

Liquidity Shocks in Over-the-Counter Markets

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List of Symbols and Notation

All symbols and notations are defined at their first appearance in the text. Some symbols contain supplements, which refer to certain models or states. These supplements do not change the original meaning of the symbol: Variables in the steady state equilibrium are characterized by the supplement (*ss*) instead of time t (e.g. $\mathbf{V}(ss)$). The superscript '*W*' (e.g. $\mu_{\sigma}^W(t)$) denotes a Walrasian market. The superscript '*s*' (e.g. $\mathbf{V}^s(t)$) assigns the particular variable to the aggregate liquidity shock model (chapter 4, 5, and 6), whereas the superscript '*s, c*' (e.g. $\mathbf{V}^{s,c}(t)$) assigns the particular variable to the completed aggregate liquidity shock model (chapter 7). The superscript '***' (e.g. $\mathbf{V}^*(t)$) or '*-*' (e.g. $\bar{\mathbf{V}}(t)$) refers in general to a function or value related to a coordinate transformation (chapter 4 and 6), except for t^* , which characterizes the time of intersection $\mu_{lo}(t^*) = \mu_{ln}(t^*)$. The following list states the main symbols and notations used in this dissertation.

A	Bold-face upper characters specify (in general) a $(n \times m)$ matrix, containing elements a_{ij} , with $i = 1, \dots, n$ and $j = 1, \dots, m$.
a	Bold-face lower characters specify (in general) a (column) vector.
$\dot{A}(t)$	$= dA(t)/dt$, first derivative with respect to time t .
\mathbf{A}^T	Transpose of A .
\mathbf{A}^{-1}	Inverse of A .
$\mathbf{A}_1(\boldsymbol{\mu}(t))$	Time-dependent coefficient matrix, associated with the system of growth rates of value functions in the liquidity shock model.
\mathbf{A}_2	Column vector containing dividends and holding costs, associated with the system of growth rates of value functions in the liquidity shock model.

$\mathbf{A}_3(\boldsymbol{\mu}(t))$	Time-dependent matrix containing coefficients of future shocks, associated with the system of growth rates of value functions in the liquidity shock model.
$\mathbf{A}_{11}^{(k)}(t)$	Left upper block matrix with dimension $(k - 1)$, associated with a partitioned matrix $\mathbf{A}_k(t)$.
$\mathbf{a}_{12}^{(k)}(t)$	Right upper column vector, associated with a partitioned matrix $\mathbf{A}_k(t)$.
$\mathbf{a}_{21}^{(k)}(t)$	Left lower column vector, associated with a partitioned matrix $\mathbf{A}_k(t)$.
$a_{22}^{(k)}(t)$	Scalar, associated with the bottom right position of a partitioned matrix $\mathbf{A}_k(t)$.
$\ \mathbf{A}\ $	Norm of a matrix or vector \mathbf{A} .
$\arg \max$	Set of values for which the argument attains its maximum.
$A(t)$	Ask price at time t .
$B(t)$	Bid price at time t .
C_t	Cumulative consumption process.
\mathbf{c}	Vector of constants, specifying the solution of a differential equation.
D	Constant dividend.
d	Disagreement point in a bargaining problem.
δ	Holding/illiquidity cost.
$\det(\mathbf{A})$	Determinant of matrix \mathbf{A} .
$\text{diag}(\dots)$	Diagonal matrix.
$E_t[\cdot]$	Expectation over $[\cdot]$.
$\exp(a)$	$= e^a$.
\mathcal{F}	Information filtration.
$f(\mathcal{S}, d)$	Bargaining solution.
Γ	$= \{lo, hn, ho, ln\}$; set of investor types.
hn	High non-owner (investor type; buyer).
ho	High owner (investor type).
\mathbf{I}_n	Identity matrix of dimension n .
$\mathbf{1}_{\{A\}}$	Indicator function, which is 1 if the event A is true, 0 otherwise.
$J(\cdot)$	Optimum value of a utility-maximization problem.
λ	Search/meeting intensity of investors—Poisson intensity.
λ_d	Poisson intensity of switching to a low type (down-switch).
λ_u	Poisson intensity of switching to a high type (up-switch).
$\lambda_i(t), \lambda_k(t)$	(Dynamic) Eigenvalue i or k , respectively.
$\hat{\lambda}_1(t), \hat{\lambda}_2(t)$	Dynamic eigenvalues, associated with the calculation of $\mathbf{q}_4(t)$.

$\bar{\lambda}_1^s, \bar{\lambda}_2^s(t)$	Dynamic eigenvalues, associated with the verification of a valid Nash bargaining solution in the liquidity shock model.
$\Lambda(t)$	(Time-varying) Diagonal matrix, containing (dynamic) eigenvalues on the main diagonal.
ln	Low non-owner (investor type).
lo	Low owner (investor type; seller).
$M(t)$	Interdealer price at time t .
$\mathcal{M}(t)$	Matching function; number of successful matches per time unit.
$\max\{a, b\}$	Maximum value of a and b .
$\min\{a, b\}$	Minimum value of a and b .
$\mu_\sigma(t)$	Fraction of type- σ investors in the total population at time t , with $\sigma \in \Gamma$.
$\mu_h(t)$	$= \mu_{ho}(t) + \mu_{hn}(t)$; fraction of high investors.
$\mu_l(t)$	$= \mu_{lo}(t) + \mu_{ln}(t)$; fraction of low investors.
$\mu_m(t)$	$= \min\{\mu_{lo}(t), \mu_{hn}(t)\}$.
$\bar{\mu}_\sigma(0)$	Agents' post-shock distribution.
$\boldsymbol{\mu}(t)$	$= [\mu_{lo}(t), \mu_{hn}(t), \mu_{ho}(t), \mu_{ln}(t)]^T$.
$\boldsymbol{\nu}_i(t)$	(Dynamic) Eigenvector i .
Ω	Set of all possible states in the world.
\mathbb{P}	Probability measure.
$\Phi(t, t_0)$	State transition matrix to a system of differential equations.
$\pi_{hn}(t)$	Probability of hn agents switching to ln agents at time t .
$\pi_{ho}(t)$	Probability of ho agents switching to lo agents at time t .
$\pi_{ln}(t)$	Probability of ln agents switching to hn agents at time t .
$\pi_{lo}(t)$	Probability of lo agents switching to ho agents at time t .
$\mathbf{P}_k(t)$	Riccati (transformation) matrix.
$\mathbf{p}_k(t)$	Solution to a system of Riccati differential equations.
$P(t)$	Interinvestor price at time t .
$\tilde{q}(t)$	Auxiliary variable (defined on p. 48), indicating a seller's market with $\tilde{q}(t) = 1$ if $\mu_{lo}(t) < \mu_{hn}(t)$ holds, a buyer's market with $\tilde{q}(t) = 0$ if $\mu_{lo}(t) > \mu_{hn}(t)$ holds, or a balanced market with $\tilde{q}(t) \in [0, 1]$ if $\mu_{lo}(t) = \mu_{hn}(t)$ holds.
q	Seller's bargaining power.
$1 - q$	Buyer's bargaining power.
$\mathbf{Q}_k(t)$	Second transformation matrix (defined on p. 86).
$\mathbf{q}_k(t)$	Solution to a system of differential equations (defined on p. 86).
$\mathbf{q}_4(t)$	$= [q_{41}(t), q_{42}(t), q_{43}(t)]^T$, specifying the 4th dynamic eigenvector (defined on pp. 93 and appendix 4C).

$\bar{q}_2(t)$	Auxiliary variable and solution to a differential equation, associated with the verification of a valid Nash bargaining condition (defined on p. 137 and 153).
r	Interest rate.
ρ	Search intensity of market makers—Poisson intensity.
\mathcal{S}	Set of feasible utility pairs in a bargaining problem.
s	Fraction of investors owning an asset.
sup	Supremum.
t	Time (continuous).
$t_{hn}(t)$	Average meeting time for a buyer to meet sellers.
$t_{lo}(t)$	Average meeting time for a seller to meet buyers.
t^*	Intersection time, describing $\mu_{lo}(t^*) = \mu_{hn}(t^*)$.
τ	Stopping time/first arrival time.
τ_i	Next stopping time when a search and bargaining between two investors is successfully completed.
τ_l	Next stopping time when an agent changes his intrinsic type.
τ_m	Next stopping time when trade occurs between an investor and a market maker.
τ_ζ	Next stopping time when an aggregate liquidity shock occurs.
θ_t	Feasible asset holding process.
trace(\mathbf{A})	$= \sum_i a_{ii}$ is the trace of matrix \mathbf{A} .
$\mathbf{T}(t)$	(Time-varying) Transformation matrix, containing (dynamic) eigenvectors on their columns.
$V_\sigma(t)$	Value function of investor type σ at time t , with $\sigma \in \Gamma$.
$\mathbf{V}_\sigma(t)$	$= [V_{lo}(t), V_{hn}(t), V_{ho}(t), V_{ln}(t)]^T$.
$\Delta V_h(t)$	$= V_{ho}(t) - V_{hn}(t)$.
$\Delta V_l(t)$	$= V_{lo}(t) - V_{ln}(t)$.
$\Delta V_n(t)$	$= V_{hn}(t) - V_{ln}(t)$.
$\Delta V_o(t)$	$= V_{ho}(t) - V_{lo}(t)$.
W_t	Value of the bank account at time t .
$\mathbf{X}(t)$	Fundamental matrix to a system of differential equations.
y	$= \lambda_u / (\lambda_u + \lambda_d)$, probability of being a high-type agent in steady state.
z	Market makers' bargaining power.
$\mathbf{0}$	Zero matrix or vector.
ζ	Poisson intensity, defining the occurrence of aggregate liquidity shocks.

ALS	Aggregate liquidity shock.
HJB	Hamilton–Jacobi–Bellman.
LTI	Linear time-invariant.
LTV	Linear time-varying.
ODE	Ordinary differential equation.
OTC	Over-the-counter.

Chapter 1

Introduction

This dissertation addresses liquidity and aggregate liquidity shocks in over-the-counter markets. Liquidity, in this context, is related to the time delay of a trade due to search frictions.¹ Lower search frictions lead to a more liquid market and reduce trading time. As a result, search frictions influence prices.

Aggregate or systemic liquidity shocks are associated with a sudden shift in agents' preferences towards asset holding, which affects a large fraction of investors simultaneously. Investors experience a sudden decrease in their liquidity, like an unexpected need for cash or consumption, which leads to a forced withdrawal of assets. A liquidity crisis results, during which asset prices either decrease and recover over time, or asset prices become temporarily nonexistent because the market breaks down.

1.1 Motivation

The 2007–2009 financial crisis drew attention to over-the-counter markets and their shortcomings.² Many assets at the core of the crisis, such as collateralized debt obligations, credit default swaps, and a lot of other derivatives, are generally traded over-the-counter (OTC). Various other more well-known assets, such as corporate and government bonds, blocks of equity shares, currencies, real estates, or fine art, are commonly traded OTC as well.³

¹ Commonly, liquidity is also associated with asymmetric information, transaction or inventory costs, or immediacy, which do not apply in this dissertation.

² See Acharya, Engle, Figlewski, Lynch, and Subrahmanyam (2009), p. 251.

³ See Duffie (2010), p. 2.

The most notable characteristic of an OTC market is its decentralized structure. No central trading device such as a stock exchange with floor and electronic trading, limit-order books, or an auction, is available. Investors intending to trade in an OTC market must search for each other in order to locate a trading partner and to learn about prices. Search frictions lead to trading delays, which implies that it takes time to find a suitable trading partner. For example, the sale of a residential house takes 111–135 days on average.⁴ Similarly, it takes, on average, between half a day and one week for an investor to find a dealer for trading a bond.⁵

Prices are commonly bargained bilaterally in OTC markets, leading to both counterparty risk and a general lack of transparency.⁶ Agents are usually unaware of comparable trades and the associated prices bargained elsewhere in the market. Due to this opacity, Duffie (2010) designates this kind of market a “dark market”.⁷

Market frictions, such as the search and bargaining properties of an OTC market, affect the liquidity level of the assets traded in these markets, which influences asset prices. Brunnermeier and Pedersen (2009, p. 2201) distinguish between “an asset’s market liquidity (i.e., the ease with which it is traded) and traders’ funding liquidity (i.e., the ease with which they can obtain funding)”. A high funding liquidity implies the possibility of easily raising capital. High market liquidity implies the easy location of a trading partner, that is, low search frictions. Kyle (1985) introduces a specification of market liquidity that is generally accepted by both academics and practitioners: (1.) market tightness, which accounts for a trader’s loss due to turning around an asset within a short time period—reflected by bid-ask spreads, (2.) market depth, describing the impact on prices through trading, and (3.) resiliency, denoting price recovery time after a decline.⁸

During the 2007–2009 financial crisis, it became apparent that market and funding liquidity are valuable but scarce market features in times of financial distress. In some OTC markets, liquidity was reduced on short notice or even disappeared entirely. For example, the market for structured investment vehicles for rolling over short-term debt to finance long-term debt nearly broke down, because investors were unwilling to lend for the short term. As a result, structured in-

⁴ This selling time refers to the market in the United States of America (USA) during the period 1992–2002 for single-family homes. See Levitt and Syverson (2008), pp. 602.

⁵ Feldhütter (2012, pp. 1165–1173) estimates this range for “noncallable, nonconvertible, straight coupon bullet bonds with maturity less than thirty years [...] [for] the period from October 1, 2004, to June 30, 2009”.

⁶ See Acharya, Engle, Figlewski, Lynch, and Subrahmanyam (2009), p. 251.

⁷ See Duffie (2010), ch. 1.

⁸ See Brunnermeier (2009), p. 92 and Kyle (1985), pp. 1316.

vestment vehicles faced the risk of low funding liquidity.⁹ Market and funding liquidity are highly interconnected, possibly causing so-called ‘liquidity spirals’: Difficulties in funding can lead to a general reduction both in holding assets and in investors’ ability to fund trading if many agents who usually provide liquidity are affected, which, in turn, influences market liquidity, and leads to a decrease in prices. Since the value of agents’ remaining positions decreases, funding difficulties may worsen.¹⁰

This effect explains another example from the 2007–2009 financial crisis: The “Quant Meltdown”, as Khandani and Lo (2011, p. 1) called it. It is a good example of a systemic liquidity shock with short recovery time. In August 2007, hedge funds were forced to unwind large asset positions at short notice, probably due to margin calls or in order to reduce risk. These huge selling positions induced losses to others and led to a “deadly feedback loop”¹¹ by reducing prices of collaterals. After a few days, prices had mostly recovered.¹² This price recovery is in line with theory, because price drops due to liquidity shocks have a tendency to revert, whereas price drops due to changes in fundamentals do not rebound in general.¹³

The analytical modeling of OTC markets, and the pricing of assets therein, is still in the early stages compared to asset pricing in centralized markets.¹⁴ Nevertheless, the total volume of OTC derivatives alone was \$346.4 trillion at year-end 2012,¹⁵ which is not negligible. Due to the pioneering and inspiring work of Duffie, Gârleanu, and Pedersen (2005, 2007), research in this area has progressed in recent years.

My objective is to shed light on the models of Duffie, Gârleanu, and Pedersen (2005, 2007). To this end, the next section reviews related literature and section 1.3 outlines the structure of my thesis.

⁹ See Brunnermeier (2009) and Acharya, Gale, and Yorulmazer (2011) for a detailed analysis.

¹⁰ See Brunnermeier and Pedersen (2009), pp. 2204 and Brunnermeier (2009), pp. 77.

¹¹ Khandani and Lo (2011), p. 3.

¹² See Khandani and Lo (2007, 2011).

¹³ See Pedersen (2009), p. 196.

¹⁴ See Duffie (2010), p. xiii.

¹⁵ This is the volume of cleared transactions, excluding foreign exchange contracts. See International Swaps and Derivatives Association (ISDA) (2013), p. 2.

1.2 Related Literature

Duffie, Gârleanu, and Pedersen (2005) model search frictions in a single-asset over-the-counter market leading to illiquidity. Potential sellers must search for potential buyers or market makers, and vice versa. After meeting an appropriate counterparty, price bargaining takes place and the trade is executed upon agreement.

There are various extensions of the basic Duffie, Gârleanu, and Pedersen (2005) model. The first are from Duffie, Gârleanu, and Pedersen (2005) themselves and the accompanying paper by Duffie, Gârleanu, and Pedersen (2007): Duffie, Gârleanu, and Pedersen (2005) analyze both the impact of monopolistic market makers and the effect of heterogeneous types of investors—sophisticated and unsophisticated ones—on the bid-ask spread. Additionally, they study endogenous search by market makers and the influence of market makers on welfare. Duffie, Gârleanu, and Pedersen (2007) examine risk-aversion and conclude that their basic risk-neutral model is a good first-order approximation to the risk-averse one. Their second contribution analyzes the negative implication of aggregate liquidity shocks on prices. Both extensions of Duffie, Gârleanu, and Pedersen (2007) are conducted without market makers.

Vayanos and Wang (2007), Vayanos and Weill (2008), and Weill (2008) extend the Duffie, Gârleanu, and Pedersen (2005) model to consider various asset markets, which differ by the extent of the prevailing search friction. They find cross-sectional differences in asset returns. Vayanos and Weill (2008) additionally consider short sellers to explain liquidity premia. Weill (2008) discusses the impact of a general matching function. Bélanger, Giroux, and Moisan-Poisson (2013) study an OTC market in line with Duffie, Gârleanu, and Pedersen (2005), but allow for several traded assets. Gârleanu (2009) and Lagos and Rocheteau (2007, 2009a) let agents hold a different amount of assets, compared to 0 or 1 in Duffie, Gârleanu, and Pedersen (2005). They find that search frictions influence the structure of an investor's portfolio in a negative way. Lagos and Rocheteau (2009a) consider general preference types. Afonso (2011) alters the framework of Duffie, Gârleanu, and Pedersen (2005) with an endogenous entry of investors in order to model a congested market. In her model, Afonso (2011) finds that reducing search frictions does not necessarily increase market liquidity.

Empirical analysis of search frictions are carried out by Ashcraft and Duffie (2007), who utilize the OTC characteristics of the federal funds market, and by

Gavazza (2011), who studies the effect of search costs in the commercial aircraft market in the USA.

Feldhütter (2012) builds on Duffie, Gârleanu, and Pedersen (2007) but restricts his model to trade intermediated by market makers. He considers only a single systemic liquidity shock. In his appendix, he extends his model to multiple aggregate liquidity shock à la Duffie, Gârleanu, and Pedersen (2007) but limits his analysis to severe shocks. Weill (2007, 2011), Lagos, Rocheteau, and Weill (2011), and Chiu and Koepl (2011) present an extension of Duffie, Gârleanu, and Pedersen (2007) in which market makers can accumulate inventories after a shock. In Weill (2007, 2011) and Lagos, Rocheteau, and Weill (2011), market makers use their inventory to overcome a single aggregate liquidity shock, while trading is also restricted to intermediation by market makers. Under various restrictions and settings, they study the effect of market makers “leaning against the wind”¹⁶. In Chiu and Koepl (2011), market makers use their inventory to respond to an adverse shock to asset quality leading to a market freeze. Biais, Hombert, and Weill (2013) utilize much of the spirit of the Duffie, Gârleanu, and Pedersen (2007) model with aggregate liquidity shocks. However, search frictions are replaced with agents who infrequently observe their own preferences. While they consider a centralized limit order market, the focus of this thesis is OTC markets.

Duffie, Gârleanu, and Pedersen’s (2005, 2007) work and my work are both related to several strands of literature. First, the work is related to search theory. Search frictions have been implemented into labor market models since at least Stigler (1961). The major contributions come from Diamond (1982a,b), Mortensen (1982a,b), and Pissarides (1984a,b, 1985), who received the Nobel Prize for their work in 2010. A good overview of the literature on the application of search frictions in many different research areas is provided by the scientific background article for the 2010 Nobel Prize, prepared by The Royal Swedish Academy of Sciences (2010). Monetary search literature, for example, began applying a comparable model in the 1990s. Known representatives are Kiyotaki and Wright (1993) and Trejos and Wright (1995). Yet, the seminal contributions of Duffie, Gârleanu, and Pedersen (2005, 2007) were the first to integrate search frictions into a model of financial markets in order to depict liquidity. They laid the foundation for this new and promising strand of research on liquidity in over-the-counter markets.

¹⁶ Weill (2007).

Second, Duffie, Gârleanu, and Pedersen's (2005, 2007) work and also my work are related to bargaining theory. The model by Duffie, Gârleanu, and Pedersen (2005, 2007) relies on the general Nash bargaining solution, introduced by Nash (1950, 1953). The alternating-offer bargaining game, introduced by Rubinstein (1982) and specified by Binmore, Rubinstein, and Wolinsky (1986), is considered in Duffie, Gârleanu, and Pedersen (2007) as well. The present thesis addresses the latter only briefly.

This thesis is also related to the general strand about pricing illiquid assets. The seminal contributions by Kyle (1985) and Glosten and Milgrom (1985), who study asymmetric information, paved the way for the liquidity theory in finance. Apart from asymmetric information, illiquidity can be due to many reasons, such as transaction or inventory costs, immediacy, or search frictions. Amihud, Mendelson, and Pedersen (2005) provide a good survey of the vast literature and discuss these aspects. The present thesis is only related to the type of illiquidity due to search frictions.

Fourth, this thesis complements the literature on liquidity shocks. Pedersen (2009, p. 177) discusses a liquidity shock and "the dangers of rushing to the exit" in order to not be forced to sell at the lowest price. He finds that asset prices are reduced due to liquidity risk. Brunnermeier and Pedersen (2009) analyze liquidity shocks in connection with the restricted funding of investors. They find that the liquidity of a market can suddenly run dry. Coval and Stafford (2007), Mitchell, Pedersen, and Pulvino (2007), Duffie (2010), Duffie and Strulovici (2012), and Acharya, Shin, and Yorulmazer (2013) address the slow movement of capital after a liquidity shock, leading to a slow market recovery. Although their model setups are different, the results of this thesis are essentially in line with theirs: Prices are reduced due to liquidity shocks but can recover over time. There are also empirical analyses of liquidity shocks. Feldhütter (2012) estimates a variant of the liquidity shock model of Duffie, Gârleanu, and Pedersen (2007) with corporate bond market data, with the aim of identifying selling pressure. Albuquerque and Schroth (2013) empirically study the pricing implications of liquidity shocks and search frictions on investors who hold blocks of shares.

Finally, this thesis is related to the strand of literature that analyzes a market freeze or a market breakdown. Longstaff (2009) considers an exogenous black-out period in which an illiquid asset cannot be traded while a liquid one can. His findings are extreme portfolio allocations and a negative price impact on the illiquid asset. Ang, Papanikolaou, and Westerfield (2013) study optimal asset allocation, where some illiquid assets can only be traded at exogenous random times.

Bruche and Suarez (2010) and Heider, Hoerova, and Holthausen (2009) consider a market freeze in the interbank money market due to counterparty risk. Acharya, Gale, and Yorulmazer (2011) study the market freeze for short-term debt that is repeatedly rolled over. This effect appeared in the 2007–2009 crisis, as noted above. Chiu and Koepl (2011), Camargo and Lester (2013), and Camargo, Kim, and Lester (2013) analyze a market for lemons à la Akerlof (1970), and how it freezes and thaws. In their studies, market freeze is due to asymmetric information, whereas market freeze in the present thesis is related to search frictions and liquidity risk, i.e. the risk of future liquidity shocks.

1.3 Structure of the Thesis

My thesis is organized as follows: Chapter 2 lays the foundations of search, matching, and bargaining theory for the models of Duffie, Gârleanu, and Pedersen (2005, 2007).

The purpose of chapter 3 is to introduce the basic steady state equilibrium model of Duffie, Gârleanu, and Pedersen (2005) for asset pricing in an illiquid over-the-counter (OTC) market. Illiquidity frictions are modeled by two-side search and bilateral trading. The intention of this chapter is to discuss the effects of search frictions on market liquidity, influencing asset prices, bid-ask spreads, and asset allocation in an OTC market. Since this model is taken as a basis for the following chapters, I refer to this model as the ‘basic model’.

The objective of chapter 4 is the discussion of aggregate liquidity shocks. This is the extension developed by Duffie, Gârleanu, and Pedersen (2007) to the basic model of chapter 3, although Duffie, Gârleanu, and Pedersen (2007) do not take market makers into account. The focus of chapter 4 is the reaction of prices to aggregate liquidity shocks in connection to the dynamics out of and towards steady state. I explicitly derive a semi-analytical solution to the resulting linear time-varying (LTV) system of differential equations, including market makers. The implications of a sudden selling pressure on both asset prices and the bid-ask spread out of and towards steady state, as well as the market recovery pattern, are studied therewith.

Chapter 5 discusses the implications of aggregate liquidity shocks on the basis of a numerical example. I adopt the example of Duffie, Gârleanu, and Pedersen (2007) to analyze their results. To study bid-ask spreads, I extend their example

to include market makers.

Chapter 6 and 7 constitute the core of my thesis. In chapter 6, I discuss the existence of a Nash bargaining solution, which ensures the feasibility of the models in chapter 3 and 4. The nonexistence of a Nash bargaining solution would result in no gains from trade but forced trading instead of an endogenously induced market freeze. I analyze both the basic model and the aggregate liquidity shock model with respect to gains from trade. Further, I examine the impact of the level of search frictions on the risk of no gains from trade. Voluntarily trading in any case can be implemented into the aggregate liquidity shock model through some model modifications.

The reason for a market freeze is addressed in chapter 7. Because the model of chapter 4 includes the possibility of further shocks, I discuss the effects of an additional aggregate liquidity shock occurring shortly after an initial one. This possibility gives flexibility to the aggregate liquidity shock model but causes inconsistency within the model. I complete the aggregate liquidity shock model to fix both issues: no gains from trade and inconsistency.

Chapter 8 summarizes the results and concludes my thesis.

Chapter 2

Search and Bargaining

This chapter gives a short introduction to the fundamentals of search and bargaining models. First, I introduce the economics of search and matching theory in chapter 2.1. I review some methods of probability theory in section 2.1.2, since search and matching models rely on these techniques. Section 2.1.3 describes the relevance of matching functions for search theory. The appropriate matching function for the OTC market considered in this thesis is derived as well. Chapter 2.2 deals with bargaining theory. It is commonly applied to the negotiation of trading conditions after search is completed and individuals are matched successfully. I give a short introduction to the basics of game theory, and bargaining theory in general, and Nash bargaining and an alternating-offer bargaining game in particular.

Most of these concepts are by now standard in economics and finance and have been treated in detail in several textbooks. For further reading, I recommend (1) McCall and McCall (2008) and Pissarides (2000) for a basic treatment of search and matching theory; (2) Rachev, Höchstötter, Fabozzi, and Focardi (2010) and Schönbucher (2003) for a thorough discussion of probability theory; (3) Osborne and Rubinstein (1990, 1994) and Myerson (1991) for a detailed examination of bargaining theory.

2.1 Search and Matching Theory

2.1.1 Search Theory

Neoclassical economics considers a centralized market for exchange in which all kinds of information are perfectly available to all individuals. Walras (1874) in-

roduces a centralized auction to find a market clearing price that matches supply and demand in a perfect market.¹⁷ But usually, as Stigler (1961) notes in the early literature about search theory, there is no centralized market, no facility for a perfect and costless allocation of resources, and no benevolent Walrasian auctioneer matching supply and demand:

“Prices change with varying frequency in all markets, and, unless a market is completely centralized, no one will know all the prices which various sellers (or buyers) quote at any given time. A buyer (or seller) who wishes to ascertain the most favorable price must canvass various sellers (or buyers)—a phenomenon I shall term ‘search’.”¹⁸

One implicit assumption derived from this specification is that trade should not be modeled between buyer / seller and ‘the market’ but directly between buyer and seller, in order to account for the time-consuming search for a trading partner. Of course, search is not restricted to just markets for goods or to financial markets, though these are the most obvious. Other examples are the labor market, housing market, or even the marriage market.¹⁹ The central idea of search theory is summarized by Pissarides (2001, p. 13760) as follows:

“The economics of search study the implications of market frictions for economic behavior and market performance. ‘Frictions’ in this context include anything that interferes with the smooth and instantaneous exchange of goods and services.”

Search frictions result in the expenditure of time, money, and other resources in order to learn about opportunities. For example, if individuals have incomplete information about the location of an item or a trading partner, potential buyers must search for the needed item and potential sellers cannot easily locate a potential buyer for the item on sale. As a result, prices are influenced by such frictions, which cannot be eliminated with price adjustments. Trade is delayed and markets do not clear at all times.²⁰

The characterization of search in a financial market can be taken further by regarding it as a substitute for modeling liquidity. Harris (2003, p. 395) defines liquidity as follows:

¹⁷ See Neus (2013), pp. 85.

¹⁸ Stigler (1961), p. 213.

¹⁹ See Diamond (1984), pp. 1.

²⁰ See Pissarides (2001), pp. 13760 and Diamond (2011), pp. 1045.

“Liquidity is the object of bilateral search. In a *bilateral search*, buyers search for sellers, and sellers search for buyers. When a buyer finds a seller who will trade at mutually acceptable terms, the buyer has found liquidity. Likewise, when a seller finds a buyer who will trade at mutually acceptable terms, the seller has found liquidity.”

Market liquidity has many different dimensions. The three dimensions established by Kyle (1985, p. 1316) are commonly stated: “[...] ‘tightness’ (the cost of turning around a position over a short period of time), ‘depth’ (the size of an order flow innovation required to change prices a given amount), and ‘resiliency’ (the speed with which prices recover from a random, uninformative shock).”²¹ Kyle (1985, pp. 1316) himself refers to Black (1971, pp. 29), who describes the bid-ask spread as a measure for market tightness. Kyle (1985, p. 1317) summarizes the definition of “a liquid market as one which is almost infinitely tight, which is not infinitely deep, and which is resilient enough so that prices eventually tend to their underlying value.”

Sometimes a fourth dimension, ‘immediacy’ (referring to the time a trade takes), is added.²² Harris (2003, p. 399) defines “liquidity [as] the ability to quickly trade large size at low cost.” The key issue of liquidity is the ability to trade, which is the core element of search and matching theory. Harris (2003, p. 399) continues by characterizing liquidity as a function, stating “the probability of trading a given size at a given price, given the time we are willing to put into [.] search.” Search models imply a natural—though slightly different—liquidity measure. The cost of a time lag due to delayed trade can be measured by the expected time it takes to find a trading partner. Liquidity in this setting is characterized by the speed of finding a trading partner.²³

The following passage provides a short overview of important contributions to the research on search theory:

Stigler (1961) is one of the first to formally model the behavior of buyers in a commodity market—from which the ‘fixed sample rule’ became known: First, a buyer chooses the optimal number of sellers to search for and then decides in favor of the seller quoting the lowest price. This model paved the way for the ‘optimal stopping rule’, and can be traced back to a model of labor markets by McCall (1970): First, a reservation price is specified, and the buyer then buys from the

²¹ Market ‘width’ is another name for ‘tightness’. See Harris (2003), pp. 398.

²² See Harris (2003), pp. 398 and Black (1971), p. 30.

²³ See Vayanos and Wang (2007), p. 75.

first seller quoting a price equal to or less than the reservation price. Therefore, search intensity and time depend on the reservation price. An exogenous price distribution is assumed within this model and a partial equilibrium problem, i.e. one-sided search with a take-it-or-leave-it price, is solved.²⁴

The next stage of development in search theory was reached by Diamond (1982a,b), Mortensen (1982a,b), and Pissarides (1984a,b, 1985). Their contributions are threefold: (1) They replaced the assumption about an exogenous price distribution with Nash bargaining, (2) they introduced two-sided search by means of an exogenous matching function, and (3) they considered the flow of agents to model a general equilibrium problem.²⁵

Search theory extends the classical, i.e. deterministic, theory of exchange by considering uncertainty.²⁶ The matching function accounts for this uncertainty; it describes how individuals come into contact through search. One input parameter is the arrival rate of trading partners within a short time interval, modeled as a stochastic process. The simplest form describing these contacts is the first arrival time τ of a Poisson process with a constant mean arrival rate λ . This arrival rate λ is denoted as 'search intensity', which is usually costly to increase. In general, it has the following properties: $\lambda \rightarrow \infty$ in a world with no frictions, $\lambda > 0$ in a market with search frictions, and $\lambda = 0$ with no search at all.²⁷

An equilibrium model is characterized by a search process that persists over time. This flow of 'new' agents searching for trading partners can be modeled by either exogenous inflows, wherein matched agents leave the market, or, as in the model considered in chapter 3–7, by exogenous and independent idiosyncratic shocks to a fraction of the population. These shocks induce search and trading impulses and are commonly driven by a Poisson process.²⁸

The following section gives a short introduction to probability theory, with the aim of modeling uncertainty in search theory. Thereafter, matching functions, particularly with regard to the model discussed subsequently, are presented. Bargaining is introduced in chapter 2.2.

²⁴ See Pissarides (2001), pp. 13761, McCall (1970), pp. 114, and McCall and McCall (2008), pp. 11.

²⁵ See Mortensen and Pissarides (1999a), p. 1173, The Royal Swedish Academy of Sciences (2010), p. 3, and McCall and McCall (2008), pp. 9 and 12.

²⁶ See McCall and McCall (2008), p. 10.

²⁷ See Diamond (1984), p. 9, Pissarides (2001), p. 13761, and Pissarides (2000), p. 127.

²⁸ See Pissarides (2001), p. 13762.

2.1.2 Basics of Probability Theory

This section starts with basic definitions of distributions and density functions. From the set of all possible distribution functions, the exponential distribution is explicitly presented, since it is of particular interest. The concepts for stopping times, hazard rates, and point processes are provided. The section concludes by defining the Poisson process and highlighting its relevance to the rest of this thesis.

First, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an information filtration $\mathcal{F}_t : t \geq 0$ is defined. Ω is the set of all possible states in the world, \mathcal{F} describes the information structure of the setup, and the probability measure \mathbb{P} attaches probabilities to the events in Ω . Notations and definitions in this section are based on Rachev, Höchstötter, Fabozzi, and Focardi (2010, Part II), Schönbucher (2003), and Duffie and Singleton (2003).

Distribution and Density Function

The distribution function $F(x)$ expresses the probability that a random variable X is equal to or less than x . It is defined as follows:

$$F(x) = \mathbf{P}(X \leq x).$$

For a continuous random variable X , the distribution function is defined by its (probability) density function $f(x)$, with

$$F(x) = \int_{-\infty}^x f(t) dt,$$

where the element of probability $dF(x) = f(x) dx$ describes the probability that the random variable X is within the infinitesimal interval $(x, x + dx)$, i.e. $f(x) dx = \mathbf{P}(x \leq X \leq x + dx)$. Another notation for the distribution function is

$$F(x) = \int_{-\infty}^{\infty} \mathbf{1}_{\{A\}}(t) f(t) dt,$$

where $\mathbf{1}_{\{A\}}(X)$ is an indicator function. The indicator function has the value 1 if the event A is true, i.e. the random variable X is an element of a set A ; the value

is 0 for all X that are not in A . Formally,

$$\mathbf{1}_{\{A\}}(X) = \begin{cases} 1, & X \in A \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

Application of an indicator function is suitable for modeling first stopping times, indicating if an event has already occurred or not.

The expected value of a function of X , $g(x)$, in terms of the (probability) density function $f(x)$, is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) dx.$$

Multivariate Probability Distribution

The multivariate distribution is the distribution of a multivariate random variable or the joint distribution of more than one random variable. It is calculated by integrating over the multivariate or joint density function

$$\begin{aligned} F(x_1, \dots, x_n) &= P(X_1 \leq x_1, \dots, X_n \leq x_n) \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} f_{x_1, \dots, x_n}(t_1, \dots, t_n) dt_1 \dots dt_n, \end{aligned}$$

where $f_{x_1, \dots, x_n}(t_1, \dots, t_n)$ is the joint density function. The random variables x_i are called independent if the joint density is

$$f(x_1, \dots, x_n) = \prod_i f_{x_i}(x_1, \dots, x_n).$$

Exponential Distribution

Of special interest are the density $f(x)$ and the distribution function $F(x)$ of an exponential distributed random variable. The density is defined as follows:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases} \quad (2.2)$$

The distribution function $F(x)$, i.e. the probability that the random variable X is equal to or less than x , is the result of integrating over the density function f . Hence,

$$\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda t} dt \\ &= 1 - e^{-\lambda x}. \end{aligned} \tag{2.3}$$

Stopping Times

The contact between two investors, possibly modeled by random search and matching, is called an ‘event’ that occurs at the random time $\tau \geq 0$. To define a stopping time, it is necessary that it is known at every time t if the event has already occurred ($\tau \leq t$) or not ($\tau > t$), given the information in \mathcal{F}_t . This means that

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0,$$

which defines the random time τ as a stopping time. It can be stated with a stochastic process using an indicator process that is defined by a switch from zero to one at the stopping time τ . This is

$$N_\tau(t) = \mathbf{1}_{\{\tau \leq t\}}.$$

Hazard Rate

Let $F(t) = \mathbf{P}(\tau \leq t)$ denote the distribution function and let $f(t) = dF(t)/dt$ describe the density function of a stopping time τ . The hazard rate is defined with

$$h(t) = \frac{f(t)}{1 - F(t)}, \tag{2.4}$$

where $S(t) = 1 - F(t) = P(\tau > t)$ is called the survivor function: The probability that an individual survives beyond time t . Translated to the concept of random search and matching, $S(t)$ is the probability of no contact between two investors. The hazard rate $h(t)$ can be interpreted as the local arrival (also: leaving, escape, defaulting) probability per unit of time of the stopping time τ or the instanta-

neous arrival rate of a contact at time t . This means that

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \mathbf{P}(\tau \leq t + \Delta t | \tau > t),$$

where $\mathbf{P}(\tau \leq t + \Delta t | \tau > t)$ describes the probability of changing the current state within the short interval $(t, t + \Delta t)$, conditional on surviving up to the beginning of the period in t .²⁹ This probability implies, for the random search and matching concept, that $\mathbf{P}(\tau \leq t + \Delta t | \tau > t)$ is the probability of a contact within the interval $(t, t + \Delta t)$, given that no meeting has occurred up to time t . Approximately for a small Δt ,

$$\mathbf{P}(\tau \leq t + \Delta t | \tau > t) \approx h(t)\Delta t.$$

The conditional hazard rate at time T , as seen from time $t \leq T$, is defined as

$$h(t, T) = \frac{f(t, T)}{1 - F(t, T)}, \quad (2.5)$$

where $F(t, T) = \mathbf{P}(\tau \leq T | \mathcal{F}_t) = \mathbf{P}(\tau \leq T | \tau > t)$ is the conditional distribution function of the stopping time τ and $f(t, T)$ is the corresponding conditional density function, both conditioned on the information \mathcal{F}_t available at time $t \leq T$. Hence,

$$h(t, T) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \mathbf{P}(\tau \leq T + \Delta t | \{\tau > T\} \wedge \mathcal{F}_t),$$

where $\mathbf{P}(\tau \leq T + \Delta t | \{\tau > T\} \wedge \mathcal{F}_t)$ describes the conditional default probability over the interval $(T, T + \Delta t)$ as seen from time $t \leq T$.

Since $d(1 - F(t))/dt = -f(t)$ and $d(1 - F(t, T))/dT = -f(t, T)$, the hazard rate of equation (2.4) and the conditional hazard rate of equation (2.5) can be written as

$$h(t) = -\frac{d \ln(1 - F(t))}{dt},$$

and

$$h(t, T) = -\frac{d \ln(1 - F(t, T))}{dT}.$$

Integrating and using $F(0) = 0$ and $F(t, t) = 0$, the unconditional probabilities

²⁹ See McCall and McCall (2008), p. 117.

$F(t)$ and $f(t)$ as well as the conditional probabilities $F(t, T)$ and $f(t, T)$ can be reconstructed with

$$F(t) = 1 - e^{-\int_0^t h(x) dx},$$

$$f(t) = h(t) e^{-\int_0^t h(x) dx},$$

and

$$F(t, T) = 1 - e^{-\int_t^T h(t, x) dx},$$

$$f(t, T) = h(t, T) e^{-\int_t^T h(t, x) dx}.$$

In many standard cases, there is a constant hazard rate, i.e. $h(t) = h$. This constant hazard rate leads to the distribution function $F(t) = 1 - \exp(-ht)$, which is equal to an exponential distribution. Hazard rates, however, can change (even stochastically) over time, in which case the distribution $F(t)$ need not be an exponential one. Other possibilities are, for example, the Weibull distribution or a log-logistic distribution.³⁰

Point Processes

A stopping time only describes one single event, whereas a point process is a sequence of multiple events, like a random collection of different stopping times: $\{\tau_i, i \in \mathbb{N}\} = \{\tau_1, \tau_2, \dots, \tau_N\}$. A counting process is a stochastic process that describes this collection of (random) numbers of points in time. This means that

$$N(t) = \sum_i \mathbf{1}_{\{\tau_i \leq t\}},$$

where $N(t)$ accumulates the number of time points that are located in an interval $[0, t]$. For all $\tau_i > 0$, $N(t)$ is a step function starting at zero and having a step size of one. Each step occurs as soon as the next τ_i is attained.

The Poisson process is the most prominent representative of a counting process. Its definition is presented in the next passage.

³⁰ See McCall and McCall (2008), pp. 118–120.

Homogeneous Poisson Process

Let $(X(t))_{t \geq 0}$ be a stochastic process and this process is called Poisson process if it has the following properties:³¹

- (i.) $X_0 = 0$.
- (ii.) X_t has independent increments: For $0 \leq t_0 < t_1 < \dots < t_n < \infty$ and any $n \in \mathbb{N}$, the random variables $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, \dots , $X_{t_n} - X_{t_{n-1}}$ are stochastic independent.
- (iii.) The stochastic process X_t is homogeneous: The random variables $X_{t+h} - X_t$ and $X_{s+h} - X_s$ are identically distributed for any $s \geq 0$, $t \geq 0$, and $h > 0$.
- (iv.) Let $X(t, t + \Delta t)$ be the number of events in an interval $(t, t + \Delta t]$. As $\Delta t \rightarrow 0+$:

$$\mathbf{P}\{N(t, t + \Delta t) = 0\} = 1 - \lambda \Delta t + o(\Delta t), \quad (2.6)$$

$$\mathbf{P}\{N(t, t + \Delta t) = 1\} = \lambda \Delta t + o(\Delta t), \quad (2.7)$$

$$\mathbf{P}\{N(t, t + \Delta t) > 1\} = o(\Delta t), \quad (2.8)$$

with λ ($\lambda > 0$) as the expected number of events per unit time, also called the rate of occurrence.

From properties (i.)–(iii.) it follows that the probability distribution of the random variable $X(t)$ is a Poisson distribution with parameter λt and

$$P(X(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \dots$$

From property (iv.) it follows that during a small time interval of length Δt an event occurs with probability $\lambda \Delta t$ (equation (2.7)). With probability $1 - \lambda \Delta t$ no event arrives (equation (2.6)), and the probability of more than one event during the time interval $[t, t + \Delta t]$ is negligible (equation (2.8)). The Poisson distribution has an expected quantity of jumps per time unit, that is

$$E[X(t)] = \lambda t,$$

where the parameter λ is called ‘intensity’.

³¹ The definition follows the one in Bening and Korolev (2002), p. 69 and Cox and Miller (1965), p. 6 and pp. 146.

Assume that τ_n (with $n \geq 1$) are the times when the jump of a Poisson process occurs. Then, the distribution of the random variable $\tau_n - \tau_{n-1}$, i.e. the time interval between two jumps, is exponential distributed with

$$F(t, T) = 1 - e^{-\lambda(T-t)}, \quad (T \geq t).$$

The probability density function is

$$f(t, T) = \lambda e^{-\lambda(T-t)}, \quad (T \geq t).$$

Stated differently, inter-arrival times of a Poisson process are constituted by an exponential distribution. One of the advantages of an exponential distribution is its property of memorylessness. It ensures the tractability of the search and matching models described in chapters 3 to 7. To predict the mean time until the next event occurs, one does not need any information about the time that has elapsed since the last event took place. The expected time until the next event occurs is $1/\lambda$.

The conditional survival probability $S(t, T)$ is defined as the probability that no event takes place between time t and T , given both \mathcal{F}_t and no occurrence until time t . Hence,

$$S(t, T) = 1 - F(t, T) = e^{-\lambda(T-t)}.$$

The conditional hazard rate is

$$h(t, T) = \lambda$$

for a constant intensity λ .

As time passes, new information is revealed and the occurrence rate of events might change over time. Such changes can be driven by an underlying state variable, which varies with the business cycle or other economic parameters. The following passage generalizes the Poisson process by considering time dependence.

Inhomogeneous Poisson Process

When the intensity parameter of the Poisson process is a (deterministic) function of time $\lambda(t)$ ($\lambda(t) \geq 0, \forall t$), the process is called an inhomogeneous Poisson pro-

cess. The properties of homogeneous and inhomogeneous Poisson processes are similar: Properties (i.) and (ii.) must hold, and property (iv.) is restated with

$$\mathbf{P}\{N(t, t + \Delta t) = 0\} = 1 - \lambda(t)\Delta t + o(\Delta t), \quad (2.9)$$

$$\mathbf{P}\{N(t, t + \Delta t) = 1\} = \lambda(t)\Delta t + o(\Delta t), \quad (2.10)$$

$$\mathbf{P}\{N(t, t + \Delta t) > 1\} = o(\Delta t). \quad (2.11)$$

It follows that the increments in (ii.) are Poisson distribution for $0 \leq t \leq T$ with

$$P(X(T) - X(t) = k) = e^{-\int_t^T \lambda(s) ds} \frac{\left(\int_t^T \lambda(s) ds\right)^k}{k!}, \quad k = 0, 1, \dots$$

Hence, the inter-arrival time has an exponential distribution. The distribution function, probability density function, and survival probability of the first stopping time, given no event has occurred since time t , are

$$F(t, T) = 1 - e^{-\int_t^T \lambda(s) ds}, \quad (T \geq t),$$

$$f(t, T) = \lambda(T)e^{-\int_t^T \lambda(s) ds}, \quad (T \geq t),$$

$$S(t, T) = e^{-\int_t^T \lambda(s) ds}, \quad (T \geq t).$$

The hazard rate is

$$h(t, T) = \lambda(T).$$

The superposition process of N independent Poisson processes is again a Poisson process. It is defined with

$$\int_0^t \Lambda(s) ds = \sum_{i=1}^N \int_0^t \lambda_i(s) ds, \quad (2.12)$$

where the intensity $\Lambda(s)$ is the sum of the individual processes.

2.1.3 Matching Function

The core element of search and matching theory is the matching function. It facilitates the modeling of frictions within a framework that is easily tractable. If there are no search frictions, random matches occur instantaneously and the rationed side determines the amount of matches. But in a world with search frictions, in-

dividuals must spend time, money, and other resources on the search for each other. All these frictions are captured in a matching function of reduced-form, reflecting the degree of mismatch in a market.³²

“The matching function summarizes a trading technology between agents [...] that eventually bring[s] them together into productive matches.”³³

Different matching technologies are possible. For example, a linear matching technology is applied if the probability of a match within a short time interval does not rely on the amount of unmatched agents. A quadratic matching technology is characterized by a proportional relationship to potential matching partners.³⁴ The most basic definition for a matching function is

$$\mathcal{M} = m(\mu_i, \mu_j),$$

where \mathcal{M} is the output of a matching function. It states the number of successful matches per unit of time—the flow rate of matches. It is the instantaneous matching rate in continuous time. The matching function depends on the number of agents μ_i and μ_j voluntarily searching for each other, i.e. the inputs into a matching function. The action of one agent of, say, type i , inevitably influences the matching probability of all other agents of type i and of type j , since the matching rate is affected. It is assumed that the matching function is nonnegative, homogeneous of degree one, increasing, and concave in both its arguments. This characterization implies that search frictions decrease with the amount of agents. Individuals must be located on both sides of a market for a successful match, i.e. $m(0, \mu_j) = m(\mu_i, 0) = 0$. If the matching function is multiplied with a scaling parameter, the argument is multiplied by this scaling parameter with power one—due to homogeneity of degree one. Without frictions, the matching function is $\mathcal{M} = \min(\mu_i, \mu_j)$ in discrete time, and goes to infinity in continuous time.³⁵

Commonly, the matching function is declared as a “black box that gives the outcome of the search process in terms of the inputs into search”³⁶. The kind of meeting process eventually determines the specific matching function. The matching

³² See Stevens (2007), p. 847, Petrongolo and Pissarides (2001), p. 390, and Pissarides (2000), p. 22.

³³ Petrongolo and Pissarides (2001), p. 391.

³⁴ See Mortensen (1982b), p. 235.

³⁵ See Petrongolo and Pissarides (2001), p. 392.

³⁶ Pissarides (2001), p. 13762.

function usually applied in the empirical labor market literature is of a Cobb–Douglas form, given by

$$\mathcal{M} = m_0 v^{1-\alpha} u^\alpha,$$

where α is the elasticity parameter ($0 < \alpha < 1$), u is the measure of unemployed agents, v is the measure of job vacancies, and m_0 is a scaling parameter ($m_0 > 0$). There is empirical evidence for this function³⁷, though the exact matching process generating it is not known, i.e. there are no microfoundations supporting a Cobb–Douglas form.³⁸ The matching function commonly applied in the theoretical labor market literature is of an exponential form with

$$\mathcal{M} = v \left(1 - e^{-\lambda u/v} \right), \quad \text{for } \lambda > 0,$$

with λ describing the search intensity of workers. The motivation of this matching function is based on the “assumption of uncoordinated random search”³⁹. For an infinitesimal period of time dt , the number of matches is $v \left(1 - e^{-\lambda u dt/v} \right)$. This leads to a Poisson matching rate $\mathcal{M} = \lambda u$ with $dt \rightarrow 0$. Two-sided search reveals a linear and symmetric matching technology: $\mathcal{M} = \lambda u + \gamma v$, where γ represents the rate of recruitment.⁴⁰

The crucial point is finding the particular matching function that best fits an OTC market with search frictions. In general, an OTC market is characterized by bilateral trade negotiations between a potential buyer and a potential seller, two-sided search for a counterparty, and search frictions. These properties are best described by a symmetric, quadratic matching technology, i.e. simultaneous search by potential buyers and potential sellers and dependence on the fraction of potential matching partners.⁴¹ Hereafter, I follow the modeling of a random matching process presented by Duffie (2012, ch. 3.1 and 3.2). Notations and definitions are primarily based on his presentation.

First, some mathematical preliminaries are specified: A probability space, as defined in chapter 2.1.2, and a measure space (G, \mathcal{G}, μ) of agents are fixed. G is the

³⁷ See Blanchard and Diamond (1989).

³⁸ See Mortensen and Pissarides (1999b), pp. 2575, Petrongolo and Pissarides (2001), p. 392, and Stevens (2007), p. 848.

³⁹ Pissarides (2001), p. 13762.

⁴⁰ See Stevens (2007), pp. 848.

⁴¹ See Stevens (2007), pp. 848, Mortensen and Pissarides (1999b), pp. 2575, and Duffie (2012), ch. 3.2.

set of agents, which could be, e.g., $G = [0, 1]$, i.e. a uniform distribution over the unit interval. It is assumed that the measure μ is nonatomic and the set of agents is a continuum, i.e. there is an infinite amount of agents and no agent has a positive mass. The set $\mu(G)$ is the total amount of agents, which is positive and can be normalized to 1. The amount of agents in a measurable subset A is described by $\mu(A)$.

The random matching process is now defined: Starting point is the specification of two representative agents, agent I and agent II , picked from the subsets A and B , respectively. Random matching assigns agent I to one single other agent at the most and agent I is not matched with himself. When agent I is matched with agent II , agent II is inevitably assigned to agent I . Let us assume that the probability of being matched to anyone is based on a Poisson process with arrival rate λ_I for agent I and λ_{II} for agent II . The probability of agent I being matched with anyone of a measurable subset B is $\lambda_I \mu(B) / \mu(G)$, i.e. proportional to the amount $\mu(B)$ of agents in this subset. For subset A and B , being disjoint, the matching function is derived as follows:

$$\begin{aligned} E \left[\int_{I \in A} \mathbf{1}_{\{I, B\}} d\mu(I) \right] + E \left[\int_{II \in B} \mathbf{1}_{\{A, II\}} d\mu(II) \right] \\ = \int_{I \in A} E \left[\mathbf{1}_{\{I, B\}} \right] d\mu(I) + \int_{II \in B} E \left[\mathbf{1}_{\{A, II\}} \right] d\mu(II) \\ = \mu(A) \lambda_I \frac{\mu(B)}{\mu(G)} + \mu(B) \lambda_{II} \frac{\mu(A)}{\mu(G)}, \end{aligned}$$

with

$$\begin{aligned} \mathbf{1}_{\{I, B\}} &= \int_{II \in B} \mathbf{1}_{\{I, II\}} d\mu(II), \\ \mathbf{1}_{\{A, II\}} &= \int_{I \in A} \mathbf{1}_{\{I, II\}} d\mu(I), \end{aligned}$$

where $\mathbf{1}_{\{I, II\}}$ is an indicator function, which has the value 1 for agent I being matched to agent II and 0 otherwise. The random variable $\mathbf{1}_{\{I, B\}}$ is measuring the event wherein agent I is matched to anyone of subset B and $\mathbf{1}_{\{A, II\}}$ is measuring the event wherein agent II is matched to anyone of subset A . This matching function describes the overall expected amount of matches between agents in the subset A and agents in the subset B . For the special case that $\lambda_I = \lambda_{II} = \lambda$ and $\mu(G)$ is normalized to 1, the matching function describing the search technology

in this OTC market is

$$\mathcal{M} = 2\lambda\mu(A)\mu(B). \quad (2.13)$$

This matching function has the property wherein meetings increase by more than two if the measure of agents in subset A and B are doubled. This feature leads to a reduction in search time.⁴² With the matching function defined in equation (2.13), an agent of subset A contacts agents of subset B with Poisson arrival intensity $\mathcal{M}/\mu(A) = 2\lambda\mu(B)$.

The characteristic of an independent random matching—as defined by Duffie (2012)—is an independent matching result for agent I and agent II . It states that the described correlation of agent I being matched to agent II and the inevitable match of agent II to agent I goes to zero in a continuum population.⁴³ With this property being valid, the exact law of large numbers, defined and proved in Duffie and Sun (2007, 2012), can be applied. It states that

“with independence, the empirical distribution is almost surely the same as the average probability distribution. [...] [The] task is dramatically simplified if agents correctly assume that the empirical cross-sectional distribution of matches is not merely approximated by its probability distribution but is actually *equal* to it.”⁴⁴

The matching function in equation (2.13) best describes the process of an independent random matching in an OTC market with a continuum population. It depicts the stochastic process that brings together potential buyers and potential sellers to undertake transactions.

In most of the literature, it is assumed that agents start to bargain over the terms of trade, i.e. the division of the trading surplus, immediately after being successfully matched to a partner. That means they bargain over the trading price.⁴⁵ Accordingly, Mortensen (1982b, p. 234) defines:

“The divisions of the surplus attributable to the existence of a match is by nature a bilateral bargaining problem. A particular solution to this problem determines the value of the match to each member of a pair.”

⁴² See Vayanos and Wang (2007), p. 70.

⁴³ Duffie and Sun (2007, 2012) constitute the mathematical foundation for this independent random matching in a continuum population.

⁴⁴ Duffie (2012), p. 30.

⁴⁵ See Rubinstein and Wolinsky (1985), p. 1133.

The next section gives an introduction to bargaining theory for modeling the division of the matching surplus.

2.2 Bargaining Theory

As bargaining theory belongs to the field of game theory, I start with a short excursion. The basic concept of game theory is to mathematically describe situations in which a conflict of agents' interests prevails. This is a situation wherein one player's choice depends on the choice of others. In other words: each player must take the decisions of others into consideration. Agents are usually called 'players' in game theory, so I will follow this practice. A player is not a gambler, but a rational decision maker and may be an individual (like a sales manager) or a group of individuals (like governments and companies). Game theory provides analytical tools for describing and analyzing the interactions of these decision makers, with the aim of better understanding the situation.⁴⁶

Game theory distinguishes between cooperative and non-cooperative games: Players of non-cooperative games cannot collaborate and cannot put up binding rules with which all players must comply. They usually play against each other, like in military decisions. Both players care only about their own advantage. In cooperative games, however, players do not have completely contrary nor perfectly identical interests. They can arrange the terms of game and can jointly put up binding and authentic agreements or strategies before the game is actually played. Players voluntarily participate in the solving of the conflict. The negotiated agreement results in a gain in utility for both players, in contrast to the status quo alternative.⁴⁷

Bargaining situations are usually characterized as games. In his seminal paper, Nash (1950) defines a bargaining situation as follows:⁴⁸

"A two-person bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit in more than one way."⁴⁹

Bargaining situations can be found in all modes of daily life. Typical examples

⁴⁶ See Myerson (1991), pp. 1 and Osborne and Rubinstein (1994), pp. 1 and 117.

⁴⁷ See Osborne and Rubinstein (1994), pp. 2 and Nash (1953), p. 128.

⁴⁸ See Binmore, Osborne, and Rubinstein (1992), p. 181.

⁴⁹ Nash (1950), p. 155.

include wage bargaining, discussions about the evening TV program, or trade negotiations. Bargaining theory offers ideas for mathematical solutions to these situations. There are diverse analytical frameworks for these solutions. The focus of the present thesis is on both the axiomatic approach, the strategic approach, and their interconnections, since these are addressed by Duffie, Gârleanu, and Pedersen (2005, 2007).

Section 2.2.1 presents the axiomatic approach of Nash's bargaining solution, commonly stated as the framework for cooperative bargaining. The discussion follows Osborne and Rubinstein (1990, ch. 2). Section 2.2.2 briefly states two other bargaining solutions to cooperative games. The strategic approach of Rubinstein (1982) is typically used for solving non-cooperative bargaining games. A short introduction to this, following Myerson (1991, ch. 8.7) and Osborne and Rubinstein (1990, ch. 3), is given in section 2.2.3. Section 2.2.4 establishes a connection between the axiomatic and the strategic approach, based on Coles and Wright (1994, ch. 3) and Osborne and Rubinstein (1990, ch. 4).

2.2.1 An Axiomatic Approach: Nash's Solution

In his seminal 1950 and 1953 articles, Nash introduces the following framework for studying bargaining situations: In general, there are i rational players, but the set is usually restricted to two, i.e. $i = 1, 2$. When both players start bargaining, they can either come to an agreement in an arbitrary set A , or the disagreement event D occurs. It is assumed that all players have complete and symmetric information. Each player can put up a preference order for all possible outcomes of the game, i.e. over the set $A \cup D$. Players evaluate the outcome of the game with a utility function u_i . Based on these definitions, the set \mathcal{S} of all utility pairs resulting from bargaining is a payoff vector in a two dimensional space, i.e. $\mathcal{S} = \{[u_1(a), u_2(a)] \subset \mathcal{R}^2 : a \in A\}$, for an agreement. In the case of failure to reach an agreement, the disagreement point (or threat point, status quo) is $d = [u_1(D), u_2(D)]$. In accordance with Nash (1950), a bargaining problem is defined as a pair $\langle \mathcal{S}, d \rangle$. It is assumed that \mathcal{S} is closed, bounded, and convex, that $d \in \mathcal{S}$, and for some $s \in \mathcal{S}$ there exist $s_i > d_i$, for $i = 1, 2$. This implies the existence of a strictly better agreement allocation than the disagreement distribution for both players, i.e. bargaining is attractive.

The aim is to find a solution to this bargaining problem. Let the function $f(\mathcal{S}, d)$ be defined as the unique outcome of every bargaining problem $\langle \mathcal{S}, d \rangle$, with

$f(S, d) \in S$. In his axiomatic approach, Nash (1950) sets up some properties that the bargaining solution should meet, instead of explicitly constructing a solution. Then, he seeks solutions complying with these properties. The properties are stated within the following four axioms.⁵⁰

A1 *Scale Invariance to equivalent utility representations*: If the bargaining problem $\langle S', d' \rangle$ is the result of a transformation of $\langle S, d \rangle$ with $s'_i = \alpha_i s_i + \beta_i$ and $d'_i = \alpha_i d_i + \beta_i$, with $\alpha_i > 0$ for $i = 1, 2$, then $f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$.

A2 *Symmetry*: The bargaining problem is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ is equivalent to $(s_2, s_1) \in S$. Then, it follows that $f_1(S, d) = f_2(S, d)$.

A3 *Independence of irrelevant alternatives*: If (S, d) and (S^*, d) are two bargaining problems with the properties $S \subset S^*$ and $f(S^*, d) \in S$, then $f(S, d) = f(S^*, d)$.

A4 *Pareto efficiency*: Let $\langle S, d \rangle$ be a bargaining problem with $s \in S$ and $\hat{s} \in S$. If $\hat{s}_i > s_i$ for $i = 1, 2$, then $f(S, d) \neq s$.

Axiom A1 states that if S' is attained from S by an affine utility transformation, then the same transformation to the solution $f(S, d)$ should lead to the solution $f(S', d')$. Axiom A2 specifies that, if the positions of both players are indistinguishable in the bargaining problem, the solution should allocate the utility symmetrically. Axiom A3 determines that the bargaining solution should not change due to the elimination of irrelevant alternatives. These alternatives would not have been chosen and therefore should not influence the solution. Axiom A4 states, there should be no bargaining outcome that can be improved in such a way that both players can reach a higher utility. This leads to an instant agreement.

Nash (1950) proves that a unique solution $f(S, d)$ to the bargaining game, which satisfies all four axioms, does exist. It is the utility pair that maximizes the product of the players' increase in utility over the disagreement outcome, given by

$$f(S, d) = \arg \max_{(d_1, d_2) \leq (s_1, s_2) \in S} (s_1 - d_1)(s_2 - d_2), \quad (2.14)$$

with maximization over $s \in S$ and the constraint $s_i \geq d_i$, for $i = 1, 2$. The solution f to this maximization problem is called the Nash bargaining solution. Occasionally, it is also characterized as 'splitting the surplus' or 'splitting the pie'. The

⁵⁰ See Osborne and Rubinstein (1990), pp. 11–17.

product on the right-hand side of equation (2.14) is called the ‘Nash product’.⁵¹

Axiom A4 can be replaced by the following axiom, A5, as shown by Roth (1977):

A5 *Strong individual rationality*: For every bargaining problem $\langle S, d \rangle$, the solution f fulfills the condition $f(S, d) > d$.

It states that no player should accept a bargaining solution that is lower or equal to the guaranteed payoff in disagreement. Consequentially, bargaining is voluntary.

Harsanyi and Selten (1972) derive the so-called generalized (or asymmetric) Nash bargaining solution by dropping axiom A2 (Symmetry) but retaining the other three.⁵² It is specified by

$$f_q(S, d) = \arg \max_{(d_1, d_2) \leq (s_1, s_2) \in S} (s_1 - d_1)^q (s_2 - d_2)^{1-q}, \quad (2.15)$$

where $q \in [0, 1]$. Usually, q is referred to as the bargaining power of player 1, though bargaining power is basically characterized by the disagreement points d_1 and d_2 .⁵³ Another interpretation of q is that it is a relative measure of bargaining power. Different values for q can, e.g., arise if one trading partner is more patient than the other or has a different opinion about the probability of a disagreement.⁵⁴

The differences $s_1 - d_1$ and $s_2 - d_2$ are the surpluses from the game and the generalized Nash solution maximizes the surpluses’ weighted geometric mean.

2.2.2 Other Bargaining Solutions to Cooperative Games

Other bargaining solutions to cooperative games are the utilitarian and the egalitarian approach. However, both the utilitarian and the egalitarian solution violate the axiom of scale invariance (axiom A1).⁵⁵

The egalitarian solution states that in a two-person game both players must experience the same increase in utility. For the bargaining problem $\langle S, d \rangle$, the egal-

⁵¹ A proof can be found in Osborne and Rubinstein (1990), pp. 13.

⁵² Kalai (1977) formalized the so-called ‘nonsymmetric’ solution in the spirit of Nash’s axioms.

⁵³ See Pissarides (2001), p. 13764 and Binmore, Rubinstein, and Wolinsky (1986), p. 186 for an analysis.

⁵⁴ See Pissarides (2000), p. 16 and Binmore, Rubinstein, and Wolinsky (1986), p. 179.

⁵⁵ See Neus (2013), pp. 80–83 and Myerson (1991), pp. 381.

itarian solution $s \in \mathcal{S}$ satisfies the condition of equal gains

$$s_1 - d_1 = s_2 - d_2,$$

where s is weakly efficient in \mathcal{S} .

The utilitarian approach states that the player with the highest usage should get the 'surplus', independent of any disagreement point. A utilitarian solution $f(\mathcal{S})$ to the bargaining problem $\langle \mathcal{S} \rangle$ in a two-person game is any solution function that chooses an allocation $s \in \mathcal{S}$ with

$$f(\mathcal{S}) = \arg \max_{(s_1, s_2) \in \mathcal{S}} (s_1 + s_2).$$

2.2.3 A Strategic Approach: Alternating-Offer Bargaining Game

Rubinstein (1982) introduces a strategic approach by explicitly modeling an infinite horizon, two-person alternating-offer bargaining game: The order of moves, the time preference, and the conditions of an agreement are all well-defined. It is assumed that both players have complete information regarding the other's preferences. The rules for the game are as follows:

A 'pie' of size one is the bargaining basis for two players ($i = 1, 2$) and time runs forever with $t \in T$ for $T = \{0, 1, 2, \dots\}$. Upon agreement, player 1 gets the share x_1 and player 2 receives x_2 . The set containing all possible agreements is

$$X = \left\{ (x_1, x_2) \in \mathcal{R}^2 : x_1 + x_2 = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2 \right\}.$$

At time $t = 0$, one player (say 1) makes an offer x_1 , which the other player can accept or reject. Accepting the offer leads to an end to the game and the agreement is realized with the payoff (x_1, x_2) . If the offer is rejected, the game continues in period $t + 1 = 1$. At this time, player 2 makes an offer and player 1 can decide whether to accept or reject the proposal, and so on. In theory, this back and forth offering can continue endlessly, since the number of periods is not limited. This endless path is generally called 'disagreement', since all offers are rejected. It is denoted with D , where the payoff is $d = (0, 0)$. Additionally, time is valuable. The payoff upon agreement depends on the offer in that period and on the time of agreement, not on the path leading to it. When reaching an agreement in period t , the outcome for player 1 is $\vartheta^t x_1$, whereas it is $\vartheta^t (1 - x_1)$ for player 2, with the discount factor $\vartheta \in [0, 1]$.

The players' preferences are as follows:

- A1 *Disagreement is the worst outcome*: Agents prefer $x_i \in X$ at $t \in T$ over D .
- A2 *Pie is desirable*: Agents prefer $x_i \in X$ at $t \in T$ over $y_i \in X$ at $t \in T$, if $x_i > y_i$.
- A3 *Time is valuable*: Agents prefer $x_i \in X$ at $t \in T$ over $x_i \in X$ at $s \in T$, if $t < s$.
- A4 *Continuity*: Assume there are sequences $\{x_n\}_{n=1}^{\infty}$ at $t \in T$ and $\{y_n\}_{n=1}^{\infty}$ at $s \in T$. Both sequences are contained in X . Furthermore, $\lim_{n \rightarrow \infty} x_n = x$ as well as $\lim_{n \rightarrow \infty} y_n = y$ hold. Then, x at time t is preferred over y at time s if x_n at time t are preferred over y_n at time s for all n .
- A5 *Stationarity*: If any agent prefers $x_i \in X$ at $t = 0$ over $y_i \in X$ at $t = 1$, then he also prefers x_i at t over y_i at $t + 1$ for any time $t \in T$.

Rubinstein (1982) proves the uniqueness of a subgame perfect equilibrium⁵⁶ for this bargaining game. It is characterized as follows: Player 1 always offers $(\bar{x}_{1,t_1}, \bar{x}_{2,t_1})$ and accepts any offer by player 2 with $x_{1,t_2} \geq \bar{x}_{1,t_2}$. Player 2 always offers $(\bar{x}_{1,t_2}, \bar{x}_{2,t_2})$ and accepts any share $x_{2,t_1} \geq \bar{x}_{2,t_1}$. The shares \bar{x}_{1,t_2} and \bar{x}_{2,t_1} are called 'reservation value', where $t_1 = 0, 2, 4, \dots$ and $t_2 = 1, 3, 5, \dots$. Player 1 will always offer player 2's reservation value and player 2 will always offer player 1's reservation value. As a result, player 1 proposes \bar{x}_{2,t_1} in period $t = 0$ to player 2, player 2 accepts immediately, and the game ends. The payoff upon agreement is $(\bar{x}_{1,t_1}, \bar{x}_{2,t_1}) = (1/(1 + \vartheta), \vartheta/(1 + \vartheta))$. This game contains a first-mover advantage for $\vartheta < 1$.

2.2.4 Connection between Axiomatic and Strategic Approach

Binmore, Rubinstein, and Wolinsky (1986) consider special modifications in order to establish a connection between Nash's general bargaining solution and the result of a subgame perfect equilibrium of an alternating-offer bargaining game. I follow Coles and Wright (1994, ch. 3), who integrate the modifications of Binmore, Rubinstein, and Wolinsky (1986) and some other generalizations into one model.

The setup is as follows: Consider a discount factor for agent i with $i = 1, 2$, which is specified by $\vartheta_i = 1/(1 + r_i \Delta)$, and with the discount rate r_i . Agent i assumes that an exogenous breakdown occurs with Poisson arrival rates λ_i . In the case

⁵⁶ "A strategy pair is a *subgame perfect equilibrium* of a bargaining game of alternating offers if the strategy pair it induces in every subgame is a Nash equilibrium of that subgame." (Osborne and Rubinstein (1990), p. 44).

of an exogenous breakdown, the utility is b_i . Assume further that agent i can meet a new bargaining partner with Poisson arrival rate α_i . Let p_i be the probability with which agent i will make the next proposal, given that no breakdown occurred. And $p_1 + p_2 = 1$. Let the average offer be $x_1 = p_1 \bar{x}_{1,t_2} + p_2 \bar{x}_{1,t_1}$ and agents cannot derive any utility while bargaining is in progress.⁵⁷ Then, the alternating-offer bargaining game approaches the generalized Nash bargaining solution with threat points⁵⁸

$$d_i = b_i \quad \text{for } i = 1, 2,$$

and the bargaining power

$$q = \frac{p_1 (r_2 + \alpha_2 + \lambda_2)}{p_1 (r_2 + \alpha_2 + \lambda_2) + p_2 (r_1 + \alpha_1 + \lambda_1)}.$$

Rubinstein and Wolinsky (1985) implement this generalized alternating-offer bargaining game into a search, matching, and bargaining model, which is comparable to the model developed in Duffie, Gârleanu, and Pedersen (2005, 2007). A breakdown in such a model arises upon arrival of a new partner. This connection implies that the arrival rate for agent 1 is equal to the breakdown rate for agent 2 and the arrival rate for agent 2 is equal to the breakdown rate for agent 1: $\alpha_1 = \lambda_2$ and $\alpha_2 = \lambda_1$. Additionally, if $r_1 = r_2$, the bargaining power q of Nash's general bargaining solution equals the probability that agent 1 makes the first offer, that is

$$q = p_1.$$

⁵⁷ See Coles and Wright (1994), p. 14.

⁵⁸ See Coles and Wright (1994), p. 21.

Chapter 3

The Basic Model

In this chapter, I introduce the basic model of Duffie, Gârleanu, and Pedersen (2005) for asset pricing in an illiquid over-the-counter market. Illiquidity frictions in this OTC market are modeled by two-side search and bilateral trading between agents. As a result, trade does not happen instantly. First, I describe the model setup of Duffie, Gârleanu, and Pedersen (2005). In the second step, I present equilibrium masses of investor types. These masses are the basis for steady state equilibrium prices, which are discussed in section 3.3. Due to illiquidity frictions, these prices are lower than in a perfect market. The numerical example in section 3.4 illustrates the results of the basic model. The appendix 3A contains the derivation of some results from section 3.3.

3.1 Model Setup

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information filtration $\mathcal{F}_t : t > 0$ in a risk-neutral world with continuous time ($t \geq 0$) are fixed in advance. Ω is the set of all possible states in the world. \mathcal{F} is the filtration of sub- σ -algebras. It describes the revealing of information to investors over time. \mathbb{P} is the probability measure on \mathcal{F} . It is assumed that probability space and the information filtration satisfy the usual hypotheses as defined by Protter (2005, p. 3).

Let us assume the economy is populated by two kinds of agents—investors and market makers—who are both risk-neutral and live infinitely. All agents have a constant and known time preference rate $r > 0$, with which they discount the future. A single nonstorable consumption good, consumed by all agents (e.g.

'cash'), is used as a numéraire.⁵⁹ All agents have access to a (risk free) bank account offering interest rate r , and to the OTC market for a special asset. The bank account is comparable with a liquid security, which can be traded without frictions. To avoid unlimited borrowing, the value W_t of the bank account is bounded from below. The asset traded in the OTC market is illiquid, as it can only be traded when a potential buyer and a potential seller can find each other. The illiquid asset pays a constant dividend rate D of consumption per time unit (e.g. one year), like a consol bond.⁶⁰ Time runs forever. Each agent can only hold a maximum of (illiquid) assets at a time, which is normalized to one. As the utility function of risk-neutral agents is linear, asset holding corresponds to either zero or one unit in equilibrium. Initially, a fraction s of all investors is endowed with one unit of this asset, implying a fixed asset supply. Short selling is not allowed.

The population of investors is segmented into four different groups: Investors can either own the asset (o), or not (n), and they all have either an intrinsic type that is high (h) or low (l). These intrinsic types can be interpreted as the investor's marginal utility from the asset. Duffie, Gârleanu, and Pedersen (2005, pp. 1818.) give some possible explanations for low-type investors: "(i) low liquidity (that is, a need for cash), (ii) high financing costs, (iii) hedging reasons to sell, (iv) a relative tax disadvantage, or (v) a low personal use of the asset." The full set of investor types is $\Gamma = \{ho, hn, lo, ln\}$.

The intrinsic type of an investor is modeled as a Markov chain. A low investor receives an exogenous idiosyncratic preference or funding shock causing a type switch from low to high with an intensity $\lambda_u > 0$. A high investor who suffers such a preference (or funding) shock switches from high to low with intensity $\lambda_d > 0$. The switching processes are random and are assumed to be pairwise independent for any two investors. Type switches generate a need for change in asset holdings, since investors' valuation, i.e. marginal utility, towards the asset changes over time and depends on investor's type. Because only low-type owners want to sell their asset, whereas high-type non-owners want to buy one in equilibrium, type switches generate trade. Ln agents and ho agents do not trade. Consequently, low owners (lo) are called potential sellers and high non-owners (hn) are called potential buyers.

High owners who are affected by an idiosyncratic preference shock switch to a

⁵⁹ See Weill (2007), p. 1332.

⁶⁰ Duffie, Gârleanu, and Pedersen (2005, 2007) normalize this dividend to $D = 1$.

low intrinsic type. Those low-type agents are exposed to a holding cost for the asset of δ per unit of time, with $\delta > 0$, leading to a utility flow of $D - \delta$. Holding costs only occur for low-type investors owning the asset, as it reflects the negative impact of an idiosyncratic liquidity shock. Since there can be gains from trade due to different utility and due to costs of holding assets, low owners want to sell their asset.

Duffie, Gârleanu, and Pedersen (2005) define a unit mass continuum of investors with measure normalized to one. Then, $\mu_\sigma(t)$ denotes for each $\sigma \in \Gamma$ the fraction at time t of type- σ investors in the total population. These fractions must add up to one at any time and must be nonnegative, leading to

$$\mu_{lo}(t) + \mu_{hn}(t) + \mu_{ln}(t) + \mu_{ho}(t) = 1, \quad (3.1)$$

$$\mu_\sigma(t) \geq 0. \quad (3.2)$$

By assumption, only a fraction $s \in [0, 1]$ of investors owns one unit of the asset. This prerequisite defines the market clearing condition, which implies for every time t that

$$\mu_{lo}(t) + \mu_{ho}(t) = s. \quad (3.3)$$

Agents who want to trade in an OTC market must search for each other, since no central trading device is available. Assume $\lambda \in [0, \infty)$ is the exogenous and constant intensity of a homogenous Poisson process. This intensity λ describes the random contact of one investor with a counterparty, and reflects search ability or efficiency in the OTC market. The search technology in this market is as follows: Assume that any agent, which is chosen from the set of all agents, is of type σ_1 . The probability for being of type σ_1 is equal to $\mu_{\sigma_1}(t)$. The probability of any agent, which is not of type σ_1 , being matched with an agent of type σ_1 is then $\lambda \mu_{\sigma_1}(t)$. All agents of, say, a set σ_2 , where set σ_2 is distinct from set σ_1 , search for agents of set σ_1 with $\lambda \mu_{\sigma_1}(t) \mu_{\sigma_2}(t)$. Simultaneously, agents of set σ_1 search for agents of set σ_2 . The matching function derived in equation (2.13) models exactly this independent search and matching process for potential buyers and sellers.⁶¹ With application to the prevailing notation, the appropriate matching function is $\mathcal{M}_\lambda(t) = 2\lambda \mu_{lo}(t) \mu_{hn}(t)$, where $\mathcal{M}(t)$ is the number of successful matches per unit of time. As soon as two agents meet, they start bargaining over the price,

⁶¹ The law of large numbers is assumed to hold throughout. See Duffie and Sun (2007, 2012) and footnote 43.

according to a bargaining process described in chapter 2.2. After completing the transaction, the *lo* agent (seller) becomes an *ln* agent, the *hn* agent (buyer) becomes an *ho* agent, and both part ways.

This search model contains, according to Vayanos and Wang (2007, p. 75), a natural liquidity measure. As investors are prevented from immediate trading, costs of delay accrue. These costs can be measured by the expected time it takes until an investor finds an adequate counterparty. This expected time is nothing else but the inverse of the measure of liquidity. An *hn* agent (potential buyer) meets *lo* agents (potential sellers) at the rate $2\lambda\mu_{lo}(t)$. The average time it takes until a potential buyer meets potential sellers is $t_{hn}(t) = 1 / (2\lambda\mu_{lo}(t))$. In the same way, the expected meeting time for a potential seller to meet potential buyers is $t_{lo}(t) = 1 / (2\lambda\mu_{hn}(t))$.⁶²

Additionally, it is assumed that there are independent nonatomic market makers in this OTC market who are of unit mass and who maximize their profit. Investors and market makers search for each other and meet with exogenous and constant intensity $\rho \geq 0$. It is the sum of investors' intensity of searching for market makers and market makers' intensity of searching for investors. This intensity captures the availability of market makers in the market. Investors and market makers also start bargaining over the price as soon as they meet. It is assumed that market makers have no inventory but immediately unload their asset on a frictionless interdealer market. Without bearing any inventory risk, market makers are matchmakers. They ration either the buy side or the sell side, depending on which one is higher. The matching function is $\mathcal{M}_\rho(t) = \rho \min\{\mu_{lo}(t), \mu_{hn}(t)\}$.

For tractability, it is assumed that all described Poisson processes are independent. The flow diagram in figure 3.1, which is in the style of Chiu and Koepl (2011, p. 7), illustrates search and bargaining in the just-specified OTC market.

⁶² See chapter 2.1.3, Vayanos and Wang (2007), p. 75, and Duffie, Gârleanu, and Pedersen (2007), p. 1878.

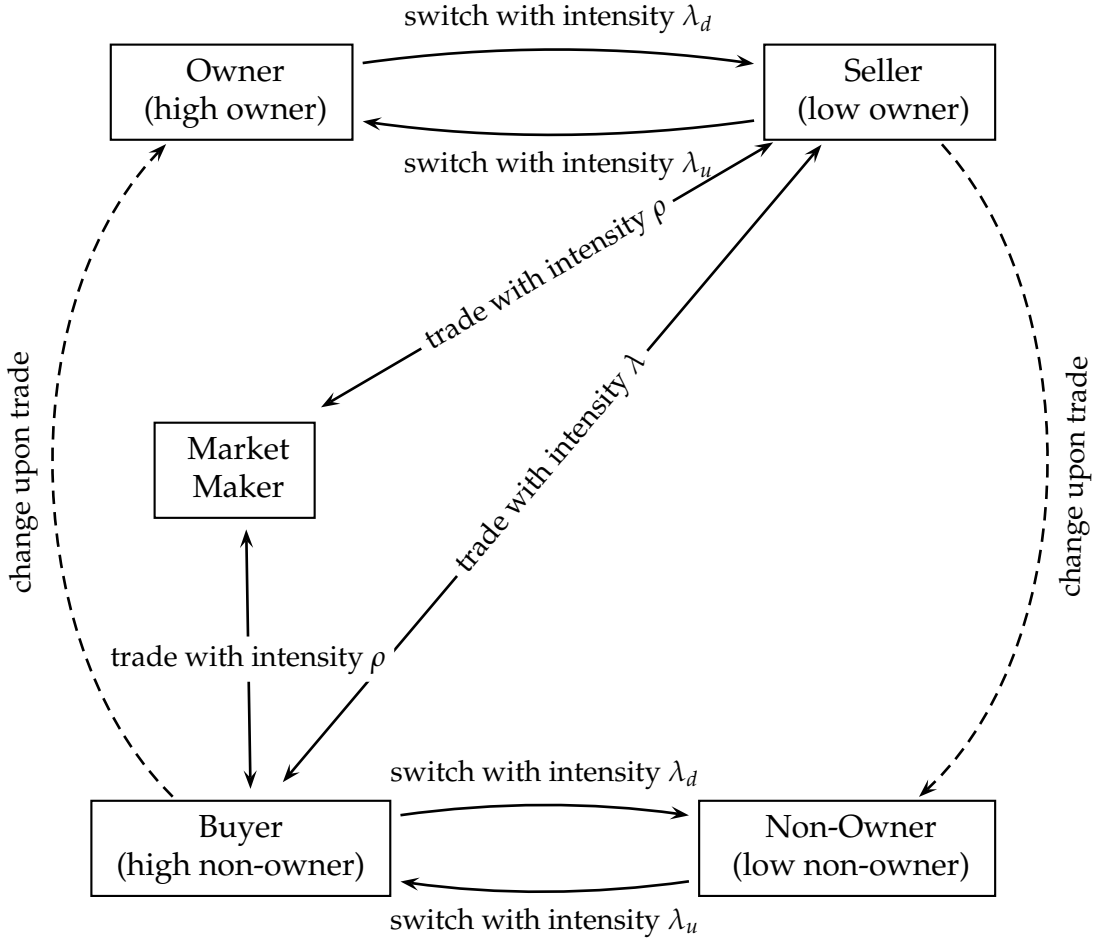


Figure 3.1: Flow diagram.

3.2 Equilibrium Masses of Investor Types

This section discusses both the dynamic and steady state equilibrium masses of investor types. The starting point is the flow equations of masses $\mu_\sigma(t)$, which are stated as follows:

$$\dot{\mu}_{lo}(t) = - (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t), \quad (3.4)$$

$$\dot{\mu}_{hn}(t) = - (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) + \lambda_u\mu_{ln}(t) - \lambda_d\mu_{hn}(t), \quad (3.5)$$

$$\dot{\mu}_{ho}(t) = (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) + \lambda_u\mu_{lo}(t) - \lambda_d\mu_{ho}(t), \quad (3.6)$$

$$\dot{\mu}_{ln}(t) = (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) - \lambda_u\mu_{ln}(t) + \lambda_d\mu_{hn}(t), \quad (3.7)$$

with $\mu_m(t) = \min\{\mu_{lo}(t), \mu_{hn}(t)\}$ and $\dot{\mu}_\sigma(t) = d\mu_\sigma(t)/dt$. The first term in the brackets of equations (3.4)–(3.7) describes mass changes based on random search, bargaining, and trade between two investors, i.e. between lo and hn agents. Bargaining theory states that trade is executed immediately, when gains from trade

can be acquired. These flow equations implicitly assume that all meetings between a potential buyer and a potential seller result in a trade. For the sake of simplicity, potential sellers and potential buyers are designated ‘seller’ and ‘buyer’ when the presumption that all meetings result in a trade is fulfilled. The total change of masses based on these *lo-hn*-meetings accounts for $2\lambda\mu_{hn}(t)\mu_{lo}(t)$, i.e. the matching function $\mathcal{M}_\lambda(t)$ between investors. Upon completion, when an *lo* agent and an *hn* agent successfully trade with each other, the *lo* agent changes his type to an *ln* agent and the *hn* agent changes his type to an *ho* agent.

The second term in the brackets illustrates mass changes due to meetings between investors and market makers: *lo* agents meet market makers at the contact rate $\rho\mu_{lo}(t)$, whereas *hn* agents meet market makers at the contact rate $\rho\mu_{hn}(t)$. As long as more *hn* investors meet market makers than *lo* agents do, i.e. $\mu_{lo}(t) \leq \mu_{hn}(t)$, all *lo* agents can sell their assets to the market makers and all *lo* investors change to *ln* investors. The *hn* investors get rationed. It follows that $\rho\mu_{lo}(t)$ is the matching function between investors and market makers. In case there are less *hn* agents than *lo* ones, there are more investors who are required to sell their asset than to buy one. But in equilibrium, supply and demand must be balanced and market makers are matchmakers. If $\mu_{lo}(t) > \mu_{hn}(t)$, trade between investors and market makers takes place at the intensity $\rho\mu_{hn}(t)$. The *lo* investors get rationed.

The last two terms of equations (3.4)–(3.7) state the intrinsic type changes: *lo* and *ln* investors mutate to high-type investors with intensity λ_u , whereas *ho* and *hn* investors mutate to low-type investors with intensity λ_d . Masses change due to these mutations.

The flow equations (3.4)–(3.7) of equilibrium masses $\mu_\sigma(t)$ are combined to two ordinary differential equations (ODEs) by defining $\mu_l(t) = \mu_{lo}(t) + \mu_{ln}(t)$ and $\mu_h(t) = \mu_{ho}(t) + \mu_{hn}(t)$. The ODEs are

$$\begin{aligned}\dot{\mu}_l(t) &= -(\lambda_u + \lambda_d)\mu_l(t) + \lambda_d, \\ \dot{\mu}_h(t) &= -(\lambda_u + \lambda_d)\mu_h(t) + \lambda_u.\end{aligned}$$

With the initial conditions $\mu_l(t_0 = 0) = 0$ and $\mu_h(t_0 = 0) = 0$, the solution to these ODEs

are⁶³

$$\mu_l(t) = \mu_l(0)e^{-(\lambda_u + \lambda_d)t} + \frac{\lambda_d}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u + \lambda_d)t} \right], \quad (3.8)$$

$$\mu_h(t) = \mu_h(0)e^{-(\lambda_u + \lambda_d)t} + \frac{\lambda_u}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u + \lambda_d)t} \right]. \quad (3.9)$$

From any feasible starting condition $\mu_\sigma(0)$, equations (3.8) and (3.9) converge monotonically to $\lim_{t \rightarrow \infty} \mu_l(t) = \lambda_d / (\lambda_u + \lambda_d)$ and $\lim_{t \rightarrow \infty} \mu_h(t) = \lambda_u / (\lambda_u + \lambda_d)$, respectively.

From here on, chapter 3 deals with steady state equilibria. This condition implies a constant mass distribution over time. The steady state solutions to equations (3.1)–(3.7) are derived as follows: Start with equation (3.4) and express all $\mu_\sigma(t)$ in terms of $\mu_{lo}(t)$ by using equations (3.1), (3.3), and the definition $\mu_l(t) = \mu_{lo}(t) + \mu_{ln}(t)$. It can be seen that equation (3.8) is independent from $\mu_{lo}(t)$ for $t \rightarrow \infty$. Hence, equation (3.4) can be written as

$$\begin{aligned} \dot{\mu}_{lo}(t) &= - [2\lambda (1 - s - \mu_{ln}(t)) \mu_{lo}(t) + \rho \mu_m(t)] - \lambda_u \mu_{lo}(t) + \lambda_d (s - \mu_{lo}(t)) \\ &= - 2\lambda [1 - s - (\mu_l(t) - \mu_{lo}(t))] \mu_{lo}(t) \\ &\quad - \rho \min \{ \mu_{lo}(t), 1 - s - \mu_l(t) + \mu_{lo}(t) \} - \lambda_u \mu_{lo}(t) + \lambda_d (s - \mu_{lo}(t)) \\ &= - 2\lambda (\mu_{lo}(t))^2 - [2\lambda (1 - s - \mu_l(t)) + \lambda_u + \lambda_d] \mu_{lo}(t) \\ &\quad + \rho \max \{ -\mu_{lo}(t), -1 + s + \mu_l(t) - \mu_{lo}(t) \} + \lambda_d s. \end{aligned}$$

The following solution function results with $\dot{\mu}_{lo}(t) = G(\mu_{lo}(t), \mu_l(t))$:

$$\begin{aligned} G(\mu_{lo}(t), \mu_l(t)) &= - 2\lambda (\mu_{lo}(t))^2 - [2\lambda (1 - s - \mu_l(t)) + \lambda_u + \lambda_d + \rho] \mu_{lo}(t) \\ &\quad + \rho \max \{ 0, s + \mu_l(t) - 1 \} + \lambda_d s. \end{aligned} \quad (3.10)$$

⁶³ A general solution to an arbitrary first-order linear differential equation of the form $\dot{x}(t) = -a(t)x(t) + b(t)$ is

$$x(t) = ce^{-\int_{t_0}^t a(\tau) d\tau} + e^{-\int_{t_0}^t a(\tau) d\tau} \int_{t_0}^t b(y) e^{\int_{t_0}^y a(\tau) d\tau} dy,$$

with an arbitrary constant c . The initial condition $x(t_0) = x_0$ specifies this constant. See Polyanin and Zaitsev (2003), p. 4.

For the steady state equilibrium, consider $\lim_{t \rightarrow \infty} \dot{\mu}_\sigma(t) = 0$, i.e. masses are constant in steady state, and denote $\lim_{t \rightarrow \infty} \mu_\sigma(t) = \mu_\sigma(ss)$. This steady state equilibrium implies, for equation (3.10), that

$$0 = 2\lambda(\mu_{lo}(ss))^2 + [2\lambda(y - s) + \rho + \lambda_u + \lambda_d] \mu_{lo}(ss) - \lambda_d s - \rho \max[0, s - y], \quad (3.11)$$

where

$$y = \lim_{t \rightarrow \infty} \mu_h(t) = \frac{\lambda_u}{\lambda_u + \lambda_d} \quad (3.12)$$

is the probability of being a high-type agent in steady state.⁶⁴ I state the solution to equation (3.11), which is the steady state equilibrium mass of low owners, by solving the quadratic formula

$$\mu_{lo}(ss) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (3.13)$$

with

$$a = 2\lambda,$$

$$b = 2\lambda(y - s) + \rho + \lambda_u + \lambda_d,$$

$$c = -\rho \max[0, s - y] - \lambda_d s.$$

There is one negative and one positive solution for equation (3.13). For economic reasons, the negative solution is excluded, since $\mu_\sigma(t) \in [0, 1]$ must hold for all t . The positive solution in $[0, 1]$ is assured with $G(\mu_{lo}(ss) = 0, \mu_l(ss) = 1 - y) > 0$ and $G(\mu_{lo}(ss) = 1, \mu_l(ss) = 1 - y) < 0$ of equation (3.10).

In steady state, the other three equilibrium masses result as follows:

$$\mu_{hn}(ss) = y - s + \mu_{lo}(ss), \quad (3.14)$$

$$\mu_{ho}(ss) = s - \mu_{lo}(ss), \quad (3.15)$$

$$\mu_{ln}(ss) = 1 - y - \mu_{lo}(ss). \quad (3.16)$$

Duffie, Gârleanu, and Pedersen (2005, pp. 1836) prove the uniqueness of the constant steady state solution stated in equations (3.13)–(3.15). For $t \rightarrow \infty$, the mass distribution $\mu_\sigma(t)$ always converges by means of equations (3.4)–(3.7) to this steady state solution, whatever the starting conditions $\mu_\sigma(t = 0)$, satisfying (3.1) and (3.3), may be.

⁶⁴ Duffie, Gârleanu, and Pedersen (2005, pp. 1836) state this result only for $s < y$.

3.3 Equilibrium Prices

Steady state equilibrium masses of investor types, derived in section 3.2, provide the basis for steady state prices. The interinvestor price $P(ss)$, determined by bargaining between two investors, is calculated in the following passage. Likewise, both the bid price $B(ss)$, which an investor receives when selling to a market maker, and the ask price $A(ss)$, which an investor has to pay when buying from a market maker, are derived. First, investors' utilities of lifetime consumption are calculated.

3.3.1 The Value Function

Each rational investor chooses an asset holding strategy that maximizes his expected utility (i.e. present value) of his lifetime consumption. Since each individual lives infinitely, a continuous and infinite consumption process has to be modeled by considering a search and matching process. At an arbitrary time t , the investor's utility depends only on his current type, $\sigma(t) \in \Gamma$, and wealth or money W_t , which he has in his bank account. The infinite horizon expected utility-maximization problem for all investor types, who are risk-neutral and measure their lifetime consumption with a utility function, can be derived by means of dynamic programming. The optimal value function $J(\cdot)$, the optimum value of the utility-maximization problem, is stated as follows:⁶⁵

$$J(W_t, \sigma(t), t) = \sup_{C, \theta} E_t \left[\int_0^{\infty} e^{-rv} dC_{t+v} \right], \quad (3.17)$$

given the dynamics

$$dW_t = rW_t dt - dC_t + \theta_t (D - \delta \mathbf{1}_{\{\sigma^\theta(t)=lo\}}) dt - \hat{P}(t) d\theta_t, \quad (3.18)$$

with the expectation E_t , conditioned on \mathcal{F}_t . An investor can freely decide over his consumption and asset holding, so that the two control processes are: (1) a cumulative consumption process C_t , and (2) a feasible asset holding process

⁶⁵ In general, it is not clear that the maximum actually exists for these processes until it is known that $J(W_t, \sigma(t), t)$ is bounded. Therefore, the supremum for a precise formulation of the utility-maximization problem is applied first. Verification that the maximum is actually attained follows in a second step, i.e. the verification that the value function is bounded. If a value function is unbounded, it can go to infinity, but it can never attain it. In such a case, the supremum is suitable. See Bellman (1954), p. 507.

$\theta_t \in \{0, 1\}$. The other input parameters are: (i) σ^θ , the type process induced by θ , and (ii) $\hat{P}(t) \in \{P(t), A(t), B(t)\}$ which is the trade price at time t , dependent on the agent's counterparty. $J(W_t, \sigma(t), t)$ is called a value function or indirect utility function. It differs from a normal utility function as it always implies an optimization process.

The core of the dynamic programming theory is Bellman's 'principle of optimality':

"An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions."⁶⁶

A recursive application of Bellman's 'principle of optimality' on equation (3.17) leads to an iterative optimization problem

$$\begin{aligned} J(W_t, \sigma(t), t) &= \sup_{C, \theta} E_t \left[\int_t^\infty e^{-r(k-t)} dC_k \right] \\ &= \sup_{C, \theta} E_t \left[\int_t^{t+dt} e^{-r(k-t)} dC_k + J(W_t + dW_t, \sigma(t) + d\sigma(t), t + dt) \right], \end{aligned} \quad (3.19)$$

with $k = t + v$. Dynamic programming approaches a dynamic optimization problem by a recursive solution technique, translating a problem composed of multi-stages into a sequence of separate states.⁶⁷ This recursive solution implies the consideration of all possible states within a final period by weighting the corresponding payoffs with the probability of their occurrence. Working backward in time leads to the optimal equilibrium path.

The first part of equation (3.19) can be approximated by the mean value theorem of integral calculus. The second part can be derived by a Taylor series expansion of function $J(\cdot)$ around the point $(W_t, \sigma(t), t)$ to approximate $J(W_t + dW_t, \sigma(t) + d\sigma(t), t + dt)$. Inserting both parts into equation (3.19), subtracting $J(W_t, \sigma(t), t)$ on both sides, dividing everything by dt , and letting $dt \rightarrow 0$, leads to the Hamilton–Jacobi–Bellman (HJB) equation. The optimal value function in continuous time dynamic programming is the solution to the HJB equation, which is in general a partial differential equation. This equation acts as a

⁶⁶ Bellman (1954), p. 504.

⁶⁷ See Bellman (1954), p. 503.

necessary and sufficient condition to ensure optimality.⁶⁸

Since agents are risk-neutral by assumption, the value function (3.17) describing investors' lifetime utility is a linear function in wealth W_t . I show this by inserting equation (3.18) into equation (3.17), leading to

$$\begin{aligned}
J(W_t, \sigma(t), t) &= \sup_{\theta \in \{0,1\}} E_t \left[\int_t^\infty e^{-r(k-t)} \cdot \left[rW_k dk - dW_k \right. \right. \\
&\quad \left. \left. + \theta_k \left(D - \delta \mathbf{1}_{\{\sigma^\theta(k)=l_0\}} \right) dk - \hat{P}(k) d\theta_k \right] \right] \\
&= \sup_{\theta} E_t \left[\int_t^\infty e^{-r(k-t)} \cdot \left(rW_k dk - \frac{dW_k}{dk} dk \right) \right] \\
&\quad + \underbrace{\sup_{\theta} E_t \left[\int_t^\infty e^{-r(k-t)} \cdot \left(\theta_k (D - \delta \mathbf{1}_{\{\sigma^\theta(k)=l_0\}}) dk - \hat{P}(k) d\theta_k \right) \right]}_{V_\sigma(t)} \\
&= \sup_{\theta} E_t \left[\int_t^\infty r e^{-r(k-t)} W_k dk - \left[e^{-r(k-t)} W_k \right]_t^\infty - \int_t^\infty r e^{-r(k-t)} W_k dk \right] \\
&\quad + V_\sigma(t) \\
&= W_t + V_\sigma(t).
\end{aligned}$$

Since the value function $J(W_t, \sigma(t), t)$ is linear in wealth, it is sufficient to solve the following maximization problem:

$$V_\sigma(t) = \sup_{\theta \in \{0,1\}} E_t \left[\int_t^\infty e^{-r(v-t)} \left(\theta_v \left(D - \delta \mathbf{1}_{\{\sigma^\theta(v)=l_0\}} \right) dv - \hat{P}(v) d\theta_v \right) \right]. \quad (3.20)$$

The utility-maximization problem is thus changed from deciding over both optimal consumption and optimal asset holding to only choosing the optimal asset holding. The transversality condition (also called the no-bubble condition) $\lim_{x \rightarrow \infty} E_t [e^{-rx} \max \{P(x), A(x), B(x)\}] = 0$ ensures that the value function is well defined.

Relating this finding to continuous time dynamic programming, the value function $V_\sigma(t)$ can be calculated by applying Bellman's 'principle of optimality' and focusing on a particular agent at a particular time t . Since agents are risk-neutral

⁶⁸ See Schöbel (1995), ch. 3.3 and Björk (2009), ch. 19.

by assumption and the value function $J(W_t, \sigma(t), t)$ is linear in holding cash W_t , the HJB equation happens to be a system of ordinary differential equations, and is derived in the following passage.

Duffie, Gârleanu, and Pedersen (2005, p. 1837) define τ_l as the next stopping time when an agent changes his intrinsic type, τ_i as the next stopping time when a search and bargaining between two investors is successfully completed, τ_m as the next stopping time when trade occurs between an investor and a market maker, and $\tau = \min\{\tau_l, \tau_i, \tau_m\}$. The optimal value functions result with

$$\begin{aligned} V_{lo}(t) = E_t & \left[\int_t^\tau e^{-r(u-t)} (D - \delta) du + e^{-r(\tau_l-t)} V_{ho}(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} \right. \\ & + e^{-r(\tau_i-t)} (V_{ln}(\tau_i) + P(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \\ & \left. + e^{-r(\tau_m-t)} (V_{ln}(\tau_m) + B(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} \right], \end{aligned} \quad (3.21)$$

$$\begin{aligned} V_{hn}(t) = E_t & \left[e^{-r(\tau_l-t)} V_{ln}(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} + e^{-r(\tau_i-t)} (V_{ho}(\tau_i) - P(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \right. \\ & \left. + e^{-r(\tau_m-t)} (V_{ho}(\tau_m) - A(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} \right], \end{aligned} \quad (3.22)$$

$$V_{ho}(t) = E_t \left[\int_t^{\tau_l} e^{-r(u-t)} D du + e^{-r(\tau_l-t)} V_{lo}(\tau_l) \right], \quad (3.23)$$

$$V_{ln}(t) = E_t \left[e^{-r(\tau_l-t)} V_{hn}(\tau_l) \right], \quad (3.24)$$

where the expectation is with respect to τ_l, τ_i, τ_m and is conditional on \mathcal{F}_t . The first term of asset owners' value functions $[V_{lo}(t), V_{ho}(t)]$ gives the dividend flow, possibly reduced by holding costs. The second term of owners' value functions $[V_{lo}(t), V_{ho}(t)]$ and the first of non-owners' value functions $[V_{ln}(t), V_{hn}(t)]$ describes the discounted value of an intrinsic type switch, given the random stopping time is $\tau = \tau_l$. The second last term of potential buyers' and potential sellers' value functions $[V_{lo}(t), V_{hn}(t)]$ is the discounted value of trading with an investor, given that the random stopping time is $\tau = \tau_i$. And the last part of potential buyers' and potential sellers' value functions $[V_{lo}(t), V_{hn}(t)]$ describes the discounted value of trading with a market maker, given that the random stopping time is $\tau = \tau_m$. As a result, investor's utility depends on his current expected utility, e.g. from holding the asset, and on his prospective expected utility.

I state the explicit equations for (3.21)–(3.24) and the derivation of the HJB equations in appendix 3A. These HJB equations, which are solved by the value functions $V_\sigma(t)$, are⁶⁹

$$\begin{aligned} \dot{V}_{lo}(t) = & rV_{lo}(t) - \lambda_u (V_{ho}(t) - V_{lo}(t)) - \rho (V_{ln}(t) + B(t) - V_{lo}(t)) \\ & - 2\lambda\mu_{hn}(t) (V_{ln}(t) + P(t) - V_{lo}(t)) - (D - \delta), \end{aligned} \quad (3.25)$$

$$\begin{aligned} \dot{V}_{hn}(t) = & rV_{hn}(t) - \lambda_d (V_{ln}(t) - V_{hn}(t)) - \rho (V_{ho}(t) - A(t) - V_{hn}(t)) \\ & - 2\lambda\mu_{lo}(t) (V_{ho}(t) - P(t) - V_{hn}(t)), \end{aligned} \quad (3.26)$$

$$\dot{V}_{ho}(t) = rV_{ho}(t) - \lambda_d (V_{lo}(t) - V_{ho}(t)) - D, \quad (3.27)$$

$$\dot{V}_{ln}(t) = rV_{ln}(t) - \lambda_u (V_{hn}(t) - V_{ln}(t)). \quad (3.28)$$

The first part on the right hand side of equations (3.25)–(3.28) corresponds to opportunity costs. The second element characterizes value changes based on expected changes in intrinsic types. For buyer (*hn*) and seller (*lo*), the third and fourth element is due to trade between investors and trade intermediated by market makers, respectively. The last term for asset owners (*lo*, *ho*) accounts for dividends and holding costs of the asset.⁷⁰

In order to consider steady state equilibria, the value changes have to be zero: $\dot{V}_\sigma(t) = 0$. I write $\lim_{t \rightarrow \infty} V_\sigma(t) = V_\sigma(ss)$ for steady state value functions. Since only steady state equilibria are considered, prices are time independent as well. Upon setting equations (3.25)–(3.28) to zero and rearranging them, the value functions in steady state are

$$V_{lo}(ss) = \frac{\lambda_u V_{ho}(ss) + (2\lambda\mu_{hn}(ss) + \rho)V_{ln}(ss) + 2\lambda\mu_{hn}(ss)P(ss) + \rho B(ss) + D - \delta}{r + \lambda_u + 2\lambda\mu_{hn}(ss) + \rho}, \quad (3.29)$$

$$V_{hn}(ss) = \frac{\lambda_d V_{ln}(ss) + (2\lambda\mu_{lo}(ss) + \rho)V_{ho} - 2\lambda\mu_{lo}(ss)P(ss) - \rho A(ss)}{r + \lambda_d + 2\lambda\mu_{lo}(ss) + \rho}, \quad (3.30)$$

$$V_{ho}(ss) = \frac{\lambda_d V_{lo}(ss) + D}{r + \lambda_d}, \quad (3.31)$$

$$V_{ln}(ss) = \frac{\lambda_u V_{hn}(ss)}{r + \lambda_u}. \quad (3.32)$$

⁶⁹ Duffie, Gârleanu, and Pedersen (2005, pp. 1839) show optimality by verifying that under complete information any agent always trades at the stated equilibrium strategy, provided others do so. Trades are always executed at proposed equilibrium prices if gains from trade are possible with the agent in contact.

⁷⁰ There is a typing error in Duffie, Gârleanu, and Pedersen (2005, p. 1823), equation (10). The Poisson arrival intensity for a buyer (*hn*) contacting sellers (*lo*) is $2\lambda\mu_{lo}(t)$.

Equation system (3.29)–(3.32) still depends on bargaining prices. The next section states the bargaining conditions for deriving these prices.

3.3.2 Bargaining over the Price

Until now, not much has been said about the structure of the price, except that (i) trade occurs only between *lo*-type investors who want to sell and *hn*-type investors who want to buy, and (ii) they start bargaining over the price as soon as they meet each other.⁷¹

It is clear from equation (3.25) that an *lo*-type investor will only accept a price that is higher than or equal to $\Delta V_l(t) = V_{lo}(t) - V_{ln}(t)$, i.e. $V_{ln}(t) + P(t) - V_{lo}(t) \geq 0$. This means that the price has to compensate the investor for the change in his utility when selling the asset. Otherwise it would be advantageous for him to keep the asset. $\Delta V_l(t)$ is the reservation value or participation constraint of a potential seller. At this point, a potential seller is indifferent to trading.

Equation (3.26) shows that an *hn*-type investor will only purchase an asset for a price which is lower than or equal to $\Delta V_h(t) = V_{ho}(t) - V_{hn}(t)$, that is $V_{ho}(t) - P(t) - V_{hn}(t) \geq 0$. Otherwise he would be better off without the asset. $\Delta V_h(t)$ is the reservation value or participation constraint of a potential buyer.

If these two types of investors, low owner and high non-owner, start bargaining over the price, the result is located somewhere between these two reservation values $\Delta V_l(t)$ and $\Delta V_h(t)$.

It is assumed that all agents have complete and symmetric information, and, under this assumption, bargaining theory states that trade happens instantly. Assume further that an *lo*-type investor (potential seller) has bargaining power $q \in [0, 1]$. The generalized Nash solution, defined in equation (2.15), applies with

$$f_q^P(t) = \arg \max_{P(t)} [V_{ln}(t) + P(t) - V_{lo}(t)]^q [V_{ho}(t) - P(t) - V_{hn}(t)]^{1-q}, \quad (3.33)$$

subject to (s. t.)

$$0 \leq V_{ln}(t) + P(t) - V_{lo}(t), \quad (3.34)$$

$$0 \leq V_{ho}(t) - P(t) - V_{hn}(t), \quad (3.35)$$

⁷¹ An introduction into bargaining theory is given in section 2.2.

for all t .⁷² The agreement point for a potential seller (*lo*-type) is $V_{ln}(t) + P(t)$, which is the value of a seller if an agreement is reached. The agreement point for a potential buyer (*ln*-type) is $V_{ho}(t) - P(t)$, which is the value of a buyer if an agreement is reached. In absence of an agreement, the seller keeps the asset and stays with the value $V_{lo}(t)$, which is his disagreement or threat point. The situation is analogous for the buyer: He remains without an asset and stays with the value $V_{hn}(t)$ if they fail to reach an agreement. The bargaining situation of buyers and sellers depends on their outside options, which are equal to their threat points in this setting.⁷³ These outside options depend on the availability of suitable counterparties over time,⁷⁴ since outside options are the expected value of waiting for a new trading partner—unless an intrinsic type switch occurs in the meantime.

The maximization of the argument in (3.33) is carried out over the price $P(t)$ and is subject to both $V_{ln}(t) + P(t) \geq V_{lo}(t)$ and $V_{ho}(t) - P(t) \geq V_{hn}(t)$. When there are gains from trade, the first order condition is

$$0 = q[V_{ln}(t) + P(t) - V_{lo}(t)]^{q-1} [V_{ho}(t) - P(t) - V_{hn}(t)]^{1-q} - [V_{ln}(t) + P(t) - V_{lo}(t)]^q (1 - q)[V_{ho}(t) - P(t) - V_{hn}(t)]^{(-q)}.$$

As a result, the price function is

$$P(t) = (1 - q) [V_{lo}(t) - V_{ln}(t)] + q [V_{ho}(t) - V_{hn}(t)], \quad (3.36)$$

$$\text{s. t. } V_{lo}(t) - V_{ln}(t) \leq P(t) \leq V_{ho}(t) - V_{hn}(t). \quad (3.37)$$

An intuitive interpretation for inequality (3.37) is that the reservation value of a buyer should be higher than a seller's reservation value. The buyer is of high intrinsic type, whereas the seller is a low-type. Because the buyer has no holding costs, the flow of dividends has a higher value for the buyer than for the seller. When the buyer switches to a low valuation type some day, he has the same trading possibilities—in steady state—as the seller had at that time. As a result, trade should be efficient.⁷⁵

⁷² The alternating-offer bargaining model of Rubinstein (1982) (see section 2.2.3) leads to a comparable result. The probability of the seller making the first offer equals the bargaining power q if there is a positive probability of a breakdown while waiting for a counteroffer. See Duffie, Gârleanu, and Pedersen (2007), pp. 1871 and the model of Rubinstein and Wolinsky (1985).

⁷³ See Binmore, Shaked, and Sutton (1989) for a discussion about disagreement points and outside options.

⁷⁴ See Duffie, Gârleanu, and Pedersen (2005), p. 1820.

⁷⁵ See Vayanos and Wang (2007), p. 73.

When investors and market makers bargain over the price, bid or ask prices result. An investor sells to the market maker at the bid price $B(t)$ and buys from the market maker at the ask price $A(t)$, with $A(t) \geq B(t)$. Both prices can be calculated analogously to the bargaining between two investors. Bid and ask prices depend both on buyers' and sellers' outside options, i.e. the availability of suitable counterparties over time. Market makers can trade the asset in the interdealer market, so that their outside option is to trade at the interdealer price $M(t)$. The bargaining power of market makers is defined with $z \in [0, 1]$. To specify the ask price, the generalized Nash solution is

$$f_z^A(t) = \arg \max_{A(t)} [A(t) - M(t)]^z [V_{ho}(t) - A(t) - V_{hm}(t)]^{1-z}, \quad (3.38)$$

subject to

$$0 \leq A(t) - M(t), \quad (3.39)$$

$$0 \leq V_{ho}(t) - A(t) - V_{hm}(t). \quad (3.40)$$

The generalized Nash solution for the bid price is

$$f_z^B(t) = \arg \max_{B(t)} [M(t) - B(t)]^z [V_{ln}(t) + B(t) - V_{lo}(t)]^{1-z}, \quad (3.41)$$

subject to

$$0 \leq M(t) - B(t), \quad (3.42)$$

$$0 \leq V_{ln}(t) + B(t) - V_{lo}(t). \quad (3.43)$$

Maximization of the argument in (3.38) over $A(t)$ and in (3.41) over $B(t)$ results in the following ask $A(t)$ and bid $B(t)$ prices, where

$$A(t) = (1 - z)M(t) + z[V_{ho}(t) - V_{hm}(t)], \quad (3.44)$$

$$\text{s. t. } M(t) \leq A(t) \leq V_{ho}(t) - V_{hm}(t), \quad (3.45)$$

and

$$B(t) = (1 - z)M(t) + z[V_{lo}(t) - V_{ln}(t)], \quad (3.46)$$

$$\text{s. t. } V_{lo}(t) - V_{ln}(t) \leq B(t) \leq M(t). \quad (3.47)$$

Equilibrium requires that supply and demand of the asset are balanced. This con-

dition affects the interdealer price $M(t)$: It depends on whether market makers meet an equal number of buyers and sellers ($\mu_{lo}(t) = \mu_{hn}(t)$), or whether there is an imbalance between potential sellers and potential buyers ($\mu_{lo}(t) \leq \mu_{hn}(t)$).

In the case that $\mu_{lo}(t) < \mu_{hn}(t)$, more potential buyers than potential sellers meet market makers. Not all potential buyers have the possibility to buy an asset. Since not all potential buyers are able to trade, market makers and buyers must be indifferent to trading. The interdealer price $M(t)$ must be equal to buyers' reservation value $V_{ho}(t) - V_{hn}(t)$ and is therefore set equal to the ask price $M(t) = A(t)$.

In the case $\mu_{lo}(t) > \mu_{hn}(t)$, more potential sellers than potential buyers meet market makers. Hence, market makers and potential sellers must be indifferent to trading. The interdealer price $M(t)$ must be equal to sellers' reservation value $V_{lo}(t) - V_{ln}(t)$, implying $M(t) = B(t)$.

In the rare event that the buy and sell side is balanced, i.e. $\mu_{lo} = \mu_{hn}$, the interdealer price is located somewhere between the bid and ask price.

As a result, the general interdealer price can be stated as follows:

$$M(t) = (1 - \tilde{q}(t)) [V_{lo}(t) - V_{ln}(t)] + \tilde{q}(t) [V_{ho}(t) - V_{hn}(t)], \quad (3.48)$$

with

$$\tilde{q}(t) \begin{cases} = 1 & \text{if } \mu_{lo}(t) < \mu_{hn}(t) \\ = 0 & \text{if } \mu_{lo}(t) > \mu_{hn}(t) \\ \in [0, 1] & \text{if } \mu_{lo}(t) = \mu_{hn}(t), \end{cases} \quad (3.49)$$

where the case $\tilde{q}(t) \in [0, 1]$ if $\mu_{lo}(t) = \mu_{hn}(t)$ denotes that, initially, $\tilde{q}(t)$ can arbitrarily be chosen from $[0, 1]$, but is then a constant for all cases $\mu_{lo}(t) = \mu_{hn}(t)$.⁷⁶

3.3.3 Steady State Value Functions and Prices

To derive steady state prices, insert the price equations (3.36), (3.44), and (3.46) into equations (3.29)–(3.32) and assume that there are always gains from trade. In the first step, the following general steady state equilibrium value functions

⁷⁶ See Duffie, Gârleanu, and Pedersen (2005), p. 1824 and Feldhütter (2010), p. 14.

result:

$$V_{lo}(ss) = \frac{D}{r} - \frac{\delta}{r} \frac{(r + \lambda_u + \lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss)))(r + \lambda_d)}{\gamma(ss)}, \quad (3.50)$$

$$V_{hn}(ss) = \frac{\delta}{r} \frac{(2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss)))(r + \lambda_u)}{\gamma(ss)}, \quad (3.51)$$

$$V_{ho}(ss) = \frac{D}{r} - \frac{\delta}{r} \frac{(r + \lambda_u + \lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss)))\lambda_d}{\gamma(ss)}, \quad (3.52)$$

$$V_{ln}(ss) = \frac{\delta}{r} \frac{(2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss)))\lambda_u}{\gamma(ss)}, \quad (3.53)$$

with

$$\gamma(ss) = (r + \lambda_d + \lambda_u) (r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)). \quad (3.54)$$

Inserting equations (3.50)–(3.53) into equations (3.36), (3.44), and (3.46), the general steady state equilibrium prices are⁷⁷

$$P(ss) = \frac{D}{r} - \frac{\delta}{r} \frac{r(1 - q) + \lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss))}{r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)}, \quad (3.55)$$

$$A(ss) = \frac{D}{r} - \frac{\delta}{r} \frac{\lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + (\rho + r)(1 - z)(1 - \tilde{q}(ss))}{r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)}, \quad (3.56)$$

$$B(ss) = \frac{D}{r} - \frac{\delta}{r} \frac{\lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + r - (1 - z)\tilde{q}(ss)r + \rho(1 - z)(1 - \tilde{q}(ss))}{r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)}. \quad (3.57)$$

All three prices bear resemblances to each other: The prices consist of the present value of the stream of dividends (D/r), also called the fundamental value of the asset, less an illiquidity discount.

Sensitivity analysis shows the following effects on prices (ceteris paribus and independent of the rationed trading side, i.e. independent of the relation of s and y to each other):⁷⁸

⁷⁷ The pricing formulas (14)–(16) in Duffie, Gârleanu, and Pedersen (2005, p. 1824) are nested in equations (3.55)–(3.57), with $\tilde{q}(ss) = 1$ and $D = 1$.

⁷⁸ Proofs are in Duffie, Gârleanu, and Pedersen (2007), pp. 1872, 1894, and Afonso (2011), appendix B.

Prices decrease with an increase in

- holding costs: $\partial P(ss)/\partial \delta < 0$,
- the fraction of asset owners: $\partial P(ss)/\partial s < 0$ (with $\partial \mu_{lo}(ss)/\partial s > 0$ and $\partial \mu_{hn}(ss)/\partial s < 0$),
- the frequency of downward preference or funding shocks: $\partial P(ss)/\partial \lambda_d < 0$ (with $\partial \mu_{lo}(ss)/\partial \lambda_d > 0$ and $\partial \mu_{hn}(ss)/\partial \lambda_d < 0$).

Prices increase with an increase in

- the bargaining power of the seller: $\partial P(ss)/\partial q > 0$,
- the frequency of upward preference or funding shocks: $\partial P(ss)/\partial \lambda_u > 0$ (with $\partial \mu_{lo}(ss)/\partial \lambda_u < 0$ and $\partial \mu_{hn}(ss)/\partial \lambda_u > 0$).

The bid-ask spread is the difference between the ask price for selling the asset to a market maker and the bid price for buying the asset from a market maker. It is the fee a market maker gains for buying the asset from an *lo* agent and selling it to an *hn* agent. The bid-ask spread offers an incentive for market makers to care about market liquidity. In this model, bid and ask prices reflect the outside options of investors, which depend on the availability of suitable counterparties: both other market makers and other investors. Thus, the bid-ask spread, denoted with

$$A(ss) - B(ss) = \frac{\delta z}{r + \lambda_d + \lambda_u + 2\lambda \mu_{hn}(ss)q + 2\lambda \mu_{lo}(ss)(1 - q) + \rho(1 - z)}, \quad (3.58)$$

is the compensation for market makers' cost of searching and matching. It is ensured that market makers' gains from trade are nonnegative, since the numerator, with holding cost δ and the market makers' bargaining power z , is always equal to or greater than zero. The denominator is always greater than zero, since all coefficients are positive.

3.3.4 Walrasian Equilibrium

In a perfect competitive market, supply equals demand and agents can sell and buy instantly. The result is a so-called Walrasian equilibrium⁷⁹, which can be obtained by eliminating search frictions, i.e. $\lambda \rightarrow \infty$ (and (ρ^i) is any sequence)

⁷⁹ See Walras (1874) and Neus (2013), pp. 85.

or $\rho \rightarrow \infty$ (and (λ^i) is any sequence). The bid-ask spread then tends to zero (for $z < 1$). Three cases are distinguished:⁸⁰

If there are more buyers than sellers in steady state, i.e. $s < \lambda_u / (\lambda_u + \lambda_d)$, all sellers can sell immediately and buyers are rationed. High-type investors are the marginal asset owners. The Walrasian price is $P^{W^*}(ss) = D/r$, which is the present value of the asset's dividends. The proof for $\lambda \rightarrow \infty$ results with $\lambda\mu_{hm} \xrightarrow{\lambda \rightarrow \infty} \infty$ and $\lambda\mu_{lo}$ is bounded if $0 < q$. Steady state equilibrium masses of investors are

$$\begin{aligned}\mu_{lo}^{W^*}(ss) &= 0, \\ \mu_{hn}^{W^*}(ss) &= \frac{\lambda_u}{\lambda_u + \lambda_d} - s, \\ \mu_{ho}^{W^*}(ss) &= s, \\ \mu_{ln}^{W^*}(ss) &= 1 - \frac{\lambda_u}{\lambda_u + \lambda_d}.\end{aligned}\tag{3.59}$$

For $\rho \rightarrow \infty$, the results converge to the ones stated in (3.59) if $z < 1$.⁸¹ The same results occur with $\lambda_u \rightarrow \infty$, i.e. when recovering from a low state is easy.⁸²

If there are more sellers than buyers in steady state, i.e. $s > \lambda_u / (\lambda_u + \lambda_d)$, all buyers can buy immediately and sellers get rationed. Low-type investors are the marginal asset owners. The Walrasian price is then $P^{W^{**}}(ss) = (D - \delta)/r$. This price corresponds to the reservation value of a marginal asset owner, who expects to stay a low owner forever. The proof for $\lambda \rightarrow \infty$ results with $\lambda\mu_{lo} \xrightarrow{\lambda \rightarrow \infty} \infty$ and $\lambda\mu_{hm}$ is bounded if $q < 1$. Steady state equilibrium masses of investors are

$$\begin{aligned}\mu_{lo}^{W^{**}}(ss) &= s - \frac{\lambda_u}{\lambda_u + \lambda_d}, \\ \mu_{hn}^{W^{**}}(ss) &= 0, \\ \mu_{ho}^{W^{**}}(ss) &= \frac{\lambda_u}{\lambda_u + \lambda_d}, \\ \mu_{ln}^{W^{**}}(ss) &= 1 - s.\end{aligned}\tag{3.60}$$

For $\rho \rightarrow \infty$, the results converge to the ones stated in (3.60) if $z < 1$.

⁸⁰ Proofs and detailed information on equilibrium masses of investors and prices are in Duffie, Gârleanu, and Pedersen (2005), pp. 1826 and Afonso (2011), p. 350.

⁸¹ Duffie, Gârleanu, and Pedersen (2005, pp. 1826) show that eliminating search frictions for a monopolistic market maker, i.e. $z = 1$, does not result in a Walrasian price.

⁸² See Duffie, Gârleanu, and Pedersen (2007), p. 1873.

If there are as many sellers as buyers in steady state, i.e. $s = \lambda_u / (\lambda_u + \lambda_d)$, all buyers and sellers can trade immediately and none of the investors get rationed. As a result, the quantity of buyers and sellers has to be zero for $\lambda \rightarrow \infty$ or $\rho \rightarrow \infty$. The Walrasian equilibrium is attained with $P^{W^{***}}(ss) = (D - \delta(1 - q)) / r$. Steady state equilibrium masses of investors are

$$\begin{aligned}\mu_{lo}^{W^{***}}(ss) &= 0, \\ \mu_{hn}^{W^{***}}(ss) &= 0, \\ \mu_{ho}^{W^{***}}(ss) &= s, \\ \mu_{ln}^{W^{***}}(ss) &= 1 - s.\end{aligned}\tag{3.61}$$

3.4 Numerical Example

In this section, the main implications of search frictions on liquidity and asset prices are explained by means of an example. I adopt the input parameters from the example in Duffie, Gârleanu, and Pedersen (2007, pp. 1883–1887), to provide a benchmark case for chapters 5–7. Interpretations follow, to a large extent, Duffie, Gârleanu, and Pedersen (2005, 2007).

	Parameter	Value
Fraction of investors owning an asset	s	0.75
Holding / illiquidity cost	δ	2.5
Constant dividend rate	D	1
Interest rate	r	0.1
Intensity of switching to a high preference type	λ_u	2
Intensity of switching to a low preference type	λ_d	0.2
Investors' meeting intensity	λ	125
Market makers' meeting intensity	ρ	125
Seller's bargaining power (between investors)	q	0.5
Market makers' bargaining power	z	0.8

Table 3.1: Input parameters for the numerical example.⁸³

All parameters are stated per period, where one period equals one year. I follow Weill (2007, p. 1334) by assuming 250 trading days per year and 10 trading hours per day.

⁸³ Feldhütter (2012, p. 1173) finds in an empirical estimation for a comparable model (with $\lambda = 0$) that ρ lies between 40 (Feldhütter calls it a very unsophisticated investor) and 372 (for a highly sophisticated investor), $z = 0.97$, $\lambda_d = 0.33$, and $\lambda_u = 3.25$. I chose $\rho = 125$, since market makers are not considered in the example of Duffie, Gârleanu, and Pedersen (2007, pp. 1883–1887).

The switching intensities imply that an investor is, on average, a high-type investor $y = \lambda_u / (\lambda_u + \lambda_d) = 90.91\%$ of the time. He stays a high-type for $1/\lambda_d = 5$ years on average. A low-type investor is, on average, of low-type 9.09% of the time and stays low for $1/\lambda_u = 0.5$ years on average. The meeting intensity $\lambda = 125$ states that each investor locates other investors every other day on average. He can expect to interact with one investor every day, since $2\lambda/250 = 1$.

Equilibrium Masses

Based on equations (3.13)–(3.16), the steady state equilibrium masses $\mu_\sigma(ss)$ for the four investor types are calculated first. Table 3.2 displays the results.

	Parameter	Fraction
Fraction of <i>lo</i> -type investors	$\mu_{lo}(ss)$	0.0009
Fraction of <i>hn</i> -type investors	$\mu_{hn}(ss)$	0.1600
Fraction of <i>ho</i> -type investors	$\mu_{ho}(ss)$	0.7491
Fraction of <i>ln</i> -type investors	$\mu_{ln}(ss)$	0.0900

Table 3.2: Steady state equilibrium masses.

With the chosen input values, 74.91% of all investors are high-type investors owning an asset. 16% are of high-type but do not own an asset, and are thus potential buyers. Only 0.09% of investors are up to selling their asset in this steady state equilibrium. There are considerably more buyers than sellers, so all *lo*-type investors meeting a market maker are able to sell their asset to him. On the other hand, not all of the *hn*-type investors can buy an asset when they meet a market maker. Thus, buyers are rationed.⁸⁴

Both the rationing of buyers by market makers and search frictions have a direct impact on liquidity, measured in trading time. On average, it takes about $[2\lambda\mu_{lo}(ss) + \rho \min\{\mu_{lo}(ss), \mu_{hn}(ss)\}]^{-1} = 2.9724$ years to buy an asset, but only $[2\lambda\mu_{hn}(ss) + \rho \min\{\mu_{lo}(ss), \mu_{hn}(ss)\}]^{-1} = 0.0249$ years (or 6 trading days) to sell an asset. The percentage asset turnover per year, calculated with $(2\lambda\mu_{lo}(ss)\mu_{hn}(ss) + \rho \min\{\mu_{lo}(ss), \mu_{hn}(ss)\})/s = 19.8\%$, is low in this steady state equilibrium. This low asset turnover is partly due to a low rate of asset misallocations to low-type investors. Only $\mu_{lo}(ss)/s = 0.12\%$ of the total asset supply s is owned by sellers.

⁸⁴ If search frictions are eliminated with $\lambda \rightarrow \infty$ or $\rho \rightarrow \infty$, then the mass distribution would be $\mu_{lo}^{W^*}(ss) = 0$, $\mu_{hn}^{W^*}(ss) = 0.1591$, $\mu_{ho}^{W^*}(ss) = 0.75$, $\mu_{ln}^{W^*}(ss) = 0.0909$.

Value Function

The optimal values in steady state are as expected: Asset owners expect a higher present value of payoffs than non-owners, whereas agents with a high preference for this asset attain a higher level of expected utility than agents with a low preference. Table 3.3 states the values in steady state.

	Parameter	Value
Value function of <i>lo</i> -type investors	$V_{lo}(ss)$	9.8341
Value function of <i>hn</i> -type investors	$V_{hn}(ss)$	0.0540
Value function of <i>ho</i> -type investors	$V_{ho}(ss)$	9.8894
Value function of <i>ln</i> -type investors	$V_{ln}(ss)$	0.0514

Table 3.3: Value functions.

The value functions of asset owners, $V_{lo}(ss)$ and $V_{ho}(ss)$, increase with increasing meeting intensities ρ and λ , because search frictions decrease and agents are matched faster. Their expected utility increases since agents do not get stuck in an undesired position for a long time upon switching to a low state. In general, the value functions of asset non-owners, $V_{hn}(ss)$ and $V_{ln}(ss)$, increase with increasing meeting intensity ρ and λ . This general result is influenced by the interrelation of

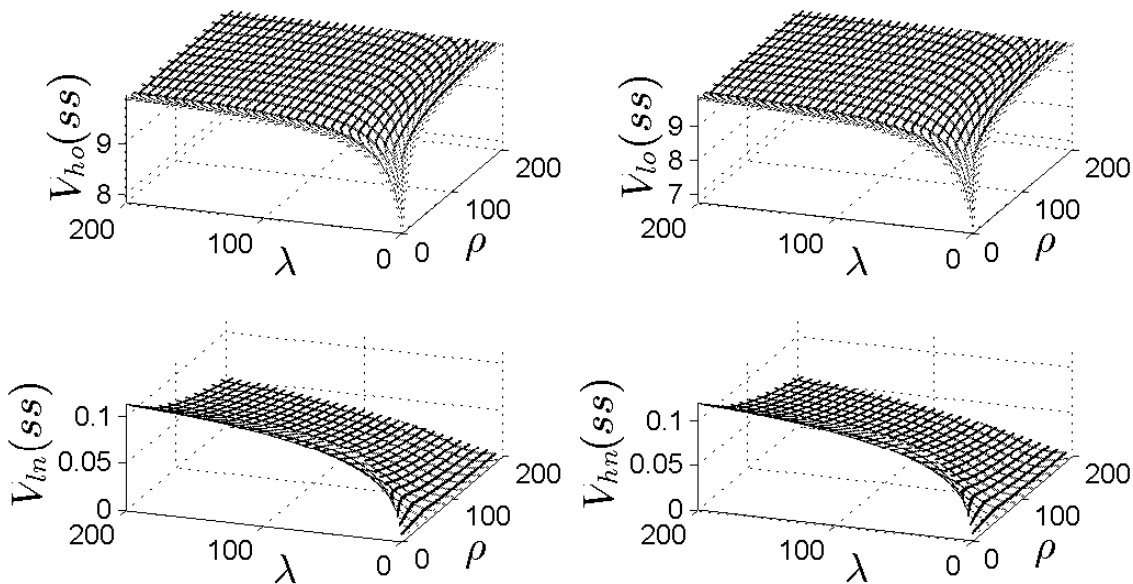


Figure 3.2: Steady state value functions as functions of λ and ρ .

λ and ρ . In the prevailing situation, potential buyers are rationed by market makers. Hence, trading opportunities between investors are more valuable to buyers than meetings with market makers. I illustrate the effects of the varying meeting parameters on steady state value functions in figure 3.2.

Prices

Without search frictions, the competitive, Walrasian market price would be $\lim_{\lambda \rightarrow \infty} P(ss) = \lim_{\rho \rightarrow \infty} P(ss) = 10$, since $s < \lambda_u / (\lambda_u + \lambda_d)$. The bid-ask spread would converge to zero. With the parameters given in table 3.1, the price is not Walrasian. The steady state ask price $A(ss)$, the bid price $B(ss)$, the interinvestor price $P(ss)$, and the bid-ask spread are stated in table 3.4.

	Parameter	Value
Interinvestor price	$P(ss)$	9.8090
Ask price	$A(ss)$	9.8354
Bid price	$B(ss)$	9.7932
Bid-ask spread	$A(ss) - B(ss)$	0.0422

Table 3.4: Steady state equilibrium prices.

Since $s < \lambda_u / (\lambda_u + \lambda_d)$, the interdealer price $M(ss)$ is set equal to the ask price $A(ss)$, which implies $\tilde{q}(ss) = 1$. The interinvestor price $P(ss)$ reveals that the illiquidity discount due to search frictions is approximately 2%. Illiquidity due to search frictions can also be stated in terms of the yield spread between liquid and illiquid asset: The yield of the illiquid asset is $1/9.8090 = 0.1019$. This exceeds the yield of the liquid security, i.e. the interest rate, by 19 basis points.

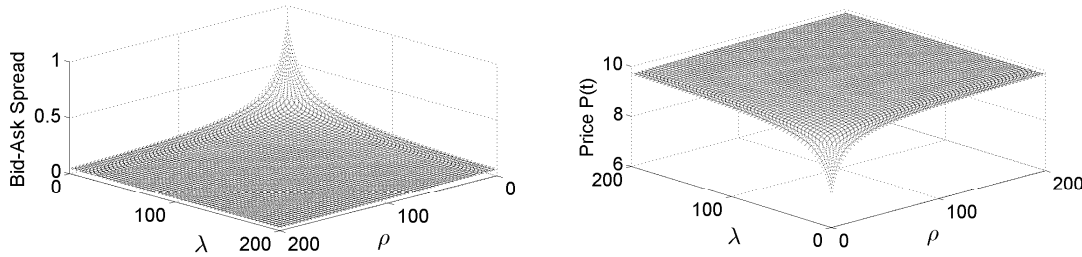


Figure 3.3: The bid-ask spread (left panel) and the interinvestor prices (right panel) with different values for λ and ρ .

The left panel of figure 3.3 shows that the steady state bid-ask spread rises sharply for very low meeting rates between investors as well as between investors and market makers. It converges to zero for very high meeting rates, i.e. when search frictions diminish. This fact can also be verified if $\lambda \rightarrow \infty$ or $\rho \rightarrow \infty$ is considered in equation (3.58), i.e. $\lim_{\lambda \rightarrow \infty} (A(ss) - B(ss)) = 0$ or $\lim_{\rho \rightarrow \infty} (A(ss) - B(ss)) = 0$ for $z < 1$. The right panel of figure 3.3 reveals a rapidly dropping steady state interinvestor price $P(ss)$ with decreasing λ and ρ . For increasing λ or ρ , the interinvestor price converges to the price in a Walrasian market.

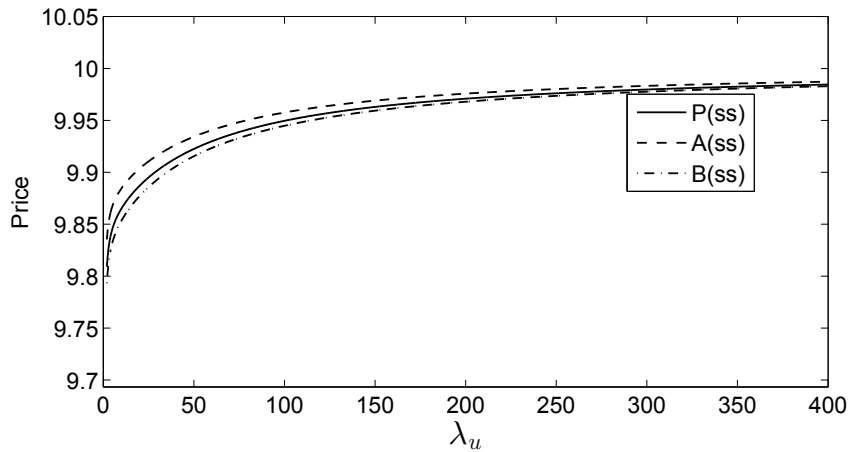


Figure 3.4: Steady state prices as a function of λ_u .

The Walrasian price is also attained with $\lambda_u \rightarrow \infty$, as denoted in section 3.3.4. Bid-ask spreads then tend to zero. Figure 3.4 shows the steady state interinvestor price $P(ss)$, the ask price $A(ss)$, and the bid $B(ss)$ price as a function of λ_u .

Negative Interinvestor Prices

The model by Duffie, Gârleanu, and Pedersen (2005) permits negative equilibrium prices.⁸⁵ A simple variation of the parameters stated above—with $\lambda_d = 0.7$ and $\rho = 0$ —leads to a steady state equilibrium price of $P(ss) = -2.3619$. This negative price implies that the seller has to pay the buyer a fee to get rid of the asset. Since $s > \lambda_u / (\lambda_u + \lambda_d)$ holds, this small change in λ_d (and ρ) alters the whole steady state equilibrium. There are more sellers than buyers in steady state and the marginal investor is of low-type. A low-type investor is of low-type 25.93% (instead of 9.09%) of the time and stays low for 1.43 (instead of 0.5) years, on average. Investors anticipate this longer period of time by implying that it will be difficult to sell the asset in the future.

Another crucial factor is the holding cost δ , measured in units of consumption. With $\delta = 2.5$, the holding cost is higher than the dividend of 1 per unit of consumption, which the asset is paying. Investors anticipate that they are probably exposed to a situation in which they will lose consumption for a relatively long time if their preference type switches from high to low. In this case, a seller is willing to pay the buyer in order to not lose consumption any longer.

Using equation (3.55), the condition for a strictly positive interinvestor price

⁸⁵ To the best of my knowledge, this has not been addressed before.

$P(ss) > 0$ is

$$D \frac{r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)}{r(1 - q) + \lambda_d + 2\lambda\mu_{lo}(ss)(1 - q) + \rho(1 - z)(1 - \tilde{q}(ss))} > \delta.$$

This condition shows the sensitivity of the model's parameters on prices.

3.5 Conclusion

This chapter introduces the basic search and bargaining model for asset pricing in an illiquid OTC market, developed by Duffie, Gârleanu, and Pedersen (2005). Illiquidity is modeled by search frictions, which imply that trade does not happen instantly. Asset prices are directly bargained between agents. In the initial step, dynamic and steady state equilibrium masses of investor types are discussed. Investors' utility is calculated in the second step. Finally, I present asset prices and bid-ask spreads for the steady state equilibrium. By deriving them in a general case, the pricing formulas in Duffie, Gârleanu, and Pedersen (2005, p. 1824) are nested therein. As a result of the basic model, prices for assets are lower—compared to prices in a perfect market—due to search frictions. The more challenging it is to find a trading partner, the higher this illiquidity discount becomes.

3A Appendix: Derivation of the Value Function

In this appendix, I derive the value functions (3.21)–(3.24) stated in section 3.3.⁸⁶ I start with the value $V_{ho}(t)$ of high owner, as stated in equation (3.23), and the value $V_{ln}(t)$ of low non-owner, as stated in equation (3.24), since these two are the easiest ones. Then, the values $V_{hn}(t)$ and $V_{lo}(t)$ of agents who trade with each other—as stated in equations (3.22) and (3.21), respectively—are determined.

Value Function of *ho*-Type Agents

I begin by restating equation (3.23)

$$V_{ho}(t) = E_t \left[\int_t^{\tau_l} e^{-r(u-t)} D du \right] + E_t \left[e^{-r(\tau_l-t)} V_{lo}(\tau_l) \right],$$

where

$$E_t \left[\int_t^{\tau_l} e^{-r(u-t)} D du \right] = E_t \left[\frac{D}{r} \left(1 - e^{-r(\tau_l-t)} \right) \right].$$

Since the random variable τ_l is the stopping time when an agent changes his intrinsic type from high to low, its density function is

$$f(\tau_l) = \begin{cases} \lambda_d e^{-\lambda_d(\tau_l-t)} & \text{for } \tau_l \geq t \\ 0 & \text{for } \tau_l < t, \end{cases}$$

where λ_d is the switching intensity from high to low. Hence, the expectation can be replaced, so that

$$\begin{aligned} E_t \left[\int_t^{\tau_l} e^{-r(u-t)} D du \right] &= \int_t^{\infty} \frac{D}{r} (1 - e^{-r(x-t)}) \cdot \lambda_d e^{-\lambda_d(x-t)} dx \\ &= \frac{D}{r + \lambda_d}. \end{aligned}$$

⁸⁶ The derivation of the value functions is in line with Feldhütter (2012), pp. 1183–1186 and the comparable model therein.

Analogously,

$$E_t \left[e^{-r(\tau_l - t)} V_{lo}(\tau_l) \right] = \int_t^\infty e^{-r(x-t)} V_{lo}(x) \lambda_d e^{-\lambda_d(x-t)} dx.$$

Differentiating with respect to t , the HJB equation is derived with

$$\begin{aligned} \dot{V}_{ho}(t) &= \frac{\partial}{\partial t} E_t \left[\int_t^{\tau_l} e^{-r(u-t)} D du \right] + \frac{\partial}{\partial t} E_t \left[e^{-r(\tau_l - t)} V_{lo}(\tau_l) \right] \\ &= \underbrace{\frac{\partial}{\partial t} \left[\frac{D}{r + \lambda_d} \right]}_{=0} + \frac{\partial}{\partial t} \left[\int_t^\infty V_{lo}(x) \lambda_d e^{-(r+\lambda_d)(x-t)} dx \right]. \end{aligned}$$

Leibniz's integration rule states that for an arbitrary integral $I(t)$, with

$$I(t) = \int_{b(t)}^{a(t)} f(t, x) dx,$$

that

$$\frac{\partial I(t)}{\partial t} = f(t, a(t)) \cdot \frac{\partial a}{\partial t} - f(t, b(t)) \cdot \frac{\partial b}{\partial t} + \int_{b(t)}^{a(t)} \frac{\partial f}{\partial t}(t, x) dx$$

holds. Here, $a(t) \rightarrow \infty$, $\frac{\partial a}{\partial t} = 0$, $b(t) = t$, $\frac{\partial b}{\partial t} = 1$, $f(t, x) = V_{lo}(x) \lambda_d e^{-(r+\lambda_d)(x-t)}$.

With Leibniz's integration rule, the derivative of the value function $V_{ho}(t)$ with respect to t is

$$\begin{aligned} \dot{V}_{ho}(t) &= -\lambda_d V_{lo}(t) e^{-(r+\lambda_d)(t-t)} + \int_t^\infty \lambda_d (r + \lambda_d) V_{lo}(x) e^{-(r+\lambda_d)(x-t)} dx \\ &= -\lambda_d V_{lo}(t) + (r + \lambda_d) \int_t^\infty \lambda_d V_{lo}(x) e^{-(r+\lambda_d)(x-t)} dx \\ &= -\lambda_d V_{lo}(t) + (r + \lambda_d) \left[V_{ho}(t) - \frac{D}{r + \lambda_d} \right] \\ &= r V_{ho}(t) - \lambda_d (V_{lo}(t) - V_{ho}(t)) - D. \end{aligned}$$

Value Function of ln -Type Agents

To derive the optimal value function of low non-owners, I take the derivative of equation (3.24) to obtain the HJB equation with

$$\begin{aligned}
\dot{V}_{ln}(t) &= \frac{\partial}{\partial t} E_t \left[e^{-r(\tau_l - t)} V_{ln}(\tau_l) \right] \\
&= \frac{\partial}{\partial t} \int_t^\infty e^{-r(x-t)} V_{ln}(x) \lambda_u e^{-\lambda_u(x-t)} dx \\
&= -\lambda_u V_{ln}(t) + (r + \lambda_u) \underbrace{\int_t^\infty \lambda_u V_{ln}(x) e^{-(r+\lambda_u)(x-t)} dx}_{=V_{ln}(t)} \\
&= rV_{ln}(t) - \lambda_u(V_{ln}(t) - V_{ln}(t)).
\end{aligned}$$

Value Function of hn -Type Agents

To derive the optimal value function of high non-owners, I start by taking the derivative of equation (3.22) to obtain the HJB equation with

$$\begin{aligned}
\dot{V}_{hn}(t) &= \frac{\partial}{\partial t} E_t \left[e^{-r(\tau_l - t)} V_{ln}(\tau_l) \mathbf{1}_{\{\tau_l = \tau\}} + e^{-r(\tau_i - t)} (V_{ho}(\tau_i) - P(\tau_i)) \mathbf{1}_{\{\tau_i = \tau\}} \right. \\
&\quad \left. + e^{-r(\tau_m - t)} (V_{ho}(\tau_m) - A(\tau_m)) \mathbf{1}_{\{\tau_m = \tau\}} \right].
\end{aligned}$$

All three events,

- i. intrinsic type change ($\lambda_{ud} \in [\lambda_d, \lambda_u]$) with its first stopping time τ_l ,
- ii. trade between two investors ($2\lambda\mu_\sigma(t) \in [2\lambda\mu_{l\sigma}(t), 2\lambda\mu_{hn}(t)]$) with its first stopping time τ_i ,
- iii. trade between investor and market maker (ρ) with its first stopping time τ_m ,

are assumed to be mutually independent and the superposition theorem, defined in equation (2.12), applies.

Each event has an exponential density function with

$$\begin{aligned}
f_{\tau_l}(t, \tau) &= \lambda_{ud} e^{-\lambda_{ud}(\tau-t)} && \text{for } \tau \geq t, \\
f_{\tau_i}(t, \tau) &= 2\lambda\mu_\sigma(\tau) e^{-\int_t^\tau 2\lambda\mu_\sigma(u) du} && \text{for } \tau \geq t, \\
f_{\tau_m}(t, \tau) &= \rho e^{-\rho(\tau-t)} && \text{for } \tau \geq t,
\end{aligned}$$

and $\tau = \min\{\tau_l, \tau_i, \tau_m\}$. The corresponding distribution functions are

$$\begin{aligned} F_{\tau_l}(t, \tau) &= 1 - e^{-\lambda_{ud}(\tau-t)} && \text{for } \tau \geq t, \\ F_{\tau_i}(t, \tau) &= 1 - e^{-\int_t^\tau 2\lambda\mu_\sigma(u) du} && \text{for } \tau \geq t, \\ F_{\tau_m}(t, \tau) &= 1 - e^{-\rho(\tau-t)} && \text{for } \tau \geq t. \end{aligned}$$

The probability that the first event occurs between $(\tau, \tau + d\tau)$, and that this event is an intrinsic type change, is the element of probability $g_{\tau_l}(\tau) d\tau$, with⁸⁷

$$\begin{aligned} g_{\tau_l}(\tau) &= \frac{dF_{\tau_l}(t, \tau)}{d\tau} \cdot [1 - F_{\tau_i}(t, \tau)] \cdot [1 - F_{\tau_m}(t, \tau)] \\ &= f_{\tau_l}(t, \tau) \cdot \left[e^{-\int_t^\tau 2\lambda\mu_\sigma(u) du} \right] \left[e^{-\rho(\tau-t)} \right] \\ &= \lambda_{ud} e^{-\int_t^\tau (\lambda_{ud} + 2\lambda\mu_\sigma(u) + \rho) du}, \end{aligned}$$

for $\tau \geq t$ and $\tau = \min\{\tau_l, \tau_i, \tau_m\} = \tau_l$. Comparable functions can be derived for both the trade between two investors

$$\begin{aligned} g_{\tau_i}(\tau) &= f_{\tau_i}(t, \tau) \cdot [1 - F_{\tau_l}(t, \tau)] \cdot [1 - F_{\tau_m}(t, \tau)] \\ &= 2\lambda\mu_\sigma(\tau) e^{-\int_t^\tau (\lambda_{ud} + 2\lambda\mu_\sigma(u) + \rho) du}, \end{aligned}$$

for $\tau \geq t$ and $\tau = \min\{\tau_l, \tau_i, \tau_m\} = \tau_i$, and the trade between investor and market makers

$$\begin{aligned} g_{\tau_m}(\tau) &= f_{\tau_m}(t, \tau) \cdot [1 - F_{\tau_l}(t, \tau)] \cdot [1 - F_{\tau_i}(t, \tau)] \\ &= \rho e^{-\int_t^\tau (\lambda_{ud} + 2\lambda\mu_\sigma(u) + \rho) du}, \end{aligned}$$

for $\tau \geq t$ and $\tau = \min\{\tau_l, \tau_i, \tau_m\} = \tau_m$.

The derivative of equation (3.22) can be written as

$$\begin{aligned} \dot{V}_{hm}(t) &= \frac{\partial}{\partial t} \int_t^\infty e^{-r(x-t)} V_{ln}(x) \lambda_d e^{-\int_t^x (\lambda_d + 2\lambda\mu_{lo}(y) + \rho) dy} dx \\ &\quad + \frac{\partial}{\partial t} \int_t^\infty e^{-r(x-t)} (V_{ho}(x) - P(x)) 2\lambda\mu_{lo}(x) e^{-\int_t^x (\lambda_d + 2\lambda\mu_{lo}(y) + \rho) dy} dx \\ &\quad + \frac{\partial}{\partial t} \int_t^\infty e^{-r(x-t)} (V_{ho}(x) - A(x)) \rho e^{-\int_t^x (\lambda_d + 2\lambda\mu_{lo}(y) + \rho) dy} dx \end{aligned}$$

⁸⁷ Freund (1961) and Duffie (2011), pp. 10.

$$\begin{aligned}
&= -\lambda_d V_{ln}(t) - 2\lambda\mu_{lo}(t)(V_{ho}(t) - P(t)) - \rho(V_{ho}(t) - A(t)) \\
&\quad + (r + \lambda_d + 2\lambda\mu_{lo}(t) + \rho)V_{hn}(t) \\
&= rV_{hn}(t) - \lambda_d(V_{ln}(t) - V_{hn}(t)) - 2\lambda\mu_{lo}(t)(V_{ho}(t) - P(t) - V_{hn}(t)) \\
&\quad - \rho(V_{ho}(t) - A(t) - V_{hn}(t)).
\end{aligned}$$

Value Function of lo -Type Agents

To derive the HJB equation for the value functions of low owners, I use equation (3.21) so that

$$\begin{aligned}
\dot{V}_{lo}(t) &= \frac{\partial}{\partial t} E_t \left[\int_t^\tau e^{-r(u-t)} (D - \delta) du + e^{-r(\tau_i-t)} V_{ho}(t) \mathbf{1}_{\{\tau_i=\tau\}} \right. \\
&\quad \left. + e^{-r(\tau_i-t)} (V_{ln}(t) + P(t)) \mathbf{1}_{\{\tau_i=\tau\}} + e^{-r(\tau_m-t)} (V_{ln}(t) + B(t)) \mathbf{1}_{\{\tau_m=\tau\}} \right] \\
&= \frac{\partial}{\partial t} E_t \left[\int_t^\tau e^{-r(u-t)} (D - \delta) du \right] \\
&\quad + \frac{\partial}{\partial t} \int_t^\infty \lambda_u V_{ho}(x) e^{-\int_t^x (r+\lambda_u+2\lambda\mu_{hm}(y)+\rho) dy} dx \\
&\quad + \frac{\partial}{\partial t} \int_t^\infty 2\lambda\mu_{hn}(x) (V_{ln}(x) + P(x)) e^{-\int_t^x (r+\lambda_u+2\lambda\mu_{hm}(y)+\rho) dy} dx \\
&\quad + \frac{\partial}{\partial t} \int_t^\infty \rho (V_{ln}(x) + B(x)) e^{-\int_t^x (r+\lambda_u+2\lambda\mu_{hm}(y)+\rho) dy} dx.
\end{aligned}$$

The superposition of independent Poisson processes is again a Poisson process (see equation (2.12)), implying that its intensity is the sum of the individual intensities. The probability that an event occurs within the infinitesimal interval $(\tau, \tau + d\tau)$ is $g_\tau(\tau) d\tau$, with

$$g_\tau(\tau) = (\lambda_{ud} + 2\lambda\mu_\sigma(\tau) + \rho) e^{-\int_t^\tau (\lambda_{ud} + 2\lambda\mu_\sigma(y) + \rho) dy},$$

for $\tau \geq t$, where this event is either an intrinsic type change (λ_{ud}), or interinvestor trade ($2\lambda\mu_\sigma(\tau)$), or trade intermediated by market maker (ρ).

The remaining expectation is

$$E_t \left[\int_t^{\tau} e^{-r(u-t)} (D - \delta) du \right] = \int_t^{\infty} (D - \delta) \frac{1}{r} \left(1 - e^{-r(x-t)} \right) (\lambda_u + 2\lambda\mu_{hn}(x) + \rho) e^{-\int_t^x (\lambda_u + 2\lambda\mu_{hn}(y) + \rho) dy} dx.$$

The derivative of the value function $V_{lo}(t)$ with respect to t is

$$\begin{aligned} \dot{V}_{lo}(t) &= \frac{\partial}{\partial t} \int_t^{\infty} \frac{D - \delta}{r} \left(1 - e^{-r(x-t)} \right) (\lambda_u + 2\lambda\mu_{hn}(x) + \rho) e^{-\int_t^x (\lambda_u + 2\lambda\mu_{hn}(y) + \rho) dy} dx \\ &\quad - \lambda_u V_{ho}(t) - 2\lambda\mu_{hn}(t)(V_{ln}(t) + P(t)) - \rho(V_{ln}(t) + B(t)) \\ &\quad + (r + \lambda_u + 2\lambda\mu_{hn}(t) + \rho) \left[\int_t^{\infty} \lambda_u V_{ho}(x) e^{-\int_t^x (r + \lambda_u + 2\lambda\mu_{hn}(y) + \rho) dy} dx \right. \\ &\quad + \int_t^{\infty} 2\lambda\mu_{hn}(x)(V_{ln}(x) + P(x)) e^{-\int_t^x (r + \lambda_u + 2\lambda\mu_{hn}(y) + \rho) dy} dx \\ &\quad \left. + \int_t^{\infty} \rho(V_{ln}(x) + B(x)) e^{-\int_t^x (r + \lambda_u + 2\lambda\mu_{hn}(y) + \rho) dy} dx \right] \\ &= (r + \lambda_u + 2\lambda\mu_{hn}(t) + \rho) \cdot E_t \left[\int_t^{\tau} e^{-r(u-t)} (D - \delta) du \right] - (D - \delta) \\ &\quad - \lambda_u V_{ho}(t) - 2\lambda\mu_{hn}(t)(V_{ln}(t) + P(t)) - \rho(V_{ln}(t) + B(t)) \\ &\quad + (r + \lambda_u + 2\lambda\mu_{hn}(t) + \rho) \left(V_{lo}(t) - E_t \left[\int_t^{\tau} e^{-r(u-t)} (D - \delta) du \right] \right) \\ &= rV_{lo}(t) - \lambda_u(V_{ho}(t) - V_{lo}(t)) - 2\lambda\mu_{hn}(t)(V_{ln}(t) + P(t) - V_{lo}(t)) \\ &\quad - \rho(V_{ln}(t) + B(t) - V_{lo}(t)) - (D - \delta). \end{aligned}$$

Chapter 4

Aggregate Liquidity Shocks

In chapter 4, I discuss the aggregate liquidity shock model introduced by Duffie, Gârleanu, and Pedersen (2007). They expand the basic search and bargaining model of chapter 3 by modeling a sudden decrease in aggregated liquidity. This model enables us to analyze the dynamics out of and towards steady state of prices and return reactions after aggregate liquidity shocks, whereas chapter 3 discusses only steady state equilibria. However, Duffie, Gârleanu, and Pedersen (2007) do not consider market makers.⁸⁸

The outline of this chapter is as follows: Section 4.1 starts with a short introduction to aggregate liquidity shocks. Section 4.2 introduces the Duffie, Gârleanu, and Pedersen (2007) model setup. Aggregate liquidity shocks are implemented in section 4.3. At this point, I reintroduce market makers. One essential extension to the analysis in Duffie, Gârleanu, and Pedersen (2007) is section 4.4, where I show a semi-analytical solution for this model. The solution method is presented in appendix 4A, while the derivation of the explicit results is stated in appendices 4B and 4C. Numerical examples are deferred to chapter 5.

4.1 Introduction

An aggregate liquidity shock hitting either the whole market, like a systemic liquidity shock, or a class of investors or assets, has negative effects on investors'

⁸⁸ Duffie, Gârleanu, and Pedersen (2007) consider only trade between investors (i.e. $\rho = 0$). Feldhütter (2012) allows only for trade between investors and market makers (i.e. $\lambda = 0$) in a setting closely related to the one applied by Duffie, Gârleanu, and Pedersen (2007). I use the framework from chapter 3, with trade between two investors ($\lambda \neq 0$) and between investors and market makers ($\rho \neq 0$), to model aggregate liquidity shocks that are comparable to Duffie, Gârleanu, and Pedersen (2007).

preference towards asset holdings. All investors, or a fraction of them, experience a sudden decrease in their liquidity. To recover from the shock, investors are either forced to liquidate their current asset position or to withdraw from the buy side. The forced liquidation leads to an abnormally high fraction of potential sellers immediately after the aggregate liquidity shock. A selling pressure results. The withdrawal from the buy side decreases the fraction of potential buyers. Combining both effects inevitably leads to prices that are under pressure. Khandani and Lo (2011) and Pedersen (2009) provide an example for an aggregate liquidity shock with short recovery time: In August 2007, one or more hedge funds were forced to unwind large asset positions at short notice, probably due to margin calls or to reduce risk. These huge selling positions induced losses to others, leading to a “deadly feedback loop”⁸⁹ by reducing prices of collaterals. After a few days, prices mostly recovered. In general, a price drop due to a liquidity shock leads to a price reversal.⁹⁰ Coval and Stafford (2007) find similar results for mutual funds suffering a liquidity shock, which is usually due to the poor performance of the fund. Price reversals are slower, though.

The aggregate liquidity shock model presented in this chapter describes a general scenario of systemic liquidity shocks. It addresses the effects of a sudden selling pressure on prices, bid-ask spreads, market recovery, and the impact that the risk of further shocks has on long term equilibrium prices.

4.2 Model Setup

Duffie, Gârleanu, and Pedersen (2007) begin with the assumptions that an aggregate liquidity shock occurs once in a while, is randomly timed, and affects many investors at the same time. The occurrence is modeled by a Poisson arrival process with mean arrival rate ζ . For tractability, the shocks’ independence from all other random variables is assumed.

Agents are affected in different ways and to different extents. A randomly chosen fraction of high-type agents experiences a sudden external force to liquidate their asset position, or to withdraw from standing ready to buy, since their intrinsic types simultaneously jump to a low status. Their preference towards asset holdings is decreased due to the aggregate preference shock. Duffie, Gârleanu, and Pedersen (2007) introduce the notation $\bar{\mu}_\sigma(0)$, that is $\bar{\mu}_{ho}(0)$ for the ho agents’ and

⁸⁹ Khandani and Lo (2011), p. 3.

⁹⁰ See Pedersen (2009), p. 196.

$\bar{\mu}_{hn}(0)$ for the *hn* agents' post-shock distribution, i.e. the distribution immediately after the shock. Time t is rescaled as the time that has elapsed since the last shock; and it is reset to $t = 0$ upon occurrence of another shock. For example, $\mu_{ho}(t)$ is the fraction of high owners t units of time after the most recent liquidity shock. A necessary simplification for tractability is the assumption that prior shocks do not influence the post-shock distribution by any aftereffects. This assumption leads to a fixed post-shock distribution, on which the fraction of high-type agents changing to a low state depends: An *ho* agent switches to an *lo* agent with probability

$$\pi_{ho}(t) = 1 - \frac{\bar{\mu}_{ho}(0)}{\mu_{ho}(t)}, \quad (4.1)$$

and remains an *ho* agent with probability $1 - \pi_{ho}(t)$. An *hn* agent switches to an *ln* agent with probability

$$\pi_{hn}(t) = 1 - \frac{\bar{\mu}_{hn}(0)}{\mu_{hn}(t)}, \quad (4.2)$$

and remains an *hn* agent with probability $1 - \pi_{hn}(t)$. Assume that the shock occurs when the system is in steady state (*ss*). The post-shock distribution $\bar{\mu}_{\sigma}(0)$ of high-type agents' intrinsic type is calculated with

$$\bar{\mu}_{ho}(0) = (1 - \pi_{ho}(ss)) \mu_{ho}(ss), \quad (4.3)$$

$$\bar{\mu}_{hn}(0) = (1 - \pi_{hn}(ss)) \mu_{hn}(ss), \quad (4.4)$$

where $\bar{\mu}_{ho}(0) < \mu_{ho}(ss)$ and $\bar{\mu}_{hn}(0) < \mu_{hn}(ss)$. The conditions $0 \leq \pi_{ho}(ss) \leq 1$ and $0 \leq \pi_{hn}(ss) \leq 1$ for a probability holds. Whenever a shock occurs, the type distribution jumps to the post-shock fractions specified in (4.3) and (4.4). These equations show that the probability of high agents switching to a low state upon occurrence of a shock in steady state, i.e. $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$, directly affects the severity of a shock.

From equations (4.3), (4.4), and (3.1) it is clear that low-type agents are affected as well, though in an indirect way. The fractions $\pi_{ho}(ss)\mu_{ho}(ss)$ and $\pi_{hn}(ss)\mu_{hn}(ss)$ of high-type agents change to low-type agents. The post-shock distribution of low-type agents increases by these amounts, with

$$\bar{\mu}_{lo}(0) = \mu_{lo}(ss) + \pi_{ho}(ss)\mu_{ho}(ss), \quad (4.5)$$

$$\bar{\mu}_{ln}(0) = \mu_{ln}(ss) + \pi_{hn}(ss)\mu_{hn}(ss). \quad (4.6)$$

The type distribution $\mu_\sigma(t)$ has to satisfy constraints (3.1)–(3.3) for all t and evolve after the shock according to the system of ordinary differential equations defined in equations (3.4)–(3.7). On the condition that no additional shock occurs in the meantime, these equations converge continuously to the steady state equilibrium for any starting condition.⁹¹

4.3 Implementing Aggregate Liquidity Shocks

Aggregate liquidity shocks—in the style of Duffie, Gârleanu, and Pedersen (2007)—are integrated into the indirect utility functions $V_\sigma^s(t)$ (where the superscript ‘s’ denotes ‘shock’). I focus again on a particular agent at a particular time t and assume that the value functions $V_\sigma^s(t)$ are well defined, i.e. bounded. The stopping times τ_l, τ_i, τ_m are defined as in chapter 3.3.1. Additionally, set τ_ζ as the stopping time for occurrence of an aggregate liquidity shock. The value functions are as follows:

$$\begin{aligned} V_{lo}^s(t) = E_t & \left[\int_t^\tau e^{-r(u-t)} (D - \delta) du + e^{-r(\tau_l-t)} V_{ho}^s(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} \right. \\ & + e^{-r(\tau_i-t)} (V_{ln}^s(\tau_i) + P^s(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \\ & \left. + e^{-r(\tau_m-t)} (V_{ln}^s(\tau_m) + B^s(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} + e^{-r(\tau_\zeta-t)} V_{lo}^s(0) \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \end{aligned} \quad (4.7)$$

$$\begin{aligned} V_{hn}^s(t) = E_t & \left[e^{-r(\tau_l-t)} V_{ln}^s(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} + e^{-r(\tau_i-t)} (V_{ho}^s(\tau_i) - P^s(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \right. \\ & + e^{-r(\tau_m-t)} (V_{ho}^s(\tau_m) - A^s(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} \\ & \left. + e^{-r(\tau_\zeta-t)} ((1 - \pi_{hn}(\tau_\zeta)) V_{hn}^s(0) + \pi_{hn}(\tau_\zeta) V_{ln}^s(0)) \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \end{aligned} \quad (4.8)$$

$$\begin{aligned} V_{ho}^s(t) = E_t & \left[\int_t^\tau e^{-r(u-t)} D du + e^{-r(\tau_l-t)} V_{lo}^s(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} \right. \\ & \left. + e^{-r(\tau_\zeta-t)} ((1 - \pi_{ho}(\tau_\zeta)) V_{ho}^s(0) + \pi_{ho}(\tau_\zeta) V_{lo}^s(0)) \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \end{aligned} \quad (4.9)$$

$$V_{ln}^s(t) = E_t \left[e^{-r(\tau_l-t)} V_{hn}^s(\tau_l) \mathbf{1}_{\{\tau_l=\tau\}} + e^{-r(\tau_\zeta-t)} V_{ln}^s(0) \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \quad (4.10)$$

where $\tau = \min\{\tau_l, \tau_i, \tau_m, \tau_\zeta\}$. The first terms are identical to equations (3.21)–(3.24). The last summand is due to the possibility of an aggregate liquidity shock. Since all low-type agents stay low, they only jump to the respective states $V_{lo}^s(0)$

⁹¹ See Duffie, Gârleanu, and Pedersen (2005, pp. 1836) for the proof of convergence.

and $V_{ln}^s(0)$ immediately after the shock upon occurrence, i.e. in $t = 0$. High-type agents stay high with probabilities $(1 - \pi_{ho}(\tau_\zeta))$ and $(1 - \pi_{hn}(\tau_\zeta))$ for ho agents and hn agents, respectively. They jump to the respective states $V_{ho}^s(0)$ and $V_{hn}^s(0)$ immediately after the shock, i.e. in $t = 0$. With probabilities $\pi_{ho}(\tau_\zeta)$ and $\pi_{hn}(\tau_\zeta)$, ho agents and hn agents, respectively, switch to low-type agents in $t = 0$.

Along the lines of appendix 3A, the growth rate of the value functions with liquidity shocks, satisfying the proper transversality condition, are

$$\begin{aligned} \dot{V}_{lo}^s(t) &= (r + \lambda_u + 2\lambda\mu_{hn}(t) + \rho + \zeta) V_{lo}^s(t) - \lambda_u V_{ho}^s(t) \\ &\quad - (2\lambda\mu_{hn}(t) + \rho) V_{ln}^s(t) - 2\lambda\mu_{hn}(t) P^s(t) - \rho B^s(t) \\ &\quad - \zeta V_{lo}^s(0) - (D - \delta), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \dot{V}_{hn}^s(t) &= (r + \lambda_d + 2\lambda\mu_{lo}(t) + \rho + \zeta) V_{hn}^s(t) - \lambda_d V_{ln}^s(t) \\ &\quad - (2\lambda\mu_{lo}(t) + \rho) V_{ho}^s(t) + 2\lambda\mu_{lo}(t) P^s(t) + \rho A^s(t) \\ &\quad - \zeta (1 - \pi_{hn}(t)) V_{hn}^s(0) - \zeta \pi_{hn}(t) V_{ln}^s(0), \end{aligned} \quad (4.12)$$

$$\begin{aligned} \dot{V}_{ho}^s(t) &= (r + \lambda_d + \zeta) V_{ho}^s(t) - \lambda_d V_{lo}^s(t) - \zeta (1 - \pi_{ho}(t)) V_{ho}^s(0) \\ &\quad - \zeta \pi_{ho}(t) V_{lo}^s(0) - D, \end{aligned} \quad (4.13)$$

$$\dot{V}_{ln}^s(t) = (r + \lambda_u + \zeta) V_{ln}^s(t) - \lambda_u V_{hn}^s(t) - \zeta V_{ln}^s(0). \quad (4.14)$$

The bargaining prices are similar to equations (3.36)–(3.48), implying

$$P^s(t) = (1 - q) (V_{lo}^s(t) - V_{ln}^s(t)) + q (V_{ho}^s(t) - V_{hn}^s(t)), \quad (4.15)$$

$$A^s(t) = (1 - z) M^s(t) + z (V_{ho}^s(t) - V_{hn}^s(t)), \quad (4.16)$$

$$B^s(t) = (1 - z) M^s(t) + z (V_{lo}^s(t) - V_{ln}^s(t)), \quad (4.17)$$

$$M^s(t) = (1 - \tilde{q}(t)) (V_{lo}^s(t) - V_{ln}^s(t)) + \tilde{q}(t) (V_{ho}^s(t) - V_{hn}^s(t)), \quad (4.18)$$

with

$$\tilde{q}(t) \begin{cases} = 1 & \text{if } \mu_{lo}(t) < \mu_{hn}(t) \\ = 0 & \text{if } \mu_{lo}(t) > \mu_{hn}(t) \\ \in [0, 1] & \text{if } \mu_{lo}(t) = \mu_{hn}(t), \end{cases} \quad (4.19)$$

and subject to

$$V_{lo}^s(t) - V_{ln}^s(t) \leq P^s(t) \leq V_{ho}^s(t) - V_{hn}^s(t), \quad (4.20)$$

$$V_{lo}^s(t) - V_{ln}^s(t) \leq B^s(t) \leq A^s(t) \leq V_{ho}^s(t) - V_{hn}^s(t). \quad (4.21)$$

Again, it is assumed that a meeting between two agents results in a trade.⁹²

Equations (4.11)–(4.21) depend on each other. I follow Duffie, Gârleanu, and Pedersen (2007) by restating them as a system of linear (time-varying) differential equations:

$$\dot{\mathbf{V}}_{\sigma}^s(t) = \mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{V}_{\sigma}^s(t) - \mathbf{A}_2 - \mathbf{A}_3(\boldsymbol{\mu}(t))\mathbf{V}_{\sigma}^s(0), \quad (4.22)$$

with

$$\mathbf{V}_{\sigma}^s(t) = \begin{bmatrix} V_{lo}^s(t) \\ V_{hn}^s(t) \\ V_{ho}^s(t) \\ V_{ln}^s(t) \end{bmatrix}, \quad \boldsymbol{\mu}(t) = \begin{bmatrix} \mu_{lo}(t) \\ \mu_{hn}(t) \\ \mu_{ho}(t) \\ \mu_{ln}(t) \end{bmatrix},$$

$$\mathbf{A}_1(\boldsymbol{\mu}(t)) = \quad (4.23)$$

$$\begin{bmatrix} \begin{pmatrix} r+\zeta+2\lambda\mu_{hn}(t)q \\ +\lambda_u+\rho(1-z)\tilde{q}(t) \end{pmatrix} & \begin{pmatrix} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} & \begin{pmatrix} -\lambda_u-2\lambda\mu_{hn}(t)q \\ -\rho(1-z)\tilde{q}(t) \end{pmatrix} & \begin{pmatrix} -2\lambda\mu_{hn}(t)q \\ -\rho(1-z)\tilde{q}(t) \end{pmatrix} \\ \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} r+\zeta+\lambda_d \\ +2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} -\lambda_d \\ -2\lambda\mu_{lo}(t)(1-q) \\ -\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} \\ -\lambda_d & 0 & r + \lambda_d + \zeta & 0 \\ 0 & -\lambda_u & 0 & r + \lambda_u + \zeta \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} D - \delta \\ 0 \\ D \\ 0 \end{bmatrix}, \quad (4.24)$$

⁹² Chapter 6 shows that this assumption is not fulfilled in general.

and

$$\mathbf{A}_3(\boldsymbol{\mu}(t)) = \begin{bmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta(1 - \pi_{hn}(t)) & 0 & \zeta\pi_{hn}(t) \\ \zeta\pi_{ho}(t) & 0 & \zeta(1 - \pi_{ho}(t)) & 0 \\ 0 & 0 & 0 & \zeta \end{bmatrix}. \quad (4.25)$$

For $t \rightarrow \infty$, matrices $\mathbf{A}_1(\boldsymbol{\mu}(t))$ and $\mathbf{A}_3(\boldsymbol{\mu}(t))$ converge to constant, time-invariant matrices, given that no additional shock has occurred.

Matrix $\mathbf{A}_1(\boldsymbol{\mu}(t))$ is time-dependent and non-diagonal. These properties imply that the system of differential equations in (4.22) cannot be solved with a textbook formula. In order to solve equation (4.22), I present a solution technique for these systems of ordinary differential equations. To focus on the primary objectives, i.e. aggregate liquidity shocks, I defer the comprehensive discussion of this solution technique to appendix 4A. The next section sketches the approach and presents the solution to equation (4.22).

4.4 Prices after Aggregate Liquidity Shocks

To solve the target system of linear (time-varying) differential equations in (4.22), I apply the solution technique introduced in appendix 4A: Assume there exists a (Lyapunov) transformation $\mathbf{T}(t)$ with

$$\mathbf{V}_\sigma^s(t) = \mathbf{T}(t)\mathbf{V}_\sigma^*(t),$$

and the derivative

$$\dot{\mathbf{V}}_\sigma^s(t) = \dot{\mathbf{T}}(t)\mathbf{V}_\sigma^*(t) + \mathbf{T}(t)\dot{\mathbf{V}}_\sigma^*(t),$$

to diagonalize the matrix $\mathbf{A}_1(\boldsymbol{\mu}(t))$. This transforms the equation (4.22) with

$$\begin{aligned} \dot{\mathbf{T}}(t)\mathbf{V}_\sigma^*(t) + \mathbf{T}(t)\dot{\mathbf{V}}_\sigma^*(t) &= \mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{T}(t)\mathbf{V}_\sigma^*(t) - \mathbf{A}_2 - \mathbf{A}_3(\boldsymbol{\mu}(t))\mathbf{T}(0)\mathbf{V}_\sigma^*(0), \\ &\Leftrightarrow \\ \dot{\mathbf{V}}_\sigma^*(t) &= \left[\mathbf{T}(t)^{-1}\mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{T}(t) - \mathbf{T}(t)^{-1}\dot{\mathbf{T}}(t) \right] \mathbf{V}_\sigma^*(t) \\ &\quad - \mathbf{T}(t)^{-1}\mathbf{A}_2 - \mathbf{T}(t)^{-1}\mathbf{A}_3(\boldsymbol{\mu}(t))\mathbf{T}(0)\mathbf{V}_\sigma^*(0). \end{aligned} \quad (4.26)$$

Define:

$$\Lambda(\boldsymbol{\mu}(t)) = \mathbf{T}(t)^{-1} \mathbf{A}_1(\boldsymbol{\mu}(t)) \mathbf{T}(t) - \mathbf{T}(t)^{-1} \dot{\mathbf{T}}(t), \quad (4.27)$$

$$\mathbf{A}_2^*(t) = \mathbf{T}(t)^{-1} \mathbf{A}_2, \quad (4.28)$$

$$\mathbf{A}_3^*(\boldsymbol{\mu}(t)) = \mathbf{T}(t)^{-1} \mathbf{A}_3(\boldsymbol{\mu}(t)) \mathbf{T}(0), \quad (4.29)$$

with \mathbf{A}_2 and $\mathbf{A}_3(\boldsymbol{\mu}(t))$ as stated in equations (4.24) and (4.25), respectively. Matrix $\Lambda(\boldsymbol{\mu}(t))$ is a diagonal matrix, which is the aim of this transformation. Inserting equations (4.27), (4.28), and (4.29) into equation (4.26), I obtain the transformed system with

$$\dot{\mathbf{V}}_\sigma^*(t) = \Lambda(\boldsymbol{\mu}(t)) \mathbf{V}_\sigma^*(t) - \mathbf{A}_2^*(t) - \mathbf{A}_3^*(\boldsymbol{\mu}(t)) \mathbf{V}_\sigma^*(0). \quad (4.30)$$

Since $\Lambda(\boldsymbol{\mu}(t))$ is still time-dependent but also a diagonal matrix, the solution to the transformed system (4.30) is

$$\mathbf{V}_\sigma^*(t) = \int_t^\infty e^{-\int_t^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} [\mathbf{A}_2^*(x) + \mathbf{A}_3^*(\boldsymbol{\mu}(x)) \mathbf{V}_\sigma^*(0)] dx. \quad (4.31)$$

Set $t = 0$ and rearrange this equation to solve for $\mathbf{V}_\sigma^*(0)$, with

$$\begin{aligned} \mathbf{V}_\sigma^*(0) &= \int_0^\infty e^{-\int_0^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} [\mathbf{A}_2^*(x) + \mathbf{A}_3^*(\boldsymbol{\mu}(x)) \mathbf{V}_\sigma^*(0)] dx \\ &= \int_0^\infty e^{-\int_0^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} \mathbf{A}_2^*(x) dx + \int_0^\infty e^{-\int_0^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} \mathbf{A}_3^*(\boldsymbol{\mu}(x)) dx \mathbf{V}_\sigma^*(0) \\ &= \left(\mathbf{I}_4 - \int_0^\infty e^{-\int_0^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} \mathbf{A}_3^*(\boldsymbol{\mu}(x)) dx \right)^{-1} \times \\ &\quad \left(\int_0^\infty e^{-\int_0^x \Lambda(\boldsymbol{\mu}(\tau)) d\tau} \mathbf{A}_2^*(x) dx \right), \end{aligned} \quad (4.32)$$

where \mathbf{I}_4 is the identity matrix of dimension four. Inverse transformation completes the solution process

$$\mathbf{V}_\sigma^s(t) = \mathbf{T}(t) \mathbf{V}_\sigma^*(t).$$

Based on this transformation method, I derive the semi-analytical solution for

calculating investors' optimal lifetime consumption with

$$\mathbf{V}_\sigma^s(t) = \int_t^\infty \mathbf{T}(t) e^{-\int_t^x \Lambda(\mu(\tau)) d\tau} \mathbf{T}(x)^{-1} [\mathbf{A}_2 + \mathbf{A}_3(\mu(x)) \mathbf{V}_\sigma^s(0)] dx, \quad (4.33)$$

where

$$\Phi(t, x) = \mathbf{T}(t) e^{-\int_t^x \Lambda(\mu(\tau)) d\tau} \mathbf{T}(x)^{-1} \quad (4.34)$$

is the state transition matrix. With the Riccati transformation method introduced in appendix 4A.5, the solution for the matrices $\mathbf{T}(t)$ and $\Lambda(\mu(t))$ is straight forward. Matrix $\mathbf{T}(t)$ contains the dynamic eigenvectors $\nu_i(t)$ (for $i = 1, 2, 3, 4$) of matrix $\mathbf{A}_1(\mu(t))$ on its columns, whereas matrix $\Lambda(\mu(t))$ contains the dynamic eigenvalues $\lambda_i(t)$ of matrix $\mathbf{A}_1(\mu(t))$ on its main diagonal. The explicit results, which I derive in appendices 4B and 4C, are

$$\mathbf{T}(t) = \begin{bmatrix} 1 & 1 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & q_{41}(t) \\ 1 & 2 & \frac{\lambda_d}{\lambda_u + \lambda_d} & q_{42}(t) \\ 1 & 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} & q_{43}(t) \\ 1 & 2 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & q_{41}(t) + q_{42}(t) - q_{43}(t) + 1 \end{bmatrix}, \quad (4.35)$$

with

$$\begin{aligned} \mathbf{q}_4(t) &= \begin{bmatrix} q_{41}(t) & q_{42}(t) & q_{43}(t) \end{bmatrix}^T, \\ &= -\frac{\lambda_d}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \hat{\lambda}_1(\tau) d\tau} \left(2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x) \right) dx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &\quad + \frac{1}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \hat{\lambda}_2(\tau) d\tau} \left(2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x) \right) dx \begin{bmatrix} -\lambda_u \\ \lambda_d \\ \lambda_d \end{bmatrix} \\ &\quad - \int_0^t e^{\int_x^t \hat{\lambda}_1(\tau) d\tau} \left(\lambda_d + 2\lambda\mu_{lo}(x)(1-q) + \rho(1-z)(1-\tilde{q}(x)) \right) dx \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \end{aligned} \quad (4.36)$$

and

$$\hat{\lambda}_1(t) = -(\lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)), \quad (4.37)$$

$$\hat{\lambda}_2(t) = -(2\lambda\mu_{hm}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)). \quad (4.38)$$

Matrix $\Lambda(\boldsymbol{\mu}(t))$ is obtained with

$$\Lambda(\boldsymbol{\mu}(t)) = \begin{bmatrix} r + \zeta & 0 & 0 & 0 \\ 0 & r + \zeta & 0 & 0 \\ 0 & 0 & r + \zeta + \lambda_d + \lambda_u & 0 \\ 0 & 0 & 0 & \left(\begin{array}{c} r + \zeta + \lambda_d + \lambda_u \\ + 2\lambda\mu_{lo}(t)(1-q) + 2\lambda\mu_{hm}(t)q \\ + \rho(1-z) \end{array} \right) \end{bmatrix}. \quad (4.39)$$

I calculate price paths with price equations (4.15)–(4.19) and the value function stated in (4.33), so that

$$P^s(t) = \begin{bmatrix} (1-q) & -q & q & -(1-q) \end{bmatrix} \mathbf{V}_\sigma^s(t), \quad (4.40)$$

$$A^s(t) = \begin{bmatrix} (1-z)(1-\tilde{q}(t)) \\ -(1-z)\tilde{q}(t) - z \\ (1-z)\tilde{q}(t) + z \\ -(1-z)(1-\tilde{q}(t)) \end{bmatrix}^T \mathbf{V}_\sigma^s(t), \quad (4.41)$$

$$B^s(t) = \begin{bmatrix} (1-z)(1-\tilde{q}(t)) + z \\ -(1-z)\tilde{q}(t) \\ (1-z)\tilde{q}(t) \\ -(1-z)(1-\tilde{q}(t)) - z \end{bmatrix}^T \mathbf{V}_\sigma^s(t). \quad (4.42)$$

I show (see appendix 4B) that the assumed transversality or no-bubble condition $\lim_{x \rightarrow \infty} E_t [e^{-rx} \max \{P^s(x), A^s(x), B^s(x)\}] = 0$ holds, i.e. the value functions $\mathbf{V}_\sigma^s(t)$ are actually well defined, if $\lambda, \rho, \lambda_d, \lambda_u < \infty$.

The extension of the basic model of chapter 3 for aggregate liquidity shocks implies that there is no closed form steady state solution that can easily be calculated. Detailed analysis cannot be given analytically anymore, since coefficients are time-dependent and there is no closed form, out-of-steady-state solution for the mass dynamics $\mu_\sigma(t)$. I discuss general characteristics of the aggregate liquidity shock model on the basis of a numerical example in chapter 5.

4.5 Concluding Remarks

This chapter analyzes the dynamics out of and towards the steady state of prices and of return reactions after aggregate liquidity shocks, initially discussed in Duffie, Gârleanu, and Pedersen (2007). Aggregate liquidity shocks are associated with a sudden shift in agents' preferences towards asset holding, affecting a large fraction of investors simultaneously. Several investors experience a sudden decrease in their liquidity, leading to a forced withdrawal of assets: The market is hit by a selling pressure.

I develop a semi-analytical solution for the resulting linear time-varying (LTV) system of differential equations, including market makers. With this solution, my analysis reveals weaknesses in the presentation of the solution in Duffie, Gârleanu, and Pedersen (2007), which I address next. I use their numbering, whereas the label in squared brackets refers to the equations in my thesis.

First, Duffie, Gârleanu, and Pedersen (2007) do not give the exact contents of $\mathbf{A}_1(\boldsymbol{\mu}(t))$, \mathbf{A}_2 , and $\mathbf{A}_3(\boldsymbol{\mu}(t))$, except for stating equations (26) [4.11–4.15] (without market maker, i.e. $\rho = 0$). When comparing solution (28), that is

$$\mathbf{V}(t) = \int_t^{\infty} e^{-\int_t^s \mathbf{A}_1(\boldsymbol{\mu}(u)) du} (\mathbf{A}_2 + \mathbf{A}_3(\boldsymbol{\mu}(s))\mathbf{V}(0)) ds,$$

where $\mathbf{V}(t) = \mathbf{V}_\sigma^s(t)$, with my transformed solution in [4.31] and the final solution in [4.33], some differences do attract attention:

If equation (28) solves the transformed system [4.31], then $\mathbf{A}_1(\boldsymbol{\mu}(t))$ would be equal to $\boldsymbol{\Lambda}(\boldsymbol{\mu}(t))$ and the ansatz $\exp(-\int_t^s \mathbf{A}_1(\boldsymbol{\mu}(u)) du)$ is correctly applied. However, \mathbf{A}_2 is then time-dependent and $\mathbf{A}_1(\boldsymbol{\mu}(t))$ in equation (27) [4.22] and $\mathbf{A}_1(\boldsymbol{\mu}(t))$ in (28) must be different. Hence, $\mathbf{V}(t)$ is not the desired solution, but the solution to the transformed system.

If equation (28) solves system [4.33], then $\mathbf{A}_1(\boldsymbol{\mu}(t))$ should be equal in both equations. However, the ansatz $\exp(-\int_t^s \mathbf{A}_1(\boldsymbol{\mu}(u)) du)$ is incorrect (see section 4A.3). For a solely numerical result, one might propose the Magnus–expansion: Magnus (1954) suggests a function $\boldsymbol{\Omega}(t)$, which depends on the coefficient matrix $\mathbf{A}_1(\boldsymbol{\mu}(t))$ of the differential equation $\dot{\mathbf{V}}(t) = \mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{V}(t)$, such that $\mathbf{V}(t) = \exp(\boldsymbol{\Omega}(t))$. Therefore, there must be differences between $\mathbf{A}_1(\boldsymbol{\mu}(t))$ in equation (27) [4.22] and $\mathbf{A}_1(\boldsymbol{\mu}(t))$ in (28).

Consequently, equation (28) is not a solution to (27) [4.22] and (26) [4.11–4.15].

4A Appendix: Linear Systems of Ordinary Differential Equations

Linear systems of ordinary differential equations (ODEs) are commonly subdivided into linear time-invariant (LTI) and linear time-varying (LTV) systems, since solution techniques differ between both. This section first reviews some basic theory about linear systems of ODEs in 4A.1 and then shows the solution approach for LTI systems in 4A.2. Both topics are treated in depth in standard textbooks about differential equations: Boyce and DiPrima (2009), Adrianova (1995), Birkhoff and Rota (1978), and Coddington and Levinson (1955). Then, I give a short introduction to LTV systems in 4A.3, following Chen (1999), Wu (1980, 1981), and Zadeh and Desoer (1963). Sections 4A.4 and 4A.5 show the solution technique for LTV systems derived in Abou-Kandil, Freiling, Ionescu, and Jank (2003), van der Kloet and Neerhoff (2001, 2004), and van der Kloet, Neerhoff, and de Anda (2001), which I utilize in this thesis.

4A.1 Basic Theory about Systems of Ordinary Differential Equations

Consider the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t), \quad (4.43)$$

with a known coefficient matrix $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$, a known vector $\mathbf{b}(t) \in \mathbb{R}^{n \times 1}$, and an unknown vector variable $\mathbf{x}(t) \in \mathbb{R}^{n \times 1}$, which are all time-dependent in general cases. System (4.43) is called linear inhomogeneous. If $\mathbf{b}(t) = 0$, system (4.43) is called linear homogeneous, i.e.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t). \quad (4.44)$$

First, a solution to the homogeneous system in (4.44) is considered. The general solution to the inhomogeneous system in (4.43) can then be found with the method of variation of parameters.

Assume that $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ constitute a set of n linearly independent solutions of system (4.44), which forms a fundamental set of solutions. The fundamental matrix $\mathbf{X}(t)$ contains the n linearly independent solutions $\mathbf{X}(t) = [\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)]$

as its columns, so that $\mathbf{X}(t)$ solves the system in (4.44) with

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t). \quad (4.45)$$

The general solution to the homogeneous system in (4.44) is

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}, \quad (4.46)$$

where \mathbf{c} is a vector of constants. Let $t = t_0$ be the starting time of system (4.46). Then, the constant vector can be calculated with $\mathbf{c} = \mathbf{X}(t_0)^{-1}\mathbf{x}(t_0)$. System (4.46) can therewith be stated in the form

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}\mathbf{x}(t_0) \quad (4.47)$$

$$= \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0), \quad (4.48)$$

where $\mathbf{\Phi}(t, t_0) = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}$ is called the state transition matrix. It has the following properties:

1. $\mathbf{\Phi}(t, t) = \mathbf{I}$,
2. $\mathbf{\Phi}(t, v) = \mathbf{\Phi}(v, t)^{-1}$,
3. $\mathbf{\Phi}(t_1, t_2) = \mathbf{\Phi}(t_1, t_0)\mathbf{\Phi}(t_0, t_2)$,
4. $d\mathbf{\Phi}(t, v)/dt = \mathbf{A}(t)\mathbf{\Phi}(t, v)$.

The general solution to the inhomogeneous system in (4.43) can be found by applying the method of variation of parameters:

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}\mathbf{x}(t_0) + \mathbf{X}(t) \int_{t_0}^t \mathbf{X}(v)^{-1}\mathbf{b}(v) dv \quad (4.49)$$

$$= \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^t \mathbf{\Phi}(t, v)\mathbf{b}(v) dv. \quad (4.50)$$

The crucial part of finding a solution to system (4.44) consists in finding a fundamental matrix $\mathbf{X}(t)$ or a state transition matrix $\mathbf{\Phi}(t, v)$. Section 4A.2 deals with a solution technique for a constant coefficient matrix \mathbf{A} . Section 4A.3 derives a (general) technique when the matrix $\mathbf{A}(t)$ is time-dependent. Usually, the state transition matrix must be calculated numerically for time-varying systems, since an analytic solution is generally not available.⁹³

⁹³ See Zadeh and Desoer (1963), p. 369.

4A.2 Linear Time-Invariant (LTI) Systems

Let the matrix $\mathbf{A}(t)$ in system (4.44) have constant coefficients. The homogeneous system under consideration is then

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t). \quad (4.51)$$

In the scalar case with $n = 1$, the general solution is $c \exp(t\mathbf{A})$. It can be shown that this approach is also applicable for an arbitrary (but finite) n . The fundamental matrix is

$$\mathbf{X}(t) = e^{\mathbf{A}t}, \quad (4.52)$$

and the state transition matrix is

$$\Phi(t, t_0) = e^{\mathbf{A}(t-t_0)},$$

where $\exp(\mathbf{A}t)$ is the exponential of a matrix. It is defined as follows:

$$e^{\mathbf{A}t} = \sum_{j=0}^{\infty} \frac{t^j}{j!} \mathbf{A}^j. \quad (4.53)$$

Differentiating element-wise shows that (4.52) is indeed a fundamental matrix:⁹⁴

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}.$$

The definition of the matrix exponential in (4.53) clarifies that the calculation of $\exp(\mathbf{A}t)$ is not an easy task in general. However, if the coefficient matrix is a diagonal matrix, the calculation of the matrix exponential is as follows: Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, then $\exp(\Lambda) = \text{diag}(\exp(\lambda_1), \dots, \exp(\lambda_n))$. For this property, assume a linear coordinate transformation of system (4.51) with

$$\mathbf{x}(t) = \mathbf{T}\mathbf{z}(t), \quad (4.54)$$

that maps the coefficient matrix \mathbf{A} of system (4.51) into the diagonal matrix Λ . The transformed system is

$$\dot{\mathbf{z}}(t) = \Lambda\mathbf{z}(t), \quad (4.55)$$

⁹⁴ See Boyce and DiPrima (2009), pp. 416.

which consists of n decoupled ODEs, since matrix \mathbf{A} is a diagonal matrix. The general solution is

$$\mathbf{z}(t) = e^{\mathbf{A}t} \mathbf{c}.$$

Inverse transformation leads to the solution of system (4.51), which is

$$\mathbf{x}(t) = \mathbf{T}e^{\mathbf{A}t} \mathbf{c}, \quad (4.56)$$

where $\mathbf{X}(t) = \mathbf{T} \exp(\mathbf{A}t)$ is the fundamental matrix. After combining (i.) equation (4.54), (ii.) equation (4.51), (iii.) the derivative of equation (4.54), and (iv.) equation (4.55), the coordinate transformation reads

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{\Lambda}, \quad (4.57)$$

since only non-trivial solutions for $\mathbf{z}(t)$ are considered. In the following, matrix $\mathbf{T} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n)$ is treated as a system of column vectors. The entries on the diagonal matrix $\mathbf{\Lambda}$ are $\lambda_1, \lambda_2, \dots, \lambda_n$. Hence, system (4.57) can be stated with

$$(\mathbf{A} - \lambda_i \mathbf{I}) \boldsymbol{\nu}_i = \mathbf{0},$$

for $i = 1, 2, \dots, n$, and with the identity matrix \mathbf{I} . This linear system of equations has a solution if

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0,$$

which is called the characteristic equation of \mathbf{A} . It is of order n and has n roots defining all λ_i for $i = 1, \dots, n$. The vectors $\boldsymbol{\nu}_i$ are called eigenvectors and λ_i are the corresponding eigenvalues. As a result, equation (4.56) solves the system in (4.51), with the columns of matrix $\mathbf{T} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n)$ consisting of n linearly independent eigenvectors of \mathbf{A} . The elements on the main diagonal of matrix $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$ are the corresponding eigenvalues of \mathbf{A} .

4A.3 Linear Time-Varying (LTV) Systems

Let the matrix $\mathbf{A}(t)$ in system (4.44) have time-dependent coefficients. An approach comparable to equation (4.52) with $\Phi(t) = \exp(\int \mathbf{A}(\tau) d\tau)$ is misleading,

since the derivative of a time-dependent matrix exponential $\exp(\mathbf{H}(t))$ is⁹⁵

$$\frac{d}{dt}e^{\mathbf{H}(t)} = \int_0^1 e^{(1-u)\mathbf{H}(t)} \dot{\mathbf{H}}(t) e^{u\mathbf{H}(t)} du.$$

The chain rule for the exponential matrix function $\exp(\int_{t_0}^t \mathbf{A}(\tau) d\tau)$ only holds for the special case where $\int_{t_0}^t \mathbf{A}(\tau) d\tau$ and $\mathbf{A}(t)$ commute for all t ,⁹⁶ i.e.

$$\left[e^{\int_{t_0}^t \mathbf{A}(\tau) d\tau} \right] \mathbf{A}(t) = \mathbf{A}(t) \left[e^{\int_{t_0}^t \mathbf{A}(\tau) d\tau} \right], \quad \forall t, \forall t_0.$$

Then, $\exp(\int_{t_0}^t \mathbf{A}(\tau) d\tau)$ is a state transition matrix to the system (4.44). This property of commutativity applies to constant matrices \mathbf{A} and to diagonal time-varying matrices $\mathbf{A}(t)$.⁹⁷ Therefore, the task is to diagonalize $\mathbf{A}(t)$, which is carried out by means of a time-varying coordinate transformation

$$\mathbf{x}(t) = \mathbf{T}(t)\mathbf{z}(t) \tag{4.58}$$

to the homogeneous system (4.44). This transforms matrix $\mathbf{A}(t)$ into a diagonal matrix $\mathbf{\Lambda}(t)$. The new LTV system

$$\dot{\mathbf{z}}(t) = \mathbf{\Lambda}(t)\mathbf{z}(t) \tag{4.59}$$

consists of n decoupled ODEs. The general solution is straight-forwardly

$$\mathbf{z}(t) = e^{\int \mathbf{\Lambda}(\tau) d\tau} \mathbf{c}.$$

Inverse transformation leads to the solution of the system (4.44), which is

$$\mathbf{x}(t) = \mathbf{T}(t)e^{\int \mathbf{\Lambda}(\tau) d\tau} \mathbf{c},$$

and $\mathbf{X}(t) = \mathbf{T}(t) \exp(\int \mathbf{\Lambda}(\tau) d\tau)$ is the fundamental matrix. After combining (i.) equation (4.58), (ii.) equation (4.44), (iii.) the derivative of (4.58), and (iv.) equation (4.59), the time-varying coordinate transformation reads

$$\mathbf{A}(t)\mathbf{T}(t) = \mathbf{T}(t)\mathbf{\Lambda}(t) + \dot{\mathbf{T}}(t), \tag{4.60}$$

⁹⁵ See Snider (1964), p. 1587 and Wilcox (1967), p. 964.

⁹⁶ There are some special cases for which the chain rule holds without commutativity. See Ma and Shekhtman (2010).

⁹⁷ See Coddington and Levinson (1955), p. 76.

since only non-trivial solutions for $\mathbf{z}(t)$ are considered. Consequently, matrix $\mathbf{A}(t)$ and $\mathbf{\Lambda}(t)$ are related to each other. This connection is called kinematic similarity.⁹⁸ Again, treat matrix $\mathbf{T}(t) = (\boldsymbol{\nu}_1(t), \dots, \boldsymbol{\nu}_n(t))$ as a system of column vectors. The entries on the diagonal matrix $\mathbf{\Lambda}(t)$ are $\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)$. Hence, system (4.60) can be stated as

$$[\mathbf{A}(t) - \lambda_i(t)\mathbf{I}] \boldsymbol{\nu}_i(t) = \dot{\boldsymbol{\nu}}_i(t),$$

for $i = 1, 2, \dots, n$, and with the identity matrix \mathbf{I} . In this context, vectors $\boldsymbol{\nu}_i(t)$ are called dynamic eigenvectors and $\lambda_i(t)$ are called the corresponding dynamic eigenvalues.^{99, 100} The crucial part is to find matrix $\mathbf{T}(t)$ and diagonal matrix $\mathbf{\Lambda}(t)$. Van der Kloet and Neerhoff (2001, 2004) and van der Kloet, Neerhoff, and de Anda (2001) suggest a transformation algorithm to obtain the matrices $\mathbf{T}(t)$ and $\mathbf{\Lambda}(t)$ or, equivalently, the modal form for n linearly independent solutions for the system (4.44)¹⁰¹

$$\mathbf{x}_i(t) = e^{\int \lambda_i(\tau) d\tau} \boldsymbol{\nu}_i(t), \quad \text{for } i = 1, 2, \dots, n.$$

Vectors $\mathbf{x}_i(t)$ are called mode-vectors.¹⁰² Before this solution technique is introduced in the following sections, some comments on the transformation matrix $\mathbf{T}(t)$ are given.¹⁰³

Lyapunov Transformation

Assume system (4.44) has bounded coefficients and transformation (4.58) should not alter the class of the system. Then, some additional restrictions on the transformation matrix $\mathbf{T}(t)$, besides non-singularity and continuity for $t \geq t_0$, are necessary:

Matrix $\mathbf{T}(t)$ is called a Lyapunov matrix if $\mathbf{T}(t)$, $\mathbf{T}(t)^{-1}$, and $\dot{\mathbf{T}}(t)$ are bounded for all t . Hence, transformation (4.58) is called a Lyapunov transformation.

Adrianova (1995, pp. 43–45) states some characteristics of Lyapunov transforma-

⁹⁸ See Adrianova (1995), p. 44.

⁹⁹ Dynamic eigenvectors and dynamic eigenvalues are also denoted as “eigenvector” and “eigenvalue” (see Wu (1980)) or as extended eigenvector (or x -eigenvector) and extended eigenvalue (or x -eigenvalue) (see Wu (1984)).

¹⁰⁰ Dynamic eigenvalues $\lambda(t)$ and dynamic eigenvectors $\boldsymbol{\nu}(t)$ are not unique. See Wu (1980, 1981) for an introduction into dynamic eigenvalues and dynamic eigenvectors.

¹⁰¹ See Wu (1984).

¹⁰² See de Anda (2012).

¹⁰³ See Adrianova (1995), pp. 43–45.

tion:

1. The inverse $\mathbf{T}(t)^{-1}$ is a Lyapunov matrix if $\mathbf{T}(t)$ is a Lyapunov matrix.
2. Two successive Lyapunov transformations constitute a Lyapunov transformation as well, i.e. $\mathbf{T}(t) = \mathbf{P}(t)\mathbf{Q}(t)$, if $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ are Lyapunov matrices.

It follows that a Lyapunov transformation does not alter the stability characteristics of the system (4.44).¹⁰⁴

4A.4 Riccati Transformation

It is well known that there is a close connection between linear systems of ordinary differential equations and the Riccati¹⁰⁵ differential equation: A system of linear differential equations can be (partially) decoupled by a transformation called Riccati transformation.^{106, 107}

The starting point for the matrix Riccati transformation is the following n -dimensional homogeneous system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad (4.61)$$

with the time-dependent system matrix $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$ and $\mathbf{x}(t) \in \mathbb{R}^{n \times 1}$. First, partition matrix $\mathbf{A}(t)$ with

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) & \mathbf{A}_{12}(t) \\ \mathbf{A}_{21}(t) & \mathbf{A}_{22}(t) \end{bmatrix}, \quad (4.62)$$

and $\mathbf{A}_{11}(t) \in \mathbb{R}^{a \times a}$, $\mathbf{A}_{12}(t) \in \mathbb{R}^{a \times b}$, $\mathbf{A}_{21}(t) \in \mathbb{R}^{b \times a}$, $\mathbf{A}_{22}(t) \in \mathbb{R}^{b \times b}$, and $a + b = n$. Transform system (4.61) with

$$\mathbf{x}(t) = \mathbf{P}(t)\mathbf{y}(t), \quad \text{for } \mathbf{P}(t) \in \mathbb{R}^{n \times n}, \quad (4.63)$$

¹⁰⁴ See Chen (1999), p. 139.

¹⁰⁵ Riccati (1724).

¹⁰⁶ See Abou-Kandil, Freiling, Ionescu, and Jank (2003), pp. 90–97, Smith (1987), Reid (1972), and for an early source Radon (1928), pp. 288–299.

¹⁰⁷ The notation and presentation of section 4A.4 are based on Abou-Kandil, Freiling, Ionescu, and Jank (2003, pp. 90–97).

where

$$\mathbf{P}(t) = \begin{bmatrix} \mathbf{I}_a & \mathbf{0} \\ \mathbf{K}(t) & \mathbf{I}_b \end{bmatrix}, \quad (4.64)$$

and the identity matrices \mathbf{I}_a and \mathbf{I}_b of dimension a and b , respectively. Transformation (4.63) converts system (4.61) in a form with an upper block triangular matrix $\mathbf{B}(t)$ with

$$\dot{\mathbf{y}}(t) = \mathbf{B}(t)\mathbf{y}(t) \quad (4.65)$$

iff $\mathbf{K}(t) \in \mathbb{R}^{b \times a}$ is a solution to the matrix Riccati differential equation¹⁰⁸

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}_{11}(t) + \mathbf{A}_{22}(t)\mathbf{K}(t) + \mathbf{A}_{21}(t) - \mathbf{K}(t)\mathbf{A}_{12}(t)\mathbf{K}(t). \quad (4.66)$$

With the connection

$$\mathbf{B}(t) = \mathbf{P}(t)^{-1}\mathbf{A}(t)\mathbf{P}(t) - \mathbf{P}(t)^{-1}\dot{\mathbf{P}}(t), \quad (4.67)$$

matrix $\mathbf{B}(t)$ reads as follows:

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) - \mathbf{A}_{12}(t)\mathbf{K}(t) & \mathbf{A}_{12}(t) \\ \mathbf{0} & \mathbf{K}(t)\mathbf{A}_{12}(t) + \mathbf{A}_{22}(t) \end{bmatrix}. \quad (4.68)$$

The subsequent second transformation

$$\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t), \quad \text{for } \mathbf{Q}(t) \in \mathbb{R}^{n \times n}, \quad (4.69)$$

converts system (4.65) into a block diagonal matrix $\mathbf{C}(t)$ with

$$\dot{\mathbf{z}}(t) = \mathbf{C}(t)\mathbf{z}(t) \quad (4.70)$$

and

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{I}_a & -\mathbf{L}(t) \\ \mathbf{0} & \mathbf{I}_b \end{bmatrix} \quad (4.71)$$

¹⁰⁸ See Reid (1972, ch. 2) for solving Riccati differential equations.

iff $\mathbf{L}(t) \in \mathbb{R}^{a \times b}$ is a solution of the Sylvester¹⁰⁹ differential equation

$$\dot{\mathbf{L}}(t) = [\mathbf{A}_{11}(t) - \mathbf{A}_{12}\mathbf{K}(t)] \mathbf{L}(t) - \mathbf{L}(t) [\mathbf{A}_{22}(t) + \mathbf{K}(t)\mathbf{A}_{12}(t)] - \mathbf{A}_{12}(t). \quad (4.72)$$

The block diagonal matrix $\mathbf{C}(t)$ reads

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) - \mathbf{A}_{12}(t)\mathbf{K}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{K}(t)\mathbf{A}_{12}(t) + \mathbf{A}_{22}(t) \end{bmatrix}, \quad (4.73)$$

and the transformation is complete. The order of system (4.61) is reduced by a partial decoupling.

Finally, it must be ensured that the stability properties of system (4.61) do not change due to the Riccati transform: Matrices $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ must be Lyapunov matrices. This characteristic implies that $\mathbf{K}(t)$, $\dot{\mathbf{K}}(t)$ and $\mathbf{L}(t)$, $\dot{\mathbf{L}}(t)$ must be bounded for all t .

4A.5 Solution for Systems of Linear Time-Varying Differential Equations

The two steps of the Riccati transformation, presented in section 4A.4, form the basis for the transformation algorithm provided by van der Kloet and Neerhoff (2001, 2004) and van der Kloet, Neerhoff, and de Anda (2001). The diagonalization of system (4.61) is performed by use of these two consecutive coordinate transformations, applied $(n - 1)$ times. A lower order of Riccati and Sylvester differential equations have to be solved in each of the $(n - 1)$ transformation rounds. In principle—i.e. if analytic solutions to the Riccati and Sylvester differential equations can be derived—an analytic solution to system (4.61) is feasible. The final result of this procedure is the fundamental matrix $\mathbf{X}(t)$, with n linearly independent solutions.

The following passage presents in detail the transformation algorithm for the two consecutive coordinate transformations. The first step transforms matrix $\mathbf{A}(t)$ in an upper block triangular matrix, the second step in a block diagonal matrix. Each transformation round disentangles one dynamic eigenvalue and the corre-

¹⁰⁹ Sylvester (1884).

sponding dynamic eigenvector, hence a mode vector. As a result, matrix $\mathbf{A}(t)$ is diagonalized.

1. Transformation Step: Upper Block Triangular Matrix

Specify the LTV system, which is defined in equation (4.61), with

$$\dot{\mathbf{x}}_k(t) = \mathbf{A}_k(t) \mathbf{x}_k(t), \quad \text{for } k = n, n-1, \dots, 2, \quad (4.74)$$

where k indicates the dimension of system matrix $\mathbf{A}_k(t) \in \mathbb{R}^{k \times k}$ and where $\mathbf{x}_k(t) \in \mathbb{R}^{k \times 1}$ is the state vector. The first transformation round starts with $k = n$. Each transformation round reduces k by one, until $k = 2$.¹¹⁰

The first transformation step is carried out via the relation that is equivalent to equation (4.63), but specified for $k = n, \dots, 2$, so that

$$\mathbf{x}_k(t) = \mathbf{P}_k(t) \mathbf{y}_k(t). \quad (4.75)$$

The transformation matrix $\mathbf{P}_k(t) \in \mathbb{R}^{k \times k}$, also called Riccati matrix, is comparable to matrix (4.64), but specified for the transformation algorithm as follows: The dimensions of the identity matrices \mathbf{I}_a and \mathbf{I}_b are $a = k - 1$ and $b = 1$, respectively. Consequently, matrix $\mathbf{K}(t)$ is a row vector of dimension $1 \times (k - 1)$. For clarity, I specify and redefine $\mathbf{K}(t) = \mathbf{p}_k^T(t)$. Hence,

$$\mathbf{P}_k(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \mathbf{0} \\ \mathbf{p}_k^T(t) & 1 \end{bmatrix}, \quad (4.76)$$

with the subvector

$$\mathbf{p}_k(t) = \begin{bmatrix} p_{k,1}(t) \\ p_{k,2}(t) \\ \vdots \\ p_{k,k-1}(t) \end{bmatrix}, \quad (4.77)$$

and the scalar functions of time $p_{k,i}(t)$ for $i = 1, 2, \dots, k - 1$. The submatrix $\mathbf{0} \in \mathbb{R}^{(k-1) \times 1}$ is a zero (column) vector. The key aspect of the first transformation is calculating vector $\mathbf{p}_k(t) \in \mathbb{R}^{(k-1) \times 1}$. This is obtained by partition matrix

¹¹⁰ If $k = 1$, system (4.74) is a scalar differential equation, which need not be diagonalized.

$\mathbf{A}_k(t)$ of equation (4.74) first, comparable to the matrix in (4.62), so that

$$\mathbf{A}_k(t) = \begin{bmatrix} \mathbf{A}_{11}^{\langle k \rangle}(t) & \mathbf{a}_{12}^{\langle k \rangle}(t) \\ [\mathbf{a}_{21}^{\langle k \rangle}(t)]^T & a_{22}^{\langle k \rangle}(t) \end{bmatrix}, \quad (4.78)$$

with matrix $\mathbf{A}_{11}^{\langle k \rangle}(t) \in \mathbb{R}^{(k-1) \times (k-1)}$, vectors $\mathbf{a}_{12}^{\langle k \rangle}(t), \mathbf{a}_{21}^{\langle k \rangle}(t) \in \mathbb{R}^{(k-1) \times 1}$, and the scalar $a_{22}^{\langle k \rangle}(t)$. Superscript $\langle k \rangle$ indicates affiliation to the k -dimensional system (4.74). The first transformation (4.75) converts system (4.74) into an upper block triangular form, comparable to the transformation (4.65) in connection with matrix (4.68), where

$$\dot{\mathbf{y}}_k(t) = \begin{bmatrix} \mathbf{A}_{11}^{\langle k \rangle}(t) + \mathbf{a}_{12}^{\langle k \rangle}(t)[\mathbf{p}_k(t)]^T & \mathbf{a}_{12}^{\langle k \rangle}(t) \\ \mathbf{0}^T & \lambda_k(t) \end{bmatrix} \mathbf{y}_k(t) \quad (4.79)$$

iff $\mathbf{p}_k(t)$ is any solution of the system of Riccati differential equations

$$\begin{aligned} \dot{\mathbf{p}}_k(t) = & -\mathbf{p}_k(t) [\mathbf{a}_{12}^{\langle k \rangle}(t)]^T \mathbf{p}_k(t) - [\mathbf{A}_{11}^{\langle k \rangle}(t)]^T \mathbf{p}_k(t) \\ & + \mathbf{p}_k(t) a_{22}^{\langle k \rangle}(t) + \mathbf{a}_{21}^{\langle k \rangle}(t), \end{aligned} \quad (4.80)$$

while $\lambda_k(t)$ is called dynamic eigenvalue and is calculated with

$$\lambda_k(t) = a_{22}^{\langle k \rangle}(t) - [\mathbf{p}_k(t)]^T \mathbf{a}_{12}^{\langle k \rangle}(t). \quad (4.81)$$

At the end of the first transformation step, it is important to emphasize that any solution of the Riccati equation (4.80) transfers equation (4.74) into the upper block triangular matrix of (4.79), since dynamic eigenvalues and -vectors are not unique.

2. Transformation Step: Block Diagonal Matrix

The second coordinate transformation converts equation (4.79) into a block diagonal form. This is obtained via the relation that is equivalent to equation (4.69) but specified again for $k = n, \dots, 2$, so that

$$\mathbf{y}_k(t) = \mathbf{Q}_k(t) \mathbf{z}_k(t). \quad (4.82)$$

The transformation matrix $\mathbf{Q}_k(t) \in \mathbb{R}^{k \times k}$ is comparable to matrix (4.71) but specified for the transformation algorithm as follows: The dimensions of the identity matrices \mathbf{I}_a and \mathbf{I}_b are further on $a = k - 1$ and $b = 1$, respectively. Consequently,

matrix $\mathbf{L}(t)$ is a column vector of dimension $(k-1) \times 1$. Again, for clarity, I specify and redefine $-\mathbf{L}(t) = \mathbf{q}_k(t)$. Hence,

$$\mathbf{Q}_k(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \mathbf{q}_k(t) \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad (4.83)$$

where

$$\mathbf{q}_k(t) = \begin{bmatrix} q_{k,1}(t) \\ q_{k,2}(t) \\ \vdots \\ q_{k,k-1}(t) \end{bmatrix}, \quad (4.84)$$

and the scalar functions of time $q_{k,i}(t)$ for $i = 1, \dots, k-1$. The submatrix $\mathbf{0}^T \in \mathbb{R}^{1 \times (k-1)}$ is a zero (row) vector. This transformation ensures that equation (4.79) goes into the following block diagonal form, comparable to the transformation (4.70) in connection with matrix (4.73), where

$$\dot{\mathbf{z}}_k(t) = \begin{bmatrix} \mathbf{A}_{11}^{\langle k \rangle}(t) + \mathbf{a}_{12}^{\langle k \rangle}(t) [\mathbf{p}_k(t)]^T & \mathbf{0} \\ \mathbf{0}^T & \lambda_k(t) \end{bmatrix} \mathbf{z}_k(t) \quad (4.85)$$

iff $\mathbf{q}_k(t)$ is any particular solution of the system of differential equations

$$\dot{\mathbf{q}}_k(t) = \left(\mathbf{A}_{11}^{\langle k \rangle}(t) + \mathbf{a}_{12}^{\langle k \rangle}(t) [\mathbf{p}_k(t)]^T - \lambda_k(t) \mathbf{I}_{k-1} \right) \mathbf{q}_k(t) + \mathbf{a}_{12}^{\langle k \rangle}(t). \quad (4.86)$$

The combination of the two consecutive transformations results in

$$\mathbf{x}_k(t) = \mathbf{P}_k(t) \mathbf{Q}_k(t) \mathbf{z}_k(t),$$

which is called Riccati transform. The second transformation step completes the first round of transformations.

Second Transformation Round

In order to decouple the whole system (4.74), these two transformation steps must be applied $(n-1)$ times in total. As a result, each round decouples an additional row in equation (4.85). The next transformation round starts with an order reduction of system (4.74) by one (i.e. $k := k-1$), leading to the starting system of the

second transformation round with

$$\dot{\mathbf{x}}_k(t) = \mathbf{A}_k(t)\mathbf{x}_k(t), \quad \text{for } k = n - 1, \quad (4.87)$$

where the updated system matrix in (4.87) is

$$\mathbf{A}_k = \mathbf{A}_{11}^{\langle k+1 \rangle}(t) + \mathbf{a}_{12}^{\langle k+1 \rangle}(t) [\mathbf{p}_{k+1}(t)]^T. \quad (4.88)$$

Dynamic Eigenvalues and the State Transition Matrix

After completing $(n - 1)$ transformation rounds, the dynamic eigenvalue $\lambda_1(t)$ is still pending. It can be calculated via the connection

$$\lambda_1(t) = \text{trace} [\mathbf{A}_n(t)] - \sum_{i=2}^n \lambda_i(t), \quad (4.89)$$

since the Riccati transform possesses the property of preserving the trace.¹¹¹ Dynamic eigenvalues “are invariant under any algebraic transformation.”¹¹²

The results of the two transformation steps, which are performed $(n - 1)$ times, are $(n - 1)$ \mathbf{P} -matrices and $(n - 1)$ \mathbf{Q} -matrices. Van der Kloet and Neerhoff (2001, 2004) combine these matrices to form one transformation matrix $\mathbf{T}(t) \in \mathbb{R}^{n \times n}$. The transformation matrix $\mathbf{T}(t)$ diagonalizes system (4.44), as shown in equation (4.58), where $\mathbf{T}(t)$ is calculated with

$$\mathbf{T}(t) = \prod_{i=1}^{(n-1)} \mathbf{S}_{n-i+1}(t), \quad (4.90)$$

and

$$\mathbf{S}_k(t) = \begin{bmatrix} \mathbf{P}_k(t)\mathbf{Q}_k(t) & \mathbf{0} \\ \mathbf{0}^T & \mathbf{I}_{n-k} \end{bmatrix}, \quad (4.91)$$

for $k = n, n - 1, \dots, 2$. The submatrix $\mathbf{0}$ is of dimension $k \times (n - k)$. As a result,

¹¹¹ See van der Kloet, Neerhoff, and de Anda (2001), p. 79, and Zhu and Johnson (1991), pp. 204–206.

¹¹² Wu (1980), p. 825.

matrix $\mathbf{T}(t)$ transforms equation (4.44) into the diagonal system

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} \lambda_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n(t) \end{bmatrix} \mathbf{z}(t),$$

which is comparable to equation (4.59), via the relation (4.60), that is

$$\mathbf{\Lambda}(t) = [\mathbf{T}(t)]^{-1} \mathbf{A}(t) \mathbf{T}(t) - [\mathbf{T}(t)]^{-1} \dot{\mathbf{T}}(t).$$

The state transition matrix $\mathbf{\Phi}(t, t_0)$ to system (4.44) results with

$$\mathbf{\Phi}(t, t_0) = \mathbf{T}(t) e^{\int_{t_0}^t \mathbf{\Lambda}(\tau) d\tau} [\mathbf{T}(t_0)]^{-1},$$

for the initial time t_0 . Any particular solution to $\mathbf{p}_k(t)$ and $\mathbf{q}_k(t)$ is feasible, and mode-vectors are not unique.

Lyapunov Transformation

Finally, it must be ensured that the stability properties do not change due to the successive Riccati transforms: Matrices $\mathbf{P}_k(t)$ and $\mathbf{Q}_k(t)$ must be Lyapunov matrices. This condition implies that the chosen solutions to the systems of Riccati differential equations $\mathbf{p}_k(t)$, $\dot{\mathbf{p}}_k(t)$ and the chosen solutions to the systems of Sylvester differential equations $\mathbf{q}_k(t)$, $\dot{\mathbf{q}}_k(t)$ have to be bounded for all t .

For the first transformation step, the solutions $\mathbf{p}_k(t)$ to the matrix Riccati equation can have the property of blowing up on a finite interval. This phenomenon is called finite escape time and it arises when there are singularities.¹¹³ Since any solution to equation (4.80) is feasible, one can try to find a bounded solution. This approach works for the system (4.22), so I do not go into details about non-blow-up conditions here. Detailed information can be found in Abou-Kandil, Freiling, Ionescu, and Jank (2003) or Freiling, Jank, and Sarychev (2000).

For the second transformation step, the solution to the systems of differential equations $\mathbf{q}_k(t)$ can be unbounded, i.e. unstable. It is well known that the stability analysis of LTI systems is characterized by its eigenvalues:

“A linear homogeneous system with constant coefficients [like equation (4.51)] is

¹¹³ See Abou-Kandil, Freiling, Ionescu, and Jank (2003), p. 91.

- 1) stable if and only if all the eigenvalues of the coefficient matrix have non-positive real parts, and simple elementary divisors correspond to the eigenvalues with zero real part,
- 2) asymptotically stable if and only if all the eigenvalues of the coefficient matrix have negative real parts.¹¹⁴ [Emphasis deleted.]

The stability analysis of LTV systems is, in general, not as easy as it is for LTI systems. Eigenvalues are misleading in the LTV case and the application of dynamic eigenvalues is not clear yet.¹¹⁵ In general, the characteristics of the state transition matrix are examined. But, in most cases, the state transition matrix must be calculated numerically, since an analytic solution is not available. The transformation algorithm provided by van der Kloet and Neerhoff (2001, 2004) and van der Kloet, Neerhoff, and de Anda (2001) shows that—in principle¹¹⁶—an analytic solution can be derived. The transformation algorithm can be applied again to find a solution of $\mathbf{q}_k(t)$ in equation (4.86). The stability of $\mathbf{q}_k(t)$ can be verified by the corresponding state transition matrix:¹¹⁷

The LTV homogeneous system (4.44) is

1. uniformly stable iff there exists a constant $D_1 > 0$ so that

$$\|\Phi(t, t_0)\| \leq D_1, \quad \text{for } 0 \leq t_0 \leq t < \infty, \quad (4.92)$$

where $\|\cdot\|$ is the matrix norm.¹¹⁸ Uniform stability implies that all solutions to the homogeneous system remain bounded for $t \geq t_0$.

2. uniformly asymptotically stable iff two constants $D_2 > 0$, $\alpha > 0$ (independent of t_0) exist so that

$$\|\Phi(t, t_0)\| \leq D_2 e^{-\alpha(t-t_0)}, \quad \text{for } 0 \leq t_0 \leq t < \infty. \quad (4.93)$$

¹¹⁴ Adrianova (1995), p. 84.

¹¹⁵ Van der Kloet, Neerhoff, and Waning (2007) suggest Lyapunov characteristic exponents, calculated by the mean value of the dynamic eigenvalues, to determine the stability of a LTV system. But the counterexample of de Anda (2012) shows that some more research has to be done on this topic.

¹¹⁶ An analytic solution for an LTV system is possible if analytic solutions to the Riccati equations (4.80) and differential equations (4.86) can be derived. This is feasible for 2×2 systems, but it can become challenging for higher order systems. Kolas (2008, ch. 7) develops a computer algorithm for calculating dynamic eigenvalues. Additionally, it can handle some singularities.

¹¹⁷ See Zadeh and Desoer (1963), ch. 7 and Adrianova (1995), ch. IV.

¹¹⁸ See the definition for matrix norm in Adrianova (1995), ch. I §2.

Uniform asymptotic stability implies that all solutions to the homogeneous system tend to zero as $t \rightarrow \infty$. This is equal to

$$\lim_{t \rightarrow \infty} \|\Phi(t, t_0)\| = 0, \quad \forall t_0.$$

Hence, uniform asymptotic stability includes uniform stability.

The corresponding inhomogeneous LTV system (4.43) is bounded, i.e. stable,

1. iff the corresponding homogeneous system is stable, and
2. iff $\int_{t_0}^t \Phi(t, v) \mathbf{b}(v) dv$ is bounded. Hence, it must be ensured that there exists a number M_1 such that

$$\|\mathbf{b}(t)\| < M_1, \quad (4.94)$$

i.e. $\mathbf{b}(t)$ is bounded $\forall t > t_0$. Additionally, it must be ensured that there are positive constants M_2 and β such that

$$\|\Phi(t, t_0)\|_{II} \leq M_2 e^{-\beta(t-t_0)}, \quad \text{for } 0 \leq t_0 \leq t < \infty. \quad (4.95)$$

¹¹⁹ Subscript II denotes that the norm $\|A(t)\|_{II} = \max_j \sum_{i=1}^n |a_{ij}|$ is used. See Adrianova (1995), p. 3 for the definition of matrix norms.

4B Appendix: Solution of the System of Differential Equations

In this appendix, I derive the solution presented in section 4.4. Starting point is the homogeneous part to equation (4.22)

$$\dot{\mathbf{V}}_{\sigma}^s(t) = \mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{V}_{\sigma}^s(t), \quad (4.96)$$

where $\mathbf{A}_1(\boldsymbol{\mu}(t)) \in \mathbb{R}^{4 \times 4}$ is stated in equation (4.23). For the diagonalization algorithm, I define $\mathbf{A}_1(\boldsymbol{\mu}(t)) = \mathbf{A}_4(t)$ (cf. equation (4.74)), since the initial matrix $\mathbf{A}_1(\boldsymbol{\mu}(t))$ is of dimension four. Three transformation rounds are necessary, which are presented in the following. Since the individual steps of each round are discussed in detail in section 4A.5, I keep the presentation of procedures and solutions brief.

First Transformation Round

1. Partition matrix $\mathbf{A}_4(t)$:

$$\mathbf{A}_{11}^{\langle 4 \rangle} = \begin{bmatrix} \begin{pmatrix} r+\zeta+2\lambda\mu_{hn}(t)q \\ +\lambda_u+\rho(1-z)\tilde{q}(t) \end{pmatrix} & \begin{pmatrix} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} & -\begin{pmatrix} \lambda_u+2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} \\ \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} r+\zeta+2\lambda\mu_{lo}(t)(1-q) \\ +\lambda_d+\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} \\ -\lambda_d & 0 & r + \lambda_d + \zeta \end{bmatrix},$$

$$\mathbf{a}_{12}^{\langle 4 \rangle} = \begin{bmatrix} -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix},$$

$$\mathbf{a}_{21}^{\langle 4 \rangle} = \begin{bmatrix} 0 \\ -\lambda_u \\ 0 \end{bmatrix},$$

$$a_{22}^{\langle 4 \rangle} = r + \zeta + \lambda_u.$$

2. Find any solution $\mathbf{p}_4(t)$ to the system of Riccati differential equations:

$$\begin{aligned} \begin{bmatrix} \dot{p}_{4,1}(t) \\ \dot{p}_{4,2}(t) \\ \dot{p}_{4,3}(t) \end{bmatrix} &= - \begin{bmatrix} p_{4,1}(t) \\ p_{4,2}(t) \\ p_{4,3}(t) \end{bmatrix} \begin{bmatrix} (-2\lambda\mu_{hn}(t)q) & (-\lambda_d - 2\lambda\mu_{lo}(t)(1-q)) & 0 \\ -\rho(1-z)\tilde{q}(t) & -\rho(1-z)(1-\tilde{q}(t)) & 0 \end{bmatrix} \begin{bmatrix} p_{4,1}(t) \\ p_{4,2}(t) \\ p_{4,3}(t) \end{bmatrix} \\ &- \begin{bmatrix} \left(\begin{array}{cc} r+\zeta+\lambda_u & 2\lambda\mu_{lo}(t)(1-q) \\ +2\lambda\mu_{hn}(t)q & +\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & -\lambda_d \\ \left(\begin{array}{cc} 2\lambda\mu_{hn}(t)q & r+\zeta+\lambda_d \\ +\rho(1-z)\tilde{q}(t) & +2\lambda\mu_{lo}(t)(1-q) \end{array} \right) & 0 \\ \left(\begin{array}{cc} -\lambda_u & -2\lambda\mu_{lo}(t)(1-q) \\ -\rho(1-z)\tilde{q}(t) & -\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & r + \lambda_d + \zeta \end{bmatrix} \begin{bmatrix} p_{4,1}(t) \\ p_{4,2}(t) \\ p_{4,3}(t) \end{bmatrix} \\ &+ \begin{bmatrix} p_{4,1}(t) \\ p_{4,2}(t) \\ p_{4,3}(t) \end{bmatrix} (r + \lambda_u + \zeta) + \begin{bmatrix} 0 \\ -\lambda_u \\ 0 \end{bmatrix}. \end{aligned}$$

Since any solution to this equation is suitable, it can be shown that

$$\mathbf{p}_4(t) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

is a particular solution to the Riccati equation.

3. Calculate the dynamic eigenvalue $\lambda_4(t)$:

$$\begin{aligned} \lambda_4(t) &= r + \zeta + \lambda_u - \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix} \\ &= r + \zeta + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z). \end{aligned}$$

4. Construct the first transformation matrix $\mathbf{P}_4(t)$:

$$\mathbf{P}_4(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

5. Construct the second transformation matrix $\mathbf{Q}_4(t)$:

$$\begin{aligned}
 & \begin{bmatrix} \dot{q}_{4,1}(t) \\ \dot{q}_{4,2}(t) \\ \dot{q}_{4,3}(t) \end{bmatrix} = \\
 & \left\{ \begin{bmatrix} \begin{pmatrix} r+\zeta+2\lambda\mu_{hn}(t)q \\ +\lambda_u+\rho(1-z)\tilde{q}(t) \end{pmatrix} & \begin{pmatrix} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} & - \begin{pmatrix} \lambda_u+2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} \\
 \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} r+\zeta+2\lambda\mu_{lo}(t)(1-q) \\ +\lambda_d+\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & - \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} \\
 -\lambda_d & 0 & r + \lambda_d + \zeta \end{bmatrix} \right. \\
 & + \begin{bmatrix} -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \\
 & - [r + \zeta + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)] \mathbf{I}_3 \left. \begin{matrix} \right\} \begin{bmatrix} q_{4,1}(t) \\ q_{4,2}(t) \\ q_{4,3}(t) \end{bmatrix} \right. \\
 & + \begin{bmatrix} -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \dot{\mathbf{q}}_4(t) = & \begin{bmatrix} \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d-\rho(1-z) \\ -2\lambda\mu_{hn}(t)q \end{pmatrix} & 0 & -\lambda_u \\
 -\lambda_d & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d-\lambda_u-\rho(1-z) \\ -2\lambda\mu_{hn}(t)q \end{pmatrix} & \lambda_d \\
 -\lambda_d & 0 & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_u-\rho(1-z) \\ -2\lambda\mu_{hn}(t)q \end{pmatrix} \end{bmatrix} \mathbf{q}_4(t) \\
 & + \begin{bmatrix} -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix}.
 \end{aligned} \tag{4.97}$$

A solution to this system of time-dependent differential equations is stated in appendix 4C. Hence, matrix $\mathbf{Q}_4(t)$ is

$$\mathbf{Q}_4(t) = \begin{bmatrix} 1 & 0 & 0 & q_{4,1}(t) \\ 0 & 1 & 0 & q_{4,2}(t) \\ 0 & 0 & 1 & q_{4,3}(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first transformation round is complete.

Second Transformation Round

The second transformation round starts with an order reduction of one: The system matrix $\mathbf{A}_3(t)$ is of dimension three.

1. Define the (sub-)matrix $\mathbf{A}_3(t)$:

$$\begin{aligned} \mathbf{A}_3(t) &= \begin{bmatrix} \left(\begin{array}{c} r+\zeta+2\lambda\mu_{mn}(t)q \\ +\lambda_u+\rho(1-z)\tilde{q}(t) \end{array} \right) & \left(\begin{array}{c} 2\lambda\mu_{mn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{array} \right) & - \left(\begin{array}{c} \lambda_u+2\lambda\mu_{mn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{array} \right) \\ \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & \left(\begin{array}{c} r+\zeta+2\lambda\mu_{lo}(t)(1-q) \\ +\lambda_d+\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) \\ -\lambda_d & 0 & r + \lambda_d + \zeta \end{bmatrix} \\ &+ \begin{bmatrix} -(2\lambda\mu_{mn}(t)q+\rho(1-z)\tilde{q}(t)) \\ -(\lambda_d+2\lambda\mu_{lo}(t)(1-q)+\rho(1-z)(1-\tilde{q}(t))) \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} r + \zeta + \lambda_u & 0 & -\lambda_u \\ -\lambda_d & r + \zeta & \lambda_d \\ -\lambda_d & 0 & r + \zeta + \lambda_d \end{bmatrix}. \end{aligned}$$

2. Partition matrix $\mathbf{A}_3(t)$:

$$\begin{aligned} \mathbf{A}_{11}^{(3)} &= \begin{bmatrix} r + \lambda_u + \zeta & 0 \\ -\lambda_d & r + \zeta \end{bmatrix}, \\ \mathbf{a}_{12}^{(3)} &= \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix}, \\ \mathbf{a}_{21}^{(3)} &= \begin{bmatrix} -\lambda_d \\ 0 \end{bmatrix}, \end{aligned}$$

$$a_{22}^{(3)} = r + \zeta + \lambda_d.$$

3. Find any solution $\mathbf{p}_3(t)$ to the Riccati differential equation:

$$\begin{aligned} \begin{bmatrix} \dot{p}_{3,1}(t) \\ \dot{p}_{3,2}(t) \end{bmatrix} &= - \begin{bmatrix} p_{3,1}(t) \\ p_{3,2}(t) \end{bmatrix} \begin{bmatrix} -\lambda_u & \lambda_d \end{bmatrix} \begin{bmatrix} p_{3,1}(t) \\ p_{3,2}(t) \end{bmatrix} - \begin{bmatrix} r+\lambda_u+\zeta & -\lambda_d \\ 0 & r+\zeta \end{bmatrix} \begin{bmatrix} p_{3,1}(t) \\ p_{3,2}(t) \end{bmatrix} \\ &+ \begin{bmatrix} p_{3,1}(t) \\ p_{3,2}(t) \end{bmatrix} (r + \lambda_d + \zeta) + \begin{bmatrix} -\lambda_d \\ 0 \end{bmatrix}. \end{aligned}$$

Since any solution to this equation is suitable, it can be shown that

$$\mathbf{p}_3(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a particular solution to the Riccati differential equation.

4. Calculate the dynamic eigenvalue $\lambda_3(t)$:

$$\begin{aligned} \lambda_3(t) &= r + \zeta + \lambda_d - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix} \\ &= r + \zeta + \lambda_u + \lambda_d. \end{aligned}$$

5. Construct the first transformation matrix $\mathbf{P}_3(t)$:

$$\mathbf{P}_3(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

6. Construct the second transformation matrix $\mathbf{Q}_3(t)$:

$$\begin{aligned} \begin{bmatrix} \dot{q}_{3,1}(t) \\ \dot{q}_{3,2}(t) \end{bmatrix} &= \left\{ \begin{bmatrix} r + \lambda_u + \zeta & 0 \\ -\lambda_d & r + \zeta \end{bmatrix} + \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right. \\ &\quad \left. - (r + \zeta + \lambda_u + \lambda_d) \mathbf{I}_2 \right\} \begin{bmatrix} q_{3,1}(t) \\ q_{3,2}(t) \end{bmatrix} + \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix} \\ &= \begin{bmatrix} -(\lambda_u + \lambda_d) & 0 \\ 0 & -(\lambda_u + \lambda_d) \end{bmatrix} \begin{bmatrix} q_{3,1}(t) \\ q_{3,2}(t) \end{bmatrix} + \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix}. \end{aligned}$$

Since any solution to this equation is suitable, it can be shown that

$$\mathbf{q}_3(t) = \begin{bmatrix} -\frac{\lambda_u}{\lambda_u + \lambda_d} \\ \frac{\lambda_d}{\lambda_u + \lambda_d} \end{bmatrix}$$

is a particular solution to the ODE. Therefore,

$$\mathbf{Q}_3(t) = \begin{bmatrix} 1 & 0 & -\frac{\lambda_u}{\lambda_u + \lambda_d} \\ 0 & 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} \\ 0 & 0 & 1 \end{bmatrix}.$$

The second transformation round is complete.

Third Transformation Round

The third transformation round starts with an order reduction of one: The system matrix $\mathbf{A}_2(t)$ is of dimension two.

1. Define the (sub-)matrix $\mathbf{A}_2(t)$:

$$\begin{aligned} \mathbf{A}_2(t) &= \begin{bmatrix} r + \lambda_u + \zeta & 0 \\ -\lambda_d & r + \zeta \end{bmatrix} + \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} r + \zeta & 0 \\ 0 & r + \zeta \end{bmatrix}. \end{aligned}$$

2. Partition matrix $\mathbf{A}_2(t)$:

$$\begin{aligned} \mathbf{A}_{11}^{(2)} &= r + \zeta, \\ \mathbf{a}_{12}^{(2)} &= 0, \\ \mathbf{a}_{21}^{(2)} &= 0, \\ a_{22}^{(2)} &= r + \zeta. \end{aligned}$$

3. Find any solution $\mathbf{p}_2(t)$ to the Riccati differential equation:

$$\begin{aligned} \dot{p}_{2,1}(t) &= -p_{2,1}(t) \cdot 0 \cdot p_{2,1}(t) - (r + \zeta) \cdot p_{2,1}(t) + p_{2,1}(t) \cdot (r + \zeta) + 0 \\ &= 0. \end{aligned}$$

$\dot{p}_{2,1}(t) = 0$ means that $p_{2,1}(t)$ is a constant. Since any solution to this equation is suitable, it can be shown that $p_2(t) = 1$ is a particular solution to the Riccati equation.

4. Calculate the dynamic eigenvalue $\lambda_2(t)$:

$$\lambda_2(t) = r + \zeta.$$

5. Calculate the first transformation matrix $\mathbf{P}_2(t)$:

$$\mathbf{P}_2(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

6. Calculate the second transformation matrix $\mathbf{Q}_2(t)$:

$$\begin{aligned} \dot{q}_{2,1}(t) &= \{r + \zeta + 0 \cdot 1 - (r + \zeta)\} q_{2,1}(t) + 0 \\ &= 0. \end{aligned}$$

$\dot{q}_{2,1}(t) = 0$ means that $q_{2,1}(t)$ is a constant. Since any solution to this equation is suitable, it can be shown that $q_2(t) = 1$ is a particular solution to the ODE. As a result, the second transformation matrix $\mathbf{Q}_2(t)$ reads

$$\mathbf{Q}_2(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The third and final transformation round is complete. The dynamic eigenvalue $\lambda_1(t)$ is calculated with

$$\begin{aligned} \lambda_1(t) &= \text{trace}(\mathbf{A}_1(\boldsymbol{\mu}(t))) - \sum_{i=2}^4 \lambda_i(t) \\ &= r + \zeta. \end{aligned}$$

State Transition Matrix

The state transition matrix is composed of the single transformation matrix $\mathbf{T}(t)$ and the diagonal matrix $\boldsymbol{\Lambda}(\boldsymbol{\mu}(t))$. Matrix $\mathbf{T}(t)$, in turn, is calculated via the Riccati transformations of each transformation round. These Riccati transformations of

round one, two, and three are calculated first, so that

$$\mathbf{P}_4(t)\mathbf{Q}_4(t) = \begin{bmatrix} 1 & 0 & 0 & q_{41}(t) \\ 0 & 1 & 0 & q_{42}(t) \\ 0 & 0 & 1 & q_{43}(t) \\ 1 & 1 & -1 & q_{41}(t) + q_{42}(t) - q_{43}(t) + 1 \end{bmatrix},$$

$$\mathbf{P}_3\mathbf{Q}_3 = \begin{bmatrix} 1 & 0 & -\frac{\lambda_u}{\lambda_u + \lambda_d} \\ 0 & 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} \\ 1 & 0 & \frac{\lambda_d}{\lambda_u + \lambda_d} \end{bmatrix},$$

$$\mathbf{P}_2\mathbf{Q}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Matrix $\mathbf{T}(t)$, the columns of which consist of dynamic eigenvectors, is calculated with

$$\mathbf{T}(t) = \mathbf{S}_4(t) \cdot \mathbf{S}_3 \cdot \mathbf{S}_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & q_{4,1}(t) \\ 0 & 1 & 0 & q_{4,2}(t) \\ 0 & 0 & 1 & q_{4,3}(t) \\ 1 & 1 & -1 & \begin{pmatrix} q_{4,1}(t) + q_{4,2}(t) \\ -q_{4,3}(t) + 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & 0 \\ 0 & 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} & 0 \\ 1 & 0 & \frac{\lambda_d}{\lambda_u + \lambda_d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The following matrix is equal to the transition matrix $T(t)$ stated in equation (4.35) of section 4.4:

$$\mathbf{T}(t) = \begin{bmatrix} 1 & 1 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & q_{41}(t) \\ 1 & 2 & \frac{\lambda_d}{\lambda_u + \lambda_d} & q_{42}(t) \\ 1 & 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} & q_{43}(t) \\ 1 & 2 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & q_{41}(t) + q_{42}(t) - q_{43}(t) + 1 \end{bmatrix},$$

where

$$\mathbf{q}_4(t) = \begin{bmatrix} q_{4,1}(t) \\ q_{4,2}(t) \\ q_{4,3}(t) \end{bmatrix},$$

and

$$\dot{\mathbf{q}}_4(t) = \hat{\mathbf{A}}_3(t)\mathbf{q}_4(t) + \hat{\mathbf{b}}(t),$$

with

$$\widehat{\mathbf{A}}_3(t) = \begin{bmatrix} \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} & 0 & -\lambda_u \\ -\lambda_d & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d - \lambda_u - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} & \lambda_d \\ -\lambda_d & 0 & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_u - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} \end{bmatrix},$$

and

$$\widehat{\mathbf{b}}(t) = \begin{bmatrix} -(2\lambda\mu_{hm}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1 - \tilde{q}(t))) \\ 0 \end{bmatrix}.$$

A solution to $\mathbf{q}_4(t) = [q_{41}(t) \quad q_{42}(t) \quad q_{43}(t)]^T$ is derived in appendix 4C.

The diagonal matrix with dynamic eigenvalues is equal to the matrix $\mathbf{\Lambda}(\boldsymbol{\mu}(t))$ stated in equation (4.39) of section 4.4:

$$\mathbf{\Lambda}(\boldsymbol{\mu}(t)) = \begin{bmatrix} r + \zeta & 0 & 0 & 0 \\ 0 & r + \zeta & 0 & 0 \\ 0 & 0 & r + \zeta + \lambda_d + \lambda_u & 0 \\ 0 & 0 & 0 & \begin{pmatrix} r + \zeta + \lambda_d + \lambda_u \\ +2\lambda\mu_{lo}(t)(1-q) + 2\lambda\mu_{hm}(t)q \\ +\rho(1-z) \end{pmatrix} \end{bmatrix}.$$

The state transition matrix is:

$$\Phi(t, x) = \mathbf{T}(t) \cdot e^{-\int_t^x \mathbf{\Lambda}(\boldsymbol{\mu}(\tau)) d\tau} \cdot \mathbf{T}(x)^{-1}, \quad (4.98)$$

where

$$\mathbf{T}(x)^{-1} = \quad (4.99)$$

$$\begin{bmatrix} \begin{pmatrix} 1+q_{41}(x) \\ -q_{42}(x)+q_{43}(x) \\ +\frac{\lambda_u}{\lambda_u+\lambda_d}[-1] \\ +q_{43}(x)-q_{41}(x) \end{pmatrix} & \begin{pmatrix} -1+q_{41}(x) \\ -q_{42}(x)+q_{43}(x) \\ +\frac{\lambda_u}{\lambda_u+\lambda_d} \\ [q_{43}(x)-q_{41}(x)] \end{pmatrix} & \begin{pmatrix} 1-q_{41}(x) \\ +q_{42}(x)-q_{43}(x) \\ +\frac{\lambda_u}{\lambda_u+\lambda_d}[1] \\ -q_{43}(x)+q_{41}(x) \end{pmatrix} & \begin{pmatrix} -q_{41}(x)+q_{42}(x) \\ -q_{43}(x)-\frac{\lambda_u}{\lambda_u+\lambda_d} \\ [q_{43}(x)-q_{41}(x)] \end{pmatrix} \\ q_{42}(x)-q_{43}(x) & 1+q_{42}(x)-q_{43}(x) & -1-q_{42}(x)+q_{43}(x) & -q_{42}(x)+q_{43}(x) \\ -1+q_{43}(x)-q_{41}(x) & q_{43}(x)-q_{41}(x) & 1-q_{43}(x)+q_{41}(x) & -q_{43}(x)+q_{41}(x) \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

and

$$\det \mathbf{T}(t) = 1.$$

Lyapunov Transformation and Stability

It must be ensured that the stability properties do not change due to the successive Riccati transforms: Matrices $\mathbf{P}_k(t)$ and $\mathbf{Q}_k(t)$ must be Lyapunov matrices. This implies that the chosen solutions to the systems of Riccati differential equations $\mathbf{p}_k(t)$, $\dot{\mathbf{p}}_k(t)$ and the chosen solutions to the systems of differential equations $\mathbf{q}_k(t)$, $\dot{\mathbf{q}}_k(t)$ have to be bounded for all t :

1. All $\mathbf{p}_k(t)$, $\dot{\mathbf{p}}_k(t)$ are bounded, since all chosen solutions $\mathbf{p}_k(t)$ are constants.
2. All $\mathbf{q}_k(t)$ and $\dot{\mathbf{q}}_k(t)$ are bounded if $\lambda, \rho, \lambda_d, \lambda_u < \infty$: The chosen solutions $\mathbf{q}_3(t)$ and $\mathbf{q}_2(t)$ are constants. The solution $\mathbf{q}_4(t)$ and $\dot{\mathbf{q}}_4(t)$ are bounded for $\lambda, \rho, \lambda_d, \lambda_u < \infty$, as shown in appendix 4C.

This ensures that $\mathbf{T}(t)$ is bounded for $\lambda, \rho, \lambda_d, \lambda_u < \infty$. The transformation is of Lyapunov type, i.e. the stability of the system (4.22) is not altered by the Riccati transformation. The homogeneous system to (4.22) is uniformly asymptotically stable, since

$$\mathbf{T}(t) \cdot e^{-\int_t^\infty \Lambda(\mu(\tau)) d\tau} \cdot \mathbf{T}(\infty)^{-1} = \mathbf{0}, \quad \forall t.$$

The inhomogeneous LTV system (4.22) is stable if the corresponding homogeneous system is stable and if $\int_t^\infty \Phi(t, x) [\mathbf{A}_2 + \mathbf{A}_3(\mu(x)) \mathbf{V}_\sigma^s(0)] dx$ is bounded for all t . Obviously, $\Phi(t, x)$ is an exponentially decreasing matrix function of time t , which is clearly bounded for $0 \leq t \leq x$. Furthermore, the condition

$$M_1 > \|\mathbf{A}_2 + \mathbf{A}_3(\mu(t)) \mathbf{V}_\sigma^s(0)\|, \quad \forall t,$$

for a number M_1 implies approximately

$$M_1 > \|\mathbf{A}_2\| + \|\mathbf{A}_3(\mu(t))\| \cdot \|\mathbf{V}_\sigma^s(0)\|,$$

where

$$\begin{aligned} \|\mathbf{A}_2\| &= D, \\ \|\mathbf{A}_3(\mu(t))\| &= \zeta \max \{ |1 - \pi_{hn}(t)| + |\pi_{hn}(t)|, |1 - \pi_{ho}(t)| + |\pi_{ho}(t)| \}, \\ \|\mathbf{V}_\sigma^s(0)\| &= \max \{ V_{lo}^s(0), V_{hn}^s(0), V_{ho}^s(0), V_{ln}^s(0) \}. \end{aligned}$$

Matrix \mathbf{A}_2 is bounded. Matrix $\mathbf{A}_3(\mu(t))$ is bounded if $\pi_{hn}(t)$ and $\pi_{ho}(t)$ are bounded. Therewith, $\mathbf{V}_\sigma^s(0)$ is bounded, since $\exp(-\int_0^x \Lambda(\mu(\tau)) d\tau)$ is an exponentially decreasing matrix function, which can be seen from equation (4.32).

As a result, the value function $\mathbf{V}_\sigma^s(t)$ is bounded for all t if $\lambda, \rho, \lambda_d, \lambda_u < \infty$, which is required for the boundedness of $\dot{\mathbf{q}}_4(t)$.

4C Appendix: Calculating Eigenvector $q_4(t)$

In this appendix, I calculate an arbitrary solution for the dynamic eigenvector $\mathbf{q}_4(t)$ defined in equation (4.97). I repeat the linear time-varying system of differential equations with

$$\dot{\mathbf{q}}_4(t) = \widehat{\mathbf{A}}_3(t)\mathbf{q}_4(t) + \widehat{\mathbf{b}}(t), \quad (4.100)$$

where

$$\widehat{\mathbf{A}}_3(t) = \begin{bmatrix} \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} & 0 & -\lambda_u \\ -\lambda_d & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_d - \lambda_u - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} & \lambda_d \\ -\lambda_d & 0 & \begin{pmatrix} -2\lambda\mu_{lo}(t)(1-q) \\ -\lambda_u - \rho(1-z) \\ -2\lambda\mu_{hm}(t)q \end{pmatrix} \end{bmatrix},$$

$$\widehat{\mathbf{b}}(t) = \begin{bmatrix} -(2\lambda\mu_{hm}(t)q + \rho(1-z)\tilde{q}(t)) \\ -(\lambda_d + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1 - \tilde{q}(t))) \\ 0 \end{bmatrix},$$

and

$$\mathbf{q}_4(t) = \begin{bmatrix} q_{4,1}(t) \\ q_{4,2}(t) \\ q_{4,3}(t) \end{bmatrix},$$

for which a solution is derived. Since the solution technique to solve equation (4.100) is identical to the one in appendix 4B, I present it in an abridged form.

First Transformation Round

1. Partition of matrix $\widehat{\mathbf{A}}_3(t)$:

$$\widehat{\mathbf{A}}_{11}^{(3)} = \begin{bmatrix} - \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) + \lambda_d \\ +2\lambda\mu_{hm}(t)q + \rho(1-z) \end{pmatrix} & 0 \\ -\lambda_d & - \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) + \lambda_d \\ +2\lambda\mu_{hm}(t)q + \lambda_u + \rho(1-z) \end{pmatrix} \end{bmatrix},$$

$$\begin{aligned}\widehat{\mathbf{a}}_{12}^{(3)} &= \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix}, \\ \widehat{\mathbf{a}}_{21}^{(3)} &= \begin{bmatrix} -\lambda_d \\ 0 \end{bmatrix}, \\ \widehat{a}_{22}^{(3)} &= -(\lambda_u + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)).\end{aligned}$$

2. Riccati differential equations:

$$\begin{aligned}\begin{bmatrix} \dot{\widehat{p}}_{3,1}(t) \\ \dot{\widehat{p}}_{3,2}(t) \end{bmatrix} &= - \begin{bmatrix} \widehat{p}_{3,1}(t) \\ \widehat{p}_{3,2}(t) \end{bmatrix} \begin{bmatrix} -\lambda_u & \lambda_d \end{bmatrix} \begin{bmatrix} \widehat{p}_{3,1}(t) \\ \widehat{p}_{3,2}(t) \end{bmatrix} \\ &+ \begin{bmatrix} (\lambda_d - \lambda_u) & \lambda_d \\ 0 & \lambda_d \end{bmatrix} \begin{bmatrix} \widehat{p}_{3,1}(t) \\ \widehat{p}_{3,2}(t) \end{bmatrix} - \begin{bmatrix} -\lambda_d \\ 0 \end{bmatrix}.\end{aligned}$$

Particular solution:

$$\widehat{\mathbf{p}}_3(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

3. Dynamic eigenvalue $\widehat{\lambda}_3(t)$:

$$\widehat{\lambda}_3(t) = -(\lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)).$$

4. First transformation matrix $\widehat{\mathbf{P}}_3(t)$:

$$\widehat{\mathbf{P}}_3(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. Second transformation matrix $\widehat{\mathbf{Q}}_3(t)$:

$$\begin{bmatrix} \dot{\widehat{q}}_{3,1}(t) \\ \dot{\widehat{q}}_{3,2}(t) \end{bmatrix} = \begin{bmatrix} \lambda_u & -\lambda_u \\ -\lambda_d & \lambda_d \end{bmatrix} \begin{bmatrix} \widehat{q}_{3,1}(t) \\ \widehat{q}_{3,2}(t) \end{bmatrix} + \begin{bmatrix} -\lambda_u \\ \lambda_d \end{bmatrix}.$$

Particular solution:

$$\widehat{\mathbf{q}}_3(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As a result,

$$\widehat{\mathbf{Q}}_3(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Riccati transformation:

$$\widehat{\mathbf{P}}_3(t)\widehat{\mathbf{Q}}_3(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Second Transformation Round

1. Submatrix $\widehat{\mathbf{A}}_2(t)$:

$$\widehat{\mathbf{A}}_2(t) = \begin{bmatrix} - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) + \lambda_d \\ + 2\lambda\mu_{hn}(t)q + \rho(1-z) \end{array} \right) & -\lambda_u \\ -\lambda_d & - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) + \lambda_u \\ + 2\lambda\mu_{hn}(t)q + \rho(1-z) \end{array} \right) \end{bmatrix}.$$

2. Partition of matrix $\widehat{\mathbf{A}}_2(t)$:

$$\begin{aligned} \widehat{\mathbf{a}}_{11}^{(2)} &= -(\lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)), \\ \widehat{\mathbf{a}}_{12}^{(2)} &= -\lambda_u, \\ \widehat{\mathbf{a}}_{21}^{(2)} &= -\lambda_d, \\ \widehat{\mathbf{a}}_{22}^{(2)} &= -(\lambda_u + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)). \end{aligned}$$

3. Riccati differential equation:

$$\begin{aligned} \dot{\widehat{p}}_{2,1}(t) &= -\widehat{p}_{2,1}(t)[- \lambda_u] \widehat{p}_{2,1}(t) - \lambda_d \\ &\quad + [\lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)] \widehat{p}_{2,1}(t) \\ &\quad - \widehat{p}_{2,1}(t) [\lambda_u + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)]. \end{aligned}$$

Particular solution: $\widehat{\mathbf{p}}_2(t) = 1$.

4. Dynamic eigenvalue $\widehat{\lambda}_2(t)$:

$$\widehat{\lambda}_2(t) = - (2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)).$$

5. First transformation matrix $\widehat{\mathbf{P}}_2(t)$:

$$\widehat{\mathbf{P}}_2(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

6. Second transformation matrix $\widehat{\mathbf{Q}}_2(t)$:

$$\dot{\widehat{q}}_{2,1}(t) = -(\lambda_u + \lambda_d)\widehat{q}_{2,1}(t) - \lambda_u.$$

Particular solution: $\widehat{q}_2(t) = -\lambda_u/(\lambda_u + \lambda_d)$. Hence,

$$\widehat{\mathbf{Q}}_2(t) = \begin{bmatrix} 1 & -\frac{\lambda_u}{\lambda_u + \lambda_d} \\ 0 & 1 \end{bmatrix}.$$

7. Riccati transformation:

$$\widehat{\mathbf{P}}_2(t)\widehat{\mathbf{Q}}_2(t) = \begin{bmatrix} 1 & -\frac{\lambda_u}{\lambda_u + \lambda_d} \\ 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} \end{bmatrix}.$$

Final Dynamic Eigenvalue and Transformation Matrix $\widehat{\mathbf{T}}(t)$

Dynamic eigenvalue $\widehat{\lambda}_1(t)$:

$$\widehat{\lambda}_1(t) = -(\lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1 - q) + \rho(1 - z)).$$

Transformation Matrix $\widehat{\mathbf{T}}(t)$:

$$\widehat{\mathbf{T}} = \widehat{\mathbf{S}}_3 \cdot \widehat{\mathbf{S}}_2,$$

$$\widehat{\mathbf{T}} = \begin{bmatrix} 1 & -\frac{\lambda_u}{\lambda_u + \lambda_d} & 1 \\ 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} & 0 \\ 1 & \frac{\lambda_d}{\lambda_u + \lambda_d} & 1 \end{bmatrix}.$$

The inverse $\widehat{\mathbf{T}}^{-1}$ is then

$$\widehat{\mathbf{T}}^{-1} = \begin{bmatrix} \frac{\lambda_d}{\lambda_u + \lambda_d} & 1 & -\frac{\lambda_d}{\lambda_u + \lambda_d} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

The diagonal matrix $\widehat{\Lambda}(t)$ is obtained with

$$\widehat{\Lambda}(t) = \begin{bmatrix} \widehat{\lambda}_1(t) & 0 & 0 \\ 0 & \widehat{\lambda}_2(t) & 0 \\ 0 & 0 & \widehat{\lambda}_3(t) \end{bmatrix},$$

where

$$\widehat{\lambda}_1(t) = -(\lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)),$$

$$\widehat{\lambda}_2(t) = -(2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)),$$

$$\widehat{\lambda}_3(t) = -(\lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)).$$

These eigenvalues are equal to $\widehat{\lambda}_1(t)$ and $\widehat{\lambda}_2(t)$ stated in equations (4.37) and (4.38) of section 4.4, and $\widehat{\lambda}_3(t) = \widehat{\lambda}_1(t)$. The state transition matrix $\widehat{\Phi}(t, t_0)$ is

$$\widehat{\Phi}(t, t_0) = \widehat{\Gamma} e^{\int_{t_0}^t \widehat{\Lambda}(\tau) d\tau} \widehat{\Gamma}^{-1}.$$

Solution

The solution can be found with

$$\mathbf{q}_4(t) = \widehat{\Gamma} e^{\int_0^t \widehat{\Lambda}(\tau) d\tau} \mathbf{c} + \int_0^t \widehat{\Gamma} e^{\int_x^t \widehat{\Lambda}(\tau) d\tau} \widehat{\Gamma}^{-1} \widehat{\mathbf{b}}(x) dx.$$

Since any solution for $\mathbf{q}_4(t)$ is suitable, the starting condition $\mathbf{q}_4(0)$ can be chosen such that $\mathbf{c} = \mathbf{0}$.¹²⁰ Then,

$$\mathbf{q}_4(t) = \int_0^t \widehat{\Gamma} e^{\int_x^t \widehat{\Lambda}(\tau) d\tau} \widehat{\Gamma}^{-1} \widehat{\mathbf{b}}(x) dx.$$

This equation is stated explicitly in equation (4.36) of section 4.4, which is similar to

$$q_{4,1}(t) = -\frac{\lambda_d}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \widehat{\lambda}_1(\tau) d\tau} (2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x)) dx \\ - \frac{\lambda_u}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \widehat{\lambda}_2(\tau) d\tau} (2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x)) dx,$$

¹²⁰ See Wu (1980), p. 826.

$$\begin{aligned}
q_{4,2}(t) &= - \int_0^t e^{\int_x^t \hat{\lambda}_1(\tau) d\tau} \left(\frac{\lambda_d}{\lambda_u + \lambda_d} (2\lambda\mu_{hm}(x)q + \rho(1-z)\tilde{q}(x)) \right. \\
&\quad \left. + \lambda_d + 2\lambda\mu_{lo}(x)(1-q) + \rho(1-z)(1-\tilde{q}(x)) \right) dx \\
&\quad + \frac{\lambda_d}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \hat{\lambda}_2(\tau) d\tau} (2\lambda\mu_{hm}(x)q + \rho(1-z)\tilde{q}(x)) dx, \\
q_{4,3}(t) &= - \frac{\lambda_d}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \hat{\lambda}_1(\tau) d\tau} (2\lambda\mu_{hm}(x)q + \rho(1-z)\tilde{q}(x)) dx \\
&\quad + \frac{\lambda_d}{\lambda_u + \lambda_d} \int_0^t e^{\int_x^t \hat{\lambda}_2(\tau) d\tau} (2\lambda\mu_{hm}(x)q + \rho(1-z)\tilde{q}(x)) dx.
\end{aligned}$$

Stability

The state transition matrix $\hat{\Phi}(t, t_0)$ for system (4.100) is

$$\hat{\Phi}(t, t_0) = \frac{1}{\lambda_u + \lambda_d} e^{\int_{t_0}^t \hat{\lambda}_2(\tau) d\tau} \times \quad (4.101)$$

$$\begin{bmatrix}
\lambda_d e^{-(\lambda_u + \lambda_d)(t-t_0)} + \lambda_u & 0 & \lambda_u \left(e^{-(\lambda_u + \lambda_d)(t-t_0)} - 1 \right) \\
\lambda_d \left(e^{-(\lambda_u + \lambda_d)(t-t_0)} - 1 \right) & e^{-(\lambda_u + \lambda_d)(t-t_0)} & -\lambda_d \left(e^{-(\lambda_u + \lambda_d)(t-t_0)} - 1 \right) \\
\lambda_d \left(e^{-(\lambda_u + \lambda_d)(t-t_0)} - 1 \right) & 0 & \lambda_u \left(e^{-(\lambda_u + \lambda_d)(t-t_0)} - 1 \right)
\end{bmatrix}.$$

Obviously,

$$\lim_{t \rightarrow \infty} \left\| \hat{\Phi}(t, t_0) \right\| = 0, \quad \forall t_0. \quad (4.102)$$

is satisfied, since $\hat{\lambda}_1(t), \hat{\lambda}_2(t) < 0$. It follows that the homogeneous system to (4.100) is uniformly asymptotically stable.

The inhomogeneous LTV system (4.100) is stable if the corresponding homogeneous system is stable and if $\int_{t_0}^t \hat{\Phi}(t, v) \hat{\mathbf{b}}(v) dv$ is bounded. For this, it must be ensured that there exists a number M_1 such that

$$\left\| \hat{\mathbf{b}}(t) \right\| < M_1,$$

which is fulfilled for $\lambda, \rho, \lambda_d < \infty$. Additionally, it must be ensured that there are positive constants M_2 and β such that

$$\|\Phi(t, t_0)\|_{II} \leq M_2 e^{-\beta(t-t_0)}, \quad \text{for } 0 \leq t_0 \leq t < \infty.$$

This condition is fulfilled, as seen from

$$\begin{aligned} \|\Phi(t, t_0)\|_{II} = & \frac{1}{\lambda_u + \lambda_d} e^{\int_{t_0}^t \widehat{\lambda}_2(\tau) d\tau} \times \\ & \max \left\{ (\lambda_d - \lambda_u) e^{-(\lambda_u + \lambda_d)(t-t_0)} + 2\lambda_u, \right. \\ & e^{-(\lambda_u + \lambda_d)(t-t_0)}, \\ & \left. \lambda_u + \lambda_d - (\lambda_d + \lambda_u) e^{-(\lambda_u + \lambda_d)(t-t_0)} \right\}. \end{aligned}$$

The inhomogeneous LTV system (4.100) is stable for $\lambda, \rho, \lambda_d < \infty$. However, since $\widehat{\Phi}(t, v)$ decreases exponentially the faster the higher λ, ρ, λ_d are, the boundedness of $\widehat{\mathbf{b}}(t)$ does not influence the overall stability. But for $\widehat{\mathbf{q}}_4(t)$ being bounded, it must be ensured that $\widehat{\mathbf{A}}_3(t)$ and $\widehat{\mathbf{b}}(t)$ are bounded, which is true for $\lambda, \rho, \lambda_d, \lambda_u < \infty$.

Chapter 5

Numerical Example (Aggregate Liquidity Shocks)

To demonstrate the effects, implications, and general characteristics of aggregate liquidity shocks, I utilize an example. First, I adopt the numerical example introduced in Duffie, Gârleanu, and Pedersen (2007, pp. 1883–1887 and p. 1878) for a thorough investigation. I extend their example in section 5.2 by market makers. Appendices 5A and 5B discuss special cases, which have not been addressed in the literature yet. Appendix 5C derives the time of intersection t^* , at which the selling pressure due to the shock alleviates.

The parameter values for the example are given in table 5.1:

	Parameter	Value
Fraction of investors owning an asset	s	0.75
Holding cost	δ	2.5
Constant dividend rate	D	1
Interest rate	r	0.1
Intensity of switching to a high preference type	λ_u	2
Intensity of switching to a low preference type	λ_d	0.2
Investors' meeting intensity	λ	125
Sellers' bargaining power (between investors)	q	0.5
Market makers' bargaining power	z	0.8
Intensity of a liquidity shock	ζ	0.1
Shock-probability of a switch $ho \rightarrow lo$	$\pi_{ho}(ss)$	0.5
Shock-probability of a switch $hn \rightarrow ln$	$\pi_{hn}(ss)$	0.5
Market makers' meeting intensity	ρ	0 / 125
Auxiliary variable for $\mu_{hn}(t^*) = \mu_{lo}(t^*)$	$\tilde{q}(t^*)$	0.25

Table 5.1: Parameters for the numerical example with aggregate liquidity shocks. For the market makers' meeting intensity, $\rho = 0$ is applied in chapter 5.1 (no market makers) and $\rho = 125$ in chapter 5.2 (with market makers).

This example treats the case $s < \lambda_u / (\lambda_u + \lambda_d)$, which implies that steady state is a seller's market, i.e. $\mu_{hn}(ss) > \mu_{lo}(ss)$. The marginal investor is a buyer, because there is an excess demand for the asset. With $\pi_{hn}(ss) = \pi_{ho}(ss) = 0.5$, the aggregate liquidity shock is severe enough that the market switches to a buyer's market after the shock, i.e. $\bar{\mu}_{hn}(0) < \bar{\mu}_{lo}(0)$. I discuss the other two cases, (1) $s < \lambda_u / (\lambda_u + \lambda_d)$ with no severe shock and (2) $s > \lambda_u / (\lambda_u + \lambda_d)$, in appendix 5A.

5.1 Example without Market Maker

Duffie, Gârleanu, and Pedersen (2007) start with the assumption that post-shock masses are calculated on the basis of an aggregate liquidity shock occurring in steady state. With equations (3.13), (3.16), (3.14), and (3.15), steady state masses $\mu_\sigma(ss)$ are calculated first. The results are stated in table 5.2.

	Parameter	Fraction
Fraction of <i>lo</i> -type investors	$\mu_{lo}(ss)$	0.0035
Fraction of <i>hn</i> -type investors	$\mu_{hn}(ss)$	0.1626
Fraction of <i>ho</i> -type investors	$\mu_{ho}(ss)$	0.7465
Fraction of <i>ln</i> -type investors	$\mu_{ln}(ss)$	0.0874

Table 5.2: Steady state masses of agents' types (without market maker).

An aggregate liquidity shock changes the fraction of investor types. First, there is a direct impact on both high-type investors *hn* and *ho*. When a shock occurs, the probability that an *ho* agent switches to an *lo* agent from steady state is assumed with $\pi_{ho}(ss) = 50\%$ and the probability of remaining an *ho* agent is $1 - \pi_{ho}(ss) = 50\%$. Similarly, with the probability of $\pi_{hn}(ss) = 50\%$ an *hn* agent switches to an *ln* agent, and he remains an *hn* agent with the probability of $1 - \pi_{hn}(ss) = 50\%$. The post-shock distribution $\bar{\mu}_\sigma(0)$ of high-type agents are calculated with equations (4.3) and (4.4). Additionally, there is an indirect impact on low agents: The fractions $\pi_{ho}(ss)\mu_{ho}(ss) = 0.3733$ and $\pi_{hn}(ss)\mu_{hn}(ss) = 0.0813$ of investors switch from high to low-type agents. The post-shock masses for low-type agents increases by these amounts. The masses $\bar{\mu}_{lo}(0)$ and $\bar{\mu}_{ln}(0)$ right after the shock are calculated with equations (4.5) and (4.6). Table 5.3 summarizes the results.

Due to the aggregate liquidity shock, the fraction of potential sellers increases from 0.35% to 37.68% right after the shock, which corresponds to a jump of

	Parameter	Fraction
Fraction of <i>lo</i> -type investors	$\bar{\mu}_{lo}(0)$	0.3768
Fraction of <i>hn</i> -type investors	$\bar{\mu}_{hn}(0)$	0.0813
Fraction of <i>ho</i> -type investors	$\bar{\mu}_{ho}(0)$	0.3732
Fraction of <i>ln</i> -type investors	$\bar{\mu}_{ln}(0)$	0.1687

Table 5.3: Post-shock masses of agents' types (without market maker).

10.67%. The fraction of potential buyers decreases from 16.26% to 8.13%. This shock causes an oversupply of potential sellers, who face only few potential buyers, i.e. $\mu_{lo}(0) > \mu_{hn}(0)$. Such an excess of asset supply over demand characterizes a buyer's market—in contrast to a seller's market in steady state ($\mu_{hn}(ss) > \mu_{lo}(ss)$). The selling pressure does not last forever, however. The post-shock distribution $\mu_{\sigma}(0)$ acts as a starting condition for the system of ODEs in (3.4)–(3.7). As stated in chapter 3, steady state masses are reached from any starting condition. Agents' intrinsic type masses converge after an aggregate liquidity shock to the steady state masses, which are stated in table 5.2, given that no further shock occurs in the meantime. Within this numerical example, it takes approximately nine years to fully recover from an aggregate liquidity shock, but a fairly normal level is reached after roughly two years.

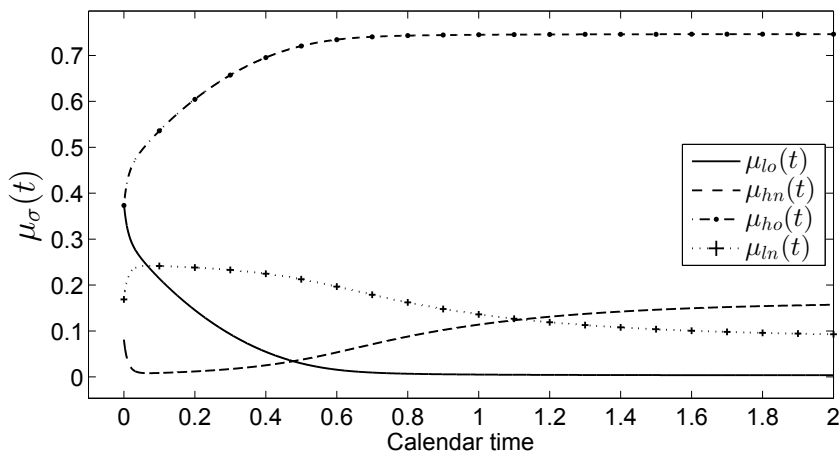


Figure 5.1: Process of mass distribution after an aggregate liquidity shock and without market makers. The solid line illustrates the fraction of sellers over time. The dashed line represents the fraction of buyers. Dashed-dotted and dotted (with plus sign) lines show the fraction of high owners and low non-owners, respectively.

Figure 5.1 shows this evolution of mass distribution, where time zero means zero years after the shock. For *ho* agents, masses start to increase, and for *lo* agents, masses decrease immediately after the shock. The situation is different for non-owners. The fraction of *hn* agents decreases and the fraction of *ln* agents increases

further before they rebound. It takes about $t^* = 0.48$ years until the fraction of potential buyers exceeds the fraction of potential sellers again. Time t^* is called the time of intersection, with its calculation being deferred to the appendix 5C. The market returns to a seller's market with $\mu_{lo}(t) < \mu_{hn}(t)$ for $t > t^* > 0$. Hence, the selling pressure is considerably reduced after 0.48 years.

Value Function

Aggregate liquidity shocks arise every $1/\zeta = 10$ years on average in this setup. This risk of future shocks is taken into account by all agents within their value functions. The impact of these recurring aggregate liquidity shocks on agents' utility depends on agents' particular intrinsic type, as seen in equations (4.7)–(4.10). Table 5.4 presents agents' utilities immediately and a long time after the shock, whereas figure 5.2 visualizes them.

	Parameter	$t = 0$	$t \rightarrow \infty$
Value function of <i>lo</i> -type investors	$V_{lo}^s(t)$	8.4736	9.1428
Value function of <i>hn</i> -type investors	$V_{hn}^s(t)$	2.2229	1.1348
Value function of <i>ho</i> -type investors	$V_{ho}^s(t)$	9.2385	9.2854
Value function of <i>ln</i> -type investors	$V_{ln}^s(t)$	1.5091	1.1003

Table 5.4: Value functions immediately and a long time after the liquidity shock (without market makers).

Potential sellers experience a 7% utility loss due to the shock, since trading opportunities worsen for quite a while. The fraction of potential sellers increases excessively and, at the same time, the fraction of potential buyers decreases. It takes longer after the shock to find a trading partner, which influences search opportunities. Once a trading partner is found, the bargaining position of potential sellers is low. Potential sellers are locked into an unfavorable situation for an unusually long time, compared with steady state. Additionally, investors willing to sell have to pay holding costs, which then accrue for a longer time period. On the other hand, for a potential seller there is a chance that his intrinsic type will switch to high owner within $1/\lambda_u = 0.5$ years in expectation, reducing his utility loss.

The potential buyers' utility increases by 95%, because buyers' trading opportunities improve due to the shock. They gain a higher bargaining position and thereby negotiate better prices. The impact of a possible down-switch to an *ln*-type is low, since this event happens only every $1/\lambda_d = 5$ years in expectation. The selling pressure, however, diminishes after about 0.48 years, i.e. when there

are again more high-type investors than asset owners. Combining both effects, a type switch occurs—in expectation—after the selling pressure is considerably reduced.

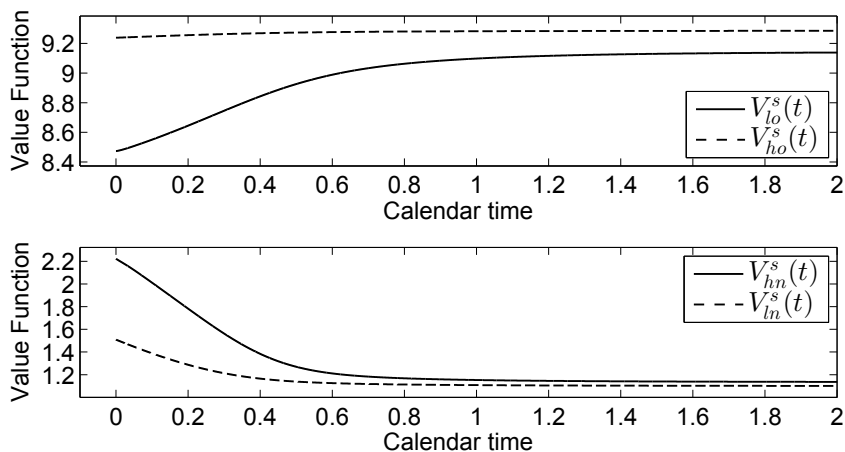


Figure 5.2: Value functions after an aggregate liquidity shock (without market makers). The upper panel shows the utility development of asset owners, the lower panel refers to non-owners.

The impact on high owners is, with a utility drop of about 0.5%, negligible. High owners stay with their type until they receive an idiosyncratic preference shock, which happens every $1/\lambda_d = 5$ years in expectation and upon which they switch to potential sellers. This impact is minor, since agents anticipate that the market will return to normality before they become a potential seller.

Low non-owners' utility increases by 37%. This increase is due to the fact that ln agents stay ln until they switch to a potential buyer. This switch happens every $1/\lambda_u = 0.5$ years in expectation. The utility increase is due to the possibility of switching to a favorable type in the near future.

The utility changes for high-type investors who shift to low-type due to the shock are analyzed next: High owners who change to low owners receive a 9% drop in their utility, because they change to the seller's side. High non-owners who switch to low non-owners receive a 33% increase in utility, comparing the ex ante with the post-shock situation. This increase is inferior in comparison to remaining a high non-owner, because potential buyers benefit the most from this shock.

Finally, the steady state values cum aggregate shocks ($\zeta = 0.1$) are compared with steady state values ex aggregate shocks ($\zeta = 0$), c.p. The utility of owning the asset decreases with the possibility of aggregate liquidity shocks, since agents anticipate the risk of being locked into an unfavorable asset position for a while.

The utility of not owning an asset increases, since these agents can take advantage of the selling pressure.¹²¹

Prices

An aggregate liquidity shock induces a sudden selling pressure, which affects interinvestor prices in a negative way: The asset price drops from a long run level of 8.0965 to 6.9901 immediately after the shock. The price is reduced by 13.67%, but recovers gradually from this shock over time. Figure 5.3 shows this price recovery path, when a shock occurs at $t = 0$.¹²²

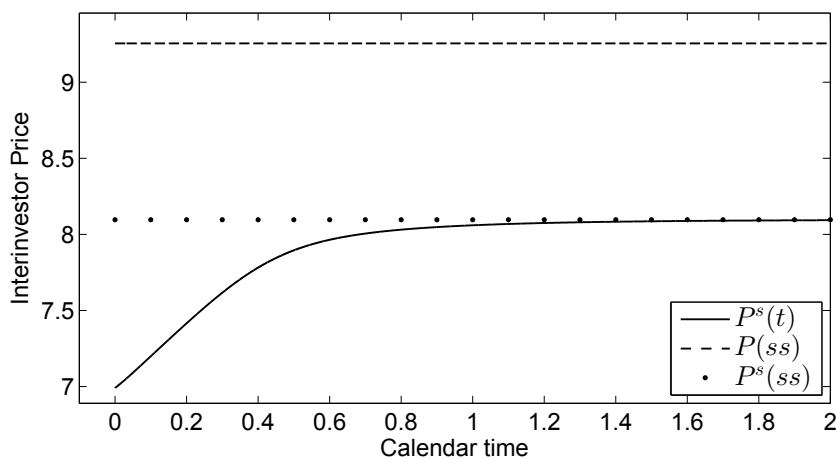


Figure 5.3: Price recovery after an aggregate liquidity shock (without market makers). The straight line shows the price path after an aggregate liquidity shock, which occurred at time $t = 0$. The dotted line describes the price a long time after the last shock, while the dashed line illustrates the steady state price without aggregate liquidity shocks.

The price recovery rate is high immediately after the shock. Half of the loss in price is even regained within 0.26 years (66 trading days). But recovery slows down as soon as the selling pressure alleviates. It takes about 1.6 years for the price to reach a fairly normal level.¹²³ However, the price does not reach the steady state price level without aggregate liquidity shocks but is reduced by 12.51%.¹²⁴ This lower price is due to the fact that market participants anticipate further shocks by factoring severity and frequency of aggregate liquidity shocks into the price.

¹²¹ The exact values for steady state ex aggregate shocks are $V_{lo}(ss) = 9.6129$, $V_{hm}(ss) = 0.4331$, $V_{ho}(ss) = 9.7419$, $V_{ln}(ss) = 0.4125$.

¹²² See figure 4 (top panel) in Duffie, Gârleanu, and Pedersen (2007, p. 1885).

¹²³ After 1.58 years, the percentage price change is less than 0.001% per day.

¹²⁴ The steady state price without aggregate liquidity shocks and without market makers is $P(ss) = 9.2546$.

The time the price takes to recover from an aggregate liquidity shock is influenced by several factors: (1) Severity of the shock, (2) search frictions, and (3) agents' individual recovery time. The severity of the shock, determined by $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$, has a negative impact on the price. The more severe the shock, the longer it takes to recover. In particular, the recovery time reacts more sensitively to the percentage of high owners turning to potential sellers than to potential buyers turning to low non-owners. Kyle (1985, p. 1316) denotes the recovery time as " 'resiliency' (the speed with which prices recover from a random [.] shock)", which is one of his three dimensions describing market liquidity. Search frictions, determined by λ , influence market liquidity as well. Higher search frictions lead to a slower recovery time, since trade is constricted. Agents' individual recovery time, denoted by λ_u , describes agents' "funding liquidity (i.e., the ease with which they can obtain funding)"¹²⁵. Duffie, Gârleanu, and Pedersen (2007, p. 1886) specify a long individual recovery time as "slow refinancing". It measures the time it takes investors to raise cash or, generally speaking, to adjust their positions. A faster individual recovery time, in turn, leads to a faster price recovery. I depict these three effects on the price recovery time in table 5.5, exemplarily and on a *ceteris paribus* basis.

$(\pi_{ho}(ss); \pi_{hn}(ss))$	(0.3;0.3)	(0.5;0.8)	(0.8;0.5)
Price recovery time (in years)	1.3	1.6	1.8
λ	20	250	375
Price recovery time (in years)	1.8	1.4	1.3
λ_u	1	5	20
Price recovery time (in years)	4.1	0.6	0.2

Table 5.5: The price recovery time for different values (c.p.) of the severity of the shock $(\pi_{ho}(ss); \pi_{hn}(ss))$, search frictions λ , and agents' individual recovery time λ_u (without market makers). I define the price recovery time as the time when the percentage price change is less than 0.001% per day.

The annualized realized instantaneous excess return of the illiquid asset over the interest rate of the liquid one is calculated with

$$\frac{\dot{P}(t) + D}{P(t)} - r.$$

¹²⁵ Brunnermeier and Pedersen (2009), p. 2201.

Figure 5.4 presents this return.¹²⁶ Agents buying right after the shock realize an annualized instantaneous excess return of over 30% for about three months.¹²⁷ The long run level of the excess return is $1/8.0962 - 10\% = 2.35\%$.

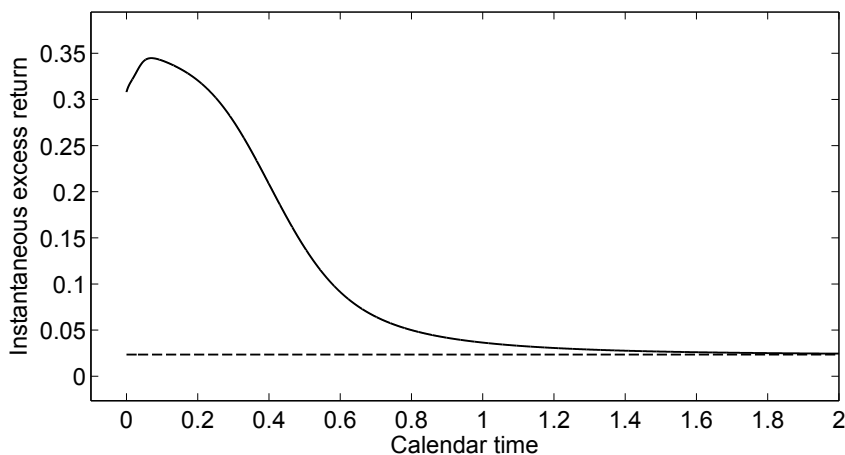


Figure 5.4: The solid line shows the annualized realized instantaneous excess return after an aggregate liquidity shock (without market makers). The dashed line shows this excess return in the long run.

5.2 Example with Market Maker

I extend the aggregate liquidity shock model by market makers. First, the fractions of investors in steady state must be determined, which are already stated in table 3.2. Market makers, the additional trading channel in this section, lead to a reduction in search frictions (in comparison with section 5.1). In a second step, I derive the type distribution immediately after the shock by means of equations (4.3), (4.4), (4.5), and (4.6). Table 5.6 summarizes the solutions.

	Parameter	$t = 0$	$t \rightarrow \infty$
Fraction of <i>lo</i> -type investors	$\mu_{lo}(t)$	0.3754	0.0009
Fraction of <i>hn</i> -type investors	$\mu_{hn}(t)$	0.0800	0.1600
Fraction of <i>ho</i> -type investors	$\mu_{ho}(t)$	0.3746	0.7491
Fraction of <i>ln</i> -type investors	$\mu_{ln}(t)$	0.1700	0.0900

Table 5.6: Masses of agents' intrinsic types immediately and a long time after the liquidity shock, with market makers.

¹²⁶ See figure 4 (bottom panel) in Duffie, Gârleanu, and Pedersen (2007, p. 1885). There, the annualized realized instantaneous excess return is mismatched with the annualized realized instantaneous return.

¹²⁷ This model is too simple to capture the effect of predatory trading to benefit from the shock. See Duffie, Gârleanu, and Pedersen (2007), p. 1884 and Brunnermeier and Pedersen (2005).

The evolution of mass distribution—with market makers—is shown in figure 5.5. A comparison with figure 5.1 (page 111) illustrates some differences: The fraction of buyers cum market makers decreases after the shock as well but stays at a very low rate until the intersection time t^* . Hereafter, the fraction of buyers starts to increase concavely until it reaches its steady state level. In comparison, buyers' fraction ex market makers does not decrease after the shock as fast and as lowly. This fraction starts to increase more intensely right after its lowest point, which is much earlier than the intersection point t^* . Implementing market makers ensures a second trading channel and therefore better matching conditions. The fraction of sellers diminishes faster cum market makers, although the selling pressure per se is not reduced earlier. In both cases, the fraction of sellers slightly decreases further after the intersection time by means of a convex curve.

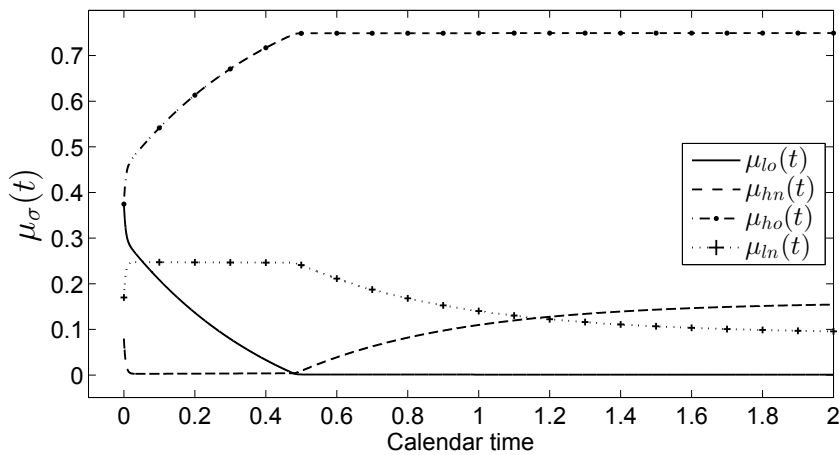


Figure 5.5: Process of mass distribution after an aggregate liquidity shock and with market makers. The solid line illustrates the fraction of sellers over time. The dashed line represents the fraction of buyers. Dotted and dashed-dotted (with plus sign) lines show the fraction of low non-owners and high owners, respectively.

After an aggregate liquidity shock, market makers contact more potential sellers than potential buyers, since $\mu_{lo}(0) > \mu_{hn}(0)$. Consequently, market makers trade with all potential buyers and must ration potential sellers. The interdealer price $M(t)$ is set equal to sellers' reservation value $B(t)$. It also takes about $t^* = 0.48$ years until the amount of potential buyers exceeds the amount of potential sellers, i.e. $\mu_{lo}(t) < \mu_{hn}(t)$ for $t > t^* > 0$. With the formula in appendix 5C, it is clear that this intersection time is not directly affected by meeting intensities λ and ρ . Implementing market makers influences only slightly the fraction of high-type agents $\mu_h(0)$. After the intersection point t^* , all sellers can trade when meeting a market maker, whereas buyers are rationed. The interdealer price $M(t)$ is then set equal to buyers' reservation value $A(t)$.

The higher the meeting intensities, the shorter the average time interval during which potential buyers and potential sellers meet. The remaining buyers after the shock are matched more quickly with potential sellers, leading to a lower overall fraction of potential buyers shortly after the shock, since all agents mutating thereafter to a potential buyer are matched faster as well. This market gets one-sided, with considerably more potential sellers than potential buyers. The effect of such a market is best explained with ‘congestion’, a term coined by Afonso (2011, p. 325), who finds a similar result:¹²⁸

“In a one-sided market with more sellers than buyers, introducing a measure that improves the efficiency of the search process makes it easier for one of the few buyers present in the market to acquire the asset. But when the buyer purchases the asset [...], the proportion of buyers to sellers falls further and the market becomes more one-sided. [...] Reducing market frictions in a distressed market thus magnifies the effect of congestion [...].”

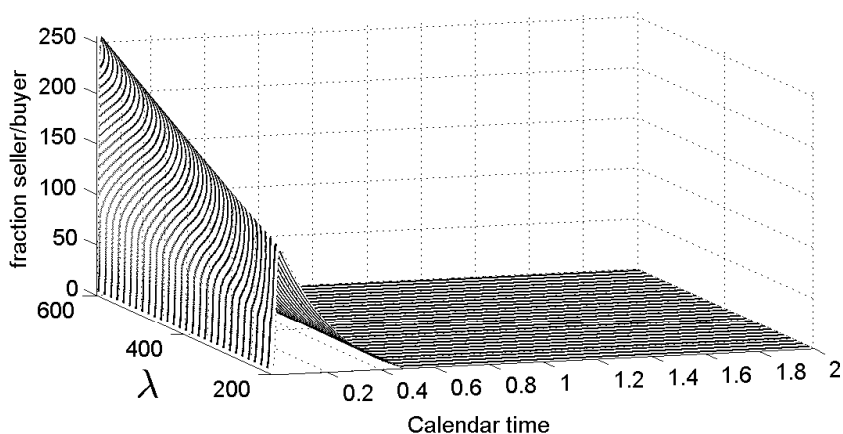


Figure 5.6: Fraction of potential sellers vs. potential buyers, i.e. $\mu_{lo}(t)/\mu_{hm}(t)$, for $\lambda \in [200, 600]$ and $\rho = 125$, c.p.

Figure 5.6 illustrates the congestion effect of a one-sided market by means of the fraction of potential sellers versus potential buyers for $\lambda \in [200, 600]$ and $\rho = 125$. The higher the value, the more potential sellers in proportion to potential buyers in the market. The focus lies on the peak immediately after the shock. It increases considerably with decreasing search frictions. The lower the search frictions, the higher the imbalance between potential sellers and potential buyers after an aggregate liquidity shock—and the more one-sided this market becomes.

¹²⁸ Afonso (2011) triggers the congestion effect by an endogenous entry of investors in a steady state setup. The situation of a one-sided market after an aggregate liquidity shock is comparable.

Trading Volume and Trading Time

Table 5.6 and figure 5.5 show that after an aggregate liquidity shock, there are more investors available who are active in searching for a good match; this is $\mu_{lo}(0) + \mu_{hn}(0) > \mu_{lo}(ss) + \mu_{hn}(ss)$. The amount of potential successful matches is increased as well: $\min\{\mu_{lo}(0), \mu_{hn}(0)\} > \min\{\mu_{lo}(ss), \mu_{hn}(ss)\}$. As a result, the trading volume is elevated right after the shock. Afonso (2011) finds a similar effect: If there are more agents intending to trade, the trading volume increases in the case that search frictions are not too high. At the same time, a price drop, e.g. due to a selling pressure, can arise. This price discount is also a measure of market liquidity. As a result, high trading volume and market illiquidity, measured by price pressure, are not mutually exclusive.

A suitable illustration of trading volume is by means of average asset turnover, where

$$\text{asset turnover} = (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho \min\{\mu_{hn}(t), \mu_{lo}(t)\}) / s. \quad (5.1)$$

It is the fraction of assets bought and sold in proportion to all assets.

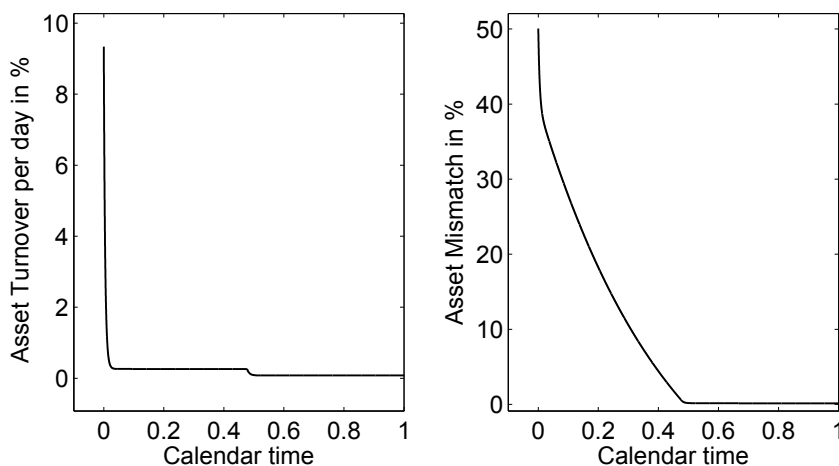


Figure 5.7: The left panel shows the average asset turnover in proportion to the asset supply. The right panel shows the asset mismatch.

The left panel of figure 5.7 shows the average asset turnover per day in proportion to the asset supply. Compared with the situation without market makers, this asset turnover is more than twice as high. The panel to the right shows the asset mismatch of all assets, defined with

$$\text{asset mismatch} = \mu_{lo}(t) / s. \quad (5.2)$$

It describes the fraction of all assets that are misallocated to investors with low preferences. As expected, there is a high fraction of asset mismatch at time $t = 0$, due to the shock. The increased trading volume following time $t = 0$ reduces this misallocation. After the worst part of the shock is over, i.e. when the fraction of high owners reaches its long run level, the asset mismatch returns to its normal level. Asset mismatch is therewith a good substitute for reflecting the turning point when markets start returning to normality.

Another measure describing investors' situation after an aggregate liquidity shock is the average time it takes to buy (t_{lo}) or sell (t_{hn}) an asset. These measures are defined with

$$t_{hn}(t) = [2\lambda\mu_{lo}(t) + \rho \min\{\mu_{hn}(t), \mu_{lo}(t)\}]^{-1}, \quad (5.3)$$

$$t_{lo}(t) = [2\lambda\mu_{hn}(t) + \rho \min\{\mu_{hn}(t), \mu_{lo}(t)\}]^{-1}. \quad (5.4)$$

Figure 5.8 shows these average trading times.

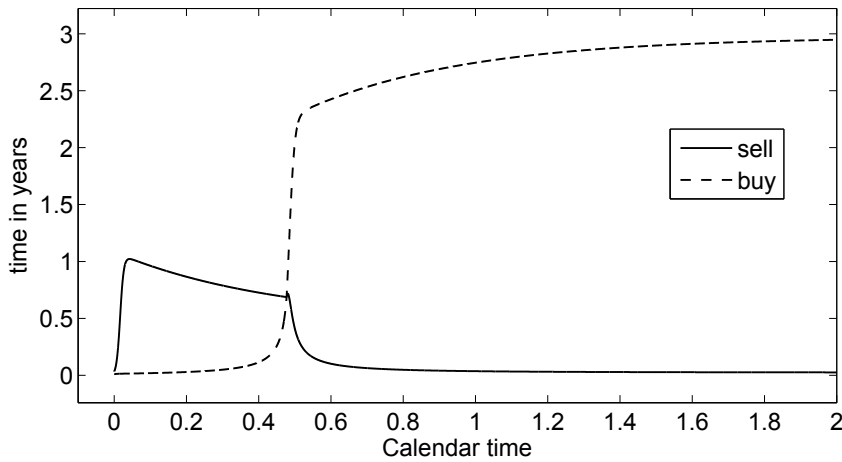


Figure 5.8: Average time in years it takes to buy ($t_{lo}(t)$, dashed line) or sell ($t_{hn}(t)$, solid line) an asset over time.

The expected times for buying or selling are almost equal and very short immediately after the shock, i.e. at time $t = 0$. Since there are more sellers than buyers, the expected time to buy an asset stays short for a while and then starts to increase as the fraction of sellers decreases. The expected time to sell an asset starts to increase sharply immediately after the shock, since then the majority of the remaining buyers after the shock is matched. The average time to sell gets shorter slowly thereafter, until the fractions of buyers and sellers are equal. After this point of intersection, the average time to sell increases for a very short time interval before it drops to its low long run level. This tiny hump-shaped characteristic

is due to different recovery patterns for buyers and sellers around the intersection time: Shortly after the intersection time, the fraction of potential sellers decreases faster than the fraction of potential buyers increases. This hump-shaped pattern only appears with market makers, because then both expected trading times depend likewise on the fraction of buyers as well as on the fraction of sellers. As a result, the shock reduces the average trading time for potential buyers, since the market is a buyer's market after the shock. Sellers' trading time drops to its low long run level as soon as the market returns to a seller's market.

Value Function

The indirect utility functions immediately and a long time after the liquidity shock are stated in table 5.7.

	Parameter	$t = 0$	$t \rightarrow \infty$
Value function of <i>lo</i> -type investors	$V_{lo}^s(t)$	8.6339	9.3670
Value function of <i>hn</i> -type investors	$V_{hn}^s(t)$	1.9854	0.8193
Value function of <i>ho</i> -type investors	$V_{ho}^s(t)$	9.3982	9.4375
Value function of <i>ln</i> -type investors	$V_{ln}^s(t)$	1.2454	0.8015

Table 5.7: Value Functions immediately and a long time after the liquidity shock, with market makers.

The general effect of an aggregate liquidity shock on the value functions is similar to section 5.1. However, the bargaining position of potential sellers improves due to facilitated search, whereas the bargaining position of potential buyers deteriorates. These effects have a positive impact on *ho* agents, since their bargaining situation in the future improves as well. *Ln* agents are negatively affected, since their future bargaining situation worsens. As a result, reducing search time due to market makers favors asset owners and puts non-owners at a disadvantage, both immediately after the shock and a long time later. This effect can be seen by comparing table 5.4 (page 112) with table 5.7. The value change due to the shock is larger with market makers—except for high owners. Figure 5.9 shows agents' value function as a function of time.

The effect on potential sellers is twofold: On one hand, they benefit from introducing market makers, since they recover substantially faster after a shock and reach their long run level earlier; on the other hand, the drop in value function is bigger. This can be seen by comparing figure 5.2 (page 113) with figure 5.9.

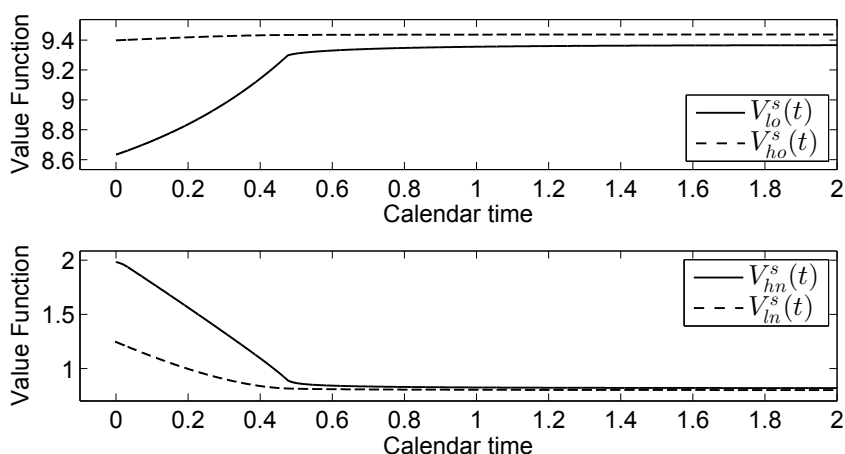


Figure 5.9: Value functions after an aggregate liquidity shock (with market makers). The upper panel shows the utility development of asset owners, the lower panel refers to non-owners.

Prices and Bid-Ask Spread

The percentage price drop due to the aggregate liquidity shock is, at 13.86%, nearly identical to the situation without market makers. The general price level is, however, higher, since a trading partner can be located more easily. Table 5.8 states the interinvestor price $P^s(t)$, the bid price $B^s(t)$, the ask price $A^s(t)$, and the bid-ask spread immediately after the shock and a long time after.

	Parameter	$t = 0$	$t \rightarrow \infty$
Interinvestor price	$P^s(t)$	7.4007	8.5918
Ask price	$A^s(t)$	7.4080	8.6182
Bid price	$B^s(t)$	7.3885	8.5760
Bid-ask spread	$A^s(t) - B^s(t)$	0.0195	0.0421

Table 5.8: Prices immediately and a long time after an aggregate liquidity shock, with market makers.

The time it takes for the price to recover from an aggregate preference shock depends on market liquidity, measured by expected search time λ and ρ . If markets are illiquid, it takes longer to recover.¹²⁹ With the introduction of market makers, market liquidity increases. As a result, prices recover faster in comparison to the model without market makers. Figure 5.10 shows the development of the interinvestor price $P^s(t)$, the bid price $B^s(t)$, and the ask price $A^s(t)$ after an aggregate liquidity shock.

¹²⁹ See Pedersen (2009), p. 191 and the analysis in table 5.5.

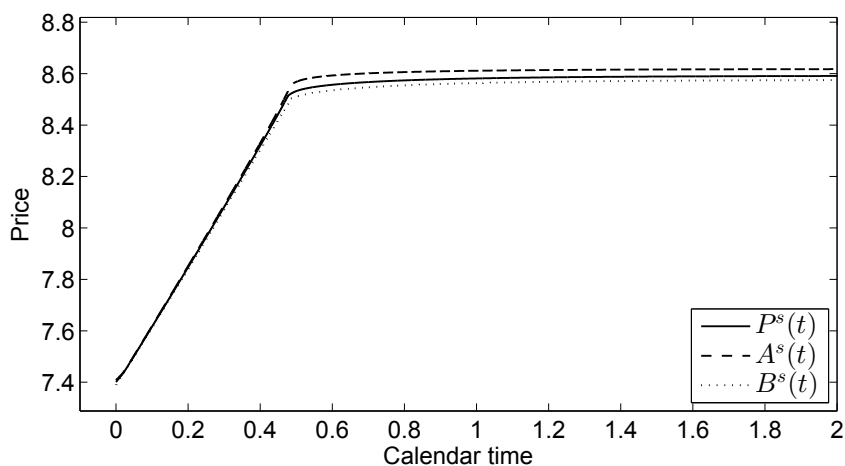


Figure 5.10: Price after an aggregate liquidity shock, with market makers. The solid line illustrates the interinvestor price $P^s(t)$. The dashed line represents the ask price $A^s(t)$ and the dotted line shows the bid price $B^s(t)$.

Half of the loss in price is regained as well within 0.26 years. However, prices reach a fairly normal level within 1.1 years, compared with 1.6 years without market makers.¹³⁰ Not even an increase of λ in section 5.1 from 125 to 375 (while $\rho = 0$), which makes meeting intensities approximately comparable, would obtain this fast recovery time. Furthermore, as appendix 5B shows, for $\lambda = 0$ and $\rho = 125$, prices reach a fairly normal level within approximately 0.9 years. These effects imply that market makers provide superior search service.

The components of the bid-ask spread are usually split into two parts: (1) transaction costs and (2) adverse selection. Part (1) includes all fixed and variable costs of a market maker running his business, like compensation for market makers' time, inventory cost and risk. Part (2) contains a compensation for losses due to trading with informed investors.¹³¹

Market makers in this model are match makers, i.e. they do not hold inventory. Additionally, there are no information asymmetries. The bid-ask spread simply captures the return of market makers providing search service, and thereby reducing the search time of buyers and sellers alike. It is not a charge for bearing inventory risk.¹³² Thus, the bid-ask spread measures the capability of market makers locating a suitable trading partner. More precisely, the bid-ask spread depends on the degree of competition. Market makers must take investors' outside options into account, because bid and ask prices reflect the availability of

¹³⁰ After 1.08 years, the percentage price change is less than 0.001% per day.

¹³¹ See Glosten and Milgrom (1985) and Harris (2003), chapter 14.

¹³² See Rubinstein and Wolinsky (1987), p. 591 and Grossman and Miller (1988), p. 628 and 620.

other market makers as well as other investors. These outside options imply that market makers compete both with each other and with the ability of investors finding each other in a direct way. Investors bargain bid and ask prices which reflect the possibility of terminating the bargaining process and finding another trading partner. If outside options are favorable, bid-ask spreads are tighter.¹³³

The bid-ask spread is calculated as the difference in the surplus of sellers and buyers over time, weighted with market makers' bargaining power z . This is

$$A^s(t) - B^s(t) = z [(V_{ho}^s(t) - V_{hn}^s(t)) - (V_{lo}^s(t) - V_{ln}^s(t))]. \quad (5.5)$$

Market makers gain the fraction z of the overall trading surplus, i.e. the difference between buyers' and sellers' reservation value. Figure 5.11 depicts the development of the bid-ask spread after an aggregate preference shock.

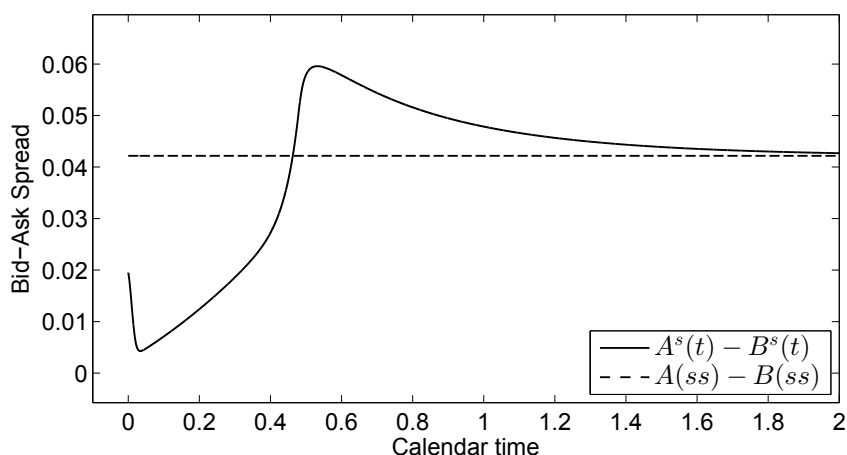


Figure 5.11: The solid line shows the bid-ask spread over time after an aggregate liquidity shock. The dashed line shows the bid-ask spread without an aggregate liquidity shock.

Immediately after the shock, it is easy for a potential buyer to meet potential sellers. Since there is an oversupply of potential sellers, buyers have a better outside option compared with steady state. For a potential seller, some potential buyers are still available. Combining both, the better outside option of potential buyers overcompensates for the reduction in outside options for potential seller. A tighter bid-ask spread results immediately after the shock, in comparison with the long run level. However, and in contrast to figure 5.11, one would expect an immediately widening bid-ask spread thereafter, since both outside options then decline. The downward hump is therefore puzzling. I defer further explanations to chapter 7, where the reason for this confusing effect is addressed.

¹³³ See Feldhütter (2012), p. 1162 and Duffie, Gârleanu, and Pedersen (2005), p. 1816.

Some final remarks about the bid-ask spreads are feasible, though: Ask and bid prices, which imply the risk of an aggregate liquidity shock, are clearly lower than the steady state prices without a shock. The level of long run bid-ask spreads, however, is nearly equal to the level of bid-ask spreads in steady state without aggregate liquidity shocks (here: 0.0421 versus 0.0422 without shocks).

5.3 Conclusion

The analysis in section 5.1 highlights general characteristics of repeated aggregate liquidity shocks: First, an aggregate liquidity shock causes a selling pressure, which results in an immediate price drop. During the recovery time, selling an asset is more time-consuming. Second, the market recovers from this shock over time, whereas the recovery time depends on search intensity. Prices reach a fairly normal level after a while. Third, this level is lower compared to the steady state price level without aggregate liquidity shocks. Agents anticipate the risk that a shock can occur in the future.

My extension of the aggregate liquidity shock model by market makers reveals additional results: Market makers provide superior search service, since prices reach their fairly normal level disproportionately faster, and an increase in market liquidity results. Aggregate liquidity shocks immediately reduce market makers' bid-ask spread, because investors face better outside options. In the long run, the bid-ask spread is only marginally affected by the risk of repeated aggregate liquidity shocks.

5A Appendix: Seller's Market vs. Buyer's Market

The steady state relation of $\mu_{hn}(ss)$ and $\mu_{lo}(ss)$ influences post-shock characteristics of the market. Two cases can be distinguished:

Case 1: $s < \lambda_u / (\lambda_u + \lambda_d)$. With this relation, the steady state equilibrium of investors' types is characterized by a seller's market with $\mu_{hn}(ss) > \mu_{lo}(ss)$. It is the situation which Duffie, Gârleanu, and Pedersen (2007) consider. Potential sellers can trade more easily and faster than potential buyers, because there are more potential buyers available in steady state than sellers. Therefore, the fraction of potential sellers is very low in steady state. Agents switching from *ho*-type to *lo*-type do not stay in the *lo* state for a long time, since finding a trading partner is easy for these agents. If $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$ are chosen in such a way that $\bar{\mu}_{hn}(0) < \bar{\mu}_{lo}(0)$, then the market switches temporarily to a buyer's market. Equations (3.4)–(3.7) ensure that it returns to a seller's market over time. This case is treated in the numerical example in chapter 5.

If $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$ are chosen in such a way that $\bar{\mu}_{hn}(0) > \bar{\mu}_{lo}(0)$, the market stays a seller's market. The aggregate liquidity shock is not severe, because there is no surplus of potential buyers over sellers. The market recovers quickly. Figure 5.12 shows the mass distribution of the second example presented in Duffie, Gârleanu, and Pedersen (2007, p. 1886), i.e. $\pi_{ho}(ss) = 0.1668$ and $\pi_{hn}(ss) = 0.1697$.¹³⁴

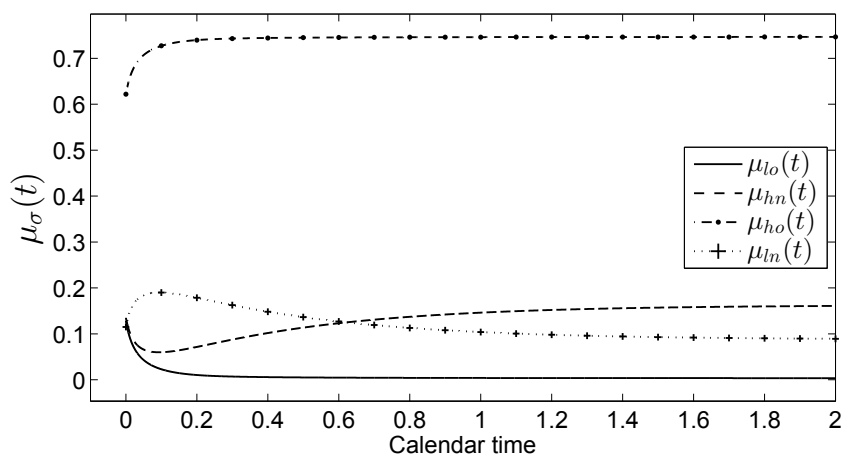


Figure 5.12: The graph shows the process of mass distribution after a small aggregate liquidity shock (without market makers).

The price drop is only 2.20%, and half of it is regained after 40 days. After one year, the percentage price change is less than 0.001% per day. The general pattern

¹³⁴ Duffie, Gârleanu, and Pedersen (2007) state $\pi_{ho}(ss) = \pi_{hn}(ss) = 0.17$, which does not exactly fit the denoted post-shock distribution $\bar{\mu}_{lo} = 0.1280$, $\bar{\mu}_{hn} = 0.1350$, $\bar{\mu}_{ho} = 0.6220$, $\bar{\mu}_{ln} = 0.1150$.

of all other variables are similar to the ones presented in sections 5.2 and 5.1, but only the values and processes after the intersection time are relevant here.

Case 2: $s > \lambda_u / (\lambda_u + \lambda_d)$. In this case, which has not yet been discussed in the literature, steady state equilibrium is a buyer's market with $\mu_{hn}(ss) < \mu_{lo}(ss)$. Potential buyers can trade faster, because there are more potential sellers than potential buyers available in steady state. Thus, the fraction of potential buyers is low in steady state. and sellers are marginal investors. Agents switching from ln -type to hn -type do not stay in the hn state for a long time, since finding a trading partner is easy for these agents. Since the fraction of potential buyers is low, the impact of an aggregate liquidity shock on both μ_{hn} and μ_{ln} is low. This market remains a buyer's market, independently of $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$. I illustrate this case by modifying $\lambda_d = 0.7$ (c.p.).

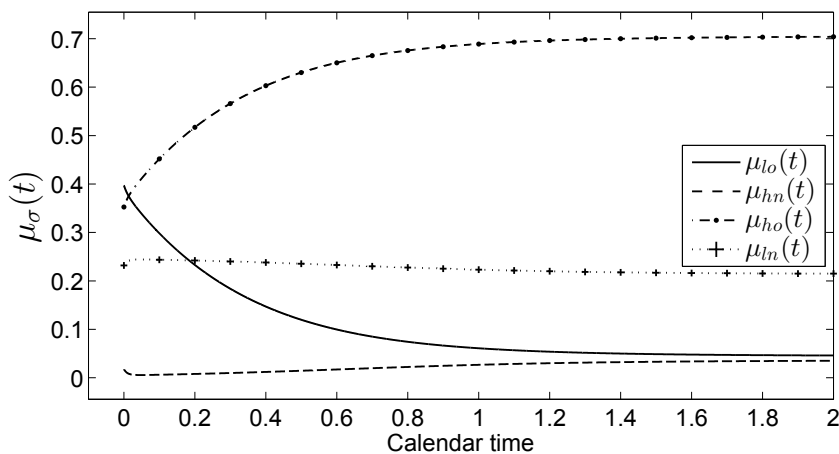


Figure 5.13: The graph shows the process of mass distribution after an aggregate liquidity shock in a buyer's market (without market makers).

A high-type investor stays high for $1/\lambda_d = 1.43$ years on average, which is considerably shorter than the five years in the initial example. Figure 5.13 shows the process of masses. The impact of this shock on potential sellers is not minor. However, there is no intersection time t^* for which $\mu_{hn}(t^*) = \mu_{lo}(t^*)$ is valid.

As highlighted in section 3.4, the price without liquidity shocks is negative for this scenario, so that it is negative with aggregate liquidity shocks, as well. The recovery is slow: Half of the price drop is not regained until 106 trading days have elapsed. After 2.32 years, the percentage price change is less than 0.001% per day. The patterns of all other variables are similar to the ones presented in sections 5.2 and 5.1. But there is one exception if market makers are present: The bid-ask spread.

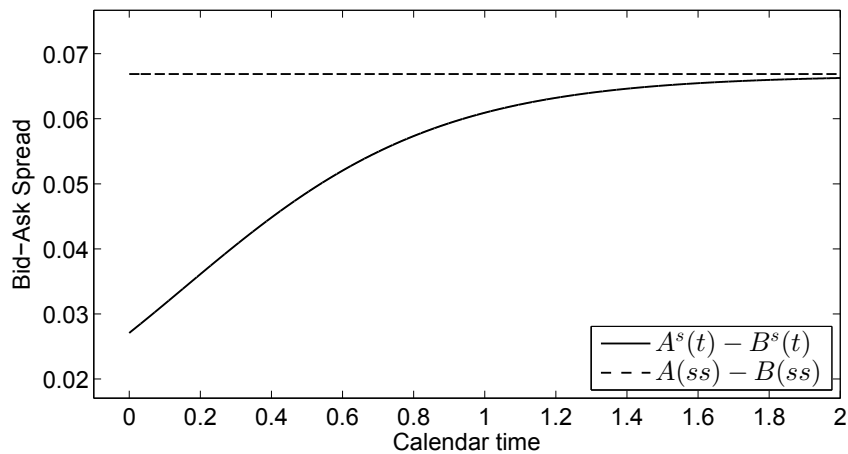


Figure 5.14: Bid-ask spread in a buyer's market (with market makers, $\rho = 125$).

Figure 5.14 illustrates the bid-ask spread after an aggregate liquidity shock in a buyer's market and with market makers. The bid-ask spread is reduced immediately after the shock and increases monotonically until it reaches its long run value. There is no hump-shaped pattern. This is due to the fact that all but buyer's value functions are barely affected by the shock. Figure 5.15 illustrates this effect.

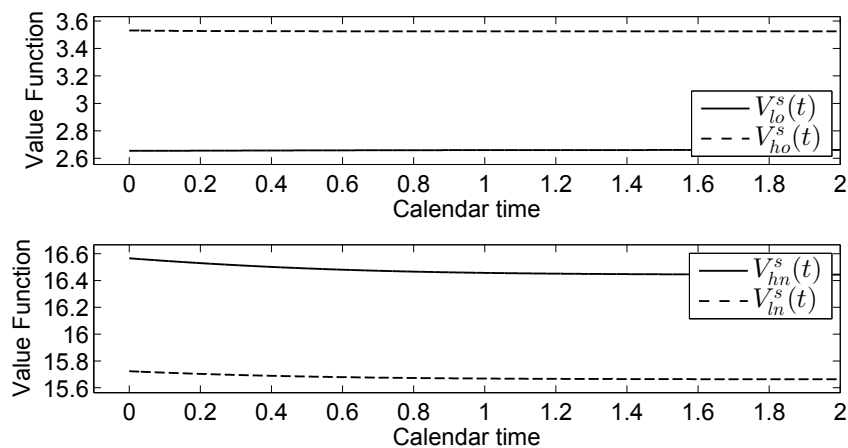


Figure 5.15: Value functions in a buyer's market (with market makers, $\rho = 125$).

Buyer's value function increases due to the shock, since it is easier for the remaining fraction of buyers to find an appropriate trading partner.

5B Appendix: Trade Intermediation by Market Makers

Makers

In this appendix, I restrict trade intermediation to market makers ($\lambda = 0, \rho = 125$, c.p.). Some effects are striking and differ from the general treatment in section 5.2: The expected search time is equal for buyers and sellers, since both are matched through market makers. In expectation it takes approximately 12.5 days after the shock until two trading parties are matched. In steady state, however, market makers find two suitable trading partners after 3.3 years in expectation. Figure 5.16 shows the expected time it takes until a trade is executed via market makers.

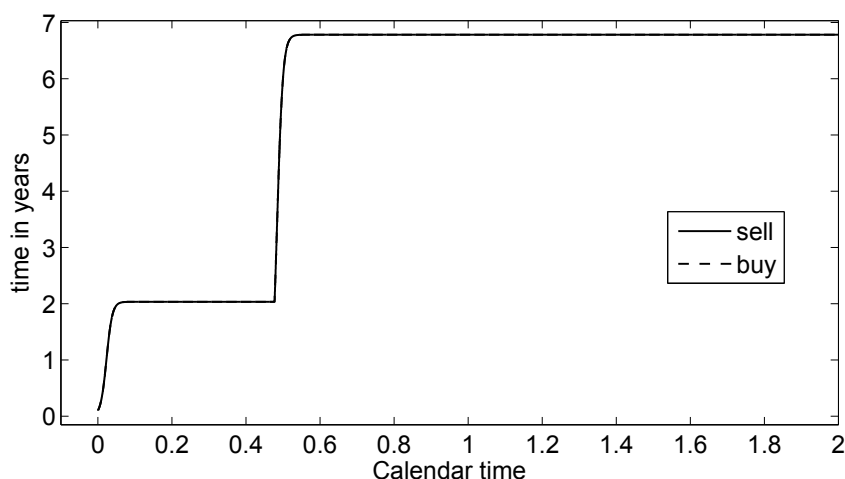


Figure 5.16: Average time in years it takes to buy or sell an asset via market makers.

The pattern resembles stairs: The expected trading time is low immediately after the shock, since there are many potential sellers in the market and there are still some potential buyers available. Agents are matched fast. After the majority of remaining buyers are matched, search takes longer (approximately two years), described by the first stair. Market makers' search is quite efficient, since agents switching from ln -type to hn -type are matched quickly. These quick matches lead to a low level of potential buyers. The second stair is reached after the intersection time t^* , i.e. when $\mu_{hn}(t^*) = \mu_{lo}(t^*)$. After t^* , the fraction of potential sellers drops quickly to its long run level, which is very low.

The price drop after the shock is almost identical to sections 5.2 and 5.1. Half of the loss in price is regained within 0.26 years as well. However, prices reach a fairly normal level within approximately 0.9 years¹³⁵, which is faster than recov-

¹³⁵ After 0.88 years, the percentage mid-price change is less than 0.001% per day.

ery with only interinvestor trade. It is even faster than the recovery time with interinvestor trade and intermediation by market maker. This effect is unexpected: The market overcomes the shock about 2.5 months earlier by cutting off the interinvestor channel and restricting trade intermediation to market makers. Simultaneously, market makers charge a considerably higher bid-ask spread when interinvestor trade is not permitted. As a result, market makers provide superior search service, for which they receive a remarkably higher compensation, since investors' outside options are reduced.

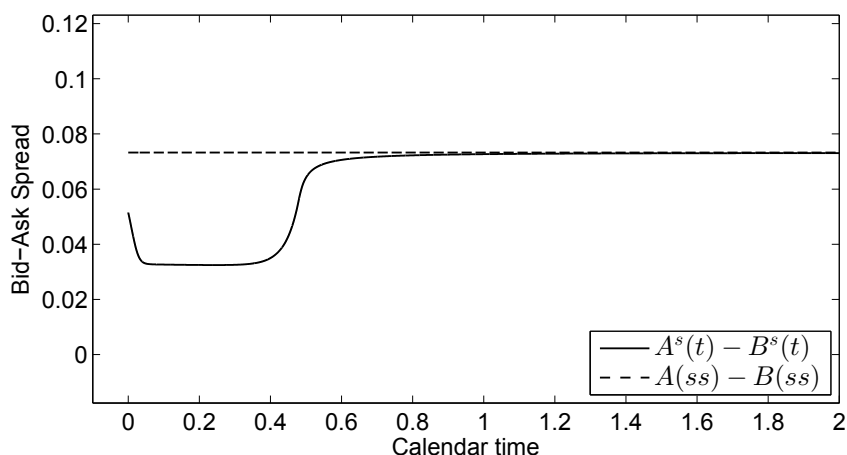


Figure 5.17: The solid line shows the change of the bid-ask spread over time after an aggregate liquidity shock. The dashed line shows the bid-ask spread without an aggregate liquidity shock.

Figure 5.17 shows the bid-ask spread as a function of time after an aggregate liquidity shock. The counterintuitive downward hump appears again, the explanation of which I defer to chapter 7. For the sake of completeness, I state prices in table 5.9.

	Parameter	$t = 0$	$t \rightarrow \infty$
Ask Price	$A^s(t)$	7.4750	8.6365
Bid price	$B^s(t)$	7.4235	8.5634
Bid-Ask Spread	$A^s(t) - B^s(t)$	0.0514	0.0731

Table 5.9: Prices immediately and a long time after the liquidity shock, where trade is only intermediated by market makers.

5C Appendix: Time of Intersection t^*

The example discussed in chapter 5 and the cases analyzed in appendix 5A show that under some circumstances there is a time t^* —after an aggregate liquidity shock—at which a market switches from a buyer’s market to a seller’s market. The other direction never occurs after this kind of shock, as the following calculation shows. I start with the difference between equation (3.4) and (3.5):¹³⁶

$$\begin{aligned}\dot{\mu}_{lo}(t) - \dot{\mu}_{hn}(t) &= -\lambda_u \mu_{lo}(t) + \lambda_d \mu_{ho}(t) - \lambda_u \mu_{ln}(t) + \lambda_d \mu_{hn}(t) \\ &= (\lambda_u + \lambda_d) \mu_h(t) - \lambda_u.\end{aligned}\quad (5.6)$$

$\mu_h(t)$ is already stated in equation (3.9) with

$$\mu_h(t) = \bar{\mu}_h(0) e^{-(\lambda_u + \lambda_d)t} + \frac{\lambda_u}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u + \lambda_d)t} \right].$$

Additionally, the integration of $\dot{\mu}_{lo}(t) - \dot{\mu}_{hn}(t)$ results in

$$\mu_{lo}(t) - \mu_{hn}(t) = \bar{\mu}_{lo}(0) - \bar{\mu}_{hn}(0) + \int_0^t (\dot{\mu}_{lo}(\tau) - \dot{\mu}_{hn}(\tau)) d\tau. \quad (5.7)$$

Plug equations (5.6) and (3.9) into (5.7), so that

$$\mu_{lo}(t) - \mu_{hn}(t) = \bar{\mu}_{lo}(0) + \bar{\mu}_{ho}(0) - \frac{\lambda_u}{\lambda_u + \lambda_d} - \left[\bar{\mu}_h(0) - \frac{\lambda_u}{\lambda_u + \lambda_d} \right] e^{-(\lambda_u + \lambda_d)t}. \quad (5.8)$$

For some starting conditions $\bar{\mu}_\sigma(0)$, there is a time t^* at which equation (5.7) changes its sign: From + to – if the market changes from a buyer’s market to a seller’s market. Or from – to +, when the market changes from a seller’s market to a buyer’s market. A solution to

$$\begin{aligned}0 &= s - \frac{\lambda_u}{\lambda_u + \lambda_d} - \left[\bar{\mu}_h(0) - \frac{\lambda_u}{\lambda_u + \lambda_d} \right] e^{-(\lambda_u + \lambda_d)t^*}, \\ \Leftrightarrow \\ t^* &= -\frac{1}{\lambda_u + \lambda_d} \ln \left[\frac{\frac{\lambda_u}{\lambda_u + \lambda_d} - s}{\frac{\lambda_u}{\lambda_u + \lambda_d} - \bar{\mu}_h(0)} \right],\end{aligned}\quad (5.9)$$

¹³⁶ See Feldhütter (2010), pp. 47.

must then exist and $t^* > 0$ must hold, i.e. the argument of the natural logarithm must be located between 0 and 1. The existence of a solution to equation (5.9) is ensured by the validity of one of the following two conditions:

$$\bar{\mu}_h(0) \leq s < \frac{\lambda_u}{\lambda_u + \lambda_d}, \quad (5.10)$$

$$\text{or } \frac{\lambda_u}{\lambda_u + \lambda_d} < s \leq \bar{\mu}_h(0). \quad (5.11)$$

The condition (5.11) never occurs due to an aggregate liquidity shock: If $\lambda_u/(\lambda_u + \lambda_d) < s$ holds, then $\mu_{hn}(ss) < \mu_{lo}(ss)$ is valid as well. The shock reduces μ_{hn} and increases μ_{lo} . As a result, there are no probabilities $0 \leq \pi_{ho} \leq 1$ and $0 \leq \pi_{hn} \leq 1$, so that $s \leq \bar{\mu}_h(0)$, which is equal to $\bar{\mu}_{hn}(0) > \bar{\mu}_{lo}(0)$, is valid.

The only valid intersection time after an aggregate liquidity shock occurs due to a change from a buyer's market to a seller's market. Condition (5.10) ensures this intersection time with $s < \lambda_u/(\lambda_u + \lambda_d)$, which is equal to $\mu_{hn}(ss) > \mu_{lo}(ss)$. In addition, this equation controls for the severity of the shock with $\bar{\mu}_h(0) \leq s$, which is equal to $\bar{\mu}_{hn}(0) \leq \bar{\mu}_{lo}(0)$.

Chapter 6

Frozen Market

In this chapter, I show that the aggregate liquidity shock model pretends that agents would trade despite no gains from trade. Agents are forced to trade when meeting a trading partner, although a bargaining solution does not exist.

Section 6.1 states the properties for a Nash bargaining solution. For the basic model, the existence of a Nash bargaining solution is studied in section 6.2. For the aggregate liquidity shock model, I analyze the existence of a Nash bargaining solution in section 6.3. The numerical example in section 6.4 illustrates the results of section 6.3. It shows that there are no gains from trade in some market situations which should lead to a ‘frozen market’.¹³⁷ Taking voluntary trading instead of forced trading into consideration, I suggest some modifications in section 6.5. Section 6.6 concludes. Appendices 6A and 6B contain the derivation of the results stated in sections 6.2 and 6.3, respectively. Appendix 6C presents some further analysis to section 6.3.

6.1 Analyzing the Nash Bargaining Solution

The discussion and analysis so far has implicitly presumed the existence of gains from trade, i.e. the existence of a Nash bargaining solution. It is assumed that all meetings intermediated by market makers as well as all meetings between sellers and buyers result in a trade. Chapter 6 scrutinizes this presumption.

A general bargaining game satisfying the axioms [A1], [A3], and [A4]—defined in section 2.2.1—has a unique solution $f(S, d)$ if the bargaining problem is well

¹³⁷ Chiu and Koepl (2011), Camargo and Lester (2013), and Camargo, Kim, and Lester (2013) also utilize the term ‘market freeze’ when there is no trading due to no gains from trade.

defined.¹³⁸ I must validate that the bargaining problems of chapter 3 and chapter 4 are indeed well defined, since the rule of splitting the surplus must then hold for all points in time. The central definition of Nash (1950, 1953) for a well-defined bargaining problem is, as stated in Osborne and Rubinstein (1990, p. 10), as follows:

“A bargaining problem is a pair $\langle \mathcal{S}, d \rangle$, where $\mathcal{S} \subset \mathcal{R}^2$ is compact (i.e. closed and bounded) and convex, $d \in \mathcal{S}$, and there exists $s \in \mathcal{S}$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems is denoted \mathcal{B} . A bargaining solution is a function $f : \mathcal{B} \rightarrow \mathcal{R}^2$ that assigns to each bargaining problem $\langle \mathcal{S}, d \rangle \in \mathcal{B}$ a unique element of \mathcal{S} .”

This definition contains several restrictions which must be controlled for:

- (1.) The sets \mathcal{S}^j for $j = P, A, B$ for the three basic bargaining problems¹³⁹ stated in chapter 3.3.2 are defined as follows:

$$\mathcal{S}^P = \{V_{ln}(t) + P(t), V_{lo}(t), V_{ho}(t) - P(t), V_{hn}(t)\},$$

$$\mathcal{S}^A = \{A(t) - M(t), 0, V_{ho}(t) - A(t), V_{hn}(t)\},$$

$$\mathcal{S}^B = \{M(t) - B(t), 0, V_{ln}(t) + B(t), V_{lo}(t)\}.$$

The transversality condition ensures the compactness, since $V_\sigma(t)$ and $V_\sigma^s(t)$ have to be bounded.

- (2.) The assumption of a convex set \mathcal{S} is fulfilled: The set \mathcal{S} is convex if it contains all pure agreements as well as all lotteries over pure agreements. Player's utility function can be represented by Von Neumann–Morgenstern utility functions.
- (3.) The disagreement or threat point d , i.e. $V_{hn}(t), V_{lo}(t), 0$, is included in the sets \mathcal{S} . Agents can agree to disagree.
- (4.) The condition $s_i > d_i$ for $i = 1, 2$ assures that both agents prefer an agreement over a disagreement, i.e. $f(\mathcal{S}, d) = \{d\}$ is never a bargaining solution by assumption. Economically stated, this means: If there is nothing on the table worthy of agreement, there is no bargaining situation at all.

I analyze restriction number (4.) in the remaining sections of chapter 6, since this is the crucial point: First, I briefly repeat the side conditions stated in inequalities (3.37), (3.45), (3.47), and (4.21), for which both agents prefer an agreement over a

¹³⁸ See section 2.2.1 for further details.

¹³⁹ The sets of the bargaining problems for the aggregate liquidity shock model are alike.

disagreement, that is

$$\begin{aligned} V_{lo}(t) - V_{ln}(t) &\leq P(t) \leq V_{ho}(t) - V_{hn}(t), \\ V_{lo}(t) - V_{ln}(t) &\leq B(t) \leq M(t), \\ M(t) &\leq A(t) \leq V_{ho}(t) - V_{hn}(t). \end{aligned}$$

Combining all three, the following inequality has to hold for a well-defined bargaining problem:

$$V_{lo}(t) - V_{ln}(t) \leq V_{ho}(t) - V_{hn}(t), \quad \forall t \geq 0. \quad (6.1)$$

This inequality states that the reservation value of a potential buyer must be equal to or higher than the reservation value of a potential seller, since high-type agents attribute a higher value to the flow of dividends.

6.2 Bargaining in the Basic Model

For the basic model, I start with inequality (6.1) and define the differences $\Delta V_n(t) = V_{hn}(t) - V_{ln}(t)$ and $\Delta V_o(t) = V_{ho}(t) - V_{lo}(t)$. The decisive constraint (6.1) reads as follows: $\Delta V_n(t) \leq \Delta V_o(t)$. Inserting equations (3.25)–(3.28), (3.36), (3.44), and (3.46), I state the ODEs of $\Delta V_n(t)$ and $\Delta V_o(t)$ with

$$\begin{aligned} \Delta \dot{V}_o(t) &= (r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \Delta V_o(t) \\ &\quad - (2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \Delta V_n(t) - \delta, \\ \Delta \dot{V}_n(t) &= - (2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \Delta V_o(t) \\ &\quad + (r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))) \Delta V_n(t). \end{aligned}$$

Combining both, a time-varying system of differential equations results with

$$\begin{aligned} \begin{bmatrix} \Delta \dot{V}_o(t) \\ \Delta \dot{V}_n(t) \end{bmatrix} &= \begin{bmatrix} \left(\begin{array}{c} r + \lambda_d + \lambda_u \\ + 2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t) \end{array} \right) & - \left(\begin{array}{c} 2\lambda\mu_{hn}(t)q \\ + \rho(1-z)\tilde{q}(t) \end{array} \right) \\ - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) \\ + \rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & \left(\begin{array}{c} r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(t)(1-q) \\ + \rho(1-z)(1-\tilde{q}(t)) \end{array} \right) \end{bmatrix} \begin{bmatrix} \Delta V_o(t) \\ \Delta V_n(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} \delta \\ 0 \end{bmatrix}. \end{aligned} \quad (6.2)$$

Based on the derivation stated in appendix 6A, constraint (6.1) is converted into

$$\delta \int_t^{\infty} e^{-\int_t^x (r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(\tau)(1-q) + 2\lambda\mu_{hm}(\tau)q + \rho(1-z)) d\tau} dx \geq 0, \quad (6.3)$$

for all t . There are always gains from trade, since $\delta > 0$ holds by assumption, i.e. the side condition (6.1) is met and the bargaining problem is well defined. As correctly assumed in chapter 3, a meeting between agents always induces a trade.

6.3 Bargaining with Aggregate Liquidity Shocks

The situation with shocks is different: Agents anticipate that a shock can suddenly change the asset valuation of all agents and such a shock can occur repeatedly. High-type agents are negatively affected by an aggregate liquidity shock, whereas low-type agents benefit. After a shock, trading conditions are time-dependent, since agents' type distribution evolves according to a system of differential equations. A potential buyer must take into account that the trading situation may be worse by the time he eventually wants to sell the asset. This poor trading situation can be the case either due to a low fraction of buyers at that time or due to a sudden selling pressure, or, at worst, both. Agents factor these illiquidity risks into the prices they demand and offer.

To verify the existence of a Nash bargaining solution in the aggregate liquidity shock model, I use again the definitions $\Delta V_n^s(t) = V_{hn}^s(t) - V_{ln}^s(t)$ and $\Delta V_o^s(t) = V_{ho}^s(t) - V_{lo}^s(t)$. Combining both $\Delta V_n^s(t)$ and $\Delta V_o^s(t)$, a linear time-varying system of differential equations is obtained with

$$\begin{aligned} \begin{bmatrix} \Delta \dot{V}_o^s(t) \\ \Delta \dot{V}_n^s(t) \end{bmatrix} &= \begin{bmatrix} \begin{pmatrix} r + \zeta + \lambda_d + \lambda_u \\ + 2\lambda\mu_{hm}(t)q + \rho(1-z)\tilde{q}(t) \end{pmatrix} & - \begin{pmatrix} 2\lambda\mu_{hm}(t)q \\ + \rho(1-z)\tilde{q}(t) \end{pmatrix} \\ - \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ + \rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} r + \zeta + \lambda_d + \lambda_u \\ + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} \end{bmatrix} \begin{bmatrix} \Delta V_o^s(t) \\ \Delta V_n^s(t) \end{bmatrix} \\ &- \begin{bmatrix} \delta \\ 0 \end{bmatrix} - \begin{bmatrix} \zeta(1 - \pi_{ho}(t)) & 0 \\ 0 & \zeta(1 - \pi_{hn}(t)) \end{bmatrix} \begin{bmatrix} \Delta V_o^s(0) \\ \Delta V_n^s(0) \end{bmatrix}. \end{aligned} \quad (6.4)$$

Appendix 6B derives a solution to the system (6.4). Based on this solution, I

examine the inequality (6.1). The inequality's validity ensures that agents trade voluntarily.

Gains from Trade Immediately after the Shock at $t = 0$

First, I analyze the inequality $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$, which is

$$\begin{aligned} \frac{\delta \zeta}{\Psi} & \left\{ \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (\pi_{hn}(x) - \pi_{ho}(x)) dx \right] \left[\int_0^\infty e^{-\bar{\lambda}_1^s x} (1 + \bar{q}_2(x)) dx \right] \right. \\ & + \left. \left[\frac{1}{\zeta} - \int_0^\infty e^{-\bar{\lambda}_1^s x} [(1 - \pi_{ho}(x))(1 + \bar{q}_2(x)) - \bar{q}_2(x)(1 - \pi_{hn}(x))] dx \right] \times \right. \\ & \left. \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} dx \right] \right\} \\ & \geq 0, \end{aligned} \quad (6.5)$$

with

$$\begin{aligned} \Psi = \zeta^2 & \left\{ \left[\frac{1}{\zeta} - \int_0^\infty e^{-\bar{\lambda}_1^s x} (1 - \pi_{ho}(x))(1 + \bar{q}_2(x)) dx \right] \times \right. \\ & \left[\frac{1}{\zeta} - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{hn}(x)) dx \right] \\ & + \left[\int_0^\infty e^{-\bar{\lambda}_1^s x} \bar{q}_2(x) (1 - \pi_{hn}(x)) dx \right] \times \\ & \left. \left[\frac{1}{\zeta} - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{ho}(x)) dx \right] \right\}, \end{aligned} \quad (6.6)$$

and

$$\bar{q}_2(t) = - \int_0^t e^{-\int_x^t (2\lambda\mu_{hn}(\tau)q + 2\lambda\mu_{lo}(\tau)(1-q) + \rho(1-z)) d\tau} (2\lambda\mu_{hn}(x)q + \rho(1-z)\bar{q}(x)) dx, \quad (6.7)$$

$$\bar{\lambda}_1^s = r + \zeta + \lambda_u + \lambda_d, \quad (6.8)$$

$$\bar{\lambda}_2^s(t) = r + \zeta + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z). \quad (6.9)$$

If the inequality $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$ holds, trade immediately after the shock is valuable since there are gains from trade.

Gains from Trade for Time $t > 0$

However, meeting the condition $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$ immediately after the shock does not ensure that $\Delta V_o^s(t) - \Delta V_n^s(t) \geq 0$ holds for all $t > 0$ as well. The condition for gains from trade for $t > 0$ is

$$\int_t^{\infty} e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [\delta + \zeta (1 - \pi_{ho}(x)) \Delta V_o^s(0) - \zeta (1 - \pi_{hn}(x)) \Delta V_n^s(0)] dx \geq 0. \quad (6.10)$$

Gains from Trade a Long Time after the Shock

The constraint a long time after an aggregate liquidity shock, given no additional shock has occurred, i.e. $\lim_{t \rightarrow \infty} (\Delta V_o^s(t) - \Delta V_n^s(t)) \geq 0$, is

$$\delta \geq \zeta [(1 - \pi_{hn}(ss)) \Delta V_n^s(0) - (1 - \pi_{ho}(ss)) \Delta V_o^s(0)]. \quad (6.11)$$

The analysis of inequalities (6.5), (6.10), and (6.11) is deferred to appendix 6C. This analysis shows that, in general, it is the function $\pi_{hn}(t)$ which provokes a market freeze. In the aggregate liquidity shock model, however, Duffie, Gârleanu, and Pedersen (2007) do not take this market freeze into account. Agents are forced to trade despite no gains from trade. I show the effect and related impacts of a forced trading in the following section by means of an example.

6.4 Example: No Gains from Trade and Forced Trading

The existence of a bargaining solution is best highlighted by a numerical example. To analyze the driving forces, I continue the example of chapter 5. The time it takes for the price to recover from an aggregate liquidity shock is influenced by several factors, as I have stated in section 5.2:

- (1.) Search frictions, determined by λ and ρ , which measure liquidity,

- (2.) agents' individual recovery time, denoted by λ_u ,
- (3.) the severity of the shock, determined by $\pi_{ho}(ss)$ and $\pi_{hm}(ss)$.

The interaction of all three parameters—condensed within $\pi_{ho}(t)$ and $\pi_{hm}(t)$ —combined particularly with the possibility of further shocks—expressed with ζ —determines whether the market in this model recovers or whether there are no gains from trade. These effects are analyzed in the following passage.

Meeting Intensities λ and ρ

The meeting intensities λ and ρ are the input parameters initiating a situation with no gains from trade in the aggregate liquidity shock model.

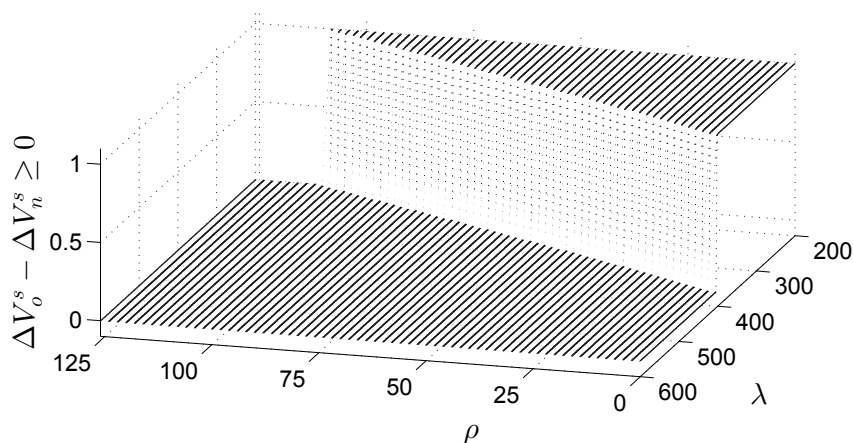


Figure 6.1: Region for gains from trade for $\lambda \in [200, 600]$ and $\rho \in [0, 125]$.

Figure 6.1 plots the observation of gains from trade for various combinations of λ and ρ . If $\Delta V_o^s(t) - \Delta V_n^s(t) \geq 0$ is valid for all $t \geq 0$, the value 1 is assigned. The value 1 reflects a well-defined bargaining problem and gains from trade. Otherwise, the value 0 is set, i.e. no gains from trade. Figure 6.1 shows that a bargaining solution does not exist for high meeting intensities λ and ρ . There is a sharp edge where an increase in ρ must come along with a decrease in λ , so that the condition (6.1) is met further on. Under fairly normal conditions, increasing meeting intensities reduces search frictions and therefore reduces illiquidity. But within the aggregate liquidity shock model, high meeting intensities might result in no gains from trade. No gains from trade should cause a market freeze with no trading at all, i.e. infinite search frictions.

The shape over time of the difference $\Delta V_o^s(t) - \Delta V_n^s(t)$ is shown in figure 6.2. I keep market makers' meeting intensity constant and vary investors' meeting

intensity with $\lambda = [200, 600]$. The left panel is without market makers, i.e. it is equal to the example of Duffie, Gârleanu, and Pedersen (2007) with $\rho = 0$. The right panel is with market makers, i.e. $\rho = 125$. This figure demonstrates the presumption of the aggregate liquidity shock model that all meetings result in a trade—even if there are no gains from trade, i.e. agents are forced to trade. This forced trading is reflected by values below the black plane. However, the market should be frozen during this time—which is not considered within this model. From an economic perspective, this aggregate liquidity shock model is flawed.

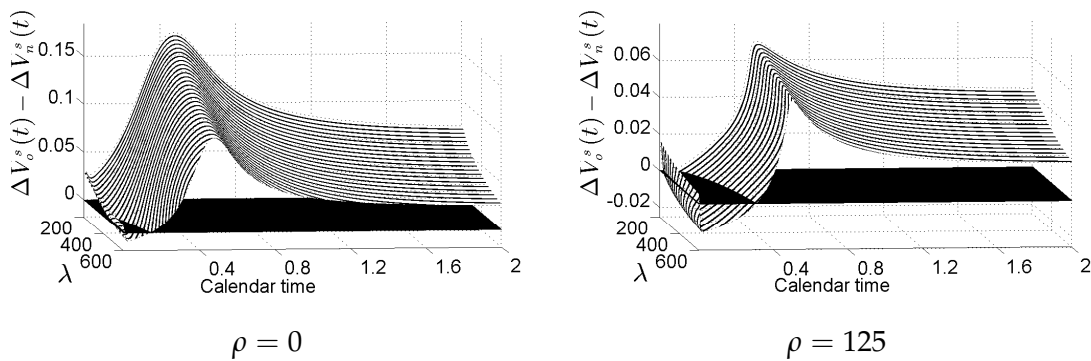


Figure 6.2: Difference $\Delta V_o^s(t) - \Delta V_n^s(t)$ for varying $\lambda \in [200, 600]$.

Figure 6.2 shows that situations of no gains from trade but forced trading prevail mainly during the time shortly after the shock, i.e. when the selling pressure is large, which is the case for $0 \leq t \leq t^*$. If agents are forced to trade despite no gains, the market would endogenously return to normal conditions, i.e. with gains from trade, for sufficiently small values for ζ . Equation (6.11) shows this effect, too.

Individual Recovery

If agents know that low intrinsic types do not have to stay in this unfavorable low state for a long time—that means λ_u is relatively high—then the market recovers quickly from an aggregate liquidity shock. Hence, a forced trading in the aggregate liquidity shock model can be avoided if investors have access to easy refunding conditions.

Severity of the Shock and the Risk of Further Shocks

The matching functions presume that all meetings actually result in a trade. These matching functions are contained in the flow equations of masses $\mu_\sigma(t)$,

which affect the evolution of probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$. With these probabilities, however, the term $\zeta (1 - \pi_{hn}(t)) \Delta V_n^s(0)$ of inequality (6.10) can temporarily outweigh the term $\zeta (1 - \pi_{ho}(t)) \Delta V_o^s(0)$. No gains from trade result.

The third crucial parameter I have announced above is the severity of a shock, represented by the probabilities of high agents switching to a low state due to the shock and based on steady state values, i.e. $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$. These probabilities control for the starting condition of agents' masses $\bar{\mu}_\sigma(0)$, as seen in equations (4.3) and (4.4). These starting conditions determine the evolution of masses over time. The probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$ in turn depend both on the starting conditions of agents' masses $\bar{\mu}_\sigma(0)$ and on the type distribution $\mu_\sigma(t)$ at time $t \geq 0$, as seen by equations (4.1) and (4.2).

Figure 6.3 shows the probability for a high owner switching to a low owner upon a shock, i.e. $\pi_{ho}(t)$ as a function of time and meeting intensity $\lambda \in [200, 600]$. This figure illustrates that the meeting intensity λ has an inferior effect on the evolution of $\pi_{ho}(t)$. In general, the type distribution of high owners $\mu_{ho}(t)$ monotonically increases after a shock, i.e. $\mu_{ho}(0) \leq \mu_{ho}(t)$. This effect can be seen in figure 5.5. The increase in the fraction of *ho* agents is due to an elevated quantity of misallocated assets right after the shock. This misallocation alleviates over time. The change in the fraction of high owners is mainly due to trade and up-shifts, while down-shifts are a secondary influence. Generally, the condition $0 \leq \pi_{ho}(t) \leq \pi_{ho}(ss)$ holds.

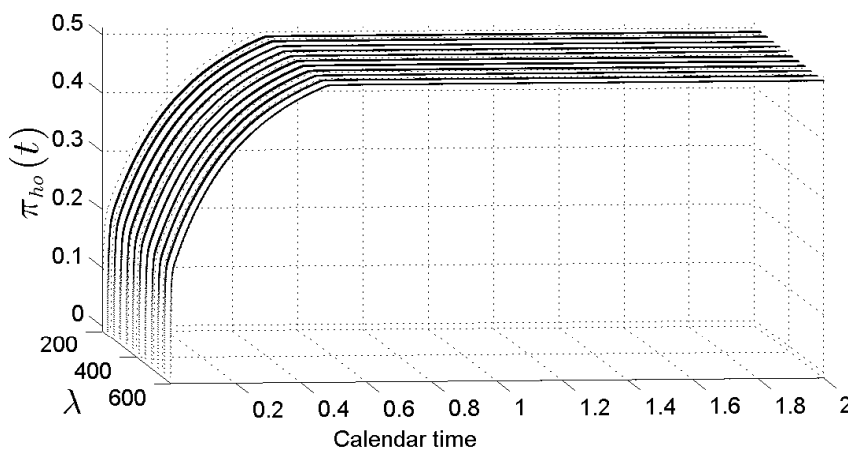


Figure 6.3: Evolution of $\pi_{ho}(t)$ with $\pi_{ho}(ss) = 0.5$ and $\rho = 125$.

Figure 6.4 shows the probability for a high non-owner switching to a low non-owner, i.e. $\pi_{hn}(t)$ as a function of time and meeting intensity $\lambda \in [200, 600]$. The pattern of the probability $\pi_{hn}(t)$ differs significantly from the pattern of the prob-

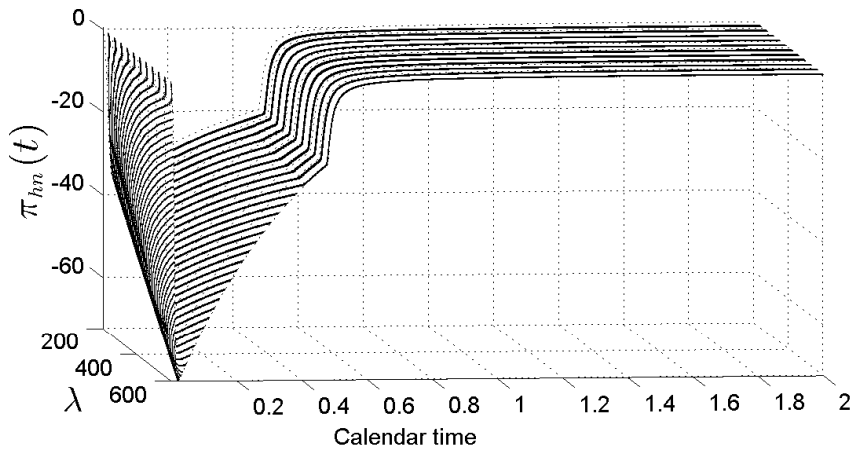


Figure 6.4: Evolution of $\pi_{hn}(t)$ with $\pi_{hn}(ss) = 0.5$ and $\rho = 125$.

ability $\pi_{ho}(t)$. For high values of λ (and ρ), the value $\pi_{hn}(t)$ drops sharply below zero shortly after the shock. As a result, $\pi_{hn}(t)$ loses its characteristic as a probability. The reason is as follows: The fraction of potential buyers $\mu_{hn}(t)$ further decreases immediately after a shock and the market becomes one-sided. The higher $\pi_{ho}(ss)$ and λ are and the lower $\pi_{hn}(ss)$ is, the more severe and the faster the reduction of potential buyers after the shock. This reduction is due to an immediate absorption of nearly all remaining potential buyers in this buyers' market, when search frictions are low, i.e. meeting intensities are high—and when all meetings actually result in a trade, even by force. The matching functions $\mathcal{M}_\lambda(t) = 2\lambda\mu_{hn}(t)\mu_{lo}(t)$ and $\mathcal{M}_\rho(t) = \rho \min\{\mu_{lo}(t), \mu_{hn}(t)\}$ precisely assume this ensured trade. With this, the fraction $\mu_{hn}(t)$ temporarily drops almost to zero and leads to $0 \geq \pi_{hn}(t)$. Thereafter, $\mu_{hn}(t)$ recovers and moves towards its long run level, given no further shock occurs. Due to this fact, the term $\zeta(1 - \pi_{hn}(t)) \Delta V_n^s(0)$ can temporarily outweigh the term $\zeta(1 - \pi_{ho}(t)) \Delta V_o^s(0)$ so that inequality (6.10) can become invalid for high meeting intensities λ (and ρ).

The next section presents some ideas for preventing the aggregate liquidity shock model from forcing agents to trade despite no gains from trade.

6.5 Trading Voluntarily

I suggest some modifications to meeting intensities and the characteristics of the shock for voluntary trading in the aggregate liquidity shock model. The aim is to control for the existence of a Nash bargaining solution. The first suggestion, a

temporarily frozen market with no trading (section 6.5.1), leads to frequent and short market closures as soon as gains from trade vanish. The market reopens immediately when there are gains from trade again. This would be the best model modification, although difficult to implement. The second suggestion assumes a severe crisis (section 6.5.2). It is already implemented in the literature, however the versatility of this model variant is limited. My suggestions are: First, in section 6.5.3, a complete trading halt which prevails until the selling pressure alleviates. Second, in section 6.5.4, the choice of an optimal search intensity so that there are always gains from trade. The numerical example of section 6.4 is continued in each section. I use the same parameters but increase meeting intensities to $\lambda = 500$ and $\rho = 250$ to trigger no gains from trade in the aggregate liquidity shock model. Other parameter modifications are stated in the particular sections.

6.5.1 Temporarily Frozen Market 1

Chiu and Koepl (2011, p. 8) suggest including a maximum-function into the value functions $V_\sigma^s(t)$, which implies the possibility of investors disagreeing, i.e. to not buy or sell the asset. Markets get temporarily frozen if there are no gains from trade in equilibrium. For this case, the meeting intensities between investors as well as between investors and market makers drop temporarily to zero, since agents do not trade: $\lambda(t) = 0$ and $\rho(t) = 0$. As soon as there are again gains from trade, meeting intensities return to their initial value. Equations (6.12) and (6.13) describe these effects analytically with

$$\lambda(t) = \begin{cases} 0 & \text{for } \Delta V_l^s(t) > \Delta V_h^s(t) \\ \lambda & \text{for } \Delta V_l^s(t) \leq \Delta V_h^s(t) \end{cases} \quad (6.12)$$

and

$$\rho(t) = \begin{cases} 0 & \text{for } \Delta V_l^s(t) > \Delta V_h^s(t) \\ \rho & \text{for } \Delta V_l^s(t) \leq \Delta V_h^s(t), \end{cases} \quad (6.13)$$

for all $t \geq 0$. This suggestion seems to be the most elegant one but implementation is difficult, since masses of investor types $\mu_\sigma(t)$ depend on the meeting intensities λ and ρ , which in turn are determined by $V_\sigma^s(t)$. The feedback from $V_\sigma^s(t)$ into the type process $\mu_\sigma(t)$ is not provided by the Duffie, Gârleanu, and Pedersen (2005, 2007) models. As a result, either the type process would pretend further on that all meetings result in a trade, or this model is intractable.

6.5.2 Severe Crisis

Feldhütter (2012, pp. 1197–1202), and Weill (2007, 2011) assume that all high-type agents simultaneously suffer a preference shock and switch to a low state upon an aggregate liquidity shock. These assumptions account for considerable model restrictions: Right after the shock, there are no potential buyers at all. This form of aggregate liquidity shock leads to a severe crisis with $\pi_{ho}(ss) = \pi_{hn}(ss) = 1$. Immediately after the shock, the fraction of high-type agents is always zero: $\bar{\mu}_{ho}(0) = \bar{\mu}_{hn}(0) = 0$, whereas $\mu_{lo}(0) = s$, $\mu_{ln}(0) = 1 - s$. As a result, this post-shock type distribution implies constant probabilities $\pi_{ho}(t) = \pi_{hn}(t) = 1$ for all time $t \geq 0$. Integrating this result into inequalities (6.11), (6.10), and (6.5) leads to the single bargaining condition

$$\delta \int_t^{\infty} e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} dx \geq 0,$$

which is valid since $\delta > 0$ holds by assumption. Feldhütter (2012) therefore avoids the issue of invalid bargaining conditions, though he does not address it. Weill (2007, 2011) considers only a single aggregate liquidity shock, while the issue of no gains from trade does not arise at all.

Example

The severity of this kind of shock leads to a long lasting crisis. It takes about 0.79 years for the selling pressure to alleviate—compared to 0.48 years in the setting so far. Figure 6.5 illustrates the effect on mass dynamics.

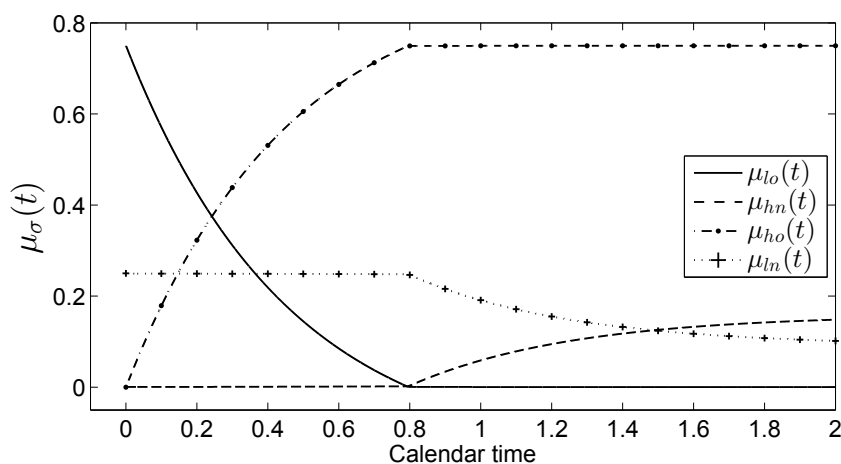


Figure 6.5: Process of masses distribution after a severe shock.

Both the fraction of high owners and the fraction of potential buyers are zero immediately after the shock. The fraction of potential buyers remains close to zero, since all ln agents who experience a preference shock are quickly matched with sellers. After 0.79 years, the market returns to a seller's market.

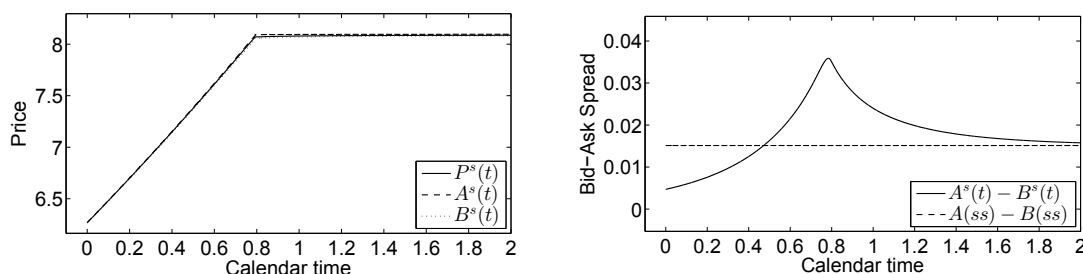


Figure 6.6: Price process (left panel) and bid-ask spread (right panel) after a severe aggregate liquidity shock.

The severity of this crisis can be seen in figure 6.6. The left panel shows the price process after a severe aggregate liquidity shock. Prices are considerably lower, compared to the shock with a fifty-fifty chance for high-type agents mutating to a low-type—which I have utilized so far. This effect is clear when comparing figure 5.10 (page 123) with the left panel in figure 6.6. Although meeting intensities are two to four times higher in figure 6.6 than in figure 5.10, prices are considerably lower right after the shock and a long time after it. This price reduction is due to the severity of this shock. Half of the loss in price is not regained until after 0.41 years. However, prices reach a fairly normal level after 1.05 years.¹⁴⁰

The panel to the right of figure 6.6 shows the development of the bid-ask spread after a severe aggregate liquidity shock. The bid-ask spread is positive, since there are always gains from trade.¹⁴¹

6.5.3 Temporarily Frozen Market 2

Afonso (2011, p. 340) suggests “trading halts [...] to slow down trading” as one feasible reaction to a one-sided market. Longstaff (2009) considers a temporarily frozen period that he calls ‘blackout’, during which the illiquid asset cannot be traded.¹⁴² I implement both concepts into the aggregate liquidity shock model

¹⁴⁰ After 1.05 years, the percentage price change is less than 0.001% per day.

¹⁴¹ There is a special case for this setting: Assume severe aggregate liquidity shocks and trade is only intermediated by market makers, i.e. $\lambda = 0$. Then, the impact on the bid-ask spread is negligible. It is only a very small parallel shift, compared to a setting without any aggregate shocks. Feldhütter (2010, p. 17) states a comparable result.

¹⁴² This blackout period is the only source of illiquidity in the model of Longstaff (2009).

in the following way: Assume that the market is characterized by a seller's market a long time after the shock, i.e. $\mu_{lo}(ss) < \mu_{hn}(ss)$. Assume further that the shock leads to a buyer's market, which implies $\bar{\mu}_{lo}(0) > \bar{\mu}_{hn}(0)$. There is a single intersection time t^* , for which $\mu_{lo}(t^*) = \mu_{hn}(t^*)$ with $0 < t^* < \infty$ holds. I suggest a trading stop after the shock if there are no gains from trade. At time t^* , this is when the market returns to a seller's market again with $\mu_{lo}(t) < \mu_{hn}(t)$ for $0 < t^* < t$, trade reopens and continues with the constant intensities λ and ρ . Summarizing all presumptions leads to two definitions for the meeting intensities λ and ρ , which are

$$\lambda(t) = \begin{cases} 0 & \text{for } 0 \leq t < t^* \\ \lambda & \text{for } t^* \leq t \end{cases} \quad (6.14)$$

and

$$\rho(t) = \begin{cases} 0 & \text{for } 0 \leq t < t^* \\ \rho & \text{for } t^* \leq t. \end{cases} \quad (6.15)$$

The aggregate liquidity shock leads to an elevated amount of potential sellers and a reduced amount of potential buyers—each compared to its steady state. The only possibility of a type change during a trading halt is an idiosyncratic preference shock with intensities λ_u and λ_d . As a result, the remaining mass of potential buyers does not soak up the surplus of potential sellers, because trade is impossible. Instead, the mass of potential buyers increases monotonically and the mass of potential sellers decreases monotonically during the frozen period.

The proof for the latter proposition is as follows: The mass dynamics of $\mu_{hn}^f(t)$ and $\mu_{lo}^f(t)$ are

$$\dot{\mu}_{lo}^f(t) = -[\lambda_u + \lambda_d]\mu_{lo}^f(t) + \lambda_d s, \quad \text{for } 0 \leq t \leq t^*, \quad (6.16)$$

$$\dot{\mu}_{hn}^f(t) = -[\lambda_u + \lambda_d]\mu_{hn}^f(t) + \lambda_u(1 - s), \quad \text{for } 0 \leq t \leq t^*, \quad (6.17)$$

where superscript ' f ' assigns the ODEs (6.16) and (6.17) to the frozen time period $0 \leq t \leq t^*$. The solutions to equations (6.16) and (6.17) are

$$\mu_{lo}^f(t) = \bar{\mu}_{lo}(0)e^{-(\lambda_u + \lambda_d)t} + \frac{\lambda_d s}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u + \lambda_d)t} \right], \quad \text{for } 0 \leq t \leq t^*, \quad (6.18)$$

$$\mu_{hn}^f(t) = \bar{\mu}_{hn}(0)e^{-(\lambda_u + \lambda_d)t} + \frac{\lambda_u(1 - s)}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u + \lambda_d)t} \right], \quad \text{for } 0 \leq t \leq t^*. \quad (6.19)$$

It is clear from equations (6.18) and (6.19) that $\mu_{lo}^f(t)$ and $\mu_{hn}^f(t)$ converge monotonically. Since the relation $\mu_{lo}(ss) < \mu_{hn}(ss)$ and $\bar{\mu}_{lo}(0) > \bar{\mu}_{hn}(0)$ hold by assumption, $\bar{\mu}_{lo}(0) > \mu_{lo}^f(t^*) = \mu_{hn}^f(t^*) > \bar{\mu}_{hn}(0)$ must then hold as well. This connection completes the proof.

As soon as $\mu_{lo}(t^*) = \mu_{hn}(t^*)$, the market reopens again. At this time, there are equally as many potential buyers as potential sellers in the market. Nearly all potential buyers and potential sellers are rapidly matched to each other, since meeting intensities are relatively high by assumption. The fraction of potential buyers are not close to zero at time t^* . With reference to equations (3.4) and (3.5), type changes due to trading generally dominate type switches then. As a result, a sharp drop of fractions $\mu_{lo}(t)$ and $\mu_{hn}(t)$ towards zero can be observed immediately after time t^* . Thereafter, masses start to converge monotonically to their long run values.

The evolution of probability $\pi_{hn}(t)$ reacts on the altered type evolution of potential buyers discussed above. Since $\mu_{hn}^f(t)$ is increasing until time t^* , the probability $\pi_{hn}(t)$ is greater than zero for $0 \leq t \leq t^*$. Function $\pi_{hn}(t)$ drops sharply below zero right after time t^* . But this time, it is only a short peak, since $\mu_{hn}(t)$ immediately starts to increase towards its steady state level after the sharp drop. Nevertheless, the higher the meeting intensities λ and ρ are, the more intense this peak is, because trade takes place quickly.

Example

Figure 6.7 illustrates the effect of this modification on mass dynamics.

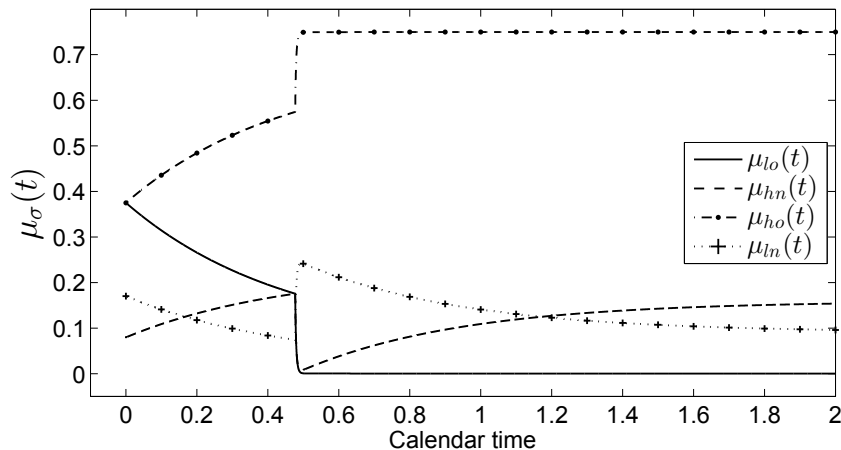


Figure 6.7: Process of mass distribution in a temporarily frozen market.

The pattern is as explained above: The fraction of potential buyers (sellers) increases (decreases) until the intersection time $t^* = 0.48$. This intersection time must be the same time as in chapter 5.2, since λ_u , λ_d , $\bar{\mu}_{ho}(0)$, and $\bar{\mu}_{hn}(0)$ are not affected by the trading halt. At time t^* , the market reopens and buyers and sellers are matched quickly. Both fractions drop sharply and finally start to converge towards their long run level.

Prices are shown in the left panel of figure 6.8. No trade takes place during the first 0.48 years after the shock. There are no prices at all. At time t^* , the market reopens and market participants bargain prices. The prices a long time after the shock are only 4.14% lower than steady state prices without aggregate liquidity shocks.¹⁴³

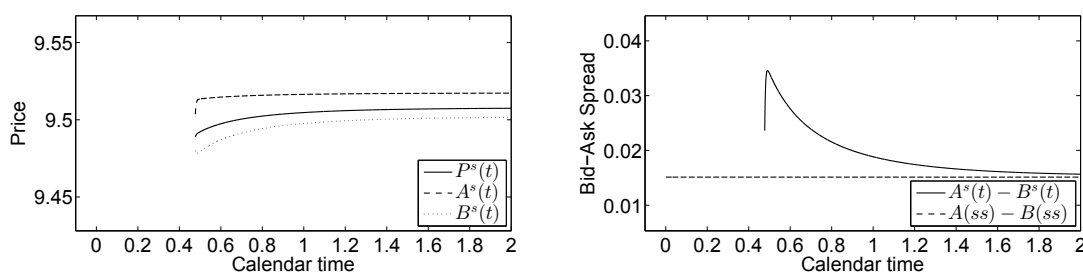


Figure 6.8: Price process (left panel) and bid-ask spread (right panel) in a temporarily frozen market.

The right panel of figure 6.8 depicts the bid-ask spread. There are no bid-ask spreads during the first 0.48 years after the shock as well. The remaining part is similar to figure 5.11 but with higher meeting intensities.

Remark

One drawback of a market freeze is, as Camargo, Kim, and Lester (2013, p. 2) emphasize, that the price discovery process is interrupted. Trading prices do not only indicate the value of an asset for particular buyers and sellers at a particular time but also imply information for other agents. A recent example from the 2007 financial crisis is the impossibility of fair valuation of assets in three BNP Paribas investment funds. BNP Paribas temporarily suspended money withdrawals from these funds, as stated on August 9, 2007: “For some of the securities there are just no prices [...] As there are no prices, we can’t calculate the value of the funds.”¹⁴⁴

¹⁴³ The steady state price without aggregate liquidity shocks is $P(ss) = 9.9183$, compared to $P^s(ss) = 9.5077$ within this modification.

¹⁴⁴ Alain Papiasse, head of BNP Paribas’s asset management and services division. See Boyd (2007) and Camargo, Kim, and Lester (2013), p. 2.

Therefore, a temporary market freeze is a suboptimal solution. The next section offers another possibility of controlling for gains from trade in the aggregate liquidity shock model.

6.5.4 Optimal Search Intensity

The aggregate liquidity shock model considers repeated shocks that occur once in a while with the Poisson arrival rate ζ . However, agents are aware of the threat of no gains from trade due to a shock. They anticipate the forced trading by reducing meeting intensities in advance and permanently. This construction is in line with the interpretation of “search intensities as based on a technology that is difficult to change”—as suggested by Duffie, Gârleanu, and Pedersen (2005, p. 1831). Agents drive meeting intensities down to an optimal level. This means they choose their constant meeting intensities λ and ρ to just meet the condition (6.1) at all points in time. However, different combinations of λ and ρ , which all just meet this condition, are possible. Solutions can be calculated only iteratively.

Example

Suppose the meeting intensities are as given above: $\lambda = 500$ and $\rho = 250$. These are the starting points. First, the validity of condition (6.1) is checked by inserting these meeting intensities. If condition (6.1) is invalid, then λ and/or ρ must be changed. I suggest a parallel shift in λ and ρ so that both λ and ρ are reduced by one. The validity of condition (6.1) is rechecked with the ‘new’ intensities. If it is valid, the iteration process stops. Otherwise, both λ and ρ are reduced by one again—and so on. Of course, if condition (6.1) is valid with the initial parameters, it is also possible to increase λ and/or ρ until condition (6.1) is just met. The optimal meeting intensities for the example given are $\lambda = 302$ and $\rho = 52$.

Figure 6.9 shows that the bid-ask spread is positive, which implies that trading is voluntary. The overall pattern is equivalent to the bid-ask spread in figure 5.11, since this modification has the least impact on the aggregate liquidity shock model. Nevertheless, $\pi_{lm}(t)$ still decreases below zero.

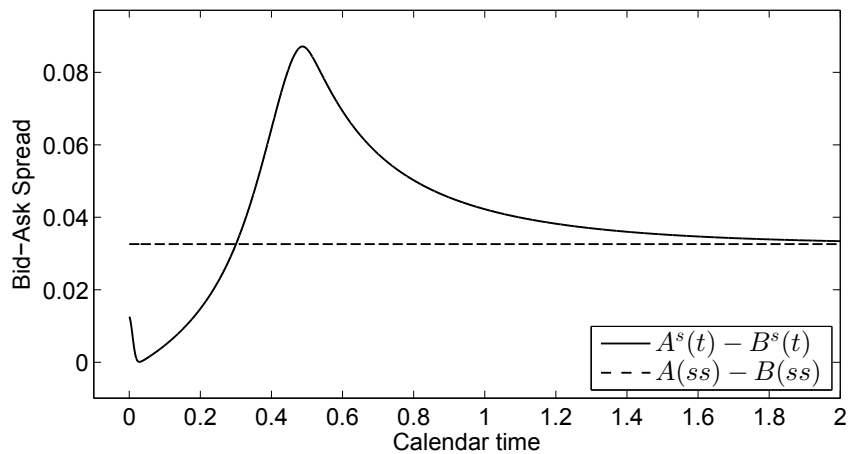


Figure 6.9: Bid-ask spread with optimal search intensities ($\lambda = 302, \rho = 52$).

6.6 Conclusion

Based on the analysis in this chapter it is not ensured for any feasible input variable that there are always gains from trade in the aggregate liquidity shock model. A Nash bargaining solution does not exist at all times. Or, stated differently, sometimes there is nothing on the table worth agreeing on. However, the aggregate liquidity shock model of Duffie, Gârleanu, and Pedersen (2007) pretends that agents would trade despite no gains from trade. Hence, agents are forced to trade when meeting a trading partner.

I suggest some modifications to the meeting intensities, as well as to the characteristics of the shock, as a means of addressing the issue of agents trading voluntarily in the aggregate liquidity shock model.

6A Appendix: Bargaining Constraint of the Basic Model

In this appendix, I derive equation (6.3) from system (6.2) in connection with constraint (6.1). First, I obtain the solution to the LTV system (6.2) by means of the solution technique presented in appendix 4A. The starting point is equation (6.2), which I restate with

$$\Delta \dot{\mathbf{V}}_{\sigma}(t) = \mathbf{B}_1(\mu(t))\Delta \mathbf{V}_{\sigma}(t) - \mathbf{B}_2, \quad (6.20)$$

where

$$\mathbf{B}_1(\mu(t)) = \begin{bmatrix} \begin{pmatrix} r+\lambda_d+\lambda_u \\ +2\lambda\mu_{hn}(t)q+\rho(1-z)\tilde{q}(t) \end{pmatrix} & - \begin{pmatrix} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{pmatrix} \\ - \begin{pmatrix} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} & \begin{pmatrix} r+\lambda_d+\lambda_u \\ 2\lambda\mu_{lo}(t)(1-q)+\rho(1-z)(1-\tilde{q}(t)) \end{pmatrix} \end{bmatrix}, \quad (6.21)$$

and

$$\mathbf{B}_2 = \begin{bmatrix} \delta \\ 0 \end{bmatrix}. \quad (6.22)$$

Assume, there exists a time-varying coordinate transformation $\bar{\mathbf{T}}(t)$ with

$$\Delta \mathbf{V}_{\sigma}(t) = \bar{\mathbf{T}}(t)\Delta \bar{\mathbf{V}}_{\sigma}(t),$$

which transforms equation (6.20) into

$$\Delta \dot{\bar{\mathbf{V}}}_{\sigma}(t) = \bar{\mathbf{\Lambda}}(\mu(t))\Delta \bar{\mathbf{V}}_{\sigma}(t) - \bar{\mathbf{B}}_2(t),$$

where

$$\begin{aligned} \bar{\mathbf{\Lambda}}(\mu(t)) &= \bar{\mathbf{T}}(t)^{-1} \mathbf{B}_1(\mu(t)) \bar{\mathbf{T}}(t) - \bar{\mathbf{T}}(t)^{-1} \dot{\bar{\mathbf{T}}}(t), \\ \bar{\mathbf{B}}_2(t) &= \bar{\mathbf{T}}(t)^{-1} \mathbf{B}_2. \end{aligned}$$

With the transformation method introduced in appendix 4A.5, a solution for the matrices $\bar{\mathbf{T}}(t)$ and $\bar{\mathbf{\Lambda}}(\mu(t))$ can straight forwardly be derived. Starting point is the homogeneous equation to (6.20) with

$$\Delta \dot{\mathbf{V}}_{\sigma}(t) = \mathbf{B}_1(\mu(t))\Delta \mathbf{V}_{\sigma}(t).$$

Only one transformation round must be performed, since system (6.20) is of dimension two. The steps are as follows:

1. Partition matrix $\mathbf{B}_1(\mu(t))$:

$$\mathbf{B}_1(\mu(t)) = \begin{bmatrix} \mathbf{B}_{11}^{(2)}(t) & \mathbf{b}_{12}^{(2)}(t) \\ [\mathbf{b}_{21}^{(2)}(t)]^T & b_{22}^{(2)}(t) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{B}_{11}^{(2)}(t) &= r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t), \\ \mathbf{b}_{12}^{(2)}(t) &= -(2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)), \\ \mathbf{b}_{21}^{(2)}(t) &= -(2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))), \\ b_{22}^{(2)}(t) &= r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t)). \end{aligned}$$

Since matrix $\mathbf{B}_1(\mu(t))$ is of dimension $\mathbb{R}^{2 \times 2}$, $\mathbf{B}_{11}^{(2)}(t)$, $\mathbf{b}_{12}^{(2)}(t)$, $\mathbf{b}_{21}^{(2)}(t)$, and $b_{22}^{(2)}(t)$ are all scalars.

2. Find any solution $\bar{p}_2(t)$ to the Riccati differential equation:

$$\begin{aligned} \dot{\bar{p}}_2(t) &= -\bar{p}_2(t) [-2\lambda\mu_{hn}(t)q - \rho(1-z)\tilde{q}(t)] \bar{p}_2(t) \\ &\quad - [r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)] \cdot \bar{p}_2(t) \\ &\quad + \bar{p}_2(t) [r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))] \\ &\quad - [2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t))]. \end{aligned}$$

Since any solution to this equation is suitable, it can be shown that $\bar{p}_2(t) = 1$ is a particular solution to the Riccati differential equation.

3. Calculate the dynamic eigenvalue $\bar{\lambda}_2(t)$:

$$\begin{aligned} \bar{\lambda}_2(t) &= r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)(1-\tilde{q}(t)) \\ &\quad - \bar{p}_2(t) (-2\lambda\mu_{hn}(t)q - \rho(1-z)\tilde{q}(t)), \\ &= r + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z). \end{aligned}$$

4. Construct the first transformation matrix $\bar{\mathbf{P}}_2(t)$:

$$\bar{\mathbf{P}}_2(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

5. Construct the second transformation matrix $\bar{\mathbf{Q}}_2(t)$:

$$\begin{aligned}\dot{\bar{q}}_2(t) &= \{r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t) \\ &\quad - [2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)]\bar{p}_2(t) \\ &\quad - [r + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)]\bar{q}_2(t) \\ &\quad - (2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)) \\ &= - [2\lambda\mu_{hn}(t)q + \rho(1-z) + 2\lambda\mu_{lo}(t)(1-q)]\bar{q}_2(t) \\ &\quad - [2\lambda\mu_{hn}(t)q + \rho(1-z)\tilde{q}(t)].\end{aligned}$$

Since any solution to this equation is suitable, a feasible solution is:

$$\bar{q}_2(t) = - \int_0^t e^{-\int_x^t (2\lambda\mu_{hn}(\tau)q + 2\lambda\mu_{lo}(\tau)(1-q) + \rho(1-z)) d\tau} (2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x)) dx. \quad (6.23)$$

Hence,

$$\bar{\mathbf{Q}}_2(t) = \begin{bmatrix} 1 & \bar{q}_2(t) \\ 0 & 1 \end{bmatrix}.$$

6. Calculate the dynamic eigenvalue $\bar{\lambda}_1(t)$:

$$\begin{aligned}\bar{\lambda}_1(t) &= \text{trace}(\mathbf{B}_1(\mu(t))) \\ &\quad - [r + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z)], \\ &= r + \lambda_u + \lambda_d.\end{aligned}$$

7. Set up the fundamental matrix $\bar{\mathbf{T}}(t)$:

$$\begin{aligned}\bar{\mathbf{T}}(t) &= \bar{\mathbf{P}}_2(t)\bar{\mathbf{Q}}_2(t), \\ &= \begin{bmatrix} 1 & \bar{q}_2(t) \\ 1 & \bar{q}_2(t) + 1 \end{bmatrix},\end{aligned} \quad (6.24)$$

and the inverse $\bar{\mathbf{T}}(t)^{-1}$:

$$\bar{\mathbf{T}}(t)^{-1} = \begin{bmatrix} 1 + \bar{q}_2(t) & -\bar{q}_2(t) \\ -1 & 1 \end{bmatrix}.$$

8. Calculate the transformed matrix $\bar{\Lambda}(\mu(t))$ and vector $\mathbf{B}_2(t)$:

$$\bar{\Lambda}(\mu(t)) = \begin{bmatrix} \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_2(t) \end{bmatrix}, \quad (6.25)$$

$$\bar{\mathbf{B}}_2(t) = \begin{bmatrix} \delta(1 + \bar{q}_2(t)) \\ -\delta \end{bmatrix}. \quad (6.26)$$

The transformed system of equation (6.2) is stated with

$$\begin{bmatrix} \Delta \dot{\bar{V}}_o(t) \\ \Delta \dot{\bar{V}}_n(t) \end{bmatrix} = \begin{bmatrix} r + \lambda_d + \lambda_u & 0 \\ 0 & \begin{pmatrix} r + \lambda_d + \lambda_u + 2\lambda\mu_{hn}(t) \\ + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z) \end{pmatrix} \end{bmatrix} \begin{bmatrix} \Delta \bar{V}_o(t) \\ \Delta \bar{V}_n(t) \end{bmatrix} - \begin{bmatrix} \delta(1 + \bar{q}_2(t)) \\ -\delta \end{bmatrix}. \quad (6.27)$$

The transformed system in equation (6.27) has the desired properties:

1. Matrix $\bar{\Lambda}(\mu(t))$ is a diagonal matrix, which decouples the whole system.
2. The dynamic eigenvalues $\bar{\lambda}_i$ ($i = 1, 2$) show up on the main diagonal.

Inverse transformation leads to

$$\Delta V_\sigma(t) = \int_t^\infty \bar{\mathbf{T}}(t) e^{-\int_t^x \bar{\Lambda}(\mu(\tau)) d\tau} \bar{\mathbf{T}}(x)^{-1} \mathbf{B}_2(x) dx,$$

$$\Delta V_o(t) = \int_t^\infty \delta e^{-\int_t^x \lambda_1 d\tau} (1 + \bar{q}_2(x)) dx - \bar{q}_2(t) \int_t^\infty \delta e^{-\int_t^x \lambda_2(\mu(\tau)) d\tau} dx, \quad (6.28)$$

$$\Delta V_n(t) = \int_t^\infty \delta e^{-\int_t^x \lambda_1 d\tau} (1 + \bar{q}_2(x)) dx - (1 + \bar{q}_2(t)) \int_t^\infty \delta e^{-\int_t^x \lambda_2(\mu(\tau)) d\tau} dx. \quad (6.29)$$

The inequality

$$\Delta V_n(t) \leq \Delta V_o(t)$$

is analyzed by means of equations (6.28) and (6.29), which imply

$$0 \leq \delta \int_t^{\infty} e^{-\int_t^x (r + \lambda_d + \lambda_u + 2\lambda\mu_{lo}(\tau)(1-q) + 2\lambda\mu_{hm}(\tau)q + \rho(1-z)) d\tau} dx.$$

This inequality is equal to the inequality (6.3) stated in section 6.2. It holds by assumption, since

$$0 < \delta.$$

6B Appendix: Bargaining Constraint of the Liquidity Shock Model

In this appendix, I examine if there are always gains from trade in the aggregate liquidity shock model. First, a solution for the LTV system defined in (6.4) is derived. The approach is equivalent to appendix 6A but applied to aggregate liquidity shocks. The algebraic transformations coincide, to a large extent, which is why I keep the presentation brief at the points of congruency. The point of departure is

$$\Delta \dot{\mathbf{V}}_{\sigma}^s(t) = \mathbf{B}_1^s(\mu(t)) \Delta \mathbf{V}_{\sigma}^s(t) - \mathbf{B}_2^s - \mathbf{B}_3^s(\mu(t)) \quad (6.30)$$

with

$$\mathbf{B}_1^s(\mu(t)) = \begin{bmatrix} \left(\begin{array}{c} r+\zeta+\lambda_d+\lambda_u \\ +2\lambda\mu_{hn}(t)q+\rho(1-z)\tilde{q}(t) \end{array} \right) & - \left(\begin{array}{c} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{array} \right) \\ - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & \left(\begin{array}{c} r+\zeta+\lambda_d+\lambda_u \\ 2\lambda\mu_{lo}(t)(1-q)+\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) \end{bmatrix},$$

$\mathbf{B}_2^s = \mathbf{B}_2$, as defined in (6.22), and

$$\mathbf{B}_3^s(\mu(t)) = \begin{bmatrix} \zeta(1 - \pi_{ho}(t)) & 0 \\ 0 & \zeta(1 - \pi_{hn}(t)) \end{bmatrix}.$$

Assume again there exists a time-varying algebraic coordinate transformation $\bar{\mathbf{T}}^s(t)$ with

$$\Delta \mathbf{V}_{\sigma}^s(t) = \bar{\mathbf{T}}^s(t) \Delta \bar{\mathbf{V}}_{\sigma}^s(t)$$

which transforms equation (6.30) into

$$\Delta \dot{\bar{\mathbf{V}}}_{\sigma}^s(t) = \bar{\mathbf{\Lambda}}^s(\mu(t)) \Delta \bar{\mathbf{V}}_{\sigma}^s(t) - \bar{\mathbf{B}}_2^s(t) - \bar{\mathbf{B}}_3^s(\mu(t)) \Delta \bar{\mathbf{V}}_{\sigma}^s(0),$$

where

$$\bar{\mathbf{\Lambda}}^s(\mu(t)) = \left[\bar{\mathbf{T}}^s(t) \right]^{-1} \mathbf{B}_1^s(\mu(t)) \bar{\mathbf{T}}^s(t) - \left[\bar{\mathbf{T}}^s(t) \right]^{-1} \dot{\bar{\mathbf{T}}}^s(t),$$

$$\bar{\mathbf{B}}_2^s(t) = \left[\bar{\mathbf{T}}^s(t) \right]^{-1} \mathbf{B}_2^s,$$

$$\bar{\mathbf{B}}_3^s(\mu(t)) = \left[\bar{\mathbf{T}}^s(t) \right]^{-1} \mathbf{B}_3^s(\mu(t)) \bar{\mathbf{T}}^s(0).$$

Since the homogeneous equation to (6.4) is similar to (6.2), it can easily be shown by straight forward calculation that $\bar{\mathbf{T}}^s(t) = \bar{\mathbf{T}}(t)$, as defined in equations (6.24), and $\bar{\mathbf{\Lambda}}^s(\mu(t)) = \bar{\mathbf{\Lambda}}(\mu(t)) + \text{diag}(\zeta, \zeta)$, where $\bar{\mathbf{\Lambda}}(\mu(t))$ is as defined in equation (6.25). Matrix $\bar{\mathbf{B}}_2^s(t) = \bar{\mathbf{B}}_2(t)$, which is stated in equation (6.26). The transformed matrix $\bar{\mathbf{B}}_3^s(\mu(t))$ is

$$\begin{aligned} \bar{\mathbf{B}}_3^s(\mu(t)) &= \begin{bmatrix} 1 + \bar{q}_2(t) & -\bar{q}_2(t) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \zeta(1 - \pi_{ho}(t)) & 0 \\ 0 & \zeta(1 - \pi_{hn}(t)) \end{bmatrix} \begin{bmatrix} 1 & \bar{q}_2(0) \\ 1 & \bar{q}_2(0) + 1 \end{bmatrix} \\ &= \begin{bmatrix} \zeta(1 - \pi_{ho}(t)) + \bar{q}_2(t)\zeta(\pi_{hn}(t) - \pi_{ho}(t)) & -\bar{q}_2(t)\zeta(1 - \pi_{hn}(t)) \\ -\zeta(\pi_{hn}(t) - \pi_{ho}(t)) & \zeta(1 - \pi_{hn}(t)) \end{bmatrix}, \end{aligned}$$

where $\bar{q}_2(t)$ is as defined in equation (6.23).

The following step is similar to the derivation of $\mathbf{V}_\sigma^*(t)$ and $\mathbf{V}_\sigma^*(0)$ on page 71. Hence,

$$\Delta \bar{\mathbf{V}}_\sigma^s(t) = \int_t^\infty e^{-\int_t^x \bar{\mathbf{\Lambda}}^s(\mu(\tau)) d\tau} \left[\bar{\mathbf{B}}_2^s(x) + \bar{\mathbf{B}}_3^s(\mu(x)) \Delta \bar{\mathbf{V}}_\sigma^s(0) \right] dx, \quad (6.31)$$

$$\begin{aligned} \Delta \bar{\mathbf{V}}_\sigma^s(0) &= \left(\mathbf{I}_2 - \int_0^\infty e^{-\int_0^x \bar{\mathbf{\Lambda}}^s(\mu(\tau)) d\tau} \bar{\mathbf{B}}_3^s(\mu(x)) dx \right)^{-1} \times \\ &\quad \left(\int_0^\infty e^{-\int_0^x \bar{\mathbf{\Lambda}}^s(\mu(\tau)) d\tau} \bar{\mathbf{B}}_2^s(x) dx \right). \end{aligned} \quad (6.32)$$

The two terms stated in the brackets of equation (6.32) are analyzed first:

$$\begin{aligned} &\left(\mathbf{I}_2 - \int_0^\infty e^{-\int_0^x \bar{\mathbf{\Lambda}}^s(\mu(\tau)) d\tau} \bar{\mathbf{B}}_3^s(\mu(x)) dx \right) \\ &= \begin{bmatrix} \left(1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s(\mu(\tau)) d\tau} \zeta [(1 - \pi_{ho}(x)) + \bar{q}_2(x)(\pi_{hn}(x) - \pi_{ho}(x))] dx \right) & \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s(\mu(\tau)) d\tau} \bar{q}_2(x) \zeta (1 - \pi_{hn}(x)) dx \\ \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (\pi_{hn}(x) - \pi_{ho}(x)) dx & 1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (1 - \pi_{hn}(x)) dx \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned}\bar{\lambda}_1^s &= r + \zeta + \lambda_u + \lambda_d, \\ \bar{\lambda}_2^s(t) &= r + \zeta + \lambda_u + \lambda_d + 2\lambda\mu_{hm}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z).\end{aligned}$$

Hence,

$$\begin{aligned}\left(\mathbf{I}_2 - \int_0^\infty e^{-\int_0^x \bar{\Lambda}^s(\mu(\tau)) d\tau} \bar{\mathbf{B}}_3^s(\mu(x)) dx \right)^{-1} &= \frac{1}{\Psi} \times \\ &\begin{bmatrix} 1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (1 - \pi_{hm}(x)) dx & - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \bar{q}_2(x) \zeta (1 - \pi_{hm}(x)) dx \\ - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (\pi_{hm}(x) - \pi_{ho}(x)) dx & \left(1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \zeta [(1 - \pi_{ho}(x)) \right. \\ &\quad \left. + \bar{q}_2(x) (\pi_{hm}(x) - \pi_{ho}(x))] dx \right)\end{bmatrix},\end{aligned}$$

with

$$\begin{aligned}\Psi &= \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \zeta [(1 - \pi_{ho}(x)) + \bar{q}_2(x) (\pi_{hm}(x) - \pi_{ho}(x))] dx \right] \times \\ &\quad \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (1 - \pi_{hm}(x)) dx \right] \\ &\quad - \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta (\pi_{hm}(x) - \pi_{ho}(x)) dx \right] \times \\ &\quad \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \bar{q}_2(x) \zeta (1 - \pi_{hm}(x)) dx \right].\end{aligned}$$

The second term in brackets of equation (6.32) reads

$$\left(\int_0^\infty e^{-\int_0^x \bar{\Lambda}^s(\mu(\tau)) d\tau} \bar{\mathbf{B}}_2^s(x) dx \right) = \begin{bmatrix} \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \delta (1 + \bar{q}_2(x)) dx \\ - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \delta dx \end{bmatrix}.$$

Inverse transformation of $\Delta \bar{\mathbf{V}}_\sigma^s(0)$ with $\Delta \mathbf{V}_\sigma^s(0) = \bar{\mathbf{T}}^s(0) \Delta \bar{\mathbf{V}}_\sigma^s(0)$ obtains

$$\begin{bmatrix} \Delta V_o^s(0) \\ \Delta V_n^s(0) \end{bmatrix} = \frac{1}{\Psi} \times \left[\begin{array}{c} \left\{ \begin{array}{l} \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta(1 - \pi_{hm}(x)) dx \right] \cdot \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \delta(1 + \bar{q}_2(x)) dx \right] \\ + \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \bar{q}_2(x) \zeta(1 - \pi_{hm}(x)) dx \right] \cdot \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \delta dx \right] \end{array} \right\} \\ \left\{ \begin{array}{l} \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta(1 - \pi_{hm}(x)) dx \right] \cdot \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \delta(1 + \bar{q}_2(x)) dx \right] \\ + \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \bar{q}_2(x) \zeta(1 - \pi_{hm}(x)) dx \right] \cdot \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \delta dx \right] \\ + \left[- \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta(\pi_{hm}(x) - \pi_{ho}(x)) dx \right] \cdot \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \delta(1 + \bar{q}_2(x)) dx \right] \\ + \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \zeta[(1 - \pi_{ho}(x)) + \bar{q}_2(x)(\pi_{hm}(x) - \pi_{ho}(x))] dx \right] \cdot \left[- \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \delta dx \right] \end{array} \right\} \end{array} \right].$$

Based on these derivations of $\Delta \mathbf{V}_\sigma^s(t)$, the conditions for gains from trade are as follows.

Gains from Trade Immediately after the Shock

I start with condition $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$, which must satisfy

$$\begin{aligned} & \frac{\delta}{\Psi} \left\{ \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} \zeta(\pi_{hm}(x) - \pi_{ho}(x)) dx \right] \times \right. \\ & \quad \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} (1 + \bar{q}_2(x)) dx \right] \\ & + \left[1 - \int_0^\infty e^{-\int_0^x \bar{\lambda}_1^s d\tau} \zeta[(1 - \pi_{ho}(x)) + \bar{q}_2(x)(\pi_{hm}(x) - \pi_{ho}(x))] dx \right] \times \\ & \quad \left. \left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} dx \right] \right\} \\ & \geq 0. \end{aligned}$$

This inequality is equal to inequality (6.5) stated in section 6.3.

Gains from Trade for Time $t > 0$

Secondly, I derive $\Delta \mathbf{V}_\sigma^s(t)$, which is equal to equation (6.31) after inverse transformation, with $\Delta \mathbf{V}_\sigma^s(t) = \bar{\mathbf{T}}^s(t) \Delta \bar{\mathbf{V}}_\sigma^s(t)$, so that

$$\Delta \mathbf{V}_\sigma^s(t) = \bar{\mathbf{T}}^s(t) \int_t^\infty e^{-\int_t^x \bar{\Lambda}^s(\mu(\tau)) d\tau} \left[\bar{\mathbf{T}}^s(x) \right]^{-1} (\mathbf{B}_2^s + \mathbf{B}_3^s(\mu(x)) \Delta \mathbf{V}_\sigma^s(0)) dx. \quad (6.33)$$

With intermediate step

$$\begin{aligned} & \left[\bar{\mathbf{T}}^s(x) \right]^{-1} (\mathbf{B}_2^s + \mathbf{B}_3^s(\mu(x)) \Delta \mathbf{V}_\sigma^s(0)) \\ &= \begin{bmatrix} \delta(1 + \bar{q}_2(t)) \\ -\delta \end{bmatrix} + \begin{bmatrix} (1 + \bar{q}_2(t))\zeta(1 - \pi_{ho}(t)) & -\bar{q}_2(t)\zeta(1 - \pi_{hm}(t)) \\ -\zeta(1 - \pi_{ho}(t)) & \zeta(1 - \pi_{hm}(t)) \end{bmatrix} \begin{bmatrix} \Delta V_o^s(0) \\ \Delta V_n^s(0) \end{bmatrix}, \end{aligned}$$

equation (6.33) reads

$$\Delta \mathbf{V}_\sigma^s(t) = \begin{bmatrix} 1 & \bar{q}_2(t) \\ 1 & \bar{q}_2(t) + 1 \end{bmatrix} \begin{bmatrix} \left(\int_t^\infty e^{-\int_t^x \bar{\lambda}_1^s(\mu(\tau)) d\tau} [\delta(1 + \bar{q}_2(x)) + (1 + \bar{q}_2(x))\zeta(1 - \pi_{ho}(x))\Delta V_o^s(0) - \bar{q}_2(x)\zeta(1 - \pi_{hm}(x))\Delta V_n^s(0)] dx \right) \\ \left(\int_t^\infty e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [-\delta - \zeta(1 - \pi_{ho}(x))\Delta V_o^s(0) + \zeta(1 - \pi_{hm}(x))\Delta V_n^s(0)] dx \right) \end{bmatrix}.$$

Hence, the condition $\Delta V_o^s(t) - \Delta V_n^s(t) \geq 0$ is obtained with

$$\int_t^\infty e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [\delta + \zeta(1 - \pi_{ho}(x))\Delta V_o^s(0) - \zeta(1 - \pi_{hm}(x))\Delta V_n^s(0)] dx \geq 0,$$

which is equal to inequality (6.10) stated in section 6.3.

Gains from Trade a Long Time after the Shock

The constraint a long time after an aggregate liquidity shock, given no additional shock has occurred, i.e. $\lim_{t \rightarrow \infty} (\Delta V_o^s(t) - \Delta V_n^s(t)) \geq 0$, is

$$\delta \geq \zeta [(1 - \pi_{hm}(ss))\Delta V_n^s(0) - (1 - \pi_{ho}(ss))\Delta V_o^s(0)].$$

This is equal to inequality (6.11) stated in section 6.3.

6C Appendix: Analyzing the Frozen Market

With an example in section 6.4, I have illustrated that the driving factors for no gains from trade are

- (1.) search frictions, determined by λ and ρ , which measure liquidity,
- (2.) agents' individual recovery time, denoted by λ_u ,
- (3.) the severity of the shock, determined by $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$.

First, I state some convergence properties of masses dynamics $\mu_\sigma(t)$, since these are influenced by both search frictions λ, ρ and agents' individual recovery time λ_u . Secondly, I analyze both the severity of the shock and the interaction of $\pi_{ho}(ss)$, $\pi_{hn}(ss)$, and $\mu_\sigma(t)$, expressed in $\pi_{ho}(t)$ and $\pi_{hn}(t)$. Finally, based on these discussions, I scrutinize inequalities (6.5), (6.10), and (6.11).

Mass Dynamics of Investor Types

Results for the Convergence of Mass Dynamics

(i) Either $\mu_{lo}(t)$ or $\mu_{hn}(t)$ converges monotonically to its steady state value after an aggregate liquidity shock. (ii) The other moves away for a while, before the sign of its derivative changes and it converges to its steady state. (iii) If $\dot{\mu}_{lo}(0) \leq 0$ and $\dot{\mu}_{hn}(0) \geq 0$ hold simultaneously, both converge monotonically, i.e. $\dot{\mu}_{lo}(t) \leq 0$ and $\dot{\mu}_{hn}(t) \geq 0$ for all $t \geq 0$.¹⁴⁵

Discussion and Proof

Before showing the proof for statement (iii) and discussing statement (i) and (ii), I start with a review of some general properties of masses dynamics $\mu_\sigma(t)$. From section (3.2), the two equations

$$\begin{aligned} \dot{\mu}_{lo}(t) = & -2\lambda(\mu_{lo}(t))^2 - [2\lambda(\mu_h(t) - s) + \lambda_u + \lambda_d + \rho] \mu_{lo}(t) \\ & + \rho \max\{0, s - \mu_h(t)\} + \lambda_d s \end{aligned} \quad (6.34)$$

¹⁴⁵ See Duffie, Gârleanu, and Pedersen (2007), pp. 1897.

and

$$\mu_h(t) = \bar{\mu}_h(0)e^{-(\lambda_u+\lambda_d)t} + \frac{\lambda_u}{\lambda_u + \lambda_d} \left[1 - e^{-(\lambda_u+\lambda_d)t} \right] \quad (6.35)$$

are already known. Since $\bar{\mu}_h(0) < \mu_h(ss)$ holds due to the shock, it can be verified with equation (6.35) and $\mu_h(ss) = \lambda_u/(\lambda_u + \lambda_d)$ that $\mu_h(t) < \mu_h(ss)$ holds for all t as well. Secondly, $\bar{\mu}_{lo}(0) > \mu_{lo}(ss)$ holds, since the fraction of sellers increases due to the shock. By assumption, $\mu_{lo}(t) > 0 \forall t$ holds. Hence, as verified by Duffie, Gârleanu, and Pedersen (2007, pp. 1897), $\mu_{lo}(t) > \mu_{lo}(ss)$ and $\mu_{hn}(t) < \mu_{hn}(ss)$ is valid for all time t .

The proof for statement (iii) is as follows: If $\dot{\mu}_{lo}(0) \leq 0$ holds, then the inequality

$$2\lambda\bar{\mu}_{hn}(0)\bar{\mu}_{lo}(0) + \rho\bar{\mu}_m(0) \geq -\lambda_u\bar{\mu}_{lo}(0) + \lambda_d\bar{\mu}_{ho}(0) \quad (6.36)$$

is equivalent to it. If $\dot{\mu}_{hn}(0) \geq 0$ holds, then

$$2\lambda\bar{\mu}_{hn}(0)\bar{\mu}_{lo}(0) + \rho\bar{\mu}_m(0) \leq \lambda_u\bar{\mu}_{ln}(0) - \lambda_d\bar{\mu}_{hn}(0) \quad (6.37)$$

is equivalent. Inequalities (6.36) and (6.37) imply that $\lambda_u\bar{\mu}_l(0) \geq \lambda_d\bar{\mu}_h(0)$ must hold, too. With equations (3.8) and (3.9), it can be shown that $\lambda_u\mu_l(t) \geq \lambda_d\mu_h(t)$ holds for all t as well, which completes the proof.

Statement (i) and (ii) are discussed next: The numerical examples in chapter 5 and section 6.4 illustrate the case in which $\mu_{lo}(t)$ converges monotonically to its steady state value, which implies $\dot{\mu}_{lo}(t) \leq 0$. At the same time, $\mu_{hn}(t)$ moves away from its steady state level for a while, i.e. $\dot{\mu}_{hn}(t) \leq 0$ for $0 \leq t \leq t_1$, where $t_1 < \infty$ is the point in time at which the sign of $\dot{\mu}_{hn}(t)$ changes. Right after the shock, trading activities dominate intrinsic type switches. This effect occurs with ‘normal’ and low search frictions (i.e. not very low λ and ρ) and/or small switching intensity λ_u . For $t > t_1$, the derivative changes its sign, i.e. $\dot{\mu}_{hn}(t) \geq 0$, and $\mu_{hn}(t)$ converges monotonically to its steady state. Equation (6.38) demonstrates these effects:

$$\begin{aligned} \dot{\mu}_{hn}(t) = & -2\lambda(\mu_{hn}(t))^2 - [2\lambda(s - \mu_h(t)) + \lambda_u + \lambda_d + \rho]\mu_{hn}(t) \\ & + \rho \max\{0, \mu_h(t) - s\} + \lambda_u(1 - s). \end{aligned} \quad (6.38)$$

For $s \leq \lambda_u/(\lambda_u + \lambda_d)$ and a large shock leading to $s \geq \bar{\mu}_h(0)$, it is very likely that $\dot{\mu}_{hn}(0) \leq 0$. This is due to the fact that $\max\{0, \bar{\mu}_h(0) - s\} = 0$ and

$2\lambda(s - \bar{\mu}_h(0)) \geq 0$. If $\bar{\mu}_{hn}(0)$ is very low, or λ_u is very high, then $\lambda_u(1-s)$ dominates the whole equation. If the shock is relatively small leading to $s < \bar{\mu}_h(0)$, the effect is not clear: $\max\{0, \bar{\mu}_h(0) - s\} = \bar{\mu}_h(0) - s$, $2\lambda(s - \bar{\mu}_h(0)) < 0$ and, in general, $\bar{\mu}_{hn}(0)$ is not very small then. As a result, the sign of (6.38) immediately after the shock depends on the size of the shock.

For $s > \lambda_u/(\lambda_u + \lambda_d)$, an aggregate liquidity shock leads to $s > \bar{\mu}_h(0)$, since this market remains a buyer's market after the shock. In a buyer's market, the fraction of $\mu_{hn}(ss)$ is low in steady state; especially for high meeting intensities: $\mu_{hn}(ss) \xrightarrow{\lambda \rightarrow \infty} 0$ or $\mu_{hn}(ss) \xrightarrow{\rho \rightarrow \infty} 0$. Therefore, potential buyers are barely affected by a shock, as illustrated in figure (5.13). This still very low fraction of potential buyers can influence equation (6.38) negatively immediately after the shock. Shortly thereafter, mass dynamics $\mu_{hn}(t)$ start to monotonically converge to their steady state level, since then $+\lambda_u(1-s)$ dominates equation (6.38) while $\mu_{hn}(t)$ remains very low.

If $\mu_{hn}(t)$ converges monotonically to its steady state, i.e. $\dot{\mu}_{hn}(t) \geq 0$, then $\mu_{lo}(t)$ might move away from its steady state level for a while, i.e. $\dot{\mu}_{lo}(t) \geq 0$ for $0 \leq t \leq t_1$. For $\dot{\mu}_{lo}(0) \geq 0$, the relation reads

$$2\lambda\bar{\mu}_{hn}(0)\bar{\mu}_{lo}(0) + \rho\bar{\mu}_m(0) \leq -\lambda_u\bar{\mu}_{lo}(0) + \lambda_d\bar{\mu}_{ho}(0), \quad (6.39)$$

and from equation (6.34) for $\dot{\mu}_{lo}(t) \geq 0$,

$$0 \leq -2\lambda(\mu_{lo}(t))^2 - [2\lambda(\mu_h(t) - s) + \lambda_u + \lambda_d + \rho]\mu_{lo}(t) + \rho \max\{0, s - \mu_h(t)\} + \lambda_d s. \quad (6.40)$$

Inequality (6.39) implies that intrinsic type switches dominate trading activities right after the shock. In general, however, there are high trading activities immediately after a shock due to a high misallocation of assets. As a result, the addressed property only appears in some extreme combinations. For example, very high search frictions combined with a high one-sided shock that hits only the buy-side. On the right hand side in (6.39), $\bar{\mu}_{ho}(0)$ is not reduced and $\bar{\mu}_{lo}(0)$ is not increased due to a shock. The high one-sided shock heavily reduces $\bar{\mu}_{hn}(0)$ so that there are only a few potential sellers and a few potential buyers in the market. With high search frictions they barely meet. This situation changes already after a short time, since $\mu_{lo}(t)$ increases due to this relation. Inequality (6.40) highlights that, in general, the prevailing sign of $\dot{\mu}_{lo}(t)$ is negative after a shock, since there is an elevated quantity of $\mu_{lo}(t)$ after a shock and, of course, $\mu_{lo}(t) \geq 0$ holds.

The Severity of a Shock

Results for the Convergence of Probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$

The convergence of mass dynamics stated in (i), (ii), and (iii) influences probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$:

- If $\dot{\mu}_{ho}(t) \geq 0$ for $t \geq 0$, both $\mu_{ho}(t)$ and $\pi_{ho}(t)$ converge monotonically to their steady state level, given no further shock occurs.
- If $\dot{\mu}_{hn}(t) \geq 0$ for $t \geq 0$, both $\mu_{hn}(t)$ and $\pi_{hn}(t)$ converge monotonically to their steady state level, given no further shock occurs.
- If $\bar{\mu}_{hn}(0) > \mu_{hn}(t)$ for $0 < t \leq t_1$, where t_1 is the time at which the sign of $\dot{\mu}_{hn}(t)$ changes, then $\pi_{hn}(t)$ drops below zero before it converges towards $\pi_{hn}(ss)$, given no further shock occurs.
- If $\bar{\mu}_{lo}(0) < \mu_{lo}(t)$ for $0 < t \leq t_1$, then $\pi_{ho}(t)$ drops below zero before it converges towards $\pi_{ho}(ss)$. This situation only occurs for some extreme parameter combinations and persists then only for a short period of time.

Discussion and Proof

These four effects are determined by the key parameters which influence the severity of an aggregate liquidity shock: Steady state probabilities of high agents switching to a low state due to a shock, i.e. $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$. These probabilities control for the starting condition of agents' masses $\bar{\mu}_\sigma(0)$ with

$$\bar{\mu}_{ho}(0) = (1 - \pi_{ho}(ss)) \mu_{ho}(ss), \quad (6.41)$$

$$\bar{\mu}_{hn}(0) = (1 - \pi_{hn}(ss)) \mu_{ho}(ss). \quad (6.42)$$

By assumption, the type distribution jumps to the fixed fractions $\bar{\mu}_\sigma(0)$ at each aggregate liquidity shock. In controlling for the starting conditions, probabilities $\pi_{ho}(ss)$ and $\pi_{hn}(ss)$ determine the evolution of masses over time as well. Both effects influence the probability of high agents switching to a low state, due to a shock which occurs out of steady state. The definitions of $\pi_{ho}(t)$ and $\pi_{hn}(t)$, restated in equations (6.43) and (6.44) with

$$\pi_{ho}(t) = 1 - \frac{\bar{\mu}_{ho}(0)}{\mu_{ho}(t)}, \quad (6.43)$$

$$\pi_{hn}(t) = 1 - \frac{\bar{\mu}_{hn}(0)}{\mu_{hn}(t)}, \quad (6.44)$$

show this effect. Furthermore, equations (6.43) and (6.44) show that statement (i), (ii), and (iii) influence probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$. Some general remarks are necessary:

If $\dot{\mu}_{lo}(t) \leq 0$ for $t \geq 0$, then $\dot{\mu}_{ho}(t) \geq 0$ for $t \geq 0$ must hold simultaneously. Both $\mu_{ho}(t)$ and $\pi_{ho}(t)$ converge monotonically to their steady state level, given no further shock occurs. Likewise, if $\dot{\mu}_{hn}(t) \geq 0$ for $t \geq 0$, then both $\mu_{hn}(t)$ and $\pi_{hn}(t)$ converge monotonically to their steady state level.

Things are different for both $\dot{\mu}_{hn}(t) \leq 0$ and $\dot{\mu}_{lo}(t) \geq 0$ for any $0 \leq t \leq t_1$, where t_1 is the point in time at which the sign of either $\dot{\mu}_{lo}(t)$ or $\dot{\mu}_{hn}(t)$ changes. If $\mu_{hn}(t)$ decreases after the shock, i.e. $\bar{\mu}_{hn}(0) \geq \mu_{hn}(t)$ for $0 \leq t \leq t_1$, then $\pi_{hn}(t)$ drops below zero for $0 < t \leq t_1$ before it monotonically converges towards $\pi_{hn}(ss)$, given no further shock occurs. The smaller the value of $\mu_{hn}(t)$, the deeper is $\pi_{hn}(t)$ below zero. Figure 6.4 illustrates this effect. As seen in section 6.4, it is a common situation that $\pi_{hn}(t)$ is negative for a while.

If $\mu_{lo}(t)$ increases after the shock, i.e. $\bar{\mu}_{lo}(0) \leq \mu_{lo}(t)$ for $0 \leq t \leq t_1$, then $\pi_{ho}(t)$ drops below zero for $0 < t \leq t_1$, before it monotonically converges towards $\pi_{ho}(ss)$. As discussed above, this situation only occurs for some extreme parameter combinations and persists only for a short period of time.

If either $\pi_{hn}(t)$ or $\pi_{ho}(t)$ is negative, the respective probability measure loses its characteristics as a probability. A negative $\pi_{hn}(t)$ or $\pi_{ho}(t)$ implies that if a further aggregate liquidity shock hits the market, the fraction of *hn* or *ho*, respectively, has to increase due to the shock. Potential buyers (or *ho* agents) must be hit by a positive liquidity shock to meet the fixed fractions $\bar{\mu}_\sigma(0)$. As a result, the fixed starting point of investors' masses after a shock is the crucial assumption. However, the model seems to be intractable without it.¹⁴⁶

The Inequalities for the Existence of a Nash Bargaining Solution

Using the results of my analysis of $\mu_\sigma(t)$, $\pi_{hn}(t)$, and $\pi_{ho}(t)$, I take a closer look at inequalities (6.5), (6.10), and (6.11) and equations (6.6)–(6.9).

¹⁴⁶ See Duffie, Gârleanu, and Pedersen (2007), p. 1882.

Results for the Existence of a Nash Bargaining Solution

Condition $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$ can be violated for high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$. The difference $\Delta V_o^s(t) - \Delta V_n^s(t)$ can temporarily become negative. The condition $\lim_{t \rightarrow \infty} (\Delta V_o^s(t) - \Delta V_n^s(t)) \geq 0$ is always met for sufficiently small ζ .

Analysis of $\bar{\lambda}_1^s, \bar{\lambda}_2^s(t)$, and $\bar{q}_2(t)$

I start the analysis with some comments about $\bar{\lambda}_1^s, \bar{\lambda}_2^s(t)$, and $\bar{q}_2(t)$. Obviously,

$$\bar{\lambda}_1^s = r + \zeta + \lambda_u + \lambda_d > 0,$$

and

$$\bar{\lambda}_2^s(t) = r + \zeta + \lambda_u + \lambda_d + 2\lambda\mu_{hn}(t)q + 2\lambda\mu_{lo}(t)(1-q) + \rho(1-z) \geq \bar{\lambda}_1^s.$$

Additionally, $\bar{\lambda}_2^s(t)$ increases with λ and ρ . It is clear from

$$\bar{q}_2(t) = - \int_0^t e^{-\int_x^t (2\lambda\mu_{hn}(\tau)q + 2\lambda\mu_{lo}(\tau)(1-q) + \rho(1-z)) d\tau} (2\lambda\mu_{hn}(x)q + \rho(1-z)\tilde{q}(x)) dx,$$

that $0 \geq \bar{q}_2(t)$, since $\bar{q}_2(t)$ decreases from $\bar{q}_2(0) = 0$ to its steady state level

$$\bar{q}_2(ss) = - \frac{2\lambda\mu_{hn}(ss)q + \rho(1-z)\tilde{q}(ss)}{2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1-q) + \rho(1-z)},$$

given no further shock occurs in the meantime. Hence,

$$0 \geq \bar{q}_2(t) \geq - \frac{2\lambda\mu_{hn}(ss)q + \rho(1-z)\tilde{q}(ss)}{2\lambda\mu_{hn}(ss)q + 2\lambda\mu_{lo}(ss)(1-q) + \rho(1-z)} \geq -1.$$

Analysis of $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$

I start with an analysis of $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$. First, Ψ is examined, which is positive for sufficiently small mean arrival rates ζ of aggregate liquidity shocks.

I partition Ψ in the following way:

$$\Psi = \zeta^2 \times \left\{ \left[\underbrace{\frac{1}{\zeta} - \int_0^{\infty} e^{-\bar{\lambda}_1^s x} (1 - \pi_{ho}(x)) (1 + \bar{q}_2(x)) dx}_{\text{Part A}} \right] \right. \\ \left. \left[\underbrace{\frac{1}{\zeta} - \int_0^{\infty} e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{hn}(x)) dx}_{\text{Part B}} \right] \right. \\ \left. + \left[\underbrace{\int_0^{\infty} e^{-\bar{\lambda}_1^s x} \bar{q}_2(x) (1 - \pi_{hn}(x)) dx}_{\text{Part C}} \right] \right. \\ \left. \left[\underbrace{\frac{1}{\zeta} - \int_0^{\infty} e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{ho}(x)) dx}_{\text{Part D}} \right] \right\}. \quad (6.45)$$

My analysis relies on a modified version of Steffensen's inequality, stated in Pečarić, Proschan, and Tong (1992, p. 182):

"Let f be a decreasing function on $(0, \infty)$, and g be a measurable function on $[0, \infty)$ such that $0 \leq g(x) \leq A$ (A is a positive real number).

Then

$$\int_0^{\infty} f(x)g(x) dx \leq A \int_0^{\lambda} f(x) dx,$$

where

$$\lambda = \frac{1}{A} \int_0^{\infty} g(x) dx."$$

Generally, part A in equation (6.45) is smaller than $1/\bar{\lambda}_1^s$, which in turn is smaller than $1/\zeta$. The exponential function $\exp(-\bar{\lambda}_1^s t)$ converges to zero and the higher $\bar{\lambda}_1^s$, the faster the convergence. More often than not $0 \leq (1 - \pi_{ho}(t)) \leq 1$ holds.

$0 \leq (1 + \bar{q}_2(t)) \leq 1$ is always valid. In general, $(1 - \pi_{ho}(t))$ and $(1 + \bar{q}_2(t))$ are decreasing functions of time t , leading to a positive part A.

There are two opposite effects in part B of equation (6.45): $(1 - \pi_{hn}(t))$ is the higher the higher λ, ρ , and $\pi_{ho}(ss)$ are, and $0 \leq (1 - \pi_{hn}(t))$ is not bounded from above. Secondly, the higher λ, ρ , and $\pi_{ho}(ss)$ are, the faster $\exp(-\int_0^t \bar{\lambda}_2^s(\mu(\tau)) d\tau)$ converges to zero. Since $1 - \pi_{hn}(0) = 1$, the highest value of $(1 - \pi_{hn}(t))$ is not right after the shock, but a short time after. The decreasing effect of $\exp(-\int_0^t \bar{\lambda}_2^s(\mu(\tau)) d\tau)$ obviously dominates this term. For sufficiently small mean arrival rates ζ , the second term in brackets is positive.

Part C in equation (6.45) is small but negative: $\bar{q}_2(t)$ stays close to zero right after the shock, while $(1 - \pi_{hn}(t))$ can increase to a very high value. The effects of $\mu_{hn}(t)$ on $\bar{q}_2(t)$ and on $(1 - \pi_{hn}(t))$ are opposed and partially offset each other. Medium and long term values are reduced by $\exp(-\bar{\lambda}_1^s t)$. In total, the third term in brackets is negative but small.

Part D in equation (6.45) is very small, since $\exp(-\int_0^t \bar{\lambda}_2^s(\mu(\tau)) d\tau)$ converges to zero faster, the higher λ, ρ , and $\pi_{ho}(ss)$ are. More often than not, $(1 - \pi_{ho}(t))$ stays between 1 and $\pi_{ho}(ss)$.

With these different effects, numerical analysis indicates that Ψ has a positive value for sufficiently small ζ .

Next, I partition equation (6.5) so that

$$\begin{aligned}
 & \left[\underbrace{\int_0^\infty e^{-\bar{\lambda}_1^s x} (1 + \bar{q}_2(x)) dx}_{\text{Part E}} \right] \left[\underbrace{\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (\pi_{hn}(x) - \pi_{ho}(x)) dx}_{\text{Part F}} \right] \\
 & + \left[\frac{1}{\zeta} - \underbrace{\int_0^\infty e^{-\bar{\lambda}_1^s x} [(1 - \pi_{ho}(x))(1 + \bar{q}_2(x)) - \bar{q}_2(x)(1 - \pi_{hn}(x))] dx}_{\text{Part G}} \right] \times \quad (6.46) \\
 & \left[\underbrace{\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} dx}_{\text{Part H}} \right].
 \end{aligned}$$

Part E in equation (6.46) lies between 0 and $1/\bar{\lambda}_1^s$, since $(1 + \bar{q}_2(t))$ is a decreasing function of t . The exponential function $\exp(-\bar{\lambda}_1^s t)$ converges to zero and the higher $\bar{\lambda}_1^s$ is, the faster the convergence.

Part F in equation (6.46) is the crucial one, because there is a chance of a negative sign for high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$. The function $\pi_{hm}(t)$ drops fast and heavily below zero. Although $\exp(-\int_0^t \bar{\lambda}_2^s(\mu(\tau)) d\tau)$ is working in the opposite direction by converging to zero faster, the higher λ, ρ , and $\pi_{ho}(ss)$ are, the sign for part E can be negative nevertheless.

Part G in equation (6.46) is not that clear: More often than not, the first term is bounded from above with $(1 - \pi_{ho}(t))(1 + \bar{q}_2(t)) \leq 1$ and it is decreasing in t . For the second term, the function $\bar{q}_2(t)$ is negative and very small for low values of time t . This compensates high values of $(1 - \pi_{hm}(t))$. The exponential function $\exp(-\bar{\lambda}_1^s t)$ converges to zero. Numerical analysis indicates that part G is smaller than $1/\zeta$ for sufficiently small values of ζ .

Part H in equation (6.46) is very small, especially for high meeting intensities. The overall impact of $(1/\zeta - \text{part G})(\text{part H})$ is very small as well.

As a result, the equation (6.46) can be negative in total, also for sufficiently small ζ . This happens especially for high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$, since $\pi_{hm}(t)$ drops fast and heavily below zero. Part F can get negative. As a result, the condition $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$ can be violated for high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$ —as shown in section 6.4.

Analysis of $\Delta V_o^s(t) - \Delta V_n^s(t)$

The next step is an analysis of $\Delta V_o^s(t) - \Delta V_n^s(t)$, that is

$$\int_t^\infty e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [\delta + \zeta (1 - \pi_{ho}(x)) \Delta V_o^s(0) - \zeta (1 - \pi_{hm}(x)) \Delta V_n^s(0)] dx \geq 0.$$

The drop below zero for $\pi_{hm}(t)$, especially for high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$, was mentioned quite a few times. This is the key effect here, too. The exponential function $\exp(-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau)$ reduces medium and long term values, but it has a smaller effect on values at time t and shortly thereafter. Thus, a very high value of $(1 - \pi_{hm}(t))$ can determine the sign of the whole term, even if ζ is small. As a result, $\Delta V_o^s(t) - \Delta V_n^s(t)$ can (temporarily) be negative.

Analysis of $\lim_{t \rightarrow \infty} (\Delta V_o^s(t) - \Delta V_n^s(t)) \geq 0$

When mass dynamics reach their steady state level, given no further shock has occurred, the condition

$$\delta \geq \zeta [(1 - \pi_{hn}(ss)) \Delta V_n^s(0) - (1 - \pi_{ho}(ss)) \Delta V_o^s(0)]$$

is always met for sufficiently small ζ and $\delta > 0$. This result is due to the fact that $\pi_{hn}(t)$ converges monotonically towards $\pi_{hn}(ss)$ (with $0 \leq \pi_{hn}(ss) \leq 1$) as soon as $\mu_{hn}(t)$ starts to converge monotonically towards its steady state level. Simultaneously, $\pi_{ho}(t)$ converges monotonically towards $\pi_{ho}(ss)$ (with $0 \leq \pi_{ho}(ss) \leq 1$) as soon as $\mu_{lo}(t)$ starts to converge monotonically towards its steady state level.

As a result, the existence of a Nash bargaining solution and, therefore, gains from trade are, in general, not ensured for the aggregate liquidity shock model: For high meeting intensities λ, ρ , and a high probability $\pi_{ho}(ss)$, the conditions $\Delta V_o^s(0) - \Delta V_n^s(0) \geq 0$ and $\Delta V_o^s(t) - \Delta V_n^s(t) \geq 0$ can be violated. For sufficiently small ζ , the inequality $\lim_{t \rightarrow \infty} (\Delta V_o^s(t) - \Delta V_n^s(t)) \geq 0$ is always met.

Chapter 7

The Completed Aggregate Liquidity Shock Model

The reason for the collapse of the aggregate liquidity shock model is not entirely characterized by trading despite no gains from trade. No gains from trade is rather a model incompleteness than a model breakdown, which I explain in section 7.1. I prevent the model from collapsing in section 7.2 by completing the aggregate liquidity shock model. Section 7.3 illustrates the completion by means of an example. Section 7.4 concludes. The validity of gains from trade in my completed aggregate liquidity model is analyzed in appendix 7A.

7.1 Model Inconsistency

Consider the following property of agents' types, which is due to an aggregate liquidity shock: In many cases, the fraction of potential buyers starts to decrease after a shock, before it converges monotonically to its long run level. Therefore, the relation $\bar{\mu}_{hn}(0) > \mu_{hn}(t)$ can temporarily arise.¹⁴⁷

Next, consider a second aggregate liquidity shock occurring shortly after the previous shock. In the aggregate liquidity shock model, this second shock hits potential buyers in such a way that their fraction must increase due to the second shock. I illustrate the effect of a second shock on the process of mass distribution in figure 7.1, by using the parameters of section 5.1.

¹⁴⁷ It is also possible that the fraction of potential sellers increases for a short period after a shock. However, this incident rarely arises (see appendix 6C). Either the fraction of potential buyers decreases after the shock or the fraction of potential sellers increases. Both effects do not occur at the same time. See Duffie, Gârleanu, and Pedersen (2007), pp. 1886 and p. 1898.

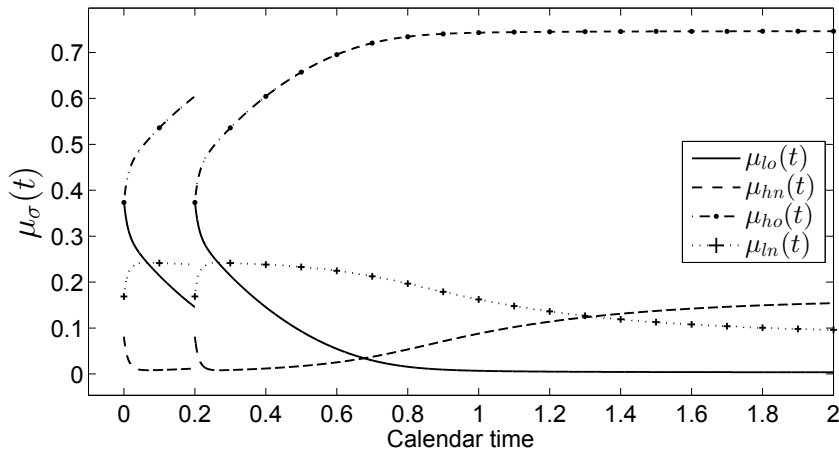


Figure 7.1: Process of mass distribution with a second aggregate liquidity shock and without market makers. The solid line illustrates the fraction of sellers over time. The dashed line represents the fraction of buyers. Dashed-dotted and dotted (with plus sign) lines show the fraction of high owners and low non-owners, respectively.

In figure 7.1, a second shock occurs 50 days after the first one, so that $t_{shock_1} = 0.2$ of the first shock is equal to the starting time $t_{shock_2} = 0$ in the second shock. Clearly, due to this second shock, some ln agents must switch to hn agents in order to fulfill the assumption that prior shocks do not influence the post-shock distribution. However, the aggregate liquidity shock model does not take the necessity of $ln \rightarrow hn$ switches correctly into account, which I show in the following passage.

The decisive factors for the inconsistency in the aggregate liquidity shock model are the probabilities $\pi_{hn}(t)$ and $\pi_{ho}(t)$: The probabilities are first defined as the probabilities of high agents switching to a low state due to a shock. Secondly, these probabilities provide the tractability of the aggregate liquidity shock model, since this model considers not only a single aggregate liquidity shock, but also contains the possibility of future aggregate liquidity shocks. Whenever an aggregate liquidity shock occurs, $\pi_{hn}(t)$ and $\pi_{ho}(t)$ ensure that there are no “ ‘after-effects’ of prior shocks”¹⁴⁸. These properties of $\pi_{hn}(t)$ and $\pi_{ho}(t)$ imply a fixed post-shock distribution $\bar{\mu}_\sigma(0)$.

However, since the fraction of potential buyers starts to decrease after a shock, $\pi_{hn}(t)$ can be negative.¹⁴⁹ Furthermore, Duffie, Gârleanu, and Pedersen (2007, p. 1882) assume that all ln agents remain ln agents due to the shock and

¹⁴⁸ Duffie, Gârleanu, and Pedersen (2007), p. 1882.

¹⁴⁹ Function $\pi_{ho}(t)$ can be negative if the fraction of potential sellers increases after the shock.

$\bar{\mu}_{lo}(0) > \mu_{lo}(t)$ and $\bar{\mu}_{ln}(0) > \mu_{ln}(t)$ have to hold.¹⁵⁰ But as figure 7.1 shows, a fraction of *ln* agents are forced to mutate to *hn* agents so that there are no after-effects. This increase in *hn* agents is not correctly reflected in the value functions (4.7)–(4.10). The forced mutation $ln \rightarrow hn$ is not included in equation (4.10) at all. Instead, the resulting negative ‘probability’ $\pi_{hn}(t)$ implies that a negative fraction of *hn*-agents switches to *ln* agents, which is—from an economic perspective—impossible.

As a result, there is something missing in equations (4.7)–(4.10) if the assumption of ‘no aftereffects’ is to be upheld. In the following section, I present my modifications to the aggregate liquidity shock model to fix these defects while keeping the assumption of no “‘aftereffects’ of prior shocks”¹⁵¹. My completed aggregate liquidity shock model results.

7.2 The Completed Model

I start by redefining the probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$, originally stated in section 4.2: An *ho* agent switches to an *lo* agent with probability

$$\pi_{ho}^{s,c}(t) = \begin{cases} 1 - \frac{\bar{\mu}_{ho}(0)}{\mu_{ho}(t)} & \text{if } \bar{\mu}_{ho}(0) \leq \mu_{ho}(t) \\ 0 & \text{otherwise} \end{cases} \quad (7.1)$$

and remains an *ho* agent with probability $1 - \pi_{ho}^{s,c}(t)$. Analogously, an *hn* agent switches to an *ln* agent with probability

$$\pi_{hn}^{s,c}(t) = \begin{cases} 1 - \frac{\bar{\mu}_{hn}(0)}{\mu_{hn}(t)} & \text{if } \bar{\mu}_{hn}(0) \leq \mu_{hn}(t) \\ 0 & \text{otherwise} \end{cases} \quad (7.2)$$

and remains an *hn* agent with probability $1 - \pi_{hn}^{s,c}(t)$.

The constant post-shock distribution $\bar{\mu}_{\sigma}(0)$ is calculated further on with equations (4.3)–(4.6), since the relations $\bar{\mu}_{ho}(0) \leq \mu_{ho}(ss)$ and $\bar{\mu}_{hn}(0) \leq \mu_{hn}(ss)$ hold due to the characteristic of the aggregate liquidity shock. As this redefinition does not affect $\pi_{ho}^{s,c}(ss) = \pi_{ho}(ss)$ and $\pi_{hn}^{s,c}(ss) = \pi_{hn}(ss)$, both are still specified exogenously.

¹⁵⁰ The assumptions $\bar{\mu}_{lo}(0) > \mu_{lo}(t)$ and $\bar{\mu}_{ln}(0) > \mu_{ln}(t)$ for low-type agents are equivalent to assuming $\bar{\mu}_{ho}(0) < \mu_{ho}(t)$ and $\bar{\mu}_{hn}(0) < \mu_{hn}(t)$ for high-type agents.

¹⁵¹ Duffie, Gârleanu, and Pedersen (2007), p. 1882.

Secondly, I define the additional probability for an *lo* agent switching to an *ho* agent with

$$\pi_{lo}^{s,c}(t) = \begin{cases} 1 - \frac{\bar{\mu}_{lo}(0)}{\mu_{lo}(t)} & \text{if } \bar{\mu}_{ho}(0) > \mu_{ho}(t) \\ 0 & \text{otherwise} \end{cases} \quad (7.3)$$

and the probability for remaining an *lo* agent with $1 - \pi_{lo}^{s,c}(t)$. Define likewise the probability for an *ln* agent switching to an *hn* agent with

$$\pi_{ln}^{s,c}(t) = \begin{cases} 1 - \frac{\bar{\mu}_{ln}(0)}{\mu_{ln}(t)} & \text{if } \bar{\mu}_{hn}(0) > \mu_{hn}(t) \\ 0 & \text{otherwise} \end{cases} \quad (7.4)$$

and the probability for remaining an *ln* agent with $1 - \pi_{ln}^{s,c}(t)$.

The probability $\pi_{lo}^{s,c}(t)$ is only valid for the period of time when there are fewer *ho* agents in the market than the post-shock distribution requires. The same applies for the probability $\pi_{ln}^{s,c}(t)$, which is defined for a fraction of *hn* agents that is less than the post-shock distribution requires. The probabilities $\pi_{ho}^{s,c}(t)$ and $\pi_{hn}^{s,c}(t)$ are valid during 'normal' times, i.e. when there are more *ho* agents and *hn* agents, respectively, in the market in comparison with the post-shock distribution $\bar{\mu}_\sigma(0)$.

Probabilities (7.3) and (7.4) imply that if an aggregate liquidity shock hits the market shortly after a previous one, then a fraction of agents with a low intrinsic type experience a positive shift in their preferences towards asset holding. Interpretations of this positive shock are, for instance, easy refunding due to financial intervention by governments and benefits from reduced prices after a shock.

The equilibrium type distribution $\mu_\sigma(t)$ evolves according to the differential equations (3.4)–(3.7), where $\bar{\mu}_\sigma(0)$ is the starting point after an aggregate liquidity shock. The redefinition of probabilities $\pi_{ho}^{s,c}(t)$ and $\pi_{hn}^{s,c}(t)$, as well as the introduction of $\pi_{lo}^{s,c}(t)$ and $\pi_{ln}^{s,c}(t)$, have no impact on agents' type distribution, since the fixed type distribution $\bar{\mu}_\sigma(0)$ is not affected by this redefinition. However, these probabilities influence agents' value functions. I integrate the probabilities of equations (7.1)–(7.4) into agents' value functions by focusing on a particular agent at a particular time t . The value functions $V_\sigma^{s,c}(t)$, where the superscript 'c'

indicates that this is my completion of the aggregate liquidity shock model, are

$$\begin{aligned}
V_{lo}^{s,c}(t) = E_t & \left[\int_t^\tau e^{-r(u-t)} (D - \delta) du + e^{-r(\tau_1-t)} V_{ho}^{s,c}(\tau_1) \mathbf{1}_{\{\tau_1=\tau\}} \right. \\
& + e^{-r(\tau_i-t)} (V_{ln}^{s,c}(\tau_i) + P^{s,c}(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \\
& + e^{-r(\tau_m-t)} (V_{ln}^{s,c}(\tau_m) + B^{s,c}(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} \\
& \left. + e^{-r(\tau_\zeta-t)} [(1 - \pi_{lo}^{s,c}(\tau_\zeta)) V_{lo}^{s,c}(0) + \pi_{lo}^{s,c}(\tau_\zeta) V_{ho}^{s,c}(0)] \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \tag{7.5}
\end{aligned}$$

$$\begin{aligned}
V_{hn}^{s,c}(t) = E_t & \left[e^{-r(\tau_1-t)} V_{ln}^{s,c}(\tau_1) \mathbf{1}_{\{\tau_1=\tau\}} + e^{-r(\tau_i-t)} (V_{ho}^{s,c}(\tau_i) - P^{s,c}(\tau_i)) \mathbf{1}_{\{\tau_i=\tau\}} \right. \\
& + e^{-r(\tau_m-t)} (V_{ho}^{s,c}(\tau_m) - A^{s,c}(\tau_m)) \mathbf{1}_{\{\tau_m=\tau\}} \\
& \left. + e^{-r(\tau_\zeta-t)} [(1 - \pi_{hn}^{s,c}(\tau_\zeta)) V_{hn}^{s,c}(0) + \pi_{hn}^{s,c}(\tau_\zeta) V_{ln}^{s,c}(0)] \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \tag{7.6}
\end{aligned}$$

$$\begin{aligned}
V_{ho}^{s,c}(t) = E_t & \left[\int_t^\tau e^{-r(u-t)} D du + e^{-r(\tau_1-t)} V_{lo}^{s,c}(\tau_1) \mathbf{1}_{\{\tau_1=\tau\}} \right. \\
& \left. + e^{-r(\tau_\zeta-t)} [(1 - \pi_{ho}^{s,c}(\tau_\zeta)) V_{ho}^{s,c}(0) + \pi_{ho}^{s,c}(\tau_\zeta) V_{lo}^{s,c}(0)] \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \tag{7.7}
\end{aligned}$$

$$\begin{aligned}
V_{ln}^{s,c}(t) = E_t & \left[e^{-r(\tau_1-t)} V_{hn}^{s,c}(\tau_1) \mathbf{1}_{\{\tau_1=\tau\}} \right. \\
& \left. + e^{-r(\tau_\zeta-t)} [(1 - \pi_{ln}^{s,c}(\tau_\zeta)) V_{ln}^{s,c}(0) + \pi_{ln}^{s,c}(\tau_\zeta) V_{hn}^{s,c}(0)] \mathbf{1}_{\{\tau_\zeta=\tau\}} \right], \tag{7.8}
\end{aligned}$$

where $\tau = \min\{\tau_l, \tau_i, \tau_m, \tau_\zeta\}$. The interpretation is as follows: The first terms of the value functions are the same as those in equations (4.7)–(4.10). For the last term, I implement the risk for and the impact of future aggregate liquidity shocks on all agents that are probably affected by the shock.

If $\bar{\mu}_{lo}(0) \geq \mu_{lo}(t)$, or, equivalently, $\bar{\mu}_{ho}(0) \leq \mu_{ho}(t)$ holds, then the value of any *lo* agent jumps upon a shock to the value of an *lo* agent at time $t = 0$. This means that all *lo* agents remain *lo* agents due to a shock—which Duffie, Gârleanu, and Pedersen (2007) mistakenly assume to be valid in general. Equation (7.5) reflects this case as follows: $\pi_{lo}^{s,c}(t)$ is zero for $\bar{\mu}_{lo}(0) \geq \mu_{lo}(t)$, which implies that equation (7.5) coincides with equation (4.7). If $\bar{\mu}_{lo}(0) < \mu_{lo}(t)$ holds, i.e. there are more potential sellers at time $t > 0$ than immediately after the shock, a positive fraction $\pi_{lo}^{s,c}(t)$ of all *lo* agents must switch to *ho* agents due to a shock in order to maintain a constant post-shock distribution. Any *lo* agent stays *lo* due to a shock

with probability $(1 - \pi_{lo}^{s,c}(t))$ and switches to an *ho* intrinsic type with probability $\pi_{lo}^{s,c}(t)$. As a result, equation (7.5) is a completed representation of equation (4.7).

Equivalently for the value function of an *hn* agent: If $\bar{\mu}_{hn}(0) \leq \mu_{hn}(t)$ holds, then, upon a shock, the value function of the *hn* agent jumps with probability $(1 - \pi_{hn}^{s,c}(t))$ to the value of an *hn* agent at time $t = 0$ and with probability $\pi_{hn}^{s,c}(t)$ to the value of an *ln* agent at time $t = 0$. This means that a positive fraction $(1 - \pi_{hn}^{s,c}(t))$ of potential buyers stays with their intrinsic type upon a shock and a positive fraction $\pi_{hn}^{s,c}(t)$ mutates to an *ln* agent. Duffie, Gârleanu, and Pedersen (2007) erroneously assume that this is valid in general. However, if $\bar{\mu}_{hn}(0) > \mu_{hn}(t)$ holds, i.e. there are more potential buyers at time $t > 0$ than immediately after the shock, all *hn* agents remain buyers. The value of all *hn* agents jumps upon a shock to the value of an *hn* agent at time $t = 0$. This effect is factored into equation (7.6) as $\pi_{hn}^{s,c}(t)$ is zero for $\bar{\mu}_{hn}(0) > \mu_{hn}(t)$, per definition. Equation (7.6) completes equation (4.8).

Interpretations for $V_{ho}^{s,c}(t)$ and $V_{ln}^{s,c}(t)$ are comparable, so that equations (7.7) and (7.8) are likewise completed representations of equations (4.9) and (4.10), respectively. Prices are as stated in equations (4.15)–(4.19), but with $V_{\sigma}^{s,c}(t)$ instead of $V_{\sigma}^s(t)$.

As a result, the system of linear (time-varying) differential equations in the completed model is

$$\dot{\mathbf{V}}_{\sigma}^{s,c}(t) = \mathbf{A}_1(\boldsymbol{\mu}(t))\mathbf{V}_{\sigma}^{s,c}(t) - \mathbf{A}_2 - \mathbf{A}_3^c(\boldsymbol{\mu}(t))\mathbf{V}_{\sigma}^{s,c}(0), \quad (7.9)$$

where $\mathbf{A}_1(\boldsymbol{\mu}(t))$ and \mathbf{A}_2 are as defined in equations (4.23) and (4.24), respectively, and

$$\mathbf{A}_3^c(\boldsymbol{\mu}(t)) = \begin{bmatrix} \zeta(1 - \pi_{lo}^{s,c}(t)) & 0 & \zeta\pi_{lo}^{s,c}(t) & 0 \\ 0 & \zeta(1 - \pi_{hn}^{s,c}(t)) & 0 & \zeta\pi_{hn}^{s,c}(t) \\ \zeta\pi_{ho}^{s,c}(t) & 0 & \zeta(1 - \pi_{ho}^{s,c}(t)) & 0 \\ 0 & \zeta\pi_{ln}^{s,c}(t) & 0 & \zeta(1 - \pi_{ln}^{s,c}(t)) \end{bmatrix}. \quad (7.10)$$

Since $\mathbf{A}_1(\boldsymbol{\mu}(t))$ remains unchanged by my completion of the aggregate liquidity shock model, the homogeneous part of system (7.9) is equal to the homogeneous part of system (4.22) in the aggregate liquidity shock model. Consequently, the state transition matrix $\boldsymbol{\Phi}(t, x)$ is unchanged as well upon my completion. The solution to system (7.9), which is based on the solution technique presented in

section 4.4 and appendix 4A, is

$$\mathbf{V}_\sigma^{s,c}(t) = \int_t^\infty \Phi(t, x) [\mathbf{A}_2 + \mathbf{A}_3^c(\mu(x)) \mathbf{V}_\sigma^{s,c}(0)] dx, \quad (7.11)$$

with

$$\mathbf{V}_\sigma^{s,c}(0) = \mathbf{T}(0) \left(\mathbf{I}_4 - \int_0^\infty e^{-\int_0^x \Lambda(\mu(\tau)) d\tau} \mathbf{A}_3^{c*}(\mu(x)) dx \right)^{-1} \times \left(\int_0^\infty e^{-\int_0^x \Lambda(\mu(\tau)) d\tau} \mathbf{A}_2^*(x) dx \right), \quad (7.12)$$

where the state transition matrix $\Phi(t, x)$ is as defined in equation (4.34), $\mathbf{T}(t)$ is defined in equation (4.35), $\Lambda(\mu(t))$ is stated in equation (4.39), $\mathbf{A}_2^*(t)$ is defined in equation (4.28), and $\mathbf{A}_3^{c*}(\mu(t)) = \mathbf{T}(t)^{-1} \mathbf{A}_3^c(\mu(t)) \mathbf{T}(0)$.

The following section analyzes my completion by means of an example. Appendix 7A investigates the validity of the Nash bargaining condition.

7.3 Example

I continue the example of section 5.2 but increase the meeting intensity λ to 625 in order to fully depict my completion of the aggregate liquidity shock model. First of all, this completion has no impact on the type distribution $\mu_\sigma(t)$, on trading volume, on asset mismatch, and on trading time. Since I choose a different value for λ , the process of mass distribution as shown in figure 5.5 is altered. I repeat this part of section 5.2 within figure 7.2.

But—as must be expected—value functions, prices, and bid-ask spreads do change. I confront the results of the aggregate liquidity shock (ALS) model, abbreviated ‘ALS Model’, with my completed aggregate liquidity shock model, denoted as ‘Completed ALS Model’, to display the change due to my completion. However, comparing the ‘ALS Model’ with my ‘Completed ALS Model’ is like comparing apples and oranges, since the ‘ALS Model’ is defective and my completion adds the missing feature.

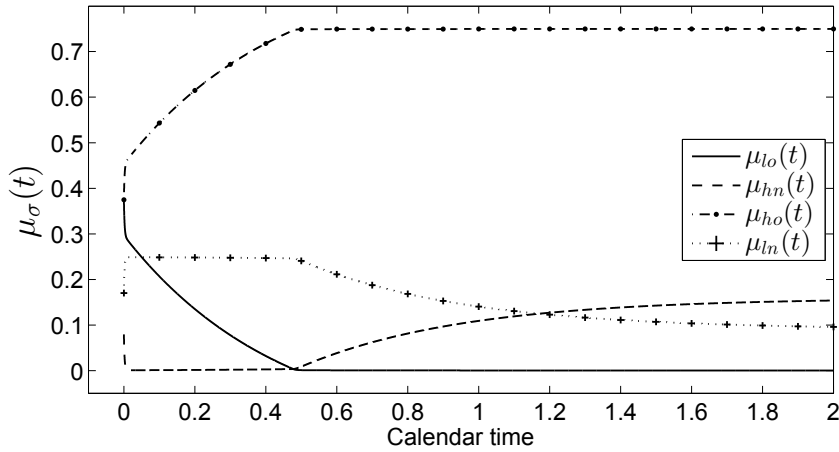


Figure 7.2: Process of mass distribution after an aggregate liquidity shock and with market makers. The solid line illustrates the fraction of sellers over time. The dashed line represents the fraction of buyers. Dotted and dashed-dotted (with plus sign) lines show the fraction of low non-owners and high owners, respectively.

Value Function

Table 7.1 shows the indirect utilities immediately and a long time after an aggregate liquidity shock for both the ‘ALS Model’ and the ‘Completed ALS Model’.

‘ALS Model’			‘Completed ALS Model’		
Parameter	$t = 0$	$t \rightarrow \infty$	Parameter	$t = 0$	$t \rightarrow \infty$
$V_{lo}^s(t)$	8.6926	9.4619	$V_{lo}^{s,c}(t)$	8.7284	9.4759
$V_{hn}^s(t)$	2.0458	0.8430	$V_{hn}^{s,c}(t)$	1.9244	0.7891
$V_{ho}^s(t)$	9.4575	9.4997	$V_{ho}^{s,c}(t)$	9.4733	9.5131
$V_{ln}^s(t)$	1.2864	0.8249	$V_{ln}^{s,c}(t)$	1.1899	0.7714

Table 7.1: Value functions in the ‘ALS Model’ and the ‘Completed ALS Model’ immediately and a long time after an aggregate liquidity shock, with market makers.

The structure of this example implies that $\bar{\mu}_{ho}(0) < \mu_{ho}(t)$ holds for all $t \geq 0$, which can be seen in figure 7.2. The value function $V_{lo}^{s,c}(t)$, stated in equation (7.5), is not influenced by $\pi_{lo}^{s,c}(t)$, since $\pi_{lo}^{s,c}(t) = 0$ for all $t > 0$. However, the fraction $\mu_{hn}(t)$ drops below $\bar{\mu}_{hn}(0)$ for approximately $t_1 = 0.79$ years. During this time, probability $\pi_{hn}^{s,c}(t)$ is zero while probability $\pi_{ln}^{s,c}(t) = 1 - \bar{\mu}_{ln}(0) / \mu_{ln}(t)$ is valid for approximately $0 < t < t_1$. If a second aggregate liquidity shock occurs at t , with $0 < t < t_1$, the fraction of potential buyers must increase due to this second shock in order to ensure a constant post-shock distribution. The value functions in the ‘Completed ALS Model’ contain this possibility correctly.

Table 7.1 shows that my modification does not solely alter the value functions of *ln* agents and *hn* agents but also influences those of *lo* agents and *ho* agents, due to their interconnections.

Prices and Bid-Ask Spread

The percentage price drop in the ‘Completed ALS Model’ is 13.43%, due to the aggregate liquidity shock, which is smaller than the 14.32% price drop in the ‘ALS Model’. The overall price level is higher in the ‘Completed ALS Model’. Table 7.2 states the interinvestor prices, the bid prices, the ask prices, and the bid-ask spreads for the ‘ALS Model’, and for the ‘Completed ALS Model’, immediately after the shock and a long time after.

‘ALS Model’			‘Completed ALS Model’		
Parameter	$t = 0$	$t \rightarrow \infty$	Parameter	$t = 0$	$t \rightarrow \infty$
$P^s(t)$	7.408p	8.6469	$P^{s,c}(t)$	7.5437	8.7143
$A^s(t)$	7.4106	8.6567	$A^{s,c}(t)$	7.5469	8.7241
$B^s(t)$	7.4062	8.6410	$B^{s,c}(t)$	7.5385	8.7084
$A^s(t) - B^s(t)$	0.0044	0.0157	$A^{s,c}(t) - B^{s,c}(t)$	0.0083	0.0157

Table 7.2: Prices in the ‘ALS Model’ and the ‘Completed ALS Model’ immediately and a long time after an aggregate liquidity shock, with market makers.

The recovery paths of these prices after an aggregate liquidity shock are shown in figure 7.3. The left panel depicts the recovery path in the ‘ALS Model’, whereas the right panel shows the price recovery path in the ‘Completed ALS Model’.

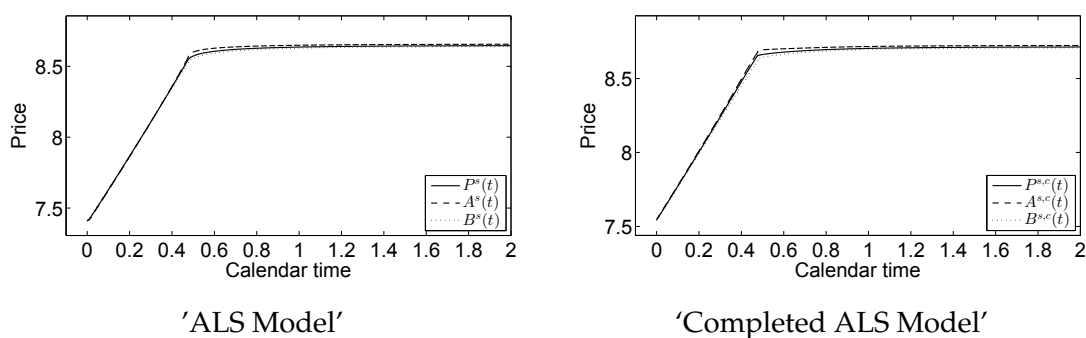


Figure 7.3: Prices in the ‘ALS Model’ and the ‘Completed ALS Model’ after an aggregate liquidity shock, with market makers.

Half of the loss in the interinvestor price is regained within 0.25 years in the ‘Completed ALS Model’, and within 0.27 years in the ‘ALS Model’. The interinvestor price reaches a fairly normal level within 1.09 years in the ‘Completed

ALS Model' and within 1.1 years in the 'ALS Model'.¹⁵² It seems as if the recovery is slightly faster with the 'Completed ALS Model'. One would like to draw the conclusion that the 'Completed ALS Model' is more efficient, because the recovery time is lower, prices are higher, and the price drop due to the shock is smaller. These facts are misleading, however, since they imply a comparison of apples and oranges: My completion adds a missing feature, without which the aggregate liquidity shock model is defective. However, the general intention of the model is not altered.

My modification also fixes some economic problems which arise due to implementing market makers: The remarkable pattern of the bid-ask spread. For the 'ALS Model', the left panel of figure 7.4 depicts the development of the bid-ask spread after an aggregate preference shock. As addressed in section 5.2, the downward hump is puzzling. One expects an immediately widening bid-ask spread shortly after the shock, since investors' outside options decline.

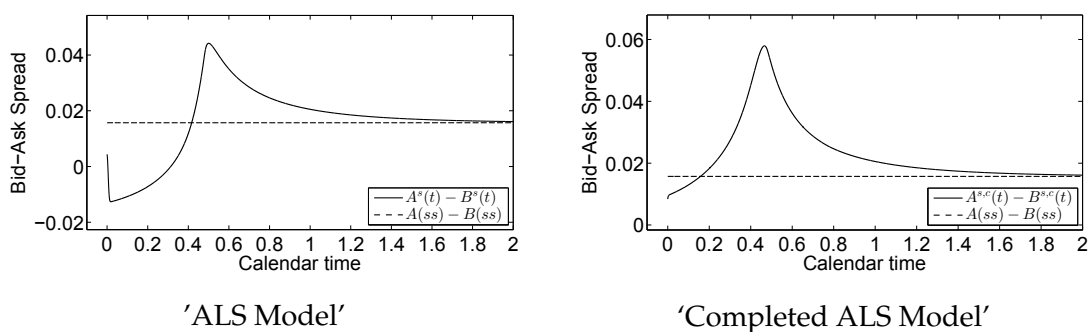


Figure 7.4: Bid-ask spread in the 'ALS Model' and the 'Completed ALS Model'. The solid line shows the bid-ask spread after an aggregate liquidity shock and over time. The dashed line shows the bid-ask spread in the basic model without aggregate liquidity shocks.

The pattern of the bid-ask spread in the 'Completed ALS Model' (right panel) is in line with theory. The bid-ask spread is lower immediately after the shock than in the long run. At this time, it is relatively easy for market makers to match suitable trading partners, since there is an elevated quantity of potential sellers and there are still some potential buyers remaining. The outside options of investors are favorable. After a short time, the majority of these remaining potential buyers are matched with potential sellers. The outside option of potential buyers decreases while potential sellers' outside options remain low. It is more time-consuming to match potential buyers and potential sellers. As a result, search costs increase. The bid-ask spread is the highest around point t^* , where as many potential sellers as potential buyers are available. However, both fractions are very low so that

¹⁵² After 1.09 years (1.1 years), the percentage price change is less than 0.001% per day.

there are nearly no agents in the market who want to trade. Search costs are the highest then. Afterwards, the market—and therefore also the bid-ask spreads—start to return to normality.

Gains from Trade

No gains from trade but forced trading—this was the result of chapter 6. It is due to the inconsistency between the underlying process of equilibrium type fractions $\mu_\sigma(t)$ and the value functions $V_\sigma^s(t)$, which are based on the process $\mu_\sigma(t)$. My completion of the aggregate liquidity shock model fixes these issues: gains from trade, voluntary trading, and consistency.

Gains from trade arise for the ‘ALS Model’ if the difference $\Delta V_o^s(t) - \Delta V_n^s(t)$ is nonnegative for all t , and for the ‘Completed ALS Model’ if the difference $\Delta V_o^{s,c}(t) - \Delta V_n^{s,c}(t)$ is nonnegative for all t . Figure 7.5 depicts the shape over time of those differences.

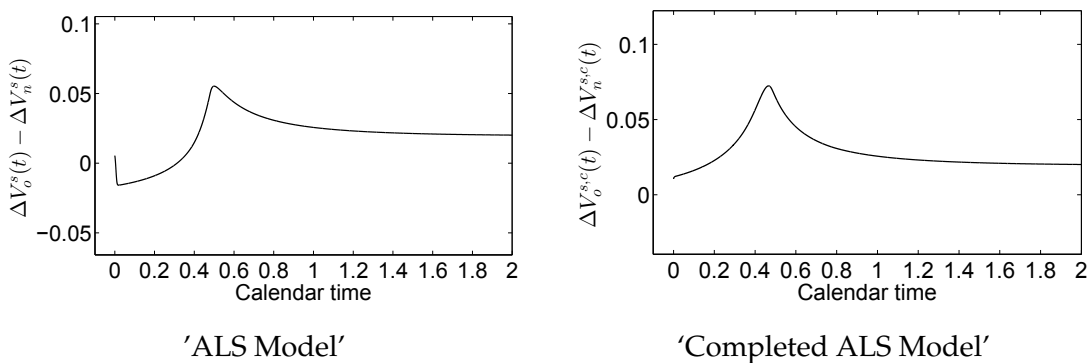


Figure 7.5: Gains from trade in the ‘ALS Model’ and the ‘Completed ALS Model’.

In the ‘ALS Model’ (left panel), there are temporarily no gains from trade, since the difference $\Delta V_o^s(t) - \Delta V_n^s(t)$ can become negative. During this time, agents are forced to trade. In the ‘Completed ALS Model’ (right panel), however, the difference $\Delta V_o^{s,c}(t) - \Delta V_n^{s,c}(t)$ is positive, i.e. there are always gains from trade and agents trade voluntarily.

Figure 7.6 shows that the difference $\Delta V_o^{s,c}(t) - \Delta V_n^{s,c}(t)$ in the ‘Completed ALS Model’ is nonnegative even for very high meeting intensities. To address this, I keep market makers’ meeting intensity constant (with $\rho = 125$) but vary investors’ meeting intensity with $\lambda = [200, 10\,000]$. As a result, there are always

gains from trade due to my completion of the aggregate liquidity shock model. Appendix 7A displays the corresponding conditions analytically.

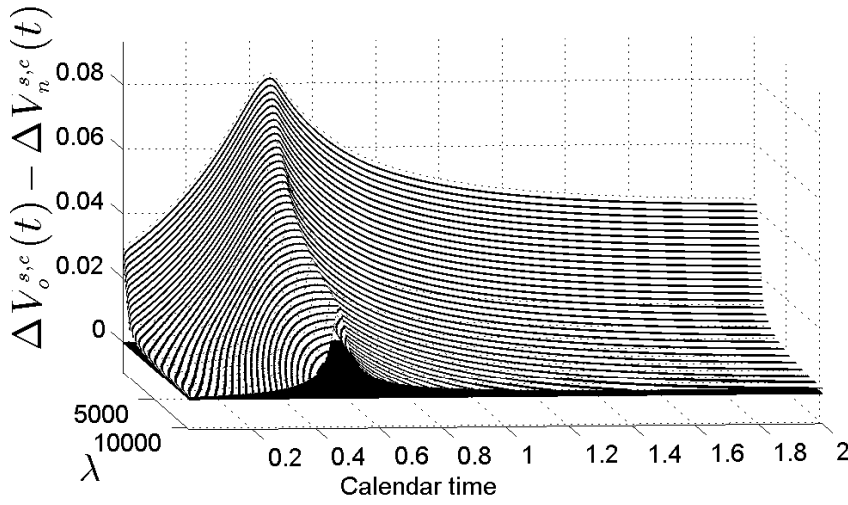


Figure 7.6: Gains from trade in the ‘Completed ALS Model’, for $\rho = 125$ and $\lambda \in [200, 10\,000]$.

For the sake of completeness, figure 7.7 shows the redefined probabilities for high investors switching to low investors upon a shock, i.e. $\pi_{ho}^{s,c}(t)$ and $\pi_{hn}^{s,c}(t)$, both as a function of time and meeting intensity λ . These probabilities are nonnegative and have values in the range between 0 and 1.

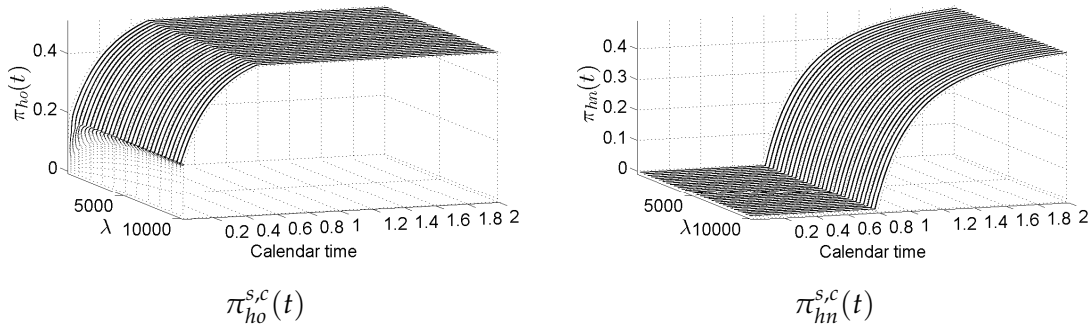


Figure 7.7: Evolution of probability $\pi_{ho}^{s,c}(t)$ and $\pi_{hn}^{s,c}(t)$ with $\rho = 125$ and $\lambda \in [200, 10\,000]$.

Figure 7.8 depicts the newly defined probabilities for completing the aggregate liquidity shock model, i.e. $\pi_{lo}^{s,c}(t)$ and $\pi_{ln}^{s,c}(t)$, likewise as a function of time and meeting intensity λ . As discussed in appendix 6C, either $\mu_{lo}(t)$ or $\mu_{hn}(t)$ converges monotonically to its steady state value after an aggregate liquidity shock and the other moves away for a while, or both converge monotonically. Therefore, either $\pi_{lo}^{s,c}(t)$ or $\pi_{ln}^{s,c}(t)$ or both are zero for all t . Within the prevailing example, $\mu_{hn}(t)$ moves away for a while, before it converges to its steady state, i.e. $\pi_{lo}^{s,c}(t) = 0 \forall t$.

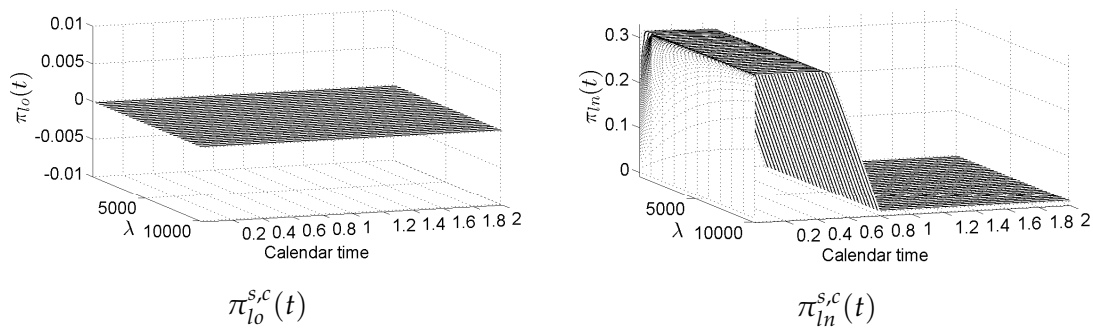


Figure 7.8: Evolution of probability $\pi_{lo}^{s,c}(t)$ and $\pi_{hn}^{s,c}(t)$ with $\rho = 125$ and $\lambda \in [200, 10000]$.

7.4 Conclusion

In this chapter, I unveil the incompleteness of the aggregate liquidity shock model: There is an economic inconsistency between the type process $\mu_\sigma(t)$ and the value functions $V_\sigma^s(t)$ via the probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$ of high agents switching to a low state due to a shock. The aggregate liquidity shock model implicitly assumes that these probabilities can become negative. However, negative probabilities imply that a negative fraction of high agents can switch to low agents, which is—from an economic perspective—impossible.

I modify and therefore complete the aggregate liquidity shock model by including the missing link in the value functions. As a result, the completed aggregate liquidity shock model fixes all issues: no gains from trade, forced trading, and inconsistency. However, my completion does not alter the general intention of the aggregate liquidity shock model but allows for reasonable interpretations—especially for the bid-ask spread. The overall findings of Duffie, Gârleanu, and Pedersen (2007) apply in general.

7A Appendix: Gains from Trade

For my completed model, I check within this appendix whether there are always gains from trade after an aggregate liquidity shock. The approach is equivalent to section 6.3. The two-dimensional system of differential equations is

$$\begin{bmatrix} \Delta \dot{V}_o^{s,c}(t) \\ \Delta \dot{V}_n^{s,c}(t) \end{bmatrix} = \begin{bmatrix} \left(\begin{array}{cc} r+\zeta+\lambda_d+\lambda_u & - \\ +2\lambda\mu_{hn}(t)q+\rho(1-z)\tilde{q}(t) & +\rho(1-z)\tilde{q}(t) \end{array} \right) & - \left(\begin{array}{c} 2\lambda\mu_{hn}(t)q \\ +\rho(1-z)\tilde{q}(t) \end{array} \right) \\ - \left(\begin{array}{c} 2\lambda\mu_{lo}(t)(1-q) \\ +\rho(1-z)(1-\tilde{q}(t)) \end{array} \right) & \left(\begin{array}{cc} r+\zeta+2\lambda\mu_{lo}(t)(1-q) & \\ +\lambda_d+\lambda_u+\rho(1-z)(1-\tilde{q}(t)) & \end{array} \right) \end{bmatrix} \begin{bmatrix} \Delta V_o^{s,c}(t) \\ \Delta V_n^{s,c}(t) \end{bmatrix} \\ - \begin{bmatrix} \delta \\ 0 \end{bmatrix} \\ - \begin{bmatrix} \zeta(1-\pi_{ho}^{s,c}(t)-\pi_{lo}^{s,c}(t)) & 0 \\ 0 & \zeta(1-\pi_{hn}^{s,c}(t)-\pi_{ln}^{s,c}(t)) \end{bmatrix} \begin{bmatrix} \Delta V_o^{s,c}(0) \\ \Delta V_n^{s,c}(0) \end{bmatrix}. \quad (7.13)$$

Calculations that proceed similarly to appendix 6B show that there are gains from trade immediately after the shock, i.e. $\Delta V_o^{s,c}(0) - \Delta V_n^{s,c}(0) \geq 0$, if the inequality

$$\begin{aligned} 0 \leq \frac{\delta \zeta}{\Psi^c} & \left\{ \underbrace{\left[\int_0^\infty e^{-\bar{\lambda}_1^s x} (1 + \bar{q}_2(x)) dx \right]}_{\text{Part E}'} \times \right. & (7.14) \\ & \left. \underbrace{\left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [-\pi_{hn}^{s,c}(x) - \pi_{ln}^{s,c}(x) + \pi_{ho}^{s,c}(x) + \pi_{lo}^{s,c}(x)] dx \right]}_{\text{Part F}'} \right. \\ & + \underbrace{\left[\frac{1}{\zeta} - \int_0^\infty e^{-\bar{\lambda}_1^s x} (1 - \pi_{ho}^{s,c}(x) - \pi_{lo}^{s,c}(x)) (1 + \bar{q}_2(x)) dx \right]}_{\text{Part Ga}'} \\ & \left. + \underbrace{\left[\int_0^\infty e^{-\bar{\lambda}_1^s x} \bar{q}_2(x) (1 - \pi_{hn}^{s,c}(x) - \pi_{ln}^{s,c}(x)) dx \right]}_{\text{Part Gb}'} \right] \left\{ \underbrace{\left[\int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} dx \right]}_{\text{Part H}'} \right\}, \end{aligned}$$

holds, where

$$\begin{aligned}
 \Psi^c = \zeta^2 \times & \left\{ \left[\underbrace{\frac{1}{\zeta} - \int_0^\infty e^{-\bar{\lambda}_1^s x} (1 - \pi_{ho}^{s,c}(x) - \pi_{lo}^{s,c}(x)) (1 + \bar{q}_2(x)) dx}_{\text{Part A'}} \right] \times \right. \\
 & \left[\underbrace{\frac{1}{\zeta} - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{hn}^{s,c}(x) - \pi_{ln}^{s,c}(x)) dx}_{\text{Part B'}} \right] \\
 & + \left[\underbrace{\int_0^\infty e^{-\bar{\lambda}_1^s x} \bar{q}_2(x) (1 - \pi_{hn}^{s,c}(x) - \pi_{ln}^{s,c}(x)) dx}_{\text{Part C'}} \right] \times \\
 & \left. \left[\underbrace{\frac{1}{\zeta} - \int_0^\infty e^{-\int_0^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} (1 - \pi_{ho}^{s,c}(x) - \pi_{lo}^{s,c}(x)) dx}_{\text{Part D'}} \right] \right\}, \tag{7.15}
 \end{aligned}$$

and $\bar{q}_2(t)$, $\bar{\lambda}_1^s$, $\bar{\lambda}_2^s(t)$ are as defined in equations (6.7), (6.8), and (6.9), respectively.

I partitioned equation (7.15) and inequality (7.14) in a way similar to equation (6.45) and (6.46), respectively. The size of the respective parts are roughly the same as in appendix 6B and their analysis in appendix 6C, except that no single input parameter increases or decreases excessively in (7.14) and (7.15). This effect is due to well-defined probabilities, with $0 \leq \pi_{\sigma}^{s,c}(t) \leq 1$. In particular, part C' of equation (6.45) and part F' of equation (6.46) stay within a reasonable range.

To proof the validity of the bargaining condition in $t = 0$, I again use the modified version of Steffensen's inequality, quoted on page 167, to estimate the integrals:

Part A', part B', |part C'|, and part D' are all smaller than $1/\bar{\lambda}_1^s$. Furthermore, |part C'| is very small, compared to $(1/\zeta - \text{part A}')$, $(1/\zeta - \text{part B}')$, and $(1/\zeta - \text{part D}')$. From this it follows that $\Psi^c > 0$ for sufficiently small ζ .

Part E', |part F'|, part Ga', |part Gb'|, and part H' are all smaller than $1/\bar{\lambda}_1^s$. |Part F'| < part H' and part E' < $(1/\zeta - \text{part Ga}' + |\text{part Gb}'|)$. It follows that the inequality $\Delta V_o^{s,c}(0) - \Delta V_n^{s,c}(0) \geq 0$ is fulfilled for sufficiently small ζ .

Condition $\Delta V_o^{s,c}(t) - \Delta V_n^{s,c}(t) \geq 0$ is obtained as

$$\int_t^\infty e^{-\int_t^x \bar{\lambda}_2^s(\mu(\tau)) d\tau} [\delta + \zeta (1 - \pi_{ho}^{s,c}(x) - \pi_{lo}^{s,c}(x)) \Delta V_o^{s,c}(0) - \zeta (1 - \pi_{hn}^{s,c}(x) - \pi_{ln}^{s,c}(x)) \Delta V_n^{s,c}(0)] dx \geq 0. \quad (7.16)$$

The interpretation has the same tendency: No single input parameter increases or decreases excessively, since all probabilities are defined with $0 \leq \pi_\sigma^{s,c}(t) \leq 1$. The condition $\Delta V_o^{s,c}(0) - \Delta V_n^{s,c}(0) \geq 0$ holds, so that condition (7.16) is met for sufficiently small ζ and $\delta > 0$.

The constraint a long time after an aggregate liquidity shock is equal to equation (6.11), since $\pi_{ln}^{s,c}(ss)$ and $\pi_{lo}^{s,c}(ss)$ are zero.

Chapter 8

Summary and Conclusion

This dissertation addressed liquidity and aggregate liquidity shocks in over-the-counter (OTC) markets. The topic was inspired by the pioneering work of Duffie, Gârleanu, and Pedersen (2005, 2007), who initiated a new strand of literature about asset pricing in OTC markets. I completed the aggregate liquidity shock model of Duffie, Gârleanu, and Pedersen (2007), since it turned out to be imperfect.

First, I introduced the basic search and bargaining model by Duffie, Gârleanu, and Pedersen (2005) for asset pricing in an illiquid OTC market. Illiquidity is modeled with search frictions, which imply that trade does not happen instantly. Upon finding a trading partner, asset prices are directly bargained between those agents. To begin, I discussed the steady state equilibrium masses of investor types. Then, I displayed the derivation of asset prices and bid-ask spreads in steady state, which are based on the equilibrium masses. The presented formulas can be applied flexibly to a buyer's market, a seller's market, as well as to a balanced market in steady state. The results of the basic model presented in chapter 3 are steady state asset prices that are lower due to search frictions, compared to prices in a perfect market. The more challenging it is to find a potential trading partner, the higher the illiquidity discount.

Secondly, I analyzed the dynamics out of and towards the steady state of prices and of return reactions after aggregate liquidity shocks, initially addressed by Duffie, Gârleanu, and Pedersen (2007). Aggregate or systemic liquidity shocks are associated with a sudden shift in agents' preferences towards asset holding, affecting a large fraction of investors simultaneously. Several investors experience a sudden decrease in their liquidity, leading to a forced withdrawal of assets: The market is hit by a selling pressure. After presenting an analytical so-

lution method for the aggregate liquidity shock model, I developed in chapter 4 the semi-analytical solution for the resulting linear time-varying (LTV) system of differential equations, including market makers. The applied solution technique is a Riccati transformation, which is specified by an algorithm derived in van der Kloet and Neerhoff (2001, 2004) and van der Kloet, Neerhoff, and de Anda (2001). My results revealed that the solution stated by Duffie, Gârleanu, and Pedersen (2007, p. 1883) does not solve the associated system of differential equations.

The effects, implications, and general characteristics of aggregate liquidity shocks on investors' types, value functions, and prices, as well as the recovery pattern of the market in general, were depicted in chapter 5 by means of a numerical example: A shock causes a selling pressure, which results in a price drop. During the recovery time, selling an asset is more time-consuming. The market recovers from this shock over time, whereas the recovery time depends on search intensity. Prices reach a normal level after a while, which is lower compared to the steady state price level without aggregate liquidity shocks. Agents anticipate the risk that a shock can occur in the future. My extension of the aggregate liquidity shock model by market makers revealed additional results: Market makers provide superior search service, since prices reach their normal level disproportionately faster; an increase in market liquidity results. Aggregate liquidity shocks immediately reduce market makers' bid-ask spread, because investors face better outside options. In the long run, the bid-ask spread is only marginally affected by the risk of repeated shocks.

The semi-analytical solution for the dynamic aggregate liquidity shock model facilitated the possibility of verifying gains from trade, which is an essential prerequisite of the Duffie, Gârleanu, and Pedersen (2005, 2007) models. After analyzing the Nash bargaining solution, which requires gains from trade, I showed in chapter 6 that the aggregate liquidity model could endogenously induce a market freeze, so that agents would not trade during all possible market conditions. However, the aggregate liquidity shock model pretends that agents trade despite no gains from trade, i.e. agents are forced to trade. No gains from trade result, in particular, with high meeting intensities and are due to the risk of future shocks. I suggested some modifications to both the meeting intensities and the characteristics of the shock in order to address the issue of voluntary trading.

I demonstrated the limits of the aggregate liquidity shock model in chapter 7 by pointing out a model incompleteness: There is an economic inconsistency between the underlying type process of investors $\mu_\sigma(t)$ and the corresponding value functions $V_\sigma^s(t)$ via the probabilities $\pi_{ho}(t)$ and $\pi_{hn}(t)$ of high agents

switching to a low state due to a shock. The aggregate liquidity shock model implicitly assumes that these probabilities can become negative, implying that a negative fraction of high agents can switch to low agents—which is impossible from an economic perspective. This effect is the reason for no gains from trade in the aggregate liquidity shock model. I completed the aggregate liquidity shock model by including the missing link into the value functions. As a result, I fixed all issues—no gains from trade, forced trading, and model inconsistency.

My completed aggregate liquidity shock model does not alter the intention of the aggregate liquidity shock model, but allows for reasonable interpretations. The overall findings and conclusions of Duffie, Gârleanu, and Pedersen (2007) apply in general: An aggregate liquidity shock causes a selling pressure, which results in an immediate price drop. Selling an asset is more time-consuming. A meeting between a potential buyer and a potential seller always results in a voluntary trade and there is no market freeze. This leads to a market recovery over time for sure. Prices reach a fairly normal level after a while. This price level is lower compared to the steady state price level without aggregate liquidity shocks, because agents anticipate the risk of shocks in the future. Finally, I showed that market makers' bid-ask spread is—in the long run—only marginally affected by the risk of aggregate liquidity shocks.

Further research can extend the completed aggregate liquidity shock model by including search friction for varying degrees of investor's sophistication in the style of Feldhütter (2012). This increases the dimension of the LTV system of differential equations and might lead to a more complex solution. An extension that allows market makers to hold inventory for overcoming systemic liquidity shocks, which occur once in a while, would be interesting as well.

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