On Weiss-Staffans Perturbations of Semigroup Generators

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Itaca ti ha dato il bel viaggio,
senza di lei mai ti saresti messo
sulla strada: che cos'altro ti aspetti?
(Itaca - Kostantinos Kavafis)

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Mein erster Dank geht an Rainer Nagel, er hat mich dazu gebracht diesen “törichten und fantastischen Flug” zu unternehmen indem er an mich geglaubt und mich inspiriert hat. Ihm widme ich folgende Verse aus Inferno Canto XXVI der Divina Commedia von Dante Alighieri.

‘‘Ihr Brüder’’, sagte ich ‘‘durch hunderttausend
Gefahren seid ihr nach Westen gelangt,
wollt nun bitte der so kurzen Wachzeit

unserer Sinne, die uns noch verbleibt,
die Erfahrung nicht verweigern,
wie hinter der Sonne die Welt ohne Menschen aussieht.

Bedenkt den Samen, den ihr in euch tragt:
Geschaffen wart ihr nicht, damit ihr lebtet wie die Tiere,
vielmehr um Tugend und Erkentniss anzustreben’’.

Meine Gefährten machte ich mit dieser kurzen Rede
so begierig auf die Fahrt,
dass ich sie danach nur mit Mühe noch hätte zurückhalten können.

Wir wendeten das Heck dem Morgen zu,
machten dem törichten Flug mit den Rudern Flügel
und gewannen dabei stets in der Richtung zur Linken.

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der mir Ehre gemacht hat.

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Introduction

The time evolution of physical phenomena can often be described by a system of linear partial differential equations. Sometimes we can rewrite such a system as an abstract Cauchy problem by introducing a linear operator $(A, D(A))$ on an appropriate state space X such that the problem takes the form

$$\text{(CP)} \qquad \begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0, & t \geq 0. \end{cases}$$

Following Hadamard one calls such a problem well–posed if

- it has a solution,
- this solution is unique,
- the solution depends continuously on the initial data.

With appropriate definition this type of well-posedness is equivalent to the fact that the operator $(A, D(A))$ generates a C_0-semigroup on X, see [**EN00**, Thm. II.6.7], hence to the Hille–Yoshida conditions of A (see [**EN00**, Thm. II.3.8]).

However, to verify these conditions for a concrete operator is often a difficult task. One approach it to split the given operator A into a sum of simpler operators, i.e., $A = A_1 + \ldots + A_n$.

Even if we assume that the operators $A_1, \ldots, A_n$ generate a C_0-semigroup, it remains to show that the "sum" $A_1 + \ldots + A_n$ is again the generator of a C_0-semigroup. This is highly nontrivial and in fact consists of two partial problems.

Given a generator A and a perturbation P on a Banach space X

1. *How should one define the "sum" $A + P$?*
2. *Under which conditions on P is this sum a generator?*

Numerous results are known in this field (see, e.g., [**EN00**, Sects. III.1–3 & related Notes]), but no unifying and general theory is yet available.

Our aim is to go a step towards a more systematic perturbation theory for such generators. To this end we choose the following setting. For the generator A with domain $D(A) \subset X$ consider perturbations

$$P : D(P) \subset X \to X_{-1}^{A},$$

where X_{-1}^{A} is the extrapolated space associated to A (see [**EN00**, Sect. II.5.a]). The sum is then defined as $A_P := (A_{-1} + P)|_X$, i.e.,

$$A_P x = A_{-1}x + Px \quad \text{for } x \in D(A_P) := \left\{ z \in D(P) : A_{-1}z + Pz \in X \right\}. \tag{0.1}$$

Then we ask for which P remains A_P a generator on X. The bounded perturbation theorem ([**EN00**, Sect. III.1]), the Desch–Schappacher ([**EN00**, Sect. III.3.a]) and the Miyadera–Voigt theorems ([**EN00**, Sect. III.3.c]) give some well-known answers in these cases.

A more general result in this direction is the Weiss–Staffans theorem on the well-posedness of perturbed linear systems, cf. [**Wei94a**, Thms. 6.1 and 7.2] and [**Sta05**, Sects. 7.1 & 7.4].

In Chapter 1 we introduce the notions of admissibility for control-, observation-, feedback- and pairs of operators. These concept are then used to formulate and prove the Weiss–Staffans theorem on the well-posedness of linear control systems with feedback in a purely operator theoretic way, see Theorem 1.2.1. All this has been published in [**ABE14**].

We conclude this chapter with a generalization of the Weiss–Staffans perturbation theorem, see Theorem 1.3.3.

In Chapter 2 we apply Theorem 1.3.3 in order to characterize the well-posedness of linear control systems by the generator property of an operator matrix.

In Chapter 3 we apply Theorem 1.2.1 in order to generalize the result of Greiner [**Gre87**] to unbounded perturbations of the boundary condition of a generator.

The results of this chapter have been published in [**ABE14**].

In Chapter 4 we consider Weiss–Staffans perturbations of analytic semigroups. We simplify the conditions appearing in Theorem 1.2.1 by using of the concept of Favard spaces and fractional powers of a generator, see Theorem 4.2.3.

The results of this chapter shall be published in a forthcoming paper together with M. Adler and K.-J. Engel.

In Chapter 5 we first introduce the concept of measurable evolution family and then use Theorem 1.3.3 to extend the Weiss–Staffans perturbation theorem to time dependent perturbations, see Theorem 5.2.4.

In Chapter 6 we apply the results of Chapter 5 to time dependent boundary perturbations. We conclude with a concrete example concerning transport on networks.

CHAPTER 1

The Weiss–Staffans perturbation theorem

When we are interested in the generator property of A_P for some perturbation $P : D(P) \subset X \to X_{-1}^A$, we can assume that the growth bound $\omega_0(A) < 0$ and hence

$$0 \in \rho(A).$$

This condition on the growth bound implies that the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable, i.e., there exists $K \geq 1$ and $\omega < 0$ such that

$$\|T(t)\| \leq Ke^{\omega t} \quad \text{for all } t \geq 0. \tag{1.1}$$

For our perturbation problem this assumption is not a restriction. If we start with the generator $(A, D(A))$ of a C_0-semigroup $(T(t))_{t\geq 0}$, we can "rescale" the operator A with $\lambda \in \mathbb{C}$ and obtain (see [**EN00**, Ex. II.2.2])

$$A^\lambda := A - \lambda I,\ \ D(A^\lambda) = D(A).$$

The Sobolev-Towers (see [**EN00**, Sec. II.5.a]) of the operators A and A^λ coincide and the following holds.

Lemma 1.0.1. *The operator $A_P = A_{-1} + P$ with domain $D(A_P) = \{x \in D(P) : A_{-1}x + Px \in X\}$ is a generator on X if and only if for every $\lambda \in \mathbb{C}$ the operator $A_P^\lambda = A_{-1}^\lambda + P$ with domain $D(A_P^\lambda) = \{x \in D(P) : A_{-1}^\lambda x + Px \in X\}$ is a generator on X.*

Proof. For every $\lambda \in \mathbb{C}$

- the operator A_P^λ is a bounded perturbation of A_P:
$$\left(A_P^\lambda, D(A_P^\lambda)\right) = \left(A_P + \lambda I, D(A_P)\right),$$
- the operator A_P is a bounded perturbation of A_P^λ:
$$\left(A_P, D(A_P)\right) = \left(A_P^\lambda - \lambda I, D(A_P^\lambda)\right).$$

By the Bounded Perturbation Theorem [**EN00**, Thm. III.1.3] we obtain the assertion. □

1.1. The setting

The classical Weiss–Staffans theorem starts from an abstract linear system[1], i.e., a quadruple $(\mathbb{T}, \Phi, \Psi, \mathbb{F})$ of operator families verifying a set of functional equations (for the precise definition see [**Wei94a**, Def. 5.1]). It states that to an admissible feedback operator K (cf. [**Wei94a**, Def. 3.5]) there corresponds a unique closed-loop system $(\mathbb{T}^K, \Phi^K, \Psi^K, \mathbb{F}^K)$. Moreover, it relates the generating operators (A, B, C, D) and (A^K, B^K, C^K, D^K) of these two systems. Since the operators A and A^K are generators of C_0-semigroups, respectively, this result implicitly contains a perturbation theorem for generators of C_0-semigroups.

However, the language of linear systems is quite specialized, and it is not so evident how to deduce a perturbation result for generators from the above Weiss–Staffans theorem.

For this reason we start directly from a triple (A, B, C) of operators and then give conditions in terms of the semigroup generated by A and the operators B and C implying that $A_P := (A_{-1} + P)|_X$ for $P = BC$ generates a C_0-semigroup.

Even though in our approach it is not necessary, it may be helpful to interpret the perturbed generator as the state operator of a control system with feedback in order to give some motivation for the various definitions of "admissibility". For this reason we use some terminology from control theory.

More precisely, choose two Banach spaces X and U called *state-* and *observation-/control space*[2], respectively. On these spaces consider the operators

- $A : D(A) \subset X \to X$, called the *state operator* (of the unperturbed system),
- $B \in \mathcal{L}(U, X_{-1}^A)$, called the *control operator*,
- $C \in \mathcal{L}(Z, U)$, called the *observation operator*,

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $D(C) = Z$ is a Banach space such that

$$X_1^A \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X,$$

[1]Here we use the notation of Weiss, cf. [**Wei94a**].

[2]We assume that the observation and control spaces coincide. This, in case of closed loop systems, is no restriction of generality and somewhat simplifies the presentation.

where "$\overset{c}{\hookrightarrow}$" denotes a continuous linear injection and X_1^A is the domain $D(A)$ equipped with the graph norm. Then consider the linear control system

$$\Sigma(A,B,C) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

with control u and observation y.

The solution of $\Sigma(A,B,C)$ is formally given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,ds. \tag{1.2}$$

Closing this system by putting $u(t) = y(t)$, one formally obtains the perturbed abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = (A_{-1} + BC)x(t), & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

which is well-posed in X if and only if A_P for $P := BC \in \mathcal{L}(Z, X_{-1}^A)$ is a generator on X, cf. [**EN00**, Sect. II.6].

Before elaborating this idea, we introduce the properties needed.

1.1.1. Admissible control operators. Taking $C = 0$ in the system $\Sigma(A,B,C)$ and considering the initial value $x_0 = 0$ it is natural to ask that for every control function $u \in \mathrm{L}^p\big([0,t_0],U\big)$ one obtains a state $x(t_0) \in X$ for some/all $t_0 > 0$. Hence formula (1.2) leads to the following definition, cf. [**Wei89a**, Def. 4.1], see also [**Eng98a**].

Definition 1.1.1. The control operator $B \in \mathcal{L}(U, X_{-1}^A)$ is called *p-admissible* for some $1 \leq p < +\infty$ if there exists $t_0 > 0$ such that

$$\int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \in X \quad \text{for all } u \in \mathrm{L}^p\big([0,t_0],U\big). \tag{1.4}$$

Note that (1.4) becomes less restrictive for growing $p \in [1, +\infty)$.

Remark 1.1.2. The range condition (1.4) in the previous definition means that the operator $\mathcal{B}_{t_0} : \mathrm{L}^p\big([0,t_0],U\big) \to X_{-1}^A$ given by

$$\mathcal{B}_{t_0}u := \int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds, \quad u \in \mathrm{L}^p\big([0,t_0],U\big) \tag{1.5}$$

has range $\operatorname{rg}(\mathcal{B}_{t_0}) \subseteq X$. Since obviously $\mathcal{B}_{t_0} \in \mathcal{L}\big(\mathrm{L}^p\big([0,t_0],U\big), X_{-1}^A\big)$, the closed graph theorem implies that for admissible B the *controllability map* $\mathcal{B}_{t_0}$ belongs to $\mathcal{L}\big(\mathrm{L}^p\big([0,t_0],U\big), X\big)$. On the other hand, using integration by parts, it follows that for every $u \in \mathrm{W}^{1,p}\big([0,t_0],U\big)$

$$\int_0^{t_0} T_{-1}(t_0-s)Bu(s)\,ds = A_{-1}^{-1}\left(T_{-1}(t_0)Bu(0) - Bu(t_0) + \int_0^{t_0} T_{-1}(t_0-s)Bu'(s)\,ds\right) \in X.$$

Since $\mathrm{W}^{1,p}\big([0,t_0],U\big)$ is dense in $\mathrm{L}^p\big([0,t_0],U\big)$, this shows that the range condition (1.4) is equivalent to the existence of some $M \geq 0$ such that

$$\left\| \int_0^{t_0} T_{-1}(t_0-s)Bu(s)\,ds \right\|_X \leq M \cdot \|u\|_p \quad \text{for all } u \in \mathrm{W}^{1,p}\big([0,t_0],U\big). \tag{1.6}$$

Using (1.1), i.e. $\omega_0(A) < 0$, one can prove the following result which is closely related to [**Wei89b**, Prop. 2.5] and was shown in [**BE14**, Lem. 3.15].

Lemma 1.1.3. *If the control operator B is p-admissible, then there exists $M_B \geq 0$ such that*

$$\left\| \int_0^{t} T_{-1}(t-r)Bu(r)\mathrm{d}r \right\|_X \leq M_B \|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \qquad \textit{for all } u \in \mathrm{L}^p\big([0,+\infty),U\big),\ t \geq 0. \tag{1.7}$$

Proof. By assumption there exists $t_0 > 0$ and $M > 0$ such that

$$\left\| \int_0^{t_0} T_{-1}(t_0-r)Bu(r)\mathrm{d}r \right\|_X \leq M \|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \qquad \text{for all } u \in \mathrm{L}^p\big([0,+\infty),U\big).$$

For $0 \leq t \leq t_0$ we denote by u_{t_0-t} the translated function

$$u_{t_0-t}(s) := \begin{cases} 0 & \text{if } 0 \leq s < t_0-t, \\ u(s-t_0+t) & \text{if } s \geq t_0-t. \end{cases} \tag{1.8}$$

Then $u_{t_0-t} \in \mathrm{L}^p\big([0,+\infty),U\big)$ and $\|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} = \|u_{t_0-t}\|_{\mathrm{L}^p\left([0,+\infty),U\right)}$. Moreover

$$\int_0^t T_{-1}(t-r)Bu(r)\mathrm{d}r = \int_0^{t_0} T_{-1}(t_0-r)Bu_{t_0-t}(r)\mathrm{d}r \in X.$$

This implies

$$\begin{aligned} \left\| \int_0^{t} T_{-1}(t-r)Bu(r)\mathrm{d}r \right\|_X &= \left\| \int_0^{t_0} T_{-1}(t_0-r)Bu_{t_0-t}(r)\mathrm{d}r \right\|_X \\ &\leq M\|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \qquad \text{for all } u \in \mathrm{L}^p\big([0,+\infty),U\big). \end{aligned} \tag{1.9}$$

For $t \geq t_0$ we write $t = nt_0 + s$ for $n \in \mathbb{N}$ and $s \in [0, t_0)$. Then we obtain

$$\begin{aligned}\int_0^t T_{-1}(t-r)Bu(r)\mathrm{d}r &= \int_0^s T_{-1}(nt_0 + s - r)Bu(r)\mathrm{d}r + \int_s^{nt_0+s} T_{-1}(nt_0+s-r)Bu(r)\mathrm{d}r \\ &=: L_1 + L_2.\end{aligned}$$

We consider the two terms of the sum separately. For the first one we get $L_1 \in X$ and

$$\|L_1\|_X \leq \|T(nt_0)\| \cdot \left\| \int_0^s T_{-1}(s-r)Bu(r)\mathrm{d}r \right\|_X \leq KM\|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)}. \tag{1.10}$$

Here we used that $(T(t))_{t\geq 0}$ is bounded and (1.9). For the second term we obtain

$$\begin{aligned} L_2 &= \sum_{k=0}^{n-1} \int_{kt_0}^{(k+1)t_0} T_{-1}(nt_0 - r)Bu(r+s)\mathrm{d}r \\ &= \sum_{k=0}^{n-1} T\big((n-(k+1))t_0\big) \cdot \int_0^{t_0} T_{-1}(t_0 - r)Bu(r + s + kt_0)\mathrm{d}r \in X. \end{aligned}$$

Moreover, using (1.1) and that B is a p-admissible control operator this gives the estimates

$$\|L_2\|_X \leq K \sum_{k=0}^{n-1} e^{\omega(n-k-1)t_0} \cdot M\|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \leq \frac{KM}{1-e^{\omega t_0}}\|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)}. \tag{1.11}$$

Summing up (1.10) and (1.11) we obtain (1.7) for $M_B := MK + \frac{MK}{1-e^{\omega t_0}}$. □

By combining the previous results we obtain the following statement.

Corollary 1.1.4. *If B is a p-admissible control operator, then for every $t \geq 0$ we have $\mathrm{rg}\big(\mathcal{B}_t\big) \subset X$ and $\mathcal{B}_t \in \mathcal{L}(\mathrm{L}^p\big([0,+\infty),U\big),X)$. Moreover, the family $(\mathcal{B}_t)_{t\geq 0}$ is strongly continuous and uniformly bounded.*

Proof. If B is a p-admissible control operator, then we conclude from Remark 1.1.2 and Lemma 1.1.3 that $\mathrm{rg}\big(\mathcal{B}_t\big) \subset X$, hence by the closed graph theorem $\mathcal{B}_t \in \mathcal{L}(\mathrm{L}^p\big([0,+\infty),U\big),X)$ for every $t \geq 0$. To show that $(\mathcal{B}_t)_{t\geq 0}$ is strongly continuous let $0 \leq r \leq t$ and $u \in \mathrm{L}^p\big([0,+\infty),U\big)$. Then

$$\begin{aligned} \|\mathcal{B}_t u - \mathcal{B}_r u\|_X &= \|\mathcal{B}_t\left(u - u_{t-r}\right)\|_X \\ &\leq \|\mathcal{B}_t\| \cdot \|u - u_{t-r}\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \\ &\leq M_B\, \|u - u_{t-r}\|_{\mathrm{L}^p\left([0,+\infty),U\right)}, \end{aligned}$$

where u_{t-r} is defined as in (1.8). Since the shift on $\mathrm{L}^p\big([0,+\infty),U\big)$ is strongly continuous, we have

$$\lim_{|t-r|\to 0} \|u - u_{t-r}\|_{\mathrm{L}^p\left([0,+\infty),U\right)} = 0$$

and the assertion follows. □

1.1.2. Admissible observation operators. Next, consider $\Sigma(A,B,C)$ with $B=0$. Then it is reasonable to ask that every initial value $x_0 \in D(A)$ gives rise to an observation $y(\bullet) = CT(\bullet)x_0 \in \mathrm{L}^p\big([0,t_0],U\big)$ for some/all $t_0>0$ which also depends continuously on x_0. This leads to the following definition, cf. [**Wei89b**, Def. 6.1], see also [**Eng98a**].

Definition 1.1.5. The observation operator $C \in \mathcal{L}(Z,U)$ is called *p-admissible* for some $1 \le p < +\infty$ if there exist $t_0>0$ and $M \ge 0$ such that

$$\int_0^{t_0} \big\|CT(s)x\big\|_U^p \, ds \le M \cdot \|x\|_X^p \quad \text{for all } x \in D(A). \tag{1.12}$$

Note that (1.12) becomes more restrictive for growing $p \in [1,+\infty)$.

Remark 1.1.6. The norm condition (1.12) in the previous definition combined with the denseness of $D(A) \subset X$ implies that there exists a unique *observability map* $\mathcal{C}_{t_0} \in \mathcal{L}\big(X, \mathrm{L}^p\big([0,t_0],U\big)\big)$ satisfying $\|\mathcal{C}_{t_0}\| \le M$ such that

$$(\mathcal{C}_{t_0}x)(s) = CT(s)x \quad \text{for all } x \in D(A),\ s \in [0,t_0]. \tag{1.13}$$

Analogously to Lemma 1.1.3 we have the following result which is closely related to [**Wei89b**, Prop. 2.3] and was shown in [**BE14**, Lem. 3.9]. Here we need again Condition (1.1), i.e. $\omega_0(A) < 0$.

Lemma 1.1.7. *If the observation operator C is p-admissible, then there exists $M_C \ge 0$ such that*

$$\int_0^{t} \big\|CT(s)x\big\|_U^p \mathrm{d}s \le M_C \|x\|_X^p \qquad \textit{for all } x \in D(A),\ t \ge 0. \tag{1.14}$$

Proof. If C is p-admissible, there exists $t_0>0$ and $M>0$ such that

$$\int_0^{t_0} \big\|CT(s)x\big\|_U^p \mathrm{d}s \le M \|x\|_X^p \qquad \text{for all } x \in D(A).$$

For $t \le t_0$ it is clear that

$$\int_0^{t} \big\|CT(s)x\big\|_U^p \mathrm{d}s \le \int_0^{t_0} \big\|CT(s)x\big\|_U^p \mathrm{d}s \le M \|x\|_X^p \qquad \text{for all } x \in D(A).$$

For $t > t_0$ we can write $t = nt_0 + r$ where $n \in \mathbb{N}$ and $0 \le r < t_0$. Using (1.1) we then obtain

$$\begin{aligned}\int_0^t \|CT(s)x\|_U^p \mathrm{d}s &\le \sum_{k=0}^n \int_{kt_0}^{(k+1)t_0} \|CT(s)x\|_U^p \mathrm{d}s \\ &= \sum_{k=0}^n \int_0^{t_0} \|CT(s)T(kt_0)x\|_U^p \mathrm{d}s \\ &\le M \sum_{k=0}^n \|T(kt_0)x\|_X^p \\ &\le MK^p \frac{1}{1-e^{p\omega t_0}} \|x\|_X^p \qquad \text{for all } x \in D(A).\end{aligned}$$

Choosing $M_C := M + MK^p \frac{1}{1-e^{p\omega t_0}}$ we obtain (1.14). This concludes the proof. □

Remark 1.1.8. Lemma 1.1.7 combined with the denseness of $D(A) \subset X$ implies that there exists a unique bounded operator $\mathcal{C}_\infty \in \mathcal{L}\big(X, \mathrm{L}^p\big([0,+\infty), U\big)\big)$ satisfying $\|\mathcal{C}_\infty\| \le M_C$ such that

$$(\mathcal{C}_\infty x)(s) = CT(s)x \quad \text{for all } x \in D(A),\ s \in [0,\infty]. \tag{1.15}$$

1.1.3. Admissible pairs. Consider the system $\Sigma(A,B,C)$ with p-admissible control and observation operators B and C. The following compatibility condition is needed to proceed, cf. [**Hel76**, Sect. II.A]. For more information and various related conditions see [**Wei94b**, Thm. 5.8] and [**Sta05**, Def. 5.1.1]. Recall that $Z = D(C)$.

Definition 1.1.9. The triple (A,B,C) (or the system $\Sigma(A,B,C)$) is called *compatible* if for some $\lambda \in \rho(A)$ we have

$$\mathrm{rg}\big(R(\lambda, A_{-1})B\big) \subset Z. \tag{1.16}$$

If the inclusion (1.16) holds for some $\lambda \in \rho(A)$, then it holds for all $\lambda \in \rho(A)$ by the resolvent identity. Moreover, the closed graph theorem implies the boundedness of the operator

$$CR(\lambda, A_{-1})B \in \mathcal{L}(U) \quad \text{for all } \lambda \in \rho(A). \tag{1.17}$$

Consider now a compatible control system $\Sigma(A,B,C)$ with initial value $x_0 = 0$. Then the input-output map of $\Sigma(A,B,C)$ mapping a control $u(\bullet)$ to the corresponding observation $y(\bullet)$ by (1.2) is formally given by

$$u(\bullet) \mapsto y(\bullet) = C \int_0^\bullet T_{-1}(\bullet - s)Bu(s)\,ds.$$

Of course, the right hand side does, in general, not make sense for arbitrary $u \in \mathrm{L}^p\big([0,t_0],U\big)$ since the integral might not give values in $Z = D(C)$. However, if

$$u \in \mathrm{W}^{2,p}_0\big([0,t_0],U\big) := \big\{u \in \mathrm{W}^{2,p}\big([0,t_0],U\big) : u(0) = u'(0) = 0\big\},$$

then integrating by parts twice and using (1.16) one obtains

$$\int_0^r T_{-1}(r-s)Bu(s)\,ds = -A_{-1}^{-1}\left(Bu(r) + A_{-1}^{-1}Bu'(r) - \int_0^r T(r-s)A_{-1}^{-1}Bu''(s)\,ds\right) \in Z. \tag{1.18}$$

At this point it is reasonable to ask that the input-output map is continuous. This gives rise to the following definition.

Definition 1.1.10. The pair $(B,C) \in \mathcal{L}(U, X_{-1}^A) \times \mathcal{L}(Z,U)$ is called *p-admissible* for some $1 \le p < +\infty$ if (A,B,C) is compatible and there exist $t_0 > 0$ and $M \ge 0$ such that

$$\int_0^{t_0} \left\| C\int_0^r T_{-1}(r-s)Bu(s)\,ds \right\|_U^p dr \le M \cdot \|u\|_p^p \qquad \text{for all } u \in \mathrm{W}^{2,p}_0\big([0,t_0],U\big). \tag{1.19}$$

The pair (B,C) (or the system $\Sigma(A,B,C)$) is called *jointly p-admissible* if in addition to (1.19) B is a p-admissible control operator and C is a p-admissible observation operator.

Remark 1.1.11. If $\Sigma(A,B,C)$ is jointly p-admissible, then there exists a bounded *input-output map*

$$\mathcal{F}_{t_0} \in \mathcal{L}\Big(\mathrm{L}^p\big([0,t_0],U\big)\Big) \quad \text{such that}$$

$$(\mathcal{F}_{t_0}u)(\bullet) = C\int_0^{\bullet} T_{-1}(\bullet - s)Bu(s)\,ds \quad \text{for all } u \in \mathrm{W}^{2,p}_0\big([0,t_0],U\big). \tag{1.20}$$

Recall that we assume the semigroup $(T(t))_{t\ge0}$ to be exponentially stable. This implies the following result shown in [**BE14**, Lem. 3.22], which is analogous to Lemma 1.1.3 and 1.1.7, and closely related to [**Sta05**, Thm.2.5.4.(ii)] [**Wei89c**, Prop. 2.1].

Lemma 1.1.12. *If the pair (B,C) is jointly p-admissible, then there exists $M_{BC} \ge 0$ such that*

$$\int_0^t \left\| C\int_0^s T_{-1}(s-r)Bu(r)\mathrm{d}r \right\|_Y^p \mathrm{d}s \le M_{BC}\|u\|^p_{\mathrm{L}^p\left([0,+\infty),U\right)} \tag{1.21}$$

for all $u \in \mathrm{W}^{2,p}_0\big([0,+\infty),U\big)$, $t \ge 0$.

Proof. If the pair (B,C) is jointly p-admissible, then we can suppose without loss of generality that $t_0 = 1$ in (1.19). Then it is clear that (1.19) also holds for all $0 \le t \le 1$.

In particular, it follows that for each $0 \le t \le 1$ there exist bounded input-output maps $\mathcal{F}_t \in \mathcal{L}\big(\mathrm{L}^p([0,t],U)\big)$ such that

$$(\mathcal{F}_t u)(\bullet) = C\int_0^{\bullet} T_{-1}(\bullet - s)Bu(s)\mathrm{d}s \quad \text{for all } u \in \mathrm{W}_0^{2,p}\big([0,+\infty),U\big)$$

and $\|\mathcal{F}_t\| \le M$.

To prove (1.21) it suffices to show that it holds for every $t = n \in \mathbb{N}$. To this end we write

$$\begin{aligned}\left(\int_0^n \left\| C\int_0^s T_{-1}(s-r)Bu(r)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}} &= \left(\sum_{k=0}^{n-1}\int_k^{k+1}\left\|C\int_0^s T_{-1}(s-r)Bu(r)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}}\\ &\le \sum_{k=0}^{n-1}\left(\int_0^1\left\|C\int_0^{s+k} T_{-1}(s+k-r)Bu(r)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}}.\end{aligned}\tag{1.22}$$

The terms of the last sum can be estimated as

$$\begin{aligned}&\left(\int_0^1\left\| C\int_0^{s+k} T_{-1}(s+k-r)Bu(r)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad= \left(\int_0^1\left\|C\left(\sum_{m=0}^{k-1}\int_m^{m+1} T_{-1}(s+k-r)Bu(r)\mathrm{d}r + \int_k^{s+k} T_{-1}(s+k-r)Bu(r)\mathrm{d}r\right)\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}}\\ &\quad\le \left(\int_0^1\left\|CT(s)\sum_{m=0}^{k-1}T(k-m-1)\int_m^{m+1} T_{-1}(m+1-r)Bu(r)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}}\\ &\qquad+\left(\int_0^1\left\|C\int_0^s T_{-1}(s-r)Bu(r+k)\mathrm{d}r\right\|_U^p \mathrm{d}s\right)^{\frac{1}{p}} =: L_1 + L_2.\end{aligned}$$

We consider the two terms of this sum separately. To this end we define for $m \in \mathbb{N}$ the operators $P_m \in \mathcal{L}\big(\mathrm{L}^p([0,+\infty),U)\big)$ by $(P_m u)(s) := \mathbb{1}_{[0,1]}(s)u(s+m)$ for $s \in [0,\infty)$. Then

$$L_2 = \left\|\mathcal{F}_1 P_k u\right\|_{\mathrm{L}^p([0,+\infty),U)} \le M\left\|P_k u\right\|_{\mathrm{L}^p([0,+\infty),U)},$$

where we used that the pair (B,C) is p-admissible.

The first term of the sum can be estimated as

$$\begin{aligned}L_1 &\le M_C^{\frac{1}{p}}\left\|\sum_{m=0}^{k-1}T(k-m-1)\int_0^1 T_{-1}(1-r)Bu(r+m)\mathrm{d}r\right\|_X\\ &\le M_C^{\frac{1}{p}}K\sum_{m=0}^{k-1}e^{\omega(k-m-1)}\left\|\mathcal{B}_1 P_m u\right\|_X\\ &\le M_C^{\frac{1}{p}}M_B K\sum_{m=0}^{k-1}e^{\omega(k-m-1)}\left\|P_m u\right\|_{\mathrm{L}^p([0,+\infty),U)}.\end{aligned}$$

Here we used that C is a p-admissible observation operator, the stability condition (1.1) and that B is a p-admissible control operator. Thus using the notation

$$l_m := \begin{cases} M & \text{if } m = 0 \\ M_C^{\frac{1}{p}} M_B K e^{\omega(m-1)} & \text{if } 1 \le m \le n-1 \end{cases}$$

we obtain that for $0 \le k \le n-1$

$$\left(\int_0^1 \left\| C \int_0^{s+k} T_{-1}(s+k-r) B u(r) \mathrm{d}r \right\|_Y^p \mathrm{d}s \right)^{\frac{1}{p}} \le \sum_{m=0}^{k} l_{k-m} \|P_m u\|_{\mathrm{L}^p\left([0,+\infty),U\right)}.$$

Summing up we obtain by (1.22) for arbitrary $n \in \mathbb{N}$ and $u \in \mathrm{W}_0^{2,p}\big([0,+\infty),U\big)$ that

$$\begin{aligned} \left(\int_0^n \left\| C \int_0^{s} T_{-1}(s-r) B u(r) \mathrm{d}r \right\|_Y^p \mathrm{d}s \right)^{\frac{1}{p}} &\le \sum_{k=0}^{n-1} \sum_{m=0}^{k} l_{k-m} \|P_m u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \\ &\le \left(\sum_{k=0}^{n-1} l_k \right) \cdot \left(\sum_{k=0}^{n-1} \|P_k u\|^p_{\mathrm{L}^p\left([0,+\infty),U\right)} \right)^{\frac{1}{p}} \\ &\le \left(M + \frac{M_C^{\frac{1}{p}} M_B K}{1-e^{\omega}} \right) \cdot \|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)} \\ &=: M_{BC} \cdot \|u\|_{\mathrm{L}^p\left([0,+\infty),U\right)}, \end{aligned}$$

where in the second estimate we used Young's inequality for the convolution of sequences. □

Remark 1.1.13. If the pair (B,C) is jointly p-admissible, then by Lemma 1.1.12 the operator

$$\begin{aligned} &\mathcal{F}_\infty : \mathrm{W}_0^{2,p}\big([0,+\infty),U\big) \subset \mathrm{L}^p\big([0,+\infty),U\big) \to \mathrm{L}^p\big([0,+\infty),U\big), \\ &(\mathcal{F}_\infty u)(\bullet) := C \int_0^{\bullet} T_{-1}(\bullet - r) B u(r) \mathrm{d}r \end{aligned}$$

has a unique bounded extension to $\mathcal{L}\big(\mathrm{L}^p\big([0,+\infty),U\big)\big)$.

1.1.4. Characterization of Admissible Pairs. The aim of this section is to characterize admissibility in terms of the Laplace transform of $\mathcal{F}_\infty$. For the admissibility of the observation operator C, cf. Subsection 1.1.2, and the admissibility of the control operator B, cf. Subsection 1.1.1), this problem was posed by Weiss in [**Wei91b**], [**Wei99**] and in the sequel has been studied by various authors. We refer to [**JP04**] for a nice survey on this matter.

Here we concentrate on $\mathcal{F}_\infty$ which is related to the admissibility of the pair (B, C). Our approach is based on the concept of Fourier multipliers, cf. [**BP05**, Sect. 5.2], [**Haa06**, App. E.1]. We recall the basic definition, denoting the Fourier transform by $\mathscr{F}$.

Definition 1.1.14. Let V, W be two Banach spaces and $1 \le p < \infty$. A function $m \in \mathrm{L}^\infty\big(\mathbb{R}, \mathcal{L}(V, W)\big)$ is called (bounded) L^p-*Fourier multiplier* if the map[3]

$$v \mapsto \mathscr{F}^{-1}\left(m\mathscr{F}v\right) \qquad \text{for } v \in \mathcal{S}\left(\mathbb{R}, V\right)$$

has a continuous extension to a bounded operator from $\mathrm{L}^p(\mathbb{R}, V)$ to $\mathrm{L}^p(\mathbb{R}, W)$.

Since by Assumption 1.1 we have $i\mathbb{R} \subset \rho(A)$ we can, using (1.17), define the map

$$m : \mathbb{R} \to \mathcal{L}(U), \quad m(\gamma) \coloneqq CR(i\gamma, A_{-1})B.$$

In order to proceed we first need the following result.

Lemma 1.1.15. *Let $(T(t))_{t\ge 0}$ a C_0-semigroup on X with generator $(A, D(A))$ and $v \in \mathrm{L}^p\big([0, +\infty), X\big)$. Then the convolution $f \coloneqq T * v$ is a bounded and continuous function on $\mathbb{R}_+$. Hence for $\operatorname{Re}\lambda > 0$ its Laplace transform exists and is given by*

$$\mathcal{L}(f)(\lambda) = R(\lambda, A)\mathcal{L}(v)(\lambda).$$

If, in addition, $v \in \mathrm{L}^1\big([0, +\infty), X\big)$, then the same formula holds for $\operatorname{Re}\lambda \ge 0$.

Proof. Boundedness of f follows easily while continuity is shown in [**ABHN11**, Prop. 1.3.4]. Now take $\operatorname{Re}\lambda > 0$. Using Assumption 1.1 the integral

$$\int_0^{+\infty}\int_0^{+\infty} e^{-\operatorname{Re}\lambda(t+r)}\big\|T(t)v(r)\big\|\mathrm{d}t\mathrm{d}r \le K\int_0^{+\infty}\int_0^{+\infty} e^{-\operatorname{Re}\lambda(t+r)}e^{\omega t}\big\|v(r)\big\|\mathrm{d}t\mathrm{d}r < +\infty \tag{1.23}$$

is finite. Hence we can use Fubini's theorem (see [**ABHN11**, Thm. 1.1.9]) to conclude that

$$\begin{aligned}
\mathcal{L}(f)(\lambda) &= \int_0^{+\infty} e^{-\lambda t}\int_0^t T(t-r)v(r)\mathrm{d}r\mathrm{d}t \\
&= \int_0^{+\infty}\int_r^{+\infty} e^{-\lambda t}T(t-r)v(r)\mathrm{d}t\mathrm{d}r \\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda(t+r)}T(t)v(r)\mathrm{d}t\mathrm{d}r \\
&= R(\lambda, A)\mathcal{L}(v)(\lambda).
\end{aligned}$$

Now assume that $v \in \mathrm{L}^1\big([0, +\infty), X\big)$. By Young's inequality [**ABHN11**, Prop.1.3.5.(a)] we obtain $f \in \mathrm{L}^1\big([0, +\infty), X\big)$. Hence (1.23) still holds for $\operatorname{Re}\lambda = 0$ and the claim follows as before. $\square$

[3]By $\mathcal{S}(\mathbb{R}, V)$ we denote the space of Schwartz functions with values in the Banach space V.

This leads to the following characterization.

Proposition 1.1.16. *Let B and C be p-admissible control and observation operators, respectively. Then the pair (B,C) is p-admissible if and only if m is a bounded Fourier multiplier.*

Proof. As we have seen in Remark 1.1.13, the pair (B,C) is p-admissible if and only if the operator $\mathcal{F}_\infty$ has a bounded extension to $\mathrm{L}^p\big([0,+\infty),U\big)$. Let $\gamma \in \mathbb{R}$ and

$$u \in W^{2,p}_{0,c}([0,\infty),U) := \Big\{u \in \mathrm{W}^{1,p}\big([0,\infty),U\big) : u(0) = u'(0) = 0 \text{ and } u \text{ has compact support}\Big\}.$$

For such u we have u, u', $u'' \in \mathrm{L}^1\big([0,+\infty),U\big)$.

Let $\gamma \in \mathbb{R}$, then by (1.18) one first obtains

$$\begin{aligned}\mathcal{L}(\mathcal{F}_\infty u)(i\gamma) &= \int_0^\infty e^{-i\gamma t} C \int_0^t T_{-1}(t-r)Bu(r)\mathrm{d}r\mathrm{d}t \\ &= \int_0^\infty e^{-i\gamma t}\bigg(-CA_{-1}^{-1}Bu(t) - CA_{-1}^{-2}Bu'(t) + CA^{-1}\int_0^t T(t-r)A_{-1}^{-1}Bu''(r)\mathrm{d}r\bigg)\mathrm{d}t.\end{aligned}$$

Hence even though $\mathrm{Re}(i\gamma) = 0$, applying the second part of Lemma 1.1.15 and [**ABHN11**, Cor. 1.6.6], which states that $\widehat{v'}(\lambda) = \lambda\hat{v}(\lambda) - v(0)$ for $v \in \mathrm{W}^{1,p}\big([0,\infty),X\big)$, we obtain

$$\begin{aligned}\mathcal{L}(\mathcal{F}_\infty u)(i\gamma) &= CA^{-1}\Big(-Id - \lambda A_{-1}^{-1} + (i\gamma)^2 R(i\gamma,A)A_{-1}^{-1}\Big)B\mathcal{L}(u)(i\gamma) \\ &= CR(i\gamma,A_{-1})B\mathcal{L}(u)(i\gamma)\end{aligned}$$

for all $\gamma \in \mathbb{R}$. It thus follows that

$$\mathscr{F}(\mathcal{F}_\infty u) = m\mathscr{F}u.$$

Using this we conclude that m is a bounded Fourier-multiplier if and only if $\mathcal{F}_\infty$ has a bounded extension to $\mathrm{L}^p\big([0,+\infty),U\big)$ if and only if the pair (B,C) is p-admissible. □

1.1.5. Admissible feedback. Closing the system $\Sigma(A,B,C)$ by means of a Feedback $F \in \mathcal{L}(U)$

$$u(t) = Fy(t) \quad \text{for all } t \geq 0,$$

one formally obtains the following problem

$$\begin{cases}\dot{x}(t) = (A_{-1} + BFC)\,x(t), & t \geq 0 \\ x(0) = x_0. \end{cases}$$

As already mentioned, the operators B and C may be unbounded, thus the feedback F combines their discontinuities. Consequently, although the feedback is a bounded operator, we need a further condition.

Definition 1.1.17. An operator $F \in \mathcal{L}(U)$ is called a *p-admissible* feedback operator for some $1 \leq p < +\infty$ if there exists $t_0 > 0$ such that $\mathrm{Id} - F\mathcal{F}_{t_0} \in \mathcal{L}\big(\mathrm{L}^p\big([0,t_0],U\big)\big)$ is invertible.

In the sequel we will mainly consider $F = I \in \mathcal{L}(U)$. In this case, using the Neumann series, $F = I$ is admissible if $\|\mathcal{F}_{t_0}\| < 1$.

For further reference we collect the previous notions in a single concept.

Definition 1.1.18. Let A be the generator of a C_0-semigroup on a Banach space X, $B \in \mathcal{L}(U, X_{-1}^A)$ and $C \in \mathcal{L}(Z,U)$ for a Banach space Z satisfying $X_1^A \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X$. Then $P := BC \in \mathcal{L}(Z, X_{-1}^A)$ is called a *Weiss–Staffans perturbation for A* if for some $1 \leq p < \infty$ the following holds.

(i) (A,B,C) is a compatible triple,

(ii) B is a p-admissible control operator,

(iii) C is a p-admissible observation operator,

(iv) (B,C) is a p-admissible pair,

(v) $\mathrm{Id} \in \mathcal{L}(U)$ is a p-admissible feedback operator.

For $\mu \geq 0$ we indicate the controllability-, observability- and input-output maps associated to the triple $(A-\mu, B, C)$ with the superscript "μ", e.g.,

$$(\mathcal{F}_\infty^\mu u)(\bullet) = C\int_0^\bullet e^{-\mu(\bullet-s)}T_{-1}(\bullet - s)Bu(s)\,ds \qquad \text{for all } u \in \mathrm{W}_0^{2,p}\big([0,+\infty),U\big).$$

The next result gives a condition such that the invertibility of $I - \mathcal{F}_{t_0}$ (see condition (v) of Theorem 1.2.1) implies the one of $I - \mathcal{F}_\infty^\mu$ for μ sufficiently large.

Lemma 1.1.19. *Let BC be a Weiss–Staffans perturbation. If for $\mu \geq 0$ and $t_0 > 0$*

$$\left\|T(t_0) + \mathcal{B}_{t_0}(1-\mathcal{F}_{t_0})^{-1}\mathcal{C}_{t_0}\right\| < e^{\mu t_0} \tag{1.24}$$

holds, then $1 \in \rho(\mathcal{F}_\infty^\mu)$.

PROOF. Inspired by [**SW04**, (2.6)] and the proof of [**Wei89c**, Prop.2.1] consider for $n \in \mathbb{N}$ the surjective isometry[4]

$$J : \mathrm{L}^p\big([0, nt_0], U\big) \to \prod_{k=1}^{n} \mathrm{L}^p\big([0, t_0], U\big), \quad u \mapsto (u_1, \dots, u_n)^T,$$

where $u_k : [0, t_0] \to U$, $u_k(s) := u\big((k-1)t_0 + s\big)$ and $\|(u_1, \dots, u_n)^T\|_p^p := \sum_{k=1}^n \|u_k\|^p$.

Then $\mathcal{F}_{nt_0}$ is isometrically isomorphic to the matrix

$$J\,\mathcal{F}_{nt_0} J^{-1} = \begin{pmatrix} \mathcal{F}_{t_0} & 0 & 0 & \dots & \dots & 0 \\ \mathcal{C}_{t_0} T(t_0)^0 \mathcal{B}_{t_0} & \mathcal{F}_{t_0} & 0 & \ddots & & \vdots \\ \mathcal{C}_{t_0} T(t_0)^1 \mathcal{B}_{t_0} & \mathcal{C}_{t_0}\mathcal{B}_{t_0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \mathcal{C}_{t_0}\mathcal{B}_{t_0} & \mathcal{F}_{t_0} & 0 \\ \mathcal{C}_{t_0} T(t_0)^{n-2} \mathcal{B}_{t_0} & \dots & \dots & \mathcal{C}_{t_0} T(t_0)\mathcal{B}_{t_0} & \mathcal{C}_{t_0}\mathcal{B}_{t_0} & \mathcal{F}_{t_0} \end{pmatrix}.$$

Since by assumption $1 - \mathcal{F}_{t_0}$ is invertible, $1 - \mathcal{F}_{nt_0}$ as well as $J(1 - \mathcal{F}_{nt_0})^{-1} J^{-1} =$

$$\begin{pmatrix} \mathcal{G} & 0 & 0 & \dots & \dots & 0 \\ \mathcal{G}\mathcal{C}_{t_0}\big(T(t_0) + \mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\big)^0 \mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} & 0 & \ddots & & \vdots \\ \mathcal{G}\mathcal{C}_{t_0}\big(T(t_0) + \mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\big)^1 \mathcal{B}_{t_0}\mathcal{G} & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} & 0 \\ \mathcal{G}\mathcal{C}_{t_0}\big(T(t_0) + \mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\big)^{n-2} \mathcal{B}_{t_0}\mathcal{G} & \dots & \dots & \mathcal{G}\mathcal{C}_{t_0}\big(T(t_0) + \mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\big)\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G}\mathcal{C}_{t_0}\mathcal{B}_{t_0}\mathcal{G} & \mathcal{G} \end{pmatrix}$$

are invertible, where we put $\mathcal{G} := (1 - \mathcal{F}_{t_0})^{-1}$. By Lemma A.1.1 applied to $J(1 - \mathcal{F}_{nt_0})^{-1} J^{-1}$ one obtains the estimate

$$\|(1 - \mathcal{F}_{nt_0})^{-1}\| \le \|\mathcal{G}\| + \|\mathcal{G}\mathcal{C}_{t_0}\| \cdot \|\mathcal{B}_{t_0}\mathcal{G}\| \cdot \sum_{l=1}^{n-1} \big\|\big(T(t_0) + \mathcal{B}_{t_0}\mathcal{G}\mathcal{C}_{t_0}\big)\big\|^{l-1}. \tag{1.25}$$

This shows that $\|(1 - \mathcal{F}_{nt_0})^{-1}\|$ remains bounded as $n \to +\infty$ if (1.24) holds for $\mu = 0$.

If the estimate (1.24) only holds for some $\mu > 0$, consider the triple $(A - \mu, B, C)$. Let $M_{\varepsilon_\mu} \in \mathcal{L}\big(\mathrm{L}^p\big([0, t_0], U\big)\big)$ be the multiplication operator defined by

$$(M_{\varepsilon_\mu} u)(s) := e^{\mu s} \cdot u(s), \qquad u \in \mathrm{L}^p\big([0, t_0], U\big).$$

Then M_{ε_μ} is invertible with inverse $M_{\varepsilon_{-\mu}}$ and a simple computation shows that

$$\mathcal{B}_{t_0}^\mu = e^{-\mu t_0}\mathcal{B}_{t_0} M_{\varepsilon_\mu}, \qquad \mathcal{C}_{t_0}^\mu = M_{\varepsilon_\mu}^{-1}\mathcal{C}_{t_0} \qquad \text{and} \qquad \mathcal{F}_{t_0}^\mu = M_{\varepsilon_\mu}^{-1}\mathcal{F}_{t_0} M_{\varepsilon_\mu}. \tag{1.26}$$

[4]Denote by v^T the transposed vector of a vector v.

By similarity this implies that $1 \in \rho(\mathcal{F}_{t_0}^\mu)$. Hence, repeating the above reasoning for $(A-\mu, B, C)$ one obtains from (1.25) that $\|(1-\mathcal{F}_{nt_0}^\mu)^{-1}\|$ remains bounded as $n \to +\infty$ if

$$\left\| e^{-\mu t_0} T(t_0) + \mathcal{B}_{t_0}^\mu (1-\mathcal{F}_{t_0}^\mu)^{-1} \mathcal{C}_{t_0}^\mu \right\| < 1. \tag{1.27}$$

Since by (1.26) one has

$$e^{-\mu t_0} T(t_0) + \mathcal{B}_{t_0}^\mu (1-\mathcal{F}_{t_0}^\mu)^{-1} \mathcal{C}_{t_0}^\mu = e^{-\mu t_0}\big(T(t_0) + \mathcal{B}_{t_0}(1-\mathcal{F}_{t_0})^{-1}\mathcal{C}_{t_0}\big),$$

the estimates (1.27) and (1.24) are equivalent. Summing up, (1.24) implies that

$$K := \sup_{n\in\mathbb{N}} \left\|(1-\mathcal{F}_{nt_0}^\mu)^{-1}\right\| < +\infty. \tag{1.28}$$

Using this fact we finally show that $1 \in \rho(\mathcal{F}_\infty^\mu)$. Observe first that $(1-\mathcal{F}_\infty^\mu)u = 0$ for some $u \in \mathrm{L}^p\big([0,+\infty), U\big)$ implies that $(1-\mathcal{F}_{nt_0}^\mu)(u|_{[0,nt_0]}) = 0$ for every $n \in \mathbb{N}$. Since $(1-\mathcal{F}_{nt_0}^\mu)$ is injective for every $n \in \mathbb{N}$, this gives that $u = 0$, i.e., $1-\mathcal{F}_\infty^\mu$ is injective.

To show surjectivity fix some $v \in \mathrm{L}^p\big([0,+\infty), U\big)$ and define

$$u_n := (1-\mathcal{F}_{nt_0}^\mu)^{-1}(v|_{[0,nt_0]}) \in \mathrm{L}^p\big([0,nt_0], U\big) \quad \text{for } n \in \mathbb{N},$$

i.e., u_n is the unique solution in $\mathrm{L}^p\big([0,nt_0], U\big)$ of the equation

$$(1-\mathcal{F}_{nt_0}^\mu)u = v|_{[0,nt_0]}. \tag{1.29}$$

However, for $m \ge n$ one has $(\mathcal{F}_{mt_0}^\mu u_m)|_{[0,nt_0]} = \mathcal{F}_{nt_0}^\mu(u_m|_{[0,nt_0]})$, hence also $u_m|_{[0,nt_0]} \in \mathrm{L}^p\big([0,nt_0], U\big)$ solves (1.29). This implies that

$$u_m|_{[0,nt_0]} = u_n.$$

Thus one can define

$$u(s) := \lim_{n\to+\infty} u_n(s), \qquad s \in [0,+\infty).$$

Since, by (1.28), $\|u_n\| \le K\cdot\|v\|$ for all $n \in \mathbb{N}$, Fatou's lemma implies that $u \in \mathrm{L}^p\big([0,+\infty), U\big)$. Moreover, by construction

$$\big((1-\mathcal{F}_\infty^\mu)u\big)|_{[0,nt_0]} = (1-\mathcal{F}_{nt_0}^\mu)u_n = v|_{[0,nt_0]} \qquad \text{for all } n \in \mathbb{N},$$

which implies $((1-\mathcal{F}_\infty^\mu)u = v$. Since $v \in \mathrm{L}^p\big([0,t_0], U\big)$ was arbitrary, this shows that $1-\mathcal{F}_\infty^\mu$ is surjective. Hence $1-\mathcal{F}_\infty^\mu$ is bijective and therefore $1 \in \rho(\mathcal{F}_\infty^\mu)$ as claimed. □

Next we show that the invertibility of $\mathrm{Id} - \mathcal{F}_\infty^\mu$ implies the invertibility of $\mathrm{Id} - CR(\lambda, A_{-1})B$ for sufficiently large λ.

For this purpose we denote by

$$(\mathcal{L}u)(\lambda) := \hat{u}(\lambda) := \int_0^{+\infty} e^{-\lambda r} u(r)\, dr$$

the Laplace transform of a function u defined on $\mathbb{R}_+$. Furthermore for a Banach space X the right shift semigroup $(S_r(t))_{t\geq 0}$ on $\mathrm{L}^p([0,+\infty),X)$ is given by

$$(S_r(t)f)(s) = \begin{cases} f(s-t) & \text{for } s-t\geq 0,\\ 0 & \text{else.}\end{cases}$$

We will make use of the following result due to G. Weiss [**Wei91a**, Thm.2.3].

Proposition 1.1.20. *Let $\mathbb{C}_0$ be the right open half-plane in $\mathbb{C}$. Suppose U and Y are Banach spaces, $1\leq p<\infty$, and $\mathcal{F}\in\mathcal{L}(\mathrm{L}^p\big([0,+\infty),U\big),\mathrm{L}^p\big([0,\infty),Y\big))$ commutes with the right shift. Then there exists a (unique) bounded analytic $\mathcal{L}(U,Y)$-valued function $\boldsymbol{H}$ defined on $\mathbb{C}_0$ such that, for any $u\in\mathrm{L}^p\big([0,+\infty),U\big)$, denoting $y=\mathcal{F}u$,*

$$\hat{y}(s)=\boldsymbol{H}(s)\hat{u}(s)\quad \textit{for all } s\in\mathbb{C}_0,$$

holds and $\sup_{s\in\mathbb{C}_0}\|\boldsymbol{H}(s)\|\leq\|\mathcal{F}\|$.

We are now ready to prove the following result.

Lemma 1.1.21. *Assume that $1\in\rho(\mathcal{F}^\mu_\infty)$ for some $\mu\geq 0$. Then $1\in\rho\big(CR(\lambda,A_{-1})B\big)$ for all $\lambda\in\mathbb{C}$ satisfying $\operatorname{Re}\lambda>\mu$ and*

$$\mathcal{L}\big((\mathrm{Id}-\mathcal{F}^\mu_\infty)^{-1}u\big)(\lambda)=\big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)^{-1}\hat{u}(\lambda)\quad \textit{for all } u\in\mathrm{L}^p\big([0,+\infty),U\big).$$

Proof. Assume first that $\mu=0$. Then it is well known that $\mathcal{F}_\infty=\mathcal{F}^\mu_\infty$ commutes with the right shift (cf. [**Wei91a**]). Then also $\mathcal{G}:=\mathrm{Id}-\mathcal{F}_\infty\in\mathcal{L}\big(\mathrm{L}^p\big([0,+\infty),U\big)\big)$ commutes with the right shift, thus by Proposition 1.1.20 and similar calculation as in the proof of Proposition 1.1.16 one obtains for $u\in\mathrm{L}^p\big([0,+\infty),U\big)$

$$\widehat{(\mathcal{G}u)}(\lambda)=\big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)\cdot\hat{u}(\lambda),\quad \operatorname{Re}\lambda>0.$$

Let $\mathcal{R}:=\mathcal{G}^{-1}\in\mathcal{L}\big(\mathrm{L}^p\big([0,+\infty),U\big)\big)$. Then clearly the right shift also commutes with $\mathcal{R}$. Hence again by Proposition 1.1.20 there exists $R(\lambda)\in\mathcal{L}(U)$ such that

$$\widehat{(\mathcal{R}u)}(\lambda)=R(\lambda)\cdot\hat{u}(\lambda),\quad \operatorname{Re}\lambda>0,\ u\in\mathrm{L}^p\big([0,+\infty),U\big).$$

Summing up one obtains for all $u\in\mathrm{L}^p\big([0,+\infty),U\big)$ that

$$\begin{aligned}\hat{u}(\lambda)&=\widehat{(\mathcal{R}\mathcal{G}u)}(\lambda)=R(\lambda)\cdot\widehat{(\mathcal{G}u)}(\lambda)\\&=R(\lambda)\cdot\big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)\cdot\hat{u}(\lambda)\\&=\widehat{(\mathcal{G}\mathcal{R}u)}(\lambda)=\big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)\cdot\widehat{(\mathcal{R}u)}(\lambda)\\&=\big(\mathrm{Id}-CR(\lambda,A_{-1})B\big)\cdot R(\lambda)\cdot\hat{u}(\lambda).\end{aligned}$$

Taking $u(s) = e^{-s}v$ for some $v \in U$, this implies

$$\begin{aligned} \tfrac{1}{1+\lambda} \cdot v &= R(\lambda) \cdot \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big) \cdot \tfrac{1}{1+\lambda} \cdot v \\ &= \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big) \cdot R(\lambda) \cdot \tfrac{1}{1+\lambda} \cdot v, \quad \operatorname{Re}\lambda > 0. \end{aligned}$$

Hence $R(\lambda) = \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1}$.

If $\mu > 0$, then by the same reasoning applied to $\mathcal{F}_\infty^\mu$ one obtains that

$$1 \in \rho\big(CR(\lambda, A_{-1} - \mu)B\big) = \rho\big(CR(\lambda + \mu, A_{-1})B\big) \qquad \text{for all } \operatorname{Re}\lambda > 0.$$

Clearly this implies our claim in case $\mu > 0$ and the proof is complete. □

1.2. The theorem

In this section we state and prove our main perturbation result. It is a purely operator theoretic version of a perturbation theorem for abstract linear systems due to Weiss [**Wei94a**, Thms. 6.1 and 7.2 (1994)] in the Hilbert space case and Staffans [**Sta05**, Thms. 7.1.2 and 7.4.5 (2005)] for Banach spaces. In particular, our approach avoids the use of abstract linear systems and Lebesgue extensions. For related results see also [**Had05**] and [**Sal87**, Thms. 4.2 and 4.3].

Our result has been published in [**ABE14**] and we follow the presentation there.

Theorem 1.2.1. *Assume that $P = BC \in \mathcal{L}(Z, X_{-1}^A)$ is a Weiss–Staffans perturbation of the generator A of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. This means that there exist $1 \leq p < +\infty$, $t_0 > 0$ and $M \geq 0$ such that*

(i) $\operatorname{rg}\big(R(\lambda, A_{-1})B\big) \subset Z$ *for some* $\lambda \in \rho(A)$,

(ii) $\displaystyle\int_0^{t_0} T_{-1}(t_0 - s)Bu(s)\,ds \in X$ *for all* $u \in \mathrm{L}^p\big([0,t_0],U\big)$,

(iii) $\displaystyle\int_0^{t_0} \|CT(s)x\|_U^p\,ds \leq M \cdot \|x\|_X^p$ *for all* $x \in D(A)$,

(iv) $\displaystyle\int_0^{t_0} \Big\| C\int_0^r T_{-1}(r-s)Bu(s)\,ds \Big\|_U^p\,dr \leq M \cdot \|u\|_p^p$ *for all* $u \in \mathrm{W}_0^{2,p}\big([0,t_0],U\big)$,

(v) $1 \in \rho(\mathcal{F}_{t_0})$, *where* $\mathcal{F}_{t_0} \in \mathcal{L}\big(\mathrm{L}^p\big([0,t_0],U\big)\big)$ *is given by* (1.20).

Then

$$A_{BC} := (A_{-1} + BC)|_X, \quad D(A_{BC}) := \big\{x \in Z : (A_{-1} + BC)x \in X\big\} \tag{1.30}$$

generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X. Moreover, the perturbed semigroup verifies the variation of parameters formula

$$S(t)x = T(t)x + \int_0^t T_{-1}(t-s)\cdot BC\cdot S(s)x\,ds \quad \text{for all } t\geq 0 \text{ and } x\in D(A_{BC}). \tag{1.31}$$

For the proof we recall the extended controllability-, observability- and input-output maps from Corollary 1.1.4, Remark 1.1.8 and Remark 1.1.13. Keep in mind that we assume $\omega_0(A) < 0$.

Lemma 1.2.2. *Let (A,B,C) be compatible and (B,C) jointly p-admissible for some $1\leq p<+\infty$. Then there exist*

(i) a strongly continuous, uniformly bounded family $(\mathcal{B}_t)_{t\geq 0}\subset \mathcal{L}\big(\mathrm{L}^p\big([0,+\infty),U\big),X\big)$,

(ii) a bounded operator $\mathcal{C}_\infty\in\mathcal{L}\big(X,\mathrm{L}^p\big([0,+\infty),U\big)\big)$, and

(iii) a bounded operator $\mathcal{F}_\infty\in\mathcal{L}\big(\mathrm{L}^p\big([0,+\infty),U\big)\big)$

such that

$$\mathcal{B}_t u := \int_0^t T_{-1}(t_0-s)Bu(s)\,ds \qquad \text{for all } u\in \mathrm{L}^p\big([0,+\infty),U\big), \tag{1.32}$$

$$(\mathcal{C}_\infty x)(s) = CT(s)x \qquad \text{for all } x\in D(A),\ s\in[0,+\infty), \tag{1.33}$$

$$(\mathcal{F}_\infty u)(\bullet) = C\int_0^\bullet T_{-1}(\bullet-s)Bu(s)\,ds \qquad \text{for all } u\in \mathrm{W}_0^{2,p}\big([0,+\infty),U\big). \tag{1.34}$$

We are now well prepared to prove the above theorem.

Proof of Theorem 1.2.1. The idea is to define an operator family $(S(t))_{t\geq 0}\subset \mathcal{L}(X)$ and then to verify that it is a C_0-semigroup with generator A_{BC}.

To this end, assume that the condition (1.24) in Lemma 1.1.19 holds for $\mu=0$. Then $\mathrm{Id}-\mathcal{F}_\infty$ is invertible, and one can define

$$S(t) := T(t) + \mathcal{B}_t(\mathrm{Id}-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty\in\mathcal{L}(X),\ \ t\geq 0. \tag{1.35}$$

Since $(T(t))_{t\geq 0}$ and $(\mathcal{B}_t)_{t\geq 0}$ are both strongly continuous and uniformly bounded, the same holds for $(S(t))_{t\geq 0}$. We proceed to compute the Laplace transform of $S(\bullet)x:[0,+\infty)\to X$ for $x\in X$. Since

$$S(\bullet)x = T(\bullet)x + T_{-1}(\bullet)B * (1-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty x, \tag{1.36}$$

the convolution theorem for the Laplace transform (or [**BE14**, Lem. 3.12]) and Lemma 1.1.21 imply for every $x \in X$ and $\operatorname{Re}\lambda > 0$

$$
\begin{aligned}
\mathcal{L}\big(S(\bullet)x\big)(\lambda) &= R(\lambda, A)x + R(\lambda, A_{-1})B \cdot \mathcal{L}\big((1-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty x\big)(\lambda) \\
&= R(\lambda, A)x + R(\lambda, A_{-1})B \cdot \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} \cdot CR(\lambda, A)x \\
&=: Q(\lambda)x. \qquad (1.37)
\end{aligned}
$$

We now show that $Q(\lambda) = R(\lambda, A_{BC})$. First note that by the compatibility condition (1.16) one has

$$\operatorname{rg}\big(Q(\lambda)\big) \subset D(A) + Z = Z = D(C).$$

Moreover,

$$
\begin{aligned}
&(\lambda - A_{-1} - BC) \cdot Q(\lambda) = \\
&\qquad = \mathrm{Id} - BCR(\lambda, A) + B \cdot \mathrm{Id} \cdot \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} CR(\lambda, A) \\
&\qquad\qquad - B \cdot CR(\lambda, A_{-1})B \cdot \big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} CR(\lambda, A) \\
&\qquad = \mathrm{Id}.
\end{aligned}
$$

This implies that $Q(\lambda)$ is a right inverse and $\operatorname{rg}\big(Q(\lambda)\big) \subset D(A_{BC})$. To show that it is also a left inverse take $x \in D(A_{BC}) \subset Z = D(C)$. Then we obtain

$$
\begin{aligned}
&Q(\lambda) \cdot (\lambda - A_{-1} - BC)x = \\
&\quad = x - R(\lambda, A_{-1})BCx + R(\lambda, A_{-1})B\big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} \cdot \mathrm{Id} \cdot Cx \\
&\qquad - R(\lambda, A_{-1})B\big(\mathrm{Id} - CR(\lambda, A_{-1})B\big)^{-1} \cdot CR(\lambda, A_{-1})B \cdot Cx \\
&\quad = x.
\end{aligned}
$$

This shows $Q(\lambda) = R(\lambda, A_{BC})$ as claimed. Summing up we showed that $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a strongly continuous family with Laplace transform $R(\lambda, A_{BC})$. By [**ABHN11**, Thm. 3.1.7] this implies that $(S(t))_{t\geq 0}$ is a C_0-semigroup with generator A_{BC}.

To verify the variation of parameters formula (1.31) one first notes that by Lemma 1.1.21 and the explicit representation of $R(\lambda, A_{BC})$ in (1.37) one has

$$\mathcal{L}\big((1-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty(\bullet)x)\big)(\lambda) = \mathcal{L}\big(CS(\bullet)x\big)(\lambda) \quad \text{for all } x \in D(A_{BC}) \text{ and } \operatorname{Re}\lambda > \mu = 0.$$

By the uniqueness of the Laplace transform this implies that

$$(1-\mathcal{F}_\infty)^{-1}\mathcal{C}_\infty(\bullet)x = CS(\bullet)x,$$

and the assertion follows from the definition of $(S(t))_{t\geq 0}$ in (1.36).

Now assume that (1.24) only holds for some $\mu > 0$. Then repeating the same reasoning for the triple $(A-\mu, B, C)$ one concludes as before that $(A-\mu)_{BC} = \big((A-\mu)_{-1} + BC\big)|_X = A_{BC} - \mu$ is a generator. Clearly this implies that also A_{BC} generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. Moreover, one obtains that the rescaled semigroups $\big(e^{-\mu t}T(t)\big)_{t\geq 0}$ and $\big(e^{-\mu t}S(t)\big)_{t\geq 0}$ verify the variation of parameters formula (1.31) which implies that this formula holds for $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ as well. □

We point out that our theorem can be consider as a generalization of

- the Miyadera-Voigt perturbation Theorem [**Miy66**] and [**Voi77**], see also [**EN00**, Cor. III.3.16] and [**TW09**, Thm. 5.4.2] as shown in [**ABE14**, Sec. 4.2],
- the Desch-Schappacher perturbation Theorem [**DS89**, Thm. 5, Prop.8], see also [**EN00**, Cor. III.3.4] and [**TW09**, Cor. 5.5.1] as shown in [**ABE14**, Sect. 4.1].

1.3. A generalization of the Weiss–Staffans perturbation

In [**Miy66**] Miyadera proved the following theorem.

Theorem 1.3.1. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Assume that for $C \in \mathcal{L}(X_1^A, X)$ there exist $1 < p < +\infty$, $t_0 > 0$ and $M \geq 0$ such that*

$$\int_0^{t_0} \|CT(t)x\|_X^p \, \mathrm{d}t \leq M \|x\|_X^p \quad \text{for all } x \in D(A). \tag{1.38}$$

Then the perturbed operator $(A_C, D(A_C)) = (A + C, D(A))$ is the generator of a C_0-semigroup on X.

Voigt [**Voi77**] generalized this result considering a perturbation $C : D \subset D(A) \to X$ where D is a $(T(t))_{t\geq 0}$-invariant core of A.

Theorem 1.3.2. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Assume that $C : D \to X$ is a linear operator and $D \subset X$ a $(T(t))_{t\geq 0}$-invariant core such that*

(a) $[0, +\infty) \ni t \mapsto CT(t)x$ is continuous for every $x \in D$,

(b) there exist $0 \leq M < 1$ and $t_0 > 0$ such that

$$\int_0^{t_0} \|CT(t)x\|_X \, \mathrm{d}t \leq M \|x\|_X \quad \text{for all } x \in D.$$

Then the closure A_C of $(A+C, D)$ generates a C_0-semigroup on X. Furthermore C admits a unique extension $\tilde{C} \in \mathcal{L}(X_1, X)$ and $(A_C, D(A_C)) = (A + \tilde{C}, D(A))$.

This means that he required the estimate (1.38) only on a core of the generator A. However, he needed the further condition (a) implying the existence of an A-bounded extension of C. In a sequent paper, jointly with Thieme [**TV09**], he analyzed under which conditions such an operator C admits a continuous extension $\tilde{C}$ to all of $X_1^A = D(A)$.

Such generalizations are useful, e.g., for so called "non-autonomous" Miyadera-Voigt perturbations see [**RRS96**] and [**RSRV00**]. That is why, in order to extend our Weiss–Staffans perturbation Theorem 1.2.1 later to nonautonomous perturbation (see Chapter 5 and 6), we first generalize it by requiring condition (iii) of Theorem 1.2.1 just on a core of the considered generator.

Furthermore, in the non autonomous setting the first factor B of the Weiss-Staffans perturbation does not act within the Sobolev tower corresponding to the given generator. This also happens when one considers operator matrices, as we do by studying linear control systems in Chapter 2. We now explain in details the situation we are going to investigate.

Let $\left(\tilde{G}, D\left(\tilde{G}\right)\right)$ be the generator of a C_0-semigroup $(\tilde{T}(t))_{t\geq 0}$ on a Banach space $\tilde{X}$ and assume it to be exponentially stable, i.e., $\omega_0\left(\tilde{G}\right) < 0$.

Let X be a second Banach space with continuous and dense embedding $X \hookrightarrow \tilde{X}$. Furthermore, we assume that X is invariant under $(\tilde{T}(t))_{t\geq 0}$ and that the restriction of $(\tilde{T}(t))_{t\geq 0}$ on X is a strongly continuous semigroup denoted by $(T(t))_{t\geq 0}$. Clearly, the generator $(G, D(G))$ of $(T(t))_{t\geq 0}$ is the part of $\tilde{G}$ in X. See [**EN00**, Example II.2.3].

Let $D \subset D(G)$ be a $(T(t))_{t\geq 0}$ invariant core. Moreover, let U and $\tilde{Z}$ be Banach spaces such that $D(G) \hookrightarrow \tilde{Z} \hookrightarrow X$ and take operators $\tilde{C} \in \mathcal{L}(\tilde{Z}, U)$ and $B \in \mathcal{L}(U, \tilde{X})$.

This situation is explained in the diagramm below.

$$
\begin{tikzcd}
\tilde{X} & \\
X \arrow[u, hook] & U \arrow[ul, "B"'] \\
\tilde{Z} \arrow[u, hook] \arrow[ur, "\tilde{C}"'] & \\
D(G) \arrow[u, hook] \arrow[d, phantom, "\cup"] & \\
D &
\end{tikzcd}
$$

We are now ready to state the main result of this section.

Theorem 1.3.3. *The operator* $G_{BC} := \left(\tilde{G} + B\tilde{C}\right)_{|X}$ *generates a* C_0*-semigroup* $(S(t))_{t\geq 0}$ *on* X *if the following conditions are satisfied.*

(**a**) $\operatorname{rg}\left(\tilde{G}^{-1}B\right) \subset \tilde{Z}$.

(**b**) *There exists* $t > 0$ *and* $M_C \geq 0$ *such that*

$$\int_0^t \left\|\tilde{C}T(s)x\right\|_U^p ds \leq M_C \|x\|_X^p \quad \text{for all } x \in D. \tag{1.39}$$

(**c**) *There exists* $t > 0$ *and* $M_B \geq 0$ *such that*

$$\left\|\int_0^t \tilde{T}(t-r)Bu(r)dr\right\|_X^p ds \leq M_B \|u\|_U^p \quad \text{for all } u \in \mathrm{L}^p\big([0,t],U\big). \tag{1.40}$$

(**d**) *There exists* $t > 0$ *and* $M_{BC} \geq 0$ *such that*

$$\int_0^t \left\|\tilde{C}\int_0^r \tilde{T}(r-s)Bu(s)ds\right\|_U^p dr \leq M_{BC} \|u\|_X^p \quad \text{for all } u \in \mathrm{W}_0^{1,p}\big([0,t],U\big), \tag{1.41}$$

where $\mathrm{W}_0^{1,p}\big([0,t],U\big) := \left\{u \in W^{1,p}\big([0,t],U\big) : u(0) = 0\right\}$.

(**e**) $1 \in \rho(\tilde{\mathcal{F}}_t)$ *for one (every)* $t > 0$*, where* $\tilde{\mathcal{F}}_t \in \mathcal{L}\left(\mathrm{L}^p\big([0,t],U\big)\right)$ *is the unique continuous extension of the map*

$$\mathrm{W}_0^{1,p}\big([0,t],U\big) \ni u \mapsto \tilde{C}\int_0^{\bullet} \tilde{T}(\bullet - r)Bu(r)dr.$$

Before proving this theorem we first discuss the assumptions appearing in it.

Remark 1.3.4. Using the integral representation of the resolvent one obtains

$$R(\lambda,\tilde{G})x = R(\lambda,G)x \quad \forall x \in X \text{ and } \forall \lambda \in \rho(\tilde{G}).$$

Furthermore, applying the resolvent equation one proves that

$$\operatorname{rg}\left(\tilde{G}^{-1}B\right) \subset \tilde{Z} \iff \operatorname{rg}\left(R(\lambda,\tilde{G})B\right) \subset \tilde{Z} \quad \forall \lambda \in \rho(\tilde{G}).$$

Remark 1.3.5. Notice that in this case condition (**b**) only holds for x in a core of G, while in Theorem 1.2.1 condition (iii) was considered on $D(G)$. Furthermore, if condition (**b**) holds for one $t > 0$, then one can prove (as in Lemma 1.1.7) that it holds for every $t > 0$ with a constant M_C not depending on t.

Thus one can define an operator $\tilde{\mathcal{C}}_\infty \in \mathcal{L}(X, \mathrm{L}^p\big([0,+\infty),U\big))$ as the continuous extension of the operator

$$D \ni x \mapsto \tilde{C}T(\bullet)x \in U.$$

Remark 1.3.6. If condition (**c**) holds, one can prove (as in Lemma 1.1.3) that it holds for every $t > 0$ with a constant M_B not depending on t. Thus for every $t > 0$, the operators given by

$$\tilde{\mathcal{B}}_t u := \int_0^t \tilde{T}(t-r)Bu(r)dr, \quad u \in U,$$

belong to $\mathcal{L}(U, X)$, and the family $(\tilde{\mathcal{B}}_t)_{t \geq 0}$ is uniformly bounded and strongly continuous.

Remark 1.3.7. The left hand side of (1.41) is well-defined since for $u \in \mathrm{W}_0^{1,p}\big([0,t], U\big)$

$$\begin{aligned}
\int_0^r \tilde{T}(r-s)Bu(s)ds &= -\tilde{G}^{-1}Bu(r) + \tilde{G}^{-1} \int_0^r \tilde{T}(r-s)Bu'(s)ds \\
&= -\tilde{G}^{-1}Bu(r) + G^{-1} \int_0^r \tilde{T}(r-s)Bu'(s)ds \in \tilde{Z},
\end{aligned} \tag{1.42}$$

where we used condition **a** and **c** in the second equality.

If condition (**d**) holds for one $t > 0$, then it holds for every $t > 0$ with a constant M_{BC} not depending on t (see Lemma 1.1.12).

Thus one can define an operator $\tilde{\mathcal{F}}_\infty \in \mathcal{L}(\mathrm{L}^p\big([0, +\infty), U\big))$ as the unique continuous extension of the operator

$$W_0^{1,p}(\mathbb{R}_+, U) \ni u \mapsto \tilde{C} \int_0^\bullet \tilde{T}(\bullet - r)Bu(r)dr \in U.$$

Furthermore, as in Lemma 1.1.19 one has that if $1 \in \rho(\tilde{\mathcal{F}}_t)$, then $1 \in \rho(\tilde{\mathcal{F}}_\infty^\mu)$ for sufficiently large $\mu \geq 0$.

We are now ready to prove Theorem 1.3.3. We could do this with the same strategy used in the proof of Theorem 1.2.1, but we present a slightly modified one.

Proof. Define the Banach space $\mathfrak{X} := L^p\big(\mathbb{R}_+, \mathcal{L}_s(X, U)\big)$ equipped with the norm $\|F\| := \sup_{\|x\|_X \leq 1} \big(\int_{\mathbb{R}} \|F(t)x\|_U^p \,\mathrm{d}t\big)^{\frac{1}{p}}$ and the operator $\mathcal{V} \in \mathcal{L}(\mathfrak{X})$ as

$$(\mathcal{V}Q(\bullet))x := \tilde{\mathcal{F}}_\infty(Q(\bullet)x) \quad \forall Q \in \mathfrak{X},\ x \in X.$$

As remarked above, condition (**d**) implies that $1 \in \rho(\tilde{\mathcal{F}}_\infty^\mu)$ for sufficiently large $\mu \geq 0$. Let us first assume that this holds for $\mu = 0$, then from

$$((I - \mathcal{V})Q(\bullet))x = (I - \tilde{\mathcal{F}}_\infty)(Q(\bullet)x) \quad \forall Q \in \mathfrak{X},\ x \in X,$$

we obtain that $1 \in \rho(\mathcal{V})$ and we can define

$$R := (I - \mathcal{V})^{-1}\tilde{\mathcal{C}}_\infty \in L^p(\mathbb{R}_+, \mathcal{L}(X)).$$

Using condition (**c**), we obtain operators $S(t) \in \mathcal{L}(X)$ by

$$\begin{aligned} S(t)x &:= T(t)x + \int_0^t \tilde{T}(t-r)BR(r)x dr \\ &= T(t)x + \tilde{\mathcal{B}}_t R(\bullet)x \quad \forall x \in X,\ t \geq 0. \end{aligned}$$

Since $(T(t))_{t\geq 0}$ and $(\tilde{\mathcal{B}}_t)_{t\geq 0}$ are both strongly continuous and uniformly bounded, the same holds for $(S(t))_{t\geq 0}$.

Apply the Laplace transform to $S(\bullet)f$ and use the convolution theorem to obtain

$$\mathcal{L}\big(S(\bullet)f\big)(\lambda) = R(\lambda, G)x + R(\lambda, \tilde{G})B\mathcal{L}\big(R(\bullet)f\big)(\lambda),\ \operatorname{Re}\lambda > 0.$$

In order to compute the Laplace transform of $R(\cdot)f$, use (**b**) and notice that

$$R(\bullet)x - \mathcal{V}R(\bullet)x = \tilde{C}T(\bullet)x \quad \forall x \in D(G). \tag{1.43}$$

For $x \in D(G)$ we obtain

$$\begin{aligned} \int_0^\infty e^{-\lambda r}\tilde{C}T(r)xdr &= \tilde{C}G^{-1}\int_0^\infty e^{-\lambda r}GT(r)xdr \\ &= \tilde{C}\int_0^\infty e^{-\lambda r}T(r)xdr \\ &= \tilde{C}R(\lambda, G)x. \end{aligned} \tag{1.44}$$

This is well-defined and admits a (unique) bounded extension to X.

Furthermore, for $u \in \mathrm{W}_0^{1,p}\big(\mathbb{R}_+, U\big)$ we obtain

$$\begin{aligned} &\int_0^\infty e^{-\lambda t}\tilde{C}\int_0^t \tilde{T}(t-r)Bu(r)drdt = \\ &= \int_0^\infty e^{-\lambda t}\tilde{C}\Big[-\tilde{G}^{-1}Bu(t)(\bullet) + \tilde{G}^{-1}\int_0^t \tilde{T}(t-r)Bu'(r)dr\Big]dt \\ &= \tilde{C}\Big[-\tilde{G}^{-1}B\int_0^\infty e^{-\lambda t}u(t)dt + \tilde{G}^{-1}\int_0^\infty e^{-\lambda t}\int_0^t \tilde{T}(t-r)Bu'(r)drdt\Big] \\ &= \tilde{C}\Big[\int_0^\infty e^{-\lambda t}\int_0^t \tilde{T}(t-r)Bu(r)drdt\Big] \\ &= \tilde{C}R(\lambda, \tilde{G})B\mathcal{L}\big(u\big)(\lambda). \end{aligned}$$

This is well-defined thanks to assumption (**d**) and admits a (unique) bounded extension to $\mathrm{L}^p\big([0, +\infty), U\big)$.

Thus we conclude that

$$\mathcal{L}(\mathcal{V}R(\bullet)x)(\lambda) = \tilde{C}R(\lambda, \tilde{G})B\mathcal{L}(R(\bullet)x)(\lambda). \tag{1.45}$$

Combining equations (1.43), (1.44) and (1.45), one obtains

$$[I - \tilde{C}R(\lambda,\tilde{G})B]\mathcal{L}(R(\bullet)x)(\lambda) = \tilde{C}R(\lambda,G)x \quad \forall x \in D(G).$$

By the same calculation as in Lemma 1.1.21 one obtains that $I - \tilde{C}R(\lambda,\tilde{G})B$ is invertible. Therefore

$$\mathcal{L}(R(\bullet)x)(\lambda) = [I - \tilde{C}R(\lambda,\tilde{G})B]^{-1}\tilde{C}R(\lambda,G)x \quad \forall x \in D(G),$$

thus

$$\mathcal{L}(S(\bullet)x)(\lambda) = R(\lambda,G)x + R(\lambda,\tilde{G})B[I - \tilde{C}R(\lambda,\tilde{G})B]^{-1}\tilde{C}R(\lambda,G)x =: Q(\lambda)x$$

for every $x \in D(G)$. This admits a (unique) bounded extension to X.

As in the proof of Theorem 1.2.1 one notices that

$$(\lambda - \tilde{G} - B\tilde{C})Q(\lambda)x = x \quad \forall x \in X.$$

On the other hand

$$Q(\lambda)(\lambda - \tilde{G} - B\tilde{C})x = x \quad \forall x \in D(G_{BC}).$$

Thus by [**ABHN11**, Theorem 3.1.7] one concludes that $(\tilde{G} + B\tilde{C})_{|X}$ generates a C_0-semigroup on X.

Let us now assume that $1 \in \rho(\tilde{\mathcal{F}}^{\mu}_{\infty})$ for $\mu > 0$, then analogously as in the proof of Theorem 1.2.1 we repeat the same reasoning for the triple $(G - \mu, B, C)$ and conclude that $(G - \mu)_{BC} = G_{BC} - \mu$ is a generator. Clearly this implies that G_{BC} generates a C_0-semigroup. □

Next we show that the semigroup $(S(t))_{t\geq 0}$ generated by G_{BC} satisfies the Variation of Parameters Formula.

Lemma 1.3.8. *If the operator G_{BC} generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X, then*

$$S(t)x = T(t)x + \int_0^t \tilde{T}(t-r)B\tilde{C}S(r)x\mathrm{d}r \quad \forall x \in D(G_{BC}).$$

Proof. We first notice that the domain of G_{BC} is a subset of $\tilde{Z} \cap D(\tilde{G}) = \tilde{Z}$. Furthermore we recall that $S(t)D(G_{BC}) \subset D(G_{BC})$ for every $t \geq 0$.

For $x \in D(G_{BC})$ we consider the function

$$[0,t] \ni r \mapsto \xi_x(r) := \tilde{T}(t-r)S(r)x \in X \subset \tilde{X}.$$

Then $\xi_x(\bullet)$ is continuously differentiable in $\tilde{X}$ with derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\xi_x(r) &= -\tilde{T}(t-r)\tilde{G}S(r)x + \tilde{T}(t-r)(\tilde{G}+B\tilde{C})S(r)x\\ &= \tilde{T}(t-r)B\tilde{C}S(r)x,\end{aligned}$$

since if a function is continuously differentiable in X the same holds in $\tilde{X}$.

Thus one concludes that

$$X \ni \xi_x(t) - \xi_x(0) = S(t)x - T(t)x = \int_0^t \xi_x'(r)\mathrm{d}r = \int_0^t \tilde{T}(t-r)B\tilde{C}S(r)x\mathrm{d}r.$$

$\square$

CHAPTER 2

Well–posed linear control systems

In this chapter we use our generalized Weiss–Staffans perturbation theorem 1.3.3 to give a semigroup proof of a result (due to [**CW89**, Thm. 5.1] and [**SW02**, Sect. 6] for the Hilbert space case, [**Sta05**, Sect. 2.7 and Thm. 4.8.3] for the Banach space case) on the well-posedness of linear control systems of the form

$$\Sigma(A,B,C) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t) & t \geq 0. \end{cases}$$

The operators A, B, C are linear and defined on Banach spaces X, Y and U, called *state-*, *observation-* and *control space*, respectively, and satisfy the following hypotheses:

- $A : D(A) \subset X \to X$, called the *state operator*, is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$,
- $B \in \mathcal{L}(U, X_{-1}^A)$ is the *control operator*,
- $C \in \mathcal{L}(Z, Y)$ is the *observation operator*,
- $X_1^A \hookrightarrow Z \hookrightarrow X$.

For the motivation, concrete examples, and a systematic treatment of such systems we refer to [**CZ95**], [**HI05**], [**HIR06**], [**SW12**], [**TW09**] and the references therein. Moreover, in Section 2.2 we illustrate our results by considering a heat equation with boundary control and point observation.

In [**BE14**] we generalized an idea of Grabowski and Callier [**GC96**], see also Engel [**Eng98b**] and associated to our system an operator matrix $\big(\mathcal{A}_{BC}, D(\mathcal{A}_{BC})\big)$ defined on an appropriate product space $\mathcal{X}^p$ depending on $p \geq 1$. We then called $\Sigma(A,B,C)$ *p-well-posed* if this operator matrix generates a C_0-semigroup $\mathcal{T} = \big(\mathcal{T}(t)\big)_{t\geq 0}$ on $\mathcal{X}^p$.

In other words, $\Sigma(A,B,C)$ is well-posed if the Cauchy problem

$$\text{(2.1)} \qquad \begin{cases} \dot{\mathcal{x}}(t) = \mathcal{A}_{BC}\mathcal{x}(t), & t \geq 0, \\ \mathcal{x}(0) = \mathcal{x}_0 \end{cases}$$

is well-posed on $\mathcal{X}^p$ in the sense of Hadamard (see [**EN00**, Sect. II.6]).

It turns out that this definition of well-posedness leads to the concept of p-admissibility of the control operator B and the observation operator C as studied, e.g., by Staffans and Weiss, see [**Wei89b**], [**Wei89a**], [**TW09**], [**Wei94b**], [**SW02**].

We mention that the semigroup $\mathcal{T}$ generated by $\mathcal{A}_{BC}$ also appears in [**TW09**], [**SW02**], [**Sta05**] and [**LP67**] where it is called the "*Lax-Phillips semigroup*".

2.1. The setting

In the following we introduce and decompose $\mathcal{A}$ in such a way that it becomes a Weiss–Staffans perturbation of a generator. We then apply Theorem 1.3.3 and obtain that the same conditions as in [**BE14**, Thm. 5.1] imply the generator property of $\mathcal{A}$.

In order to do this, we first fix some $1 \le p < \infty$ and introduce the spaces

- $E_1^p := \mathrm{L}^p\big((-\infty,0],Y\big)$,
- $E_2^p := \mathrm{L}^p\big([0,+\infty),U\big)$,
- $\mathcal{X}^p = E_1^p \times X \times E_2^p$,

and the operators

- $D_1 := \frac{\mathrm{d}}{\mathrm{d}s} : D(D_1) \subset E_1^p \to E_1^p$ with domain $D(D_1) := \mathrm{W}_0^{1,p}\big((-\infty,0],Y\big) := \big\{y \in \mathrm{W}^{1,p}\big((-\infty,0],Y\big) : y(0) = 0\big\}$,
- $D_2 := \frac{\mathrm{d}}{\mathrm{d}s} : D(D_2) \subset E_2^p \to E_2^p$ with domain $D(D_2) := \mathrm{W}^{1,p}\big([0,\infty),U\big)$.

It is well-known that

- D_1 is the generator of the left shift semigroup $\big(S_1(t)\big)_{t\ge0}$ on E_1^p,
- D_2 is the generator of the left shift semigroup $\big(S_2(t)\big)_{t\ge0}$ on E_2^p.

On $\mathcal{X}^p$ (equipped with an arbitrary product norm) we define for some fixed $\lambda > 0$ the operator matrix

$$\mathcal{A}_{BC} := \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}s} - \lambda & 0 & 0 \\ 0 & A_{-1} - \lambda & B\delta_0 \\ 0 & 0 & D_2 - \lambda \end{pmatrix}, \tag{2.2}$$

$$D(\mathcal{A}_{BC}) := \left\{ \left(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right) \in \mathcal{E} : A_{-1}x + Bu(0) \in X,\ y(0) = Cx \right\}, \tag{2.3}$$

where $\delta_0 : \mathrm{W}^{1,p}\big([0,\infty),U\big) \subset E_2^p \to U$ denotes the point evaluation given by $\delta_0 u := u(0)$ and

$$\mathcal{E} := \mathrm{W}^{1,p}\big((-\infty,0],Y\big) \times Z \times \mathrm{W}^{1,p}\big([0,\infty),U\big).$$

We then make the following definition.

Definition 2.1.1. The linear control system $\Sigma(A,B,C)$ is p-well-posed if $\mathcal{A}_{BC}$ generates a C_0-semigroup on $\mathcal{X}^p$.

In order to write $\mathcal{A}_{BC}$ as a generalized Weiss-Staffans perturbation of a generator, on the space $\tilde{\mathcal{X}}^p := (E_1^p)_{-1}^{D_1} \times X_{-1}^A \times E_2^p$ we introduce the generator

$$\tilde{\mathcal{A}} := \begin{pmatrix} D_{1,-1} - \lambda & 0 & 0 \\ 0 & A_{-1} - \lambda & 0 \\ 0 & 0 & D_2 - \lambda \end{pmatrix}, \quad D(\tilde{\mathcal{A}}) := E_1^p \times X \times D(D_2),$$

furthermore, letting $(\varepsilon_\lambda \otimes y)(s) := e^{\lambda s} y$ for every $s \in \mathbb{R}_-$ and $\tilde{\mathcal{Z}} := E_1^p \times Z \times D(D_2)$, we define the operator

$$L := \begin{pmatrix} 0 & (D_{1,-1} - \lambda)(-\varepsilon_\lambda \otimes C) & 0 \\ 0 & 0 & B\delta_0 \\ 0 & 0 & 0 \end{pmatrix} : \tilde{\mathcal{Z}} \longrightarrow \tilde{\mathcal{X}}^p.$$

Clearly $\tilde{\mathcal{A}}$ generates a C_0-semigroup $(\tilde{\mathcal{S}}(t))_{t\ge 0}$ given by

$$\tilde{\mathcal{S}}(t) = \begin{pmatrix} e^{\lambda t} S_{1,-1}(t) & 0 & 0 \\ 0 & e^{\lambda t} T_{-1}(t) & 0 \\ 0 & 0 & e^{\lambda t} S_2(t) \end{pmatrix}, \quad t \ge 0 \tag{2.4}$$

on $\tilde{\mathcal{X}}^p$ with $\omega_0 < 0$ for sufficiently large $\lambda > 0$. Furthermore $\mathcal{X}^p \hookrightarrow \tilde{\mathcal{X}}^p$ and $\mathcal{X}^p$ is $\tilde{\mathcal{S}}(t)$ invariant.

Then one can show the following.

Lemma 2.1.2. *The operator $\mathcal{A}_{BC}$ can be decomposed as an additive perturbation of a generator, namely*

$$\mathcal{A}_{BC} = \tilde{\mathcal{A}} + L$$

with domain $D(\tilde{\mathcal{A}} + L) = \Big\{ \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) \in \tilde{\mathcal{Z}} : \tilde{\mathcal{A}}_{-1} \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) + L \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) \in \mathcal{X}^p \Big\}$.

Proof. A simple computation shows that $D(\mathcal{A}_{BC}) = D(\tilde{\mathcal{A}} + L)$ and that

$$\mathcal{A}_{BC} \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) = \big(\tilde{\mathcal{A}} + L\big) \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) \quad \text{for all } \big(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\big) \in D(\mathcal{A}_{BC}).$$

$\square$

Furthermore, defining

$$\mathcal{U} := E_1^p \times X \times U,$$

L can be decomposed as $L = \mathcal{B} \circ \mathcal{C}$ and these two factors are given by

$$\mathcal{C} := \begin{pmatrix} 0 & -\varepsilon_\lambda \otimes C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta_0 \end{pmatrix} \in \mathcal{L}\left(\tilde{\mathcal{Z}}, \mathcal{U}\right),$$

$$\mathcal{B} := \begin{pmatrix} (D_{1,-1} - \lambda) & 0 & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}\left(\mathcal{U}, \tilde{\mathcal{X}}^p\right).$$

Clearly

$$\mathcal{X}_1^p \hookrightarrow \tilde{\mathcal{Z}} \hookrightarrow \mathcal{X}^p.$$

Thus we are in the situation introduced in Section 1.3 and in order to conclude that $\mathcal{A}_{BC}$ generates a C_0-semigroup on $\mathcal{X}^p$ we need the assumptions **(a)**–**(e)** of Theorem 1.3.3.

Condition (a). $\operatorname{rg}(\tilde{\mathcal{A}}_{-1}^{-1}\mathcal{B}) \subset \mathcal{Z} \Longleftrightarrow \operatorname{rg}(A_{-1}^{-1}B) \subset Z$.

In order to show this take $\left(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right) \in \mathcal{U}$. Then

$$\tilde{\mathcal{A}}_{-1}^{-1}\mathcal{B}\left(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right) = \begin{pmatrix} R(\lambda, D_{1,-1}) & 0 & 0 \\ 0 & R(\lambda, A_{-1}) & 0 \\ 0 & 0 & R(\lambda, D_{2,-1}) \end{pmatrix} \begin{pmatrix} (D_{1,-1} - \lambda)P & 0 & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} Py \\ R(\lambda, A_{-1})Bu \\ 0 \end{pmatrix} \in \tilde{\mathcal{Z}} \Longleftrightarrow R(\lambda, A_{-1})Bu \in Z.$$

Condition (b). $\mathcal{C}$ is p-admissible for $\mathcal{A} \Longleftrightarrow C$ is p-admissible for A.

Take $\mathcal{x} = \left(\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right) \in D(\tilde{\mathcal{A}})$ and $t_0 > 0$. Then

$$\begin{aligned} \int_0^{t_0} \|\mathcal{C}\mathcal{S}(t)\mathcal{x}\|_{\mathcal{U}}^p \mathrm{d}t &= \int_0^{t_0} \left(\|(-\varepsilon_\lambda \otimes C)e^{-\lambda t}T(t)x\|_{E_1^p}^p + \|e^{-\lambda t}u(t)\|_U^p\right)\mathrm{d}t \\ &= \int_0^{t_0} \left(\int_{-\infty}^0 \|e^{\lambda s}Ce^{-\lambda t}T(t)x\|_Y^p \mathrm{d}s + \|e^{-\lambda t}u(t)\|_U^p\right)\mathrm{d}t \\ &= \int_0^{t_0} \left(\left\|\frac{1}{p\lambda}Ce^{-\lambda t}T(t)x\right\|_Y^p + \|e^{-\lambda t}u(t)\|_U^p\right)\mathrm{d}t. \end{aligned}$$

Hence

$$\int_0^{t_0} \|\mathcal{C}\mathcal{S}(t)\mathcal{x}\|_{\mathcal{U}}^p \mathrm{d}t \le M\|\mathcal{x}\|_{\mathcal{X}^p} \Longleftrightarrow \int_0^{t_0} \|Ce^{-\lambda t}T(t)x\|_Y^p \mathrm{d}t \le \tilde{M}\|x\|_X.$$

Condition (c). $\mathcal{B}$ is p-admissible for $\mathcal{A} \Longleftrightarrow B$ is p-admissible for A.

In order to show this one can consider $\tilde{\mathcal{U}} := D(D_1) \times X \times U$, a dense subspace of $\mathcal{U}$, and, by a similar argument as in (1.6), check Condition (**c**) for $f := \begin{pmatrix} y(\bullet) \\ x(\bullet) \\ u(\bullet) \end{pmatrix} \in \mathrm{L}^p\big([0,\infty), \tilde{\mathcal{U}}\big)$. Then for $t_0 > 0$ we obtain

$$\Big\| \int_0^{t_0} \mathcal{S}_{-1}(t_0 - r)\mathcal{B}f(r)\mathrm{d}r \Big\|_{\mathcal{X}^p} = \left\| \int_0^{t_0} \begin{pmatrix} e^{-\lambda(t_0-r)}S_{1,-1}(t_0-r)(D_{1,-1}-\lambda)y(r) \\ e^{-\lambda(t_0-r)}T_{-1}(t_0-r)Bu(r) \\ 0 \end{pmatrix} \mathrm{d}r \right\|_{\mathcal{X}^p}$$

$$= \left\| \int_0^{t_0} e^{-\lambda(t_0-r)} S_1(t_0 - r)(D_1 - \lambda)y(r)\mathrm{d}r \right\|_{E_1^p} + \left\| \int_0^{t_0} e^{-\lambda(t_0-r)} T_{-1}(t_0 - r)Bu(r)\mathrm{d}r \right\|_X$$

$$\le M\|f\|_{\mathrm{L}^p\left([0,\infty),\mathcal{U}\right)} \Longleftrightarrow \left\| \int_0^{t_0} e^{-\lambda(t_0-r)} T_{-1}(t_0 - r)Bu(r)\mathrm{d}r \right\|_X \le \tilde{M}\|u\|_{E_2^p}.$$

Condition (d). The pair $(\mathcal{B}, \mathcal{C})$ is p-admissible for $\mathcal{A} \Leftrightarrow$ the pair (B, C) is p-admissible for A.

Take $f := \begin{pmatrix} y(\bullet) \\ x(\bullet) \\ u(\bullet) \end{pmatrix} \in \mathrm{W}_0^{1,p}\big([0,\infty), \mathcal{U}\big)$. Then for $t_0 > 0$

$$\int_0^{t_0} \left\| \mathcal{C} \int_0^t \mathcal{S}_{-1}(t - r)\mathcal{B}f(r)\mathrm{d}r \right\|_{\mathcal{U}}^p \mathrm{d}t = \int_0^{t_0} \left\| \begin{pmatrix} -\varepsilon_\lambda \otimes C\int_0^t e^{-\lambda(t-r)}T_{-1}(t-r)Bu(r)\mathrm{d}r \\ 0 \\ 0 \end{pmatrix} \right\|_{\mathcal{U}}^p \mathrm{d}t$$

$$= \int_0^{t_0} \frac{1}{p\lambda} \left\| C \int_0^t e^{-\lambda(t-r)} T_{-1}(t - r)Bu(r)\mathrm{d}r \right\|_Y^p \mathrm{d}t.$$

Thus

$$\int_0^{t_0} \left\| \mathcal{C} \int_0^t \mathcal{S}_{-1}(t - r)\mathcal{B}f(r)\mathrm{d}r \right\|_{\mathcal{U}}^p \mathrm{d}t \le M\|f\|_{\mathrm{L}^p\left([0,\infty),\mathcal{U}\right)}$$

$$\Longleftrightarrow \int_0^{t_0} \left\| C \int_0^t e^{\lambda(t-r)} T_{-1}(t - r)Bu(r)\mathrm{d}r \right\|_Y^p \mathrm{d}t \le \tilde{M}\|u\|_{E_2^p}.$$

Condition (e). $1 \in \rho(\mathcal{F}_\infty)$.

We notice that $\mathcal{F}_\infty$ is the continuous extension of

$$\mathcal{C} \int_0^\bullet \mathcal{S}_{-1}(\bullet - r)B\mathrm{d}r = \begin{pmatrix} 0 & 0 & -\varepsilon_\lambda \otimes C\int_0^\bullet e^{-\lambda(\bullet-r)}T_{-1}(\bullet - r)B\mathrm{d}r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

defined on $\mathrm{W}_0^{1,p}\big(\mathbb{R}_+, W^{1,p}(\mathbb{R}, U)\big)$.

Thus it is clear that $1 \in \rho(\mathcal{F}_\infty)$ since $I - \mathcal{F}_\infty$ is a bounded upper triangular matrix with invertible entries on the diagonal.

Similarly as in [**BE14**, Thm.5.1] one obtains the following result.

Theorem 2.1.3. *If* $\mathrm{rg}(A_{-1}^{-1}B) \subset Z$*,* B *is a p-admissible control operator,* C *is a p-admissible observation operator and the pair* (B,C) *is p-admissible, then* $\mathcal{A}$ *generates a strongly continuous semigroup on* $\mathcal{X}^p$*.*

Thus, using Proposition 1.1.16, the result above can be reformulated as follows.

Theorem 2.1.4. *The linear control system* $\Sigma(A,B,C)$ *is p-well-posed if* $\mathrm{rg}(A_{-1}^{-1}B) \subset Z$*,* B *is a p-admissible control operator,* C *is a p-admissible observation operator, and* $m(\bullet) := CR(i\bullet, A_{-1})B$ *is a bounded Fourier multiplier.*

As a corollary we characterize the 2-well-posedness of the system $\Sigma(A,B,C)$ in case that all the spaces X, Y and U are Hilbert spaces. Using the Plancherel Theorem (see [**ABHN11**, Thm.1.8.2]) one can first prove the following.

Lemma 2.1.5. *Let* V,W *be two Hilbert spaces, then every* $m \in \mathrm{L}^\infty\big(\mathbb{R}, \mathcal{L}(V,W)\big)$ *is a (bounded)* L^2*-Fourier multiplier.*

Combining Proposition 2.1.4 and Lemma 2.1.5 we immediately obtain our next result.

Corollary 2.1.6. *Let* X*,* Y *and* U *be Hilbert spaces. Then the system* $\Sigma(A,B,C)$ *is* 2*-well-posed if* B *and* C *are* 2*-admissible and* $m(\bullet) = CR(i\bullet, A_{-1})B \in \mathrm{L}^\infty\big(\mathbb{R}, \mathcal{L}(U,Y)\big)$*.*

Remark 2.1.7. The semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ generated by $\mathcal{A}$ already appears in Staffans and Weiss [**SW02**, Prop. 6.2] and is called the *Lax-Phillips semigroup* (of index 0) referring to the paper [**LP67**] by Lax and Phillips.

This semigroup describes the solutions of the well-posed system $\Sigma(A,B,C)$ as follows. For $\mathcal{x} = \big(y(\bullet), x, u(\bullet)\big)^t \in \mathcal{X}^p$

- the first component of $\mathcal{T}(\bullet)\mathcal{x}$ gives the *past output*,
- the second component of $\mathcal{T}(\bullet)\mathcal{x}$ represents the *present state*,
- the third component of $\mathcal{T}(\bullet)\mathcal{x}$ can be interpreted as the *future input*

of the system.

Remark 2.1.8. If the semigroup $(T(t))_{t\geq 0}$ generated by the state operator A is not exponentially stable, as needed in Assumption 1.1 (i.e., if the growth bound $\omega_0(A) \geq 0$, cf. [**EN00**, Def. I.5.6]), then we choose $\lambda_0 > \omega_0(A)$ and for the rescaled generator $A - \lambda_0$ we obtain $\omega_0(A - \lambda_0) < 0$. Moreover, on the product space $\mathcal{X}^p$ we introduce the operator matrix $\mathcal{A}^{\lambda_0}$ associated to the control problem $\Sigma(A - \lambda_0, B, C, D)$. This operator can be written as

$$\mathcal{A}^{\lambda_0} = \mathcal{A} - \lambda_0 \mathcal{P}_2 \qquad \text{for} \qquad \mathcal{P}_2 := \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{X}^p).$$

If there exists $\lambda \in \rho(A)$ such that $\operatorname{rg}\big(R(\lambda, A_{-1})B\big) \subset D(C)$, then this holds for every $\lambda \in \rho(A)$. Hence $\operatorname{rg}\big(R(\mu, A_{-1}-\lambda_0)B\big) = \operatorname{rg}\big(R(\mu+\lambda_0, A_{-1})B\big) \subset D(C)$ for every $\mu \in \rho(A-\lambda_0)$. This shows that A satisfies the compatibility assumption (1.16) if and only if $A - \lambda_0$ does, leading to the following result.

Theorem 2.1.9. *Let $\lambda_0 \in \rho(A)$. Then the following are equivalent.*

(a) $\mathcal{A}$ is the generator of a C_0-semigroup on $\mathcal{X}^p$,

(b) $\mathcal{A}^{\lambda_0}$ is the generator of a C_0-semigroup on $\mathcal{X}^p$,

(c) B, C and the pair (B, C) are p-admissible with respect to $A - \lambda_0$ (or A),

(d) B and C are p-admissible with respect to $A - \lambda_0$ (or A) and $m^{\lambda_0} := CR(\lambda_0 + i\,\bullet, A_{-1})B$ is a bounded Fourier-multiplier.

Proof. (a) $\Longleftrightarrow$ (b). Since $\mathcal{A}$ and $\mathcal{A}^{\lambda_0}$ differ only by a bounded operator, this equivalence holds by the bounded perturbation theorem, cf. [**EN00**, Thm.III.1.3].

(b) $\Longleftrightarrow$ (c). This equivalence holds by Theorem 2.1.3.

(c) $\Longleftrightarrow$ (d). It is clear that B and C are p-admissible with respect to $A - \lambda_0$ if and only if they are p-admissible with respect to A. By Theorem 1.1.16 the pair (B, C) is p-admissible with respect to $A - \lambda_0$ if and only if $m^{\lambda_0} = CR(i\,\bullet, A_{-1} - \lambda_0)B = CR(\lambda_0 + i\,\bullet, A_{-1})B$ is a bounded Fourier-multiplier. $\square$

2.2. Example: A heat equation with boundary control and point observation

To illustrate our results we consider a metal bar of length π modeled as a segment $[0, \pi]$. Our aim is to control its temperature by putting controls $u_0(t)$ and $u_1(t)$ at the edges 0 and π. Moreover, we observe the system by measuring its temperature at the center $\frac{\pi}{2} \in [0, \pi]$.

This was discussed in [**ET00**, Example 2.1] and later in [**BE14**, Section 6]. We now treat this example by applying the theory developed in the previous section without using the concept of Lebesgue extension.

As state space we choose the Hilbert space $X = \mathrm{L}^2[0,\pi]$ and consider the *state function* $x(s,t)$ representing the temperature in the point $s \in [0,\pi]$ at time $t \geq 0$.

If we start from the temperature profile $x_0 \in X$, the time evolution of our system can be described by a heat equation with boundary control and point observation, more precisely by the equations

$$
\begin{cases}
\frac{\partial x(s,t)}{\partial t} = \frac{\partial^2 x(s,t)}{\partial s^2}, & t \geq 0,\ s \in [0,\pi], \\
x(s,0) = x_0(s), & s \in [0,\pi], \\
\frac{\partial x}{\partial s}(0,t) = u_0(t), & t \geq 0, \\
\frac{\partial x}{\partial s}(\pi,t) = u_1(t), & t \geq 0, \\
y(t) = x\left(\frac{\pi}{2},t\right), & t \geq 0.
\end{cases} \tag{2.5}
$$

Here the boundary conditions in $s = 0$ and $s = \pi$ involving $u_0(\bullet)$ and $u_1(\bullet)$ describe the forced heat exchange between the ends of the bar and the environment.

In order to write (2.5) as a linear control system of the form $\Sigma(A,B,C)$ we use the approach for boundary control problems developed in [**EKFK⁺10**, Sect. 2]. To this end we define the following operators and spaces.

- The *maximal system operator*

$$A_m := \frac{\mathrm{d}^2}{\mathrm{d}s^2} \qquad \text{with domain} \qquad D(A_m) := W^{2,2}[0,\pi] \subset X = \mathrm{L}^2[0,\pi];$$

- the *boundary space* $\partial X := \mathbb{C}^2$ and the *boundary operator*[1]

$$Q : [D(A_m)] \to \partial X, \quad Qf := \big(f'(0),\ f'(\pi)\big)^t;$$

- the *control space* $U := \mathbb{C}^2$ and the *control operator* $\tilde{B} := Id \in \mathcal{L}(U, \partial X)$;
- the *observation space* $Y := \mathbb{C}$ and the *observation operator*[2] $C := \delta_{\frac{\pi}{2}}$.

With this notation (2.5) can be rewritten as an abstract Boundary Control System

$$
\begin{cases}
\dot{x}(t) = A_m x(t), & t \geq 0, \\
Qx(t) = \tilde{B}u(t), & t \geq 0, \\
y(t) = Cx(t), & t \geq 0, \\
x(0) = x_0.
\end{cases} \tag{aBCS}
$$

[1]Here $[D(A_m)]$ indicates the space $D(A_m)$ endowed with the graph norm $\|\bullet\|_{A_m}$.

[2]By $\delta_{\frac{\pi}{2}}$ we indicate the point evaluation in $\frac{\pi}{2}$.

We note that

(a) the operator $A \subset A_m$ with domain

$$D(A) := \ker(Q) = \left\{h \in \mathrm{W}^{2,2}[0,\pi] : f'(0) = f'(\pi) = 0\right\}$$

is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, and its spectrum is given by $\sigma(A) = \left\{-n^2 : n \in \mathbb{N}\right\}$ (see [**EN00**, Sect. II.3.30]);

(b) the boundary operator Q is surjective,

i.e., the Main Assumptions 2.3 in [**EKFK**$^+$**10**] are satisfied.

In order to use the abstract theory for boundary control systems developed in [**EKFK**$^+$**10**, Sect. 2] we need the following result due to Greiner in [**Gre87**, Lem. 1.2].

Lemma 2.2.1. *Let the above assumptions* (*a*) *and* (*b*) *be satisfied. Then for each* $\lambda \in \rho(A)$ *the operator* $Q_{|\ker(\lambda - A_m)}$ *is invertible and* $Q_\lambda = \left(Q_{|\ker(\lambda-A_m)}\right)^{-1} : \partial X \to \ker(\lambda - A_m) \subset X$ *is bounded.*

The operator

$$Q_\lambda = \left(Q_{|\ker(\lambda-A_m)}\right)^{-1} : \partial X \to \ker(\lambda - A_m) \text{ for } \lambda \in \rho(A)$$

is called *Dirichlet operator.*

Since[3] $\ker(\lambda - A_m) = \operatorname{span}\left\{\cosh\left(\sqrt{\lambda}\,\bullet\right), \cosh\left(\sqrt{\lambda}(\pi - \bullet)\right)\right\}$, a simple computation shows that

$$Q_\lambda = \left(q_0(\bullet), q_1(\bullet)\right),$$

where for $s \in [0,\pi]$

$$q_0(s) := -\frac{\cosh\left(\sqrt{\lambda}(\pi - s)\right)}{\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\pi\right)}, \qquad q_1(s) := \frac{\cosh\left(\sqrt{\lambda}s\right)}{\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\pi\right)}.$$

Let $B_\lambda := Q_\lambda \tilde{B} = Q_\lambda$. Then, by [**EKFK**$^+$**10**, Sect. 2], the system (aBCS) is equivalent to $\Sigma(A,B,C)$ for the operators

$$\begin{aligned} B &:= (\lambda - A_{-1})Q_\lambda \in \mathcal{L}(U, X_{-1}^A), \\ C &:= \delta_{\frac{\pi}{2}} \in \mathcal{L}([D(A_m)], Y). \end{aligned}$$

In order to prove 2-well-posedness of the system $\Sigma(A,B,C)$ we transform it into an isomorphic problem on ℓ^2.

[3]With $\operatorname{span}\{f,g\}$ we denote the linear vector space generated by f and g.

To this end we first note that A is self-adjoint and has compact resolvent. Hence its normalized eigenvectors given by

$$e_n(s) = \sqrt{\tfrac{w_n}{\pi}}\cos(ns) \qquad \text{where} \qquad w_n = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n \geq 1 \end{cases}$$

form an orthonormal basis of X. Using this basis we define the surjective isometry

$$J : X \to \ell^2, \quad Jf := \big(\langle f, e_n\rangle\big)_{n\in\mathbb{N}},$$

which associates to a function $f \in X$ the sequence of its Fourier coefficients relatively to $(e_n)_{n\in\mathbb{N}}$.

Next we put $z(t) := Jx(t)$. Then the system $\Sigma(A, B, C)$ transforms to

$$\Sigma\big(JAJ^{-1}, JB, CJ^{-1}\big) = \Sigma\big(JAJ^{-1}, J(\lambda - A_{-1})Q_\lambda, \delta_{\frac{\pi}{2}}J^{-1}\big).$$

In particular, the differential operator A transforms into the multiplication operator

$$JAJ^{-1} =: M_\alpha =: M : D(M) \subset \ell^2 \to \ell^2,$$

where $\alpha = (-n^2)_{n\in\mathbb{N}}$ and

$$D(M) = \Big\{(a_n)_{n\in\mathbb{N}} \in \ell^2 : (-n^2 a_n)_{n\in\mathbb{N}} \in \ell^2\Big\}.$$

This gives for $\lambda > 0$ the extrapolation space

$$X_{-1}^M = \Big\{(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \Big(\frac{a_n}{\lambda + n^2}\Big)_{n\in\mathbb{N}} \in \ell^2\Big\}.$$

Moreover, the Dirichlet operator Q_λ becomes the operator

$$JQ_\lambda = \left(\Big(-\frac{\sqrt{w_n/\pi}}{\lambda + n^2}\Big)_{n\in\mathbb{N}}, \Big(\frac{(-1)^n\sqrt{w_n/\pi}}{\lambda + n^2}\Big)_{n\in\mathbb{N}}\right).$$

Thus the control operator B transforms into

$$\hat{B} := J(\lambda - A_{-1})Q_\lambda = (\lambda - M)JQ_\lambda = \left(\Big(-\sqrt{\tfrac{w_n}{\pi}}\Big)_{n\in\mathbb{N}}, \Big((-1)^n\sqrt{\tfrac{w_n}{\pi}}\Big)_{n\in\mathbb{N}}\right), \tag{2.6}$$

while the observation operator C transforms into the operator

$$\hat{C} := CJ^{-1} = \Big(e_n\big(\tfrac{\pi}{2}\big)\Big)_{n\in\mathbb{N}}, \tag{2.7}$$

where

$$e_n\big(\tfrac{\pi}{2}\big) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}}\sqrt{\tfrac{w_n}{\pi}} & \text{if } n \text{ is even}. \end{cases}$$

Summing up, the Control System (2.5) is isometrically isomorphic to

$$
\begin{cases} \dot{z}(t) = Mz(t) + \hat{B}u(t), & t \geq 0, \\ y(t) = \hat{C}z(t), & t \geq 0, \\ z(0) = z_0, \end{cases} \tag{2.8}
$$

where $z(t) := Jx(t) \in \ell^2$ and $z_0 := Jx_0$.

Our aim is now to prove the 2-well-posedness of the system $\Sigma(M, \hat{B}, \hat{C})$ in (2.8). Since $\omega_0(A) = \omega_0(M) = 0$ we consider $M - 1$ instead of M, cf. Remark 2.1.8 and Theorem 2.1.9.

First we verify the compatibility condition (1.16).

Lemma 2.2.2. *For every $\gamma \in \mathbb{R}$ we have*

$$
\operatorname{rg}\big(R(1 + i\gamma, M_{-1})\hat{B}\big) \subset D(\hat{C}). \tag{2.9}
$$

Moreover, $m(\bullet) := \hat{C}R(1 + i\bullet, M_{-1})\hat{B} \in \mathrm{L}^\infty\big(\mathbb{R}, \mathcal{L}(U, Y)\big) = \mathrm{L}^\infty\big(\mathbb{R}, \mathcal{L}(\mathbb{C}^2, \mathbb{C})\big)$.

Proof. Since

$$
\begin{aligned}
\operatorname{rg}\big(R(1 + i\gamma, M_{-1})\hat{B}\big) &= \operatorname{rg}\big(R(1 + i\gamma, M_{-1})(\lambda - M)JQ_\lambda\big) \\
&\subset \operatorname{rg}(JQ_\lambda) \subset J\big(D(A_m)\big) = D(\hat{C}),
\end{aligned}
$$

the range condition is satisfied.

Let $u := \binom{u_1}{u_2} \in U = \mathbb{C}^2$ and $\gamma \in \mathbb{R}$. Then it follows

$$
R(1 + i\gamma, M_{-1})\hat{B}u = \left(\frac{1}{1 + n^2 + i\gamma} \Big(-\sqrt{\tfrac{w_n}{\pi}}\, u_1 + (-1)^n \sqrt{\tfrac{w_n}{\pi}}\, u_2 \Big) \right)_{n \in \mathbb{N}} =: (r_n)_{n\in\mathbb{N}}.
$$

Since

$$
\left| e_n\big(\tfrac{\pi}{2}\big) \cdot r_n \right| \leq \frac{4}{(1 + n^2)\pi} \cdot \big(|u_1| + |u_2| \big) \qquad \text{for all } n \in \mathbb{N},\ \gamma \in \mathbb{R},
$$

the series

$$
\sum_{n=0}^{\infty} e_n\big(\tfrac{\pi}{2}\big) \cdot r_n
$$

converges and

$$
\left| \hat{C}R(1 + i\gamma, M_{-1})\hat{B}u \right| \leq \frac{4\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{1}{1 + n^2} \cdot \|u\|_2 \qquad \text{for all } \gamma \in \mathbb{R},\ u \in U.
$$

Since this implies that $m(\bullet)$ is bounded, the proof is complete. $\square$

Next we verify the 2-admissibility of the operators $\hat{C}$ and $\hat{B}$. To this end we denote by $(S(t))_{t\geq 0}$ the semigroup generated by $M-1$.

Proposition 2.2.3. *The observation operator $\hat{C}$ is 2-admissible with respect to $M-1$.*

Proof. Let $t_0 > 0$ and $z = (z_n)_{n\in\mathbb{N}} \in D(M)$. Then by the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\int_0^{t_0} \left|\hat{C}\, S(s)z\right|^2 \mathrm{d}s &= \int_0^{t_0} \left|\hat{C}\, S(s)z\right|^2 \mathrm{d}s \\ &= \int_0^{t_0} \left|\sum_{n=0}^{\infty} e_n\left(\tfrac{\pi}{2}\right) e^{-(1+n^2)s} z_n\right|^2 \mathrm{d}s \\ &\leq \frac{2}{\pi} \sum_{n=0}^{+\infty} \int_0^{+\infty} e^{-2(1+n^2)s} \mathrm{d}s \cdot \sum_{n=0}^{+\infty} |z_n|^2 \\ &\leq \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{1}{1+n^2} \cdot \|z\|_{\ell^2}^2,\end{aligned}$$

hence by definition $\hat{C}$ is an admissible observation operator. □

Proposition 2.2.4. *The control operator $\hat{B} = (b_1, b_2)$ is 2-admissible with respect to $M-1$.*

Proof. Clearly $\hat{B}$ is 2-admissible if and only if b_1, $b_2 : \mathbb{C} \to X_{-1}^M$ are both 2-admissible. Let $t_0 > 0$ and $u \in \mathrm{L}^2[0, +\infty)$. Then by Young's inequality (cf. [**ABHN11**, Prop. 1.3.5.(a)]) we obtain for $i = 1, 2$

$$\begin{aligned}\left\|\int_0^{t_0} S_{-1}(t_0 - r) b_i u(r) \mathrm{d}r\right\|_{\ell^2}^2 &\leq \frac{2}{\pi} \sum_{n=0}^{+\infty} \left(\int_0^{t_0} e^{-(1+n^2)(t_0-r)} \left|u(r)\right| \mathrm{d}r\right)^2 \\ &\leq \frac{2}{\pi} \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} e^{-2(1+n^2)r} \mathrm{d}r\right)^2 \cdot \left(\int_0^{+\infty} \left|u(r)\right|^2 \mathrm{d}r\right)^2 \\ &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^2} \cdot \|u\|_{\mathrm{L}^2[0,+\infty)}^2,\end{aligned}$$

hence each b_i is an admissible control operator. □

Remark 2.2.5. For multiplication semigroups and finite dimensional observation/control spaces there exists a characterization for the admissibility of an observation/control operator via a *Carleson measure criteria*. For the details we refer to [**TW09**, Thm. 5.3.2] and [**HR83**, Cor. 2.5], [**Wei88**, Thm. 1.2], respectively.

Finally, from Lemmas 1.1.16, 2.1.5 and 2.2.2 we obtain the following.

Corollary 2.2.6. *The pair $(\hat{B}, \hat{C})$ is 2-admissible.*

Summing up, we obtain by Theorem 2.1.9 the main result of this section.

Corollary 2.2.7. *The system* $\Sigma(M, \hat{B}, \hat{C})$*, hence also the Heat Equation* (2.5)*, is* 2*-well-posed.*

CHAPTER 3

Unbounded boundary perturbations

In this chapter we apply Theorem 1.2.1 to boundary perturbations generalizing Greiner's approach from [**Gre87**] to unbounded boundary operators Φ. The results of this section have been published in [**ABE14**, Sec. 4.3].

3.1. The setting

We start from

- two Banach spaces[1] X and ∂X, the latter called "boundary space";
- a closed, densely defined "maximal" operator[2] $A_m : D(A_m) \subseteq X \to X$;
- the Banach space $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ where $\|f\|_{A_m} := \|f\| + \|A_m f\|$ is the graph norm;
- two "boundary" operators $L, \Phi \in \mathcal{L}([D(A_m)], \partial X)$.

This yields two restricted operators A, $A^\Phi \subset A_m$ with

$$
\begin{aligned}
D(A) &:= \{f \in D(A_m) : Lf = 0\} = \ker L, \\
D(A^\Phi) &:= \{f \in D(A_m) : Lf = \Phi f\}.
\end{aligned}
$$

In many applications X, ∂X and $D(A_m)$ are function spaces and L is a "trace-type" operator which restricts a function in $D(A_m)$ to (a part of) the boundary of its domain. Hence one can consider A^Φ with boundary condition $Lf = \Phi f$ as a perturbation of the operator A with abstract "Dirichlet type" boundary condition $Lf = 0$.

In order to treat this setup within our framework we make the following assumptions.

(i) The operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X;

(ii) the boundary operator $L : D(A_m) \to \partial X$ is surjective.

---

[1] In this section we denote the elements of X by f instead of x.

[2] "maximal" concerns the size of the domain, e.g., a differential operator without boundary conditions.

The following lemma, shown by Greiner [**Gre87**, Lem. 1.2], is the key to write A^Φ as a Weiss–Staffans type perturbation of A.

Lemma 3.1.1. *Let the above assumptions (i) and (ii) be satisfied. Then for each* $\lambda \in \rho(A)$ *the operator* $L|_{\ker(\lambda - A_m)}$ *is invertible and* $D_\lambda := (L|_{\ker(\lambda - A_m)})^{-1} : \partial X \to \ker(\lambda - A_m) \subseteq X$ *is bounded.*

Using this so-called *Dirichlet operator* D_λ one obtains the following representation of A^Φ where, for simplicity, we assume A to be invertible.

Lemma 3.1.2. *If* $0 \in \rho(A)$*, then*

$$A^\Phi = \big(A_{-1} - A_{-1}D_0 \cdot \Phi\big)|_X, \tag{3.1}$$

i.e., $A^\Phi = A_{BC}$ *for* $U := \partial X$*,* $Z := [D(A_m)]$ *and*

$$B := -A_{-1}D_0 \in \mathcal{L}(U, X_{-1}^A), \qquad C := \Phi \in \mathcal{L}(Z, U).$$

Proof. Denote the operator on the right-hand side of (3.1) by $\tilde{A}^\Phi$. Then

$$\begin{aligned} f \in D\big(\tilde{A}^\Phi\big) &\iff f - D_0\Phi f \in D(A) \\ &\iff Lf = LD_0\Phi f = \Phi f \\ &\iff f \in D\big(A^\Phi\big). \end{aligned}$$

Moreover, for $f \in D(A^\Phi)$ we have

$$\tilde{A}^\Phi f = A(f - D_0\Phi f) = A_m(f - D_0\Phi f) = A_m f = A^\Phi f$$

as claimed. ☐

We mention that Greiner [**Gre87**, Thm. 2.1] assumes that the boundary perturbation $\Phi \in \mathcal{L}(X, U)$ is bounded and gives a condition on L implying that $A_{-1}D_0$ is a 1-admissible control operator. Hence in his case A^Φ is a generator due to the Desch–Schappacher theorem [**EN00**, Thm. III.3.1].

Our Theorem 1.2.1 now allows to deal also with unbounded Φ.

Proposition 3.1.3. *Assume that for some* $1 \le p < +\infty$ *the pair* $(A_{-1}D_0, \Phi)$ *is jointly* p*-admissible and that* $\mathrm{Id} \in \mathcal{L}(\partial X)$ *is a* p*-admissible feedback operator for* A*. Then* A^Φ *is the generator of a* C_0*-semigroup on* X*.*

Proof. One only has to show the compatibility condition (1.16). This, however, follows immediately from

$$\operatorname{rg}\big(R(\lambda, A_{-1})B\big) = \operatorname{rg}\big((\mathrm{Id} - \lambda R(\lambda, A))D_0\big) \subset \ker(A_m) + D(A) \subseteq D(A_m) = Z. \qquad \square$$

Remark 3.1.4. We note that in [**HMR15**, Thm. 4.1] the authors study a similar problem in the context of regular linear systems.

Example 3.1.5. As a simple but typical example for Proposition 3.1.3 consider the space $X := \mathrm{L}^p[0,1]$ and the first derivative $A_m := \frac{d}{ds}$ with domain $D(A_m) := \mathrm{W}^{1,p}[0,1]$ (c.f. [**Gre87**, Expl. 1.1.(c)]). As boundary space choose $\partial X = \mathbb{C}$, as boundary operator the point evaluation $L = \delta_1$ and as boundary perturbation some $\Phi \in \big(\mathrm{W}^{1,p}[0,1]\big)'$. This gives rise to the differential operators A, $A^\Phi \subset \frac{d}{ds}$ with domains

$$D(A) := \big\{f \in \mathrm{W}^{1,p}[0,1] : f(1) = 0\big\},$$
$$D(A^\Phi) := \big\{f \in \mathrm{W}^{1,p}[0,1] : f(1) = \Phi f\big\}.$$

Then the assumptions (i) and (ii) made above are satisfied since A generates the nilpotent left-shift semigroup given by

$$\big(T(t)f\big)(s) = \begin{cases} f(s+t) & \text{if } s+t \le 1, \\ 0 & \text{else.} \end{cases}$$

However, A^Φ is not always a generator. For example, if $\Phi = \delta_1$, then $A^\Phi = A_m$ and $\sigma(A^\Phi) = \mathbb{C}$, hence A^Φ is not a generator. Thus one needs an additional assumption on Φ.

Definition 3.1.6. A bounded linear functional $\Phi : \mathrm{C}[0,1] \to \mathbb{C}$ has *little mass* in $r = 1$ if there exist $q < 1$ and $\delta > 0$ such that

$$|\Phi f| \le q \cdot \|f\|_\infty$$

for every $f \in \mathrm{C}[0,1]$ satisfying $\operatorname{supp} f \subset [1-\delta, 1]$.

Note that $\mathrm{W}^{1,p}[0,1] \overset{c}{\hookrightarrow} \mathrm{C}[0,1]$ and hence $(\mathrm{C}[0,1])' \subset [D(A_m)]'$. Now the following holds.

Corollary 3.1.7. *If $\Phi \in \big(\mathrm{C}[0,1]\big)'$ has little mass in $r = 1$, then for all $1 \le p < +\infty$ the operator A^Φ is the generator of a strongly continuous semigroup on $\mathrm{L}^p[0,1]$.*

Proof. By Proposition 3.1.3 it suffices to show that for the triple $(A, A_{-1}D_0, \Phi)$ the conditions (ii)–(v) of Theorem 1.2.1 are satisfied. To this end, note that $0 \in \rho(A)$ and that the Dirichlet operator $D_0 : \mathbb{C} \to \mathrm{L}^p[0,1]$ is given by $D_0\alpha = \alpha \cdot \mathbb{1}$ where $\mathbb{1}(s) = 1$ for all $s \in [0,1]$.

(ii) By Remark 1.1.2 it suffices to verify estimate (1.6) where we may assume that $u \in \mathrm{W}_0^{1,p}[0,t_0]$ for some $0 < t_0 \le 1$. Using integration by parts and [**Nei81**, Thm. 4.2] we conclude[3] that

$$\begin{aligned}
\int_0^{t_0} T_{-1}(t_0-s)Bu(s)\,ds &= -\int_0^{t_0} T_{-1}(t_0-s)A_{-1}D_0u(s)\,ds \\
&= D_0u(t_0) - \int_0^{t_0} T(t_0-s)D_0u'(s)\,ds \\
&= u(t_0)\cdot \mathbb{1} - \int_0^{t_0} \big(T(t_0-s)\mathbb{1}\big)\cdot u'(s)\,ds \\
&= u(t_0)\cdot \mathbb{1} - \int_{\max\{0,\bullet+t_0-1\}}^{t_0} u'(s)\,ds \\
&= u\big(\max\{0,\bullet+t_0-1\}\big) \\
&= \tilde{u}(\bullet+t_0-1). \qquad (3.2)
\end{aligned}$$

This implies $\|\mathcal{B}_{t_0}u\|_X = \|\mathcal{B}_{t_0}u\|_p \le \|u\|_p$ for all $u \in \mathrm{W}_0^{1,p}[0,t_0]$ which shows (ii).

(iii) By the Riesz–Markov representation theorem there exists a regular complex Borel measure μ on $[0,1]$ such that

$$\Phi f = \int_0^1 f(r)\,d\mu(r) \quad \text{for all } f \in \mathrm{C}[0,1]. \qquad (3.3)$$

Using Fubini's theorem and Hölder's inequality one obtains for $0 < t_0 \le 1$ and $f \in D(A)$

$$\begin{aligned}
\int_0^{t_0} \big|CT(s)f\big|^p\,ds &= \int_0^{t_0} \big|\Phi\tilde{f}(\bullet+s)\big|^p\,ds \\
&\le \int_0^{t_0} \Big(\int_0^1 \big|\tilde{f}(r+s)\big|\,d|\mu|(r)\Big)^p\,ds \\
&\le \int_0^{t_0} \big(|\mu|[0,1]\big)^{p-1}\cdot \int_0^1 \big|\tilde{f}(r+s)\big|^p\,d|\mu|(r)\,ds \\
&= \|\mu\|^{p-1}\cdot \int_0^1 \int_0^{t_0} \big|\tilde{f}(r+s)\big|^p\,ds\,d|\mu|(r) \\
&\le \|\mu\|^p \cdot \|f\|_p^p, \qquad (3.4)
\end{aligned}$$

where $\|\mu\| := |\mu|[0,1]$ (which coincides with $\|\Phi\|_\infty$). This proves (iii).

[3] For a function g defined on an interval we denote in the sequel by $\tilde{g}$ its extension to $\mathbb{R}$ by the value 0.

(iv) From (3.2) one obtains for $0 < t_0 \le 1$ and $u \in \mathrm{W}_0^{1,p}[0,t_0]$ by similar arguments as in (iii) that

$$\begin{aligned}\int_0^{t_0} \Big| C \int_0^r T_{-1}(r-s)Bu(s)\,ds \Big|^p dr &= \int_0^{t_0} \big|\Phi\,\tilde{u}(\bullet + r - 1)\big|^p dr \\ &= \int_0^{t_0} \Big| \int_{1-r}^1 u(s+r-1)\,d\mu(s) \Big|^p dr \\ &\le \int_0^{t_0} \big(|\mu|[1-r,1]\big)^{p-1} \cdot \int_{1-r}^1 \big|u(s+r-1)\big|^p d|\mu|(s)\,dr \\ &\le \big(|\mu|[1-t_0,1]\big)^{p-1} \cdot \int_{1-t_0}^1 \int_{1-s}^1 \big|u(s+r-1)\big|^p dr\,d|\mu|(s) \\ &\le \big(|\mu|[1-t_0,1]\big)^{p} \cdot \|u\|_p^p. \qquad (3.5)\end{aligned}$$

This shows (iv).

(v) Since, by assumption, Φ has little mass in $r = 1$, it follows that $|\mu|[1-t_0,1] < 1$ for sufficiently small $t_0 > 0$. Hence from Estimate (3.5) and the denseness of $\mathrm{W}_0^{1,p}[0,t_0]$ in $\mathrm{L}^p[0,t_0]$ it follows that $\|\mathcal{F}_{t_0}\| \le |\mu|[1-t_0,1] < 1$ for $0 < t_0 \le 1$ sufficiently small. This implies $1 \in \rho(\mathcal{F}_{t_0})$ as claimed. □

Remarks 3.1.8. (i) Corollary 3.1.7 can be generalized (with essentially the same proof) to the first derivative on $\mathrm{L}^p\big([0,1],\mathbb{C}^n\big)$. One can even go further and prove a similar result on $\mathrm{L}^p\big([0,1],E\big)$ for a (possibly infinite dimensional) Banach space E provided the boundary operator Φ has a representation as a Riemann-Stieltjes integral as in (3.3). See also [**HMR15**, Example 5.1].

(ii) In most cases the admissibility of the identity as a feedback operator follows from an estimate $\|\mathcal{F}_{t_0}\| < 1$ for sufficiently small $t_0 > 0$. Choosing $\Phi = \alpha\delta_1$, by (3.2) one obtains that $\mathcal{F}_{t_0} = \alpha\,\mathrm{Id}$ for all $t_0 > 0$, hence $1 \in \rho(\mathcal{F}_{t_0})$ if and only if $\alpha \ne 1$. This provides an example where our perturbation theorem is applicable even if $\|\mathcal{F}_{t_0}\| > 1$ for all $t_0 > 0$. Note that for $\alpha = 1$ one obtains $A^\Phi = A_m$, hence in this case A^Φ cannot be a generator.

3.2. More examples

In [**HMR15**, Sec. 5] the authors consider some example in the context of linear control systems. We reinterpret these examples by means of our perturbation theorem.

Example 1: Difference equations

Starting from a Banach space U we define for $p \in (1,+\infty)$ and $r > 0$ the space $X := L^p([-r,0],U)$. Clearly $Z := W^{1,p}([-r,0],U) \hookrightarrow X$ with continuous embedding.

For a function $f : [-r, +\infty] \to U$ take $t \geq 0$ and define its *history function* (for more details see [**BP05**, Chapt. 3.1]) by

$$f_t(s) := f(s+t) \in U \quad \text{for } s \in [-r, 0].$$

Let $\mu : [-r, 0] \to \mathcal{L}(U)$ be a function of bounded variation with $\mu(0) = 0$ and consider the following difference equation

$$\text{(DE)} \qquad \begin{cases} f(t) = \int_{-r}^{0} \mathrm{d}\mu(s) f(s+t), \quad t \geq 0, \\ f(s) = f_0(s) \text{ for a.e. } s \in [-r, 0], \end{cases}$$

for some function $f_0 \in X$.

By [**BP05**, Lem. 3.4], if $f \in W^{1,p}([-r, +\infty), U)$, then the function $x : [0, +\infty) \times [-r, 0] \to U$ given by $x(t, s) := f_t(s) = f(s+t)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} x(t,s) = \frac{\partial}{\partial s} x(t,s), \quad (t,s) \in [0, +\infty) \times [-r, 0], \\ x(t, 0) = f(t), \quad t \geq 0, \\ x(0, \cdot) = f_0(\cdot). \end{cases}$$

Thus system (DE) is equivalent to

$$\text{(DDE)} \qquad \begin{cases} \frac{\partial}{\partial t} x(t,s) = \frac{\partial}{\partial s} x(t,s), \quad (t,s) \in [0, +\infty) \times [-r, 0], \\ x(t, 0) = \int_{-r}^{0} \mathrm{d}\mu(s) x(t,s), \quad t \geq 0, \\ x(0, \cdot) = f_0(\cdot). \end{cases}$$

In order to use the results obtained in Section 3.1 we introduce the operators

- $A_m := \frac{\mathrm{d}}{\mathrm{d}s} \in \mathcal{L}(Z, X)$ the maximal operator,
- $L := \delta_0 \in \mathcal{L}(Z, U)$ the boundary operator
- $\Phi \in \mathcal{L}(Z, U)$ given by $\Phi f := \int_{-r}^{0} \mathrm{d}\mu(s) f(s)$ the boundary perturbation.

Then the system (DDE) can be rewritten as

$$\begin{cases} \dot{x}(t) = A_m x(t), \quad t \geq 0, \\ L x(t) = \Phi x(t), \quad t \geq 0, \\ x(0) = f_0, \end{cases}$$

which is again equivalent to

$$\begin{cases} \dot{x}(t) = A^{\Phi}x(t), & t \geq 0, \\ x(0) = f_0, \end{cases}$$

where $A := A_m|_{\ker L}$ and A^{Φ} is a boundary perturbation of A.

One notices that we are in the situation of Example 3.1.5. Thus, in order to show the generator property of A^{Φ}, we just have to apply Corollary 3.1.7 and Remark 3.1.8.

Since μ is a function of bounded variation with $\mu(0) = 0$, the operator $\Phi \in C\left([-r,0],U\right)'$ defined by

$$\Phi f := \int_{-r}^{0} \mathrm{d}\mu(s) f(s)$$

has little mass in 0. This allows us to conclude that the system (DE) and (DDE) are well–posed.

Example 2: One–dimensional heat equation with Neumann boundary conditions

Given $\mu : [-\pi, 0] \to \mathbb{R}$ a function of bounded variation with $\mu(0) = 0$ we consider the following one–dimensional heat equation with perturbed Neumann boundary conditions.

$$\text{(HN)} \qquad \begin{cases} \frac{\partial f}{\partial t}(x,t) = \frac{\partial^2 f}{\partial x^2}(x,t), & 0 \leq x \leq \pi,\ t \geq 0, \\ \frac{\partial f}{\partial x}(0,t) = \int_0^{\pi} \int_{-\pi}^{0} \mathrm{d}\mu(\theta) f(x, t+\theta)\mathrm{d}x, & t \geq 0, \\ f(\pi, t) = 0, & t \geq 0, \\ f(x,\theta) = h(x,\theta), & 0 \leq x \leq \pi,\ -\pi \leq \theta \leq 0, \\ f(x,0) = f_0(x), & 0 \leq x \leq \pi. \end{cases}$$

As in Example 3.2 we introduce the history function $v(x,t,\theta) = f(x,t+\theta) = f_t(x,\theta)$ for $t \geq 0,\ 0 \leq x \leq \pi,\ -\pi \leq \theta \leq 0$, which again satisfies

$$\begin{cases} \frac{\partial f}{\partial t}(x,t,\theta) = \frac{\partial v}{\partial \theta}(x,t,\theta), & 0 \leq x \leq \pi,\ t \geq 0, -\pi \leq \theta \leq 0, \\ v(x,t,0) = f(x,t), & 0 \leq x \leq \pi,\ t \geq 0, \\ v(x,0,\theta) = h(x,\theta), & 0 \leq x \leq \pi,\ -\pi \leq \theta \leq 0. \end{cases}$$

Introducing the new variable

$$w(x,t) = \begin{pmatrix} f(x,t) \\ f_t(x,\bullet) \end{pmatrix}, \quad 0 \leq x \leq \pi,\ t \geq 0,$$

we can rewrite equation(HN) into the following equivalent problem.

$$\text{(HNH)}\qquad \begin{cases} \frac{\partial w}{\partial t}(x,t) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix} w(x,t), \quad 0 \le x \le \pi,\ t \ge 0, \\ \begin{pmatrix} \frac{\partial f}{\partial x}(0,t) \\ f_t(x,0) \end{pmatrix} = \begin{pmatrix} \int_0^\pi \int_{-\pi}^0 \mathrm{d}\mu(\theta) f(x,t+\theta)\mathrm{d}x \\ f(x,t) \end{pmatrix}, \quad 0 \le x \le \pi,\ t \ge 0, \\ f(\pi,t) = 0, \quad t \ge 0, \\ w(x,0) = \begin{pmatrix} f_0(x) \\ h(x,\bullet) \end{pmatrix}, 0 \le x \le \pi. \end{cases}$$

Thus, introducing the spaces $X_0 := L^2[0,\pi]$, $\mathcal{X} := X_0 \times L^2([-\pi,0],X_0)$ and on $\mathcal{X}$ the operator

$$\mathcal{A} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix},$$

$$D(\mathcal{A}) := \Big\{ \binom{\phi}{\varphi} \in H^2[0,\pi] \times W^{1,2}\big([-\pi,0],L^2[0,\pi]\big) : \phi(\pi) = 0,\ \varphi(0) = \phi,$$

$$\phi'(0) = \int_0^\pi \int_{-\pi}^0 d\mu(\theta)\varphi(x,\theta)dx \Big\},$$

equation (HNH) is equivalent to the abstract Cauchy problem

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t), \quad t \ge 0, \\ w(0) = w_0. \end{cases}$$

This implies that in order to analize if our starting problem (HN) is well–posed, one has to show that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{X}$.

We first notice that $\mathcal{A}$ is a boundary perturbation of the operator

$$\mathcal{A}^0 := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix},$$

$$D(\mathcal{A}^0) := \Big\{ \binom{\phi}{\varphi} \in H^2[0,\pi] \times W^{1,2}([-\pi,0],L^2[0,\pi]) : \phi(\pi) = 0,\ \phi'(0) = 0,\ \varphi(0) = 0 \Big\}.$$

Thus, following Lemma 3.1.2, we write $\mathcal{A}$ as a Weiss–Staffans perturbation of $\mathcal{A}^0$.

In order to do so and to simplify the notation one first introduces the operators

- $L_0 \in \mathcal{L}(\mathbb{C}, X_0), \quad L_0(\beta) = \beta \cdot (\bullet - \pi),$

- $D_0 \in \mathcal{L}(X_0, L^2([-\pi, 0], X_0)), \quad D_0 f = \mathbb{1} \otimes f,$

- $C \in \mathcal{L}(W^{1,2}([-\pi, 0], X_0), \mathbb{C}), \quad C\varphi := \int_0^\pi \int_{-\pi}^0 d\mu(\theta)\varphi(x,\theta)dx.$

By simple calculations one notices that $\mathcal{A}$ can be written as

$$\mathcal{A} = \left(\mathcal{A}^0_{-1} - \mathcal{A}^0_{-1}\mathcal{D}_0\Phi\right)|_{\mathcal{X}},$$

with

$$\mathcal{D}_0 := \begin{pmatrix} L_0 & 0 \\ 0 & D_0 \end{pmatrix} : \mathbb{C} \times X_0 \to X_0 \times L^2([-\pi, 0], X_0),$$

$$\Phi := \begin{pmatrix} 0 & C \\ Id_{X_0} & 0 \end{pmatrix} : X_0 \times W^{1,2}([-\pi, 0], X_0) \to \mathbb{C} \times X_0 \; =: \partial X.$$

By Proposition 3.1.3 it suffices to show that $\mathcal{A}^0$ is a generator, the couple $(-\mathcal{A}^0_{-1}\mathcal{D}_0, \Phi)$ is jointly 2-admissible with respect to $\mathcal{A}^0$ and that $Id_{\mathbb{C}\times X_0}$ is an admissible feedback for $(\mathcal{A}^0, -\mathcal{A}^0_{-1}\mathcal{D}_0, \Phi)$.

Defining

- $A = \frac{d^2}{dx^2}$ with domain $D(A) = \left\{\phi \in W^{2,2}([-\pi, 0], X_0) : \phi(-\pi) = 0,\; \phi'(0) = 0\right\}$,
- $Q = \frac{d}{dx}$ with domain $D(Q) = \{\varphi \in W^{1,2}([-\pi, 0], X_0) : \varphi(0) = 0\}$,

one notices that the operator $\mathcal{A}^0 =: \left(\begin{smallmatrix} A & 0 \\ 0 & Q \end{smallmatrix}\right)$ is a diagonal matrix with diagonal domain. This suggests to split the problem into two parts.

Part 1. We show that A generates a C_0-semigroup and that the couple $(-A_{-1}L_0, Id_{X_0})$ is jointly 2-admissible with respect to A.

• The operator $A = \frac{d^2}{dx^2}$ with domain $D(A) = \left\{\phi \in W^{2,2}([-\pi, 0], X_0) : \phi(-\pi) = 0,\; \phi'(0) = 0\right\}$ is self-adjoint and negative definite with spectrum

$$\sigma(A) = \left\{-\tfrac{(2k+1)^2}{4} : k \in \mathbb{N}\right\},$$

thus it generates an analytic semigroup $(T(t))_{t\geq 0}$ on X_0.

• Of course, the operator Id_{X_0} is a 2-admissible observation operator for A.

• We now show that $-A_{-1}L_0 : \mathbb{C} \to X_{0,-1}$ is a 2-admissible control operator with respect to A. Analogously as in Section 2.2 we first notice that A is a self-adjoint operator with compact resolvent, thus its normalized eigenvectors

$$e_k(s) := \sqrt{\tfrac{2}{\pi}} \cos\left(\tfrac{2k+1}{2}\, s\right)$$

form an orthormal basis for X_0. Thus we can define a surjective isometry

$$J : X_0 \to l^2, \qquad Jf = \big(\langle f, e_k\rangle\big)_{k\in\mathbb{N}},$$

that maps every function $f \in X_0$ to the sequence of its Fourier-coefficients with respect to $(e_k)_{k\in\mathbb{N}}$. Then A is transformed into the multiplication operator

$$JAJ^{-1} =: M = M_{\lambda_k} : D(M) \subset l^2 \to l^2,$$

for $\lambda_k := \frac{(2k+1)^2}{4}$, $k \in \mathbb{N}$ and $D(M) := \left\{(x_k)_{k\in\mathbb{N}} \in l^2 : \left(\frac{(2k+1)^2}{4} x_k\right)_{k\in\mathbb{N}} \in l^2\right\}$.

Furthermore $-A_{-1}L_0$ becomes

$$-JA_{-1}L_0 = -M_{-1}JL_0 := B = \left(-\sqrt{\tfrac{2}{\pi}}\right)_{k\in\mathbb{N}} : \mathbb{C} \to (l^2)_{-1}^M := \left\{(x_k)_{k\in\mathbb{N}} : x_k \in \mathbb{C} \text{ and } \left(\frac{4}{(2k+1)^2} x_k\right)_{k\in\mathbb{N}} \in l^2\right\}.$$

This is a 2-admissible control operator with respect to M since, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left\| \int_0^{t_0} e^{M_{-1}(t_0-r)} Bu(r)dr \right\|_{l^2}^2 &\le \frac{2}{\pi} \sum_{k=0}^{\infty} \left(\int_0^{t_0} e^{-\frac{(2k+1)^2}{4}(t_0-r)} |u(r)|\, dr \right)^2 \\
&\le \frac{2}{\pi} \sum_{k=0}^{\infty} \left(\int_0^{\infty} e^{-\frac{(2k+1)^2}{2} r} dr \right) \cdot \left(\int_0^{\infty} |u(r)|^2\, dr \right) \\
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2} \|u\|_{L^2}^2
\end{aligned}$$

for every $u \in L^2[0, t_0]$.

• Since Id_{X_0} is a bounded operator, the couple $(-A_{-1}L_0, Id_{X_0})$ is jointly 2-admissible with respect to A.

Part 2. We show that Q generates a C_0-semigroup and that the couple $(-Q_{-1}D_0, C)$ is 2-admissible with respect to Q.

• The operator $Q = \frac{d}{dx}$ with domain $D(Q) = \{\varphi \in W^{1,2}([-\pi, 0], X_0) : \varphi(0) = 0\}$ generates the nilpotent left-shift semigroup $(S(t))_{t\ge 0}$ on $L^2([-\pi, 0], X_0)$.

• By the proof of Corollary 3.1.7, part (ii), the operator $-Q_{-1}D_0$ is a 2-admissible control operator for Q.

• The operator C given by $C\varphi = \int_0^{\pi} \int_{-\pi}^{0} d\mu(\vartheta)\varphi(x, \vartheta)dx \in \mathbb{C}$ for $\varphi \in W^{1,2}([-\pi, 0], X_0)$ is a 2-admissible observation operator with respect to Q. Indeed, using Hölder's inequality and Fubini-Tonelli's theorem (here we follow the calculation in [**HMR15**, (5.9)]) we have

$$
\begin{aligned}
\int_0^{t_0} |CS(t)\phi|^2 \mathrm{d}t &= \int_0^{t_0} \left| \int_0^{\pi} \int_{-\pi}^{0} \mathrm{d}\mu(s)(S(t)\phi)(x,s)\mathrm{d}x \right|^2 \mathrm{d}t \\
&= \int_0^{t_0} \left| \int_0^{\pi} \int_{-\pi}^{t} \mathrm{d}\mu(s)\phi(x,s+t)\mathrm{d}x \right|^2 \mathrm{d}t \\
&\le \int_0^{t_0} \left(\int_0^{\pi} \int_{-\pi}^{t} |\phi(x,s+t)| \,\mathrm{d}\,|\mu|\,(s)\mathrm{d}x \right)^2 \mathrm{d}t \\
&\le \pi\,|\mu|\,([-\pi,0]) \int_0^{t_0} \int_0^{\pi} \int_{-\pi}^{t} |\phi(x,s+t)|^2 \,\mathrm{d}\,|\mu|\,(s)\mathrm{d}x\mathrm{d}t \\
&= \pi\,|\mu|\,([-\pi,0]) \int_0^{t_0} \int_{-\pi}^{t} \int_0^{\pi} |\phi(x,s+t)|^2 \,\mathrm{d}x\mathrm{d}\,|\mu|\,(s)\mathrm{d}t \\
&= \pi\,|\mu|\,([-\pi,0]) \int_0^{t_0} \int_{-\pi}^{t} \|\phi(\bullet,s+t)\|_{X_0}^2 \,\mathrm{d}\,|\mu|\,(s)\mathrm{d}t \\
&= \pi\,|\mu|\,([-\pi,0]) \int_0^{-s} \int_{-\pi}^{t} \|\phi(\bullet,s+t)\|_{X_0}^2 \,\mathrm{d}\,|\mu|\,(s)\mathrm{d}t \\
&\le \pi\,(|\mu|\,([-\pi,0]))^2 \,\|\phi\|_{L^2([-\pi,0],X_0)}^2 .
\end{aligned}
$$

- The couple $(-Q_{-1}D_0, C)$ is 2-admissible with respect to Q since by Equation (3.2)

$$
-\int_0^t S_{-1}(t-r)Q_{-1}\big(\mathbb{1}\otimes u(r)\big)(\cdot)dr = u\big(\max\{0,\cdot+t\}\big)
$$

for $u \in W_0^{2,2}([0,t_0],X_0)$. Choosing $t_0 = \pi$, one obtains the estimate

$$
\begin{aligned}
& \int_0^{\pi} \left| \int_0^{\pi} \int_{-\pi}^{0} d\mu(\vartheta)u\big(\max\{0,\vartheta+t\}\big)(x)dx \right|^2 dt \\
= \;& \int_0^{\pi} \left| \int_0^{\pi} \int_{-t}^{0} d\mu(\vartheta)u(t+\vartheta)(x)dx \right|^2 dt \\
\le \;& \int_0^{\pi} \left(\int_0^{\pi} \int_{-t}^{0} d\,|\mu|\,(\vartheta)\,|u(t+\vartheta)(x)|\,dx \right)^2 dt \\
= \;& \int_0^{\pi} \left(\int_{-t}^{0} \int_0^{\pi} |u(t+\vartheta)(x)|\,dx d\,|\mu|\,(\vartheta) \right)^2 dt \qquad (3.6)\\
\le \;& \int_0^{\pi} |\mu|[-t,0] \int_{-t}^{0} \int_0^{\pi} d|\mu|(\vartheta)\,|u(t+\vartheta)(x)|^2\,dx\,dt \qquad (3.7)\\
\le \;& |\mu|[-\pi,0] \int_0^{\pi} \int_{-t}^{0} \|u(t+\vartheta)\|_{X_0}^2 \,d|\mu|(\vartheta)\,dt \\
= \;& |\mu|[-\pi,0] \int_{-\pi}^{0} \int_0^{\pi+\vartheta} \|u(t)\|_{X_0}^2 \,dt\,d|\mu|(\vartheta) \qquad (3.8)\\
\le \;& \big(|\mu|[-\pi,0]\big)^2 \cdot \|u\|_{L^2([0,\pi],X_0)}^2 .
\end{aligned}
$$

Here in (3.7) we used Cauchy–Schwarz's inequality while in (3.6) & (3.8) we applied Fubini-Tonelli's theorem.

Putting together Part 1 and 2 one concludes that $\mathcal{A}^0$ is the generator of a C_0-semigroup and the couple $(-\mathcal{A}^0_{-1}\mathcal{D}_0, \Phi)$ is jointly 2-admissible with respect to $\mathcal{A}^0$.

Part 3. It remains to show that $Id_{\mathbb{C}\times X_0}$ is a 2-admissible feedback operator for $(\mathcal{A}^0, -\mathcal{A}^0_{-1}\mathcal{D}_0, \Phi)$.

For $u_1 \in W^{2,1}_0[0,t_0]$ and $u_2 \in W^{2,1}_0([0,t_0], X_0)$ we have

$$\begin{aligned}\mathcal{F}_{t_0}\begin{pmatrix}u_1\\u_2\end{pmatrix}(t) &= -\begin{pmatrix}0 & C\\ I & 0\end{pmatrix}\int_0^t \begin{pmatrix}T_{-1}(t-r) & 0\\ 0 & S_{-1}(t-r)\end{pmatrix}\begin{pmatrix}A_{-1} & 0\\ 0 & Q_{-1}\end{pmatrix}\begin{pmatrix}D_0 & 0\\ 0 & L_0\end{pmatrix}\begin{pmatrix}u_1(r)\\u_2(r)\end{pmatrix}dr\\ &=: \begin{pmatrix}0 & F_{t_0}(t)\\ G_{t_0}(t) & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}F_{t_0}(t)u_2 &= -C\int_0^t S_{-1}(t-r)Q_{-1}D_0u_2(r)dr,\\ G_{t_0}(t)u_1 &= -\int_0^t T_{-1}(t-r)A_{-1}L_0u_1(r)dr \quad \text{for } t\in[0,t_0].\end{aligned}$$

Since Id_{X_0} is a bounded observation operator, by the proof of [**ABE14**, Thms. 14 und 16],we obtain that $\|G_{t_0}\| \to 0$ for $t_0 \searrow 0$.

Furthermore, since $\|F_t\| \le \|F_{t_0}\| \le (|\mu|\,[-\pi,0])^2$ for every $t\in[0,\pi]$, it follows that $\|F_{t_0}G_{t_0}\| < 1$ for $t_0 > 0$ small enough.

Thus $1 \in \rho(F_{t_0}G_{t_0})$ for $t_0 > 0$ small enough which by [**Eng99**, Lemma 2.1] is equivalent to the invertibility of

$$\begin{aligned}Id - \mathcal{F}_{t_0} &= Id_{\mathbb{C}\times X_0} - \begin{pmatrix}0 & F_{t_0}\\ G_{t_0} & 0\end{pmatrix}\\ &= \begin{pmatrix}I_{\mathbb{C}} & -F_{t_0}\\ 0 & I_{X_0}\end{pmatrix}\begin{pmatrix}I_{\mathbb{C}}-F_{t_0}G_{t_0} & 0\\ 0 & I_{X_0}\end{pmatrix}\begin{pmatrix}I_{\mathbb{C}} & 0\\ -G_{t_0} & I_{X_0}\end{pmatrix}.\end{aligned}$$

The results of *Part 1*, *Part 2* and *Part 3* together permit to conclude that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{X}$.

CHAPTER 4

Weiss–Staffans perturbation of analytic semigroups

In this chapter we study Weiss–Staffans perturbations of generators of analytic semigroups. The results will appear in a forthcoming joint paper with M. Adler and K.-J. Engel.

4.1. Analytic semigroups

We first introduce the basic concepts on analytic semigroups. For more details see [**EN00**, Chap. II.4] and [**Lun95**, Chap. 2].

Definition 4.1.1. Let $(A, D(A))$ be a closed, densely defined operator on a Banach space X. Then A is called *sectorial* of angle $\delta \in \left(0, \frac{\pi}{2}\right]$ if the sector

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \setminus \{0\}$$

is contained in the resolvent set $\rho(A)$, and if for every $\epsilon \in (0, \delta)$ there exists $M_\epsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\epsilon}{|\lambda|} \qquad \text{for all } 0 \neq \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\epsilon}.$$

Definition 4.1.2. A family of bounded operators $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on X is called an *analytic semigroup* of angle $\delta \in \left(0, \frac{\pi}{2}\right]$ if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.

(ii) The map $z \mapsto T(z)$ is analytic in Σ_δ.

(iii) $\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ for all $x \in X$ and $0 < \delta' < \delta$.

If, in addition,

(iv) $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$,

we call $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ a *bounded analytic semigroup*.

These two concept are related as the following result states ([**EN00**, Thm. II.4.6]).

Theorem 4.1.3. *Let $(A, D(A))$ be an operator on a Banach space X. Then he following are equivalent.*

(a) A generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on X.

(b) There exists $\theta \in \left(0, \frac{\pi}{2}\right)$ such that the operators $e^{\pm i\theta}A$ generate bounded strongly continuous semigroups on X.

(c) A generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X such that $\operatorname{rg}(T(t)) \subset D(A)$ for all $t > 0$ and

$$\sup_{t>0} \|tAT(t)\| < \infty.$$

(d) A generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X and there exists $C > 0$ such that

$$\|R(r + is, A)\| \leq \frac{C}{|s|}$$

for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

(e) A is sectorial.

In order to formulate our perturbation result, we need the following tool (for more informations see [**EN00**, Sec. II.5.b]).

Definition 4.1.4. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with growth bound $\omega_0 < 0$. For each $\alpha \in (0, 1]$ the space

$$F_\alpha := \left\{x \in X : \sup_{t>0} \left\| \frac{1}{t^\alpha} (T(t)x - x) \right\| < \infty \right\}$$

with norm

$$\|x\|_{F_\alpha} := \sup_{t>0} \left\| \frac{1}{t^\alpha} (T(t)x - x) \right\|$$

is called the *Favard space of order α* corresponding to $(T(t))_{t\geq 0}$.

One can characterize these spaces also by means of the generator $(A, D(A))$ of the C_0-semigroup $(T(t))_{t\geq 0}$ as follows (see [**EN00**, Prop. II.5.12]).

Proposition 4.1.5. *Assume that $(T(t))_{t\geq 0}$ is a C_0-semigroup with $\omega_0 < 0$. For $\alpha \in (0, 1]$ the Favard space of order α is complete and coincides with*

$$F_\alpha = \left\{x \in X : \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty \right\}.$$

Moreover, the Favard norm $\|\cdot\|_{F_\alpha}$ is equivalent to

$$|||x|||_{F_\alpha} := \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|.$$

4.2. The perturbation theorem

In order to state the main result of this chapter we first need a technical lemma which recalls Young's inequality.

Lemma 4.2.1. *Let X, F be Banach spaces, $K : (0,1] \to \mathcal{L}(F,X)$ strongly continuous, and $1 \leq p, q, r$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.*

If $k(\bullet) := \|K(\bullet)\|_{\mathcal{L}(F,X)} \in L^q(0,1)$ and $v \in C([0,1],F)$ then $K * v \in L^r((0,1),X)$ and

$$\|K * v\|_r \leq \|k\|_q \|v\|_p .$$

Proof. We follow the proof of [**ABHN11**, Prop. 1.3.5]. Let $0 < t \leq 1$ and $v \in C([0,1],F)$. Using the uniform boundedness principle one can show that for $s \in (0,t)$ and $(0,t) \ni s_n \overset{n\to\infty}{\longrightarrow} s$, for $n \in \mathbb{N}$ big enough

$$\begin{aligned}
&\|K(t-s_n)v(s_n) - K(t-s)v(s)\|_X \\
&\leq \|K(t-s_n)(v(s_n) - v(s))\|_X + \|K(t-s_n)v(s) - K(t-s)v(s)\|_X \\
&\leq M\|v(s_n) - v(s)\|_F + \|K(t-s_n)v(s) - K(t-s)v(s)\|_X \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

since v is continuous and K strongly continuous.

Thus the function

$$s \mapsto b(s) := K(t-s)v(s)$$

is continuous on $(0,t)$ and so also measurable.

By assumption $k(\bullet) \in L^q(0,1)$, thus $k(t-\bullet) \in L^q(0,t) \subset L^1(0,t)$.

Since

$$\|b(s)\|_X = \|K(t-s)v(s)\|_X \leq k(t-s)\|v\|_\infty ,$$

$\|b(\bullet)\|$ is integrable on $[0,t]$. By Bochner's Theorem (see [**ABHN11**, Thm. 1.1.4]) $b(\bullet)$ is Bochner integrable and so $(K * v)(t)$ exists for every $t \in [0,1]$.

We now show that $t \mapsto (K * v)(t)$ is continuous on $[0,1]$.

Let $t \in [0,1]$ and $h \in \mathbb{R}$ such that $t + h \in [0,1]$. By assumption we know that $k(\bullet) \in L^q(0,1) \subset L^1(0,1)$ and $v \in C[0,1]$ is uniformly continuous. This allows us to perform the following computations

$$
\begin{aligned}
&\left\|(K*v)(t+h)-(K*v)(t)\right\|_X \\
&=\left\|\int_0^{t+h}K(s)v(t+h-s)\mathrm{d}s-\int_0^t K(s)v(t-s)\mathrm{d}s\right\|_X \\
&\le \int_0^t k(s)\left\|v(t+h-s)-v(t-s)\right\|_F\mathrm{d}s+\int_t^{t+h}k(s)\left\|v(t+h-s)\right\|_F\mathrm{d}s \\
&\le \|k\|_1\sup_{s\in[0,t]}\left\|v(t+h-s)-v(t-s)\right\|_F+\|v\|_\infty\int_t^{t+h}k(s)\mathrm{d}s \xrightarrow{h\to 0} 0.
\end{aligned}
$$

Thus $K*v\in C([0,1],X)\subset L^r((0,1),X)$ and by the scalar–valued Young's inequality

$$\|K*v\|_r\le \left\|k*\|v(\bullet)\|_F\right\|_r\le \|k\|_q\|v\|_p.$$

□

We also need the following lemma describing the relation between the domain and the Favard space of a given generator and the "rotated" operator $A_\phi := e^{i\phi}A$ for some $\phi\in[0,\pi)$.

Lemma 4.2.2. *Let A be the generator of an analytic semigroup of angle $0<\theta\le\frac{\pi}{2}$.*

Then A_ϕ generates an analytic semigroup for every $\phi\in(-\theta,\theta)$. Furthermore, for all $\alpha\in(0,1]$ A^α, the α power of A (see [**EN00**, Def. II.5.31]*), satisfies*

$$D(A^\alpha)=D(A_\phi^\alpha)\quad \textit{and } Fav_\alpha^A=Fav_\alpha^{A_\phi}.$$

Proof. By Theorem 4.1.3 part (b), A_ϕ is the generator of a bounded C_0-semigroup. For $\alpha>0$ using Cauchy's integral theorem (see [**DS88**, Sect. III.14]) one easily obtains

$$A_\phi^{-\alpha}=e^{-i\phi\alpha}A^{-\alpha}. \tag{4.1}$$

Since $D(A^\alpha)=\mathrm{rg}(A_\phi^{-\alpha})$, Equation (4.1) implies

$$D(A^\alpha)=D(A_\phi^\alpha).$$

In order to prove that the two Favard spaces coincide we use of the characterization given by Proposition 4.1.5.

For sake of clarity we point out that

$$R(\lambda,A_\phi)=e^{-i\phi}R(e^{-i\phi}\lambda,A),$$

thus

$$A_\phi R(\lambda,A_\phi)=AR(e^{-i\phi}\lambda,A).$$

To show that $F_\alpha^A \subset F_\alpha^{A_\phi}$ let $x \in F_\alpha^A$. Then

$$
\begin{aligned}
&\sup_{\lambda>0} \|\lambda^\alpha A_\phi R(\lambda, A_\phi)x\|_X \\
&= \sup_{\lambda>0} \|\lambda^\alpha AR(e^{-i\phi}\lambda, A)x\|_X \\
&\le \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|_X + \sup_{\lambda>0} \|\lambda^{\alpha+1}(1-e^{-i\phi})AR(\lambda, A)R(e^{-i\phi}\lambda, A)x\|_X \qquad (4.2)\\
&\le \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|_X + \sup_{\lambda>0} \|(1-e^{-i\phi})e^{i\phi}\lambda R(e^{-i\phi}\lambda, A)\|_{\mathcal{L}(X)} \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|_X \\
&\le \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|_X + M \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|_X,
\end{aligned}
$$

for $M \ge 0$, where in (4.2) we used the resolvent equation. Since $x \in F_\alpha^A$,

$$\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty.$$

This implies that

$$\sup_{\lambda>0} \|\lambda^\alpha A_\phi R(\lambda, A_\phi)x\| < \infty,$$

thus

$$F_\alpha^A \subset F_\alpha^{A_\phi}. \tag{4.3}$$

In order to show that $F_\alpha^{A_\phi} \subset F_\alpha^A$, one just notice that $A = e^{-i\phi}A_\phi$ and then apply (4.3) for A and A_ϕ interchanged and $-\Phi$ instead of Φ.

It remains to show that A_ϕ generates an analytic semigroup.

Since $\phi \in (-\theta, \theta)$, there exists $\epsilon > 0$ such that $|\phi \pm \epsilon| < \theta$. Thus $e^{i(\phi\pm\epsilon)A} = e^{\pm i\epsilon}A_\phi$ is generate bounded C_0-semigroup and by Theorem [**EN00**, Thm. II.4.6] A_ϕ generates an analytic semigroup. □

We are now ready to state and prove the main result of this chapter.

Theorem 4.2.3. *Let A be the generator of an analytic semigroup $(T(t))_{t\ge0}$ of angle $\theta \in \left(0, \frac{\pi}{2}\right]$ on a Banach space X. For Banach spaces Z, U such that $X_1 \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X_{-1}$ we take $C \in \mathcal{L}(Z, U)$ and $B \in \mathcal{L}(U, X_{-1})$. If there exists $\beta \ge 0$ and $\gamma > 0$ such that*

(i) $\operatorname{rg}(A_{-1}^{-1}B) \subset F_{1-\beta}^A$,

(ii) $D(A^\gamma) \overset{c}{\hookrightarrow} Z$,

(iii) $\beta + \gamma < 1$,

then

(a) *the triple* (A, B, C) *is compatible,*

(b) B *is* p*-admissible for every* $p > \frac{1}{1-\beta}$*; if* $\beta = 0$*, then also for* $p = 1$*,*

(c) C *is* p*-admissible for every* $p < \frac{1}{\gamma}$*,*

(d) (B, C) *is* p*-admissible for every* $\frac{1}{1-\beta} < p < \frac{1}{\gamma}$*; if* $\beta = 0$*, then also for* $p = 1$*,*

(e) *for every* $0 < \epsilon < 1 - (\beta + \gamma)$ *and* $\frac{1}{1-\beta} \leq p < \frac{1}{\gamma}$ *there exists* $M \geq 0$ *such that*

$$\|\mathcal{F}_t\|_p \leq M t^{\epsilon}, \quad 0 < t \leq 1.$$

This means that every $F \in \mathcal{L}(U)$ *is a* p*-admissible feedback for* (A, B, C)*, and* $(A_{-1} + BFC)|_X$ *generates an analytic semigroup.*

Proof. (a) Let $0 < \delta < \alpha < 1$ then by [**EN00**, Prop. II.5.14 and Prop. II.5.33]

$$D(A^{\alpha}) \overset{c}{\hookrightarrow} X_{\alpha}^{A} \overset{c}{\hookrightarrow} F_{\alpha}^{A} \overset{c}{\hookrightarrow} D(A^{\delta}).$$

Hypothesis (iii) implies that $1 - \beta > \gamma$, then applying (i) and (ii) we obtain

$$\mathrm{rg}(A_{-1}^{-1} B) \subset F_{1-\beta}^{A} \subset D(A^{\gamma}) \subset Z.$$

(c) Let $t > 0$, then using (ii) one obtains

$$\|CT(t)\|_{\mathcal{L}(X,U)} \leq \|CA^{-\gamma}\|_{\mathcal{L}(X,U)} \|A^{\gamma} T(t)\|_{\mathcal{L}(X)}.$$

Furthermore by [**RR93**, Lem. 11.36]

$$\|A^{\gamma} T(t)\|_{\mathcal{L}(X)} \leq M t^{-\gamma} \quad \text{for every } t \in (0,1]. \tag{4.4}$$

This permits us to conclude that for every $p < \frac{1}{\gamma}$ $\|A^{\gamma} T(\bullet) x\| \in L^p(0,1)$ for every $x \in D(A)$, hence C is p-admissible.

(b) The closed graph theorem together with condition (i) imply that $A_{-1}^{-1} B \in \mathcal{L}(U, F_{1-\beta}^{A})$, thus,

$$v := A_{-1}^{-1} B u \in \mathrm{L}^p\left((0,1), F_{1-\beta}^{A}\right)$$

for every $u \in \mathrm{L}^p\left((0,1), U\right)$.

Since $\mathrm{rg}\left(T(t)\right) \subset D\left(A^{\infty}\right) := \bigcap_{n \in \mathbb{N}} D\left(A^n\right)$ for every $t > 0$, we can define

$$K : (0,1] \to \mathcal{L}\left(F_{1-\beta}^{A}, X\right), \quad t \mapsto AT(t).$$

Then K is strongly continuous on $(0,1]$ and by [**EN00**, Prop. II.5.13] there exists $M > 0$ such that

$$\|t^\beta K(t)x\|_X \le \sup_{s\in(0,1]} \|s^\beta AT(s)x\|_X \le M\|x\|_{F^A_{1-\beta}}$$

for every $x \in F^A_{1-\beta}$. It thus follows that

$$k(t) := \|K(t)\|_{\mathcal{L}\left(F^A_{1-\beta},X\right)} \le Mt^{-\beta} \quad \text{for all } t \in (0,1], \tag{4.5}$$

this imply

$$k \in L^q(0,1) \text{ if } \begin{cases} q < \frac{1}{\beta} \text{ and } \beta > 0, \\ q \ge 1 \text{ and } \beta = 0. \end{cases}$$

Choosing $r = \infty$ in Lemma 4.2.1 one obtains that for $q = \frac{p}{p-1}$ there exists $M \ge 0$ such that for every $u \in C([0,1],U)$

$$\begin{aligned} &\left\| \int_0^1 T_{-1}(1-s)Bu(s)\mathrm{d}s \right\|_X \\ =& \left\| \int_0^1 AT_{-1}(1-s)A^{-1}_{-1}Bu(s)\mathrm{d}s \right\|_X \\ =& \|(K * v)(1)\|_X \le \|K * v\|_\infty \le \|k\|_q \|u\|_p, \end{aligned}$$

provided

$$\begin{cases} q = \frac{p}{1-p} < \frac{1}{\beta} \text{ and } \beta > 0 \Leftrightarrow p > \frac{1}{1-\beta} \text{ and } \beta > 0, \\ q = \frac{p}{1-p} \ge 1 \text{ and } \beta = 0 \Leftrightarrow p \ge 1 \text{ and } \beta = 0. \end{cases}$$

Since $C([0,1],U)$ is dense in $\mathrm{L}^p([0,1],U)$ the proof is concluded.

(d) Again, since $\mathrm{rg}(T(t)) \subset D(A^\infty)$ for every $t > 0$, we can define the following strongly continuous function

$$L : (0,1] \to \mathcal{L}\left(F^A_{1-\beta}, X\right), \quad t \mapsto A^{1+\gamma}T(t).$$

Then by (4.4) and (4.5) there exists $\tilde{M} \ge 0$ such that

$$\begin{aligned} l(t) :=& \|L(t)\|_{\mathcal{L}\left(F^A_{1-\beta},X\right)} \\ \le& \left\| A^\gamma T\left(\frac{t}{2}\right)\right\|_{\mathcal{L}(X)} \left\| AT\left(\frac{t}{2}\right)\right\|_{\mathcal{L}\left(F^A_{1-\beta},X\right)} \\ \le& \tilde{M}t^{-(\beta-\gamma)}. \end{aligned}$$

Choosing $p = \frac{1}{1-\beta} \le r < \frac{1}{\gamma}$ in Lemma 4.2.1, we obtain $\frac{1}{q} = \beta + \frac{1}{r} > \beta + \gamma$ and thus

$$q(\beta+\gamma) < 1 \text{ implying } l \in \mathrm{L}^q(0,1).$$

Let $u \in C([0,1],U)$, then by Lemma 4.2.1 there exists $\hat{M} \geq 0$ such that

$$\begin{aligned}
&\left(\int_0^1 \left\| C\int_0^t T_{-1}(t-s)Bu(s)\mathrm{d}s\right\|_X^r \mathrm{d}t\right)^{\frac{1}{r}}\\
&\leq \|CA^{-\gamma}\|_{\mathcal{L}(X,U)} \left(\int_0^1 \left\|\int_0^t A^{1+\gamma}T(t-s)A_{-1}^{-1}Bu(s)\mathrm{d}s\right\|_X^r \mathrm{d}t\right)^{\frac{1}{r}}\\
&\leq \|CA^{-\gamma}\|_{\mathcal{L}(X,U)} \|(L*v)\|_r\\
&\leq \hat{M}\,\|l\|_q\,\|u\|_{\frac{1}{1-\beta}}\,. \qquad (4.6)
\end{aligned}$$

This implies that for $\frac{1}{1-\beta} \leq r < \frac{1}{\gamma}$ the input-output map has a unique continuous extension

$$\mathcal{F}_t : \mathrm{L}^{\frac{1}{1-\beta}}((0,t),U) \to \mathrm{L}^r((0,t),U)\,.$$

Since $\mathrm{L}^r((0,t),U) \overset{c}{\hookrightarrow} \mathrm{L}^{\frac{1}{1-\beta}}((0,t),U)$ for $r \geq \frac{1}{1-\beta}$, considering also (b) and (c), one can conclude that (B,C) is jointly p-admissible for every $p \in \left(\frac{1}{1-\beta}, \frac{1}{\gamma}\right)$, if $\beta = 0$ then also for $p = 1$.

(e) Jensen's inequality implies that for $1 \leq p \leq r < \infty$ and $u \in \mathrm{L}^r([0,t],U) \subset \mathrm{L}^p([0,t],U)$

$$\|u\|_p \leq t^{\frac{1}{p}-\frac{1}{r}}\|u\|_r.$$

This together with (4.6) implies that for $\frac{1}{1-\beta} \leq p \leq r < \frac{1}{\gamma}$ and $u \in \mathrm{L}^r([0,t],U)$ there exists $M_1 \geq 0$ such that

$$t^{-\frac{1}{p}+\frac{1}{r}}\|\mathcal{F}_t u\|_p \leq \|\mathcal{F}_t u\|_r \leq M_1\|u\|_{\frac{1}{1-\beta}} \leq M_1 t^{1-\beta-\frac{1}{p}}\|u\|_p.$$

Let $0 < \epsilon < 1-(\beta+\gamma)$ and define $r := \frac{1}{1-\beta-\epsilon} \in \left(\frac{1}{1-\beta}, \frac{1}{\gamma}\right)$. Then since $\mathrm{L}^r([0,t],U)$ is dense in $\mathrm{L}^p([0,t],U)$

$$\|\mathcal{F}_t\|_p \leq M_1 t^{1-\beta-\frac{1}{r}} \leq M_1 t^{\epsilon}.$$

Thus we can apply Theorem 1.2.1 and conclude that $(A_{-1}+BFC)_{|X}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X for every $F \in \mathcal{L}(U)$.

It remains to prove that $(S(t))_{t\geq 0}$ is analytic.

Since A generates an analytic semigroup of angle $\theta \in \left(0, \frac{\pi}{2}\right]$ then by Lemma 4.2.2 for $\phi \in (-\theta,\theta)$ also $A_{\pm\phi}$ generate analytic semigroups.

Using again Lemma 4.2.2 one obtains that

$$(A_{\phi,-1}+BFC)_{|X} \text{ and } (A_{-\phi,-1}+BFC)_{|X}$$

are also generators for every $F \in \mathcal{L}(U)$. Hence replacing F by $e^{i\phi}F$ also

$$\left(A_{\phi,-1} + e^{i\phi}BFC\right)_{|X} \quad \text{and} \quad \left(A_{-\phi,-1} + e^{-i\phi}BFC\right)_{|X}$$

are generators. This together with Theorem 4.1.3 part (b) permits us to conclude that $(A_{-1} + BFC)_{|X}$ generates an analytic semigroup for every $F \in \mathcal{L}(U)$. □

If the Banach space Z is the domain of a closed operator K, then the condition $D(A^\gamma) \subset Z$ can be verified by the following result, see [**RR93**, Lem. 11.39].

Lemma 4.2.4. *Let A be the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X and let K be a closed linear operator such that $D(A) \subset Z := D(K)$. If for $\alpha \in (0,1)$ and every $\rho \geq \rho_0 > 0$ there exists $M \geq 0$ such that*

$$\|Kx\| \leq M \cdot \left(\rho^\alpha \|x\| + \rho^{\alpha-1}\|Ax\|\right) \quad \text{for all } x \in D(A),$$

then $D(A^\gamma) \subset Z$ for every $\gamma > \alpha$.

4.3. Example

We conclude this chapter with an application of Theorem 4.2.3.

Let us consider a heated metal bar of length π modeled as a segment $[0,\pi]$.

As state space we choose the Banach space $X := L^1([0,\pi])$, since its norm represents the total heat at time $t \geq 0$, and consider the state function $x(s,t)$ representing the temperature in the point $s \in [0,\pi]$ at time $t \geq 0$. Clearly, as in section 2.2, the "maximal" operator describing the system is given by

$$A_m := \frac{\mathrm{d}^2}{\mathrm{d}s^2} \qquad \text{with domain} \qquad D(A_m) := W^{2,1}[0,\pi] \subset X = \mathrm{L}^1[0,\pi].$$

We first consider the case where there is no heat exchange between the ends of the bar and the environment. This corresponds to the boundary operator

$$L : W^{2,1}[0,\pi] \to U, \quad Lf := \left(f'(0),\ f'(\pi)\right)^t,$$

for $U := \mathbb{C}^2$ and the system operator given by

$$A := A_m|_{\ker L}.$$

We now modify our system, such that the heat exchange between the ends of the bar and the environment is equal to the temperature in $s = \frac{\pi}{2}$. That means, introducing the operator

$$\Phi : W^{1,1}[0,\pi] =: Z \to U, \quad \Phi f := \left(f\left(\frac{\pi}{2}\right),\ f\left(\frac{\pi}{2}\right)\right)^t,$$

the system operator becomes

$$A^\Phi := A_m \qquad \text{with domain} \qquad D(A^\Phi) := \{x \in D(A_m) : Lx = \Phi x\}.$$

Clearly A^Φ is a boundary perturbation of the generator A, thus similarly as in Lemma 3.1.2, using the Dirichlet-operator D_λ (see Lemma 3.1.1), we can write A^Φ as a Weiss-Staffans perturbation of the generator A

$$A^\Phi = (A_{-1} + (2 - A_{-1})D_2\Phi)\,|_X,$$

for $B := (2 - A_{-1})D_2 \in \mathcal{L}(U, X_{-1})$ and $C := \Phi \in \mathcal{L}(Z, U)$.

Since A is the generator of an analytic semigroup on X, in order to obtain the generator property of A^ϕ, one can apply Theorem 4.2.3 and show that conditions $(i), (ii)$ and (iii) are satisfied.

Condition (i). We show that $\ker(2 - A_m) \subset F_1^A$, where

$$\ker(2 - A_m) = \langle f, g \rangle$$

for $f(s) := e^{-\sqrt{2}s}$ and $g(s) := s^2$.

Modifying f and g in a small neighborhood of the endpoints $s = 0$ and $s = \pi$, one obtains sequences of functions $\left(h_n^{(f)}\right)_{n\in\mathbb{N}} \subset D(A)$ and $\left(h_n^{(g)}\right)_{n\in\mathbb{N}} \subset D(A)$ converging with respect to the norm of X to f and g respectively with

$$\sup_{n\in\mathbb{N}} \left\{ \left\| A h_n^{(f)} \right\|_X \right\} < \infty \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \left\{ \left\| A h_n^{(g)} \right\|_X \right\} < \infty.$$

Thus, $f,\ g \in F_1^A$, and condition (i) of Theorem 4.2.3 is satisfied for $\beta = 0$.

Conditions (ii) and (iii). Since $K := \frac{\mathrm{d}}{\mathrm{d}s}$ with domain $D(K) = Z$ is a closed operator on X we can use Lemma 4.2.4 and show

$$D(A^\gamma) \subset Z \quad \text{for all } \gamma > \frac{1}{2}.$$

Namely, following [**EN00**, Expl. III.2.2], for every $f \in D(A)$ and $\epsilon > 0$ we have

$$\|f'\|_X \le \frac{9}{\epsilon}\|f\|_X + \epsilon\|Af\|_X,$$

thus, choosing $\rho := \epsilon^{-1} \ge 9$

$$\|f'\|_X \le 9\left(\rho^{\frac{1}{2}}\|f\|_X + \rho^{1-\frac{1}{2}}\|Af\|_X\right).$$

Thus for every $\gamma > \frac{1}{2}$, $D(A^\gamma) \subset Z$ and conditions (ii) and (iii) of Theorem 4.2.3 are satisfied, and A^Φ generates an analytic semigroup.

CHAPTER 5

Non autonomous Weiss–Staffans perturbation

In this section we shall apply Theorem 1.3.3 to the following problem.

Given the generator A of a C_0-semigroup $(e^{tA})_{t\geq 0}$ on X and a family of (unbounded) operators $(P(t))_{t\in\mathbb{R}}$. How can we associate a time evolution to the (in a suitable way defined) sums "$A(t) := A + P(t)$"?

5.1. Evolution semigroups

For our approach to this question, we start from a σ-finite measure space (Ω, Σ, μ) and a separable Banach space X.

Definition 5.1.1. Let $1 \leq p \leq \infty$. An operator $\mathcal{M} \in \mathcal{L}(\mathrm{L}^p(\Omega, X))$ is called a *bounded multiplication operator* if there exists an operator valued function $M(\cdot) \in L^\infty(\Omega, \mathcal{L}_s(X))$ such that

$$(\mathcal{M}f)(s) = M(s)f(s) \quad \forall f \in \mathrm{L}^p(\mathbb{R}, X), \text{ a.e. } s \in \Omega. \tag{5.1}$$

In particular, each $\phi \in \mathrm{L}^\infty(\Omega, \mathbb{C})$ yields the multiplication operator $M_\phi \in \mathcal{L}(\mathrm{L}^p(\Omega, X))$ defined by

$$(M_\phi f)(s) = \phi(s)f(s) \quad \text{for a.e. } s \in \Omega.$$

For a systematic investigation of such operators see, e.g., [**How74**], [**Eva76**], [**AT05**] and [**Hey14**].

Clearly, every operator $\mathcal{M} \in \mathcal{L}(\mathrm{L}^p(\Omega, X))$ of the form (5.1) commutes with M_ϕ for every $\phi \in L^\infty(\Omega, \mathbb{C})$. Surprisingly the opposite also holds as proved by Evans in [**Eva76**, Theorem 5.7] in the case $\Omega = \mathbb{R}^n$ and by [**AT05**, Thm. 2.3] in the general case.

Theorem 5.1.2. *Let X be a separable Banach space and $\mathcal{M} \in \mathcal{L}(\mathrm{L}^p(\Omega, X))$ such that*

$$\mathcal{M}M_\phi = M_\phi\mathcal{M} \quad \forall \phi \in L^\infty(\Omega, \mathbb{C}).$$

Then for every $s \in \Omega$ there exists $M(s) \in \mathcal{L}(X)$ such that

- *$M(\cdot)x$ is measurable for every $x \in X$,*
- *$(\mathcal{M}f)(s) = M(s)f(s)$ for a.e. $s \in \Omega$ and every $f \in \mathrm{L}^p(\Omega, X)$.*

The operators $M(s)$ are determined up to a set of measure zero and $\|\mathcal{M}\| = \operatorname{esssup}_{s\in\Omega} \|M(s)\|$.

We now introduce a second concept.

Definition 5.1.3. A family of bounded linear operators $(U(t,s))_{t\geq s}$ on X is called an *exponentially bounded, strongly measurable evolution family* if

- $U(t,t) = I_X \quad \forall t \in \mathbb{R}$,
- $U(t,r)U(r,s) = U(t,s) \quad \forall s \leq r \leq t$,
- $(t,s) \mapsto U(t,s)$ from $\{(t,s) \in \mathbb{R}^2 : t \geq s\}$ into $\mathcal{L}(X)$ is strongly measurable,
- there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)} \quad \forall t \geq s$.

The second property is also known as *Chapman-Kolmogorov equation*.

Given an exponentially bounded strongly measurable evolution family $(U(t,s))_{t\geq s}$ on X one can define on the space $\mathcal{X} = \mathrm{L}^p(\mathbb{R}, X),\ 1 \leq p < \infty$, a family of bounded linear operators as

$$(T(t)f)(s) := U(s, s-t)f(s-t) \quad \forall f \in \mathrm{L}^p(\mathbb{R}, X),\ s \in \mathbb{R},\ t \geq 0. \tag{5.2}$$

If the bounded evolution family is strongly continuous one can prove that the family $(T(t))_{t\geq 0}$ is a C_0-semigroup (see [**CL99**, Chapter 3.2] and [**Eva76**, Section 6]).

We thus make the following definition.

Definition 5.1.4. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X} = \mathrm{L}^p(\mathbb{R}, X)$ is called an *evolution semigroup* if there exists an exponentially bounded, strongly measurable evolution family $(U(t,s))_{t\geq s}$ on X such that (5.2) holds.

Remark 5.1.5. Denoting by $(S(t))_{t\geq 0}$ the right-shift semigroup on $\mathcal{X}$, one notices that for an evolution semigroup $(T(t))_{t\geq 0}$

$$T(t)S(-t)f(s) = U(s, s-t)f(s) \quad \forall f \in \mathcal{X}.$$

Thus for every $t \geq 0$ the operator $T(t)S(-t)$ is a multiplication operator on $\mathcal{X}$.

Using Theorem 5.1.2 and following the proof of [**RRS96**, Theorem 3.4] and [**How74**, Theorem 1] one can characterize evolution semigroup as follows.

Theorem 5.1.6. *Let X be a separable Banach space and $(T(t))_{t\geq 0}$ a C_0-semigroup on $\mathcal{X} = \mathrm{L}^p(\mathbb{R}, X)$ with generator $(G, D(G))$.*

Then the following are equivalent.

(1) $(T(t))_{t\geq 0}$ *is an evolution semigroup.*

(2) $T(t)(\phi f) = (S(t)\phi)T(t)f \quad \forall t \geq 0, f \in \mathcal{X}, \phi \in L^\infty(\mathbb{R})$.

(3) *For every $\phi \in C_c^1(\mathbb{R})$ and $f \in D(G)$, $\phi f \in D(G)$ and*

$$G(\phi f) = -\phi' f + \phi G f.$$

Proof. (1) ⇒ (3): Let $\phi \in C_c^1(\mathbb{R})$ and $f \in D(G)$, then

$$\frac{T(t)(\phi f) - \phi f}{t} = \frac{(S(t)\phi)(T(t)f) - \phi f}{t} \xrightarrow{t \searrow 0} -\phi' f + \phi G f.$$

(3) ⇒ (2): For $\phi \in C_c^1(\mathbb{R})$ and $f \in D(G)$ let us define

$$u(t) := (S(t)\phi)(T(t)f), \quad t \geq 0.$$

Then $u(t) \in D(G)$ for every $t \geq 0$ and

$$Gu(t) = -\phi' T(t)f + (S(t)\phi)GT(t)f.$$

Furthermore $u(\cdot)$ is continuously differentiable with

$$\dot{u}(t) = -\phi' T(t)f + (S(t)\phi)GT(t)f.$$

Thus $u(\cdot)$ is the solution of the wellposed Cauchy-Problem

$$\begin{cases} \dot{u}(t) = Gu(t), & t \geq 0, \\ u(0) = \phi f, \end{cases}$$

hence $(S(t)\phi)T(t)f = T(t)\phi f$ for every $\phi \in C_c^1(\mathbb{R})$ and $f \in D(G)$. By a density argument one obtains (2).

(2) ⇒ (1) : First introduce the space $\mathfrak{X} := \mathrm{L}^p(\mathbb{R}, \mathcal{X}) = \mathrm{L}^p(\mathbb{R}^2, X)$ where the second equality holds since we can identify every element $f(s,t) \in \mathrm{L}^p(\mathbb{R}^2, X)$ with $f(t) = f(\cdot, t) \in \mathrm{L}^p(\mathbb{R}, \mathcal{X})$ for a.e. $t \in \mathbb{R}$.

In this way we can consider

$$\mathcal{M}(t) := \begin{cases} T(t)S(-t) & \text{if } t \geq 0, \\ 0 & \text{else} \end{cases}$$

as an operator acting on $\mathrm{L}^p(\mathbb{R}^2, X)$. By hypothesis $\mathcal{M}$ commutes with multiplication by all bounded measurable function $t \mapsto \psi(t)$ and $s \mapsto \phi(s)$, hence, since $\mathrm{L}^\infty(\mathbb{R}) \otimes \mathrm{L}^\infty(\mathbb{R})$ is dense in $\mathrm{L}^\infty(\mathbb{R}^2)$ with respect to the weak-$*$ topology, $\mathcal{M}$ commutes with multiplication by any function $(t,s) \mapsto \psi(t,s)$ in $\mathrm{L}^\infty(\mathbb{R}^2)$. Thus we can apply Theorem 5.1.2 and conclude that there exists a strongly measurable function $M(\bullet,\bullet)$ on $\mathbb{R}^2$ such that

$$(\mathcal{M}f)(s,t) = M(s,t)f(s,t) \quad \text{for all } s,t \in \mathbb{R} \text{ and } f \in \mathrm{L}^p(\mathbb{R}^2, X).$$

Define

$$U(t,s) = M(t, t-s) \quad \text{for } t \geq s.$$

Since $(t,s) \mapsto (t, t-s)$ is a bijective, Borel-measure-preserving function on $\mathbb{R}^2$, $U(\bullet,\bullet)$ is strongly measurable.

It remains to show that $(U(t,s))_{t\geq s}$ satisfies the Chapman-Kolmogorov equation.

To this aim let $f \in \mathcal{X}$, then

$$\begin{aligned}
(T(t+r)f)(s) &= (\mathcal{M}(t+r)S(t+r)f)(s) \\
&= M(t+r,s)f(s-t-r) \\
&= U(s, s-t-r)f(s-t-r).
\end{aligned}$$

On the other hand

$$\begin{aligned}
(T(t)T(r)f)(s) &= (\mathcal{M}(t)S(t)\mathcal{M}(r)S(r)f)(s) \\
&= M(t,s)M(r,s-t)f(s-t-r) \\
&= U(s,s-t)U(s-t,s-t-r)f(s-t-r).
\end{aligned}$$

Therefore

$$U(t,r)U(r,s) = U(t,s) \quad \text{for all } t \geq r \geq s.$$

□

5.2. Non autonomous Weiss–Staffans perturbations

Let $(A, D(A))$ be the generator of an exponentially stable C_0-semigroup $(e^{tA})_{t\geq 0}$ on the Banach space X.

Further let U and Z be Banach spaces such that $X_1 \hookrightarrow Z \hookrightarrow X$.

Given operators

$$\begin{aligned}
\mathcal{C} &:= C(\cdot) \in L^\infty(\mathbb{R}, \mathcal{L}(Z,U)), \\
\mathcal{B} &:= B \in L^\infty(\mathbb{R}, \mathcal{L}(U, X_{-1})),
\end{aligned}$$

we define (perturbed) operators on X by

$$\begin{aligned} D(A_P(t)) &:= \{x \in Z : A_{-1}x + BC(t)x \in X\}, \\ A_P(t) &:= A_{-1} + BC(t) \quad \text{for a.e. } t \in \mathbb{R}. \end{aligned}$$

We are interested in conditions on the operators $\mathcal{C}$ and $\mathcal{B}$ such that the operator family $(A_P(t))_{t\in\mathbb{R}}$ generates a time evolution. In order to do this, we perturb the generator G of the "evolution semigroup" $(T(t))_{t\geq 0}$ associated to A on the space $\mathcal{X} = \mathrm{L}^p(\mathbb{R}, X)$, $1 \leq p < \infty$, and apply Theorem 1.3.3 to this situation.

Let D be the generator of the right-shift semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$ and $\mathcal{A}$ the multiplication operator $(\mathcal{A}f)(s) := Af(s)$ for every $s \in \mathbb{R}$ defined on $\mathcal{X}_1^A := \mathrm{L}^p(\mathbb{R}, X_1)$.

Then $\mathcal{A}$ generates a semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{X}$ given by

$$(e^{t\mathcal{A}}f)(s) = e^{tA}f(s) \quad \forall s \in \mathbb{R}.$$

Since the two semigroups $(S(t))_{t\geq 0}$ and $(e^{t\mathcal{A}})_{t\geq 0}$ commute, their product

$$T(t) := e^{t\mathcal{A}}S(t) = S(t)e^{t\mathcal{A}}, \quad t \geq 0,$$

defines a C_0-semigroup on $\mathcal{X}$ (see [**EN00**, I.5.15]) and is called the *evolution semigroup* associated to A. The idea behind this definition is to add the time variable to the original problem.

The generator G of this semigroup is formally given by the sum $\mathcal{A} + D$ (for more details see [**Nag95**, Theorem 4.3]).

It is clear that, starting from $\mathcal{X}$, we can define at least three Sobolev towers, each one corresponding to one of the generators introduced above. This is visualized the picture below.

$$\begin{tikzcd}
\mathcal{X}_{-1}^{\mathcal{A}} = \mathrm{L}^p(\mathbb{R}, X_{-1}) & \mathcal{X}_{-1}^G & \mathcal{X}_{-1}^D \\
& \mathcal{X} = \mathrm{L}^p(\mathbb{R}, X) \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] & \\
\mathcal{X}_1^D \arrow[ur, hook] & X_1^G \arrow[u, hook] & \mathcal{X}_1^{\mathcal{A}} = \mathrm{L}^p(\mathbb{R}, X_1) \arrow[ul, hook]
\end{tikzcd}$$

FIGURE 1. Sobolev Towers.

In our case it will be useful to concentrate on the one corresponding to $\mathcal{A}$, i.e., on the black "skew" tower.

On the extrapolated space with respect to $\mathcal{A}$, given by $\mathcal{X}_{-1}^{A} := \mathrm{L}^p(\mathbb{R}, X_{-1})$, we can again define a right-shift semigroup $(\tilde{S}(t))_{t\geq 0}$ and then the product semigroup

$$\tilde{T}(t) := e^{tA_{-1}}\tilde{S}(t) = \tilde{S}(t)e^{tA_{-1}}, \quad t \geq 0,$$

with generator $\tilde{G}$, again formally given by $\tilde{G} = \mathcal{A}_{-1} + D$.

If we define the spaces

$$\begin{aligned} \mathcal{Z} &:= \mathrm{L}^p(\mathbb{R}, Z) \\ \mathcal{U} &:= \mathrm{L}^p(\mathbb{R}, U), \end{aligned}$$

the operators $\mathcal{C}$ and $\mathcal{B}$ from above satisfy $\mathcal{C} \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ and $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}^{A})$. See the diagram below.

$$\begin{tikzcd}
\mathcal{X}_{-1}^{\mathcal{A}} = \mathrm{L}^p(\mathbb{R}, X_{-1}) & & & & \\
& \mathcal{X} = \mathrm{L}^p(\mathbb{R}, X) \arrow[ul, hook] & & & \mathcal{U} = \mathrm{L}^p(\mathbb{R}, U) \arrow[ullll, "\mathcal{B}"'] \\
& & \mathcal{Z} = \mathrm{L}^p(\mathbb{R}, Z) \arrow[ul, hook] \arrow[urr, "\mathcal{C}"'] & & \\
& \mathcal{X}_1^{G} \arrow[uu, hook] & & \mathcal{X}_1^{\mathcal{A}} = \mathrm{L}^p(\mathbb{R}, X_1) \arrow[ul, hook] & \\
& & \mathcal{D} = D(D) \cap D(\mathcal{A}) \arrow[ul, phantom, "\supset"] \arrow[ur, phantom, "\subset"] & &
\end{tikzcd}$$

FIGURE 2. The setting.

Our aim is now to define the sum

$$G_P := \tilde{G} + \mathcal{B}\mathcal{C}$$

on a suitable domain and give conditions on the operators $\mathcal{B}$ and $\mathcal{C}$ such that this sum becomes the generator of a C_0-semigroup.

In order to do this we first need the following lemma.

Lemma 5.2.1. *If there exists $t > 0$ and $M \geq 0$ such that*

$$\left\| \int_0^t \tilde{T}(t-r)\mathcal{B}u(r)dr \right\|_{\mathcal{X}}^p ds \leq M \|u\|_{\mathrm{L}^p([0,t],\mathcal{U})}^p \quad \forall u \in \mathrm{L}^p([0,t],\mathcal{U}), \tag{5.3}$$

then

$$R(\lambda,\tilde{G})\mathcal{B}\in\mathcal{L}(\mathcal{U},\mathcal{X}) \tag{5.4}$$

for every $\lambda>0$.

Proof. Remark 1.3.6 allows us to compute the Laplace-transform of $t\mapsto\tilde{\mathcal{B}}_t u$ where $u\in \mathrm{L}^p\big([0,\infty),\mathcal{U}\big)$ as

$$\mathcal{L}(\tilde{\mathcal{B}}_\bullet u)(\lambda)=R(\lambda,\tilde{G})\mathcal{B}\mathcal{L}(u)(\lambda),\quad \text{for } \lambda>0$$

satisfying

$$\|R(\lambda,\tilde{G})\mathcal{B}\mathcal{L}(u)(\lambda)\|_{\mathcal{X}}\le\frac{M_B}{\lambda}\|u\|_{\mathrm{L}^p\big([0,\infty),\mathcal{U}\big)}. \tag{5.5}$$

For $u_0\in\mathcal{U}$ and $\mu>\lambda$ apply (5.5) to $u(\bullet)=\mu e^{(\lambda-\mu)\bullet}u_0$ and obtain

$$\|R(\lambda,\tilde{G})\mathcal{B}u_0\|_{\mathcal{X}}\le\frac{M_B\mu}{\lambda((\mu-\lambda)^p)^{1/p}}\|u_0\|_{\mathcal{U}}.$$

Thus for every $\lambda>0$

$$R(\lambda,\tilde{G})\mathcal{B}\in\mathcal{L}(\mathcal{U},\mathcal{X}).$$

□

This result can be used to prove the following fact.

Corollary 5.2.2. *If in addition to condition* (5.3) *the operator* B *be is injective and such that* $\mathrm{rg}(B)\cap X=\{0\}$, *then one can define a norm on* $\mathrm{rg}\big(R(\lambda,\tilde{G})\mathcal{B}\big)$ *making it into a Banach space. Furthermore one can define the direct sum*

$$\tilde{Z}:=\mathrm{rg}\big(R(\lambda,\tilde{G})\mathcal{B}\big)\oplus D(G)$$

and therefore $D(G)\hookrightarrow\tilde{Z}\hookrightarrow\mathcal{X}$.

Proof. Since $\mathcal{B}$ is injective, the operator $\Psi(\lambda):=R(\lambda,\tilde{G})\mathcal{B}$ is continuous, injective and surjective on its image, thus it is invertible and its inverse

$$\Psi(\lambda)^{-1} \text{ defined on } \mathrm{rg}(R(\lambda,\tilde{G})\mathcal{B})$$

is a closed operator. Thus $\mathrm{rg}(R(\lambda,\tilde{G})\mathcal{B})$ endowed with the graph norm of $\Psi(\lambda)^{-1}$ is a Banach space.

Using the fact that $\mathrm{rg}(B)\cap X=\{0\}$ we can define

$$\tilde{Z}:=\mathrm{rg}(R(\lambda,\tilde{G})\mathcal{B})\oplus D(G)$$

for $\lambda>0$.

Using the resolvent equality, it is clear that this definition does not depend on the choice of λ and thanks to Lemma 5.2.1 we have $\tilde{Z} \hookrightarrow \mathcal{X}$. $\square$

Analogously to Lemma 5.2.1 one can prove the following.

Lemma 5.2.3. *If there exists $t > 0$ and $M \geq 0$ such that*

$$\int_0^t \left\| \mathcal{C} \int_0^r \tilde{T}(r-s)\mathcal{B}u(s)ds \right\|_{\mathcal{U}}^p dr \leq M \|u\|_{\mathrm{L}^p([0,t],\mathcal{U})}^p \qquad \forall u \in \mathrm{W}_0^{1,p}\big([0,t], W^{1,p}(\mathbb{R}, U)\big)$$

then $\mathcal{C}R(\lambda, \tilde{G})\mathcal{B}$ is well defined on $W^{1,p}(\mathbb{R}, U)$ and admits a bounded extension

$$\Gamma(\lambda) \in \mathcal{L}(\mathcal{U}) \tag{5.6}$$

for every $\lambda > 0$.

Proof. Analogously to Theorem 1.3.3 for $\lambda > 0$ one can compute the Laplace transform of $(\tilde{\mathcal{F}}_\infty u)(\bullet) = \mathcal{C}\int_0^\bullet \tilde{T}(\bullet - r)\mathcal{B}u(r)dr$ for every $u \in W_0^{1,p}(\mathbb{R}_+, W^{1,p}(\mathbb{R}, U))$ obtaining

$$\mathcal{L}(\tilde{\mathcal{F}}_\infty u)(\lambda) = \mathcal{C}R(\lambda, \tilde{G})\mathcal{B}\mathcal{L}(u)(\lambda) \quad \text{ for } \lambda > 0.$$

Applying Hölder's inequality, we obtain that there exists a constant M_λ depending on $\lambda > 0$ such that

$$\|\mathcal{C}R(\lambda, \tilde{G})\mathcal{B}\mathcal{L}(u)(\lambda)\|_U \leq M_\lambda \|u\|_{\mathrm{L}^p\left([0,\infty),\mathcal{U}\right)}$$

for every $u \in W_0^{1,p}(\mathbb{R}_+, W^{1,p}(\mathbb{R}, U))$.

Now let $u_0 \in W^{1,p}(\mathbb{R}, U)$ and define $t \mapsto \bar{u}(t) = (\lambda+1)^2 t e^{-t} u_0 \in W_0^{1,p}(\mathbb{R}_+, W^{1,p}(\mathbb{R}, U))$. Then

$$\begin{aligned} \|\mathcal{C}R(\lambda, \tilde{G})\mathcal{B}\mathcal{L}(\bar{u})(\lambda)\|_U &= \|\mathcal{C}R(\lambda, \tilde{G})\mathcal{B}u_0\|_U \\ &\leq M_\lambda \|\bar{u}\|_{\mathrm{L}^p\left([0,\infty),\mathcal{U}\right)} \\ &= \frac{M_\lambda(\lambda+1)(p!)^{1/p}}{p} \|u_0\|_{\mathcal{U}}. \end{aligned}$$

Thus $\mathcal{C}R(\lambda, \tilde{G})\mathcal{B} : W^{1,p}(\mathbb{R}, U) \to \mathcal{U}$ can be extended to a bounded operator $\Gamma(\lambda)$ on $\mathcal{U}$. $\square$

Using Theorem 1.3.3 we prove the main result of this section.

Theorem 5.2.4. *Let the operator $B \in \mathcal{L}(U, X)$ be injective and such that* $\mathrm{rg}(B) \cap X = \{0\}$. *Let the following conditions be satisfied.*

a': *There exists $\lambda \in \rho(A)$ such that $R(\lambda, A_{-1})B \subset Z$.*

b': *There exists $t > 0$ and $M \geq 0$ such that*

$$\int_0^t \|\mathcal{C}T(s)f\|_{\mathcal{U}}^p \, ds \leq M \, \|f\|_{\mathcal{X}}^p, \quad f \in \mathcal{X}_1^A. \tag{5.7}$$

c': *There exists $t > 0$ and $M \geq 0$ such that*

$$\left\| \int_0^t \tilde{T}(t-r)\mathcal{B}u(r)dr \right\|_{\mathcal{X}}^p \leq M \, \|u\|_{\mathrm{L}^p([0,t],\mathcal{U})}^p, \quad u \in \mathrm{L}^p\big([0,t],\mathcal{U}\big). \tag{5.8}$$

d': *There exists $t > 0$ and $M \geq 0$ such that*

$$\int_0^t \left\| \mathcal{C} \int_0^r \tilde{T}(r-s)\mathcal{B}u(s)ds \right\|_{\mathcal{U}}^p dr \leq M \, \|u\|_{\mathrm{L}^p([0,t],\mathcal{U})}^p, \quad u \in \mathrm{W}_0^{1,p}\big([0,t], W^{1,p}(\mathbb{R},U)\big), \tag{5.9}$$

where $\mathrm{W}_0^{1,p}\big([0,t], W^{1,p}(\mathbb{R},U)\big) := \big\{u \in W^{1,p}\big([0,t], W^{1,p}(\mathbb{R},U)\big) : u(0) = 0\big\}$.

e': $1 \in \rho(\tilde{\mathcal{F}}_t)$ *for one $t > 0$. Here $\tilde{\mathcal{F}}_t \in \mathcal{L}\left(\mathrm{L}^p\big([0,t],\mathcal{U}\big)\right)$ is the continuous extension of*

$$\mathrm{W}_0^{1,p}\big([0,t], W^{1,p}(\mathbb{R},U)\big) \ni u \mapsto \mathcal{C}\int_0^{\bullet} \tilde{T}(\bullet - r)\mathcal{B}u(r)dr.$$

Then $\mathcal{C}$ admits a bounded extension $\tilde{\mathcal{C}}$ onto the Banach space $\tilde{Z} = D(G) \oplus \mathrm{rg}(R(\lambda,\tilde{G})B)$ and the operator

$$D(G_P) := \left\{ f \in \tilde{Z} : \tilde{G}f + \mathcal{B}\tilde{\mathcal{C}}f \in \mathcal{X} \right\},$$
$$G_P := \tilde{G} + \mathcal{B}\tilde{\mathcal{C}}$$

generates an evolution semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$.

We describe the situation of Theorem 5.2.4 by inserting $\tilde{\mathcal{Z}}$ and $\tilde{C}$ in diagram 2.

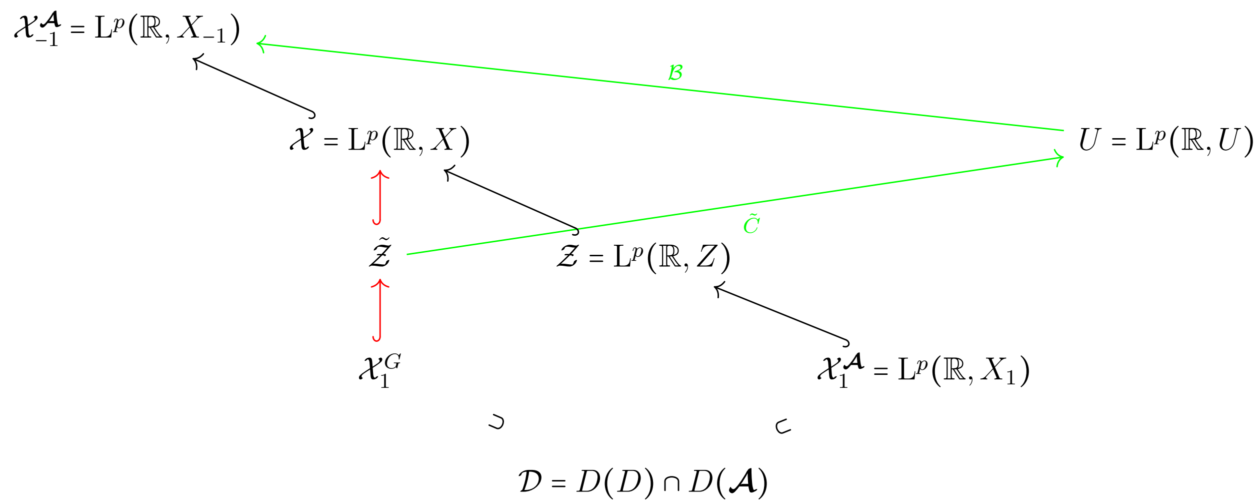

FIGURE 3. The new setting.

Remarks.

- The left part of (5.7) is well-defined since $\mathcal{X}_1^A$ is $T(t)$ invariant, and $\mathcal{X}_1^A \hookrightarrow \mathcal{Z}$.
- The left part of (5.9) is well-defined since, letting $(S_U(t))_{t\geq 0}$ be the right-shift on $\mathcal{U}$ with generator D_U, then for $u \in \mathrm{W}_0^{1,p}\big([0,t], W^{1,p}(\mathbb{R}, U)\big)$

$$\left(\int_0^r \tilde{T}(r-s)\mathcal{B}u(s)ds\right)(\cdot) = \int_0^r e^{(r-s)A_{-1}}\mathcal{B}S_U(r-s)u(s)(\cdot)ds$$
$$= -A_{-1}^{-1}Bu(r)(\cdot) + \mathcal{A}_{-1}^{-1}\int_0^r e^{(r-s)A_{-1}}\mathcal{B}\frac{d}{ds}(S_U(r-s)u(s)(\cdot))ds$$
$$= -A_{-1}^{-1}Bu(r)(\cdot) + \mathcal{A}^{-1}\int_0^r \tilde{T}(r-s)\mathcal{B}[u'(s)(\cdot) - D_U u(s)(\cdot)]ds.$$

Using conditions **a'** and **d'**, one concludes that $\left(\int_0^r \tilde{T}(r-s)\mathcal{B}u(s)ds\right)(\cdot) \in \mathcal{Z}$.

Thus one can define the operator $\tilde{\mathcal{F}}_\infty \in \mathcal{L}(\mathrm{L}^p\big([0,\infty),\mathcal{U}\big))$ as the continuous extension of the operator

$$W_0^{1,p}(\mathbb{R}_+, W^{1,p}(\mathbb{R},U)) \ni u \mapsto \mathcal{C}\int_0^\bullet \tilde{T}(\bullet - r)\mathcal{B}u(r)dr.$$

We can now start with the proof of Theorem 5.2.4.

Proof. We first show that all the conditions in Theorem 1.3.3 are satisfied.

As the core needed in Theorem 1.3.3 we take the space $\mathcal{D} := D(\mathcal{A}) \cap D(D)$.

We show that $\mathcal{C}$ admits a continuous extension $\tilde{\mathcal{C}}$ on $\tilde{\mathcal{Z}}$. Then each condition of Theorem 5.2.4 implies the corresponding condition of Theorem 1.3.3 and the conclusions follows.

To do so we notice that

$$\mathcal{C}T(\bullet)f = \mathcal{C}\mathcal{A}^{-1}T(\bullet)\mathcal{A}f$$

is a continuous function for $f \in \mathcal{X}_1^A \supset \mathcal{D}$. This together with assumption **b'** permits us to apply [**TV09**, Cor. 1.6], which, with an argument based on the Laplace transform, imply that $\mathcal{C}$ admits a G-bounded extension $\mathcal{C}'$ on $D(G)$.

We now show that $\mathcal{C}$ can be extended to a continuous operator $\mathcal{C}''$ on $\mathrm{rg}(R(\lambda,\tilde{G})B)$ for a $\lambda > 0$.

Since

$$\Psi(\lambda)^{-1} \in \mathcal{L}(\mathrm{rg}(\Psi(\lambda)), U)$$

it follows that

$$\Gamma(\lambda)\Psi(\lambda)^{-1} \in \mathcal{L}(\mathrm{rg}(\Psi(\lambda)), U).$$

For $x = \Psi(\lambda)u \in \Psi(\lambda)\left(W^{1,p}(\mathbb{R}, U)\right)$ one has that

$$\Gamma(\lambda)\Psi(\lambda)^{-1}x = \mathcal{C}R(\lambda, \tilde{G})\mathcal{B}u = \mathcal{C}x.$$

Using that $\Psi(\lambda)\left(W^{1,p}(\mathbb{R}, U)\right)$ is dense in $\mathrm{rg}(\Psi(\lambda))$ it follows that $\mathcal{C}$ has a bounded extension

$$\mathcal{C}'' \in \mathcal{L}(\mathrm{rg}(R(\lambda, \tilde{G})\mathcal{B}), U).$$

On the space $\tilde{Z} = D(\tilde{G}) \oplus \mathrm{rg}(R(\lambda, \tilde{G})\mathcal{B})$ define the operator $\tilde{\mathcal{C}}$ as

$$\begin{aligned} &\tilde{\mathcal{C}}x := C'x_1 + C''x_2 \\ &\text{for } x = x_1 + x_2 \quad \text{with } x_1 \in D(\tilde{G}) \text{ and } x_2 \in \mathrm{rg}(R(\lambda, \tilde{G})\mathcal{B}) \end{aligned}$$

Clearly $\tilde{\mathcal{C}} \in \mathcal{L}(\tilde{Z}, U)$ and, by the resolvent equation, it does not depend on the choice of $\lambda > 0$.

It remains to prove that the semigroup $(S(t))_{t\geq 0}$ generated by G_P is an evolution semigroup. We do this using Theorem 5.1.6

To this aim we first show for every $f \in \tilde{Z}$ and $\phi \in C_c^1(\mathbb{R})$ that

$$\begin{aligned} \phi f \in \tilde{Z} \quad &\text{and} \\ (\tilde{G} + \mathcal{B}\tilde{\mathcal{C}})\phi f &= -\phi' f + \phi(\tilde{G} + \mathcal{B}\tilde{\mathcal{C}})f. \end{aligned}$$

Let $f \in \mathcal{D}$ and $\phi \in C_c^1(\mathbb{R})$, then clearly $\phi f \in \mathcal{D}$ and

$$\begin{aligned} (\tilde{G} + \mathcal{B}\tilde{\mathcal{C}})\phi f &= G\phi f + \mathcal{B}\mathcal{C}'\phi f && (5.10) \\ &= \phi' f + \phi G f + \mathcal{B}\mathcal{C}\phi f && (5.11) \\ &= \phi' f + \phi G f + \phi\mathcal{B}\mathcal{C} f && (5.12) \\ &= \phi' f + \phi(G + \mathcal{B}\mathcal{C}')f. \end{aligned}$$

In (5.11) we used that G is the generator of an evolution semigroup and in (5.12) that $\mathcal{C}$ and $\mathcal{B}$ are both multiplication operators. Since $\mathcal{D}$ is dense in $D(G)$, the same formula holds for every $f \in D(G)$ and $\phi \in C_c^1(\mathbb{R})$.

With similar arguments we obtain for $f \in \Psi(\lambda)(W^{1,p}(\mathbb{R}, U))$ and $\phi \in C_c^1(\mathbb{R})$ that $\phi f \in \mathcal{Z}$ and

$$(\tilde{G} + \mathcal{B}\tilde{\mathcal{C}})\phi f = \phi' f + \phi(G + \mathcal{B}\mathcal{C}'')f. \tag{5.13}$$

As before one just uses a density argument and obtains that (5.13) holds for every $f \in \mathrm{rg}(R(\lambda, \tilde{G})\mathcal{B})$ and $\phi \in C_c^1(\mathbb{R})$.

Summing up, we conclude that for every $f \in D(G_P)$ and $\phi \in C_c^1(\mathbb{R})$, $\phi f \in D(G_P)$ and

$$G_P \phi f = -\phi' f + \phi G_P f,$$

therefore $(S(t))_{t \geq 0}$ is an evolution semigroup. □

We conclude this section by proving a lemma giving a "pointwise" condition implying **b'**. The proof follows the one in [**RRS96**, Theorem 4.2].

Lemma 5.2.5. *If there exists $t > 0$ and $M \geq 0$ such that*

$$\int_0^t \|C(s+r)e^{rA}x\|_U^p \mathrm{d}r \leq M\|x\|_X^p \tag{5.14}$$

for a.e. $s \in \mathbb{R}$ and every $x \in D(A)$, then

$$\int_0^t \|\mathcal{C}T(r)f\|_{\mathcal{U}}^p \mathrm{d}r \leq M\|f\|_{\mathcal{X}} \tag{5.15}$$

for every $f \in D(\mathcal{A})$.

Proof. For $f \in D(\mathcal{A})$ we obtain

$$\begin{aligned}
\int_0^t \|\mathcal{C}T(r)f\|_{\mathcal{U}}^p \mathrm{d}r &= \int_0^t \int_{-\infty}^{+\infty} \|C(s)\,(T(r)f)\,(s)\|_U^p \mathrm{d}s\mathrm{d}r \\
&= \int_0^t \int_{-\infty}^{+\infty} \|C(s)e^{rA}f(s-r)\|_U^p \mathrm{d}s\mathrm{d}r \\
&= \int_0^t \int_{-\infty}^{+\infty} \|C(s+r)e^{rA}f(s)\|_U^p \mathrm{d}s\mathrm{d}r \\
&= \int_{-\infty}^{+\infty} \int_0^t \|C(s+r)e^{rA}f(s)\|_U^p \mathrm{d}r\mathrm{d}s \qquad (5.16)\\
&\leq \int_{-\infty}^{+\infty} M\|f(s)\|_X^p \mathrm{d}s \qquad (5.17)\\
&= M\|f\|_{\mathcal{X}}^p,
\end{aligned}$$

hence (5.15) holds. In (5.16) we used the Fubini-Tonelli Theorem and in (5.17) the assumption (5.14). □

We use this lemma in Example 6.2, where it will simplify some computations.

Instead of using this semigroup approach, one could choose a direct approach as Schnaubelt did in [**Sch02**]. There, under "strong" hypotheses (see [**Sch02**, Def. 3.8]), he obtains a strongly continuous evolution family [**Sch02**, Thm .4.4]. We decided to avoid this procedure since the conditions appearing in [**Sch02**, Def. 3.8] are not easy to verify, at least not in the cases that we will treat in Chapter 6.

CHAPTER 6

Non autonomous boundary perturbations

In this chapter we apply the results of Chapter 5 to time-dependent boundary perturbations generalizing the problem considered in Chapter 3 to nonautonomous perturbations.

In order to proceed we first recall (see also Chapter 3) Greiner's approach [**Gre87**] transforming a perturbation of the domain of a generator into a multiplicative and so by [**EN00**, Sect. III.3.d] into an additive perturbation.

On two Banach spaces X and ∂X we consider linear operators

$$\begin{aligned} A_m &: D(A_m) \subset X \to X, \\ L &: D(A_m) \to \partial X. \end{aligned}$$

Assume that $Z = (D(A_m), \|\cdot\|)$ is a third Banach space continuously embedded in X and

$$\begin{aligned} A_m &\in \mathcal{L}(Z, X), \\ L &\in \mathcal{L}(Z, \partial X). \end{aligned}$$

On these operators we make the following assumptions throughout this section.

- $A := A_{m|\ker(L)}$ generates a C_0-semigroup $(e^{tA})_{t\geq 0}$ on X.
- L is surjective.

For an operator $\Phi \in \mathcal{L}(Z, \partial X)$ we consider the perturbed operator

$$\begin{aligned} D(A_\Phi) &:= \{x \in Z : Lx = \Phi x\}, \\ A_\Phi x &:= A_m x, \quad \text{for every } x \in D(A_\Phi). \end{aligned}$$

The question under which conditions A_Φ generates a C_0-semigroup on X has been treated in Chapter 3.

In this chapter we consider the case where Φ is also time-dependent, i.e.,

$$t \longmapsto \Phi(t) \in \mathcal{L}(Z, \partial X), \quad t \in \mathbb{R}.$$

We then obtain a family of perturbed operators

$$D(A_\Phi(t)) := \{x \in Z : Lx = \Phi(t)x\}\,, \quad t \in \mathbb{R},$$
$$A_\Phi(t)x := A_m x, \quad t \in \mathbb{R},$$

and look for conditions such that these operators generate a time evolution.

6.1. The setting

To treat this problem we follow the approach of Greiner, see Lemma 3.1.2, and write each $A_\Phi(t)$ as an additive perturbation of the generator A.

For simplicity we assume the semigroup $(e^{tA})_{t\geq 0}$ to be exponentially stable, i.e., $\omega(A) < 0$. As in Lemma 3.1.1 we note by D_0 the Dirichlet operator

$$D_0 := \left(L|_{\ker A_m}\right)^{-1} : \partial X \to X.$$

Then, as in Lemma 3.1.2, $A_\Phi(t)$ can be written as an additive perturbation.

Lemma 6.1.1. *Let $x \in X$ and $t \in \mathbb{R}$. Then $x \in D(A_\Phi(t))$ if and only if $(I - D_0\Phi(t))x \in D(A)$. Furthermore*

$$A_\Phi(t) = (A_{-1} - A_{-1}D_0\Phi(t))_{|X}, \quad t \in \mathbb{R}.$$

We now proceed by assuming

$$\Phi(\cdot) \in \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(Z, \partial X))$$

and setting $U := \partial X$. Then we are exactly in the setting of Section 5.2, with

$$\mathcal{C} = C(\cdot) = \Phi(\cdot) \in \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(Z, \partial X)),$$
$$\mathcal{B} = B = -A_{-1}D_0 \in \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(\partial X, X_{-1})),$$

and B is injective such that $\mathrm{rg}(B) \cap X = \{0\}$.

Thus, applying Theorem 5.2.4, we can associate a time evolution to $(A_\Phi(t))_{t\in\mathbb{R}}$. The following result shows that in this case the “compatibility” condition **a’** of Theorem 5.2.4 is automatically fulfilled.

Corollary 6.1.2. *Assume that there exists $1 \leq p < \infty$ such that conditions **b’**–**e’** of Theorem 5.2.4 are satisfied. Then there exists an evolution family $(U(t,s))_{t\geq s}$ associated to $(A_\Phi(t))_{t\in\mathbb{R}}$.*

PROOF. The only condition to show is **a'**. Indeed, for $\lambda \in \rho(A)$ we obtain

$$\begin{aligned}\operatorname{rg}(R(\lambda, A_{-1})B) &= \operatorname{rg}(-R(\lambda, A_{-1})A_{-1}D_0)\\ &= \operatorname{rg}(D_0 - \lambda R(\lambda, A_{-1})D_0)\\ &\subset \ker(A_m) + D(A)\\ &\subset D(A_m) = Z.\end{aligned}$$

□

6.2. Nonautonomous flows in networks

A concrete example for the abstract setting above is given by flows in time dependent networks.

We describe the situation.

Definition 6.2.1. A directed graph G is a triple $G = (V, E, \varphi)$ where for some $n, m \in \mathbb{N}$

- $V := \{v_1, \ldots, v_n\}$ is the set of the vertices,
- $E := \{e_1, \ldots, e_m\}$ is the set of the edges,
- $\varphi : E \longrightarrow V \times V$ is the incidence function.

For an edge $e \in E$, $\varphi(e) = (v_i, v_j)$ means that the edge e connects the vertex v_i to v_j.

The following matrices describe the graph completely (see also [**KS05**, Section 1 and 2], [**Bay12**, Section 6]).

(a) The outgoing incidence matrix $\mathbf{\Phi}^- = (\phi^-_{ij})_{n\times m}$, where

$$\phi^-_{ij} = \begin{cases} 1, & \text{if } \exists v \in V \text{ such that } \varphi(e_j) = (v_i, v),\\ 0, & \text{else.}\end{cases}$$

(b) The incoming incidence matrix $\mathbf{\Phi}^+ = (\phi^+_{ij})_{n\times m}$, where

$$\phi^+_{ij} = \begin{cases} 1, & \text{if } \exists v \in V \text{ such that } \varphi(e_j) = (v, v_i),\\ 0, & \text{else.}\end{cases}$$

(c) The adjacency matrix $\mathbb{A} = (a_{ik})_{n\times n} := \mathbf{\Phi}^+(\mathbf{\Phi}^-)^T$.

(d) The adjacency matrix of the line graph $\mathbb{B} = (b_{ik})_{n\times n} := (\mathbf{\Phi}^-)^T\mathbf{\Phi}^+$.

We parametrize every edge by the interval $[0,1]$. We denote by t and $\tau \in \mathbb{R}$ the time variables, and by $s \in [0,1]$ the space variable.

- On every edge we consider the following transport equation
$$\frac{\mathrm{d}}{\mathrm{d}t} x_j(t,s) = c_j \frac{\mathrm{d}}{\mathrm{d}s} x_j(t,s), \quad t \geq \tau,$$
where $c_j > 0$ is the (constant) velocity of the material along the edge e_j.

- We denote by $0 \leq \omega_{ij}(t) \leq 1$ the proportion of material going from vertex v_i into edge e_j at time $t \in \mathbb{R}$, and we assume that $\sum_{j=1}^{m} \omega_{ij}(t) = 1$ for every $t \in \mathbb{R}$, i.e., there is no loss of material in the vertices.

- In every vertex the following Kirchhoff law holds
$$\phi_{ij}^{-} x_j(t,1) = \omega_{ij}(t) \sum_{k=1}^{m} \phi_{ik}^{+} x_k(t,0) \quad \forall t \geq \tau.$$

- The initial condition is given by
$$x_j(\tau,s) = f_j(s), \quad s \in [0,1].$$

In order to reformulate the problem on the space $X =: \mathrm{L}^1([0,1],\mathbb{C}^m)$ endowed with the norm $\|f\|_1 = \sum_{i=1}^{m} \int_0^1 |f_i(s)|\, \mathrm{d}s$ we consider the operators
$$A(t) := \operatorname{diag}\left(c_j \frac{\mathrm{d}}{\mathrm{d}s}\right)_{j=1,\dots,m}, \quad t \in \mathbb{R},$$
with domain
$$\begin{aligned} D(A(t)) &= \left\{ g \in W^{1,1}([0,1],\mathbb{C}^m) \middle| \ g(1) \in \operatorname{rg}(\mathbf{\Phi}_{\omega}^{-}(t))^T \text{ and } \mathbf{\Phi}^{-} g(1) = \mathbf{\Phi}^{+} g(0) \right\} \\ &= \left\{ g \in W^{1,1}([0,1],\mathbb{C}^m) \middle| \ g(1) = \mathbb{B}_{\omega}(t) g(0) \right\}. \end{aligned}$$

Here $\mathbf{\Phi}_{\omega}^{-}(t) = (\omega_{ij}(t))_{n \times m}$ is the weighted outgoing incidence matrix, and $\mathbb{B}_{\omega}(t) := (\mathbf{\Phi}_{\omega}^{-}(t))^T \mathbf{\Phi}^{+}$ the weighted adjacency matrix of the line graph at time $t \in \mathbb{R}$. This allows us to reformulate our problem as a nonautonomous abstract Cauchy problem as follows
$$(nACP) \begin{cases} \dot{x}(t) = A(t)x(t), & t \geq s_0, \\ x(\tau) = f_\tau. \end{cases}$$

In order to use the result of Section 6.1 we take

- $Z := W^{1,1}([0,1],\mathbb{C}^m)$ and $\partial X = \mathbb{C}^m$,

- $L := \delta_1 \in \mathcal{L}(Z, \partial X)$ the point evaluation at 1,

- $\Phi(t) := \mathbb{B}_\omega(t)\delta_0 \in \mathcal{L}(Z, \partial X), \quad t \in \mathbb{R}$,

- $A_m := \operatorname{diag}(c_j \frac{\mathrm{d}}{\mathrm{d}s})_{j=1,\dots,m}$ with domain $D(A_m) := Z$.

Then

- L is surjective,

- $(A_m, \ker L) = (A, D(A))$ is the generator of the nilpotent semigroup $(e^{tA})_{t\geq 0}$ given by

$$(e^{tA}f)_j(s) := \begin{cases} f_j(s + c_j t) & \text{if } 0 \leq s + c_j t \leq 1 \\ 0 \quad \text{else} \end{cases}$$

  for $f \in X$, $s \in [0,1]$, $j \in \{1, \dots, m\}$,

- $A(t) = A_\Phi(t), \quad t \in \mathbb{R}$.

By Corollary 6.1.2 $(A(t))_{t\in\mathbb{R}}$ yields an associated measurable evolution family if conditions **b'–e'** of Theorem 5.2.4 are satisfied. In our case we choose $p = 1$.

In order to prove **b'** we show that condition (5.14) of Lemma 5.2.5 holds for $t = 1$. For $x = (x_1(\cdot), \dots, x_m(\cdot))^T \in D(A)$, it follows that

$$\begin{aligned} \int_0^1 \|C(s+r)e^{rA}x\|_{\mathbb{C}^m}\mathrm{d}r &= \int_0^1 \left\| \mathbb{B}_\omega(s+r)\delta_0 \begin{pmatrix} x_1(\cdot + c_1 r) \\ \vdots \\ x_m(\cdot + c_m r) \end{pmatrix} \right\|_{\mathbb{C}^m} \mathrm{d}r \\ &= \int_0^1 \left\| \mathbb{B}_\omega(s+r) \begin{pmatrix} x_1(c_1 r) \\ \vdots \\ x_m(c_m r) \end{pmatrix} \right\|_{\mathbb{C}^m} \mathrm{d}r. \end{aligned}$$

Since $\mathbb{B}_\omega(t)$ is column stochastic for every $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
\int_0^1 \left\| \mathbb{B}_\omega(s+r) \begin{pmatrix} x_1(c_1 r) \\ \vdots \\ x_m(c_m r) \end{pmatrix} \right\|_{\mathbb{C}^m} \mathrm{d}r &\leq \int_0^1 \sum_{j=1}^m |x_j(c_j r)|\,\mathrm{d}r \\
&= \sum_{j=1}^m \int_0^{\frac{1}{c_j}} \frac{1}{c_j} |x_j(r)|\,\mathrm{d}r \\
&\leq \frac{1}{\min\{c_j : j = 1,\dots,m\}} \|x\|_X .
\end{aligned}$$

In order to check the other conditions we first need to compute the following expression.

Let $t = \frac{1}{\min\{c_j : j=1,\dots,m\}}$ and $u \in W_0^{1,1}([0,t], W^{1,1}(\mathbb{R},\mathbb{C}^m))$, then denoting by $\cdot$ the variable in $\mathbb{R}$ and by $\star$ the variable in $[0,t]$

$$\begin{aligned}
\int_0^t \tilde{T}(t-r)Bu(r)(\cdot)\mathrm{d}r &= \int_0^t e^{(t-r)A_{-1}} Bu(r)(\cdot - t + r)\mathrm{d}r \\
&= -A_{-1}^{-1}Bu(t)(\cdot) + A_{-1}^{-1}\int_0^t e^{(t-r)A_{-1}} B \frac{d}{dr}(u(r)(\cdot - t + r))\mathrm{d}r \\
&= D_0 u(t)(\cdot) - \int_0^t e^{(t-r)A} D_0 \frac{d}{dr}(u(r)(\cdot - t + r))\mathrm{d}r \\
&= \mathbb{1}_{[0,1]} \otimes u(t)(\cdot) - \int_0^t e^{(t-r)A_{-1}} \mathbb{1}_{[0,1]} \otimes \frac{d}{dr}(u(r)(\cdot - t + r))\mathrm{d}r \\
&= u(t)(\cdot) - \int_0^t \left(\tfrac{d}{dr}(u_j(r)(\cdot - t + r))(\star + c_j(t-r)) \right)_{j=1,\dots,m} \mathrm{d}r \\
&= u(t)(\cdot) - \left(\int_{\max\left\{0, \frac{c_j t + \star - 1}{c_j}\right\}}^t \tfrac{d}{dr}(u_j(r)(\cdot - t + r))(\star)\mathrm{d}r \right)_{j=1,\dots,m} .
\end{aligned}$$

Since $t = \frac{1}{\min\{c_j : j=1,\dots,m\}}$, we have $\frac{c_j t + \star - 1}{c_j} \geq 0$, thus

$$\int_0^t \tilde{T}(t-r)Bu(r)(\cdot)\mathrm{d}r = \left(u(\tfrac{c_j t - 1 + \star}{c_j})(\cdot - t + \tfrac{c_j t - 1 + \star}{c_j}) \right)_{j=1,\dots,m} .$$

Using this one obtains

$$
\begin{aligned}
\left\| \int_0^t \tilde{T}(t-r)Bu(r)(\cdot)\mathrm{d}r \right\|_{\mathcal{X}} &= \sum_{j=1}^m \int_{-\infty}^{\infty} \left(\int_0^1 \left| u_j(\frac{c_j t - 1 + l}{c_j})(s - t + \frac{c_j t - 1 + l}{c_j}) \right| \mathrm{d}l \right) \mathrm{d}s \\
&= \sum_{j=1}^m \int_{-\infty}^{\infty} \left(\int_{\frac{c_j t - 1}{c_j}}^{t} |u_j(l)(s-t+l)| \,\mathrm{d}l \right) \mathrm{d}s \\
&= \sum_{j=1}^m \int_{\frac{c_j t - 1}{c_j}}^{t} \left(\int_{-\infty}^{\infty} |u_j(l)(s-t+l)| \,\mathrm{d}s \right) \mathrm{d}l \\
&= \sum_{j=1}^m \int_{\frac{c_j t - 1}{c_j}}^{t} \left(\int_{-\infty}^{\infty} |u_j(l)(s)| \,\mathrm{d}s \right) \mathrm{d}l \\
&\leq \|u\|_{\mathrm{L}^1([0,t],\mathrm{L}^1(\mathbb{R},\mathbb{C}^m))}
\end{aligned}
$$

and condition **c'** follows.

For $t = \frac{1}{\max\{c_j : j=1,\dots,m\}}$ and $u \in W_0^{1,1}([0,t], W^{1,1}(\mathbb{R},\mathbb{C}^m))$ we have

$$
\begin{aligned}
(\tilde{\mathcal{F}}_t u)(r) &= \mathbb{B}_\omega(\cdot) \left(u(\max\left\{0, \tfrac{c_j r - 1 + \star}{c_j}\right\})(\cdot - r + \max\left\{0, \tfrac{c_j r - 1 + \star}{c_j}\right\}) \right)_{j=1,\dots,m} \\
&= 0.
\end{aligned}
$$

Thus conditions **d'** and **e'** hold and Theorem 5 allows to conclude that there exists a strongly measurable evolution family on X associated to (nACP).

CHAPTER 7

Conclusions and open questions

In this thesis work we present a perturbation result and illustrate it by many examples. However, some questions remain open.

In Chapter 5 we start from a family of unbounded operators $(A_P(t))_{t\in\mathbb{R}}$ on the Banach space X and translate the associated nonautonomous Cauchy problem into an autonomous one on $\mathcal{X} = \mathrm{L}^p(\mathbb{R}, X)$, $1 \le p < \infty$, by considering the respective evolution semigroup. This strategy is well-known and has been used by e.g., [**How74**], [**Eva76**], [**CL99**], [**Nei81**], [**RRS96**], [**RSRV00**], [**Nag95**], [**NN02**], [**Nic96**], [**Nic97**], [**Nic00**] and many others.

Applying Theorem 1.3.3 we were able to obtain an evolution semigroup in Theorem 5.2.4. This evolution semigroup yields a strongly measurable evolution family associated to $(A_P(t))_{t\in\mathbb{R}}$.

However, for the solutions of a nonautonomous Cauchy problem one expects at least continuity. In order to obtain this, as explained in [**CL99**, Prop. 3.11], one needs an evolution semigroup on the space $C_0(\mathbb{R}, X)$.

However, if one modifies the setting in Chapter 5 by considering $C_0(\mathbb{R}, X)$ instead of $\mathrm{L}^p(\mathbb{R}, X)$, one runs into serious problems and needs stronger hypotheses (e.g. [**RRS96**, Sect. 3]).

APPENDIX A

A.1. Estimating the p-Norm of a triangular Toeplitz matrix

For the proof of Lemma 1.1.19 we needed the following result.

Lemma A.1.1. *For a Banach space X endow $\mathcal{X} := X^n$, $n \in \mathbb{N}$, with the p-norm*

$$\left\|(x_1,\dots,x_n)^T\right\|_p := \Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{\frac{1}{p}}$$

for some $1 \le p \le +\infty$. Moreover, let $T_0,\dots,T_{n-1} \in \mathcal{L}(X)$. Then the norm of the Toeplitz operator matrix

$$\mathcal{T} := \big(T_{j-i}\big)_{i,j=1}^{n} = \begin{pmatrix} T_0 & 0 & 0 & \dots & \dots & 0 \\ T_1 & T_0 & 0 & \ddots & & \vdots \\ T_2 & T_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & T_1 & T_0 & 0 \\ T_{n-1} & \dots & \dots & T_2 & T_1 & T_0 \end{pmatrix}_{n\times n} \in \mathcal{L}(\mathcal{X})$$

can be estimated as

$$\|\mathcal{T}\| \le \sum_{j=0}^{n-1} \|T_j\|.$$

Proof. Let $\mathcal{x} = (x_1, \dots, x_n)^T \in \mathcal{X}$. Then one can estimate

$$\begin{aligned}
\|\mathcal{T}\mathcal{x}\|_p &= \Big(\sum_{j=1}^{n}\Big\|\sum_{i=1}^{j} T_{j-i}x_i\Big\|^p\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{j} \|T_{j-i}\| \cdot \|x_i\|\Big)^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{j=1}^{n}\Big(\big((\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|) * (\|x_1\|, \|x_2\|, \dots, \|x_n\|)\big)(j)\Big)^p\Big)^{\frac{1}{p}} \\
&= \Big\|\big(\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|\big) * \big(\|x_1\|, \|x_2\|, \dots, \|x_n\|\big)\Big\|_p \\
&\leq \Big\|\big(\|T_0\|, \|T_1\|, \dots, \|T_{n-1}\|\big)\Big\|_1 \cdot \Big\|\big(\|x_1\|, \|x_2\|, \dots, \|x_n\|\big)\Big\|_p \\
&= \sum_{j=0}^{n-1} \|T_j\| \cdot \|\mathcal{x}\|_p
\end{aligned}$$

where the second last step follows from Young's inequality applied to the convolution of sequences. $\square$

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Deutsche Zusammenfassung

In dieser Arbeit wird folgende Problematik untersucht:

Gegeben der Generator $(A, D(A))$ einer C_0-Halbgruppe $(T(t))_{t\geq 0}$ auf einem Banachraum X. Für welche Operatoren P auf X ist die Summe $A_P := A + P$ wieder ein Generator?

Viele Ergebnisse sind dazu bekannt (siehe [**EN00**, Sects. III.1-3]), aber es gibt noch keine allgemeine alle Spezialfälle umfassende Theorie.

Um die Summe A_P in vernünftiger Weise zu definieren, brauchen wir zunächst Voraussetzungen an den Operator $(P, D(P))$. Was wir annehmen ist, dass er auf dem zu A gehörigen Sobolevturm (siehe [**EN00**, Sect. II.5.a]) wirkt. Besser gesagt, dass ein Banachraum Z existiert, sodass

$$X_1^A \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X \text{ und } P \in \mathcal{L}(Z, X_{-1}^A),$$

wobei X_1^A der Definitionsbereich von A versehen mit der Graphennorm ist, während X_{-1}^A der extrapolierte Raum ist, der zu A gehört (siehe [**EN00**, Sect. II.5.a]).

In diesem Fall definiere ich $A_P := (A_{-1} + P)|_X$, wobei A_{-1} der Generator der extrapolierten Halbgruppe ist (siehe [**EN00**, Sect. II.5.a]), genauer

$$A_P x := A_{-1}x + Px \quad \text{für } x \in D(A_P) := z \in Z : A_{-1}z + Pz \in X.$$

Dies deckt drei bekannte Situationen ab:

- beschränkte Störungen [**EN00**, Sect. III.1],
- Desch–Schappacher Störungen [**EN00**, Sect. III.3.a],
- Miyadera–Voigt Störungen [**EN00**, Sect. III.3.c].

In Arbeiten über Wohlgestelltheit von lineare Kontrollsysteme mit Feedback haben Weiss und Staffans ein allgemeineres Ergebnis zu dieser Problematik dargestellt (siehe [**Wei94a**, Thms. 6.1 and 7.2] und [**Sta05**, Sects. 7.1 & 7.4]).

In Kapitel 1 dieser Arbeit bearbeiten wir den Weiss–Staffans Ansatz, indem wir das Ergebnis in einer rein operatortheoretischen Perspektive diskutieren.

Die Idee dahinter ist, mit Hilfe eines zusäztlichen Banachraumes U die Störung in zwei Teile zu spalten, d.h. P zu schreiben als

$$P = B \circ C \text{ wobei } C \in \mathcal{L}(Z, U) \text{ und } B \in \mathcal{L}(U, X_{-1}^{A}).$$

Es ist dann so, als ob wir einen "Miyadera–Voigt Teil" C und einen "Desch–Schappacher Teil" B hätten. Diese Interpretation spiegelt sich weiter in den Voraussetzungen, die wir für Theorem 1.2.1 gemacht haben.

Die Ergebnissen dieses Kapitel, die zu Theorem 1.2.1 führen, sind Teil einer Zusammenarbeit mit M. Adler und K.-J. Engel und wurden in [**ABE14**] publiziert.

Das Kapitel wird abgeschlossen mit einer Verallgemeinerung von Theorem 1.2.1, siehe Theorem 1.3.3.

Im Kapitel 2 verwenden wir Theorem 1.3.3, um die Wohlgestelltheit von linearen Kontrollsystemen durch die Generatoreigenschaft einer Operatormatrix zu charakterisieren (vgl. dazu [**BE14**]).

Im Kapitel 3 wird eine Verallgemeinerung des Greinerschen Ansatzes zu Randstörungen [**Gre87**] vorgestellt. Mit Hilfe von Theorem 1.2.1 konnten wir dort unbeschränkten Randstörungen handeln. Die Ergebnisse dieses Kapitels sind Teil einer Zusammenarbeit mit M. Adler und K.-J. Engel und wurden in [**ABE14**] veröffentlicht.

Im Kapitel 4 betrachten wir den Spezialfall, indem der zu störende Generator A eine analytische Halbgruppe $(T(t))_{t\geq 0}$ erzeugt. In diesem Fall können wir mit Hilfe von Favardräumen und abstrakten Hölderräumen die Bedingungen, die in Theorem 1.2.1 vorkommen, vereinfachen und Theorem 4.2.3 beweisen.

Die Ergebnisse dieses Kapitels sind Teil einer Zusammenarbeit mit M. Adler und K.-J. Engel und werden in einem gemeinsamen Paper erscheinen.

Im Kapitel 5 wird Theorem 1.3.3 angewendet, um zeitabhängige Weiss–Staffans Störungen zu betrachten. Dies geschieht mit Hilfe sogenannter Evolutionshalbgruppen. Unter Ausnutzung verschiedener Sobolevtürme und unserer Verallgemeinerung des Weiss–Staffans Störungssatzes (Theorem 1.3.3) gelingt es uns in Theorem 5.2.4 eine Evolutionshalbgruppe zu erhalten, die einen nichtautonomen Cauchy-Problem entspricht.

Die Arbeit wird abgeschlossen mit Kapitel 6, in dem die Ergebnisse von Kapitel 5 auf zeitabhängigen Randstörungen angewendet werden. Dies wird durch ein abschließendes Beispiel über zeitabhängige Flüsse auf Netzwerke illustriert.

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