposition 5.2.8. If necessary, we prove or disprove the existence of a $\mathbb{K}^*$-action. Finally, we sort out isomorphic surfaces.

(I) Surfaces of type $\mathrm{Bl}\, \mathbb{P}_2(\star^4)$. By Propositions 5.2.8 and 5.1.2, these are blow ups of the surface $\mathrm{Bl}\, \mathbb{P}_2(\star^3\ i)$ or $\mathrm{Bl}\, \mathbb{P}_2(\star^3\ ii)$ in a point in one of the exceptional divisors.



Recall that (4) stands for a fourfold iterated blow up. We first consider blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii)$. As seen in the proof of Proposition 5.2.8, the exceptional divisors are

$$V(X_1;\, T_4), \qquad V(X_1;\, T_5), \qquad V(X_1;\, T_7).$$

The curve $V(X_1;\, T_5)$ is a parabolic fixed point curve. Consequently, by Remark 5.3.5, it suffices to consider the points

$$\begin{array}{lll@{\qquad}lll}
q_1 & := & [-1,1,0,0,1,1,1], & q_2 & := & [-1,1,1,0,1,1,1], \\
q_3 & := & [-1,1,1,0,0,1,1], & q_4 & := & [1,1,1,\lambda,0,\lambda-1,1], \\
q_5 & := & [1,1,1,1,0,1,0], & q_6 & := & [1,1,1,1,1,1,0], \\
q_7 & := & [1,1,1,1,1,0,0], & q_8 & := & [1,0,1,1,0,1,1] \ \in \ X_1
\end{array}$$

where all points exist by Lemma 5.2.16 or by an application of Algorithm 2.3.8. Observe that we cover all necessary points on $V(X_1;\, T_5)$. Pulling back generators for the irrelevant ideal of X_1 we obtain $\langle T - 1\rangle \subseteq \mathbb{K}[T]$, compare Algorithm 2.3.11. Hence, $q_4 \in \widehat{X}_1$ for each $\lambda \in \mathbb{K} \setminus \{1\}$. Write $p\colon \widehat{X}_1 \to X_1$ for the characteristic space and $\iota\colon \mathbb{K} \setminus \{1\} \to V(T_5)$ for the assignment $\lambda \mapsto (1,1,1,\lambda,0,\lambda-1,1)$. Define $D := V(X_1;\, T_5)$ and $\varphi := p \circ \iota$. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{P}_1 \arrow[drr, "\overline{\varphi}"'] & \supseteq & \mathbb{K} \setminus \{1\} \arrow[r, "\iota"] \arrow[dr, "\varphi"] & V(T_5) \arrow[d, "p"] & \subseteq & \widehat{X}_1 \arrow[d, "p"] \\
 & & \mathbb{P}_1 \arrow[r, "\cong"'] & D & \subseteq & X_1
\end{tikzcd}$$

where φ and the lifted morphism $\overline{\varphi}$ are non-constant. This means $\overline{\varphi}$ is surjective and the image of φ comprises D up to at most two points. Since $q_5, q_8 \in D$ are distinct points not contained in $\mathrm{Im}(\varphi)$ it suffices to consider the listed points q_i.

Using Algorithm 4.5.9, we compute the Cox rings of the resulting surfaces which we call $\mathrm{Bl}\ \mathbb{P}_2(\star^4\ i)$ to $\mathrm{Bl}\ \mathbb{P}_2(\star^4\ viii)$. By Remark 5.3.6, the computation could also be carried out for the case $\mathrm{Bl}\ \mathbb{P}_2(\star^4\ iv)$. All obtained Cox rings are either listed in the table of Theorem 5.3.1 or in the following one; we will show at the end of this proof that the following ones are redundant.

X	Cox ring $\mathcal{R}(X)$	degree matrix
$\mathrm{Bl}\ \mathbb{P}_2(\star^4\ i)$	$\mathbb{K}[T_1, \ldots, T_8]/I$ with I generated by $-T_3^2 T_4 T_8^3 + T_1 T_2 + T_6 T_7$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$

$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ ii)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ with I generated by $T_1T_2+T_5T_6-T_3T_7^2T_8^3$	$\begin{bmatrix} 1&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&3&-1\\ 0&0&1&-2&0&0&-2&1\\ 0&0&0&0&1&0&2&-1\\ 0&0&0&0&0&1&1&-1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ iii)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ with I generated by $-T_3^2T_4T_8+T_1T_2+T_6T_7$	$\begin{bmatrix} 1&0&0&0&0&2&-1&1\\ 0&1&0&0&0&3&-2&1\\ 0&0&1&0&0&-3&3&-2\\ 0&0&0&1&0&-1&1&-1\\ 0&0&0&0&1&1&-1&0 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ vii)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ with I generated by $T_3^2T_4-T_1T_2-T_6T_7T_8^2$	$\begin{bmatrix} 1&0&0&1&0&0&-1&1\\ 0&1&0&1&0&0&-3&2\\ 0&0&1&-2&0&0&2&-1\\ 0&0&0&0&1&0&-2&1\\ 0&0&0&0&0&1&1&-1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ viii)$	$\mathbb{K}[T_1,\ldots,T_8]/I$ with I generated by $T_3^2T_4-T_6T_7-T_1T_2T_8$	$\begin{bmatrix} 1&0&0&0&0&-2&2&-1\\ 0&1&0&0&0&-1&1&-1\\ 0&0&1&0&0&5&-3&2\\ 0&0&0&1&0&3&-2&1\\ 0&0&0&0&1&1&-1&0 \end{bmatrix}$

However, for the surfaces $Y_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^4\ ii)$ and $Y_1' := \mathrm{Bl}\,\mathbb{P}_2(\star^4\ vi)$ Algorithm 4.5.9 returned a ring different from the one listed in the table. We now prove that these rings are isomorphic. For Y_1, Algorithm 4.5.9 returns the Cox ring and degree matrix

$$\mathcal{R}(Y_1) \;=\; \mathbb{K}[T_1,\ldots,T_8] \Big/ \left\langle \begin{array}{c} T_1T_2+T_5T_6-T_8T_3T_2^2T_4^2T_6^2 \\ +2T_3T_2T_4T_6T_7T_8^2-T_3T_7^2T_8^3, \end{array} \right\rangle,$$

$$Q_1 \;=\; \begin{bmatrix} 1&0&0&0&0&1&2&-1\\ 0&1&0&0&0&1&5&-3\\ 0&0&1&0&0&0&1&-1\\ 0&0&0&1&0&0&3&-2\\ 0&0&0&0&1&-1&-3&2 \end{bmatrix}.$$

We claim that Y_1 is isomorphic to the $\mathbb{K}^*$-surface Y_2 with the same degree matrix $Q_2 := Q_1$ and Cox ring

$$\mathcal{R}(Y_2) \;=\; \mathbb{K}[T_1,\ldots,T_8] \,/\, \langle T_1T_2+T_5T_6-T_3T_7^2T_8^3\rangle.$$

By Lemma 5.1.5, it suffices to show that the Cox rings $\mathcal{R}(Y_2)$ and $\mathcal{R}(Y_1)$ are isomorphic as graded algebras. We choose the homomorphism $\psi\colon \mathcal{R}(Y_2)\to\mathcal{R}(Y_1)$ induced by the homomorphism $\mathbb{K}[T_1,\ldots,T_8]\to\mathbb{K}[T_1,\ldots,T_8]$ with

$$T_i \;\mapsto\; \begin{cases} T_5-T_2^2T_3T_4^2T_6T_8+2T_2T_3T_4T_7T_8^2, & i=5,\\ T_i, & i\neq 5. \end{cases}$$

One directly verifies that (ψ,id) is an isomorphism of $\mathbb{Z}^5$-graded algebras. Hence, Y_1 is isomorphic to the $\mathbb{K}^*$-surface Y_2. Similarly, for $Y_1' = \mathrm{Bl}\,\mathbb{P}_2(\star^4\ vi)$, Algorithm 4.5.9 delivers its Cox ring and degree matrix

$$\mathcal{R}(Y_1') \;=\; \mathbb{K}[T_1,\ldots,T_8] \,/\, \langle T_3^2T_4-T_1T_2-T_6T_8T_2T_3T_4T_5+T_6T_8^2T_7\rangle,$$

$$Q_1' \;=\; \begin{bmatrix} 1&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&3&-1\\ 0&0&1&-2&0&0&-2&1\\ 0&0&0&0&1&0&2&-1\\ 0&0&0&0&0&1&1&-1 \end{bmatrix}.$$

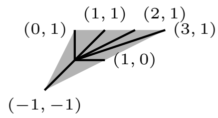
We have a $\mathbb{Z}^5$-graded isomorphism (ψ',id) between the Cox ring $\mathcal{R}(Y_2')$ of a $\mathbb{K}^*$-surface Y_2' and $\mathcal{R}(Y_1')$ where

$$\mathcal{R}(Y_2') \;:=\; \mathbb{K}[T_1,\ldots,T_8] \,/\, \langle T_3^2T_4-T_1T_2+T_6T_8^2T_7\rangle$$

and the degree matrix is again Q_1'. The graded homomorphism ψ' is induced by the homomorphism $\mathbb{K}[T_1, \ldots, T_8] \to \mathbb{K}[T_1, \ldots, T_8]$ assigning $T_1 + T_3T_4T_5T_6T_8$ to T_1 and $T_i \mapsto T_i$ otherwise. Thus, Y_1' is isomorphic to Y_2'.

We come to the blow up of the toric variety Bl $\mathbb{P}_2(\star^3\ i)$ in a point in the union of the exceptional divisors. It is given by the toric orbits through the points

$$
\begin{aligned}
q_1 &:= [1,1,0,0,1,1], & q_2 &:= [1,1,1,0,1,1],\\
q_3 &:= [1,1,1,0,0,1], & q_4 &:= [1,1,1,1,0,1],\\
q_5 &:= [1,1,1,1,0,0], & q_6 &:= [1,1,1,1,1,0],\\
q_7 &:= [1,0,1,1,1,0].
\end{aligned}
$$



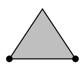
All points exist by Lemma 5.2.16 or by Algorithm 2.3.8. The fan of Bl $\mathbb{P}_2(\star^3\ i)$ is drawn on the right. Observe that the equivariant automorphism

$$\widehat{Z}_1 \to \widehat{Z}_1, \qquad z \mapsto (z_1,\, z_2,\, z_3 - z_2z_5z_6^2,\, z_4,\, z_5,\, z_6)$$

maps q_2 to q_1; compare Lemma 5.1.5. Using Algorithm 4.5.9, we compute the Cox rings of the remaining surfaces. We denote the results by Bl $\mathbb{P}_2(\star^4\ ix)$ and Bl $\mathbb{P}_2(\star^4\ xi)$ to Bl $\mathbb{P}_2(\star^4\ xv)$. The results can be found in the table of Theorem 5.3.1 or in the following one.

X	Cox ring $\mathcal{R}(X)$	degree matrix
Bl $\mathbb{P}_2(\star^4\ xi)$	$\mathbb{K}[T_1, \ldots, T_7]$	$\begin{bmatrix} 1&0&0&0&2&-1&0\\ 0&1&0&0&1&-1&0\\ 0&0&1&0&0&1&-1\\ 0&0&0&1&1&0&-1\\ 0&0&0&0&3&-1&-1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ xii)$	$\mathbb{K}[T_1, \ldots, T_8]/I$ with I generated by $T_3^2T_4 - T_1T_2T_6 - T_7T_8$	$\begin{bmatrix} 1&0&0&0&0&-1&-2&2\\ 0&1&0&0&0&-1&-1&1\\ 0&0&1&0&0&2&5&-3\\ 0&0&0&1&0&1&3&-2\\ 0&0&0&0&1&0&1&-1 \end{bmatrix}$

(II) Blow ups of type Bl $\mathbb{P}_2(\star^3\star)$. By Propositions 5.2.8 and 5.1.2, the surfaces of type Bl $\mathbb{P}_2(\star^3\star)$ are blow ups of Bl $\mathbb{P}_2(\star^3\ ii)$ or of Bl $\mathbb{P}_2(\star^3\ i)$ in a point that maps to $[0,1,0] \in \mathbb{P}_2$ under the first three blow ups.

(3) 

We first treat blow ups of $X_1 :=$ Bl $\mathbb{P}_2(\star^3\ ii)$. Recall from the proof of Proposition 5.2.8 the blow up sequence

$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{P}_2(\star^2)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star^2) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{P}_2(\star) \xrightarrow{\pi_1} \mathbb{P}_2$$

where, as in Setting 4.2.9, the embedding $\bar{\iota}_1 \colon \mathbb{K}^5 \to \mathbb{K}^6$ is given by $z \mapsto (z, h_1(z))$ with the polynomial $h_1 := T_3^2T_4 - T_1T_2$ in $\mathbb{K}[T_1, \ldots, T_5]$. The blow ups are

$$
\begin{aligned}
\pi_3([z]) &= [z_1, \ldots, z_4, z_5z_7, z_6z_7],\\
\pi_2([z]) = [z_1, z_2z_5, z_3, z_4z_5], &\qquad \pi_1([z]) = [z_1, z_2z_4, z_3z_4].
\end{aligned}
$$

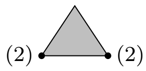
Using Algorithm 2.3.8, we see that the following point p exists on X_1 and its blow up yields the desired surface:

$$\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3(p) = [0,1,0] \in \mathbb{P}_2, \qquad p := [0,1,0,1,1,0,1] \in X_1.$$

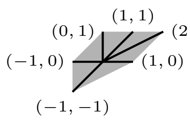
In a similar manner, for the toric variety $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^3\ i)$ the point $[0,1,0,1,1,1] \in X_1$ exists and projects to $[0,1,0] \in \mathbb{P}_2$. In each of the two cases we use Algorithm 4.5.9 to blow up X_1 in p. We obtain surfaces $\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\ i)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\ ii)$ with the following Cox rings.

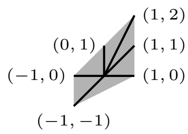
X	Cox ring $\mathcal{R}(X)$	degree matrix
$\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\ i)$	$\mathbb{K}[T_1,\dots,T_8]/I$ with I generated by $T_3^2T_4T_8 - T_1T_2 - T_6T_7$	$\begin{bmatrix} 1&0&0&0&0&0&1&1\\ 0&1&0&0&0&1&0&1\\ 0&0&1&0&0&1&-1&-2\\ 0&0&0&1&0&1&-1&-1\\ 0&0&0&0&1&1&-1&0 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\ ii)$	$\mathbb{K}[T_1,\dots,T_7]$	$\begin{bmatrix} 1&0&0&0&0&1&2\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&-1&-3\\ 0&0&0&1&0&-1&-2\\ 0&0&0&0&1&-1&-1 \end{bmatrix}$

(III) Surfaces of type $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2)$. These are blow ups of $Z_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\ i)$ or $Z_2 := \mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\ ii)$ in a point in the second exceptional divisor, i.e.



The toric varieties Z_i have the following fans and ray generators





$$\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & \multirow{2}{*}{v} \\ -1 & 0 & 1 & 1 & 0 & \end{bmatrix}$$

with $v = (2,1)$ or $v = (1,2)$ respectively. On both Z_1 and Z_2 the second exceptional divisor is $V(Z_i; T_5)$ and consists of the toric orbits through the points

$$q_1 \ := \ [0,1,1,1,0,1], \qquad q_2 \ := \ [1,1,1,1,0,1], \qquad q_3 \ := \ [1,1,0,1,0,1].$$

For both i, the point $q_2 \in Z_i$ is mapped to q_1 by the respective equivariant automorphism

$$\begin{aligned} \widehat{Z}_1 \ &\to\ \widehat{Z}_1, \qquad & z \ &\mapsto\ (z_1 - z_3z_4z_6,\ z_2,\ \dots,\ z_6), \\ \widehat{Z}_2 \ &\to\ \widehat{Z}_2, \qquad & z \ &\mapsto\ (z_1 - z_3z_4z_6^2,\ z_2,\ \dots,\ z_6), \end{aligned}$$

compare Lemma 5.1.5. Denote by $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ i)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ ii)$ the surfaces obtained as blow up of Z_1 in q_1 and q_3. The blow ups of Z_2 in q_1 and q_3 are called $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iii)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iv)$. The table of the theorem lists the Cox rings of the latter two whereas the ones of the former two are as follows.

X	Cox ring $\mathcal{R}(X)$	degree matrix
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ i)$	$\mathbb{K}[T_1,\dots,T_7]$	$\begin{bmatrix} 1&0&0&0&1&0&-1\\ 0&1&0&0&1&0&0\\ 0&0&1&0&-2&0&1\\ 0&0&0&1&-1&0&1\\ 0&0&0&0&0&1&1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ ii)$	$\mathbb{K}[T_1,\dots,T_7]$	$\begin{bmatrix} 1&0&0&0&1&1&0\\ 0&1&0&0&1&0&0\\ 0&0&1&0&1&0&-1\\ 0&0&0&1&2&0&-1\\ 0&0&0&0&3&1&-1 \end{bmatrix}$

(IV) Surfaces of type $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star)$. Let $Z_1 = \mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\ i)$ and $Z_2 = \mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\ ii)$ be as in the previous paragraph (III). By Proposition 5.1.2, we have to consider the configurations

By Proposition 5.2.8, this means that the surfaces of type $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ )$ are blow ups of Z_1 or Z_2 in a point that maps to $[0,0,1] \in \mathbb{P}_2$ or to $[1,1,0] \in \mathbb{P}_2$ under the maps of the first three blow ups. Recall from the proof of Proposition 5.2.8 that the blow up sequence of both Z_i is

$$Z_i \xrightarrow{\pi_{i,3}} \mathrm{Bl}\,\mathbb{P}_2(\star\star) \xrightarrow{\pi_2} \mathrm{Bl}\,\mathbb{P}_2(\star) \xrightarrow{\pi_1} \mathbb{P}_2$$

where the blow ups π_1, π_2 and $\pi_{i,3}$ are

$$\begin{aligned} \pi_{1,3}([z]) &= [z_1, z_2z_6, z_3, z_4z_6, z_5], & \pi_{2,3}([z]) &= [z_1, z_2, z_3z_6, z_4z_6, z_5], \\ \pi_2([z]) &= [z_1z_5, z_2, z_3z_5, z_4], & \pi_1([z]) &= [z_1, z_2z_4, z_3z_4]. \end{aligned}$$

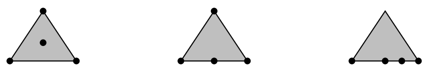
In our case, the points $p := [0,0,1,1,1,1] \in Z_i$ and $p' := [1,1,0,1,1,1] \in Z_i$ are directly seen to exist and satisfy

$$\pi_1 \circ \pi_2 \circ \pi_{i,3}(p) \ = \ [0,0,1] \ \in \ \mathbb{P}_2, \qquad \pi_1 \circ \pi_2 \circ \pi_{i,3}(p') \ = \ [1,1,0] \ \in \ \mathbb{P}_2.$$

The blow ups of Z_1 in p and p' are called $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ i)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ ii)$ whereas $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ iii)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ iv)$ are the blow ups of Z_2 in p and p'. The results are listed in the table except for the Cox ring and degree matrix

$$\mathcal{R}\left(\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\ iii)\right) \ = \ \mathbb{K}[T_1, \dots, T_7], \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

(V) Surfaces of type $\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star)$. Proposition 5.1.2 lists the three point configurations we have to consider, i.e.,



The blow ups of the first two configurations can be obtained as a blow up of $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\ i)$. As seen in the proof of Proposition 5.2.8, X_1 was obtained as blow up

$$\pi_{3,2,1}\colon X_1 \ \to \ \mathbb{P}_2, \qquad [z] \ \mapsto \ [z_1z_5z_6,\ z_2z_4z_6,\ z_3z_4z_5].$$

Under $\pi_{3,2,1}$, the points $[1,\dots,1]$ and $[1,1,0,1,1,1] \in X_1$ project to $[1,1,1] \in \mathbb{P}_2$ and $[1,1,0] \in \mathbb{P}_2$ respectively. Note that the second point exists by Lemma 5.2.16. The blow ups of X_1 in these points with Algorithm 4.5.9 yield surfaces $\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\ i)$ and $\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\ ii)$. Their Cox rings can be found in the table.

For the third configuration, we want to blow up $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\ ii)$ in a point projecting to $[1,\lambda,0] \in \mathbb{P}_2$, where $\lambda \in \mathbb{K}^* \setminus \{1\}$, under the blow ups

$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\,\mathbb{P}_2(\star\star\ \ i)' \xleftarrow{\iota_1} \mathrm{Bl}\,\mathbb{P}_2(\star\star\ \ i) \xrightarrow{\pi_{2,1}} \mathbb{P}_2$$

$$\pi_3([z]) \ = \ [z_1, z_2, z_3z_7, z_4, z_5, z_6z_7], \qquad \pi_{2,1}([z]) \ = \ [z_1z_5, z_2z_4, z_3z_4z_5]$$

and the embedding ι_1 is given by $[z] \mapsto [z, h_1(z)]$ with the polynomial $h_1 := T_2T_4 - T_1T_5$ in $\mathbb{K}[T_1,\dots,T_5]$. We have

$$\pi_1 \circ \iota_1^{-1} \circ \pi_2(q) \ = \ [1,\lambda,0] \qquad q \ := \ [1,\lambda,0,1,1,\lambda-1,1] \ \in \ X_1$$

and q exists in X_1 by Lemma 5.2.16. Using Algorithm 4.5.9 with Remark 5.3.6, we obtain the listed surface $\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\ iii)$.

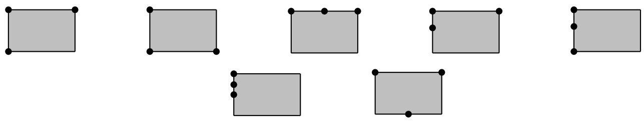
(VI) Surfaces of type Bl $\mathbb{F}_a(\star\star\star)$. We now treat the blow ups of the Hirzebruch surface $\mathbb{F}_a$. According to Proposition 5.2.8, each blow up of type Bl $\mathbb{F}_a(\star\star\star)$ is obtained by blowing up one of the surfaces

$$\mathrm{Bl}\,\mathbb{F}_a(\star\star\ i), \qquad \mathrm{Bl}\,\mathbb{F}_a(\star\star\ ii), \qquad \mathrm{Bl}\,\mathbb{F}_a(\star\star\ v)$$

in a point that is not contained in the union of the two exceptional divisors. This rules out the configurations

$$\mathbb{F}_a(\star\star\star\, vii), \quad \mathbb{F}_a(\star\star\star\, viii), \quad \mathbb{F}_a(\star\star\star\, xi), \quad \mathbb{F}_a(\star\star\star\, xii)$$

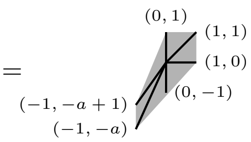
found in Proposition 5.1.4. Additionally, observe that we also need not consider $\mathbb{F}_a(\star\star\star\ vi)$. The blow up of this configuration is isomorphic to the blow up of Bl $\mathbb{F}_a(\star\star\ iii)$ in a point projecting to $[0,1,1,1]$. As seen in the proof of Proposition 5.2.8, there is an isomorphism $\varphi\colon Z_1 \to Z_2$ from the toric variety $Z_1 := \mathrm{Bl}\ \mathbb{F}_{a+1}(\star\star\ i)$ to the toric variety $Z_2 := \mathrm{Bl}\ \mathbb{F}_a(\star\star\ iii)$. In terms of fans, φ is given by an invertible matrix sending the ray $\mathbb{Q}_{\geq 0}\cdot(-1,-a-1)$ corresponding to $V(Z_1;T_1)$ to the ray $\mathbb{Q}_{\geq 0}\cdot(-1,-a)$ corresponding to $V(Z_2;T_1)$. In particular, Bl $\mathbb{F}_a(\star\star\star\, vi)$ is isomorphic to the blow up of $\mathbb{F}_a(\star\star\star\, i)$ in $[0,1,1,1]$, i.e., to Bl $\mathbb{F}_a(\star\star\star\, iv)$. The remaining cases are



There are further reductions: as seen in the proof of Proposition 5.2.8, Bl $\mathbb{F}_a(\star\star\ ii)$ is isomorphic to the surface Bl $\mathbb{F}_{a+1}(\star^2\star\ i)$. Therefore, the surfaces Bl $\mathbb{F}_a(\star\star\star\, i)$, Bl $\mathbb{F}_a(\star\star\star\ ii)$ and Bl $\mathbb{F}_a(\star\star\star\ v)$ are redundant and do not appear in the table of the theorem.

We first consider blow ups of the toric variety $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\star\ i)$. This includes Bl $\mathbb{F}_a(\star\star\star\, iii)$, Bl $\mathbb{F}_a(\star\star\star\, iv)$ and Bl $\mathbb{F}_a(\star\star\star\, x)$. The rays of the fan Σ_1 of X_1 are generated by the columns of

$$P_1 := \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ -a & 0 & 1 & -1 & -a+1 & 1 \end{bmatrix}, \qquad \Sigma_1 =$$



For Bl $\mathbb{F}_a(\star\star\star\, iii)$, we want to blow up X_1 at $q := [1,1,0,1,1,1]$, a point which exists by Lemma 5.2.16. The steps are the same as in, e.g., the proof of Proposition 5.2.8 for case Bl $\mathbb{P}_2(\star^3\ ii)$. In Setting 4.2.9, choose the embedding

$$\bar{\imath}\colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_1T_5 - T_2T_6 \in \mathbb{K}[T_1,\ldots,T_6].$$

As in Algorithm 4.3.2 we obtain a new CEMDS X_1' with degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of the toric ambient variety Z_1':

$$Q_1' = \begin{bmatrix} 1 & 0 & 0 & a-3 & -2 & -1 & -1 \\ 0 & 1 & 0 & a-3 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-2 & -1 & -1 & -1 \end{bmatrix}, \quad P_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 & -a \end{bmatrix}.$$

For the blow up of $\iota(q) = [1,1,0,1,1,1,0] \in X_1'$ we work in Setting 4.2.5. We perform the stellar subdivision $\Sigma_2 \to \Sigma_1'$ of Σ_1' at the sum $v := (-1,-1,-a+1)$ of the third and seventh column of P_1'; this determines the toric modification $\pi\colon Z_2 \to Z_1'$. Set $P_2 := [P_1', v]$. The vanishing ideal I_2 of $\overline{X}_2$ is generated by

$$g := p_2^\star(p_1)_\star(T_7 - h_1) = T_7T_8 - T_1T_5 + T_2T_6 \in \mathbb{K}[T_1,\ldots,T_8].$$

By Lemma 5.2.17, g is prime, all variables define prime elements and one directly verifies that $\dim V(\mathbb{K}^8; g, T_i, T_j) = 5$ for all $i \neq j$. Hence, by Theorem 4.2.6, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1,\ldots,T_8]/\langle g\rangle$. The degree matrix listed in the table is obtained as a Gale dual matrix of P_2 using, e.g., Lemma 5.2.6. The ideal $\langle T_3, T_7, h_1\rangle$ of $\mathbb{K}[T_1, \dots, T_7]$ is prime by Lemma 5.2.17, its zero set contains $\bar\imath((1, 1, 0, 1, 1, 1))$ and it is four-dimensional. Therefore, the modification was the desired blow up by Lemma 5.2.15.

The case Bl $\mathbb{F}_a(\star\star\star\ iv)$ is similar to the previous one. Here, we want to blow up X_1 in $q := [0, 1, 1, 1, 1, 1]$. By Lemma 5.2.16, $q \in X_1$ exists. Choose $h_1 := T_2^a T_4 T_6^{a-1} - T_3T_5 \in \mathbb{K}[T_1, \dots, T_6]$ for the embedding $\bar\imath\colon \mathbb{K}^6 \to \mathbb{K}^7$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \begin{bmatrix} 1 & 0 & 0 & a-3 & -2 & -1 & -2 \\ 0 & 1 & 0 & a-3 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & a-2 & -1 & -1 & -1 \end{bmatrix}, \quad P_1' = \begin{bmatrix} 1 & a-1 & 0 & 1 & 1 & a-2 & -1 \\ 0 & a & 0 & 1 & 0 & a-1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

For the blow up of X_1' in $\imath(q) = [0, 1, 1, 1, 1, 1, 0]$ we perform the stellar subdivision of Σ_1' at the sum $v := (0, -1, -1)$ of the first and seventh column of P_1'. Set $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_8]$ of $\overline{X}_2$ is generated by

$$p_2^\star (p_1)_\star (T_7 - h_1) \ = \ T_7T_8 - T_2^a T_4 T_6^{a-1} + T_3T_5 \ \in \ \mathbb{K}[T_1, \dots, T_8].$$

Using Lemma 5.2.17, similar to before, the requirements for Theorem 4.2.6 are fulfilled. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1, \dots, T_8]/I_2$. Its degree matrix is a a Gale dual matrix of P_2. By Lemma 5.2.15, we did perform a blow up since $\langle T_1, T_7, h_1\rangle \subseteq \mathbb{K}[T_1, \dots, T_7]$ is prime by Lemma 5.2.17 and its vanishing set is of dimension four while containing $\bar\imath((0, 1, 1, 1, 1, 1))$.

We proceed for Bl $\mathbb{F}_a(\star\star\star\ x)$ in the same manner. Here, we want to blow up X_1 in $q := [1, 1, 1, 0, 1, 1]$ which exists by Lemma 5.2.16. Choose $h_1 := T_1T_5 - T_2T_6 \in \mathbb{K}[T_1, \dots, T_6]$ for the embedding $\bar\imath\colon \mathbb{K}^6 \to \mathbb{K}^7$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \begin{bmatrix} 1 & 0 & 0 & a-3 & -2 & -1 & -1 \\ 0 & 1 & 0 & a-3 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-2 & -1 & -1 & -1 \end{bmatrix}, \quad P_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 & -a \end{bmatrix}.$$

For the blow up of $\imath(q) = [1, 1, 1, 0, 1, 1, 0] \in X_1'$, we insert the ray through $v := (-1, -1, -a-1)$ into Σ_1' by performing the stellar subdivision at v. Set $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_8]$ of $\overline{X}_2$ is generated by

$$p_2^\star (p_1)_\star (T_7 - h_1) \ = \ T_7T_8 - T_1T_5 + T_2T_6 \ \in \ \mathbb{K}[T_1, \dots, T_8].$$

Using Lemma 5.2.17, the requirements for Theorem 4.2.6 are fulfilled. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1, \dots, T_8]/I_2$. Its degree matrix is as listed in the table. By Lemma 5.2.15, the modification was a blow up as $\langle T_4, T_7, h_1\rangle \subseteq \mathbb{K}[T_1, \dots, T_7]$ is prime by Lemma 5.2.17, its zero set contains $\bar\imath((1, 1, 1, 0, 1, 1))$ and is four-dimensional.

We now consider blow ups of the variety $X_1 :=$ Bl $\mathbb{F}_a(\star\star\ v)$. The surface Bl $\mathbb{F}_a(\star\star\star\ ix)$ is a blow up of X_1 in a point q projecting to $[0, 1, 1, \lambda] \in \mathbb{F}_a$ under the previous blow ups where $\lambda \in \mathbb{K}^* \setminus \{1\}$. By the proof of Proposition 5.2.8, the blow up sequence is

$$X_1 \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i)' \xleftarrow{\imath_1} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where $\imath_1([z]) := [z, h_1(z)]$ with $h_1 := T_2^a T_4 - T_3T_5$ and the blow ups π_1 and π_2 are given by

$$\pi_2([z]) \ = \ [z_1z_7, z_2, z_3, z_4, z_5, z_6z_7], \qquad \pi_1([z]) \ = \ [z_1z_5, z_2, z_3z_5, z_4].$$

In our case, the point $q := [0, 1, 1, \lambda, 1, \lambda - 1, 1] \in X_1$ exists by Lemma 5.2.16 and satisfies $\pi_1 \circ \imath_1^{-1} \circ \pi_2(q) = [0, 1, 1, \lambda]$. We now perform the same steps as in the previous cases. In $\mathbb{K}[T_1, \dots, T_7]$, choose $h_2 := (\lambda - 1)T_2^a T_4 - \lambda T_6T_7$ for the embedding $\bar\imath\colon \mathbb{K}^7 \to \mathbb{K}^8$. Let Q_1 be the degree matrix of X_1. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ a \\ 0 \\ 1 \end{smallmatrix} \end{array}\right], \qquad P_1' \;=\; \left[\begin{smallmatrix} 1 & a-1 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{smallmatrix}\right].$$

For the blow up of $\iota(q) = [0,1,1,\lambda,1,\lambda-1,1,0] \in X_1'$, we consider the toric morphism corresponding to the stellar subdivision of Σ_1' at the ray through $v := (0,-1,-1,-1)$. Set $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\ldots,T_9]$ of $\overline{X}_2$ is generated by the modified equations defining $\overline{X}_1'$, i.e.,

$$\begin{aligned} p_2^\star\,(p_1)_\star\,(T_6T_7 - T_2^aT_4 + T_3T_5) \;&=\; T_6T_7 - T_2^aT_4 + T_3T_5, \\ p_2^\star\,(p_1)_\star\,(T_8 - h_2) \;&=\; T_8T_9 - (\lambda-1)T_2^aT_4 + \lambda T_6T_7. \end{aligned}$$

After scaling, e.g., T_8 by a suitable element of $\mathbb{K}^*$, Remark 1.5.6 applies and, therefore, I_2 is prime. In particular, $I_2 = I_2 : T_9^\infty$. We now show that the variable T_9 defines a prime element in $R_2 = \mathbb{K}[T_1,\ldots,T_9]/I_2$. Note that, by 1.5.3, for suitable choices of P and A we have $R_2 = R(P,A)$, i.e., R_2 is the Cox ring of a $\mathbb{K}^*$-surface and T_9 defines a prime element. Alternatively, we may show that the image of the ideal $I_2 + \langle T_9\rangle$ under the isomorphism $T_6 \mapsto T_6(\lambda-1)/\lambda$, namely

$$I \;:=\; \langle T_9,\,(\lambda-1)T_6T_7 - \lambda T_2^aT_4 + \lambda T_3T_5,\,-T_2^aT_4 + T_6T_7\rangle \;\subseteq\; \mathbb{K}[T_1,\ldots,T_9]$$

is a prime ideal using the binomial trick 5.3.4. This means we consider the ideal generated by the image $\psi_A(I)$ under the homomorphism

$$\begin{aligned} \psi_A\colon \mathbb{K}[T_1,\ldots,T_9] \;&\to\; \mathbb{K}[Y_1,\ldots,Y_9] \\ T^\nu \;&\mapsto\; Y^{A^t\cdot\nu}, \end{aligned} \qquad A \;:=\; \left[\begin{smallmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&a&1&0&0 \\ 0&0&0&a&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{smallmatrix}\right].$$

Then $\langle\psi_A(I)\rangle \subseteq \mathbb{K}[Y_1,\ldots,Y_9]$ is generated by Y_9 and $-Y_4^aY_6^aY_5Y_7 + \lambda Y_2Y_3$ and is a prime ideal by Lemma 5.2.17. In turn, since $-T_2^aT_4 + T_6T_7$ is prime by Lemma 5.2.17, Lemma 5.3.4 tells us that T_9 is a prime element in R_2. Furthermore, each two variables T_i, T_j are pairwise non-associated since $\deg(T_i) \neq \deg(T_j)$ for $i \neq j$. Also, $T_9 \nmid T_i$ for all $i < 9$, since each of the following intersections is of codimension two in $\overline{X}_2$:

$$\begin{aligned} \overline{X}_2 \cap V(T_1,T_9) \;&=\; V(T_1,\;T_9,\;T_6T_7 - T_2^aT_4 + T_3T_5,\;(\lambda-1)T_2^aT_4 - \lambda T_6T_7) \\ &=\; V(T_1,\;T_9,\;T_2^aT_4 - \lambda T_3T_5,\;(\lambda-1)T_2^aT_4 - \lambda T_6T_7), \\ \overline{X}_2 \cap V(T_2,T_9) \;&=\; V(T_2,\;T_9,\;T_3T_5,\;T_6T_7), \\ \overline{X}_2 \cap V(T_3,T_9) \;&=\; V(T_3,\;T_9,\;T_6T_7,\;T_2T_4), \\ \overline{X}_2 \cap V(T_4,T_9) \;&=\; V(T_4,\;T_9,\;T_3T_5,\;T_6T_7), \\ \overline{X}_2 \cap V(T_5,T_9) \;&=\; V(T_5,\;T_9,\;T_6T_7,\;T_2T_4), \\ \overline{X}_2 \cap V(T_6,T_9) \;&=\; V(T_6,\;T_9,\;T_3T_5,\;T_2T_4), \\ \overline{X}_2 \cap V(T_7,T_9) \;&=\; V(T_7,\;T_9,\;T_3T_5,\;T_2T_4), \\ \overline{X}_2 \cap V(T_8,T_9) \;&=\; V(T_8,\;T_9,\;T_6T_7 - T_2^aT_4 + T_3T_5,\;(\lambda-1)T_2^aT_4 - \lambda T_6T_7) \\ &=\; V(T_8,\;T_9,\;T_2^aT_4 - \lambda T_3T_5,\;(\lambda-1)T_2^aT_4 - \lambda T_6T_7). \end{aligned}$$

This can be seen directly or, for $i \in \{1,8\}$, we write the exponent vectors of the binomial generators into the rows of a matrix as in Lemma 5.3.3 and obtain

$$\left[\begin{smallmatrix} 0 & a & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{smallmatrix}\right]$$

which is of rank two. Hence, by Lemma 5.3.3, the dimension of $\overline{X}_2 \cap V(T_i, T_9)$ is five on $\mathbb{T}^9 \cdot (0, 1, \ldots, 1, 0)$ or $\mathbb{T}^9 \cdot (1, \ldots, 1, 0, 0)$ for $i = 1, 8$ respectively. One directly checks that on the smaller tori the dimension is also at most six. For instance setting the second component to zero, we obtain the variety $V(T_i,\, T_9,\, T_2,\, T_3T_5,\, T_6T_7)$ in $\mathbb{K}^9$ of dimension four. By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1, \ldots, T_9]/I_2$ is the Cox ring of the performed modification with its degree matrix as listed in the table.

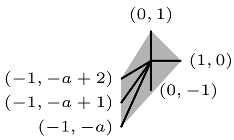
We now show that the modification was the desired blow up. Note that the factor ring $\mathbb{K}[T_1, \ldots, T_8]/I'$ with $I' := \langle T_1,\, T_8,\, g_1,\, g_2\rangle$ is isomorphic to the integral domain $\mathbb{K}[T_1, \ldots, T_9]/(I + \langle T_9\rangle)$. Hence, I' is prime. Since $(0, 1, 1, \lambda, 1, \lambda - 1, 1, 0) \in V(I')$, by Lemma 5.2.15, the performed modification was the claimed blow up using

$$\dim\left(V(\mathbb{K}^8;\, I')\right) \;=\; -1 \,+\, \dim\left(\overline{X}_2 \cap V\left(\mathbb{K}^9;\, T_8,\, T_9\right)\right) \;=\; 4.$$

(VII) Surfaces of type Bl $\mathbb{F}_a(\star^2\star)$. These are blow ups of a surface of type Bl $\mathbb{F}_a(\star^2)$ in a point not belonging to one of the two exceptional divisors. By Proposition 5.2.8, the only such surfaces of Picard number four are Bl $\mathbb{F}_a(\star^2\ i)$ and Bl $\mathbb{F}_a(\star^2\ ii)$. As the contraction of the exceptional divisors must lead to a configuration listed in Proposition 5.1.4, we only have to take the configurations

into account where (2) stands for an iterated blow up. Let $X_1 :=$ Bl $\mathbb{F}_a(\star^2\ i)$. As a toric variety, the fan Σ_1 of X_1 has its rays generated by the columns of

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 \end{bmatrix}, \qquad \Sigma_1 \;=$$



and, by the proof of Proposition 5.2.8, the blow ups π_i are

$$X_1 \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

$$\pi_2([z]) \;=\; [z_1, z_2, z_3z_6, z_4, z_5z_6], \qquad \pi_1([z]) \;=\; [z_1z_5, z_2, z_3z_5, z_4].$$

The surfaces Bl $\mathbb{F}_a(\star^2\star\ \ i)$ to Bl $\mathbb{F}_a(\star^2\star\ \ iv)$ are obtained as blow ups of X_1 in points $q_i \in X_1$ such that

$$\begin{aligned} \pi_1 \circ \pi_2(q_1) \;&=\; [1, 0, 0, 1], \qquad & \pi_1 \circ \pi_2(q_2) \;&=\; [0, 1, 1, 0], \\ \pi_1 \circ \pi_2(q_3) \;&=\; [1, 0, 1, 0], \qquad & \pi_1 \circ \pi_2(q_4) \;&=\; [0, 1, 1, 1] \;\in\; \mathbb{F}_a. \end{aligned}$$

We may choose the following ones. Note that their existence can be seen by an inspection of Σ_1 or by Lemma 5.2.16

$$\begin{aligned} q_1 \;&:=\; [1, 0, 0, 1, 1, 1], \qquad & q_2 \;&:=\; [0, 1, 1, 0, 1, 1], \\ q_3 \;&:=\; [1, 0, 1, 0, 1, 1], \qquad & q_4 \;&:=\; [0, 1, 1, 1, 1, 1] \;\in\; X_1. \end{aligned}$$

Therefore, the first three blow ups are toric and hence are performed by the stellar subdivision $\Sigma_2 \to \Sigma_1$ at $v \in \mathbb{Z}^2$ where the respective vectors are

$$v \;=\; (1, 1), \qquad v \;=\; (-1, -a-1), \qquad v \;=\; (1, -1).$$

We come to the blowup Bl $\mathbb{F}_a(\star^2\star\ \ iv)$ of X_1 in q_4. The steps are as before. Choose in $\mathbb{K}[T_1, \ldots, T_6]$ the polynomial $h_1 := T_2^aT_4 - T_3T_5T_6^2$ for the embedding $\bar{\imath}\colon \mathbb{K}^6 \to \mathbb{K}^7$. We obtain a new CEMDS X_1' with degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -a+2 & a-1 & a \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}, \qquad P_1' = \begin{bmatrix} 1 & a-1 & 0 & 1 & 1 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 2 & -1 \end{bmatrix}.$$

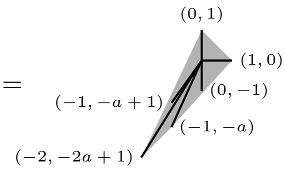
For the blow up of X_1' at $\iota(q_4) = [0,1,1,1,1,1,0]$, we perform the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (0,-1,-1)$. Set $P_2 := [P_1', v]$. The ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_8]$ of $\overline{X}_2$ is generated by

$$p_2^\star\,(p_1)_\star\,(T_7 - h_1) \;=\; T_7T_8 - T_2^aT_4 + T_3T_5T_6^2 \;\in\; \mathbb{K}[T_1, \dots, T_8].$$

As before, one directly checks that the requirements for Theorem 4.2.6 are fulfilled. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1, \dots, T_8]/I_2$. Its degree matrix is as listed in the table. By Lemma 5.2.15, we did perform the claimed blow up since $\langle T_1, T_7, h_1 \rangle \subseteq \mathbb{K}[T_1, \dots, T_7]$ is prime by Lemma 5.2.17 and its zero set contains $\bar\iota((0,1,1,1,1,1))$ while being four-dimensional.

We now treat the blow ups of $X_1 := \mathrm{Bl}\,\mathbb{F}_a(\star^2\ ii)$. As a toric variety, the fan Σ_1 of X_1 has its rays generated by the columns of

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 \end{bmatrix}, \qquad \Sigma_1 \;=$$



and, by the proof of Proposition 5.2.8, the previous blow ups π_i are

$$X_1 \xrightarrow{\pi_2} \mathrm{Bl}\,\mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

$$\pi_2([z]) \;=\; [z_1z_6, z_2, z_3, z_4, z_5z_6], \qquad \pi_1([z]) \;=\; [z_1z_5, z_2, z_3z_5, z_4].$$

The surfaces $\mathrm{Bl}\,\mathbb{F}_a(\star^2\star\ v)$ to $\mathrm{Bl}\,\mathbb{F}_a(\star^2\star\ viii)$ are the blow ups of X_1 in the same points $q_i \in X_1$ as defined in the previous case, i.e.,

$$\begin{array}{llll} q_1 \;:=\; [1,0,0,1,1,1], & & q_2 \;:=\; [0,1,1,0,1,1], & \\ q_3 \;:=\; [1,0,1,0,1,1], & & q_4 \;:=\; [0,1,1,1,1,1] & \in\; X_1. \end{array}$$

Again, their existence can be seen by inspecting Σ_1 or using Lemma 5.2.16. Therefore, the first three blow ups are toric and hence are performed by stellar subdivision of Σ_1 at $v \in \mathbb{Z}^2$ where the respective vectors are

$$v \;=\; (1,1), \qquad v \;=\; (-1,-a-1), \qquad v \;=\; (1,-1).$$

We now treat the blow up $\mathrm{Bl}\,\mathbb{F}_a(\star^2\star\ viii)$ of X_1 in q_4. The steps are as in previous cases. Choose $h_1 := T_2^aT_4 - T_3T_5T_6 \in \mathbb{K}[T_1, \dots, T_6]$ for the embedding $\bar\iota\colon \mathbb{K}^6 \to \mathbb{K}^7$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

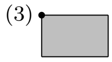
$$Q_1' \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2a-1 & -a+1 & a \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 \end{bmatrix}, \quad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 0 & 1 & 1 & 2 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

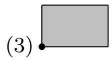
For the blow up of $\iota(q_4) = [0,1,1,1,1,1,0] \in X_1'$, we perform the stellar subdivision of Σ_1' at $v := (0,-1,-1) \in \mathbb{Z}^3$. Set $P_2 := [P_1', v]$. The vanishing ideal I_2 of $\overline{X}_2$ is generated by

$$p_2^\star\ (p_1)_\star\ (T_7 - h_1) \;=\; T_7T_8 - T_2^aT_4 + T_3T_5T_6 \;\in\; \mathbb{K}[T_1, \dots, T_8].$$

By a direct check, using, e.g., Lemma 5.2.17, the requirements for Theorem 4.2.6 are fulfilled. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1, \dots, T_8]/I_2$. Its degree matrix is as listed in the table. The performed modification was a blow up, for $\langle T_1, T_7, h_1 \rangle \subseteq \mathbb{K}[T_1, \dots, T_7]$ is prime by Lemma 5.2.17 and its zero set contains $\bar\iota((0,1,1,1,1,1))$ while being four-dimensional, see Lemma 5.2.15.

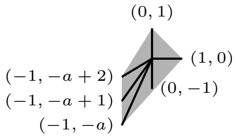
(VIII) Surfaces of type $\mathrm{Bl}\,\mathbb{F}_a(\star^3)$. Each surface $\mathrm{Bl}\,\mathbb{F}_a(\star^3)$ is a blow up of a surface of type $\mathrm{Bl}\,\mathbb{F}_a(\star^2)$ in a point in the union of the two exceptional divisors. By Proposition 5.1.4 and Proposition 5.2.8, we need only consider the following configurations where (3) stands for the threefold iterated blow up.





Thus, we blow up the two surfaces $\mathrm{Bl}\ \mathbb{F}_a(\star^2\ i)$ and $\mathrm{Bl}\ \mathbb{F}_a(\star^2\ ii)$ listed in Proposition 5.2.8 in a point in the union of the exceptional divisors. Let $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star^2\ i)$. As a toric variety the fan Σ_1 of X_1 has its rays generated by the columns of

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 \end{bmatrix}, \qquad \Sigma_1 \;=$$



The exceptional divisors are $V(X_1;\, T_5)$ and $V(X_1;\, T_6)$ and their union consists of the toric orbits through the points

$$q_1 \;:=\; [1,1,0,1,1,0], \qquad q_2 \;:=\; [1,1,1,1,1,0], \qquad q_3 \;:=\; [1,1,1,1,0,0],$$
$$q_4 \;:=\; [1,1,1,1,0,1], \qquad q_5 \;:=\; [0,1,1,1,0,1] \;\in\; X_1.$$

Note that all points exist by Lemma 5.2.16 or by an inspection of Σ_1. The automorphism $z \mapsto (z_1, z_2, z_3 - z_1^2 z_2^{a-2} z_4 z_5, z_4, \ldots, z_6)$ of $\widehat{X}_1 \subseteq \mathbb{K}^6$ maps the point q_2 to q_1; compare Lemma 5.1.5. The blow ups of X_1 in q_1 (and thus q_2), q_3 as well as q_5 are toric. They are determined by the stellar subdivision $\Sigma_2 \to \Sigma_1$ at $v \in \mathbb{Z}^2$ where

$$v = (-1, -a+3), \qquad v = (-2, -2a+3), \qquad v = (-2, -2a+1),$$

respectively. The resulting toric surfaces are called $\mathrm{Bl}\ \mathbb{F}_a(\star^3\ i)$, $\mathrm{Bl}\ \mathbb{F}_a(\star^3\ iii)$ and $\mathrm{Bl}\ \mathbb{F}_a(\star^3\ v)$; their Cox rings can be found in the table.

We now consider the blow up $\mathrm{Bl}\ \mathbb{F}_a(\star^3\ iv)$ of X_1 in q_4. The steps are as in previous cases. Choose in $\mathbb{K}[T_1, \ldots, T_6]$ the polynomial $h_1 := T_1 T_2^{a-1} T_4 - T_3 T_6$ for the embedding $\bar{\iota}\colon \mathbb{K}^6 \to \mathbb{K}^7$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{matrix} 1 \\ a-1 \\ 0 \\ 1 \end{matrix} \end{array}\right], \qquad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 0 & 1 & 0 & 0 & -1 \\ 0 & a & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

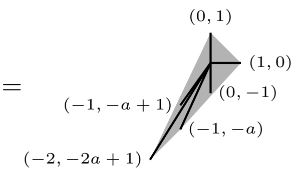
For the blow up of X_1' in the point $\iota(q) = [1,1,1,1,0,1,0]$ we perform the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (-1,-2,-1)$. Set $P_2 := [P_1', v]$. The vanishing ideal I_2 of $\overline{X}_2$ is generated by

$$p_2^\star\,(p_1)_\star\,(T_7 - h_1) \;=\; T_7 T_8 - T_1 T_2^{a-1} T_4 + T_3 T_6 \;\in\; \mathbb{K}[T_1, \ldots, T_8].$$

Using Lemma 5.2.17, the requirements for Theorem 4.2.6 are fulfilled. Hence, the Cox ring of the performed modification is $R_2 = \mathbb{K}[T_1, \ldots, T_8]/I_2$. Its degree matrix is as listed in the table. By Lemma 5.2.15, we have performed a blow up since $\langle T_5, T_7, h_1 \rangle \subseteq \mathbb{K}[T_1, \ldots, T_7]$ is prime by Lemma 5.2.17 and its zero set contains $\bar{\iota}((1,1,1,1,0,1))$ while being four-dimensional.

We now treat the blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star^2\ ii)$. Its fan Σ_1 and generators for its rays are

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 \end{bmatrix}, \qquad \Sigma_1 \;=$$



The exceptional divisors are $V(X_1; T_5)$ and $V(X_1; T_6)$ and their union consists of the toric orbits through the points

$$q_1 \;:=\; [1,1,0,1,0,1], \qquad q_2 \;:=\; [1,1,1,1,0,1], \qquad q_3 \;:=\; [1,1,1,1,0,0],$$
$$q_4 \;:=\; [1,1,1,1,1,0], \qquad q_5 \;:=\; [0,1,1,1,1,0] \;\in\; X_1.$$

Note that all points exist by Lemma 5.2.16 or by an inspection of Σ_1. The automorphism $z \mapsto (z_1, z_2, z_3 - z_1 z_2^{a-1} z_4 z_6, z_4, \ldots, z_6)$ of $\widehat{X}_1 \subseteq \mathbb{K}^6$ maps q_2 to q_1. The blow ups of X_1 in q_1 (and hence q_2), q_3 and q_5 are toric. They are determined by the stellar subdivision of $\Sigma_2 \to \Sigma_1$ at $v \in \mathbb{Z}^2$ where

$$v \;=\; (-1, -a+2), \qquad v \;=\; (-3, -3a+2), \qquad v \;=\; (-3, -3a+1),$$

respectively. The resulting toric surfaces are called Bl $\mathbb{F}_a(\star^3\ vi)$, Bl $\mathbb{F}_a(\star^3\ viii)$ and Bl $\mathbb{F}_a(\star^3\ x)$. The latter two can be found in the table; the former will be isomorphic to another surface and sorted out at the end of this proof.

We now consider the blow up Bl $\mathbb{F}_a(\star^3\ ix)$ of X_1 in q_4. The steps are the same as in previous cases. Choose in $\mathbb{K}[T_1, \ldots, T_6]$ the polynomial $h_1 := T_1 T_2^{2a-1} T_4^2 - T_3^2 T_5$ for the embedding $\bar{\imath} \colon \mathbb{K}^6 \to \mathbb{K}^7$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[Q_1 \;\middle|\; \begin{matrix} 1 \\ 2a-1 \\ 0 \\ 2 \end{matrix} \right], \qquad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 1 & 1 & 1 & 1 & -1 \\ 0 & a & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

For the blow up of X_1' in the point $\imath(q_4) = [1,1,1,1,1,0,0]$ we insert the ray through $v := (0,-2,-1)$ into Σ_1' by performing the stellar subdivision at $v \in \mathbb{Z}^3$. Set $P_2 := [P_1', v]$. The vanishing ideal I_2 of $\overline{X}_2$ is generated by

$$p_2^{\star}\,(p_1)_{\star}\,(T_7 - h_1) \;=\; T_7 T_8 - T_1 T_2^{2a-1} T_4^2 + T_3^2 T_5 \;\in\; \mathbb{K}[T_1, \ldots, T_8].$$

As before, one directly checks that the requirements for Theorem 4.2.6 are fulfilled. This leaves us with the Cox ring $R_2 = \mathbb{K}[T_1, \ldots, T_8]/I_2$ of the performed modification. Its degree matrix is as listed in the table. The ideal $\langle T_6, T_7, h_1 \rangle \subseteq \mathbb{K}[T_1, \ldots, T_7]$ is prime by Lemma 5.2.17, its zero set contains $\bar{\imath}((1,1,1,1,1,0))$ and is four-dimensional. By Lemma 5.2.15 the modification was the wanted blow up.

Isomorphisms: We now remove redundancies between the obtained surfaces. More precisely, we will show that the surfaces not listed in the table of Theorem 5.3.1 are isomorphic to surfaces appearing in the table. Keep in mind that also these redundant Cox rings have been presented explicitly throughout this proof. We first treat the toric then the non-toric $\mathbb{K}^*$-surfaces. Note that, in our case, exactly the surfaces without a relation in their Cox ring are toric as the total coordinate spaces of the other ones are singular.

Given toric surfaces Z_1, Z_2, we write primitive generators for the rays of their fans into the columns of matrices P_{Z_1}, P_{Z_2}. By Remark 5.2.10, we then have

$$Z_1 \;\cong\; Z_2 \qquad \Leftrightarrow \qquad A \cdot P_{Z_1} \;\risingdotseq\; P_{Z_2} \quad \text{with} \quad A \in \mathrm{GL}(2, \mathbb{Z})$$

in the notation of 5.2.9. We will reuse the matrices P_{Z_i} from the proof if possible. Otherwise, we will use Gale dual matrices computed with Algorithm 2.1.25 from the respective degree matrices.

$Z_1 \cong Z_2$	$AP_{Z_1} \risingdotseq P_{Z_2}$ with $A \in \mathrm{GL}(2,\mathbb{Z})$
Bl $\mathbb{P}_2(\star^4\ xi)$ $\cong$ Bl $\mathbb{F}_3(\star^3\ iii)$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 3 & 3 \\ -1 & 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} \risingdotseq \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -2 \\ -3 & 0 & 1 & -1 & -2 & -1 & -3 \end{bmatrix}$

$\mathrm{Bl}\,\mathbb{P}_2(\star^3\,\star\;\; ii) \cong \mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star^2\;\; ii)$	$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 2 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star^2\;\; i) \cong \mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star\,\star\; i)$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 2 & -2 \\ -1 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix} =: \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star^2\;\; ii) \cong \mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star^2\;\; iii)$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 2 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 1 & -2 \\ -1 & 0 & 1 & 1 & 0 & 2 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star\,\star\; i) \cong \mathrm{Bl}\,\mathbb{P}_2(\star^2\,\star\,\star\; iii)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 2 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 2 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; v) \cong \mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; vi)$	$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -2 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 & -2a+1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 & -a+2 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^2\,\star\;\; i) \cong \mathrm{Bl}\,\mathbb{F}_{a-1}(\star^2\,\star\;\; iii)$	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & 1 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 & 1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & 1 \\ -a+1 & 0 & 1 & -1 & -a+2 & -a+3 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; i) \cong \mathrm{Bl}\,\mathbb{F}_{a-1}(\star^2\,\star\;\; ii)$	$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 & -a+3 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -1 \\ -a+1 & 0 & 1 & -1 & -a+2 & -a+3 & -a \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; iii) \cong \mathrm{Bl}\,\mathbb{F}_{a-1}(\star^2\,\star\;\; vi)$	$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -2 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 & -2a+3 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -a+1 & 0 & 1 & -1 & -a+2 & -2a+3 & -a \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; vi) \cong \mathrm{Bl}\,\mathbb{F}_{a-2}(\star^3\;\; xiii)$	$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 & -a+2 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -2 \\ -a+2 & 0 & 1 & -1 & -a+1 & -a & -2a+1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_{a+1}(\star^2\,\star\;\; v) \cong \mathrm{Bl}\,\mathbb{F}_a(\star^2\,\star\;\; vii)$	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & 1 \\ -a-1 & 0 & 1 & -1 & -a & -2a-1 & 1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & 1 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 & -1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; v) \cong \mathrm{Bl}\,\mathbb{F}_a(\star^3\;\; vi)$	$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & -2 \\ -a & 0 & 1 & -1 & -a+1 & -a+2 & -2a+1 \end{bmatrix} =: \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -a & 0 & 1 & -1 & -a+1 & -2a+1 & -a+2 \end{bmatrix}$

We show that the listed toric surfaces are pairwise non-isomorphic by comparing the respective self intersection numbers D_i^2 where $D_i := V(X; T_i)$.

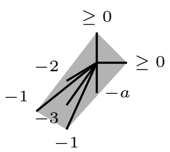
X	D_1^2	D_2^2	D_3^2	D_4^2	D_5^2	D_6^2	D_7^2
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ ix)$	≥ 0	-2	-1	-3	-2	-1	-1
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xiii)$	≥ 0	-2	≥ 0	-2	-3	-2	-1
$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xv)$	≥ 0	-3	≥ 0	-2	-2	-2	-1
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iii)$	-1	≥ 0	-2	-2	-2	-1	-1
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iv)$	≥ 0	≥ 0	-3	-2	-2	-1	-1
$\mathrm{Bl}\,\mathbb{P}_2(\star^2\star\star\ i)$	-1	-2	-1	-2	-1	-1	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^2\star\ i)$	-1	-1	≥ 0	$-a$	-2	-1	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^2\star\ v)$	-2	-1	≥ 0	$-a$	-2	-1	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\ i)$	-1	≥ 0	≥ 0	$-a$	-2	-2	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\ iii)$	-1	≥ 0	≥ 0	$-a$	-3	-2	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\ v)$	-2	≥ 0	≥ 0	$-a$	-3	-1	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\ viii)$	-2	≥ 0	≥ 0	$-a$	-3	-2	-1
$\mathrm{Bl}\,\mathbb{F}_a(\star^3\ x)$	-3	≥ 0	≥ 0	$-a$	-2	-2	-1

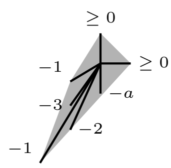
Counting the number of $(-k)$-curves we can rule out all isomorphisms except for the following.

$$\mathrm{Bl}\,\mathbb{P}_2(\star^4\ xiii) \ \to\ \mathrm{Bl}\,\mathbb{P}_2(\star^4\ xv)$$

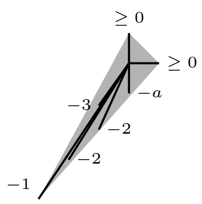
$$\begin{array}{rclrcl} \mathrm{Bl}\,\mathbb{P}_2(\star^4\ ix) & \to & \mathrm{Bl}\,\mathbb{F}_3(\star^2\star\ v), & \mathrm{Bl}\,\mathbb{P}_2(\star^2\star^2\ iv) & \to & \mathrm{Bl}\,\mathbb{F}_3(\star^3\ i), \\ \mathrm{Bl}\,\mathbb{F}_a(\star^3\ iii) & \to & \mathrm{Bl}\,\mathbb{F}_a(\star^3\ v), & \mathrm{Bl}\,\mathbb{F}_a(\star^3\ viii) & \to & \mathrm{Bl}\,\mathbb{F}_a(\star^3\ x). \end{array}$$

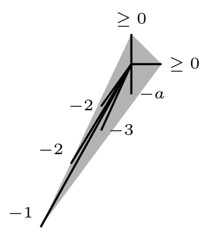
The first three ones are not possible by Algorithm 5.2.11. To rule out the other isomorphisms we compare the intersection behavior of negative curves, i.e., their exceptional graphs. To this end, consider the fans $\Sigma(\star^3\ iii)$ and $\Sigma(\star^3\ v)$ of the surfaces $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ iii)$ and $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ v)$ where the self-intersection numbers of the divisors D_ϱ corresponding to rays ϱ are drawn beside the rays.

$\Sigma(\star^3\ iii) \ =$ 

$\Sigma(\star^3\ v) \ =$ 

Thus, on $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ v)$, a (-2)-curve has non-trivial intersection with a (-1)-curve and a $(-a)$-curve which is not the case on $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ iii)$. We proceed similarly for $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ viii)$ and $\mathrm{Bl}\,\mathbb{F}_a(\star^3\ x)$. Their fans $\Sigma(\star^3\ viii)$ and $\Sigma(\star^3\ x)$ and self-intersection numbers of the D_ϱ are

$\Sigma(\star^3\ iii) \ =$ 

$\Sigma(\star^3\ x) \ =$ 

On $\mathrm{Bl}\,\mathbb{F}_a(\star^3\star\ x)$ there is a (-3)-curve that meets the $(-a)$-curve. This is not the case on the surface $\mathrm{Bl}\,\mathbb{F}_a(\star^3\star\ viii)$.

We come to isomorphisms between the found non-toric $\mathbb{K}^*$-surfaces. By Algorithm 5.2.14 and Remark 5.2.13, two $\mathbb{K}^*$-surfaces X_1, X_2 are isomorphic if and only if the degree matrices Q_i of $\mathcal{R}(X_i)$ coincide up to multiplication by admissible matrices, i.e.,

$$A \cdot Q_1 \ = \ Q_2 \cdot U \qquad \text{with} \qquad A \in \mathrm{GL}(5, \mathbb{Z})$$

admissible and U is a block-invariant, admissible permutation matrix. We use the notation of 5.2.9 where U is given implicitly.

$X_1 \cong X_2$	$AQ_1 \mathrel{\dot=} Q_2$ with $A \in \mathrm{GL}(5,\mathbb{Z})$
Bl $\mathbb{P}_2(\star^4\ i)$ $\cong$ Bl $\mathbb{P}_2(\star^4\ ii)$	$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 5 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -3 & 2 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ v)$ $\cong$ Bl $\mathbb{P}_2(\star^4\ viii)$	$\begin{bmatrix} -2 & 2 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 5 & -3 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 5 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ vi)$ $\cong$ Bl $\mathbb{P}_2(\star^4\ vii)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ i)$ $\cong$ Bl $\mathbb{F}_3(\star\star\star\ iv)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ iii)$ $\cong$ Bl $\mathbb{F}_3(\star^3\ iv)$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 0 & 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{P}_2(\star^4\ xii)$ $\cong$ Bl $\mathbb{P}_2(\star^4\ v)$	$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \mathrel{\dot=} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

$$\begin{array}{c} \mathrm{Bl}\ \mathbb{P}_2(\star^3 \star\ i) \\ \cong \\ \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star \star\ ii) \end{array} \quad \begin{array}{l} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \\ \quad =: \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 & -1 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \mathrm{Bl}\ \mathbb{F}_a(\star \star \star\ iii) \\ \cong \\ \mathrm{Bl}\ \mathbb{F}_{a-1}(\star \star \star\ x) \end{array} \quad \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -a+2 & a-1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -a+3 & a-2 \end{bmatrix} \\ \quad =: \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & a-1 & -a+2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & a-2 & -a+3 \end{bmatrix} \end{array}$$

By Remark 5.2.13, there are no further isomorphisms between the listed $\mathbb{K}^*$-surfaces except possibly between $\mathrm{Bl}\ \mathbb{P}_2(\star \star \star \star\ ii)$ and $\mathrm{Bl}\ \mathbb{F}_a(\star \star \star\ iii)$. However, this cannot be the case as, by the blow up sequence, the self intersection numbers of the $D_i := V(X; T_i)$ are different for each $a \geq 3$:

X	D_1^2	D_2^2	D_3^2	D_4^2	D_5^2	D_6^2	D_7^2	D_8^2
$\mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\ ii)$	-1	-1	-2	-1	-1	-1	$\bullet$	-1
$\mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ iii)$	-1	-1	≥ 0	$-a$	-1	-1	$\bullet$	-1

□

Remark 5.3.7. In the proof of Theorem 5.3.1, we presented the self-intersection numbers of all occurring toric surfaces. The self-intersection numbers for the $\mathbb{K}^*$-surfaces X listed in Theorem 5.3.1 are as follows. Let $D_i := V(X; T_i)$. We write $\bullet$ if we do not know the value from the blow up sequence.

X	D_1^2	D_2^2	D_3^2	D_4^2	D_5^2	D_6^2	D_7^2	D_8^2	D_9^2
$\mathrm{Bl}\ \mathbb{P}_2(\star^4\ iv)$	≥ 0	-1	≥ 0	-2	-3	$\bullet$	-1	$\bullet$	-1
$\mathrm{Bl}\ \mathbb{P}_2(\star^4\ v)$	≥ 0	-1	≥ 0	-2	-3	$\bullet$	-2	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star^4\ vi)$	≥ 0	-1	≥ 0	-2	-2	-2	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star^4\ xiv)$	≥ 0	-2	≥ 0	-2	-2	-2	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\star\ ii)$	≥ 0	-1	-2	-2	-1	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\star\ iv)$	≥ 0	≥ 0	-3	-2	-1	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\ ii)$	-1	-1	-2	-1	-1	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\ iii)$	≥ 0	≥ 0	-3	-1	-1	$\bullet$	-1	$\bullet$	-1
$\mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ iii)$	-1	-1	≥ 0	$-a$	-1	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ iv)$	-2	-1	≥ 0	$-a$	-1	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ ix)$	-3	≥ 0	≥ 0	$-a$	-1	$\bullet$	-1	$\bullet$	-1
$\mathrm{Bl}\ \mathbb{F}_a(\star^2\star\ iv)$	-2	≥ 0	≥ 0	$-a$	-2	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{F}_a(\star^2\star\ viii)$	-3	≥ 0	≥ 0	$-a$	-2	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{F}_a(\star^3\ iv)$	-1	≥ 0	≥ 0	$-a$	-3	-1	$\bullet$	-1	
$\mathrm{Bl}\ \mathbb{F}_a(\star^3\ ix)$	-2	≥ 0	≥ 0	$-a$	-2	-2	$\bullet$	-1	

4. Smooth rational surfaces with $\varrho(X) = 6$

In this section, building on Proposition 5.2.8 and Theorem 5.3.1, we present and prove the central theorem of this chapter, Theorem 5.4.1. We show that each smooth rational surface of Picard number six is a Mori dream space and classify the Cox rings of the families without a non-trivial $\mathbb{K}^*$-action. All Cox rings are given explicitly. Each such surface can be obtained as a blow up of a smooth rational surface of Picard number five as classified in Theorem 5.3.1. Theorem 5.4.1 (as well as the proof of one of the cases) has been stated together with J. Hausen and A. Laface in [57, Sec. 6].

The Cox rings of blow ups of $\mathbb{P}_2$ will be determined computationally using Algorithm 4.5.9 with option `verify`. For blow ups of $\mathbb{F}_a$ we apply the algorithm in a formal way by the steps explained in Remark 5.2.1.

Theorem 5.4.1. *Each smooth rational surface X with Picard number $\varrho(X) \le 6$ is a Mori dream space. Moreover, the following statements hold.*

(i) *If $\varrho(X) \le 5$ holds, then either X admits a non-trivial $\mathbb{K}^*$-action or is isomorphic to $\overline{M}_{0,5}$, the blow up of $\mathbb{P}_2$ in four general points. The Cox ring of X is listed in Theorem 5.3.1 or Propositions 5.2.8, 5.2.5, 5.2.4.*

(ii) *If $\varrho(X) = 6$ holds, then X admits a non-trivial $\mathbb{K}^*$-action or is isomorphic to exactly one of the following surfaces where $a \in \mathbb{Z}_{\ge 3}$.*

X	*Cox ring $\mathcal{R}(X)$*	*degree matrix*
$\mathrm{Bl}\,\mathbb{P}_2(\star^5\ i)$	$\mathbb{K}[T_1,\dots,T_{10}]/I$ *with I generated by* $T_3^2T_4 - T_1T_2 - T_6T_7T_8T_{10}$, $T_1T_2^2T_3T_4T_5 - T_6^2T_7 - T_9T_{10}$	$\begin{bmatrix} 1&0&0&1&0&0&2&0&3&-1\\ 0&1&0&1&0&0&3&0&5&-2\\ 0&0&1&-2&0&0&-1&0&-2&1\\ 0&0&0&0&1&0&1&0&2&-1\\ 0&0&0&0&0&1&-2&0&-1&1\\ 0&0&0&0&0&0&0&1&1&-1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3\star\ \star\ i)$	$\mathbb{K}[T_1,\dots,T_{10}]/I$ *with I generated by* $T_3T_5T_8 - T_2T_6 - T_9T_{10}$, $T_1T_5 + T_7T_8 - T_2T_6^2T_4T_{10}$	$\begin{bmatrix} 1&0&0&0&0&0&1&0&-1&1\\ 0&1&0&0&0&0&-1&1&2&-1\\ 0&0&1&0&0&0&1&-1&0&0\\ 0&0&0&1&0&0&0&0&1&-1\\ 0&0&0&0&1&0&2&-1&-1&1\\ 0&0&0&0&0&1&-1&1&3&-2 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3\star\ \star\ ii)$	$\mathbb{K}[T_1,\dots,T_{11}]/I$ *with I generated by* $T_3^2T_4T_5^2T_8 - T_2T_7 - T_{11}T_{10}$, $T_2^2T_4T_6^2T_{11} - T_5T_9 + T_8T_{10}$, $T_1T_5 + T_7T_8 - T_2T_4T_6^2T_{11}^2$, $T_3^2T_4T_5T_8^2 + T_1T_2 - T_9T_{11}$, $T_3^2T_4^2T_5T_8T_2T_6^2T_{11} - T_7T_9$ $-T_1T_{10}$	$\begin{bmatrix} 1&0&0&0&1&0&2&0&0&1&1\\ 0&1&0&0&1&0&1&0&1&2&0\\ 0&0&1&0&0&0&1&-1&0&1&0\\ 0&0&0&1&1&0&2&-1&0&2&0\\ 0&0&0&0&2&0&3&-1&-1&2&1\\ 0&0&0&0&0&1&0&0&1&1&-1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3\star\ \star\ iii)$	$\mathbb{K}[T_1,\dots,T_{10}]/I$ *with I generated by* $T_1T_5 + T_7T_8 - T_2T_4T_6T_{10}$, $T_3T_5T_7T_8^2 - T_2^2T_4 - T_9T_{10}$	$\begin{bmatrix} 1&0&0&0&0&0&2&-1&-1&1\\ 0&1&0&0&0&0&-2&2&3&-1\\ 0&0&1&0&0&0&1&-1&0&0\\ 0&0&0&1&0&0&-1&1&2&-1\\ 0&0&0&0&1&0&3&-2&-1&1\\ 0&0&0&0&0&1&0&0&1&-1 \end{bmatrix}$
$\mathrm{Bl}\,\mathbb{P}_2(\star\star\ \star\star\ i)$ $\lambda \in \mathbb{K}^* \setminus \{1\}$	$\mathbb{K}[T_1,\dots,T_{13}]/I$ *with I generated by* $T_1T_{11} - T_4T_3T_9 - T_8T_{12}$, $T_1T_7 - T_2T_8 + T_3T_9T_{13}$, $T_2T_6 + T_7T_{10} - T_3T_5T_{13}$, $T_1T_6 + T_8T_{10} - T_3T_4T_{13}$, $T_2T_{11} - \lambda T_5T_3T_9 - T_7T_{12}$, $(\lambda-1)T_1T_5 - T_{10}T_9 - T_{12}T_{13}$, $(\lambda-1)T_5T_8 + T_6T_9 - T_{11}T_{13}$, $T_{10}T_{11} - (\lambda-1)T_4T_3T_5 + T_6T_{12}$, $(\lambda-1)T_4T_7 + \lambda T_6T_9 - T_{11}T_{13}$, $(\lambda-1)T_2T_4 - \lambda T_{10}T_9 - T_{12}T_{13}$,	$\begin{bmatrix} 1&0&0&0&-1&0&0&1&0&0&-1&-1&1\\ 0&1&0&0&1&0&0&-1&0&1&0&1&0\\ 0&0&1&0&0&0&0&0&0&0&1&1&-1\\ 0&0&0&1&1&0&0&0&1&0&2&2&-1\\ 0&0&0&0&0&1&0&0&-1&1&-1&-1&1\\ 0&0&0&0&0&0&1&1&1&-1&1&0&0 \end{bmatrix}$

Bl $\mathbb{P}_2(\star\star\star\star\star\, ii)$ $\lambda \neq \mu \in \mathbb{K}^* \setminus \{1\}$

$\mathbb{K}[T_1, \ldots, T_{16}]/I$

with I generated by

$T_6T_{12} + \lambda T_7T_{14} - T_8T_{13}$,
$T_5T_{12} - \mu T_7T_{15} - T_9T_{13}$,
$T_4T_{13} - \lambda T_5T_{14} - \mu T_6T_{15}$,
$T_4T_{12} - \mu T_8T_{15} - \lambda T_9T_{14}$,
$T_3T_{11} + T_7T_{14} - T_8T_{13}$,
$T_1T_{13} - T_2T_{14} - T_3T_{15}$,
$T_1T_{11} - T_8T_{15} - T_9T_{14}$,
$T_2T_{11} - T_7T_{15} - T_9T_{13}$,
$(\lambda - \mu)T_3T_5 + \mu T_7T_{10} - T_{13}T_{16}$,
$(-\lambda + 1)T_5T_{14} + (-\mu + 1)T_6T_{15} + T_{10}T_{11}$,
$(\lambda - 1)T_5T_8 + (-\mu + 1)T_6T_9 - T_{11}T_{16}$,
$(\lambda - 1)T_4T_7 + (\lambda - \mu)T_6T_9 - T_{11}T_{16}$,
$(\mu - 1)T_3T_4 - \mu T_8T_{10} + T_{14}T_{16}$,
$(-\lambda + 1)T_2T_{14} + (-\mu + 1)T_3T_{15} + T_{10}T_{12}$,
$(\lambda\mu - \mu)T_2T_8 + (-\lambda\mu + \lambda)T_3T_9 - T_{12}T_{16}$,
$(\lambda - \mu)T_2T_6 + \lambda T_7T_{10} - T_{13}T_{16}$,
$(\lambda - 1)T_2T_4 - \lambda T_9T_{10} - T_{15}T_{16}$,
$(\lambda\mu - \mu)T_1T_7 + (\lambda - \mu)T_3T_9 - T_{12}T_{16}$,
$(\mu - 1)T_1T_6 - T_8T_{10} + T_{14}T_{16}$,
$(\lambda - 1)T_1T_5 - T_9T_{10} - T_{15}T_{16}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Bl $\mathbb{P}_2(\star\star\star\star\star\, iii)$

$\mathbb{K}[T_1, \ldots, T_{11}]/I$

with I generated by

$T_6T_2T_4 + T_5T_9 - T_8T_{10}$,
$T_3T_4T_8 - T_1T_6 - T_9T_{11}$,
$T_3T_4T_5 + T_6T_7 - T_{11}T_{10}$,
$T_1T_5 + T_7T_8 - T_2T_4T_{11}$,
$T_3T_4^2T_2 - T_7T_9 - T_1T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Bl $\mathbb{P}_2(\star^2\star\star\star\, iv)$

$\mathbb{K}[T_1, \ldots, T_{10}]/I$

with I generated by

$T_3T_5T_8 - T_2T_6 - T_9T_{10}$,
$T_1T_5 + T_7T_8 - T_2T_4T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

Bl $\mathbb{F}_a(\star\star\star\star\, vi)$

$\mathbb{K}[T_1, \ldots, T_{10}]/I$

with I generated by

$T_1T_5T_{10} - T_2T_6 - T_7T_8$,
$T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5$
$-T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -a+1 & a & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -a+2 & a-1 & -1 & 1 \end{bmatrix}$$

Bl $\mathbb{F}_a(\star\star\star\star\, xiv)$ $\kappa \in \mathbb{K}^*$ ††

$\mathbb{K}[T_1, \ldots, T_{11}]/I$

with I generated by

$T_7T_8 - T_2^aT_4T_6^{a-1}T_{11}^a + T_3T_5$,
$T_9T_{11} - T_1^aT_4T_5^aT_8^{a-1} - \kappa T_6T_7$,
$T_{10}T_{11} - T_1^aT_4T_5^{a-1}T_8^a + \kappa T_3T_6$,
$-\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9 - T_5T_{10}$,
$T_1^aT_2^aT_4^2T_5^{a-1}T_6^{a-1}T_8^{a-1}T_{11}^{a-1} - T_3T_9 - T_7T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & a-1 & 0 & 1 & 2a-3 & 2a-2 & -a+2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & a-1 & 1 & 0 & 2a-2 & 2a-3 & -a+2 \\ 0 & 0 & 0 & 0 & 0 & a & -1 & 1 & 2a-2 & 2a-1 & -a+1 \end{bmatrix}$$

Bl $\mathbb{F}_a(\star^3\star\, i)$

$\mathbb{K}[T_1, \ldots, T_{10}]/I$

with I generated by

$T_2^aT_4 - T_3T_5T_6^2T_{10} - T_7T_8$,
$T_1T_2^{a-1}T_4T_8 - T_3T_6 - T_9T_{10}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2a-1 & -a+1 & -a & a \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 3 & -2 \end{bmatrix}$$

Bl $\mathbb{F}_a(\star^3\star$ $ii)$	$\mathbb{K}[T_1,\dots,T_{10}]/I$ *with I generated by* $T_2^aT_4 - T_3T_6T_{10}T_5 - T_7T_8$, $T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5$ $-T_9T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3a-1 & -2a+1 & -a & a \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^4\ i)$ †	$\mathbb{K}[T_1,\dots,T_9]/I$ *with I generated by* $T_1T_2^{2a-1}T_4^2 - T_3^2T_5$ $-T_1T_2^{a-1}T_3T_4T_5T_6T_7T_9$ $+T_7T_8T_9^2$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 & 2a-1 & 0 & 0 & 4a-3 & -a+1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$
Bl $\mathbb{F}_a(\star^2\star^2$ $i)$ †	$\mathbb{K}[T_1,\dots,T_9]/I$ *with I generated by* $T_2^aT_4 - T_3T_5T_6^2$ $-T_7T_9T_1T_2^{a-1}T_4T_5T_6$ $+T_7T_9^2T_8$	$\begin{bmatrix} 1 & 1 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

All surfaces except possibly the surfaces marked with a single † do not admit a non-trivial $\mathbb{K}^*$-action. The surface marked with †† has the listed ring as its Cox ring for $a \le 15$; for $a > 15$ it is a Mori dream surface having the H_2-equivariant normalization of $\mathbb{K}[T_1,\dots,T_{11}]/(I : (T_1\cdots T_{11})^\infty)$ as its Cox ring.

For the following remark, recall from [5, Thm. V.2.1.7] that a *weak del Pezzo surface* is a surface that is ismorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, to $\mathbb{F}_2$ or to a blow up

$$X_r \longrightarrow X_{r-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \mathbb{P}_2$$

of $\mathbb{P}_2$ in $0 \le r \le 8$ points $p_1,\dots,p_r$ in *almost general position*, i.e., $p_i \in X_{i-1}$, no four points are mapped to the same line in $\mathbb{P}_2$ and for each i, the total transform of the exceptional divisor over $p_i \in X_{i-1}$ is a chain of rational curves where the last one is a (-1)-curve and the remaining ones are (-2)-curves. The *degree* of a weak del Pezzo surface is $9 - r$.

Remark 5.4.2. In Theorem 5.4.1, the weak del Pezzo surfaces of degree four, i.e., with $r = 5$, are

$$\mathrm{Bl}\,\mathbb{P}_2(\star^3\star\star\ ii), \qquad \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ i),$$
$$\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ ii), \qquad \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ iii), \qquad \mathrm{Bl}\,\mathbb{P}_2(\star^2\star\star\star\ iv).$$

Their Cox rings have been predicted in [32, Sec 6.4]. All other surfaces listed in the table of Theorem 5.4.1 contain $(-k)$-curves with $k \ge 3$. This can be seen from Remark 5.3.7 and the blow up sequence shown in the proof of Theorem 5.4.1. Therefore, these surfaces do not appear in [32].

Lemma 5.4.3 (Serre's criterion)**.** *Let $f_1,\dots,f_s \in \mathbb{K}[T_1,\dots,T_r]$ be homogeneous polynomials with respect to a pointed grading of $\mathbb{K}[T_1,\dots,T_r]$ by a lattice $\mathbb{Z}^n$. Write $I := \langle f_1,\dots,f_s\rangle$ and $\overline{X} := V(\mathbb{K}^r; I)$. Then I is prime if there is an open subset $U \subseteq \overline{X}$ such that*

$$\operatorname{codim}_{\overline{X}}\left(\overline{X}\setminus U\right) \;\ge\; 2, \qquad \operatorname{rank}\left(\frac{\partial f_i}{\partial T_j}(u)\right)_{i,j} \;=\; s \qquad \textit{for all } u \in U.$$

Proof. We first show that $\overline{X}$ is connected. Let $w_i \in \mathbb{Z}^n$ be the degree of T_i. Write $\vartheta := \operatorname{cone}(w_1,\dots,w_r) \subseteq \mathbb{Q}^n$ for the weight cone and choose an element $u \in (\vartheta^\vee)^\circ$. Since ϑ is pointed, we obtain a $\mathbb{Z}_{\ge 0}$-grading of the polynomial ring $\mathbb{K}[T_1,\dots,T_r]$ by setting $\deg(T_i) := a_i$ where $a_i := u(w_i) > 0$. Note that I is homogeneous. For each two points $x, x' \in \overline{X}$ there are one-parameter subgroups $\mathbb{K}^* \to \mathbb{T}^r$ given by

$$t \ \mapsto \ (t^{a_1}x_1, \ldots, t^{a_r}x_r)\,, \qquad t \ \mapsto \ (t^{a_1}x_1', \ldots, t^{a_r}x_r')\,,$$

respectively, that leave $\overline{X}$ invariant and have limit 0 for $t \to 0$ since $a_i > 0$ for all i. This shows that $\overline{X}$ is connected. By Serre's criterion [73, 6.2], I is radical, $R := \mathbb{K}[T_1, \ldots, T_r]/I$ a complete intersection and $\overline{X}$ a normal variety. By [36, Thm. 18.15], R is a product of integral domains. Since $\overline{X}$ is connected, R is an integral domain, i.e., $\sqrt{I} = I$ is prime. □

Lemma 5.4.4. *Let $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$ be an ideal such that $T_k - f \in I$ for some $1 \le k \le r$ and $f \in \mathbb{K}[T_i;\ i \ne k] =: R_k$. Let $I_k \subseteq R_k$ be the ideal generated by the image of I under*

$$\mathbb{K}[T_1, \ldots, T_r] \ \to \ R_k, \qquad T_k \ \mapsto \ f, \qquad T_i \ \mapsto \ T_i \ \ \text{for} \ \ i \ne k.$$

Then $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$ is prime if and only if $I_k \subseteq R_k$ is prime. Moreover, we have $\dim(V(\mathbb{K}^r; I)) = \dim(V(\mathbb{K}^{r-1}; I_k))$.

Proof. This statement is due to the observation that $\mathbb{K}[T_1, \ldots, T_r]/I$ is isomorphic to R_k/I_k. □

Amongst others, the following lemma describes the behavior of the irrelevant ideal under a toric blow up $Z_2 \to Z_1$. Let $\Sigma_i \subseteq \mathbb{Q}^d$ be the fan corresponding to Z_i. Assume the rays of Σ_i are $\varrho_1^i, \ldots, \varrho_r^i$. Define

$$\nu_i\colon \Sigma_i \ \to \ \{0,1\}^r, \qquad \nu_i(\sigma)_j \ := \ \begin{cases} 1, & \varrho_j^i \not\subseteq \sigma, \\ 0, & \varrho_j^i \subseteq \sigma. \end{cases}$$

Lemma 5.4.5. *In Setting 4.2.5, let Z_1 be a smooth toric variety with dense torus $\mathbb{T}_{Z_1}$ and fan Σ_1 with rays $\varrho_1, \ldots, \varrho_r$. Assume $\pi\colon Z_2 \to Z_1$ arises from the barycentric subdivision $\Sigma_2 \to \Sigma_1$ of a cone $\sigma \in \Sigma_1$. Suppose $X_1 \cap (\mathbb{T}_{Z_1} \cdot z(\sigma)) = \{[p_1]\}$ with $p_1 \in \widehat{Z}_1 \subseteq \mathbb{K}^r$ and π induces a blow up of X_1 in $[p_1]$. Let $V(Z_2; T_{r+1})$ be the exceptional divisor of the blow up π.*

(i) *The ideal of $\overline{Z}_2 \setminus \widehat{Z}_2$ in the ring $\mathbb{K}[T_1, \ldots, T_{r+1}]$ is*

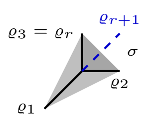
$$\left\langle T^{\nu_1(\sigma')}T_{r+1};\ \sigma' \in \Sigma_1^{\max} \setminus \{\sigma\} \right\rangle + \left\langle T^{\nu_1(\sigma)}T_i;\ \varrho_i \subseteq \sigma,\ 1 \le i \le r \right\rangle.$$

(ii) *The set $(\widehat{X}_1 \times \{1\}) \cap \overline{X}_2$ is contained in $\widehat{X}_2$.*

(iii) *Let $p_2 \in \overline{X}_2 \subseteq \mathbb{K}^{r+1}$. Then p_2 is contained in $\widehat{X}_2$ if there is $1 \le j \le r$ with*

$$(p_2)_j \ \ne \ 0, \qquad (p_1)_j \ = \ 0, \qquad \prod_{(p_1)_i \ne 0} (p_2)_i \ \ne \ 0.$$

Proof. First, recall from [5, Prop. II.1.3.3] that the vanishing ideal of $\overline{Z}_i \setminus \widehat{Z}_i$ in $\mathbb{K}[T_1, \ldots, T_r]$ is $J_i := \langle T^{\nu_i(\sigma')};\ \sigma' \in \Sigma_i^{\max} \rangle$. Let $\varrho_{r+1} \in \Sigma_2$ be the ray corresponding to the exceptional divisor $V(Z_2; T_{r+1})$. For the first statement, each maximal cone $\sigma' \in \Sigma_2$ either is a maximal cone of Σ_1 or contains ϱ_{r+1}. In the former case, $\nu_2(\sigma') = (\nu_1(\sigma'), 1)$ whereas in the latter case $\sigma' \subseteq \sigma$ which, by regularity of Σ_i, contains exactly one more of the rays ϱ_i than σ' where $1 \le i \le r$. This shows (i).



For (ii), consider $q_2 := (q_1, 1) \in \overline{X}_2 \subseteq \mathbb{K}^{r+1}$ with $q_1 \in \widehat{X}_1$. Since $q_1 \notin V(J_1)$, we have $T^{\nu_1(\sigma')}(q_1) \neq 0$ for a maximal cone $\sigma' \in \Sigma_1$. If $\varrho_{r+1} \not\subseteq \sigma'$, then $(q_1, 1) \notin V(J_2)$ as

$$T^{\nu_2(\sigma')}(q_1, 1) \quad = \quad T^{\nu_1(\sigma')}(q_1) \quad \neq \quad 0$$

by the first statement. On the other hand, if $\varrho_{r+1} \subseteq \sigma'$, we have $\sigma' = \sigma$. Assume $(T^{\nu_1(\sigma)} T_i)(q_1, 1) = 0$ for all rays $\varrho_i \subseteq \sigma$ with $1 \leq i \leq r$. Since $T^{\nu_1(\sigma)}(q_1) \neq 0$, we have $(q_1)_i = 0$ for all $\varrho_i \subseteq \sigma$. This means, q_1 lies in the toric orbit $\mathbb{T}_{Z_1} \cdot z(\tau)$ corresponding to a cone $\tau \in \Sigma_1$ with $\sigma \preceq \tau$. By assumption, we have

$$\{[q_1]\} \ \subseteq \ X_1 \cap (\mathbb{T}_{Z_1} \cdot z(\tau)) \ \subseteq \ X_1 \cap (\mathbb{T}_{Z_1} \cdot z(\sigma))) \ = \ \{[p_1]\},$$

and thus $[p_1] = [q_1] \in X_1$. Because $\pi([q_1, 1]) = [q_1] = [p_1]$ this implies that $[q_1, 1]$ is an element of the exceptional divisor $V(Z_2; T_{r+1})$, a contradiction.

We come to (iii). Since $\sigma \in \Sigma_1$, the monomial $T^{\nu_1(\sigma)} = \prod_{(p_1)_i \neq 0} T_i$ is an element of the ideal J_1. Since $(p_1)_j = 0$, we have $\varrho_j \subseteq \sigma$. By (i) and the requirements on p_2 we conclude

$$T_j \cdot \prod_{(p_1)_i \neq 0} T_i \ \in \ J_2, \qquad p_2 \ \in \ \overline{X}_2 \setminus V\left(\mathbb{K}^{r+1};\, J_2 + I_2\right) \ = \ \widehat{X}_2.$$

□

The following proposition will be used to identify non-isomorphic surfaces in the proof of Theorem 5.4.1. We call the Cox ring $\mathcal{R}(X)$ of a Mori dream space X *minimally presented* if the $\mathrm{Cl}(X)$-grading is pointed and no generator of $\mathcal{R}(X)$ may be omitted.

Proposition 5.4.6. *Consider Mori dream spaces X_1, X_2 sharing the same class group $\mathrm{Cl}(X_i) = \mathbb{Z}^n$. Assume both Cox rings $\mathcal{R}(X_i)$ are generated by $r \in \mathbb{Z}_{>0}$ elements and are minimally presented. If $X_1 \cong X_2$, then the following assertions hold.*

(i) *There is a permutation $\sigma \in \mathrm{Sym}(r)$ such that for the corresponding permutation matrix $U_\sigma \in \mathrm{GL}(r, \mathbb{Z})$ the $n \times r$ degree matrices Q_i satisfy*

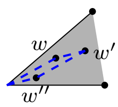
$$S \cdot Q_1 \cdot U_\sigma \ = \ Q_2 \qquad \textit{for some} \qquad S \in \mathrm{GL}(n, \mathbb{Z}).$$

(ii) *Let L_i be the lists consisting of the sorted absolute values of all $n \times n$ minors of the Q_i. Then $L_1 = L_2$.*

(iii) *There is $\sigma \in \mathrm{Sym}(r)$ such that the Hermite normal forms of $Q_1 \cdot U_\sigma$ and of Q_2 are equal up to units.*

The idea of the proof of Proposition 5.4.6 is to track the permutation of certain "minimal weight vectors"; this concept is used in an ongoing project together with J. Hausen. Let K be a finitely generated abelian group. Consider an affine $\mathbb{K}$-algebra $R = \mathbb{K}[T_1, \ldots, T_r]/I$ with a pointed K-grading $R = \bigoplus_{w \in K} R_w$ and a K-homogeneous ideal $I \subseteq \mathbb{K}[T_1, \ldots, T_r]$. We order the elements of K by

$$w \ \leq \ w' \quad :\Leftrightarrow \quad w' \ = \ w + w'' \tag{6}$$
$$\text{for some } w'' \in K.$$



Definition 5.4.7. Let $S(R) := \{w \in K;\, R_w \neq \{0\}\}$. A vector $w \in S(R)$ is *originary* if it is minimal in $S(R)$ with respect to the relation $\leq$ defined in (6). We denote the set of originary vectors by $\mathrm{orig}(R) \subseteq S(R)$.

For instance, if $w \in S(R)$ and w is an element of a Hilbert basis for the weight cone $\vartheta \subseteq K_{\mathbb{Q}}$ of R or w is a primitive generator of a ray $\varrho \preceq \vartheta$, then w is originary.

Remark 5.4.8. In Definition 5.4.7, the originary vectors are the elements $w \in S(R)$ such that

$$f \ \notin \ \mathbb{K}\left[\bigcup_{w_0<w} R_{w_0}\right] \qquad \text{for some} \qquad f \ \in \ R_w.$$

Let K_1, K_2 be finitely generated abelian groups and R_i two $\mathbb{K}$-algebras that are graded by K_i. Recall, e.g., from [5, Sec. I.1.1], that an *isomorphism of graded algebras* is a pair (ψ, α) with an isomorphism $\psi\colon R_1 \to R_2$ of algebras and an isomorphism $\alpha\colon K_1 \to K_2$ of groups such that

$$\psi((R_1)_w) \ = \ (R_2)_{\alpha(w)} \qquad \text{for all} \qquad w \ \in \ K.$$

Moreover, a homomorphism $\psi\colon R_1 \to R_2$ of algebras is called *graded* if there is a homomorphism $\alpha\colon K_1 \to K_2$ of groups such that $\psi((R_1)_w) \subseteq (R_2)_{\alpha(w)}$ for all $w \in K$.

Lemma 5.4.9. *Let K, K' be finitely generated abelian groups and R, R' affine $\mathbb{K}$-algebras with respective pointed K- and K'-gradings. Each isomorphism $(\psi, \alpha)\colon R \to R'$ satisfies $\alpha(\mathrm{orig}(R)) = \mathrm{orig}(R')$.*

Proof. As (ψ, α) is an isomorphism, we have $\alpha(S(R_1)) = S(R_2)$. Note that α is an isomorphism of posets, i.e., $w \le w'$ in $S(R)$ if and only if $\alpha(w) \le \alpha(w')$ in $S(R')$. In particular, the minimal elements of $S(R)$ are mapped to the minimal elements of $S(R')$, as claimed. ☐

Lemma 5.4.10. *Consider a Mori dream space X with Cox ring $\mathcal{R}(X)$ that is minimally presented by the generators $f_1, \ldots, f_s$. Then the set of originary vectors of $\mathcal{R}(X)$ is $\{\deg(f_1), \ldots, \deg(f_s)\}$.*

Proof. We write $R := \mathcal{R}(X)$. Assume there were $1 \le j \le s$ with $\deg(f_j)$ not originary. This means we have

$$f_j \quad \in \quad \mathbb{K}\left[\bigcup_{w_0<\deg(f_j)} R_{w_0}\right]$$

In particular, f_j can be removed from the presentation of $\mathcal{R}(X)$, a contradiction. For the reverse inclusion, assume $w \in \mathrm{orig}(R)$ is given such that $w \ne \deg(f_j)$ for all $1 \le j \le s$. Since no generator f_j is an element of R_w, any $0 \ne f \in R_w$ is a combination $f = \sum h_i f_i$ with $h_i \in R$. In particular, w is not originary. ☐

Proof of Proposition 5.4.6. If X_1 and X_2 are isomorphic, there is an isomorphism (ψ, β) of $\mathbb{Z}^n$-graded $\mathbb{K}$-algebras $\mathcal{R}(X_1) \to \mathcal{R}(X_2)$; compare [5, 6]. Note that $\beta\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is given by a matrix $S \in \mathrm{GL}(n, \mathbb{Z})$. Since $\mathcal{R}(X_i)$ is minimally presented, by Lemmas 5.4.10 and 5.4.9, generator degrees in $\mathrm{Cl}(X_1)$ are mapped to generator degrees in $\mathrm{Cl}(X_2)$ under β, i.e.,

$$S \cdot Q_1 \cdot U_\sigma \ = \ Q_2 \qquad \text{for some} \qquad \sigma \ \in \ \mathrm{Sym}(r).$$

In particular, the Hermite normal forms of $Q_1 \cdot U_\sigma$ and Q_2 coincide up to units. This shows (i) and (iii). Statement (ii) follows from (i) and the fact that up to sign the maximal minors of a matrix are invariant with respect to multiplication by an invertible matrix. ☐

The following observations will be useful in the proof of Theorem 5.4.1, to identify surfaces that do admit a non-trivial $\mathbb{K}^*$-action. The next lemma uses a result from P. Orlik and P. Wagreich [86].

Lemma 5.4.11. *Compare* [86]. *Consider a surface X_1 admitting a non-trivial $\mathbb{K}^*$-action. Let X_2 be the blow up of a point $x \in X_1$ with Cox coordinates $z \in \widehat{X}_1 \subseteq \mathbb{K}^r$.*

If x is a fixed point, then X_2 admits a non-trivial $\mathbb{K}^*$-action. In particular, this is the case if one of the following conditions hold.

(i) *X_1 is toric and x is not contained in in the big $\mathbb{T}_{X_1}$-orbit of X_1.*
(ii) *X_1 is non-toric, embedded equivariantly into its canonical toric ambient variety and there are $i \neq j$ such that $z_i = z_j = 0$.*

Proof. If x is a fixed point, X_2 admits again a non-trivial $\mathbb{K}^*$-action by [86]. For (ii), the intersection of the $\mathbb{K}^*$-invariant divisors $V(X_1; T_i)$ and $V(X_1; T_j)$ is again $\mathbb{K}^*$-invariant, i.e., is a hyperbolic fixed point, see Proposition 1.5.10.

For the first statement, the dense torus $\mathbb{T}_{X_1}$ of X_1 can be obtained as $\mathbb{T}_{X_1} = \operatorname{Spec}\mathbb{K}[E]$ with a lattice E, see Remark 1.1.2. By assumption, x belongs to a torus orbit $\mathbb{T} \cdot z(\sigma)$ with the distinguished point $z(\sigma) \in X_1$ of a cone $\{0\} \neq \sigma \subseteq \mathbb{Q} \otimes F$ of the fan of X_1. Consider the sublattice $L := F \cap \operatorname{lin}_{\mathbb{Q}}(\sigma) \subseteq F$. The inclusion $L \to F$ corresponds to an epimorphism $Q\colon E \to K$ of the dual lattices. By [5, Prop. II.1.4.2], we have

$$\operatorname{rank}(K_\sigma) \ \geq \ 1 \qquad \text{where} \qquad K_\sigma \ := \ K \,/\, Q(\sigma^\perp \cap E).$$

Hence, the isotropy group $H_{z(\sigma)} = \operatorname{Spec}\mathbb{K}[K_\sigma]$ is a subgroup of $\mathbb{T}_{X_1}$ of dimension at least one. This means, x lies on a fixed point curve of a $\mathbb{K}^*$-action and X_2 is a $\mathbb{K}^*$-surface. □

Lemma 5.4.12. *Let X be a $\mathbb{K}^*$-surface. Consider negative curves $D, D' \subseteq X$ that each intersect at least three other negative curves on X non-trivially. Given a negative curve $E \subseteq X$, we have*

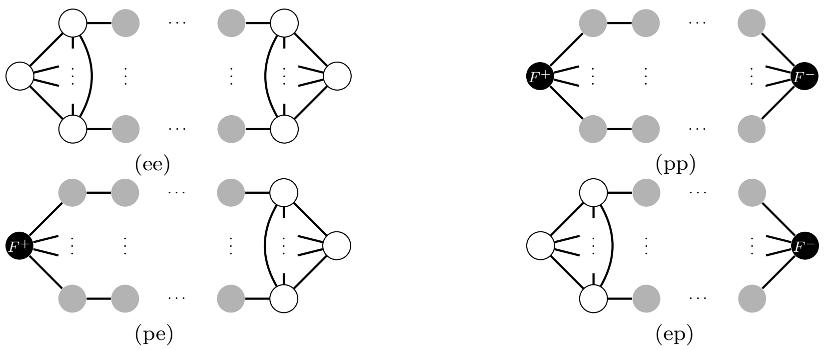
$$D \cap E \ = \ \emptyset \qquad \text{or} \qquad D' \cap E \ = \ \emptyset.$$

Proof. By Proposition 1.5.10, D and D' must be sink and source of the $\mathbb{K}^*$-action on X. In particular, they must not meet, i.e., $D \cap D' = \emptyset$. If $E \notin \{D, D'\}$, then the contraction $X \to X'$ of E yields again a $\mathbb{K}^*$-surface X'. In terms of P-matrices as in 1.5.2 this means deleting a column. Let Z and Z' be the canonical toric ambient varieties of X and X' as in 1.5.7. Since the fan of Z' is obtained by contraction of a ray of the fan of Z we have a toric contraction

$$(\varphi, \tilde\varphi)\colon Z \ \to \ Z' \qquad \text{where} \qquad \tilde\varphi \ = \ \mathrm{id}\colon \mathbb{T}_Z \ \to \ \mathbb{T}_{Z'},$$

φ is proper and the contraction is equivariant. Thus, on X', sink and source have non-trivial intersection. □

Remark 5.4.13. Let X be a smooth $\mathbb{K}^*$-surface. By Proposition 1.5.10, depending on the type of fixed points, the graph of exceptional curves G_X is of shape



where gray and black vertices are negative curves and white vertices form a complete subgraph of not necessarily negative curves. The graph of case (pp) is called the *Orlik Wagreich graph* defined in [86].

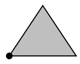
Remark 5.4.14. See [37, Cor. 2.6] and compare [64], [50, Kap. 2.1]. Given a binomial ideal $I \subseteq \mathbb{K}[T_1, \dots, T_r]$, let $T_{i_1}, \dots, T_{i_s}$ be the variables that are contained in I. Write the exponents of binomial generators of I into a matrix B as in Lemma 5.3.3. Then I is prime if and only if the entries of the Smith normal form of B are elements of $\{0, \pm 1\}$ and

$$I \;=\; \langle T_{i_1}, \dots, T_{i_s} \rangle \,+\, (I : f^{\infty})\,, \qquad f \;:=\; \prod_{i \notin \{i_1, \dots, i_s\}} T_i.$$

Proof of Theorem 5.4.1. Each surface X of Picard number six can be obtained as blow up of a surface of Picard number five listed in Theorem 5.3.1 such that the contraction of the exceptional divisors on X leads to one of the configurations of Propositions 5.1.2 and 5.1.4, see Remark 5.2.3. We compute the resulting surfaces grouped into originating surface of Picard number five.

Blow ups that lead to (toric or non-toric) $\mathbb{K}^*$-surfaces will be omitted; they are known to be Mori dream spaces, see, e.g., [5, 61]. Algorithm 4.5.9 will be used directly for blow ups of $\mathbb{P}_2$ and formally for blow ups of $\mathbb{F}_a$. We will prove or disprove the existence of a non-trivial $\mathbb{K}^*$-action directly when we encounter the surfaces. In the final step, we present or rule out isomorphisms between the listed surfaces.

(I) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^4\ iv)$*.* As seen in the proofs of Proposition 5.2.8 and Theorem 5.3.1, the point configuration and blow up sequence for X_1 are

(4) 

$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii)' \xleftarrow{\iota_2} \mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii) \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{P}_2(\star^2)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star^2) \xrightarrow{\pi_1 \circ \pi_2} \mathbb{P}_2$$

The embeddings ι_i are as in Setting 4.2.9 with

$$\begin{array}{llll} \bar{\iota}_1 \colon \mathbb{K}^5 \to \mathbb{K}^6, & z \mapsto (z, h_1(z)), & h_1 \;:=\; T_3^2 T_4 - T_1 T_2, \\ \bar{\iota}_2 \colon \mathbb{K}^7 \to \mathbb{K}^8, & z \mapsto (z, h_2(z)), & h_2 \;:=\; (\lambda - 1) T_3^2 T_4 - \lambda T_6 T_7 \end{array}$$

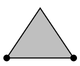
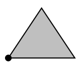
where $\lambda \in \mathbb{K}^* \setminus \{1\}$, $h_1 \in \mathbb{K}[T_1, \dots, T_5]$ and $h_2 \in \mathbb{K}[T_1, \dots, T_7]$. The blow ups are given by

$$\begin{array}{lll} \pi_4([z]) \;=\; [z_1, \dots, z_4, z_5 z_9, z_6, z_7, z_8 z_9], & & \pi_3([z]) \;=\; [z_1, \dots, z_4, z_5 z_7, z_6 z_7], \\ \pi_2([z]) \;=\; [z_1, z_2 z_5, z_3, z_4 z_5], & & \pi_1([z]) \;=\; [z_1, z_2 z_4, z_3 z_4]. \end{array}$$

The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1; T_4), \qquad V(X_1; T_5), \qquad V(X_1; T_7), \qquad V(X_1; T_9).$$

By Proposition 5.1.2 and Theorem 5.3.1, we want to blow up X_1 in a point $q \in X_1$ which, together with the exceptional divisors, projects to one of the configurations

(4)  (5) 

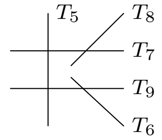
For the first configuration, we blow up X_1 in the point q where

$$q := [0,1,0,1,1,0,1,0,1] \in X_1, \qquad \pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3 \circ \iota_2^{-1} \circ \pi_4(q) = [0,1,0] \in \mathbb{P}_2$$

and the existence of q can be seen by an application of Algorithm 2.3.8. By an application of Algorithm 4.5.9 with Remark 5.3.6, the result is again a $\mathbb{K}^*$-surface; compare Lemma 5.4.11.

We come to the second configuration, i.e., blow ups of X_1 of type $\mathrm{Bl}\ \mathbb{P}_2(\star^5)$. Note that the curve $V(X_1; T_5)$ is a parabolic fixed point curve. Therefore, by Lemma 5.4.11, blowing up any point q lying in $V(X_1; T_5)$ or in the intersection of two invariant divisors $V(X_1; T_i)$ and $V(X_1; T_j)$ leads to a surface with at least a $\mathbb{K}^*$-action and thus will be omitted. By Remark 5.3.5, it suffices to consider

$$
\begin{aligned}
q_1 &:= [-1, 1, 1, 0, 1, 1, 1, -\lambda, 1], \\
q_2 &:= [1, 1, 1, 1, 1, 0, \lambda - 1, 1], \\
q_3 &:= [1, 1, 1, \lambda, 1, \lambda - 1, 1, 1, 0].
\end{aligned}
$$



By Lemma 5.2.16, all points exist. Using Algorithm 4.5.9 with Remark 5.3.6, we obtain the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix
q_1	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $(\lambda-1)T_1T_2 - T_5T_6 - T_7T_8$, $(\lambda-1)T_2^2T_3T_4^2T_6^2T_8^2T_{10}$ $+(-2\lambda+2)T_2T_3T_4T_6T_8T_9T_{10}^2$ $+(\lambda-1)T_3T_9^2T_{10}^3 - \lambda T_5T_6 - T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 5 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -3 & 2 \end{bmatrix}$
q_2	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $T_3^2T_4 - T_1T_2\lambda + T_7T_8$, $T_6T_{10}T_2T_3T_4T_5T_8 - T_6T_{10}^2T_9$ $+(-\lambda+1)T_1T_2 + T_7T_8$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}$
q_3	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $T_3^2T_4 - T_1T_2 - T_6T_7$, $T_8T_{10}T_2T_3T_4T_5T_7 - \lambda T_8T_9T_{10}^2$ $+(-\lambda^2+\lambda)T_1T_2 + \lambda T_6T_7$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$

We now show that all three surfaces are $\mathbb{K}^*$-surfaces since their Cox rings are isomorphic to the following Cox rings belonging to $\mathbb{K}^*$-surfaces with the same grading:

$$
\begin{aligned}
&\mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} (\lambda-1)T_1T_2 - T_5T_6 - T_7T_8, \\ (\lambda-1)T_3T_9^2T_{10}^3 - \lambda T_5T_6 - T_7T_8 \end{array} \right\rangle, \\
&\mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} T_3^2T_4 - T_1T_2\lambda + T_7T_8, \\ -T_6T_{10}^2T_9 + (-\lambda+1)T_1T_2 + T_7T_8 \end{array} \right\rangle, \\
&\mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} T_3^2T_4 - T_1T_2 - T_6T_7, \\ -\lambda T_8T_9T_{10}^2 + (-\lambda^2+\lambda)T_1T_2 + \lambda T_6T_7 \end{array} \right\rangle.
\end{aligned}
$$

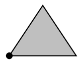
By Lemma 5.1.5, it suffices to provide $\mathbb{Z}^6$-graded isomorphisms $\mathbb{K}[T_1, \ldots, T_{10}] \to \mathbb{K}[T_1, \ldots, T_{10}]$ that induce isomorphisms of the respective Cox rings. For q_1, q_2 and q_3 we respectively choose

$$
T_i \mapsto \begin{cases} T_1 + T_2T_3T_4^2T_6^2T_8^2T_{10} - 2T_3T_4T_6T_8T_9T_{10}^2, & i = 1, \\ T_7 + (\lambda-1)T_2^2T_3T_4^2T_6^2T_8T_{10} - (2\lambda-2)T_2T_3T_4T_6T_9T_{10}^2, & i = 7, \\ T_i, & \text{else,} \end{cases}
$$

$$
T_i \mapsto \begin{cases} T_1 - T_6T_{10}T_3T_4T_5T_8, & i = 1, \\ T_7 - \lambda T_6T_{10}T_2T_3T_4T_5, & i = 7, \\ T_i, & \text{else,} \end{cases}
$$

$$T_i \mapsto \begin{cases} T_1 + \lambda^{-2} T_8 T_{10} T_3 T_4 T_5 T_7, & i = 1, \\ T_6 - \lambda^{-2} T_8 T_{10} T_2 T_3 T_4 T_5, & i = 6, \\ T_i, & \text{else.} \end{cases}$$

(II) Blow ups of $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^4\ v)$. As seen in the proofs of Proposition 5.2.8 and Theorem 5.3.1, the point configuration and blow up sequence for X_1 are

(4) 

$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\,\mathbb{P}_2(\star^3\ ii) \xrightarrow{\pi_3} \mathrm{Bl}\,\mathbb{P}_2(\star^2)' \xleftarrow{\iota_1} \mathrm{Bl}\,\mathbb{P}_2(\star^2) \xrightarrow{\pi_1 \circ \pi_2} \mathbb{P}_2$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_3^2 T_4 - T_1 T_2$$

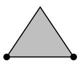
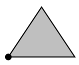
where $h_1 \in \mathbb{K}[T_1, \ldots, T_5]$ and the blow ups are

$$\begin{aligned} \pi_4([z]) &= [z_1, \ldots, z_4, z_5 z_8, z_6, z_7 z_8], & \pi_3([z]) &= [z_1, \ldots, z_4, z_5 z_7, z_6 z_7], \\ \pi_2([z]) &= [z_1, z_2 z_5, z_3, z_4 z_5], & \pi_1([z]) &= [z_1, z_2 z_4, z_3 z_4]. \end{aligned}$$

The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1; T_4), \qquad V(X_1; T_5), \qquad V(X_1; T_7), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

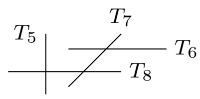
(4)  (5) 

For the first configuration, we use Algorithm 4.5.9 to blow up X_1 in the following point q; it exists by Algorithm 2.3.8 and we then obtain a $\mathbb{K}^*$-surface

$$q := [0, 1, 0, 1, 1, 0, 1, 1] \in X_1, \qquad \pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3 \circ \pi_4(q) = [0, 1, 0] \in \mathbb{P}_2.$$

We come to the second configuration, i.e., blow ups of X_1 of type $\mathrm{Bl}\,\mathbb{P}_2(\star^5)$. On X_1, the curve $V(X_1; T_5)$ is a parabolic fixed point curve. By Lemma 5.4.11, blow ups of points lying in $V(X_1; T_5)$ or in the intersection $V(X_1; T_i) \cap V(X_1; T_j)$ of two invariant divisors yield surfaces with a non-trivial $\mathbb{K}^*$-action and thus may be omitted. By Remark 5.3.5, it suffices to consider the following points.

$$\begin{aligned} q_1 &:= [1, 1, 1, 0, 1, 1, 1, -1], \\ q_2 &:= [1, 1, 1, 1, 1, 1, 1, 0], \\ q_3 &:= [1, 1, 1, 1, 1, 1, 0, 1]. \end{aligned}$$



By Lemma 5.2.16, all points exist. Using Algorithm 4.5.9, we computed the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix
q_1	$\mathbb{K}[T_1, \ldots, T_9]/I$ with I generated by $T_5 T_6 T_7 + T_1 T_2 - T_3 T_9 T_2^2 T_4^2 T_6^2 T_7^4$ $+2 T_3 T_9^2 T_2 T_4 T_6 T_7^2 T_8 - T_3 T_9^3 T_8^2$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -3 & 2 \end{bmatrix}$
q_2	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $T_3^2 T_4 - T_1 T_2 - T_6 T_7 T_8 T_{10}$, $T_1 T_2^2 T_3 T_4 T_5 - T_6^2 T_7 - T_9 T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & 0 & 5 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$

q_3	$\mathbb{K}[T_1, \ldots, T_9]/I$ with I generated by $T_3^2T_4 - T_1T_2 - T_6T_7^2T_9T_2T_3T_4T_5$ $+T_6T_7T_9^2T_8$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & -2 \end{bmatrix}$

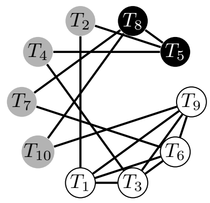
Both the blow up of X_1 in q_1 and the blow up of X_1 in q_3 are $\mathbb{K}^*$-surfaces since their Cox rings are isomorphic to the respective Cox rings

$$\mathbb{K}[T_1, \ldots, T_9] \ / \ \langle T_5T_6T_7 + T_1T_2 - T_3T_9^3T_8^2 \rangle,$$
$$\mathbb{K}[T_1, \ldots, T_9] \ / \ \langle T_3^2T_4 - T_1T_2 + T_6T_7T_9^2T_8 \rangle$$

of $\mathbb{K}^*$-surfaces with the same degree matrices as listed in the table. By Lemma 5.1.5, it suffices to consider the isomorphisms induced by the $\mathbb{Z}^6$-graded homomorphisms $\mathbb{K}[T_1, \ldots, T_9] \to \mathbb{K}[T_1, \ldots, T_9]$ given by

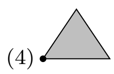
$$\begin{array}{lcllcll} T_5 & \mapsto & T_5 + T_3T_9T_2^2T_4^2T_6T_7^3 - 2T_3T_9^2T_2T_4T_7T_8, & T_i & \mapsto & T_i & \text{for } i \neq 5, \\ T_1 & \mapsto & T_1 - T_6T_7^2T_9T_3T_4T_5, & T_i & \mapsto & T_i & \text{for } i \neq 1, \end{array}$$

respectively. Denote by $X_2 := \mathrm{Bl}\ \mathbb{P}_2(\star^5\ i)$ the blow up of X_1 in q_2. By an application of Algorithm 2.3.27, its graph G_{X_2} of exceptional curves is



where gray and black vertices stand for negative curves. Black vertices correspond to curves intersecting at least three other negative curves. By Lemma 5.4.12, X_2 is not a $\mathbb{K}^*$-surface.

(III) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^4\ vi)$. As seen in the proofs of Proposition 5.2.8 and Theorem 5.3.1, the point configuration and blow up sequence for X_1 are



$$\begin{tikzcd} W_1' \arrow[d, "\pi_4"] & & W_1 \arrow[ll, "\iota_3"'] & & X_1 \arrow[ll, "\varphi"', "\cong"] \\ \mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii)' & \mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii) \arrow[l, "\iota_2"'] \arrow[r, "\pi_3"] & \mathrm{Bl}\ \mathbb{P}_2(\star^2)' & \mathrm{Bl}\ \mathbb{P}_2(\star^2) \arrow[l, "\iota_1"'] \arrow[r, "\pi_1 \circ \pi_2"] & \mathbb{P}_2 \end{tikzcd}$$

Here, the embeddings ι_i are as in Setting 4.2.9 with

$$\begin{array}{lcllcl} \bar{\iota}_1 \colon \mathbb{K}^5 & \to & \mathbb{K}^6, \quad z \mapsto (z, h_1(z)), & h_1 & := & T_3^2T_4 - T_1T_2, \\ \bar{\iota}_2 \colon \mathbb{K}^7 & \to & \mathbb{K}^8, \quad z \mapsto (z, h_2(z)), & h_2 & := & T_2T_3T_4T_5 - T_6, \\ \bar{\iota}_3 \colon \mathbb{K}^8 & \to & \mathbb{K}^9, \quad z \mapsto (z_1, \ldots, z_5, z_2z_3z_4z_5 - z_8z_9, z_6, z_7, z_8) & & & \end{array}$$

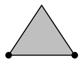
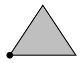
where $h_1 \in \mathbb{K}[T_1, \ldots, T_5]$ and $h_2 \in \mathbb{K}[T_1, \ldots, T_6]$, $\bar{\iota}_3$ eliminates a fake relation as in Algorithm 4.3.3, the isomorphism φ maps $z \in \mathbb{K}^8$ to $(z_1 + z_3z_4z_5z_6z_8, z_2, \ldots, z_8) \in \mathbb{K}^8$ and the blow ups are

$$\begin{array}{lcllcl} \pi_4([z]) & = & [z_1, \ldots, z_4, z_5, z_6, z_7z_9, z_8z_9], & \pi_3([z]) & = & [z_1, \ldots, z_4, z_5z_7, z_6z_7], \\ \pi_2([z]) & = & [z_1, z_2z_5, z_3, z_4z_5], & \pi_1([z]) & = & [z_1, z_2z_4, z_3z_4]. \end{array}$$

Note that after having eliminated redundant equations with Algorithm 4.3.3 the exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1;\, T_4), \qquad V(X_1;\, T_5), \qquad V(X_1;\, T_6), \qquad V(X_1;\, T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

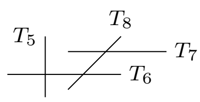
(4)  (5) 

For the first configuration, Algorithm 4.5.9 returns a $\mathbb{K}^*$-surface where as input we used $q := [0,1,0,1,1,1,0,1] \in X_1$, a point which exists by Algorithm 2.3.8 and satisfies

$$\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3 \circ \iota_2^{-1} \circ \pi_4 \circ \iota_3 \circ \varphi(q) \;=\; [0,1,0] \;\in\; \mathbb{P}_2.$$

We come to the second configuration, i.e., blow ups of X_1 of type $\mathrm{Bl}\,\mathbb{P}_2(\star^5)$. Note that the curve $V(X_1;\, T_5)$ is a parabolic fixed point curve. This means that blowing up any point q lying in $V(X_1;\, T_5)$ or in the intersection of two invariant divisors $V(X_1;\, T_i) \cap V(X_1;\, T_j)$ leads to a surface with a non-trivial $\mathbb{K}^*$-action and thus will be left out, see Lemma 5.4.11. By Remark 5.3.5, it suffices to consider the points

$$q_1 := [1,1,1,0,1,1,1,1], \qquad q_2 := [1,1,1,1,1,0,1,1], \qquad q_3 := [1,1,1,1,1,1,1,0].$$



All points exist by Lemma 5.2.16. Using Algorithm 4.5.9, we computed the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix
q_1	$\mathbb{K}[T_1,\ldots,T_9]/I$ with I generated by $T_5T_6T_7^2 - T_1T_2 - T_3T_9T_2^2T_4^2T_5^2T_7^2$ $+2T_3T_9^2T_2T_4T_5T_7T_8 - T_3T_9^3T_8^2$	$\left[\begin{array}{rrrrrrrrr} 1&0&0&0&0&1&0&-1&1\\ 0&1&0&0&0&1&0&2&-1\\ 0&0&1&0&0&0&0&1&-1\\ 0&0&0&1&0&0&0&3&-2\\ 0&0&0&0&1&1&-1&0&0\\ 0&0&0&0&0&2&-1&-3&2 \end{array}\right]$
q_2	$\mathbb{K}[T_1,\ldots,T_{10}]/I$ with I generated by $T_2T_3T_4T_5 - T_7T_8 - T_9T_{10}$, $T_3^2T_4 - T_1T_2 + T_6T_7T_8^2T_{10}$	$\left[\begin{array}{rrrrrrrrrr} 1&0&0&1&0&0&0&1&2&-1\\ 0&1&0&1&0&0&0&2&5&-3\\ 0&0&1&-2&0&0&0&-1&-3&2\\ 0&0&0&0&1&0&0&1&3&-2\\ 0&0&0&0&0&1&0&0&1&-1\\ 0&0&0&0&0&0&1&-1&-1&1 \end{array}\right]$
q_3	$\mathbb{K}[T_1,\ldots,T_9]/I$ with I generated by $T_3^2T_4 - T_1T_2 + T_6^2T_7^2T_9^2T_2^2T_4T_5^2$ $-T_6T_7^2T_9^3T_8$	$\left[\begin{array}{rrrrrrrrr} 1&0&0&1&0&0&0&1&0\\ 0&1&0&1&0&0&0&4&-1\\ 0&0&1&-2&0&0&0&-3&1\\ 0&0&0&0&1&0&0&3&-1\\ 0&0&0&0&0&1&0&2&-1\\ 0&0&0&0&0&0&1&1&-1 \end{array}\right]$

Both the blow up of X_1 in q_1 and the blow up of X_1 in q_3 are $\mathbb{K}^*$-surfaces since their respective Cox rings are isomorphic to the Cox rings

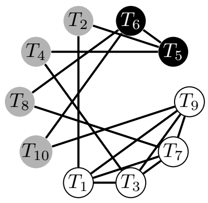
$$\mathbb{K}[T_1,\ldots,T_9] \;/\; \langle T_5T_6T_7^2 - T_1T_2 - T_3T_9^3T_8^2 \rangle,$$
$$\mathbb{K}[T_1,\ldots,T_9] \;/\; \langle T_3^2T_4 - T_1T_2 - T_6T_7^2T_9^3T_8 \rangle$$

of $\mathbb{K}^*$-surfaces with the same degree matrices as listed in the table; see Lemma 5.1.5. The isomorphisms are induced by the graded homomorphisms $\mathbb{K}[T_1,\ldots,T_9] \to$

$\mathbb{K}[T_1, \ldots, T_9]$ where

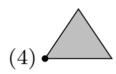
$$\begin{array}{lllll} T_1 & \mapsto & T_1 - T_2T_3T_4^2T_5^2T_7^2T_9 + 2T_3T_4T_5T_7T_8T_9^2, & T_i \ \mapsto \ T_i & \text{for } i \neq 1, \\ T_1 & \mapsto & T_1 + T_2T_4T_5^2T_6^2T_7^2T_9^2, & T_i \ \mapsto \ T_i & \text{for } i \neq 1, \end{array}$$

respectively. Denote by $X_2 := \mathrm{Bl}\ \mathbb{P}_2(\star^5\ ii)$ the blow up of X_1 in q_2. Using Algorithm 2.3.27 its graph of exceptional curves G_{X_2} is



where gray and black vertices stand for negative curves. Black vertices stand for curves intersecting at least three other negative curves. Since both of them have non-trivial intersection, by Lemma 5.4.12, X_2 is not a $\mathbb{K}^*$-surface.

(IV) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^4\ xiv)$. As seen in the proofs of Proposition 5.2.8 and Theorem 5.3.1, the point configuration and blow up sequence for X_1 are



$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star^3\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star^3\ i) \xrightarrow{\pi_1 \circ \pi_2 \circ \pi_3} \mathbb{P}_2$$

Here, the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_1^2T_2 - T_3^3T_4^2T_5$$

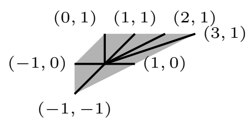
where $h_1 \in \mathbb{K}[T_1, \ldots, T_6]$ and the blow ups are

$$\begin{array}{lll} \pi_4([z]) = [z_1, \ldots, z_4, z_5, z_6z_8, z_7z_8], & & \pi_3([z]) = [z_1, z_2z_6, z_3, z_4, z_5z_6], \\ \pi_2([z]) = [z_1, z_2z_5, z_3, z_4z_5], & & \pi_1([z]) = [z_1, z_2z_4, z_3z_4]. \end{array}$$

The exceptional divisors of the first, second, third and fourth blow up are $V(X_1;\, T_i)$ with $i = 4, 5, 6, 8$. On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

For the first configuration, the direct blow up of X_1 needs higher multiplicities in Algorithm 4.5.9, see Example 4.5.11 for this very instance. To keep our description concise, we instead blow up the toric variety Z_1 with fan Σ_1 and ray generators

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 3 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad \Sigma_1 \ =$$



in a point q in the exceptional divisor $V(Z_1;\, T_6)$. The choice $q := [1, 1, 1, 1, 1, 0] \in Z_1$ is viable by Lemma 5.2.16. Algorithm 4.5.9 then delivers the Cox ring of a $\mathbb{K}^*$-surface.

Next, we treat the second configuration, i.e., blow ups of X_1 of type $\mathrm{Bl}\ \mathbb{P}_2(\star^5)$. Note that the curve $V(X_1;\, T_6)$ is a parabolic fixed point curve. This means that blowing up any point q lying in $V(X_1;\, T_6)$ or in the intersection of two invariant divisors $V(X_1;\, T_i) \cap V(X_1;\, T_j)$ leads to a surface with a non-trivial $\mathbb{K}^*$-action and thus will be left out, see Lemma 5.4.11. By Remark 5.3.5, it suffices to consider the points

$$q_1 \;:=\; [1,1,1,1,1,1,1,0], \qquad q_2 \;:=\; [1,1,1,1,0,1,-1,1],$$
$$q_3 \;:=\; [1,1,1,0,1,1,-1,1] \;\in\; X_1$$

all of which exist by Lemma 5.2.16. Using Algorithm 4.5.9, we computed the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix
q_1	$\mathbb{K}[T_1,\ldots,T_9]/I$ with I generated by $T_3^3T_4^2T_5 - T_1^2T_2 + T_7T_9^2T_8$ $-T_7T_9T_1T_2T_3T_4T_5T_6$	$\left[\begin{array}{rrrrrrrrr} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$
q_2	$\mathbb{K}[T_1,\ldots,T_{10}]/I$ with I generated by $T_1T_2T_6T_8 - T_3^2T_4 - T_9T_{10}$, $T_1^2T_2 + T_7T_8 - T_3^3T_4^2T_5T_{10}$	$\left[\begin{array}{rrrrrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$
q_3	$\mathbb{K}[T_1,\ldots,T_9]/I$ with I generated by $T_1^2T_2 + T_6T_7 + T_3^2T_4T_9^5T_8^3$ $-T_3^2T_4^4T_9^2T_2^3T_5^6T_7^6$ $+3T_3^2T_4^3T_9^3T_2^2T_5^4T_7^4T_8$ $-3T_3^2T_4^2T_9^4T_2T_5^2T_7^2T_8^2$	$\left[\begin{array}{rrrrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 9 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 7 & -4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -5 & 3 \end{array}\right]$

Note that by the blow up sequence the blow up X_2 of X_1 in q_1 is a weak del Pezzo surface. Using Algorithm 2.3.27 in conjunction with Algorithm 2.3.48, we obtain the graph of exceptional curves G_{X_2}. The subgraph of (-2)-curves is

$$\begin{tikzcd} T_7 \arrow[r, dash] & T_6 \arrow[r, dash] \arrow[d, dash] & T_5 \arrow[r, dash] & T_4 \\ & T_2 & & \end{tikzcd}$$

In particular, X_2 has ADE singularity type D_5. By [33, Sec. 5.5], the Cox ring of X_2 is isomorphic to the Cox ring of a $\mathbb{K}^*$-surface

$$\mathbb{K}[T_1,\ldots,T_9] \,/\, \langle T_3^3T_4^2T_5 - T_1^2T_2 + T_7T_9^2T_8 \rangle.$$

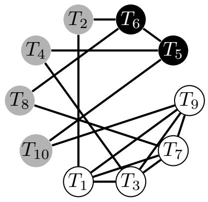
Also, the blow up of X_1 in q_1 is a $\mathbb{K}^*$-surface since its Cox ring is isomorphic to the Cox ring of a $\mathbb{K}^*$-surface Y_2 where

$$\mathcal{R}(Y_2) \;=\; \mathbb{K}[T_1,\ldots,T_9] \;/\; \langle T_1^2T_2 + T_6T_7 + T_3^2T_4T_9^5T_8^3 \rangle$$

with the same $\mathbb{Z}^6$-grading as $\mathcal{R}(X_2)$, see Lemma 5.1.5. The isomorphism is induced by the $\mathbb{Z}^6$-graded isomorphism $\mathbb{K}[T_1,\ldots,T_9] \to \mathbb{K}[T_1,\ldots,T_9]$ with

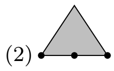
$$T_i \;\mapsto\; \begin{cases} T_6 + T_3^2T_4^4T_9^2T_2^3T_5^6T_7^5 - 3T_3^2T_4^3T_9^3T_2^2T_5^4T_7^3T_8 \\ \quad +3T_3^2T_4^2T_9^4T_2T_5^2T_7T_8^2, & i = 6, \\ T_i, & i \neq 6. \end{cases}$$

Redefine $X_2 := \mathrm{Bl}\,\mathbb{P}_2(\star^5\ iii)$ as the blow up of X_1 in q_2. Using Algorithm 2.3.27 its graph of exceptional curves G_{X_2} is



As before, gray and black vertices stand for negative curves. If X_2 were a $\mathbb{K}^*$-surface, the black vertices correspond to sink and source. Since they have non-trivial intersection, by Lemma 5.4.12, X_2 is not a $\mathbb{K}^*$-surface.

(V) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\star \ \ ii)$. We recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star \ \ ii)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star \ \ ii) \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{P}_2(\star\star\ ) \xrightarrow{\pi_1 \circ \pi_2} \mathbb{P}_2$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 \ := \ T_2T_4T_6^2 - T_1T_5$$

where $h_1 \in \mathbb{K}[T_1, \dots, T_6]$ and the blow ups π_i are

$$\begin{array}{rclrcl} \pi_4([z]) & = & [z_1, z_2, z_3z_8, z_4, z_5, z_6, z_7z_8], & \pi_3([z]) & = & [z_1, z_2z_6, z_3, z_4z_6, z_5], \\ \pi_2([z]) & = & [z_1z_5, z_2, z_3z_5, z_4], & \pi_1([z]) & = & [z_1, z_2z_4, z_3z_4]. \end{array}$$

The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1; T_4), \qquad V(X_1; T_5), \qquad V(X_1; T_6), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

Let $\lambda \in \mathbb{K}^* \setminus \{1\}$. For the first and second configuration, we choose in X_1 the following points q_i which exist by Algorithm 2.3.8

$$\begin{array}{rclrcl} q_1 & := & [1, \lambda, 0, 1, 1, 1, \lambda - 1, 1], & \pi_1 \circ \pi_2 \circ \pi_3 \circ \iota_1^{-1} \circ \pi_4(q_1) & = & [1, \lambda, 0] \ \in \ \mathbb{P}_2, \\ q_2 & := & [0, 0, 1, 1, 1, 1, 0, 1], & \pi_1 \circ \pi_2 \circ \pi_3 \circ \iota_1^{-1} \circ \pi_4(q_2) & = & [0, 0, 1] \ \in \ \mathbb{P}_2. \end{array}$$

Since $V(X_1; T_3)$ is a parabolic fixed point curve, by Lemma 5.4.11, the blow up of X_1 in q_1 will be a toric surface or a $\mathbb{K}^*$-surface. Using Algorithm 4.5.9, we see that the same is true for the blow up in q_2.

For the third configuration, we want to blow up a point in the second exceptional divisor, i.e., $V(X_1; T_5)$. By Lemma 5.4.11, we need not consider points whose Cox coordinates have two vanishing entries. By Remark 5.3.5, this leaves us with $q_3 := [1, 1, 1, 1, 0, 1, 1, 1]$. It exists by Lemma 5.2.16. Algorithm 4.5.9 returns the ring and degree matrix

$$R_2 \ := \ \mathbb{K}[T_1, \dots, T_9] \,/\, \langle T_1T_3T_5^2 - T_4T_9T_2T_3T_5T_7 + T_4T_9^2T_8 - T_6T_7 \rangle,$$

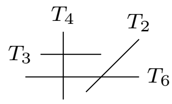
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}.$$

Observe that the blow up of X_1 in q_3 is in fact a $\mathbb{K}^*$-surface; by Lemma 5.1.5, the graded homomorphism $T_6 \mapsto T_6 - T_4T_9T_2T_3T_5$ induces an isomorphism from the Cox ring R_2' of a $\mathbb{K}^*$-surface with the same degree matrix to R_2 where

$$R_2' \;:=\; \mathbb{K}[T_1, \ldots, T_9] \,/\, \langle T_1T_3T_5^2 + T_4T_9^2T_8 - T_6T_7 \rangle.$$

We come to the fourth configuration, i.e., blow ups of X_1 of type Bl $\mathbb{P}_2(\star^3 \star\star)$. By Lemma 5.4.11, we need not consider points that lie in the fixed point curve $V(X_1; T_3)$ or in the intersection of two invariant curves $V(X_1; T_i) \cap V(X_1; T_j)$. Thus, by Remark 5.3.5, it suffices to consider

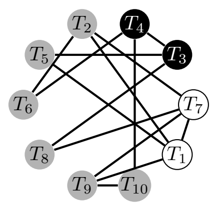
$$q_1 \;:=\; [-1, 1, 1, 0, 1, 1, 1, 1], \qquad q_2 \;:=\; [-1, 1, 1, 1, 1, 0, 1, 1],$$

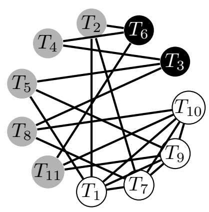


By Lemma 5.2.16, all points exist. Using Algorithm 4.5.9, we computed the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix
q_1	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $T_3T_5T_8 - T_2T_6 - T_9T_{10}$, $T_1T_5 + T_7T_8 - T_2T_6^2T_4T_{10}$	$\left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 3 & -2 \end{smallmatrix}\right]$
q_2	$\mathbb{K}[T_1, \ldots, T_{11}]/I$ with I generated by $T_3^2T_4T_5^2T_8 - T_2T_7 - T_{11}T_{10}$, $T_2^2T_4T_6^2T_{11} - T_5T_9 + T_8T_{10}$, $T_1T_5 + T_7T_8 - T_2T_4T_6^2T_{11}^2$, $T_3^2T_4T_5T_8^2 + T_1T_2 - T_9T_{11}$, $T_3^2T_4^2T_5T_8T_2T_6^2T_{11} - T_7T_9 - T_1T_{10}$	$\left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 3 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \end{smallmatrix}\right]$

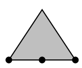
Denote by Bl $\mathbb{P}_2(\star^3 \star\star \;\; i)$ and Bl $\mathbb{P}_2(\star^3 \star\star \;\; ii)$ the blow ups of X_1 in q_2 and q_4. By an application of Algorithm 2.3.27, their respective graphs of exceptional curves are





where gray and black vertices stand for negative curves. Black vertices correspond to curves that meet at least three other negative curves. By Lemma 5.4.12, neither Bl $\mathbb{P}_2(\star^3 \star\star\, i)$ nor Bl $\mathbb{P}_2(\star^3 \star\star\, ii)$ is a $\mathbb{K}^*$-surface.

(VI) Blow ups of $X_1 :=$ Bl $\mathbb{P}_2(\star^2 \star\star \;\; iv)$. As in the previous case, we recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence

(2) 

$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star \;\; ii)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star \;\; ii) \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{P}_2(\star\star) \xrightarrow{\pi_1 \circ \pi_2} \mathbb{P}_2$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_2T_4T_6 - T_1T_5$$

where $h_1 \in \mathbb{K}[T_1, \dots, T_6]$ and the blow ups π_i are

$$\begin{aligned} \pi_4([z]) &= [z_1, z_2, z_3z_8, z_4, z_5, z_6, z_7z_8], & \pi_3([z]) &= [z_1, z_2, z_3z_6, z_4z_6, z_5], \\ \pi_2([z]) &= [z_1z_5, z_2, z_3z_5, z_4], & \pi_1([z]) &= [z_1, z_2z_4, z_3z_4]. \end{aligned}$$

The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1; T_4), \qquad V(X_1; T_5), \qquad V(X_1; T_6), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

For the first and second configuration the following points $q_i \in X_i$ exist by an application of Algorithm 2.3.8 and satisfy

$$\begin{aligned} q_1 &:= [1, \lambda, 0, 1, 1, 1, \lambda - 1, 1], & \pi_1 \circ \pi_2 \circ \pi_3 \circ \iota_1^{-1} \circ \pi_4(q_1) &= [1, \lambda, 0] \in \mathbb{P}_2, \\ q_2 &:= [0, 0, 1, 1, 1, 1, 0, 1], & \pi_1 \circ \pi_2 \circ \pi_3 \circ \iota_1^{-1} \circ \pi_4(q_2) &= [0, 0, 1] \in \mathbb{P}_2. \end{aligned}$$

Since $V(X_1; T_3)$ is a parabolic fixed point curve, by Lemma 5.4.11, the blow up of X_1 in q_1 will be a toric surface or a $\mathbb{K}^*$-surface. Also, the blow up in q_2 admits a non-trivial $\mathbb{K}^*$-action as can be seen by an inspection of the output of Algorithm 4.5.9.

For the third configuration, we want to blow up a point in the second exceptional divisor, i.e., $V(X_1; T_5)$. By Lemma 5.4.11 we need not consider points whose Cox coordinates have two vanishing entries. By Remark 5.3.5, this leaves us with $q_3 := [1, 1, 1, 1, 0, 1, 1, 1] \in X_1$ which exists by Lemma 5.2.16. Algorithm 4.5.9 returns the ring and degree matrix

$$R_2 := \mathbb{K}[T_1, \dots, T_9] \,/\, \langle T_1T_3T_5 - T_6T_7 + T_4T_9^2T_8 - T_4T_9T_2T_3T_5^2T_7 \rangle,$$

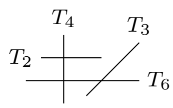
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}.$$

Observe that the blow up of X_1 in q_3 is in fact a $\mathbb{K}^*$-surface: the graded homomorphism $T_6 \mapsto T_6 - T_4T_9T_2T_3T_5^2$ induces an isomorphism from the Cox ring R_2' of a $\mathbb{K}^*$-surface with the same degree matrix to R_2 where

$$R_2' = \mathbb{K}[T_1, \dots, T_9] \,/\, \langle T_1T_3T_5 - T_6T_7 + T_4T_9^2T_8 \rangle,$$

see Lemma 5.1.5. We come to the fourth configuration, i.e., we consider blow ups of X_1 of type Bl $\mathbb{P}_2(\star^3 \star \star)$. By Lemma 5.4.11 we need not consider points that lie in the fixed point curve $V(X_1; T_3)$ or in the intersection of two invariant curves. Thus, by Remark 5.3.5, it suffices to consider the points

$$\begin{aligned} q_1 &:= [-1, 1, 1, 1, 1, 0, 1, 1], \\ q_2 &:= [-1, 1, 1, 0, 1, 1, 1, 1]. \end{aligned}$$



By Lemma 5.2.16, all points exist. Using Algorithm 4.5.9, we computed the following Cox rings.

q_i	Cox ring $\mathcal{R}(X)$	degree matrix

q_1	$\mathbb{K}[T_1, \ldots, T_{10}]/I$ with I generated by $T_1T_5 + T_7T_8 - T_2T_4T_6T_{10}$, $T_3T_5T_7T_8^2 - T_2^2T_4 - T_9T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$
q_2	$\mathbb{K}[T_1, \ldots, T_9]/I$ with I generated by $T_1T_4 + T_6T_7 + T_3T_5T_9^2T_8$ $-T_3T_5^2T_9T_2T_4T_7$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}$

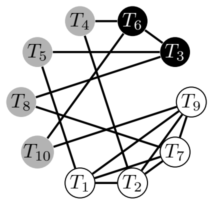
The blow up of X_1 in q_2 is a $\mathbb{K}^*$-surface since its Cox ring is isomorphic to the Cox ring of a $\mathbb{K}^*$-surface

$$\mathbb{K}[T_1, \ldots, T_9] \,/\, \langle T_1T_4 + T_6T_7 + T_3T_5T_9^2T_8 \rangle$$

with the same degree matrix as listed in the table, see Lemma 5.1.5. The isomorphism is induced by the $\mathbb{Z}^6$-graded homomorphism

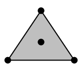
$$\mathbb{K}[T_1, \ldots, T_9] \ \to \ \mathbb{K}[T_1, \ldots, T_9], \qquad T_i \ \mapsto \begin{cases} T_1 + T_3T_5^2T_9T_2T_7, & i = 1, \\ T_i, & i \neq 1. \end{cases}$$

Denote by $X_2 := \mathrm{Bl}\ \mathbb{P}_2(\star^3 \star\star\ iii)$ the blow up of X_1 in q_2. Using Algorithm 2.3.27 we obtain the graph G_{X_2} of exceptional curves

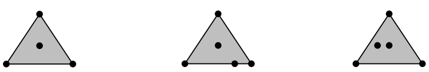


Gray and black vertices stand for negative curves where the latter correspond to curves intersecting at least three other negative curves; if X_2 were a $\mathbb{K}^*$-surface they must be sink and source. By Lemma 5.4.12, $\mathrm{Bl}\ \mathbb{P}_2(\star^3\star\star\ iii)$ cannot be a $\mathbb{K}^*$-surface.

(VII) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\ i)$. Recall from Theorem 5.3.1 that X_1 was obtained from the point configuration



By Proposition 5.1.2 and Theorem 5.3.1, we have to blow up the point configurations

(2) 

For the first configuration, by Proposition 5.2.8, instead of blowing up X_1 we may also blow up a general point in $Z_1 := \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\star\ i)$ which we obtained as

$$Z_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\ i) \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{P}_2(\star\star\ ) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{P}_2(\star) \xrightarrow{\pi_1} \mathbb{P}_2$$

with the blow ups π_i given by

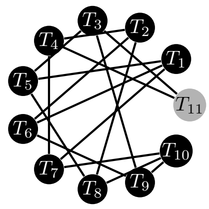
$$\begin{array}{rclrcl} \pi_4([z]) & = & [z_1z_7, z_2z_7, z_3, \ldots, z_6], & \pi_3([z]) & = & [z_1, z_2z_6, z_3, z_4z_6, z_5], \\ \pi_2([z]) & = & [z_1z_5, z_2, z_3z_5, z_4], & \pi_1([z]) & = & [z_1, z_2z_4, z_3z_4]. \end{array}$$

Clearly, the point $p_1 := [1,1,1,1,1,1,1] \in Z_1$ projects to $[1,1,1] \in \mathbb{P}_2$ under the map $\pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4$. Algorithm 4.5.9 then returns the surface $X_2 := \mathrm{Bl}\ \mathbb{P}_2(\star^2 \star\star\star\ i)$ with Cox ring and degree matrix

$$\mathbb{K}[T_1,\ldots,T_{11}] \Big/ \left\langle \begin{array}{l} T_4T_6T_8 + T_5T_9 - T_7T_{10},\ T_2T_6T_9 + T_1T_8 - T_3T_{10}, \\ T_3T_4T_6 - T_1T_7 - T_9T_{11},\ T_2T_6T_7 - T_3T_5 - T_8T_{11}, \\ T_2T_4T_6^2 - T_1T_5 - T_{11}T_{10} \end{array} \right\rangle,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Since X_2 can be obtained as a blow up of the surface $\mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\ i)$ without $\mathbb{K}^*$-action, X_2 cannot be a $\mathbb{K}^*$-surface: equivariant contractions preserve $\mathbb{K}^*$-actions. Alternatively, this can be seen by an inspection of G_{X_2} and Lemma 5.4.12:



where, in this case, all vertices are negative curves. If the surface were a $\mathbb{K}^*$-surface, each black vertex had to be sink or source, a contradiction to Lemma 5.4.12.

For the second and the third configuration we recall from the proofs of Theorem 5.3.1 and Proposition 5.2.8 the blow up sequence

$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\ \mathbb{P}_2(\star\star\star\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{P}_2(\star\star\star\ i) \xrightarrow{\pi_{3,2,1}} \mathbb{P}_2$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^9, \qquad z \mapsto (z, h_1(z), h_2(z), h_3(z)),$$
$$h_1 := T_3T_5 - T_2T_6, \qquad h_2 := T_3T_4 - T_1T_6, \qquad h_3 := T_2T_4 - T_1T_5$$

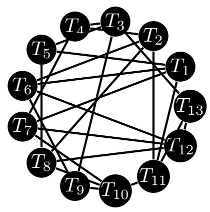
where $h_i \in \mathbb{K}[T_1,\ldots,T_6]$ and the blow ups are

$$\pi_4([z]) = [z_1,\ldots,z_6, z_7z_{10}, z_8z_{10}, z_9z_{10}], \qquad \pi_{3,2,1}([z]) = [z_1z_5z_6, z_2z_4z_6, z_3z_4z_5].$$

For the second configuration, for each $\lambda \in \mathbb{K}^* \setminus \{1\}$, the following point $q \in X_1$ exists by Lemma 5.2.16 where

$$\pi_{3,2,1} \circ \iota_1^{-1} \circ \pi_4(q) \ = \ [1,\lambda,0] \ \in \ \mathbb{P}_2, \qquad q \ := \ [1,\lambda,0,1,1,1,-\lambda,-1,\lambda-1,1].$$

By an application of Algorithm 4.5.9 with Remark 5.3.6 we obtain the surface $X_2 := \mathrm{Bl}\ \mathbb{P}_2(\star\star\star\star\star\ i)$ as listed in the table. By the same reasoning as before, X_2 cannot be a $\mathbb{K}^*$-surface since it is a blow up of X_1. Alternatively, this can be seen by an inspection of G_{X_2} and Lemma 5.4.12:

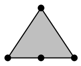


For the third configuration, let λ, μ be distinct elements of $\mathbb{K}^* \setminus \{1\}$. The following point $q \in X_1$ exists and satisfies

$$\pi_{1,2,3} \circ \iota^{-1} \circ \pi_4(q) \ = \ [1,\lambda,\mu] \ \in \ \mathbb{P}_2, \qquad q \ := \ [1,\lambda,\mu,1,1,1,\mu-\lambda,\mu-1,\lambda-1,1].$$

Using Algorithm 4.5.9, we obtain the surfaces $X_2 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\; ii)$ depending on λ and μ. By the same reasoning as before, X_2 cannot be a $\mathbb{K}^*$-surface.

(VIII) Blow ups of $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\; ii)$*.* Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_4} \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\; i)' \xleftarrow{\iota_1} \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\; i) \xrightarrow{\pi_{3,2,1}} \mathbb{P}_2$$

where the embedding ι_1 is as in Setting 4.2.9 and the blow ups, in the situation of Setting 4.2.5, are

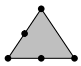
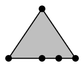
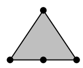
$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_2T_4 - T_1T_5,$$

$$\pi_4([z]) = [z_1, z_2, z_3z_8, z_4, z_5, z_6, z_7z_8], \qquad \pi_{3,2,1}([z]) = [z_1z_5z_6, z_2z_4z_6, z_3z_4z_5]$$

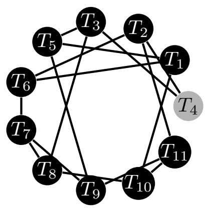
with $h_1 \in \mathbb{K}[T_1, \ldots, T_6]$. The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1; T_4), \qquad V(X_1; T_5), \qquad V(X_1; T_6), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

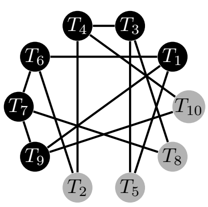


(2) 

For the first configuration, the point $q_1 := [1, 0, 1, 1, 1, 1, -1, 1] \in X_1$ exists by Lemma 5.2.16 and is mapped to $[1, 0, 1] \in \mathbb{P}_2$ under $\pi_{3,2,1} \circ \iota_1^{-1} \circ \pi_4$. Using Algorithm 4.5.9 we obtain the surface $X_2 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\; iii)$ listed in the table. An inspection of its graph of exceptional curves G_{X_2}, shows that Lemma 5.4.12 applies, compare Algorithm 2.3.27. This means that X_2 cannot be a $\mathbb{K}^*$-surface as there are adjacent black vertices, i.e., sink and source would meet.

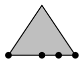


For the second configuration, we consider the points $q_2 := [1, \lambda, 0, 1, 1, 1, \lambda - 1, 1]$ in X_1 where for each $\lambda \in \mathbb{K}^* \setminus \{1\}$ the point exist by Lemma 5.2.16 and is mapped to $[1, \lambda, 0] \in \mathbb{P}_2$ under $\pi_{3,2,1} \circ \iota_1^{-1} \circ \pi_4$. Since $V(X_1; T_3)$ is a parabolic fixed point curve, by Lemma 5.4.11, the blow up of X_1 in q_2 will admit a non-trivial $\mathbb{K}^*$-action.

For the third configuration, we want to blow up a point in the first exceptional divisor $V(X_1; T_4)$. By Lemma 5.4.11 we need not consider points whose Cox coordinates have two vanishing entries. By Remark 5.3.5, this leaves us with the point $[1, 1, 1, 0, 1, 1, -1, 1] \in X_1$. It exists by Lemma 5.2.16. Algorithm 4.5.9 delivers the Cox ring of the surface $\mathrm{Bl}\,\mathbb{P}_2(\star^2\star\star\star\; iv)$ listed in the table. Arguing as before, by Lemma 5.4.12, its graph of exceptional curves cannot belong to a $\mathbb{K}^*$-surface:



(IX) Blow ups of $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\ iii)$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$\begin{tikzcd}
\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\ ii) \arrow[rr, "\iota_2"'] \arrow[d, "\pi_3"] & & \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\ ii)' & & X_1 \arrow[ll, "\pi_4"'] \\
\mathrm{Bl}\,\mathbb{P}_2(\star\star\ i)' & & \mathrm{Bl}\,\mathbb{P}_2(\star\star\ i) \arrow[ll, "\iota_1"'] \arrow[rr, "\pi_{2,1}"] & & \mathbb{P}_2
\end{tikzcd}$$

where the embeddings ι_i are as in Setting 4.2.9 with

$$\begin{aligned}
\bar{\iota}_1 \colon \mathbb{K}^5 &\to \mathbb{K}^6, & z &\mapsto (z, h_1(z)), & h_1 &:= T_2T_4 - T_1T_5,\\
\bar{\iota}_2 \colon \mathbb{K}^7 &\to \mathbb{K}^8, & z &\mapsto (z, h_2(z)), & h_2 &:= (\lambda-1)T_2T_4 - \lambda T_6T_7
\end{aligned}$$

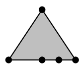
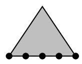
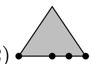
where $h_1 \in \mathbb{K}[T_1,\ldots,T_5]$ and $h_2 \in \mathbb{K}[T_1,\ldots,T_7]$. The blow ups π_i are

$$\pi_4([z]) \;=\; [z_1, z_2, z_3z_9, z_4, \ldots, z_7, z_8z_9],$$
$$\pi_3([z]) \;=\; [z_1, z_2, z_3z_7, z_4, z_5, z_6z_7], \qquad \pi_{2,1}([z]) \;=\; [z_1z_5, z_2z_4, z_3z_4z_5].$$

The exceptional divisors of the first, second, third and fourth blow up are

$$V(X_1;\, T_4), \qquad V(X_1;\, T_5), \qquad V(X_1;\, T_7), \qquad V(X_1;\, T_9).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

  (2) 

The first configuration has already been dealt with in part (VIII) of this proof. For the second configuration, let $\mu \in \mathbb{K}^* \setminus \{1, \lambda\}$. Then each point

$$q_1 \;:=\; [1, \mu, 0, 1, 1, \mu - 1, 1, \lambda - \mu, 1] \;\in\; X_1,$$

exists by Lemma 5.2.16 and projects to $[1, \mu, 0] \in \mathbb{P}_2$ under the map $\pi_{2,1} \circ \iota_1^{-1} \circ \pi_3 \circ \iota_2^{-1} \circ \pi_4$. Since $V(X_1; T_3)$ is a parabolic fixed point curve, by Lemma 5.4.11, the blow up of X_1 in q_1 will admit a non-trivial $\mathbb{K}^*$-action.

For the third configuration, we want to blow up a point in the first exceptional divisor $V(X_1; T_4)$. By Lemma 5.4.11 we need not consider points whose Cox coordinates have two vanishing entries. By Remark 5.3.5, this leaves us with the point $q_3 := [1, 1, 1, 0, 1, -1, 1, \lambda, 1] \in X_1$ which exists by Lemma 5.2.16. Consider the following Cox rings R_1 and R_2 sharing the same degree matrix. The first one is returned by Algorithm 4.5.9 for the blow up X_2 of X_1 in q_3 and the second one is the Cox ring of a $\mathbb{K}^*$-surface:

$$R_1 \;:= \mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} T_7T_8 - (\lambda-1)T_3T_{10}T_2T_4T_6T_8 + (\lambda-1)T_3T_{10}^2T_9 + \lambda T_5T_6, \\ \lambda T_1T_4 - T_7T_8 - T_3T_{10}T_2T_4T_6T_8 + T_3T_{10}^2T_9 \end{array} \right\rangle,$$

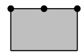
$$R_2 \;:= \mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} T_7T_8 + (\lambda-1)T_3T_{10}^2T_9 + \lambda T_5T_6, \\ \lambda T_1T_4 - T_7T_8 + T_3T_{10}^2T_9 \end{array} \right\rangle,$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & -1\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 5 & -2\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1
\end{bmatrix}.$$

Then X_2 is a $\mathbb{K}^*$-surface, since R_2 is the Cox ring of a $\mathbb{K}^*$-surface. From Lemma 5.1.5 we infer that also X_1 admits a non-trivial $\mathbb{K}^*$-action since we have an isomorphism between the two rings that is induced by the $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1,\ldots,T_{10}] \to \mathbb{K}[T_1,\ldots,T_{10}], \qquad T_i \mapsto \begin{cases} T_5 + T_2T_3T_4T_8T_{10}, & i=5 \\ T_7 - T_2T_3T_4T_6T_{10}, & i=7 \\ T_i, & \text{else.} \end{cases}$$

(X) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ iii)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ i) \xrightarrow{\pi_{2,1}} \mathbb{F}_a$$

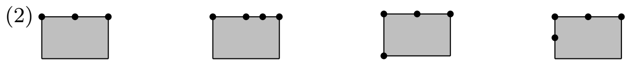
where the embedding ι_1 is as in Setting 4.2.9 and the blow ups, in the situation of Setting 4.2.5, are

$$\bar\iota_1\colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_1T_5 - T_2T_6,$$
$$\pi_3([z]) = [z_1, z_2, z_3z_8, z_4, \ldots, z_6, z_7z_8], \qquad \pi_{2,1}([z]) = [z_1z_5, z_2z_6, z_3z_5z_6, z_4]$$

with $h_1 \in \mathbb{K}[T_1,\ldots,T_6]$. The exceptional divisors of the first, second and third blow up are

$$V(X_1; T_5), \qquad V(X_1; T_6), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations



For the first configuration, we want to blow up a point in the first exceptional divisor, i.e., $V(X_1; T_5)$. By Lemma 5.4.11 we need not consider points whose Cox coordinates have two vanishing entries. By Remark 5.3.5, this leaves us with $q_1 := [1,1,1,1,0,-1,1,1] \in X_1$ which exists by Lemma 5.2.16. The $\mathbb{Z}^5$-graded homomorphism

$$\mathbb{K}[T_1,\ldots,T_8] \to \mathbb{K}[T_1,\ldots,T_8], \quad T_i \mapsto \begin{cases} T_3 + (-1)^a T_1T_2^{a-2}T_4T_6^{a-3}T_7, & i=3, \\ T_i, & i \neq 3 \end{cases}$$

induces an automorphism of $\mathcal{R}(X_1)$. Thus, instead of blowing up q_1 we may blow up the point $[1,1,0,1,0,-1,1,1] \in X_1$. By Lemma 5.4.11, the blow up will be a $\mathbb{K}^*$-surface.

We need not consider the second configuration, since $V(X_1; T_3)$ is a parabolic fixed point curve; by Lemma 5.4.11 the blow up will result in a surface with non-trivial $\mathbb{K}^*$-action. For the third configuration, the point $[0,1,1,0,1,1,-1,1] \in X_1$ projects to $[0,1,1,0] \in \mathbb{F}_a$ under $\pi_{2,1}\circ\iota_1^{-1}\circ\pi_3$ and exists by iteratively applying Lemma 5.4.5 and Lemma 5.2.16. By Lemma 5.4.11, the blow up will admit a non-trivial $\mathbb{K}^*$-action.

For the fourth configuration the steps are as in the proof of Theorem 5.3.1 for, e.g., case $\mathrm{Bl}\ \mathbb{P}_2(\star^3\ ii)$. Here, we want to blow up X_1 in q_4 where

$$\pi_{2,1}\circ\iota_1^{-1}\circ\pi_3(q_4) = [0,1,1,1] \in \mathbb{F}_a, \qquad q_4 := [0,1,1,1,1,1,-1,1] \in X_1.$$

Note that $q_4 \in X_1$ exists by, e.g., Lemma 5.2.16. Choose in $\mathbb{K}[T_1,\ldots,T_8]$ the polynomial $h_2 := T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5$ for the embedding $\bar\iota_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1'

whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \left[Q_1 \left| \begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{smallmatrix} \right. \right], \qquad P_1' = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & a & a-1 & -1 \end{smallmatrix} \right].$$

For the blow up of X_1' in $\iota(q_4) = [0,1,1,1,1,1,-1,1,0]$ we consider the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (1,0,-1,-1) \in \mathbb{Z}^4$. Let $P_2 := [P_1', v]$ be the enlarged matrix. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 &:= p_2^\star (p_1)_\star (T_7T_8 - T_1T_5 + T_2T_6) = T_7T_8 - T_1T_5T_{10} + T_2T_6, \\ g_2 &:= p_2^\star (p_1)_\star (T_9 - h_2) = T_9T_{10} - T_2T_4T_7^{a-1}T_8^{a-2} + T_3T_5. \end{aligned}$$

We show that I_2 is saturated with respect to T_{10} by showing that I_2 is prime. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U := \left\{ x \in \overline{X}_2;\ x_5x_6 \neq 0 \text{ or } x_2x_{10} \neq 0 \right\} \subseteq \overline{X}_2 = V(\mathbb{K}^{10}; I_2).$$

Let $J := (\partial g_i / \partial T_j)_{i,j}$ be the Jacobian matrix. Inspecting the submatrices of J with indices $i = 1,2$ and $j = 2,3$ as well as $i = 1,2$ and $j = 6,9$, respectively, we see that the rank of $J(u)$ is two for all $u \in U$. Furthermore, $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$V\left(\mathbb{K}^{10}; T_5, T_2\right), \quad V\left(\mathbb{K}^{10}; T_5, T_{10}\right), \quad V\left(\mathbb{K}^{10}; T_6, T_2\right), \quad V\left(\mathbb{K}^{10}; T_6, T_{10}\right).$$

We claim that in $\mathbb{K}^{10}$ each of the following intersections is of dimension six.

$$\begin{aligned} \overline{X}_2 \cap V(T_5, T_2) &= V(T_5,\, T_2,\, T_7T_8,\, T_9T_{10}), \\ \overline{X}_2 \cap V(T_5, T_{10}) &= V(T_5,\, T_{10},\, T_7T_8 + T_2T_6,\, T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5), \\ \overline{X}_2 \cap V(T_6, T_2) &= V(T_6,\, T_2,\, T_7T_8 - T_1T_5T_{10},\, T_9T_{10} + T_3T_5), \\ \overline{X}_2 \cap V(T_6, T_{10}) &= V(T_6,\, T_{10},\, T_7T_8,\, T_3T_5), \end{aligned}$$

For all but the second one, this could be done computationally since the equations are independent of a. For $\overline{X}_2 \cap V(T_5, T_{10})$, we used Lemma 5.3.3 with the matrix

$$\left[\begin{smallmatrix} 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & a-1 & a-2 & 0 & 0 \end{smallmatrix} \right]$$

of rank two to see that its dimension is six on the torus $\mathbb{T}^{10} \cdot (1,1,1,1,0,1,1,1,1,0)$. Also, on the smaller tori, the dimension does not exceed six; for instance, for $\mathbb{T}^{10} \cdot (1,0,1,1,0,1,1,1,1,0)$ we consider the zero set $V(\mathbb{K}^{10}; T_2, T_5, T_{10}, T_7T_8, T_3T_5)$ which is of dimension five. Therefore, $\dim(\overline{X}_2 \backslash U) \leq 6$ and, since $\overline{X}_2$ is of dimension at least eight, the codimension of $\overline{X}_2 \setminus U$ in $\overline{X}_2$ is at least two. By Lemma 5.4.3, the ideal I_2 is prime.

We now show that the variable T_{10} defines a prime element in the ring $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$. Removing monomial generators and shifting variable indices, instead of showing that $I_2 + \langle T_{10} \rangle$ is prime, we may show that I_0 has the same property where

$$\begin{aligned} I_2 + \langle T_{10} \rangle &= \langle T_{10},\, T_2T_6 + T_7T_8,\, T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5 \rangle \subseteq \mathbb{K}[T_1, \dots, T_{10}], \\ I_0 &:= \langle T_1T_5 + T_6T_7,\, T_1T_3T_6^{a-1}T_7^{a-2} - T_2T_4 \rangle \subseteq \mathbb{K}[T_1, \dots, T_7]. \end{aligned}$$

The extension of I_0 to $\mathbb{K}[T_1^{\pm 1}, \dots, T_7^{\pm 1}]$ is prime since the matrix with the exponents of the binomial generators as its rows

$$\left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & a-1 & a-2 \end{smallmatrix} \right]$$

has a Smith normal form of shape $[E_2, 0, \dots, 0]$ where E_2 is the 2×2 unit matrix, compare [37, Thm. 2.1]. Now, by Remark 5.4.14, I_0 is prime if $I_0 = I_0 : (T_1 \cdots T_7)^\infty$.

We will prove this by showing that

$$\mathcal{G} \ := \ \{f_1, f_2, f_3\} \ := \ \begin{cases} \left\{ \begin{array}{l} T_1T_5 + T_6T_7, \\ T_1T_3T_6^{a-1}T_7^{a-2} - T_2T_4, \\ T_3T_6^aT_7^{a-1} + T_2T_4T_5 \end{array} \right\}, & T_1T_5 > T_6T_7, \\ \left\{ \begin{array}{l} T_1T_5 + T_6T_7, \\ T_1^{a-1}T_3T_6T_5^{a-2} + (-1)^{a-1}T_2T_4, \\ T_1^aT_3T_5^{a-1} - (-1)^{a-1}T_2T_4T_7 \end{array} \right\}, & T_1T_5 < T_6T_7 \end{cases}$$

is a Gröbner basis for I_0 with respect to the degree reverse lexicographical ordering for any ordering $T_1 > \ldots > T_{i-1} > T_{i+1} > \ldots > T_7 > T_i$ with $1 \le i \le 7$. First, observe that in the case $T_1T_5 > T_6T_7$ we have $I_0 = \langle \mathcal{G} \rangle$ since $f_3 = T_3T_6^{a-1}T_7^{a-2}f_1 - T_5f_2$. In the second case, let g_1 and g_2 be the generators for I_0. Using the relation $T_1T_5 = -T_6T_7$, we have

$$\begin{aligned} f_1 &= g_1, \\ f_2 &= T_1^{a-2}T_3T_5^{a-3}T_6g_1 + (-1)^a g_2, \\ f_3 &= T_1^{a-1}T_3T_5^{a-2}g_1 + (-1)^{a+1}T_7g_2, \end{aligned}$$

i.e., $\langle \mathcal{G} \rangle \subseteq I_0$. The first and second equation also show $I_0 \subseteq \langle \mathcal{G} \rangle$. We compute the S-polynomials. They are

$$\begin{aligned} S(f_1,\, f_2) &= \begin{cases} T_3T_6^aT_7^{a-1} + T_2T_4T_5, & T_1T_5 > T_6T_7, \\ T_1^aT_3T_5^{a-1} - (-1)^{a-1}T_2T_4T_7, & T_1T_5 < T_6T_7, \end{cases} \\ S(f_1,\, f_3) &= \begin{cases} T_3T_6^{a+1}T_7^a - T_1T_2T_4T_5^2, & T_1T_5 > T_6T_7, \\ T_1^{a+1}T_3T_5^a + (-1)^{a-1}T_2T_4T_6T_7^2, & T_1T_5 < T_6T_7, \end{cases} \\ S(f_2,\, f_3) &= \begin{cases} -T_1T_2T_4T_5 - T_2T_4T_6T_7, & T_1T_5 > T_6T_7, \\ (-1)^{a-1}T_2T_4T_6T_7 + (-1)^{a-1}T_1T_2T_4T_5, & T_1T_5 < T_6T_7, \end{cases} \end{aligned}$$

Note that this holds for all $a \ge 3$. Applied to $S(f_i, f_j)$ and f_1, f_2, f_3, the division algorithm, see [26, Ch. 2, Thm. 3], returns the combinations

$$\begin{aligned} S(f_1,\, f_2) &= \begin{cases} f_3, & T_1T_5 > T_6T_7, \\ f_3, & T_1T_5 < T_6T_7, \end{cases} \\ S(f_1,\, f_3) &= \begin{cases} -T_2T_4T_5f_1 + T_6T_7f_3, & T_1T_5 > T_6T_7, \\ T_2T_4T_7f_1 + T_1T_5f_3, & T_1T_5 < T_6T_7, \end{cases} \\ S(f_2,\, f_3) &= \begin{cases} -T_2T_4f_1, & T_1T_5 > T_6T_7, \\ (-1)^{a-1}T_2T_4f_1, & T_1T_5 < T_6T_7. \end{cases} \end{aligned}$$

By the Buchberger criterion, see [26, Ch. 2, Thm. 6], $\mathcal{G}$ is a Gröbner basis for I_0 with respect to the chosen ordering. From [90, Lem. 12.1], we infer that

$$\left\{ \frac{f}{T_i^{k_i(f)}};\ f \in \mathcal{G} \right\} \ = \ \mathcal{G}, \qquad k_i(f) \ := \ \max\big(n \in \mathbb{Z}_{\ge 0};\ T_i^n \mid f\big)$$

is a Gröbner basis for $I_0 : T_i^\infty$ for each $1 \le i \le 7$. In particular, $I_0 = I_0 : T_i^\infty$ for each i. As observed in [90, p. 114], the claim follows from

$$I_0 : (T_1 \cdots T_7)^\infty \ = \ ((\cdots (I_0 : T_1^\infty) \cdots) : T_7^\infty) \ = \ I_0.$$

Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \ne \deg(T_j)$ for all $i \ne j$. We have $T_{10} \nmid T_i$ for each $i < 10$ because each of the following intersections is six-dimensional:

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, -T_2T_6 - T_7T_8, T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_7T_8, T_3T_5),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, -T_2T_6 - T_7T_8, T_2T_4T_7T_8),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3T_5, -T_2T_6 - T_7T_8),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_3T_5, T_2T_6),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3T_5, T_2T_6),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, -T_2T_6 - T_7T_8, T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5),
\end{aligned}$$

where $\overline{X}_2 \cap V(T_5, T_{10})$ and $\overline{X}_2 \cap V(T_6, T_{10})$ have already been treated above. To compute the dimension of $\overline{X}_2 \cap V(T_i, T_{10})$ for $i \in \{1, 9\}$ we use Lemma 5.3.3 with the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & a-1 & a-2 & 0 & 0 \end{bmatrix}$$

of rank two. This shows that on $\mathbb{T}^{10} \cdot (0, 1, \ldots, 1, 0)$ or $\mathbb{T}^{10} \cdot (1, \ldots, 1, 0, 0)$, respectively, the dimension is six. One directly checks that also on the smaller tori the dimension is at most six.

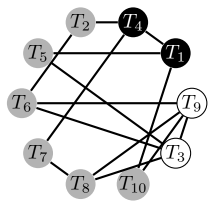
By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with its degree matrix as listed in the table. We now show that we performed the desired blow up. The factor ring $\mathbb{K}[T_1, \ldots, T_9]/I'$ where

$$I' := \langle T_1, T_9, h_2, T_7T_8 - T_1T_5 + T_2T_6 \rangle = \langle T_1, T_9, h_2, T_2T_6 + T_7T_8 \rangle$$

is isomorphic to the integral domain $\mathbb{K}[T_1, \ldots, T_{10}]/(I_2 + \langle T_{10} \rangle)$. Thus, I' is prime. Given Cox coordinates $z := (0, 1, 1, 1, 1, 1, -1, 1, 0) \in \mathbb{K}^9$ for $\iota(q_1) \in X_1'$ we have $z \in V(\mathbb{K}^9\ I')$. By the previous dimension computations

$$\dim\big(V(\mathbb{K}^9;\ I')\big) = -1 + \dim\big(\overline{X}_2 \cap V(T_1, T_{10})\big) = 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. The Cox ring and degree matrix of the resulting surface $X_2 := \mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ vi)$ are listed in the table. Note that X_2 is not a $\mathbb{K}^*$-surface by Lemma 5.4.12 if we can show that its graph of exceptional curves G_{X_2} is



We prove only the part of the graph needed for the argument, i.e., the subgraph induced by the vertices T_i with $i \in \{1, 2, 4, 5, 7, 10\}$. Note that this could be done using the blow up sequence and Remark 5.3.7. Instead, we give a direct argument. Write $w_i := \deg(T_i)$ for the degrees of the generators of $\mathcal{R}(X_2)$ and $Q := Q_2 = [w_1, \ldots, w_{10}]$ for its degree matrix. First, note that $\mathbb{Q}_{\geq 0} \cdot w_3$ and $\mathbb{Q}_{\geq 0} \cdot w_9$ are non-extremal rays of $Q(\mathbb{Q}^{10}_{\geq 0})$:

$$\begin{aligned}
w_3 &= (a-2)w_1 + w_2 + w_4 + (a-3)w_5 + w_7 + (a-2)w_{10},\\
w_9 &= w_1 + (a-2)w_2 + w_4 + w_5 + (a-1)w_6 + w_7.
\end{aligned}$$

All other rays $\mathbb{Q}_{\geq 0} \cdot w_i$ are extremal since w_i can be separated from the other w_j by the following linear forms $u_i \in \mathrm{Hom}(\mathbb{Z}^6, \mathbb{Z}) \cong \mathbb{Z}^6$:

i	$u_i \in \mathrm{Hom}(\mathbb{Z}^6, \mathbb{Z})$	$u_i(w_1), \ldots, u_i(w_{10})$
1	$(-2, 0, 0, 1, 1, 0)$	$-2, 0, 0, 1, 1, 0, 0, 0, 0, 1$
2	$(0, -1, 0, 1, 0, 1)$	$0, -1, 0, 1, 0, 1, 0, 0, 0, 0$
4	$(1, 1, 0, -a, 0, 0)$	$1, 1, 0, -a, 0, 0, 1, 0, 0, 0$
5	$(1, 0, 1, 0, -1, 0)$	$1, 0, 1, 0, -1, 0, 0, 0, 0, 0$
6	$(0, 1, 1, 0, 0, -1)$	$0, 1, 1, 0, 0, -1, 0, 0, 1, 0$
7	$(0, 0, 0, 1, 0, 0)$	$0, 0, 0, 1, 0, 0, -1, 1, 0, 0$
8	$(0, 0, 1, 0, 0, 0)$	$0, 0, 1, 0, 0, 0, 1, -1, 1, 0$
10	$(1, 0, 0, 0, 0, 0)$	$1, 0, 0, 0, 0, 0, 0, 0, 1, -1$

We now claim that $w := (4, 6a-3, 6, 6, 6, 6a-3)$ is an element of the relative interior $\mathrm{Mov}(Q)^\circ$. It suffices to show that for each $1 \le i \le 10$ with $\mathbb{Q}_{\ge 0} \cdot w_i$ a ray of $Q(\mathbb{Q}^{10}_{\ge 0})$ we can find an expression

$$\mathbb{Q}_{\ge 0} \cdot w \ = \ \mathbb{Q}_{\ge 0} \cdot \sum_{\substack{j \ne i,\ \mathbb{Q}_{\ge 0} \cdot w_j \\ \text{extremal}}} \alpha_j w_j, \qquad \alpha_j \in \mathbb{K}^*,$$

because then, by [87, Thm. 6.5], w is an element of

$$\bigcap_{\substack{\mathbb{Q}_{\ge 0} \cdot w_i \\ \text{extremal}}} \mathrm{cone}\,(w_j;\ j \ne i)^\circ \ = \ \left(\bigcap_{\substack{\mathbb{Q}_{\ge 0} \cdot w_i \\ \text{extremal}}} \mathrm{cone}\,(w_j;\ j \ne i) \right)^\circ \ = \ \mathrm{Mov}(Q)^\circ.$$

Here, for $i \in \{1, 2, 4, \ldots, 8, 10\}$, we found the following combinations.

$$\begin{aligned}
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (0, 2, 16, 4, 4, 16, 4a+1, 4a+3+12, 16, 4), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (6a+1, 0, 8, 6, 6a-3, 12, 2, 14, 22, 6a+11)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (2, 2, 34, 0, 8, 44, 3, 45, 50, 24)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (8, 12a-3, 4, 24, 0, 12a-9, 12a-2, 12a-8, 8, 4)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (12a+6, 6, 6, 24, 12a, 0, 12a-9, 12a-15, 6, 12a)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (6a+1, 1, 4, 6, 6a-3, 7, 0, 6, 14, 6a+7)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (12a+6, 12a-9, 6, 24, 12a, 12a-15, 6, 0, 6, 12a)), \\
\mathbb{Q}_{\ge 0} \cdot w \ &= \ \mathbb{Q}_{\ge 0} \cdot (Q \cdot (8, 12a-3, 12, 28, 4, 12a-7, 16a-1, 16a-5, 8, 0)).
\end{aligned}$$

Now, we show that the edges

$$(T_4, T_1), \quad (T_4, T_2), \quad (T_4, T_7), \quad (T_1, T_5), \quad (T_1, T_{10})$$

exist in G_{X_2}. Let $\gamma_{i,j} := \mathrm{cone}(e_k;\ k \notin \{i,j\}) \preceq \gamma$ with $\gamma := \mathbb{Q}^{10}_{\ge 0}$. By Algorithm 2.3.27, this means we have to show that the faces

$$\gamma_{1,4},\ \gamma_{2,4},\ \gamma_{7,4},\ \gamma_{1,5},\ \gamma_{1,10}, \qquad \gamma_{i,j} \ := \ \mathrm{cone}(e_k;\ k \notin \{i,j\}) \ \preceq \ \gamma$$

are I_2-faces in the sense of Chapter 3 and the respective projection $Q(\gamma_{i,j})$ contains w in its relative interior. For the former, define as in Section 1 of Chapter 3 the torus and ideal

$$\mathbb{T}^{10}_{\gamma_{i,j}} \ := \ \left\{ t_{\gamma_{i,j}};\ t \in \mathbb{T}^{10} \right\}, \qquad I_2^{\gamma_{i,j}} \ := \ \{ f_{\gamma_{i,j}};\ f \in I_2 \},$$

where the k-th entry of $z_{\gamma_{i,j}}$ equals z_k if $e_k \in \gamma_{i,j}$ and zero otherwise. Then $\gamma_{i,j}$ is an I_2-face if and only if $V(\mathbb{T}^{10}_{\gamma_{i,j}};\ I_2^{\gamma_{i,j}}) \ne \emptyset$. We directly list elements of the respective vanishing sets.

i, j	$V(\mathbb{T}^{10}_{\gamma_{i,j}};\ I_2^{\gamma_{i,j}})$	contained element
$1, 4$	$V(-T_2T_6 - T_7T_8,$ $-T_3T_5 - T_9T_{10})$	$(0, -1, -1, 0, 1, 1, 1, 1, 1, 1)$
$2, 4$	$V(T_1T_5T_{10} - T_7T_8,$ $-T_3T_5 - T_9T_{10})$	$(1, 0, -1, 0, 1, 1, 1, 1, 1, 1)$

$7,4$	$V(T_1T_5T_{10} - T_2T_6,$ $-T_3T_5 - T_9T_{10})$	$(1,1,-1,0,1,1,0,1,1,1)$
$1,5$	$V(-T_2T_6 - T_7T_8,$ $T_2T_4T_7^{a-1}T_8^{a-2} - T_9T_{10})$	$(0,1,1,1,0,-1,1,1,1,1)$
$1,10$	$V(-T_2T_6 - T_7T_8,$ $T_2T_4T_7^{a-1}T_8^{a-2} - T_3T_5)$	$(0,1,1,1,1,-1,1,1,1,0)$

We now show that for each $\gamma_{i,j}$ the relative interior $Q(\gamma_{i,j})^\circ$ contains the vector $w = (4, 6a-3, 6, 6, 6, 6a-3)$. For this, we present expressions

$$\mathbb{Q}_{\geq 0} \cdot w \;=\; \mathbb{Q}_{\geq 0} \cdot \sum_{e_k \in \gamma_{i,j}} \alpha_k w_k, \qquad \alpha_k \in \mathbb{K}^*.$$

For the respective $\gamma_{i,j}$ they are given by

$$\begin{aligned}
\mathbb{Q}_{\geq 0} \cdot w &= \mathbb{Q}_{\geq 0} \cdot (Q \cdot (0,1,16,0,4,19,2,20,20,8)),\\
\mathbb{Q}_{\geq 0} \cdot w &= \mathbb{Q}_{\geq 0} \cdot (Q \cdot (1,0,15,0,4,18,2,20,21,10)),\\
\mathbb{Q}_{\geq 0} \cdot w &= \mathbb{Q}_{\geq 0} \cdot (Q \cdot (1,2,15,0,4,20,0,18,21,10)),\\
\mathbb{Q}_{\geq 0} \cdot w &= \mathbb{Q}_{\geq 0} \cdot (Q \cdot (0,1,12,4,0,15,4a+2,4a+16,20,8)),\\
\mathbb{Q}_{\geq 0} \cdot w &= \mathbb{Q}_{\geq 0} \cdot (Q \cdot (0,4a+1,16,8,4,4a+11,4a+2,4a+12,12,0)).
\end{aligned}$$

(XI) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ iv)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence

$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ i) \xrightarrow{\pi_{2,1}} \mathbb{F}_a$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 \;:=\; T_2^aT_4T_6^{a-1} - T_3T_5$$

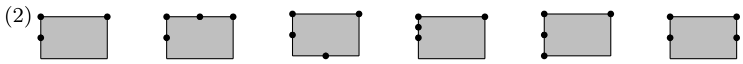
where $h_1 \in \mathbb{K}[T_1,\ldots,T_6]$ and the blow ups π_i are given by

$$\pi_3([z]) \;=\; [z_1z_8, z_2, z_3, z_4, \ldots, z_6, z_7z_8], \qquad \pi_{2,1}([z]) \;=\; [z_1z_5, z_2z_6, z_3z_5z_6, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1;\, T_5), \qquad V(X_1;\, T_6), \qquad V(X_1;\, T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations



For the first configuration, we want to blow up a point in the first exceptional divisor, i.e., $V(X_1;\, T_5)$. By Lemma 5.4.11 and Remark 5.3.5 it suffices to consider the point $q := [1,1,1,1,0,1,1,1] \in X_1$. It exists by Lemma 5.2.16. However, using Lemma 5.1.5, the existence of the automorphism of $\widehat{X}_1$ induced by

$$\mathbb{K}[T_1,\ldots,T_8] \;\to\; \mathbb{K}[T_1,\ldots,T_8], \qquad T_i \;\mapsto\; \begin{cases} T_3 - T_1T_2^{a-1}T_4T_6^{a-2}T_8, & i = 3,\\ T_i, & \text{else}\end{cases}$$

shows that the blow up of X_1 in q is isomorphic to the blow up of X_1 in the point $[1,1,0,1,0,1,1,1]$ which will result in a $\mathbb{K}^*$-surface by Lemma 5.4.11.

The second configuration has already been dealt with in part (X) of this proof. We now treat the third configuration which leads to the surface $\mathrm{Bl}\ \mathbb{F}_a(\star\star\star\star\ vii)$. Here, the point $q_3 := [1,1,1,0,1,1,-1,1] \in X_1$ exists by Lemma 5.2.16 and projects to $[1,1,1,0] \in \mathbb{F}_a$ under $\pi_{2,1} \circ \iota_1^{-1} \circ \pi_3$. Blowing up X_1 in q_3 is done by the same steps as before. Choose $h_2 := T_1T_5T_8 - T_2T_6 \in \mathbb{K}[T_1,\ldots,T_8]$ for the embedding $\bar\imath_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0\\0\\1\\0\\1 \end{smallmatrix} \end{array}\right], \qquad P_1' = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & a \end{smallmatrix}\right].$$

For the blow up of X_1' in $\imath_2(q_3) = [1,1,1,0,1,1,-1,1,0]$ we perform the stellar subdivision of Σ_1' at the vector $v := (-1,-1,0,a+1) \in \mathbb{Z}^4$. Let $P_2 := [P_1', v]$ be the enlarged matrix. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 := p_2^\star (p_1)_\star (T_7T_8 - T_2^aT_4T_6^{a-1} + T_3T_5) = T_7T_8 - T_2^aT_4T_6^{a-1}T_{10} + T_3T_5,$$
$$g_2 := p_2^\star (p_1)_\star (T_9 - h_2) = T_9T_{10} - T_1T_5T_8 + T_2T_6.$$

We show that I_2 is saturated with respect to T_{10} by showing that I_2 is prime. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U := \left\{x \in \overline{X}_2;\ x_1x_5 \neq 0 \text{ or } x_3x_{10} \neq 0\right\} \subseteq \overline{X}_2 = V\left(\mathbb{K}^{10}; I_2\right).$$

Inspecting the indices $i = 1,2$ and $j = 3,8$ as well as $i = 1,2$ and $j = 5,9$ we see that the rank of the Jacobian matrix $(\partial g_i/\partial T_j)_{i,j}(u)$ is two for all $u \in U$. Furthermore, $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$V\left(\mathbb{K}^{10}; T_1, T_3\right), \quad V\left(\mathbb{K}^{10}; T_1, T_{10}\right), \quad V\left(\mathbb{K}^{10}; T_5, T_3\right), \quad V\left(\mathbb{K}^{10}; T_5, T_{10}\right).$$

Note that each of the following intersections is of dimension six

$$\begin{aligned} \overline{X}_2 \cap V(T_1, T_3) &= V(T_1, T_3, T_2T_6 + T_9T_{10}, T_2^aT_6^{a-1}T_{10}T_4 - T_7T_8), \\ \overline{X}_2 \cap V(T_1, T_{10}) &= V(T_1, T_{10}, T_2T_6, T_3T_5 + T_7T_8), \\ \overline{X}_2 \cap V(T_5, T_3) &= V(T_5, T_3, T_2T_6 + T_9T_{10}, T_2^aT_6^{a-1}T_{10}T_4 - T_7T_8), \\ \overline{X}_2 \cap V(T_5, T_{10}) &= V(T_5, T_{10}, T_2T_6, T_7T_8), \end{aligned}$$

where for the first and third variety we write the binomials into a matrix as in Lemma 5.3.3 and obtain the matrix

$$\left[\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 1 \end{smallmatrix}\right]$$

of rank two. By Lemma 5.3.3, the dimensions of $\overline{X}_2 \cap \mathbb{T}^{10} \cdot (0,1,0,1,\ldots,1)$ and of $\overline{X}_2 \cap \mathbb{T}^{10} \cdot (1,1,0,1,0,1,\ldots,1)$ are six respectively. Also, on the smaller tori, we are in dimension at most six. Therefore, $\dim(\overline{X}_2 \setminus U) \leq 6$ and since $\overline{X}_2$ is of dimension at least eight $\operatorname{codim}_{\overline{X}_2}(\overline{X}_2 \setminus U) \geq 2$. By Lemma 5.4.3, the ideal I_2 is prime. We now show that the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\ldots,T_{10}]/I_2$. This is the case since the ideal

$$I_2 + \langle T_{10}\rangle = \langle T_{10}, T_1T_5T_8 - T_2T_6, T_3T_5 + T_7T_8\rangle \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$$

is prime by a computation, see Algorithm 2.2.10. Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for all i, j. Also, $T_i \nmid T_{10}$ for all $i < 10$ because the intersections

$$\begin{aligned} \overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_3T_5 + T_7T_8, -T_1T_5T_8), \\ \overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_7T_8, -T_1T_5T_8 + T_2T_6), \\ \overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3T_5 + T_7T_8, -T_1T_5T_8 + T_2T_6), \\ \overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_3T_5 + T_7T_8, -T_1T_5T_8), \\ \overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_3T_5, -T_1T_5T_8 + T_2T_6), \\ \overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3T_5, T_2T_6), \\ \overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_3T_5 + T_7T_8, -T_1T_5T_8 + T_2T_6) \end{aligned}$$

are all of dimension six as can be seen by computations. By Theorem 4.2.6, the Cox ring and degree matrix of the surface $X_2 = \mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ vii)$ are

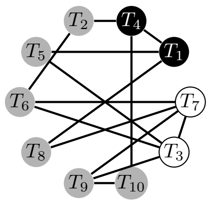
$$\mathcal{R}(X_2) \;=\; \mathbb{K}[T_1,\dots,T_{10}] \Big/ \left\langle \begin{array}{c} T_1T_5T_8 - T_2T_6 - T_9T_{10}, \\ T_2^aT_6^{a-1}T_{10}T_4 - T_3T_5 - T_7T_8 \end{array} \right\rangle,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & a+1 & -a \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & a & -a+1 \end{bmatrix}.$$

We now show that we have performed a blow up. Since we have an isomorphism from

$$\mathbb{K}[T_1,\dots,T_9]\,/\,I', \qquad I' \;:=\; \langle T_4,\, T_9,\, h_2,\, T_7T_8 + T_3T_5\rangle$$

to the integral domain $\mathbb{K}[T_1,\dots,T_{10}]/(I_2+\langle T_{10}\rangle)$ the ideal I' is prime. Consider Cox coordinates $z := (1,1,1,0,1,1,-1,1,0) \in \mathbb{K}^9$ for $\iota(q_3) \in X_1'$. Then $z \in V(\mathbb{K}^9;\, I')$ and $\dim(V(\mathbb{K}^9;\, I')) = 5$ by a computation. Using Lemma 5.2.15, we see that the performed modification was the desired blow up. To show that X_2 is not a $\mathbb{K}^*$-surface we claim that the graph of exceptional curves G_{X_2} is as follows. The assertion then follows from Lemma 5.4.12.



It suffices to prove the existence of the subgraph induced by the vertices T_i with $i \in \{1,2,4,5,8,10\}$. By Remark 5.3.7 and the fact that $V(T_{10})$ is the exceptional divisor of the last blow up, we know that the curves corresponding to the vertices are negative. The existence of the edges, i.e., the fact that the curves meet, is directly seen from the blow up sequence of X_2 as explained above.

We come to the fourth and fifth configurations. Let $\lambda \in \mathbb{K}^* \setminus \{1\}$. For the following points q_4 and $q_5 \in X_1$ we have

$$\begin{aligned} \pi_{2,1}\circ\iota_1^{-1}\circ\pi_3(q_4) &= [0,1,1,\lambda] \in \mathbb{F}_a, & q_4 &:= [0,1,1,\lambda,1,1,\lambda-1,1], \\ \pi_{2,1}\circ\iota_1^{-1}\circ\pi_3(q_5) &= [0,1,1,0] \in \mathbb{F}_a, & q_5 &:= [0,1,1,0,1,1,-1,1]. \end{aligned}$$

The point q_4 exists by Lemma 5.2.16 whereas for q_5 we iteratively use Lemma 5.4.5 and Lemma 5.2.16. The blow ups of X_1 in q_4 and of X_1 in q_5 will admit again a $\mathbb{K}^*$-action by Lemma 5.4.11; for q_4 this is due to the fact that $V(X_1;\, T_1)$ is a parabolic fixed point curve.

For the sixth configuration, let $\kappa \in \mathbb{K}^*$. In X_1, the point $q_6 := [1,0,1,\kappa,1,1,-1,1]$ exists by Lemma 5.2.16 and projects to $[0,1,1,\kappa] \in \mathbb{F}_a$ under $\pi_{2,1}\circ\iota_1^{-1}\circ\pi_3$. We will perfom the blow up X_1 in q_6 by steps similar to before but we will only show finite generation of the Cox ring for $a > 15$. This is done by carrying out Algorithm 4.5.15 in a formal way. Consider the embedding

$$\bar\iota_2\colon \mathbb{K}^8 \;\to\; \mathbb{K}^{10}, \qquad z \;\mapsto\; (z, h_2(z), h_3(z)),$$
$$h_2 \;:=\; T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7, \qquad h_3 \;:=\; T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6$$

with $h_i \in \mathbb{K}[T_1,\dots,T_8]$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We obtain a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' = \left[Q_1 \left| \begin{array}{rr} 1 & 0 \\ 1 & -1 \\ 2a-3 & 2 \\ 2 & 0 \\ 2a-2 & 1 \end{array} \right. \right], \quad P_1' = \left[\begin{array}{rrrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & 0 & -a+1 & 1 & -1 & a-1 \\ 0 & a & 0 & 1 & 0 & 0 & -a & -1 & 0 & a-1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array} \right].$$

For the blow up of X_1' in $\iota_2(q_6) = [1, 0, 1, \kappa, 1, 1, -1, 1, 0, 0]$ we consider the stellar subdivision of Σ_1' at the vector $v := (2a-3, 2a-1, 0, 0, -1) \in \mathbb{Z}^5$. Write P_2 for the enlarged matrix $[P_1', v]$. The extension $I_2' \subseteq \mathbb{K}[T_1, \ldots, T_{10}, T_{11}^{\pm 1}]$ of the vanishing ideal of $\overline{X}_2 \subseteq \mathbb{K}^{11}$ is generated by

$$\begin{aligned} g_1 &:= p_2^\star (p_1)_\star (T_7T_8 - h_1) = T_7T_8 - T_2^aT_4T_6^{a-1}T_{11}^a + T_3T_5, \\ g_2 &:= p_2^\star (p_1)_\star (T_9 - h_2) = T_9T_{11} - T_1^aT_4T_5^aT_8^{a-1} - \kappa T_6T_7, \\ g_3 &:= p_2^\star (p_1)_\star (T_{10} - h_3) = T_{10}T_{11} - T_1^aT_4T_5^{a-1}T_8^a + \kappa T_3T_6. \end{aligned}$$

The next step is to compute the saturated ideal $I_2' : T_{11}^\infty \subseteq \mathbb{K}[T_1, \ldots, T_{11}]$. We claim that it is given by

$$\begin{aligned} I_2' : (T_1 \cdots T_{11})^\infty = \langle & g_1, g_2, g_3, \\ & -\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9 - T_5T_{10}, \\ & T_1^aT_2^aT_4^2T_5^{a-1}T_6^{a-1}T_8^{a-1}T_{11}^{a-1} - T_3T_9 - T_7T_{10} \rangle \\ =: I_2 & \end{aligned}$$

For $3 \leq a \leq 15$ we verified algorithmically that this equality holds. Moreover, we checked that the ideal

$$\begin{aligned} I_2 + \langle T_{11} \rangle = \langle & T_{11}, -T_1^aT_4T_5^aT_8^{a-1} - \kappa T_6T_7, -T_1^aT_4T_5^{a-1}T_8^a + \kappa T_3T_6, \\ & T_7T_8 + T_3T_5, T_8T_9 - T_5T_{10}, T_3T_9 + T_7T_{10} \rangle \end{aligned}$$

in $\mathbb{K}[T_1, \ldots, T_{10}]$ is saturated with respect to $T_1 \cdots T_{10}$ for $a \leq 15$. Since the exponent matrix as in Lemma 5.3.3

$$\left[\begin{array}{rrrrrrrrrrr} 0 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{array} \right]$$

has a Smith normal form of shape $[E_5, 0]$ with the 5×5 unit matrix E_5 Remark 5.4.14 tells us that T_{11} is a prime element for $3 \leq a \leq 15$. We will later compute $\dim(V(T_i, T_j) \cap \overline{X}_2)$ for all $i \neq j$. This will show that no two variables divide one another and are pairwise non-associated. Thus, the Cox ring of the performed modification is $\mathbb{K}[T_1, \ldots, T_{11}]/I_2$ for $3 \leq a \leq 15$ with its degree matrix as listed in the table. We now show that we did perform the desired blow up. The factor ring $\mathbb{K}[T_1, \ldots, T_{10}]/I'$ where

$$\begin{aligned} I' &:= \langle T_2, T_9, T_{10}, h_2, h_3, T_7T_8 - h_1 \rangle \\ &= \langle T_2, T_9, T_{10}, T_7T_8 + T_3T_5, -T_1^aT_4T_5^aT_8^{a-1} - \kappa T_6T_7, \\ &\qquad -T_1^aT_4T_5^{a-1}T_8^a + \kappa T_3T_6 \rangle \end{aligned}$$

is isomorphic to the integral domain $\mathbb{K}[T_1, \ldots, T_{11}]/(I_2 + \langle T_{11} \rangle)$. Thus, I' is prime. Given Cox coordinates

$$z := (1, 0, 1, \kappa, 1, 1, -1, 1, 0, 0) \in \mathbb{K}^{11} \qquad \text{for} \qquad \iota_2(q_6) \in X_1'$$

we have $z \in V(\mathbb{K}^{10}; I')$. By a computation, $\dim V(\mathbb{K}^{10}; I') = 5$ and an application of Lemma 5.2.15 shows that the performed modification was the claimed blow up.

We now show that $\mathcal{R}(X)$ is finitely generated for any $a \geq 3$ using Algorithm 4.5.15 in a formal way. To this end, we first have to show that T_{11} defines a prime element in the ring $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}, T_{11}]/I_2$. Consider the ideal

$$\begin{aligned} J &:= \langle -T_1^a T_4 T_5^a T_8^{a-1} - \kappa T_6 T_7,\ -T_1^a T_4 T_5^{a-1} T_8^a + \kappa T_3 T_6, \\ &\qquad T_7 T_8 + T_3 T_5,\ T_8 T_9 - T_5 T_{10},\ T_3 T_9 + T_7 T_{10} \rangle \\ &\subseteq \ \mathbb{K}[T_1, \ldots, T_{10}] \end{aligned}$$

obtained from $I_2 + \langle T_{11} \rangle \subseteq \mathbb{K}[T_1, \ldots, T_{11}]$ by removing the monomial generator T_{11}. Using the fifth and third generator, in the Laurent polynomial ring $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}]$, we substitute

$$T_{10} \ = \ -\frac{T_3 T_9}{T_7}, \qquad\qquad T_7 \ = \ -\frac{T_3 T_5}{T_8}$$

into the other generators of J and obtain in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}]$ the ideal

$$\begin{aligned} J' &:= \left\langle T_5 \left(-T_1^a T_4 T_5^{a-1} T_8^a + \kappa T_3 T_6 \right),\ -T_1^a T_4 T_5^{a-1} T_8^a + \kappa T_3 T_6, \right. \\ &\qquad \left. T_{10} + \frac{T_3 T_9}{T_7},\ T_7 + \frac{T_3 T_5}{T_8} \right\rangle \\ &= \left\langle -T_1^a T_4 T_5^{a-1} T_8^a + \kappa T_3 T_6,\ T_{10} + \frac{T_3 T_9}{T_7},\ T_7 + \frac{T_3 T_5}{T_8} \right\rangle. \end{aligned}$$

Then $J' \subseteq \mathbb{K}[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}]$ is a prime ideal if $f := -T_1^a T_4 T_5^{a-1} T_8^a + \kappa T_3 T_6$ is a prime element in $\mathbb{K}[T_i^{\pm 1};\ i \notin \{7, 10\}]$ since

$$\mathbb{K}\left[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}\right] / J' \quad \cong \quad \mathbb{K}\left[T_i^{\pm 1};\ i \notin \{7, 10\}\right] / \langle f \rangle.$$

By Lemma 4.3.5, $f \in \mathbb{K}\left[T_i^{\pm 1};\ i \notin \{7, 10\}\right]$, is a prime element because $\langle f \rangle$ is saturated with respect to the product over all variables and defines a prime ideal in $\mathbb{K}[T_i;\ i \notin \{7, 10\}]$ by Lemma 5.2.17. In turn, this shows that T_{11} defines a prime element in $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{10}^{\pm 1}, T_{11}]/I_2$. The next step in Algorithm 4.5.15 is to show that

$$\dim \left(\overline{X}_2 \cap V\left(\mathbb{K}^{11};\, T_i,\, T_j\right) \right) \ \leq \ \dim \left(\overline{X}_2 \right) - 2 \qquad \text{for all} \qquad i \neq j.$$

The dimension of $V(\mathbb{T}^{11};\, I_2') = V(\mathbb{T}^{11};\, I_2)$ is at least eight as it is defined by three equations. Therefore, $V\left(\mathbb{K}^{11};\, I_2\right)$ is of at least eight-dimensional and it suffices to show that the dimension of $\overline{X}_2 \cap V\left(\mathbb{K}^{11};\, T_i,\, T_j\right)$ is at most six for all $i < j$. For $i = 1$ and $j = 2$, the variety

$$\begin{aligned} \overline{X}_2 \cap V(T_1,\, T_2) \ &= \ V(T_1,\, T_2,\, T_3 T_5 + T_7 T_8,\ -\kappa T_6 T_7 + T_9 T_{11}, \\ &\qquad -\kappa T_3 T_6 - T_{11} T_{10},\ T_8 T_9 - T_5 T_{10},\ T_3 T_9 + T_7 T_{10}) \end{aligned}$$

in $\mathbb{K}^{11}$ is of dimension six by a computer check. For $i = 1$ and $j = 3$, we decompose the vanishing set

$$\begin{aligned} \overline{X}_2 \cap V(T_1,\, T_3) \ &= \ V(T_1,\, T_3,\, T_2^a T_4 T_{11}^a T_6^{a-1} - T_7 T_8,\, \kappa T_6 T_7 - T_9 T_{11}, \\ &\qquad T_{11} T_{10},\, T_7 T_{10},\, -\kappa T_2^a T_4 T_6^a T_{11}^{a-1} + T_8 T_9 - T_5 T_{10}) \\ &= \ V(T_1,\, T_3,\, T_7,\, T_{11},\, T_8 T_9 - T_5 T_{10}) \ \cup \\ &\qquad V(T_1,\, T_3,\, T_{10},\, T_2^a T_4 T_{11}^a T_6^{a-1} - T_7 T_8, \\ &\qquad \kappa T_6 T_7 - T_9 T_{11},\, -\kappa T_2^a T_4 T_6^a T_{11}^{a-1} + T_8 T_9) \end{aligned}$$

in $\mathbb{K}^{11}$ into two components. The first one clearly is of dimension six. For the second component, we consider the matrix with the exponent vectors of the three occurring binomials as its rows

$$\begin{bmatrix} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \end{bmatrix},$$

which is of rank two. By Lemma 5.3.3, the component is of dimension six on $\mathbb{T}^{11} \cdot (0,1,0,1,\dots,1,0,1)$. Note that also on the smaller tori the dimension is at most six; e.g., for $\mathbb{T}^{11} \cdot (0,0,0,1,\dots,1,0,1)$ we receive the variety

$$V(T_1,\ T_2,\ T_3,\ T_{10},\ T_7T_8,\ \kappa T_6T_7 - T_9T_{11},\ T_8T_9) \ \subseteq\ \mathbb{K}^{11}$$

of dimension five. The remaining dimension arguments are similar. We restrict ourselves to listing the vanishing sets and the exponent matrices of Lemma 5.3.3 for components with unclear dimension.

i,j	$\overline{X}_2 \cap V(T_i,T_j)$	dimension argument (some components)
$1,4$	$V(T_1,\, T_4,\, -T_3T_5 - T_7T_8,$ $\kappa T_6T_7 - T_9T_{11},$ $-\kappa T_3T_6 - T_{11}T_{10},$	$T_8T_9 - T_5T_{10},$ $-T_3T_9 - T_7T_{10})$ *Method:* computation.
$1,5$	$V(T_1,\, T_5,\, \kappa T_6T_7 - T_9T_{11},$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,$ $-\kappa T_3T_6 - T_{11}T_{10},$	$-\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9,$ $-T_3T_9 - T_7T_{10})$
	$\begin{bmatrix} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$	
$1,6$	$V(T_1,\, T_6,\, T_9T_{11},\, T_{11}T_{10},$ $-T_3T_5 - T_7T_8,$	$T_8T_9 - T_5T_{10},$ $-T_3T_9 - T_7T_{10})$ *Method:* computation.
$1,7$	$V(T_1,\, T_7,\, T_9,$ $-\kappa T_3T_6 - T_{11}T_{10},$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,$	$-\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10})$ $\cup\ V(T_1,\, T_7,\, T_3,\, T_{11},$ $T_8T_9 - T_5T_{10})$
	$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \end{bmatrix}$	
$1,8$	$V(T_1,\, T_8,\, \kappa T_6T_7 - T_9T_{11},$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,$ $-\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10},$	$-\kappa T_3T_6 - T_{11}T_{10},$ $-T_3T_9 - T_7T_{10})$
	$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$	
$1,9$	$V(T_1,\, T_9,\, T_7,\, -\kappa T_3T_6 - T_{11}T_{10},$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,$ $-\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10})$	$\cup\ V(T_1,\, T_9,\, T_6,\, T_{10},$ $-T_3T_5 - T_7T_8)$
	$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \end{bmatrix}$	
$1,10$	$V(T_1,\, T_{10},\, T_3,\, \kappa T_6T_7 - T_9T_{11},$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,$ $-\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9)$	$\cup\ V(T_1,\, T_{10},\, T_6,\, T_9,$ $-T_3T_5 - T_7T_8)$
	$\begin{bmatrix} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \end{bmatrix}$	

$1, 11$ — $V(T_1,\, T_{11},\, T_6T_7,\, T_3T_6,\, T_3T_5 + T_7T_8,\, T_8T_9 - T_5T_{10},\, T_3T_9 + T_7T_{10})$

Method: computation.

$2, 3$ — $V(T_2,\, T_3,\, T_7,\, T_8T_9 - T_5T_{10},\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10}) \cup V(T_2,\, T_3,\, T_8,\, T_{10},\, T_9T_{11} - \kappa T_6T_7)$

$$\begin{bmatrix} a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$2, 4$ — $V(T_2,\, T_4,\, -T_3T_5 - T_7T_8,\, \kappa T_6T_7 - T_9T_{11},\, -\kappa T_3T_6 - T_{11}T_{10},\, T_8T_9 - T_5T_{10},\, -T_3T_9 - T_7T_{10})$

Method: computation.

$2, 5$ — $V(T_2,\, T_5,\, T_7T_8,\, \kappa T_6T_7 - T_9T_{11},\, -\kappa T_3T_6 - T_{11}T_{10},\, T_8T_9,\, -T_3T_9 - T_7T_{10})$

Method: computation.

$2, 6$ — $V(T_2,\, T_6,\, T_8T_9 - T_5T_{10},\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10},\, -T_3T_5 - T_7T_8,\, -T_3T_9 - T_7T_{10})$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$2, 7$ — $V(T_2,\, T_7,\, T_3,\, T_8T_9 - T_5T_{10},\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10}) \cup V(T_2,\, T_7,\, T_5,\, T_9,\, -\kappa T_3T_6 - T_{11}T_{10})$

$$\begin{bmatrix} a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$2, 8$ — $V(T_2,\, T_8,\, T_3T_5,\, \kappa T_6T_7 - T_9T_{11},\, -\kappa T_3T_6 - T_{11}T_{10},\, T_5T_{10},\, -T_3T_9 - T_7T_{10})$

Method: computation.

$2, 9$ — $V(T_2,\, T_9,\, T_{10},\, -T_3T_5 - T_7T_8,\, T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,\, T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6) \cup V(T_2,\, T_9,\, T_5,\, T_7,\, -\kappa T_3T_6 - T_{11}T_{10})$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \end{bmatrix}$$

$2, 10$ — $V(T_2,\, T_{10},\, T_9,\, -T_3T_5 - T_7T_8,\, T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,\, T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6) \cup V(T_2,\, T_{10},\, T_8,\, T_3,\, -\kappa T_6T_7 - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a)$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \end{bmatrix}$$

$2,11$ $V(T_2,\, T_{11},\, -T_3T_5 - T_7T_8,\, T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,\, T_8T_9 - T_5T_{10},\, T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6,\, -T_3T_9 - T_7T_{10})$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$3,4$ $V(T_3,\, T_4,\, T_7T_8,\, T_7T_{10},\, T_{11}T_{10},\, \kappa T_6T_7 - T_9T_{11},\, T_8T_9 - T_5T_{10})$

Method: computation.

$3,5$ $V(T_3,\, T_5,\, T_{10},\, \kappa T_6T_7 - T_9T_{11},\, T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9)$ $\cup$ $V(T_3,\, T_5,\, T_7,\, T_{11},\, T_8T_9)$

$$\begin{bmatrix} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \end{bmatrix}$$

$3,6$ $V(T_3,\, T_6,\, T_7,\, T_8T_9 - T_5T_{10},\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10})$ $\cup$ $V(T_3,\, T_6,\, T_8,\, T_{10},\, T_9T_{11})$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \end{bmatrix}$$

$3,7$ $V(T_3,\, T_7,\, T_2T_4T_{11}T_6,\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10},\, T_8T_9 - T_5T_{10})$

$$\begin{bmatrix} a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$3,8$ $V(T_3,\, T_8,\, T_{10},\, T_2T_4T_{11}T_6,\, -\kappa T_6T_7 + T_9T_{11})$ $\cup$ $V(T_3,\, T_8,\, T_7,\, T_{11},\, T_5)$

Method: computation.

$3,9$ $V(T_9,\, T_3,\, T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,\, T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10},\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10},\, T_1^aT_2^aT_4^2T_5^{a-1}T_6^{a-1}T_8^{a-1}T_{11}^{a-1} - T_7T_{10})$

$$\begin{bmatrix} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \\ a & a & 0 & 2 & a-1 & a-1 & -1 & a-1 & 0 & -1 & a-1 \end{bmatrix}$$

$3,10$ $V(T_{10},\, T_3,\, T_1T_4T_5T_8,\, -\kappa T_6T_7 + T_9T_{11},\, T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \end{bmatrix}$$

$3,11$ $V(T_{11},\, T_3,\, T_7,\, T_1T_4T_5T_8,\, T_8T_9 - T_5T_{10})$ $\cup$ $V(T_{11},\, T_3,\, T_8,\, T_{10},\, T_6)$

Method: computation.

$4,5$ $V(T_5,\, T_4,\, T_7T_8,\, T_8T_9,\, -\kappa T_6T_7 + T_9T_{11},\, -\kappa T_3T_6 - T_{11}T_{10},\, -T_3T_9 - T_7T_{10})$

Method: computation.

$4,6$	$V(T_6,\, T_4,\, T_9T_{11},\, T_{11}T_{10},$ $T_3T_5 + T_7T_8,$	$T_8T_9 - T_5T_{10},$ $T_3T_9 - T_7T_{10})$ *Method:* computation.
$4,7$	$V(T_7,\, T_4,\, T_3T_5,\, T_9T_{11},$ $T_3T_9,\, -\kappa T_3T_6 - T_{11}T_{10},$	$T_8T_9 - T_5T_{10})$ *Method:* computation.
$4,8$	$V(T_8,\, T_4,\, T_3T_5,$ $T_5T_{10},\, -\kappa T_6T_7 + T_9T_{11},$	$-\kappa T_3T_6 - T_{11}T_{10},$ $-T_3T_9 - T_7T_{10})$ *Method:* computation.
$4,9$	$V(T_9,\, T_4,\, T_6T_7,\, T_5T_{10},$ $T_7T_{10}, T_3T_5 + T_7T_8,$	$-\kappa T_3T_6 - T_{11}T_{10})$ *Method:* computation.
$4,10$	$V(T_{10},\, T_4,\, T_3T_9,\, T_8T_9,$ $T_3T_6,\, -T_3T_5 - T_7T_8,$	$\kappa T_6T_7 - T_9T_{11})$ *Method:* computation.
$4,11$	$V(T_{11},\, T_4,\, T_6T_7,\, T_3T_6,$ $T_3T_5 + T_7T_8,$	$T_8T_9 - T_5T_{10},$ $T_3T_9 + T_7T_{10})$ *Method:* computation.
$5,6$	$V(T_6,\, T_5,\, T_7T_8,\, T_9T_{11},$ $T_{11}T_{10},\, T_8T_9,$	$T_3T_9 + T_7T_{10})$ *Method:* computation.
$5,7$	$V(T_7,\, T_5,\, T_3T_9,$ $T_9T_{11},\, T_8T_9,$	$T_2T_4T_6T_{11},$ $-\kappa T_3T_6 - T_{11}T_{10})$ *Method:* computation.
$5,8$	$V(T_8,\, T_5,\, T_2T_4T_6T_{11},$ $-\kappa T_6T_7 + T_9T_{11},$	$-\kappa T_3T_6 - T_{11}T_{10},$ $T_3T_9 + T_7T_{10})$ *Method:* computation.
$5,9$	$V(T_9,\, T_5,\, T_6T_7,$ $T_7T_{10},\, T_7T_8,$	$T_2T_4T_6T_{11},$ $-\kappa T_3T_6 - T_{11}T_{10})$ *Method:* computation.
$5,10$	$V(T_{10},\, T_5,\, T_3T_9,\, T_3T_6,$ $T_2^aT_4T_{11}^aT_6^{a-1} - T_7T_8,$ $\left[\begin{array}{rrrrrrrrrrr} 0 & a & 0 & 1 & 0 & a-1 & -1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \end{array}\right]$	$-\kappa T_6T_7 + T_9T_{11},$ $-\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9)$
$5,11$	$V(T_{11},\, T_5,\, T_7T_8,$ $T_6T_7,\, T_3T_6,\, T_8T_9,$	$T_3T_9 - T_7T_{10})$ *Method:* computation.

6, 7 $V(T_7,\, T_6,\, T_3T_5,\, T_3T_9,\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - T_{11}T_{10},\, T_8T_9 - T_5T_{10})$

$$\begin{bmatrix} a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & 0 & 1 & a-1 & 0 & 0 & a & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

6, 8 $V(T_8,\, T_6,\, T_3T_5,\, T_9T_{11},\, T_{11}T_{10},\, T_5T_{10},\, T_3T_9 - T_7T_{10})$

Method: computation.

6, 9 $V(T_9,\, T_6,\, T_5T_{10},\, T_1T_4T_5T_8,\, T_{11}T_{10},\, T_7T_{10},\, T_3T_5 + T_7T_8,\,)$

Method: computation.

6, 10 $V(T_{10},\, T_6,\, T_3T_5 + T_7T_8,\, T_9T_{11},\, T_1T_4T_5T_8,\, T_8T_9,\, T_3T_9)$

Method: computation.

6, 11 $V(T_{11},\, T_6,\, T_1T_4T_5T_8,\, T_3T_5 + T_7T_8,\, T_8T_9 - T_5T_{10},\, T_3T_9 + T_7T_{10})$

Method: computation.

7, 8 $V(T_8,\, T_7,\, T_9T_{11},\, T_3T_9,\, -\kappa T_3T_6 - T_{11}T_{10},\, T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10})$

$$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \end{bmatrix}$$

7, 9 $V(T_9,\, T_7,\, T_1T_4T_5T_8,\, T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,\, -\kappa T_3T_6 - T_{11}T_{10},\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10})$

$$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \end{bmatrix}$$

7, 10 $V(T_{10},\, T_7,\, T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,\, T_1^aT_4T_5^aT_8^{a-1} - T_9T_{11},\, T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6,\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} + T_8T_9,\, T_1^aT_2^aT_4^2T_5^{a-1}T_6^{a-1}T_8^{a-1}T_{11}^{a-1} - T_3T_9)$

$$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ a & 0 & 0 & 1 & a & 0 & 0 & a-1 & -1 & 0 & -1 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & a & 0 & -1 & -1 & 0 & a-1 \\ a & a & -1 & 2 & a-1 & a-1 & 0 & a-1 & -1 & 0 & a-1 \end{bmatrix}$$

7, 11 $V(T_{11},\, T_7,\, T_3T_5,\, T_3T_9,\, T_3T_6,\, T_1T_4T_5T_8,\, T_8T_9 - T_5T_{10})$

Method: computation.

8, 9 $V(T_9,\, T_8,\, T_7T_{10},\, T_6T_7,\, T_2^aT_4T_{11}^aT_6^{a-1} - T_3T_5,\, -\kappa T_3T_6 - T_{11}T_{10},\, -\kappa T_2^aT_4T_6^aT_{11}^{a-1} - T_5T_{10})$

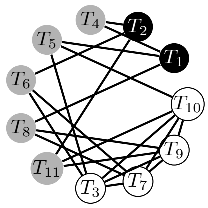
$$\begin{bmatrix} 0 & a & -1 & 1 & -1 & a-1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & a & 0 & 0 & 0 & -1 & a-1 \end{bmatrix}$$

8, 10 $V(T_{10},\, T_8,\, T_3T_9,\, T_3T_6,\, T_2T_4T_6T_{11},\, T_3T_5,\, -\kappa T_6T_7 + T_9T_{11})$

Method: computation.

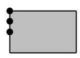
$8, 11$	$V(T_{11},\, T_8,\, T_3T_5,\, T_6T_7,$ $T_3T_6,\, T_5T_{10},$	$T_3T_9 - T_7T_{10})$ *Method:* computation.
$9, 10$	$V(T_{10},\, T_9,\, T_2T_4T_6T_{11},$ $T_3T_5 + T_7T_8,$	$T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,$ $T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6)$
	$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \end{bmatrix}$	
$9, 11$	$V(T_{11},\, T_9,\, T_7T_{10},$ $T_5T_{10},\, T_3T_5 + T_7T_8,$	$T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,$ $T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6)$
	$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \end{bmatrix}$	
$10, 11$	$V(T_{11},\, T_{10},\, T_3T_9,$ $T_8T_9,\, T_3T_5 + T_7T_8,$	$T_1^aT_4T_5^aT_8^{a-1} + \kappa T_6T_7,$ $T_1^aT_4T_5^{a-1}T_8^a - \kappa T_3T_6)$
	$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & a & -1 & -1 & a-1 & 0 & 0 & 0 \\ a & 0 & -1 & 1 & a-1 & -1 & 0 & a & 0 & 0 & 0 \end{bmatrix}$	

Thus, by Algorithm 4.5.15, the surface X_2 is a Mori dream surface and the statement about its Cox ring holds. The fact that it does not admit a $\mathbb{K}^*$-action can be seen from its graph G_{X_2} of exceptional curves



where the subgraph induced by the vertices T_i with $i \in \{1, 2, 4, 5, 6, 8, 11\}$ exists by the blow up sequence, compare Remark 5.3.7. By Lemma 5.4.12, the surface cannot be a $\mathbb{K}^*$-surface because sink and source, marked black, meet in the common negative curve $V(X_2;\, T_4)$.

(XII) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star\star\star\ ix)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ v)' \xleftarrow{\iota_2} \mathrm{Bl}\ \mathbb{F}_a(\star\star\ v) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where the embeddings ι_i are as in Setting 4.2.9 with

$$\bar\iota_1 \colon \mathbb{K}^5 \to \mathbb{K}^6, \qquad z \mapsto (z, h_1(z)), \qquad h_1 := T_2^aT_4 - T_3T_5,$$
$$\bar\iota_2 \colon \mathbb{K}^7 \to \mathbb{K}^8, \qquad z \mapsto (z, h_2(z)), \qquad h_2 := (\lambda - 1)T_2^aT_4 - \lambda T_6T_7$$

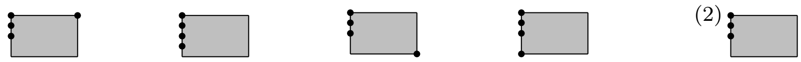
where $\lambda \in \mathbb{K}^* \setminus \{1\}$ and the blow ups π_i are

$$\pi_3([z]) \;=\; [z_1z_9, z_2, \ldots, z_7, z_8z_9],$$
$$\pi_2([z]) \;=\; [z_1z_7, z_2, \ldots, z_5, z_6z_7], \qquad \pi_1([z]) \;=\; [z_1z_5, z_2, z_3z_5, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1;\, T_5), \qquad V(X_1;\, T_7), \qquad V(X_1;\, T_9).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations



The first configuration has already been dealt with in part (XI) of this proof. For the second configuration, we choose the point

$$q_2 \ := \ [0,1,1,\mu,1,\mu-1,1,\lambda-\mu,1] \ \in \ X_1, \qquad \mu \ \in \ \mathbb{K}^* \setminus \{1,\lambda\}.$$

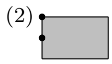
It exists by Lemma 5.2.16 and $\pi_1 \circ \iota_1^{-1} \circ \pi_2 \circ \iota_2^{-1} \circ \pi_3(q_2)$ equals $[0,1,1,\mu] \in \mathbb{F}_a$. Since $V(X_1;T_1)$ is a parabolic fixed point curve, the blow up of X_1 in q_2 will admit a $\mathbb{K}^*$-action, see Lemma 5.4.11. The blow ups of the third and fourth configurations are blow ups of X_1 in the points

$$q_3 \ := \ [1,0,1,0,1,-1,1,\lambda,1], \qquad q_4 \ := \ [0,1,1,0,1,-1,1,\lambda,1] \ \in \ X_1$$

which project to the respective points $[1,0,1,0]$ and $[0,1,1,0] \in \mathbb{F}_a$ under $\pi_1 \circ \iota_1^{-1} \circ \pi_2 \circ \iota_2^{-1} \circ \pi_3$. Both points exist by an iterated application of Lemma 5.4.5 and Lemma 5.2.16. By Lemma 5.4.11, both the blow up of X_1 in q_3 and the blow up of X_1 in q_4 will admit non-trivial $\mathbb{K}^*$-actions.

The last configuration means blowing up X_1 in a point in the exceptional divisor $V(X_1;T_5)$. Since it can also be obtained as the blow up of Bl $\mathbb{F}_a(\star^2 \star \ \ iv)$ or Bl $\mathbb{F}_a(\star^2 \star \ \ viii)$ in the respective divisor $V(T_1)$ we will treat this case in parts $(XIII)$ and (XIV) of this proof.

(XIII) Blow ups of X_1 := Bl $\mathbb{F}_a(\star^2 \star \ \ iv)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ i) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \ \to \ \mathbb{K}^7, \qquad z \ \mapsto \ (z, h_1(z)), \qquad h_1 \ := \ T_2^a T_4 - T_3 T_5 T_6^2$$

where $h_1 \in \mathbb{K}[T_1,\ldots,T_6]$ and the blow ups π_i are given by

$$\pi_3([z]) \ = \ [z_1 z_8, z_2, \ldots, z_6, z_7 z_8],$$
$$\pi_2([z]) \ = \ [z_1, z_2, z_3 z_6, z_4, z_5 z_6], \qquad \pi_1([z]) \ = \ [z_1 z_5, z_2, z_3 z_5, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1;T_5), \qquad V(X_1;T_6), \qquad V(X_1;T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

For the first configuration, we choose in X_1 the point $q_1 := [0,1,1,\lambda,1,1,\lambda-1,1]$ with $\lambda \in \mathbb{K}^* \setminus \{1\}$. It exists by Lemma 5.2.16 and satisfies

$$\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3(q_1) \quad = \quad [0,1,1,\lambda] \quad \in \quad \mathbb{F}_a.$$

Since $V(X_1;T_1)$ is a parabolic fixed point curve the blow up of X_1 in q_1 will admit a $\mathbb{K}^*$-action, see Lemma 5.4.11. For the second, third and fourth configurations, we blow up the points

$$q_2 \;:=\; [1,0,0,1,1,1,0,1], \qquad q_3 \;:=\; [1,0,1,0,1,1,-1,1]$$
$$q_4 \;:=\; [0,1,1,0,1,1,-1,1] \;\in\; X_1$$

which project to $[1,0,0,1]$, $[1,0,1,0]$ and $[0,1,1,0] \in \mathbb{F}_a$ under $\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3$. A stepwise application of Lemma 5.4.5 and Lemma 5.2.16 shows that all points exist. By Lemma 5.4.11, all three surfaces will be $\mathbb{K}^*$-surfaces.

For the fifth configuration, we want to blow up a point in the union of the exceptional divisors $V(X_1; T_5)$ and $V(X_1; T_6)$. By Remark 5.3.5, it suffices to consider the points

$$q_5 \;:=\; [1,1,1,1,0,1,1,1], \qquad q_5' \;:=\; [1,1,1,1,1,0,1,1] \;\in\; X_1$$

both of which exist by Lemma 5.2.16. We first blow up X_1 in q_5. The steps are as before, compare Remark 5.2.1. Choose in $\mathbb{K}[T_1,\dots,T_8]$ the polynomial $h_2 := T_1T_2^{a-1}T_4T_8 - T_3T_6$ for the embedding $\bar\iota_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{smallmatrix} \end{array}\right], \qquad P_1' \;=\; \left[\begin{array}{rrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{array}\right].$$

For the blow up of X_1' in $\iota(q_5) = [1,1,1,1,0,1,1,1,0]$ we stellarly subdivide Σ_1' at the vector $v := (-1,0,-1,2)$. Write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 \;:=\; p_2^{\star}\,(p_1)_{\star}\,(T_2^aT_4 - T_3T_5T_6^2 - T_7T_8) \;=\; T_2^aT_4 - T_3T_5T_6^2T_{10} - T_7T_8,$$
$$g_2 \;:=\; p_2^{\star}\,(p_1)_{\star}\,(T_9 - h_2) \;=\; T_9T_{10} - T_1T_2^{a-1}T_4T_8 + T_3T_6.$$

We show that I_2 is prime. In particular, I_2 is saturated with respect to T_{10}. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U \;:=\; \left\{x \in \overline{X}_2;\; x_8x_9 \neq 0 \text{ or } x_7x_{10} \neq 0\right\} \;\subseteq\; \overline{X}_2 \;=\; V\left(\mathbb{K}^{10}; I_2\right).$$

Inspecting the indices $i = 1,2$ and $j = 7,10$ or $i = 1,2$ and $j = 8,9$ respectively we see that the rank of the Jacobian matrix $(\partial g_i/\partial T_j)_{i,j}(u)$ is two for all $u \in U$. Furthermore, $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$V\left(\mathbb{K}^{10}; T_8, T_7\right), \quad V\left(\mathbb{K}^{10}; T_8, T_{10}\right), \quad V\left(\mathbb{K}^{10}; T_9, T_7\right), \quad V\left(\mathbb{K}^{10}; T_9, T_{10}\right).$$

We claim that in $\mathbb{K}^{10}$ each of the following intersections is of dimension six.

$$\begin{aligned}
\overline{X}_2 \cap V(T_8, T_7) &= V(T_8,\, T_7,\, T_2^aT_4 - T_3T_5T_6^2T_{10},\, T_9T_{10} + T_3T_6), \\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_8,\, T_{10},\, T_2T_4,\, T_3T_6), \\
\overline{X}_2 \cap V(T_9, T_7) &= V(T_9,\, T_7,\, T_2^aT_4 - T_3T_5T_6^2T_{10},\, T_1T_2^{a-1}T_4T_8 - T_3T_6), \\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_9,\, T_{10},\, T_2^aT_4 - T_7T_8,\, T_1T_2^{a-1}T_4T_8 - T_3T_6).
\end{aligned}$$

The second one is clearly of dimension six whereas for the others, as in Lemma 5.3.3, we consider the exponent matrices

$$\left[\begin{array}{rrrrrrrrrr} 0 & a & -1 & 1 & -1 & -2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{array}\right],$$
$$\left[\begin{array}{rrrrrrrrrr} 0 & a & -1 & 1 & -1 & -2 & 0 & 0 & 0 & -1 \\ 1 & a-1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right],$$
$$\left[\begin{array}{rrrrrrrrrr} 0 & a & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & a-1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

all of which are of rank two. Thus, on the respective tori, the dimension is six by Lemma 5.3.3. One directly checks that the dimension is at most six on the smaller tori. Therefore, $\dim(\overline{X}_2 \setminus U) \leq 6$ and, since $\overline{X}_2$ is of dimension at least eight, the codimension of $\overline{X}_2 \setminus U$ in $\overline{X}_2$ is at least two. By Lemma 5.4.3, the ideal I_2 is prime. Consider the ideals

$$\begin{aligned} I_2 + \langle T_{10} \rangle \; &= \; \langle T_{10},\, T_2^a T_4 - T_7 T_8,\; T_1 T_2^{a-1} T_4 T_8 - T_3 T_6 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_{10}], \\ I_0 \; &:= \; \langle T_2^a T_4 - T_6 T_7,\; T_1 T_2^{a-1} T_4 T_7 - T_3 T_5 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_7]. \end{aligned}$$

Then the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \ldots, T_{10}]/I_2$, i.e., $I_2 + \langle T_{10} \rangle$ is prime, if and only if I_0 is a prime ideal in $\mathbb{K}[T_1, \ldots, T_7]$. The matrix consisting of the exponents of the binomial generators

$$\begin{bmatrix} 0 & a & 0 & 1 & 0 & -1 & -1 \\ 1 & a-1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

has a Smith normal form of shape $[E_2, 0, \ldots, 0]$ where E_2 is the 2×2 unit matrix. Hence, the ideal $\langle I_0 \rangle \subseteq \mathbb{K}[T_1^{\pm 1}, \ldots, T_7^{\pm 1}]$ is prime, compare [37]. By Remark 5.4.14, I_0 is prime if $I_0 = I_0 : T_1 \cdots T_7^{\infty}$. To this end we first prove that

$$\begin{aligned} \mathcal{G} \; &:= \; \{f_1,\, f_2,\, f_3\} \\ &:= \; \{T_1 T_6 T_7^2 - T_2 T_3 T_5,\; T_2^a T_4 - T_6 T_7,\; T_1 T_2^{a-1} T_4 T_7 - T_3 T_5\} \end{aligned}$$

is a Gröbner basis for I_0 with respect to any degree reverse lexicographical ordering with $T_1 > \ldots > T_{i-1} > T_{i+1} > \ldots > T_7 > T_i$ for a $1 \leq i \leq 7$. First, observe that we have $I_0 = \langle \mathcal{G} \rangle$ since f_2 and f_3 are the generators of I_0 and $f_1 = -T_1 T_7 f_2 + T_2 f_3$. The S-polynomials are

$$\begin{aligned} S(f_1,\, f_2) \; &= \; -\, T_2^{a+1} T_3 T_4 T_5 + T_1 T_6^2 T_7^3, \\ S(f_1,\, f_3) \; &= \; -\, T_2^{a} T_3 T_4 T_5 + T_3 T_5 T_6 T_7, \\ S(f_2,\, f_3) \; &= \; -\, T_1 T_6 T_7^2 + T_2 T_3 T_5. \end{aligned}$$

The division algorithm, see [26, Ch. 2, Thm. 3], returns the combinations

$$S(f_1, f_2) \; = \; T_6 T_7 f_1 - T_2 T_3 T_5 f_2, \quad S(f_1, f_3) \; = \; -T_3 T_5 f_2, \quad S(f_1, f_4) \; = \; -f_1.$$

By the Buchberger criterion, see [26, Ch. 2, Thm. 6], $\mathcal{G}$ is a Gröbner basis for I_0 with respect to the chosen orderings. By [90, Lem. 12.1], we know that

$$\left\{ \frac{f}{T_i^{k_i(f)}};\; f \in \mathcal{G} \right\} \; = \; \mathcal{G}, \qquad k_i(f) \; := \; \max\bigl(n \in \mathbb{Z}_{\geq 0};\, T_i^n \mid f\bigr)$$

is a Gröbner basis for $I_0 : T_i^{\infty}$. In particular, $I_0 = I_0 : T_i^{\infty}$ for each $1 \leq i \leq 7$. As in [90, p. 114], the claim follows from

$$I_0 : (T_1 \cdots T_7)^{\infty} \; = \; ((\cdots (I_0 : T_1^{\infty}) \cdots) : T_7^{\infty}) \; = \; I_0.$$

Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for all $i \neq j$ and we have $T_{10} \nmid T_i$ for all $i < 10$ since

$$\begin{aligned} \overline{X}_2 \cap V(T_1,\, T_{10}) \; &= \; V(T_{10},\, T_1,\, T_3 T_6,\, T_2^a T_4 - T_7 T_8), \\ \overline{X}_2 \cap V(T_2,\, T_{10}) \; &= \; V(T_{10},\, T_2,\, T_7 T_8,\, T_3 T_6), \\ \overline{X}_2 \cap V(T_3,\, T_{10}) \; &= \; V(T_{10},\, T_3,\, T_2^a T_4 - T_7 T_8,\, T_1 T_2^{a-1} T_4 T_8), \\ \overline{X}_2 \cap V(T_4,\, T_{10}) \; &= \; V(T_{10},\, T_4,\, T_7 T_8,\, T_3 T_6), \\ \overline{X}_2 \cap V(T_5,\, T_{10}) \; &= \; V(T_{10},\, T_5,\, T_2^a T_4 - T_7 T_8,\, T_1 T_2^{a-1} T_4 T_8 - T_3 T_6), \\ \overline{X}_2 \cap V(T_6,\, T_{10}) \; &= \; V(T_{10},\, T_6,\, T_2^a T_4 - T_7 T_8,\, T_1 T_2 T_4 T_8), \\ \overline{X}_2 \cap V(T_7,\, T_{10}) \; &= \; V(T_{10},\, T_7,\, T_2 T_4,\, T_3 T_6) \end{aligned}$$

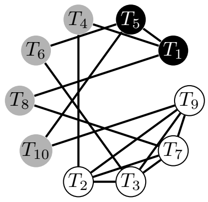
are all of dimension six; this can be seen as in the dimension computations of $\overline{X}_2 \cap V(T_8,\, T_{10})$ and $\overline{X}_2 \cap V(T_9,\, T_{10})$ above. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with its degree matrix as listed in the table. We now show that we performed the desired blow up. The ideal

$$\begin{aligned} I' &:= \langle T_5,\, T_9,\, h_2,\, h_1 - T_7T_8\rangle \\ &= \langle T_5,\, T_9,\, T_2^aT_4 - T_7T_8,\, T_1T_2^{a-1}T_4T_8 - T_3T_6\rangle \ \subseteq\ \mathbb{K}[T_1,\ldots,T_9] \end{aligned}$$

is a prime ideal since the ring $\mathbb{K}[T_1,\ldots,T_9]/I'$ is isomorphic to the integral domain $\mathbb{K}[T_1,\ldots,T_{10}]/(I_2 + \langle T_{10}\rangle)$. Given Cox coordinates $z := (1,1,1,1,0,1,1,1,0) \in \mathbb{K}^9$ for the point $\iota(q_5) \in X_1'$ we have $z \in V(\mathbb{K}^9; I')$ and by the previous dimension arguments

$$\dim\left(V\left(\mathbb{K}^9; I'\right)\right) \ = \ -1 + \dim\left(\overline{X}_2 \cap V(T_5,\, T_{10})\right) \ = \ 5.$$

An application of Lemma 5.2.15 shows that the performed modification was the claimed blow up. The Cox ring and degree matrix of the resulting surface $X_2 = \mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ i)$ are listed in the table. We claim that its graph of exceptional curves G_{X_2} is as follows. By Lemma 5.4.12, X_2 then cannot be a $\mathbb{K}^*$-surface.



It suffices to prove the existence of the subgraph induced by the vertices T_i with $i \in \{1,4,5,6,8,10\}$. By Remark 5.3.7 and the fact that $V(X_2; T_{10})$ is the exceptional divisor of the last blow up, we know that the curves $V(X_2; T_i)$ are negative. The existence of the edges, i.e., the fact that the curves meet, is directly seen from the blow up sequence of X_2.

We now blow up X_1 in $q_5' = [1,1,1,1,1,0,1,1]$ by the same steps. Choose in $\mathbb{K}[T_1,\ldots,T_8]$ the polynomial $h_2 := T_1^2T_2^{a-2}T_4T_5T_8^2 - T_3$ for the embedding $\overline{\iota}_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1', whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \ = \ \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ -1 \\ -2 \\ 1 \\ 1 \end{smallmatrix} \end{array}\right], \qquad P_1' \ = \ \left[\begin{array}{rrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & -1 & 1 \end{array}\right].$$

For the blow up of X_1' in the point $\iota(q_5') = [1,1,1,1,1,0,1,1,0]$ we determine the stellar subdivision of Σ_1' at $v := (-2,0,-1,3) \in \mathbb{Z}^4$ and write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 \ &:= \ p_2^\star\,(p_1)_\star\,(T_2^aT_4 - T_3T_5T_6^2 - T_7T_8) \ = \ T_2^aT_4 - T_3T_5T_6^2T_{10}^2 - T_7T_8, \\ g_2 \ &:= \ p_2^\star\,(p_1)_\star\,(T_9 - h_2) \ = \ T_9T_{10} - T_1^2T_2^{a-2}T_4T_5T_8^2 + T_3. \end{aligned}$$

We show that I_2 is saturated with respect to T_{10} by showing that I_2 is prime. By Lemma 5.4.4, the ideal I_2 is prime if the ideal

$$I_2' \ := \ \langle T_2^aT_3 - T_9^2T_4^2T_5^2T_1^2T_2^{a-2}T_3T_7^2 + T_9^3T_4T_5^2T_8 - T_6T_7\rangle \ \subseteq\ \mathbb{K}[T_1,\ldots,T_9]$$

obtained by substitution of $T_3 = -g_2 + T_3$ in g_1 and relabeling all T_i with $i > 3$ by T_{i-1} is prime. The latter follows from Lemma 5.2.17. In similar manner, by Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\ldots,T_{10}]/I_2$ since the ideal

$$I_2' + \langle T_9\rangle \ = \ \langle T_9,\ T_2^aT_3 - T_6T_7\rangle \ \subseteq\ \mathbb{K}[T_1,\ldots,T_9]$$

is prime by Lemma 5.2.17. Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for all $i \neq j$. Also, $T_{10} \nmid T_i$ for all $i < 10$: the dimension of each zero set

$$
\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_3, T_1, T_2^a T_4 - T_7 T_8),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_3, T_2, T_7 T_8),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_2^a T_4 - T_7 T_8, T_1 T_2 T_4 T_5 T_8),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_3, T_2^a T_4 - T_7 T_8),\\
\overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_2^a T_4 - T_7 T_8, -T_1^2 T_2^{a-2} T_4 T_5 T_8^2 + T_3),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_2 T_4, T_3),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3, T_2 T_4),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_2^a T_4 - T_7 T_8, -T_1^2 T_2^{a-2} T_4 T_5 T_8^2 + T_3)
\end{aligned}
$$

is six. In the parameter-free cases this can be seen by computations whereas otherwise, Lemma 5.3.3 is used. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we have performed the desired blow up. The ideal

$$
\begin{aligned}
I' &:= \langle T_6, T_9, h_2, h_1 - T_7 T_8 \rangle\\
&= \langle T_6, T_9, T_2^a T_4 - T_7 T_8, -T_1^2 T_2^{a-2} T_4 T_5 T_8^2 + T_3 \rangle \subseteq \mathbb{K}[T_1, \ldots, T_9]
\end{aligned}
$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,0,1,1,1,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5') \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$ and by the previous computations

$$
\dim\left(V\left(\mathbb{K}^9; I'\right)\right) = -1 + \dim\left(\overline{X}_2 \cap V(T_6, T_{10})\right) = 5.
$$

By Lemma 5.2.15, the performed modification was the claimed blow up. As all requirements for Algorithm 4.3.3 are fulfilled we may eliminate the equation $T_3 = g_2 + T_3$ from I_2. We obtain the Cox ring $R_2' := \mathbb{K}[T_1, \ldots, T_9]/I_2'$ describing the blow up of X_1' in $\iota(q_5')$ with the degree matrix Q_2' given by

$$
Q_2' = \begin{bmatrix}
1 & 1 & -a & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0\\
-2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
2 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

Observe that the surface X_2 is a $\mathbb{K}^*$-surface. Its Cox ring $\mathcal{R}(X_2) = R_2'$ is isomorphic to the Cox ring of a $\mathbb{K}^*$-surface Y given by

$$
\mathcal{R}(Y) = \mathbb{K}[T_1, \ldots, T_9] / \langle T_2^a T_3 + T_9^3 T_4 T_5^2 T_8 - T_6 T_7 \rangle
$$

with the same degree matrix Q_2'. Using Lemma 5.1.5, the isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the $\mathbb{Z}^6$-graded homomorphism

$$
\mathbb{K}[T_1, \ldots, T_9] \to \mathbb{K}[T_1, \ldots, T_9], \qquad T_i \mapsto \begin{cases} T_6 + T_9^2 T_4^2 T_5^2 T_1^2 T_2^{a-2} T_3 T_7, & i = 6,\\ T_i, & \text{else.} \end{cases}
$$

We come to the blow up of the last configuration. This is the blow up of X_1 in a point in the last exceptional divisor, i.e., $V(X_1; T_8)$. By Remark 5.3.5, it suffices to consider the point $q_6 := [1,1,1,1,1,1,1,0] \in X_1$. As before, choose in $\mathbb{K}[T_1, \ldots, T_8]$ the polynomial $h_2 := T_1 T_2^{a-1} T_4 T_5 T_6 - T_7$ for the embedding $\overline{\iota}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$
Q_1' = \left[\begin{array}{c|c} Q_1 & \begin{matrix} 0\\0\\0\\1\\0 \end{matrix} \end{array}\right], \qquad P_1' = \begin{bmatrix}
1 & a-1 & 0 & 1 & 1 & 1 & 0 & 0 & -1\\
0 & a & 0 & 1 & 0 & 0 & 0 & -1 & -1\\
0 & 0 & 1 & 0 & 1 & 2 & 0 & -1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}.
$$

For the blow up of $\iota_2(q_6) = [1,1,1,1,1,1,1,0,0] \in X_1'$, we determine the stellar subdivision of Σ_1' at the vector $v := (-1,-2,-2,-1) \in \mathbb{Z}^4$ and write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 \ := \ p_2^\star\,(p_1)_\star\,(T_2^aT_4 - T_3T_5T_6^2 - T_7T_8) \ = \ T_2^aT_4 - T_3T_5T_6^2 - T_7T_8T_{10},$$
$$g_2 \ := \ p_2^\star\,(p_1)_\star\,(T_9 - h_2) \ = \ T_9T_{10} - T_1T_2^{a-1}T_4T_5T_6 + T_7.$$

We show that I_2 is prime. In particular, I_2 then is saturated with respect to T_{10}. By Lemma 5.4.4, the ideal I_2 is prime if the ideal

$$I_2' \ := \ \langle T_2^aT_4 - T_3T_5T_6^2 - T_7T_9T_1T_2^{a-1}T_4T_5T_6 + T_7T_9^2T_8\rangle \ \subseteq \ \mathbb{K}[T_1, \dots, T_9]$$

obtained by substitution of $T_7 = -g_2 + T_7$ in g_1 and relabeling all T_i with $i > 7$ by T_{i-1} is prime. The latter follows from Lemma 5.2.17. By the same reasoning, making again use of Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$ since

$$I_2' + \langle T_9\rangle \ = \ \langle T_9,\, T_2^aT_4 - T_3T_5T_6^2\rangle \ \subseteq \ \mathbb{K}[T_1, \dots, T_9].$$

is prime by Lemma 5.2.17. Moreover, no two variables T_i, T_j are associated because their degrees $\deg(T_i)$, $\deg(T_j)$ are different for all $i \neq j$. Also, $T_{10} \nmid T_i$ for all $i < 10$: the dimensions of all intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_1,\, T_2^aT_4 - T_3T_5T_6^2),\\
\overline{X}_2 \cap V(T_2, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_2,\, T_3T_5T_6),\\
\overline{X}_2 \cap V(T_3, T_{10}) \ &= \ V(T_{10},\, T_3,\, T_2T_4,\, T_7),\\
\overline{X}_2 \cap V(T_4, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_4,\, T_3T_5T_6),\\
\overline{X}_2 \cap V(T_5, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_5,\, T_2T_4),\\
\overline{X}_2 \cap V(T_6, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_6,\, T_2T_4),\\
\overline{X}_2 \cap V(T_7, T_{10}) \ &= \ V(T_{10},\, T_7,\, T_2^aT_4 - T_3T_5T_6^2,\, T_1T_2T_4T_5T_6),\\
\overline{X}_2 \cap V(T_8, T_{10}) \ &= \ V(T_{10},\, T_8,\, T_2^aT_4 - T_3T_5T_6^2,\, -T_1T_2^{a-1}T_4T_5T_6 + T_7),\\
\overline{X}_2 \cap V(T_9, T_{10}) \ &= \ V(T_{10},\, T_9,\, T_2^aT_4 - T_3T_5T_6^2,\, -T_1T_2^{a-1}T_4T_5T_6 + T_7)
\end{aligned}$$

are six. This is done by computations or with Lemmas 5.3.3 and 5.4.4 By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$ is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we performed the desired blow up. The ideal

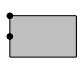
$$\begin{aligned}
I' \ &:= \ \langle T_8,\, T_9,\, h_2,\, h_1 - T_7T_8\rangle\\
&= \ \langle T_8,\, T_9,\, T_2^aT_4 - T_3T_5T_6^2,\, T_1T_2^{a-1}T_4T_5T_6 - T_7\rangle \ \subseteq \ \mathbb{K}[T_1, \dots, T_9]
\end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,1,1,1,0,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_6) \in X_1'$. Then $z \in V(\mathbb{K}^9;\, I')$ and, as g_2 is a fake relation, we have

$$\dim\left(V\left(\mathbb{K}^9;\, I'\right)\right) \ = \ -1 + \dim\left(V\left(\mathbb{K}^9;\, T_8,\, T_9,\, T_2^aT_4 - T_3T_5T_6^2\right)\right) \ = \ 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. By Algorithm 4.3.3 we are allowed to remove the redundant equation $T_7 = -g_2 + T_7$. Hence, the Cox ring of the blow up X_2 of X_1 in q_6 is $\mathcal{R}(X_2) = \mathbb{K}[T_1, \dots, T_9]/I_2'$. Its degree matrix is given by removing the seventh column of a Gale dual matrix of P_2. Note that at the moment we are uncertain whether X_2 is a $\mathbb{K}^*$-surface or not.

(XIV) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star^2 \star\ viii)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence

(2) 

$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\,\mathbb{F}_a(\star^2\ ii)' \xleftarrow{\iota_1} \mathrm{Bl}\,\mathbb{F}_a(\star^2\ ii) \xrightarrow{\pi_2} \mathrm{Bl}\,\mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1\colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 \ :=\ T_2^a T_4 - T_3 T_5 T_6$$

where $h_1 \in \mathbb{K}[T_1, \ldots, T_6]$ and the blow ups π_i are given by

$$\pi_3([z]) \ =\ [z_1 z_8, z_2, \ldots, z_6, z_7 z_8],$$

$$\pi_2([z]) \ =\ [z_1 z_6, z_2, \ldots, z_4, z_5 z_6], \qquad \pi_1([z]) \ =\ [z_1 z_5, z_2, z_3 z_5, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1;\, T_5), \qquad V(X_1;\, T_6), \qquad V(X_1;\, T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

For the first configuration, we choose the point $q_1 := [0, 1, 1, \lambda, 1, 1, \lambda - 1, 1]$ in X_1 where $\lambda \in \mathbb{K}^* \setminus \{1\}$. It exists by Lemma 5.2.16 and satisfies

$$\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3(q_1) \ =\ [0, 1, 1, \lambda] \ \in\ \mathbb{F}_a.$$

Since $V(X_1;\, T_1)$ is a parabolic fixed point curve the blow up of X_1 in q_1 will admit a $\mathbb{K}^*$-action, see Lemma 5.4.11. For the configurations two, three and four we want to blow up the points

$$[1, 0, 0, 1, 1, 1, 0, 1], \qquad [1, 0, 1, 0, 1, 1, -1, 1], \qquad [0, 1, 1, 0, 1, 1, -1, 1] \ \in\ X_1,$$

which project to $[1, 0, 0, 1]$, $[1, 0, 1, 0]$ and $[0, 1, 1, 0] \in \mathbb{F}_a$ under $\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3$ respectively. By an iterative application of Lemma 5.4.5 and Lemma 5.2.16, we see that all points exist. By Lemma 5.4.11, all three surfaces will be $\mathbb{K}^*$-surfaces.

For the fifth configuration, we want to blow up a point in the union of the exceptional divisors $V(X_1;\, T_5)$ and $V(X_1;\, T_6)$. By Remark 5.3.5 it suffices to consider the points

$$q_5 \ :=\ [1, 1, 1, 1, 0, 1, 1, 1], \qquad q_5' \ :=\ [1, 1, 1, 1, 1, 0, 1, 1] \ \in\ X_1$$

both of which exist by Lemma 5.2.16. We first blow up X_1 in q_5. Choose the polynomial $h_2 := T_1 T_2^{a-1} T_4 T_6 T_8 - T_3 \in \mathbb{K}[T_1, \ldots, T_8]$ for the embedding $\bar{\iota}_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \ =\ \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix} \end{array}\right], \qquad P_1' \ =\ \begin{bmatrix} 1 & a-1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

For the blow up of $\iota(q_5) = [1, 1, 1, 1, 0, 1, 1, 1, 0] \in X_1'$ we determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (-1, 0, -1, 2) \in \mathbb{Z}^4$ and write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 \ :=\ p_2^\star\,(p_1)_\star\,(T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8) \ =\ T_2^a T_4 - T_3 T_5 T_6 T_{10} - T_7 T_8,$$

$$g_2 \ :=\ p_2^\star\,(p_1)_\star\,(T_9 - h_2) \ =\ T_9 T_{10} - T_1 T_2^{a-1} T_4 T_6 T_8 + T_3.$$

We show that I_2 is prime. In particular, I_2 is saturated with respect to T_{10}. By Lemma 5.4.4, the ideal I_2 is prime if the ideal

$$I_2' \ :=\ \langle T_2^a T_3 - T_9 T_4 T_5^2 T_1 T_2^{a-1} T_3 T_7 + T_9^2 T_4 T_5 T_8 - T_6 T_7 \rangle \ \subseteq\ \mathbb{K}[T_1, \ldots, T_9]$$

obtained by substitution of $T_3 = -g_2 + T_3$ in g_1 and replacing all T_i with T_{i-1} if $i > 3$ is prime. The latter follows from Lemma 5.2.17. By the same reasoning, using Lemmas 5.4.4 and 5.2.17, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \ldots, T_{10}]/I_2$ since we have a prime ideal

$$I_2' + \langle T_9 \rangle \;=\; \langle T_9,\, T_2^a T_3 - T_6 T_7 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_9].$$

Moreover, no two variables T_i, T_j are associated since their degrees $\deg(T_i)$, $\deg(T_j)$ are different for all $i \neq j$. Also, $T_{10} \nmid T_i$ for all $i < 10$: each of the intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, T_3, T_2^a T_4 - T_7 T_8),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_2^a T_4 - T_7 T_8, T_1 T_2 T_4 T_6 T_8),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_2^a T_4 - T_7 T_8, -T_1 T_2^{a-1} T_4 T_6 T_8 + T_3),\\
\overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_3, T_2^a T_4 - T_7 T_8),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_2 T_4, -T_1 T_2^{a-1} T_4 T_6 T_8 + T_3),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3, T_2 T_4),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_2^a T_4 - T_7 T_8, -T_1 T_2^{a-1} T_4 T_6 T_8 + T_3)
\end{aligned}$$

is six-dimensional. This can be seen directly or with Lemma 5.4.4. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we have performed the desired blow up. The ideal

$$\begin{aligned}
I' \;&:=\; \langle T_5, T_9, h_2, h_1 - T_7 T_8 \rangle\\
&=\; \langle T_5, T_9, T_2^a T_4 - T_7 T_8, T_1 T_2^{a-1} T_4 T_6 T_8 - T_3 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_9]
\end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,0,1,1,1,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5) \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$ and, as above, we have

$$\dim\left(V\left(\mathbb{K}^9; I'\right)\right) \;=\; -1 \,+\, \dim\left(\overline{X}_2 \cap V(T_5, T_{10})\right) \;=\; 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. Using Algorithm 4.3.3, we eliminate the equation $T_3 = -g_2 + T_3$ and obtain the Cox ring $R_2' := \mathbb{K}[T_1, \ldots, T_9]/I_2'$ of the blow up of X_1 in q_5. Its degree matrix Q_2' is given by removing the third column of a Gale dual matrix of P_2, i.e.,

$$Q_2' \;=\; \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 2a-1 & -a+1 & -a & a\\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1\\
0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}.$$

Observe that the blow up X_2 is isomorphic to a $\mathbb{K}^*$-surface Y since its Cox ring $\mathcal{R}(X_2) = R_2'$ is isomorphic to

$$\mathcal{R}(Y) \;:=\; \mathbb{K}[T_1, \ldots, T_9] \,/\, \langle T_2^a T_3 + T_9^2 T_4 T_5 T_8 - T_6 T_7 \rangle$$

with the same degree matrix Q_2', compare Lemma 5.1.5. The isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1, \ldots, T_9] \;\to\; \mathbb{K}[T_1, \ldots, T_9], \qquad T_i \;\mapsto\; \begin{cases} T_6 + T_9 T_4 T_5^2 T_1 T_2^{a-1} T_3, & i = 6,\\ T_i, & \text{else.} \end{cases}$$

We now blow up X_1 in q_5' by similar steps. Choose in $\mathbb{K}[T_1, \ldots, T_8]$ the polynomial $h_2 := T_1 T_2^{a-1} T_4 T_7 T_8^2 - T_3^2 T_5$ for the embedding $\bar{\imath}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\, Q_1 \left|\begin{array}{c} 0\\0\\2\\0\\1 \end{array}\right. \right], \qquad P_1' \;=\; \left[\begin{array}{rrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & 0 & 1 & 2 & -1\\ 0 & a & 0 & 1 & 0 & 0 & -1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1\\ 0 & 0 & 0 & 0 & 1 & 2 & -2 & -2 & 1 \end{array}\right].$$

For the blow up of X_1' in $\iota(q_5') = [1,1,1,1,1,0,1,1,0]$ we determine the stellar subdivision of Σ_1' at $v := (-1,0,-2,3) \in \mathbb{Z}^4$ and write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 &:= p_2^\star (p_1)_\star (T_2^a T_4 - T_3T_5T_6 - T_7T_8) \;=\; T_2^aT_4 - T_3T_5T_6T_{10} - T_7T_8,\\ g_2 &:= p_2^\star (p_1)_\star (T_9 - h_2) \;=\; T_9T_{10} - T_1T_2^{a-1}T_4T_7T_8^2 + T_3^2T_5. \end{aligned}$$

We show that $I_2 = I_2 : T_{10}^\infty$ by showing that I_2 is prime. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U \;:=\; \left\{x \in \overline{X}_2;\; x_2x_4x_7x_8 \neq 0 \text{ or } x_3x_5x_{10} \neq 0\right\} \;\subseteq\; \overline{X}_2 \;=\; V\left(\mathbb{K}^{10};\, I_2\right).$$

Inspecting the entries with indices $i = 1,2$ and $j = 1,7$ as well as $i = 1,2$ and $j = 6,9$, we see that the rank of the Jacobian matrix $(\partial g_i/\partial T_j)_{i,j}(u)$ is two for all $u \in U$. Furthermore, $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$\begin{array}{llll} V\left(\mathbb{K}^{10}; T_2, T_3\right), & V\left(\mathbb{K}^{10}; T_2, T_5\right), & V\left(\mathbb{K}^{10}; T_2, T_{10}\right), & V\left(\mathbb{K}^{10}; T_4, T_3\right),\\ V\left(\mathbb{K}^{10}; T_4, T_5\right), & V\left(\mathbb{K}^{10}; T_4, T_{10}\right), & V\left(\mathbb{K}^{10}; T_7, T_3\right), & V\left(\mathbb{K}^{10}; T_7, T_5\right),\\ V\left(\mathbb{K}^{10}; T_7, T_{10}\right), & V\left(\mathbb{K}^{10}; T_8, T_3\right), & V\left(\mathbb{K}^{10}; T_8, T_5\right), & V\left(\mathbb{K}^{10}; T_8, T_{10}\right). \end{array}$$

We directly see that each of the following intersections is of dimension six.

$$\begin{aligned} \overline{X}_2 \cap V(T_2, T_3) &= V(T_3, T_2, T_9T_{10}, T_7T_8),\\ \overline{X}_2 \cap V(T_2, T_5) &= V(T_5, T_2, T_9T_{10}, T_7T_8),\\ \overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_7T_8, T_3T_5),\\ \overline{X}_2 \cap V(T_4, T_3) &= V(T_4, T_3, T_9T_{10}, T_7T_8),\\ \overline{X}_2 \cap V(T_4, T_5) &= V(T_5, T_4, T_9T_{10}, T_7T_8),\\ \overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_7T_8, T_3T_5),\\ \overline{X}_2 \cap V(T_7, T_3) &= V(T_7, T_3, T_9T_{10}, T_2T_4),\\ \overline{X}_2 \cap V(T_7, T_5) &= V(T_7, T_5, T_9T_{10}, T_2T_4),\\ \overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_3T_5, T_2T_4),\\ \overline{X}_2 \cap V(T_8, T_3) &= V(T_8, T_3, T_9T_{10}, T_2T_4),\\ \overline{X}_2 \cap V(T_8, T_5) &= V(T_8, T_5, T_9T_{10}, T_2T_4),\\ \overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3T_5, T_2T_4). \end{aligned}$$

Therefore, $\dim(\overline{X}_2 \setminus U) \leq 6$ and, since $\overline{X}_2$ is of dimension at least eight, the codimension of $\overline{X}_2 \setminus U$ in $\overline{X}_2$ is at least two. By Lemma 5.4.3, the ideal I_2 is prime. We now show that the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\dots,T_{10}]/I_2$ by showing that the ideal

$$I_2 + \langle T_{10}\rangle = \langle T_{10},\; T_2^aT_4 - T_7T_8,\; T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5\rangle \;\subseteq\; \mathbb{K}[T_1,\dots,T_{10}]$$

is a prime ideal. Since $\mathbb{K}[T_1,\dots,T_{10}]/(I_2 + \langle T_{10}\rangle)$ is isomorphic to $\mathbb{K}[T_1,\dots,T_7]/I_0$ this is equivalent to

$$I_0 \;:=\; \langle T_2^aT_4 - T_6T_7,\; T_1T_2^{a-1}T_4T_6T_7^2 - T_3^2T_5\rangle \;\subseteq\; \mathbb{K}[T_1,\dots,T_7]$$

being prime. By a computation, we verified the cases $3 \leq a \leq 4$. Let $a \geq 5$. The ideal $\langle I_0\rangle \subseteq \mathbb{K}[T_1^{\pm 1},\dots,T_7^{\pm 1}]$ is prime since the matrix consisting of the exponents of the binomial generators

$$\begin{bmatrix} 0 & a & 0 & 1 & 0 & -1 & -1 \\ 1 & a-1 & -2 & 1 & -1 & 1 & 2 \end{bmatrix}$$

has a Smith normal form of shape $[E_2, 0, \ldots, 0]$ where E_2 is the 2×2 unit matrix, compare [37]. By Remark 5.4.14, I_0 is prime if $I_0 = I_0 : (T_1 \cdots T_7)^\infty$. To this end, we first show that

$$\begin{aligned} \mathcal{G} &:= \{f_1,\ f_2,\ f_3\} \\ &:= \left\{T_1T_6^2T_7^3 - T_2T_5T_3^2,\ T_2^aT_4 - T_6T_7,\ T_1T_2^{a-1}T_4T_6T_7^2 - T_5T_3^2\right\} \end{aligned}$$

is a Gröbner basis for I_0 with respect to any degree reverse lexicographical ordering with $T_1 > \ldots > T_{i-1} > T_{i+1} > \ldots > T_7 > T_i$ for a $1 \le i \le 7$. Since f_2, f_3 are the generators of I_0 and $f_1 = -T_1T_6T_7^2f_2 + T_2f_3$, we have $\langle \mathcal{G} \rangle = I_0$. The S-polynomials are

$$\begin{aligned} S(f_1,\ f_2) &= -T_2^{a+1}T_3^2T_4T_5 + T_1T_6^3T_7^4, \\ S(f_1,\ f_3) &= -T_2^aT_3^2T_4T_5 + T_3^2T_5T_6T_7, \\ S(f_2,\ f_3) &= -T_1T_6^2T_7^3 + T_2T_5T_3^2. \end{aligned}$$

The division algorithm, see [26, Ch. 2, Thm. 3], returns the combinations

$$S(f_1, f_2) = T_6T_7f_1 - T_2T_3^2T_5f_2, \quad S(f_1, f_3) = -T_3^2T_5f_2, \quad S(f_2, f_3) = -f_1.$$

By the Buchberger criterion, see [26, Ch. 2, Thm. 6], $\mathcal{G}$ is a Gröbner basis for I_0 with respect to each of the chosen orderings. From [90, Lem. 12.1], we infer that

$$\left\{\frac{f}{T_i^{k_i(f)}};\ f \in \mathcal{G}\right\} = \mathcal{G}, \qquad k_i(f) := \max\bigl(n \in \mathbb{Z}_{\ge 0};\ T_i^n \mid f\bigr)$$

is a Gröbner basis for $I_0 : T_i^\infty$ for each $1 \le i \le 7$, i.e., we have $I_0 = I_0 : T_i^\infty$ for each i. As in [90, p. 114], the claim follows from

$$I_0 : T_1 \cdots T_7^\infty = ((\cdots (I_0 : T_1^\infty) \cdots) : T_7^\infty) = I_0.$$

Furthermore, no two variables T_i, T_j are associated since $\deg(T_i) \ne \deg(T_j)$ for all $i \ne j$ and $T_{10} \nmid T_i$ for all $i < 10$: each of the intersections

$$\begin{aligned} \overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, T_3T_5, T_2^aT_4 - T_7T_8), \\ \overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_2^aT_4 - T_7T_8, T_1T_2T_4T_7T_8), \\ \overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_2^aT_4 - T_7T_8, T_1T_2T_4T_7T_8), \\ \overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_2^aT_4 - T_7T_8, T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5), \\ \overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_2^aT_4 - T_7T_8, T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5) \end{aligned}$$

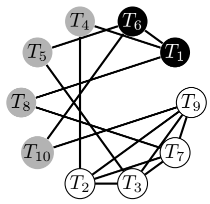
is six-dimensional; here, one uses Lemma 5.3.3. The missing cases have been treated before. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with its degree matrix as listed in the table. Observe that we performed the desired blow up. The factor ring $\mathbb{K}[T_1, \ldots, T_9]/I'$ where

$$\begin{aligned} I' &:= \langle T_6,\ T_9,\ h_2,\ h_1 - T_7T_8 \rangle \\ &= \langle T_6,\ T_9,\ T_2^aT_4 - T_7T_8,\ T_1T_2^{a-1}T_4T_7T_8^2 - T_3^2T_5 \rangle \subseteq \mathbb{K}[T_1, \ldots, T_9] \end{aligned}$$

is isomorphic to the integral domain $\mathbb{K}[T_1, \ldots, T_{10}]/(I_2 + \langle T_{10} \rangle)$. Hence, the ideal I' is prime. Let $z := (1, 1, 1, 1, 1, 0, 1, 1, 0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5') \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$ and by the previous dimension computations

$$\dim\left(V\left(\mathbb{K}^9;\ I'\right)\right) = -1 + \dim\left(\overline{X}_2 \cap V(T_6, T_{10})\right) = 5.$$

The performed modification then was the claimed blow up, see Lemma 5.2.15. The Cox ring and degree matrix of the resulting surface $X_2 = \mathrm{Bl}\ \mathbb{F}_a(\star^3 \star\ ii)$ are listed in the table of Theorem 5.4.1. We claim that the graph of exceptional curves G_{X_2} is



By Lemma 5.4.12, X_2 then cannot be a $\mathbb{K}^*$-surface. It suffices to prove the existence of the subgraph induced by the vertices T_i with $i \in \{1, 4, 5, 6, 8, 10\}$. By Remark 5.3.7 and the fact that $V(X_2; T_{10})$ is the exceptional divisor of the last blow up, we know that the curves corresponding to the vertices are negative. The existence of the edges is directly seen from the blow up sequence.

We come to the last configuration. Here, we want to blow up a point in the last exceptional divisor $V(X_1; T_8)$. By Remark 5.3.5, it suffices to consider the point $q_6 := [1, 1, 1, 1, 1, 1, 1, 0] \in X_1$. Similar to before, we choose in $\mathbb{K}[T_1, \ldots, T_8]$ the polynomial $h_2 := T_1 T_2^{a-1} T_4 T_5 T_6^2 - T_7$ for the embedding $\bar{\imath}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|r} & 1 \\ & 3a-1 \\ Q_1 & -2 \\ & 3 \\ & -1 \end{array}\right], \qquad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 0 & 1 & 1 & 2 & 0 & 0 & -1 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

For the blow up of X_1' in $\iota(q_6) = [1, 1, 1, 1, 1, 1, 1, 0, 0]$ we determine the stellar subdivision of Σ_1' at $v := (-1, -2, -2, -1) \in \mathbb{Z}^4$ and write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 \;&:=\; p_2^\star (p_1)_\star \left(T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8\right) \;=\; T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8 T_{10}, \\ g_2 \;&:=\; p_2^\star (p_1)_\star \left(T_9 - h_2\right) \;=\; T_9 T_{10} - T_1 T_2^{a-1} T_4 T_5 T_6^2 + T_7. \end{aligned}$$

We show that I_2 is saturated with respect to T_{10} by proving that I_2 is prime. By Lemma 5.4.4, the latter is the case if the ideal

$$I_2' \;:=\; \langle T_2^a T_4 - T_3 T_5 T_6 - T_7 T_9 T_1 T_2^{a-1} T_4 T_5 T_6^2 + T_7 T_9^2 T_8 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_9]$$

obtained by substitution of $T_7 = -g_2 + T_7$ in g_1 and replacing all T_i with $i > 7$ by T_{i-1} is prime. By Lemma 5.2.17, this is the case. In a similar manner, using Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \ldots, T_{10}]/I_2$ since

$$I_2' + \langle T_9 \rangle \;=\; \langle T_9,\, T_2^a T_4 - T_3 T_5 T_6 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_9]$$

is prime by Lemma 5.2.17. Moreover, as the degrees $\deg(T_i)$ are pairwise different, no two variables T_i, T_j are associated for $i \neq j$. Also, $T_{10} \nmid T_i$ for all $i < 10$: each of the intersections

$$\begin{aligned} \overline{X}_2 \cap V(T_1, T_{10}) \;&=\; V(T_{10}, T_1, T_7, T_2^a T_4 - T_3 T_5 T_6), \\ \overline{X}_2 \cap V(T_2, T_{10}) \;&=\; V(T_{10}, T_2, T_7, T_3 T_5 T_6), \\ \overline{X}_2 \cap V(T_3, T_{10}) \;&=\; V(T_{10}, T_3, T_2 T_4, T_7), \\ \overline{X}_2 \cap V(T_4, T_{10}) \;&=\; V(T_{10}, T_4, T_7, T_3 T_5 T_6), \\ \overline{X}_2 \cap V(T_5, T_{10}) \;&=\; V(T_{10}, T_5, T_7, T_2 T_4), \\ \overline{X}_2 \cap V(T_6, T_{10}) \;&=\; V(T_{10}, T_6, T_7, T_2 T_4), \\ \overline{X}_2 \cap V(T_7, T_{10}) \;&=\; V(T_{10}, T_7, T_2^a T_4 - T_3 T_5 T_6, T_1 T_2 T_4 T_5 T_6), \\ \overline{X}_2 \cap V(T_8, T_{10}) \;&=\; V(T_{10}, T_8, T_2^a T_4 - T_3 T_5 T_6, -T_1 T_2^{a-1} T_4 T_5 T_6^2 + T_7), \end{aligned}$$

$$\overline{X}_2 \cap V(T_9,\, T_{10}) \;=\; V(T_{10},\, T_9,\, T_2^aT_4 - T_3T_5T_6,\, -T_1T_2^{a-1}T_4T_5T_6^2 + T_7)$$

is of dimension six; here, Lemma 5.4.4 can be used. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification. Its degree matrix is a Gale dual matrix of P_2. Observe that we performed the desired blow up. The ideal

$$\begin{aligned} I' \;&:=\; \langle T_8,\, T_9,\, h_2,\, h_1 - T_7T_8\rangle \\ &=\; \langle T_8,\, T_9,\, T_2^aT_4 - T_3T_5T_6, -T_1T_2^{a-1}T_4T_5T_6^2 + T_7\rangle \;\subseteq\; \mathbb{K}[T_1,\dots,T_9] \end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,1,1,1,0,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_6) \in X_1'$. Then $z \in V(\mathbb{K}^9;\, I')$. Moreover, since T_7 is linear in g_2 we have

$$\dim\big(V\left(\mathbb{K}^9;\, I'\right)\big) \;=\; -1 + \dim\big(V\left(\mathbb{K}^9;\, T_8,\, T_9,\, T_2^aT_4 - T_3T_5T_6\right)\big) \;=\; 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. Using Algorithm 4.3.3, we substitute the equation $T_7 = -g_2 + T_7$ into g_1. We obtain the Cox ring $\mathcal{R}(X_2) = \mathbb{K}[T_1,\dots,T_9]/I_2'$ of the blow up X_2 of X_1 in q_6. Its degree matrix Q_2' is given by removing the seventh column of a Gale dual matrix of P_2, i.e.,

$$Q_2' \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & a & 0 & 5a-2 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

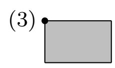
Observe that X_2 is isomorphic to a $\mathbb{K}^*$-surface Y. By Lemma 5.1.5 it suffices to show that $\mathcal{R}(X_2)$ is isomorphic to

$$\mathcal{R}(Y) \;:=\; \mathbb{K}[T_1,\dots,T_9]\,/\,\langle T_2^aT_4 - T_3T_5T_6 + T_7T_9^2T_8\rangle$$

where the degree matrix of $\mathcal{R}(Y)$ is again Q_2'. The isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the graded homomorphism

$$\mathbb{K}[T_1,\dots,T_9] \;\to\; \mathbb{K}[T_1,\dots,T_9], \qquad T_i \;\mapsto\; \begin{cases} T_3 + T_7T_9T_1T_2^{a-1}T_4T_6, & i = 3, \\ T_i, & \text{else.} \end{cases}$$

(XV) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star^3\ iv)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence



$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ i)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ i) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1 \colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 \;:=\; T_1T_2^{a-1}T_4 - T_3T_6$$

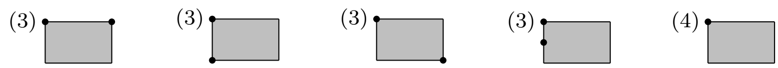
where $h_1 \in \mathbb{K}[T_1,\dots,T_6]$ and the blow ups π_i are given by

$$\pi_3([z]) \;=\; [z_1,\dots,z_4, z_5z_8, z_6, z_7z_8],$$
$$\pi_2([z]) \;=\; [z_1, z_2, z_3z_6, z_4, z_5z_6], \qquad \pi_1([z]) \;=\; [z_1z_5, z_2, z_3z_5, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1;\, T_5), \qquad V(X_1;\, T_6), \qquad V(X_1;\, T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations



The blow ups of the first three configurations are the blow ups of X_1 in the points

$$[1,0,0,1,1,1,0,1], \quad [0,1,1,0,1,1,-1,1], \quad [1,0,1,0,1,1,-1,1] \ \in \ X_1$$

which project under $\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3$ to $[1,0,0,1]$, $[0,1,1,0]$ and $[1,0,1,0] \in \mathbb{F}_a$ respectively. Note that all points exist by a stepwise application of Lemmas 5.4.5 and 5.2.16. By Lemma 5.4.11, all three surfaces will be $\mathbb{K}^*$-surfaces.

We come to the fourth configuration. The main steps are as in previous cases. We want to blow up of X_1 in the point

$$q_4 \ := \ [0,1,1,1,1,1,-1,1] \in X_1, \qquad \pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3(q_4) \ = \ [0,1,1,1] \ \in \ \mathbb{F}_a.$$

Note that q_4 exists by Lemma 5.2.16. Choose in $\mathbb{K}[T_1,\ldots,T_8]$ the polynomial $h_2 := T_2^a T_4 + T_5 T_6 T_7 T_8^2$ for the embedding $\bar{\iota}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \ = \ \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ a \\ 0 \\ 1 \\ 0 \end{smallmatrix} \end{array}\right], \qquad P_1' \ = \ \left[\begin{array}{rrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & -1 \end{array}\right].$$

For the blow up of X_1' in $\iota(q_4) = [0,1,1,1,1,1,-1,1,0]$, we determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (1,-1,0,-1) \in \mathbb{Z}^4$. Write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 \ := \ p_2^\star (p_1)_\star (T_1 T_2^{a-1} T_4 - T_3 T_6 - T_7 T_8) \ = \ T_1 T_2^{a-1} T_4 T_{10} - T_3 T_6 - T_7 T_8,$$
$$g_2 \ := \ p_2^\star (p_1)_\star (T_9 - h_2) \ = \ T_9 T_{10} - T_2^a T_4 - T_5 T_6 T_7 T_8^2.$$

We show that I_2 is prime. In particular, I_2 is saturated with respect to T_{10}. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U \ := \ \left\{x \in \overline{X}_2;\ x_6 x_9 \neq 0 \text{ or } x_7 x_{10} \neq 0\right\} \ \subseteq \ \overline{X}_2 \ = \ V\left(\mathbb{K}^{10};\, I_2\right).$$

Inspecting the indices $i = 1,2$ and $j = 3,10$ as well as $i = 1,2$ and $j = 8,9$ we see that the rank of the Jacobian matrix $(\partial g_i / \partial T_j)_{i,j}(u)$ is two for all $u \in U$. Furthermore, $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$V\left(\mathbb{K}^{10};\, T_6,\, T_7\right), \quad V\left(\mathbb{K}^{10};\, T_6,\, T_{10}\right), \quad V\left(\mathbb{K}^{10};\, T_9,\, T_7\right), \quad V\left(\mathbb{K}^{10};\, T_9,\, T_{10}\right).$$

Furthermore, each of the following intersections is of dimension six.

$$\begin{aligned} \overline{X}_2 \cap V(T_6,\, T_7) \ &= \ V(T_7,\, T_6,\, T_2^a T_4 - T_9 T_{10},\, T_1 T_2 T_4 T_{10}), \\ \overline{X}_2 \cap V(T_6,\, T_{10}) \ &= \ V(T_{10},\, T_6,\, T_7 T_8,\, T_2 T_4), \\ \overline{X}_2 \cap V(T_9,\, T_7) \ &= \ V(T_9,\, T_7,\, T_2 T_4,\, T_3 T_6), \\ \overline{X}_2 \cap V(T_9,\, T_{10}) \ &= \ V(T_{10},\, T_9,\, T_3 T_6 - T_7 T_8,\, T_2^a T_4 + T_5 T_6 T_7 T_8^2). \end{aligned}$$

For $\overline{X}_2 \cap V(T_9,\, T_{10})$, we used Lemma 5.3.3 with the exponent matrix

$$\left[\begin{array}{rrrrrrrrrr} 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & a & 0 & 1 & -1 & -1 & -1 & -2 & 0 & 0 \end{array}\right]$$

to see that the dimension is six on $\mathbb{T}^{10} \cdot (1,\ldots,1,0,0)$. One directly verifies that the dimension is at most six on all smaller tori. Therefore, $\dim(\overline{X}_2 \setminus U) \leq 6$ and, since $\overline{X}_2$ is of dimension at least eight, the codimension of $\overline{X}_2 \setminus U$ in $\overline{X}_2$ is at least two. By Lemma 5.4.3, the ideal I_2 is prime. We now show that the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\ldots,T_{10}]/I_2$ by proving that

$$I_2 + \langle T_{10} \rangle \ = \ \langle T_{10},\ T_3 T_6 + T_7 T_8,\ T_2^a T_4 + T_5 T_6 T_7 T_8^2 \rangle \ \subseteq \ \mathbb{K}[T_1,\ldots,T_{10}]$$

is a prime ideal. Since $\mathbb{K}[T_1,\ldots,T_{10}]/(I_2 + \langle T_{10} \rangle)$ is isomorphic to $\mathbb{K}[T_1,\ldots,T_7]/I_0$ this is equivalent to

$$I_0 \ := \ \langle T_2 T_5 + T_6 T_7,\ T_1^a T_3 + T_4 T_5 T_6 T_7^2 \rangle \ \subseteq \ \mathbb{K}[T_1,\ldots,T_7]$$

being prime. By a computation, we verified the cases $3 \le a \le 4$. Assume $a \ge 5$. The ideal $\langle I_0 \rangle \subseteq \mathbb{K}[T_1^{\pm 1}, \dots, T_7^{\pm 1}]$ is prime since the matrix consisting of the exponents of the binomial generators

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ a & 0 & 1 & -1 & -1 & -1 & -2 \end{bmatrix}$$

has a Smith normal form of shape $[E_2, 0, \dots, 0]$ where E_2 is the 2×2 unit matrix, compare [37]. By Remark 5.4.14, I_0 is prime if $I_0 = I_0 : (T_1 \cdots T_7)^\infty$. To show this, we first prove that

$$\mathcal{G} \ := \ \{f_1,\ f_2\} \ := \ \begin{cases} \left\{ T_2T_5 + T_6T_7,\ T_1^aT_3 + T_4T_5T_6T_7^2 \right\}, & T_2T_5 > T_6T_7, \\ \left\{ T_6T_7 + T_5T_2,\ T_1^aT_3 - T_4T_5^2T_7T_2 \right\}, & T_2T_5 < T_6T_7 \end{cases}$$

is a Gröbner basis for I_0 with respect to any degree reverse lexicographical ordering with $T_1 > \dots > T_{i-1} > T_{i+1} > \dots > T_7 > T_i$ for any $1 \le i \le 7$. Let g_i' be the generators of I_0. In the case $T_2T_5 < T_6T_7$ we have $f_2 = -T_4T_5T_7g_1' + g_2'$. In particular, we have $\langle \mathcal{G} \rangle = I_0$ in both cases. The single S-polynomial is

$$S(f_1,\ f_2) \ = \ \begin{cases} T_1^aT_3T_6T_7 - T_2T_4T_5^2T_6T_7^2, & T_2T_5 > T_6T_7, \\ T_1^aT_3T_5T_2 + T_2T_4T_5^2T_6T_7^2, & T_2T_5 < T_6T_7. \end{cases}$$

The division algorithm, see [26, Ch. 2, Thm. 3], returns the combinations

$$S(f_1,\ f_2) \ = \ \begin{cases} -T_4T_5T_6T_7^2f_1 + T_6T_7f_2, & T_2T_5 > T_6T_7, \\ T_4T_5^2T_7T_2f_1 + T_5T_2f_2, & T_2T_5 < T_6T_7. \end{cases}$$

By the Buchberger criterion, see [26, Ch. 2, Thm. 6], $\mathcal{G}$ is a Gröbner basis for I_0 with respect to each of the chosen orderings. By [90, Lem. 12.1], we know that

$$\left\{ \frac{f}{T_i^{k_i(f)}};\ f \in \mathcal{G} \right\} \ = \ \mathcal{G}, \qquad k_i(f) \ := \ \max\bigl(n \in \mathbb{Z}_{\ge 0};\ T_i^n \mid f\bigr)$$

is a Gröbner basis for $I_0 : T_i^\infty$ for each $1 \le i \le 7$. In particular, $I_0 = I_0 : T_i^\infty$ for each i. As in [90, p. 114], the claim follows from

$$I_0 : (T_1 \cdots T_7)^\infty \ = \ ((\cdots (I_0 : T_1^\infty) \cdots) : T_7^\infty) \ = \ I_0.$$

Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \ne \deg(T_j)$ for all $i \ne j$. Also, $T_{10} \nmid T_i$ for all $i < 10$ since each of the intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) \ &= \ V(T_{10}, T_1, T_3T_6 + T_7T_8, T_2^aT_4 + T_5T_6T_7T_8^2), \\
\overline{X}_2 \cap V(T_2, T_{10}) \ &= \ V(T_{10}, T_2, T_3T_6 + T_7T_8, T_5T_6T_7T_8), \\
\overline{X}_2 \cap V(T_3, T_{10}) \ &= \ V(T_{10}, T_3, T_7T_8, T_2T_4), \\
\overline{X}_2 \cap V(T_4, T_{10}) \ &= \ V(T_{10}, T_4, T_3T_6 + T_7T_8, T_5T_6T_7T_8), \\
\overline{X}_2 \cap V(T_5, T_{10}) \ &= \ V(T_{10}, T_5, T_3T_6 + T_7T_8, T_2T_4), \\
\overline{X}_2 \cap V(T_7, T_{10}) \ &= \ V(T_{10}, T_7, T_3T_6, T_2T_4), \\
\overline{X}_2 \cap V(T_8, T_{10}) \ &= \ V(T_{10}, T_8, T_3T_6, T_2T_4)
\end{aligned}$$

is of dimension six. As in previous cases, this is done by a computer check or using Lemma 5.3.3. The missing cases have been treated before. By Theorem 4.2.6, R_2 is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. Observe that we performed the desired blow up. The ideal

$$\begin{aligned}
I' \ &:= \ \langle T_1, T_9, h_2, h_1 - T_7T_8 \rangle \\
&= \ \langle T_1, T_9, T_3T_6 + T_7T_8, T_2^aT_4 + T_5T_6T_7T_8^2 \rangle \ \subseteq \ \mathbb{K}[T_1, \dots, T_9]
\end{aligned}$$

is prime since $\mathbb{K}[T_1,\dots,T_9]/I'$ is isomorphic to $\mathbb{K}[T_1,\dots,T_{10}]/(I_2+\langle T_{10}\rangle)$ which is an integral domain. Let $z := (0,1,1,1,1,1,-1,1,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_4) \in X_1'$. Then $z \in V(\mathbb{K}^9;\, I')$ and

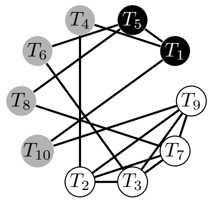
$$\dim\left(V\left(\mathbb{K}^9;\, I'\right)\right) \;=\; -1 + \dim\left(\overline{X}_2 \cap V(T_1,\, T_{10})\right) \;=\; 5$$

by the previous dimension arguments. By Lemma 5.2.15, the performed modification was the claimed blow up. The Cox ring and degree matrix of the resulting surface $X_2 = \mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ iii)$ are

$$\mathcal{R}(X_2) \;=\; \mathbb{K}[T_1,\dots,T_{10}] \Big/ \left\langle \begin{array}{c} T_1T_2^{a-1}T_4T_{10} - T_3T_6 - T_7T_8, \\ T_2^aT_4 + T_5T_6T_7T_8^2 - T_9T_{10} \end{array} \right\rangle,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -a & a & 2a-1 & -a+1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & -2 & -1 & 1 \end{bmatrix}.$$

Observe that X_2 is not a $\mathbb{K}^*$-surface. To this end, we claim that the graph of exceptional curves G_{X_2} is as follows. However, note that it suffices to prove the existence of the subgraph induced by the vertices T_i with $i \in \{1,4,5,6,8,10\}$.



By Remark 5.3.7 and the fact that $V(X_2;\, T_{10})$ is the exceptional divisor of the last blow up, we know that the curves corresponding to the vertices are negative. The existence of the edges, i.e., the fact that the curves meet, is directly seen from the blow up sequence. By Lemma 5.4.12, X_2 cannot be a $\mathbb{K}^*$-surface.

For the fifth configuration, we treat blow ups of X_1 in a point in the union of the exceptional divisors

$$V(X_1;\, T_5) \;\;\cup\;\; V(X_1;\, T_6) \;\;\cup\;\; V(X_1;\, T_8).$$

Note that we do not have to blow up a point in the parabolic fixed point curve $V(X_1;\, T_5)$ by Lemma 5.4.11. By Remark 5.3.5, it suffices to consider the points

$$q_5 \;:=\; [1,1,1,1,1,0,1,1], \qquad q_5' \;:=\; [1,1,1,1,1,1,1,0] \;\in\; X_1.$$

Lemma 5.2.16 ensures their existence. We first blow up X_1 in q_5. Choose in $\mathbb{K}[T_1,\dots,T_8]$ the polynomial $h_2 := T_1^2T_2^{a-2}T_4T_5T_8 - T_3$ for the embedding $\bar\iota_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix} \end{array}\right], \qquad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & a & 0 & 1 & -1 & 0 & -2 & -3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

For the blow up of X_1' in $\iota(q_5) = [1,1,1,1,1,0,1,1,0]$, we determine the stellar subdivision of Σ_1' at $v := (0,1,-1,2) \in \mathbb{Z}^4$. Write $P_2 = [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 \;&:=\; p_2^\star\,(p_1)_\star\,(T_1T_2^{a-1}T_4 - T_3T_6 - T_7T_8) \;=\; T_1T_2^{a-1}T_4 - T_3T_6T_{10} - T_7T_8, \\ g_2 \;&:=\; p_2^\star\,(p_1)_\star\,(T_9 - h_2) \;=\; T_9T_{10} - T_1^2T_2^{a-2}T_4T_5T_8 + T_3. \end{aligned}$$

We show that I_2 is saturated with respect to T_{10} by showing that I_2 is prime. Consider the ideal

$$I_2' \;:=\; \langle T_1T_2^{a-1}T_3 - T_9T_5T_1^2T_2^{a-2}T_3T_4T_7 + T_9^2T_5T_8 - T_6T_7\rangle \;\subseteq\; \mathbb{K}[T_1,\dots,T_9]$$

obtained by substitution of $T_3 = -g_2 + T_3$ in g_1 and replacing all T_i with T_{i-1} if $i > 3$. By Lemma 5.2.17, I_0 is prime. This implies primality of I_2, see Lemma 5.4.4. Similarly, by Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$ since the ideal

$$I_2' + \langle T_9 \rangle \;=\; \langle T_9,\, T_1 T_2^{a-1} T_3 - T_6 T_7 \rangle \;\subseteq\; \mathbb{K}[T_1, \dots, T_9]$$

is prime by Lemma 5.2.17. Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for all $i \neq j$. Also, we have $T_{10} \nmid T_i$ for each $i < 10$: all intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_1 T_2^{a-1} T_4 - T_7 T_8, T_1 T_2 T_4 T_5 T_8),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3, T_7 T_8),\\
\overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_3, T_1 T_2^{a-1} T_4 - T_7 T_8),\\
\overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_1 T_2^{a-1} T_4 - T_7 T_8, -T_1^2 T_2^{a-2} T_4 T_5 T_8 + T_3),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_1 T_2 T_4, T_3),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3, T_1 T_2 T_4),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_1 T_2^{a-1} T_4 - T_7 T_8, -T_1^2 T_2^{a-2} T_4 T_5 T_8 + T_3)
\end{aligned}$$

are six-dimensional; this can be seen using Lemmas 5.4.4 and Lemma 5.3.3. Theorem 4.2.6 shows that $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$ is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we performed the desired blow up. The ideal

$$\begin{aligned}
I' &:= \langle T_6, T_9, h_2, h_1 - T_7 T_8 \rangle\\
&= \langle T_6, T_9, T_1 T_2^{a-1} T_4 - T_7 T_8, T_1^2 T_2^{a-2} T_4 T_5 T_8 - T_3 \rangle \;\subseteq\; \mathbb{K}[T_1, \dots, T_9]
\end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1, 1, 1, 1, 1, 0, 1, 1, 0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5) \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$ and

$$\dim\left(V\left(\mathbb{K}^9; I'\right)\right) \;=\; -1 + \dim\left(\overline{X}_2 \cap V(T_6, T_{10})\right) \;=\; 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. Using Algorithm 4.3.3, we eliminate the equation $T_3 = -g_2 + T_3$ and obtain the graded ring $R_2' := \mathbb{K}[T_1, \dots, T_9]/I_2'$ as Cox ring of the blow up X_2 of X_1 in q_5. Its degree matrix Q_2' is obtained by removing the third column of a Gale dual matrix of P_2, i.e.,

$$Q_2' \;=\; \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 3 & -2 & -1 & 1\\
0 & 1 & 0 & 0 & 0 & 2a-3 & -a+2 & -a+1 & a-1\\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -1\\
0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 & 1\\
0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1
\end{bmatrix}.$$

Note that X_2 is isomorphic to a $\mathbb{K}^*$-surface Y. By Lemma 5.1.5 it suffices to show that $\mathcal{R}(X_2)$ is isomorphic to

$$\mathcal{R}(Y) \;:=\; \mathbb{K}[T_1, \dots, T_9]/\langle T_1 T_2^{a-1} T_3 + T_9^2 T_5 T_8 - T_6 T_7 \rangle$$

with the same degree matrix Q_2'. The isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1, \dots, T_9] \;\to\; \mathbb{K}[T_1, \dots, T_9], \qquad T_i \;\mapsto\; \begin{cases} T_6 + T_9 T_5 T_1^2 T_2^{a-2} T_3 T_4, & i = 6,\\ T_i, & \text{else.} \end{cases}$$

For the blow up of X_1 in q_5' we choose in $\mathbb{K}[T_1, \dots, T_8]$ the polynomial $h_2 := T_1^2 T_2^{a-2} T_4 T_5 T_6 - T_7$ for the embedding $\bar{\iota}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|r} Q_1 & \begin{array}{r} 3 \\ 2a-3 \\ -1 \\ 2 \\ 1 \end{array} \end{array}\right], \qquad P_1' \;=\; \left[\begin{array}{rrrrrrrrr} 1 & a-1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & a & 0 & 1 & -1 & -1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right].$$

For the blow up of X_1' in $\iota(q_5') = [1,1,1,1,1,1,1,0,0]$ we determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at the sum $v := (-2,-3,-2,-1) \in \mathbb{Z}^4$ of the last two columns of P_1'. Define $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 \;&:=\; p_2^\star\,(p_1)_\star\,(T_1T_2^{a-1}T_4 - T_3T_6 - T_7T_8) \;=\; T_1T_2^{a-1}T_4 - T_3T_6 - T_7T_8T_{10}, \\ g_2 \;&:=\; p_2^\star\,(p_1)_\star\,(T_9 - h_2) \;=\; T_9T_{10} - T_1^2T_2^{a-2}T_4T_5T_6 + T_7. \end{aligned}$$

We show that I_2 is prime. Note that this implies that I_2 is saturated with respect to T_{10}. Consider the ideal

$$I_2' \;:=\; \langle T_1T_2^{a-1}T_4 - T_3T_6 - T_7T_9T_1^2T_2^{a-2}T_4T_5T_6 + T_7T_9^2T_8\rangle \;\subseteq\; \mathbb{K}[T_1,\ldots,T_9]$$

obtained by substitution of $T_7 = -g_2 + T_7$ in g_1 and replacing all T_i with T_{i-1} if $i > 7$. By Lemma 5.4.4, the ideal I_2 is prime if the ideal I_2' is. The latter follows from Lemma 5.2.17. Using again Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\ldots,T_{10}]/I_2$ since the ideal

$$I_2' + \langle T_9\rangle \;=\; \langle T_9,\, T_1T_2^{a-1}T_4 - T_3T_6\rangle \;\subseteq\; \mathbb{K}[T_1,\ldots,T_9].$$

is prime, see Lemma 5.2.17. Furthermore, as the degrees $\deg(T_i)$ are pairwise distinct, no two variables T_i, T_j are associated for $i \neq j$. Also, $T_i \nmid T_{10}$ for all $i < 10$: the vanishing sets

$$\begin{aligned} \overline{X}_2 \cap V(T_1,\,T_{10}) \;&=\; V(T_{10},\,T_1,\,T_7,\,T_3T_6), \\ \overline{X}_2 \cap V(T_2,\,T_{10}) \;&=\; V(T_{10},\,T_2,\,T_7,\,T_3T_6), \\ \overline{X}_2 \cap V(T_3,\,T_{10}) \;&=\; V(T_{10},\,T_3,\,T_1T_2T_4,\,T_7), \\ \overline{X}_2 \cap V(T_4,\,T_{10}) \;&=\; V(T_{10},\,T_4,\,T_7,\,T_3T_6), \\ \overline{X}_2 \cap V(T_5,\,T_{10}) \;&=\; V(T_{10},\,T_5,\,T_7,\,T_1T_2^{a-1}T_4 - T_3T_6), \\ \overline{X}_2 \cap V(T_6,\,T_{10}) \;&=\; V(T_{10},\,T_6,\,T_7,\,T_1T_2T_4), \\ \overline{X}_2 \cap V(T_7,\,T_{10}) \;&=\; V(T_{10},\,T_7,\,T_1T_2^{a-1}T_4 - T_3T_6,\,T_1T_2T_4T_5T_6), \\ \overline{X}_2 \cap V(T_8,\,T_{10}) \;&=\; V(T_{10},\,T_8,\,T_1T_2^{a-1}T_4 - T_3T_6,\,-T_1^2T_2^{a-2}T_4T_5T_6 + T_7), \\ \overline{X}_2 \cap V(T_9,\,T_{10}) \;&=\; V(T_{10},\,T_9,\,T_1T_2^{a-1}T_4 - T_3T_6,\,-T_1^2T_2^{a-2}T_4T_5T_6 + T_7) \end{aligned}$$

are all six-dimensional; this can be seen using Lemmas 5.4.4 and 5.3.3. By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1,\ldots,T_{10}]/I_2$ is the Cox ring of the performed modification with a Gale dual matrix of P_2 as its degree matrix. We now show that we performed the desired blow up. The ideal

$$\begin{aligned} I' \;&:=\; \langle T_8,\,T_9,\,h_2,\,h_1 - T_7T_8\rangle \\ &=\; \langle T_8,\,T_9,\,T_1T_2^{a-1}T_4 - T_3T_6,\,-T_1^2T_2^{a-2}T_4T_5T_6 + T_7\rangle \;\subseteq\; \mathbb{K}[T_1,\ldots,T_9] \end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,1,1,1,0,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5') \in X_1'$. Then $z \in V(\mathbb{K}^9;\ I')$ and

$$\dim\left(V\left(\mathbb{K}^9;\,I'\right)\right) \;=\; -1 + \dim\left(\overline{X}_2 \cap V(T_9,\,T_{10})\right) \;=\; 5$$

where we used the previous dimension computations. By Lemma 5.2.15, the performed modification was the claimed blow up. Using Algorithm 4.3.3, we eliminate the equation $T_7 = -g_2 + T_7$ and obtain the $\mathbb{Z}^6$-graded ring $R_2' := \mathbb{K}[T_1,\ldots,T_9]/I_2'$ as

Cox ring of the blow up X_2 of X_1 in q_5'. Its degree matrix Q_2' is given by removing the seventh column of a Gale dual matrix of P_2, i.e.,

$$Q_2' \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & -2 \\ 0 & 1 & 0 & 0 & 0 & a-1 & 0 & 3a-5 & -a+2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

We show that X_2 is isomorphic to a $\mathbb{K}^*$-surface Y. By Lemma 5.1.5 it suffices to given an isomorphism between $\mathcal{R}(X_2)$ and

$$\mathcal{R}(Y) \;:=\; \mathbb{K}[T_1,\dots,T_9] \,/\, \langle T_1T_2^{a-1}T_4 - T_3T_6 + T_7T_9^2\,T_8\rangle$$

where the degree matrix of $\mathcal{R}(Y)$ is again Q_2'. The isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1,\dots,T_9] \;\to\; \mathbb{K}[T_1,\dots,T_9], \qquad T_i \;\mapsto\; \begin{cases} T_3 + T_7T_9T_1^2T_2^{a-2}T_4T_5, & i = 3, \\ T_i, & \text{else.} \end{cases}$$

(XVI) Blow ups of $X_1 := \mathrm{Bl}\ \mathbb{F}_a(\star^3\ ix)$. Let $a \geq 3$. Recall from the proofs of Proposition 5.2.8 and Theorem 5.3.1 the point configuration and blow up sequence

(3) 

$$X_1 \xrightarrow{\pi_3} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ ii)' \xleftarrow{\iota_1} \mathrm{Bl}\ \mathbb{F}_a(\star^2\ ii) \xrightarrow{\pi_2} \mathrm{Bl}\ \mathbb{F}_a(\star\ i) \xrightarrow{\pi_1} \mathbb{F}_a$$

where the embedding ι_1 is as in Setting 4.2.9 with

$$\bar{\iota}_1\colon \mathbb{K}^6 \to \mathbb{K}^7, \qquad z \mapsto (z, h_1(z)), \qquad h_1 \;:=\; T_1T_2^{2a-1}T_4^2 - T_3^2T_5$$

where $h_1 \in \mathbb{K}[T_1,\dots,T_6]$ and the blow ups π_i are

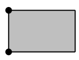
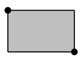
$$\pi_3([z]) \;=\; [z_1,\dots,z_5, z_6z_8, z_7z_8],$$

$$\pi_2([z]) \;=\; [z_1z_6, z_2, z_3, z_4, z_5z_6], \qquad \pi_1([z]) \;=\; [z_1z_5, z_2, z_3z_5, z_4].$$

The exceptional divisors of the first, second and third blow up are

$$V(X_1; T_5), \qquad V(X_1; T_6), \qquad V(X_1; T_8).$$

On X_1, we want to blow up a point which, together with the exceptional divisors, projects to one of the configurations

(3)  (3)  (3)  (3)  (4) 

For the first three configurations we blow up X_1 in the points

$$[1,0,0,1,1,1,0,1], \quad [0,1,1,0,1,1,-1,1], \quad [1,0,1,0,1,1,-1,1] \;\in\; X_1,$$

which project under $\pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3$ to $[1,0,0,1]$, $[0,1,1,0]$ and $[1,0,1,0] \in \mathbb{F}_a$ respectively. By a stepwise application of Lemma 5.4.5 and Lemma 5.2.16, all points exist. By Lemma 5.4.11, all three surfaces will be $\mathbb{K}^*$-surfaces.

The blow up of the fourth configuration is the blow up of X_1 in the point

$$q_4 \;:=\; [0,1,1,1,1,1,-1,1] \;\in\; X_1, \qquad \pi_1 \circ \pi_2 \circ \iota_1^{-1} \circ \pi_3(q_4) \;=\; [0,1,1,1] \;\in\; \mathbb{F}_a.$$

Note that q_4 exists by Lemma 5.2.16. The main steps are as in previous cases. Choose in $\mathbb{K}[T_1,\dots,T_8]$ the polynomial $h_2 := T_2^aT_4 - T_3T_5T_6T_8$ for the embedding $\bar{\iota}_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ a \\ 0 \\ 1 \\ 0 \end{smallmatrix} \end{array}\right], \qquad P_1' \;=\; \begin{bmatrix} 1 & a-1 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & -2 \end{bmatrix}.$$

For the blow up of X_1' in the point $\iota(q_4) = [0,1,1,1,1,1,-1,1,0]$ we determine the stellar subdivision of Σ_1' at $v := (2,-1,1,-2) \in \mathbb{Z}^4$. Define the enlarged matrix $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 &:= p_2^\star (p_1)_\star \left(T_1T_2^{2a-1}T_4^2 - T_3^2T_5 - T_7T_8\right) = T_1T_2^{2a-1}T_4^2T_{10} - T_3^2T_5 - T_7T_8, \\ g_2 &:= p_2^\star (p_1)_\star (T_9 - h_2) = T_9T_{10} - T_2^aT_4 + T_3T_5T_6T_8. \end{aligned}$$

We show that $I_2 = I_2 : T_{10}^\infty$ by proving that I_2 is a prime ideal. The grading is pointed by Remark 4.2.10. Consider the open subset

$$U := \left\{x \in \overline{X}_2;\ x_8x_9 \neq 0 \text{ or } x_7x_{10} \neq 0\right\} \subseteq \overline{X}_2 = V\left(\mathbb{K}^{10}; I_2\right).$$

Inspecting the indices $i = 1,2$ and $j = 7,10$ as well as $i = 1,2$ and $j = 8,9$ we see that the rank of the Jacobian matrix $(\partial g_i/\partial T_j)_{i,j}(u)$ is two for all $u \in U$. The complement $\overline{X}_2 \setminus U$ is contained in the union of the 8-dimensional subspaces

$$V\left(\mathbb{K}^{10}; T_8, T_7\right), \quad V\left(\mathbb{K}^{10}; T_8, T_{10}\right), \quad V\left(\mathbb{K}^{10}; T_9, T_7\right), \quad V\left(\mathbb{K}^{10}; T_9, T_{10}\right).$$

Each of the following intersections is of dimension six

$$\begin{aligned} \overline{X}_2 \cap V(T_8, T_7) &= V(T_8, T_7, T_1T_2^{2a-1}T_4^2T_{10} - T_3^2T_5, T_9T_{10} - T_2^aT_4), \\ \overline{X}_2 \cap V(T_8, T_{10}) &= V(T_8, T_{10}, T_3T_5, T_2T_4), \\ \overline{X}_2 \cap V(T_9, T_7) &= V(T_9, T_7, T_1T_2^{2a-1}T_4^2T_{10} - T_3^2T_5, -T_2^aT_4 + T_3T_5T_6T_8), \\ \overline{X}_2 \cap V(T_9, T_{10}) &= V(T_9, T_{10}, -T_3^2T_5 - T_7T_8, -T_2^aT_4 + T_3T_5T_6T_8). \end{aligned}$$

Note that for the first, third and fourth variety we used Lemma 5.3.3 with the respective exponent matrices

$$\begin{bmatrix} 1 & 2a-1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -a & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2a-1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & -1 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 2 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & a & -1 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

to see that the dimension is six on the respective tori

$$\mathbb{T}^{10} \cdot (1,\dots,1,0,0,1,1), \qquad \mathbb{T}^{10} \cdot (1,\dots,1,0,1,0), \qquad \mathbb{T}^{10} \cdot (1,\dots,1,0,0)$$

and then directly checks that the dimension is at most six on all smaller tori. Therefore, $\dim(\overline{X}_2 \setminus U) \leq 6$. Since $\overline{X}_2$ is of dimension at least eight, the codimension of $\overline{X}_2 \setminus U$ in $\overline{X}_2$ is at least two. By Lemma 5.4.3, this shows that I_2 is prime. We now prove that the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\dots,T_{10}]/I_2$. Consider the ideals

$$\begin{aligned} I_2 + \langle T_{10}\rangle &= \langle T_{10},\ -T_3^2T_5 - T_7T_8,\ -T_2^aT_4 + T_3T_5T_6T_8\rangle \subseteq \mathbb{K}[T_1,\dots,T_{10}], \\ I_0 &:= \langle T_2^2T_4 + T_6T_7,\ T_1^aT_3 - T_2T_4T_5T_7\rangle \subseteq \mathbb{K}[T_1,\dots,T_7]. \end{aligned}$$

Since $\mathbb{K}[T_1,\dots,T_{10}]/(I_2 + \langle T_{10}\rangle)$ is isomorphic to $\mathbb{K}[T_1,\dots,T_7]/I_0$ it suffices to show that I_0 is prime. The ideal $\langle I_0\rangle \subseteq \mathbb{K}[T_1^{\pm 1},\dots,T_7^{\pm 1}]$ is prime since the matrix consisting of the exponents of the binomial generators

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 & -1 & -1 \\ a & -1 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

has a Smith normal form of shape $[E_2, 0, \dots, 0]$ where E_2 is the 2×2 unit matrix, compare [37]. By Remark 5.4.14, I_0 is prime if $I_0 = I_0 : (T_1 \cdots T_7)^\infty$. In a first step we show that the set of generators

$$\mathcal{G} := \{f_1,\ f_2\} := \{T_2^2T_4 + T_6T_7,\ T_1^aT_3 - T_2T_4T_5T_7\}$$

already is a Gröbner basis for I_0 with respect to the degree reverse lexicographical ordering for any ordering of the variables of the kind $T_1 > \ldots > T_{i-1} > T_{i+1} >$

$\ldots > T_7 > T_i$ with $1 \le i \le 7$. We verified the case $3 \le a \le 4$ by a computer check, so assume $a \ge 5$. The single S-polynomial is

$$S(f_1,\, f_2) \;=\; T_1^a T_3 T_6 T_7 + T_2^3 T_4^2 T_5 T_7.$$

The division algorithm, see [26, Ch. 2, Thm. 3], returns the combination

$$S(f_1,\, f_2) \;=\; T_2 T_4 T_5 T_7 f_1 + T_6 T_7 f_2.$$

Using the Buchberger criterion, see [26, Ch. 2, Thm. 6], $\mathcal{G}$ is a Gröbner basis for I_0 with respect to each of the specified orderings. By [90, Lem. 12.1], we know that

$$\left\{ \frac{f}{T_i^{k_i(f)}};\ f \in \mathcal{G} \right\} \;=\; \mathcal{G}, \qquad k_i(f) \;:=\; \max\bigl(n \in \mathbb{Z}_{\ge 0};\ T_i^n \mid f\bigr)$$

is a Gröbner basis for $I_0 : T_i^\infty$ for each $1 \le i \le 7$. In particular $I_0 = I_0 : T_i^\infty$ for each i. As in [90, p. 114], the claim follows from

$$I_0 : T_1 \cdots T_7^\infty \;=\; ((\cdots (I_0 : T_1^\infty) \cdots) : T_7^\infty) \;=\; I_0.$$

We have shown that T_{10} is prime. Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for $i \neq j$. Also, observe that $T_{10} \nmid T_i$ for all $i < 10$. The intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1,\, T_{10}) &= V(T_{10},\, T_1,\, -T_3^2 T_5 - T_7 T_8,\, -T_2^a T_4 + T_3 T_5 T_6 T_8),\\
\overline{X}_2 \cap V(T_2,\, T_{10}) &= V(T_{10},\, T_2,\, -T_3^2 T_5 - T_7 T_8,\, T_3 T_5 T_6 T_8),\\
\overline{X}_2 \cap V(T_3,\, T_{10}) &= V(T_{10},\, T_3,\, T_7 T_8,\, T_2 T_4),\\
\overline{X}_2 \cap V(T_4,\, T_{10}) &= V(T_{10},\, T_4,\, -T_3^2 T_5 - T_7 T_8,\, T_3 T_5 T_6 T_8),\\
\overline{X}_2 \cap V(T_5,\, T_{10}) &= V(T_{10},\, T_5,\, T_7 T_8,\, T_2 T_4),\\
\overline{X}_2 \cap V(T_6,\, T_{10}) &= V(T_{10},\, T_6,\, -T_3^2 T_5 - T_7 T_8,\, T_2 T_4),\\
\overline{X}_2 \cap V(T_7,\, T_{10}) &= V(T_{10},\, T_7,\, T_3 T_5,\, T_2 T_4)
\end{aligned}$$

are all six-dimensional; as in previous cases, this can be seen by computer checks or using Lemma 5.3.3. The missing cases have been treated before. By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1, \ldots, T_{10}]/I_2$ is the Cox ring of the performed modification. Its degree matrix is listed below. We now show that we performed the desired blow up. The ideal

$$\begin{aligned}
I' \;&:=\; \langle T_1,\, T_9,\, h_2,\, h_1 - T_7 T_8 \rangle\\
&=\; \langle T_1,\, T_9,\, T_3^2 T_5 + T_7 T_8,\, T_2^a T_4 - T_3 T_5 T_6 T_8 \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_9]
\end{aligned}$$

is prime since $\mathbb{K}[T_1, \ldots, T_9]/I'$ is isomorphic to $\mathbb{K}[T_1, \ldots, T_{10}]/(I_2 + \langle T_{10} \rangle)$ which is an integral domain. Let $z := (0, 1, 1, 1, 1, 1, -1, 1, 0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_4) \in X_1'$. Then $z \in V(\mathbb{K}^9;\, I')$ and

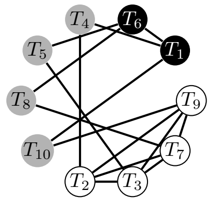
$$\dim\bigl(V\left(\mathbb{K}^9;\, I'\right)\bigr) \;=\; -1 + \dim\bigl(\overline{X}_2 \cap V(T_1,\, T_{10})\bigr) \;=\; 5$$

by the previous dimension computations. By Lemma 5.2.15, the performed modification was the claimed blow up. The Cox ring and degree matrix of the resulting surface $X_2 = \mathrm{Bl}\, \mathbb{F}_a(\star^3 \star\ iv)$ are

$$\mathcal{R}(X_2) \;=\; \mathbb{K}[T_1, \ldots, T_{10}] \Big/ \left\langle \begin{array}{l} T_1 T_2^{2a-1} T_4^2 T_{10} - T_3^2 T_5 - T_7 T_8, \\ T_9 T_{10} - T_2^a T_4 + T_3 T_5 T_6 T_8 \end{array} \right\rangle,$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 1 & 0 & 0 & 0 & 0 & -a & a & 3a-1 & -2a+1\\
0 & 0 & 1 & 0 & 0 & 0 & 3 & -1 & -2 & 2\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 3 & -2\\
0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & -1 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0
\end{bmatrix}.$$

We claim that its graph G_{X_2} of exceptional curves is as follows. Note that it suffices to prove the existence of the subgraph induced by the vertices T_i with $i \in \{1, 4, 5, 6, 8, 10\}$.



By Remark 5.3.7 and the fact that $V(X_2; T_{10})$ is the exceptional divisor of the last blow up, we know that the curves corresponding to the vertices are negative. The existence of the edges, i.e., the fact that the curves meet, is directly seen from the blow up sequence. By Lemma 5.4.12, X_2 then cannot be a $\mathbb{K}^*$-surface.

For the fifth configuration we want to blow up a point in the union of the exceptional divisors

$$V(X_1; T_5) \ \cup \ V(X_1; T_6) \ \cup \ V(X_1; T_8).$$

Note that we need not treat blow ups of points on the parabolic fixed point curve $V(X_1; T_6)$, see Lemma 5.4.11. Hence, by Remark 5.3.5, it suffices to consider

$$q_5 \ := \ [1,1,1,1,0,1,1,1], \qquad q_5' \ := \ [1,1,1,1,1,1,1,0] \ \in \ X_1.$$

Both points exist by Lemma 5.2.16. We first blow up X_1 in q_5. Choose in $\mathbb{K}[T_1,\dots,T_8]$ the polynomial $h_2 := T_1T_2^{a-1}T_4T_6T_8 - T_3$ for the embedding $\bar{\imath}_2 \colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \ = \ \left[\begin{array}{c|c} Q_1 & \begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix} \end{array}\right], \qquad P_1' \ = \ \begin{bmatrix} 1 & a-1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & a & 0 & 1 & 0 & -1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

For the blow up of X_1' in $\imath(q_5) = [1,1,1,1,0,1,1,1,0]$ we perform the stellar subdivision of Σ_1' at $v := (-1,1,-1,3) \in \mathbb{Z}^4$. Define the enlarged matrix $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1,\dots,T_{10}]$ of $\overline{X}_2$ is generated by

$$\begin{aligned} g_1 \ &:= \ p_2^\star\,(p_1)_\star\,(T_1T_2^{2a-1}T_4^2 - T_3^2T_5 - T_7T_8) \ = \ T_1T_2^{2a-1}T_4^2 - T_3^2T_5T_{10} - T_7T_8, \\ g_2 \ &:= \ p_2^\star\,(p_1)_\star\,(T_9 - h_2) \ = \ T_9T_{10} - T_1T_2^{a-1}T_4T_6T_8 + T_3. \end{aligned}$$

We show that I_2 is prime. In particular, I_2 is saturated with respect to T_{10}. By Lemma 5.4.4, the ideal I_2 is prime if the ideal

$$\begin{aligned} I_2' \ := \ \langle & T_1T_2^{2a-1}T_3^2 - T_9T_4T_1^2T_2^{2a-2}T_3^2T_5^2T_7^2 \\ & + 2T_9^2T_4T_1T_2^{a-1}T_3T_5T_7T_8 - T_9^3T_4T_8^2 - T_6T_7\rangle \ \subseteq \ \mathbb{K}[T_1,\dots,T_9] \end{aligned}$$

obtained by substitution of $T_3 = -g_2 + T_3$ in g_1 and replacing all T_i with T_{i-1} if $i > 3$ is prime. This follows from Lemma 5.2.17. In a similar manner, by Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1,\dots,T_{10}]/I_2$ since the ideal

$$I_2' + \langle T_9\rangle \ = \ \langle T_9,\, T_1T_2^{2a-1}T_3^2 - T_6T_7\rangle \ \subseteq \ \mathbb{K}[T_1,\dots,T_9].$$

is prime, see Lemma 5.2.17. Moreover, no two variables are associated because the degrees $\deg(T_i) \in \mathbb{Z}^6$ are pairwise different. Also, $T_i \nmid T_{10}$ for all $i < 10$ since the dimension of each of the intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, T_3, T_7T_8),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_3, T_7T_8),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_1T_2^{2a-1}T_4^2 - T_7T_8, T_1T_2T_4T_6T_8),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_3, T_7T_8),\\
\overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_1T_2^{a-1}T_4T_6T_8 - T_3, T_1T_2^{2a-1}T_4^2 - T_7T_8),\\
\overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_3, T_1T_2^{2a-1}T_4^2 - T_7T_8),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_3, T_1T_2T_4),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_3, T_1T_2T_4),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_1T_2^{a-1}T_4T_6T_8 - T_3, T_1T_2^{2a-1}T_4^2 - T_7T_8)
\end{aligned}$$

is six. This can be seen using Lemma 5.4.4. By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1, \ldots, T_{10}]/I_2$ is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we performed the desired blow up. The ideal

$$\begin{aligned}
I' &:= \langle T_5, T_9, h_2, h_1 - T_7T_8\rangle\\
&= \langle T_5,\ T_9,\ T_1T_2^{2a-1}T_4^2 - T_7T_8,\ T_1T_2^{a-1}T_4T_6T_8 - T_3\rangle \ \subseteq\ \mathbb{K}[T_1, \ldots, T_9]
\end{aligned}$$

is prime by Lemmas 5.4.4 and 5.2.17. Let $z := (1,1,1,1,0,1,1,1,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5) \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$. By the previous dimension arguments

$$\dim\left(V\left(\mathbb{K}^9; I'\right)\right) \;=\; -1 + \dim\left(\overline{X}_2 \cap V(T_5, T_{10})\right) \;=\; 5.$$

Thus, the performed modification was the claimed blow up, see Lemma 5.2.15. Using Algorithm 4.3.3, we substitute the equation $T_3 = -g_2 + T_3$ and obtain the graded ring $R_2' := \mathbb{K}[T_1, \ldots, T_9]/I_2'$ as the Cox ring of the blow up X_2 of X_1 in q_5. Its degree matrix Q_2' is given by removing the third column of a Gale dual matrix of P_2, i.e.,

$$Q_2' \;=\; \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1\\
0 & 1 & 0 & 0 & 0 & 3a-2 & -a+1 & -2a+1 & 2a-1\\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & -2\\
0 & 0 & 1 & 0 & 0 & 3 & -1 & -2 & 2\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0
\end{bmatrix}.$$

Note that the blow up of X_1 in q_5 is isomorphic to a $\mathbb{K}^*$-surface Y. By Lemma 5.1.5 it suffices to show that $R_2' = \mathcal{R}(X_2)$ is isomorphic to

$$\mathcal{R}(Y) \;:=\; \mathbb{K}[T_1, \ldots, T_9] \,/\, \langle T_1T_2^{2a-1}T_3^2 - T_9^3T_4T_8^2 - T_6T_7\rangle$$

with the same degree matrix Q_2'. The isomorphism $\mathcal{R}(Y) \to \mathcal{R}(X_2)$ is induced by the $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1, \ldots, T_9] \;\to\; \mathbb{K}[T_1, \ldots, T_9],$$

$$T_i \;\mapsto\; \begin{cases} T_6 + T_9T_4T_1^2T_2^{2a-2}T_3^2T_5^2T_7 - 2T_9^2T_4T_1T_2^{a-1}T_3T_5T_8, & i = 6,\\ T_i, & \text{else.}\end{cases}$$

For the blow up X_1 in q_5' we choose the polynomial $h_2 := T_1T_2^{a-1}T_3T_4T_5T_6 - T_7$ in $\mathbb{K}[T_1, \ldots, T_8]$ for the embedding $\bar{\imath}_2\colon \mathbb{K}^8 \to \mathbb{K}^9$. Let Q_1 be the degree matrix of $\mathcal{R}(X_1)$. We have a new degree matrix Q_1' and a matrix P_1' whose columns are generators for the rays of the fan Σ_1' of Z_1':

$$Q_1' \;=\; \left[\begin{array}{c|c} Q_1 & \begin{matrix} 2\\ 3a-2\\ -1\\ 3\\ 1\end{matrix}\end{array}\right], \qquad P_1' \;=\; \begin{bmatrix}
1 & a-1 & 1 & 1 & 1 & 1 & 0 & 0 & -1\\
0 & a & 1 & 1 & 0 & -1 & 0 & -2 & -1\\
0 & 0 & 2 & 0 & 1 & 0 & 0 & -1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}.$$

For the blow up of X_1' in $\iota(q_5') = [1,1,1,1,1,1,1,0,0]$, we determine the stellar subdivision $\Sigma_2 \to \Sigma_1'$ at $v := (-1,-3,-2,-1) \in \mathbb{Z}^4$. Write $P_2 := [P_1', v]$. The vanishing ideal $I_2 \subseteq \mathbb{K}[T_1, \dots, T_{10}]$ of $\overline{X}_2$ is generated by

$$g_1 \;:=\; p_2^\star (p_1)_\star (T_1T_2^{2a-1}T_4^2 - T_3^2T_5 - T_7T_8) \;=\; T_1T_2^{2a-1}T_4^2 - T_3^2T_5 - T_7T_8T_{10},$$

$$g_2 \;:=\; p_2^\star (p_1)_\star (T_9 - h_2) \;=\; T_9T_{10} - T_1T_2^{a-1}T_3T_4T_5T_6 + T_7.$$

We show that I_2 is saturated with respect to T_{10} by showing that I_2 is prime. Consider the ideal

$$I_2' \;:=\; \langle T_1T_2^{2a-1}T_4^2 - T_3^2T_5 - T_1T_2^{a-1}T_3T_4T_5T_6T_7T_9 + T_7T_8T_9^2\rangle \;\subseteq\; \mathbb{K}[T_1, \dots, T_9]$$

obtained by substitution of $T_7 = -g_2 + T_7$ in g_1 and replacing all T_i with T_{i-1} for $i > 7$. By Lemma 5.4.4, the ideal I_2 is prime if I_2' is prime. The latter follows from Lemma 5.2.17. In a similar manner, by Lemma 5.4.4, the variable T_{10} defines a prime element in $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$, since the ideal

$$I_2' + \langle T_9\rangle \;=\; \langle T_9,\, T_1T_2^{2a-1}T_4^2 - T_3^2T_5\rangle \;\subseteq\; \mathbb{K}[T_1, \dots, T_9].$$

is prime by Lemma 5.2.17. Moreover, no two variables T_i, T_j are associated since $\deg(T_i) \neq \deg(T_j)$ for all $i \neq j$. Also, $T_{10} \nmid T_i$ for all $i < 10$ since each of the intersections

$$\begin{aligned}
\overline{X}_2 \cap V(T_1, T_{10}) &= V(T_{10}, T_1, T_7, T_3T_5),\\
\overline{X}_2 \cap V(T_2, T_{10}) &= V(T_{10}, T_2, T_7, T_3T_5),\\
\overline{X}_2 \cap V(T_3, T_{10}) &= V(T_{10}, T_3, T_7, T_1T_2T_4),\\
\overline{X}_2 \cap V(T_4, T_{10}) &= V(T_{10}, T_4, T_7, T_3T_5),\\
\overline{X}_2 \cap V(T_5, T_{10}) &= V(T_{10}, T_5, T_7, T_1T_2T_4),\\
\overline{X}_2 \cap V(T_6, T_{10}) &= V(T_{10}, T_6, T_7, T_1T_2^{2a-1}T_4^2 - T_3^2T_5),\\
\overline{X}_2 \cap V(T_7, T_{10}) &= V(T_{10}, T_7, T_1T_2^{2a-1}T_4^2 - T_3^2T_5, T_1T_2T_3T_4T_5T_6),\\
\overline{X}_2 \cap V(T_8, T_{10}) &= V(T_{10}, T_8, T_1T_2^{2a-1}T_4^2 - T_3^2T_5, T_1T_2^{a-1}T_3T_4T_5T_6 - T_7),\\
\overline{X}_2 \cap V(T_9, T_{10}) &= V(T_{10}, T_9, T_1T_2^{2a-1}T_4^2 - T_3^2T_5, T_1T_2^{a-1}T_3T_4T_5T_6 - T_7)
\end{aligned}$$

is six-dimensional; here, Lemma 5.4.4 can be used. By Theorem 4.2.6, $R_2 = \mathbb{K}[T_1, \dots, T_{10}]/I_2$ is the Cox ring of the performed modification with a Gale dual matrix of P_2 as degree matrix. We now show that we performed the desired blow up. The ideal

$$\begin{aligned}
I' \;&:=\; \langle T_8, T_9, h_2, h_1 - T_7T_8\rangle\\
&=\; \langle T_8, T_9, T_1T_2^{2a-1}T_4^2 - T_3^2T_5, -T_1T_2^{a-1}T_3T_4T_5T_6 + T_7\rangle\\
&\subseteq\; \mathbb{K}[T_1, \dots, T_9]
\end{aligned}$$

is prime by Lemma 5.4.4 and Lemma 5.2.17. Let $z := (1,1,1,1,1,1,1,0,0) \in \mathbb{K}^9$ be Cox coordinates for $\iota(q_5') \in X_1'$. Then $z \in V(\mathbb{K}^9; I')$ and

$$\dim\left(V\left(\mathbb{K}^9; I'\right)\right) \;=\; -1 + \dim\left(\overline{X}_2 \cap V(T_9, T_{10})\right) \;=\; 5.$$

By Lemma 5.2.15, the performed modification was the claimed blow up. Using Algorithm 4.3.3, we eliminate the equation $T_7 = -g_2 + T_7$ and obtain the graded ring $R_2' := \mathbb{K}[T_1, \dots, T_9]/I_2'$. The Cox ring of the blow up X_2 of X_1 in q_5' then is $\mathcal{R}(X_2) = R_2'$. Its degree matrix is given by removing the seventh column of a Gale dual matrix of P_2. Note that it is not obvious whether X_2 is a $\mathbb{K}^*$-surface or not.

Isomorphisms: We now remove redundancies between the found surfaces without a non-trivial $\mathbb{K}^*$-action. Note that the Cox rings of all surfaces are either listed in the table of Theorem 5.4.1 or directly when encountered in this proof. To rule out isomorphisms, we will first apply Proposition 5.4.6 formally to compare the lists L_i of the absolute values of the maximal minors of the respective degree matrices. Note that for blow ups of $\mathbb{F}_a$, the L_i also could be computed with symbolic parameter $a \in \mathbb{Z}_{\geq 3}$. For a concise presentation, given integers k, $n \in \mathbb{Z}$, we shortly write k^n for the sequence $k, \dots, k$ of n copies of k.

X	list of 6×6 minors
$\mathrm{Bl}\,\mathbb{P}_2(\star^5\ i)$	$0^{47}, 1^{70}, 2^{51}, 3^{17}, 4^{11}, 5^7, 6^3, 7^1, 8^1, 10^1, 11^1$
$\mathrm{Bl}\,\mathbb{P}_2(\star^5\ ii)$	$0^{49}, 1^{97}, 2^{44}, 3^{13}, 4^4, 5^1, 6^1, 7^1$
$\mathrm{Bl}\,\mathbb{P}_2(\star^5\ iii)$	$0^{47}, 1^{70}, 2^{51}, 3^{17}, 4^{11}, 5^7, 6^3, 7^1, 8^1, 10^1, 11^1$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\star\ i)$	$0^{54}, 1^{127}, 2^{20}, 3^8, 5^1$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\star\ ii)$	$0^{128}, 1^{202}, 2^{100}, 3^{28}, 4^4$
$\mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\star\ iii)$	$0^{49}, 1^{97}, 2^{44}, 3^{13}, 4^4, 5^1, 6^1, 7^1$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\star\ i)$	$0^{168}, 1^{266}, 2^{28}$
$\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ i)$	$0^{756}, 1^{920}, 2^{40}$
$\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ ii)$	$0^{3960}, 1^{4032}, 2^{16}$
$\mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ iii)$	$0^{168}, 1^{266}, 2^{28}$
$\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\star\ iv)$	$0^{62}, 1^{135}, 2^{12}, 3^1$
$\mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ vi)$	$0^{52}, 1^{116}, 2^6, \lvert a-2\rvert^{10}, \lvert -3+a\rvert^2, \lvert 2a-1\rvert^1, \lvert -5+2a\rvert^1, \lvert -3+2a\rvert^2,$ $\lvert a-1\rvert^{14}, \lvert a\rvert^6$
$\mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ vii)$	$0^{52}, 1^{116}, 2^6, \lvert a+1\rvert^6, \lvert a-2\rvert^2, \lvert a-1\rvert^{10}, \lvert 2a-1\rvert^2, \lvert a\rvert^{14},$ $\lvert 1+2a\rvert^1, \lvert -3+2a\rvert^1$
$\mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ xiii)$, ††	$0^{124}, 1^{166}, 2^{28}, \lvert a\rvert^{64}, \lvert 2a-2\rvert^8, \lvert -2+a\rvert^4, \lvert a-1\rvert^{32}, \lvert 2a-3\rvert^4,$ $\lvert -4+4a\rvert^4, \lvert 2a-1\rvert^{28}$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ i)$	$0^{47}, 1^{89}, 2^{17}, 3^7, 5^1, 3\lvert a\rvert^1, 2\lvert a\rvert^2, \lvert -3+2a\rvert^1, \lvert 2a-1\rvert^6, \lvert -2+3a\rvert^2,$ $\lvert -3+5a\rvert^1, \lvert a-2\rvert^2, \lvert a\rvert^{27}, \lvert a-1\rvert^7$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ ii)$	$0^{47}, 1^{68}, 2^{30}, 3^{11}, 4^2, 5^1, 6^1, 7^1, \lvert -3+7a\rvert^1, \lvert 2a-1\rvert^6, 3\lvert a\rvert^1, \lvert -3+4a\rvert^1,$ $\lvert -2+6a\rvert^1, \lvert -2+4a\rvert^1, \lvert a-1\rvert^2, 2\lvert a\rvert^8, \lvert a\rvert^{21}, \lvert -2+5a\rvert^1, \lvert 3a-1\rvert^5,$ $\lvert -2+3a\rvert^1$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ iii)$	$0^{47}, 1^{89}, 2^{17}, 3^7, 5^1, \lvert a\rvert^{27}, \lvert 2a-1\rvert^6, \lvert -3+5a\rvert^1, 2\lvert a\rvert^2, \lvert a-2\rvert^2,$ $\lvert -3+2a\rvert^1, \lvert -2+3a\rvert^2, \lvert a-1\rvert^7, 3\lvert a\rvert^1$
$\mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ iv)$	$0^{47}, 1^{68}, 2^{30}, 3^{11}, 4^2, 5^1, 6^1, 7^1, \lvert a-1\rvert^2, \lvert -2+4a\rvert^1, \lvert a\rvert^{21}, \lvert -3+7a\rvert^1,$ $3\lvert a\rvert^1, \lvert 3a-1\rvert^5, \lvert 2a-1\rvert^6, \lvert -2+6a\rvert^1, \lvert -2+5a\rvert^1, 2\lvert a\rvert^8, \lvert -3+4a\rvert^1,$ $\lvert -2+3a\rvert^1$
$\mathrm{Bl}\,\mathbb{F}_a(\star^4\ i)$, †	$0^{10}, 1^{24}, 2^{21}, 3^3, 4^7, \lvert a-1\rvert^1, \lvert -4+4a\rvert^1, \lvert 4a-3\rvert^2, 2\lvert a\rvert^2, \lvert 2a-1\rvert^7,$ $\lvert -2+3a\rvert^1, \lvert -2+4a\rvert^2, \lvert a\rvert^3$
$\mathrm{Bl}\,\mathbb{F}_a(\star^2 \star^2\ i)$, †	$0^{10}, 1^{35}, 2^{15}, 3^2, 4^2, \lvert -4+4a\rvert^1, \lvert -2+3a\rvert^2, \lvert 2a-1\rvert^1, \lvert a\rvert^{10},$ $\lvert a-1\rvert^2, \lvert a-2\rvert^2, 2\lvert a\rvert^2$

By the second statement of Proposition 5.4.6, only the following isomorphisms are possible. In fact, all of them turn out to be isomorphisms.

$$\mathrm{Bl}\,\mathbb{P}_2(\star^5\ i) \ \to\ \mathrm{Bl}\,\mathbb{P}_2(\star^5\ iii), \qquad \mathrm{Bl}\,\mathbb{P}_2(\star^5\ ii) \ \to\ \mathrm{Bl}\,\mathbb{P}_2(\star^3 \star\star\ iii)$$
$$\mathrm{Bl}\,\mathbb{P}_2(\star^2 \star\star\star\ i) \to \mathrm{Bl}\,\mathbb{P}_2(\star\star\star\star\star\ iii), \quad \mathrm{Bl}\,\mathbb{F}_{a+1}(\star\star\star\star\ vi) \to \mathrm{Bl}\,\mathbb{F}_a(\star\star\star\star\ vii),$$
$$\mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ i) \ \to\ \mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ iii), \qquad \mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ ii) \ \to\ \mathrm{Bl}\,\mathbb{F}_a(\star^3 \star\ iv).$$

For the first isomorphism, write $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^5\ i)$ and $X_2 := \mathrm{Bl}\,\mathbb{P}_2(\star^5\ iii)$. Recall from the beginning of this proof the Cox rings $\mathcal{R}(X_i)$ and their degree matrices Q_i.

Y	Cox ring $\mathcal{R}(Y)$	degree matrix
X_1	$\mathbb{K}[T_1, \dots, T_{10}]/I_1$ with I_1 generated by $T_3^2T_4 - T_1T_2 - T_6T_7T_8T_{10}$, $T_1T_2^2T_3T_4T_5 - T_6^2T_7 - T_9T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 & 0 & 5 & -2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$

X_2	$\mathbb{K}[T_1,\ldots,T_{10}]/I_2$ with I_2 generated by $T_1T_2T_6T_8 - T_3^2T_4 - T_9T_{10}$, $T_1^2T_2 + T_7T_8 - T_3^3T_4^2T_5T_{10}$	$\left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 5 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{smallmatrix}\right]$

Write $I_i \subseteq \mathbb{K}[T_1,\ldots,T_{10}]$ for the ideal of relations of $\mathcal{R}(X_i)$. Substituting $T_3^2T_4 = T_1T_2T_6T_8 - T_9T_{10}$ into I_2 does not change the ideal, i.e.,

$$\begin{aligned} I_2 \quad = \quad & \langle T_1T_2T_6T_8 - T_3^2T_4 - T_9T_{10}, \\ & T_1^2T_2 + T_7T_8 - T_1T_2T_3T_4T_5T_6T_8T_{10} + T_3T_4T_5T_9T_{10}^2\rangle. \end{aligned}$$

The $\mathbb{Z}^6$-graded homomorphism

$$\mathbb{K}[T_1,\ldots,T_{10}] \ \to \ \mathbb{K}[T_1,\ldots,T_{10}], \qquad T_i \ \mapsto \ \begin{cases} T_7 - T_1T_2T_3T_4T_5T_6T_{10}, & i = 7, \\ T_i, & \text{else} \end{cases}$$

induces an isomorphism $R_2' \to \mathcal{R}(X_2)$ of $\mathbb{Z}^6$-graded algebras where R_2' also has Q_2 as degree matrix and is given by

$$R_2' \ := \ \mathbb{K}[T_1,\ldots,T_{10}] \,/\, I_2', \qquad I_2' \ := \ \left\langle \begin{array}{c} T_1T_2T_6T_8 - T_3^2T_4 - T_9T_{10}, \\ T_1^2T_2 + T_7T_8 + T_3T_4T_5T_9T_{10}^2 \end{array} \right\rangle.$$

By Lemma 5.1.5, the surfaces X_1 and X_2 are isomorphic if R_2' is isomorphic to $\mathcal{R}(X_1)$ as $\mathbb{Z}^6$-graded algebra. Consider the homomorphism of algebras $\psi\colon R_2' \to \mathcal{R}(X_1)$ induced by

$$\begin{gathered} \mathbb{K}[T_1,\ldots,T_{10}] \ \to \ \mathbb{K}[T_1,\ldots,T_{10}], \\ T_1 \mapsto T_6, \qquad T_2 \mapsto T_7, \qquad T_3 \mapsto T_3, \qquad T_4 \mapsto T_4, \\ T_5 \mapsto T_5, \qquad T_6 \mapsto T_8, \qquad T_7 \mapsto T_9, \qquad T_8 \mapsto T_{10}, \\ T_9 \mapsto -T_1, \qquad T_{10} \mapsto T_2. \end{gathered}$$

Then $\langle \psi(I_2')\rangle = I_1$ and the homomorphism ψ is a well-defined isomorphism of $\mathbb{K}$-algebras. To see that ψ is also an isomorphism of $\mathbb{Z}^6$-graded algebras we consider the homomorphism of abelian groups

$$\alpha\colon \mathbb{Z}^6 \to \mathbb{Z}^6, \quad e_i \mapsto A\cdot e_i, \quad A := \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(6,\mathbb{Z}).$$

This turns the pair (ψ,α) into an isomorphism $\mathcal{R}(X_1) \to \mathcal{R}(X_2)$ of $\mathbb{Z}^6$-graded algebras: for all $w \in \mathbb{Z}^6$ the image $\psi((R_2')_w)$ is contained in the component $\mathcal{R}(X_1)_{\alpha(w)}$ because

$$A\cdot Q_2 \ = \ \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 & 3 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 5 & -2 & 0 & 1 \\ 0 & -1 & 1 & -2 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix} \ =: \ Q_1.$$

For the second isomorphism, let $X_1 := \mathrm{Bl}\,\mathbb{P}_2(\star^5\ ii)$ and $X_2 := \mathrm{Bl}\,\mathbb{P}_2(\star^3\star\star\ iii)$. We claim that we have an isomorphism given by

$$\varphi\colon \overline{X}_2 \to \overline{X}_1, \qquad (z_1,\ldots,z_{10}) \mapsto (-z_9, z_{10}, z_2, z_4, z_6, z_3, -z_7, z_8, z_1, z_5).$$

For this, by Lemma 5.1.5, it suffices to show that the comorphism $\psi := \varphi^* \colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ is a $\mathbb{Z}^6$-graded isomorphism that is induced by

$$\mathbb{K}[T_1, \ldots, T_{10}] \ \to \ \mathbb{K}[T_1, \ldots, T_{10}],$$
$$T_1 \mapsto -T_9, \qquad T_2 \mapsto T_{10}, \qquad T_3 \mapsto T_2, \qquad T_4 \mapsto T_4,$$
$$T_5 \mapsto T_6, \qquad T_6 \mapsto T_3, \qquad T_7 \mapsto -T_7, \qquad T_8 \mapsto -T_8,$$
$$T_9 \mapsto T_1, \qquad T_{10} \mapsto T_5.$$

Write $I_i \subseteq \mathbb{K}[T_1, \ldots, T_{10}]$ for the ideal of relations of $\mathcal{R}(X_i)$. Then $\langle \psi(I_1) \rangle = I_2$ and the homomorphism ψ is a well-defined isomorphism of $\mathbb{K}$-algebras. Observe that ψ is also an isomorphism of $\mathbb{Z}^6$-graded algebras. Consider the homomorphism of abelian groups

$$\alpha \colon \mathbb{Z}^6 \to \mathbb{Z}^6, \quad e_i \mapsto A \cdot e_i, \quad A := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 2 \\ 3 & -1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 3 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

This turns the pair (ψ, α) into an isomorphism $\mathcal{R}(X_1) \to \mathcal{R}(X_2)$ of $\mathbb{Z}^6$-graded algebras: for all $w \in \mathbb{Z}^6$ the image $\psi(\mathcal{R}(X_1)_w)$ is contained in the component $\mathcal{R}(X_2)_{\alpha(w)}$ because

$$A \cdot Q_1 \quad = \quad \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 3 & -2 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad =: \quad Q_2.$$

For the third isomorphism define $X_1 := \mathrm{Bl}\, \mathbb{P}_2(\star^2 \star\star\star\ i)$ and $X_2 := \mathbb{P}_2(\star\star\star\star\star\ iii)$. As seen by the blow up sequences of the respective surfaces the curves $V(X_1; T_6)$ and $V(X_2; T_4)$ are (-1)-curves. Their respective contractions lead to surfaces X_i' fitting into the diagram

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[d] \\ X_1' & X_2' \arrow[l, "\varphi", "\cong"'] \end{tikzcd}$$

where φ will be specified below. The Cox rings $\mathcal{R}(X_i')$ and their degree matrices Q_i' are listed in the following table.

Y	$\mathcal{R}(Y)$	degree matrix
X_1'	$\mathbb{K}[T_1, \ldots, T_5, T_7, \ldots, T_{11}] \,/\, I_1'$ with I generated by $T_4T_8 + T_5T_9 - T_7T_{10}$, $T_2T_9 + T_1T_8 - T_3T_{10}$, $T_3T_4 - T_1T_7 - T_9T_{11}$, $T_2T_7 - T_3T_5 - T_8T_{11}$, $T_2T_4 - T_1T_5 - T_{11}T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$
X_2'	$\mathbb{K}[T_1, \ldots, T_3, T_5, \ldots, T_{11}] \,/\, I_2'$ with I generated by $T_6T_2 + T_5T_9 - T_8T_{10}$, $T_3T_8 - T_1T_6 - T_9T_{11}$, $T_3T_5 + T_6T_7 - T_{11}T_{10}$, $T_1T_5 + T_7T_8 - T_2T_{11}$, $T_3T_2 - T_7T_9 - T_1T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix}$

Note that $\overline{X}'_1$ is a subset of $\bigoplus_{i\neq 6}\mathbb{K}\cdot e_i \cong \mathbb{K}^{10}$ whereas $\overline{X}'_2$ is a subset of $\bigoplus_{i\neq 4}\mathbb{K}\cdot e_i \cong \mathbb{K}^{10}$. We claim that the isomorphism $\varphi\colon \overline{X}'_2 \to \overline{X}'_1$ of the total coordinate spaces of the contracted surfaces and its inverse φ^{-1} are given by

$$\begin{array}{rcl} \overline{X}'_1 & \longleftrightarrow & \overline{X}'_2 \\ (z_1,z_8,z_2,z_3,z_6,z_{10},z_5,z_7,z_{11},z_9) & \leftarrow\!\shortmid & (z_1,z_2,z_3,z_5,\ldots,z_{11}), \\ (z_1,\ldots,z_5,z_7,\ldots,z_{11}) & \mapsto & (z_1,z_3,z_4,z_8,z_5,z_9,z_2,z_{11},z_7,z_{10}) \end{array}$$

To this end, it suffices to show that the comorphism $\psi := \varphi^*\colon \mathcal{R}(X'_1) \to \mathcal{R}(X'_2)$ is a $\mathbb{Z}^5$-graded isomorphism induced by

$$\begin{array}{c} \mathbb{K}[T_1,\ldots,T_5,T_7,\ldots,T_{11}] \;\to\; \mathbb{K}[T_1,\ldots,T_3,T_5,\ldots,T_{11}], \\ T_i \;\mapsto\; T_{\sigma(i)}, \qquad \sigma \;:=\; \left[\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 11 \\ 1 & 8 & 2 & 3 & 6 & 10 & 5 & 7 & 11 & 9 \end{smallmatrix}\right] \end{array}$$

where σ stands for the bijective function $\{1,\ldots,11\}\setminus\{6\} \to \{1,\ldots,11\}\setminus\{4\}$ mapping the i-th element of the first row to the i-th element in the second row; see Lemma 5.1.5. Note that $\langle\psi(I'_1)\rangle = I'_2$ and ψ is a well-defined isomorphism of $\mathbb{K}$-algebras. Observe that ψ is also an isomorphism of $\mathbb{Z}^5$-graded algebras. For this purpose, consider the homomorphism of abelian groups

$$\alpha\colon \mathbb{Z}^5 \to \mathbb{Z}^5, \qquad e_i \mapsto A\cdot e_i, \qquad A := \left[\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right] \in \mathrm{GL}(5,\mathbb{Z}).$$

This turns the pair (ψ,α) into an isomorphism $\mathcal{R}(X'_1) \to \mathcal{R}(X'_2)$ of $\mathbb{Z}^5$-graded algebras: for all $w \in \mathbb{Z}^5$, the image $\psi(\mathcal{R}(X'_1)_w)$ is contained in the component $\mathcal{R}(X'_2)_{\alpha(w)}$ as

$$A\cdot Q'_1 = \left[\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{smallmatrix}\right] =: Q'_2.$$

Switching back to the original surfaces X_i, X_1 can be obtained as the blow up of X'_1 in the point p_1 and X_2 as the blow up of X'_2 in p_2 where

$$\begin{array}{rcl} p_1 & := & [1,0,1,0,1,1,1,1,1,-1] \in X'_1, \\ p_2 & := & [-1,0,0,1,1,1,1,1,1,1] \in X'_2. \end{array}$$

Both points exist by an application of Algorithm 2.3.8. We claim that we have an automorphism

$$\begin{array}{rcl} \eta\colon \overline{X}'_2 & \to & \overline{X}'_2 \\ (z_1,z_2,z_3,z_5,\ldots,z_{11}) & \mapsto & (z_9,z_8,z_3,z_5,z_7,z_6,z_2,z_1,z_{11},z_{10}). \end{array}$$

Then, by definition of φ^{-1} and η, we have

$$\eta\circ\varphi^{-1}(p_1) = [\eta((1,1,0,1,1,1,0,-1,1,1))] = p_2 \in X'_2.$$

Since φ^{-1} is an isomorphism and η an automorphism, using uniqueness of the blow up we then may conclude that X_1 is isomorphic to X_2, see Proposition 1.4.5 or [48]. By Lemma 5.1.5, it remains to show that the comorphism $\kappa := \eta^*\colon \mathcal{R}(X'_2) \to \mathcal{R}(X'_2)$ is an isomorphism of $\mathbb{Z}^5$-graded algebras induced by

$$\begin{array}{c} \mathbb{K}[T_1,\ldots,T_3,T_5,\ldots,T_{11}] \;\to\; \mathbb{K}[T_1,\ldots,T_3,T_5,\ldots,T_{11}], \\ T_i \;\mapsto\; T_{\sigma'(i)}, \qquad \sigma' \;:=\; \left[\begin{smallmatrix} 1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 8 & 3 & 5 & 7 & 6 & 2 & 1 & 11 & 10 \end{smallmatrix}\right] \end{array}$$

where σ' stands for the bijective function $\{1,\ldots,11\}\setminus\{4\} \to \{1,\ldots,11\}\setminus\{4\}$ mapping the i-th element of the first row to the i-th element of the second row.

Note that κ is well-defined since $\langle \kappa(I_2') \rangle = I_2'$. Consider the homomorphism of abelian groups

$$\beta\colon \mathbb{Z}^5 \to \mathbb{Z}^5, \qquad e_i \mapsto B \cdot e_i, \qquad B := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{GL}(5, \mathbb{Z}).$$

Similar to before, (κ, β) is an isomorphism $\mathcal{R}(X_2') \to \mathcal{R}(X_2')$ of $\mathbb{Z}^5$-graded algebras since we have $\kappa(\mathcal{R}(X_2')_w) \subseteq \mathcal{R}(X_2')_{\beta(w)}$ for all $w \in \mathbb{Z}^5$:

$$B \cdot Q_2' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \eqqcolon Q_2'.$$

We come to the fourth isomorphism. Let $a \geq 3$. Redefine $X_1 := \mathrm{Bl}\, \mathbb{F}_{a+1}(\star\star\star\star\ vi)$ and $X_2 := \mathrm{Bl}\, \mathbb{F}_a(\star\star\star\star\ vii)$. We have an isomorphism of $\mathbb{K}$-algebras $\psi\colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ induced by

$$\begin{gathered} \mathbb{K}[T_1, \ldots, T_{10}] \to \mathbb{K}[T_1, \ldots, T_{10}], \\ T_7 \mapsto T_{10}, \quad T_8 \mapsto T_9, \quad T_9 \mapsto T_7, \quad T_{10} \mapsto T_8, \quad T_i \mapsto T_i \ \text{else}. \end{gathered}$$

Comparing the degree matrices of $\mathcal{R}(X_i)$ we see that (ψ, id) is an isomorphism $\mathcal{R}(X_1) \to \mathcal{R}(X_2)$ of $\mathbb{Z}^6$-graded algebras. By Lemma 5.1.5, this shows that X_1 and X_2 are isomorphic.

For the fifth isomorphism, write $X_1 := \mathrm{Bl}\, \mathbb{F}_a(\star^3 \star\ i)$ and $X_2 := \mathrm{Bl}\, \mathbb{F}_a(\star^3 \star\ iii)$. Denote by Q_i the degree matrices of their Cox rings. We claim that we have an isomorphism $\varphi\colon \overline{X}_1 \to \overline{X}_2$ with its comorphism $\psi := \varphi^*\colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ induced by

$$\begin{gathered} \mathbb{K}[T_1, \ldots, T_{10}] \to \mathbb{K}[T_1, \ldots, T_{10}], \\ T_i \mapsto T_{\sigma(i)}, \qquad \sigma := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 7 & 4 & 5 & 8 & 9 & 10 & 3 & 6 \end{bmatrix} \in \mathrm{Sym}(10). \end{gathered}$$

Note that the ideals of relations of $\mathcal{R}(X_i)$ are mapped to each other. Then ψ is an isomorphism of $\mathbb{K}$-algebras. To see that ψ is $\mathbb{Z}^6$-graded consider the homomorphism of abelian groups

$$\gamma\colon \mathbb{Z}^6 \to \mathbb{Z}^6, \qquad e_i \mapsto C \cdot e_i, \qquad C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -a & 0 & 0 & a \\ 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 3 & 0 & 0 & -2 \end{bmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

Thus, (ψ, γ) is an isomorphism $\mathcal{R}(X_1) \to \mathcal{R}(X_2)$ of $\mathbb{Z}^6$-graded algebras since ψ maps the component $\mathcal{R}(X_1)_w$ into the component $\mathcal{R}(X_2)_{\gamma(w)}$ for all $w \in \mathbb{Z}^6$:

$$C \cdot Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -a & 0 & 0 & a & 2a-1 & -a+1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -2 & -1 & 1 & 0 & 1 \end{bmatrix} \eqqcolon Q_2.$$

Therefore, again by Lemma 5.1.5, we conclude that the surfaces X_1 and X_2 are isomorphic.

For the sixth isomorphism, let $X_1 := \mathrm{Bl}\, \mathbb{F}_a(\star^3 \star\ ii)$ and $X_2 := \mathrm{Bl}\, \mathbb{F}_a(\star^3 \star\ iv)$. As seen by the blow up sequences of the respective surfaces, the curves $V(X_1; T_{10})$ and $V(X_2; T_8)$ are (-1)-curves. Their contraction leads to surfaces X_1' and X_2'. We have a diagram

$$
\begin{tikzcd}
 & \overline{X}_1 \arrow[d] & & \overline{X}_2 \arrow[d] & \\
 & \overline{X}_1' & & \overline{X}_2' & \\
\mathbb{K}^8 \arrow[r, phantom, "\supseteq"] & \overline{X}_1'' \arrow[u, "\bar{\imath}_1"] & & \overline{X}_2'' \arrow[u, "\bar{\imath}_2"'] \arrow[ll, "\cong", "\varphi"'] \arrow[r, phantom, "\subseteq"] & \mathbb{K}^8
\end{tikzcd}
$$

where we consider $\overline{X}_1'$ as a subset of $\bigoplus_{i=1}^{9} \mathbb{K} \cdot e_i$ and $\overline{X}_2'$ as a subset of $\bigoplus_{i \neq 8} \mathbb{K} \cdot e_i \cong \mathbb{K}^9$. Moreover, the embeddings $\bar{\imath}_i$ are as in Setting 4.2.9 and satisfy

$$
\begin{aligned}
\bar{\imath}_1 \colon \mathbb{K}^8 &\to \mathbb{K}^9, & (x_1, \ldots, x_8) &\mapsto (x_1, \ldots, x_8, h_1(x)),\\
\bar{\imath}_2 \colon \mathbb{K}^8 &\to \mathbb{K}^9, & (x_1, \ldots, x_6, x_9, x_{10}) &\mapsto (x_1, \ldots, x_6, h_2(x), x_9, x_{10}),\\
h_1 &:= T_1 T_2^{a-1} T_4 T_7 T_8^2 - T_3^2 T_5 \in \mathbb{K}[T_1, \ldots, T_8],\\
h_2 &:= T_1 T_2^{2a-1} T_4^2 T_{10} - T_3^2 T_5 \in \mathbb{K}[T_1, \ldots, T_8].
\end{aligned}
$$

The Cox rings and degree matrices Q_i' and Q_i'' of the surfaces X_i' and X_i'' are as follows.

Y	$\mathcal{R}(Y)$	degree matrix
X_1'	$\mathbb{K}[T_1, \ldots, T_9] \,/\, I$ with I generated by $T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8,\ T_9 - h_1$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & a & 3a-1 & -2a+1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$
X_2'	$\mathbb{K}[T_1, \ldots, T_7, T_9, T_{10}] \,/\, I$ with I generated by $T_2^a T_4 - T_3 T_5 T_6 - T_9 T_{10},\ T_7 - h_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & a & 0 & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$
X_1''	$\mathbb{K}[T_1, \ldots, T_8] \,/\, I$ with I generated by $T_2^a T_4 - T_3 T_5 T_6 - T_7 T_8$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & a & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$
X_2''	$\mathbb{K}[T_1, \ldots, T_6, T_9, T_{10}] \,/\, I$ with I generated by $T_2^a T_4 - T_3 T_5 T_6 - T_9 T_{10}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & a & 3a-1 & -2a+1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$

Inspecting the degree matrices Q_i'' and the ideals of $\mathcal{R}(X_i'')$, we see that we have an isomorphism $\mathcal{R}(X_1'') \to \mathcal{R}(X_2'')$ of $\mathbb{Z}^5$-graded algebras that arises from

$$
\mathbb{K}[T_1, \ldots, T_8] \;\to\; \mathbb{K}[T_1, \ldots, T_6, T_9, T_{10}], \qquad T_i \;\mapsto\; \begin{cases} T_9, & i = 7,\\ T_{10}, & i = 8,\\ T_i, & i \neq 7, 8. \end{cases}
$$

This yields an isomorphism $\varphi \colon \overline{X}_2'' \to \overline{X}_1''$. The surface X_1 is obtained as the blow up of X_1' at the point $[\bar{\imath}_1(q_1)] \in X_1'$ where

$$
q_1 := (1, 1, 1, 1, 1, 0, 1, 1) \in \widehat{X}_1'', \qquad \bar{\imath}_1(q_1) \in \widehat{X}_1',
$$

compare Lemmas 5.1.5 and 5.2.16. Then $\varphi^{-1}(q_1) = (1,1,1,1,1,0,1,1) \in \widehat{X}_2''$ and the blow up of X_2' at the point $[\bar{\imath}_2(\varphi^{-1}(q_1))]$ delivers X_2. We conclude $X_2 \cong X_1$.

Further note that no isomorphisms between cases with fixed discrete parameter $a \in \mathbb{Z}_{\geq 3}$ but different continuous parameters are possible. For the blow ups of $\mathbb{P}_2$, this is due to the fact that the respective originating point configurations are not isomorphic. For blow ups of $\mathbb{F}_a$ we only need to show that the point configuration $\mathbb{F}_a(\star\star\star\star\ xiv)$ of Proposition 5.1.4 with fixed $a \in \mathbb{Z}_{\geq 3}$ and fixed $\kappa \in \mathbb{K}^*$ cannot be moved to the same configuration with continuous parameter $\kappa' \in \mathbb{K}^* \setminus \{\kappa\}$ by an automorphism. To this end, let

$$v_1 := (-1,-a), \quad v_2 := (1,0), \quad v_3 := (0,1), \quad v_4 := (0,-1) \in \mathbb{Z}^2$$

be primitive generators of the rays of the fan Σ_a of $\mathbb{F}_a$. Recall from [27] that a *root* of Σ_a is an element $u \in \mathbb{Z}^2$ such that there is $1 \leq i(u) \leq 4$ with

$$\langle u, v_{i(u)} \rangle = 1, \qquad \langle u, v_j \rangle \leq 0 \quad \text{for } i(u) \neq j.$$

By [27, Cor. 4.7] the group of equivariant automorphisms of $\widehat{\mathbb{F}}_a$ is generated by the maximal torus, (certain) permutations of coordinates and automorphisms corresponding to one-parameter subgroups

$$y_u(\lambda)\colon \mathcal{R}(\mathbb{F}_a) \to \mathcal{R}(\mathbb{F}_a), \quad T_j \mapsto \begin{cases} T_j + \lambda \prod_{k \neq j} T_k^{\langle -u, v_k \rangle}, & j = i(u), \\ T_j, & j \neq i(u), \end{cases} \quad \lambda \in \mathbb{K}$$

of graded automorphisms of $\mathcal{R}(\mathbb{F}_a) = \mathbb{K}[T_1,\ldots,T_4]$ where $u \in \mathbb{Z}^2$ runs through all roots. Since the roots of $\mathbb{F}_a$ are

$$(-1,0),\ (1,0),\ (-b,1) \quad \text{where} \quad 0 \leq b \leq a,$$

these automorphisms are as in Lemma 5.1.6. In particular, two configurations of type $\mathbb{F}_a(\star\star\star\star\ xiv)$ with same $a \in \mathbb{Z}_{\geq 3}$ but different continuous parameter cannot be mapped to each other by an automorphism. $\square$

Remark 5.4.15. To compute the Cox ring of the †† case Bl $\mathbb{F}_a(\star\star\star\star\ xiv)$ for $a > 15$ one may proceed by the following steps.

- Prove that the ideal I_2 shown in the table is saturated with respect to T_{11}. As in other cases, this can be done by providing a Gröbner basis depending on the parameter $a > 15$.
- Show that the binomial ideal $I_2 + \langle T_{11} \rangle$ is prime. Again, this can be done by providing a Gröbner basis and using Remark 5.4.14; see also Lemma 5.3.4.

However, note that in experiments with fixed $a > 15$, a Gröbner basis for the first step contained more than 800 elements.

APPENDIX A

Procedures of the MDSpackage

We describe an implementation of the algorithms that we have developed throughout this thesis. The *MDSpackage* is currently available for the computer algebra system `Maple` in joint work with J. Hausen [54, 55]. We make use of the `convex`-package [40] by M. Franz. Some of the algorithms have already been implemented in [20, 70] and [16]. This appendix mainly serves as a manual.

The structure of this chapter will be similar to Chapter 2. Section 1 describes procedures on AGs and AGHs whereas Section 2 introduces functions on GRs. Section 3, 4 and 5 present procedures on MDSs, complexity-one T-varieties and miscellaneous functions. We first shortly recall the involved data types from Chapter 2. Corollary 1.3.9 and the description of Construction 1.3.6 serve as a theoretical foundation.

Data types A.0.16. We store a Mori dream space X in a *MDS*, i.e., a pair (R, Φ) where R is a graded ring, *GR* for short, and Φ a collection of overlapping cones in $\mathrm{Cl}(X) \otimes \mathbb{Q}$, called a *BUN*. The GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ encodes the Cox ring $\mathcal{R}(X)$ in $\mathbb{K}[T_1, \ldots, T_r]/\langle G \rangle$ with a list of polynomials $G \subseteq \mathbb{K}[T_1, \ldots, T_r]$ and the grading of R by $K := \mathrm{Cl}(X)$ in the form of a degree map, i.e., a homomorphism of finitely generated abelian groups

$$Q \colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ \deg(T_i).$$

We encode Q in a data type *AGH* and K in a data type *AG* for finitely generated abelian groups. It is useful to also store the projection $Q^0 \colon \mathbb{Z}^r \to K^0 = K/K^{\mathrm{tor}}$ onto the free part as a matrix as well as a Gale dual matrix P of the homomorphism Q.

AG
- U: matrix
- L: matrix

AGH
- F: AG
- K: AG
- A: matrix

GR
- G: polynomials
- TT: list of variables T_i
- $[Q, Q^0]$: AGH, matrix
- P: matrix
- $F_{\mathfrak{F}}$: list of sets: $\mathfrak{F}$-faces

BUN
- C: list of CONEs

MDS
- R: GR
- Φ: BUN

1. Procedures on finitely generated abelian groups

In this section, we describe procedures on finitely generated abelian groups (AGs) and their homomorphisms (AGHs). This describes an implementation of the algorithms of Section 1 of Chapter 2. Here is an overview:

- *Creation and stored data:* create an AG (Procedure A.1.1), return the stored data of an AG (Procedure A.1.2), create an AGH (Procedure A.1.3), return the stored data of an AGH (Procedure A.1.4).

- *Compare AGs:* test for isomorphy (Procedure A.1.5), test for containment (Procedure A.1.6), test for equality (Procedure A.1.7).
- *Construct new AGs:* factor groups (Procedure A.1.8), product groups (Procedure A.1.9), free part (Procedure A.1.10), intersection (Procedure A.1.11).
- *Image and kernel of AGHs:* image (Procedure A.1.12), preimage (Procedure A.1.13), kernel (Procedure A.1.14), test for being injective (Procedure A.1.15), test for being surjective (Procedure A.1.16), complete an exact sequence (Procedure A.1.17).
- *AGHs as degree maps, etc.:* gradiator (Procedure A.1.18), K-degree (Procedure A.1.19), test for being homogeneous (Procedure A.1.20), test for being almost free (Procedure A.1.21), section (Procedure A.1.22).
- *Gale duality:* compute the degree map Q out of P (Procedure A.1.23), compute P out of Q (Procedure A.1.24), compute the projection Q^0 onto the free part out of Q (Procedure A.1.25).

Recall that given an integral $r \times n$ matrix A we write $\mathrm{lin}_{\mathbb{Z}}(A)$ for the sublattice of $\mathbb{Z}^r$ spanned by the n columns of A. In the following procedures one should use the option `'nocheck'` if possible to speed up the computations.

Procedure A.1.1 (`createAG`). Constructor for the data type AG.

Input: there are four input types:

- An integer r. This will create the AG $\mathbb{Z}^r$.
- An integer r and a list $[a_1, \dots, a_k]$ with $a_i \in \mathbb{Z}$. This will create the AG $\mathbb{Z}^r \oplus \bigoplus_i \mathbb{Z}/a_i\mathbb{Z}$.
- An integral $r \times s$ matrix L. This will create an AG representing $\mathbb{Z}^r/\mathrm{lin}_{\mathbb{Z}}(L)$.
- An integral $r \times n$ matrix U and an integral $r \times s$ matrix L. This will create an AG representing $(\mathrm{lin}_{\mathbb{Z}}(U) + \mathrm{lin}_{\mathbb{Z}}(L))/\mathrm{lin}_{\mathbb{Z}}(L)$.

Output: an AG. Also prints an integer r and a list of integers $[a_1, \dots, a_k]$ such that the returned group is isomorphic to $\mathbb{Z}^r \oplus \bigoplus_i \mathbb{Z}/a_i\mathbb{Z}$ as a $\mathbb{Z}$-module.

Example: `> createAG(2, [3]); #` creates $\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$

$$AG(2, [3])$$

`> L := linalg[matrix]([[0],[3]]):`
`> createAG(L); #` creates $\mathbb{Z}^2/\mathrm{lin}_{\mathbb{Z}}(L)$

$$AG(1, [3])$$

`> U := linalg[matrix]([[3,0],[0,3]]):`
`> createAG(U, L); #` creates $(\mathrm{lin}_{\mathbb{Z}}(U) + \mathrm{lin}_{\mathbb{Z}}(L))/\mathrm{lin}_{\mathbb{Z}}(L)$.

$$AG(1, [])$$

Procedure A.1.2 (`AGdata`). Returns the stored information of the given AG G, i.e., a list $[U, L, r, [a_1, \dots, a_s]]$ with integral matrices U, L and r, $a_i \in \mathbb{Z}_{\geq 0}$ such that

$$G \quad \cong \quad (\mathrm{lin}_{\mathbb{Z}}(U) + \mathrm{lin}_{\mathbb{Z}}(L))/\mathrm{lin}_{\mathbb{Z}}(L) \quad \cong \quad \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/a_i\mathbb{Z}.$$

Input: an AG $G = (U, L)$.

Output: a list $[U, L, r, [a_1, \dots, a_s]]$ with matrices U and L an integer r and integers $a_i \in \mathbb{Z}_{>0}$ as explained above.

Example:
```
> U1 := linalg[matrix]([[2,0],[0,3]]):
> L1 := linalg[matrix]([[0],[3]]):
> H1 := createAG(U1, L1);
```

$$H1 := AG(1, [])$$

```
> AGdata(H1);
```

$$\left[\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}, 1, []\right]$$

Procedure A.1.3 (`createAGH`)**.** Constructor for the data type AGH. Represents a homomorphism $Q\colon G_1 \to G_2$ of AGs G_1 and G_2.

Input: an AG $G_1 = (U_1, L_1)$, an AG $G_2 = (U_2, L_2)$ and an integral matrix A. Throws an error if $A \cdot \mathrm{lin}_{\mathbb{Z}}(L_1) \not\subseteq \mathrm{lin}_{\mathbb{Z}}(L_2)$ or if $A \cdot (\mathrm{lin}_{\mathbb{Z}}(U_1) + \mathrm{lin}_{\mathbb{Z}}(L_1))$ is not contained in $\mathrm{lin}_{\mathbb{Z}}(U_2) + \mathrm{lin}_{\mathbb{Z}}(L_2)$.

Output: the AGH (G_1, G_2, A). Also prints integers r_i and lists $[a_{i1}, \ldots, a_{ik_i}]$ of integers such that the G_i are isomorphic to $\mathbb{Z}^{r_i} \oplus \bigoplus_j \mathbb{Z}/a_{ij}\mathbb{Z}$ as $\mathbb{Z}$-modules.

Example:
```
> G1 := createAG(3,[3]);
```

$$G1 := AG(3, [3])$$

```
> G2 := createAG(2,[2]);
```

$$G2 := AG(2, [2])$$

```
> A := linalg[matrix]([[1,1,0,0],[0,1,0,0],[0,0,1,2]]);
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{bmatrix}$$

```
> Q := createAGH(G1, G2, A);
```

$$Q := AGH([3, [3]], [2, [2]])$$

Procedure A.1.4 (`AGHdata`)**.** Returns the stored data of the given AGH $\varphi = (G_1, G_2, A)$, i.e., returns a list $[G_1, G_2, A]$ with G_i AGs and A an integral matrix.

Input: an AGH φ.

Output: a list $[G_1, G_2, A]$ with AGs G_1, G_2 and a matrix A as explained above.

Example: let `Q` be as in Procedure A.1.3.

```
> AGHdata(Q);
```

$$\left[AG(3, [3]), AG(2, [2]), \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{bmatrix}\right]$$

Procedure A.1.5 (`AGareisom`)**.** Implements Algorithm 2.1.4.

Input: an AG G_1 and an AG G_2.

Output: *true* if G_1 is isomorphic to G_2 and *false* otherwise.

Example:
```
> U1 := linalg[matrix]([[2,0],[0,3]]):
> L1 := linalg[matrix]([[0],[3]]):
> H1 := createAG(U1, L1);
```

$$H1 := AG(1, [])$$

```
> H2 := createAG(1);
```

$$H2 := AG(1, [])$$

```
> AGareisom(H1, H2);
```

$$true$$

Procedure A.1.6 (`AGcontains`)**.** Implements Algorithm 2.1.6.

Input: there are two input possibilities:

- An AG G_1 and an AG G_2.
- An AG G and a vector w.

Output: *true* if G_1 contains G_2 or, for the second input type, if $w \in G$. Returns *false* otherwise.

Example: `> L := linalg[matrix]([[0],[3]]);`

$$L := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

`> U1 := linalg[matrix]([[2,0],[0,1]]);`

$$U1 := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

`> H1 := createAG(U1, L);`

$$H1 := AG(1, [3])$$

`> U2 := linalg[matrix]([[2,0],[0,3]]);`

$$U2 := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

`> H2 := createAG(U2, L);`

$$H2 := AG(1, [])$$

`> AGcontains(H2, H1); AGcontains(H1, H2);`

$$true$$
$$false$$

`> AGcontains(H2, [2,0]);`

$$true$$

Procedure A.1.7 (`AGareequal`)**.** Implements Algorithm 2.1.8.

Input: there are two input types:

- An AG G_1 and an AG G_2.
- Vectors w, $w' \in \mathbb{Z}^r$ and an AG $G = (U, L)$ such that $w + \mathrm{lin}_{\mathbb{Z}}(L)$ and $w' + \mathrm{lin}_{\mathbb{Z}}(L)$ are elements of G.

Output: *true* if $G_1 = G_2$ or, for the second input type, if $w = w' \in G$. Returns *false* otherwise.

Options: `'nocheck'`: do not check whether w, $w' \in G$.

Example: let `H1` and `H2` be as in Procedure A.1.6.

`> AGareequal(H1, H1); AGareequal(H1, H2);`

$$true$$

$$false$$

`> AGareequal([2,4],[2,7], H1);`

$$true$$

Procedure A.1.8 (`AGfactgrp`)**.** Implements Algorithm 2.1.9.

Input: an AG G and an AG H such that $H \leq G$.

Output: an AG representing the factor group G/H.

Options: `'nocheck'`: do not check whether $H \leq G$.

Example: `> U1 := linalg[matrix]([[2,0],[0,3]]);`

$$U1 := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

```
> U2 := linalg[matrix]([[4,0],[0,3]]);
```

$$U2 := \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

```
> L := linalg[matrix]([[0],[3]]);
```

$$L := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

```
> G := createAG(U1, L);
```

$$G := AG(1, [])$$

```
> H := createAG(U2, L);
```

$$H := AG(1, [])$$

```
> GH := AGfactgrp(G, H); AGdata(GH);
```

$$GH := AG(0, [2])$$

$$\left[\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, 0, [2] \right]$$

Procedure A.1.9 (`AGprodgrp`). Implements Algorithm 2.1.10.
Input: an AG G_1 and an AG G_2.
Output: an AG representing the product $G_1 \times G_2$.
Example: `> U1 := linalg[matrix]([[2,0],[0,2]]);`

$$U1 := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> L1 := linalg[matrix]([[0],[3]]);
```

$$L1 := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

```
> G1 := createAG(U1, L1);
```

$$G1 := AG(1, [3])$$

```
> U2 := linalg[matrix]([[3,0],[0,2]]);
```

$$U2 := \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> L2 := linalg[matrix]([[0],[3]]);
```

$$L2 := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

```
> G2 := createAG(U2, L2);
```

$$G2 := AG(1, [3])$$

```
> G12 := AGprodgrp(G1, G2); AGdata(G12);
```

$$G12 := AG(2, [3, 3])$$

$$\left[\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}, 2, [3, 3] \right]$$

Procedure A.1.10 (`AGfreered`). Implements Algorithm 2.1.11.
Input: an AG G.
Output: an AG representing the free part of G, i.e., the lattice G/G^{tor}.
Example: `> U := linalg[matrix]([[2,0],[0,2]]);`

$$U := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> L := linalg[matrix]([[0],[3]]);
```

$$L := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

```
> G := createAG(U, L);
```

$$G := AG(1, [3])$$

```
> G0 := AGfreered(G); AGdata(G0);
```

$$G0 := AG(1, [])$$

$$\left[\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1, [] \right]$$

Procedure A.1.11 (`AGintersect`). Implements Algorithm 2.1.13.

Input: an AG G_1 and an AG G_2.

Output: an AG G representing the intersection $G_1 \cap G_2$.

Options: `'nocheck'` prevents checks for the groups to have the same torsion part.

Example: consider the AGs `G1` and `G2` defined in the example of Procedure A.1.9.

```
> G12 := AGintersect(G1, G2); AGdata(G12);
```

$$G12 := AG(1, [3])$$

$$\left[\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, 1, [3] \right]$$

Procedure A.1.12 (`AGHim`). Implements Algorithm 2.1.17.

Input: an AGH $Q = (G_1, G_2, A)$ and an AG $H_1 \leq G_1$. If the subgroup H_1 is left out, $H_1 = G_1$ will be used.

Output: an AG representing $Q(H_1) \leq G_2$.

Example: `> G1 := createAG(2);`

$$G1 := AG(2, [])$$

```
> G2 := createAG(1,[4]);
```

$$G2 := AG(1, [4])$$

```
> A := linalg[matrix]([[2,0],[0,2]]);
```

$$A := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> Q := createAGH(G1, G2, A);
```

$$Q := AGH([2, []], [1, [4]])$$

`> H1 := AGHim(Q); AGdata(H1); #` the image is $2\mathbb{Z} \oplus 2\mathbb{Z}/4\mathbb{Z}$

$$H1 := AG(1, [2]);$$

$$\left[\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, 1, [2] \right]$$

Procedure A.1.13 (`AGHpreim`). Implements Algorithm 2.1.18.

Input: an AGH $Q = (G_1, G_2, A)$ and an AG $H_2 \leq G_2$.

Output: an AG representing the preimage $Q^{-1}(H_2) \leq G_1$.

Options: `'nocheck'`: do not check whether $H_2 \leq G_2$.

Example: `> G1 := createAG(2); G2 := createAG(1,[4]);`

$$G1 := AG(2, [])$$

$$G2 := AG(1, [4])$$

```
> A := linalg[matrix]([[1,0],[0,1]]);
```

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> Q := createAGH(G1, G2, A);
```

$$Q := AGH([2, []], [1, [4]])$$

```
> U := linalg[matrix]([[2,0],[0,3]]);
```

$$U := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

```
> L := linalg[matrix]([[0],[4]]);
```

$$L := \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

```
> H2 := createAG(U, L);
```

$$H2 := AG(1, [4]);$$

```
> H1 := AGHpreim(Q, H2); AGdata(H1);
```

$$H1 := AG(2, [])$$

$$\left[\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 2, [] \right]$$

Procedure A.1.14 (`AGHker`). Implements Algorithm 2.1.19.
Input: an AGH $Q = (G_1, G_2, A)$.
Output: returns an AG representing $\ker(Q) \leq G_1$.
Example: define the AGH Q as in A.1.13.

```
> H1 := AGHker(Q); AGdata(H1);
```

$$H1 := AG(1, [])$$

$$\left[\begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1, [] \right]$$

Procedure A.1.15 (`AGHisinj`). Implements Algorithm 2.1.20.
Input: an AGH Q.
Output: returns *true* if Q is injective and *false* otherwise.
Example: `> G1 := createAG(1, [2]); G2 := createAG(1, [4]);`

$$G1 := AG(1, [2])$$
$$G2 := AG(1, [4])$$

```
> A1 := linalg[matrix]([[1,0], [0,2]]);
```

$$A1 := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> phi1 := createAGH(G1, G2, A1);
```

$$phi1 := AGH([1, [2]], [1, [4]])$$

```
> AGHisinj(phi1);
```

$$true$$

```
> A2 := linalg[matrix]([[1,0], [0,4]]);
```

$$A2 := \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

```
> phi2 := createAGH(G1, G2, A2);
```

$$phi2 := AGH([1, [2]], [1, [4]])$$

```
> AGHisinj(phi2);
```

$$false$$

Procedure A.1.16 (`AGHissurj`)**.** Implements Algorithm 2.1.21.
Input: an AGH Q.
Output: returns *true* if Q is surjective and *false* otherwise.
Example: let `phi1` and `G1` be as in A.1.15.

```
> AGHissurj(phi1);
```

$$false$$

```
> G3 := createAG(0, [2,2]);
```

$$G3 := AG(0, [2, 2])$$

```
> A2 := linalg[matrix]([[1,0], [0,1]]);
```

$$A2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> phi2 := createAGH(G1, G3, A2);
```

$$phi2 := AGH([1, [2]], [0, [2, 2]])$$

```
> AGHissurj(phi2);
```

$$true$$

Procedure A.1.17 (`AGHcompleteseq`)**.** Implements Algorithm 2.1.22.
Input: an AGH $\varphi = (G_1, G_2, A)$ that is either injective or surjective.
Output: an AGH ψ completing the respective exact sequence

$$0 \longleftarrow G_2 \xleftarrow[\varphi]{} G_1 \xleftarrow[\psi]{} G \longleftarrow 0$$

$$0 \longleftarrow G \xleftarrow{\psi} G_2 \xleftarrow{\varphi} G_1 \longleftarrow 0$$

Options: `'inj'` or `'surj'` assumes φ is injective or surjective without further tests.
Example: `> U1 := linalg[matrix]([[0],[2]]);`

$$U1 := \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

```
> L1 := linalg[matrix]([[0],[4]]);
```

$$L1 := \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

`> G1 := createAG(U1, L1); #` represents $\{0\} \oplus 2\mathbb{Z}/4\mathbb{Z}$

$$G1 := AG(0, [2])$$

`> G2 := createAG(1,[4]); #` represents $\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

$$G2 := AG(1, [4])$$

```
> U3 := linalg[matrix]([[1,0],[0,2]]);
```

$$U3 := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> L3 := linalg[matrix]([[0],[4]]);
```

$$L3 := \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

```
> G3 := createAG(U3, L3); # represents Z ⊕ 2Z/4Z
```

$$G3 := AG(1, [2])$$

```
> A := linalg[matrix]([[1,0],[0,2]]);
```

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> phi := createAGH(G2, G3, A);
```

$$phi := AGH([1, [4]], [1, [2]])$$

```
> iota := AGHcompleteseq(phi, 'surj'); AGHdata(iota);
```

$$iota := AGH([0, [2]], [1, [4]])$$

$$\left[AG(0, [2]), AG(1, [4]), \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

```
> pi := AGHcompleteseq(iota, 'inj'); AGHdata(pi);
```

$$pi := AGH([1, [4]], [1, [2]])$$

Procedure A.1.18 (`AGHgradiator`). Implements Algorithm 2.1.29.

Input: a list of polynomials $[f_1, \ldots, f_s]$ and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: a list $[Q, P]$ with an AGH $Q\colon \mathbb{Z}^r \to K$ representing the *gradiator*, i.e., the maximal quasi-torus action keeping $V(f_1, \ldots, f_s)$ invariant. The second element is a Gale dual matrix P of Q.

Example:
```
> RL := [T[1]*T[2] + T[3]*T[4] + T[5]*T[6],
T[1]*T[2] + T[3]^2 + T[5]^2];
```

$$RL := \left[T[1]T[2] + T[3]T[4] + T[5]T[6], T[1]T[2] + T[3]^2 + T[5]^2\right]$$

```
> TT := vars(6);
```

$$TT := [T[1], T[2], T[3], T[4], T[5], T[6]]$$

```
> L := AGHgradiator(RL, TT);
```

$$L := \left[AGH([6, []], [2, [2]]), \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \end{bmatrix} \right]$$

```
> AGHdata(L[1]);
```

$$\left[AG(6, []), AG(2, [2]), \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \right]$$

Procedure A.1.19 (`AGHdeg`). Computes the K-degrees of a list of polynomials.

Input: an AGH $Q = (\mathbb{Z}^r, K, A)$ with $K = (U, L)$, a list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: a list of vectors $[w_1, \ldots, w_s]$ with $w_i \in \mathbb{Z}^r$ representing the degree $\deg(f_i) = w_i + \mathrm{lin}_{\mathbb{Z}}(L) \in K$.

Example:
```
> G1 := createAG(4);
```

$$G1 := AG(4, [])$$

```
> K := createAG(1,[3]);
```

$$K := AG(1, [3])$$

```
> A := linalg[matrix]([[1,1,0,0],[1,1,1,1]]);
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> Q := createAGH(G1, K, A);
```

$$Q := AGH([4,[]],[1,[3]])$$

```
> TT := vars(4);
```

$$TT := [T[1], T[2], T[3], T[4]]$$

```
> RL := [T[1]*T[3] + T[2]*T[4], T[1]^3 + T[2]^3];
```

$$RL := \left[T[1]T[3] + T[2]T[4], T[1]^3 + T[2]^3\right]$$

`> AGHdeg(Q, RL, TT); #` interpreted as $(1, \overline{2})$ and $(3, \overline{3}) \in K$:

$$[[1,2],[3,3]]$$

Procedure A.1.20 (`AGHishomog`)**.** Implements Algorithm 2.1.31.

Input: an AGH or matrix Q, a list of polynomials $[f_1, \ldots, f_s]$ and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: *true* if all f_i are homogeneous with respect to the grading $\deg(T_i) = Q(e_i)$. Returns *false* otherwise.

Example: `> E := createAG(4); K := createAG(1, [3]);`

$$E := AG(4,[])$$
$$K := AG(1,[3])$$

```
> B := linalg[matrix]([[1,0,-1,2],[1,2,2,2]]);
```

$$B := \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}$$

```
> Q := createAGH(E, K, B);
```

$$Q := AGH([4,[]],[1,[3]])$$

```
> RL := [T[1]*T[3] + T[2]^3, T[3]*T[4] + T[1]];
```

$$RL := [T[1]T[3] + T[2]^3, T[3]T[4] + T[1]];$$

```
> TT := [T[1], T[2], T[3], T[4]];
```

$$TT := [T[1], T[2], T[3], T[4]]$$

```
> AGHishomog(Q, RL, TT);
```

$$true$$

Procedure A.1.21 (`AGHisalmostfree`)**.** Implements Algorithm 2.1.32.

Input: an AGH or a matrix Q.

Output: *true* if the grading of $\mathbb{K}[T_1, \ldots, T_r]$ given by $\deg(T_i) = Q(e_i)$ is almost free and *false* otherwise.

Example: `> B := linalg[matrix]([[1,0,1],[0,2,2]]);`

$$B := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

`> AGHisalmostfree(B); #` B is not surjective

$$false$$

`> E := createAG(3); K := createAG(1, [3]); #` $K = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$

$$E := AG(3,[])$$
$$K := AG(1,[3])$$

```
> Q := createAGH(E, K, B);
```

$$Q := AGH([3,[]],[1,[3]])$$

```
> AGHisalmostfree(Q);
```

$$true$$

Procedure A.1.22 (`AGHsection`). Implements Algorithm 2.1.34.

Input: a surjective AGH $Q = (G_1, G_2, A)$ or a surjective matrix.

Output: a pair $[b, \psi]$. If the algorithm found a section, then b is *true* and ψ a section. Otherwise, b is *false* and $\varphi = []$. In the first case, ψ is given as AGH $\psi\colon G_2 \to G_1$.

Example: `> Q := createAGH(createAG(1), createAG(0,[2]), linalg[matrix]([[1]]));`

$$Q := AGH([1, []], [0, [2]])$$

`> AGHsection(Q);#` no homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ exists:

$$[false, []]$$

`> P := linalg[matrix]([[1,0,1],[0,2,1]]);`

$$P := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

`> S := AGHsection(P); AGHdata(S[2]);`

$$S := [true, AGH([2, []], [3, []])]$$

$$\left[AG(2, []), AG(3, []), \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

Procedure A.1.23 (`AGHP2Q`). Implements Algorithm 2.1.24.

Input: an integral $n \times r$ matrix P.

Output: an AGH $Q = (\mathbb{Z}^r, K, A)$ such that $K \cong \mathbb{Z}^r/\mathrm{Im}(P^*)$.

Options: `'nocheck'`: skip the test of Q being surjective.

Example: `> P := linalg[matrix]([[1,0,2],[0,2,2]]);`

$$P := \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

`> Q := AGHP2Q(P);`

$$Q := AGH([3, []], [1, [2]])$$

`> AGHdata(Q);`

$$\left[AG(3, []), AG(1, [2]), \begin{bmatrix} -2 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right]$$

Procedure A.1.24 (`AGHQ2P`). Implements Algorithm 2.1.25.

Input: a surjective AGH $Q = (\mathbb{Z}^r, K, A)$.

Output: a Gale dual matrix P for Q, i.e., P is dual to the inclusion $\ker(Q) \to \mathbb{Z}^r$.

Example: `> E := createAG(4); K := createAG(2, [2]);`

$$E := AG(4, [])$$
$$K := AG(2, [2])$$

`> B := linalg[matrix]([[1,0,1,0],[0,1,0,1],[1,1,0,0]]);`

$$B := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

`> Q := createAGH(E, K, B);`

$$Q := AGH([4, []], [2, [2]])$$

`> P := AGHQ2P(Q);`

$$P := \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -2 \end{bmatrix}$$

Procedure A.1.25 (`AGHQ2Q0`). Implements Algorithm 2.1.26.

Input: a surjective AGH $Q = (\mathbb{Z}^r, K, A)$.

Output: a matrix Q^0 representing the projection $\mathbb{Z}^r \to K/K^{\mathrm{tor}}$.

Example:
```
> E := createAG(4); K := createAG(2, [2]);
```
$$E := AG(4, [])$$
$$K := AG(2, [2])$$
```
> B := linalg[matrix]([[1,0,1,0],[0,1,0,1],[1,1,0,0]]);
```
$$B := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$
```
> Q := createAGH(E, K, B);
```
$$Q := AGH([4, []], [2, [2]])$$
```
> Q0 := AGHQ2Q0(Q);
```
$$Q0 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

2. Procedures on graded rings

In this section, we describe the implementation of algorithms that work on rings that are graded by a finitely generated abelian group (GRs). See mainly Section 2 of Chapter 2. Here is an overview:

- *Creation on stored data:* create a BUN (Procedure A.2.1), return the stored data of a BUN (Procedure A.2.2), create a GR (Procedure A.2.3), return the stored data of a GR (Procedure A.2.4).
- *Grading:* homogeneous components (Procedure A.2.5), dimension of a homogeneous component (Procedure A.2.6).
- *GIT:* GIT-fan (Procedure A.2.7), $(H, 2)$-maximal sets (Procedure A.2.8).
- *Tropical algorithms:* tropical variety for one equation (Procedure A.2.9), containment in the tropical variety (Procedure A.2.10).

Procedure A.2.1 (`createBUN`). Constructor for the data type BUN. Represents a true $\mathfrak{F}$-bunch Φ in $K_{\mathbb{Q}}$.

Input: there are five types of input:

- A vector $w \in \mathrm{Mov}(Q^0)^\circ \subseteq K_{\mathbb{Q}}$ and a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.
- A vector $w \in \mathrm{Mov}(Q^0)^\circ \subseteq K_{\mathbb{Q}}$, a list of polynomials $G \subseteq \mathbb{K}[T_1, \ldots, T_r]$, a list of variables $[T_1, \ldots, T_r]$ and an integral matrix Q^0.
- A list of cones $[\vartheta_1, \ldots, \vartheta_s]$ in $K_{\mathbb{Q}}$, a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.
- A list of cones $[\vartheta_1, \ldots, \vartheta_s]$ in $K_{\mathbb{Q}}$, a list of polynomials $G \subseteq \mathbb{K}[T_1, \ldots, T_r]$, a list of variables $[T_1, \ldots, T_r]$ and an integral matrix Q^0.
- A list of cones $[\vartheta_1, \ldots, \vartheta_s]$ in $K_{\mathbb{Q}}$.

Output: a BUN Φ in $K_{\mathbb{Q}}$. The printed information is the number of stored cones. Depending on the input type, Φ is given by

$$\Phi \;=\; \begin{cases} \left\{Q^0(\gamma_0);\ \gamma_0\ \mathfrak{F}\text{-face},\ w \in \left(Q^0(\gamma_0)\right)^\circ\right\}, & \text{in case one or two,} \\ \left\{Q^0(\gamma_0);\ \gamma_0\ \mathfrak{F}\text{-face},\ \vartheta_i^\circ \subseteq \left(Q^0(\gamma_0)\right)^\circ\ \text{for some } i\right\}, & \text{in case three or four,} \\ \{\vartheta_1, \ldots, \vartheta_s\}, & \text{in case five.} \end{cases}$$

Options: `'nocheck'`: in cases four and five, do not check whether the cones satisfy $\vartheta_i^\circ \cap \vartheta_j^\circ \neq \emptyset$; in cases one and two, do not check whether $w \in \mathrm{Mov}(Q^0)^\circ$.

Example: In the following example `B1` to `B4` all define the same BUN. Its cones are the elements of `CL3`; compare Procedure A.2.2.

```
> RL := [T[1]*T[2] + T[3]*T[4] + T[5]^2];
```

$$RL := [T[1]T[2] + T[3]T[4] + T[5]^2]$$

```
> TT := vars(5);
```

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

```
> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0],[1, 1, 1, 1, 1]]);
```

$$Q0 := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> w := [-1,2];
```

$$w := [-1, 2]$$

```
> R := createGR(RL, TT, [Q0]);
```

$$R := GR(5, 1, [2, []])$$

```
> B1 := createBUN(w, R);
```

$$B1 := BUN(5)$$

```
> B2 := createBUN(w, RL, TT, Q0);
```

$$B2 := BUN(5)$$

```
> CL3 := [poshull([-2,1],[1,1]), poshull([-2,1],[2,1]),
poshull([2,1],[-1,1]), poshull([-1,1],[1,1]), poshull([-2,1],[1,1])];
```

$$CL3 := [CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2),$$
$$CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2)]$$

```
> B3 := createBUN(CL3);
```

$$B3 := BUN(5)$$

```
> CL4 := [poshull([-1,1],[1,1])];
```

$$CL4 := [CONE(2, 2, 0, 2, 2)]$$

```
> B4 := createBUN(CL4, RL, TT, Q0);
```

$$B4 := BUN(5)$$

Procedure A.2.2 (`BUNdata`). Returns the data stored in a given BUN Φ.

Input: a BUN

Output: a list with its only entry a list of CONEs in $K_{\mathbb{Q}}$.

Example: Let `B4` be the BUN defined in the example of Procedure A.2.1.

```
> L4 := BUNdata(B4);
```

$$L4 := [[CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2),$$
$$CONE(2, 2, 0, 2, 2), CONE(2, 2, 0, 2, 2)]]$$

```
> map(b -> rays(b), L4[1]);
```

$$[[[-2, 1], [1, 1]], \quad [[2, 1], [-1, 1]], \quad [[-2, 1], [1, 1]],$$
$$[[-1, 1], [1, 1]], \quad [[-2, 1], [2, 1]]]$$

Procedure A.2.3 (`createGR`). Constructor for the data type GR.

Input: there are multiple ways to use this function:

- a list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$, a pair $[Q, Q^0]$ with an AGH $Q = (\mathbb{Z}^r, K, A)$ and a matrix Q^0, a matrix P.

- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list $[Q]$ with an AGH $Q = (\mathbb{Z}^r, K, A)$, a matrix P. Here, Q^0 will be computed from Q using Algorithm 2.1.26.
- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list $[Q^0]$ with a matrix Q^0, a matrix P. The AGH $Q := Q^0 \colon \mathbb{Z}^r \to \mathbb{Z}^n$ will be used.
- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list $[Q]$ with an AGH $Q = (\mathbb{Z}^r, K, A)$. Both Q^0 and P will be computed from Q using Algorithms 2.1.25 and 2.1.26.
- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list $[Q^0]$ with a matrix Q^0. Here, $Q := Q^0 \colon \mathbb{Z}^r \to \mathbb{Z}^n$ will be used and P will be computed from Q using Algorithm 2.1.25.
- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list $[Q]$ with an AGH $Q \colon \mathbb{Z}^r \to K$. Here, both Q^0 and P will be computed from Q using Algorithms 2.1.25 and 2.1.26.
- a list of polynomials $[f_1, \dots, f_s]$, a list of variables $[T_1, \dots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a matrix P. Both Q and Q^0 will be computed from P using Algorithms 2.1.24 and 2.1.26.
- An integral matrices P and A as in Construction 1.5.11. The procedure then returns the graded ring $R(P, A)$, see 1.5.3.

In each case, we require Q to be surjective, P to be of full rank, the grading of $\mathbb{K}[T_1, \dots, T_r]$ given by $\deg(T_i) = Q(e_i)$ to be almost free and all f_i must be K-homogeneous. Moreover, all variables must be K-prime, $P(Q^0)^t = 0$ and $K \cong \mathbb{Z}^r/\mathrm{Im}(P^*)$.

Output: a GR $R = (\{f_1, \dots, f_s\}, Q, Q^0, P, F_{\mathfrak{F}})$. Then R represents the ring $\mathbb{K}[T_1, \dots, T_r]/I$ with the ideal $I = \langle f_1, \dots, f_s \rangle$ and degree map $Q \colon \mathbb{Z}^r \to K$. It fixes matrices P, Q^0 such that $K \cong \mathbb{Z}^r/\mathrm{Im}(P^*)$ and the matrix Q^0 represents the projection $\mathbb{Z}^r \to K/K^{\mathrm{tor}}$. See Chapter 2 for details.

Internally, the list of all $\mathfrak{F}$-faces is stored unless the option `'noffaces'` was given. The printed information is the number of variables, the number s of generators for the ideal of relations and information about the AG K.

Options:
- `'nocheck'`: skips tests for the parameters.
- `'noffaces'`: postpones the computation and storage of $\mathfrak{F}$-faces.
- `'Singular'`: use the software `Singular` for the computations of $\mathfrak{F}$-faces; this only possible on UNIX-based machines where `Singular` is available on the command line. Writes temporary files to the current directory.

Example:
```
> RL := [T[1]*T[6] + T[2]*T[5] + T[3]*T[4] + T[7]*T[8]];
```

$$RL := [T[1]T[6] + T[2]T[5] + T[3]T[4] + T[7]T[8]]$$

```
> TT := vars(8);
```

$$TT := [T[1], T[2], T[3], T[4], T[5], T[6], T[7], T[8]]$$

```
> A := linalg[matrix]([[1,1,0,0,-1,-1,2,-2], [0,1,1,-1,-1,0,1,-1],
[1,1,1,1,1,1,1,1], [1,0,1,0,1,0,1,0]]);
```

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

```
> F := createAG(8); K := createAG(3, [2]);
```

$$F := AG(8, [])$$
$$K := AG(3, [2])$$

```
> Q := createAGH(F, K, A);
```

$$Q := AGH([8,[]],[3,[2]])$$

```
> Q0 := linalg[delrows](A, 4..4);
```

$$Q0 := \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> P := AGHQ2P(Q);
```

$$P := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 & -2 & -5 \\ 0 & 0 & 0 & 0 & 4 & 4 & -2 & -6 \end{bmatrix}$$

```
> R1 := createGR(RL, TT, [Q, Q0], P);
```

$$R1 := GR(8,1,[3,[2]])$$

The printed information means that `R1` represents $\mathbb{K}[T_1,\ldots,T_8]/I$ with I generated by a single polynomial and the grading group K is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$. The function `GRdata` in Procedure A.2.4 shows all stored information. We enter another example with a free grading group:

```
> RL := [T[1]^2 + T[2]^2 - T[3]*T[4]];
```

$$RL := [T[1]^2 + T[2]^2 - T[3]T[4]]$$

```
> TT := vars(4);
```

$$TT := [T[1], T[2], T[3], T[4]]$$

```
> Q0 := linalg[matrix]([[0,0,-1, 1], [1,1, 1, 1]]);
```

$$Q0 := \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> R2 := createGR(RL, TT, [Q0], 'Singular');
```

$$R2 := GR(4,1,[2,[]])$$

In the next example, we enter a toric variety:

```
> RL := []; TT := vars(4);
```

$$RL := []$$
$$TT := [T[1], T[2], T[3], T[4]]$$

```
> P := linalg[matrix]([[-1,1,0,1], [-1,0,1,1]]);
```

$$P := \begin{bmatrix} -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

```
> R3 := createGR(RL, TT, P);
```

$$R3 := GR(4,0,[2,[]])$$

We now enter a $\mathbb{K}^*$-surface with $\mathbb{Z}^4$-graded Cox ring $\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6\rangle$ by providing its P-matrix and a list A of integers:

```
> P := matrix([[-1,-1,1,1,0,0], [-1,-1,0,0,1,1]]);
```

$$P := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

```
> A := [[1,0],[1,1],[0,1]];
```

$$A := [[1,0],[1,1],[0,1]]$$

```
> R := createGR(P, A);
```

$$R := GR(6,1,[4,[]])$$

Procedure A.2.4 (`GRdata`). Returns the stored information on the given GR R.

Input: a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ with AGH $Q = (\mathbb{Z}^r, K, A)$.

Output: a list $[G, [T_1, \dots, T_r], [Q, Q^0], P, F]$ with a list G of K-homogeneous polynomials in $\mathbb{K}[T_1, \dots, T_r]$, integral matrices Q^0 and P as well as a list of all $\mathfrak{F}$-faces F such that

$$R \quad = \quad \mathbb{K}[T_1, \dots, T_r] \,/\, \langle f_1, \dots, f_s \rangle,$$

the matrix P is dual to the inclusion $\ker(Q) \to \mathbb{Z}^r$ and Q^0 is a matrix describing the projection $\mathbb{Z}^r \to K/K^{\mathrm{tor}}$. Moreover, Q is surjective and the grading given by $\deg(T_i) = Q(e_i)$ is almost free.

Example: we enter the GR `R2` of the example of Procedure A.2.3.

```
> GRdata(R2);
```

$$\Bigg[\left[T[1]^2 + T[2]^2 - T[3]T[4]\right], [T[1], T[2], T[3], T[4]],$$

$$\left[AGH([4, []], [2, []]), \left[\begin{array}{cccc} 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]\right], \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right]$$

$$\{\{\}, \{1,2,3\}, \{3\}, \{1,3,4\}, \{4\}, \{1,2,4\}, \{1,2\}, \{1,2,3,4\}, \{2,3,4\}\}\Bigg]$$

Procedure A.2.5 (`GRgradedcomp`). Implements Algorithm 2.2.3.

Input: there are two possible input types:

- A graded ring $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and a vector $w \in Q^0(\mathbb{Q}^r_{\geq 0})$.
- A list $[f_1, \dots, f_s]$ of polynomials $f_i \in \mathbb{K}[T_1, \dots, T_r]$, a list of variables $[T_1, \dots, T_r]$, a $r \times k$ matrix Q^0 and $w \in Q^0(\mathbb{Q}^r_{\geq 0})$ where the f_i are homogeneous with respect to the grading $\deg(T_i) = Q^0(e_i)$.

In both cases, the cone over the columns of Q^0 must be pointed and Q^0 must not contain zero-columns.

Output: a basis for the vector space $\langle G \rangle_w$ as a list of polynomials.

Example: `> RL := [T[1]*T[2] + T[3]*T[4] + T[5]^2];`

$$RL := [T[1]T[2] + T[3]T[4] + T[5]^2]$$

```
> TT := vars(5);
```

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

```
> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0],[1, 1, 1, 1, 1]]);
```

$$Q0 := \left[\begin{array}{ccccc} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

```
> w := [-1,2];
```

$$w := [-1, 2]$$

```
> R := createGR(RL, TT, [Q0]);
```

$$R := GR(5, 1, [2, []])$$

```
> w := [-1,3];
```

$$w := [-1, 3]$$

```
> C := GRgradedcomp(R, w);
```

$$C := \left[T[1]T[2]T[3] + T[3]^2T[4] + T[3]T[5]^2\right]$$

Procedure A.2.6 (`GRgradedcompdim`). Implements Algorithm 2.2.5.

Input: there are two possible input types:

- A graded ring $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and a vector $w \in Q^0(\mathbb{Q}^r_{\geq 0})$.

- A list $[f_1, \ldots, f_s]$ of polynomials $f_i \in \mathbb{K}[T_1, \ldots, T_r]$, a list of variables $[T_1, \ldots, T_r]$, a $r \times k$ matrix Q^0 and $w \in Q^0(\mathbb{Q}^r_{\geq 0})$ where the f_i are homogeneous with respect to the grading $\deg(T_i) = Q^0(e_i)$.

In both cases, the cone over the columns of Q^0 must be pointed and Q^0 must not contain zero-columns.

Output: the dimension of the vector space $\mathbb{K}[T_1, \ldots, T_r]_w / \langle G \rangle_w$.

Example: enter the GR `R` from the example of Procedure A.2.5.

```
> w := [-1,3];
```

$$w := [-1, 3]$$

```
> C := GRgradedcompdim(R, w);
```

$$3$$

Procedure A.2.7 (`GRgitfan`). Uses Algorithm 3.2.9 to compute the GIT-fan of the action of the torus $H^0 := \operatorname{Spec} \mathbb{K}[K^0]$ on $X := \operatorname{Spec} R$. This procedure is as in [70].

Input: a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

Output: the GIT-fan $\Lambda(X, H^0)$ given as a list of maximal CONEs.

Options: `'FAN'`: return a FAN instead of a list of its maximal CONEs.

Example: Consider the GR `R1` as in the example of Procedure A.2.3:

```
> GRgitfan(R1, 'FAN');
```

$$FAN(3, 0, [0, 0, 37])$$

Procedure A.2.8 (`GRH2max`). Implements Algorithm 3.3.4.

Input: a GR R or a MDS X.

Output: a list of BUNs. They correspond to the $(H, 2)$-maximal sets of X.

Example: `> Q0 := linalg[matrix]([[1,0,1,0], [0,1,0,1], [0,0,1,1]]);`

$$Q0 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

```
> R := createGR([], vars(4), [Q0]);
```

$$R := GR(4, 0, [3, []])$$

```
> GRH2max(R);
```

$$[BUN(1), BUN(1), BUN(1), BUN(1), BUN(1), BUN(1), BUN(1), BUN(1), BUN(3), BUN(3), BUN(3), BUN(3), BUN(3), BUN(3), BUN(3), BUN(3), BUN(3)]$$

Procedure A.2.9 (`GRtrop`). Implements Algorithm 2.2.7. Also works for arbitrary ideals if `gfan` [63] is available.

Input: There are three input possibilities:

- A GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.
- A MDS $X = (R, \Phi)$ with a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.
- A list of polynomials $G = [f_1, \ldots, f_s]$ and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

In the first two cases, let P be the integral $n \times r$ matrix dual to the degree matrix Q.

Output: a FAN in $\mathbb{Q}^n$ with support $P(\operatorname{trop}(\langle G \rangle))$ or a FAN in $\mathbb{Q}^r$ with support $\operatorname{trop}(\langle G \rangle)$ if the option `'F'` was specified or the third input type was used.

Options:

- `'F'`: return a FAN in $\mathbb{Q}^r$ with support $\mathrm{trop}(\langle G\rangle)$.
- `'CONEs'`: return a list of CONEs instead of a FAN. These represent the maximal cones of the fan.
- `'gfan'`: if the software `gfan` is installed on a UNIX-based machine, then arbitrary ideals may be entered. Writes temporary files in the current directory.

Example: `> RL := [T[1]*T[2] + T[3]*T[4] + T[5]^2];`

$$RL := [T[1]T[2] + T[3]T[4] + T[5]^2]$$

`> TT := vars(5);`

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

`> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0],[1, 1, 1, 1, 1]]);`

$$Q0 := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

`> R := createGR(RL, TT, [Q0]);`

$$R := GR(5, 1, [2, []])$$

`> GRtrop(R); #` after projection under $P\colon \mathbb{Z}^5 \to \mathbb{Z}^3$.

$$FAN(3, 1, [0, 3, 0])$$

`> GRtrop(R, 'F', 'CONEs'); #` in $\mathbb{Q}^5$.

$$[CONE(5, 4, 3, 1, 1), CONE(5, 4, 3, 1, 1), CONE(5, 4, 3, 1, 1)]$$

`> RL := [T[2]^3 -3*T[1]^2*T[2]^3 + 1, T[2]+T[1]+T[1]*T[2]^2 -1];`

$$RL := \left[T[2]^3 - 3T[1]^2T[2]^3 + 1, T[2] + T[1] + T[1]T[2]^2 - 1\right]$$

`> TT := vars(2);`

$$TT := [T[1], T[2]]$$

`> F := GRtrop(RL, TT, 'gfan');`

$$F := FAN(2, 1, [1, 0])$$

Procedure A.2.10 (`GRtropcontains`). Implements Algorithm 2.2.8.

Input: there are two input types:

- A GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ or a MDS $X = (R, \Phi)$ with P of size $n \times r$ and a vector $f \in \mathbb{Z}^r$.
- A GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ or a MDS $X = (R, \Phi)$ with P of size $n \times r$ and a vector $v \in \mathbb{Z}^n$.

Output: *true* if $\mathbb{Q}_{\geq 0} \cdot v \subseteq P(\mathrm{trop}(\langle G\rangle))$ or if $\mathbb{Q}_{\geq 0} \cdot f \subseteq \mathrm{trop}(\langle G\rangle)$ respectively. Returns *false* otherwise.

Example: enter the GR `R` from the example of Procedure A.2.9.

`> f := [0,0,0,-2,-1];`

$$f := [0, 0, 0, -2, -1]$$

`> GRtropcontains(R, f); #` $\mathbb{Q}_{\geq 0} \cdot f$ is contained in $\mathrm{trop}(\langle G\rangle)$.

true

`> v := [0, -2, -1];`

$$v := [0, -2, -1]$$

`> GRtropcontains(R, v); #` $\mathbb{Q}_{\geq 0} \cdot v$ is contained in $P(\mathrm{trop}(\langle G\rangle))$.

true

3. Procedures on Mori dream spaces

In this section, we describe an implementation of our algorithms that work on Mori dream spaces (MDS). The algorithms have been stated mainly in Section 3 of Chapter 2. The special case of complexity-one T-varieties will be treated in the next section. Here is an overview:

- *Creation and stored data:* create an MDS (Procedure A.3.1), return the stored data of an MDS (Procedure A.3.2).
- *Orbit data:* relevant $\mathfrak{F}$-faces (Procedure A.3.3), covering collection (Procedure A.3.4), toric ambient variety and completions (Procedure A.3.5).
- *Miscellanea:* Mori chamber decomposition (Procedure A.3.6), dimension (Procedure A.3.7), existence of points (Procedure A.3.8), irrelevant ideal (Procedure A.3.9), stratum (Procedure A.3.10), degree matrix (Procedure A.3.11), graph of exceptional curves (Procedure A.3.12).
- *Cones of divisor classes:* semiample cone (Procedure A.3.13), effective cone (Procedure A.3.14), moving cone (Procedure A.3.15).
- *Groups:* divisor class group (Procedure A.3.16), local divisor class groups (Procedure A.3.17), Picard group (Procedure A.3.18), Picard index (Procedure A.3.19).
- *Complete intersection Cox rings:* anticanonical divisor class (Procedure A.3.20), test for being ($\mathbb{Q}$)-Gorenstein (Procedures A.3.21 and A.3.22), Gorenstein index (Procedure A.3.23), test for being Fano (Procedure A.3.24), intersection numbers (Procedure A.3.25).
- *Singularities, further properties:* test for being ($\mathbb{Q}$-)factorial (Procedures A.3.26 and A.3.27), test for being quasismooth (Procedure A.3.28), test for being smooth (Procedure A.3.29), singularities (Procedure A.3.30), test for being (quasi-)projective (Procedure A.3.31 and A.3.32), test for being complete (Procedure A.3.33).

Procedure A.3.1 (`createMDS`). Constructor for the data type MDS. Represents Mori dream spaces in terms of bunched rings.

Input: there are two types of input:

- A GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and a vector $w \in K_{\mathbb{Q}}$. This input will return the result of `createMDS` with parameters R and the BUN
$$\Phi(w) \;=\; \left\{Q^0(\gamma_0);\, \gamma_0 \text{ is an } \mathfrak{F}\text{-face and } w \in \left(Q^0(\gamma_0)\right)^\circ\right\}.$$
- a GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ and a BUN Φ in $K_{\mathbb{Q}}$.

Output: `'nocheck'`: do not perform checks. Use this option if you know that the input is valid.

Options: the MDS $X = (R, \Phi)$. The printed information of an MDS is the number of variables, the number of relations, its dimension and information about the grading group.

Example: We reenter the GR `R`, the BUN `B1` and the vector `w` as in the example of Procedure A.2.1. Then `X` and `Y` describe the same Mori dream surface:

```
> RL := [T[1]*T[2] + T[3]*T[4] + T[5]^2];
```

$$RL := [T[1]T[2] + T[3]T[4] + T[5]^2]$$

```
> TT := vars(5);
```

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

```
> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0],[1, 1, 1, 1, 1]]);
```

$$Q0 := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> w := [-1,2];
```

$$w := [-1, 2]$$

```
> R := createGR(RL, TT, [Q0]);
```

$$R := GR(5, 1, [2, []])$$

```
> B := createBUN(w, R);
```

$$B := BUN(5)$$

```
> X := createMDS(R, B);
```

$$X := MDS(5, 1, 2, [2, []])$$

```
> Y := createMDS(R, w);
```

$$Y := MDS(5, 1, 2, [2, []])$$

Procedure A.3.2 (`MDSdata`). Returns the stored data of the given MDS.

Input: an MDS $X = (R, \Phi)$.

Output: a list $[R, \Phi]$ with a GR R and a BUN Φ.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

```
> MDSdata(X);
```

$$[GR(5, 1, [2, []]), BUN(5)]$$

Procedure A.3.3 (`MDSrlv`). Implements Algorithm 2.3.5.

Input: there are three types of input:

- An MDS $X = (R, \Phi)$.
- A BUN Φ, a matrix Q^0, a list F of $\mathfrak{F}$-faces.
- A BUN Φ, a matrix Q^0, a list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_r]$.

Output: in the first case, the list $\mathrm{rlv}(X)$ of all relevant $\mathfrak{F}$-faces is returned. In the second case, all $\gamma_0 \in F$ such that $Q^0(\gamma_0) \in \Phi$ is returned. In the third case, all $\langle f_1, \ldots, f_s \rangle$-faces $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $Q^0(\gamma_0) \in \Phi$ are returned.

Example: consider the MDS `X`, the matrix `Q0` and the BUN `B` as in Procedure A.3.1. Then the lists `FFr1` to `FFr3` coincide:

```
> FFr1 := MDSrlv(X);
```

$$FFr1 := [\{1,2,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4\}, \{3,4,5\}, \{1,3,4,5\},$$
$$\{1,2,4,5\}, \{1,2,5\}, \{1,2,3,5\}, \{2,3\}, \{1,4\}]$$

```
> FFr2 := MDSrlv(B, Q0, RL, TT);
```

$$FFr2 := [\{1,2,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4\}, \{3,4,5\}, \{1,3,4,5\},$$
$$\{1,2,4,5\}, \{1,2,5\}, \{1,2,3,5\}, \{2,3\}, \{1,4\}]$$

```
> FFr3 := MDSrlv(B, Q0, ffaces(RL, TT));
```

$$FFr3 := [\{1,2,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4\}, \{3,4,5\}, \{1,3,4,5\},$$
$$\{1,2,4,5\}, \{1,2,5\}, \{1,2,3,5\}, \{2,3\}, \{1,4\}]$$

Procedure A.3.4 (`MDScov`). Implements Algorithm 2.3.6.

Input: there are three types of input:

- An MDS $X = (R, \Phi)$.
- A BUN Φ, a matrix Q^0, a list of $\mathfrak{F}$-faces F.
- A BUN Φ, a matrix Q^0, a list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: in the first, second, third input case, the algorithm returns

- the list $\mathrm{cov}(X)$ of all minimal relevant $\mathfrak{F}$-faces.
- all $\gamma_0 \in F$ such that $Q^0(\gamma_0) \in \Phi$ and γ_0 is minimal with this property.
- all $\langle f_1, \ldots, f_s \rangle$-faces $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $Q^0(\gamma_0) \in \Phi$ and γ_0 is minimal with this property.

Example: consider the MDS `X`, the matrix `Q0` and the BUN `B` as in Procedure A.3.1. Compare also Procedure A.3.3.

```
> C1 := MDScov(X);
```

$$C1 := [\{1,2,5\}, \{3,4,5\}, \{1,4\}, \{2,3\}]$$

```
> C2 := MDScov(B, Q0, RL, TT);
```

$$C2 := [\{1,2,5\}, \{3,4,5\}, \{1,4\}, \{2,3\}]$$

```
> C3 := MDScov(B, Q0, ffaces(RL, TT));
```

$$C3 := [\{1,2,5\}, \{3,4,5\}, \{1,4\}, \{2,3\}]$$

Procedure A.3.5 (`MDSambtorvar`)**.** Implements Algorithm 2.3.9.

Input: an MDS X.

Output: a FAN or a list of FANs if `'completions'` was given.

Options:

- `'completions'`: return a list of FANs representing all possible completions for projective X as in Algorithm 2.3.9.
- `'nocheck'`: do not test whether X is projective.
- `'CONEs'`: return lists of maximal cones, i.e., lists of CONEs, instead of FANs.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

```
> Z := MDSambtorvar(X);
```

$$Z := FAN(3,0,[0,2,2])$$

```
> ZL := MDSambtorvar(X, 'completions');
```

$$ZL := [FAN(3,0,[0,0,6]),\ FAN(3,0,[0,0,5]),\ FAN(3,0,[0,0,6])]$$

```
> map(iscomplete, ZL); ZL[1] &>= Z; ZL[2] &>= Z; ZL[3] &>= Z;
```

$$[true, true, true]$$
$$true$$
$$true$$
$$true$$

Procedure A.3.6 (`MDSchambers`)**.** Implements Algorithm 3.2.11 to compute the Mori chamber decomposition; compare also [70].

Input: an MDS $X = (R, \Phi)$ with $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

Output: the Mori chamber decomposition as a list of maximal CONEs or a FAN if the option `'FAN'` was used.

Options: `'FAN'`: return a FAN instead of a list of its maximal CONEs.

Example: consider the MDS `X` as in Procedure A.3.1:

```
> MDSchambers(X, 'FAN');
```

$$FAN(3,0,[0,0,37])$$

Procedure A.3.7 (`MDSdim`). Implements Algorithm 2.3.4.
Input: an MDS $X = (R, \Phi)$.
Output: an integer $d \in \mathbb{Z}_{\geq 0}$ such that $\dim(X) = d$.
Example: consider the MDS `X` as in the example of Procedure A.3.1:

```
> MDSdim(X);
```

$$2$$

Procedure A.3.8 (`MDSpointex`). Implements Algorithm 2.3.8.
Input: an MDS X and a vector $z \in \mathbb{K}^r$.
Output: *true* if $z \in \widehat{X}$ and *false* otherwise. This means $[z] \in X$.
Example: consider the MDS `X` as in the example of Procedure A.3.1:

```
> z := [1,1,-1,1,0];
```

$$z := [1, 1, -1, 1, 0]$$

```
> MDSpointex(X, z);
```

$$true$$

Procedure A.3.9 (`MDSirrel`). Implements Algorithm 2.3.11.
Input: an MDS $X = (R, \Phi)$.
Output: a list of generators $g_1, \ldots, g_n \in \mathbb{K}[T_1, \ldots, T_r]$ for the vanishing ideal of $\overline{X} \setminus \widehat{X}$ in $\mathbb{K}^r$.
Example: consider the MDS `X` as in Procedure A.3.1:

```
> MDSirrel(X);
```

$$\left[T[2]T[3],\ T[1]T[4],\ T[3]T[4]T[5],\ T[1]T[2]T[5],\ T[1]T[2] + T[3]T[4] + T[5]^2\right]$$

Procedure A.3.10 (`MDSstrat`). Implements Algorithm 2.3.39.
Input: an MDS $X = (R, \Phi)$. Optional: a relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$.
Output: Depending on the input type:

- If a second parameter γ_0 was given: computes a list of generators $G_{\gamma_0} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ for the ideal $I_{\gamma_0} \subseteq \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ of the stratum $X(\gamma_0) \subseteq X$. Returns the pair $[G_{\gamma_0}, [T_1, \ldots, T_n]]$.
- If no second parameter was given: generators $G_{\gamma_0} \subseteq \mathbb{K}[T_1, \ldots, T_r]$ for the ideals $I_{\gamma_0} \subseteq \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ of all strata $X(\gamma_0)$ are returned as a pair $[L, [T_1, \ldots, T_n]]$ where the list L consists of all pairs $[\gamma_0, G_{\gamma_0}]]$ with γ_0 running through $\mathrm{rlv}(X)$.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

```
> gam0 := {2, 3, 4, 5};
```

$$gam0 := \{2, 3, 4, 5\}$$

```
> MDSstrat(X, gam0);
```

$$\left[[T[2] + T[3]^2], [T[1], T[2], T[3]]\right]$$

```
> MDSstrat(X);
```

$$\Big[\Big[[3, 4, 5, [T[2] + T[3]^2]],\ [1, 3, 4, 5, [T[2] + T[3]^2]],\ [1, 2, 4, 5, [T[1] + T[3]^2]],$$
$$[1, 2, 5, [T[1] + T[3]^2]],\ [1, 2, 3, 5, [T[1] + T[3]^2]],\ [1, 2, 3, 4, 5, [T[1] + T[2] + T[3]^2]],$$
$$[2, 3, 4, 5, [T[2] + T[3]^2]],\ [2, 3, []],\ [1, 4, []],\ [1, 2, 3, 4, [T[1] + T[2]]]\Big],\ [T[1], T[2], T[3]]\Big]$$

Procedure A.3.11 (`MDSdegmat`). This is a special version of Procedure A.1.23. First, A.1.23 will be applied to the P-matrix of the given MDS X to obtain an AGH $Q' = (\mathbb{Z}^r, K', A)$ where

$$Q' \colon \mathbb{Z}^r \ \to \ K', \qquad K' \ := \ \mathbb{Z}^d \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/a_i\mathbb{Z}$$

and the columns of A represent the degrees $\deg(T_i) \in K'$. Afterwards, A is returned together with the integers $a_1, \ldots, a_k$.

Input: an MDS $X = (R, \Phi)$ or a GR R where $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$.

Output: a pair $[A, [a_1, \ldots, a_k]]$ with an integral matrix A and $a_i \in \mathbb{Z}$. The entries of the last k rows of A must be interpreted as elements of $\mathbb{Z}/a_i\mathbb{Z}$; for this, Algorithm 2.1.24 is used.

Example: `> A := linalg[matrix]([[ 1, 0, 1],[ 1, 1, 0]]);`

$$A := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

`> Q := createAGH(createAG(3), createAG(1, [3]), A);`

$$Q := AGH([3, []], [1, [3]])$$

`> R := createGR([], vars(3), [Q]);`

$$R := GR(3, 0, [1, [3]])$$

`> X := createMDS(R, [1]);`

$$X := MDS(3, 0, 2, [1, [3]])$$

`> MDSdegmat(X); #` the last row must be interpreted as elements of $\mathbb{Z}/3\mathbb{Z}$.

$$\left[\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, [3] \right]$$

Procedure A.3.12 (`MDSintersgraph`). Implements Algorithm 2.3.27.

Input: an MDS $X = (R, \Phi)$ of dimension two.

Output: the graph of exceptional curves G_X (without intersection numbers).

Options: `'latex'`: prints LaTeX code to draw G_X. Gray vertices stand for negative curves and black vertices for negative curves which are incident with at least three other curves. Also, the non-negative curves among the $V(X;\, T_i)$ are drawn in white.

Example: consider the MDS `X` as in Procedure A.3.1:

`> G := MDSintersgraph(X); #` also draws a representation of the graph.

$$G := G$$

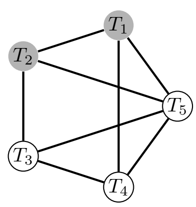
`> networks[vertices](G); #` shows the names of all vertices of G.

$$\{"T1", "T2"\}$$

`> networks[edges]("T1", "T2", G); #` there is an edge between the vertices representing $V(T_1)$ and $V(T_2)$

$$\{e1\}$$

`> MDSintersgraph(X, 'latex'); #` prints code for G_X



Procedure A.3.13 (`MDSsample`). Implements Algorithm 2.3.15.

Input: an MDS X.

Output: the CONE SAmple(X) in the vector space $K_{\mathbb{Q}}$.

Example: consider the MDS `X` defined in Procedure A.3.1:

```
> c := MDSsample(X); rays(c);
```

$$c := CONE(2, 2, 0, 2, 2)$$
$$[[1, 1], [-1, 1]]$$

Procedure A.3.14 (`MDSeff`). Implements Algorithm 2.3.13.

Input: an MDS $X = (R, \Phi)$.

Output: the CONE Eff(X) in the vector space $K_{\mathbb{Q}}$.

Example: consider the MDS `X` as in Procedure A.3.1:

```
> c := MDSeff(X); rays(c);
```

$$CONE(2, 2, 0, 2, 2)$$
$$[[-2, 1], [2, 1]]$$

Procedure A.3.15 (`MDSmov`). Implements Algorithm 2.3.14.

Input: there are three input possibilities:

- An MDS $X = (R, \Phi)$.
- A GR R where the matrix Q^0 has columns $q_1, \ldots, q_r \in K_{\mathbb{Q}}$.
- An integral matrix with columns $q_1, \ldots, q_r$.

Output: the CONE $\mathrm{Mov}(X) \subseteq K_{\mathbb{Q}}$ or, for the other input types, the CONE

$$\bigcap_{i=1}^{r} \mathrm{cone}\,(q_j;\ j \neq i) \quad \subseteq \quad K_{\mathbb{Q}}.$$

Example: consider the MDS `X` and the matrix Q^0 in the example of Procedure A.3.1:

```
> c1 := MDSmov(X); rays(c1);
```

$$c1 := CONE(2, 2, 0, 2, 2)$$
$$[[-1, 1], [1, 1]]$$

```
> c2 := MDSmov(Q0); c1 &= c2;
```

$$c2 := CONE(2, 2, 0, 2, 2)$$
$$true$$

Procedure A.3.16 (`MDSclassgrp`). Returns the divisor class group.

Input: an MDS X.

Output: an AG representing the class group Cl(X).

Example: consider the MDS `X` defined in Procedure A.3.1:

```
> MDSclassgrp(X);
```

$$AG(2, [])$$

Procedure A.3.17 (`MDSlocclassgrp`). Implements Algorithm 2.3.18.

Input: an MDS $X = (R, \Phi)$ and a point $x \in X$ which is given either in Cox coordinates $z \in \mathbb{K}^r$ or as a relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $x \in X(\gamma_0)$.

Output: an AG representing the local class group Cl(X, x).

Example: we computationally verify part of [5, Ex. III.3.3.5]:

```
> RL := [T[1]*T[2] + T[3]^2 + T[4]*T[5]];
```

$$RL := [T[1]T[2] + T[3]^2 + T[4]T[5]]$$

```
> TT := vars(5);
```

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

```
> Q0 := linalg[matrix]([[1, -1, 0, -1, 1], [1, 1, 1, 0, 2]]);
```

$$Q0 := \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

```
> R := createGR(RL, TT, [Q0]);
```

$$R := GR(5, 1, [2, []])$$

```
> w := [0,3];
```

$$w := [0, 3]$$

```
> Y := createMDS(R, w);
```

$$Y := MDS(5, 1, 2, [2, []])$$

> `x0 := [0,1,0,0,1];` # Cox coordinates for a point in $X(\mathrm{cone}(e_2, e_5))$.

$$x0 := [0, 1, 0, 0, 1]$$

> `MDSlocclassgrp(Y, x0);` # will be $\mathbb{Z}/3\mathbb{Z}$.

$$AG(0, [3])$$

```
> gam123 := {1,2,3};
```

$$gam123 := \{1, 2, 3\}$$

> `MDSlocclassgrp(Y, gam123);` # will be the trivial group:

$$AG(0, [])$$

Procedure A.3.18 (`MDSpic`)**.** Implements Algorithm 2.3.20.

Input: an MDS X.

Output: the AG $\mathrm{Pic}(X)$ as a subgroup of $\mathrm{Cl}(X)$.

Example: consider the MDS `X` as in Procedure A.3.1:

```
> Pic := MDSpic(X); AGdata(Pic);
```

$$Pic := AG(2, [])$$

$$\left[\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 2, [] \right]$$

Procedure A.3.19 (`MDSpicind`)**.** Implements Algorithm 2.3.21.

Input: an MDS X.

Output: the Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$.

Example: consider the MDS `X` as in Procedure A.3.1:

```
> MDSpicind(X);
```

$$18$$

Procedure A.3.20 (`MDSantican`)**.** Implements Algorithm 2.3.41.

Input: an MDS $X = (R, \Phi)$ such that R is a complete intersection.

Output: a vector $w \in \mathbb{Z}^r$. If $K = (U, L)$ is the grading group then the anticanonical divisor class $-w_X^{\mathrm{can}} \in K$ satisfies $-w_X^{\mathrm{can}} = w + \mathrm{lin}_{\mathbb{Z}}(L)$.

Example: consider the MDS `X` as in Procedure A.3.1:

`> MDSantican(X); #` represents an element of $K = \mathbb{Z}^2$:

$$[0, 3]$$

Procedure A.3.21 (`MDSisgorenstein`)**.** Implements Algorithm 2.3.44.

Input: an MDS $X = (R, \Phi)$ such that R is a complete intersection.

Output: *true* if X is Gorenstein and *false* otherwise.

Example: let `Y` be as in the example of Procedure A.3.17. We computationally verify part of [5, Ex. III.3.3.5]:

`> MDSisgorenstein(Y);`

$$true$$

Procedure A.3.22 (`MDSisQgorenstein`)**.** Implements Algorithm 2.3.43.

Input: an MDS $X = (R, \Phi)$ such that R is a complete intersection.

Output: *true* if X is $\mathbb{Q}$-Gorenstein and *false* otherwise.

Example: let `Y` be as in the example of Procedure A.3.17. We computationally verify part of [5, Ex. III.3.3.5]:

`> MDSisQgorenstein(Y); #` since $(0, 5) - (0, 2) \in Q(\mathrm{cone}(e_2, e_5))$:

$$true$$

Procedure A.3.23 (`MDSgorensteinind`)**.** Implements Algorithm 2.3.45.

Input: an MDS $X = (R, \Phi)$ such that R is a complete intersection.

Output: *true* if X is $\mathbb{Q}$-Gorenstein and *false* otherwise.

Example: let `Y` be as in the example of Procedure A.3.17. We computationally verify part of [5, Ex. III.3.3.5].

`> MDSgorensteinind(Y); #` Y is Gorenstein:

$$1$$

Procedure A.3.24 (`MDSisfano`)**.** Implements Algorithm 2.3.46.

Input: an MDS $X = (R, \Phi)$ such that R is a complete intersection.

Output: *true* if X is Fano and *false* otherwise.

Example: let `Y` be as in the example of Procedure A.3.17. We computationally verify part of [5, Ex. III.3.3.5]:

`> MDSisfano(Y); #` since $(0, 5) - (0, 2) \in \vartheta^\circ$ for each $\vartheta \in \Phi$:

$$true$$

Procedure A.3.25 (`MDSintersno`)**.** Implements Algorithm 2.3.48.

Input: a quasiprojective MDS $X = (R, \Phi)$ with R having a principal ideal of relations and two elements $w, w' \in K$.

Output: the intersection number $D \cdot D' \in \mathbb{Q}$ where D, D' are divisors on X with classes $[D] = w$ and $[D'] = w'$.

Options: `'allself'`: return the list of all $V(X; T_i)^2$.

Example: let `Y` be as in the example of Procedure A.3.17. We computationally verify [5, Ex. III.3.3.5]:

`> w := -MDSantican(Y);`

$$w := [0, -3]$$

`> n := MDSintersno(Y, w, w);` # the self-intersection number $(w_Y^{\mathrm{can}})^2$

$$n := 6$$

Procedure A.3.26 (`MDSisfact`). Implements Algorithm 2.3.30.

Input: an MDS $X = (R, \Phi)$. Optional second parameter: a point $x \in X$ given in Cox coordinates $z \in \mathbb{K}^r$ or as a relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $x \in X(\gamma_0)$.

Output: if only X was given: *true* if X is factorial and *false* otherwise. If a point $x \in X$ was given: *true* if $x \in X$ is factorial and *false* otherwise.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

`> MDSisfact(X);` # since $Q(\mathrm{lin}_{\mathbb{Z}}(\mathrm{cone}(e_1, e_2, e_5) \cap \mathbb{Z}^8)) \neq \mathbb{Z}^2$:

$$false$$

`> gam0 := {1,2,5};`

$$gam0 := \{1, 2, 5\}$$

`> MDSisfact(X, gam0);` # no point $x \in X(\gamma_0)$ is factorial:

$$false$$

`> MDSisfact(X, [-1,1,0,0,1]);` # the point $[-1, 1, 0, 0, 1] \in X$ is not factorial:

$$false$$

Procedure A.3.27 (`MDSisQfact`). Implements Algorithm 2.3.31.

Input: an MDS $X = (R, \Phi)$.

Output: *true* if X is $\mathbb{Q}$-factorial and *false* otherwise.

Example: consider the MDS `X` as defined in Procedure A.3.1:

`> MDSisQfact(X);` # all cones of Φ are full-dimensional.

$$true$$

Procedure A.3.28 (`MDSisquasismooth`). Implements Algorithm 2.3.23.

Input: an MDS $X = (R, \Phi)$. Optional second parameter: a point $x \in X$ given in Cox coordinates $z \in \mathbb{K}^r$ or as a relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $x \in X(\gamma_0)$.

Output: if only X was given: *true* if $\widehat{X}$ is smooth and *false* otherwise. If also a point $x \in X$ was given, *true* if $x \in \widehat{X}^{\mathrm{reg}}$ and *false* otherwise.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

`> MDSisquasismooth(X);` # $\widehat{X}$ is smooth since $\overline{X}^{\mathrm{sing}} = \{0\}$ in $\mathbb{K}^5$:

$$true$$

`> MDSisquasismooth(X, {2,3});` # this means $\overline{X(\mathrm{cone}(e_2, e_3))} \cap \widehat{X}$ is smooth.

$$true$$

`> MDSisquasismooth(X, [-2,1,1,1,1]);` # $(-2, 1, 1, 1, 1) \in \widehat{X}$ is smooth.

$$true$$

Procedure A.3.29 (`MDSissmooth`). Implements Algorithm 2.3.24.

Input: an MDS $X = (R, \Phi)$. Optional second parameter: a point $x \in X$ given in Cox coordinates $z \in \mathbb{K}^r$ or as a relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ such that $x \in X(\gamma_0)$.

Output: if only X was given: *true* if X is smooth and *false* otherwise. If a point $x \in X$ was given, *true* if $x \in X$ is smooth and *false* otherwise.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

`> MDSissmooth(X); #` $\widehat{X}$ is smooth, the toric ambient variety is singular:

$$false$$

`> MDSissmooth(X, {2,3}); #` the stratum $X(\mathrm{cone}(e_2, e_3))$ is singular:

$$false$$

`> MDSissmooth(X, [-2,1,1,1,1]); #` the point $[-2,1,1,1,1] \in X$ is smooth:

$$true$$

Procedure A.3.30 (`MDSsing`). Implements Algorithm 2.3.25.

Input: an MDS X.

Output: a list of lists $[[J, [T_1, \dots, T_r]], F]$ where

- J is a list of polynomials in $\mathbb{K}[T_1, \dots, T_r]$ such that the vanishing set $V(J) \subseteq \mathbb{K}^r$ equals $\overline{X}^{\mathrm{sing}}$,
- F is a list of all relevant $\mathfrak{F}$-faces such that $X(\gamma_0)$ is singular.

Example: `> RL := [2*T[1]^3*T[2]^2 + T[3]^3*T[4]^2 + T[5]^3*T[6]^2];`

$$RL := \left[2T[1]^3T[2]^2 + T[3]^3T[4]^2 + T[5]^3T[6]^2\right]$$

`> TT := vars(6);`

$$TT := [T[1], T[2], T[3], T[4], T[5], T[6]]$$

`> Q0 := linalg[matrix]([[ 0, 1, 0, 1, 0, 1 ],[ 1, 0, 1, 0, 1, 0 ]]);`

$$Q0 := \left[\begin{array}{cccccc} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

`> R := createGR(RL, TT, [Q0]);`

$$R := GR(6, 1, [2, []])$$

`> Y := createMDS(R, [2,3]);`

$$Y := MDS(6, 1, 3, [2, []])$$

`> MDSsing(Y); #` both $\overline{X}$ and $\widehat{X}$ are singular

$$\Big[\big[[6T[1]^2T[2]^2,\ 4T[1]^3T[2],\ 3T[3]^2T[4]^2,\ 2T[3]^3T[4],\ 3T[5]^2T[6]^2,\ 2T[5]^3T[6],$$
$$2T[1]^3T[2]^2 + T[3]^3T[4]^2 + T[5]^3T[6]^2],\ [T[1], T[2], T[3], T[4], T[5], T[6]]\big],$$
$$[\{1,6\},\ \{3,6\},\ \{1,3,6\},\ \{2,3,6\},\ \{1,4,6\},\ \{2,5\},$$
$$\{2,3,5\},\ \{4,5\},\ \{1,4,5\},\ \{2,4,5\},\ \{2,3\}, \{1,4\}]\Big]$$

Procedure A.3.31 (`MDSisquasiproj`). Implements Algorithm 2.3.35.

Input: an MDS $X = (R, \Phi)$.

Output: *true* if X is quasiprojective and *false* otherwise.

Example: consider the MDS `X` as in Procedure A.3.1:

`> MDSisquasiproj(X); #` Φ was defined by a vector $w \in K_{\mathbb{Q}}$:

$$true$$

Procedure A.3.32 (`MDSisproj`). Implements Algorithm 2.3.36.

Input: an MDS X.

Output: *true* if X is projective and *false* otherwise.

Example: consider the MDS `X` as in the example of Procedure A.3.1:

`> MDSisproj(X); #` X is quasiprojective and the grading is pointed

$$true$$

Procedure A.3.33 (`MDSiscomplete`). Implements Algorithm 2.3.33.

Input: an MDS $X = (R, \Phi)$ where the ideal of relations of R is principal.

Output: *true* if X is complete and *false* otherwise.

Example: consider the MDS `X` as in Procedure A.3.1:

`> MDSiscomplete(X); #` $|\Sigma| \supseteq \mathrm{trop}(f)$ where Z_Σ is as in Construction 1.3.12.

$$true$$

4. Procedures on complexity-one T-varieties

In this section, we describe our implementation of algorithms for complexity-one T-varieties. See Section 4 of Chapter 2 for the algorithms. Here is an overview:

- *Automorphisms:* horizontal and vertical Demazure P-roots (Procedure A.4.1 and A.4.2), roots of $\mathrm{Aut}(X)^0$ (Procedure A.4.3).
- *Singularities:* resolution of singularities (Procedure A.4.4).
- *Anticanonical complex:* anticanonical polytope (Procedure A.4.5), anticanonical complex (Procedure A.4.6), test for being (ϵ-log-) terminal (Procedures A.4.7 and A.4.8).

In this section, we call an MDS $X = (R, \Phi)$ or also its GR R *of complexity one* if X is a complexity-one T-variety with Cox ring $R = R(P, A)$ as in Construction 1.5.3. This means the GR R was obtained by a call to Procedure A.2.3 with input P and A.

Procedure A.4.1 (`MDShdemazure`). Implements Algorithm 2.4.2.

Input: there are three input types:

- An MDS $X = (R, \Phi)$ of complexity one.
- A GR R of complexity one.
- Integral matrices P and A as in Construction 1.5.2. If instead of A the number of blocks of P is given, the procedure chooses A.

In the first two cases, we require that $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ has been entered as `createGR(P,A)`.

Output: a list of all horizontal Demazure P-roots of $R = R(P, A)$.

Example: we computationally verify [6, Ex. 5.3]:

```
> P := cols2matrix([[-1,-1,-1],[-3,-3,-2],[3,0,1],[0,2,1]]);
```

$$P := \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}$$

```
> A := [[0,1],[-1,-1],[1,0]];
```

$$A := [[0, 1], [-1, -1], [1, 0]];$$

```
> R := createGR(P, A);
```

$$R := GR(4, 1, [1, []])$$

```
> HDEM := MDShdemazure(R);
```

$$HDEM := [[[[-1, -2, 3]], 2, 3, [1, 1, 1]]]$$

Let us consider another example and input type:

```
> P := cols2matrix([[-1,-1,-1],[-3,-3,-2],[2,0,1],[0,1,1],
[0,0,1],[0,0,-1]]);
```

$$P := \begin{bmatrix} -1 & -3 & 2 & 0 & 0 & 0 \\ -1 & -3 & 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 1 & 1 & -1 \end{bmatrix}$$

`> HDEM := MDShdemazure(P, 3); #` there are three blocks in P; no horizontal Demazure P-root exists:

$$HDEM := [[[], 1, 2, [1, 1, 1]], [[], 1, 2, [2, 1, 1]], [[], 2, 3, [1, 1, 1]]]$$

Procedure A.4.2 (`MDSvdemazure`). Implements Algorithm 2.4.1.

Input: there are three input types:

- An MDS $X = (R, \Phi)$ of complexity one.
- A GR R of complexity one.
- Integral matrices P and A as in Construction 1.5.2. If instead of A the number of blocks of P is given, the procedure chooses A.

In the first two cases, we require that $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ has been entered as `createGR(P,A)`.

Output: a list of all vertical Demazure P-roots of $R = R(P, A)$.

Example: as in Procedure A.4.1, we continue the verification [6, Ex. 5.3]:

```
> P := cols2matrix([[-1,-1,-1],[-3,-3,-2],[3,0,1],[0,2,1]]);
```

$$P := \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix}$$

```
> A := [[0,1],[-1,-1],[1,0]];
```

$$A := [[0, 1], [-1, -1], [1, 0]];$$

```
> R := createGR(P, A);
```

$$R := GR(4, 1, [1, []])$$

`> VDEM := MDSvdemazure(R); #` There are no vertical P-roots:

$$VDEM := [[[], 0]]$$

In the next example there are vertical P-roots. Note that we use another input type:

```
> P := cols2matrix([[-1,-1,-1],[-3,-3,-2],[2,0,1],[0,1,1],
[0,0,1],[0,0,-1]]);
```

$$P := \begin{bmatrix} -1 & -3 & 2 & 0 & 0 & 0 \\ -1 & -3 & 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 1 & 1 & -1 \end{bmatrix}$$

`> VDEM := MDSvdemazure(P, 3); #` there are three blocks in P; $(v, 2)$ with $v = (0, -1, 1)$ is the only vertical Demazure P-root:

$$VDEM := \left[[[], 1], [[[0, -1, 1]], 2]\right]$$

Procedure A.4.3 (`MDSautroots`). Implements Algorithm 2.4.6.

Input: there are two input types:

- An MDS $X = (R, \Phi)$ of complexity one.
- A GR R of complexity one.

In both cases, the GR $R = (G, Q, Q^0, P, F_{\mathfrak{F}})$ must have been obtained from Procedure A.2.3 as `createGR(P,A)`.

Output: the roots of the unit component $\mathrm{Aut}(X)^0$ as a set of integral vectors. These are the P-roots of X.

Example: we verify Example 2.4.7:

```
> P := cols2matrix([[-2,-2,-1,-1], [1,0,0,0], [1,0,1,0], [0,1,0,1],
[0,1,0,0]]);
```

$$P := \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> A := [[1,0],[0,1],[-1,-1]];
```

$$A := [[1,0],[0,1],[-1,-1]]$$

```
> R := createGR(P, A);
```

$$R := GR(5,1,[1,[]])$$

```
> Roots := autoroots(R); # root system B2:
```

$$Roots := \{[1,-1],[1,1],[-1,-1],[-1,1],[0,-1],[0,1],[1,0],[-1,0]\}$$

Procedure A.4.4 (`MDSresolvesing`). Implements Algorithm 2.4.8.

Input: there are two input types:

- An MDS $X = (R,\Phi)$ of complexity one, i.e., $R = R(P,A)$ has been entered as `createGR(P,A)`.
- An MDS $X = (R,\Phi)$ where the ideal of relations of R is principal.

Output: a pair $Y = (R',\Phi')$. If X is of complexity one or if in the second case the `'verify'`-tests succeeded, Y is a smooth MDS such that $Y \to X$ is a resolution of singularities.

Options:

- `'verify'`: tries to verify that Y is a smooth MDS; this is not needed if X is of complexity one.
- `'minimal'`: compute a minimal resolution if X is a surface.
- `'noffaces'`: skip the computation of $\mathfrak{F}$-faces.
- `'noMDS'`: do not return a data type MDS but only a list of generators for the defining ideal, a list of variables and the new matrix P'; this usually is much quicker.

Example: we algorithmically verify [5, Ex. III.4.4.10]:

```
> A := [[0,1],[-1,-1],[1,0]];
```

$$A := [[0,1],[-1,-1],[1,0]];$$

```
> P := linalg[matrix]([[-3,-1,3,0],[-3,-1,0,2],[-2,-1,1,1]]);
```

$$P := \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ -3 & -1 & 1 & 1 \end{bmatrix}$$

```
> R := createGR(P, A); GRdata(R);
```

$$R := GR(4,1,[1,[]])$$

$$\left[\left[T[4]^2 + T[3]^3 + T[1]T[2]\right], [T[1],T[2],T[3],T[4]], [AGH([4,[]],[1,[]]), \begin{bmatrix} 1 & 3 & 2 & 3 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ -3 & -1 & 1 & 1 \end{bmatrix}, [\{\},\{1,2,4\},\{1,3,4\},\{1,2,3\},\{1\},\{2\},\{3,4\},\{1,2,3,4\},\{2,3,4\}]\right]$$

```
> w := relint(MDSmov(R));
```

$$w := [1]$$

```
> X := createMDS(R, w);
```

$$X := MDS(4,1,2,[1,[]])$$

```
> Y := MDSresolvesing(X, 'noffaces');
```

$$Y := MDS(13,1,2,[10,[]])$$

```
> GRdata(MDSdata(Y)[1])[1]; # print the defining equation:
```

$$\left[T[5]T[8]^3T[10]T[11]^2 + T[2]T[4]^2T[6]T[9]^3 + T[7]T[12]^2T[13]\right]$$

Procedure A.4.5 (`MDSanticanpoly`). Implements Algorithm 2.4.13.

Input: an MDS $X = (R, \Phi)$ of complexity one, i.e., $R = R(P, A)$ has been entered as `createGR(P,A)`.

Output: the anticanonical polytope A_X.

Example: we enter the E_6-singular cubic, see [61, Ex. 7.3]:

```
> P := cols2matrix([[-3,-3,-2],[-1,-1,-1],[2,0,1],[0,3,1]]);
```

$$P := \begin{bmatrix} -3 & -1 & 2 & 0 \\ -3 & -1 & 0 & 3 \\ -2 & -1 & 1 & 1 \end{bmatrix}$$

```
> A := [[-1,-1],[1,0],[0,1]]
```

$$A := [[-1, -1], [1, 0], [0, 1]]$$

```
> R := createGR(P, A);
```

$$R := GR(4, 1, [1, []])$$

```
> X := createMDS(R, relint(MDSmov(R)));
```

$$X := MDS(4, 1, 2, [1, []])$$

```
> AX := MDSanticanpoly(X);
```

$$AX := POLYTOPE(3, 3, 6, 8)$$

```
> vertices(AX);
```

$$[[-3, -3, -2], [-1, -1, -1], [2, 0, 1], [0, 3, 1], [0, 0, -1/5], [0, 0, 1]]$$

Procedure A.4.6 (`MDSanticancomp`). Implements Algorithm 2.4.14.

Input: an MDS $X = (R, \Phi)$ of complexity one, i.e., $R = R(P, A)$ has been entered as `createGR(P,A)`.

Output: the anticanonical complex $\mathcal{A}_X$ of X.

Example: enter the example of Procedure A.4.5. Then:

```
> AXC := MDSanticancomp(X);
```

$$AXC := PCOMPLEX(3, [0, 0, 7, 0])$$

```
> vertices(AXC);
```

$$\{[0, 0, -1/5], [0, 0, 0], [-3, -3, -2], [-1, -1, -1], [2, 0, 1], [0, 3, 1], [0, 0, 1]\}$$

Procedure A.4.7 (`MDSisterminal`). Implements Algorithm 2.4.16.

Input: an MDS $X = (R, \Phi)$ of complexity one, i.e., $R = R(P, A)$ has been entered as `createGR(P,A)`.

Output: *true* if X is terminal and *false* otherwise.

Example: enter the example of Procedure A.4.5. Then:

```
> MDSisterminal(X);
```

$$false$$

Procedure A.4.8 (`MDSisepslogterminal`). Implements Algorithm 2.4.15.

Input: an MDS $X = (R, \Phi)$ of complexity one, i.e., $R = R(P, A)$ has been entered as `createGR(P,A)` and a rational number $0 < \varepsilon \le 1$.

Output: *true* if X is ε-log-terminal (for $\varepsilon < 1$) and *false* otherwise.

Example: we continue the example of Procedure A.4.5:

```
> MDSisepslogterminal(X, 1/2);
```

$$true$$

5. Miscellanea

This section contains information on several procedures that do not fit into one of the previous categories. Most of them have been described throughout Chapter 2. Here is an overview:

- *Algebraic operations:* closure computation (Procedure A.5.1), primality test (Procedure A.5.2), prime variables (Procedure A.5.3).
- *Polyhedral operations:* dualize fans or bunches of cones (Procedures A.5.5 and A.5.4), fiber polyhedron (Procedure A.5.6), lattice points or interior points of a polytope (Procedure A.5.7).
- *Modifying equations, etc.:* $\star$-pullback (Procedure A.5.8), $\star$-pushforward (Procedure A.5.9), variables (Procedure A.5.10).

Procedure A.5.1 (`closure`). Implements Algorithm 2.2.14.

Input: a list of polynomials $[f_1, \ldots, f_s]$ and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: a list $[[g_1, \ldots, g_m], [T_1, \ldots, T_r]]$ such that $g_i \in \mathbb{K}[T_1, \ldots, T_r]$ and the closure $\overline{V(\mathbb{T}^r;\, f_1, \ldots, f_s)}$ in $\mathbb{K}^r$ is given by $V(\mathbb{K}^r;\, g_1, \ldots, g_s)$.

Example: `> RL := [T[1]*T[10]-T[3]*T[7]+T[4]*T[6], T[1]*T[8]-T[2]*T[6]+T[3]*T[5], T[1]*T[9]-T[2]*T[7]+T[4]*T[5]];`

$$RL := [T_1T_{10} - T_3T_7 + T_4T_6,\, T_1T_8 - T_2T_6 + T_3T_5,\, T_1T_9 - T_2T_7 + T_4T_5]$$

`> TT := vars(10);`

$$TT := [T[1], T[2], T[3], T[4], T[5], T[6], T[7], T[8], T[9], T[10]]$$

`> closure(RL, TT);` # the affine cone over $G(2, 5)$.

$$\Big[[T_7T_8 - T_6T_9 + T_5T_{10},\, T_4T_8 - T_3T_9 + T_2T_{10},\, T_1T_{10} - T_3T_7 + T_4T_6,\\ T_1T_9 - T_2T_7 + T_4T_5,\, T_1T_8 - T_2T_6 + T_3T_5],\\ [T[1], T[2], T[3], T[4], T[5], T[6], T[7], T[8], T[9], T[10]]\Big]$$

Procedure A.5.2 (`isprimeideal`). Implements Algorithm 2.2.10 for the case of a free class group.

Input: a list $[f_1, \ldots, f_s]$ of polynomials and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{Q}[T_1, \ldots, T_r]$.

Output: *true* if the ideal $\langle f_1, \ldots, f_s\rangle \subseteq \overline{\mathbb{Q}}[T_1, \ldots, T_r]$ is prime and *false* otherwise.

Example: `> RL := [T[1]^2 +1];`

$$\left[T[1]^2 + 1\right]$$

`> isprimeideal(RL, vars(1));` # the ideal $\langle T_1^2 + 1\rangle \subseteq \overline{\mathbb{Q}}[T_1]$ is not prime:

$$false$$

Procedure A.5.3 (`primevars`). Successively applies Procedure A.5.2 to test whether T_i defines a prime element in $R = \mathbb{K}[T_1, \ldots, T_r]/\langle f_1, \ldots, f_s\rangle$.

Input: a list $[f_1, \ldots, f_s]$ of polynomials and a list of variables $[T_1, \ldots, T_r]$ such that $f_i \in \mathbb{Q}[T_1, \ldots, T_r]$.

Output: the set of all indices $1 \le i \le r$ such that T_i defines a prime element in R.

Example: `> RL := [T[1]*T[2] + T[2]*T[3]+T[4]*T[5]];`

$$RL := [T[1]T[2] + T[2]T[3] + T[4]T[5]]$$

`> primevars(RL, vars(5));` # In $\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_2T_3 + T_4T_5\rangle$ only T_1 and T_3 define prime elements:

$$\{1, 3\}$$

Procedure A.5.4 (`w2fan`). Consider a surjective integral $k \times r$ matrix Q^0 and a Gale dual matrix P of size $n \times r$. Let either a vector $w \in \mathrm{cone}(Q^0)$ or a BUN Φ in $\mathbb{Q}^k$ for the ring $\mathbb{K}[T_1, \dots, T_r]$ with grading $\deg(T_i) := Q(e_i)$ be given. In $\mathbb{Q}^n$ we have fans

$$\begin{aligned} \Sigma(w) &= \left\{P(\gamma_0^*);\ \gamma_0 \preceq \mathbb{Q}^r_{\geq 0} \text{ and } w \in \left(Q^0(\gamma_0)\right)^\circ\right\}, \\ \Sigma(\Phi) &= \left\{P(\gamma_0^*);\ \gamma_0 \preceq \mathbb{Q}^r_{\geq 0} \text{ and } Q^0(\gamma_0) \in \Phi\right\}. \end{aligned}$$

Input: in the above notation, there are three input types:

- A vector $w \in \mathbb{Q}^k$, a matrix Q^0, a matrix P.
- A CONE $\lambda \subseteq \mathbb{Q}^k$, a matrix Q^0, a matrix P. Then $w \in \lambda^\circ$ will be chosen.
- A BUN Φ in $\mathbb{Q}^k$, a matrix Q^0, a matrix P.

Output: a FAN $\Sigma \subseteq \mathbb{Q}^n$. For the first two input cases, we have $\Sigma = \Sigma(w)$ whereas $\Sigma = \Sigma(\Phi)$ holds for the third input case.

Options: `'CONEs'`: return a list of maximal CONEs instead of a FAN.

Example: in the following example, the fans `Sigw`, `Siglam` and `SigB` coincide.

```
> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0], [1, 1, 1, 1, 1]]);
```

$$Q0 := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> P := linalg[matrix]([[3, 1, -4, 0, 0], [2, 0, -3, 1, 0],
[1, 0, -2, 0, 1]]);
```

$$P := \begin{bmatrix} 3 & 1 & -4 & 0 & 0 \\ 2 & 0 & -3 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{bmatrix}$$

```
> w := [-1,2];
```

$$w := [-1, 2]$$

```
> Sigw := w2fan(w, Q0, P);
```

$$Sigw := FAN(3, 0, [0, 0, 6])$$

```
> lam := poshull([-1,1],[0,1]);
```

$$lam := CONE(2, 2, 0, 2, 2)$$

```
> Siglam := w2fan(lam, Q0, P);
```

$$Siglam := FAN(3, 0, [0, 0, 6])$$

```
> B := createBUN(w, [], vars(5), Q0);
```

$$B := BUN(8)$$

```
> SigB := w2fan(B, Q0, P);
```

$$SigB := FAN(3, 0, [0, 0, 6])$$

Procedure A.5.5 (`fan2w`). Let $\Sigma \subseteq \mathbb{Q}^n$ be a fan and P an integral $n \times r$ matrix such that the columns of P are pairwise different primitive generators for the rays of Σ and the columns of P generate $\mathbb{Q}^n$ as a cone. Let Q^0 be a matrix that describes the map $\mathbb{Z}^r \to K^0$ that is dual to the inclusion $\ker(P) \to \mathbb{Z}^r$. In $K^0_{\mathbb{Q}}$, we have a BUN $\Phi(\Sigma)$ in the ring $\mathbb{K}[T_1, \dots, T_r]$ that is graded by $\deg(T_i) := Q(e_i)$ where

$$\Phi(\Sigma) = \left\{Q(\delta_0^*);\ \delta_0 \preceq \mathbb{Q}^r_{\geq 0} \text{ and } P(\delta_0) \in \Sigma\right\}.$$

Input: a FAN $\Sigma \subseteq \mathbb{Q}^n$ as well as integral matrices P and Q^0 as explained above.

Output: the BUN $\Phi(\Sigma)$ in $K^0_{\mathbb{Q}}$.

Options: `'w'`: instead of a BUN return a vector $w \in \left(\bigcap_\vartheta \vartheta\right)^\circ$ where ϑ ranges over the elements of $\Phi(\Sigma)$.

Example: consider the setting of the example of Procedure A.5.4.

```
> w := fan2w(Sigw, P, Q0, 'w');
```

$$w := [-1, 2]$$

```
> B := fan2w(Sigw, P, Q0);
```

$$B := BUN(8)$$

Procedure A.5.6 (`fiberpoly`). Consider an integral surjective $k \times r$ matrix Q^0 and a vector w inside the cone over the columns of Q^0. The *fiber polyhedron* is

$$B_w \; := \; \left(Q^0\right)^{-1}(w) \; \cap \; \mathbb{Q}^r_{\geq 0} \; \subseteq \; \mathbb{Q}^r.$$

Input: there are two input types:

- A matrix Q^0 and a vector $w \in \mathbb{Q}^k$.
- A matrix Q^0 and a CONE $\lambda \subseteq \mathbb{Q}^k$. Here, $w \in \lambda^\circ$ will be chosen.

In both cases, a Gale dual matrix P of Q^0 may be given as a third parameter.

Output: the POLYHEDRON $B_w \subseteq \mathbb{Q}^r$. If a third parameter P was given then $(P^*)^{-1}(B_w) \subseteq \mathbb{Q}^n$ will be returned.

Example:
```
> Q0 := linalg[matrix]([[-2, 2, -1, 1, 0],
[1, 1, 1, 1, 1]]);
```

$$Q0 := \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

```
> P := linalg[matrix]([[3, 1, -4, 0, 0], [2, 0, -3, 1, 0],
[1, 0, -2, 0, 1]]);
```

$$P := \begin{bmatrix} 3 & 1 & -4 & 0 & 0 \\ 2 & 0 & -3 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{bmatrix}$$

```
> w := [-1,2];
```

$$w := [-1, 2]$$

```
> Bw := fiberpoly(Q0, w);
```

$$Bw := POLYTOPE(5, 3, 6, 5)$$

```
> MBw := fiberpoly(Q0, w, P);
```

$$MBw := POLYTOPE(3, 3, 6, 5)$$

Procedure A.5.7 (`intpoints`). Implements Algorithm 2.2.2.

Input: a POLYTOPE $B \subseteq \mathbb{Q}^r$.

Output: `'relint'`: return the list of elements of $B^\circ \cap \mathbb{Z}^r$ instead of $B \cap \mathbb{Z}^r$.

Options: a list of the elements of $B \cap \mathbb{Z}^r$.

Example:
```
> B := cube(3);
```

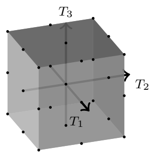
$$B := POLYTOPE(3, 3, 8, 6)$$

```
> intpoints(B);
```

$$[[-1,-1,-1], [-1,-1,0], [-1,-1,1], [-1,0,-1], [-1,0,0], [-1,0,1], [-1,1,-1], [-1,1,0], [-1,1,1], [0,-1,-1], [0,-1,0], [0,-1,1], [0,0,-1], [0,0,0], [0,0,1], [0,1,-1], [0,1,0], [0,1,1], [1,-1,-1], [1,-1,0], [1,-1,1], [1,0,-1], [1,0,0], [1,0,1], [1,1,-1], [1,1,0], [1,1,1]]$$

```
> intpoints(B, 'relint');
```

$$[[0, 0, 0]]$$



Procedure A.5.8 (`pull`). Implements Algorithm 2.2.12.

Input: there are two input types:

- A list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_n]$ with $f_i \in \mathbb{K}[T_1, \ldots, T_n]$, an integral $n \times r$ matrix P.
- A polynomial f_1, a list of variables $[T_1, \ldots, T_n]$ with $f_1 \in \mathbb{K}[T_1, \ldots, T_n]$, an integral $n \times r$ matrix P.

Output: a list $[G, [T_1, \ldots, T_r]]$ where

- for the first input type, $G = [p^\star f_1, \ldots, p^\star f_s]$ with $p^\star f_i \in \mathbb{K}[T_1, \ldots, T_r]$,
- for the second input type, $G = p^\star f_1 \in \mathbb{K}[T_1, \ldots, T_r]$.

Example:
```
> f := T[1]*T[2] + 3*T[3]^2;
```
$$f := T[1]T[2] + 3T[3]^2$$
```
> TT := vars(3);
```
$$TT := [T[1], T[2], T[3]]$$
```
> P := linalg[matrix]([[1, 0, 3, 0], [0, 1, 0, 4], [1, 1, 1, 1]]);
```
$$P := \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
```
> pull(f, TT, P);
```
$$\left[T_1T_3^3T_2T_4^4 + 3T_1^2T_2^2T_3^2T_4^2,\ [T[1], T[2], T[3], T[4]]\right]$$
```
> pull([T[1] + T[2], T[2] + T[3]], TT, P);
```
$$\left[[T_1T_3^3 + T_2T_4^4, T_2T_4^4 - 3T_1T_2T_3T_4], [T[1], T[2], T[3], T[4]]\right]$$

Procedure A.5.9 (`push`). Implements Algorithm 2.2.13.

Input: there are two input types:

- A list of polynomials $[f_1, \ldots, f_s]$, a list of variables $[T_1, \ldots, T_r]$ with $f_i \in \mathbb{K}[T_1, \ldots, T_r]$, an integral $n \times r$ matrix P of rank n.
- A polynomial f_1, a list of variables $[T_1, \ldots, T_r]$ with $f_1 \in \mathbb{K}[T_1, \ldots, T_r]$, an integral $n \times r$ matrix P of rank n.

We require the polynomials f_i to be K-homogeneous with respect to the $K := \mathbb{Z}^r/\mathrm{Im}(P^*)$-grading $\deg(T_i) = Q(e_i)$ where $Q\colon \mathbb{Z}^r \to K$ is as in Algorithm 2.1.24.

Output: a list $[G, [T_1, \ldots, T_n]]$ where

- for the first input case, $G = [p_\star f_1, \ldots, p_\star f_s]$ with $p_\star f_i \in \mathbb{K}[T_1, \ldots, T_n]$,
- for the second input case, $G = p_\star f_1 \in \mathbb{K}[T_1, \ldots, T_n]$.

Example:
```
> f1 := T[1]*T[2] + T[3]*T[5];
```
$$f1 := T[1]T[2] + T[3]T[5]$$
```
> f2 := T[1]^2 + T[2]^2 + T[3]*T[4]*T[5]^2;
```
$$f2 := T[1]^2 + T[2]^2 + T[3]T[4]T[5]^2$$

```
> TT := vars(5);
```

$$TT := [T[1], T[2], T[3], T[4], T[5]]$$

```
> P := linalg[matrix]([[-1,1,0,0,0],[2,0,-1,1,0],[-2,0,1,0,1]]);
```

$$P := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ -2 & 0 & 1 & 0 & 1 \end{bmatrix}$$

```
> push(f1, TT, P);
```

$$\left[T[1]+T[3], 1+T[1]^2+T[2]T[3]^2,\ [T[1], T[2], T[3]]\right]$$

```
> push([f1, f2], TT, P);
```

$$\left[[T[1]+T[3], 1+T[1]^2+T[2]T[3]^2],\ [T[1], T[2], T[3]]\right]$$

Procedure A.5.10 (vars)**.** Returns a list of variables.

Input: an integer $r \in \mathbb{Z}_{\geq 1}$ or a list of polynomials $[f_1, \ldots, f_s]$.

Output: the list $[T_1, \ldots, T_r]$ if an integer r was given; assumes that T has not yet been assigned. If a list of polynomials was given, then a list of all found variables is returned.

Example: `> TT := vars(3);`

$$TT := [T[1], T[2], T[3]]$$

```
> TTf := vars([T[1]*Y[3]^2 + 7*S[2] + 1, -2*S[1] - 9]);
```

$$TTf := [S[1], T[1], S[2], Y[3]]$$

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