Construction of constant mean curvature tori in $\mathbb{R}^3$ of arbitrary spectral genus via Whitham flows

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Abstrakt

Die vorliegende Arbeit behandelt eine Deformation von Spektralkurven von Flächen konstanter mittlerer Krümmung (Constant mean curvature - CMC) in $\mathbb{R}^3$. Zuerst werden die grundlegenden Fakten über konforme Immersionen von Flächen und insbesondere von CMC Tori nach $\mathbb{R}^3$ zusammengetragen. Es wird ein Bezug zwischen CMC Tori und doppelt periodischen Lösungen der sinh-Gordon Gleichung aufgezeigt. Darauf aufbauend wird eine Riemannsche Fläche, die Spektralkurve definiert und die Theorie der Spektralkurven für die doppelt periodische Lösungen der sinh-Gordon Gleichung vorgestellt. Das Geschlecht g dieser Spektralkurve heißt Spektralgeschlecht. Es werden die extrinsischen Schließungsbedingungen erklärt, die nötig sind, um eine Immersion zu einem Torus in $\mathbb{R}^3$ zu schließen.

Ausgehend von den Spektralkurven wird die Theorie der Withamdeformationen entwickelt, die die Spektralkurven so deformiert, dass die intrinsischen Schließungsbedingungen erhalten werden, d.h. die Spektralkurve weiterhin eine Spektralkurve einer doppelt periodischen Lösung der sinh-Gordon Gleichung bleibt. Die Deformation wird weiterhin um die extrinsischen Schließungsbedingungen ergänzt. Diese Deformation erzeugt ausgehend von einer Spektralkurve eines CMC Torus in $\mathbb{R}^3$ eine Familie von CMC Zylindern mit doppelt periodischer Metrik. Weiterhin schließen sich diese Zylinder zu CMC Tori auf einer abzählbar dichten Teilmenge des Existenzintervalls der Deformation.

Die Withamdeformation wird nun benutzt, um Spektralkurven von CMC Zylindern mit möglichen Doppelpunkten zu finden. Diese Spektralkurven werden für eine Bifurkation von Spektralgeschlecht g zu $g+1$ benutzt. Da die Withamdeformation am Bifurkationspunkt singulär ist, wird eine Prozedur zur Desingularisierung vorgestellt. Dabei wird zuerst eine Potenzreihenentwicklung einer möglichen Lösung konstruiert und ihre Eindeutigkeit bewiesen. Im zweiten Schritt wird die Konvergenz einer derartigen Lösung gezeigt. Auf diese Weise ist eine Bifurkation der Withamdeformation in ein höheres Spektralgeschlecht möglich, die die Schließungsbedingungen erhält. Es wird wieder eine Familie von CMC Zylindern mit doppelt periodischer Metrik erzeugt. Auch hier schließen sich diese Zylinder zu CMC Tori auf einer dichten Teilmenge des Existenzintervalls, nun mit Spektralgeschlecht $g+1$. Diese Desingularisierung wird benutzt, um einen bekannten Satz über die Existenz von Tori mit beliebig hohem Spektralgeschlecht neu zu beweisen. Da der Beweis mittels einer Deformation erbracht wird, ist es möglich eine Familie von CMC Zylindern explizit zu konstruieren, die einen bekannten Torus wie den Wente Torus mit einem neuen Torus von beliebig hohem Spektralgeschlecht verbindet.

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1 Introduction

The surfaces with constant mean curvature (CMC) have been of big interest to mathematicians since the middle of the nineteenth century. In the mid 20th century two important results about the global properties of CMC surfaces were found by Hopf and Aleksandrov. In 1955 Hopf [18] showed that any compact genus zero immersed CMC surface must be a round sphere and as such must be embedded. Aleksandrov [2] used the maximum principle to show that any compact embedded CMC surface must be a round sphere.

In the eighties of the twentieth century this topic regained new interest with the discovery of the first immersed but not embedded CMC tori in $\mathbb{R}^3$ by Wente [26]. The theory was developed further by Hitchin [17], by Pinkall and Sterling [23] and by Bobenko [4, 5, 6]. In particular the latter worked out the relation between CMC tori in different space forms, doubly periodic solutions of the sinh-Gordon equation and hyperelliptic Riemann surfaces. For a CMC surface with a doubly periodic metric, the metric is closely related to doubly periodic solutions of the sinh-Gordon equation and those doubly periodic solutions admit an additional structure in form of a hyperelliptic Riemann surface. These surfaces are called spectral curves. Krichever [22] and later Grinevich and Schmidt [15] started to study deformation properties of these spectral curves. The deformations they are using are called Whitham deformation since they are similar in spirit to those studied by Whitham [27].

One interesting question to be studied was for which genera g of those spectral curves there exist CMC tori in $\mathbb{R}^3$. Bobenko [4] already showed that there are no tori for $g = 0, 1$. The example of Wente was $g = 2$. Ercolani et al. [12] proved that for any even spectral genus g there must exist a CMC torus with a spectral curve of such genus. Later Jaggy [20] used deformation techniques to prove that for every spectral genus $g \geqslant 2$ there exist tori in $\mathbb{R}^3$ with such spectral genus. Although his proof relied on deformation techniques, it used the implicit function theorem at a crucial point. Thus the proof is not constructive in the sense that it does not allow to construct the tori whose existence it shows.

The Whitham deformations are very appealing since they allow to study the moduli space of the sinh-Gordon solutions and also of CMC tori and CMC cylinders with a doubly periodic metric. Carberry and Schmidt [9, 10] obtained interesting results about the denseness of spectral curves for CMC tori in $\mathbb{S}^3$ and $\mathbb{R}^3$ inside a suitably chosen space of admissible spectral curves. Hauswirth et al. [16] used them to study minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. Calini and Ivey [8] used similar deformations to study space curves in $\mathbb{R}^3$ corresponding to periodic and quasi periodic finite-gap solutions of the nonlinear Schrödinger equation. The two last papers had to deal with singularities of the Whitham deformation which also will be a large part of our work. The author of this thesis has already used Whitham deformation in his previous work [21] to prove a particular case where the spectral curve of a Clifford torus, a CMC torus in $\mathbb{S}^3$, was deformed through spectral curves of CMC tori in $\mathbb{S}^3$ in such a way that at the end of the deformation it arrived at the spectral curve of a Wente torus in $\mathbb{R}^3$.

In the current thesis we will study Whitham deformations of CMC tori and doubly periodic CMC cylinders in $\mathbb{R}^3$. In contrast to $\mathbb{S}^3$ where CMC tori come in families it is not possible to obtain deformations of CMC tori in $\mathbb{R}^3$. Therefore we will define a flow through doubly periodic CMC cylinders which will close up to CMC tori on a dense subset of the time interval. This will be achieved by preserving the double periodicity of the metric and of the periodicity of the translation and rotation period of the CMC surface along one of the period directions of the metric. Thus, if those translation and rotation periods vanished (i.e. the CMC surface is topologically a cylinder in $\mathbb{R}^3$) at the starting point of the flow, we would flow through doubly periodic CMC cylinders. The second period direction of the metric also gives rise to translation and rotation periods in $\mathbb{R}^3$. Our flow has the additional property that the translation period is preserved. Therefore if we started at a CMC torus, we would flow through doubly periodic CMC cylinders without any translation periods in $\mathbb{R}^3$, which will close up to CMC tori for rational angles of the rotational period. For a non-trivial flow this will happen at a dense subset of its existence interval. Using this deformation, we will find a spectral curve of a cylinder with a double point, i.e. a singular spectral curve. We will then use our flow to open the double point and bifurcate to a spectral curve of genus $g+1$. The main part of the work will be to show that it is possible to desingularize the flow at such a singular spectral curve and preserve the closing conditions on the way so we can properly flow along Whitham deformation after that. It will become clear that there will be only one direction in which the flow desingularizes. Similar desingularizations where performed by Hauswirth et al. [16] and Calini and Ivey [8]. In case of curves in $\mathbb{R}^3$ and the nonlinear Schrödinger equation the desingularization procedure is much easier then in our case. The main reason for this is that for closed the curves in $\mathbb{R}^3$ one needs to control only one period and the deformation equations for curves decouple in a way which allows a good analysis of the behavior around such bifurcation points.

The following work is divided into three main parts. We start with chapter 2, where we recall some facts about conformal immersions mainly to make what follows self contained and to fix the notations.

Chapter 3 will introduce spectral curves and explain some of their properties. We will discuss intrinsic closing conditions and how they ensure the existence of doubly periodic solutions of the sinh-Gordon equation. We will also explain the extrinsic closing conditions which are necessary to obtain cylinder and tori.

Chapter 4 contains the main part of this thesis. We introduce the Whitham deformation which preserves the intrinsic closing conditions. Then we will show how the extrinsic closing conditions can be incorporated and how they affect the freedom of the deformation. Here we will see why it is necessary to deform through cylinders. The rest of the chapter will analyze the deformation equations in great detail. We will see what are the singularities of the flow, how bifurcating to higher spectral genus introduces singularities of the flow, and how to find those points where bifurcation to higher spectral genus is possible. For a certain type of bifurcation we will show the existence of formal power series solutions to the singular deformation equation and then also show their convergence. Finally, we will use these results to prove the existence of tori of arbitrary high spectral genus. In contrast to the proof given in Jaggy [20], our proof involves only Whitham deformations and not an implicit function theorem. As such our proof is a constructive approach. At the end of this chapter we will demonstrate this by a numerical example of a Whitham flow with a Wente torus of spectral genus 2 at the start and a Dobriner torus of spectral genus 3 at the end of the deformation.

2 Conformal immersions

In the following we will gather some useful standard facts about conformal immersions into the three dimensional space forms $\mathbb{S}^3$ and $\mathbb{R}^3$. In particular we will look at an equation for the mean curvature of these immersions. Then we will define an extended frame and illustrate its relation to the sinh-Gordon equation. After that we will recall the Sym-Bobenko formulas which construct an immersion from a given extended frame. At the end of the chapter we will use monodromy of the extended frame to define the spectral curve. The exposition follows Schmitt et al. [25] and Bobenko [6].

2.1 Conformal immersions into $\mathbb{R}^3$

We will look at the matrix Lie group SU_2. The Lie algebra $\mathfrak{su}_2$ of this group is equipped with a commutator $[\cdot,\cdot]$. Let $\alpha,\beta \in \Omega^1(\mathbb{R}^2,\mathfrak{su}_2)$ be smooth 1-forms on $\mathbb{R}^2$ with values in $\mathfrak{su}_2$. We define a $\mathfrak{su}_2$-valued 2-form by

$$[\alpha \wedge \beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)] \tag{2.1}$$

for $X,Y \in T\mathbb{R}^2$. Let $L_g : h \mapsto gh$ be the left multiplication in SU_2. By left translation we obtain an isomorphism of the tangential bundle $T\,\mathrm{SU}_2 \cong \mathrm{SU}_2 \times \mathfrak{su}_2$. We also have a Maurer-Cartan form

$$\theta : T\,\mathrm{SU}_2 \to \mathfrak{su}_2, v_g \mapsto (\mathrm{d}L_{g^{-1}})_g v_g$$

which satisfies the Maurer-Cartan equation

$$2\mathrm{d}\theta + [\theta \wedge \theta] = 0. \tag{2.2}$$

For a map $F : \mathbb{R}^2 \to \mathrm{SU}_2$, the pullback $\alpha = F^*\theta$ satisfies (2.2) as well. The converse is also true: every solution $\alpha \in \Omega^1(\mathbb{R}^2,\mathfrak{su}_2)$ of (2.2) integrates to a smooth map $F : \mathbb{R}^2 \to \mathrm{SU}_2$ with $\alpha = F^*\theta$.

We now complexify the tangent bundle $T\mathbb{R}^2$ and decompose $(T\mathbb{R}^2)^{\mathbb{C}} = T'\mathbb{R}^2 \oplus T''\mathbb{R}^2$ into $(1,0)$ and $(0,1)$ tangent spaces and write $\mathrm{d} = \partial + \bar{\partial}$. We also decompose

$$\Omega^1(\mathbb{R}^2,\mathfrak{sl}_2(\mathbb{C})) = \Omega'(\mathbb{R}^2,\mathfrak{sl}_2(\mathbb{C})) \oplus \Omega''(\mathbb{R}^2,\mathfrak{sl}_2(\mathbb{C}))$$

using $\mathfrak{su}_2{}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. We split $\omega \in \Omega^1(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C}))$ accordingly into the $(1,0)$ part ω' and the $(0,1)$ part ω'' writing $\omega = \omega' + \omega''$. Finally, we set the $*$-operator on $\Omega^1(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C}))$ to $*\omega = -i\omega' + i\omega''$.

For further computations we fix the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{2.3}$$

Moreover let $\langle \cdot, \cdot \rangle$ be the billinear extension of the Ad-invariant inner product on $\mathfrak{su}_2$ to $\mathfrak{su}_2{}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, such that $\langle \epsilon, \epsilon \rangle = 1$. For $X, Y \in \mathfrak{su}_2$ we further have

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr} XY, \quad \|X\| = \sqrt{\det X}, \quad X \times Y = \frac{1}{2}[X, Y]. \tag{2.4}$$

So the following relations arise

$$\begin{gathered} \langle \epsilon_-, \epsilon_- \rangle = \langle \epsilon_+, \epsilon_+ \rangle = 0, \quad \overline{\epsilon_-}^{\mathsf{T}} = \epsilon_+, \\ [\epsilon, \epsilon_-] = 2i\epsilon_-, \quad [\epsilon_+, \epsilon] = 2i\epsilon_+, \quad [\epsilon_-, \epsilon_+] = i\epsilon. \end{gathered} \tag{2.5}$$

We will identify the three space $\mathbb{R}^3$ with the Lia algebra $\mathfrak{su}_2$. Under this identification the isometry group will be doubly covered by SU_2. Let $\mathbb{T}$ be the stabilizer of ϵ under the adjoint action of SU_2 on $\mathfrak{su}_2$. We view the two-sphere as $\mathbb{S}^3 = \mathrm{SU}_2 / \mathbb{T}$.

We will now take a closer look on conformally parametrized surfaces into $\mathbb{R}^3 = \mathfrak{su}_2$.

Let $f : \mathbb{R}^2 \to \mathfrak{su}_2$ be a conformal immersion. Let $U \subset \mathbb{R}^2$ be an open simply connected set with a coordinate $z : U \to \mathbb{C}$. Write $\mathrm{d}f' = f_z \mathrm{d}z$ and $\mathrm{d}f'' = f_{\bar{z}} \mathrm{d}\bar{z}$. The conformality of the map f is equivalent to $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ and the existence of a function $u \in C^\infty(U, \mathbb{R})$, such that $\langle f_z, f_{\bar{z}} \rangle = \frac{1}{2} e^u$. We accompany f_z and $f_{\bar{z}}$, which are tangential to the surface immersion, by the Gauss map $N : U \to \mathbb{S}^2 = \mathrm{SU}_2 / \mathbb{T}$. There exist a lift $F : U \to \mathrm{SU}_2$ such that

$$f_z = e^{\frac{u}{2}} F \epsilon_- F^{-1}, \quad f_{\bar{z}} = e^{\frac{u}{2}} F \epsilon_+ F^{-1}, \quad N = F \epsilon F^{-1}. \tag{2.6}$$

The mean curvature is $H = 2e^{-u} \langle f_{z\bar{z}}, N \rangle$ and the Hopf differential Qdz^2 is given by $Q = \langle f_{zz}, N \rangle$. Using these relations we obtain

$$f_{zz} = u_z f_z + QN, \quad f_{\bar{z}\bar{z}} = u_{\bar{z}} f_z + \bar{Q}N, \quad f_{z\bar{z}} = \frac{1}{2} e^u HN, \tag{2.7}$$

and

$$\alpha := F^{-1}\mathrm{d}F = \Big(-(\frac{1}{2}e^{\frac{u}{2}}H\mathrm{d}z + e^{-\frac{u}{2}}\bar{Q}\mathrm{d}\bar{z})i\epsilon_- \\ +(e^{-\frac{u}{2}}Q\mathrm{d}z + \frac{1}{2}e^{\frac{u}{2}}H\mathrm{d}\bar{z})i\epsilon_+ \\ -\frac{1}{4}(u_z\mathrm{d}z - u_{\bar{z}}\mathrm{d}\bar{z})i\epsilon\Big). \tag{2.8}$$

We observe that an conformal immersion f gives rise to a triple (u,H,Q) which in turn defines an 1-form α. This α defines a differential equation $\mathrm{d}F = F\alpha$ which can clearly be solved when α comes from a given frame F.

We finish this section by proving a formula for the curvature of a conformal immersion.

Lemma 2.1. *The mean curvature H of a conformal immersion $f : \mathbb{R}^2 \to \mathfrak{su}_2$ is given by* $2\mathrm{d}*\mathrm{d}f = H[df \wedge df]$.

Proof. We compute $[\mathrm{d}f \wedge \mathrm{d}f] = 2ie^u N\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Another computation shows $\mathrm{d}*\mathrm{d}f = ie^u HN\mathrm{d}z\wedge\mathrm{d}\bar{z}$ and by combining both results we have proved the claim. □

2.2 The sinh-Gordon equation

Previous section showed us how a conformally parametrized surfaces gives rise to an integrable triple (u,H,Q). In this section we will derive the integrability conditions of such triples and illustrate the relation between conformal immersions with constant mean curvature and solutions of the sinh-Gordon equation. We will introduce a spectral parameter and define extended frames.

As shown in previous chapter a triple (u,H,Q) of functions $C^\infty(U,\mathbb{R})$ gives rise to a 1-form α via

$$\alpha = \frac{1}{4}\begin{pmatrix} -u_z\mathrm{d}z + u_{\bar{z}}\mathrm{d}\bar{z} & 4ie^{-\frac{u}{2}}Q\mathrm{d}z + 2ie^{\frac{u}{2}}H\mathrm{d}\bar{z} \\ 2ie^{\frac{u}{2}}H\mathrm{d}z + 4ie^{-\frac{u}{2}}\bar{Q}\mathrm{d}\bar{z} & u_z\mathrm{d}z - u_{\bar{z}}\mathrm{d}\bar{z} \end{pmatrix}. \tag{2.9}$$

The equation

$$\mathrm{d}F = F\alpha \tag{2.10}$$

has an solution $F : U \to \mathrm{SU}_2$ exactly when $2\mathrm{d}\alpha + [\alpha \wedge \alpha] = 0$ holds.

We decompose $\alpha = \alpha'\mathrm{d}z + \alpha''\mathrm{d}\bar{z}$ into the corresponding $(1,0)$ and $(0,1)$ parts. The equation $2\mathrm{d}\alpha + [\alpha \wedge \alpha] = 0$ is then equivalent to $\bar{\partial}\alpha' - \partial\alpha'' = [\alpha', \alpha'']$. A computation shows that this equation is equivalent to a system of equations

$$u_{z\bar{z}} + \frac{1}{2}e^{u}H^{2} - 2e^{-u}Q\bar{Q} = 0, \tag{2.11a}$$

$$\frac{1}{2}e^{u}H_{z} = Q_{\bar{z}}, \tag{2.11b}$$

$$\frac{1}{2}e^{u}H_{\bar{z}} = \bar{Q}_{z}, \tag{2.11c}$$

to the well known Gauss-Codazzi equations.

The equations (2.11b) and (2.11c) show us that in case of CMC immersion, i.e. if $H \equiv$ const, Q must be a holomorphic function. The quadratic holomorphic differential Qdz^2 is called the Hopf differential. When we deal with CMC tori, we observe that than $Q \equiv$ const. In this case it is possible to scale H by scaling the Euclidean space and to scale Q by a change of coordinate. With this in mind we set $H \equiv 1$ and $Q \equiv \frac{1}{2}$ and the equation (2.11a) becomes

$$u_{z\bar{z}} + \sinh(u) = 0, \tag{2.12}$$

the sinh-Gordon equation.

We will now introduce a spectral parameter $\lambda \in \mathbb{S}^1$ to our equations by defining

$$\alpha_\lambda := \frac{1}{4}\begin{pmatrix} -u_z \mathrm{d}z + u_{\bar{z}}\mathrm{d}\bar{z} & 4i\lambda^{-1}e^{-\frac{u}{2}}Q\mathrm{d}z + 2ie^{\frac{u}{2}}H\mathrm{d}\bar{z} \\ 2ie^{\frac{u}{2}}H\mathrm{d}z + 4i\lambda e^{-\frac{u}{2}}\bar{Q}\mathrm{d}\bar{z} & u_z\mathrm{d}z - u_{\bar{z}}\mathrm{d}\bar{z} \end{pmatrix}. \tag{2.13}$$

A short computation shows that for $H \equiv$ const and $Q_{\bar{z}} = 0$ the integrability condition $2\mathrm{d}\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ yields the same condition on u as in the case of $2\mathrm{d}\alpha + [\alpha \wedge \alpha] = 0$ namely the equation (2.11a) which is the sinh-Gordon equation in case of $H \equiv 1$ and $Q \equiv \frac{1}{2}$. It is then possible to solve $dF_\lambda = F_\lambda \alpha_\lambda$.

Proposition 2.2. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Let $\lambda \in \mathbb{S}^1$. Let*

$$\alpha_\lambda := \frac{1}{4}\begin{pmatrix} -u_z \mathrm{d}z + u_{\bar{z}}\mathrm{d}\bar{z} & 2i\lambda^{-1}e^{-\frac{u}{2}}\mathrm{d}z + 2ie^{\frac{u}{2}}\mathrm{d}\bar{z} \\ 2ie^{\frac{u}{2}}\mathrm{d}z + 2i\lambda e^{-\frac{u}{2}}\mathrm{d}\bar{z} & u_z\mathrm{d}z - u_{\bar{z}}\mathrm{d}\bar{z} \end{pmatrix}.$$

Then the equation $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda]$ holds exactly if u solves the sinh-Gordon-equation

$$u_{z\bar{z}} + \sinh(u) = 0.$$

In this case the equation

$$dF_\lambda = F_\lambda \alpha_\lambda.$$

has a unique solution $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathrm{SU}_2$ with the initial data $F_\lambda(0) = \mathbb{1}$. F_λ is called an extended frame.

2.3 The Sym-Bobenko formulas

In the following we will explain the Sym-Bobenko formulas which allow us to compute a conformal immersion from a given extended frame into $\mathbb{R}^3$.

Let $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathfrak{su}_2$ be an extended frame with $\alpha_\lambda = F_\lambda^{-1} \mathrm{d}F_\lambda$.

Proposition 2.3. *Let M be a simply connected Riemann surface and F_λ an extended frame. Let $H \in \mathbb{R}^*$. For every $\lambda \in \mathbb{S}^1$ the map $f_\lambda : M \to \mathbb{R}^3$ defined by*

$$f_\lambda = -2i\lambda H^{-1}(\partial_\lambda F_\lambda)F_\lambda^{-1} - H^{-1}F_\lambda \epsilon F_\lambda^{-1}$$

is a conformal immersion $M \to \mathbb{R}^2$ with constant mean curvature H

Proof. We can then decompose α_λ in the following way

$$\alpha_\lambda = \alpha'_\lambda + \alpha''_\lambda = (\alpha'_1 + \lambda\alpha''_1)\epsilon_- + (\lambda^{-1}\alpha'_2 + \alpha''_2)\epsilon_+ + (\alpha'_3 + \alpha''_3)\epsilon. \tag{2.14}$$

We compute

$$\begin{aligned} \mathrm{d}f'_\lambda &= -2i\lambda H^{-1}F_\lambda \partial_\lambda \alpha'_\lambda F_\lambda^{-1} - H^{-1}F_\lambda[\alpha'_\lambda, \epsilon]F_\lambda^{-1} \\ &= 2i\lambda^{-1}H^{-1}F_\lambda \alpha'_2 \epsilon_+ F_\lambda^{-1} + 2iH^{-1}F_\lambda \alpha'_1 \epsilon_- F_\lambda^{-1} - 2i\lambda^{-1}H^{-1}F_\lambda \alpha'_2 \epsilon_+ F_\lambda^{-1} \\ &= 2iH^{-1}F_\lambda \alpha'_1 \epsilon_- F_\lambda^{-1} \\ &= e^{\frac{u}{2}} F \epsilon_- F^{-1} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \mathrm{d}f''_\lambda &= -2i\lambda H^{-1}F_\lambda \partial_\lambda \alpha''_\lambda F_\lambda^{-1} - H^{-1}F_\lambda[\alpha''_\lambda, \epsilon]F_\lambda^{-1} \\ &= -2i\lambda H^{-1}F_\lambda \alpha''_1 \epsilon_- F_\lambda^{-1} + 2i\lambda H^{-1}F_\lambda \alpha''_1 \epsilon_- F_\lambda^{-1} - 2iH^{-1}F_\lambda \alpha''_2 \epsilon_+ F_\lambda^{-1} \\ &= -2iH^{-1}F_\lambda \alpha''_2 \epsilon_+ F_\lambda^{-1} \\ &= e^{\frac{u}{2}} F \epsilon_+ F^{-1}. \end{aligned} \tag{2.16}$$

We can see now $\left\langle \mathrm{d}f'_\lambda, \mathrm{d}f'_\lambda \right\rangle = 0$, $\left\langle \mathrm{d}f''_\lambda, \mathrm{d}f''_\lambda \right\rangle = 0$, and $\left\langle \mathrm{d}f'_\lambda, \mathrm{d}f''_\lambda \right\rangle = \frac{1}{2}e^u$ so the conformality is clear. We have again $N = F\epsilon F^{-1}$. Further since the equations (2.15) and (2.16) are the same as in (2.6) the extended frame which comes from f_λ is indeed the same extended frame F_λ as we stated with. So with the same computations as in section 2.1 we obtain again H for the mean curvature and thus proving the claim. □

The Sym-Bobenko formulas are very useful when one knows the extended frame since their allow to replace a integration, which is usually needed to compute an immersion from its frame, by a differentiation with respect to spectral parameter λ. These formulas can also be used to great effect in numerical computations of surfaces. Here one is mostly only interested in f_λ for one particular value of λ, usually

$\lambda = 1$. One can then compute the needed derivative in the same step as the extended frame at a given value of λ.

Different spectral parameters λ yield different associated surfaces with the same metric and mean curvature but different Hopf differentials. When one refers to a particular surface in this family with a particular value of λ, one calls this λ the Sym point.

Here we showed only the formula for $\mathbb{R}^3$. Similar formulas exist also for $\mathbb{S}^3$ and $\mathbb{H}^3$.

2.4 Monodromy, closing conditions and the spectral curve

In the following we will take a closer look into doubly periodic solutions of the sinh-Gordon equation. The doubly periodicity of the sinh-Gordon solution is obviously a prerequisite for a doubly periodic metric of a CMC immersion and thus for CMC tori.

Let u be a doubly periodic solution of the sinh-Gordon equation $u_{z\bar{z}} + \sinh(u) = 0$ with period lattice $\Gamma \subset \mathbb{R}^2$. Then the 1-form α_λ is also doubly periodic and via $dF_\lambda = F_\lambda \alpha_\lambda$. The extended frame fulfills the following quasi periodicity condition

$$F_\lambda(z + p) = F_\lambda(z_0 + p) F_\lambda(z) \quad \text{for all } z \in \mathbb{R}^2$$

for a period $p \in \Gamma$ and a base point z_0. We use this relation to define the monodromy of the extended frame.

Definition 2.4 (Monodromy)**.** The monodromy $M : \Gamma \times \mathbb{C}^* \to \mathrm{SU}_2$ of F_λ is $M(p, \lambda) := F(z_0 + p, \lambda)$.

An important property of the monodromy is that despite its dependence on the choice of the base point z_0 the eigenvalues μ_1, μ_2 of M do not depend on that choice.

We can now use the monodromy of a doubly periodic sinh-Gordon solution with its eigenvalues μ_1, μ_2 to define the spectral curve of this solution.

Definition 2.5 (Spectral curve)**.** The Riemann surface Σ the defined by

$$\Sigma = \{(\mu_1, \lambda) \in \mathbb{C}^* \times \mathbb{C}^* : \mu_1^2 - \mathrm{tr}(M(p, \lambda))\mu_1 + 1 = 0\}$$

parameterizing the eigenvalues of M is called the spectral curve.

We state now some facts about the so defined spectral curve. Further details can be found in Hitchin [17]. Σ does not depend on the choice of basis vector $p \in \Gamma$. We used μ_1 to define Σ, using μ_2 yields the same spectral curve. Σ completes to hyperelliptic Riemann surface $\lambda : \Sigma \to \mathbb{CP}^1$. This is achieved by adding branch points over $\lambda = 0$ and $\lambda = \infty$. This is possibly when zeros of μ_j do no accumulate there. The genus g of Σ is called the spectral genus. A result of Pinkall and Sterling [23] shows that CMC tori in $\mathbb{R}^3$ have a finite spectral genus g allowing the above completion. Solutions of sinh-Gordon equation with finite spectral genus are also called finite-gap solutions.

The version of spectral curve we defined here is also called the multiplier spectral curve. One can also define a spectral curve based on the holonomy of a flat connection coming from the immersion. Bohle [7] has worked out in detail when those two spectral curves coincide. For our purposes the definition of multiplier spectral curve will suffice.

So far a doubly periodic solution of sinh-Gordon equation only ensures the doubly periodicity of the metric of a CMC immersion. In order to obtain CMC cylinder and tori we need global closing conditions, i.e. periodicity of the immersion f_λ. We compute

$$\begin{aligned} f_\lambda(z+p) =& -2i\lambda H^{-1}(\partial_\lambda F_\lambda)(z+p)F_\lambda^{-1}(z+p) - H^{-1}F_\lambda(z+p)\epsilon F_\lambda^{-1}(z+p) \\ =& -2i\lambda H^{-1}(\partial_\lambda M(p,\lambda)F_\lambda)(z)F_\lambda^{-1}(z)M^{-1}(p,\lambda) \\ & - H^{-1}M(p,\lambda)F_\lambda(z)\epsilon F_\lambda^{-1}(z+p)M^{-1}(p,\lambda) \\ =& -2i\lambda H^{-1}\partial_\lambda M(p,\lambda)M^{-1}(p,\lambda) \\ & -2i\lambda H^{-1}M(p,\lambda)(\partial_\lambda F_\lambda)(z)F_\lambda^{-1}(z)M^{-1}(p,\lambda) \\ & - H^{-1}M(p,\lambda)F_\lambda(z)\epsilon F_\lambda^{-1}(z+p)M^{-1}(p,\lambda) \\ =& -2i\lambda H^{-1}\partial_\lambda M(p,\lambda)M^{-1}(p,\lambda) + M(p,\lambda)f_\lambda(z)M^{-1}(p,\lambda). \end{aligned} \tag{2.17}$$

We see that in order to have $f_\lambda(z+p) = f_\lambda(z)$. we need

$$M(p,\lambda) = \pm\mathbb{1} \quad \text{and} \quad \partial_\lambda M(p,\lambda) = 0. \tag{2.18}$$

Those conditions will usually be fulfilled for only one λ. From now on we will use $\lambda = 1$ as the Sym point. Since the spectral curve gives only access to eigenvalues of M we need to express these conditions in terms of eigenvalues. Further it will be beneficial to deal with $\ln\mu_j$ instead of μ_j. This leads to the following statement.

Proposition 2.6. *The spectral curve of a CMC torus fulfills additional properties, namely the (extrinsic) closing conditions*

$$\ln\mu_j\Big|_{\lambda=1} \in \pi i\mathbb{Z} \quad \textit{and} \quad \partial_\lambda \ln\mu_j\Big|_{\lambda=1} = 0,$$

where μ_j are the eigenvalues of the monodromy with respect to lattice basis $p_1, p_2 \in \Gamma$.

The equation (2.17) indicates that in case of $M(p,\lambda) = \pm\mathbb{1}$ one is left with a translation period coming from $\partial_\lambda M(p,\lambda)$ and in case of $\partial_\lambda M(p,\lambda) = 0$ on has a rotational period coming from $M(p, \lambda)$.

In case these conditions are fulfilled only for one μ_j, we have a CMC cylinder, as the surface closes only in one direction.

Remark. Since we used the Sym Bobenko formula in order to establish the closing conditions it is clear that their form depend on the space form the immersion is mapping into.

Finally we mention that it is possible to recover the sinh-Gordon solution from its spectral curve. The same is also true for the extended frame. There are several ways to do it. One is via theta functions and Baker-Akhiezer functions. It is developed in detail in Bobenko [4]. Another way is by using a polynomial killing field. This method is described in Pinkall and Sterling [23] for tori in $\mathbb{R}^3$ and Ferus et al. [13] for minimal tori in $\mathbb{S}^4$. Finally one can also use methods developed in Dorfmeister et al. [11]. We wont go here in detail since our further work will be entirely on the level of spectral curves and we wont relay on any particular reconstruction method. In order to obtain immersions of CMC surfaces from the spectral curves which we will construct in later chapters one of course needs to use one of the mentioned reconstruction methods.

3 Spectral curves

In the previous chapter we recalled the connection between conformal immersions of constant mean curvature and solutions of the sinh-Gordon equation. Since we will be dealing with tori and doubly periodic cylinders in the rest of this thesis, we are particularly interested in doubly periodic solutions of sinh-Gordon equation. Those solutions admit an additional structure in form of a hyperelliptic Riemann surface, the so called spectral curve of the solution. Those Riemann surfaces were for example studied in detail by Bobenko [4, 5].

In this chapter we will recall the properties of the spectral curves. We will fix the notion of the spectral curve for our purposes. We will explain what are the intrinsic closing conditions and how the spectral curve encodes them. We will study the extrinsic closing conditions need to produce compact surfaces. We also will define the spectral data which will endow the spectral curve with an additional structure in form of two meromorphic differentials in a vector form suitable for further work. These spectral data will be used in the following chapter to define deformations.

3.1 Spectral curve of constant mean curvature cylinders and tori

In section 2.4 of the last chapter we defined the spectral curve and captured some of its properties. Here we will restate them in form of a proposition to have a basis for further analysis. Proofs to these properties can be found in Hitchin [17] and Bobenko [4, 5]. We will also make a coordinate change from the spectral parameter λ to κ. The relation between these parameters is the following

$$\lambda = \frac{i-\kappa}{i+\kappa}, \quad \text{and} \quad \kappa = i\frac{1-\lambda}{1+\lambda} \tag{3.1}$$

The points on the real line in κ correspond to the unit circle in λ. We do this to simplify the computations that will follow in the this thesis. One of the reasons for this is that $\lambda = \infty$ is mapped to $\kappa = -i$ and we will therefore not have to deal with branch points over a point with ∞ as a coordinate.

Let Σ be a hyperelliptic Riemann surface $\kappa : \Sigma \to \mathbb{CP}^1$ whose branch locus includes the two points $y^{\pm}$ over $\kappa = \pm i$ and let σ be the hyperelliptic involution of Σ. Further we recall that for a doubly periodic solution $u : \mathbb{R}^2 \to \mathbb{R}$ of the sinh-Gordon equation we have monodromy with its eigenvalues μ_j and a lattice $\Gamma \subset \mathbb{R}^2$ with a basis p_1, p_2 over which u is periodic.

Proposition 3.1 (Spectral Curve)**.** *If Σ is a spectral curve of a doubly periodic solution of sinh-Gordon equation the following holds.*

- *Σ possesses two additional anti-holomorphic involutions η and $\rho = \eta \circ \sigma = \sigma \circ \eta$. η has no fixed points and $\eta(y^+) = y^-$.*
- *There exists two non vanishing holomorphic functions μ_1, μ_2 on $\Sigma \setminus \{y^+, y^-\}$ with $\sigma^*\mu_j = \mu_j^{-1}$, $\eta^*\bar{\mu}_j = \mu_j$, $\rho^*\bar{\mu}_j = \mu_j^{-1}$. These are the eigenvalues of the monodromy with respect to lattice basis $p_1, p_2 \in \Gamma$*
- *The 1-forms $\theta_j := \mathrm{d}\ln\mu_j$ are meromorphic differentials with double poles on $y^\pm$ and zero residues.*
- *The principal parts at y^+ respectively y^- of θ_j are linearly independent.*

With above properties there is still some freedom left for κ. We fix $kappa$ such that it fulfills

$$\sigma^*\kappa = \kappa,\ \eta^*\bar{\kappa} = \kappa,\ \rho^*\bar{\kappa} = \kappa.$$

The spectral curve which is a hyperelliptic surface is then given by the equation

$$\nu^2 = (\kappa^2 + 1)a(\kappa).$$

Here $a(\kappa)$ is a polynomial defined by

$$a(\kappa) = \prod_{l=1}^{g}(\kappa - \alpha_l)(\kappa - \bar{\alpha}_l)$$

with pairwise different branch points $\alpha_1, \dots, \alpha_g \in \{\kappa \in \mathbb{C} : \mathrm{Im}(\kappa) > 0\}$. Therefore we have $\eta^*\bar{a} = a$ and $\rho^*\bar{a} = a$. There are no branch points on the real line and for κ on the real line we have $a(\kappa) > 0$. Finally the following transformations hold

$$\eta^*\bar{\nu} = -\nu,\ \rho^*\bar{\nu} = \nu,\ \rho^*\nu = -\nu.$$

On can also see that a is a polynomial with only real coefficients and $\deg(a) = 2g$ and a leading coefficient $a_{2g} = 1$.

3.2 Intrinsic closing conditions

Definition 3.2 (Adapted canonical basis)**.** Let $A_1, \dots, A_g, B_1, \dots, B_g \in H_1(\Sigma, \mathbb{Z})$ be a canonical basis for the homology of Σ such that $\rho^*(A_l) \equiv -A_l, \rho^*(B_l) \equiv B_l$

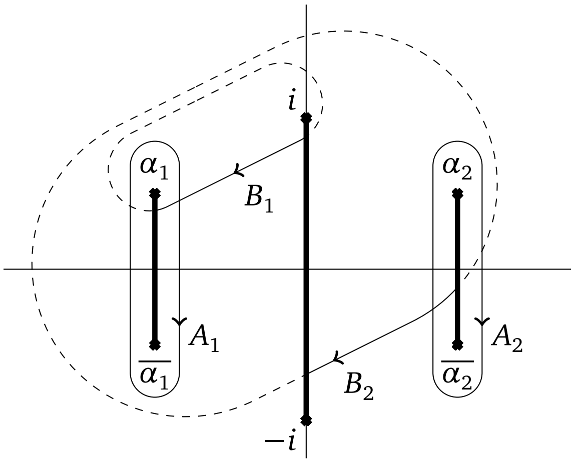


Figure 3-1: Branch points and cuts of the spectral curve Σ in the κ-plane together with the adapted canonical basis for the homology. Thick lines represent cuts and dashed lines are parts of the cycles which are on the second sheet of the hyperelliptic Riemann surface.

mod $\langle A_1, \ldots A_g \rangle$ and the projection of each A_l to $\mathbb{CP}^1$ has winding number one around $\alpha_l, \bar{\alpha}_l$ and zero the around other roots of a. $A_1, \ldots, A_g, B_1, \ldots, B_g$ will be called adapted canonical basis.

We will now investigate the space of 1-forms on Σ.

Definition 3.3. Let $\mathfrak{B}(\Sigma)$ be the vector space of meromorphic differentials on Σ which have double poles only on $y^{\pm}$ and are holomorphic elsewhere.

By Riemann-Roch the space $\mathfrak{B}(\Sigma)$ of such differentials on the hyperelliptic surface Σ has complex dimension $g+2$. The 1-forms

$$\frac{\kappa^l}{\nu(\kappa^2+1)}\mathrm{d}\kappa, \quad l = 0, \ldots, g+1 \tag{3.2}$$

form a basis for this vector space.

So we can write all those 1-forms as

$$\theta_j = \frac{i b_j(\kappa)}{\nu(\kappa^2+1)}\mathrm{d}\kappa \tag{3.3}$$

with some complex polynomial b_j of degree $\deg(b_j) = g+1$. The reason for the factor i will become apparent in a moment.

Our aim is to investigate Riemann surfaces with attached differentials which are potential candidates for spectral curves to double periodic solutions of the sinh-

Gordon equations. This mandates that the 1-forms θ_j are equal to $d\ln\mu_j$ for holomorphic functions μ_j on $\Sigma\setminus\{y^+,y^-\}$. This is equivalent to the following conditions

Definition 3.4 (Intrinsic closing conditions)**.**

$$\int_{A_l}\theta_j\in 2\pi i\mathbb{Z},\ \int_{B_l}\theta_j\in 2\pi i\mathbb{Z} \tag{3.4}$$

We recall the transformation behavior of μ_j under the involutions σ,η,ρ. If θ_j are equal to $d\ln\mu_j$ this would lead to corresponding transformation behavior of θ_j under those involutions. So we pose this transformation behavior as a condition on θ_j

$$\sigma^*\theta_j=-\theta_j,\ \eta^*\bar{\theta}_j=\theta_j,\ \rho^*\bar{\theta}_j=-\theta_j. \tag{3.5}$$

This in turn leads to a transformation behavior of b_j, in particular we obtain $\overline{b_j(\bar{\kappa})}=b_j(\kappa)$. We conclude that all coefficients of b_j needs to be real. This reduces the vector space of possible b_j from complex $g+2$-dimensional vector space to a real $g+2$-dimensional vector space.

We now combine the transformation behavior of $\theta_j\in\mathfrak{B}(\Sigma)$ with the transformation behavior of the cycles A_l and we obtain

$$\overline{\int_{A_l}\theta}=\int_{A_l}\overline{\theta}=-\int_{A_l}\rho^*\theta=-\int_{\rho^*A_j}\theta=\int_{A_l}\theta. \tag{3.6}$$

So $\int_{A_l}\theta\in\mathbb{R}$. This means in order to satisfy (3.4), we need in fact

$$\int_{A_l}\theta=\int_{A_l}\frac{ib_j(\kappa)}{\nu(\kappa^2+1)}\mathrm{d}\kappa=0. \tag{3.7}$$

These are g linearly independent real conditions on the $(g+2)$-dimensional space of real polynomials b_j with $\deg(b_j)=g+1$.

Definition 3.5. Let $\mathfrak{B}_A(\Sigma)$ be the subspace of $\mathfrak{B}(\Sigma)$ which contains all 1-forms that fulfill the transformation conditions (3.5) and have zero periods around the A-cycles (3.7).

The space $\mathfrak{B}_A(\Sigma)$ is a real 2-dimensional space.

Looking at the cycles B_l and $\theta_j\in\mathfrak{B}_A(\Sigma)$ we obtain

$$\overline{\int_{B_l}\theta}=\int_{B_l}\overline{\theta}=-\int_{B_l}\rho^*\theta=-\int_{\rho^*B_l}\theta. \tag{3.8}$$

Using $\int_{A_l} \theta = 0$, we have

$$\overline{\int_{B_l} \theta} = -\int_{\rho^* B_l} \theta = \int_{B_l} \theta \tag{3.9}$$

since $\rho^*(B_l) \equiv B_l \mod \langle A_1, \dots A_g \rangle$. So $\int_{B_l} \theta \in i\mathbb{R}$. So all $\theta_j \in \mathfrak{B}_{\mathfrak{A}}(\Sigma)$ have purely imaginary periods around B-cycles.

If the spectral curve corresponds to a doubly periodic solution of the sinh-Gordon equation, so (3.4) is fulfilled, this space is spanned by b_1, b_2 whose periods lie in $2\pi i\mathbb{Z}$.

Furthermore we have seen that a spectral curve, which consists of a hyperelliptic surfaces and two differentials on this surface is completely determined by the coefficients of the polynomials a and b_j. These are $2g + 2(g+2) = 4g + 4$ real parameters.

Definition 3.6 (Spectral Data)**.** We call the vector $(a, b_1, b_2) \in \mathbb{R}^{4g+4}$ consisting of the real coefficients of the polynomials a and b_j spectral data.

Remark. If (a, b_1, b_2) fulfills intrinsic closing conditions (3.4), then $(a, n_1 b_1, n_2 b_2)$, $n_1, n_2 \in \mathbb{Z}$ will also fulfill those closing conditions. This corresponds to taking $n_1 p_1$ and $n_2 p_2$ as the basis vectors for the fundamental domain instead of p_1 and p_2.

3.3 Extrinsic closing conditions

A spectral curve fulfilling (3.4) corresponds to a doubly periodic solution of the sinh-Gordon equation. It gives rise to a patch of a CMC surface via the Sym-Bobenko formula (Section 2.3). We are interested in CMC tori and CMC cylinders with doubly periodic metric so we need additional closing conditions which we already have established in section 2.4. For CMC immersions to $\mathbb{R}^3$ the so called extrinsic closing conditions amount to

$$\ln \mu_j \Big|_{\kappa=0} \in \pi i \mathbb{Z} \quad \text{and} \quad \partial_\kappa \ln \mu_j \Big|_{\kappa=0} = 0. \tag{3.10}$$

These conditions correspond to similar conditions on θ_j. In order to write those down we need a path γ on the Riemann surface Σ between the two points over $\kappa = 0$. The conditions (3.10) are then equivalent to

$$\int_\gamma \theta_j \in \pi i \mathbb{Z} \quad \text{and} \quad \theta_j \Big|_{\kappa=0} = 0. \tag{3.11}$$

Remark. If (a, b_1, b_2) fulfills the intrinsic (3.4) and extrinsic (3.10) closing conditions, then $(a, n_1 b_1, n_2 b_2)$, $n_1, n_2 \in \mathbb{Z}$ will also fulfill those closing conditions. Again this corresponds to taking n_1p_1 and n_2p_2 as the basis vectors for the fundamental domain instead of p_1 and p_2. The geometric meaning for the immersion is that we have a surface which will be multiply wrapped around the directions of its periods.

3.4 Branch points of the spectral curve and integrals over cycles

In the previous sections we have established how a hyperelliptic Riemann surface gives rise to a space of meromorphic differentials of certain kind and what are the intrinsic and extrinsic closing conditions on those differentials so that the Riemann surfaces becomes a spectral curve of a doubly periodic solution of the sinh-Gordon equation or even a CMC cylinder or a CMC torus. We will investigate this relationship further by using a map similar to the one defined by Jaggy [20].

We can use the results of the last two section to obtain a map from the set of branch points of the spectral curve $\{\alpha_1, \ldots, \alpha_g\}$ (we do not inlcude $y^{\pm}$ here) to a two plane of differentials which have zero periods around A-cycles

$$\{\alpha_1, \ldots, \alpha_g\} \mapsto \mathfrak{B}_A(\Sigma) \tag{3.12}$$

On the other hand there is a map from $\theta_j \in \mathfrak{B}_A(\Sigma)$ to a vector in $\mathbb{R}^{g+2}$. We define such a vector by

$$v_j := i(\theta_j(0), \int_\gamma \theta_j, \int_{B_1} \theta_j, \ldots, \int_{B_g} \theta_j). \tag{3.13}$$

The path γ connects the two points over $\kappa = 0$ on the Riemann surface Σ as before. The vectors v_j depend linearly on θ_j. The two vectors v_1 and v_2 are themselves linearly independent when θ_j are linearly independent.

We can now combine these two maps to a map from a set of branch points $\{\alpha_1, \ldots, \alpha_g\}$ to a two plane spanned by v_1 and v_2. We obtain a map

$$\begin{aligned} \mathfrak{f} : \mathbb{C}^g &\to Gr(2, \mathbb{R}^{g+2}) \\ \{\alpha_1, \ldots, \alpha_g\} &\mapsto \mathbb{R}v_1 \oplus \mathbb{R}v_2. \end{aligned} \tag{3.14}$$

One observes that this definition relies on cycles A_l,B_l. Further $\mathfrak{f}$ is multivalued function generally. In order to deal with that we restrict $\mathfrak{f}$ to a small open neighborhood $U \subset \mathbb{C}^g$ of $\{\alpha_1, \ldots, \alpha_g\}$, where all α_l are distinct. The cycles A_l,B_l can then be chosen in such a way that they are constant in κ-plane for all $\alpha \in U$. The function f is then nicely defined and is single valued.

First important observation is that if $\{\alpha_1, \ldots, \alpha_g\}$ maps to a $\mathbb{R}v_1 \oplus \mathbb{R}v_2$ which contains two linearly independent vectors $\tilde{v}_1, \tilde{v}_2 \in \mathbb{Q}^2$ and both v_1 and v_2 have a 0 in their first entry, then all intrinsic and extrinsic closing conditions are fulfilled and thus the Riemann surface Σ corresponding to $\{\alpha_1, \dots, \alpha_g\}$ is the spectral curve of a torus in $\mathbb{R}^3$.

The real dimension of $U \subset \mathbb{C}^g$ is $2g$ and this equals to the real dimension $\dim(Gr(2, \mathbb{R}^{g+2})) = 2(g+2-2) = 2g$. So there is a hope that the map f is at least locally invertible. In fact we have the following result which shows the necessary conditions.

Theorem 3.7. *Let $\alpha = \{\alpha_1, \dots, \alpha_g\}$ be in U. Let Σ be a hyperelliptic surface defined by*

$$\Sigma : \nu^2 = (\kappa^2 + 1) \prod_{l=1}^{g} (\kappa - \alpha_l)(\kappa - \bar{\alpha}_l) \tag{3.15}$$

and assume that for $\theta_1, \theta_2 \in \mathfrak{B}_A(\Sigma)$ the following conditions hold

- *θ_1 and θ_2 have a simple zero at $\kappa = 0$,*
- *θ_1 and θ_2 have no further common zeros,*
- *$\theta_j(\alpha_l) \neq 0$ for all $l \in 1, \cdots, g$,*

then $\mathrm{d}\mathfrak{f}(\alpha)$ *is invertible for* $\mathfrak{f}$ *as defined in (3.14).*

This theorem is stated by Jaggy [20], the proof is due to Kirchver [22], Bikbaev and Kuskin[3].

An immediate consequence of this theorem is that arbitrary close to any spectral curve fulfilling the assumptions of the theorem there exist a spectral curve which additionally fulfill the extrinsic and intrinsic closing condition and thus is a spectral curve of a torus.

Loosely speaking the big assumption of the theorem is that the differentials have a common root at $\kappa = 0$ and thus already fulfill this closing condition, as this is not a dense condition. The other assumptions are of technical nature. And since f is locally invertible around such a point we can fulfill all the other closing conditions, as those are dense conditions.

In the following chapter we will use the map $\mathfrak{f}$ to introduce a deformation of spectral curves known as Whitham deformation. The Theorem 3.7 will not be necessary as we will establish the non singular nature of the deformation for initial values satisfying similar assumptions to those in the theorem without relying on this result and with different methods compared to those used in proving the theorem.

4 Whitham deformations

The Whitham deformation was first introduced for the Korteweg–de Vries equation by Whitham [27]. Krichever [22] used it on spectral curves. This theory was further developed and applied to the sinh-Gordon equation and to CMC cylinders and CMC tori by Grinevich and Schmidt [15].

In this chapter we will introduce the Whitham deformation as a natural way to define a deformation on spectral data in such a way that one has control over various closing conditions. Generically one would expect that such deformations are a hard to study object involving integrals over cycles for the definition. We will see that in case of the sinh-Gordon equation with finite spectral genus these equations will become differential algebraic equation on spectral parameters.

We will work out the conditions for when those deformations become singular. We will see what are the conditions for a possible bifurcation to higher spectral genus and how to obtain such spectral data by Whitham deformations. Then we will desingularize the spectral data at a bifurcation point allowing us to deform to higher spectral genus. At the end of the chapter we will reprove the result of Jaggy [20] using our results about Whitham deformations. We will close this treatment by demonstrating the constructive nature of our proof by showing a numerical example of a flow connecting a Wente torus of spectral genus 2 with a Dobriner torus of spectral genus 3.

4.1 Deformations of spectral curves

As a motivation we recall from the last chapter 3.4 the map

$$\begin{aligned} \mathfrak{f} : \mathbb{C}^g &\to Gr(2, \mathbb{R}^{g+2}) \\ \{\alpha_1, \dots, \alpha_g\} &\mapsto \mathbb{R}v_1 \oplus \mathbb{R}v_2, \end{aligned} \tag{4.1}$$

with $v_j = i(\theta_j(0), \int_\gamma \theta_j, \int_{B_1} \theta_j, \dots, \int_{B_g} \theta_j)$ and the Theorem 3.7 which established that $d\mathfrak{f}$ is invertible in some neighborhood of α which satisfies certain conditions.

One way to understand this is in terms of ordinary differential equations. Namely for initial conditions $\alpha(t_0)$ satisfying the assumptions of Theorem 3.7 one can prescribe a path in the space of $\mathbb{R}v_1 \oplus \mathbb{R}v_2$ starting with $\mathfrak{f}(\alpha(t_0))$. Then it is possible to integrate this path to a path $\alpha(t)$ in the domain of $\mathfrak{f}$.

In the following we will choose a basis θ_1, θ_2 of $\mathfrak{B}_A(\Sigma)$ and accordingly the map $\mathfrak{f}$ will be considered as a map $\mathfrak{f} : \mathbb{C}^g \to \mathbb{R}^{2(g+2)}$, $\{\alpha_1, \ldots, \alpha_g\} \mapsto (v_1, v_2)$. We will now write down the differential equations which define a flow as the one sketched above. The data we are operating on is the spectral data consisting of the polynomials a encoding the branch points α_l and the polynomials b_j encoding the differentials θ_j in the way defined in section 3.2.

The map $\mathfrak{f}$ was defined by using $\theta_1, \theta_2 \in \mathfrak{B}_A(\Sigma)$ so we need to make sure our chosen basis of differentials stays in this subspace

$$\partial_t \int_{A_l} \theta_j = 0. \tag{4.2}$$

Then we need to follow the path $(v_1(t), v_2(t))$ and obtain

$$i(\partial_t \theta_j(0), \partial_t \int_\gamma \theta_j, \partial_t \int_{B_1} \theta_j, \ldots, \partial_t \int_{B_g} \theta_j) = \partial_t v_j. \tag{4.3}$$

The Theorem 3.7 ensures that for given data (a, b_1, b_2) and $(\partial_t v_1, \partial_t v_2)$ there is a unique $(\partial_t a, \partial_t b_1, \partial_t b_2)$ which defines an ordinary differential equation. The relation between $(a, b_1, b_2, \partial_t v_1, \partial_t v_2)$ and $(\partial_t a, \partial_t b_1, \partial_t b_2)$ governed by those equations involves integrals around cycles on the Riemann surfaces and are generally very complicated. A possible way to understand them would be to build on the work of Bobenko [4] and write them in terms of theta-functions. The following will show that those equations simplify considerably for an interesting case of the so called Whitham deformations.

The idea of Whitham deformation is to deform spectral data by only allowing deformations which preserve intrinsic closing conditions. In our language this means that $(\partial_t v_1, \partial_t v_2)$ consists almost entirely of zeros and only those components which correspond to $\theta_j(0)$ and $\int_\gamma \theta_j$, namely to the extrinsic closing conditions, are allowed to have non zero entries.

In the case of Whitham deformations the equations (4.2) and (4.3) become

$$\partial_t \int_{A_l} \theta_j = 0, \qquad \partial_t \int_{B_l} \theta_j = 0, \tag{4.4}$$

$$i\partial_t \theta_j(0) = (\partial_t v_j)_1, \qquad i\partial_t \int_\gamma \theta_j = (\partial_t v_j)_2, \tag{4.5}$$

where $(\partial_t v_j)_1$ is the first component of the vector $(\partial_t v_j)$. Recalling that $\theta_j = \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} d\kappa$, we rewrite the equations as

$$\partial_t \int_{A_l} \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = 0, \qquad \partial_t \int_{B_l} \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = 0, \tag{4.6}$$

$$i\partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)}\Big|_{\kappa=0} = (\partial_t \nu_j)_1, \qquad i\partial_t \int_{\gamma} \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = (\partial_t \nu_j)_2. \tag{4.7}$$

Since we are dealing with these equations locally, we can analyze them purely in the κ-coordinate. Furthermore, locally the A and B cycles do not depend on the deformation parameter t and we can interchange ∂_t with integration.

$$\int_{A_l} \partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = 0, \qquad \int_{B_l} \partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = 0, \tag{4.8}$$

$$i\partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)}\Big|_{\kappa=0} = (\partial_t \nu_j)_1, \qquad i\int_{\gamma} \partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = (\partial_t \nu_j)_2. \tag{4.9}$$

We observe that this equations pose a nice condition on $\partial_t \theta_j = \partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa$. Since A_l, B_l is a basis for the homology of Σ and $\partial_t \theta_j$ are zero around all these cycles and holomorphic outside of the branch points, there exists global meromorphic functions $\phi_j(\kappa)$ such that $\mathrm{d}\phi_j = \partial_t \theta_j$. All those functions can be written in the following form

$$\phi(\kappa) = \frac{ic_j(\kappa)}{\nu} \tag{4.10}$$

with polynomials c_j of degree $\deg(c_j) = g+1$. Since the integrability conditions $\partial^2_{\kappa t}\theta_j = \partial^2_{t\kappa}\theta_j$ have to hold, we obtain

$$\partial_t \frac{ib_j(\kappa)}{\nu(\kappa^2+1)} = \partial_\kappa \frac{ic_j(\kappa)}{\nu}. \tag{4.11}$$

We examine this equation further by using $\nu^2 = (\kappa^2+1)a(\kappa)$ to obtain

$$\begin{aligned} 2\dot{b}_j(\kappa)a(\kappa) - b_j(\kappa)\dot{a}(\kappa) = &- 2\kappa a(\kappa)c_j(\kappa) \\ &+ (\kappa^2+1)(2a(\kappa)c_j'(\kappa) - a'(\kappa)c_j(\kappa)) \end{aligned} \tag{4.12}$$

As we want to maintain the reality of the coefficients of a and b_j, we see that we need also the coefficients of c_j to be real. Further, we see that for each j (4.12) is an equation on polynomials in κ of degree $3g+1$. By equating the coefficients we obtain $3g+2$ equations on the coefficients of a, b_j, and c_j. In total $6g+4$ equations on $6g+8$ real parameters. The remaining 4-dimensional freedom corresponds to the freedom of setting $(\partial_t \nu_j)_1$ and $(\partial_t \nu_j)_1$.

This shows that in case of Whitham deformations the equations (4.2) and (4.3) which are non-algebraic, transform to (4.12) which is a system of differential algebraic equations. The above argument can be reversed by assuming that there exist polynomials c_j such that the integrability conditions (4.11) are fulfilled. It follows then that such a deformation must obey (4.6) and thus is a Whitham deformation. We will take this approach and show in this work that there exist polynomials c_j which solve (4.12) and equivalently (4.11) under the same assumptions as in Theorem 3.7. This means that our further treatment will not rely on the work of Jaggy [20] and in particular on the Theorem 3.7 to show the existence of the Whitham flow. Nevertheless we used this theorem as it provides a very nice motivation for the definition of the Whitham flow and explains differential algebraic nature of Whitham flow equations.

4.2 Whitham deformations of cylinders

We have seen that the integrability conditions on a, b_j, and c_j give rise to $6g+4$ equations on $6g+8$ real parameters. These equations come from intrinsic closing conditions, we did not include any of the extrinsic closing conditions so far.

Preserving the extrinsic closing conditions (3.10) amounts to

$$\partial_t \theta_j \Big|_{\kappa=0} = 0 \quad \text{and} \quad \partial_\kappa \theta_j \Big|_{\kappa=0} = 0. \tag{4.13}$$

This translates directly to conditions on b_j and c_j

$$c_j(0) = 0 \quad \text{and} \quad b_j(0) = 0. \tag{4.14}$$

If we want to preserve all closing conditions, we obtain an ODE whose only solution is the constant one. The proof of this statement is a straight forward application of the Theorem 4.3 which we will proof when we investigate the Whitham equations. This indicates that the space of the CMC tori in $\mathbb{R}^3$ is discrete.

So there are no non-trivial deformations along CMC tori in $\mathbb{R}^3$, but we can try to relax the extrinsic closing conditions thus deforming along CMC cylinders with doubly periodic metric. The goal will be to control the closing conditions in such a way that one can prove that for some time t after starting the flow all closing conditions will be fulfilled again and we arrive at a spectral curve of a CMC torus.

A good candidate for a flow through CMC cylinders with a control on closing condition would be the following ansatz

$$\begin{aligned} b_1(0) &= 0 \quad b_2(0) = 0 \\ c_1(0) &= 0 \quad c_2(0) = \nu \neq 0. \end{aligned} \tag{4.15}$$

We maintain three of the four conditions. For the last closing condition we pick some function ν which can vary with t but fulfills $\nu(t) \neq 0$ for all t This means that we vary $\theta_2|_{\kappa=0}$. The condition $\theta_2|_{\kappa=0} \in \pi i\mathbb{Z}$ is in fact a rational condition if we allow for integer multiples of θ_j. So a flow which continuously varies $\theta_2|_{\kappa=0}$ and ensures that all the other conditions are fulfilled, will produce spectral curves fulfilling all closing conditions on a dense subset of its interval of existence.

4.3 Matrix equation of the flow

We will now take a closer look at the deformation equations and bring them to a form which is better suited for further analysis.

First look at the equations which come from the integrability conditions:

$$-b_j(\kappa)\dot{a}(\kappa) + 2a(\kappa)\dot{b}_j(\kappa) + 2\kappa a(\kappa)c_j(\kappa) - (\kappa^2+1)2a(\kappa)c_j'(\kappa) \\ + (\kappa^2+1)a'(\kappa)c_j(\kappa) = 0. \tag{4.16}$$

We observe that (4.16) is a sum of products, each product having two different polynomials as factors, one coming from the spectral data a, a', b_j and one from the unknowns $\dot{a}, \dot{b}_j, c, c'$.

We now compute each product in terms of κ^i and the coefficients of the other involved polynomials. Taking the degrees of the monomials into consideration, we express them together with their polynomials as

$$a(\kappa) = \kappa^{2g} + \sum_{i=0}^{2g-1} a_i\kappa^i \qquad \dot{a}(\kappa) = \sum_{i=0}^{2g-1} \dot{a}_i\kappa^i \qquad a'(\kappa) = 2g\kappa^{2g-1} + \sum_{i=0}^{2g-2}(i+1)a_{i+1}\kappa^i \tag{4.17}$$

$$b_j(\kappa) = \sum_{i=0}^{g+1} b_{j,i}\kappa^i \qquad \dot{b}_j(\kappa) = \sum_{i=0}^{g+1} \dot{b}_{j,i}\kappa^i \tag{4.18}$$

$$c_j(\kappa) = \sum_{i=0}^{g+1} c_{j,i}\kappa^i \qquad c_j'(\kappa) = \sum_{i=0}^{g}(i+1)c_{j,i+1}\kappa^i \tag{4.19}$$

We already know that all the coefficients of the above polynomials are real.

Note that a is always a monic polynomial, so that we can put $a_{2g} = 1$ and $\dot{a}_{2g} = 0$

and thus

$$a(\kappa) = \sum_{i=0}^{2g} a_i \kappa^i \qquad a'(\kappa) = \sum_{i=0}^{2g-1} (i+1) a_{i+1} \kappa^i. \tag{4.20}$$

$$\dot{a}(\kappa) = \sum_{i=0}^{2g-1} \dot{a}_i \kappa^i$$

We now start the computations of terms in (4.16):

$$-b_j(\kappa)\dot{a}(\kappa) = \sum_{i=0}^{3g} \Big(\sum_{l=\max(0,i-(2g-1))}^{\min(g+1,i)} -b_{j,l}\dot{a}_{i-l}\kappa^i \Big). \tag{4.21}$$

The next summand is

$$2a(\kappa)\dot{b}_j(\kappa) = \sum_{i=0}^{3g+1} \Big(\sum_{l=\max(0,i-(g+1))}^{\min(2g,i)} 2a_l \dot{b}_{j,i-l}\kappa^i \Big). \tag{4.22}$$

The last three summands compute to

$$2a(\kappa)c_j(\kappa) = \sum_{i=0}^{3g+1} \Big(\sum_{l=\max(0,i-(g+1))}^{\min(2g,i)} 2a_l c_{j,i-l}\kappa^i \Big), \tag{4.23}$$

$$-2a(\kappa)c_j'(\kappa) = \sum_{i=0}^{3g} \Big(\sum_{l=\max(0,i-g)}^{\min(2g,i)} -2(i-l+1)a_l c_{j,i-l+1}\kappa^i \Big), \tag{4.24}$$

$$a'(\kappa)c_j(\kappa) = \sum_{i=0}^{3g} \Big(\sum_{l=\max(0,i-(g+1))}^{\min(2g-1,i)} (l+1)a_{l+1} c_{j,i-l}\kappa^i \Big). \tag{4.25}$$

The equation (4.16), demanding that a polynomial in κ is zero, is equivalent to a set of equations on the coefficients of this polynomial, namely that those are zero. We will now construct these equations by collecting all coefficients of κ^i for a fixed i using (4.21)–(4.28).

The coefficients of κ^i in (4.21) and (4.22) involve sums of products of coefficients from two polynomials, one from the given spectral data and one unknown polynomial. The last three expressions (4.26)– (4.28) involve coefficients from given polynomial a and the unknown polynomial c. We will combine these three expressions using additional factors κ and (κ^2+1) as they occur in (4.16):

$$2\kappa a(\kappa)c_j(\kappa) = \sum_{i=1}^{3g+2} \Big(\sum_{l=\max(1,i-(g+1))}^{\min(2g+1,i)} 2a_{l-1} c_{j,i-l}\kappa^i \Big), \tag{4.26}$$

$$
\begin{aligned}
-2(\kappa^2+1)a(\kappa)c_j'(\kappa) = &\sum_{i=2}^{3g+2}\Big(\sum_{l=\max(1,i-(g+1))}^{\min(2g+1,i-1)} -2(i-l-1)a_{l-1}c_{j,i-l}\kappa^i\Big) \\
&+\sum_{i=0}^{3g}\Big(\sum_{l=\max(-1,i-(g+1))}^{\min(2g,i-1)} -2(i-l+1)a_{l+1}c_{j,i-l}\kappa^i\Big),
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
(\kappa^2+1)a'(\kappa)c_j(\kappa) = &\sum_{i=2}^{3g+2}\Big(\sum_{l=\max(2,i-(g+1))}^{\min(2g+1,i)} (l-1)a_{l-1}c_{j,i-l}\kappa^i\Big) \\
&+\sum_{i=0}^{3g}\Big(\sum_{l=\max(0,i-(g+1))}^{\min(2g-1,i)} (l+1)a_{l+1}c_{j,i-l}\kappa^i\Big).
\end{aligned} \tag{4.28}
$$

Therefore

$$
\begin{aligned}
&2\kappa a(\kappa)c_j(\kappa) - 2(\kappa^2+1)a(\kappa)c_j'(\kappa) + (\kappa^2+1)a'(\kappa)c_j(\kappa) \\
&\quad = \sum_{i=1}^{3g+2}\Big(\sum_{l=\max(1,i-(g+1))}^{\min(2g+1,i)} (-2i+3l+1)a_{l-1}c_{j,i-l}\kappa^i\Big) \\
&\quad + \sum_{i=0}^{3g}\Big(\sum_{l=\max(-1,i-(g+1))}^{\min(2g,i)} (-2i+3l+1)a_{l+1}c_{j,i-l}\kappa^i\Big).
\end{aligned} \tag{4.29}
$$

We observe that for a given κ^i the coefficients depend linearly on the vector of coefficients of the polynomials in the spectral data and also linearly on the vector of coefficients of the unknown polynomial describing the t-derivative in the flow ODE.

Combining the previous results, we can rewrite the two equations from the integrability condition (4.16) as a matrix equation.

Proposition 4.1. *The two equations*

$$
\begin{aligned}
2\dot{b}_j(\kappa)a(\kappa) - b_j(\kappa)\dot{a}(\kappa) = &-2\kappa a(\kappa)c_j(\kappa) \\
&+(\kappa^2+1)(2a(\kappa)c_j'(\kappa) - a'(\kappa)c_j(\kappa))
\end{aligned} \tag{4.30}
$$

are equivalent to the matrix equation

$$
M(X)Y = 0 \tag{4.31}
$$

Where

- *X is the vector of coefficients $(a_i, b_{1,i}, b_{2,i})$ of the polynomials (a, b_1, b_2)*
- *Y is the vector of coefficients $(\dot{a}_i, \dot{b}_{1,i}, \dot{b}_{2,i}, c_{1,i}, c_{2,i})$ of the polynomials $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$*

- *M is a $(6g+4)\times(6g+8)$ block matrix depending linearly on X*

$$M=\begin{pmatrix} B_1 & A_1 & 0 & A_2 & 0 \\ B_2 & 0 & A_1 & 0 & A_2 \end{pmatrix}. \tag{4.32}$$

Here B_j is the $(3g+2)\times(2g)$-matrix

$$B_j=-\begin{pmatrix} b_{j,0} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ b_{j,g+1} & \cdots & b_{j,0} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & b_{j,g+1} & \cdots & b_{j,0} & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & b_{j,g+1} & \cdots & \cdots & b_{j,0} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_{j,g+1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \tag{4.33}$$

A_1 the $(3g+2)\times(g+2)$-matrix

$$A_1=2\begin{pmatrix} a_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ a_{2g-1} & \cdots & a_0 & 0 & \cdots & \cdots & \cdots & 0 \\ 2 & a_{2g-1} & \cdots & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & a_{2g-1} & \cdots & a_0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & 2 & a_{2g-1} & \cdots & \cdots & a_0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2 & a_{2g-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2 \end{pmatrix}, \tag{4.34}$$

and A_2 the $(3g+2)\times(g+2)$-matrix

$$A_2=\begin{pmatrix} 1a_1 & -2a_0 & 0 & \cdots & 0 & 0 & 0 \\ 2(a_3+a_0) & -1a_1 & -4a_0 & \ddots & 0 & 0 & 0 \\ 3(a_4+a_2) & 0(a_3+a_0) & -3a_1 & & 0 & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 2g+2 & (2g-1)a_{2g-1} & (2g-4)(1+a_{2g-2}) & & 0 & & 0 \\ 0 & 2g & (2g-3)a_{2g-1} & \ddots & 0 & & 0 \\ 0 & 0 & 2g-2 & & 0 & & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \ddots & -(2g-2)a_0 & 0 & 0 \\ 0 & 0 & 0 & & -(2g-3)a_1 & -2ga_0 & 0 \\ 0 & 0 & 0 & & -(2g-4)(a_3+a_0) & -(2g-1)a_1 & -(2g+2)a_0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & & 3a_{2g-1} & 0(1+a_{2g-2}) & -3(a_{2g-1}+a_{2g-3}) \\ 0 & 0 & 0 & & 4 & 1a_{2g-1} & -2(1+a_{2g-2}) \\ 0 & 0 & 0 & \cdots & 0 & 2 & -1a_{2g-1} \end{pmatrix}. \tag{4.35}$$

Since the coefficients of $M(X)$ depend linearly on the entries of the vector X we obtain the following useful representation of this matrix.

Proposition 4.2. *The matrix $M(X)$ can be written as $M(X) = M_0 + M_1(X)$, where M_0 is a $(6g+4)\times(6g+8)$-matrix with constant entries of $M(X)$, and M_1 is a constant $(6g+4)\times(6g+8)\times(4g+4)$ tensor capturing the linear dependence of M on X.*

It is easy to incorporate the extrinsic closing conditions into this matrix equation. The first three conditions

$$b_1(0) = 0, \quad b_2(0) = 0, \quad c_1(0) = 0 \tag{4.36}$$

are represented by three additional rows with 1 on positions corresponding to $\dot{b}_{1,0}, \dot{b}_{2,0}, c_{1,0}$ and 0 elsewhere. With those additional rows we obtain a new $(6g + 7) \times (6g + 8)$ matrix. Generally this matrix will have a 1-dimensional kernel and by setting the fourth condition $c_2(0) = v$, or equivalently $c_{2,0} = v$, we pick one element from this kernel as Y and obtain a well defined ODE.

Equivalently we can also remove the columns corresponding to $\dot{b}_{1,0}, \dot{b}_{2,0}, c_{1,0}$ and set $b_{1,0} = b_{2,0} = c_{1,0} = 0$ obtaining a $(6g+4)\times(6g+5)$ matrix with the same kernel as the $(6g+7)\times(6g+8)$ matrix after omitting $\dot{b}_{1,0}, \dot{b}_{2,0}, c_{1,0}$. We will usually use this $(6g+4)\times(6g+5)$ matrix in later computations and still call it M if there is no chance of confusion.

4.3.1 Determinant of the flow equation

The equation for the Whitham deformation flow can be written as $M(X)Y = 0$ where $M(X)$ is $(6g + 7) \times (6g + 8)$ matrix. We choose a function v and asking for $c_{2,0} = v$. In order for the equation to reflect this, we have to add a row to $M(X)$ with 1 on the position corresponding to $c_{2,0}$ and 0 elsewhere. We can then write $M(X)Y = V$ where $V = (0, \dots, 0, v)$. Since $M(X)$ is now a $(6g + 8) \times (6g + 8)$ square matrix we can look at $\det(M(X))$. The flow equations determine a unique $\dot{X}$ only when $M(X)$ is invertible, i.e. $\det(M(X)) \neq 0$. If $\det(M(X)) = 0$ there is no unique $\dot{X}$ and we will call such X singular points of the Whitham flow. Naturally it is important to analyze and understand $\det(M(X))$.

Obviously $\det(M(X))$ can be computed by using Laplace expansion. We will neglect possible sign changes of $\det(M(X))$ in the following operations to simplify the argument, since we are only interested in $\det(M(X))$ up to some constant factor. When doing the Laplace expansion we observe that $M(X)$ contains 4 rows with a 1 as the sole non zero entry. Those rows are coming from the closing conditions $\dot{b}_{1,0} = \dot{b}_{2,0} = c_{1,0} = 0$ and $c_{2,0} = v$, which involve only one element from Y. This observation let us remove those 4 rows and the 4 columns where those rows contain a 1. The 4 columns are the first columns of both A_1 blocks and the first of the A_2

blocks as defined in Proposition 4.1. Thous we arrive at a $(6g+4)\times(6g+4)$ matrix with the same determinant as the original $(6g+8)\times(6g+8)$.

In the next step we use the condition $b_{1,0} = b_{2,0} = 0$. This means that the first rows of the B_1 and B_2 block contains only zeros. Since we also remove the first columns of the A_1 blocks, those have also only zeros in the first row. The A_2 blocks have 2 non zero entries in their first row, but they also had the first row removed in the first step. This means the only non zero element of the first row in both half of $M(X)$ is $-2a_0$. We remove those rows and the second columns of A_2 which contained those non zero entries. Lets us call the resulting $(6g+2)\times(6g+2)$ matrix $\tilde{M}(X)$. We observe that $\det(M(X)) \sim a_0^2 \det(\tilde{M}(X))$, where by $\sim$ we mean up to a non-zero constant factor.

An equivalent way to define $\tilde{M}(X)$ is by remembering that $M(X)$ captures the coefficients of $\dot{a}$, $\dot{b}_1$, $\dot{b}_2$, c_1, c_2 in the expression

$$\begin{aligned} m_j(\kappa) := &-b_j(\kappa)\dot{a}(\kappa) + 2a(\kappa)\dot{b}_j(\kappa) \\ &+2\kappa a(\kappa)c_j(\kappa) + (\kappa^2+1)(a'(\kappa)c_j(\kappa) - 2a(\kappa)c_j'(\kappa)) \end{aligned} \tag{4.37}$$

After setting $b_{1,0} = b_{2,0} = 0$ and $\dot{b}_{1,0} = \dot{b}_{2,0} =$ and $c_{1,0} = c_{1,1} = 0$, $c_{2,0} = c_{2,1} = 0$ one can write $b_j = \kappa\tilde{b}_j$, $\dot{b}_j = \kappa\dot{\tilde{b}}_j$, $c_j = \kappa^2\tilde{c}_j$ and we see that we can now divide out a common factor κ from (4.37) to obtain

$$\begin{aligned} \tilde{m}_j(\kappa) := &-\tilde{b}_j(\kappa)\dot{a}(\kappa) + 2a(\kappa)\dot{\tilde{b}}_j(\kappa) \\ &-2(\kappa^2+2)a(\kappa)\tilde{c}_j(\kappa) + \kappa(\kappa^2+1)(a'(\kappa)\tilde{c}_j(\kappa) - 2a(\kappa)\tilde{c}_j'(\kappa)) \end{aligned} \tag{4.38}$$

We see that $\tilde{M}(X)$ as defined above is capturing the coefficients of $\dot{a}$, $\dot{\tilde{b}}_1$, $\dot{\tilde{b}}_2$, $\tilde{c}_1$, $\tilde{c}_2$ in that expression. The fact that $\det(M(X)) \sim a_0^2 \det(\tilde{M}(X))$ means that (4.38) exhibits the same singular behavior as (4.37), the only possible exception being at $a_0 = 0$ where (4.37) is singular and (4.38) might be regular.

So we have a $(6g+2)\times(6g+2)$ matrix $\tilde{M}(X)$ with entries which are linear in the entries of X. We observe that $\det(\tilde{M}(X))$ is at most a polynomial of degree $6g+2$ in the entries of X. In the following, we will determine this polynomial up to a constant factor by finding several polynomials of lower degree which divide $\det(\tilde{M}(X))$. Those polynomials will be mostly resultants and discriminants of polynomials $a(\kappa)$, $b_j(\kappa)$. A short overview of some general facts about resultants res and discriminants Δ can be found in [14]. In particular, we will need the fact that the determinant of the Sylvester matrix of two polynomials equals to the resultant of these two polynomials. We will also use the following relations

$$\begin{aligned}\operatorname{res}(f\,g,h) &= \operatorname{res}(f,h)\operatorname{res}(g,h),\\ \Delta(f) &\sim \operatorname{res}(f,f'),\\ \Delta(f\,g) &\sim \Delta(f)\,\Delta(g)\operatorname{res}(f,g)^2.\end{aligned} \tag{4.39}$$

We check $\tilde{m}_j$ at $\kappa = 0$ with $a_0 = 0$ and obtain $\tilde{m}_j(0) = -\tilde{b}_j(0)\dot{a}(0)$. This leads us to

$$\tilde{b}_2(0)\tilde{m}_1(0) - \tilde{b}_1(0)\tilde{m}_2(0) = 0. \tag{4.40}$$

So we found a linear combination of rows in $\tilde{M}(X)$ to create a zero row. This means that $\tilde{M}(X)$ is singular if $a_0 = 0$. Which in turn means that the resultant $\operatorname{res}(a(\kappa),\kappa)$ has to be a factor of $\det(\tilde{M}(X))$.

Now we check what happens if $\kappa = \pm i$ is a root of $a(\kappa)$. Under the assumption $a(\pm i) = 0$ we obtain $\tilde{m}_j(\pm i) = -\tilde{b}_j(\pm i)\dot{a}(\pm i)$ and in turn

$$\tilde{b}_2(\pm i)\tilde{m}_1(\pm i) - \tilde{b}_1(\pm i)\tilde{m}_2(\pm i) = 0. \tag{4.41}$$

So $\tilde{M}(X)$ is here singular as well and $\operatorname{res}(a(\kappa),(\kappa^2+1))$ has to be a factor of $\det(\tilde{M}(X))$.

Lastly, we assume $\kappa = \tilde{\kappa}$ is a root of $a(\kappa)$ and $a'(\kappa)$. Under the assumption $a(\tilde{\kappa}) = a'(\tilde{\kappa}) = 0$, we obtain $\tilde{m}_j(\tilde{\kappa}) = -\tilde{b}_j(\tilde{\kappa})\dot{a}(\tilde{\kappa})$ and in turn

$$\tilde{b}_2(\tilde{\kappa})\tilde{m}_1(\tilde{\kappa}) - \tilde{b}_1(\tilde{\kappa})\tilde{m}_2(\tilde{\kappa}) = 0. \tag{4.42}$$

So $\tilde{M}(X)$ is singular when $a(\kappa)$ has a root with a multiplicity higher than 1, and $\operatorname{res}(a(\kappa),a'(\kappa))$ has to be a factor of $\det(\tilde{M}(X))$.

We also observe that rows and columns of $\tilde{M}(X)$ can be reordered in such a way that it becomes a block matrix $\left(\begin{smallmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{smallmatrix}\right)$ where M_{11} is the Sylvester matrix of the polynomials $\tilde{b}_1(\kappa)$ and $\tilde{b}_2(\kappa)$. Since $\det(\tilde{M}(X)) = \det(M_{11})\det(M_{22} - M_{12}M_{11}^{-1}M_{21})$ and $\det(M_{11}) = \operatorname{res}(\tilde{b}_1(\kappa),\tilde{b}_2(\kappa))$ is the resultant of those polynomials, we see that $\operatorname{res}(\tilde{b}_1(\kappa),\tilde{b}_2(\kappa))$ is a factor of $\det(\tilde{M}(X))$.

To conclude this argument, we collect all the factors of $\det(\tilde{M}(X))$. We found that $\operatorname{res}(a(\kappa),\kappa)$, $\operatorname{res}(a(\kappa),(\kappa^2+1))$, $\operatorname{res}(a(\kappa),a'(\kappa))$ and $\operatorname{res}(\tilde{b}_1(\kappa),\tilde{b}_2(\kappa))$ are all factors of $\det(\tilde{M}(X))$. We can combine the first three resultants to $\operatorname{res}(a(\kappa),\kappa(\kappa^2+1)a'(\kappa))$ seeing that

$$\det(\tilde{M}(X)) = c\operatorname{res}(a(\kappa),\kappa(\kappa^2+1)a'(\kappa))\operatorname{res}(\tilde{b}_1(\kappa),\tilde{b}_2(\kappa)) \tag{4.43}$$

Since $\deg(a(\kappa)) = 2g$ and $\deg(\tilde{b}_j(\kappa)) = g$ we see that the degrees of those resultants in the coefficients of X are $4g+2$ and $2g$. On the other hand $\det(\tilde{M}(X))$ has a degree $6g+2$ and thus c can only be some scale factor. We have shown

$$\det(\tilde{M}(X)) \sim \operatorname{res}(a(\kappa), \kappa(\kappa^2+1)a'(\kappa)) \operatorname{res}(\tilde{b}_1(\kappa), \tilde{b}_2(\kappa)) \tag{4.44}$$

and

$$\det(M(X)) \sim \operatorname{res}(a(\kappa), \kappa^3(\kappa^2+1)a'(\kappa)) \operatorname{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa). \tag{4.45}$$

This expression is well defined since we assumed $b_{1,0} = b_{2,0} = 0$. This way we proved the following proposition.

Proposition 4.3. *The determinant of the matrix $M(X)$ defined in Proposition 4.1 and extended by*

$$b_1(0) = 0, \quad b_2(0) = 0, \quad c_1(0) = 0, \quad c_2(0) = \nu \tag{4.46}$$

fulfills the following proportionality condition

$$\det(M(X)) \sim \operatorname{res}(a(\kappa), \kappa^3(\kappa^2+1)a'(\kappa)) \operatorname{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa) \tag{4.47}$$

where by $\sim$ we mean up to a non-zero constant factor.

We observe that $\det(M(X)) \neq 0$ for spectral data which fulfills the following properties: the polynomial a has no multiple zeros and $a(0) \neq 0$, $a(\pm i) \neq 0$, neither of the polynomials b_j has common zeros with a, the polynomials b_j have no common zeros except for $\kappa = 0$, where both b_j do have a common zero. Since $M(X)$ is invertible in this case, we obtain $Y = 0$ if and only if $c_2(0) = \nu = 0$. This proves the existence of Whitham flow with generic initial spectral data and also the statement from section 4.2 that there are non non-trivial deformations along CMC tori in $\mathbb{R}^3$.

Our main use of the formula for $\det(M(X))$ will be during the desingularization procedure at a bifurcation point to higher genus which we will perform in the section 4.5. During this procedure we will use a particular factorization of the polynomials a and b_j and we also will the first three derivatives of $\det(M(X))$ at a particular point in the subsection 4.5.3. In order to keep the computation involving $\det(M(X))$ at one place we will show these computations in the remainder of the section. We will explain the exact reason for the need of these computations in the later sections where we will use these results.

Consider the following factorization of polynomials a and b_j:

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1} \quad \text{and} \quad b_j = (\kappa - \beta_j)b_j^{g-1} \tag{4.48}$$

with $\alpha_r, \alpha_i, \beta_j \in \mathbb{R}$.

We first observe

$$\begin{aligned} \det(M(X)) &\sim \operatorname{res}(a(\kappa), \kappa^3(\kappa^2+1)a'(\kappa)) \operatorname{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa) \\ &\sim \operatorname{res}(a(\kappa), \kappa^3(\kappa^2+1))\, \Delta(a(\kappa)) \operatorname{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa). \end{aligned} \tag{4.49}$$

Next we use

$$\begin{aligned}\Delta(a(\kappa)) \sim\, & \Delta((\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1}(\kappa)) \\ \sim\, & \Delta(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)\,\Delta(a^{g-1}(\kappa))\,\mathrm{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, a^{g-1}(\kappa))^2 \\ \sim\, & \alpha_i\,\Delta(a^{g-1}(\kappa))\,\mathrm{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, a^{g-1}(\kappa))^2\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}\mathrm{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa) \sim\, & \mathrm{res}((\kappa - \beta_1)b_1^{g-1}(\kappa)/\kappa, (\kappa - \beta_2)b_2^{g-1}(\kappa)/\kappa) \\ \sim\, & \mathrm{res}(\kappa - \beta_1, \kappa - \beta_2)\,\mathrm{res}(\kappa - \beta_1, b_2^{g-1}(\kappa)/\kappa) \\ & \mathrm{res}(b_1^{g-1}(\kappa)/\kappa, \kappa - \beta_2)\,\mathrm{res}(b_1^{g-1}(\kappa)/\kappa, b_2^{g-1}(\kappa)/\kappa) \\ \sim\, & (\beta_1 - \beta_2)\,\mathrm{res}(\kappa - \beta_1, b_2^{g-1}(\kappa)/\kappa) \\ & \mathrm{res}(b_1^{g-1}(\kappa)/\kappa, \kappa - \beta_2)\,\mathrm{res}(b_1^{g-1}(\kappa)/\kappa, b_2^{g-1}(\kappa)/\kappa)\end{aligned} \tag{4.51}$$

to obtain the following result.

Lemma 4.4. *The determinant of matrix $M(X)$ defined in Proposition 4.1 and extended by*

$$b_1(0) = 0, \quad b_2(0) = 0, \quad c_1(0) = 0, \quad c_2(0) = \nu \tag{4.52}$$

with respect to the factorization

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1} \quad \text{and} \quad b_j = (\kappa - \beta_j)b_j^{g-1} \tag{4.53}$$

fulfills the following proportionality condition

$$\begin{aligned}\det(M(X)) \sim\, & \mathrm{res}(a(\kappa), \kappa^3(\kappa^2+1))\,\Delta(a(\kappa))\,\mathrm{res}(b_1(\kappa)/\kappa, b_2(\kappa)/\kappa) \\ \sim\, & \mathrm{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, \kappa^3(\kappa^2+1))\,\mathrm{res}(a^{g-1}(\kappa), \kappa^3(\kappa^2+1)) \\ & \alpha_i(\beta_1 - \beta_2)\,\mathrm{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, a^{g-1}(\kappa))^2 \\ & \mathrm{res}(\kappa - \beta_1, b_2^{g-1}(\kappa)/\kappa)\,\mathrm{res}(b_1^{g-1}(\kappa)/\kappa, \kappa - \beta_2) \\ & \Delta(a^{g-1}(\kappa))\,\mathrm{res}(b_1^{g-1}(\kappa)/\kappa, b_2^{g-1}(\kappa)/\kappa).\end{aligned} \tag{4.54}$$

We will now use this result to compute the derivatives of $\det(M(X))$ with respect to X at a point which we will call X_{dp}. We will use the X_{dp} as a bifurcation point to higher spectral genus. Such a point is characterized by $\alpha_r = \beta_j = \kappa_{dp} \neq 0$ and $\alpha_i = 0$ with respect to the previous factorization $a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1}$ and $b_j = (\kappa - \beta_j)b_j^{g-1}$ of the polynomials a and b_j. Further denote by $\tilde{X}$ the spectral data obtained from $(a^{g-1}, b_1^{g-1}, b_2^{g-1})$ at such X_{dp}. And we assume that $\det(M(\tilde{X})) \neq 0$.

From the previous Lemma 4.4 we see that

$$\det(M(X_{dp})) = 0 \tag{4.55}$$

due to $\alpha_i = 0$ and $\beta_1 = \beta_2$. We can write $\det(M(X)) \sim \alpha_i(\beta_1 - \beta_2)f(X)$ where

$$\begin{aligned} f(x) :=& \operatorname{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, \kappa^3(\kappa^2+1)) \\ & \operatorname{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, a^{g-1}(\kappa))^2 \\ & \operatorname{res}(\kappa - \beta_1, b_2^{g-1}(\kappa)/\kappa) \operatorname{res}(b_1^{g-1}(\kappa)/\kappa, \kappa - \beta_2) \\ & \operatorname{res}(a^{g-1}(\kappa), \kappa^3(\kappa^2+1))\, \Delta(a^{g-1}(\kappa)) \operatorname{res}(b_1^{g-1}(\kappa)/\kappa, b_2^{g-1}(\kappa)/\kappa) \\ \sim & \operatorname{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, \kappa^3(\kappa^2+1)) \\ & \operatorname{res}(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i, a^{g-1}(\kappa))^2 \\ & \operatorname{res}(\kappa - \beta_1, b_2^{g-1}(\kappa)/\kappa) \operatorname{res}(b_1^{g-1}(\kappa)/\kappa, \kappa - \beta_2) \\ & \det(M(\tilde{X})) \end{aligned} \tag{4.56}$$

contains the remaining terms. From our assumptions it follows that $f(X)\big|_{X=X_{dp}} \neq 0$. We also see that

$$\begin{aligned} & \mathrm{d}\det(M(X))\big|_{X=X_{dp}} \\ & \sim (\beta_1 - \beta_2)f(X)\mathrm{d}\alpha_i + \alpha_i f(X)(\mathrm{d}\beta_1 - \mathrm{d}\beta_2) + \alpha_i(\beta_1-\beta_2)\mathrm{d}f(X)\big|_{X=X_{dp}} \\ & = 0. \end{aligned} \tag{4.57}$$

Let us now check $\mathrm{d}^2\det(M(X))$. We obtain

$$\begin{aligned} & \mathrm{d}^2\det(M(X))\big|_{X=X_{dp}} \\ & \sim f(X)\mathrm{d}\alpha_i(\mathrm{d}\beta_1 - \mathrm{d}\beta_2) + (\beta_1-\beta_2)\mathrm{d}\alpha_i\mathrm{d}f(X)\big|_{X=X_{dp}} \\ & + f(X)\mathrm{d}\alpha_i(\mathrm{d}\beta_1 - \mathrm{d}\beta_2) + \alpha_i(\mathrm{d}\beta_1 - \mathrm{d}\beta_2)\mathrm{d}f(X)\big|_{X=X_{dp}} \\ & + \alpha_i(\mathrm{d}\beta_1 - \mathrm{d}\beta_2)\mathrm{d}f(X) + (\beta_1-\beta_2)\mathrm{d}\alpha_i\mathrm{d}f(X)\big|_{X=X_{dp}} \\ & + \alpha_i(\beta_1-\beta_2)\mathrm{d}^2 f(X)\big|_{X=X_{dp}} \\ & = f(X)\mathrm{d}\alpha_i(\mathrm{d}\beta_1 - \mathrm{d}\beta_2)\big|_{X=X_{dp}} \end{aligned} \tag{4.58}$$

This shows us that $\mathrm{d}^2\det(M(X))\big|_{X=X_{dp}} \neq 0$ if $\mathrm{d}\alpha_i \neq 0$ and $\mathrm{d}\beta_1 - \mathrm{d}\beta_2 \neq 0$.

Lemma 4.5. *The determinant of matrix $M(X)$ defined in Proposition 4.1 and extended by*

$$b_1(0) = 0, \quad b_2(0) = 0, \quad c_1(0) = 0, \quad c_2(0) = \nu \tag{4.59}$$

with respect to the factorization

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1} \quad \textit{and} \quad b_j = (\kappa - \beta_j)b_j^{g-1} \tag{4.60}$$

fulfills the following conditions at X_{dp}

$$\begin{aligned} \det(M(X))\Big|_{X=X_{dp}} &= 0 \\ \mathrm{d}\det(M(X))\Big|_{X=X_{dp}} &= 0 \\ \mathrm{d}^2\det(M(X))\Big|_{X=X_{dp}} &\neq 0 \end{aligned} \tag{4.61}$$

for $\mathrm{d}\alpha_i \neq 0$ *and* $\mathrm{d}\beta_1 - \mathrm{d}\beta_2 \neq 0$*, under the assumptions that* $\det(M(\tilde{X}) \neq 0$ *for* $\tilde{X}$ *formed with coefficients from* a^{g-1}, b_j^{g-1} *and* $\alpha_r = \beta_j = \kappa_{dp} \neq 0$ *and* $\alpha_i = 0$ *for* X_{dp}.

4.4 Spectral curves with double points

In this section we will derive necessary conditions on a spectral curve for the case that two branch points meet at $\kappa_{dp} \in \mathbb{R}$. Then we will proceed to show how to find spectral curves which meet those conditions.

We recall from section 3.2 that for any $l = 1, \ldots, g$ the cycle B_l in the κ-plane crosses only the cuts between $\kappa = i$ and $\kappa = -i$ and between $\kappa = \alpha_l$ and $\kappa = \bar{\alpha}_l$.

When we assume that $\sqrt{k}$ has the cut going from 0 to $-\infty$ we can write

$$\int_{B_l} \theta_j = \int_{B_l} \frac{i b_j(\kappa)}{\tilde{\nu}(\kappa^2+1)(\mathrm{Im}(\alpha_l)\sqrt{\frac{i(\kappa-\alpha_l}{\mathrm{Im}(\alpha_l)}}\sqrt{\frac{\kappa-\bar{\alpha}_l}{\mathrm{Im}(\alpha_l)}})} \mathrm{d}\kappa \tag{4.62}$$

and obtain the cuts in the way we prescribed before.

Now we look at what happens when $\alpha_l \to \kappa_{dp}$ and thus $\mathrm{Im}(\alpha_l) \to 0$. We denote by $\tilde{\theta}_j$, $\tilde{B}_l$ the results of this limit. After taking this limit and desingularizing the spectral curve we obtain

$$\int_{\tilde{B}_l} \tilde{\theta}_j = \int_{\tilde{B}_l} \frac{i\tilde{b}_j(\kappa)}{\tilde{\nu}(\kappa^2+1)} \mathrm{d}\kappa. \tag{4.63}$$

Here the path along $\tilde{B}_l$ crosses only one cut between $\kappa = i$ and $\kappa = -i$ so it is not closed anymore, it connects on Σ the two points $\alpha_l^{\pm}$ over $\mathrm{Re}(\alpha_l) = \kappa_{dp} \in \mathbb{R}$. With

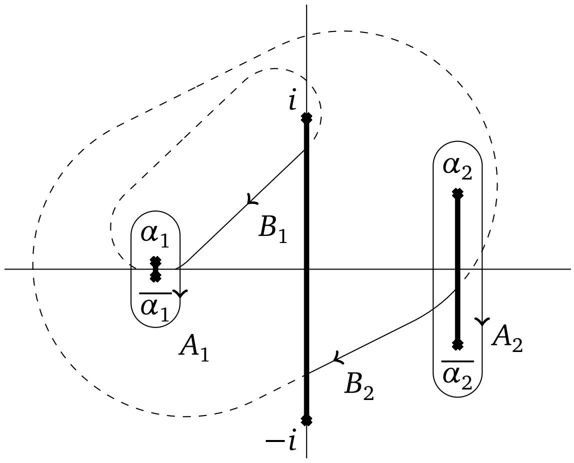


Figure 4-1: Branch points and cuts of a spectral curve Σ in the κ-plane. The branch points α_1, $\bar{\alpha}_1$ are close to the real line. When they are on the real line they form a double point.

this observation and using $\ln \mu_j(\alpha_l^+) = -\ln \mu_j(\alpha_l^-)$ we obtain

$$\int_{\tilde{B}_l} \frac{i b_j(\kappa)}{\tilde{\nu}(\kappa^2+1)} \mathrm{d}\kappa = 2 \ln \mu_j(\kappa_{dp}). \tag{4.64}$$

We conclude that if we start with a spectral curve where $\int_{B_l} \theta_j \in 2\pi i\mathbb{Z}$, after taking the limit $\mathrm{Im}(\alpha_j) \to 0$ we must have $\ln \mu_j(\kappa_{dp}) \in \pi i\mathbb{Z}$. This is the necessary condition on κ_{dp} to be a putative double point. A spectral curve having such a point can be the result of a limiting procedure of spectral curves where two branch points collide.

Definition 4.6. We say a spectral curve Σ with $\kappa_{dp} \in \mathbb{R}$ such that $\ln \mu_j(\kappa_{dp}) \in \pi i\mathbb{Z}$ has a putative double point at κ_{dp}.

Remark. From the extrinsic closing conditions for a torus we see that the Sym-point $\kappa = 0$ is always a putative double point.

Later we will use such spectral curves with a putative double point to bifurcate to higher spectral genus. In general a spectral curve of a torus may only have such a point at the Sym-point. We will need spectral curves where such a point exists and is not equal to the Sym-point. So we need a way to find such spectral curves, possibly in such a way that we still maintain a control on the closing conditions.

We will now show that we can find such spectral curves by a procedure involving Whitham deformations. We define

$$\begin{aligned} \phi &: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\ &(\kappa, t) \to (i \ln \mu_1(\kappa, t)), i \ln \mu_2(\kappa, t)). \end{aligned} \tag{4.65}$$

This function does indeed map into $\mathbb{R}^2$ since $\partial_\kappa(i \ln \mu_j(\kappa, t)) = b_j(\kappa)\nu^{-1}(\kappa^2+1)^{-1}$. We are looking for (κ_{dp}, t_{dp}) such that $\phi(\kappa_{dp}, t_{dp}) \in \mathbb{Z}^2$. As we have seen in sections 3.2 and 3.3 we allow for integer multiplies of $\ln \mu_j$ since with those multiplies we still have a closed surface, we just allow it to be multiple wrapped around the directions of its periods. This allows us to look in fact for (κ_{dp}, t_{dp}) such that $\phi(\kappa_{dp}, t_{dp}) \in \mathbb{Q}^2$.

Lemma 4.7. *For every t_0 and every $\epsilon > 0$ there is $t_{dp} \in (t_0, t_0+\epsilon)$ such that a $\kappa_{dp} \in \mathbb{R}$ exist so that $\phi(\kappa_{dp}, t_{dp}) \in \mathbb{Q}^2$.*

Proof. We look at t-derivative of ϕ

$$\partial_t \phi(\kappa, t) = (i\partial_t \ln \mu_1(\kappa, t)), i\partial_t \ln \mu_2(\kappa, t)) \tag{4.66}$$

If we have a non-constant deformation, we have $\partial_t \phi(\kappa, t_0) \not\equiv 0$. So there is κ_0 such that $\partial_t \phi(\kappa_0, t_0) \neq 0$. There exist also small $\delta, \epsilon > 0$ such that $\partial_t \phi(\kappa, t) \neq 0$ for all $\kappa \in (\kappa_0 - \delta, \kappa_0 + \delta)$ and $t \in (t_0, t_0 + \epsilon)$. By continuity of $\phi(\kappa, t)$ there must exist $\kappa_{dp} \in (\kappa_0 - \delta, \kappa_0 + \delta)$ and $t_{dp} \in (t_0, t_0 + \epsilon)$ such that $\phi(\kappa_{dp}, t_{dp}) \in \mathbb{Q}^2$. □

Remark (Procedure for finding spectral curves with double points). An obvious way to find such a spectral curve would be the following procedure:

(i) Find $\kappa_0 \in \mathbb{R}$ such that $\ln \mu_1(\kappa_0, t_0) \in \mathbb{Q}$ and $\partial_t \ln \mu_2(\kappa_0, t_0) \neq 0$.

(ii) Define $\dot{\kappa}_0$ by $\partial_t \ln \mu_1(\kappa_0(t), t) = 0$.

(iii) Find $t_{dp} \in (t_0, t_0 + \epsilon)$ such that $\ln \mu_2(\kappa_0(t_{dp}), t_{dp}) \in \mathbb{Q}$.

(iv) Now $\kappa_{dp} = \kappa_0(t_{dp})$ and $\phi(\kappa_{dp}, t_{dp}) \in \mathbb{Q}^2$.

This procedure gives us a spectral curve with a putative double point, but since we used the Whitham deformation to get there, we obtain a spectral curve of a CMC cylinder with a doubly periodic metric.

4.5 Bifurcation to higher spectral genus

In the previous section we have derived necessary conditions for the existence of a putative double point on spectral curve. Moreover, we also described a way to generate spectral curves with such points. In the following we will analyze how to use such a curve as an initial condition for our flow to move this curve into a spectral curve of higher spectral genus.

Let Σ^{g-1} be spectral curve with a putative double point κ_{dp}. The spectral data of Σ^{g-1} should be $(a^{g-1}, b_1^{g-1}, b_2^{g-1})$. We define a new singular Riemann surface Σ with two new differentials on it by changing the spectral data in the following way

$$a = (\kappa - \kappa_{dp})^2 a^{g-1} \quad \text{and} \quad b_j = (\kappa - \kappa_{dp}) b_j^{g-1}. \tag{4.67}$$

The new Riemann surface Σ has arithmetic genus $g+1$ and corresponds to the same solution of the sinh-Gordon equation as Σ^{g-1} since (a, b_1, b_2) give rise to the same differentials θ_j as $(a^{g-1}, b_1^{g-1}, b_2^{g-1})$.

Definition 4.8. Denote by X_{dp} the spectral data (a, b_1, b_2) obtained from the spectral data $(a^{g-1}, b_1^{g-1}, b_2^{g-1})$ and a putative double point $\kappa_{dp} \in \mathbb{R}$.

$$a = (\kappa - \kappa_{dp})^2 a^{g-1} \quad \text{and} \quad b_j = (\kappa - \kappa_{dp}) b_j^{g-1}. \tag{4.68}$$

For further computations we will make some more assumptions about X_{dp}.

Assumption 4.9. *Let X be the vector of spectral data (a, b_1, b_2) of the following form:*

- *The polynomial a has no multiple zeros and $a(0) \neq 0$, $a(\pm i) \neq 0$.*
- *Neither of the polynomials b_j has common zeros with a.*
- *The polynomials b_j have no common zeros except for $\kappa = 0$, where both b_j do have a common zero.*
- *At least one b_j has the full degree $g+1$ where g is the genus of Σ.*

We observe that the above assumptions on X are of such a type that if they hold for spectral data X and X is deformed continuously, then the assumptions will hold in a ϵ-neighborhood of X. The assumptions do not hold for X_{dp}, but they are only violated because a and b_j have a common zero at κ_{dp} and a has a double zero there. If there is a continuous flow defined on a interval $(t_0 - \epsilon, t_0 + \epsilon)$ and this flow opens the double point at κ_{dp}, the above assumptions will be fulfilled for $(t_0 - \epsilon, t_0 + \epsilon) \setminus \{\epsilon\}$.

4.5.1 Possible directions of the flow at a bifurcation point

In this section we will start building formal solutions of the flow ODE by investigating the space of possible directions Y of those solutions. Later we will show which of those directions lead to formal solutions of the flow ODE.

We have written the flow ODE with extrinsic closing conditions as a matrix equation $M(X)Y = 0$, ehere $M(X)$ is a $(6g+4) \times (6g+5)$ matrix. In the following we will compute the null space and the rank of the $M(X_{dp})$ in the case of spectral data X_{dp} with one double point (as constructed in 4.5) fulfilling the assumptions 4.9. We will represent those spaces by matrices formed of basis vectors.

We denote by $P_l(X_{dp})$ respectively $P_r(X_{dp})$ the left kernel resp. the right kernel of $M(X_{dp})$.

$$P_l(X_{dp}) M(X_{dp}) = 0 \quad \text{and} \quad M(X_{dp}) P_r(X_{dp}) = 0. \tag{4.69}$$

Our first observation is that for the spectral data of the form

$$a = (\kappa - \kappa_{dp})^2 \hat{a} \quad \text{and} \quad b_j = (\kappa - \kappa_{dp}) \hat{b}_j \tag{4.70}$$

both integrability equations

$$\begin{aligned} -b_j(\kappa)\dot{a}(\kappa) + 2a(\kappa)\dot{b}_j(\kappa) + 2\kappa a(\kappa)c_j(\kappa) - (\kappa^2+1)2a(\kappa)c_j'(\kappa) \\ + (\kappa^2+1)a'(\kappa)c_j(\kappa) = 0 \end{aligned} \tag{4.71}$$

are independently fulfilled when evaluated at $\kappa = \kappa_{dp}$. Having the computations in the section 4.3 in mind we remember that the first $3g+2$ rows of $M(X_{dp})$ come form the integrability equation with b_1 and the following $3g+2$ rows from b_2. Those rows are formed by the coefficients of k^i in those integrability equations. Knowing that the integrability conditions are independently of each other fulfilled at $\kappa = \kappa_{dp}$ lets us know at least two vectors in the left kernel $P_l(X_{dp})$ of $M(X_{dp})$, namely $(\kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1}, 0, \dots, 0)$ and $(0, \dots, 0, \kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1})$. So the left kernel $P_l(X_{dp})$ of $M(X_{dp})$ is at least 2-dimensional and therefore the right kernel $P_r(X_{dp})$ of $M(X_{dp})$ has to be at least 3-dimensional.

We have found the lower bound for the rank of $M(X_{dp})$. We will now compute the upper bound and then the exact rank. In order to do so we will rewrite the integrability equations. We define the following vectors of polynomials

$$\begin{aligned} A &= (&& a(\kappa), \quad (\kappa^2+1)a'(\kappa), \quad \dot{a}(\kappa) \ ) \\ B_1 &= (\ 2\dot{b}_1(\kappa) + 2\kappa c_1(\kappa) - 2(\kappa^2+1)c_1'(\kappa), && c_1(\kappa), \quad -b_1(\kappa) \ ) \\ B_2 &= (\ 2\dot{b}_2(\kappa) + 2\kappa c_2(\kappa) - 2(\kappa^2+1)c_2'(\kappa), && c_2(\kappa), \quad -b_2(\kappa) \ ) \end{aligned} \tag{4.72}$$

We define, analogously to the cross product in $\mathbb{R}^3$, a product of two of those vectors resulting in another vector of polynomials with three entries:

$$B_1 \times B_2 = \begin{pmatrix} b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) \\ \\ 2b_2(\kappa)\dot{b}_1(\kappa) + 2\kappa b_2(\kappa)c_1(\kappa) - 2(\kappa^2+1)b_2(\kappa)c_1'(\kappa) \\ - 2b_1(\kappa)\dot{b}_2(\kappa) - 2\kappa b_1(\kappa)c_2(\kappa) + 2(\kappa^2+1)b_1(\kappa)c_2'(\kappa) \\ \\ 2\dot{b}_1(\kappa)c_2(\kappa) - 2(\kappa^2+1)c_1'(\kappa)c_2(\kappa) \\ - 2\dot{b}_2(\kappa)c_1(\kappa) + 2(\kappa^2+1)c_2'(\kappa)c_1(\kappa) \end{pmatrix} \tag{4.73}$$

Lemma 4.10. *The two integrability equations*

$$\begin{aligned} 2\dot{b}_j(\kappa)a(\kappa) - b_j(\kappa)\dot{a}(\kappa) = &- 2\kappa a(\kappa)c_j(\kappa) \\ &+ (\kappa^2+1)(2a(\kappa)c_j'(\kappa) - a'(\kappa)c_j(\kappa)) \end{aligned} \tag{4.74}$$

together with

$$b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) = f_0\kappa(\kappa - f_1)a(\kappa) \tag{4.75}$$

are equivalent to the following system of three equations

$$B_1 \times B_2 = f_0\kappa(\kappa - f_1)A \tag{4.76}$$

Proof. Denote the integrability equations (4.74) and (4.75) by

$$\begin{aligned} int_j :=& 2\dot{b}_j(\kappa)a(\kappa) - b_j(\kappa)\dot{a}(\kappa) \\ &+ 2\kappa a(\kappa)c_j(\kappa) - (\kappa^2+1)(2a(\kappa)c_j'(\kappa) - a'(\kappa)c_j(\kappa)), \\ w :=& b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) - f_0\kappa(\kappa - f_1)a(\kappa), \end{aligned} \tag{4.77}$$

and denote by

$$cross_j := (B_1 \times B_2 - f_0\kappa(\kappa - f_1)A)_j \tag{4.78}$$

the 3 entries of the vector equation (4.76). Explicitly we obtain

$$\begin{aligned} cross_1 =& b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) \\ &- f_0\kappa(\kappa - f_1)a(\kappa), \\ cross_2 =& 2b_2(\kappa)\dot{b}_1(\kappa) + 2\kappa b_2(\kappa)c_1(\kappa) - 2(\kappa^2+1)b_2(\kappa)c_1'(\kappa) \\ &- 2b_1(\kappa)\dot{b}_2(\kappa) - 2\kappa b_1(\kappa)c_2(\kappa) + 2(\kappa^2+1)b_1(\kappa)c_2'(\kappa) \\ &- f_0\kappa(\kappa - f_1)(\kappa^2+1)a'(\kappa), \\ cross_3 =& 2\dot{b}_1(\kappa)c_2(\kappa) - 2(\kappa^2+1)c_1'(\kappa)c_2(\kappa) \\ &- 2\dot{b}_2(\kappa)c_1(\kappa) + 2(\kappa^2+1)c_2'(\kappa)c_1(\kappa) \\ &- f_0\kappa(\kappa - f_1)\dot{a}(\kappa). \end{aligned} \tag{4.79}$$

We note $w = cross_1$. As short computation shows

$$b_2(\kappa)int_1 - b_1(\kappa)int_2 + (\kappa^2+1)a'(\kappa)w = a(\kappa)cross_2 \tag{4.80}$$

and

$$c_2(\kappa)int_1 - c_1(\kappa)int_2 + (\kappa^2+1)\dot{a}(\kappa)w = a(\kappa)cross_3 \tag{4.81}$$

Thus we see that if $int_1 = int_2 = w = 0$, meaning that integrability conditions are fulfilled, then $a(\kappa) = 0$, or $cross_2 = 0$ and $cross_3 = 0$. So for $a(\kappa) \neq 0$ all solutions to (4.74) and (4.75) are solutions to (4.76).

As similar computation shows

$$b_j cross_3 - c_j cross_2 - 2(\kappa c_j + \dot{b}_j - (\kappa^2+1)c_j')cross_1 = f_0\kappa(\kappa - f_1)int_j \tag{4.82}$$

We see that if $cross_1 = cross_2 = cross_3 = 0$, then $f_0\kappa(\kappa - f_1) = 0$ or $int_1 = 0$ and

$int_2 = 0$. So for $f_0\kappa(\kappa - f_1) \neq 0$ all solutions to (4.76) are solutions to (4.74) and (4.75).

□

We will now use this representation of the equations to determine the space of possible solutions $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$ for given (a, b_1, b_2). of the form

$$a = (\kappa - \kappa_{dp})^2 \hat{a} \quad \text{and} \quad b_j = (\kappa - \kappa_{dp})\hat{b}_j. \tag{4.83}$$

The equation from the first entries of $B_1 \times B_2 = f_0\kappa(\kappa - f_1)A$ is

$$b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) = f_0\kappa(\kappa - f_1)a(\kappa). \tag{4.84}$$

The degree of b_j and c_j is $g + 1$ and the degree of a is $2g$. Since b_j have a common zero at 0, we can divide out the factor κ from both sides of the equation. The degree of c_j are now 1 higher than those of b_j. The polynomials b_j have a common zero at κ_{dp} in addition to the zero at $\kappa = 0$. These two facts together lead to the space of possible c_j being 3-dimensional. For the flow we are interested in, we have the restriction that $c_1(0) = 0$ and $c_1'(0) = 0$ and this cuts the space of possible c_j to a 1-dimensional space for every given (f_0, f_1).

We use the second equation from $B_1 \times B_2 = f_0\kappa(\kappa - f_1)A$ to determine $\dot{b}_j$ from the computed c_j of the first equation. This equation has the form

$$b_1(\kappa)\dot{b}_2(\kappa) - b_2(\kappa)\dot{b}_1(\kappa) = p(\kappa) \tag{4.85}$$

with p being a polynomial determined by (a', b_j, c_j). Here b_j have a common zero at 0 as before but $\dot{b}_j$ also have a common zero there. Together with the common zero of b_j at κ_{dp} we obtain a 2-dimensional space of possible $\dot{b}_j$.

Finally the last equation from $B_1 \times B_2 = f_0\kappa(\kappa - f_1)A$ reads as

$$f_0\kappa(\kappa - f_1)\dot{a}(\kappa) = p(\kappa). \tag{4.86}$$

Here p is determined by $(\dot{b}_j, c_j)$ and there is no freedom in choosing an $\dot{a}$. It is unique if the equation has a solution. So we have 1-dimensional freedom to choose f_0, f_1 since different values of f_0 lead to the same $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$ multiplied by some constant factor. We have a 1-dimensional freedom for c_1, c_2 from the first equation and a 2-dimensional freedom for $\dot{b}_1, \dot{b}_2$ from second equation. By combining these results, we see that for a given (a, b_1, b_2) with a common zero at κ_{dp} the space of possible $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$ is at most 4-dimensional.

We will now see that the solubility of the third equation restricts this space to a

3-dimensional space. The third equation reads as

$$f_0\kappa(\kappa - f_1)\dot{a}(\kappa) = 2\dot{b}_1(\kappa)c_2(\kappa) - 2(\kappa^2+1)c_1'(\kappa)c_2(\kappa) \\ - 2\dot{b}_2(\kappa)c_1(\kappa) + 2(\kappa^2+1)c_2'(\kappa)c_1(\kappa). \tag{4.87}$$

The restriction in solubility of the third equation comes from the fact that the left hand side has degree $2g+1$ but the right hand side has degree $2g+2$. Unravelling the coefficient of κ^{2g+2} from the right hand side, we obtain the following condition on solubility

$$\dot{b}_{1,g+1}c_{2,g+1} - c_{1,g+1}c_{2,g} - \dot{b}_{2,g+1}c_{1,g+1} + c_{1,g}c_{2,g+1} = 0. \tag{4.88}$$

This condition involves coefficients of $\dot{b}_j$ and c_j which are obtained from solving the first and second equation. To be precise we need the coefficients of κ^{2g+2} and κ^{2g+1} from the first equation

$$f_0\kappa(\kappa - f_1)a(\kappa) = b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa). \tag{4.89}$$

Those are

$$b_{1,g+1}c_{2,g+1} - b_{2,g+1}c_{1,g+1} = f_0 \tag{4.90}$$

and

$$b_{1,g+1}c_{2,g} + b_{1,g}c_{2,g+1} - b_{2,g}c_{1,g+1} - b_{2,g+1}c_{1,g} = -f_0f_1 + f_0a_{2g-1}. \tag{4.91}$$

We also need the coefficients of κ^{2g+2} and κ^{2g+1} from the second equation

$$f_0\kappa(\kappa - f_1)(\kappa^2+1)a'(\kappa) = 2b_2(\kappa)\dot{b}_1(\kappa) + 2\kappa b_2(\kappa)c_1(\kappa) - 2(\kappa^2+1)b_2(\kappa)c_1'(\kappa) \\ - 2b_1(\kappa)\dot{b}_2(\kappa) - 2\kappa b_1(\kappa)c_2(\kappa) + 2(\kappa^2+1)b_1(\kappa)c_2'(\kappa). \tag{4.92}$$

Those are

$$-2gf_0f_1 + (2g-1)f_0a_{2g-1} = 2b_{2,g+1}\dot{b}_{1,g+1} - 2gb_{2,g}c_{1,g+1} - 2(g-1)b_{2,g+1}c_{1,g} \\ - 2b_{1,g+1}\dot{b}_{2,g+1} + 2gb_{1,g}c_{2,g+1} + 2(g-1)b_{1,g+1}c_{2,g}. \tag{4.93}$$

Solving (4.90), (4.91), (4.93) for $b_{1,g+1}c_{1,g+1}c_{2,g+1}$ and substituting them in (4.88) we obtain

$$f_0b_{2,g+1}(2c_{2,g} - a_{2g-1}c_{2,g+1} + \dot{b}_{2,g+1}) = 0. \tag{4.94}$$

Similarly, solving for $b_{2,g+1}c_{1,g+1}c_{2,g+1}$, we obtain

$$f_0 b_{1,g+1}(2c_{1,g} - a_{2g-1}c_{1,g+1} + \dot{b}_{1,g+1}) = 0. \tag{4.95}$$

Those equations are fulfilled when $f_0 = 0$. When $f_0 \neq 0$ they give a linear condition on the space of possible $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$, since both $b_{1,g+1}$ and $b_{2,g+1}$, as leading coefficients of b_1 and b_2, can not equal 0 at the same time under the assumptions 4.9. This means that (4.88) reduces the solution space by one dimension, and we arrive at the conclusion that the space of solutions of $M(X_{dp})Y = 0$ is exactly 3-dimensional

Proposition 4.11. *The space of solutions $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$ for $M(X_{dp})Y = 0$, where X_{dp} is defined by $a = (\kappa - \kappa_{dp})^2\hat{a}$ and $b_j = (\kappa - \kappa_{dp})\hat{b}_j$, $\kappa_{dp} \neq 0$, and $\hat{a}$, $\hat{b}_j$ fulfill assumptions 4.9, is exactly 3-dimensional.*

An immediate consequence of this result is $\operatorname{rank}(P_l(X_{dp})) = 2$ and $\operatorname{rank}(P_r(X_{dp})) = 3$. This means that $(\kappa_{dp}^0, \ldots, \kappa_{dp}^{3g+1}, 0, \ldots, 0), (0, \ldots, 0, \kappa_{dp}^0, \ldots, \kappa_{dp}^{3g+1})$ is indeed a basis for the left null space of $M(X_{dp})$.

We will now construct a basis for the space of the solutions of $M(X_{dp})Y = 0$. There are two basis vectors with $f_0 = 0$. One comes from a solution with $c_j = 0$ and corresponds to a Y direction where only the k_{dp} is varied along the real axis. The second solution is when $c_j \neq 0$, in this case the double point is opened to two branch points. The last basis vector comes from solving the equation with $f_0 \neq 0$.

We now factor out form a and b_j the terms which affect the double point

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1} \quad \text{and} \quad b_j = (\kappa - \beta_j)b_j^{g-1}. \tag{4.96}$$

At the singular spectral data with the double point κ_{dp} we have

$$\alpha_r = \kappa_{dp}, \quad \alpha_i = 0, \quad \beta_j = \kappa_{dp}. \tag{4.97}$$

So this factorization at the singular data is exactly the same as in Proposition 4.11.

Obviously $\dot{a}$ and $\dot{b}_j$ are determined by $\dot{a}^{g-1}$, $\dot{b}_j^{g-1}$ and $\dot{\alpha}_r$ $\dot{\alpha}_i$ $\dot{\beta}_j$

$$\begin{aligned}
\dot{a} &= (-2\kappa\dot{\alpha}_r + 2\alpha_r\dot{\alpha}_r + \dot{\alpha}_i)a^{g-1} + (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)\dot{a}^{g-1},\\
\dot{b}_j &= -\dot{\beta}_j b_j^{g-1} + (\kappa - \beta_j)\dot{b}_j^{g-1},\\
a' &= (\kappa - \alpha_r)a^{g-1} + (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)(a^{g-1})',\\
b_j' &= b_j^{g-1} + (\kappa - \beta_j)(b_j^{g-1})'.
\end{aligned} \tag{4.98}$$

We rewrite

$$\begin{aligned}
2\dot{b}_j(\kappa)a(\kappa) - b_j(\kappa)\dot{a}(\kappa) = &- 2\kappa a(\kappa)c_j(\kappa)\\
&+ (\kappa^2 + 1)(2a(\kappa)c_j'(\kappa) - a'(\kappa)c_j(\kappa))
\end{aligned} \tag{4.99}$$

with those factored polynomials and obtain

$$
\begin{aligned}
&(2a^{g-1}\dot{b}_j^{g-1} - b_j^{g-1}\dot{a})(\kappa-\beta_j)(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i) \\
&+ 2a^{g-1}b_j^{g-1}\dot{\beta}_j(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)) - a^{g-1}b_j^{g-1}(\kappa-\beta)(-2\kappa\dot{\alpha}_r + 2\alpha_r\dot{\alpha}_r + \dot{\alpha}_i) \\
&+ ((a^{g-1})'c_j - 2a^{g-1}c_j')(\kappa^2+1)(\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i) \\
&+ 2a^{g-1}c_j((2\kappa^2 - \kappa\alpha_r + 1)(\kappa - \alpha_r) + \kappa\alpha_i) = 0.
\end{aligned}
\tag{4.100}
$$

Proposition 4.12. *Let the spectral data X_{dp} be defined by*

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1}, \quad b_j = (\kappa - \beta_j)b_j^{g-1} \tag{4.101}$$

and

$$\alpha_r = \kappa_{dp}, \quad \alpha_i = 0, \quad \beta_j = \kappa_{dp} \tag{4.102}$$

with $\kappa_{dp} \neq 0$. Let $a^{g-1} = \hat{a}$ and $b_j^{g-1} = \hat{b}_j$ where $\hat{X}$ is coming from $\hat{a}$ and $\hat{b}_j$ and fulfills the assumptions 4.9. Then the space of solutions $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2)$ for $M(X_{dp})Y = 0$ is spanned by the following three solutions: one solution with

$$\dot{\alpha}_r = 1, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 1, \quad \dot{a}^{g-1} = 0, \quad \dot{b}_j^{g-1} = 0, \quad c_j = 0, \tag{4.103}$$

a second solution with

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 0, \quad \dot{a}^{g-1} = \dot{\hat{a}}, \quad \dot{b}_j^{g-1} = \dot{\hat{b}}_j, \quad c_j = (\kappa - \kappa_{dp})\hat{c}_j, \tag{4.104}$$

where $\dot{\hat{a}}$, $\dot{\hat{b}}_j$ and $\hat{c}_j$ are solutions to $M(\hat{X})\hat{Y} = 0$ the flow of the spectral data without the double point, and finally a third solution

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 1, \quad \dot{\beta}_j = \dot{\beta}_j^o, \quad \dot{a}^{g-1} = \dot{a}^o \quad \dot{b}_j^{g-1} = \dot{b}_j^o, \quad c_j = c_j^o, \tag{4.105}$$

which does not lie in the space spanned by the first two solutions and which opens the double point.

Proof. In the following we look at the integrability condition (4.100) right at the spectral data where $\alpha_r = \kappa_{dp}$,$\alpha_i = 0$ and $\beta_j = \kappa_{dp}$.

Setting

$$\dot{\alpha}_r = 1, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 1, \quad \dot{a}^{g-1} = 0, \quad \dot{b}_j^{g-1} = 0, \quad c_j = 0, \tag{4.106}$$

one can immediately see that (4.99) is fulfilled. We also can see that $f_0 = 0$, since in

$$b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) = f_0\kappa(\kappa - f_1)a(\kappa) \tag{4.107}$$

the above choice of c_j makes the left side identically to zero.

Now we set

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 0, \quad \dot{a}^{g-1} = \dot{\hat{a}}, \quad \dot{b}_j^{g-1} = \dot{\hat{b}}_j, \quad c_j = (\kappa - \kappa_{dp})\hat{c}_j \tag{4.108}$$

and we obtain

$$(\kappa - \kappa_{dp})^3 (2\dot{\hat{b}}_j(\kappa) a^{g-1}(\kappa) - b_j^{g-1}(\kappa)\dot{\hat{a}}(\kappa) + 2\kappa a^{g-1}(\kappa)\hat{c}_j(\kappa) \\ - (\kappa^2+1)(2a^{g-1}(\kappa)\hat{c}_j'(\kappa) - (a^{g-1})'(\kappa)\hat{c}_j(\kappa))) = 0. \tag{4.109}$$

This shows that $\dot{\hat{a}}$, $\dot{\hat{b}}_j$ and $\hat{c}_j$ need to be a solutions to $M(\hat{X})\hat{Y} = 0$, the flow of the spectral data without the double point, if (4.99) is fulfilled. Here $f_0 \neq 0$.

Finally, we set

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 1, \quad \dot{\beta}_j = \dot{\beta}_j^o, \quad \dot{a}^{g-1} = \dot{a}^o \quad \dot{b}_j^{g-1} = \dot{b}_j^o, \quad c_j = c_j^o, \tag{4.110}$$

Obviously these data do not lie in the space spanned by the first two solutions. We will now show that this kind of data solves (4.99). In a first step we differentiate the integrability condition (4.100) with respect to κ and evaluate it at $\kappa = \kappa_{dp}$. We obtain

$$a^{g-1}(\kappa_{dp})(b_j^{g-1}(\kappa_{dp})\dot{\alpha}_i^o - 2(\kappa_{dp}^2+1)c_j^o(\kappa_{dp})) = 0. \tag{4.111}$$

Solving this for $c_j^o(\kappa_{dp})$ leads to

$$c_j^o(\kappa_{dp}) = \frac{b_j^{g-1}(\kappa_{dp})\dot{\alpha}_i^o}{2(\kappa_{dp}^2+1)}. \tag{4.112}$$

Now we differentiate the integrability condition (4.100) a second time with respect to κ and evaluate again at $\kappa = \kappa_{dp}$. We solve the result for $\dot{\beta}_j$ and use (4.112) to obtain

$$\dot{\beta}_j^o = \dot{\alpha}_r^o + \dot{\alpha}_i^o \frac{3\kappa_{dp}}{2(\kappa_{dp}^2+1)} + \dot{\alpha}_i^o \frac{(a^{g-1})'(\kappa_{dp})}{4a^{g-1}(\kappa_{dp})} - \dot{\alpha}_i^o \frac{(b_j^{g-1})'(\kappa_{dp})}{2b_j^{g-1}(\kappa_{dp})} \tag{4.113}$$

We see that if

$$c_j^o(\kappa_{dp}) = \frac{b_j^{g-1}(\kappa_{dp})}{2(\kappa_{dp}^2+1)} \quad , \quad \dot{\beta}_j^o = \frac{3\kappa_{dp}}{2(\kappa_{dp}^2+1)} + \frac{(a^{g-1})'(\kappa_{dp})}{4a^{g-1}(\kappa_{dp})} - \frac{(b_j^{g-1})'(\kappa_{dp})}{2b_j^{g-1}(\kappa_{dp})} \tag{4.114}$$

it follows that $\dot{\alpha}_r = 0$, $\dot{\alpha}_i = 1$. Since $b_j^{g-1}(\kappa_{dp}) \neq 0$ from our assumptions and $(b_1^{g-1})(\kappa_{dp})c_2^o(\kappa_{dp}) - (b_2^{g-1})(\kappa_{dp})c_1^o(\kappa_{dp}) = 0$ with the above data, we can always pick c_j^o from the one-dimensional space of possible c_j for $b_1(\kappa)c_2(\kappa) - b_2(\kappa)c_1(\kappa) = 0$ such that $c_j^o(\kappa_{dp})$ are as above. For these c_j^o setting $\dot{\beta}_j^o$ as above amounts again to a particular choice in the space of solutions of

$$\begin{aligned} &2b_2(\kappa)\dot{b}_1(\kappa) + 2\kappa b_2(\kappa)c_1(\kappa) - 2(\kappa^2+1)b_2(\kappa)c_1'(\kappa) \\ &- 2b_1(\kappa)\dot{b}_2(\kappa) - 2\kappa b_1(\kappa)c_2(\kappa) + 2(\kappa^2+1)b_1(\kappa)c_2'(\kappa) = 0. \end{aligned} \tag{4.115}$$

So setting $c_j^o(\kappa_{dp})$ and $\dot{\beta}_j^o$ leads to the desired $\dot{\alpha}_r = 0$, $\dot{\alpha}_i = 1$ and such a solution exists for $f_0 = 0$.

Since the three solutions we found are linearly independent, we have constructed a basis for the solutions of $M(X_{dp})Y = 0$. □

One important observation we can make here is that under the assumption that $\dot{\alpha}_r = 0$ and $\dot{\alpha}_i = 1$, we have

$$\frac{(b_1^{g-1})'(\kappa_{dp})}{b_1^{g-1}(\kappa_{dp})} \neq \frac{(b_2^{g-1})'(\kappa_{dp})}{b_2^{g-1}(\kappa_{dp})} \Leftrightarrow \dot{\beta}_1^o \neq \dot{\beta}_2^o \tag{4.116}$$

In addition to $\dot{\alpha}_i \neq 0$ this will be a necessary condition for properly open a double point. To reflect this we extend the assumption 4.9 on X to the following assumption on X_{dp}:

Assumption 4.13. *Let X_{dp} be the vector of the spectral data (a, b_1, b_2) obtained from the spectral data $(a^{g-1}, b_1^{g-1}, b_2^{g-1})$ and a putative double point $\kappa_{dp} \in \mathbb{R}$ by*

$$a = (\kappa - \kappa_{dp})^2 a^{g-1} \quad \textit{and} \quad b_j = (\kappa - \kappa_{dp}) b_j^{g-1}. \tag{4.117}$$

We assume the following:

- *The polynomial a^{g-1} has no multiple zeros and $a^{g-1}(0) \neq 0$, $a^{g-1}(\pm i) \neq 0$.*
- *Neither of the polynomials b_j^{g-1} has common zeros with a^{g-1}.*
- *The polynomials b_j^{g-1} have no common zeros except for $\kappa = 0$, where both b_j^{g-1} do have a common zero.*
- *At least one b_j^{g-1} has the full degree $g+1$ where g is the genus of Σ^{g-1}.*
- $(b_1^{g-1})'(\kappa_{dp}) b_2^{g-1}(\kappa_{dp}) \neq (b_2^{g-1})'(\kappa_{dp}) b_1^{g-1}(\kappa_{dp})$.

The procedure in section 4.4 allows to find κ_{dp} anywhere on the real line. Since b_1^{g-1} and b_2^{g-1} are distinct polynomials, the last assumption is easy to meet.

4.5.2 Power series expansion of the flow at a double point

We have seen in Proposition 4.1 that we can write the Whitham flow equation as a matrix equation $M(X)Y = 0$. Additionally, the matrix M has the form $M(X) = M_0 + M_1(X)$ as shown in Proposition 4.2. These results enable us to compute t-derivatives of the Whitham flow equation and by doing so construct a power series solution to this ODE.

First of all we remember that M and consequently M_0 and M_1 do not depend on t where X and Y are functions of t. We observe therefore

$$\begin{aligned} M(X(t)) &= M_0 + M_1(X(t)) \\ \partial_t(M(X(t))) &= M_1(\partial_t X(t)) \\ \partial_t^n(M(X(t))) &= M_1(\partial_t^n X(t)). \end{aligned} \tag{4.118}$$

By taking t-derivatives of $M(X)Y = 0$ a further computation shows

$$\begin{aligned} (M_0 + M_1(X(t)))Y(t) &= 0 \\ (M_0 + M_1(X(t)))\partial_t Y(t) + M_1(\partial_t X(t))Y(t) &= 0 \\ (M_0 + M_1(X(t)))\partial_t^2 Y(t) + 2M_1(\partial_t X(t))\partial_t Y(t) + M_1(\partial_t^2 X(t))Y(t) &= 0 \\ (M_0 + M_1(X(t)))\partial_t^n Y(t) + \sum_{i=0}^{n-1} \binom{n}{i} M_1(\partial_t^{(n-i)} X(t))\partial_t^i Y(t) &= 0 \end{aligned} \tag{4.119}$$

From the previous section we know that $M(X_{dp})$ has nontrivial left kernel $P_l(X_{dp})$. So we conclude that Y_{dp} has to fulfill the following equations.

$$(M_0 + M_1(X_{dp}))Y_{dp} = 0 \tag{4.120a}$$

$$P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0 \tag{4.120b}$$

For $\partial_t Y_{dp}$ the equations read as follows

$$(M_0 + M_1(X_{dp}))\partial_t Y_{dp} = -M_1(\partial_t X_{dp})Y_{dp} \tag{4.121a}$$

$$2P_l(X_{dp})M_1(\partial_t X_{dp})\partial_t Y_{dp} + P_l(X_{dp})M_1(\partial_t^2 X_{dp})Y_{dp} = 0 \tag{4.121b}$$

Finally for $\partial_t^n Y$

$$(M_0 + M_1(X_{dp}))\partial_t^n Y_{dp} = -\sum_{i=0}^{n-1} \binom{n}{i} M_1(\partial_t^{(n-i)} X_{dp})\partial_t^i Y_{dp} \tag{4.122a}$$

$$\begin{aligned} nP_l(X_{dp})M_1(\partial_t X_{dp})\partial_t^n Y + P_l(X_{dp})M_1(\partial_t^{n+1} X_{dp})Y_{dp} = \\ -\sum_{i=1}^{n-1} \binom{n+1}{i} P_l(X)M_1(\partial_t^{(n+1-i)} X_{dp})\partial_t^i Y_{dp} \end{aligned} \tag{4.122b}$$

We defined X and Y in such a way that $\partial_t^{n+1}X$ is just the first part of $\partial_t^n Y$. More precisely, there exists a projection $P_{xy} := [I, 0]$ and $\partial_t^{n+1}X = P_{xy}\partial_t^n Y$. This means that with the exception of the equations for Y_{dp}, all those equations are linear systems on the components of $\partial_t^n Y_{dp}$ which are the unknowns. Looking at the first equation of the general system (4.122a) for $\partial_t^n Y_{dp}$, we see that right hand side has to lie in the image of $M(X_{dp})$ in order for the equation to be solvable. This is equivalent to

$$\sum_{i=0}^{n-1}\binom{n}{i}M_1(\partial_t^{(n-i)}X_{dp})\partial_t^i Y_{dp} \perp \ker(M(X_{dp})^\intercal). \tag{4.123}$$

The latter in turn is equivalent to (4.122b) for $\partial_t^{n-1}Y_{dp}$. So if $\partial_t^n Y_{dp}$ solves (4.122b) then (4.122a) is solvable for $\partial_t^{n+1}Y_{dp}$. On the other hand (4.122b) is always linearly independent from (4.122a), so it has always a solution in the solution space of (4.122a). We conclude the following proposition.

Proposition 4.14. *For every Y_{dp} which solves*

$$\begin{aligned}(M_0 + M_1(X_{dp}))Y_{dp} &= 0\\ P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} &= 0\end{aligned} \tag{4.124}$$

there exists a unique formal solution $(Y_{dp}, \partial_t Y_{dp}, \partial_t^2 Y_{dp}, \dots)$ which solves

$$\begin{aligned}(M_0 + M_1(X_{dp}))\partial_t^n Y_{dp} &= -\sum_{i=0}^{n-1}\binom{n}{i}M_1(\partial_t^{(n-i)}X_{dp})\partial_t^i Y_{dp}\\ nP_l(X_{dp})M_1(\partial_t X_{dp})\partial_t^n Y_{dp} + P_l(X_{dp})M_1(\partial_t^{n+1}X_{dp})Y_{dp} &=\\ -\sum_{i=1}^{n-1}\binom{n+1}{i}&P_l(X_{dp})M_1(\partial_t^{(n+1-i)}X_{dp})\partial_t^i Y_{dp}\end{aligned} \tag{4.125}$$

for every $\partial_t^n Y_{dp}$.

In 4.5.1 we computed the space of solutions of

$$M(X_{dp})Y_{dp} = (M_0 + M_1(X_{dp})Y_{dp} = 0.$$

It remains to check which of those are solutions to

$$P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0.$$

We recall that $P_l(X_{dp})$ has only two rows, namely $(\kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1}, 0, \dots, 0)$ and $(0, \dots, 0, \kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1})$. Since $P_l(X_{dp})M_1(\partial_t X_{dp}) = 0$ amounts to evaluation of the integrability conditions at κ_{dp}, we can write $P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0$ as follows:

$$\partial_t \left(\begin{array}{c} -b_j(\kappa)\dot{a}(\kappa) + 2a(\kappa)\dot{b}_j(\kappa) + 2\kappa a(\kappa)c_j(\kappa) \\ -(\kappa^2+1)2a(\kappa)c_j'(\kappa) + (\kappa^2+1)a'(\kappa)c_j(\kappa) \end{array} \right)\Bigg|_{\kappa=\kappa_{dp}} = 0. \tag{4.126}$$

As before we factor out the terms which affect the double point form a and b_j

$$a = (\kappa^2 - 2\alpha_r\kappa + \alpha_r^2 + \alpha_i)a^{g-1} \quad \text{and} \quad b_j = (\kappa - \beta_j)b_j^{g-1}. \tag{4.127}$$

At the singular spectral data with the double point we have

$$\alpha_r = \kappa_{dp}, \quad \alpha_i = 0, \quad \beta_j = \kappa_{dp}. \tag{4.128}$$

Obviously $\dot{a}$ and $\dot{b}_j$ are determined by $\dot{a}^{g-1}$, $\dot{b}_j^{g-1}$ and $\dot{\alpha}_r$ $\dot{\alpha}_i$ $\dot{\beta}_j$.

The basis of the solutions for of $M(X_{dp})Y = 0$ as computed in 4.5.1 can be characterized as follows. One solution with

$$\dot{\alpha}_r = 1, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 1, \quad \dot{a}^{g-1} = 0, \quad \dot{b}_j^{g-1} = 0, \quad c_j = 0, \tag{4.129}$$

a second solution with

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 0, \quad \dot{\beta}_j = 0, \quad \dot{a}^{g-1} = \dot{\hat{a}}, \quad \dot{b}_j^{g-1} = \dot{\hat{b}}_j, \quad c_j = (\kappa - \kappa_{dp})\hat{c}_j, \tag{4.130}$$

where $\dot{\hat{a}}$, $\dot{\hat{b}}_j$ and $\hat{c}_j$ are solutions to $M(\hat{X})\hat{Y} = 0$ the flow of the spectral data without the double point, and finally a third solution

$$\dot{\alpha}_r = 0, \quad \dot{\alpha}_i = 1, \quad \dot{\beta}_j = \dot{\beta}_j^o, \quad \dot{a}^{g-1} = \dot{a}^o \quad \dot{b}_j^{g-1} = \dot{b}_j^o, \quad c_j = c_j^o, \tag{4.131}$$

which does not lie in the space spanned by the first two solutions and which opens the double point.

In a lengthy computation we rewrite the differentiated integrability condition (4.126) with the factored polynomials (4.127) and insert linear combinations of the solutions we recalled above (4.129), (4.130), (4.131) scaled by f_1, f_2 and f_3 respectively. We obtain the following condition

$$f_3(f_1h_1 + f_2h_2 + f_3h_3) = 0 \tag{4.132}$$

where

$$h_1 = - a^{g-1}(\kappa_{dp})b_j^{g-1}(\kappa_{dp}) + 2(k_{dp}^1 + 1)a^{g-1}(\kappa_{dp})c_j^0(\kappa_{dp}) \tag{4.133}$$

$$h_2 = - 2(k_{dp}^1 + 1)a^{g-1}(\kappa_{dp})\hat{c}_j(\kappa_{dp}) \tag{4.134}$$

$$\begin{aligned} h_3 =& (k_{dp}^2 + 1)((a^{g-1})'(\kappa_{dp})c_j^0(\kappa_{dp}) - a^{g-1}(c_j^0)'(\kappa_{dp})) \\ &+ 2\kappa_{dp}a^{g-1}(\kappa_{dp})c_j^0(\kappa_{dp}) - a^{g-1}(\kappa_{dp})b_j^{g-1}(\kappa_{dp})\dot{\beta}_j^o \end{aligned} \tag{4.135}$$

By having this result, we can see which of the solutions to $M(X_{dp})Y_{dp} = 0$ are solutions to $P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0$. From the above condition either $f_3 = 0$, in this case any linear combination of (4.133) and (4.134) solves (4.132) and thus any linear combination of (4.129) and (4.130) solves $P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0$. Or $f_3 \neq 0$, then there exists a unique up to a scaling combination of (4.133), (4.134), and (4.135) which solves (4.132) and thus a unique up to a scaling combination of (4.129), (4.130), and (4.131) which solves $P_l(X_{dp})M_1(\partial_t X_{dp})Y_{dp} = 0$. The first solution is any linear combination of moving the closed double point along the real axis and the flow of the lower spectral genus, the flow equation stays singular in all of these directions. The second solution gives a unique direction which opens the double point and the flow equation is not singular immediately after opening the double point.

Proposition 4.15. *The equation $M(X_{dp})Y_{dp} = 0$ has one unique formal solution $(Y_{dp}, \partial_t Y_{dp}, \partial_t^2 Y_{dp}, \dots)$ with the property that this solution opens the double point. The uniqueness is up to a scale of each of the $\partial_t^n Y_{dp}$.*

As we have seen before $\partial_t^{n+1} X_{dp} = P_{xy}\partial_t^n Y_{dp}$. We can now define the formal power series solution of the flow ODE near a singular spectral data.

Definition 4.16. Let

$$X_{dp}(t) := X_{dp} + \sum_{n=1} \partial_t^n X_{dp} \frac{t^n}{n!} \tag{4.136}$$

be the unique formal solution for the Whitham flow which opens the double point in the spectral data X_{dp}.

So far we have proven the existence and the uniqueness of a formal solution. It remains to show that this is an actual solution to the Whitham flow that is (4.136) converges.

4.5.3 Vector fields with zeros and convergence of formal solutions

In 4.5.2 we showed the existence of a formal power series solution for the Whitham flow at singular spectral data with a double point. In this section we will recall a result about convergence of power series solutions from the general theory for ODEs and use it to prove that the solution constructed in 4.5.2 is an actual solution for the Whitham flow ODE.

We start with the ODE as defined in the Proposition 4.1. We extend M by one row adding the equation $c_2(0) = 1$ to our flow and thereby picking a particular element for the kernel of M or, equivalently, fix the t-dependence of the flow. The flow equation is now

$$M(X)Y = V \tag{4.137}$$

where $Y = [\dot{X}, C]$ as in the Proposition 4.1 and $V = (0, \dots, 0, 1)$.

This is a differential algebraic equation with a singularity. Most of the theory concerning such equations deals with so called standard singular points (e.g. Rabier [24]). Without going into further detail, we remark that the singular points in our case are not standard singular points and we can not use these results. On the other hand, we do not try to understand the geometry around the singular point in full detail, we are merely interested in the convergence of a certain formal solution that we have already constructed. In the following we will see that we can relate our differential system to a system of the form

$$t\dot{X} = f(t, X) \tag{4.138}$$

where $t \in \mathbb{C}$ and $f : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ is an analytic function. There are strong results for these systems. In particular, Theorem V-2-7 from Hsieh [19] shows that every formal solution to the system (4.138) is convergent.

In the following, we will use some standard facts about determinants, adjugate matrices and their respective derivatives. We denote by

$$\delta(X) := \det(M(X)), \tag{4.139}$$

then the usual property of the adjugate matrix of $M(X)$ gives

$$M(X)\,\mathrm{adj}(M(X)) = \mathrm{adj}(M(X))M(X) = \det(M(X))I = \delta(X)I. \tag{4.140}$$

Since M is affine in X, we see that $\mathrm{adj}(M(X))$ and $\delta(X)$ are analytic in X, they are even polynomial. Away from singular X we also know that $\delta(X) \neq 0$. Thus, we can multiply (4.137) by $\mathrm{adj}(M(X))$ and preserve the behavior of the flow outside of singular points. We obtain

$$\delta(X)Y = \mathrm{adj}(M(X))V \tag{4.141}$$

We now can use the projection $P_{xy} = [I, 0]$ satisfying $\dot{X} = P_{xy}Y$ to drop the C-part of Y. We arrive at a new system

$$\delta(X)\dot{X} = P_{xy}\,\mathrm{adj}(M(X))V =: f(X). \tag{4.142}$$

We observe that f is analytic in X.

Let us now assume that at X_{dp} we have the following behavior for $\delta(X)$ and $f(X) = P_{xy}\,\mathrm{adj}(M(X))V$

$$\delta(X)\Big|_{X=X_{dp}} = 0, \quad \mathrm{d}\delta(X)\Big|_{X=X_{dp}} = 0, \quad \mathrm{d}^2\delta(X)\Big|_{X=X_{dp}} \neq 0, \tag{4.143}$$
$$f(X)\Big|_{X=X_{dp}} = 0, \quad \mathrm{d}f(X)\Big|_{X=X_{dp}} = 0, \tag{4.144}$$

and in particular also

$$\partial_t^2\delta(X_{dp})\Big|_{t=0} \neq 0. \tag{4.145}$$

Essentially, we want that the right hand side of (4.142) vanishes to at least the same order as the left hand side.

The above assumptions ensure that the power series expansion of $\delta(X_{dp}(t))$ in t starts with a quadratic term. Therefore we can make a formal change of coordinates from t to s such that

$$\delta(X_{dp}(t(s))) = s^2. \tag{4.146}$$

The new formal solution $X_{dp}(s)$ is now a formal solution to $s^2\dot{X} = f(X)$.

Let us now move X_{dp} to the origin, so that we have

$$X_{dp}(0) = 0, \quad f(X)\Big|_{X=0} = 0, \quad \mathrm{d}f(X)\Big|_{X=0} = 0. \tag{4.147}$$

We can then write $X_{dp}(s) = s\hat{X}(s)$ and accordingly $\dot{X}(s) = s\dot{\hat{X}}(s) + \hat{X}(s)$. The new formal series $\hat{X}(s)$ is now a formal solution to

$$s^3\dot{\hat{X}}(s) = f(s\hat{X}(s)) - s^2\hat{X}(s). \tag{4.148}$$

Define now a new function g by

$$g(s,W) := \frac{f(sW)}{s^2} - W. \tag{4.149}$$

We observe that $g(0,W)$ is well defined since f vanishes to at least second order at 0. So we can write a new differential equation

$$s\dot{W} = g(s,W) \tag{4.150}$$

and immediately see that $W = \hat{X}(s)$ solves this equation. For this fact we see that $g(0,\hat{X}(0)) = 0$. We can again shift by $\hat{X}(0)$ to get $g(0,0) = 0$. The makes all assumptions of Theorem V-2-7 from Hsieh [19] hold and thus $\hat{X}(s)$ converges. This means that also $X_{dp}(t)$ converges after some change of coordinates in t. Since the system we are interested in is an autonomous system, we are free to make any change of coordinates in t.

It remains to check the assumptions (4.143), (4.144) and (4.145) to finally establish the convergence of our formal solution. We have already proved the assumption (4.143) in Lemma 4.5. The assumption (4.145) follows directly from (4.143) and the property of $X_{dp}(t)$ that it opens the double point. Let us now prove the validity of (4.144). We first notice that $M(X)$ has nullity rank of 2 at X_{dp}. From this it follows directly that $\operatorname{adj}(M(X))\big|_{X=X_{dp}} = 0$. This proves $f(X)\big|_{X=X_{dp}} = 0$, the first part of (4.144). To prove $\mathrm{d}f(X)\big|_{X=X_{dp}} = 0$ a little more work is required. We split M as

$$M = \begin{pmatrix} \hat{M} & B \\ 0 & 1 \end{pmatrix} \tag{4.151}$$

such that $\binom{B}{1}$ is the column corresponding to the only non-zero entry of V. We compute

$$\operatorname{adj}(M) = \begin{pmatrix} \operatorname{adj}(\hat{M}) & -\operatorname{adj}(\hat{M})B \\ 0 & \det(\hat{M}) \end{pmatrix}. \tag{4.152}$$

This leads us to

$$f(X) = P_{xy}\operatorname{adj}(M(X))V = P_{xy}\begin{pmatrix} -\operatorname{adj}(\hat{M}(X))B(X) \\ \det(\hat{M}(X)) \end{pmatrix}. \tag{4.153}$$

We want to show $\mathrm{d}f(X)\big|_{X=X_{dp}} = 0$. We already know $\mathrm{d}\det(\hat{M}(X))\big|_{X=X_{dp}} = 0$ so it remains to show $\mathrm{d}(\operatorname{adj}(\hat{M})(X)B(X))\big|_{X=X_{dp}} = 0$ or $\mathrm{d}\operatorname{adj}(\hat{M})(X)B(X)\big|_{X=X_{dp}} = 0$ as $\mathrm{d}B(X)\big|_{X=X_{dp}} = 0$ since $B(X)$ is linear in X. We use

$$\operatorname{adj}(\hat{M}(X))\hat{M}(X) = \det(\hat{M}(X))I. \tag{4.154}$$

By differentiating and evaluating at X_{dp} we obtain

$$\mathrm{d}\operatorname{adj}(\hat{M}(X))\hat{M}(X)\big|_{X=X_{dp}} + \operatorname{adj}(\hat{M}(X))\mathrm{d}\hat{M}(X)\big|_{X=X_{dp}} = \mathrm{d}\det(\hat{M}(X))I\big|_{X=X_{dp}}. \tag{4.155}$$

Again, we already know $\operatorname{adj}(\hat{M}(X))\big|_{X=X_{dp}} = 0$ and $\mathrm{d}\det(\hat{M}(X))\big|_{X=X_{dp}} = 0$, and so we have

$$\mathrm{d}\operatorname{adj}(\hat{M}(X))\hat{M}(X)\big|_{X=X_{dp}} = 0. \tag{4.156}$$

We use this equation to ensure

$$\mathrm{d}\operatorname{adj}(\hat{M})(X)B(X)\big|_{X=X_{dp}} = 0 \tag{4.157}$$

by showing

$$B(X)\big|_{X=X_{dp}} \in \operatorname{Im}(\hat{M}(X))\big|_{X=X_{dp}}. \tag{4.158}$$

The vector $B(X_{dp})$ contains the coefficients of $c_{2,0}$ of the flow equations. Those coefficients are the coefficients of the polynomial $\kappa a(\kappa) + (\kappa^2 + 1)a'(\kappa)$. Both $a(\kappa)$ and $a'(\kappa)$ have a root at κ_{dp}. We remember that $(\kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1}, 0, \dots, 0)$ and $(0, \dots, 0, \kappa_{dp}^0, \dots, \kappa_{dp}^{3g+1})$ are the basis of left kernel $P_l(X_{dp})$ of $M(X_{dp})$. Using the last two fact we see that

$$B(X_{dp}) \perp \ker(\hat{M}(X_{dp})^\perp) \tag{4.159}$$

or equivalently

$$B(X_{dp}) \in Im(\hat{M}(X_{dp})). \tag{4.160}$$

So we have shown $\mathrm{d}\,\mathrm{adj}(\hat{M})(X)B(X)\big|_{X=X_{dp}} = 0$ and thus also $\mathrm{d}f(X)\big|_{X=X_{dp}} = 0$. This was the last assumption for the proof of the main proposition of the section giving us the convergence of the formal solution to the Whitham flow at a singular point.

4.6 Algorithm for higher spectral genus

Let us summarize our results so far. We have seen in section 4.2 that we can use a Whitham deformation with a torus as initial data and flow though cylinders hitting new tori on a dense subset of every time interval. In section 4.4 we have proven Lemma 4.7, which ensures that on such a flow we can always find cylinders with putative double points. In section 4.5 we have shown that one can bifurcate to higher spectral genus by opening such double points under the assumptions 4.13 on the spectral data. This opening procedure ensured that the closing conditions which were preserved by the Whitham flow were preserved during the bifurcation. This allows to continue the Whitham flow into higher spectral genus, still producing spectral data of CMC tori on a dense subset of the existence interval of the flow. Using such a torus as initial data for a new deformation, one obtains every time a torus of spectral genus one higher than before.

The obvious algorithm is the following. We start at a torus with spectral genus g, spectral curve Σ_g, and spectral data X_g:

(i) Use the Whitham deformation to flow from Σ_g to $\Sigma_{g,dp}$ with a putative double point at κ_{dp}. For details see section 4.4.

(ii) Add a double point to $\Sigma_{g,dp}$ at κ_{dp} obtaining a singular curve with spectral data $X_{g+1,dp}$.

(iii) Use the procedures from sections 4.5.1 and 4.5.2 to construct a formal solution to the Whitham flow desingularizing the flow at the bifurcation point. The treatment in section 4.5.3 ensures that this formal solution converges.

(iv) Use the Whitham deformation to flow further and stop at a torus with spectral genus $g + 1$, spectral curve Σ_{g+1} and spectral data X_{g+1}.

(v) Repeat starting from **(i)** until g reaches desired spectral genus.

To start the procedure one can use a known lower spectral genus torus like the Wente torus. Using the Wente torus has an advantage for numerical experiments since it is easy to compute the spectral data for this torus to an arbitrary chosen precision. One can do this by the description given by Abresch [1] in terms of elliptic functions. This problem is well understood numerically and there exist several implementation in different numerical software packages.

The above algorithm does not require long time existence of the Whitham flow since every step can be completed in an ϵ-time interval of the flow. Thus our algorithm proves the existence of tori of arbitrary high spectral genus g as long as $g \geqslant g_0$, where g_0 is spectral genus of a known torus at the start of the procedure. When we use the Wente torus of spectral genus 2 with explicitly given spectral data [5], [1]as a starting CMC torus our algorithm produces an alternative proof of the result by Jaggy [20] for the existence of tori with arbitrary spectral genus $g \geqslant 2$.

Theorem 4.17. *For every $g \geqslant 2$ there exist a torus in $\mathbb{R}^3$ such that its spectral curve has genus g.*

4.7 Numerical Example

In this section we will describe a numerical experiment showing the procedure described in the previous section 4.6. We will start with the well known Wente torus of spectral genus 2. Using the method developed before, we first deform the spectral data such that we obtain spectral data of spectral genus 2 with a putative double point. Then we will open this double point by desingularizing the Whitham flow at this point. After that we continue the Whitham flow into spectral genus 3 and finally arrive at the spectral data of the Dobriner torus.

We used the Wolfram Mathematica software package to compute the Whitham flow by using the built in ODE solver of this package. The desingularization was performed by constructing the first few terms of the power series expansion and using the truncated series to compute non singular spectral data near bifurcation point. After that the ODE solver was used again to continue the flow. The reconstruction of the extended frame from the spectral data obtained by the flow was performed by a Killing field. References for reconstruction methods can be found at the end of section 2.4. The immersion into $\mathbb{R}^3$ was then computed by the Sym-Bobenko formula. The computation was done by an external library written for this purpose and the visualization was then performed again in Mathematica.

We will show the spectral curve by plotting the branch points inside the unit circle in the λ-plane, which is more common in literature compared to the κ-plane we used to simplify our computations. We will omit the branch points outside of the unit circle since those are obtained by just reflecting at unit circle. The branch point at the origin and the Sym point at $\lambda = 1$ is also included in the plots.

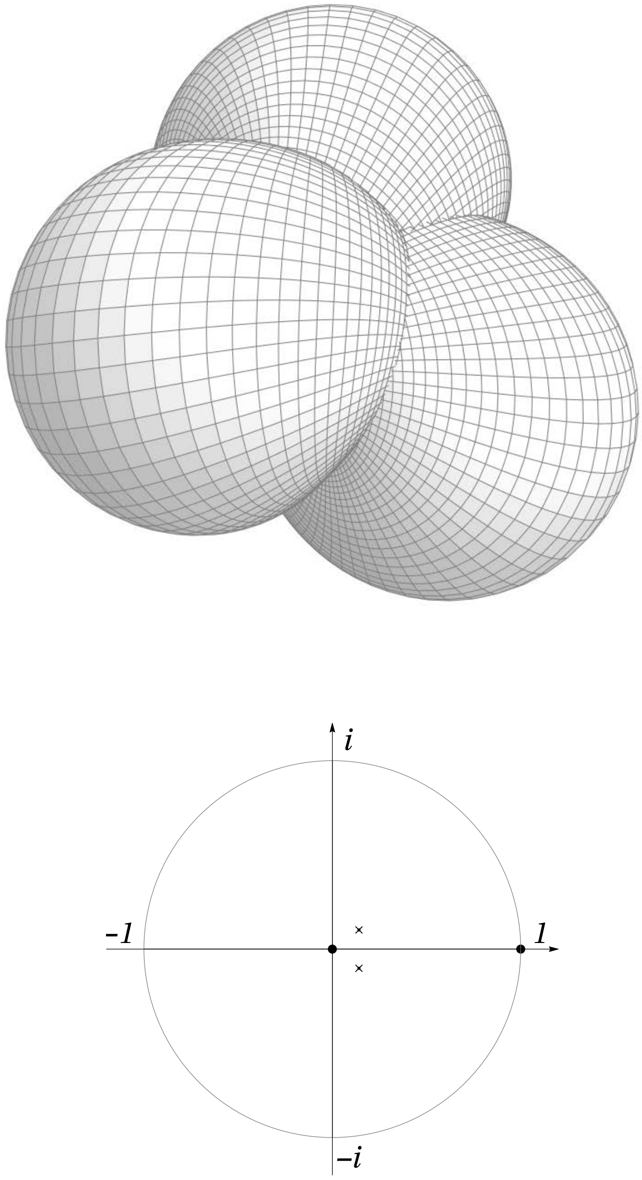


Figure 4-2: The starting point of the deformation in spectral genus 2: the Wente torus. Shown are the CMC torus in $\mathbb{R}^3$ and the spectral curve. The spectral curve is characterized by its branch points in the λ-plane denoted by small crosses. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.

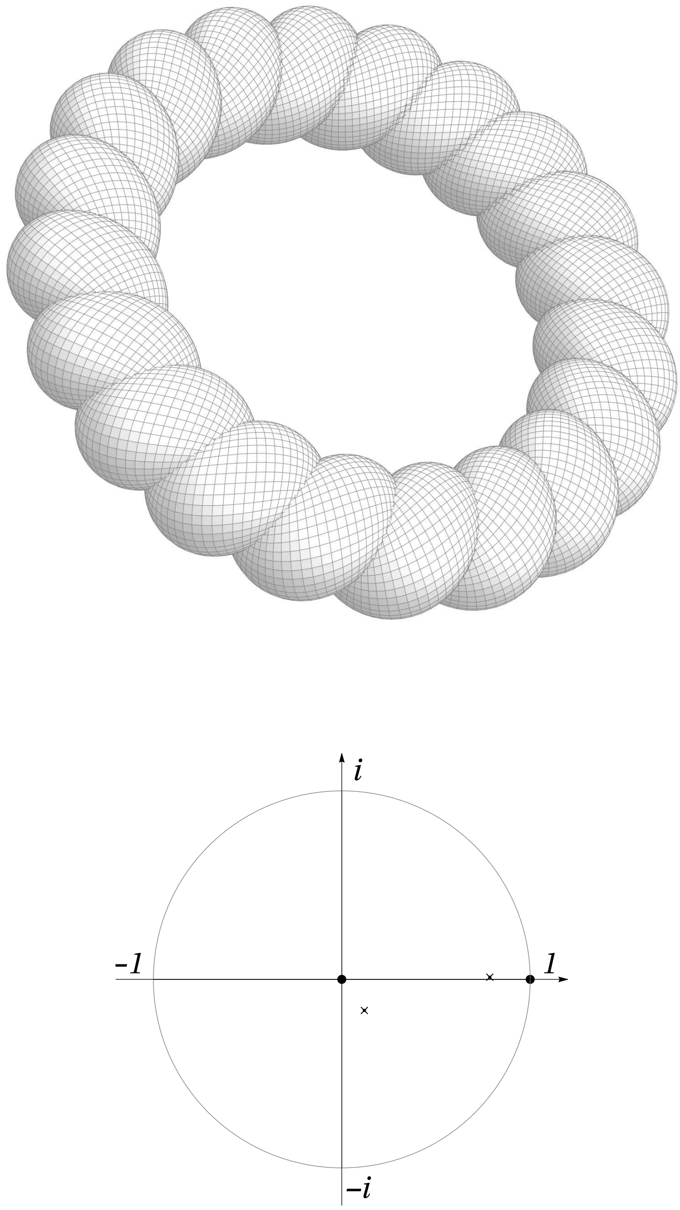


Figure 4-3: A twisted torus of spectral genus 2 near a possible bifurcation point to spectral genus 3. Shown are the CMC torus in $\mathbb{R}^3$ and the spectral curve. The spectral curve is characterized by its branch points in the λ-plane denoted by small crosses. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.

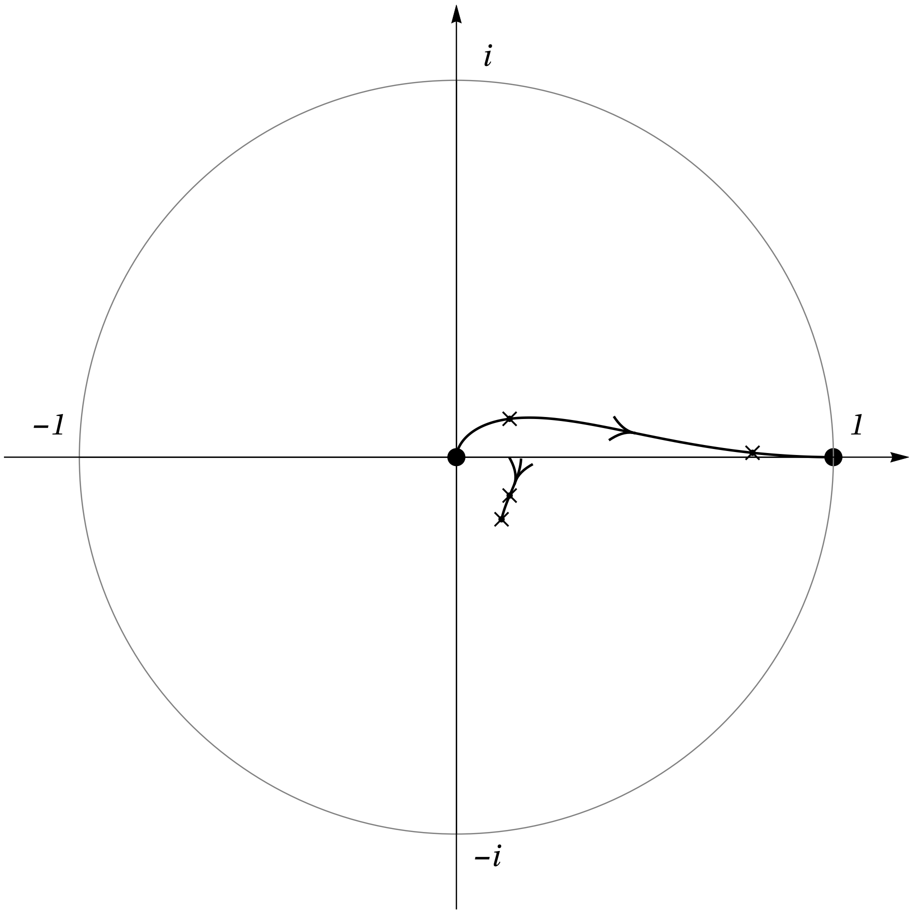
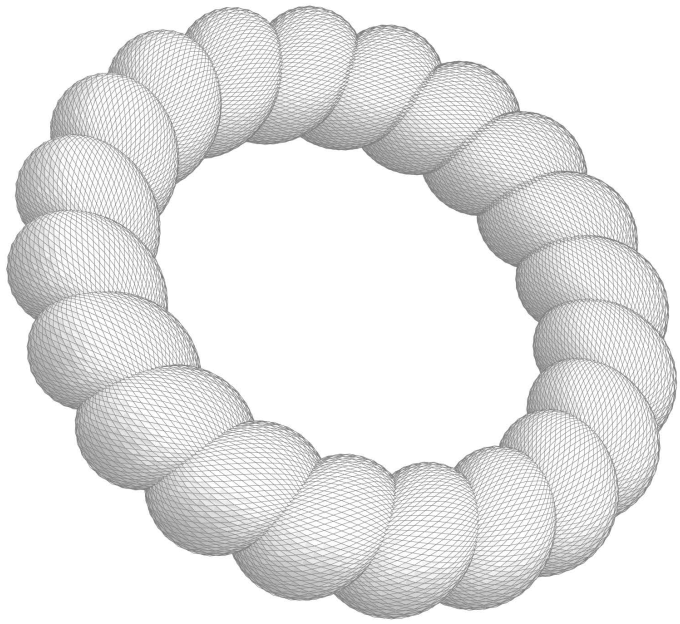


Figure 4-4: The traces of the branch points in the λ-plane during the deformation from torus in figure 4-2 to the torus in figure 4-3. The branch points in the λ-plane are denoted by small crosses. Their direction during the deformation is shown by an arrow. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.



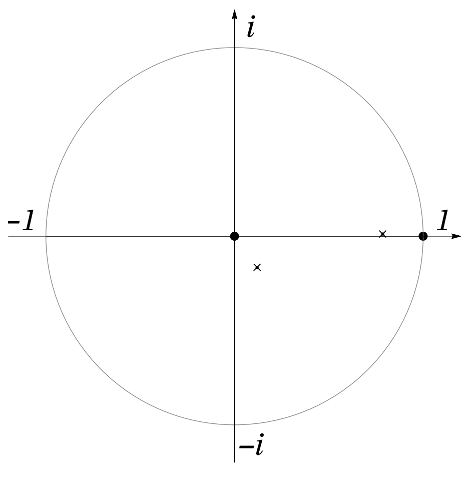


Figure 4-5: A twisted torus of spectral genus 2 near a possible bifurcation point to spectral genus 3. Shown are the CMC torus in $\mathbb{R}^3$ and the spectral curve. The spectral curve is characterized by its branch points in the λ-plane denoted by small crosses. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.

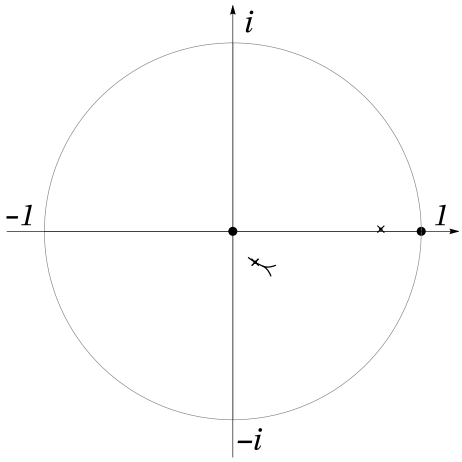


Figure 4-6: A cylinder of spectral genus 2 at a possible bifurcation point to spectral genus 3. Shown are the cylinder and the traces of the branch points of the spectral curve in the λ-plane during the deformation from the twisted torus in figure 4-5 to this cylinder. The direction of the movement of the branch point above the real axis is roughly towards the axis.

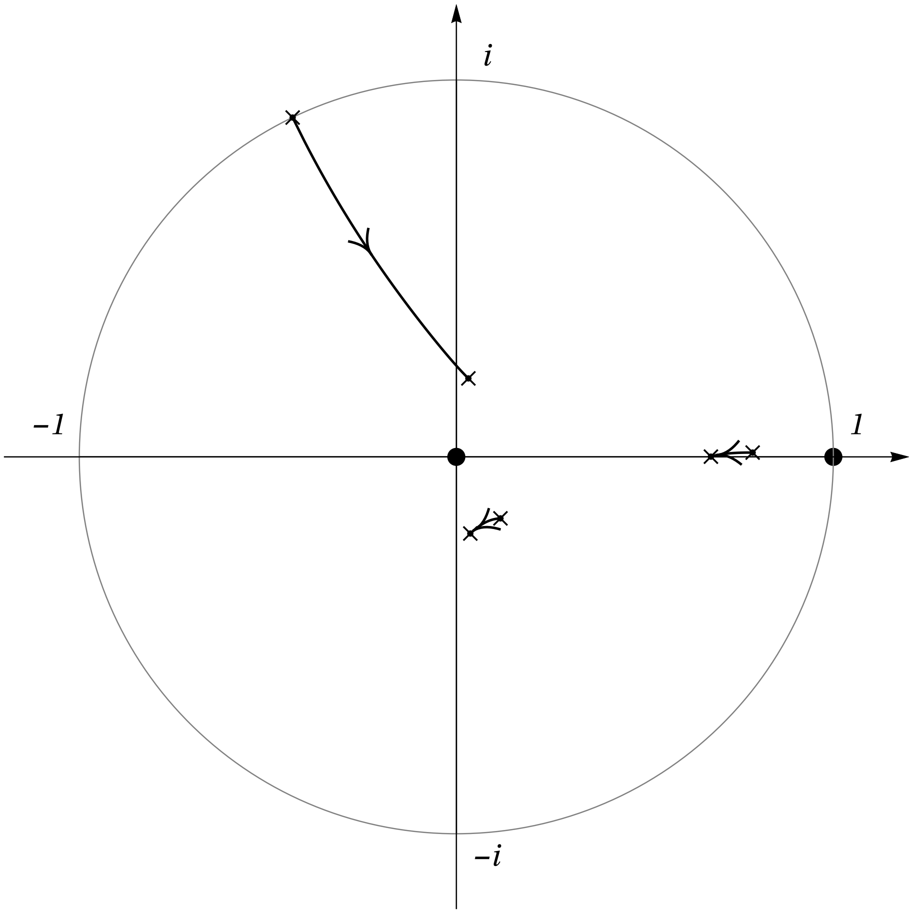


Figure 4-7: The traces of the branch points in the λ-plane during the deformation after bifurcating to spectral genus 3 from cylinder in figure 4-6. The branch points in the λ-plane are denoted by small crosses. Their direction during the deformation is shown by an arrow. A double point on the unit circle in the upper right quadrant is opened and the resulting branch point moves towards its final position in the upper left quadrant. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.

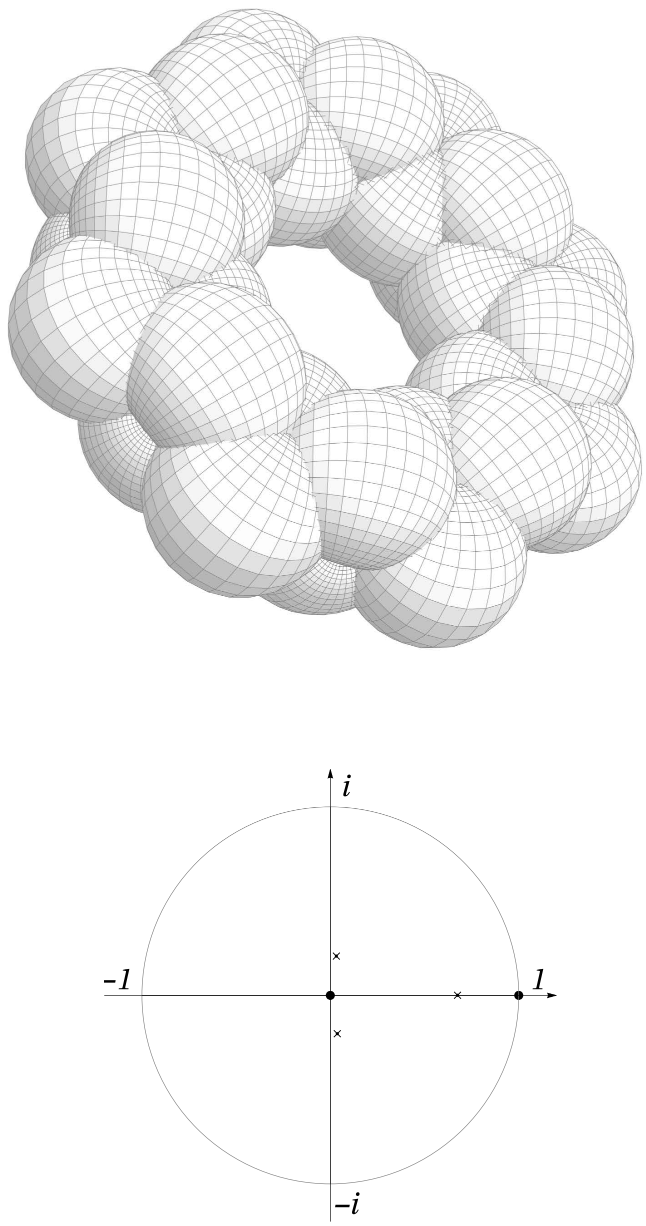


Figure 4-8: The end point of the deformation at spectral genus 3: the Dobriner torus. Shown are the CMC torus in $\mathbb{R}^3$ and the spectral curve. The spectral curve is characterized by its branch points in the λ-plane denoted by small crosses. The branch point at $\lambda = 0$ and the Sym point at $\lambda = 1$ are shown as well.

5 Conclusion

The presented work reproves a statement by Jaggy [20] about the existence of spectral curves for CMC tori with arbitrary high spectral genus using a deformation allowing to construct the resulting CMC tori. This proof deepens the understanding about the structure of the underlying moduli space of CMC tori and CMC cylinders with doubly periodic metric. It also allows to use the Witham deformation to obtain new examples of tori and gives a possible guideline to a proof of their existence since it also gives a deformation path starting at a torus known to exist. In this sense it gives some understanding about the connectedness of the moduli space.

Still there is a loot of work to do. The techniques used in this thesis can probably be expanded to understand when it is possible to open several double points at once and to the case where a double point is opened at the Sym point. These cases are harder because the conditions for when this is possible are not dense anymore. So further control on spectral data during the deformation would be desirable. Also the question of long term existence of Whitham flow should be tackled in order to obtain a further understanding of the structure of the moduli space.

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