GENERAL PROOF THEORY

Celebrating 50 Years of Dag Prawitz's "Natural Deduction"

Proceedings of the Conference held in
Tübingen, 27–29 November 2015

Edited by

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2016

Contents

Preface

General proof theory studies how proofs are structured and how they relate to each other, and not primarily what can be proved in particular formal systems. It has been developed within the framework of Gentzen-style proof theory, as well as in categorial proof theory.

As Dag Prawitz's monograph *Natural Deduction* (1965) paved the way for this development – he also proposed the term "General Proof Theory" – it is most appropriate to use this topic to celebrate 50 years of this work.

The conference took place 27–29 November, 2015 in Tübingen at the Department of Philosophy. The proceedings collect abstracts, slides and papers of the presentations given, as well as contributions from two speakers who were unable to attend.

The conference and its proceedings were supported by the French-German ANR-DFG project "Beyond Logic: Hypothetical Reasoning in Philosophy of Science, Informatics, and Law", DFG grant Schr 275/17-1.

We would like to thank Marine Gaudefroy-Bergmann for the organisation of the conference, and those who assisted her, in particular Giang Bui.

Thomas Piecha
Peter Schroeder-Heister

Photo: Giang Bui

Standing, left to right:
Heinrich Wansing, Giulio Guerrieri, Peter Schroeder-Heister, Helmut Schwichtenberg, Shawn Standefer, Nicolas Guenot, Luca Tranchini, Taus Brock-Nannestad, Greg Restall, Ulrik Buchholtz, Göran Sundholm, Per Martin-Löf, Roy Dyckhoff, Laura Tesconi, Kosta Došen, Alberto Naibo, Reinhard Kahle, Angeliki Koutsoukou-Argyraki, Daniel Wessel, Mario Piazza, Federico Aschieri, Dag Prawitz, David Binder, Mattia Petrolo, Tiago Rezende de Castro Alves, Gabriele Pulcini, Marianna Girlando, Dmitrij Smeljanskij, Michael Arndt, René Gazzari, Wolfgang Keller, Joachim Klappenecker, Stephen Read, Grigory Olkhovikov, Wilfried Keller

Front row, left to right:
Eugenio Orlandelli, Luiz Carlos Pereira, Wagner de Campos Sanz, Clayton Peterson, Hermógenes Oliveira, Marine Gaudefroy-Bergmann, Enrico Moriconi

Programme

Friday, 27 November

17.00–18.00	*Registration*
18.00–18.15	Peter Schroeder-Heister: *Opening: General Proof Theory*
18.15–19.15	Luiz Carlos Pereira and Edward Hermann Haeusler: *The Russell-Prawitz translation and schematic rules: a view from proof-theory*
20.00	*Conference dinner*

Saturday, 28 November

9.00–10.00	Per Martin-Löf: *The two interpretations of natural deduction: how do they fit together?*
10.00–10.30	Federico Aschieri: *On Natural Deduction in Classical First-Order-Logic*
10.30–11.00	*Coffee break*
11.00–11.30	Giulio Guerrieri and Alberto Naibo: *Postponement of RAA and Glivenko's theorem, revisited*
11.30–12.00	Norbert Gratzl and Eugenio Orlandelli: *Logicality, Double-line Rules, and Harmony*
12.00–14.30	*Lunch break*
14.30–15.30	Kosta Došen: *Adjunction and Normalization in Categories of Logic*
15.30–16.00	Clayton Peterson: *Monoidal logics: De Morgan negations and classical systems*
16.00–16.30	*Coffee break*
16.30–17.00	Zoran Petrić: *The natural deduction normal form and coherence*
17.00–18.00	Helmut Schwichtenberg: *Decorating natural deduction*

Sunday, 29 November

9.00–10.00	Heinrich Wansing: *A more general general proof theory*
10.00–10.30	Reinhard Kahle: *Is there a "Hilbert thesis"?*
10.30–11.00	*Coffee break*
11.00–11.30	Nissim Francez: *On distinguishing proof-theoretic consequence from derivability*
11.30–12.00	Danko Ilik: *High-school sequent calculus and an intuitionistic formula hierarchy preserving identity of proofs*
12.00–14.30	*Lunch break*
14.30–15.00	Angeliki Koutsoukou-Argyraki: *New Applications of Proof Mining to Nonlinear Analysis*
15.00–15.30	Mario Piazza and Gabriele Pulcini: *On the maximality of classical logic*
15.30–16.00	Roy Dyckhoff and Sara Negri: *Idempotent Coherentisation for First-Order Logic*
16.00–16.30	*Coffee break*
16.30–17.30	Dag Prawitz: *Gentzen's justification of inferences*

Abstracts

The Russell-Prawitz translation and schematic rules: a view from proof-theory

Luiz Carlos Pereira and Edward Hermann Haeusler
PUC Rio de Janeiro

Translations have been put to many uses in logic. In the first half of last century, translations played an important role in foundational matters: the reduction of foundational questions in classical environments to foundational questions in constructive environments. In the sixties translations acquired theoretical autonomy: several general approaches to translations were proposed (proof-theoretical, algebraic). Another distinctive use of translation is related to schematic rules of inference. What's an introduction rule for an operator ϕ? What's an elimination rule for ϕ? In 1978 Dag Prawitz proposed an answer to these questions by means of schematic introduction and elimination rules. Prawitz also proposed a constructive version of the well-known classical truth-functional completeness: if the introduction and elimination rules for an operator ϕ are instances of the schematic introduction and elimination rules, then ϕ is intuitionistically definable. Peter Schroeder-Heister showed how to obtain this completeness result for generalized schemata. These schematic introduction and elimination rules can also be used to show that logics whose operators satisfy the schematic rules and whose derivations satisfy the sub-formula principle can be translate into minimal implicational logic. Fernando Ferreira and Gilda Ferreira proposed still another use for translations: to use the Russell-Prawitz translation in order to study the proof-theory of intuitionistic propositional and first-order logic. This study is done in the system *Fat* for atomic polymorphism. This system can be characterized as a second order propositional logic in the language $\{\forall_1, \forall_2, \to\}$ such that $\forall_2$-elimination is restricted to atomic instantiations. The aim of the present paper is twofold: [1] to use atomic polymorphism to study the proof theory of schematic systems and [2] to produce high-level translations for a large class of logics.

The two interpretations of natural deduction: how do they fit together?

Per Martin-Löf
Stockholm University

Natural deduction admits of two different interpretations, which it is natural to refer to as the ancient and the modern one. According to the ancient interpretation, which goes back to Aristotle, a natural deduction, or syllogism, in his terminology, is interpreted as a demonstration that the truth of the final proposition follows from the truth of the initial propositions, which is to say, the hypotheses, or assumptions. According to the modern interpretation, a natural deduction is seen as a complex logical object, a proof object, which depends on arbitrary proofs of the assumptions, named by variables, and itself is a proof of the final proposition of the deduction. Conceptual priority is with the modern interpretation. This will be substantiated by showing how the ancient interpretation is derived from the modern interpretation by the special abbreviative device of suppressing proof objects

On Natural Deduction in Classical First-Order-Logic

Federico Aschieri
Vienna University of Technology

The goal of this talk is to endow classical first-order natural deduction with a natural set of reduction rules that also allows a natural proof of Herbrand's Theorem. Instead of using the double negation elimination principle as primitive axiom, the law of the excluded middle will be used. Treating the excluded middle as primitive, rather than deriving it from the double negation elimination, has a key consequence: classical proofs can be described as programs that make hypotheses, test and correct them when they are learned to be wrong. As a corollary, one obtains a simple and meaningful computational interpretation of classical logic.

Postponement of RAA and Glivenko's theorem, revisited

Giulio Guerrieri and Alberto Naibo
Université Paris Diderot • Université Paris 1 – Panthéon-Sorbonne

In the mid-Eighties, Seldin established a normalization strategy for classical logic, which can be considered as a kind of "dual" of the standard normalization strategy given by Prawitz in the mid-Sixties: if a sequent $\Gamma \vdash A$ is derivable in $\mathrm{NK} \setminus \{\forall_i\}$, then there exists a derivation π in $\mathrm{NK} \setminus \{\forall_i\}$ using at most one instance of the *reductio ad absurdum* rule – namely, the last one – and where the remaining part of π corresponds to a derivation π in $\mathrm{NJ} \setminus \{\forall_i\}$. We give here a new and simpler proof of Seldin's result showing that if π is $\rightarrow_i$-free, then the derivation π' is in $\mathrm{NM} \setminus \{\forall_i\}$. As a consequence, we prove a strengthened and more general form of Glivenko's theorem embedding first-order classical logic not only into the fragment $\{\neg, \wedge, \vee, \bot, \rightarrow, \exists\}$ of intuitionistic logic, but also into the fragment $\{\neg, \wedge, \vee, \bot, \exists\}$ of minimal logic.

Logicality, Double-line Rules, and Harmony

Norbert Gratzl and Eugenio Orlandelli
MCMP Munich · Università di Bologna

The inversion principle, and the related notion of harmony, has been extensively discussed in proof-theoretic semantics at least since (Prawitz, 1965). In our paper we make use of multi-conclusion sequent-style calculi to provide a harmonious proof-theoretic analysis of modal logics, and we concentrate on Standard Deontic Logic (SDL) as a running example. Our approach is based on combining display calculi (DSDL) with Došen's characterization of logicality in purely structural terms. We present a genuinely new double-line version of DSDL, i.e. DdlSDL, and we show that DSDL and DdlSDL are deductively equivalent. This equivalence allows us to show that the rules of DSDL are harmonious: the left introduction rules are inverse of the right one inasmuch as they are deductively equivalent to the bottom-up elimination rule of DdlSDL.

Adjunction and Normalization in Categories of Logic

Kosta Došen
University of Belgrade and SANU

Logic inspired by category theory is usually given by "the logic of a category", which is tied to the subobjects in a category of mathematical objects. In categorial proof theory we deal instead with "the category of a logic", where the objects are logical formulae and the arrows are deductions. Logical constants are tied to operations on arrows given by functors. These deductive categories, which are of kinds found important for category theory and mathematics independently of logic, are interesting when they are not preorders, i.e. when it is not the case that there is at most one arrow with the same source and the same target; in other words, when there can be more than one deduction with the same premise and the same conclusion. The same requirement is implied by the BHK interpretation of intuitionistic logic, though it is usually not explicitly mentioned.

Lawvere's characterization of intuitionistic logical constants through adjoint functors, which has a central place in categorial proof theory, is closely related to normalization of deductions in the style of Gentzen and Prawitz. This relationship with one of the central notions of category theory, and of mathematics in general, sheds much light on normalization. Classical deductive logic can also be understood in an interesting way in terms of categorial proof theory, but for it the relationship with adjunction is not to be found to the same extent.

Monoidal logics: De Morgan negations and classical systems

Clayton Peterson
LMU Munich

Monoidal logics (cf. Peterson 2014a,b, 2015) assume category theory as a foundation and define logical systems using specific rules and axiom schemata in order to make explicit their categorical (monoidal) structure. Monoidal logics can be compared to substructural logics (cf. Restall 2000) and, more generally, to display logics (cf. Goré 1998). It is possible to define a translation t from the language of monoidal logics to the language of display logics such that a proof $\varphi \to \psi$ is derivable within specific monoidal deductive systems if and only if $t(\varphi) \vdash t(\psi)$ is derivable within their respective display counterparts. One upshot of this comparison is that monoidal logics can be proven to be weaker and more flexible than substructural logics. For example, in substructural logics, the elimination of double negation(s) is generally thought to be accompanied by the satisfaction of the de Morgan dualities. In contrast, the elimination of double negation(s) can be proven to be independent from the de Morgan dualities in monoidal logics. In this talk, we show how this result can be understood in light of the relationship between the notions of weak distributivity (a.k.a. linear distributivity) and classical deductive systems.

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The natural deduction normal form and coherence

Zoran Petrić
Mathematical Institute, SANU

Coherence results serve to describe the canonical arrows of categories of a particular kind. Some of these results are obtained by using proof-theoretical techniques, in particular Gentzen's cut elimination. The aim of this talk is to show how the expanded normal form of natural deduction derivations is useful for coherence proofs.

Decorating natural deduction

Helmut Schwichtenberg
LMU Munich

Natural deduction provides a perfect link between logic and computation, in its purest form of Church's lambda-calculus: introduction and elimination rules correspond to abstraction and application, and formulas to types. Following Kolmogorov we can view a formula A as a problem, and a constructive proof of A as a solution. Such a solution is a computable function of a certain type determined by A. Computable content arises where the formula A contains inductively defined predicates. The clauses of the inductive definition determine the data type of the solution (or "realizer"). We address the delicate question of how to "decorate" natural deduction proofs in order to optimize their computational content. It is shown that a unique optimal such decoration exists, and some examples are discussed.

A more general general proof theory

Heinrich Wansing
Ruhr-University Bochum

In the early 1970s Dag Prawitz introduced general proof theory, now often also called "structural proof theory," as "a study of proofs in their own right where one is interested in general questions about the nature and structure of proofs." (Prawitz 1974, p. 66) At about the same time, Georg Kreisel used the term "theory of proofs."

In this talk I will suggest to broaden Prawitz's understanding of structural proof theory. The idea is to consider in addition to verifications also falsifications, so as to obtain a theory of proofs and dual proofs. I will motivate this more comprehensive view by taking up some remarks Prawitz has made on falsificationist theories of meaning in (Prawitz 2007) and by considering the natural deduction proof systems for Nelson's constructive proposition logic from his dissertation (Prawitz 1965). In that context, I will discuss the notion of co-implication within Heyting-Brouwer Logic (Rauszer 1980) and the bi-intuitionistic logic 2Int from (Wansing 2013, 2015).

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Is there a "Hilbert thesis"?

Reinhard Kahle
Universidade Nova de Lisboa

In his introductory paper to first-order logic, Jon Barwise writes in the Handbook of Mathematical Logic (1977):

> . . . the informal notion of provable used in mathematics is made precise by the formal notion *provable in first-order logic*. Following a sug[g]estion of Martin Davis, we refer to this view as *Hilbert's Thesis*.

The relation of informal and formal notion(s) of proof is currently under discussion due to the challenges which modern computer provers issue to mathematics, and one can assume that some kind of "Hilbert's Thesis" is widely accepted by the scientific community.

In the first part of our talk we discuss the nature of the thesis advocated by Barwise (including its attribution to Hilbert). We will compare it, in particular, with Church's Thesis about computability. While Church's Thesis refers to one particular model of computation, Hilbert's Thesis is open to arbitrary axiomatic implementations. We will draw some first conclusions from this difference, in particular, concerning the question, how a "Hilbert thesis" could be formulated more specifically.

Church's Thesis receives some evidence from the fact that it blocks any diagonalization argument by the use of *partial functions*. In the second part of the talk, we pose the question whether there could be something like a *partial proof*, giving a formal counterpart to a partial function. The relation would arise from an extension of the Curry-Howard isomorphism to untyped λ terms. In this perspective, the question fits squarely in the tradition of *General Proof Theory*.

On distinguishing proof-theoretic consequence from derivability

Nissim Francez
The Technion, Haifa

According to the common conception of *logical consequence*, it can be defined in two main ways:

– Model-theoretically: For a suitable notion of a model, consequence is taken to be *preservation* (also called *propagation or transmission*) of truth over all models.

$$\Gamma \models \varphi \text{ iff for every model } \mathcal{M}, \text{ if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models \varphi \tag{1}$$

– Proof-theoretically: For a suitable complete proof-system $\mathcal{N}$, consequence is taken as *derivability* in $\mathcal{N}$, denoted '$\vdash_{\mathcal{N}}$'.

$$\Gamma \vdash_{\mathcal{N}} \varphi \text{ iff there exists a derivation } \mathcal{D} \text{ of } \varphi \text{ from (open) assumptions } \Gamma \tag{2}$$

The idea that logical consequence involves preservation of *something*, not necessarily of truth, has been suggested by many. Some examples:

– *Information:* Propagation of the information (in a situation) is underlying consequence of Relevant Logic.

– *Ambiguity:* a proposition p is taken as *ambiguously* between two different propositions, p_t and p_f. A *measure* of ambiguity of an inconsistent Γ is defined as the minimal number of proposition in Γ the treatment of which as ambiguous renders Γ consistent. Propagation of ambiguity is used for defining consequence for paraconsistent logics.

– *Precisification:* In the context of *vagueness*, Logical consequence is preservation of *super-truth* (i.e., truth in all *precisifications*).

A natural question arising is, what is common to all the "things" being suggested as preserved, or propagated, by the various consequence relations mentioned above?

I want to argue that they all serve (either explicitly or implicitly) as *central concepts on which theories of meaning are based.*

Two of the main theories of meaning are the following.

– In MODEL-THEORETIC SEMANTICS (MTS), The central concept is *truth* (in arbitrary models). Meaning is defined as truth-conditions.
– In PROOF-THEORETIC SEMANTICS (PTS), The central concept is *proof*, or more generally, *canonical derivation*, in appropriate meaning-conferring proof-systems.:
  Meaning is *determined* (either implicitly or, as I prefer, explicitly) by the rules of the meaning-conferring system.
– The other propagated "things" mentioned above have a similar role as theories of meaning for Relevant Logic, general paraconsistent logics and for languages with vagueness.
  Consequently, I suggest the following informal principle:

MEANING-BASED LOGICAL CONSEQUENCE: In a theory of meaning $\mathcal{T}$, logical consequence is based on the propagation of the central concept of $\mathcal{T}$.

In this paper I argue that, in spite of the coextensiveness in many logics of derivability and preservation of truth in models, if one adheres to the proof-theoretic semantics theory of meaning then (2) is not the right definition of proof-theoretic consequence. While (1) is faithful to the usual model-theoretic conception of meaning, (2) is not faithful to the PTS conception of meaning.

Instead, I suggest the following conception of proof-theoretic consequence.
– The *(proof-theoretic) meaning* $[\![\varphi]\!]$ of φ, is given by: $[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^c_\Gamma$, a function from contexts Γ to the (possibly empty) collection of *canonical derivations* of φ from Γ ($\Gamma \vdash^c \varphi$).
– The definition of *proof-theoretic consequence* (pt-consequence) rests on the notion of *grounds for assertion* for φ, closely related to $[\![\varphi]\!]$, given by: $GA[\![\varphi]\!] = \{\Gamma \mid \Gamma \vdash^c \varphi\}$.
– Proof-theoretic consequence: ψ is a *proof-theoretic consequence* of Γ ($\Gamma \Vdash \psi$) iff $GA[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$.

The paper studies two definitions of $GA[\![\Gamma]\!]$, based on conjunction (additive) and on fusion (multiplicative).

I show that for intuitionistic logic, but not for classical logic, proof-theoretic consequence coincides with derivability.

High-school sequent calculus and an intuitionistic formula hierarchy preserving identity of proofs

Danko Ilik
Inria Saclay - Île-de-France

We propose to revisit intuitionistic proof systems from the point of view of formula isomorphism and high-school identities. We first isolate a fragment of the intuitionistic propositional sequent calculus (LJ), which we name high-school sequent calculus (HS), such that any proof of LJ can be mapped to one in HS that does not use invertible proof rules. This defines one precise criterion for identity of proofs, a problem open since the early days of intuitionistic proof theory. Second, we extend HS for first-order quantifiers and show how it gives rise to an arithmetical hierarchy for intuitionistic logic. An even neater hierarchy can be obtained if one allows higher types and the intuitionistic axiom of choice.

New Applications of Proof Mining to Nonlinear Analysis

Angeliki Koutsoukou-Argyraki
TU Darmstadt
(joint work with Ulrich Kohlenbach)

Proof mining is a research program in applied proof theory that involves the extraction of new quantitative, constructive information by logical analysis of mathematical proofs that appear to be nonconstructive. The information is 'hidden' behind an implicit use of quantifiers in the proof, and its extraction is guaranteed by certain logical metatheorems, if the statement proved is of a certain logical form. The program was initiated by Georg Kreisel in the 1950's under the name *Unwinding of Proofs*, and its scope can be summarized by Kreisel's general question:

> "What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"

Within the past 15 years, proof mining has been applied by Ulrich Kohlenbach and his collaborators to various fields of Mathematics (see [2] for a general review).

In this talk we will very briefly present two recent applications of proof mining to nonlinear analysis (both involving the study of one-parameter nonexpansive semigroups). The first work ([3]) is also the first application of proof mining to the theory of partial differential equations. It involves the extraction of explicit, effective bounds with respect to the convergence (rates of metastability) of the solution of abstract Cauchy problems generated by multivalued operators, that fulfill certain accretivity properties, in general Banach spaces. Our results were obtained by logical analysis of the proof of a theorem in [1]. The second work ([4]) is another contribution of proof mining to fixed point theory. We give explicit bounds on the computation of approximate common fixed points of one-parameter nonexpansive semigroups on a subset C of a general Banach space. Moreover, we provide the first explicit and highly uniform rate of convergence for an iterative procedure to compute such points for convex C. Our results were obtained by logical analysis of the proof of a theorem in [5].

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On the maximality of classical logic

Mario Piazza and Gabriele Pulcini
University of Chieti-Pescara · State University of Campinas

As is well known, classical propositional logic LK_0 is Post-complete, or maximal: whenever a nontautological formula α is added to it as a new axiom schema, the extended system LK_0^α becomes inconsistent. In other words, the only nontrivial extensions of LK_0 are by *proper* axioms, i.e. formulas that are not closed under uniform substitution. In [2] such extensions of LK_0 are called *supraclassical*. Although cut elimination does not hold in general for supraclassical logics [1] or, it does, but without necessarily entailing the subformula property [3], we show how to fill the gap between classical and supraclassical systems for the propositional fragment. In particular, we show how to *decompose* a proper axiom α into a finite set of atomic, classically underivable, sequents $\mathcal{S}_\alpha$ such that:

1. $\mathsf{LK}_0^{\mathcal{S}\alpha}$ enjoys both cut-elimination and subformula property,
2. LK_0^α is consistent if, and only if, the empty sequent '$\vdash$' is not in $\mathcal{S}_\alpha$,
3. $\mathcal{S}_\alpha$ is the *minimal* axiomatic decomposition allowing cut elimination.

We conclude by showing a way to make extensions infinite while preserving nontriviality.

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Idempotent Coherentisation for First-Order Logic

Roy Dyckhoff and Sara Negri
University of St Andrews · University of Helsinki

Skolem showed in 1920 that, by use of new relation symbols, every first-order sentence is in some sense replaceable by a single $\forall\exists$-sentence; the precise sense can easily be seen as the construction of something slightly stronger than a conservative extension. A variation on the argument shows that the new sentence can be ensured to be a finite conjunction of "special coherent implications", i.e. sentences that are universally quantified implications with, as antecedent, a conjunction of atoms and, as succedent, a positive formula, i.e. wlog a disjunction of existentially quantified conjunctions of atoms. [Such sentences are also called "geometric axioms".] Thus, every first-order theory has a coherent conservative relational extension. We survey some of the history of this folklore result, including what seems to be its first appearance in the unpublished Montréal Master's thesis of Antonius (1975), and present a new coherentisation algorithm with the idempotence property, i.e. there is no use of normal form (e.g. NNF or PNF) and sentences that are [almost] of the right form are left [almost] unchanged. Examples and applications will be presented if time permits. The proof theory of theories axiomatised by sentences of the described form is well-known to be particularly simple (using "dynamical proofs"); the result shows that, modulo some extra relation symbols, all first-order theories are of this form.

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Adjunction and Normalization in Categories of Logic

Miloš Adžić
University of Belgrade
[unable to attend]

In the course of his conversations with Hao Wang (Hao Wang, *A Logical Journey: From Gödel to Philosophy*), Kurt Gödel expanded on, among others, two very interesting topics. One is the notion of the *absolute proof* and the other one being his intensional *Theory of concepts*, which should, when developed, further secure his conceptual realism.

Regarding the notion of the absolute proof, already in his 1946 *Princeton Bicentennial Lecture* Gödel suggested that the its analysis should be akin to Turing's analysis of the concept of computability:

> Tarski has sketched in his lecture the great importance (and I think justly) of the concept of general recursiveness (or Turing computability). It seems to me that this importance is largely due to the fact that with this concept one has succeeded in giving a absolute definition of an interesting epistemological notion, i.e. one not depending on the formalism chosen. (Kurt Gödel, *Complete Works II*, p. 150)

On the other hand

> In all other cases treated previously, such as demonstrability or definability, one has been able to define them only relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or decidability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion. This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability). It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism or the paradox of Richard. But close examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defining them, or at least, that certain very closely related concepts may be definable in an absolute sense. (*Ibid.*, p. 150)

As pointed out in his conversations with Hao Wang (*A Logical Journey: From Gödel to Philosophy*, p. 270), some of the difficulties involved in defining the concept of proof have to do with intensional paradoxes, which, at least in principle, his theory of concepts should help us understand better.

We take some of the remarks Gödel made regarding these topics as an inspiration and offer a way in which one could interpret them from the standpoint of the present day logic. For instance, his interest in the concept of proof and its intensional underpinnings is also shared by those working in the field of categorial proof theory.

On Proof Compressions in Sequent Calculi and Natural Deductions

Lev Gordeev
University of Tübingen, University of Ghent and PUC Rio de Janeiro
[unable to attend]

In traditional Proof Theory finite proofs/derivations/deductions are presented as rooted trees whose nodes are labeled with proof objects (e.g. sequents or formulas). A more general approach allows more liberal dag-like presentation ('dag' = directed acyclic graph). This opens up several size reducing tree-to-dag proof compression opportunities. The most natural proof compression idea is to merge distinct nodes labeled with identical proof objects.

Theorem 1. In a given sequent calculus S, any tree-like deduction d of a given sequent s is constructively compressible to a dag-like deduction d' of s in which sequents occur at most once. Thus, in d', distinct nodes are supplied with distinct sequents (that also occur in d).

Consequently, the size of d' can't exceed the total number of distinct sequents occurring in d, which in several cases can exponentially reduce the size of d. Moreover, if S is cutfree, then by the subformula property we conclude that sequents occurring in d' contain only subformulas of s. Analogous dag-like compressions of Prawitz-style tree-like natural deductions are more involved. This is due to the assumption's discharging operation. On the other hand, since natural deductions' nodes are labeled with single formulas, there is hope to get polynomial control over the size of d', provided that the subformula property holds.

Theorem 2. A natural embedding of Hudelmaier's sequent calculus for purely implicational logic into analogous Prawitz-style tree-like natural deduction calculus followed by appropriate dag-like "horizontal compression" allows us to obtain polynomial-size dag-like natural deductions d' of arbitrary tree-like inputs d.

A suitable formalization of the last theorem should prove the conjecture NP = PSPACE.

Presentations

General Proof Theory

Tübingen, 27-29 November 2015

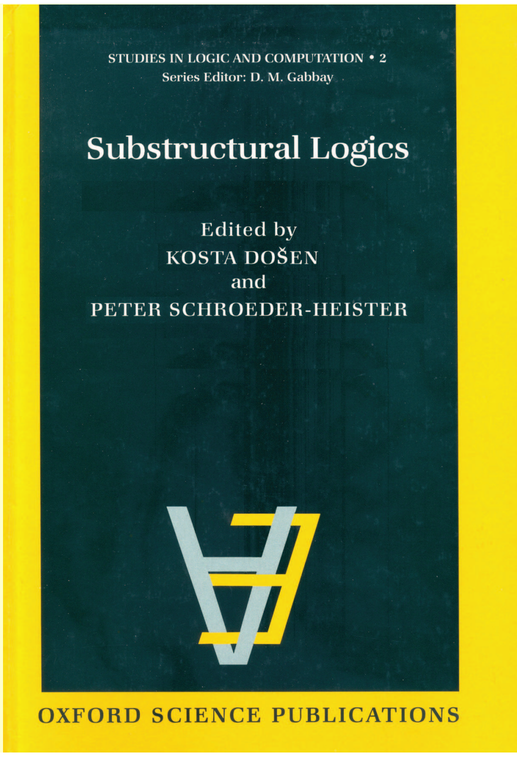
Opening

Peter Schroeder-Heister

Our internal competitor

Substructural Logics

25 years after



GPT Opening 27.11.2015 – p. 3

Why General Proof Theory?

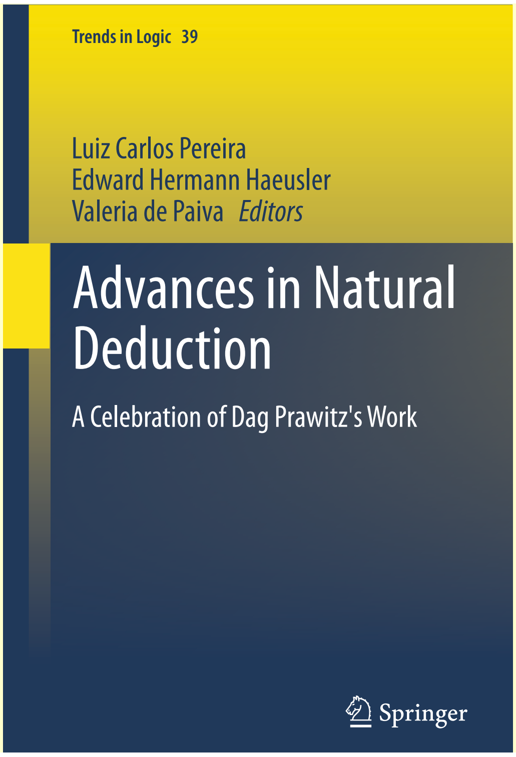
- Its significance is still underestimated, both in mathematical and philosophical proof theory.
- General Proof Theory is the study of proofs in their own right, not merely under the aspect of what proofs prove or can prove.
- This is the “intensional” aspect of proofs.
- In proof-theoretic semantics (including my own work) it is underrepresented, actually almost non-existent.
- The term was coined by Prawitz in 1971, but the topic was already implicitly put on the agenda by Prawitz in his book on natural deduction of 1965.

GPT Opening 27.11.2015 – p. 4

Celebrating 50 years of
Dag Prawitz's 'Natural Deduction'

NATURAL
DEDUCTION

STOCKHOLM 1965
RIO 2001
TÜBINGEN 2015



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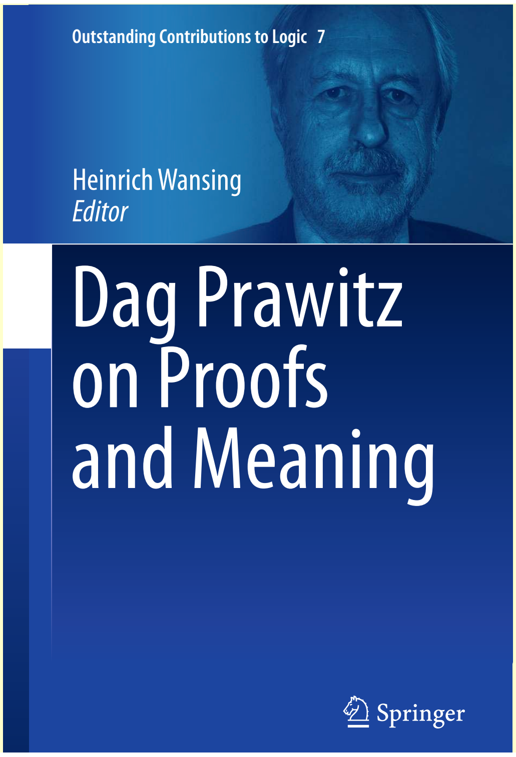

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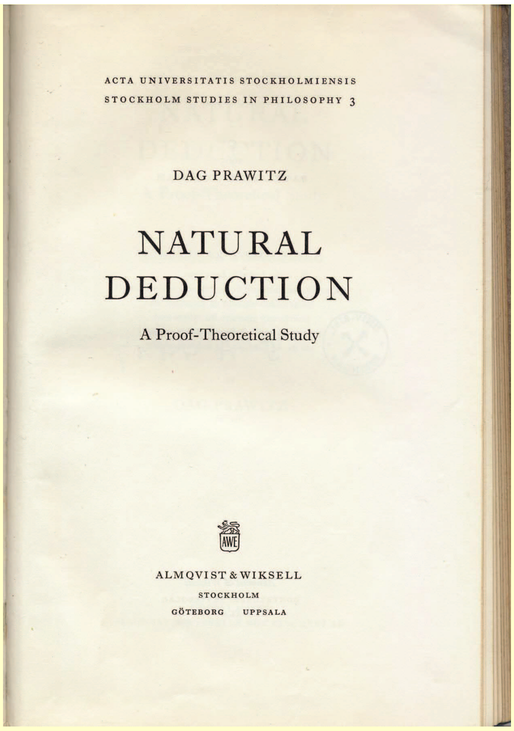


ACTA UNIVERSITATIS STOCKHOLMIENSIS
STOCKHOLM STUDIES IN PHILOSOPHY 3

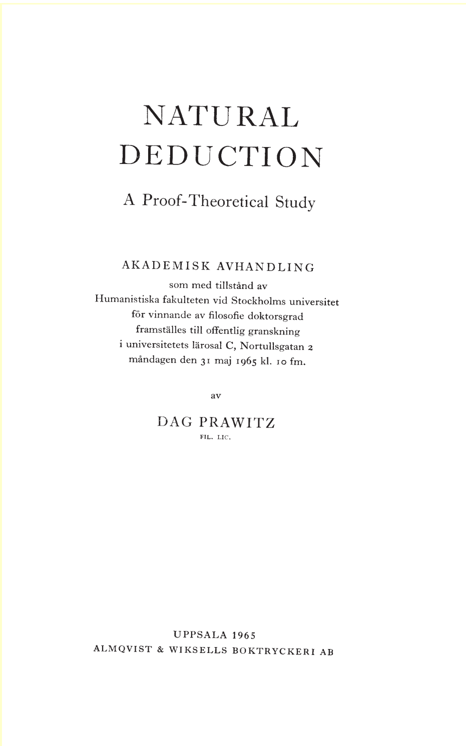
DAG PRAWITZ

NATURAL
DEDUCTION

A Proof-Theoretical Study

ALMQVIST & WIKSELL
STOCKHOLM
GÖTEBORG UPPSALA

GPT Opening 27.11.2015 – p. 9



NATURAL
DEDUCTION

A Proof-Theoretical Study

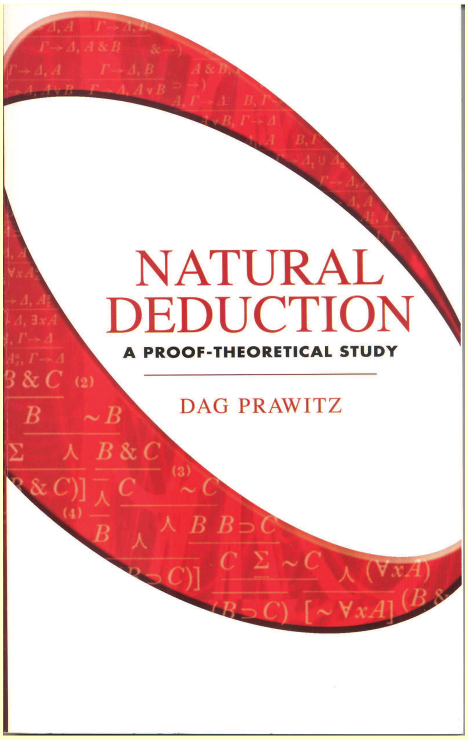
AKADEMISK AVHANDLING
som med tillstånd av
Humanistiska fakulteten vid Stockholms universitet
för vinnande av filosofie doktorsgrad
framställes till offentlig granskning
i universitetets lärosal C, Nortullsgatan 2
måndagen den 31 maj 1965 kl. 10 fm.

av

DAG PRAWITZ
FIL. LIC.

UPPSALA 1965
ALMQVIST & WIKSELLS BOKTRYCKERI AB

GPT Opening 27.11.2015 – p. 10



GPT Opening 27.11.2015 – p. 11

Why is “Natural Deduction” so important?

- Because natural deduction is so important
  - The handling of assumptions (Gentzen-Jaśkowski)
  - The systematics of introductions and eliminations
- Because the metatheory of deduction as developed in “Natural Deduction” is so important
  - inversion
  - normalisation
  - the handling of reduction

GPT Opening 27.11.2015 – p. 12

The gap in Gentzen

- Philosophically, the calculus of natural deduction is the "natural" basic system
- Technically, all results are dealt with in the sequent calculus, which is not considered philosophically significant
- Consequently, before "Natural Deduction", natural deduction had not (or almost not) been taken notice of

The contribution of "Natural Deduction"

- Prawitz closes this gap, by giving natural deduction its autonomous standing
- In doing so, by his theory of reduction of proofs, he implicitly develops a notion of identity of proofs
- This is a big step beyond Gentzen
- It means that *General Proof Theory* is the main topic of the "Natural Deduction"

Therefore a conference on *General Proof Theory* is the right way of honouring Prawitz and the work done in "Natural Deduction".

The impact of “Natural Deduction”

- Type theory
  - more generally, the computation as deduction, or deduction as computation paradigm

- The development of proof-theoretic semantics

GPT Opening 27.11.2015 – p. 15

The didactic point of view

Not very many books in mathematical logic are so well readable.

One can give “Natural Deduction” to anybody who has only a very elementary knowledge in logic.

This is an aspect Prawitz’s work shares with that of Gentzen

GPT Opening 27.11.2015 – p. 16

Personal remarks

Announcement

There will be a book on general proof theory, called

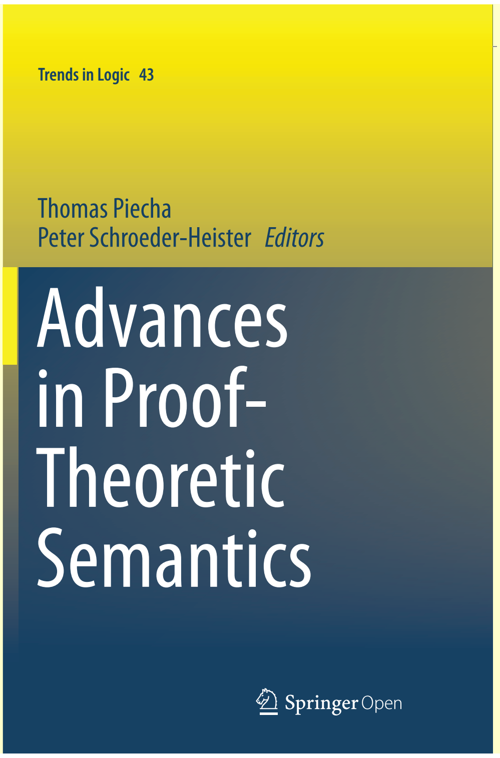
“General Proof Theory”

"Proof-theoretic semantics"

will be postponed

Some open questions in connection with natural deduction

- Inversion: What is the proper (intensional) notion of harmony
- Reduction: What is a proper reduction (in contradistinction to just a shortcut)?
- Sequent calculus: What is the proper model of hypothetical reasoning?
  $x : A \vdash t : B$ or $f : A \vdash B$?
- Atomic base: What is the proper treatment of atoms?



GPT Opening 27.11.2015 – p. 21

Acknowledgements

Programme:

- Kosta Došen, Thomas Piecha

Organisation:

- Marine Gaudefroy-Bergmann, Thomas Piecha

GPT Opening 27.11.2015 – p. 22

I wish you

a good conference

GPT Opening 27.11.2015 – p. 23

The Russell-Prawitz translation and schematic rules: a view from proof-theory

Luiz Carlos Pereira Edward Hermann Haeusler

PUC Rio de Janeiro

Tübingen 2015

Translations

In the late twenties and early thirties of last century several results were obtained concerning some relations between classical logic (CL) and intuitionistic logic (IL), and between classical arithmetic (PA) and intutionistic arithmetic (HA).

In 1927 Glivenko proved two important results relating classical propositional logic (CPL) to intuitionistic propositional logic (IPL).

In 1925 Kolmogorov proved that CPL could be translated into IPL (in fact he showed that a certain fragment of CPL could be translated in the same fragment of MPL).

In 1933 Gödel defined an interpretation of PA into HA, and in the same year Gentzen defined a different interpretation of PA into HA.

These interpretations/translations/embeddings were defined in a foundational environment.

Translations

Main idea: to reduce a foundational problem in a classical environment to a foundational problem in a constructive environment.

The concrete case: the consistency problem

If Heyting's Arithmetic (HA) is consistent, so is Peano's Arithmetic (PA)
– Gödel, Gentzen

A first general proof-theoretical approach to translations:
Prawitz and Malmnäs 1968:
A survey of some connections between classical, intuitionistic and minimal logic.

Prawitz & Malmnäs

1. Interpretability

Let S_1 and S_2 be two logical systems. A function from the language of S_1 to the language of S_2 will be called a translation. If F is a translation from $L[S_1]$ into $L[S_2]$ such that

$$S_1 \vdash A \quad \text{iff} \quad S_2 \vdash F[A]$$

then S_1 is said to be interpretable in S_2.

2. Interpretability with respect to derivability

If for each set $\Gamma \cup \{A\}$ of formulas of S we have:

$$\Gamma \vdash_{S_1} A \quad \text{iff} \quad F[\Gamma] \vdash_{S_2} F[A]$$

we say that S_1 is interpretable in S_2 with respect to derivability.

Prawitz & Malmnäs

3. Schematically interpretable

We are given a number of schemata:

(i) Defining the value of F for atomic formulas.

(ii) For each logical constant c we use the schemata to define inductively the value of *F* for formulas with *c* as the principal sign.

Example: $F[(A \wedge B)] = \varphi(F[A], F[B])$

4. Literal translation

We use the same constant *c*:
Example: $F[(A \wedge B)] = (F[A] \wedge F[B])$

Prawitz & Malmnäs

THEOREM A.[3] Let $A^{\sim\sim}$ be the result obtained from A by inserting two negation signs in front of each part of A, and let $\Gamma^{\sim\sim}$ be the result obtained from Γ by carrying out this transformation on each member of Γ. Then:

$$\Gamma \vdash_{\mathsf{C}} A \quad \text{if and only if} \quad \Gamma^{\sim\sim} \vdash_{\mathsf{I}} A^{\sim\sim}.$$

Also

$$\Gamma \vdash_{\mathsf{C}} A \quad \text{if and only if} \quad \Gamma^{\sim\sim} \vdash_{\mathsf{M}} A^{\sim\sim}.$$

COROLLARY 1.[4] Classical logic is interpretable (also interpretable with respect to derivability) in intuitionistic and minimal logic by the translation * defined schematically as follows:

(1) $(Pt_1t_2\ldots t_n)^* = \sim\sim Pt_1t_2\ldots t_n$.
(2) $(A \vee B)^* = \sim(\sim A^* \& \sim B^*)$.
(3) $(\exists x A)^* = \sim\forall x \sim A^*$.
(4) Translation of absurdity, conjunctions, implications and universal formulas: Literal.

Prawitz & Malmnäs

Theorem B.[7] Let A' be the result obtained from A by simultaneously replacing each part B of A by $B \vee \wedge$, and let Γ' be the result obtained from Γ by carrying out this transformation on each member of Γ. Then:

$$\Gamma \vdash_{\mathrm{I}} A \quad \text{if and only if} \quad \Gamma' \vdash_{\mathrm{M}} A'.$$

Corollary 1. Intuitionistic logic is interpretable in minimal logic by the translation $^{\times}$ defined schematically as follows:

(1) $(A \supset B)^{\times} = A^{\times} \supset (B^{\times} \vee \wedge)$.
(2) $(\forall x A)^{\times} = \forall x (A^{\times} \vee \wedge)$.
(3) Literal translation in other cases.

Theorem C. Minimal logic is interpretable in intuitionistic logic by the translation $''$ where A'' is the formula obtained from A by replacing every occurrence of $\wedge$ by P, where P is the alphabetically first 0-place predicate symbol that does not occur in A.

Kolmogorov's translation

(1) $\mathrm{Ko}[A] = \neg\neg A$, $\quad A$ atomic.

(2) $\mathrm{Ko}[\bot] = \bot$

(3) $\mathrm{Ko}[A \wedge B] = \neg\neg(\mathrm{Ko}[A] \wedge \mathrm{Ko}[B])$

(4) $\mathrm{Ko}[A \vee B] = \neg\neg(\mathrm{Ko}[A] \vee \mathrm{Ko}[B])$

(5) $\mathrm{Ko}[A \to B] = \neg\neg(\mathrm{Ko}[A] \to \mathrm{Ko}[B])$

(6) $\mathrm{Ko}[\forall x A(x)] = \neg\neg\forall x\, \mathrm{Ko}[A(x)]$

(7) $\mathrm{Ko}[\exists x A(x)] = \neg\neg\exists x\, \mathrm{Ko}[A(x)]$

Gödel's translation

(1) $\mathrm{G\ddot{o}}[A] = \neg\neg A$

(2) $\mathrm{G\ddot{o}}[\bot] = \bot$

(3) $\mathrm{G\ddot{o}}[A \wedge B] = \mathrm{G\ddot{o}}[A] \wedge \mathrm{G\ddot{o}}[B]$

(4) $\mathrm{G\ddot{o}}[A \vee B] = \neg(\neg\,\mathrm{G\ddot{o}}[A] \wedge \neg\,\mathrm{G\ddot{o}}[B])$

(5) $\mathrm{G\ddot{o}}[A \rightarrow B] = \neg(\mathrm{G\ddot{o}}[A] \wedge \neg\,\mathrm{G\ddot{o}}[B])$

(6) $\mathrm{G\ddot{o}}[\forall x A(x)] = \forall x\,\mathrm{G\ddot{o}}[A(x)]$

(7) $\mathrm{G\ddot{o}}[\exists x A(x)] = \neg\forall x \neg\,\mathrm{G\ddot{o}}[A(x)]$

Gentzen's translation

(1) $\mathrm{Ge}[A] = \neg\neg A$

(2) $\mathrm{Ge}[\bot] = \bot$

(3) $\mathrm{Ge}[A \wedge B] = \mathrm{Ge}[A] \wedge \mathrm{Ge}[B]$

(4) $\mathrm{Ge}[A \vee B] = \neg(\neg\,\mathrm{Ge}[A] \wedge \neg\,\mathrm{Ge}[B])$

(5) $\mathrm{Ge}[A \rightarrow B] = (\mathrm{Ge}[A] \rightarrow \mathrm{Ge}[B])$

(6) $\mathrm{Ge}[\forall x A(x)] = \forall x\,\mathrm{Ge}[A(x)]$

(7) $\mathrm{Ge}[\exists x A(x)] = \neg\forall x \neg\,\mathrm{Ge}[A(x)]$

General approaches

Other general approaches:

1. Wójcicki, 1988
2. Epstein, 1990
3. Da Silva, D'Ottaviano and Sette, 1999

Normalization and translations

Double-negation translations can be easily justified by Prawitz' Normalization strategy for classical logic.

1. Restrict the language of classical first order logic to the fragment $\{\neg, \wedge, \rightarrow, \forall\}$. Of course nothing is lost with this restriction.
2. Reduce all applications of the classical absurd rule to atomic applications, i.e., to applications with atomic conclusions.
3. Apply your favorite normalization strategy for intuitionistic first order logic.

Normalization and translations

1. Restrict the language of classical first order logic to the fragment $\{\neg, \wedge, \vee, \rightarrow, \bot, \exists\}$. Again, nothing is lost with this restriction.
2. Show that every derivation Π of $\Gamma \vdash A$ in this fragment can be transformed into a derivation Π' of $\Gamma \vdash A$ such that Π' contains at most one application of the classical absurd rule, and in case this application does occur, it is the last rule applied in Π'.
3. Apply your favorite normalization strategy for intuitionistic first order logic.

General result

Theorem 1
Let G be a translation/interpretation of L_2 into L_1 and let L_3 be an intermediate logic between L_1 and L_2. Then G is also a translation of L_2 into L_3.

Theorem 2
The translation G from L_2 into L_3 CAN NOT be a translation from L_3 into L_1!

Another use for translations: to study the proof-theory of other systems!

This could be considered as a case of "reductive proof-theory".

Russell-Prawitz translation

$\forall X.X$

$\forall X((A \to (B \to X)) \to X)$

$\forall X((A \to X) \to ((B \to X) \to X))$

$\forall X(\forall x(A \to X) \to X)$

The second order system F_{at}

The system F_{at} was introduced by Fernando Ferreira and developed by F. Ferreira and Gilda Ferreira.

F. Ferreira and G. Ferreira

- The faithfulmess of F_{at}: a proof-theoretic proof (2015)
- The faithfulness of of atomic polymorphism, *Trends in Logic* (2014)
- Atomic polymorphism, *JSL*, vol. 76 (2013)
- Commuting conversions vs. the standard conversions of the "good" connectives, *Studia Logica*, vol. 92 (2009).

G. Ferreira

- Strong Normalization for IPC via atomic F (2015)

T. Sandqvist

- A note on definability of logical operators in second-order logic (manuscript – 2008)

The second order system F_{at}

Main idea

Language: second order propositional restricted to $\forall$ and $\rightarrow$.

Rules: Natural deduction with the restriction that in the conclusion $A[C/X]$ of $\forall X.A$ ($\forall$-elim), C is atomic.

Remark:
With the predicative restriction (atomic instantiations) we get the subformula principle and good "inductive measures"

Overflow

Remark 2:
Overflow in F_{at}: For formulas of the form:

$$\forall X.X$$
$$\forall X((A \rightarrow (B \rightarrow X)) \rightarrow X)$$
$$\forall X((A \rightarrow X) \rightarrow ((B \rightarrow X) \rightarrow X))$$

It is possible to deduce

$$C$$
$$((A \rightarrow (B \rightarrow C)) \rightarrow C)$$
$$((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C))$$

for any C.

The case of implication

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall X((A \to X) \to ((B \to X) \to X))}{(A \to D_2) \to ((B \to D_2) \to D_2)}\,\text{I.H.} \qquad \dfrac{\dfrac{\dfrac{[A \to D_1 \to D_2)] \qquad [A]}{D_1 \to D_2} \qquad [D_1]}{D_2}}{A \to D_2}}{(B \to D_2) \to D_2} \qquad \mathscr{D}}{\dfrac{D_2}{D_1 \to D_2}}}{\dfrac{(B \to (D_1 \to D_2)) \to (D_1 \to D_2)}{(A \to (D_1 \to D_2)) \to ((B \to (D_1 \to D_2)) \to (D_1 \to D_2))}}
$$

where $\mathscr{D}$ is the deduction

$$
\dfrac{\dfrac{\dfrac{[B \to (D_1 \to D_2)] \qquad [B]}{D_1 \to D_2} \qquad [D_1]}{D_2}}{B \to D_2}
$$

Prawitz' translation

1. $\mathrm{Pr}[P] = P$, for P atomic.
2. $\mathrm{Pr}[\bot] = \forall X.X$
3. $\mathrm{Pr}[(A \to B)] = (\mathrm{Pr}[A] \to \mathrm{Pr}[B])$
4. $\mathrm{Pr}[(A \wedge B)] = \forall X((\mathrm{Pr}[A] \to (\mathrm{Pr}[B] \to X)) \to X)$
5. $\mathrm{Pr}[(A \vee B)] = \forall X((\mathrm{Pr}[A] \to X) \to (\mathrm{Pr}[B] \to X) \to X))$

Example

Disjunction introduction

$$\dfrac{A}{A \vee B}$$

is translated into:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \qquad [A \to X]}{X}}{((B \to X) \to X)}}{((A \to X) \to ((B \to X) \to X))}}{\forall X((A \to X) \to ((B \to X) \to X))}$$

Example

Disjunction elimination

$$\dfrac{A \vee B \qquad \begin{matrix}[A] \\ | \\ C\end{matrix} \qquad \begin{matrix}[B] \\ | \\ C\end{matrix}}{C}$$

is translated as:

$$\dfrac{\dfrac{\dfrac{\forall X((A \to X) \to ((B \to X) \to X))}{((A \to C) \to ((B \to C) \to C))} \qquad \dfrac{\begin{matrix}[A] \\ | \\ C\end{matrix}}{(A \to C)}}{((B \to C) \to C)} \qquad \dfrac{\begin{matrix}[B] \\ | \\ C\end{matrix}}{(B \to C)}}{C}$$

Russell-Prawitz translation

$$\vdash_{\mathrm{I}} A \quad \text{iff} \quad \vdash_{\mathrm{F_{at}}} \mathrm{Pr}[A]$$

Strong normalization for $\mathrm{F_{at}}$

Definition
We say that a term $t : A$ is reducible *iff $t \in Red(A)$, with $Red(A)$ defined by induction on the complexity of the type {formula} A as follows:*

1. *For C an atomic type, $t \in Red(C)$ iff t is strongly normalizable.*
2. *$t \in Red(A \to B)$ iff for all q, if $q \in Red(A)$, then $tq \in Red(B)$.*
3. *$t \in Red(\forall X.A)$ iff for all atomic types C, $tC \in Red(A[C/X])$.*

Strong normalization for I

1. Embedding I into F_{at}.
   This result depends on "instantiation overflow".

2. Beta-conversions translated into beta-eta conversions.

Moving to some generalisation

From "concrete systems" to "abstract systems"!

Re-visiting schematic rules

In what sense is natural deduction natural?

Two traditional answers:

1. Faithfull representation

2. Meaning theory

Prawitz, *Natural Deduction*

Systems of natural deduction, invented by Jáskowski and by Gentzen in the early 1930's, constitute a form for the development of logic that is natural in many respects. In the first place, there is a similarity between natural deduction and intuitive, informal reasoning. The inference rules of the systems of natural deduction correspond closely to procedures common in intuitive reasoning, and when informal proofs—such as are encountered in mathematics for example—are formalized within these systems, the main structure of the informal proofs can often be preserved. This in itself gives the systems of natural deduction an interest as an explication of the informal concept of logical deduction.

Prawitz, *Natural Deduction*

Gentzen's variant of natural deduction is natural also in a deeper sense. His inference rules show a noteworthy systematization, which, among other things, is closely related to the interpretation of the logical signs. Futhermore, as will be shown in this study, his rules allow the deduction to proceed in a certain direct fashion, affording an interesting normal form for deductions. The result that every natural deduction can be transformed into this normal form is equivalent to what is known as the *Hauptsatz* or the *normal form theorem*, a basic result in proof theory, which was established by Gentzen for the calculi of sequents. The proof of this result for systems of natural deduction is in many ways simpler and more illuminating.

"With one exception, the inference rules are of two kinds, viz., introduction rules and elimination rules."

This lead us to two new questions!

Schematic rules

What's an introduction rule for an operator φ?

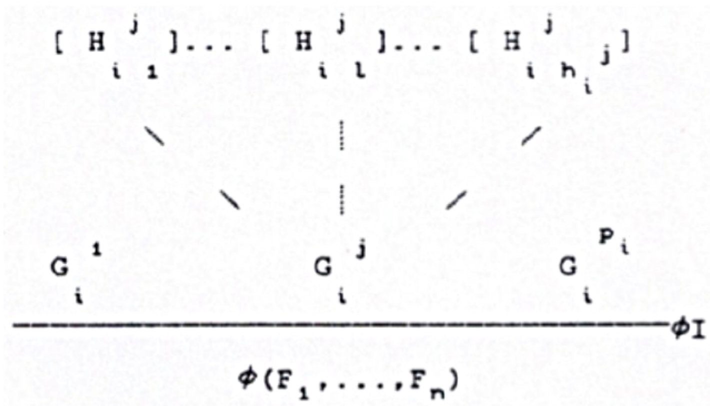
"The introduction rule for a logical constant allows an inference to a formula that has the constant as principal sign."

What's an elimination rule for an operator φ?

"The elimination rule for a constant, on the other hand, allows an inference from a formula that has the constant as principal sign".

In 1978 Dag Prawitz proposed an answer to these questions by means of schematic introduction and elimination rules.

Schematic rules



Schematic rules

$$\dfrac{\Phi(F_1,\dots,F_n) \quad H^{1}_{1,1} \quad \dots \quad H^{p_s}_{s,h_s} \quad \begin{matrix}\Gamma_1\\ \vdots \\ A\end{matrix} \quad \dots \quad \begin{matrix}\Gamma_i\\ \vdots \\ A\end{matrix} \quad \dots \quad \begin{matrix}\Gamma_s\\ \vdots \\ A\end{matrix}}{A}\,\Phi E$$

Schematic rules

The *elimination rule* for an *n*-ary sentential operation φ whose introduction rule is the one stated in Section IV.1 above is given by one rule schema of the form

$$\dfrac{\varphi(F_1, F_2, \dots, F_n) \quad \begin{matrix}\Gamma_1\\ A\end{matrix} \dots \begin{matrix}\Gamma_i\\ A\end{matrix} \dots \begin{matrix}\Gamma_{pi}\\ A\end{matrix}}{A}$$

with *r* minor premisses of the form *A* and where assumptions occurring above the *i*:th ($i \leqslant r$) minor premiss may be bound when they are of one of the forms shown in Γ_i

$$\Gamma_i = \bigwedge\nolimits_{i\, k \leq h_1} H^i_{1,k} \to G^i_1, \bigwedge\nolimits_{i\, k \leq h_2} H^i_{2,k} \to G^i_2, \dots, \bigwedge\nolimits_{i\, k \leq h_{p_i}} H^i_{p_i,k} \to G^i_{p_i}$$

Prawitz also proposed a constructive version of the well- known classical truth-functional completeness:

If the introduction and elimination rules for an operator φ are instances of the schematic introduction and elimination rules, then φ is intuitionistically definable.

Weak completeness

Definition of the function * (Wong Hao Chi)

$A^* = A, \quad$ for A atomic

$[(A \text{ op } B)]^* = ([A]^* \text{ op } [B]^*)$

$[\varphi(F_1, \ldots, F_n)]^* = \vee_{i(\wedge j(\wedge k([H(I,j,k)]^* \rightarrow [G(I,j)]^*)}$

Theorem
Let A and Γ be in $L[\mathrm{I} + \varphi]$. Then, $\Gamma \vdash_{\mathrm{I}+\varphi} \implies \Gamma^* \vdash_{\mathrm{I}} A^*$.

Schematic rules and F_{at}

Connecting schematic rules with atomic polymorphism

Let φ be an operator given by the rules:

φ-introduction:

$$\frac{\begin{array}{cc} [\Gamma_1] & [\Gamma_2] \\ | & | \\ P_1 & P_2 \end{array}}{\varphi(P_1, P_2, -)}$$

φ-elimination:

$$\frac{\varphi(P_1, P_2, -) \qquad \begin{array}{c} [\Gamma_1 \Rightarrow P_1] \ [\Gamma_2 \Rightarrow P_2] \\ | \\ R \end{array}}{R}$$

F_{at}-translation

$\forall X(((\Gamma_1 \to P_1) \to X) \to (((\Gamma_2 \to P_2) \to X) \to X))$

Main idea
Use F_{at} to do the proof-theory of φ, as it was done for intuitionistic propositional logic!

Schematic F_{at}
A generalized instantiation overflow.

Generalized instantiation overflow

Conjunctive-forms: $\forall X((R_1 \to (R_2 \to (\ldots \to (R_n \to X)\ldots) \to X)$

Disjunctive-forms: $\forall X((R_1^* \to X) \to ((R_2^* \to X) \to ((R_n^* \to X) \to X)\ldots)$

From

$$\forall X((R_1 \to (R_2 \to (\ldots \to (R_n \to X)\ldots) \to X)$$

and

$$\forall X((R_1^* \to X) \to ((R_2^* \to X) \to ((R_n^* \to X) \to X)\ldots)$$

derive

$$((R_1^* \to (R_2^* \to (\ldots \to (R_n^* \to C)\ldots) \to C)$$

and

$$((R_1^* \to C) \to ((R_2^* \to C) \to ((R_n^* \to C) \to C)\ldots)$$

for any C.

The flattening problem

$*$-introduction:

$$\dfrac{\begin{matrix}[P_1]\\ | \\ P_2\end{matrix}}{*(P_1, P_2, P_3)} \qquad\qquad \dfrac{P_3}{*(P_1, P_2, P_3)}$$

$**$-introduction:

$$\dfrac{\begin{matrix}[P_1]\\ | \\ P_3\end{matrix}}{**(P_1, P_2, P_3)} \qquad\qquad \dfrac{P_2}{**(P_1, P_2, P_3)}$$

The flattening problem

$*$-elimination:

$$\dfrac{*(P_1, P_2, P_3) \quad P_1 \quad \begin{matrix}[P_2]\\ | \\ R\end{matrix} \quad \begin{matrix}[P_3]\\ | \\ R\end{matrix}}{R}$$

$**$-elimination:

$$\dfrac{**(P_1, P_2, P_3) \quad P_1 \quad \begin{matrix}[P_2]\\ | \\ R\end{matrix} \quad \begin{matrix}[P_3]\\ | \\ R\end{matrix}}{R}$$

The flattening problem

The same rule! But the two operators are not intuitionistically equivalent!

The operator would be:

$$\forall X(P_1 \to ((P_2 \to X) \to ((P_3 \to X) \to X)))$$

The moral is: We do need high-level rules!

The flattening problem

$$\dfrac{*(P_1, P_2, P_3) \qquad \begin{matrix}[P_1 \Rightarrow P_2] \\ | \\ R\end{matrix} \qquad \begin{matrix}[P_3] \\ | \\ R\end{matrix}}{R}$$

The operator $*$ is:

$$*(P_1, P_2, P_3) \leftrightarrow ((P_1 \to P_2) \vee P_3)$$

$$\forall X(((P_1 \to P_2) \to X) \to ((P_3 \to X) \to X))$$

The flattening problem

$$\dfrac{**(P_1, P_2, P_3) \qquad \begin{matrix}[P_1 \Rightarrow P_3] \\ | \\ R\end{matrix} \qquad \begin{matrix}[P_2] \\ | \\ R\end{matrix}}{R}$$

The operator $**$ is:

$$**(P_1, P_2, P_3) \leftrightarrow ((P_1 \rightarrow P_3) \vee P_2)$$

$$\forall X(((P_1 \rightarrow P_3) \rightarrow X) \rightarrow ((P_2 \rightarrow X) \rightarrow X))$$

The flattening problem

Wrong direction

1. Take the set of introduction rules for $*$ and $**$.
2. Take the generalized elimination rule.
3. We have a new operator $***$.

$$***(P_1, P_2, P_3) \leftrightarrow (P_1 \rightarrow (P_3 \vee P_2))$$

$$\forall X(P_1 \rightarrow ((P_2 \rightarrow X) \rightarrow ((P_3 \rightarrow X) \rightarrow X)))$$

Small questions

1. The image of a normal ICP derivation does not need to be normal in F_{at}. Consider a normal derivation Π in IPC and its not normal image $Pr[\Pi]$. This not normal image can be reduced to a normal derivation $Pr[\Pi]^*$.
   What is the relationship between $Pr[\Pi]^*$ and Π?

2. Modifying the question: consider two different normal derivations Π_1 and Π_2 in IPC and their not normal images $Pr[\Pi_1]$ and $Pr[\Pi_2]$ in F_{at}.
   Do the normal forms $Pr[\Pi_1]^*$ and $Pr[\Pi_2]^*$ of $Pr[\Pi_1]$ and $Pr[\Pi_2]$ have to be different?

The future

1. Use the F_{at} system to study the identity problem (as in Widebäck) for disjunction.

2. The use of schematic elimination rules naturally introduces permutative reductions. Use F_{at} to eliminate these "undesirable" reductions.

3. Use the Prawitz-Russell translation to analyse the i-axioms and the epsilon-axioms.

4. Extensions to schematic rules with restrictions.

A general translation based on the subformula principle

We shall consider logics whose operators admit the schematic definitions and whose derivations satisfy the subformula principle.

$$\dfrac{\begin{array}{c}[\phi^1_1],\ldots,[\phi^1_{j_1}]\\ \vert \\ \beta_1\end{array} \qquad \ldots \qquad \begin{array}{c}[\phi^n_1],\ldots,[\phi^n_{j_n}]\\ \vert\\ \beta_n\end{array}}{\mathbf{c}(\beta_1,\ldots,\beta_n,\phi^1_1,\ldots,\phi^1_{j_1},\ldots,\phi^n_1,\ldots,\phi^n_{j_1},\gamma_1,\ldots,\gamma_m)}$$

Figure 1: $\mathbf{c}$-introduction rule schema

Definition 5 (ι-axiom) *Consider an introduction rule r for an operator* $\mathbf{c}$ *as shown in Figure 1. The ι-axiom concerning this rule schema r instantiated to formulas β_i and $\phi^i_1,\ldots,\phi^i_{j_i}$, denoted by $\iota(r,\beta_i,\phi^j,\gamma_k)$, is the following implicational formula:*

$$(\phi^1_1 \to (\ldots \to (\phi^1_{j_1} \to \beta_1))) \to \ldots(\phi^n_1 \to (\ldots \to (\phi^n_{j_n} \to \beta_n))) \to p_{\mathbf{c}(\beta_i,\phi^j,\gamma_k)})$$

A general translation based on the subformula principle

$$\dfrac{\mathbf{c}(\beta_i,\phi^j) \quad (\phi^j_i)_{i,j} \quad \begin{array}{c}[\beta_1]\\ \vert\ldots\\ \chi\end{array} \quad \begin{array}{c}[\beta_n]\\ \ldots\vert\\ \chi\end{array}}{\chi}$$

Figure 2: $\mathbf{c}$-elimination rules schemata

Definition 6 (ε-axiom) *Consider a formula χ of a logic $\mathscr{L}$ and an elimination rule r for* $\mathbf{c}$*, as shown right in Figure 2 instantiated to ϕ^j, β_i and χ. The ε-axiom concerning this rule schema and χ, denoted by $\varepsilon(r,\chi,\beta_i,\phi^j)$, is the following implicational formula:*

$$\phi^1_1 \to (\ldots(\phi^n_{j_n} \to (..(\beta_1 \to \chi) \to \ldots(\beta_n \to \chi) \to (p_{\mathbf{c}(\beta_i,\phi^j)} \to \chi))))$$

Definition 7 (Atomizing Operators) *The mapping $\mathscr{M}$ from the language of $\mathscr{L}$ into the one of $\mathbf{M}_{\to}$ is defined inductively, as follows:* **Atoms** *$\mathscr{M}(p) = p$, if p is a propositional letter;* **Implication** *$\mathscr{M}(\alpha_1 \to \alpha_2) = \mathscr{M}(\alpha_1) \to \mathscr{M}(\alpha_2)$;* **Operators** *$\mathscr{M}(\mathbf{c_m}(\beta_i,\phi^j,\gamma_k)) = p_{\mathbf{c_m}(\beta_i,\phi^j,\gamma_k)}$, if $\mathbf{c_m}$ is an operator of $\mathscr{L}$.*

A general translation based on the subformula principle

Definition 8 (Axiomatizing Operators) *Given a formula α in $\mathscr{L}$, the mapping $\mathscr{A}^{\alpha}$ from the language of $\mathscr{L}$ into (finite) sets of formulas in the language of $\mathbf{M}_{\rightarrow}$ is defined inductively, as follows.*

- **Atoms** $\mathscr{A}^{\alpha}(p) = \emptyset$;
- **Implication** $\mathscr{A}^{\alpha}(\alpha_1 \rightarrow \alpha_2) = \mathscr{A}^{\alpha}(\alpha_1) \cup \mathscr{A}^{\alpha}(\alpha_2)$;
- **Operators** $\mathscr{A}^{\alpha}(\mathbf{c_m}(\beta_i, \phi^j, \gamma_k)) =$

$$\{\iota(r, \mathscr{M}(\beta_i), \mathscr{M}(\phi^j), \mathscr{M}(\gamma_k))/r \text{ is a } \mathbf{c_m}\text{-intro rule}\}$$
$$\cup$$
$$\{\varepsilon(r, \mathscr{M}(\chi), \mathscr{M}(\beta_i), \mathscr{M}(\phi^j))/r \text{ is a } \mathbf{c_m}\text{-elim rule and } \chi \in sub(\alpha)\}$$

Lemma 1 *Let $\mathscr{L}$ be a logic having a Natural Deduction system satisfying sub-formula property. Let Π be a proof of α from Γ in $\mathscr{L}$. There is a derivation Π' of $\mathscr{M}(\alpha)$ from $\Gamma' \subseteq \mathscr{M}(\Gamma) \cup \bigcup_{\gamma\in\Gamma} \mathscr{A}^{\gamma}(\Gamma, \alpha) \cup \mathscr{A}^{\alpha}(\Gamma, \alpha)$ in $\mathbf{M}_{\rightarrow}$. We use the notation $\mathscr{A}^{\alpha}(\Gamma)$ to denote $\bigcup_{\gamma\in\Gamma} \mathscr{A}^{\alpha}(\gamma)$ and "Γ, α" to denote $\Gamma \cup \{\alpha\}$.*

A general translation based on the subformula principle

Theorem 1

If L satisfies the subformula principle and is decidable then the problem of knowing whether a formula of L is provable or not is in PSPACE. If L includes [M→] then this problem, also known as Validity, is PSPACE-complete.

Proposition 2 *Let $\mathscr{L}$ be a propositional logic satisfying the sub-formula principle. Consider the following translation $\star$ from formulas of $\mathscr{L}$ into formulas of $\mathbf{M}_{\rightarrow}$: Let $\mathscr{A}^{\alpha}(\alpha)$ be $\{\varphi_1, \ldots, \varphi_k\}$ $\alpha^{\star}$ is defined as $\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_k \rightarrow \mathscr{M}(\alpha)))$. Thus, $\vdash_{\mathscr{L}} \alpha$ if and only if $\vdash_{\mathbf{M}_{\rightarrow}} \alpha^{\star}$.*

A general translation based on the subformula principle

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[A]^1$}
\AxiomC{$[(A \to B) \wedge (B \to C)]^2$}
\UnaryInfC{$A \to B$}
\BinaryInfC{B}
\AxiomC{$[(A \to B) \wedge (B \to C)]^2$}
\UnaryInfC{$B \to C$}
\BinaryInfC{C}
\BinaryInfC{$A \wedge C$}
\LeftLabel{1}
\UnaryInfC{$A \to (A \wedge C)$}
\LeftLabel{2}
\UnaryInfC{$((A \to B) \wedge (B \to C)) \to (A \to (A \wedge C))$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[(A \to B) \wedge (B \to C)]^2$}
\UnaryInfC{$A \to B$}
\BinaryInfC{B}
\AxiomC{$[(A \to B) \wedge (B \to C)]^2$}
\UnaryInfC{$B \to C$}
\BinaryInfC{C}
\AxiomC{$[A]^1$}
\AxiomC{$A \to (C \to p_{(A \wedge C)})$}
\BinaryInfC{$C \to p_{(A \wedge C)}$}
\BinaryInfC{$p_{(A \wedge C)}$}
\LeftLabel{1}
\UnaryInfC{$A \to p_{(A \wedge C)}$}
\LeftLabel{2}
\UnaryInfC{$((A \to B) \wedge (B \to C)) \to (A \to p_{(A \wedge C)})$}
\end{prooftree}
$$

A general translation based on the subformula principle

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[q]^2$}
\AxiomC{$q \to (A \to B)$}
\BinaryInfC{$A \to B$}
\BinaryInfC{B}
\AxiomC{$[q]^2$}
\AxiomC{$q \to (B \to C)$}
\BinaryInfC{$B \to C$}
\BinaryInfC{C}
\AxiomC{$[A]^1$}
\AxiomC{$A \to (C \to p)$}
\BinaryInfC{$C \to p$}
\BinaryInfC{p}
\LeftLabel{1}
\UnaryInfC{$A \to p$}
\LeftLabel{2}
\UnaryInfC{$q \to (A \to p)$}
\end{prooftree}
$$

$$p_{(A \to B) \wedge (B \to C)} = q, \quad p_{(A \wedge C)} = p$$

A general translation based on the subformula principle

$$
\dfrac{\dfrac{[A]^1 \quad \dfrac{[q]^2 \quad [q \to (A \to B)]^3}{A \to B}}{B} \quad \dfrac{[q]^2 \quad [q \to (B \to C)]^4}{B \to C}}{C}
\qquad
\dfrac{[A]^1 \quad [A \to (C \to p)]^5}{C \to p}
$$

$$
\dfrac{C \qquad C \to p}{p}
$$

$$
5\,\dfrac{4\,\dfrac{3\,\dfrac{2\,\dfrac{1\,\dfrac{p}{(A \to p)}}{q \to (A \to p)}}{(q \to (A \to B)) \to (q \to (A \to p))}}{(q \to (B \to C)) \to (q \to (A \to B)) \to (q \to (A \to p))}}{(A \to (C \to p)) \to (q \to (B \to C)) \to (q \to (A \to B)) \to (q \to (A \to p))}
$$

THANK YOU!

On Natural Deduction in Classical First-Order Logic

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Tübingen, 27–29 November 2015

The Magnificent Seven

$$\bot, \neg, \wedge, \vee, \rightarrow, \forall, \exists$$

$$\Downarrow$$

$$\bot, \wedge, \rightarrow, \forall$$

$$\exists := \neg\forall\neg$$

$$\vee := \neg \wedge \neg$$

$$\Downarrow$$

$$\bot, \rightarrow, \forall$$

First Reason for Keeping $\vee$ and $\exists$

$$\mathrm{EM}_1 := \forall x\, P(x) \;\vee\; \exists x\, \neg P(x) \qquad (\,P \text{ propositional})$$

$$\mathrm{EM}_2 := \forall x\, \exists y\; P(x, y) \vee\; \exists x\, \forall y\; \neg P(x, y) \qquad (\,P \text{ propositional})$$

$$\mathrm{IL} + \mathrm{EM}_n \nvdash \exists x\, A \leftrightarrow \neg \forall x \neg\, A$$

$$\mathrm{IL} + \mathrm{EM}_1 \;\vdash\; \exists x\, A \implies x \text{ learnable in the limit}$$

Second Reason for Keeping $\vee$ and $\exists$

Interactive Learning-Based Realizability

$t \Vdash_s \exists x\, A$ iff $\pi_0 t[s] = n$ and $\pi_1 t \Vdash_s A[n/x]$

Third Reason for Keeping $\vee$ and $\exists$

Theorem (Weak Herbrand Theorem)

$$\mathrm{CL} \vdash \exists x\, \mathrm{P} \implies \mathrm{PCL} \vdash \mathrm{P}[t_1/x] \vee \cdots \vee \mathrm{P}[t_k/x]$$

(P propositional)

Theorem (Strong Herbrand Theorem (Miller))

$\mathrm{CL} \vdash \exists x\, A \implies$ *A has an expansion tree winning strategy*

Expansion Tree Winning Strategies

$$\begin{tikzcd}[column sep=small]
P(t_1, a_1, t_4) & P(t_2, a_2, t_5) & \\
\exists z\, P(t_1, a_1, z) \arrow[u, no head, "t_4"] & \exists z\, P(t_2, a_2, z) \arrow[u, no head, "t_5"] & \exists z\, P(t_3, a_3, z) \\
\forall y\, \exists z\, P(t_1, y, z) \arrow[u, no head, "a_1"] & \forall y\, \exists z\, P(t_2, y, z) \arrow[u, no head, "a_2"] & \forall y\, \exists z\, P(t_3, y, z) \arrow[u, no head, "a_3"] \\
& \exists x\, \forall y\, \exists z\, P(x, y, z) \arrow[ul, no head, "t_1"] \arrow[u, no head, "t_2"] \arrow[ur, no head, "t_3"'] &
\end{tikzcd}$$

Ordered: $t_1\ a_1\ t_2\ a_2\ t_3\ a_3\ t_4\ t_5$

Winning: $P(t_1, a_1, t_4) \vee P(t_2, a_2, t_5)$

Natural Deduction

$$\cfrac{\cfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \to B} \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}$$

converts to

$$\begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix}$$

Natural Deduction

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

converts to

$$\begin{matrix}\vdots\\ A\\ \vdots\\ C\end{matrix}$$

Natural Deduction

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A_1\end{matrix} \qquad \begin{matrix}\vdots \\ A_2\end{matrix}}{A_1 \wedge A_2}}{A_i}\ i \in \{1,2\} \qquad \text{converts to:} \qquad \begin{matrix}\vdots \\ A_i\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\pi \\ A\end{matrix}}{\forall x\, A}}{A[m/x]} \qquad \text{converts to:} \qquad \begin{matrix}\pi[m/x] \\ A[m/x]\end{matrix}$$

Natural Deduction

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A[t/x]\end{matrix}}{\exists x\, A} \qquad \begin{matrix}[A] \\ \pi \\ C\end{matrix}}{C} \qquad \text{converts to:} \qquad \begin{matrix}\vdots \\ A[t/x] \\ \pi[t/x] \\ C\end{matrix}$$

Excluded Middle

$$\dfrac{\begin{matrix}[\forall x\, A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\exists x\, A^{\perp}] \\ \vdots \\ C\end{matrix}}{C}\; EM_n$$

A is a prenex formula with $n-1$ quantifiers

Two Kinds of Premises

- Assumption
  - a sentence whose logical consequences we want to determine; we use it just as starting point of the reasoning, with no further questions about its validity.
  - Discharged by intuitionistic inference rules.

Two Kinds of Premises

- Working Hypothesis
  - J. Dewey: A sentence which is neither true nor false but "provisional, working mean of advancing investigation".
  - Discharged by excluded middle rules.
  - C. Pierce: "A hypothesis (...) we may not believe to be altogether true, but which is useful in enabling us to conceive of what takes place".

Logical Reductions: EM_n

$$\dfrac{\begin{matrix}[\forall x\, A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\exists x\, A^{\perp}] \\ \vdots \\ C\end{matrix}}{C}\, EM_n \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ C\end{matrix}$$

Logical Reductions: EM_n

$$
\begin{prooftree}
\AxiomC{$\forall x\, A$}
\UnaryInfC{$A[t/x]$}
\UnaryInfC{$\vdots$}
\AxiomC{$\forall x\, A$}
\UnaryInfC{$\vdots$}
\BinaryInfC{C}
\AxiomC{$\exists x\, A^{\perp}$}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{C}
\RightLabel{EM_n}
\BinaryInfC{C}
\end{prooftree}
$$

converts to (provided the variables of t are free in the proof of C):

$$
\begin{prooftree}
\AxiomC{$A^{\perp}[t/x]$}
\UnaryInfC{$\exists x\, A^{\perp}$}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{C}
\AxiomC{$A[t/x]$}
\UnaryInfC{$\vdots$}
\AxiomC{$\forall x A$}
\UnaryInfC{$\vdots$}
\BinaryInfC{C}
\AxiomC{$\exists x\, A^{\perp}$}
\noLine
\UnaryInfC{$\vdots$}
\UnaryInfC{C}
\RightLabel{EM_n}
\BinaryInfC{C}
\RightLabel{EM_{n-1}}
\BinaryInfC{C}
\end{prooftree}
$$

Logical Reductions: EM_n

$$
\begin{prooftree}
\AxiomC{$[\forall x\, A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B \to C$}
\AxiomC{$[\exists x\, A^{\perp}]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B \to C$}
\RightLabel{EM_n}
\BinaryInfC{$B \to C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{B}
\BinaryInfC{C}
\end{prooftree}
$$

converts to

$$
\begin{prooftree}
\AxiomC{$[\forall x\, A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B \to C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{B}
\BinaryInfC{C}
\AxiomC{$[\exists x\, A^{\perp}]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$B \to C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{B}
\BinaryInfC{C}
\RightLabel{EM_n}
\BinaryInfC{C}
\end{prooftree}
$$

Strong Normalization and Herbrand Disjunction Extraction

$$\pi : A \implies \pi \in \mathsf{SN}$$

$$\pi : \exists x\, \mathsf{P} \implies \pi \mapsto^* (t_0, u_0) \mid (t_1, u_1) \mid \ldots \mid (t_k, u_k)$$

$$\mathsf{CL} \vdash \mathsf{P}[t_0/x] \vee \mathsf{P}[t_1/x] \vee \cdots \vee \mathsf{P}[t_k/x]$$

$\pi : A \implies$ We can find an expansion tree for A?

Postponement of RAA and Glivenko's theorem, revisited

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General Proof Theory
Tübingen, November 27-29, 2015

Outline

Outline

Glivenko's theorem

Theorem (Glivenko, 1929)

1. If A is provable in classical propositional logic, then $\neg\neg A$ is provable in intuitionistic propositional logic.
2. (more generally) If A is provable from Γ in classical propositional logic then:
   - $\neg\neg A$ is provable from Γ in intuitionistic propositional logic;
   - $\bot$ is provable from $\Gamma, \neg A$ in intuitionistic propositional logic.

The converses hold trivially since A is equivalent to $\neg\neg A$ in classical logic.

Corollary

A theory in consistent in classical propositional logic if and only if it is consistent in intuitionistic propositional logic.

Remark: Glivenko's theorem holds neither in full 1^{st}-order, nor in minimal logic.

Glivenko's theorem has been widely studied: there is a lot of simple proofs (with a semantical or syntactic approach) and refinements of it.

$\rightsquigarrow$ What else is there to say?

Interlude: *reductio ad absurdum* (*raa*) in natural deduction

Consider the following rules in natural deduction:

$$\textit{ex falso quodlibet}\ \dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\,\textit{efq} \qquad\qquad \textit{reductio ad absurdum}\ \dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\\ \bot\end{matrix}}{A}\,\textit{raa}^1$$

Remark: *efq* is just the special case of *raa* where no assumption is discharged.

An instance *raa* is said to be discharging if it is not an instance of *efq* (*i.e.* it discharges at least one assumption).

NM = first-order minimal natural deduction *i.e.* all introduction and elimination rules for all connectives (except $\bot$) and quantifiers

NJ = NM $\cup$ {*efq*} = first-order intuitionstic natural deduction

NK = NM $\cup$ {*raa*} = first-order classical natural deduciton

Remark: NM $\subsetneq$ NJ $\subsetneq$ NK

A way to prove Glivenko's theorem: postponement of *raa*

Let A be provable from Γ in propositional classical logic. So, there is a derivation

$$\begin{matrix}\Gamma\\ \vdots\,\pi\\ A\end{matrix}$$

in classical propositional natural deduction, with possibly many instances of *raa*.

Suppose that it is possible to transform π in a derivation π' of A from Γ where discharging *raa* is used only at the last rule (postponement of *raa*). Then,

$$\pi' = \quad \dfrac{\begin{matrix}\Gamma, \ulcorner\neg A\urcorner^1\\ \vdots\,\pi''\\ \bot\end{matrix}}{A}\,\textit{raa}^1$$

where π'' contains no discharging instance of *raa*.

(continue)

A way to prove Glivenko's theorem: postponement of *raa*

Therefore, if we replace the instance of *raa* with $\neg_i$ in the last rule of π' we get

$$\pi''' = \begin{array}{c} \Gamma, \ulcorner\neg A\urcorner^1 \\ \vdots\, \pi'' \\ \dfrac{\bot}{\neg\neg A}\neg_i^1 \end{array}$$

π''' is a derivation of $\neg\neg A$ from Γ in propositional intuitionistic natural deduction (since π'' and hence π''' contain no discharging instance of *raa*).

To resume: If we can prove the postponement of *raa* (PR), we also have a proof of Glivenko's theorem.

Actually, PR holds for NK $\smallsetminus \{\forall_i\}$! (Seldin, 1986; von Plato, 2013).

The proof of Glivenko's theorem sketched above is interesting because:

1. PR is an interesting result *per se* (that we aim to refine);
2. deducing Glivenko's theorem from PR is more illuminating (than a "boring" proof by induction on the derivation of A from Γ in classical logic):
   - it relates Glivenko's theorem to Kuroda (1951) negative translation;
   - it gives refinements of Glivenko theorem (for 1^{st}- and 2^{nd}-order, into minimal).

Outline

Postponement of *raa*: the naive (wrong) way

Goal: Transforming a derivation in NK so that any possible instance of *raa* is pushed downwards until it vanishes or occurs only in the last rule, *preserving the same conclusion* and *without adding new non-discharged assumptions*.

The naive (wrong) approach: we consider two reduction steps as follows ($B \neq \bot$)

$$
\begin{matrix}\dfrac{\dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A}\,raa^1 \qquad \vdots\,\pi''}{B}\,s\\ \vdots\,\pi\end{matrix}
\quad\rightsquigarrow\quad
\begin{matrix}\dfrac{\dfrac{\ulcorner\neg B\urcorner^1 \qquad \dfrac{\ulcorner A\urcorner^2 \qquad \vdots\,\pi''}{B}\,s}{\bot}\,\neg_e}{\neg A}\,\neg_i^2\\ \vdots\,\pi'\\ \dfrac{\bot}{B}\,raa^1\\ \vdots\,\pi\end{matrix}
\qquad\text{and}\qquad
\begin{matrix}\dfrac{\dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A}\,raa^1 \qquad \vdots\,\pi''}{\bot}\,s\\ \vdots\,\pi\end{matrix}
\quad\rightsquigarrow\quad
\begin{matrix}\dfrac{\dfrac{\ulcorner A\urcorner^2 \qquad \vdots\,\pi''}{\bot}\,s}{\neg A}\,\neg_i^2\\ \vdots\,\pi'\\ \bot\\ \vdots\,\pi\end{matrix}
$$

Problem: It works only if no assumption of π' is discharged at s in the two derivations on the LHS of $\rightsquigarrow$, otherwise $\rightsquigarrow$ would change the set of non-discharged assumptions.

Postponement of *raa*: Seldin (1986) approach (1 of 2)

Goal: Transforming a derivation in NK so that any possible *discharging* instance of *raa* is pushed downwards until it vanishes or occurs only in the last rule, *preserving the same conclusion* and *without adding new non-discharged assumptions*.

Seldin (1986) approach: he defines two reduction steps as follows ($B \neq \bot$)

$$
\begin{matrix}\dfrac{\dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A}\,raa^1 \qquad \vdots\,\pi''}{B}\,s\\ \vdots\,\pi\end{matrix}
\quad\rightsquigarrow\quad
\begin{matrix}\dfrac{\ulcorner\neg B\urcorner^1 \qquad \dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\ulcorner\neg B\urcorner^1 \qquad \dfrac{\ulcorner A\urcorner^2 \qquad \vdots\,\pi''}{B}\,s}{\bot}\,\neg_e}{\neg A}\,\neg_i^2\\ \vdots\,\pi'\\ \bot\end{matrix}}{A}\,efq \qquad \vdots\,\pi''}{B}\,s}{\dfrac{\bot}{B}\,raa^1}\,\neg_e\\ \vdots\,\pi\end{matrix}
\tag{1}
$$

Postponement of *raa*: Seldin (1986) approach (2 of 2)

Goal: Transforming a derivation in NK so that any possible *discharging* instance of *raa* is pushed downwards until it vanishes or occurs only in the last rule, *preserving the same conclusion* and *without adding new non-discharged assumptions*.

Seldin (1986) approach: he defines two reduction steps as follows

$$\begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\ \pi' \\ \dfrac{\dfrac{\bot}{A}\ raa^1 \qquad \vdots\ \pi''}{\bot}\ s \\ \vdots\ \pi \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\dfrac{\ulcorner A \urcorner^2 \qquad \vdots\ \pi''}{\bot}\ s}{\neg A}\ \neg_i^2 \\ \vdots\ \pi' \\ \dfrac{\dfrac{\bot}{A}\ efq \qquad \vdots\ \pi''}{\bot}\ s \\ \vdots\ \pi \end{array} \tag{2}$$

Idea: By applying repeatedly the reduction steps (1) and (2) following a suitable strategy, any derivation in $\mathrm{NK} \smallsetminus \{\forall_i\}$ is transformed in the desired form.

A problem in Seldin's approach: the rule $\forall_i$

Seldin's approach does not work when the instance of *raa* that one would push down is immediately followed by an instance of the rule $\forall_i$.

Indeed, the natural way to treat the $\forall_i$ case would be the following reduction step:

$$\Pi = \begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\ \pi' \\ \dfrac{\dfrac{\bot}{A}\ raa^1}{\forall x A}\ \forall_i \\ \vdots\ \pi \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\dfrac{\ulcorner \neg \forall x A \urcorner^1 \qquad \dfrac{\ulcorner A \urcorner^2}{\forall x A}\ \forall_i}{\bot}\ \neg_e}{\neg A}\ \neg_i^2 \\ \vdots\ \pi' \\ \dfrac{\bot}{\forall x A}\ raa^1 \\ \vdots\ \pi \end{array} = \Pi'$$

but Π' is not a derivation in NK (nor in other subsystems of NK) because in Π' the rule $\forall_i$ is not correctly instantiated, since the variable x may occur free in A, and A is a non-discharged assumption when the rule $\forall_i$ is applied in Π'.

A posteriori, it is unsurprisingly that the postponement of *raa* does not hold when in the derivation in NK there is an instance of the rule $\forall_i$, otherwise Glivenko's theorem would hold for full first-order logic (not only propositional), that is false: for instance, if $A = \neg(\forall x \neg\neg P(x) \wedge \neg \forall x P(x))$, then $\vdash_{\mathrm{NK}} A$ but $\nvdash_{\mathrm{NJ}} \neg\neg A$.

Postponement of *raa*: our approach

Seldin's reduction steps introduce a *new* instance of *efq* (its occurrence in the RHS of "$\rightsquigarrow$" has no corresponding in the LHS).

Question: This new instance of *efq* is really necessary? *No*, in most cases.

Instead of Seldin's reduction steps, we define our reduction steps case by case, depending on the inference rule instantiated immediately below the instance of *raa* we want to push downwards in the derivation (see also Stålmarck, 1991; von Plato & Siders, 2012; von Plato 2013).

Goal: To transform a derivation in $\mathsf{NK} \smallsetminus \{\forall_i\}$ so that any possible instance of *raa* is pushed downwards until it vanishes or occurs only in the last rule, *preserving the same conclusion* and *without adding new non-discharged assumptions*. Moreover, we avoid introducing new instances of *efq* when possible.

- Pros: Our reduction steps allow to get a more informative version of postponement of *raa*.
- Cons: Our reduction steps are defined in a non-uniform way (less elegant).

Our reduction steps: *raa* vs. $\neg_e$ and *raa* vs. $\neg_i$

$$\dfrac{\dfrac{\begin{matrix}\ulcorner\neg\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{\neg A}\,raa^1 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{\begin{matrix}\bot\\ \vdots\,\pi\end{matrix}}\,\neg_e \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner\neg A\urcorner^2 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{\bot}\,\neg_e}{\neg\neg A}\,\neg_i^2\\ \vdots\,\pi'\\ \bot\\ \vdots\,\pi\end{matrix} \tag{2}$$

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\ulcorner A\urcorner^2, \ulcorner\neg\bot\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{\bot}\,raa^1}{\neg A}\,\neg_i^2\\ \vdots\,\pi\end{matrix} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\begin{matrix}\ulcorner A\urcorner^2, \dfrac{\ulcorner\bot\urcorner^3}{\neg\bot}\,\neg_i^3\\ \vdots\,\pi'\\ \bot\end{matrix}}{\neg A}\,\neg_i^2\\ \vdots\,\pi\end{matrix} \tag{3}$$

Our reduction steps: *raa* vs. $\to_{\mathsf{i}}$

$$
\begin{array}{c}
\ulcorner A \urcorner^2, \ulcorner \neg B \urcorner^1 \\
\vdots\ \pi' \\
\dfrac{\dfrac{\bot}{B}\,\mathit{raa}^1}{A \to B}\,{\to_{\mathsf{i}}^2} \\
\vdots\ \pi
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{\dfrac{\ulcorner \neg(A \to B) \urcorner^1 \qquad \dfrac{\dfrac{\begin{array}{c} \ulcorner A \urcorner^2 \qquad \dfrac{\dfrac{\ulcorner \neg(A \to B) \urcorner^1 \qquad \dfrac{\ulcorner B \urcorner^3}{A \to B}\,{\to_{\mathsf{i}}^0}}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^3 \\ \vdots\ \pi' \\ \bot \end{array}}{B}\,\mathit{efq}}{A \to B}\,{\to_{\mathsf{i}}^2}}{\bot}\,\neg_{\mathsf{e}}}{A \to B}\,\mathit{raa}^1 \\
\vdots\ \pi
\end{array}
\tag{4}
$$

This is the only reduction step where a new instance of *efq* is introduced.

The reduction step in the case "*raa* vs. $\to_{\mathsf{i}}$" is different from the case "*raa* vs. $\neg_{\mathsf{i}}$"
$\Rightarrow$ For our purposes, $\neg A$ is not considered as a shorthand for $A \to \bot$, the connective $\neg$ and its introduction and elimination rules are considered as primitive.

Postponement of *raa*, revisited (1 of 2)

Theorem (postponement of *raa*, version 1)

Let $\pi \colon \Gamma \vdash A$ be a derivation in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$.

1. Then, $\pi \rightsquigarrow^* \pi'$ for some derivation $\pi' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$ with at most one discharging instance of the rule *raa*; this instance, if any, is the last rule of π', the rest of π' being a derivation in NJ.
2. If π contains no instance of the rule $\to_{\mathsf{i}}$, then $\pi \rightsquigarrow^* \pi'$ for some derivation $\pi' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ with at most one instance of the rule *raa*; this instance, if any, is the last rule of π', the rest of π' being a derivation in NM.

Proof (sketch).

- For any derivation π, we define $\mathsf{size}_{\mathsf{RAA}}(\pi)$ (resp. $\mathsf{size}^+_{\mathsf{RAA}}(\pi)$) as the sum of the distances of all instances (resp. discharging instances) of *raa* in π from the last rule of π.
- $\mathsf{size}_{\mathsf{RAA}}(\pi) = 0$ (resp. $\mathsf{size}^+_{\mathsf{RAA}}(\pi) = 0$) if and only if π has no instance (resp. discharging instance) of *raa*, except possibly at the last rule.
- If $\pi \rightsquigarrow \pi'$ by applying the reduction step to an instance (resp. discharging instance) of *raa* in π with maximal distance from the last rule of π then $\mathsf{size}_{\mathsf{RAA}}(\pi) > \mathsf{size}_{\mathsf{RAA}}(\pi')$ (resp. $\mathsf{size}^+_{\mathsf{RAA}}(\pi) > \mathsf{size}^+_{\mathsf{RAA}}(\pi')$). □

Postponement of *raa*, revisited (2 of 2)

Corollary (postponement of *raa*, version 2)

Let A be provable from Γ in NK.

1. If A and the formulas in Γ do not contain any occurrence of $\forall$, then there exists a derivation $\pi' \colon \Gamma \vdash A$ in NK containing at most one discharging instance of the rule *raa*; this instance, if any, is the last rule of π', the rest of π' being a derivation in NJ.
2. If A and the formulas in Γ do not contain any occurrence of $\forall$ and $\to$, then there exists a derivation $\pi' \colon \Gamma \vdash A$ in NK containing at most one instance of the rule *raa*; this instance, if any, is the last rule of π', the rest of π' being a derivation in NM.

Proof (sketch).

- Thanks to the normalization theorem and the subformula property for NK proved by Stålmarck (1991), if A and the formulas in Γ do not contain any occurrence of $\forall$ (resp. $\forall$ and $\to$), then there exists a derivation $\pi \colon \Gamma \vdash_{\mathsf{NK}} A$ with no instance of the rule $\forall_{\mathsf{i}}$ (resp. $\forall_{\mathsf{i}}$ and $\to_{\mathsf{i}}$).
- By applying Theorem 1 (resp. Theorem 2) above, we conclude the proof. □

Outline

Intuitionistic and minimal translations

We define a translation $(\cdot)^{\mathsf{j}}$ (resp. $(\cdot)^{\mathsf{m}}$) on formulas that just redefines the universal quantifier (resp. the implication and the universal quantifier) in a standard classical way. All other connectives and the existential quantifier are not modified by $(\cdot)^{\mathsf{j}}$ (resp. $(\cdot)^{\mathsf{m}}$).

$$
\begin{array}{ccc}
P(t_1,\ldots,t_n)^{\mathsf{j}} = P(t_1,\ldots,t_n) & \top^{\mathsf{j}} = \top & \bot^{\mathsf{j}} = \bot \\
(A \wedge B)^{\mathsf{j}} = A^{\mathsf{j}} \wedge B^{\mathsf{j}} & (A \vee B)^{\mathsf{j}} = A^{\mathsf{j}} \vee B^{\mathsf{j}} & (\neg A)^{\mathsf{j}} = \neg A^{\mathsf{j}} \\
(A \to B)^{\mathsf{j}} = A^{\mathsf{j}} \to B^{\mathsf{j}} & (\forall x A)^{\mathsf{j}} = \neg \exists x \neg A^{\mathsf{j}} & (\exists x A)^{\mathsf{j}} = \exists x A^{\mathsf{j}}
\end{array}
$$

$$
\begin{array}{ccc}
P(t_1,\ldots,t_n)^{\mathsf{m}} = P(t_1,\ldots,t_n) & \top^{\mathsf{m}} = \top & \bot^{\mathsf{m}} = \bot \\
(A \wedge B)^{\mathsf{m}} = A^{\mathsf{m}} \wedge B^{\mathsf{m}} & (A \vee B)^{\mathsf{m}} = A^{\mathsf{m}} \vee B^{\mathsf{m}} & (\neg A)^{\mathsf{m}} = \neg A^{\mathsf{m}} \\
(A \to B)^{\mathsf{m}} = \neg A^{\mathsf{m}} \vee B^{\mathsf{m}} & (\forall x A)^{\mathsf{m}} = \neg \exists x \neg A^{\mathsf{m}} & (\exists x A)^{\mathsf{m}} = \exists x A^{\mathsf{m}}
\end{array}
$$

Given a set of formulas Γ, we set $\Gamma^{\mathsf{m}} = \{A^{\mathsf{m}} \mid A \in \Gamma\}$ and $\Gamma^{\mathsf{j}} = \{A^{\mathsf{j}} \mid A \in \Gamma\}$.

Remark: The difference between $(\cdot)^{\mathsf{m}}$ and $(\cdot)^{\mathsf{j}}$ is only in the translation of $A \to B$.

Kuroda negative translation vs. our translations

One can prove that

$$
\begin{array}{ccccc}
\vdash_{\mathsf{NK}} A & \iff & \vdash_{\mathsf{NK}} A^{\mathsf{j}} & \iff & \vdash_{\mathsf{NK}} A^{\mathsf{m}} \\
\vdash_{\mathsf{NK}} A & \iff & \vdash_{\mathsf{NJ}} \neg\neg A^{\mathsf{j}} & \iff & \vdash_{\mathsf{NM}} \neg\neg A^{\mathsf{m}}
\end{array}
$$

Our minimal and intuitionistic translations are deeply related to *Kuroda negative translation*.
More precisely, if $(\cdot)^{\mathsf{m}'}$ and $(\cdot)^{\mathsf{j}'}$ are the translations defined above except for

$$
(\forall x A)^{\mathsf{m}'} = \forall x \neg\neg A^{\mathsf{m}'} \qquad\qquad (\forall x A)^{\mathsf{j}'} = \forall x \neg\neg A^{\mathsf{j}'}
$$

then the negative translation $A \mapsto \neg\neg A^{\mathsf{j}'}$ is the one defined by Kuroda (1951), while the negative translation $A \mapsto \neg\neg A^{\mathsf{m}'}$ is a variant of Kuroda's one introduced by Ferreira and Oliva (2012).

Remark:

- In $(\cdot)^{\mathsf{m}}$ and $(\cdot)^{\mathsf{j}}$ we translate $\forall$ by means of $\exists$ and $\neg$ in order to avoid a case "*raa* vs. $\forall_{\mathsf{i}}$" for the postponement of *raa*.
- In $(\cdot)^{\mathsf{m}}$ we translate $\to$ by means of $\vee$ and $\neg$ in order to can apply our reduction steps for postponement of *raa* without introducing new instances of *efq* in the case "*raa* vs. $\to_{\mathsf{i}}$".

Towards a revisited Glivenko's theorem

Remark: For every formula A, by induction on A we can prove that:

1. A^{m} contains no occurrences of $\to$ and $\forall$; A^{j} contains no occurrences of $\forall$;
2. $A^{\mathsf{m}} = A$ if A contains no occurrences of $\to$ and $\forall$; $A^{\mathsf{j}} = A$ if A contains no occurrences of $\forall$.

Lemma (Preservation of provability in NK via translations)

For every derivation $\pi \colon \Gamma \vdash A$ in NK there exist a derivation $\pi' \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in NK $\smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ and a derivation $\pi'' \colon \Gamma^{\mathsf{j}} \vdash A^{\mathsf{j}}$ in NK $\smallsetminus \{\forall_{\mathsf{i}}\}$.

Glivenko's theorem, revisited (1 of 2)

Via the translations $(\cdot)^{\mathsf{j}}$ and $(\cdot)^{\mathsf{m}}$, we can embed full first-order classical logic:

- into the fragment $\{\bot, \top, \neg, \wedge, \vee, \to, \exists\}$ of intuitionistic logic;
- into the fragment $\{\bot, \top, \neg, \wedge, \vee, \exists\}$ of minimal logic.

Glivenko's theorem revisited, intuitionistic version

1. If $\Gamma \vdash_{\mathsf{NK}} A$, then $\Gamma^{\mathsf{j}} \vdash_{\mathsf{D}} \neg\neg A^{\mathsf{j}}$ and $\Gamma^{\mathsf{j}}, \neg A^{\mathsf{j}} \vdash_{\mathsf{D}} \bot$ where $\mathsf{D} = \mathsf{NJ} \smallsetminus \{\forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$.
2. If $\forall$ occurs neither in A nor in any formula of Γ, then

$$\Gamma \vdash_{\mathsf{NK}} A \iff \Gamma \vdash_{\mathsf{NJ}} \neg\neg A \iff \Gamma, \neg A \vdash_{\mathsf{NJ}} \bot.$$

Glivenko's theorem revisited, minimal version

1. If $\Gamma \vdash_{\mathsf{NK}} A$, then $\Gamma^{\mathsf{m}} \vdash_{\mathsf{D}} \neg\neg A^{\mathsf{m}}$ and $\Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash_{\mathsf{D}} \bot$ where $\mathsf{D} = \mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \to_{\mathsf{e}}, \forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$.
2. If $\to$ and $\forall$ occur neither in A nor in any formula of Γ, then

$$\Gamma \vdash_{\mathsf{NK}} A \iff \Gamma \vdash_{\mathsf{NM}} \neg\neg A \iff \Gamma, \neg A \vdash_{\mathsf{NM}} \bot.$$

Glivenko's theorem, revisited (2 of 2)

Proof (sketch) For both versions (intuitionistic and minimal)

1. Use our result about postponement of *raa* in oder to get a derivation π in NK whose last rule is an instance of *raa*, the rest of π being a derivation in NJ or NM, this instance of *raa* can be replaced by an instance of the rule $\neg_{\mathsf{i}}$.
2. Immediate consequence of the previous point and of the properties of the translations $(\cdot)^{\mathsf{j}}$ and $(\cdot)^{\mathsf{m}}$. $\square$

Corollary (consistency of a theory)

1. If $\forall$ does not occur in any formula of Γ: $\Gamma \vdash_{\mathsf{NK}} \bot \iff \Gamma \vdash_{\mathsf{NJ}} \bot$.
2. If $\to$ and $\forall$ do not occur in any formula of Γ: $\Gamma \vdash_{\mathsf{NK}} \bot \iff \Gamma \vdash_{\mathsf{NM}} \bot$.

Remark: The fact our results are restricted to the fragments $\{\bot, \top, \neg, \wedge, \vee, \exists\}$ or $\{\bot, \top, \neg, \wedge, \vee, \to, \exists\}$ of first-order classical logic is not limiting because these fragments are as expressive as full first-order classical logic.

What is the problem between $\to$ and minimal logic?

Starting from a derivation π in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$, to postpone *raa* and get a derivation in NM except possibly the last rule, we require that there is no instance of $\to_{\mathsf{i}}$ in π.

Actually, we can redefine the "*raa* vs. $\to_{\mathsf{i}}$" case in order to avoid introducing new instances of *efq* (and then get a derivation in NM except possibly the last rule)

$$
\begin{array}{c}
\ulcorner A \urcorner^2, \ulcorner \neg B \urcorner^1 \\
\vdots\, \pi' \\
\dfrac{\dfrac{\bot}{B}\, raa^1}{A \to B}\, \to_{\mathsf{i}}^2 \\
\vdots\, \pi
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{\dfrac{\ulcorner \neg(A \to B) \urcorner^1 \qquad \dfrac{\ulcorner B \urcorner^3}{A \to B}\, \to_{\mathsf{i}}^0}{\dfrac{\bot}{\neg B}\, \neg_{\mathsf{i}}^3}\, \neg_{\mathsf{e}} \qquad \ulcorner A \urcorner^2}{} \\
\vdots\, \pi' \\
\dfrac{\ulcorner \neg(A \to B) \urcorner^1 \qquad \dfrac{\dfrac{\dfrac{\bot}{\neg\neg B}\, \neg_{\mathsf{i}}}{B}\, *}{A \to B}\, \to_{\mathsf{i}}^2}{\dfrac{\bot}{A \to B}\, raa^1}\, \neg_{\mathsf{e}} \\
\vdots\, \pi
\end{array}
$$

but this derivation is correct only if B is a *negative formula* (i.e. atomic formulas occur only negated in B, and B does not contain $\vee$ nor $\exists$).

Outline

About our revisiting of postponement of *raa*

Why our postponement of *raa* (PR) is an improvement w.r.t. Seldin (1986)?

1. Our reduction steps are a refinement of the ones defined by Seldin (1986): they introduce new instances of *efq* only when "needed"
   $\Rightarrow$ this allows two different results for PR:
   - starting from a derivation in $\mathsf{NK} \setminus \{\forall_\mathsf{i}\}$, we get a derivation in NJ except possibly the last rule (as already proved by Seldin 1986)
   - starting from a derivation in $\mathsf{NK} \setminus \{\forall_\mathsf{i}, \to_\mathsf{i}\}$, we get a derivation in NM except possibly the last rule (a novelty with respect to Seldin 1986).
2. We have good reason to think that our reduction steps for PR are strongly normalizing (Seldin 1986 proved only weak normalization).
3. Our proof of weak normalization is simpler than Seldin's one.

Moreover:

- Our reduction steps can be seen as an "algorithm" transforming whatever derivation (without $\forall_\mathsf{i}$) in a derivation where *raa* is postponed.
- PR gives another way to prove normalization (i.e. *no detours*) for $\mathsf{NK} \setminus \{\forall_\mathsf{i}\}$ (see also von Plato & Siders 2012): for any derivation π in $\mathsf{NK} \setminus \{\forall_\mathsf{i}\}$,
  1. transform π into a derivation π' where *raa* is postponed
  2. apply Prawitz's procedure for normalization in NJ (or NM) to π'.
- PR holds also in $\mathsf{NK}^2 \setminus \{\forall_\mathsf{i}, \forall^2_\mathsf{i}\}$ (same proof).

About our revisiting of Glivenko's theorem

All our results about Glivenko's theorem are (more or less) well-known:

1. Glivenko (1929): from propositional classical logic into propositional intuitionistic logic;
2. Umezawa (1959), Gabbay (1972): from first-order classical logic (without $\forall$) into first-order (without $\forall$) intuitionistic logic;
3. Ertola & Sagastume (2008): from propositional classical logic (without $\to$) into propositional minimal logic (without $\to$);
4. Zdanowski (2009): from propositional second-order classical logic (without $\forall^2$) into propositional second-order intuitionistic logic (without $\forall^2$);
5. ...

All these results are proved using different approaches (semantic, syntactic, etc.)

Our proof-theoretic approach, via postponement of *raa*, gives a unique, uniform and modular proof of all these versions of Glivenko's theorem.

Moreover, our approach shows that Glivenko theorem and Kuroda negative translation are deeply related, for intuitionistic and minimal logic (see also Farahani & Ono 2012, but only for the intuitionistic case).

THANK YOU!

Outline

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Logicality, Double-line Rules, and Harmony

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28-11-2015

Goal

- Proof-theoretic semantics for modal logics:

define the meaning of modal operators by referring to the rules of inference governing those operators.

- We concentrate on standard deontic logic **KD** and on multi-succedent sequent calculi.
- We show that:
  1. Display logic allows for a double-line presentation of many normal modal logics (**KD** included);
  2. Thus it allows us to give a Došen-style analysis of the logicality of displayable modalities;
  3. We conjecture that it can be used to give an harmonious proof-theoretic definition of the modal operators.

Standard Deontic Logic

- The language of SDL is:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A$$

  where
  - $\Box A$ stands for ‘it ought to be that A’
- SDL is axiomatized by the normal modal logic **KD**, i.e. **K**+ the deontic axiom $D := \Box A \rightarrow \Diamond A$.
- The semantics for SDL is given by serial Kripke-frames, i.e. $\mathcal{F} =< W, R >$ where W is a non-empty set and R is a serial binary relation over W.
- PTS: meaning is correct use, where correctness is defined as satisfaction of some criteria (eliminating tonkish operators).

Wansing's (1998) desiderata

- Local criteria:
  - **Weakly explicit**: the principal operator † occurs in the conclusion only.
  - **Explicit**: weakly explicit+ † occurs only once.
  - **Separated**: No connective other than † occurs in a rule introducing †.
  - **Weakly symmetric**: Every rule introducing † is either a left or a right introduction rule.
  - **Symmetric**: † has both a left and a right introduction rule.
- Global criteria:
  - **Cut-elimination**: if the rules are separated, symmetric and weakly explicit, it implies the subformula property.
  - **Uniqueness**: each operator is uniquely characterized: there cannot be two operators † and ‡ that have same rules and that are not (provably) equivalent.

LKSDL and proof-theoretic semantics

LKSDL is **LKp** + $\dfrac{\Pi \Longrightarrow A}{\Box\Pi \Longrightarrow \Box A}\ LR\Box$ and $\dfrac{\Pi \Longrightarrow}{\Box\Pi \Longrightarrow}\ LD$

It doesn't meet Wansing's desiderata:

- The rule
$$\frac{\Pi \Longrightarrow A}{\Box\Pi \Longrightarrow \Box A}\ LR\Box$$
is not weakly symmetical;
- it is not explicit (the rule for modalities like 4 are not even weakly explicit);
- If both $\Box$ and $\Diamond$ are primitive, we must replace $LR\Box$ with:
$$\frac{\Pi, \Sigma_1 \Longrightarrow \Sigma_2, A}{\Box\Pi, \Diamond\Sigma_1, \Longrightarrow \Diamond\Sigma_2, \Box A}\ LR\Box^*$$
which is not separated.
- Uniqueness is not satisfied.

Display calculi (Wansing 1998)

- Natural generalization of LK-style sequent calculi.
- Better control of the structural elements of deductions:
  - Gentzen's comma is replaced by a structural connective $\circ$;
  - The new structural connectives $\mathbf{I}$, $*$ and $\bullet$ are introduced.
- Display sequents are expressions

$$X \Longrightarrow Y$$

  where X and Y are structures:

$$X ::= A \mid \mathbf{I} \mid X \circ X \mid *X \mid \bullet X$$

  instead of sets/multisets/sequences of formulas.
- In this way we have:
  - Display property: each connective can be introduced in isolation (i.e. empty context on the principal side).
  - General 'essentials only' proof of cut-elimination.

Display rules 1/3: display equivalences

Display equivalences (DE):

$$\dfrac{\dfrac{X \circ Z \Longrightarrow Y}{X \Longrightarrow Y \circ *Z}}{Z \Longrightarrow *X \circ Y} \qquad \dfrac{\dfrac{X \Longrightarrow Y}{*Y \Longrightarrow *X}}{X \Longrightarrow **Y} \qquad \dfrac{\dfrac{X \Longrightarrow Y \circ Z}{X \circ *Z \Longrightarrow Y}}{*Y \circ X \Longrightarrow Z} \qquad \dfrac{X \Longrightarrow \bullet Y}{\bullet X \Longrightarrow Y}$$

They allow to prove the:

Theorem (Display property)

Each substructure Z of a display sequent $X \Longrightarrow Y$ can be displayed as either the whole antecedent ($Z \Longrightarrow W$) or as the whole succedent ($W \Longrightarrow Z$) of a display-equivalent sequent.

Display rules 2/3: structural rules

Structural rules capture structural elements of deductions that in Gentzen-style calculi may be either

- implicit, e.g. associativity of comma:

$$\frac{X \circ (Y \circ Z) \Longrightarrow U}{(X \circ Y) \circ Z \Longrightarrow U}\,A$$

- explicit, e.g. commutativity of comma (permutation):

$$\frac{X \circ Y \Longrightarrow U}{Y \circ X \Longrightarrow U}\,P$$

- inexpressible, e.g. the intensional structural rule:

$$\frac{\bullet X \circ \bullet Y \Longrightarrow *\mathbf{I}}{X \Longrightarrow *Y}\,D$$

which corresponds to the deontic axiom $D := \Box A \rightarrow \Diamond A$.

Display rules 3/3: operational rules

Operational rules replace structural connectives with logical operators.
For each logical operator † we have a left and a right rule that are

- separate: don't exhibit any logical operator other than †;
- symmetric: † only in the antecedent or in the succedent;
- explicit: † occurs only once (and only in the conclusion).

$$\frac{A \Longrightarrow X}{\Box A \Longrightarrow \bullet X}\,\mathrm{L}\Box \qquad \frac{\bullet X \Longrightarrow A}{X \Longrightarrow \Box A}\,\mathrm{R}\Box$$

$$\frac{* \bullet * A \Longrightarrow Y}{\Diamond A \Longrightarrow Y}\,L\Diamond \qquad \frac{X \Longrightarrow A}{* \bullet * X \Longrightarrow \Diamond A}\,R\Diamond$$

$$\frac{*A \Longrightarrow X}{\neg A \Longrightarrow X}\text{L}\neg \qquad \frac{Y \Longrightarrow *A}{Y \Longrightarrow \neg A}\text{R}\neg$$

$$\frac{X \Longrightarrow A \qquad B \Longrightarrow Y}{A \to B \Longrightarrow *X \circ Y}\text{L}\to \qquad \frac{X \circ A \Longrightarrow B}{X \Longrightarrow A \to B}\text{R}\to$$

$$\frac{A \Longrightarrow X \qquad B \Longrightarrow Y}{A \vee B \Longrightarrow X \circ Y}\text{L}\vee \qquad \frac{X \Longrightarrow A \circ B}{X \Longrightarrow A \vee B}\text{R}\vee$$

$$\frac{A \circ B \Longrightarrow X}{A \wedge B \Longrightarrow X}\text{L}\wedge \qquad \frac{X \Longrightarrow A \qquad Y \Longrightarrow B}{X \circ Y \Longrightarrow A \wedge B}\text{R}\wedge$$

$$\frac{A \Longrightarrow X}{\Box A \Longrightarrow \bullet X}\text{L}\Box \qquad \frac{\bullet X \Longrightarrow A}{X \Longrightarrow \Box A}\text{R}\Box$$

$$\frac{* \bullet *A \Longrightarrow Y}{\Diamond A \Longrightarrow Y}L\Diamond \qquad \frac{X \Longrightarrow A}{* \bullet *X \Longrightarrow \Diamond A}R\Diamond$$

Cut elimination and Uniqueness

- The structural rule of Cut:

$$\frac{X \Longrightarrow A \qquad A \Longrightarrow Y}{X \Longrightarrow Y}\,Cut$$

can be eliminated from every displayable calculus satisfying 8 conditions (most of which can be shown to hold simply by inspecting the rules).

- Uniqueness is satisfied.

Limitations of Wansing's desiderata

- Each of Wansing's criteria seems to be a necessary condition, but no argument has been given for their (joint) sufficiency.
- We would like to have only local conditions, but Wansing's global criteria are not dispensable since
  - cut-elimination is needed to rule out Prior's *tonk* operator:

$$\frac{A \circ B \Longrightarrow X}{A\text{-}tonk\text{-}B \Longrightarrow X}\,\mathrm{L}t \qquad \frac{X \Longrightarrow A \circ B}{X \Longrightarrow A\text{-}tonk\text{-}B}\,\mathrm{R}t$$

  - uinqueness is needed to rule out the operator *knot*, which is the dual of Prior's *tonk*:

$$\frac{A \Longrightarrow X \qquad B \Longrightarrow Y}{A\text{-}knot\text{-}B \Longrightarrow X \circ Y}\,\mathrm{L}k \qquad \frac{X \Longrightarrow A \qquad Y \Longrightarrow B}{X \circ Y \Longrightarrow A\text{-}knot\text{-}B}\,\mathrm{R}k$$

(Došen 1989) analysis of logical symbols

A Logic is the science of formal deductions;

B Basic formal deductions are structural deductions;

C Any constant of the object language on whose presence the description of a nonstructural formal deduction depends can be ultimately analyzed in structural terms;

A constant is logical if, and only if, it can be ultimately analyzed in structural terms. (Došen 1989: 368)

In a nuthshell, the idea is that if we can prove that a sequent where an object language operator ‡ occurs is equivalent to a purely structural sequent, then ‡ is a logical operator.

DdlSDL

- W.r.t. *DSDL*, we replace atomic initial sequents $p \Longrightarrow p$ with arbitrary ones $A \Longrightarrow A$
- We replace the operational rules with the following double-line ones:

$$\frac{*A \Longrightarrow X}{\neg A \Longrightarrow X}\,\neg \qquad \frac{A \circ B \Longrightarrow X}{A \wedge B \Longrightarrow X}\,\wedge$$

$$\frac{X \circ A \Longrightarrow B}{X \Longrightarrow A \to B}\,\to \qquad \frac{X \Longrightarrow A \circ B}{X \Longrightarrow A \vee B}\,\vee$$

$$\frac{\bullet X \Longrightarrow A}{X \Longrightarrow \Box A}\,\Box \qquad \frac{* \bullet * A \Longrightarrow X}{\Diamond A \Longrightarrow X}\,\Diamond$$

Equivalence of *DSDL* and *DdlSDL*

Theorem

$$DSDL \vdash X \Longrightarrow Y \qquad iff \qquad DdlSDL \vdash X \Longrightarrow Y$$

Proof (sketch).

It is enough to show that, for each and every logical operator †, the rule † ↑ is equivalent to the *DSDL*-rule for † on the other side of the sequent arrow. E.g. for $\Box$ we have:

$$\frac{\dfrac{\dfrac{\Box A \Longrightarrow \Box A}{\bullet \Box A \Longrightarrow A}\,\Box\uparrow \qquad A \Longrightarrow X}{\bullet \Box A \Longrightarrow X}\,Cut}{\Box A \Longrightarrow \bullet X}\,DE \qquad \frac{\dfrac{X \Longrightarrow \Box A \qquad \dfrac{A \Longrightarrow A}{\Box A \Longrightarrow \bullet A}\,L\Box}{X \Longrightarrow \bullet A}\,Cut}{\bullet X \Longrightarrow A}\,DE$$

□

Došen principle

The double-line rule [□] can serve to characterize various sorts of [modalities...] What in these characterizations distinguishes various [modalities] is not [□], which is always the same, but assumptions concerning structural deductions (Došen 1989: 366)

Double-line display calculi allow us to prove the logicality of all displayable modalities, whereas the sequents of higher levels of (Došen 1985) work only for the modal logics **S5** and **S4**.

Harmony and the Inversion Principle

Harmony (in one sense) is some kind of equilibrium of deductive power between introduction and elimination rules.

It is expressed as conservativeness (Belnap)/ded. eq. (Tennant)/ reduction (Prawitz). It's based on the Inversion Principle:

> *The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. (Gentzen 1935: 80)*
>
> *an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an elimination rule one essentially only restores what had already been established by the major premise of the application of an introduction rule. (Prawitz 1965: 33).*

Double-line rules and Harmony

- In SC the meaning of † is defined by its left and right introduction rules and it is natural to claim that

Harmony is some kind of equilibrium of deductive power between left and right introduction rules.

- The double-line rule for † obeys the Inversion Principle.
- We have shown that the standard single-line left and right introduction rules for † are harmonious according to the following principle:

The single-line rules for † are harmonious iff they can be shown to be equivalent to the double-line rule for †

If this holds, the two rules defining the meaning of † are related by the Inversion Principles and, therefore, are harmonious.

Conclusion & future works

- Display calculi allow us to give an harmonious presentation of *SDL* inasmuch as:

The left and right introduction rules are in a kind of deductive equilibrium which is based on the Inversion Principle.

- By changing the structural rules we can give an harmonious presentation of many other normal modal logics.
- The harmony-as-double-line-rules hypothesis is still a work in progress which has to be made more precise;
- We aim at extending this approach to weaker modalities (non-normal; dyadic. . .)
- We have to investigate the relations with other approaches to PTS based on SC or on ND such as:
  - The approach to inversion by definitional reflection in (de Campos Sanz & Piecha 2009).
  - The approach by multiple-conclusion ND in (Francez 2014).

- de Campos Sanz W., Piecha T. (2009). 'Inversion by Definitional Reflection and the Admissibility of Logical Rules'. *RSL* 2: 550–569.
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- Wansing, H. (1998). *Displaying Modal Logic*. Dordrecht: Kluwer.

Harmony and Sequent calculi

Whereas in the context of natural deduction, the idea of inversion has been intensively discussed following Prawitz's (1965) adaptation of Lorenzen's term 'inversion principle' to explicate Gentzen's remark, there has been no comparable investigation of the relationship between right introduction and left introduction rules of the sequent calculus. (de Campos Sanz & Piecha 2009: 551)

Double-line rules and the Inversion Principle

A double-line rule is made of an introduction rule and some elimination rules respecting the Inversion Principle.

> *The asymmetry of natural deduction with respect to premises and conclusion is most unfortunate when one has to formulate precisely what Prawitz calls the Inversion Principle.[..]*
> *With Gentzen's plural (multiple-conclusion) sequents we overcome this asymmetry, and we may formulate rules for the logical constants as double-line rules, i.e. invertible rules, going both ways, from the premises to the conclusion and back. The inversion of the Inversion Principle is now really inversion.*
> *(Došen 2015: 151)*

Analysis and definition in (Došen 1989)

To analyze an expression α of a language $\mathcal{L}$, we have to find a language $\mathcal{M}$ not containing α s.t.:

1. a sentence **A** in $\mathcal{M} + \alpha$ where α occurs once is equivalent to a sentence **B** of $\mathcal{M}$;
2. the analysis of α has to be sound and complete;
3. α has to be uniquely characterized.

We get a definition if it also holds that:

1. *Pascal's condition:* the definitional equivalence should allows us find for every sentence of $\mathcal{M} + \alpha$ a sentence of $\mathcal{M}$ with the same meaning;
2. the addition of α is conservative.

> *Though an analysis doesn't give the meaning of an expression, as an explicit definition would, it follows from conditions (2) and (3) that an analysis is very closely tied to the meaning of the expression analyzed (Došen 1989: 372)*

Monoidal logics

De Morgan negations and classical systems

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LMU Munich

November 28, 2015

Motivation

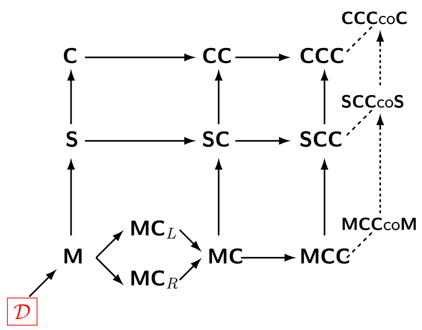
Aim: Provide categorical foundations for logic.

Inspired by some of the developments made in categorical logic:

1) **Adjoint functors** play a fundamental role in logic (Lawvere).
2) Conceptual equivalence between categorical and logical notions (Lambek).
3) **Classification** of deductive systems via their categorical structure and the functorial properties of their connectives (Joyal and Reyes).
4) Show that some problems and paradoxes are related to specific types of deductive systems.

Monoidal logics

Monoidal logics



$\mathcal{L} =$
$\{Prop, (,), \otimes, 1, \multimap, \triangleright, \oplus, 0, \ltimes, \rtimes, *, \star\}$

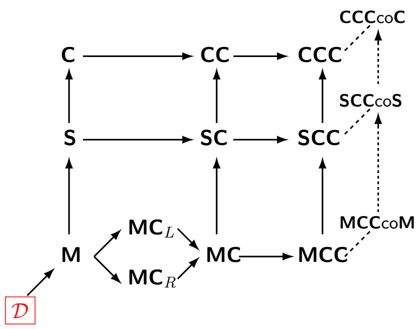
- (1) is the identity axiom
- (cut) expresses the transitivity of the consequence relation

Definition

A **deductive system** $\mathcal{D}$ is composed of a collection of formulas and a collection of equivalence classes of proofs, satisfying (1) and (cut).

$$\frac{}{\varphi \longrightarrow \varphi}\ (1) \qquad\qquad \frac{\varphi \longrightarrow \psi \qquad \psi \longrightarrow \rho}{\varphi \longrightarrow \rho}\ (\text{cut})$$

Monoidal logics



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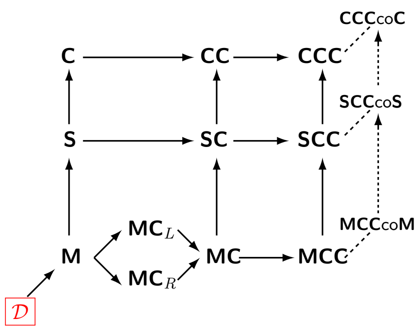

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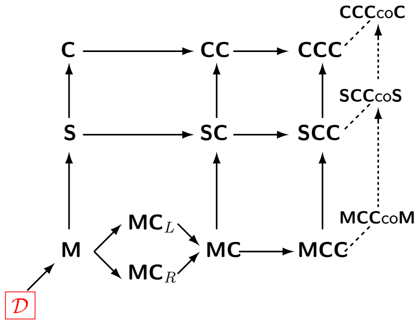

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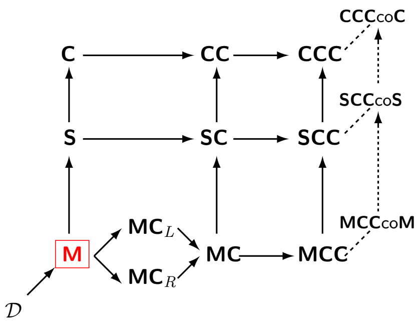
Monoidal logics



- 1 is the unit of the tensor product
- $\otimes$ is associative
- $\otimes$ respects increasing monotony

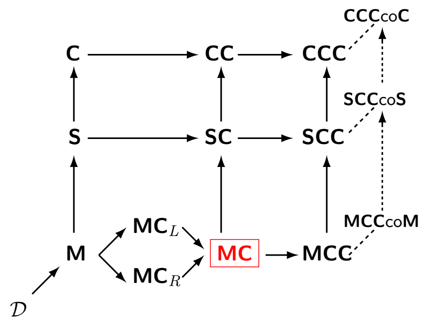
Definition

A **monoidal deductive system M** is a deductive system satisfying (t), (r), (l) and (a).

$$\frac{\varphi \longrightarrow 1 \otimes \psi}{\varphi \longrightarrow \psi}\ (\mathrm{l}) \qquad \frac{\varphi \longrightarrow \psi \otimes 1}{\varphi \longrightarrow \psi}\ (\mathrm{r})$$

$$\frac{\varphi \longrightarrow \psi \qquad \rho \longrightarrow \tau}{\varphi \otimes \rho \longrightarrow \psi \otimes \tau}\ (\mathrm{t}) \qquad \frac{\tau \longrightarrow (\varphi \otimes \psi) \otimes \rho}{\tau \longrightarrow \varphi \otimes (\psi \otimes \rho)}\ (\mathrm{a})$$

Monoidal logics



- $\multimap / \triangleright$ are adjoints to $\otimes$
- (t) is provable from (cl) and (cl')
- Negations are defined by

$$\sim \varphi =_{df} \varphi \multimap * \qquad \neg\varphi =_{df} \varphi \triangleright *$$

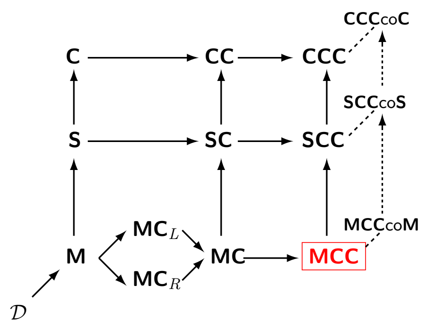
- Intuitionistic negations

Definition

A **monoidal closed deductive system MC** is a **M** satisfying (cl) and (cl').

$$\frac{\varphi \otimes \psi \longrightarrow \rho}{\varphi \longrightarrow \psi \multimap \rho}\,(\text{cl}) \qquad \frac{\varphi \otimes \psi \longrightarrow \rho}{\psi \longrightarrow \varphi \triangleright \rho}\,(\text{cl}')$$

Monoidal logics



- $\varphi \cong \neg \sim \varphi \cong \sim \neg\varphi$
- $\varphi \longrightarrow \psi$ iff $\neg\psi \longrightarrow \neg\varphi$
- $\varphi \longrightarrow \psi$ iff $\sim \psi \longrightarrow\ \sim \varphi$

Definition

A **monoidal closed deductive system with classical negations MCC** is a **MC** satisfying $(\neg \sim)$ and $(\sim \neg)$.

$$\frac{}{\sim \neg\varphi \longrightarrow \varphi}\,(\sim \neg) \qquad \frac{}{\neg \sim \varphi \longrightarrow \varphi}\,(\neg \sim)$$

Monoidal logics

$$\begin{tikzcd}
 & & & & & \mathbf{CCCcoC} \\
 & \mathbf{C} \arrow[rr] & & \mathbf{CC} \arrow[r] & \mathbf{CCC} \arrow[ur, dotted, no head] & \\
 & & & & & \mathbf{SCCcoS} \arrow[uu, dotted] \\
 & \mathbf{S} \arrow[uu] \arrow[rr] & & \mathbf{SC} \arrow[uu] \arrow[r] & \boxed{\mathbf{SCC}} \arrow[uu] \arrow[ur, dotted, no head] & \\
 & & \mathbf{MC}_L \arrow[dr] & & & \mathbf{MCCcoM} \arrow[uu, dotted] \\
 & \mathbf{M} \arrow[uu] \arrow[ur] \arrow[dr] & & \mathbf{MC} \arrow[uu] \arrow[r] & \mathbf{MCC} \arrow[uu] \arrow[ur, dotted, no head] & \\
 & & \mathbf{MC}_R \arrow[ur] & & & \\
\mathcal{D} \arrow[uur] & & & & &
\end{tikzcd}$$

- $\otimes$ is commutative
- (r) can be proven from (l)
- (cl) can be proven from (cl')

Definition

A **symmetric deductive system** **S** is a **M** satisfying (b).

$$\frac{\varphi \longrightarrow \psi \otimes \tau}{\varphi \longrightarrow \tau \otimes \psi}\ \text{(b)}$$

Monoidal logics

$$\begin{tikzcd}
 & & & & & \mathbf{CCCcoC} \\
 & \mathbf{C} \arrow[rr] & & \mathbf{CC} \arrow[r] & \boxed{\mathbf{CCC}} \arrow[ur, dotted, no head] & \\
 & & & & & \mathbf{SCCcoS} \arrow[uu, dotted] \\
 & \mathbf{S} \arrow[uu] \arrow[rr] & & \mathbf{SC} \arrow[uu] \arrow[r] & \mathbf{SCC} \arrow[uu] \arrow[ur, dotted, no head] & \\
 & & \mathbf{MC}_L \arrow[dr] & & & \mathbf{MCCcoM} \arrow[uu, dotted] \\
 & \mathbf{M} \arrow[uu] \arrow[ur] \arrow[dr] & & \mathbf{MC} \arrow[uu] \arrow[r] & \mathbf{MCC} \arrow[uu] \arrow[ur, dotted, no head] & \\
 & & \mathbf{MC}_R \arrow[ur] & & & \\
\mathcal{D} \arrow[uur] & & & & &
\end{tikzcd}$$

- **C** is a conservative extension of **S**

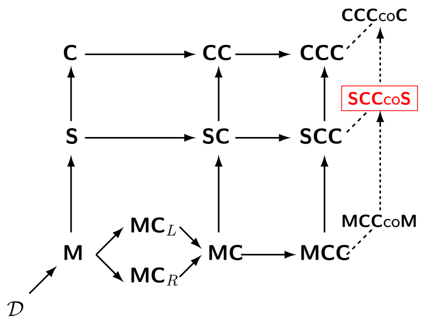
Definition

A **Cartesian deductive system** **C** is a **M** system satisfying ($\otimes$-in) and ($\otimes$-out).

$$\frac{\varphi \longrightarrow \psi \otimes \rho}{\varphi \longrightarrow \psi}\ (\otimes\text{-out}) \qquad \frac{\varphi \longrightarrow \psi \otimes \rho}{\varphi \longrightarrow \rho}\ (\otimes\text{-out})$$

$$\frac{\varphi \longrightarrow \psi \qquad \varphi \longrightarrow \rho}{\varphi \longrightarrow \psi \otimes \rho}\ (\otimes\text{-in})$$

Monoidal logics



The rules for co-deductive systems are obtained by

- reversing the arrows
- replacing $\otimes$ by $\oplus$
- replacing $\multimap$ / $\triangleright$ by $\ltimes$ / $\rtimes$
- replacing 1 by 0
- replacing $*$ by $\star$

Definition

A **co- deductive system**...

Substructural and Display logics

Substructural and Display logics

Given the language of display logics (cf. Goré 1998), it is possible to define a translation function $t'(\varphi \longrightarrow \psi) = t'_1(\varphi) \vdash t'_2(\psi)$ s.t.

Theorem

Monoidal logics with $*$ are sound and complete with respect to associative display logics.

$$* \cong 0 \qquad (*)$$

Substructural and Display logics

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Theorem

Monoidal logics with $*$ are sound and complete with respect to associative display logics.

$$* \cong 0 \qquad (*)$$

This yields the following correspondences:

Monoidal logics	Substructural logics
MC	**Lambek's syntactic calculus**
MCco**MC**	**Lambek's BL1**
SCco**SC**	Commutative BL1
CCco**C**	Intuitionistic logic
CCCco**C**	Classical logic

Classical systems

Classical systems

Classical substructural and display logics are characterized by the satisfaction of the elimination of double negation(s).

- For example, Ono (1993) or Goré (1998).

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- For example, Ono (1993) or Goré (1998).

Characteristics of **classical** systems:

1) de Morgan negation(s)

$$
\begin{array}{lll}
\neg\psi \otimes \neg\varphi \longrightarrow \neg(\varphi \oplus \psi) & \sim\psi \otimes \sim\varphi \longrightarrow \sim(\varphi \oplus \psi) & (\mathbf{dm1}) \\
\neg(\varphi \oplus \psi) \longrightarrow \neg\psi \otimes \neg\varphi & \sim(\varphi \oplus \psi) \longrightarrow \sim\psi \otimes \sim\varphi & (\mathbf{dm2}) \\
\neg\varphi \oplus \neg\psi \longrightarrow \neg(\psi \otimes \varphi) & \sim\varphi \oplus \sim\psi \longrightarrow \sim(\psi \otimes \varphi) & (\mathbf{dm3}) \\
\neg(\psi \otimes \varphi) \longrightarrow \neg\varphi \oplus \neg\psi & \sim(\psi \otimes \varphi) \longrightarrow \sim\varphi \oplus \sim\psi & (\mathbf{dm4})
\end{array}
$$

Classical systems

Classical substructural and display logics are characterized by the satisfaction of the elimination of double negation(s).

- For example, Ono (1993) or Goré (1998).

Characteristics of **classical** systems:

2) the law of excluded middle

$$
1 \longrightarrow \varphi \oplus \sim\varphi \qquad\qquad 1 \longrightarrow \neg\varphi \oplus \varphi \qquad (\mathbf{lem})
$$

Classical systems

Intuitionistic substructural or display logics are the ones that do not satisfy the elimination of double negation(s).

Lambek (1993), for instance, wrote

> *I guess intuitionistic bilinear logic is just the syntactic calculus.*

Classical systems

Intuitionistic substructural or display logics are the ones that do not satisfy the elimination of double negation(s).

Lambek (1993), for instance, wrote

> *I guess intuitionistic bilinear logic is just the syntactic calculus.*

- The elimination of double negation(s) is seen as a **sufficient condition** to go from an intuitionistic to a classical system.

Classical systems

Consider a translation from the language of display logics to the language of monoidal logics, with $t(\Gamma \vdash \Sigma) = t_1(\Gamma) \longrightarrow t_2(\Sigma)$ s.t.

$$\begin{aligned} t_1(\varphi) &= \varphi & t_2(\varphi) &= \varphi \\ t_1(\varnothing) &= 1 & t_2(\varnothing) &= 0 \\ t_1(\Gamma;\Sigma) &= t_1(\Gamma) \otimes t_1(\Sigma) & t_2(\Gamma;\Sigma) &= t_2(\Gamma) \oplus t_2(\Sigma) \\ t_1(\Sigma < \Gamma) &= t_1(\Gamma) \ltimes t_1(\Sigma) & t_2(\Sigma < \Gamma) &= t_2(\Gamma) \multimap t_2(\Sigma) \\ t_1(\Gamma > \Sigma) &= t_1(\Gamma) \rtimes t_1(\Sigma) & t_2(\Gamma > \Sigma) &= t_2(\Gamma) \triangleright t_2(\Sigma) \end{aligned}$$

Classical systems

Lambek's (displayed) bilinear logics:

BL1 is defined from the syntactic calculus with $\oplus$ and $\ltimes / \rtimes$

$$\frac{\Gamma \vdash_{\mathcal{G}} \varphi \qquad \Sigma;\varphi;\Pi \vdash_{\mathcal{G}} \Delta}{\Sigma;\Gamma;\Pi \vdash_{\mathcal{G}} \Delta}\ (\mathrm{CUT}_3) \qquad \frac{\Gamma \vdash_{\mathcal{G}} \Sigma;\varphi;\Pi \qquad \varphi \vdash_{\mathcal{G}} \Delta}{\Gamma \vdash_{\mathcal{G}} \Sigma;\Delta;\Pi}\ (\mathrm{CUT}_4)$$

- The translations of these rules hold in any **M**co**M**.

Classical systems

Lambek's (displayed) bilinear logics:

BL1(a) is BL1 with some of Grishin's rules:

$$\frac{\Pi \vdash (\Gamma; \Sigma) < \Delta}{\Pi \vdash \Gamma; (\Sigma < \Delta)} \text{(G3)} \qquad \frac{\Pi \vdash \Gamma > (\Sigma; \Delta)}{\Pi \vdash (\Gamma > \Sigma); \Delta} \text{(G4)}$$

- Within a **MC**co**M**, t(G3) and t(G4) imply **lem**.

Classical systems

Lambek's (displayed) bilinear logics:

BL1(b) is BL1 with the following cut rules:

$$\frac{\varphi; \Gamma \vdash_{\mathcal{G}} \Delta \qquad \Sigma \vdash_{\mathcal{G}} \Pi; \varphi}{\Sigma; \Gamma \vdash_{\mathcal{G}} \Pi; \Delta} (\mathsf{CUT}_1) \qquad \frac{\Gamma; \varphi \vdash_{\mathcal{G}} \Delta \qquad \Sigma \vdash_{\mathcal{G}} \varphi; \Pi}{\Gamma; \Sigma \vdash_{\mathcal{G}} \Delta; \Pi} (\mathsf{CUT}_2)$$

- Lambek noted the correspondence of these rules with the weak distributivity conditions (linear distributivity or mixed associativity):

$$\varphi \otimes (\psi \oplus \rho) \longrightarrow (\varphi \otimes \psi) \oplus \rho \qquad (\psi \oplus \rho) \otimes \varphi \longrightarrow \psi \oplus (\rho \otimes \varphi) \quad \textbf{(wd)}$$

Classical systems

Lambek's (displayed) bilinear logics:

Classical bilinear logic BL2 is BL1(ab).

Lambek and Grishin noted that BL2 can alternatively be defined via an **inter-definition** of the logical connectives.

- In the commutative case, for example:

$$\varphi \multimap \psi =_{df} \neg(\varphi \otimes \neg\psi) \quad \varphi \oplus \psi =_{df} \neg(\neg\varphi \otimes \neg\psi) \quad \varphi \ltimes \psi =_{df} \neg\varphi \otimes \psi$$

Classical systems

Assumption:
The elimination of double negation(s) is **sufficient** to go from an intuitionistic to a classical system.

For example, Goré (1998) uses Ono's (1993) conception of classical substructural logics and writes

> *A substructural logic is* **classical if** *the elimination of double negation(s) is satisfied.*

Classical systems

Assumption:
The elimination of double negation(s) is **sufficient** to go from an intuitionistic to a classical system.

However, adding the elimination of double negation(s), even if considered with (**lem**), does not imply the de Morgan dualities.

Theorem

The elimination of double negations does not imply the law of excluded middle.

Classical systems

Assumption:
The elimination of double negation(s) is **sufficient** to go from an intuitionistic to a classical system.

However, adding the elimination of double negation(s), even if considered with (**lem**), does not imply the de Morgan dualities.

Theorem

*Neither (**dm1**), (**dm2**), (**dm3**) nor (**dm4**) is derivable within* **SCC***co***S** $+$ *(**lem**).*

Classical systems

Assumption:
The elimination of double negation(s) is **sufficient** to go from an intuitionistic to a classical system.

However, adding the elimination of double negation(s), even if considered with (**lem**), does not imply the de Morgan dualities.

Conclusion:
Neither **MCC**co**M** nor **MCC**co**MCC** are **classical**.

Classical systems

Assumption:
The elimination of double negation(s) is **sufficient** to go from an intuitionistic to a classical system.

However, adding the elimination of double negation(s), even if considered with (**lem**), does not imply the de Morgan dualities.

Conclusion:
Neither **MCC**co**M** nor **MCC**co**MCC** are **classical**.

Therefore, the syntactic calculus is **not** intuitionistic (bilinear) logic.

- What is an **intuitionistic** system?

Classical systems

Consider how $\oplus$ can be defined on the grounds of $\multimap$ and $\triangleright$:

$$\begin{array}{rlr} \varphi \oplus \psi \longrightarrow \sim \varphi \triangleright \psi & \varphi \oplus \psi \longrightarrow \neg\psi \multimap \varphi & (\oplus 1) \\ \sim \varphi \triangleright \psi \longrightarrow \varphi \oplus \psi & \neg\psi \multimap \varphi \longrightarrow \varphi \oplus \psi & (\oplus 2) \end{array}$$

Remark

$(\oplus 1)$ is derivable within intuitionistic logic (**CC**co**C**).

Classical systems

1) Given **MC**co**M**:

$$\begin{aligned} \textbf{wd} + * &\Rightarrow \oplus 1 \\ \oplus 1 &\Rightarrow \textbf{dm1} \\ \textbf{dm1} &\Leftrightarrow \textbf{dm3} \end{aligned}$$

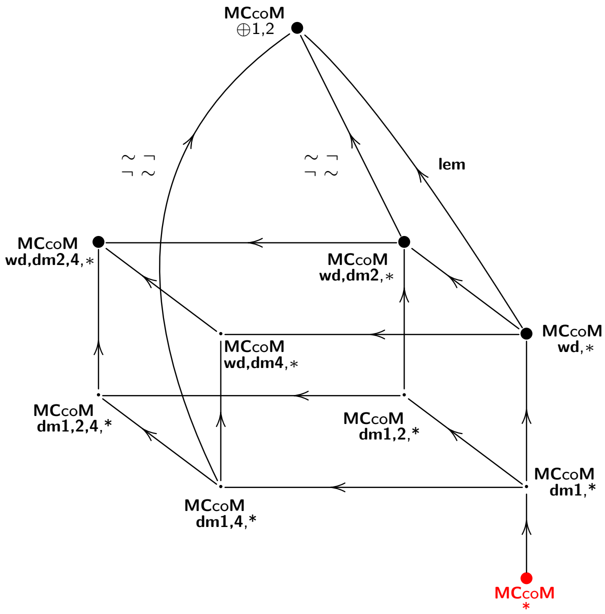
2) Given **MC**co**M** with $\oplus 1$:

$$t(\mathsf{G3}) \text{ and } t(\mathsf{G4}) \Leftrightarrow \oplus 2$$

3) Given **MCC**co**M**:

$$\begin{aligned} \oplus 1 &\Leftrightarrow \textbf{dm1} \\ \oplus 2 &\Leftrightarrow \textbf{dm2} \\ \textbf{dm2} &\Leftrightarrow \textbf{dm4} \end{aligned}$$

Classical systems



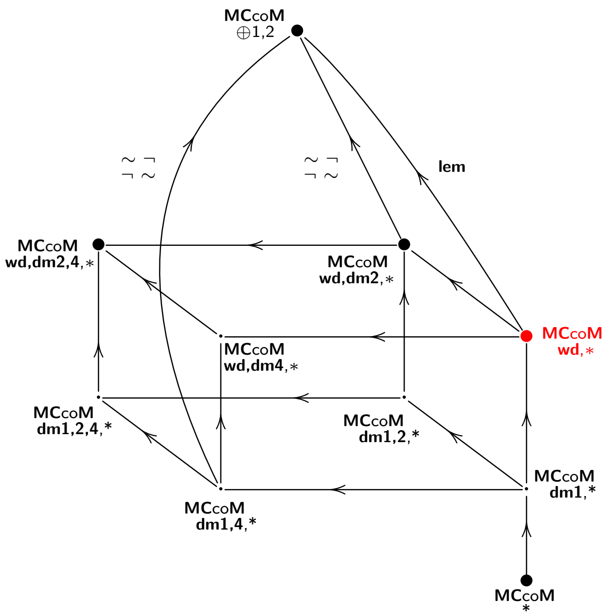
Correspondence

- Syntactic calculus
- BL1 (if co-closed)

Properties

- Intuitionistic negations

Classical systems



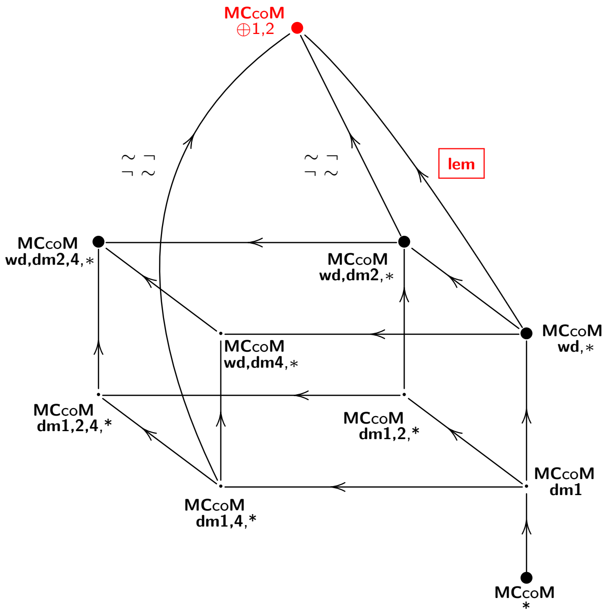
Correspondence

- non-commutative FILL
- BL1(b) (if co-closed)

Properties

- $\oplus$ is strong w.r.t. $\otimes$
- CUT
- $\oplus 1$, **dm1**, **dm3**

Classical systems



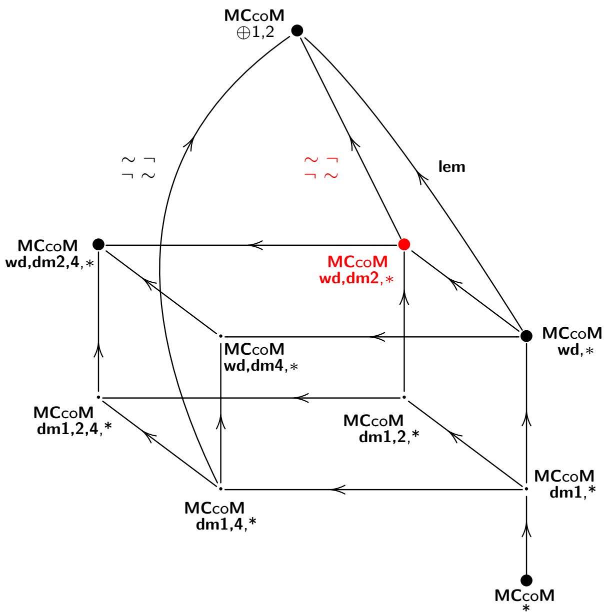
Correspondence

- **Classical** system
- BL2

Properties

- $\oplus$ adjoint of $\otimes$

Classical systems



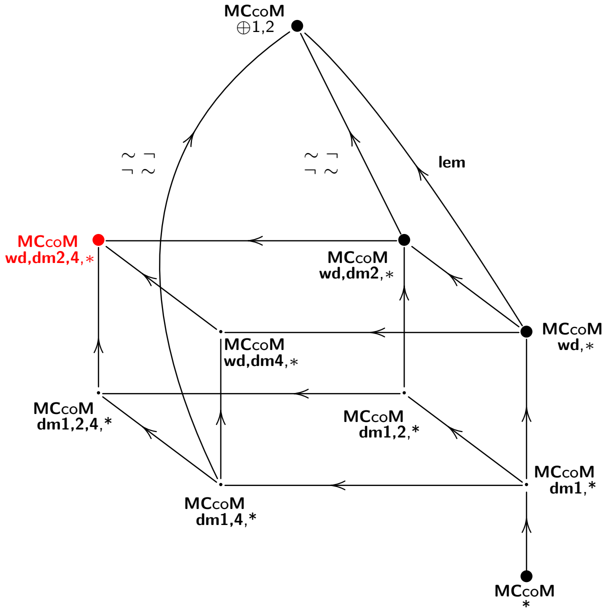
Proposal

- **Intuitionistic** system

Properties

- **dm2**

Classical systems



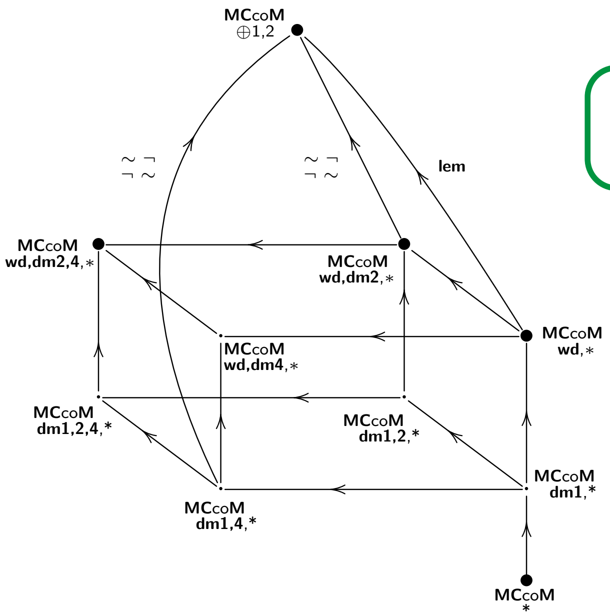
Proposal

- **de Morgan** system

Properties

- **dm4**

Classical systems



Thank you!

claytonpeterson.webs.com

Appendix

Rules

$$\frac{}{p_i \vdash p_i}\ (\mathsf{Id}) \qquad \frac{\Gamma \vdash \varphi \qquad \varphi \vdash \Sigma}{\Gamma \vdash \Sigma}\ (\mathsf{cut}_{\mathcal{G}}) \qquad \frac{\Gamma \vdash \varphi \qquad \Sigma \vdash \psi}{\Gamma; \Sigma \vdash \varphi \otimes \psi}\ (\otimes\ \mathsf{I})$$

$$\frac{\varphi \vdash \Gamma \qquad \psi \vdash \Sigma}{\varphi \oplus \psi \vdash \Gamma; \Sigma}\ (\oplus\ \mathsf{I}) \qquad \frac{\varphi; \psi \vdash \Gamma}{\varphi \otimes \psi \vdash \Gamma}\ (\mathsf{I}\ \otimes) \qquad \frac{\Gamma \vdash \varphi; \psi}{\Gamma \vdash \varphi \oplus \psi}\ (\mathsf{I}\ \oplus)$$

$$\frac{}{0 \vdash \varnothing}\ (0\vdash) \qquad \frac{}{\varnothing \vdash 1}\ (\vdash 1) \qquad \frac{\Gamma \vdash_{\mathcal{G}} \varnothing}{\Gamma \vdash 0}\ (\vdash 0) \qquad \frac{\varnothing \vdash_{\mathcal{G}} \Gamma}{1 \vdash \Gamma}\ (1\vdash)$$

$$\frac{\Delta \vdash \varphi > \psi}{\Delta \vdash \varphi \triangleright \psi}\ (\vdash \triangleright) \qquad \frac{\Delta \vdash \psi < \varphi}{\Delta \vdash \varphi \multimap \psi}\ (\vdash \multimap) \qquad \frac{\varphi \vdash \Gamma \qquad \Sigma \vdash \psi}{\psi \multimap \varphi \vdash \Gamma < \Sigma}\ (\multimap \vdash)$$

$$\frac{\Gamma \vdash \varphi \qquad \psi \vdash \Sigma}{\varphi \triangleright \psi \vdash \Gamma > \Sigma}\ (\triangleright \vdash) \qquad \frac{\varphi > \psi \vdash \Delta}{\varphi \rtimes \psi \vdash \Delta}\ (\rtimes \vdash) \qquad \frac{\psi < \varphi \vdash \Delta}{\varphi \ltimes \psi \vdash \Delta}\ (\ltimes \vdash)$$

$$\frac{\varphi \vdash \Gamma \qquad \Sigma \vdash \psi}{\Gamma > \Sigma \vdash \varphi \rtimes \psi}\ (\vdash \rtimes) \qquad \frac{\Gamma \vdash \varphi \qquad \psi \vdash \Sigma}{\Gamma < \Sigma \vdash \psi \ltimes \varphi}\ (\vdash \ltimes)$$

Display postulates

$$\begin{array}{c} \Gamma;\varnothing\vdash\Sigma \\ \hline\hline \Gamma\vdash\Sigma \\ \hline\hline \varnothing;\Gamma\vdash\Sigma \end{array}\ (\varnothing\vdash) \qquad \begin{array}{c} \Gamma\vdash\Sigma;\varnothing \\ \hline\hline \Gamma\vdash\Sigma \\ \hline\hline \Gamma\vdash\varnothing;\Sigma \end{array}\ (\vdash\varnothing)$$

$$\begin{array}{c} \Gamma\vdash\Delta<\Sigma \\ \hline\hline \Gamma;\Sigma\vdash\Delta \\ \hline\hline \Sigma\vdash\Gamma>\Delta \end{array}\ (\mathsf{dp}) \qquad \begin{array}{c} \Gamma>\Delta\vdash\Sigma \\ \hline\hline \Delta\vdash\Gamma;\Sigma \\ \hline\hline \Delta<\Sigma\vdash\Gamma \end{array}\ (\mathsf{dp})$$

Structural rules

$$\frac{\Gamma\vdash_{\mathcal{G}}\Delta}{\Gamma;\Sigma\vdash_{\mathcal{G}}\Delta}\ (\mathsf{Wk}\vdash) \qquad \frac{\Gamma\vdash_{\mathcal{G}}\Delta}{\Sigma;\Gamma\vdash_{\mathcal{G}}\Delta}\ (\mathsf{Wk}\vdash) \qquad \frac{\Gamma\vdash_{\mathcal{G}}\Delta}{\Gamma\vdash_{\mathcal{G}}\Delta;\Sigma}\ (\vdash\mathsf{Wk})$$

$$\frac{\Gamma\vdash_{\mathcal{G}}\Delta}{\Gamma\vdash_{\mathcal{G}}\Sigma;\Delta}\ (\vdash\mathsf{Wk})$$

$$\frac{\Gamma;\Gamma\vdash_{\mathcal{G}}\Delta}{\Gamma\vdash_{\mathcal{G}}\Delta}\ (\mathsf{Ctr}\vdash) \qquad \frac{\Gamma\vdash_{\mathcal{G}}\Delta;\Delta}{\Gamma\vdash_{\mathcal{G}}\Delta}\ (\vdash\mathsf{Ctr}) \qquad \frac{\Gamma;\Sigma\vdash_{\mathcal{G}}\Delta}{\Sigma;\Gamma\vdash_{\mathcal{G}}\Delta}\ (\mathsf{Com}\vdash)$$

$$\frac{\Gamma\vdash_{\mathcal{G}}\Sigma;\Delta}{\Gamma\vdash_{\mathcal{G}}\Delta;\Sigma}\ (\vdash\mathsf{Com})$$

$$\begin{array}{c} \Gamma;(\Sigma;\Delta)\vdash_{\mathcal{G}}\Pi \\ \hline\hline (\Gamma;\Sigma);\Delta\vdash_{\mathcal{G}}\Pi \end{array}\ (\mathsf{Ass}\vdash) \qquad \begin{array}{c} \Pi\vdash_{\mathcal{G}}\Gamma;(\Sigma;\Delta) \\ \hline\hline \Pi\vdash_{\mathcal{G}}(\Gamma;\Sigma);\Delta \end{array}\ (\vdash\mathsf{Ass})$$

Translation

Definition of structures:

$$\Gamma := \varphi \mid \varnothing \mid \Sigma ; \Delta \mid \Sigma < \Delta \mid \Sigma > \Delta$$

The translation $t(\Gamma \vdash_{\mathcal{G}} \Sigma) = t_1(\Gamma) \longrightarrow t_2(\Sigma)$is given by

$$\begin{aligned} t_1(\varphi) &= \varphi & t_2(\varphi) &= \varphi \\ t_1(\varnothing) &= 1 & t_2(\varnothing) &= 0 \\ t_1(\Gamma;\Sigma) &= t_1(\Gamma) \otimes t_1(\Sigma) & t_2(\Gamma;\Sigma) &= t_2(\Gamma) \oplus t_2(\Sigma) \\ t_1(\Sigma < \Gamma) &= t_1(\Gamma) \ltimes t_1(\Sigma) & t_2(\Sigma < \Gamma) &= t_2(\Gamma) \multimap t_2(\Sigma) \\ t_1(\Gamma > \Sigma) &= t_1(\Gamma) \rtimes t_1(\Sigma) & t_2(\Gamma > \Sigma) &= t_2(\Gamma) \triangleright t_2(\Sigma) \end{aligned}$$

Translation

Let $T_1'(\varphi) = \varphi$ and $T_2'(\psi) = \psi$, except for $*$ and $\star$ in which case $T_1'(*) = T_2'(*) = 0$ and $T_1'(\star) = T_2'(\star) = 1$.

The translation $t'(\varphi \longrightarrow \psi) = t_1'(\varphi) \vdash_{\mathcal{G}} t_2'(\psi)$ is given by

$$\begin{aligned} t_1'(p_i) &= p_i & t_2'(p_i) &= p_i \\ t_1'(1) &= 1 & t_2'(1) &= 1 \\ t_1'(0) &= 0 & t_2'(0) &= 0 \\ t_1'(\star) &= 1 & t_2'(\star) &= 1 \\ t_1'(*) &= 0 & t_2'(*) &= 0 \\ t_1'(\varphi \otimes \psi) &= T_1'(\varphi); T_1'(\psi) & t_2'(\varphi \otimes \psi) &= T_2'(\varphi) \otimes T_2'(\psi) \\ t_1'(\varphi \oplus \psi) &= T_1'(\varphi) \oplus T_1'(\psi) & t_2'(\varphi \oplus \psi) &= T_2'(\varphi); T_2'(\psi) \\ t_1'(\varphi \triangleright \psi) &= T_1'(\varphi) \triangleright T_1'(\psi) & t_2'(\varphi \triangleright \psi) &= T_2'(\varphi) > T_2'(\psi) \\ t_1'(\varphi \multimap \psi) &= T_1'(\varphi) \multimap T_1'(\psi) & t_2'(\varphi \multimap \psi) &= T_2'(\psi) < T_2'(\varphi) \\ t_1'(\varphi \rtimes \psi) &= T_1'(\varphi) > T_1'(\psi) & t_2'(\varphi \rtimes \psi) &= T_2'(\varphi) \rtimes T_2'(\psi) \\ t_1'(\varphi \ltimes \psi) &= T_1'(\psi) < T_1'(\varphi) & t_2'(\varphi \ltimes \psi) &= T_2'(\varphi) \ltimes T_2'(\psi) \end{aligned}$$

Category

A category $\mathcal{C}$ is composed of:

1. *$\mathcal{C}$-objects*;
2. *$\mathcal{C}$-arrows*;
3. an operation assigning to each arrow a domain and a codomain within the $\mathcal{C}$-objects;
4. composition of arrow gf for each pair $f : \varphi \longrightarrow \psi$ and $g : \psi \longrightarrow \rho$ that respects associativity, i.e., given $h : \rho \longrightarrow \tau$, $h(gf) = (hg)f$;
5. an identity arrow 1_ψ for each $\mathcal{C}$-object ψ such that $1_\psi f = f$ and $g1_\psi = g$ for each pair $f : \varphi \longrightarrow \psi$ and $g : \psi \longrightarrow \rho$.

$$
\begin{tikzcd}
\varphi \arrow[r, "f"] \arrow[dr, "1_\psi f=f"'] & \psi \arrow[d, "1_\psi"] \arrow[dr, "g=g1_\psi"] & \\
& \psi \arrow[r, "g"'] & \rho
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi \arrow[r, "f"] \arrow[d, "h(gf)=(hg)f"'] & \psi \arrow[d, "g"] \\
\tau & \rho \arrow[l, "h"]
\end{tikzcd}
$$

Functor

A functor $F : \mathcal{C} \longrightarrow \mathcal{B}$ is a morphism between two categories such that:

1. there is $F(\varphi)$ in $\mathcal{B}$ for each φ in $\mathcal{C}$;
2. there is $F(\varphi) \xrightarrow{F(f)} F(\psi)$ in $\mathcal{B}$ for each $\varphi \xrightarrow{f} \psi$ in $\mathcal{C}$;
3. $F(1_\varphi) = 1_{F(\varphi)}$;
4. $F(gf) = F(g)F(f)$.

Natural transformation

Let $f : \varphi \longrightarrow \psi$ be a $\mathcal{C}$-arrow and $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A natural transformation $\eta : F \longrightarrow G$ is a family of arrows such that for every φ of $\mathcal{C}$ there is $\eta_\varphi : F(\varphi) \longrightarrow G(\varphi)$ in $\mathcal{D}$ making the following diagram commute.

$$\begin{tikzcd}
F(\varphi) \arrow[r, "\eta_\varphi"] \arrow[d, "F(f)"'] & G(\varphi) \arrow[d, "G(f)"] \\
F(\psi) \arrow[r, "\eta_\psi"'] & G(\psi)
\end{tikzcd}$$

Adjunction

An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ together with two natural transformations $\eta : 1_\mathcal{C} \longrightarrow GF$ and $\zeta : FG \longrightarrow 1_\mathcal{D}$ such that $\zeta_F F(\eta) = 1_F$ and $G(\zeta)\eta_G = 1_G$.

$$\begin{tikzcd}
F(\varphi) \arrow[r, "F(\eta_\varphi)"] \arrow[dr, "1_{F(\varphi)}"'] & FGF(\varphi) \arrow[d, "\zeta_{F(\varphi)}"] \\
& F(\varphi)
\end{tikzcd}
\qquad
\begin{tikzcd}
G(\varphi) \arrow[r, "\eta_{G(\varphi)}"] \arrow[dr, "1_{G(\varphi)}"'] & GFG(\varphi) \arrow[d, "G(\zeta_\varphi)"] \\
& G(\varphi)
\end{tikzcd}$$

Adjunction

Alternative definition:

$$\mathsf{Hom}_{\mathcal{C}}(\varphi \otimes \psi, \rho) \cong \mathsf{Hom}_{\mathcal{C}}(\varphi, \psi \multimap \rho)$$

For each arrow $\varphi \otimes \psi \longrightarrow \rho$ there is an arrow $\varphi \longrightarrow \psi \multimap \rho$, and vice versa.

Strong

$\oplus 1_\rho$ is strong with respect to $1_\varphi \otimes$ when there are natural transformations from $1_\varphi \otimes \oplus 1_\rho$ to $\oplus 1_\rho 1_\varphi \otimes$ and from $\otimes 1_\varphi 1_\psi \oplus$ and $1\psi \oplus \otimes 1_\varphi$.

Monoidal category

A monoidal category is a category together with an associative tensor product $\otimes$ and a unit object 1 satisfying the following natural isomorphisms for every φ, ψ and ρ and respecting the triangle and pentagon identities.

$$
\begin{aligned}
& a_{\varphi,\psi,\rho} : (\varphi \otimes \psi) \otimes \rho \longrightarrow \varphi \otimes (\psi \otimes \rho) \\
& l_x : 1 \otimes \varphi \longrightarrow \varphi \\
& r_x : \varphi \otimes 1 \longrightarrow \varphi
\end{aligned}
$$

Monoidal category

Triangle and pentagon identities

$$
\begin{tikzcd}
(\varphi \otimes 1) \otimes \psi \arrow[rr, "a_{\varphi,1,\psi}"] \arrow[dr] & & \varphi \otimes (1 \otimes \psi) \arrow[dl] \\
& \varphi \otimes \psi &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (\tau \otimes \varphi) \otimes (\psi \otimes \rho) \arrow[r] & \tau \otimes (\varphi \otimes (\psi \otimes \rho)) \\
((\tau \otimes \varphi) \otimes \psi) \otimes \rho \arrow[ur] \arrow[dr] & & \\
& (\tau \otimes (\varphi \otimes \psi)) \otimes \rho \arrow[r] & \tau \otimes ((\varphi \otimes \psi) \otimes \rho) \arrow[uu]
\end{tikzcd}
$$

Symmetric category

A symmetric monoidal category is a monoidal category with a natural isomorphism $\beta_{\varphi,\psi} : \varphi \otimes \psi \longrightarrow \psi \otimes \varphi$ which is its own inverse. It has to satisfy the hexagon identities.

Symmetric category

Hexagon identities

$$
\begin{tikzcd}
\varphi \otimes (\psi \otimes \rho) \arrow[r] \arrow[d] & (\varphi \otimes \psi) \otimes \rho \arrow[r] & (\psi \otimes \varphi) \otimes \rho \arrow[d] \\
(\psi \otimes \rho) \otimes \varphi & \psi \otimes (\rho \otimes \varphi) \arrow[l] & \psi \otimes (\varphi \otimes \rho) \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\varphi \otimes \psi) \otimes \rho \arrow[r] \arrow[d] & \varphi \otimes (\psi \otimes \rho) \arrow[r] & \varphi \otimes (\rho \otimes \psi) \arrow[d] \\
\rho \otimes (\varphi \otimes \psi) & (\rho \otimes \varphi) \otimes \psi \arrow[l] & (\varphi \otimes \rho) \otimes \psi \arrow[l]
\end{tikzcd}
$$

Monoidal closed categoriy

A closed monoidal category is a monoidal category where the tensor product $- \otimes \varphi$ has right adjoint $\varphi \multimap -$.

Cartesian category

A Cartesian category is a category where $\top$ is a terminal object and the tensor product respects the universal property defined by the following commutative diagram, which means that for all f and g such that $f : \rho \longrightarrow \varphi$ and $g : \rho \longrightarrow \psi$ there is one and only one arrow $\langle f, g\rangle : \rho \longrightarrow \varphi \otimes \psi$ making the diagram commute.

$$\begin{tikzcd}
 & \rho \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
\varphi & \varphi \otimes \psi \arrow[l, "pr_\varphi"] \arrow[r, "pr_\psi"'] & \psi
\end{tikzcd}$$

Appendix

The natural deduction normal form and coherence

General Proof Theory, 28 November 2015, Tübingen

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This talk is about coherence, a notion originated in category theory, and its proof theoretical counterpart. Everything will be explained through an example recently obtained in a joint work with Kosta Došen.

Turning disjunction into conjunction

$$\vee$$

$$\wedge$$

Turning disjunction into conjunction

The same holds for derivations

$$\frac{\Phi}{\Phi \vee \Theta}$$

$$\frac{\Phi}{\Phi \vee \Theta}$$

The same holds for derivations

The goal

$$\dfrac{\dfrac{\dfrac{\dfrac{p_1\wedge p_2}{p_1}\quad\dfrac{p_1\wedge p_2}{p_2}}{p_1\wedge p_2}}{p_1}\qquad\qquad\dfrac{p_1\wedge p_2}{p_1}}{p_1\wedge p_1}$$

is faithfully represented by (where $\Pi = p\vee p\vee p$)

$$\dfrac{\Pi\vee\Pi\qquad\quad \dfrac{p}{p\vee\Pi}\quad\dfrac{p}{p\vee\Pi}\quad\dfrac{p}{p\vee\Pi}\quad\dfrac{p}{\Pi\vee p}\quad\dfrac{p}{\Pi\vee p}\quad\dfrac{p}{\Pi\vee p}}{p\vee p\vee p\vee p}\;\text{5 times}\vee\text{elim.}$$

In the language of category theory

A skeleton of the category with finite coproducts freely generated by a single object has a subcategory isomorphic to a skeleton of the category with finite products freely generated by a countable set of objects.

The conjunctive system

Consider conjunction separated from other connectives.

alphabet: $p_1, p_2, \ldots, \wedge$

rules of inference: $\dfrac{A \quad B}{A \wedge B} \qquad \dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

reductions:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix} \quad \begin{matrix}\mathcal{E}\\ B\end{matrix}}{A \wedge B}}{A} \xrightarrow{\beta} \begin{matrix}\mathcal{D}\\ A\end{matrix} \qquad\qquad \begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix} \xrightarrow{\eta} \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{A} \quad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{B}}{A \wedge B}$$

Equality of derivations

Single premise and single conclusion derivations. The reductions are turned into equalities.

The following derivations from $p_1 \wedge p_2$ to $p_1 \wedge p_1$ are equal.

$$\dfrac{\dfrac{\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \quad \dfrac{p_1 \wedge p_2}{p_2}}{p_1 \wedge p_2}}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1} = \dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

The disjunctive system

Consider disjunction separated from other connectives.

alphabet: $p, \vee$

rules of inference: as usual

reductions: as usual

The formulae (up to associativity) may be identified with finite ordinals.

The representation of formulae

Let F be a mapping from conjunctive formulae to disjunctive formulae:

$$p_i \mapsto \underbrace{p \vee \ldots \vee p}_{\mathbf{p}_i},$$

where $\mathbf{p}_i$ is the i-th prime number, and if A and B are mapped respectively to

$$\underbrace{p \vee \ldots \vee p}_{m} \text{ and } \underbrace{p \vee \ldots \vee p}_{n},$$

then $A \wedge B$ is mapped to

$$\underbrace{p \vee \ldots \vee p}_{m \cdot n}.$$

Examples

$$p_1 \wedge p_2 \mapsto \underbrace{p \vee p \vee p \vee p \vee p \vee p}_{2\cdot 3},$$

$$p_1 \wedge p_1 \mapsto \underbrace{p \vee p \vee p \vee p}_{2\cdot 2}.$$

Derivability

For $m, n \geq 1$ it is always the case that

$$\underbrace{p \vee \ldots \vee p}_{m} \vdash \underbrace{p \vee \ldots \vee p}_{n}.$$

If we are interested just in derivability, then our mapping F is not conclusive since it is not true that

$$A \vdash B \Leftrightarrow FA \vdash FB.$$

For example, let A be p_1 and let B be p_2—there is a derivation from $p \vee p$ to $p \vee p \vee p$, but there is no derivation from p_1 to p_2. Hence, when one starts representing the derivations, it will not be the case that every disjunctive derivation represents a conjunctive derivation.

Derivability

The following definition of $\vdash$ at the right-hand side of

$$A \vdash B \Leftrightarrow FA \vdash FB.$$

makes this equivalence true. For $m, n \geq 1$,

$$\underbrace{p \vee \ldots \vee p}_{m} \vdash \underbrace{p \vee \ldots \vee p}_{n},$$

when every prime that divides n divides m, too. This gives an arithmetical characterization of derivability in the conjunctive system.

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{\color{red}{p_1}} \qquad \dfrac{p_1 \wedge p_2}{\color{blue}{p_1}}}{p_1 \wedge p_1}$$

Coherence

$$\dfrac{\dfrac{p_1 \wedge p_2}{\color{red}{p_1}} \qquad \dfrac{p_1 \wedge p_2}{\color{blue}{p_1}}}{\color{red}{p_1} \wedge \color{blue}{p_1}}$$

Coherence

$$\cfrac{\cfrac{p_1 \wedge p_2}{p_1} \qquad \cfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Coherence

$$\cfrac{\cfrac{p_1 \wedge p_2}{p_1} \qquad \cfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

For every conjunctive derivation in normal form there is a function (from the letter occurrences in the conclusion to the letter occurrences in the premise). Two different normal forms correspond to different functions. Hence, our derivation is completely determined by the following picture.

$$p_1 \wedge p_2 \vdash p_1 \wedge p_1.$$

Coherence

Dually, every disjunctive derivation from

$$\underbrace{p \vee p \vee p \vee p \vee p \vee p}_{m} \quad \text{to} \quad \underbrace{p \vee p \vee p \vee p \vee p \vee p}_{n}$$

is identified with a function from the ordinal m to the ordinal n.

For every function $f \colon m \to n$ there is a derivation identified with f.

Extending F to derivations

We have to find a function from $2 \cdot 3$ to $2 \cdot 2$ that faithfully represents our derivation

$$p_1 \wedge p_2 \vdash p_1 \wedge p_1,$$

which can be determined also by the following triple

$$(\;, p_1 \wedge p_2, p_1 \wedge p_1).$$

Faithfulness means that two different derivations should be mapped to different functions.

Brauerian representation

Brauerian representation of

$$(\,\cdot\!\diagdown\!\cdot, p_1 \wedge p_2, p_1 \wedge p_1)$$

is a function from $2 \cdot 3$ to $2 \cdot 2$ defined as follows.

Identify the elements of the ordinal 6 with the elements of cartesian product 2×3 lexicographically ordered. Do the same with 4.

00 01 02 10 11 12

00 01 10 11

Brauerian representation

Brauerian representation of

$$(\,\cdot\!\diagdown\!\cdot, p_1 \wedge p_2, p_1 \wedge p_1)$$

is a function from the ordinal $2 \cdot 3$ to the ordinal $2 \cdot 2$ defined as follows.

Identify the elements of the ordinal 6 with the elements of cartesian product 2×3 lexicographically ordered. Do the same with 4.

00 01 02 10 11 12

00 01 10 11

Brauerian representation

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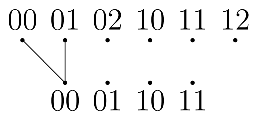

Brauerian representation

Brauerian representation of

$$(\mid\!\diagdown, p_1 \wedge p_2, p_1 \wedge p_1)$$

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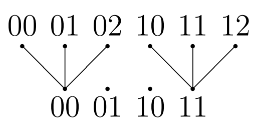
Identify the elements of the ordinal 6 with the elements of cartesian product 2×3 lexicographically ordered. Do the same with 4.



Different triples are represented by different functions.

Representing conjunctive by disjunctive derivations

Let $\mathcal{D}$ be a conjunctive derivation. Use the following steps in order to represent it by a disjunctive derivation.

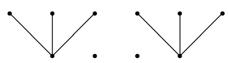
(1) Normalize $\mathcal{D}$.

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

(2) Find its triple. $(\; , p_1 \wedge p_2, p_1 \wedge p_1)$

Representing conjunctive by disjunctive derivations

(3) Transform it into a function using brauerian representation.



(4) Find a disjunctive derivation identified with that function.

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p}}{p \vee p \vee p \vee p} \; \text{5 times} \vee \text{elim.}$$

Representing conjunctive by disjunctive derivations

It is not the case that the conjunctive inference rules are derivable from the disjunctive inference rules.

"Composition", which corresponds to the cut rule in sequent systems is preserved by this representation.

Composition

Take the derivations

$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1} \qquad \text{and} \qquad \dfrac{p_1 \wedge p_1}{p_1}$$

and paste them together

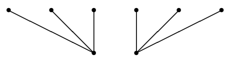
$$\dfrac{\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}}{p_1}$$

Composition

The normal form of the result is $\dfrac{p_1 \wedge p_2}{p_1}$.

The corresponding triple is (, $p_1 \wedge p_2, p_1$).

Its brauerian representation is given by:



A disjunctive derivation identified with this function is

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p}\ \text{5 times} \vee \text{elim.}$$

Composition

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p}$$

is equal to the derivation obtained by pasting together

$$\dfrac{\Pi \vee \Pi \qquad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{p \vee \Pi} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p} \quad \dfrac{p}{\Pi \vee p}}{p \vee p \vee p \vee p}\ \text{5 times} \vee \text{elim.}$$

and

$$\dfrac{p \vee p \vee p \vee p \qquad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p} \quad \dfrac{p}{p \vee p}}{p \vee p}\ \text{3 times} \vee \text{elim.}$$

Substitution

How can we treat $F(p_1) = p \vee p$ and $F(p_2) = p \vee p \vee p$ as variables in the formula $p \vee p \vee p \vee p \vee p \vee p$?

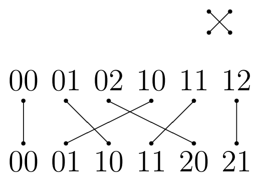
How to substitute $F(A)$ for $F(p_1)$ and $F(B)$ for $F(p_2)$ in the representation of our derivation

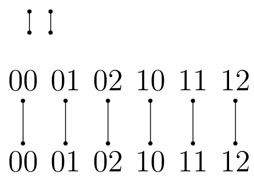
$$\dfrac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_1 \wedge p_1}$$

Substitution

The image of our conjunctive system in the disjunctive system has the universal property with respect to $\{F(p_1), F(p_2), \ldots\}$ in the sense that every mapping of that set to the set of disjunctive formulae extends in a unique way to a function that maps all the disjunctive formulae to the disjunctive formulae and all the derivations in the image of our representation to the disjunctive derivations. This function imitates substitution. However, it is not the operation of replacing words by words.

$$\frac{\dfrac{p_1 \wedge p_2}{p_2} \qquad \dfrac{p_1 \wedge p_2}{p_1}}{p_2 \wedge p_1} \qquad\qquad \frac{\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_2}}{p_1 \wedge p_2}$$





The talk was based on: K. Došen and Z. Petrić, *Representing conjunctive deductions by disjunctive deductions*, (available at: arXiv)

Decorating natural deduction

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General Proof Theory, Tübingen, 27. - 29. November 2015

1 / 21

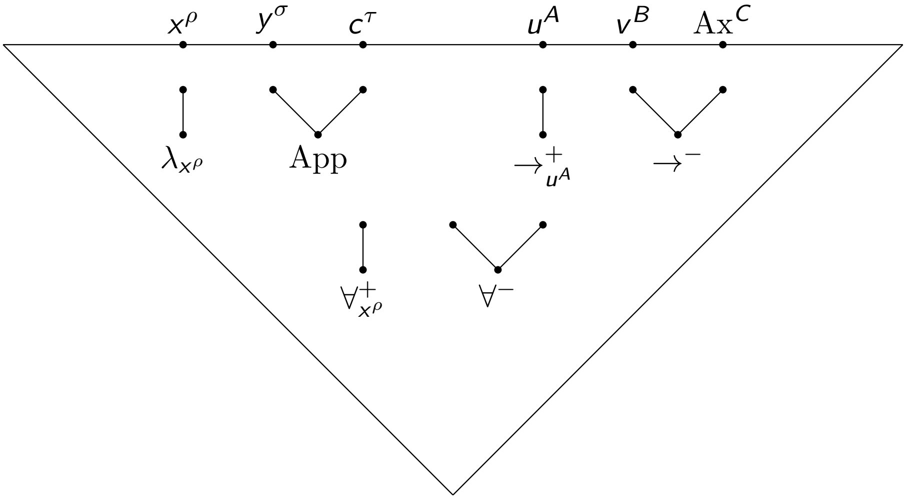
- Proofs may have computational content, which can be extracted (via realizability).
- Proofs (as opposed to programs) can easily be checked for correctness.

Issues:

- Why proofs in natural deduction?
- Complexity.

2 / 21

Proof terms in natural deduction



The realizability interpretation transforms such a proof term directly into an object term.

Logic

- Use $\to$, $\forall$ only, defined by introduction and elimination rules.
- View $\exists_x A$, $A \vee B$, $A \wedge B$ as inductively defined predicates (with parameters A, B).
- In addition, define classical existence and disjunction by

$$\tilde{\exists}_x A := \neg\forall_x \neg A,$$
$$A \mathbin{\tilde{\vee}} B := \neg(\neg A \wedge \neg B)$$

where $\neg A := (A \to \mathbf{F})$ and $\mathbf{F} := (0 = 1)$.

Example: disjunction

$A \vee B$ is inductively defined by the clauses (introduction axioms)

$$A \to A \vee B, \qquad B \to A \vee B$$

with least-fixed-point (elimination) axiom

$$A \vee B \to (A \to C) \to (B \to C) \to C.$$

Decoration

- Goal: fine tune the computational content of a proof.
- Tool: distinguish $\to^{\mathrm{c}}$, $\forall^{\mathrm{c}}$ (computational) and $\to^{\mathrm{nc}}$, $\forall^{\mathrm{nc}}$ (non-computational).

The rules for $(\to^{\mathrm{nc}})^+$, $(\forall^{\mathrm{nc}})^+$ are restricted: the abstracted (object or assumption) variable must not be "used computationally".

Remark: Coq uses Set and Prop instead (but this is less flexible).

Example: computational variants of disjunction

We have four possibilities to decorate the two clauses for $\vee$:

$$\begin{cases} A \to^{\mathrm{c}} A \vee^{\mathrm{d}} B \\ B \to^{\mathrm{c}} A \vee^{\mathrm{d}} B \end{cases} \quad \begin{cases} A \to^{\mathrm{c}} A \vee^{\mathrm{l}} B \\ B \to^{\mathrm{nc}} A \vee^{\mathrm{l}} B \end{cases} \quad \begin{cases} A \to^{\mathrm{nc}} A \vee^{\mathrm{r}} B \\ B \to^{\mathrm{c}} A \vee^{\mathrm{r}} B \end{cases} \quad \begin{cases} A \to^{\mathrm{nc}} A \vee^{\mathrm{u}} B \\ B \to^{\mathrm{nc}} A \vee^{\mathrm{u}} B \end{cases}$$

Elimination axioms:

$$\begin{aligned} A \vee^{\mathrm{d}} B \to^{\mathrm{c}} (A \to^{\mathrm{c}} C) \to^{\mathrm{c}} (B \to^{\mathrm{c}} C) \to^{\mathrm{c}} C, \\ A \vee^{\mathrm{l}} B \to^{\mathrm{c}} (A \to^{\mathrm{c}} C) \to^{\mathrm{c}} (B \to^{\mathrm{nc}} C) \to^{\mathrm{c}} C, \\ A \vee^{\mathrm{r}} B \to^{\mathrm{c}} (A \to^{\mathrm{nc}} C) \to^{\mathrm{c}} (B \to^{\mathrm{c}} C) \to^{\mathrm{c}} C, \\ A \vee^{\mathrm{u}} B \to^{\mathrm{c}} (A \to^{\mathrm{nc}} C) \to^{\mathrm{c}} (B \to^{\mathrm{nc}} C) \to^{\mathrm{c}} C. \end{aligned}$$

7 / 21

Formulas as computational problems

- Kolmogorov (1932) proposed to view a formula A as a computational problem, of type $\tau(A)$, the type of a potential solution or "realizer" of A.
- Example: $\forall^{\mathrm{c}}_n \exists_{m>n} \mathrm{Prime}(m)$ has type $\mathbf{N} \to \mathbf{N}$.
- $A \mapsto \tau(A)$, a type or the "nulltype" symbol $\circ$.
- In case $\tau(A) = \circ$ proofs of A have no computational content; such formulas A are called non-computational (n.c.) or Harrop formulas; the others computationally relevant (c.r.).

8 / 21

Decoration can simplify extracts

- Suppose that a proof M uses a lemma $L^{\mathrm{d}}\colon A \vee^{\mathrm{d}} B$.
- Then the extract $\mathrm{et}(M)$ will contain the extract $\mathrm{et}(L^{\mathrm{d}})$.
- Suppose that the only computationally relevant use of L^{d} in M was which one of the two alternatives holds true, A or B.
- Express this by using a weakened lemma $L\colon A \vee^{\mathrm{u}} B$.
- Since $\mathrm{et}(L)$ is a boolean, the extract of the modified proof is "purified": the (possibly large) extract $\mathrm{et}(L^{\mathrm{d}})$ has disappeared.

Decoration algorithm

Goal: Insert as few as possible decorations $\forall^{\mathrm{c}}, \to^{\mathrm{c}}$ into a proof.

- $\mathrm{Seq}(M)$ of a proof M consists of its context and end formula.
- The uniform proof pattern $\mathrm{P}(M)$ of a proof M is the result of changing in c.r. formulas of M (i.e., not above a n.c. formula) all $\to^{\mathrm{c}}$, $\forall^{\mathrm{c}}$ into $\to^{\mathrm{nc}}$, $\forall^{\mathrm{nc}}$ (some restrictions apply on axioms and theorems).
- A formula D extends C if D is obtained from C by changing some $\to^{\mathrm{nc}}$, $\forall^{\mathrm{nc}}$ into $\to^{\mathrm{c}}$, $\forall^{\mathrm{c}}$.
- A proof N extends M if (i) N and M are the same up to variants of $\to$, $\forall$ in their formulas, and (ii) every c.r. formula in M is extended by the corresponding one in N.

Decoration algorithm (ctd.)

- Assumption: For every axiom or theorem A and every decoration variant C of A we have another axiom or theorem whose formula D extends C, and D is the least among those extensions.
- Example: Induction

  $$A'(0) \to^{\mathrm{c/nc}} \forall_n^{\mathrm{c/nc}}(A''(n) \to^{\mathrm{c/nc}} A'''(n{+}1))) \to^{\mathrm{c/nc}} \forall_n^{\mathrm{c/nc}} A''''(n).$$

  Let A be the lub (w.r.t. deco) of $A', \dots, A''''$. Extended axiom:

  $$A(0) \to^{\mathrm{c}} \forall_n^{\mathrm{c}}(A(n) \to^{\mathrm{c}} A(n+1))) \to^{\mathrm{c}} \forall_n^{\mathrm{c}} A(n).$$

11 / 21

Decoration algorithm (ctd.)

Theorem (Ratiu & S., 2010)

Under the assumption above, for every uniform proof pattern U and every extension of its sequent $\mathrm{Seq}(U)$ we can find a decoration M_∞ of U such that

(a) *$\mathrm{Seq}(M_\infty)$ extends the given extension of $\mathrm{Seq}(U)$, and*

(b) *M_∞ is optimal in the sense that any other decoration M of U whose sequent $\mathrm{Seq}(M)$ extends the given extension of $\mathrm{Seq}(U)$ has the property that M also extends M_∞.*

12 / 21

Case $(\to^{\mathrm{nc}})^-$. Consider a proof pattern

$$\dfrac{\begin{matrix}\Phi, \Gamma \\ \mid U \\ A \to^{\mathrm{nc}} B\end{matrix} \qquad \begin{matrix}\Gamma, \Psi \\ \mid V \\ A\end{matrix}}{B}\,(\to^{\mathrm{nc}})^-$$

Given: extension $\Pi, \Delta, \Sigma \Rightarrow D$ of $\Phi, \Gamma, \Psi \Rightarrow B$. Alternating steps:

- $\mathrm{IH}_a(U)$ for extension $\Pi, \Delta \Rightarrow A \to^{\mathrm{nc}} D \mapsto$ decoration M_1 of U whose sequent $\Pi_1, \Delta_1 \Rightarrow C_1 \to D_1$ extends $\Pi, \Delta \Rightarrow A \to^{\mathrm{nc}} D$ $(\to \in \{\to^{\mathrm{nc}}, \to^{\mathrm{c}}\})$. Suffices if A is n.c.: extension $\Delta_1, \Sigma \Rightarrow C_1$ of V is a proof (in n.c. parts of a proof $\to^{\mathrm{nc}}$, $\forall^{\mathrm{nc}}$ and $\to^{\mathrm{c}}$, $\forall^{\mathrm{c}}$ are identified). For A c.r:
- $\mathrm{IH}_a(V)$ for the extension $\Delta_1, \Sigma \Rightarrow C_1 \mapsto$ decoration N_2 of V whose sequent $\Delta_2, \Sigma_2 \Rightarrow C_2$ extends $\Delta_1, \Sigma \Rightarrow C_1$.
- $\mathrm{IH}_a(U)$ for $\Pi_1, \Delta_2 \Rightarrow C_2 \to D_1 \mapsto$ decoration M_3 of U whose sequent $\Pi_3, \Delta_3 \Rightarrow C_3 \to D_3$ extends $\Pi_1, \Delta_2 \Rightarrow C_2 \to D_1$.
- $\mathrm{IH}_a(V)$ for the extension $\Delta_3, \Sigma_2 \Rightarrow C_3 \mapsto$ decoration N_4 of V whose sequent $\Delta_4, \Sigma_4 \Rightarrow C_4$ extends $\Delta_3, \Sigma_2 \Rightarrow C_3$. ...

13 / 21

Example: Euler's φ, or avoiding factorization

Let $P(n)$ mean "n is prime". Consider

`Fact`:	$\forall^{\mathrm{c}}_n(P(n) \vee^{\mathrm{r}} \exists_{m,k>1}(n = mk))$	factorization,
`PTest`:	$\forall^{\mathrm{c}}_n(P(n) \vee^{\mathrm{u}} \exists_{m,k>1}(n = mk))$	prime number test.

Euler's φ has the properties

$$\begin{cases} \varphi(n) = n - 1 & \text{if } P(n), \\ \varphi(n) < n - 1 & \text{if } n \text{ is composed.} \end{cases}$$

Using factorization and these properties we obtain a proof of

$$\forall^{\mathrm{c}}_n(\varphi(n) = n - 1 \vee^{\mathrm{u}} \varphi(n) < n - 1).$$

Goal: get rid of the expensive factorization algorithm in the computational content, via decoration.

14 / 21

Example: Euler's φ, or avoiding factorization (ctd.)

How could the better proof be found? Recall that we assumed

$$\texttt{Fact}\colon \forall^{c}_{n}(P(n) \vee^{r} \exists_{m,k>1}(n = mk)),$$
$$\texttt{PTest}\colon \forall^{c}_{n}(P(n) \vee^{u} \exists_{m,k>1}(n = mk))$$

and have a proof of $\forall^{c}_{n}(\varphi(n) = n - 1 \vee^{u} \varphi(n) < n - 1)$ from Fact.

- The decoration algorithm arrives at Fact with goal

$$P(n) \vee^{u} \exists_{m,k>1}(n = mk).$$

- PTest fits as well, and it has $\vee^{u}$ rather than $\vee^{r}$, hence is preferred.


```
(define decnproof (fully-decorate nproof "Fact" "PTest"))
(proof-to-expr-with-formulas decnproof) =>
Elim: allnc n((C n -> F) oru C n ->
 ((C n -> F) -> phi n=n--1 oru phi n<n--1) ->
 (C n --> phi n=n--1 oru phi n<n--1) ->
 phi n=n--1 oru phi n<n--1)
PTest: all n((C n -> F) oru C n)
Intro: allnc n(phi n=n--1 -> phi n=n--1 oru phi n<n--1)
EulerPrime: allnc n((C n -> F) -> phi n=n--1)
Intro: allnc n(phi n<n--1 -> phi n=n--1 oru phi n<n--1)
EulerComp: allnc n(C n -> phi n<n--1)

(lambda (n)
  ((((Elim n) (PTest n))
     (lambda (u1542) ((Intro n) ((EulerPrime n) u1542))))
    (lambda (u1544) ((Intro n) ((EulerComp n) u1544)))))

(pp (nt (proof-to-extracted-term decnproof))) => cPTest
```

Example: Maximal Scoring Segment (MSS)

- Let X be linearly ordered by $\preceq$. Given $\mathrm{seg}\colon \mathbf{N} \to \mathbf{N} \to X$.
  Want: maximal segment

$$\forall^{\mathrm{c}}_n \exists_{i \le k \le n} \forall_{i' \le k' \le n} (\mathrm{seg}(i', k') \preceq \mathrm{seg}(i, k)).$$

- Example: Regions with high G, C content in DNA.

$$\begin{aligned}
&X := \{G, C, A, T\},\\
&g\colon \mathbf{N} \to X \quad \text{(gene)},\\
&f\colon \mathbf{N} \to \mathbf{Z}, \quad f(i) := \begin{cases} 1 & \text{if } g(i) \in \{G, C\},\\ -1 & \text{if } g(i) \in \{A, T\}, \end{cases}\\
&\mathrm{seg}(i, k) = f(i) + \dots + f(k).
\end{aligned}$$

Example: MSS (ctd.)

Prove the existence of a maximal segment by induction on n, simultaneously with the existence of a maximal end segment.

$$\begin{aligned}
\forall^{\mathrm{c}}_n (&\exists_{i \le k \le n} \forall_{i' \le k' \le n} (\mathrm{seg}(i', k') \preceq \mathrm{seg}(i, k)) \land\\
&\exists_{j \le n} \forall_{j' \le n} (\mathrm{seg}(j', n) \preceq \mathrm{seg}(j, n)))
\end{aligned}$$

In the step:

- Compare the maximal segment i, k for n with the maximal end segment $j, n + 1$ proved separately.
- If $\preceq$, take the new i, k to be $j, n + 1$. Else take the old i, k.

Depending on how the existence of a maximal end segment was proved, we obtain a quadratic or a linear algorithm.

Example: MSS (ctd.)

Two proofs of the existence of a maximal end segment for $n+1$:
$\forall^{c}_{n}\exists_{j\le n+1}\forall_{j'\le n+1}(\mathrm{seg}(j', n+1) \preceq \mathrm{seg}(j, n+1))$.

- Introduce an auxiliary parameter m; prove by induction on m

$$\forall^{c}_{n}\forall^{c}_{m\le n+1}\exists_{j\le n+1}\forall_{j'\le m}(\mathrm{seg}(j', n+1) \preceq \mathrm{seg}(j, n+1)).$$

- Use $\mathtt{ES}_n$: $\exists_{j\le n}\forall_{j'\le n}(\mathrm{seg}(j', n) \preceq \mathrm{seg}(j, n))$ and the additional assumption of monotonicity

$$\forall_{i,j,n}(\mathrm{seg}(i, n) \preceq \mathrm{seg}(j, n) \to \mathrm{seg}(i, n+1) \preceq \mathrm{seg}(j, n+1)).$$

  Proceed by cases on $\mathrm{seg}(j, n+1) \preceq \mathrm{seg}(n+1, n+1)$.
  If $\preceq$, take $n+1$, else the previous j.

Example: MSS (ctd.)

Could decoration help to find the better proof? Have lemmas $\mathtt{L}$:

$$\forall^{c}_{n}\forall^{c}_{m\le n+1}\exists_{j\le n+1}\forall_{j'\le m}(\mathrm{seg}(j', n{+}1) \preceq \mathrm{seg}(j, n{+}1))$$

and $\mathtt{LMon}$:

$$\mathtt{Mon} \to \forall^{c}_{n}(\mathtt{ES}_n \to^{c} \forall^{nc}_{m\le n+1}\exists_{j\le n+1}\forall_{j'\le m}(\mathrm{seg}(j', n{+}1) \preceq \mathrm{seg}(j, n{+}1))).$$

- The decoration algorithm arrives at $\mathtt{L}$ with goal

$$\forall^{nc}_{m\le n+1}\exists_{j\le n+1}\forall_{j'\le m}(\mathrm{seg}(j', n{+}1) \preceq \mathrm{seg}(j, n{+}1)).$$

- $\mathtt{LMon}$ fits as well, its assumptions $\mathtt{Mon}$ and $\mathtt{ES}_n$ are in the context, and it is less extended ($\forall^{nc}_{m\le n+1}$ rather than $\forall^{c}_{m\le n+1}$), hence is preferred.

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A more general general proof theory

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General Proof Theory
Celebrating 50 Years of Dag Prawitz's "Natural Deduction"
27–29 November 2015
Tübingen, Germany

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

Abstract

In this talk, it is suggested to generalize our understanding of general (alias structural) proof theory and to consider it as a general theory of two kinds of derivations, namely proofs and dual proofs.

The proposal is substantiated by some considerations on assertion and denial, the idea of compositionality in proof-theoretic semantics, and some thoughts about falsification and co-implication.

The main result is a normal form theorem for the natural deduction proof system for the bi-intuitionistic logic 2Int. The proof uses the faithful embedding of 2Int into intuitionistic logic with respect to validity and shows that conversions of dual proofs can be sidestepped

In the 1970s Dag Prawitz introduced general proof theory as "a study of proofs in their own right where one is interested in general questions about the nature and structure of proofs" (1974, p. 66).

In his seminal paper on "Ideas and results in proof theory" (1971), Prawitz listed what he considered to be obvious topics in general proof theory:

> *2.1. The basic question of defining the notion of proof, including the question of the distinction between different kinds of proofs such as constructive proofs and classical proofs.*
>
> *2.2. Investigation of the structure of (different kinds of) proofs, including e.g. questions concerning the existence of certain normal forms.*

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
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> *2.3. The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.*
>
> *2.4. Applications of insights about the structure of proofs to other logical questions that are not formulated in terms of the notion of proof.*

I will suggest to broaden Prawitz's understanding of general proof theory. The idea is to consider in addition to verifications also falsifications, so as to obtain a theory of proofs and what I call dual proofs.

General proof theory would thus be a study of proofs and dual proofs in their own right where one is interested in general questions about the nature and structure of both proofs and dual proofs.

The topic can be approached from different perspectives. I shall consider (i) the speech acts of assertion and denial and (ii) the problem of compositionality in proof-theoretic semantics, and (iii) the idea of supplementing intuitionistic implication with an operation of co-implication.

Assertion and denial

Assertion and denial are usually seen to correspond to certain propositional attitudes. Ripley (2011) uses the terms "'deny' and 'denial' exclusively to pick out a certain type of speech act: the sort someone is engaged in when they deny something" and uses "'reject' and 'rejection' to pick out a certain type of attitude: the sort someone has to a content when they reject it."

Timothy Williamson (2000) explains that "we can regard assertion as the verbal counterpart of judgement and judgement as the occurrent form of belief."

If assertion is the verbal expression of the attitude of belief towards a propositional content, then denial is the verbal expression of the attitude of rejection The attitude verbalized by a denial may be understood as disbelief or perhaps a weaker form of rejection.

In the context of general proof theory, the speech acts of assertion and denial are considered in Per Martin-Löf's 1983 lectures "On the Meanings of the Logical Constants and the Justifications of the Logical Laws", where he takes up Frege's understanding of inferences as transitions between judgements or assertions. A presentation of the Fregean conception of inference can also be found in a recent paper by Prawitz (2015):

> *[A] reflective inference contains at least a number of assertions or judgements made in the belief that one of them, the conclusion, say $\mathcal{B}$, is supported by the other ones, the premisses, say $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$. An inference in the course of an argument or proof is not an assertion or judgement to the effect that $\mathcal{B}$ "follows" from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, but is first of all a transition from some assertions (or judgements) to another one. ...*

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

> *This is how Frege saw an inference, as a transition between assertions or judgements. To make an assertion is to use a declarative sentence A with assertive force, which we may indicate by writing $\vdash A$, using the Fregean assertion sign. We may also say with Frege that a sentence A expresses a thought or proposition p, while $\vdash A$, the assertion of A, is an act in which p is judged to be true.*

This is, however, only a first characterization of inferences, and Prawitz develops a more sophisticated conception by considering assertions under assumptions, open assertions, and inference figures.

The Fregean notion of inferences as transitions between judgements differs from the prevailing understanding of inference as logical consequence (or logical deduction).

Usually, logical consequence is regarded not as a relation between concrete actions but as a relation between sets of formulas of a formal language and single or multiple conclusions satisfying the well-known conditions due to Alfred Tarski and Dana Scott.

It may therefore be useful to draw a notational distinction between Frege's judgement stroke |—— that goes back to the *Begriffsschrift* and the derivability symbol $\vdash$, the "turnstile", introduced by Kleene and Rosser in the 1930s.

Since assertion and denial appear to be equally respectable and equally relevant notions, the following questions arise quite naturally:

- If the external action type of assertion corresponds to the internal action type of judgement, to which internal action type does the external action type of denial correspond?
- If we refer to the internal action type corresponding to denial as dual judgement, which subject-independent and non-agentive notion stands to dual judgement as the turnstile $\vdash$ stands to Frege's judgement stroke |——?

According to Frege, the first question can be answered by saying that denying A amounts to asserting A's negation.

The occurrent form of the attitude of the rejection of A thus is the type of action in which the negation of A is judged to be true.

According to G. Restall (2005), "[d]enial is not to be analysed as the assertion of a negation," whereas B. Brown (2002) has a modest proposal: "negation is denial in the object language."

In (Wansing 2010) I suggested to associate the denial of A with the provability of the strong negation $\sim A$ of A. In Nelson's constructive logics with strong negation a notion of disprovability can be defined as follows:

$$\vdash^{dis}_{\mathsf{N4}} A \text{ iff } \vdash_{\mathsf{N4}} \sim A.$$

For strong negation free-fragments, we would loose the possibility of expressing denial.

Moreover, a distinction between the turnstile and a subject-independent and non-agentive counterpart $\vdash^d$ of dual judgement can be drawn also in the absence of strong negation.

Compositionality in proof-theoretic semantics

In a recent paper, Nissim Francez and Gilad Ben-Avi (2011) note that

> *[i]n spite of the vast literature on the subject, there is nowhere an explicit definition for a semantic value as determined by the I-rules of the ND-system; something of the form*
>
> $$\| \star \| = \ \ldots$$
>
> *where '$\star$' is a logical constant.*

Francez and Ben-Avi call a derivation canonical iff its last rule application is an application of an introduction rule. If A is a formula and Γ a (finite) set of formulas, then

$$\| A \|^c_\Gamma := \{ D^c \mid D^c \text{ is a canonical derivation of } A \text{ from } \Gamma \}$$

The proof-theoretic meaning $\| A \|$ of A is then defined as follows:

$$\| A \| := \lambda\Gamma.\ \| A \|^c_\Gamma$$

Francez and and Ben-Avi emphasize that $\| A \|$ "is a proof-theoretic object, a function from contexts to the collection of (canonical) derivations from that context, not to be confused with model-theoretic denotations (of truth values, in this case)."

The proof-theoretic meaning $\| A \|$ of A thus defined depends on the introduction rules, but I think that for securing compositionality restrictions are needed.

Namely, compositionality for single-conclusion natural deduction requires harmonious rules (so as to exclude Prior's `tonk`) and that for any compound expression e:

- in addition to parametric formulas and e, the introduction rules for e display only immediate subformulas of e, and every immediate subformula of e is displayed in some introduction rule for e;
- in addition to parametric formulas and e, the elimination rules for e display only immediate subformulas of e, and every immediate subformula of e is displayed in some elimination rule for e.

Compositionality does not hold in Prawitz's (1965) or other natural deduction calculi for Nelson's logics, where we have, for example:

$$\frac{A \quad {\sim}B}{{\sim}(A \to B)} \qquad \frac{{\sim}A \quad {\sim}B}{{\sim}(A \vee B)} \qquad \frac{{\sim}(A \wedge B) \quad \begin{array}{c}[{\sim}A] \\ C\end{array} \quad \begin{array}{c}[{\sim}B] \\ C\end{array}}{C}$$

The problem with these rule is not only that they are holistic and display two connectives at once, but that they spoil the subformula property.

One possible reaction is to liberalize the notion of a subformula. Arnon Avron (in conversation) suggested a rather drastic generalization of the concept of a subformula. The idea is to first translate a formula into Polish notation and then to consider any well-formed order-preserving substring of it as a subformula.

Consider, as an example, the purely implicational formula $(p \to (q \to r)) \to s$. Rewriting it in Polish notation, we obtain $\to\to p \to qrs$. The set of well-formed order-preserving substrings contains the formula $\to qs$.

If meaning assignments by means of holistic natural deduction introduction rules are admitted, we could thus in a compositional manner make the meaning of $(p \to (q \to r)) \to s$ dependent on the meaning of $q \to s$.

Another solution consists of replacing the truth-preserving transitions to or from negated formulas by falsity preserving transitions to or from non-negated formulas:

$$\begin{array}{c}\overline{A} \qquad \overline{\overline{B}} \\ \hline\hline (A \to B)\end{array} \qquad \begin{array}{c}\overline{\overline{A}} \qquad \overline{\overline{B}} \\ \hline\hline (A \vee B)\end{array} \qquad \begin{array}{c}\overline{\overline{(A \wedge B)}} \quad \begin{array}{c}[\![A]\!] \\ \hline\hline C\end{array} \quad \begin{array}{c}[\![B]\!] \\ \hline\hline C\end{array} \\ \hline\hline C\end{array}$$

This way we obtain two kinds of derivations, proofs ending with a formula under a single line, and dual proofs ending with a formula under a double line.

What happens to strong negation?

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

Strong negation operates as a switch between proofs and dual proofs, and since strong negation expresses definite falsity, the dual proofs my be seen as disproofs:

$$\begin{array}{c}\overline{\overline{A}} \\ \hline {\sim}A\end{array} \qquad \begin{array}{c}\overline{A} \\ \hline\hline {\sim}A\end{array} \qquad \begin{array}{c}\overline{{\sim}A} \\ \hline\hline A\end{array} \qquad \begin{array}{c}\overline{\overline{{\sim}A}} \\ \hline A\end{array}$$

Not only is compositionality regained, the resulting relation of dual provability may be taken to be the relation that stands to dual judgement as the turnstile $\vdash$ stands to the Fregean judgement stroke |——.[1]

[1]Note on terminology: Often proofs are defined as derivations from the empty set of assumptions (hypotheses). Here I will consider proofs and dual proofs from pairs consisting of a possibly non-empty, finite set of assumptions and a possibly non-empty, finite set of counterassumptions, and I will speak of both proofs and dual proofs as derivations.

Co-implication

In Prawitz (2007), Dag Prawitz briefly considers *falsificationist* theories of meaning as suggested by Dummett (1993).
He draws a distinction between *obvious rules for the falsification of a sentence* and *the standard way of falsifying a compound sentence* and explains that (2007, p. 475)

> *[t]here are obvious rules for the falsification of a sentence such as inferring $\neg(A \wedge B)$ from either $\neg A$ or $\neg B$, $\neg(A \vee B)$ from the two premises $\neg A$ and $\neg B$, $\neg(A \rightarrow B)$ from the premises A and $\neg B$ and so on. The notion of falsity that results if we take such rules as introduction rules is, however, quite different from our usual one, since $\neg(A \wedge \neg A)$ becomes assertible only if either A or $\neg A$ is assertible.*

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

These rules result in the notion of constructible falsity assumed in David Nelson's constructive logics with strong negation, logics for which Prawitz (1965) presented natural deduction proof systems.

The falsification of an implication $(A \rightarrow B)$ in Nelson's logics is understood as the falsification of the strong negation of A together with a falsification of B. The falsification of an implication can thus be explained *completely in terms of falsifications*.

However, accounting for the falsification of a strongly negated formula $\sim A$ as a verification of A leads to considering both verification and falsification. If $\sim$ toggles between proofs and dual proofs, we need them both.

The second-mentioned way of falsifying a compound sentence A consists of deriving a constantly false sentence or, if it in the language, the falsity constant $\perp$ from A. Here Prawitz sees the problem we have just noted for strong negation as a toggle between proofs and dual proofs.

A falsification of an implication $(A \to B)$ requires in addition to a *falsification* of B also a *verification* of A, which is different from a falsification of A's negation.

Prawitz draws the quite reserved conclusion that a "falsificationist meaning theory seems thus to have to mix different ideas of meaning in an unfavorable way" (2007, p. 476).

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

The noted complication, however, is not specific to the falsificationist approach to the meaning of the logical connectives as compared to the more familiar verificationist approach. If the falsificationist has to specify falsification conditions for *implications*, then it may be claimed that the verificationist must specify verification conditions for *co-implications*.

The role of implication in verificationism is dual to the role of co-implication in falsificationism. These connectives may be seen to internalize a semantical relation of entailment into the logical object language.

There are, however, different notions of entailment that can be considered, in particular different notions of entailment co-implication may internalize.

Co-implication in Heyting-Brouwer logic

The language $\mathcal{L}_{\mathsf{BiInt}}$ of Heyting-Brouwer logic (see, e.g., (Rauszer 1980), (Goré 2000)), also called "bi-intuitionistic logic", BiInt, extends the language of intuitionistic logic, Int, by a primitive binary co-implication connective $\prec$ and is defined in Backus–Naur form as follows:

$$A ::= \quad p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A).$$

where p is a propositional variable from some fixed denumerably infinite set Φ of sentential variables (atomic formulas).

The language $\mathcal{L}_{\mathsf{DualInt}}$ of dual intuitionistic propositional logic, DualInt, is $\mathcal{L}_{\mathsf{BiInt}}$ restricted to the connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\prec$. I will use $\equiv$ to denote the syntactic identity relation between formulas.

In BiInt, co-implication internalizes the preservation of non-truth from the conclusion of a valid inference (understood as logical consequence) to its premises. In the relational semantics for BiInt, a state x from a Kripke model $\mathcal{M} = \langle I, \leq, v \rangle$ supports the truth of a co-implication $(A \prec B)$ ("B co-implies A") iff there is an "earlier" state x' such that x' supports the truth of A but fails to support the truth of B:

$$\mathcal{M}, x \models (A \prec B) \text{ iff } \text{ there exists } x' \leq x \text{ with } \mathcal{M}, x' \models A \text{ and } \mathcal{M}, x' \not\models B.$$

The support of truth clause for implication is the intuitionistic one:

$$\mathcal{M}, x \models (A \rightarrow B) \text{ iff } \text{ for every } x' \geq x : \mathcal{M}, x' \not\models A \text{ or } \mathcal{M}, x' \models B.$$

Every (no) state supports the truth of the truth (falsity) constant $\top$ ($\bot$).

A formula A is valid in a model for BiInt $\mathcal{M} = \langle I, \leq, v \rangle$ iff for every $x \in I$, $\mathcal{M}, x \models A$, and A is valid in BiInt iff A is valid in every model for BiInt.

We write $\models_{\mathsf{BiInt}} A$ to mean that A is valid in BiInt. Let $\Delta \cup \{A\}$ be a set of formulas. Δ entails A in BiInt ($\Delta \models_{\mathsf{BiInt}} A$) iff for every model for BiInt $\mathcal{M} = \langle I, \leq, v \rangle$ and every $x \in I$, it holds that if the truth of every element of Δ is supported by x, then the truth of A is supported by x.

We may note that
$A \models_{\mathsf{BiInt}} B$ iff $\top \models_{\mathsf{BiInt}} A \to B$ iff $A \prec B \models_{\mathsf{BiInt}} \bot$. In this sense co-implication in BiInt internalizes preservation of non-truth from the conclusion of a valid inference (understood as logical consequence) to the premises. Moreover, in the following sense co-implication in BiInt is the residual of disjunction with respect to entailment: $A \models_{\mathsf{BiInt}} B \vee C$ iff $A \prec B \models_{\mathsf{BiInt}} C$ iff $A \prec C \models_{\mathsf{BiInt}} B$.

Co-implication in 2Int

The bi-intuitionistic logic 2Int introduced in (Wansing 2013, Wansing 2015a) contains a co-implication connective that internalizes a notion of entailment different from preservation of non-truth from the conclusion of valid inferences (seen as as deductions) to the premises.

The language $\mathcal{L}_{\mathsf{2Int}}$ of 2Int is that of of BiInt, but the co-implication connective has a different meaning. In both systems, BiInt and 2Int, the intuitionistic negation $\neg A$ of A is defined as $A \to \bot$, and the co-negation $-A$ of A is defined as $\top \prec A$.

Definition

A model for 2Int *is a structure* $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$*, where* $\langle I, \leq \rangle$ *is a pre-order and* v^+*,* v^- *are functions from the set of atomic formulas to subsets of the non-empty set of states* I*. For* $x \in I$ *the relations* $\mathcal{M}, x \models^+ A$ *("x supports the truth of A in* $\mathcal{M}$*") and* $\mathcal{M}, x \models^- A$ *("x supports the falsity of A in* $\mathcal{M}$*") are inductively defined as shown below. Moreover, support of truth and support of falsity are required to be persistent. For every atomic formula p, and all states* x, x'*: if* $x' \geq x$ *and* $\mathcal{M}, x \models^+ p$*, then* $\mathcal{M}, x' \models^+ p$ *and if* $x' \geq x$ *and* $\mathcal{M}, x \models^- p$*, then* $\mathcal{M}, x' \models^- p$*.*

$$\begin{array}{lll} \mathcal{M}, x \models^+ p & \text{iff} & x \in v^+(p) \\ \mathcal{M}, x \models^- p & \text{iff} & x \in v^-(p) \end{array}$$

$$\begin{array}{ll} \mathcal{M}, x \models^+ \top & \mathcal{M}, x \not\models^+ \bot \\ \mathcal{M}, x \not\models^- \top & \mathcal{M}, x \models^- \bot \end{array}$$

$$\begin{array}{lll} \mathcal{M}, x \models^+ (A \wedge B) & \text{iff} & \mathcal{M}, x \models^+ A \text{ and } \mathcal{M}, x \models^+ B \\ \mathcal{M}, x \models^- (A \wedge B) & \text{iff} & \mathcal{M}, x \models^- A \text{ or } \mathcal{M}, x \models^- B \\[4pt] \mathcal{M}, x \models^+ (A \vee B) & \text{iff} & \mathcal{M}, x \models^+ A \text{ or } \mathcal{M}, x \models^+ B \\ \mathcal{M}, x \models^- (A \vee B) & \text{iff} & \mathcal{M}, x \models^- A \text{ and } \mathcal{M}, x \models^- B \\[4pt] \mathcal{M}, x \models^+ (A \to B) & \text{iff} & \text{for every } x' \geq x : \mathcal{M}, x' \not\models^+ A \text{ or } \mathcal{M}, x' \models^+ B \\ \mathcal{M}, x \models^- (A \to B) & \text{iff} & \mathcal{M}, x \models^+ A \text{ and } \mathcal{M}, x \models^- B \\[4pt] \mathcal{M}, x \models^+ \neg A & \text{iff} & \text{for every } x' \geq x : \mathcal{M}, x' \not\models^+ A \\ \mathcal{M}, x \models^- \neg A & \text{iff} & \mathcal{M}, x \models^+ A \\[4pt] \mathcal{M}, x \models^+ -A & \text{iff} & \mathcal{M}, x \models^- A \\ \mathcal{M}, x \models^- -A & \text{iff} & \text{for every } x' \geq x : \mathcal{M}, x' \not\models^- A \\[4pt] \mathcal{M}, x \models^+ (A \prec B) & \text{iff} & \mathcal{M}, x \models^+ A \text{ and } \mathcal{M}, x \models^- B \\ \mathcal{M}, x \models^- (A \prec B) & \text{iff} & \text{for every } x' \geq x : \mathcal{M}, x' \not\models^- B \text{ or } \mathcal{M}, x' \models^- A. \end{array}$$

Definition

An $\mathcal{L}_{2\mathsf{Int}}$-formula A is valid in a model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *iff for every $x \in I$, $\mathcal{M}, x \models^+ A$ (iff for every $x \in I$, $\mathcal{M}, x \models^- \neg A$); A is valid in* 2Int *($\models_{2\mathsf{Int}} A$) iff A is valid in every model for* 2Int.

An $\mathcal{L}_{2\mathsf{Int}}$-formula A is dually valid in a model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *iff for every $x \in I$, $\mathcal{M}, x \models^- A$ (iff for every $x \in I$, $\mathcal{M}, x \models^+ -A$); A is dually valid in* 2Int *($\models^d_{2\mathsf{Int}} A$) iff A is dually valid in every model for* 2Int.

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

Definition

Let $\Delta \cup \{A\}$ be a set of $\mathcal{L}_{2\mathsf{Int}}$-formulas. The set Δ entails A ($\Delta \models A$) iff for every model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *and every $x \in I$, it holds that if the truth of every element of Δ is supported by x, then the truth of A is supported by x.*
Let $\Delta \cup \{A\}$ be a set of $\mathcal{L}_{2\mathsf{Int}}$-formulas. The set Δ dually entails A ($\Delta \models^d A$) iff for every model for 2Int $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ *and every $x \in I$, it holds that if the falsity of every element of Δ is supported by x, then the falsity of A is supported by x.*

We may note that $A \models^d_{2\mathsf{Int}} B$ iff $\perp \models^d_{2\mathsf{Int}} B \prec A$. In this sense co-implication in 2Int internalizes preservation of falsity from the premises to the conclusion of a dually valid deduction. In the following sense co-implication in 2Int is the residual of disjunction with respect to dual entailment: $A \vee B \models^d_{2\mathsf{Int}} C$ iff $A \models^d_{2\mathsf{Int}} C \prec B$ iff $B \models^d_{2\mathsf{Int}} C \prec A$.

In (Goré 2000), the set of validities of DualInt is defined as the set of all $\mathcal{L}_{\text{DualInt}}$-formulas that are valid in BiInt. Entailment in DualInt can be defined as entailment in BiInt restricted to $\mathcal{L}_{\text{DualInt}}$.

We can define a relation $\models^{d}_{\text{DualInt}}$ of dual entailment for DualInt by requiring that $\Delta \models^{d}_{\text{DualInt}} A$ iff for every model for BiInt $\mathcal{M} = \langle I, \leq, v \rangle$ and every $x \in I$, it holds that if the truth of no element of Δ is supported by x, then the truth of A is not supported by x, cf. (Wansing 2013).

Dual entailment in DualInt thus preserves non-truth from the premises to the conclusion of a dually valid deduction.

The natural deduction proof system N2Int for 2Int uses single-line rules for proofs and double-line rules for dual proofs. Derivations in N2Int may contain proofs and dual proofs as subderivations.

The conclusion of a derivation therefore depends on an ordered pairs $(\Delta; \Gamma)$ of finite sets, a set of assumptions Δ and a set of counterassumptions Γ. Single square brackets $[\,]$ are used to indicate that assumptions may be cancelled, and double-square brackets $[\![\,]\!]$ are used to indicate that counterassumptions may be discharged. We write $[A]$ instead of $[\overline{A}]$ and $[\![A]\!]$ instead of $[\![\overline{\overline{A}}]\!]$.

Then we draw a distinction between rules for introducing connectives into proofs, *Ip* rules, and for eliminating them from proofs, *Ep* rule,s and rules for introducing connectives into dual proofs, *Idp* rules, and for eliminating them from dual proofs, *Edp* rules.

The proof rules for the connectives $\top$, $\bot$, $\wedge$, $\vee$, and $\rightarrow$ are basically those of intuitionistic logic. The rules for introducing (eliminating) the connectives of intuitionistic logic into (from) dual proofs are obtained by a dualization of their introduction and elimination rules for proofs.

In 2Int the rules for introducing (eliminating) implications into (from) dual proofs are chosen in accordance with the usual understanding of the falsification conditions of implications, i.e., an implication $A \rightarrow B$ is false iff A is true and B is false. This is not the only option, see (Wansing 2008, 2015a, 2015b).

The rules for introducing (eliminating) co-implications into (from) proofs are such that the provability of $A \,{-\!\!\!<}\, B$ amounts to the dual provability of $A \rightarrow B$.

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

The introduction and elimination rules can be applied to certain basic building blocks of derivations. We consider $\overline{A}$ as a proof of A from $(\{A\}; \varnothing)$ and $\overline{\overline{A}}$ as a dual proof of A from $(\varnothing; \{A\})$. Moreover $\overline{\top}$ is a proof of $\top$ from $(\varnothing; \varnothing)$ and $\overline{\overline{\bot}}$ is a dual proof of $\bot$ from $(\varnothing; \varnothing)$.

We write $(\Delta; \Gamma) \vdash A$ if there is a proof of A from $(\Delta; \Gamma)$; and we write $(\Delta; \Gamma) \vdash^{d} A$ if there is a dual proof of A from $(\Delta; \Gamma)$. Moreover, we assume that if $(\Delta; \Gamma) \vdash A$, $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$ for finite sets of $\mathcal{L}_{\mathsf{2Int}}$-formulas Δ' and Γ', then $(\Delta'; \Gamma') \vdash A$. Similarly, we assume that if $(\Delta; \Gamma) \vdash^{d} A$, $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$ for finite sets of $\mathcal{L}_{\mathsf{2Int}}$-formulas Δ' and Γ', then $(\Delta'; \Gamma') \vdash^{d} A$.

The system N2Int comprises the following proof rules:

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ \bot\end{matrix}}{A}\,(\bot Ep) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}(\Delta';\Gamma')\\ \vdots \\ B\end{matrix}}{(A \wedge B)}\,(\wedge Ip) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ (A \wedge B)\end{matrix}}{A}\,(\wedge Ep)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ (A \wedge B)\end{matrix}}{B}\,(\wedge Ep) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ A\end{matrix}}{(A \vee B)}\,(\vee Ip) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ B\end{matrix}}{(A \vee B)}\,(\vee Ip)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ (A \vee B)\end{matrix} \qquad \begin{matrix}([A],\Delta';\Gamma')\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}([B],\Delta'';\Gamma'')\\ \vdots \\ C\end{matrix}}{C}\,(\vee Ep)$$

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

$$\dfrac{\begin{matrix}([A],\Delta;\Gamma)\\ \vdots \\ B\end{matrix}}{(A \to B)}\,(\to Ip) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}(\Delta';\Gamma')\\ \vdots \\ (A \to B)\end{matrix}}{B}\,(\to Ep)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ A\end{matrix} \qquad \begin{matrix}(\Delta';\Gamma')\\ \vdots \\ \overline{\overline{B}}\end{matrix}}{(A \mathbin{-\!\!<} B)}\,(\mathbin{-\!\!<} Ip) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ (A \mathbin{-\!\!<} B)\end{matrix}}{A}\,(\mathbin{-\!\!<} Ep) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots \\ (A \mathbin{-\!\!<} B)\end{matrix}}{\overline{B}}\,(\mathbin{-\!\!<} Ep)$$

Moreover, we have the following dual proof rules.

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ \top\end{matrix}}{A}\ (\top Edp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ A\end{matrix} \quad \begin{matrix}(\Delta';\Gamma')\\ \vdots\\ B\end{matrix}}{(A \vee B)}\ (\vee Idp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ (A \vee B)\end{matrix}}{A}\ (\vee Edp)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ (A \vee B)\end{matrix}}{B}\ (\vee Edp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ A\end{matrix}}{(A \wedge B)}\ (\wedge Idp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ B\end{matrix}}{(A \wedge B)}\ (\wedge Idp)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ (A \wedge B)\end{matrix} \quad \begin{matrix}(\Delta;\Gamma,[\![A]\!])\\ \vdots\\ C\end{matrix} \quad \begin{matrix}(\Delta;\Gamma,[\![B]\!])\\ \vdots\\ C\end{matrix}}{C}\ (\wedge Edp)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma,[\![A]\!])\\ \vdots\\ B\end{matrix}}{(B \prec A)}\ (\prec Idp) \qquad \dfrac{\begin{matrix}(\Delta';\Gamma')\\ \vdots\\ (B \prec A)\end{matrix} \quad \begin{matrix}(\Delta;\Gamma)\\ \vdots\\ A\end{matrix}}{B}\ (\prec Edp)$$

$$\dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ A\end{matrix} \quad \begin{matrix}(\Delta';\Gamma')\\ \vdots\\ B\end{matrix}}{(A \to B)}\ (\to Idp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ (A \to B)\end{matrix}}{A}\ (\to Edp) \qquad \dfrac{\begin{matrix}(\Delta;\Gamma)\\ \vdots\\ (A \to B)\end{matrix}}{B}\ (\to Edp)$$

We note the following facts.

Observation

Let $\neg\Theta := \{\neg A \mid A \in \Theta\}$ *and* $-\Theta := \{-A \mid A \in \Gamma\}$ *for a set of formulas* Θ*. If* $\Theta = \varnothing$*, then* $\neg\Theta := -\Theta := \varnothing$*.*

1. $(\Delta;\Gamma) \vdash A$ *iff* $(\Delta;\Gamma) \vdash^d \neg A$*;* $(\Delta;\Gamma) \vdash^d A$ *iff* $(\Delta;\Gamma) \vdash -A$*.*
2. $(\Delta;\Gamma) \vdash A$ *iff* $(\Delta \cup -\Gamma;\varnothing) \vdash A$ *and* $(\Delta;\Gamma) \vdash^d A$ *iff* $(\varnothing;\Gamma \cup \neg\Delta) \vdash^d A$*.*
3. $(\Delta;\Gamma) \vdash \neg A$ *iff* $(\Delta;\Gamma) \vdash^d -\neg A$*;* $(\Delta;\Gamma) \vdash^d \neg A$ *iff* $(\Delta;\Gamma) \vdash -\neg A$*.*
4. $(\Delta;\Gamma) \vdash -A$ *iff* $(\Delta;\Gamma) \vdash^d \neg - A$*;* $(\Delta;\Gamma) \vdash^d -A$ *iff* $(\Delta;\Gamma) \vdash \neg - A$*.*

Observation

1. N2Int *restricted to* $\mathcal{L}_{\mathsf{Int}}$*, sets of assumptions, and the Ip and Ep rules is a natural deduction proof system* NInt *for* Int*.*
2. *Refer to the restriction of* N2Int *to* $\mathcal{L}_{\mathsf{DualInt}}$*, sets of counterassumptions, and the Idp and Edp rules as* NDualInt*. There is an isomorphism between proofs in* NInt *and dual proofs in* NDualInt*.*
3. *There is an isomorphism between dual proofs in* NDualInt *and derivations in L.Tranchini's (2012) multiple-conclusion natural deduction proof system for DualInt.*

Proof. 1.: consider the standard natural deduction proof system for Int, for example in (Prawitz 1965).

2.: The function δ from $\mathcal{L}_{\mathsf{Int}}$ to $\mathcal{L}_{\mathsf{DualInt}}$ is defined as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\delta(p) & = & p & & & \\
\delta(\top) & = & \bot & \delta(\bot) & = & \top \\
\delta((A \wedge B)) & = & (\delta(A) \vee \delta(B)) & \delta((A \vee B)) & = & (\delta(A) \wedge \delta(B)) \\
\delta((A \to B)) & = & (\delta(B) \prec \delta(A)). & & &
\end{array}$$

We extend δ to a bijection between the proofs in NInt and the dual proofs in NDualInt: $\delta(\overline{A}) = \overline{\overline{A}}$, every *Ip* rule is mapped to the *Idp* rules for the dual connective, and every *Ep* rule is mapped to the *Edp* rule for the dual connective.We have, for example,

$$\delta\left(\begin{array}{c} [A], \Delta \\ \vdots \\ \dfrac{B}{(A \to B)}\ (\to Ip) \end{array}\right) \quad = \quad \begin{array}{c} \delta(\Delta), [\![\delta(A)]\!] \\ \vdots \\ \overline{\overline{\dfrac{\delta(B)}{(\delta(B) \prec \delta(A))}}}\ (\prec Idp) \end{array}$$

where $\delta(\Delta) = \{\delta(B) \mid B \in \Delta\}$.

The function δ is an isomorphism between proofs in NInt and dual proofs in NDualInt. Π is a proof of A from Δ in NInt iff $\delta(\Pi)$ is a dual proof of $\delta(A)$ from $\delta(\Delta)$ in NDualInt; $\delta^{-1}(\delta(\pi)) = \pi$ for every proof π in NInt.

3.: We combine the isomorphism between derivations in the multiple-conclusion natural deduction system for DualInt in (Tranchini 2012) and proofs in NInt with the isomorphism δ between between proofs in NInt and dual proofs in NDualInt. $\Box$

Item 1 of the first observation reveals the difference with strong negation as a switch between provability and dual provability; the back–and-forth transition between proofs and dual proofs is accomplished not by a single negation operation but by a division of labour between intuitionistic negation and co-negation:

$$\begin{array}{c} \overline{\overline{A}} \\ \hline -A \end{array} \qquad \begin{array}{c} \overline{A} \\ \hline\hline \neg A \end{array} \qquad \begin{array}{c} \overline{-A} \\ \hline\hline A \end{array} \qquad \begin{array}{c} \overline{\overline{\neg A}} \\ \hline A \end{array}$$

The proof system N2Int is sound and complete with respect to its relational semantics.

Theorem

Let A be an $\mathcal{L}_{2\mathrm{Int}}$-formula and let $\{A_1, \ldots, A_k\}$, $\{B_1, \ldots, B_m\}$ be finite, possibly empty sets of $\mathcal{L}_{2\mathrm{Int}}$-formulas.

1. $(\{A_1, \ldots, A_k\}; \{B_1, \ldots, B_m\}) \vdash A$ *iff* $\{A_1, \ldots, A_k, -B_1, \ldots, -B_m\} \models A$;
2. $(\{A_1, \ldots, A_k\}; \{B_1, \ldots, B_m\}) \vdash^d A$ *iff* $\{\neg A_1, \ldots, \neg A_k, B_1, \ldots, B_m\} \models^d A$.

Proof. See (Wansing 2013, Wansing 2015a). $\square$

Definition

Let $\Phi' = \{p' \mid p \in \Phi\}$. The translation τ from $\mathcal{L}_{2\mathrm{Int}}$ into $\mathcal{L}_{\mathrm{Int}}$ based on the set of atomic formulas $\Phi \cup \Phi'$ is defined as follows :

$$\begin{array}{rcl@{\qquad}rcl}
\tau(p) & := & p & \tau(-p) & = & p' \\
\tau(\top) & := & \top & \tau(-\top) & := & \bot \\
\tau(\bot) & := & \bot & \tau(-\bot) & := & \top \\
\tau(A \wedge B) & := & \tau(A) \wedge \tau(B) & \tau(-(A \wedge B)) & := & \tau(-A) \vee \tau(-B) \\
\tau(A \vee B) & := & \tau(A) \vee \tau(B) & \tau(-(A \vee B)) & := & \tau(-A) \wedge \tau(-B) \\
\tau(A \rightarrow B) & := & \tau(A) \rightarrow \tau(B) & \tau(-(A \rightarrow B)) & := & \tau(A) \wedge \tau(-B) \\
\tau(A \prec B) & := & \tau(A) \wedge \tau(-B), & \tau(-(A \prec B)) & := & \tau(-B) \rightarrow \tau(-A) \\
 & & \text{if } A \not\equiv \top & & &
\end{array}$$

Definition

Let $\Phi' = \{p' \mid p \in \Phi\}$. The translation ζ from $\mathcal{L}_{\mathsf{2Int}}$ into $\mathcal{L}_{\mathsf{DualInt}}$ based on the set of atomic formulas $\Phi \cup \Phi'$ is defined as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\zeta(p) & := & p & \zeta(\neg p) & = & p' \\
\zeta(\top) & := & \top & \zeta(\neg\top) & := & \bot \\
\zeta(\bot) & := & \bot & \zeta(\neg\bot) & := & \top \\
\zeta(A \wedge B) & := & \zeta(A) \wedge \zeta(B) & \zeta(\neg(A \wedge B)) & := & \zeta(\neg A) \vee \zeta(\neg B) \\
\zeta(A \vee B) & := & \zeta(A) \vee \zeta(B) & \zeta(\neg(A \vee B)) & := & \zeta(\neg A) \wedge \zeta(\neg B) \\
\zeta(A \prec B) & := & \zeta(A) \prec \zeta(B) & \zeta(\neg(A \prec B)) & := & \zeta(\neg A) \vee \zeta(B) \\
\zeta(A \to B) & := & \zeta(\neg A) \vee \zeta(B), & \zeta(\neg(A \to B)) & := & \zeta(\neg B) \prec \zeta(\neg A) \\
 & & \text{if } B \not\equiv \bot & & &
\end{array}$$

Theorem

Let A be any formula from $\mathcal{L}_{\mathsf{2Int}}$. Then 1. $\models_{\mathsf{2Int}} A$ iff $\models_{\mathsf{Int}} \tau(A)$, and 2. $\models^d_{\mathsf{2Int}} A$ iff $\models^d_{\mathsf{DualInt}} \zeta(A)$.

Normalization for 2Int

Detour conversions

In addition to the intuitionistic detour (or β) conversions for proofs there are now detour conversions for dual proofs. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}_1$, $\mathcal{D}_2$ stand for derivations in N2Int. The derivations on the left hand side of $\rightsquigarrow$ are converted into the derivations on the right hand side of $\rightsquigarrow$; $i \in \{1, 2\}$:

$\wedge$-conversions:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A_1} \quad \dfrac{\mathcal{D}_2}{A_2}}{A_1 \wedge A_2}}{A_i} \quad \rightsquigarrow \quad \dfrac{\mathcal{D}_i}{A_i} \qquad\qquad \dfrac{\dfrac{\dfrac{\mathcal{D}}{A_i}}{A_1 \wedge A_2} \quad \begin{matrix}[\![A_1]\!] \\ \mathcal{D}_1 \\ \hline\hline C\end{matrix} \quad \begin{matrix}[\![A_2]\!] \\ \mathcal{D}_2 \\ \hline\hline C\end{matrix}}{C} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D} \\ \hline\hline A_i \\ \mathcal{D}_i \\ \hline\hline C\end{matrix}$$

$\vee$-conversions:

$$\frac{\dfrac{\mathcal{D}}{A_i}\qquad \begin{matrix}[A_1]\\ \mathcal{D}_1\\ \hline C\end{matrix}\qquad \begin{matrix}[A_2]\\ \mathcal{D}_2\\ \hline C\end{matrix}}{C} \;\rightsquigarrow\; \begin{matrix}\mathcal{D}\\ \hline A_i\\ \mathcal{D}_i\\ \hline C\end{matrix} \qquad\qquad \frac{\dfrac{\dfrac{\mathcal{D}_1}{A_1}\quad \dfrac{\mathcal{D}_2}{A_2}}{A_1 \vee A_2}}{A_i} \;\rightsquigarrow\; \frac{\mathcal{D}_i}{A_i}$$

$\rightarrow$-conversions:

$$\frac{\dfrac{\begin{matrix}[A]\\ \mathcal{D}\end{matrix}}{B}}{A \rightarrow B}\;\; \frac{\mathcal{D}_1}{A}\Big/ B \quad\text{i.e.}\quad \frac{A \rightarrow B \qquad \dfrac{\mathcal{D}_1}{A}}{B} \;\rightsquigarrow\; \begin{matrix}\mathcal{D}_1\\ \hline A\\ \mathcal{D}\\ \hline B\end{matrix} \qquad\qquad \frac{\dfrac{\dfrac{\mathcal{D}_1}{A}\quad \dfrac{\mathcal{D}_2}{B}}{A \rightarrow B}}{A} \;\rightsquigarrow\; \frac{\mathcal{D}_1}{A}$$

$\prec$-conversions:

$$\frac{\dfrac{\dfrac{\mathcal{D}_1}{A}\quad \dfrac{\mathcal{D}_2}{B}}{A \prec B}}{A} \;\rightsquigarrow\; \frac{\mathcal{D}_1}{A} \qquad\qquad \frac{\dfrac{\dfrac{\mathcal{D}_1}{A}\quad \dfrac{\mathcal{D}_2}{B}}{A \prec B}}{B} \;\rightsquigarrow\; \frac{\mathcal{D}_2}{B}$$

Permutation conversions

Depending on whether $\vee$-eliminations from proofs or $\wedge$-introductions into dual proofs are permuted over eliminations from proofs or dual proofs, we get four different kinds of permutation conversions.

$$\frac{\dfrac{\dfrac{\mathcal{D}}{A \vee B}\quad \dfrac{\mathcal{D}_1}{C}\quad \dfrac{\mathcal{D}_2}{C}}{C} \qquad \mathcal{D}'}{D}\; Ep \text{ rule} \;\rightsquigarrow\; \frac{\dfrac{\mathcal{D}}{A \vee B}\qquad \dfrac{\dfrac{\mathcal{D}_1}{C}\quad \mathcal{D}'}{D}\qquad \dfrac{\dfrac{\mathcal{D}_2}{C}\quad \mathcal{D}'}{D}}{D}$$

$$\frac{\dfrac{\dfrac{\mathcal{D}}{A \wedge B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{C} \qquad \mathcal{D}'}{D}\ Ep \text{ rule} \quad \rightsquigarrow \quad \frac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\dfrac{\mathcal{D}_1}{C} \quad \mathcal{D}'}{D} \quad \dfrac{\dfrac{\mathcal{D}_2}{C} \quad \mathcal{D}'}{D}}{D}$$

$$\frac{\dfrac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{C} \qquad \mathcal{D}'}{D}\ Edp \text{ rule} \quad \rightsquigarrow \quad \frac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\dfrac{\mathcal{D}_1}{C} \quad \mathcal{D}'}{D} \quad \dfrac{\dfrac{\mathcal{D}_2}{C} \quad \mathcal{D}'}{D}}{D}$$

$$\frac{\dfrac{\dfrac{\mathcal{D}}{A \wedge B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{C} \qquad \mathcal{D}'}{D}\ Edp \text{ rule} \quad \rightsquigarrow \quad \frac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\dfrac{\mathcal{D}_1}{C} \quad \mathcal{D}'}{D} \quad \dfrac{\dfrac{\mathcal{D}_2}{C} \quad \mathcal{D}'}{D}}{D}$$

Simplification conversions

Next to simplification conversions arising from $\vee$-eliminations (from proofs) in which no assumptions are discharged, we also consider simplification conversions arising from $\wedge$-introductions (into dual proofs) in which no counterassumptions are cancelled.

$$\frac{\dfrac{\mathcal{D}}{A \vee B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{D} \quad \rightsquigarrow \quad \frac{\mathcal{D}_i}{C} \qquad\qquad \frac{\dfrac{\mathcal{D}}{A \wedge B} \quad \dfrac{\mathcal{D}_1}{C} \quad \dfrac{\mathcal{D}_2}{C}}{D} \quad \rightsquigarrow \quad \frac{\mathcal{D}_i}{C}$$

where no assumptions are cancelled by $\vee Ep$ or $\wedge Edp$ in $\mathcal{D}_i$ for $i \in \{1, 2\}$.

We refer to the relation defined by the above conversions that exhibit only proofs (dual proofs) as $\rightsquigarrow_{\mathsf{Int}}$ ($\rightsquigarrow_{\mathsf{DualInt}}$).

Observation

Let $\mathcal{D}$, $\mathcal{D}'$ be derivations in N2Int. *If $\mathcal{D} \rightsquigarrow_{\mathsf{Int}} \mathcal{D}'$, then $\delta(\mathcal{D}) \rightsquigarrow_{\mathsf{DualInt}} \delta(\mathcal{D}')$.*

Definition

A derivation in N2Int *is in normal form iff there is no derivation to which it can be converted.*

Observation

Let C^{σ} be the result of replacing every occurrence of an atom p' in C by $-p$, and let A be a formula of $\mathcal{L}_{2\mathrm{Int}}$ based on Φ. The following derivability statements can be verified by presenting normal proofs, respectively normal dual proofs:

1. $(\{A\}; \varnothing) \vdash (\tau(A))^{\sigma}$; 2. $(\varnothing; \{A\}) \vdash (\tau(-A))^{\sigma}$;
3. $(\{(\tau(A))^{\sigma}\}; \varnothing) \vdash A$;
4. $(\{(\tau(-A))^{\sigma}\}; \varnothing) \vdash^{d} A$.

Theorem

For every derivation of a formula A in N2Int from a pair $(\Delta; \Gamma)$ of finite sets of assumptions and counterassumptions there exists a normal derivation of A from $(\Delta; \Gamma)$

Proof. Suppose first that $(\Delta; \Gamma) \vdash A$ and assume without loss of generality that $\Delta = \{A_1, \ldots, A_k\}$ and $\Gamma = \{B_1, \ldots, B_m\}$.

$$
\begin{array}{ll}
 & (\{A_1, \ldots, A_k\}; \{B_1, \ldots, B_m\}) \vdash A \\
\text{iff} & (\{A_1, \ldots, A_k, -B_1, \ldots, -B_m\}; \varnothing) \vdash A \\
\text{iff} & (\varnothing; \varnothing) \vdash (A_1 \to (A_2 \to (\ldots(-B_1 \to (\ldots(-B_m \to A)\ldots))\ldots))) \\
\text{iff} & \models (A_1 \to (A_2 \to (\ldots(-B_1 \to (\ldots(-B_m \to A)\ldots))\ldots))) \\
\text{iff} & \models_{\mathsf{Int}} \tau((A_1 \to (A_2 \to (\ldots(-B_1 \to (\ldots(-B_m \to A)\ldots))\ldots)))) \\
\text{iff} & \varnothing \vdash_{\mathsf{NInt}} \tau((A_1 \to (A_2 \to (\ldots(-B_1 \to (\ldots(-B_m \to A)\ldots))\ldots)))) \\
\text{iff} & \varnothing \vdash_{\mathsf{NInt}} (\tau(A_1) \to (\ldots(\tau(-B_1) \to (\ldots(\tau(-B_m) \to \tau(A))\ldots))\ldots)) \\
\text{iff} & \{\tau(A_1), \ldots, \tau(A_k), \tau(-B_1), \ldots, \tau(-B_m)\} \vdash_{\mathrm{NInt}} \tau(A) \\
\text{only if} & \text{there is normal proof of } \tau(A) \text{ from} \\
 & \{\tau(A_1), \ldots, \tau(A_k), \tau(-B_1), \ldots, \tau(-B_m)\} \text{ in NInt}
\end{array}
$$

Any such normal proof π' in NInt is a normal proof of $\tau(A)$ from $(\{\tau(A_1), \ldots, \tau(A_k), \tau(-B_1), \ldots, \tau(-B_m)\}; \varnothing)$ in N2Int and $\mathcal{L}_{\mathsf{2Int}}$ based on $\Phi \cup \Phi'$. We transform π' into a normal proof π of A from $(\{A_1, \ldots, A_k\}; \{B_1, \ldots, B_m\})$ in N2Int and $\mathcal{L}_{\mathsf{2Int}}$ based on Φ. In a first step, we replace every occurrence of an atom p' in π' by $-p$. The result is a normal proof π'' in N2Int and $\mathcal{L}_{\mathsf{2Int}}$. In a second step, we replace every assumption $(\tau(A_i))^\sigma$ in π'' (for $i \in \{1, \ldots, k\}$) by a normal proof of $(\tau(A_i))^\sigma$ from $(\{A_i\}; \varnothing)$ in N2int. In a third step, we replace every assumption $(\tau(-B_j))^\sigma$ (for $j \in \{1, \ldots, m\}$) by a normal proof of $(\tau(-B_j))^\sigma$ from $(\varnothing; \{B_j\})$ in N2int. In a forth and final step, we replace the conclusion $(\tau(A))^\sigma$ of by a normal proof of A from $(\{(\tau(A))^\sigma\}; \varnothing)$. The result is a normal proof π of A from $(\{A_1, \ldots, A_k\}; \{B_1, \ldots, B_m\})$ in N2Int and $\mathcal{L}_{\mathsf{2Int}}$.

Suppose now that $(\Delta; \Gamma) \vdash^d A$. Then $(\Delta; \Gamma) \vdash -A$. As shown in the previous case, there exists a normal proof π of $-A$ from $(\Delta; \Gamma)$. One application of $(\multimap Ep)$ to the conclusion of π gives us a normal dual proof of A from $(\Delta; \Gamma)$. $\square$

Conclusion

If (i) assertion and denial as well as their internal counterparts, judgement and dual judgement, are taken to be generic actions that are on a par, (ii) a compositional proof theoretic semantics for logics of constructible falsity is desired, and (iii) co-implication is taken seriously as a dual of implication, then it makes much sense to consider in addition to proofs also dual proofs.

General proof theory can be generalized to a more comprehensive theory of derivability, namely to a study of proofs and dual proofs in their own right where one is interested in general questions about their nature and structure.

We may also conclude that logic is not as pair $(\mathcal{L}, \vdash)$ consisting of a language and a consequence relation, but as a triple $(\mathcal{L}, \vdash, \vdash^d)$ consisting of a language, a consequence relation, and a dual consequence relation (or a quintuple $(\mathcal{L}, \vdash, \models, \vdash^d, \models^d)$ instead of a triple $(\mathcal{L}, \vdash, \models)$).

Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

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Assertion and denial, compositionality, and co-implication
Natural deduction for 2Int
Faithful embeddings
Normal forms for 2Int

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55 / 55

Is there a "Hilbert Thesis"?

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General Proof Theory

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29.11.2015

This work was partially supported by the Portuguese Science Foundation, FCT, through the projects
UID/MAT/00297/2013 (Centro de Matemática e Aplicações) and
PTDC/MHC-FIL/5363/2012 (The Notion of Mathematical Proof);
and by the project *Método axiomática e teoria de categorias*
of the cooperation Portugal/France in the *Programa PESSOA – 2015/2016*.

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

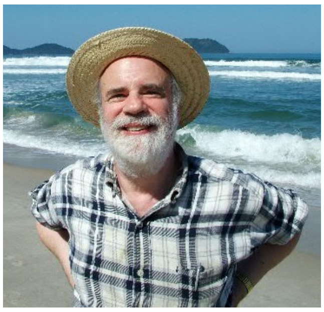
Jon Barwise, Handbook of Mathematical Logic (1977)

... the informal notion of provable used in mathematics is made precise by the formal notion provable in first-order logic. Following a sug[g]estion of Martin Davis, we refer to this view as Hilbert's Thesis.

Saul Kripke, The Church-Turing "Thesis" as a Special Corollary (2013)

Now I shall state another thesis which I shall call "Hilbert's thesis," namely, that the steps of any mathematical argument can be given in a language based on first-order logic (with identity).

There is a footnote added to "Hilbert's thesis" (see below).



Stewart Shapiro, The Open Texture of Computability (2013)

We might define "*Hilbert's thesis*" to be the statement that a text constitutes a proof if and only if it corresponds to a formal proof (although any of half a dozen other names would have done just as well—including that of Alonzo Church).

Multiple other references:

- Lon A. Berk. *Hilbert's Thesis: Some Considerations about Formalization of Mathematics.*
  PhD Thesis, MIT, September 1982, supervised by George Boolos.
- Boolos, Burgess, Jeffrey. Computability and Logic.
  §*14.3. Other Proof Procedures and Hilbert's Thesis*
- Dershowitz, Gurevich (2008) already refer to the use of "Hilbert's thesis" by Kripke.
- Stewart Shapiro: Necessity, Meaning, and Rationality (2008)
- ...

Various forms of Hilbert's thesis

1. Every mathematical proof can be formalized in ZFC.
   "ZFC version"
2. Every mathematical proof can be formalized in first-order logic.
   $\rightarrow$ Barwise
3. Every mathematical statement can be formalized in first-order logic.
   "First-order version (statement)"
4. Every mathematical proof can be formalized.
   $\rightarrow$ Shapiro
5. Every mathematical proof needs only finitly many steps.
   "Finiteness version"

Kripke's Footnote

- Martin Davis originated the term "Hilbert's thesis"; see Barwise (1977, 41).
- The version stated here, however, is weaker. Rather than referring to provability, it is simply that any mathematical *statement* can be formulated in a first-order langugage. Thus, it is about statability, rather than provability.
- Very possibly the weaker thesis about statability might have originally been intended. Certainly Hilbert and Ackermann's famous textbook still regards the completeness of conventional predicate logic as an open problem.
- Had Gödel not solved the problem in the affirmative a stronger formalism would have been necessary, or conceivably no complete system would have been possible.
- Note also that Hilbert and Ackermann do present the "restricted calculus," as they call it, as a fragment of the second-order calculus, and ultimately of the logic of order ω. However, they seem to identify even the second-order calculus with set theory, and mention the paradoxes. Little depends on these exact historical points.

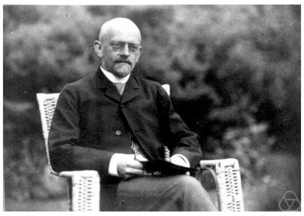
Various forms of Hilbert's thesis

1. Every mathematical proof can be formalized in ZFC.
   "ZFC version"
2. Every mathematical proof can be formalized in first-order logic.
   $\rightarrow$ Barwise
3. Every mathematical statement can be formalized in first-order logic.
   $\rightarrow$ Kripke
4. Every mathematical proof can be formalized.
   $\rightarrow$ Shapiro
5. Every mathematical proof needs only finitly many steps.
   "Finiteness version"

Hilbert's Thesis — Finiteness version

Every mathematical proof needs only finitly many steps.

- Seems to be trivial; rather part of a definition of proof than a "Thesis".
- Only version of which we have hard textual evidence in Hilbert's texts.
- Excludes, in some sense, Second-Order Logic.
  - Of course, Hilbert could not have been aware of it (before 1931).
  - Gives, in this way, evidence for the first-order versions.



- Challenges the ω rule (suggested by Hilbert himself!).

Hilbert's Thesis — ZFC version

Every mathematical proof can be formalized in ZFC.

- In this form, not forwarded by anybody.
- At best, an instance of the thesis:

  Every mathematical proof can be formalized in set theory.

- Could be considered for a "statability version".
- Added here, because it is the only version corresponding properly to *Church's Thesis* . . .

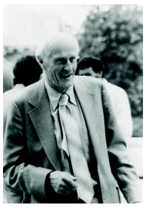
  . . . or not, as we will argue.

Church's Thesis

Every intuitively computable function is λ definable.

- You may replace λ definability by your favorite (Turing-complete) notion of computability:
  - μ recursion
  - Turing computable
  - ...


- What's *your* reason to believe in Church's Thesis?
- *My* reason is Kleene's:



Kleene, 1981

When Church proposed this thesis, I sat down to disprove it by diagonalizing out of the class of the λ-definable functions. But, quickly realizing that the diagonalisation cannot be done effectively, I became overnight a supporter of the thesis.

- Diagonalization is blocked by *partiality* (here in a strict setting):
  - $(\lambda x.xx)(\lambda x.xx) \uparrow$
  - There is no $\lambda x.t$ such that $(\lambda x.t)\, s \downarrow$, if $s \uparrow$.

Is Hilbert's Thesis immune to diagonalization?

1. ZFC version
2. First-order verison (proof)
3. First-order verison (statement)
4. Formalization version
5. Finiteness version

2–4 Too vague, in some sense.

5 Immune, but very weak Thesis.

1 The only form comparable with Church Thesis, as it refers to a *concrete* system (as CT to λ definability) ...
...but clearly subject to diagonalization:
   - Just add a new large cardinal to ZFC (and any reasonable extension), closed under the cardinal construction principle used before (echo of Cantor's Absolute).

New question

Is there a concept of "partial proof" corresponding to "partial function"?

Extended Curry-Howard isomorphism?

Recall (our) reason for Church's Thesis:

- Intuitive Computability corresponds to λ terms.
- Diagonalization is blocked by partiality.

Curry-Howard isomorphism

- *Typed* λ terms can be considered as notations for natural deduction derivations (in an intuitionistic setting).

Untyped λ calculus

- Partiality comes in by use of (untyped/untypable) *self-application*: $\lambda x.xx$.
- The key tool is the Curry's *paradoxical combinator*: $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

Extended Curry-Howard isomorphism?

Goal

Extending the Curry-Howard isomorphism covering the Y combinator.

- Y should give rise to "partial proofs".
- 1st application: Y, Second Recursion Th. & Diagonalizationlemma

  There is a recognizable parallel between
  - Y
  - The proof of Kleene's second recursion theorem
  - The proof of the Diagonalizationlemma (Gödel/Carnap).

  This correspondence should become formal!
- 2nd application: Partial Functions in Martin-Löf Type Theory.
  - Partial Functions are necessary for an *Extended Predicative Mahlo Universe*.
  - Not yet formalizable in Martin-Löf Type Theory.

  $\rightarrow$ Kahle/Setzer: The Limits of the Curry-Howard Isomorphism
  Functions, Proofs, Constructions. Tübingen, 21.2.2014

On distinguishing proof-theoretic consequence from derivability

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Introduction

– **Common conception**: *logical consequence* can be defined in two main ways.
- **Model-theoretically:** Consequence is taken, to be *preservation* (also called *propagation* or transmission) of truth over all models:
$\Gamma \models \varphi$ iff for every model $\mathcal{M}$, if $\mathcal{M} \models \Gamma$ then $\mathcal{M} \models \varphi$
– **Two characteristics**:
necessity: Here manifested by the universal quantification over *all* models.
formality: The truth in models is in virtue of (logical) *form*, depending on the *logical constants* and their arrangement.
- **Proof-Theoretically:** For a suitable proof-system $\mathcal{N}$ (which I will take here as ND), consequence is taken as *derivability* in $\mathcal{N}$, denoted $\vdash_{\mathcal{N}}$; $\Gamma \vdash_{\mathcal{N}} \varphi$ iff there exists an $\mathcal{N}$-derivation of φ from (open) assumptions Γ.
Typically, (strong) soundness and completeness theorems, that is $\Gamma \vdash_{\mathcal{N}} \varphi$ iff $\Gamma \models \varphi$, establish the *coextensiveness* of those two notions of consequence.

1

What else could be propagated?

– That logical consequence involves preservation of *something*, not necessarily of truth, has been suggested by many.
- **Information:** propagation of the information (in a situation) is underlying consequence of Relevant Logic(s).
- **Ambiguity:** This notion suggests to treat a proposition p *ambiguously* as two different propositions, p_t and p_f. A *measure* of ambiguity of an *inconsistent* Γ is the *minimal* number of proposition in Γ the treatment of which as ambiguous renders Γ consistent. Propagation of ambiguity is used for defining consequence for *paraconsistent logics*.
- **Precisification:** In the context of *vagueness*, there is an appeal to propagation of *super-truth* (i.e., truth in all *precisifications*) in defining logical consequence.

2

Propagation and meaning

– **A natural question:** what is common to all the "things" being suggested as propagated by the various consequence relations mentioned above?
- I want to argue that they all serve (either explicitly or implicitly) as *central concepts on which theories of meaning are based*.
- Two of the main theories of meaning are:
- **Model-Theoretic Semantics** (MTS): The central concept of MTS is *truth* (in arbitrary models). Meaning is defined as truth-conditions. **Proof-Theoretic Semantics** (PTS): The central concept of PTS is *proof*, or more generally, *canonical derivation* in appropriate meaning-conferring proof-systems. Meaning is defined (implicitly or explicitly) by the rules of the meaning-conferring system.
- The other propagated "things" mentioned above have a similar role in theories of meaning for Relevant Logic, general paraconsistent logics and for languages with vagueness.

3

A principle

–Consequently, I suggest the following informal principle.
meaning-based logical consequence: In a theory of meaning $\mathcal{T}$, logical consequence is based on the propagation of the central concept of $\mathcal{T}$.

By being faithful to a theory of meaning $\mathcal{T}$ I mean relating the notion of φ being a logical consequence (logically following from) Γ to the meanings of φ and Γ as determined by that theory of meaning $\mathcal{T}$.
– I want to argue that, in spite of the *coextensiveness* for many logics of derivability and preservation of truth in models, if one adheres to PTS then derivability is not the right definition of proof-theoretic consequence. While MTS is faithful to the usual model-theoretic conception of meaning, PTS is not faithful to its conception of meaning.
- I suggest another definition of proof-theoretic consequence that *is* faithful to PTS, sometimes (i.e., for some logics) coextensive with derivability, and sometimes – not.

4

Meaning according to PTS

– In the PTS literature, meaning is conceived as *implicitly defined* by the I-rules, not appealing to any *proof-theoretic semantic value* as an *explicit* definition.
- For several purposes, e.g., the construction of a *grammar*, an explicit definition is needed. In MTS, an explicit definition of meaning is usually taken as the following semantic value:
$[\![\varphi]\!] = \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$
- Note that MTS is not interested in assigning semantic values to the logical constants themselves, so usually no definitions of $[\![\wedge]\!]$, $[\![\forall]\!]$ are considered.
- But what exactly can be taken as an *explicitly* defined *proof-theoretic* semantic value within PTS as the result of the determination of meaning via the meaning-conferring I-rules?
I have suggested such a proof-theoretic semantic value as an explicit definition of meaning. I recapitulate this proposal below.

5

Canonicity - I

– An important concept in PTS is that of a *canonical proof* in $\mathcal{N}$, where a proof of some compound φ is a derivation of φ from no open assumptions (an empty Γ).

definition(**canonical proof**): A proof $\mathcal{D}$ in $\mathcal{N}$ is *canonical* iff the *last rule applied* in $\mathcal{D}$ is an I-rule.

-To define proof-theoretic consequence, there is a need to extend canonicity to arbitrary $\mathcal{N}$-derivations, including ones with open assumptions.

Definition (canonical derivation from open assumptions): A $\mathcal{N}$-derivation $\mathcal{D}$ for $\Gamma \vdash \psi$ (for a compound ψ) is *canonical* iff it satisfies one of the following two conditions.

(1) The last rule applied in $\mathcal{D}$ is an I-rule (for the main operator of ψ).

The last rule applied in $\mathcal{D}$ is an *assumption-discharging* E-rule, the major premise of which is some φ in Γ, and its encompassed sub-derivations $\mathcal{D}_1, \cdots, \mathcal{D}_n$ are all canonical derivations of ψ.

6

Canonicity - II

– Denote by $\vdash^c_{\mathcal{N}}$ canonical derivability in $\mathcal{N}$. Let $[\![\varphi]\!]^c_\Gamma$ the (possibly empty) collection of canonical derivations of φ from Γ, and $[\![\varphi]\!]^*_\Gamma$ the (also possibly empty) collection of *all* derivations of φ from Γ.

-For Γ empty, the definition reduces to that of a canonical proof.

- The important observation regarding the recursion is that it *always terminates via the first clause*, namely by an application of an I-rule. Call such an application of an I-rule an *essential* application, the outcome of which is propagated throughout the canonical sub-derivation.

-Note also that, similarly to the case of canonical proofs, there are no canonical derivations for *an atomic sentence*, which by definition has no introducible operators. Traditionally, the PTS programme views the meaning of an atomic sentence to be *given*, possibly externally. To overcome this non-specificity, I take here the rule of assumption $\Gamma, p \vdash p$ to constitute the canonical way an atomic sentence is introduced into a derivation.

7

Reified meaning, Grounds for assertion

– **Definition (sentential semantic values):**
(1) For a compound φ, its *meaning* (contributed semantic value) $[\![\varphi]\!]$, is $[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^c_\Gamma$
(2) For an arbitrary (atomic or compound) φ, its *contributing semantic value* is given as $[\![\varphi]\!]^* = \lambda\Gamma.[\![\varphi]\!]^*_\Gamma$
– The definition of *proof-theoretic consequence* (pt-consequence) rests on the notion of *grounds for assertion* for φ, closely related to $[\![\varphi]\!]$.
Definition (grounds for assertion): $GA[\![\varphi]\!] = \{\Gamma \mid \Gamma \vdash^c \varphi\}$
-Thus, any Γ that canonically derives φ serves as grounds for assertion of φ.
– The notion of grounds considered here is different than, though in the same spirit as, the grounds considered by Prawitz. The grounds here are *formal constructs*, collections of sentences (assumptions) canonically deriving a sentence (conclusion). On the other hand Prawitz considers grounds as *mental* counterparts, associated with *possession* of the formal grounds and justifying the *epistemic acts* of inference, assertion.

8

Collective grounds

– Next, I extend the definition of the grounds for assertion of a single sentence to grounds for the *collective assertion* of a finite, non-empty collection of sentences, say Δ.
- There are two natural options distinguished by the way assumptions are combined.
common grounds: $GA_c[\![\Delta]\!] =^{\text{df.}} \cap_{\psi\in\Delta} GA[\![\psi]\!]$.
joint grounds: $GA_j[\![\Delta]\!] =^{\text{df.}} \circ_{\psi\in\Delta} GA[\![\psi]\!]$, where '$\circ$' is *fusion*, known also as intensional conjunction.
- The difference between conjunction and fusion originates in the I-rules for conjunction being *additive* (or shared context), while the I-rules for fusion are *multiplicative* (context free).
- When I want to remain neutral regarding this difference in combining grounds, I will speak generically of "collective grounds", with a generic notation $GA[\![\Delta]\!]$ (without a qualifying superscript).

9

Proof-Theoretic consequence

– Based on the definitions of grounds combination, I define two notions of proof-theoretic *consequence* (pt-consequence).

Definition (proof-theoretic consequences):

- *conjunctive pt-consequence*: ψ is a *conjunctive proof-theoretic consequence* of Γ ($\Gamma \Vdash_c \psi$) iff $GA_c[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$. [*explosion!*]

-*fused pt-consequence:* ψ is a *fused proof-theoretic consequence* of Γ ($\Gamma \Vdash_j \psi$) iff $GA_j[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$. [*Relevant Grounds*]

Thus, both pt-consequences are based on *grounds propagation*: every collective grounds for collectively asserting all of Γ (depending on the mode of combination of grounds employed) are already grounds for asserting ψ.

By this definition, ψ is a pt-consequence of Γ *according to ψ's meaning* as pt-consequence involves canonical derivability.

10

Relating pt-consequence to derivability

Definition (smoothness): An ND-system $\mathcal{N}$ is *proof-theoretically smooth* iff for every Γ and every *compound* φ: $\Gamma \vdash_{\mathcal{N}} \varphi$ iff $\Gamma \vdash^c_{\mathcal{N}} \varphi$

-That is, a compound φ is $\mathcal{N}$-derivable from Γ iff it is *canonically* derivable.

- In a smooth $\mathcal{N}$, derivability *is coextensive* with pt-consequence.

- This is another formulation of Dummett's *Fundamental Assumption (FA)*, extended from proofs to derivations from open assumptions.

Proposition: The ND-system NJ (for propositional intuitionistic logic is proof-theoretically smooth.

As for classical logic, suppose we consider the version of NK obtained by adding to NJ the rule for double-negation elimination.

$$\frac{\Gamma \vdash_{NK} \neg\neg\varphi}{\Gamma \vdash_{NK} \varphi}\,(DNE)$$

Proposition: The ND-system NK (for propositional classical logic) is not proof-theoretically smooth.

11

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High-school sequent calculus and an intuitionistic formula hierarchy preserving identity of proofs

General Proof Theory, Tubingen, 29 Nov 2015

Danko Ilik (Inria, France)

Contents

1. Exp-log normal form
2. Application: Identity of proofs
3. Application: Intuitionistic "arithmetical" hierarchy
4. Application: High-school sequent calculus

1

Exp-log normal form

Types / Propositional formulas / Exponential polynomials

A common language $\mathcal{E}$ for products, coproducts, and exponentials:

$$\mathcal{E} \ni F, G ::= F \times G \mid F + G \mid F \to G \mid X_i$$

$$\mathcal{E} \ni F, G ::= F \wedge G \mid F \vee G \mid F \supset G \mid X_i$$

$$\mathcal{E} \ni F, G ::= FG \mid F + G \mid G^F \mid X_i$$

where X_i are type variables / atomic propositions / variables.

Tarski's high-school algebra problem

Can all **true** exponential polynomial equations,

$$\mathbb{N}^+ \vDash f = g,$$

be **derived** using the high-school identities:

$$\begin{aligned} f &\doteq f & 1f &\doteq f \\ f+g &\doteq g+f & f^1 &\doteq f \\ (f+g)+h &\doteq f+(g+h) & 1^f &\doteq 1 \\ fg &\doteq gf & f^{g+h} &\doteq f^g f^h \\ (fg)h &\doteq f(gh) & (fg)^h &\doteq f^h g^h \\ f(g+h) &\doteq fg+fh & (f^g)^h &\doteq f^{gh} \end{aligned}$$

?

Isomorphism of types / Strong equivalence of formulas

Definition (Type isomorphism)

Types F and G are isomorphic,

$$F \cong G,$$

if there exist $S : F \to G$ and $T : G \to F$ such that

$$\lambda x.S(Tx) =_{\beta\eta} \lambda x.x \quad \text{and} \quad \lambda y.T(Sy) =_{\beta\eta} \lambda y.y.$$

Strong equivalence / Equality of exponential polynomials

Theorem

Strong equivalence generalizes equality of multivariate exponential polynomials:

$$F \cong G \text{ implies } \mathbb{N}^{+} \models F = G.$$

Strong equivalence / Equality of exponential polynomials

Theorem

Strong equivalence generalizes equality of multivariate exponential polynomials:

$$F \cong G \text{ implies } \mathbb{N}^{+} \models F = G.$$

A **non**-strong-equivalence procedure, assuming we can solve $F = G$,

- like in absence of implication (canonical polynomials),
- or in absense of disjunction (Curried types),

but not clear in simultaneous presence of $\supset$ and $\vee$.

A quasi-normal form for propositional formulas (Du Bois Reymond, Hardy 1910)

Decompose exponentiation in terms of unary exp and log,

$$G^F = \mathrm{e}^{F \log G} = \exp\left(F \log G\right),$$

i.e. decompose implication in terms of two ‘negations’,

$$F \supset G = \neg_{\exp}(F \wedge \neg_{\log} G).$$

A quasi-normal form for propositional formulas (Du Bois Reymond, Hardy 1910)

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$$G^F = \mathrm{e}^{F \log G} = \exp\left(F \log G\right),$$

i.e. decompose implication in terms of two ‘negations’,

$$F \supset G = \neg_{\exp}(F \wedge \neg_{\log} G).$$

And we can keep things abstract since exp and log can be seen as homomorphisms between the additive and multiplicative group in exponential fields.

In other words, do not treat $\neg_{\exp}$ and $\neg_{\log}$ as logical connectives but rather as macros:

$$\exp(F+G) = (\exp F)(\exp G) \qquad \log(FG) = \log F + \log G$$
$$\exp(\log F) = F \qquad \log(\exp F) = F$$

Exp-log normal form (ENF)

Operational characterization

Operationally, the exp-log transformation is just *orienting* the high-school identities,

$$F^{G+H} \mapsto F^G F^H$$
$$(FG)^H \mapsto (F^H)(G^H)$$
$$(F^G)^H \mapsto F^{GH},$$

or logically,

$$G \vee H \supset F \mapsto (G \supset F) \wedge (H \supset F)$$
$$H \supset F \wedge G \mapsto (H \supset F) \wedge (H \supset G)$$
$$H \supset G \supset F \mapsto G \wedge H \supset F.$$

Exp-log normal form (ENF)

Inductive characterization

Theorem

A formula F is in exp-log normal form if and only if $F \in \mathcal{D}$, where:

$$\mathcal{D} \ni D ::= C \mid C \vee D$$
$$\mathcal{C} \ni C ::= A \mid A \wedge C$$
$$\mathcal{A} \ni A ::= X_i \mid C \supset X_i \mid C \supset C' \vee D$$

The inductive characterization is slightly flexibility, for instance, working with n-ary $\wedge$ and $\vee$ gives:

$$\mathcal{D} \ni D ::= X_i \mid C_1 \vee \cdots \vee C_n \qquad (n \geq 2)$$
$$\mathcal{C} \ni C ::= (C_1 \supset D_1) \wedge \cdots \wedge (C_n \supset D_n) \qquad (n \geq 0)$$

2

Application: Identity of proofs

λ-Calculus with sums

Syntax and $=_{\beta\eta}$

$$M, N ::= x^\tau \mid (M^{\tau\to\sigma} N^\tau)^\sigma \mid (\pi_1 M^{\tau\times\sigma})^\tau \mid (\pi_2 M^{\tau\times\sigma})^\sigma \mid \delta(M^{\tau+\sigma}, x_1^\tau.N_1^\rho, x_2^\sigma.N_2^\rho)^\rho \mid$$
$$\mid (\lambda x^\tau.M^\sigma)^{\tau\to\sigma} \mid \langle M^\tau, N^\sigma\rangle^{\tau\times\sigma} \mid (\iota_1 M^\tau)^{\tau+\sigma} \mid (\iota_2 M^\sigma)^{\tau+\sigma}$$

$$(\lambda x.N)M =_\beta N\{M/x\} \qquad (\beta_\to)$$
$$\pi_i\langle M_1, M_2\rangle =_\beta M_i \qquad (\beta_\times)$$
$$\delta(\iota_i M, x_1.N_1, x_2.N_2) =_\beta N_i\{M/x_i\} \qquad (\beta_+)$$
$$N =_\eta \lambda x.Nx \qquad x \notin \mathrm{FV}(N) \qquad (\eta_\to)$$
$$N =_\eta \langle \pi_1 N, \pi_2 N\rangle \qquad (\eta_\times)$$
$$N\{M/x\} =_\eta \delta(M, x_1.N\{\iota_1 x_1/x\}, x_2.N\{\iota_2 x_2/x\}) \qquad x_1, x_2 \notin \mathrm{FV}(N) \qquad (\eta_+)$$

λ-Calculus with sums

Trouble

Example (at type $\tau + \sigma \to (\tau + \sigma \to \rho) \to \rho$)

$$\lambda x.\lambda y.y\delta(x, z.\iota_1 z, z.\iota_2 z)$$
$$\lambda x.\lambda y.\delta(x, z.y(\iota_1 z), z.y(\iota_2 z))$$
$$\lambda x.\delta(x, z.\lambda y.y(\iota_1 z), z.\lambda y.y(\iota_2 z))$$

λ-Calculus with sums

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Example (at type $\tau + \sigma \to (\tau + \sigma \to \rho) \to \rho$)

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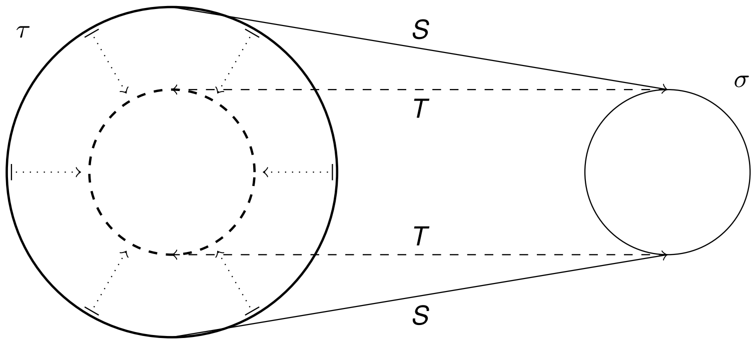
$$\lambda x.\delta(x, z.\lambda y.y(\iota_1 z), z.\lambda y.y(\iota_2 z))$$

Example (at type $(\tau_1 \to \tau_2) \to (\tau_3 \to \tau_1) \to \tau_3 \to \tau_4 + \tau_5 \to \tau_2$)

$$\lambda xyzu.x(yz)$$
$$\lambda xyzu.\delta(\delta(u, x_1.\iota_1 z, x_2.\iota_2(yz)), y_1.x(yy_1), y_2.xy_2).$$

$\beta\eta$-Congruence classes at ENF type

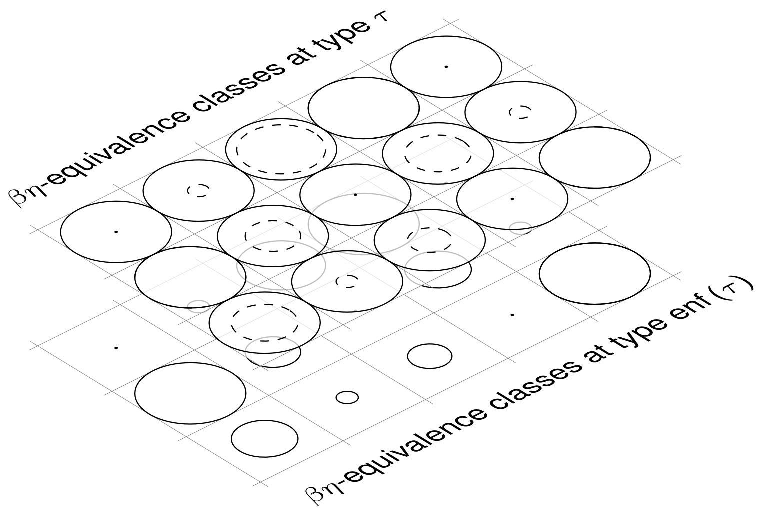
When the isomorphism $\tau \cong \sigma$ is witnessed by λ-terms S, T, **and when** σ **is a *smaller* type than** τ, we can decide $=_{\beta\eta}$ at σ rather than at τ:



In particular, when σ contains no more sum types, we collapse τ to a singleton.

$\beta\eta$-Congruence classes at ENF type

Allows to choose the plane at which we prefer to look at λ-term representations:



In particular, there may be *better* planes than the one of *quasi*-normal form
(ex. ones that involve concrete datatypes)

Canonical representations of the two examples

Example (at type $\tau + \sigma \to (\tau + \sigma \to \rho) \to \rho$)

$$\lambda x.\lambda y.y\delta(x, z.\iota_1 z, z.\iota_2 z)$$
$$\lambda x.\lambda y.\delta(x, z.y(\iota_1 z), z.y(\iota_2 z))$$
$$\lambda x.\delta(x, z.\lambda y.y(\iota_1 z), z.\lambda y.y(\iota_2 z))$$

Canonical representative: $\langle \lambda x.(\pi_1(\pi_2 x))(\pi_1 x), \lambda x.(\pi_2(\pi_2 x))(\pi_1 x)\rangle$

Example (at type $(\tau_1 \to \tau_2) \to (\tau_3 \to \tau_1) \to \tau_3 \to \tau_4 + \tau_5 \to \tau_2$)

$$\lambda xyzu.x(yz)$$
$$\lambda xyzu.\delta(\delta(u, x_1.\iota_1 z, x_2.\iota_2(yz)), y_1.x(yy_1), y_2.xy_2).$$

Canonical representative:
$\langle \lambda x.(\pi_1 x)((\pi_1\pi_2 x)(\pi_1\pi_2\pi_2 x)), \lambda x.(\pi_1 x)((\pi_1\pi_2 x)(\pi_1\pi_2\pi_2 x))\rangle$

Terms of ENF type

WLOG, we need only consider terms of ENF type:

$$\begin{aligned} M, N ::= {} & x^d \mid (M^{c\to p} N^c)^p \mid (M^{c\to c'+d} N^c)^{c'+d} \mid \\ & (\pi_1 M^{a\times c})^a \mid (\pi_2 M^{a\times c})^c \mid \delta(M^{c+d}, x_1^c.N_1^{d'}, x_2^d.N_2^{d'})^{d'} \mid \\ & (\lambda x^c.M^p)^{c\to p} \mid (\lambda x^c.M^{c'+d})^{c\to c'+d} \mid \\ & \langle M^a, N^c\rangle^{a\times c} \mid (\iota_1 M^c)^{c+d} \mid (\iota_2 M^d)^{c+d} \end{aligned}$$

Theorem

Let P, Q be terms of type τ and let $S : \tau \to \mathrm{enf}(\tau)$, $T : \mathrm{enf}(\tau) \to \tau$ be a witnessing pair of terms for the isomorphism $\tau \cong \mathrm{enf}(\tau)$. Then, $P =_{\beta\eta} Q$ if and only if $SP =^{\mathrm{e}}_{\beta\eta} SQ$ and if and only if $T(SP) =_{\beta\eta} T(SQ)$.

Equations at ENF type

An equational theory $=^{\mathrm{e}}_{\beta\eta}$ refining $=_{\beta\eta}$:

$$
\begin{array}{rcll}
(\lambda x^{c'}.N^{p}_{c+d})M & =^{\mathrm{e}}_{\beta} & N\{M/x\} & (\beta^{\mathrm{e}}_{\rightarrow}) \\
\pi_i\langle M_1^{a}, M_2^{c}\rangle & =^{\mathrm{e}}_{\beta} & M_i & (\beta^{\mathrm{e}}_{\times}) \\
\delta(\iota_i M, x_1.N_1, x_2.N_2)^{d} & =^{\mathrm{e}}_{\beta} & N_i\{M/x_i\} & (\beta^{\mathrm{e}}_{+}) \\
N^{c\to p}_{c'\to c+d} & =^{\mathrm{e}}_{\eta} & \lambda x.Nx & x \notin \mathrm{FV}(N) \quad (\eta^{\mathrm{e}}_{\rightarrow}) \\
N^{a\times c} & =^{\mathrm{e}}_{\eta} & \langle \pi_1 N, \pi_2 N\rangle & (\eta^{\mathrm{e}}_{\times}) \\
N^{p}_{c+d}\{M/x\} & =^{\mathrm{e}}_{\eta} & \delta(M, x_1.N\{\iota_1 x_1/x\}, x_2.N\{\iota_2 x_2/x\}) & x_1, x_2 \notin \mathrm{FV}(N) \quad (\eta^{\mathrm{e}}_{+}) \\
\pi_i\delta(M, x_1.N_1, x_2.N_2) & =^{\mathrm{e}}_{\eta} & \delta(M, x_1.\pi_i N_1, x_2.\pi_i N_2)^{c} & (\eta^{\mathrm{e}}_{\pi}) \\
\lambda y.\delta(M, x_1.N_1, x_2.N_2) & =^{\mathrm{e}}_{\eta} & \delta(M, x_1.\lambda y.N_1, x_2.\lambda y.N_2)^{c\to p}_{c'\to c+d} & y \notin \mathrm{FV}(M) \quad (\eta^{\mathrm{e}}_{\lambda})
\end{array}
$$

Danko Ilik – High-school sequent calculus and an intuitionistic formula hierarchy preserving identity of proofs 21

3

Application: Intuitionistic “arithmetical” hierarchy

Danko Ilik – High-school sequent calculus and an intuitionistic formula hierarchy preserving identity of proofs 22

From propositional towards predicate calculus

What is the exponential polynomial representation of intuitionistic $\forall$ and $\exists$?

The classical arithmetical hierarchy

Theorem

Every formula $F \in \Sigma_n^0 \cup \Pi_n^0$, where the classes Σ_n^0 and Π_n^0 are based on classical equivalence $\leftrightarrow^c$:

$$\begin{aligned} F \in \Pi_{n+1}^0 &\text{ iff there exists } G \in \Sigma_n^0 \text{ such that } F \leftrightarrow^c \forall xG \\ F \in \Sigma_{n+1}^0 &\text{ iff there exists } G \in \Pi_n^0 \text{ such that } F \leftrightarrow^c \exists xG \\ \Pi_0^0 &= \Sigma_0^0 \end{aligned}$$

Prenex rules.

$$\begin{array}{ccc} \forall xF \vee G \leftrightarrow^c \forall x(F \vee G) & \exists xF \wedge G \leftrightarrow^c \exists x(F \wedge G) & \text{where } x \notin \mathrm{FV}(G) \\ \forall xF \wedge G \leftrightarrow^c \forall x(F \wedge G) & \exists xF \vee G \leftrightarrow^c \exists x(F \vee G) & \\ \neg\exists xF \leftrightarrow^c \forall x\neg F & \neg\forall xF \leftrightarrow^c \exists x\neg F, & \end{array}$$

$\square$

Prenex rules seen intuitionistically

Half of the rules are not valid intuitionistically, but half are *strong* equivalences!

$$\begin{array}{rcl}
\forall xF \vee G \not\leftrightarrow^i \forall x(F \vee G) & & \exists xF \wedge G \not\leftrightarrow^i \exists x(F \wedge G) \\
\forall xF \wedge G \cong \forall x(F \wedge G) & & \exists xF \vee G \cong \exists x(F \vee G) \\
(\exists xF) \supset G \cong \forall x(F \supset G) & & (\forall xF) \supset G \not\leftrightarrow^i \exists x(F \supset G)
\end{array}$$

where $x \notin \mathrm{FV}(G)$.

Polynomial notation for intuitionistic anti-prenexing

So, instead of pushing quantifiers outside, in intuitionistic logic we should be pushing them inside!

$$\begin{array}{rcrcl}
\forall x(F \wedge G) \cong (\forall xF) \wedge (\forall xG) & & (FG)^x = F^x G^x & & \\
\exists x(F \vee G) \cong (\exists xF) \vee (\exists xG) & & x(F+G) = xF + xG & & \\
\forall x(F \supset G) \cong (\exists F) \supset G & & (G^F)^x = G^{xF} & & (x \notin \mathrm{FV}(G))
\end{array}$$

Exp-polynomial notation is retrieved if we reserve lowercase $x, y, \ldots$ for variables and uppercase $F, G, X, \ldots$ for formulas.

The exp-log normal form extended to quantifiers

Recall the inductive characterization for propositional formulas:

$$\begin{array}{ll} \Sigma \ni C ::= \top \mid A_1 \wedge \cdots \wedge A_n & n \geq 1 \\ \Pi \ni A ::= C \supset P \mid C \supset (C_1 \vee \cdots \vee C_n) & n \geq 2 \end{array}$$

The exp-log normal form extended to quantifiers

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Extending with the quantifier isomorphisms gives:

$$\begin{array}{ll} \Sigma \ni C ::= \top \mid A_1 \wedge \cdots \wedge A_n \mid \exists x C & n \geq 1 \\ \Pi \ni A ::= C \supset P \mid C \supset (C_1 \vee \cdots \vee C_n) \mid C \supset \exists x C_1 \mid \forall x A \mid \forall x \exists y C & n \geq 2 \end{array}$$

Retrieving the classes Π and Σ

The obtained hierarchy is "arithmetical" for it does not allow a linear indexing: Π and Σ have more to do with Π- and Σ-types of Martin-Löf than with the classical hierarchy.

Theorem

Every formula is intuitionistically strongly equivalent (isomorphic) to one in $\Sigma \cup \Pi$, where

$$\begin{array}{ll} \Sigma \ni C ::= \top \mid A_1 \wedge \cdots \wedge A_n \mid \exists x C & n \geq 1 \\ \Pi \ni A ::= C \supset P \mid C \supset (C_1 \vee \cdots \vee C_n) \mid C \supset \exists x C_1 \mid \forall x A \mid \forall x \exists y C & n \geq 2 \end{array}$$

Π and Σ in presence of intuitionistic choice rules Experimental!

In presence of further isomorphisms for Choice,

$$\forall x \exists y F(x,y) \cong \exists f \forall x F(x, fx) \qquad (yF)^x = y^x F^x$$

the hierarchy can be further simplified to:

$$\begin{array}{ll} \Sigma \ni C ::= \top \mid A_1 \wedge \cdots \wedge A_n \mid \exists x C & n \geq 1 \\ \Pi \ni A ::= C \supset P \mid C \supset (C_1 \vee \cdots \vee C_n) \mid \forall x A & n \geq 2 \end{array}$$

This would be the case in Martin-Löf Type Theory.

4

Application: High-school sequent calculus

Analyze sequent calculi as polynomial transformations

Conventions:

- Turnstile "$\vdash$" is the top-most $\supset$ (implication is right-associative)

  Ex. $\Gamma \supset F \supset G$ instead of $\Gamma \vdash F \supset G$

- Context comma "," is conjunction $\wedge$

  Ex. $F \wedge G \wedge \Gamma$ instead of F, G, Γ

- Conjunction and disjunction are *n*-ary
  - This avoids having to deal with associativity isomorphisms

Invertible (asynchronous) rules of LJ

$$\frac{F, \Gamma \to G}{\Gamma \to F \to G} \qquad \frac{G^{F\Gamma}}{(G^F)^\Gamma} \qquad (\to_r)$$

$$\frac{\Gamma \to F \quad \Gamma \to G}{\Gamma \to F \wedge G} \qquad \frac{F^\Gamma G^\Gamma}{(FG)^\Gamma} \qquad (\wedge_r)$$

$$\frac{F, \Gamma \to H \quad G, \Gamma \to H}{F \vee G, \Gamma \to H} \qquad \frac{H^{F\Gamma} H^{G\Gamma}}{H^{(F+G)\Gamma}} \qquad (\vee_l)$$

Further invertible rules of LJ (G4ip)

$$\frac{F \to G \to H, \Gamma \to I}{F \wedge G \to H, \Gamma \to I} \qquad \frac{I^{(H^G)^F\Gamma}}{I^{H^{FG}\Gamma}} \qquad (\to_l^\wedge)$$

$$\frac{F \to H, G \to H, \Gamma \to I}{F \vee G \to H, \Gamma \to I} \qquad \frac{I^{H^F H^G \Gamma}}{I^{H^{F+G}\Gamma}} \qquad (\to_l^\vee)$$

$$\frac{F, G, \Gamma \to H}{F \wedge G, \Gamma \to H} \qquad \frac{H^{FG\Gamma}}{H^{FG\Gamma}} \qquad (\wedge_l')$$

Further invertible rules of LJ (G4ip)

$$\frac{F \to G \to H, \Gamma \to I}{F \wedge G \to H, \Gamma \to I} \qquad \frac{I^{(H^G)^F \Gamma}}{I^{H^{FG}\Gamma}} \qquad (\to_l^{\wedge})$$

$$\frac{F \to H, G \to H, \Gamma \to I}{F \vee G \to H, \Gamma \to I} \qquad \frac{I^{H^F H^G \Gamma}}{I^{H^{F+G}\Gamma}} \qquad (\to_l^{\vee})$$

$$\frac{F, G, \Gamma \to H}{F \wedge G, \Gamma \to H} \qquad \frac{H^{FG\Gamma}}{H^{FG\Gamma}} \qquad (\wedge_l')$$

All of the invertible (asynchronous) rules shown are formula isomorphisms.

But, not all possible formula isomorphisms are accounted for as proof rules.

Synchronous rules of LJ (G4ip)

$$\frac{}{P, \Gamma \to P} \qquad \frac{}{P^{P\Gamma}} \qquad P\text{-prime} \qquad (\text{init})$$

$$\frac{\Gamma \to F}{\Gamma \to F \vee G} \qquad \frac{F^{\Gamma}}{(F+G)^{\Gamma}} \qquad (\vee_r^1)$$

$$\frac{\Gamma \to G}{\Gamma \to F \vee G} \qquad \frac{G^{\Gamma}}{(F+G)^{\Gamma}} \qquad (\vee_r^2)$$

$$\frac{F, P, \Gamma \to G}{P \to F, P, \Gamma \to G} \qquad \frac{G^{FP\Gamma}}{G^{F^P P\Gamma}} \qquad P\text{-prime} \qquad (\to_l^P)$$

$$\frac{G \to H, \Gamma \to F \to G \quad H, \Gamma \to I}{(F \to G) \to H, \Gamma \to I} \qquad \frac{(G^F)^{H^G \Gamma} \; I^{H\Gamma}}{I^{H^{G^F}\Gamma}} \qquad (\to_l^{\to})$$

Reasons for working with G4ip

- Practical: minimal set of non-invertible rules (less cases to consider when doing meta-proofs)
- Possible applications: absence of contraction leaves some hope for verifying the **in**-equality interpretation:

For all rules but $\rightarrow_l^{\rightarrow}$, we have that $F \leq G$,
whenever F represents the premise(s) and G the conclusion.

Taus Brock-Nannestad showed that $\leq$-interpretation of $\rightarrow_l^{\rightarrow}$ holds as well, when $G, H, I \geq 2$!

Applying ENF to LJ

Recall the inductive hierarchy of propositional formulas:

$$\mathcal{D} \ni D ::= P \mid C_1 \vee \cdots \vee C_n \qquad (n \geq 2)$$
$$\mathcal{C} \ni C ::= (C_1 \rightarrow D_1) \wedge \cdots \wedge (C_n \rightarrow D_n) \qquad (n \geq 0)$$

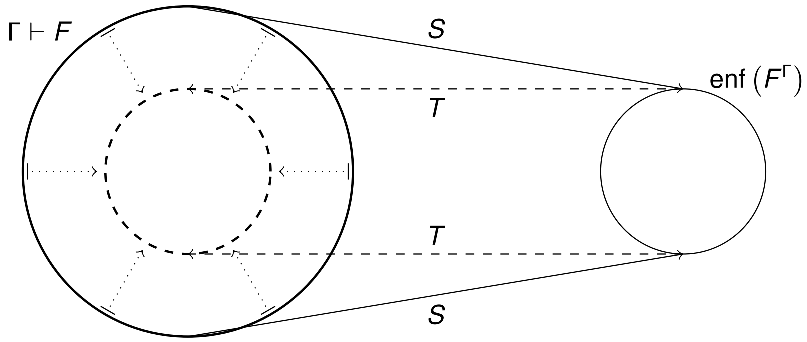
i.e.

$$\mathcal{D} \ni D ::= P \mid C_1 + \cdots + C_n \qquad (n \geq 2)$$
$$\mathcal{C} \ni C ::= D_1^{C_1} \cdots D_n^{C_n} \qquad (n \geq 0)$$

And denote by $\mathrm{enf}\,(F)$ the formula normalization operation.

Isolating a complete fragment HS of LJ

Use the same ideas as for treating λ-calculus with sums:



A complete fragment HS of LJ

Theorem

Every derivation of F in LJ can be transformed to a derivation of $\mathrm{enf}\,(F)$ *in HS.*

Proof.

- Invertible (asynchronous) rules $\frac{F}{G}$ are mapped to $\frac{\mathrm{enf}(F)}{\mathrm{enf}(G)}$ — but note that actually $\mathrm{enf}\,(F) = \mathrm{enf}\,(G)$ — **so no invertible rules are needed in the target HS!**
- Non-invertible (synchronous) rules are mapped to special forms of themselves *[next slide]*

$\square$

The rules of High-school sequent calculus (HS)

$$\frac{}{P \wedge C \to P} \qquad \frac{}{P^{PC}} \qquad P\text{-prime} \qquad \text{(init)}$$

$$\frac{C \to C_i}{C \to C_1 \vee \cdots \vee C_n} \qquad \frac{C_i^C}{(C_1 + \cdots + C_n)^C} \qquad n \geq 2 \qquad (\vee_r^i)$$

$$\frac{D_1, P, C \to D_2}{P \to D_1, P, C \to D_2} \qquad \frac{D_2^{D_1 PC}}{D_2^{D_1^P PC}} \qquad P\text{-prime} \qquad (\to_l^P)$$

$$\frac{C_1, P_1 \to P_2, C_2 \to P_1 \quad P_2, C_2 \to D}{(C_1 \to P_1) \to P_2, C_2 \to D} \qquad \frac{P_1^{C_1 P_2^{P_1} C_2} \; D^{P_2 C_2}}{D^{P_2^{P_1^{C_1}}} C_2} \qquad P\text{-prime} \qquad (\to_l^{\to})$$

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New Applications of Proof Mining to Nonlinear Analysis

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Tübingen, 29/11/2015

Funded by DFG Project KO 1737/5-2 and IRTG 1529

Origin of proof interpretations

- Hilbert's 2nd problem (1900): Is arithmetic consistent?
- Gödel (1931) : Impossible to prove the consistency of a theory $\mathcal{T}$ within $\mathcal{T}$.
- Let theories $\mathcal{T}_1$, $\mathcal{T}_2$ with languages $\mathcal{L}(\mathcal{T}_1)$, $\mathcal{L}(\mathcal{T}_2)$. $\mathcal{T}_2$ is consistent relative to $\mathcal{T}_1$ if it can be proved that if $\mathcal{T}_1$ is consistent then $\mathcal{T}_2$ is consistent.
- A theorem $\phi \in \mathcal{L}(\mathcal{T}_1)$ transformed into $\phi' \in \mathcal{L}(\mathcal{T}_2)$; the proof p of ϕ transformed in a proof p' of ϕ'. This often gives new quantitative information. Also: p' using restricted version of the assumptions of ϕ, thus proving a more general result ϕ'.
- Gödel's functional "Dialectica" Interpretation (1958): consistency of PA reduced to a quantifier-free calculus of primitive recursive functionals of finite type.
- Gödel's motivation: obtain a relative consistency proof for HA (and hence for PA).

Proof Mining

G. Kreisel (1950's): *Unwinding of proofs*

"What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"

Possible to obtain new quantitative/ qualitative information by logical analysis of proofs of statements of certain logical form. Extraction of constructive information from non-constructive proofs.
Within past $\approx$ 15 years, U. Kohlenbach *et al* have applied proof mining to : approximation theory, ergodic theory, fixed point theory, nonlinear analysis, and (recently) PDE theory.

Applications described as instances of logical phenomena by general logical metatheorems.

Herbrand normal form

- In general, for a Π_3^0 sentence, i.e. of the form

$$A \equiv \forall k \; \exists n \; \forall m \; A_0(k, n, m)$$

  where A_0 is quantifier-free, it is not possible to compute a bound on n.
- However: possible to compute a bound on n for A^H, the *Herbrand normal form* of A;

$$A^H :\equiv \forall k \; \exists n \; A_0(k, n, g(n))$$

  where g is the Herbrand index function (in theories allowing function variables and function quantifiers it would be $A^H :\equiv \forall g, k \; \exists n \; A_0(k, n, g(n))$)

Herbrand normal form - Metastability

An instance in analysis-convergence statements

- considering a statement of the form

$$\lim_{t\to\infty} P(t) = 0$$

- written as

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \geq n\ (|P(t)| < 2^{-k}),$$

- by considering the metastable version

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \in [n, n+g(n)]\ (|P(t)| < 2^{-k}),$$

  possible to find a computable bound (*rate of metastability*: a term by T. Tao) $\Phi(k, g, \cdot)$ depending on general uniform bounds on the input data, so that $n \leq \Phi(k, g, \cdot)$.

Application to the Cauchy problem generated by accretive operators

In the following X is a real Banach space with dual X^*. A mapping $A : X \to 2^X$ will be called an operator on X.

Definition

A is said to be accretive if for all $\lambda \geq 0$, $z \in Ax$, $w \in Ay$

$$\|x - y + \lambda(z - w)\| \geq \|x - y\|$$

equiv. $\langle z - w, x - y\rangle_+ \geq 0$
where: $\langle y, x\rangle_+ := \max\{\langle y, j\rangle : \ j \in J(x)\}$,

$$J(x) := \{j \in X^*; \langle x, j\rangle = \|x\|^2,\ \|j\| = \|x\|\}.$$

Application to the Cauchy problem generated by accretive operators

Definition

(J. García-Falset, 2005)[a] Let $\phi : X \to [0, \infty)$ continuous with $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ so that for every sequence (x_n) in X such that $(\|x_n\|)$ is nonincreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$. A with $0 \in Az$ is said to be *ϕ-accretive at zero* if

$$\forall (x, u) \in A \, (\langle u, x - z \rangle_+ \geq \phi(x - z)).$$

[a](García-Falset, J.: The asymptotic behavior of the solutions of the Cauchy problem generated by ϕ-accretive operators, J. Math. Anal. Appl. 310 594–608 (2005))

Application to the Cauchy problem generated by accretive operators

Preliminaries

We introduce the property of *uniform accretivity at zero* for an operator $A : D(A) \to 2^X$ with $0 \in Az$ as follows:

Definition

(Kohlenbach, K.-A., 2015) An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is called uniformly accretive at zero if

$$\forall k \in \mathbb{N} \, \forall K \in \mathbb{N}^* \, \exists m \in \mathbb{N} \, \forall (x, u) \in A$$

$$(\|x - z\| \in [2^{-k}, K] \to \langle u, x - z \rangle_+ \geq 2^{-m})(*).$$

Any function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$ is called a modulus of accretivity at zero for A if $m := \Theta_K(k)$ satisfies $(*)$.

Application to the Cauchy problem generated by accretive operators

Preliminaries

It is known that the following initial value problem (P1):

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$

$$u(0) = x$$

where $f \in L^1(0, \infty, X)$ for each $x \in \overline{D(A)}$ has a unique integral solution u so that $u(t) \in \overline{D(A)}$ for all t.
Moreover, it is known that for $x_0 \in \overline{D(A)}$ (P2):

$$u'(t) + A(u(t)) \ni 0, t \in [0, \infty)$$

$$u(0) = x_0$$

has a unique integral solution given by Crandall-Liggett :

$$u(t) := S(t)(x_0) = \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}(x_0).$$

Application to the Cauchy problem generated by accretive operators

Theorem

(J. García-Falset, 2005) Let A be an ϕ-accretive at zero operator on X so that $\forall \lambda > 0$ $(\overline{D(A)} \subset R(I + \lambda A))$. If

$$v'(t) + A(v(t)) \ni 0, t \in [0, \infty), v(0) = x_0$$

has a strong solution for each $x_0 \in D(A)$, Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty), u(0) = x$$

where $f(\cdot) \in L^1(0, \infty, X)$ converges strongly to the zero z of A as $t \to \infty$.

Application to the Cauchy problem generated by accretive operators

Theorem

(Kohlenbach, K.-A., 2015) Same as above except that A is a uniformly accretive at zero operator on X with a modulus of accretivity Θ. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty), u(0) = x$$

where $f(\cdot) \in L^1(0, \infty, X)$ satisfies

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, M, \Theta, B)$$

$$\forall t \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(t) - z\| < 2^{-k})$$

with rate of metastability

$$\Psi(k, \bar{g}, M, B, \Theta) = \tilde{g}^{(M \cdot 2^{k+1})}(0) + h(\tilde{g}^{(M \cdot 2^{k+1})}(0))$$

where

$$\tilde{g}(n) := g(n) + n,$$
$$(\ g^{(0)}(k) := k$$
$$g^{(i+1)}(k) := g(g^{(i)}(k))\ ),$$
$$g(n) := \bar{g}(n + h(n)) + h(n),$$
$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$K(n) := \ulcorner \sqrt{2(B(n) + 1)} \urcorner,$$

$B(n)$ is a nondecreasing upper bound :$B(n) \geq \frac{1}{2}\|u(n) - z\|^2$,

$$\mathbb{N} \ni M \geq I := \int_0^\infty \|f(\xi)\| d\xi.$$

One-parameter Nonexpansive Semigroups

Definition

Given a Banach space E and $C \subseteq E$, a mapping T on C is nonexpansive if $\forall x, y \in C$ $\|Tx - Ty\| \leq \|x - y\|$.

Definition

A family $\{T(t) : t \geq 0\}$ of $T(t) : C \to C$ is called a one-parameter nonexpansive semigroup on $C \subseteq E$ if :

1. for all $t \geq 0$, $T(t)$ is a nonexpansive mapping on C,
2. $T(s + t) = T(s) \circ T(t)$,
3. for each $x \in C$, the mapping $t \to T(t)x$ from $[0, \infty)$ into C is continuous.

Theorem

(Suzuki (2006))[a] *Let* $\{T(t) : t \geq 0\}$ *a nonexpansive semigroup on* $C \subseteq E$. *Let* $F(T(t))$ *the set of fixed points of* $T(t)$. *Let* $\alpha, \beta \in \mathbb{R}^+$, $0 < \alpha < \beta$, $\alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. *Let* $\lambda \in (0, 1)$. *Then :*

$$\bigcap_{t \geq 0} F(T(t)) = F(\lambda T(\alpha) + (1 - \lambda) T(\beta)).$$

[a] Suzuki, T. : *Common fixed points of one-parameter nonexpansive semigroups*, Bull. London Math. Soc. 38 1009–1018 (2006).

Main Idea

$\bigcap_{t\geq 0} F(T(t)) \subseteq F(\lambda T(\alpha) + (1-\lambda)T(\beta))$ is trivial.
We will extract a bound from (the proof of)

$$\bigcap_{t\geq 0} F(T(t)) \supseteq F(\lambda T(\alpha) + (1-\lambda)T(\beta));$$

Set

$$S := \lambda T(\alpha) + (1-\lambda)T(\beta).$$

The above gives for $q \in C$

$$Sq = q \to \forall t \geq 0 \ \ T(t)q = q$$

i.e.

$$\forall m \in \mathbb{N} \ \forall M \in \mathbb{N} \ \forall t \in [0, M] \ \exists k \in \mathbb{N}$$
$$(\|Sq - q\| \leq 2^{-k} \to \|T(t)q - q\| < 2^{-m}),$$

which is a $\forall\exists$ statement.

Main Idea

By proof mining on the proof of Suzuki's theorem (applying a tool from logic due to Kohlenbach (2005)), we will extract a computable bound $\Psi > 0$ depending on bounds on the input data so that (where for given $b \in \mathbb{N}$ let $C_b := \{q \in C : \|q\| \leq b\}$)

$$\forall b \in \mathbb{N} \ \forall q \in C_b \ \forall M \in \mathbb{N} \ \forall m \in \mathbb{N}$$

$$(\|Sq - q\| \leq \Psi(M, b, m, \ldots) \to \forall t \in [0, M] \ \|T(t)q - q\| < 2^{-m}).$$

For that we will make use of a stronger notion of uniform **equi**continuity in the following sense:

Uniform Equicontinuity for Nonexpansive Semigroups

Definition

$\{T(t) : t \geq 0\}$ on $C \subseteq E$ is uniformly equicontinuous if $t \to T(t)q$ is uniformly continuous on each $[0, K]$ for all $K \in \mathbb{N}$ with a common modulus of uniform continuity for all $q \in C_b$. i.e. if there exists $\omega : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that

$$\forall b \in \mathbb{N}\ \forall q \in C_b\ \forall m \in \mathbb{N}\ \forall K \in \mathbb{N}\ \forall t, t' \in [0, K]$$

$$(|t - t'| < 2^{-\omega_{K,b}(m)} \to \|T(t)q - T(t')q\| < 2^{-m}).$$

We call ω a modulus of uniform equicontinuity for $\{T(t) : t \geq 0\}$.

'Quantifying' Irrationality

Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Then

$$\forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ \exists z \in \mathbb{N}\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{z}).$$

The above gives rise to the following definition:

Definition

Let $\gamma \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. The function f_γ as in

$$\exists f_\gamma : \mathbb{N} \to \mathbb{N}\ \forall p \in \mathbb{N}\ \forall p' \in \mathbb{Z}^+\ (|\gamma - \frac{p'}{p}| \geq \frac{1}{f_\gamma(p)})$$

is called an *effective irrationality measure* for γ.

Theorem

(Kohlenbach, K.-A.(2015))
In addition to Suzuki's assumptions, assume that $\{T(t) : t \geq 0\}$ *is uniformly equicontinuous with a modulus* ω. *Let* f_γ *be the effective irrationality measure for* γ, $\Lambda \in \mathbb{N}$ *so that* $1/\Lambda \leq \lambda, 1-\lambda$, $N \in \mathbb{N}$ *so that* $\beta \geq 1/N$, $\mathbb{N} \ni D \geq \beta$. *Then*

$$\forall b \in \mathbb{N}\ \forall M \in \mathbb{N}\ \forall q \in C_b\ \forall m \in \mathbb{N}$$

$$(\|Sq - q\| \leq \Psi \rightarrow \forall t \in [0, M]\ \|T(t)q - q\| < 2^{-m})$$

Theorem

with

$$\Psi(m, M, N, \Lambda, D, b, f_\gamma, \omega) = \frac{2^{-m}}{8(\sum_{i=1}^{\phi(k,f_\gamma)-1} \Lambda^i + 1)(1 + MN)}$$

where

$$k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1} \in \mathbb{N}$$

and

$$\phi(k, f) := \max\{2f(i-j) + 6\ :\ 0 \leq j < i \leq k+1\} \in \mathbb{N}.$$

THANK YOU !

References:

- Kohlenbach, U. and Koutsoukou-Argyraki, A.: *Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators*, J. Math. Anal. Appl. vol. 423, 1089–1112 (2015).
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APPENDIX: Mathematical Definitions

Definition

A continuous function $u : [0, \infty) \to X$ is said to be a *strong solution* of (P2) if it is Lipschitz on every bounded subinterval of $[0, \infty)$, a.e. differentiable on $[0, \infty)$, $u(t) \in D(A)$ a.e., $u(0) = x_0$ and $u'(t) + A(u(t)) \ni 0$ for almost every $t \in [0, \infty)$.

Definition

A continuous function $u : [0, \infty) \to X$ is an *integral solution* of (P1) if $u(0) = x$ and for $s \in [0, t]$ and $(w, y) \in A$

$$\|u(t) - w\|^2 - \|u(s) - w\|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - w\rangle_+ d\tau.$$

On the maximality of classical logic

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General Proof Theory Conference
Tübingen, November 27-29, 2015

INTRODUCTION

General purpose

- *Nonmonotonic logics*
  - D. Makinson. Bridges from classical to nonmonotonic logics. College Publications, 2005.
  - Bridges = deductive extensions of propositional classical logic = supraclassical logics
  - **Goal:** give a proof-theoretic 'dignity' to nonmonotonic logics (cut-elimination $\Rightarrow$ subformula)
- *same extend-to-constrain strategy for* paraconsistency
- *proof-theoretical treatment of proper, extra-logical, axioms*:
  - proper axioms may hamper standard cut elimination procedures (Girard, 1987)
  - **Goal:** define a proof theory for supraclassical logics in a way to get both **cut elimination** and the **subformula** property

Logic of pivotal assumptions

- Fix a set of formulas $\Psi = \{\alpha_1, \ldots, \alpha_n\}$
- extend the classical consequence relation $\vdash$ to $\vdash_\Psi$ as follows:

$$\Gamma \vdash_\Psi \beta \Leftrightarrow \Gamma, \Psi \vdash \beta$$

- $\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n} = \mathsf{LK}_0 + \alpha_1 + \ldots + \alpha_n$ added as **new axioms**
- clearly: $\vdash_\Psi \; = \; \vdash_{\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n}}$

Remark

$\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n} \approx \mathsf{LK}_0^{\alpha_1\wedge\ldots\wedge\alpha_n}$, so we can reduce to one-axiom extensions without loss of generality

Post-completeness

Post-completeness

LK_0 is maximal w.r.t. the set of its theorems:

- given any *proper* axiomatic extension, **structurality** (i.e., provability is preserved by uniform substitution) and **consistency** are mutually excluding properties
- any axiom α **properly** extending (without trivializing) LK_0 must be:

  non-logical = proper = *not* closed under uniform substitution

Example

1. p := 'Bob is an inhabitant of Flatland'
2. q := 'Bob is either a polygon or a circle'
3. given the extra-logical information provided by the book *Flatland*, you have: $p \to q$

Remark

- In LK_0, any atom is a variable
- In $\mathsf{LK}_0^{p\to q}$, p and q may occur as **constants**, as **names** to all intents and purposes
- lack of structurality $\Rightarrow$ linguistic commitment (points (1) and (2))

Cut-elimination

Proper axioms, once added to LK_0, may hamper cut-elimination (even at the very propositional level!)

Example

Consider the extension $LK_0^{p \to q}$:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,ax. \qquad \dfrac{}{q \vdash q}\,ax.}{p, p \to q \vdash q}\,{\to}{\vdash} \qquad \dfrac{}{\vdash p \to q}\,\textit{proper ax.}}{p \vdash q}\,\textit{cut}$$

Clearly, $p \vdash q$ is *not* derivable in $LK_0^{p \to q}$ without resorting to cut applications

Related works

Negri & von Plato, Cut Elim. in the Presence of Axioms, BSL '98

- two ways of decomposing a formula α into a set of 'elementary' sequents:
  1. by doing proof-search on α within a suitable sequent system for LK_0 (logical rules are reversible + no structural rules)
  2. by stressing the notion of *conjunctive normal form*

$\Rightarrow$ we stress this second kind of decomposition

- Negri & von Plato are mainly interested in axiomatic theories, so they focus on first order logic

$\Rightarrow$ we are interested in propositional extensions

- proper axioms $\rightsquigarrow$ inference rules: cut-elimination is preserved, but it does not necessarily implies the subformula property

$\Rightarrow$ decompostions are deductively closed under the cut rule and it suffices to get both cut-elimination and the subformula property

Agenda

1. For any proper axiom α, there is a decompostion $\mathscr{S}_\alpha$ such that:
   - $\mathsf{LK}_0^\alpha \approx \mathsf{LK}_0^{\mathscr{S}_\alpha}$
   - $\mathsf{LK}_0^{\mathscr{S}_\alpha}$ enjoys both cut-elimination and the subformula property.
2. fully syntactical decision procedure for the consistency problem of supraclassical systems
3. uniqueness: given any proper axiom α, there is exactly one axiomatic (i.e. minimal) decomposition, which allows for cut elimination
4. final philosophical discussion: *which is the right way to extend classical propositional logic?*

DECOMPOSITIONS

Sequent system LK_0

$$\dfrac{}{\alpha \vdash \alpha}\ ax. \qquad \dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \alpha \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}\ weak. \vdash \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha, \Delta} \vdash weak.$$

$$\dfrac{\Gamma \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg \vdash \qquad \dfrac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta} \vdash \neg$$

$$\dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge \vdash \qquad \dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma' \vdash \beta, \Delta'}{\Gamma, \Gamma' \vdash \alpha \wedge \beta, \Delta, \Delta'} \vdash \wedge$$

$$\dfrac{\Gamma, \alpha \vdash \Delta \qquad \Gamma', \beta \vdash \Delta'}{\Gamma, \Gamma', \alpha \vee \beta \vdash \Delta, \Delta'}\ \vee \vdash \qquad \dfrac{\Gamma \vdash \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \vdash \vee$$

$$\dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \beta \vdash \Delta'}{\Gamma, \Gamma', \alpha \to \beta \vdash \Delta, \Delta'} \to\vdash \qquad \dfrac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} \vdash\to$$

Complementary sequents

Definition

A sequent $\Gamma \vdash \Delta$ is *complementary* if:

- Γ and Δ display only atoms
- $\Gamma \cap \Delta = \varnothing$

Example

$p, q \vdash r$ is complementary

Remark

- complementary sequents are classically unprovable
- complementary sequents are called **basic** by Gentzen and **regular** by Negri&von Plato
- complementary sequents are the axioms of complementary classical logic

Decomposition

Notation

- $\mathscr{S}, \mathscr{T}, \ldots$ sets of sequents
- $\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n}$ system obtained from LK_0 by adding $\alpha_1, \ldots, \alpha_n$ as new axioms
- $\mathsf{LK}_0^{\mathscr{S}}$ system obtained from LK_0 by adding the sequents in $\mathscr{S}$ as new axioms

Definition

A set $\mathscr{S}$ is a *decomposition* for a formula α in case:

- all the sequents in $\mathscr{S}$ are complementary
- $\mathsf{LK}_0^{\mathscr{S}} \approx \mathsf{LK}_0^{\alpha}$

Decomposition procedure

1. $\alpha \rightsquigarrow \texttt{cnf}(\alpha) = \alpha_1 \wedge \ldots \wedge \alpha_n$
2. Build a new sequent for each one of the clauses of $\texttt{cnf}(\alpha)$:
$$\alpha_i = \ell_1 \vee \ldots \vee \ell_k \Rightarrow \vdash \ell_1, \ldots, \ell_k.$$
3. Shift negative literals on the left and remove negations:
$$\vdash p_1, \ldots, p_n, \neg q_1, \ldots, \neg q_m \Rightarrow \; q_1, \ldots, q_m \vdash p_1, \ldots, p_n.$$
4. Remove identity sequents, i.e. sequents $\Gamma \vdash \Delta$ such that $\Gamma \cap \Delta \neq \varnothing$.

Example (step 1)

- $\alpha = ((p_0 \leftrightarrow p_1) \rightarrow (p_2 \rightarrow p_3)) \wedge (p_1 \rightarrow \neg(p_0 \wedge p_2))$

$$\Downarrow$$

- **step 1**: $\texttt{cnf}(\alpha) = (p_0 \vee p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge$
$\wedge (p_0 \vee \neg p_0 \vee \neg p_2 \vee p_3) \wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge$
$\wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2)$

Example (step 2)

- **step 1**: $\texttt{cnf}(\alpha) = (p_0 \vee p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge$
$\wedge (p_0 \vee \neg p_0 \vee \neg p_2 \vee p_3) \wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge$
$\wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2)$

$$\Downarrow$$

- **step 2**: $\{ \vdash p_0 \vee p_1 \vee \neg p_2 \vee p_3$
$\vdash p_1 \vee \neg p_1 \vee \neg p_2 \vee p_3$
$\vdash p_0 \vee \neg p_0 \vee \neg p_2 \vee p_3$
$\vdash \neg p_0 \vee \neg p_1 \vee \neg p_2 \vee p_3$
$\vdash \neg p_0 \vee \neg p_1 \vee \neg p_2 \}$.

Example (step 3)

- **step 2**: $\{\vdash p_0 \vee p_1 \vee \neg p_2 \vee p_3$
  $\vdash p_1 \vee \neg p_1 \vee \neg p_2 \vee p_3$
  $\vdash p_0 \vee \neg p_0 \vee \neg p_2 \vee p_3$
  $\vdash \neg p_0 \vee \neg p_1 \vee \neg p_2 \vee p_3$
  $\vdash \neg p_0 \vee \neg p_1 \vee \neg p_2\}.$

$$\Downarrow$$

- **step 3**: $\{\vdash p_0, p_1, \neg p_2, p_3$
  $\vdash p_1, \neg p_1, \neg p_2, p_3$
  $\vdash p_0, \neg p_0, \neg p_2, p_3$
  $\vdash \neg p_0, \neg p_1, \neg p_2, p_3$
  $\vdash \neg p_0, \neg p_1, \neg p_2\}.$

Example (step 4)

- **step 3**: $\{\vdash p_0, p_1, \neg p_2, p_3$
  $\vdash p_1, \neg p_1, \neg p_2, p_3$
  $\vdash p_0, \neg p_0, \neg p_2, p_3$
  $\vdash \neg p_0, \neg p_1, \neg p_2, p_3$
  $\vdash \neg p_0, \neg p_1, \neg p_2\}.$

$$\Downarrow$$

- **step 4**: $\{ p_2 \vdash p_0, p_1, p_3$
  $p_1, p_2 \vdash p_1, p_3$
  $p_0, p_2 \vdash p_0, p_3$
  $p_0, p_1, p_2 \vdash p_3$
  $p_0, p_1, p_2 \vdash \}.$

Example (step 5)

- **step 4**: $\left\{ \begin{array}{l} p_2 \vdash p_0, p_1, p_3 \\ p_1, p_2 \vdash p_1, p_3 \\ p_0, p_2 \vdash p_0, p_3 \\ p_0, p_1, p_2 \vdash p_3 \\ p_0, p_1, p_2 \vdash \end{array} \right\}$

$$\Downarrow$$

- **step 5**: $\left\{ \begin{array}{l} p_2 \vdash p_0, p_1, p_3 \\ p_0, p_1, p_2 \vdash p_3 \\ p_0, p_1, p_2 \vdash \end{array} \right\}$

CUT-ELIMINATION & SUBFORMULA

Closure under cut

Definition

- $\mathscr{S}$ is a set of complementary sequents
- its *closure under cut* $\mathscr{S}^*$ is the **smallest supraset** $\mathscr{S}^*$ of $\mathscr{S}$ such that:
  1. $\Gamma \vdash \Delta, p \in \mathscr{S}^*$ and $\Gamma', p \vdash \Delta' \in \mathscr{S}^*$ are both in $\mathscr{S}^*$
  2. $(\Gamma \cup \Gamma') \cap (\Delta \cup \Delta') = \varnothing$,

  $\Rightarrow$ $\Gamma, \Gamma' \vdash \Delta, \Delta' \in \mathscr{S}^*$

Example

$$\mathscr{S} = \{p \vdash q \,;\; q \vdash p \,;\; q \vdash r\}$$
$$\Downarrow *$$
$$\mathscr{S}^* = \{p \vdash q \,;\; q \vdash p \,;\; q \vdash r \,;\; p \vdash r\}.$$

Hauptsatz

Hauptsatz

Let LK_{0^-} be LK_0 *without* the cut rule: for any α, $\mathrm{LK}_{0^-}^{\mathscr{S}^*_\alpha} \approx \mathrm{LK}_0^{\mathscr{S}^*_\alpha}$.

Key case

$\Gamma, \Gamma' \cap \Delta, \Delta' = \varnothing$:

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, p}\,p.ax. \qquad \dfrac{}{\Gamma', p \vdash \Delta'}\,p.ax.}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \longrightarrow \quad \dfrac{}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,p.ax.$$

Remark

We stress here the fact that, if $\Gamma \vdash \Delta, p \in \mathscr{S}^*$ and $\Gamma', p \vdash \Delta' \in \mathscr{S}^*$, then $\Gamma, \Gamma' \vdash \Delta, \Delta' \in \mathscr{S}^*$.

Decision of consistency

Consistency problem

Given a formula α: is LK_0^{α} consistent?

Semantics

Equivalent to check whether α is a contradiction.

Syntax (corollary to the subformula property)

Equivalent to check whether $\vdash \ \in \mathscr{S}^*_\alpha$

Example

$$\mathscr{S}_{p\wedge\neg p} = \{\vdash p\ ;\ p\vdash\}$$
$$\Downarrow *$$
$$\mathscr{S}^*_{p\wedge\neg p} = \{\vdash p\ ;\ p\vdash\ ;\ \vdash\}$$
$$\Downarrow$$
$$\mathsf{LK}_0^{p\wedge\neg p} \text{ is inconsistent}$$

AXIOMATICITY & UNIQUENESS

Axiomatic decomposition

Definition

A set of complementary sequents $\mathscr{T}$ is said to be *axiomatic* in case:

- $\mathsf{LK}_0^{\mathscr{T}} \approx \mathsf{LK}_{0-}^{\mathscr{T}}$
- no sequent in $\mathscr{T}$ can be derived from the others within LK_{0-}

Remark

An axiomatic set $\mathscr{T}$ is a minimal set allowing for cut-elimination.

Reduct under weakening

Definition

Given a set of complementary sequents $\mathscr{T}$ its reduct under weakening is the **largest subset** $\mathscr{T}^{\star}$ of $\mathscr{T}$ such that,

- if $\Gamma \vdash \Delta \in \mathscr{T}$ and $\Gamma, \Gamma' \vdash \Delta, \Delta' \in \mathscr{T}$
- then $\Gamma, \Gamma' \vdash \Delta, \Delta' \notin \mathscr{T}^{\star}$.

Example

$$\mathscr{T} = \{p_2 \vdash p_0, p_1, p_3 \ ; \ p_0, p_1, p_2 \vdash p_3 \ ; \ p_0, p_1, p_2 \vdash\}$$
$$\Downarrow \star$$
$$\mathscr{T}^{\star} = \{p_2 \vdash p_0, p_1, p_3 \ ; \ p_0, p_1, p_2 \vdash\}.$$

Theorem

If α is a contradiction, then $\mathscr{S}_{\alpha}^{\star\star} = \{ \vdash \}$

Uniqueness

Theorem
If both $\mathscr{S}$ and $\mathscr{T}$ are axiomatic and $\mathsf{LK}_0^{\mathscr{S}} \approx \mathsf{LK}_0^{\mathscr{T}}$, *then* $\mathscr{S} = \mathscr{T}$.

Theorem
For any α, $\mathscr{S}_\alpha^{**}$ is an axiomatic decomposition for α.

Conclude:

- for any α, there is exactly one axiomatic decomposition
- such a decomposition can be achieved by computing:

$$\alpha \rightsquigarrow \mathscr{S}_\alpha \rightsquigarrow \mathscr{S}_\alpha^{*} \rightsquigarrow \mathscr{S}_\alpha^{**}$$

Circularity?

- Decompositions are closed under cut in order to get cut elimination:

  is this move circular?

- **NO!** we close under cut a set of complementary sequents $\mathscr{S}$ in order to prove cut elimination for the **whole system** $\mathsf{LK}_0^{\mathscr{S}}$

  this gap needs to be filled by a **proof** that conveys the information saying that cuts can be pushed upwards

Naturalness?

Girard, *Proof Theory and Logical Complexity*, 1987

"If the Post system is closed under cut, then it will be easy to see that the calculus enjoys the *Hauptsatz*. [...] We have not chosen this possibility because

(*i*) the pleasure of being able to state a full cut-elimination theorem is spoiled by the artificial character of the axioms one has to consider,

(*ii*) more essentially, we do not see the mathematical gain in this change."

Remark

$\alpha \leftrightarrow \beta \Leftrightarrow \mathscr{S}^{\star\star}_{\alpha} = \mathscr{S}^{\star\star}_{\beta}$

Reply to point (*i*)

Girard does not seem to consider the possibility to close **decompositions**, instead of **proper axioms**, under cut.

- uniqueness $\Rightarrow$ decompositions are not arbitrary syntactical devices
- decomposition allows for the removal of the redundant 'information' contained in proper axioms
- decomposition as a way to **analyse** the **hidden content** of proper axioms

Example

- Suppose you want to admit abduction for two atoms p and q:

$$((p \to q) \land q) \to p$$

- $\mathscr{S}^{**}_{((p \to q) \land q) \to p} = \{q \vdash p\}$

hence:

axiomatic accounts of abduction are ‘empty’: allowing abduction on p and q means imposing their equivalence $p \leftrightarrow q$

FUTURE WORK

Nonmonotonic logics

- go a step further into nonmonotonic logics:
  apply this kind of proof-theoretic achievements so as to offer a well-behaved proof theory for an as wide as possible range of nonmonotonic systems
- logics of default assumptions

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Idempotent Coherentisation for First-Order Logic

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General Proof Theory Conference
University of Tübingen
27–29 November, 2015

Preamble

Gentzen systems, both sequent calculus and natural deduction, are "natural", in the sense that proofs use logical constants in a simple (syntax-directed) way. By contrast, Hilbert systems and (from a different world) Resolution are "unnatural".

An extension of Gentzen's ideas is the notion of "coherent" (aka "geometric") logic and the associated notion of "dynamical proof", as developed and popularised by various authors (Joyal, Reyes, Simpson, Negri, Lombardi, Bezem, Coquand). In such proofs, logical constants are invisible—their effect has disappeared into the notation.

Not all (first-order) theories are coherent. A folklore result from the 1970s shows that every theory has, by use of extra relation symbols, a coherent conservative relational extension, i.e. a theory can be "coherentised". Algorithms to do this tend to use "atomisation" (every formula becomes equivalent to an atom) or preprocessing to PNF (then CNF or DNF) or NNF, generating many new axioms.

We'll recall some of the history of this result and present a new coherentisation algorithm with the virtue of being "idempotent".

Coherent and Geometric Implications

A formula is positive, *aka "coherent"*, iff built from atoms (e.g. $\top, \bot, t = t', t \leqslant t', P(\mathbf{t}), \dots$) using only $\vee$, $\wedge$ and $\exists$.
Warning: model theory also allows $\forall$.

A sentence is a coherent implication (CI) iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are positive. [Neither coherent nor an implication . . . .]

A sentence is a special coherent implication (SCI) iff of the form $\forall \mathbf{x}.C \supset D$ where C is a conjunction of atoms and D is a finite disjunction of existentially quantified conjunctions of atoms.

Some restrict the notion of "coherent implication" to mean an SCI.

Old Theorem: Any coherent implication is intuitionistically equivalent to a finite conjunction of SCIs.

A formula is geometric iff built from atoms (as before) using only $\vee$, $\wedge$, $\exists$ and **infinitary** disjunctions. A sentence is a geometric implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are geometric. Similar terminology (SGI) and result for the infinitary (geometric) case.

Examples

Universal formulae $\forall \mathbf{x}.A$ (where A is quantifier-free) are equivalent to finite conjunctions of SCIs, just by putting A into CNF, distributing $\forall$ past $\wedge$ and rewriting (e.g. $\neg P \vee Q$ as $P \supset Q$). (No $\exists$ is involved. $\top$ and $\bot$ may be useful.)

Theory of *fields* is axiomatised by SCIs, including $\forall x.\top \supset (x = 0 \vee \exists y.xy = 1)$.

Theory of *real-closed fields* is axiomatised by countably many SCIs, including $\forall \mathbf{a}.\top \supset (a_{2n+1} = 0 \vee \exists x.a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \dots + a_0 = 0)$.

Theory of *local rings* (rings with just one maximal ideal) is axiomatised by SCIs, including $\forall x.\top \supset (\exists y(xy = 1) \vee \exists y((1 - x)y = 1))$.

Theory of *transitive relations* is axiomatised by SCI: $\forall xyz.(Rxy \wedge Ryz) \supset Rxz$.

Theory of *partial order* is axiomatised by SCIs, e.g. $\forall xy.(x \leqslant y \wedge y \leqslant x) \supset x = y$.

Theory of *strongly directed relations* is axiomatised by SCI: $\forall xyz.(Rxy \wedge Rxz) \supset \exists u.Ryu \wedge Rzu$.

(Infinitary) theory of *torsion abelian groups* is axiomatised by SGIs, including $\forall x.\top \supset \bigvee_{n>0}(nx = 0)$. [$nx$ stands for "sum of n copies of x"].

(Infinitary) theory of *fields of non-zero characteristic* is axiomatised by SGIs, including $\forall x.\top \supset \bigvee_{p>0}(px = 0)$.

Theory about [and advantages of] Coherent Theories

A *coherent theory* is one axiomatised by [special] coherent implications. "Geometric" and "coherent" are often used synonymously.

1. *"Barr's Theorem":* Coherent implications form a "Glivenko Class", i.e. if a sequent $I_1, \ldots, I_n \Rightarrow I_0$ is classically provable, then it is intuitionistically provable, provided each I_i is a coherent implication.

2. Coherent theories are those whose class of models is closed under filtered co-limits (calculated in **Set**) (Keisler (1960).

3. Coherent theories are "exactly the theories expressible by natural deduction rules in a certain simple form in which only atomic formulas play a critical part" (Simpson 1994).

4. Similarly, in a sequent calculus context, SCIs can be converted directly to inference rules (using and generating only atomic formulas) so that admissibility of the structural rules is unaffected (Negri 2003).

Conversion of Coherent Implications to Rules

When adding axioms to a first-order theory, one option for formalisation (in a sequent calculus such as **G3c**) is to include them all in the antecedents of all sequents.

But **if they are SCIs** it is more elegant to convert them to inference rules (Negri 2003); Simpson (1994).

We'll sometimes write an SCI without universal quantifiers; free variables are then schematic, i.e. instantiable as any terms we like.

Such an axiom $(P_1(\mathbf{x}) \wedge P_2(\mathbf{x}) \wedge \cdots \wedge P_n(\mathbf{x})) \supset D(\mathbf{x})$ is then converted to the rule

$$\frac{D(\mathbf{t}), P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t}), \Gamma \Rightarrow \Delta}{P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t}), \Gamma \Rightarrow \Delta}$$

in which the atoms $P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t})$ are atoms in the conclusion's antecedent; the instance $D(\mathbf{t})$ can then (as we grow the derivation in a root-first fashion) be added to the antecedent.

Better still, analyse $D(\mathbf{t})$ immediately, using branching for analysis of $\vee$, fresh variables for analysis of $\exists$ and commas for analysis of $\wedge$.

Other Kinds of Sentences and Theories

Not all sentences, and not all theories, are coherent implications, or even equivalent to (resp. axiomatised by) a set of such things.

- The "McKinsey condition" (a frame condition for modal logic, related to the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$)

$$\forall x \exists y.\ xRy \wedge (\forall z.\ yRz \supset y = z)$$

  is not a coherent implication. [We can't shift the $\forall z$ out past $\exists y$.]
- The "strict seriality condition"

$$\forall x.\ \exists y.\ xRy \wedge \neg(yRx)$$

  is likewise not a coherent implication, because of the negation.
- The axioms for Henselian local rings provide (so far as I know) another example of an axiom that is almost a coherent implication.
- The frame condition for Kreisel-Putnam logic (see later) is almost a coherent implication.

Coherentisation Theorem

Theorem. Every first-order theory has a coherent conservative extension.

We'll look at two variants of this theorem, and at several methods of proof.

Proof. First, for variant 1, using "functional Skolemisation" (Skolem (1928)), i.e. by adding new function symbols, we can replace each axiom by a $\forall$-sentence, and put the body in CNF. Each conjunct is a clause, easily representable as an SCI. We'll call this a "functional extension". Well known to be "equi-satisfiable"; and in fact a conservative extension. $\Box$

We neglect this approach as (i) destroying the formula structure and (ii) introducing function symbols that, regrettably, ensure the Herbrand universe is infinite.

Remaining approaches (for variant 2) introduce no new function symbols but just new relation symbols: a "relational extension", using "relational Skolemisation" (Skolem (1920)).

Atomisation

Theorem. Every first-order theory has a coherent conservative relational extension.

Proof. We use the technique of "atomisation" (aka "Morleyisation", aka "Wajsberg's depth-reduction technique from 30's"). We introduce new relation symbols and axioms in such a way that each subformula of each axiom is made to be equivalent to an atomic formula.

For example, if $\psi(\mathbf{x})$ is a subformula of an axiom and is of the form $\forall \mathbf{y}.\phi(\mathbf{x}, \mathbf{y})$, we introduce new relation symbols R_ψ and R_ϕ and the (almost "coherent") axioms $\forall \mathbf{x}.R_\psi(\mathbf{x}) \supset \forall \mathbf{y}.R_\phi(\mathbf{x}, \mathbf{y})$ and $\forall \mathbf{x}.\ R_\psi(\mathbf{x}) \vee \exists \mathbf{y}.\neg R_\phi(\mathbf{x}, \mathbf{y})$. The first is easily dealt with by moving $\forall \mathbf{y}$ outwards; the second needs an extra trick to get rid of the negation. $\square$

We neglect this approach as (i) concealing the formula structure and (ii) introducing too many axioms, albeit very simple ones.

Essentially this approach is used in the "Structural Clause Form Transformation", of Baaz et al (1994) (c.f. also Egly et al (2000)) allowing resolution to simulate **LK** or tableaux. In that context there can undoubtedly be efficiency advantages.

Polarised Atomisation

Theorem. Every first-order theory has a coherent conservative relational extension.

Proof. (Johnstone (2002); Bezem and Coquand (2003).) We use a technique based on atomisation, but with **two** relation symbols per sub-formula. We introduce new relation symbols and axioms in such a way that, according to its polarity, each subformula of each axiom is now implied by or implies an atomic formula.

For example, if $\psi(\mathbf{x}, \mathbf{y})$ is a positively occurring subformula of an axiom and is of the form $\forall \mathbf{x}.\phi(\mathbf{x}, \mathbf{y})$, we introduce new relation symbols R^+_ψ and R^+_ϕ and the (almost "coherent") axiom $\forall \mathbf{y}.\ R^+_\psi(\mathbf{y}) \supset \forall \mathbf{x}.R^+_\phi(\mathbf{x}, \mathbf{y})$.

Formula is made into an SCI by shifting $\forall \mathbf{x}$ outwards.

Similarly, if $\psi(\mathbf{x})$ is $\neg\phi(\mathbf{x})$, we get, according to polarity, one of $\forall \mathbf{x}.\ R^+_\psi(\mathbf{x}) \supset R^-_\phi(\mathbf{x})$ and $\forall \mathbf{x}.\ R^-_\psi(\mathbf{x}) \supset R^+_\phi(\mathbf{x})$.

$\square$

We neglect this approach as (i) concealing the formula structure and (ii) introducing too many axioms, albeit very simple ones. Johnstone does it for **every** formula of the language.

Antonius' Translation

Antonius (1975) defined, for each formula A of a first-order language $\mathcal{L}$, a positive formula $\overline{A}$ in an extended language $\mathcal{L}'$, approximately thus:

$$\begin{array}{rcl}
\overline{P(\mathbf{x})} & = & P(\mathbf{x}) \\
\overline{B \wedge C} & = & \overline{B} \wedge \overline{C} \\
\overline{B \vee C} & = & \overline{B} \vee \overline{C} \\
\overline{\exists \mathbf{x} B} & = & \exists \mathbf{x} \overline{B} \\
\overline{B \supset C} & = & \overline{\neg B} \vee \overline{C} \\
\overline{\forall \mathbf{x} B} & = & \overline{\neg \exists \mathbf{x} \neg B} \\
\overline{\neg B(\mathbf{x})} & = & N_{B(\mathbf{x})}(\mathbf{x})
\end{array}$$

where $P(\mathbf{x})$ is atomic and $N_{B(\mathbf{x})}$ is a new relation symbol whose arity is the number of free variables $\mathbf{x}$ of $B(\mathbf{x})$.

Let $\mathcal{T}$ be a theory, $\mathcal{T}'$ the theory got by replacing each axiom A of $\mathcal{T}$ by $\top \supset \overline{A}$, and $\overline{\mathcal{T}}$ the theory got by adding to $\mathcal{T}'$ just the coherent implications $\forall \mathbf{x}. \top \supset (N_{B(\mathbf{x})}(\mathbf{x}) \vee \overline{B(\mathbf{x})})$ and $\forall \mathbf{x}. (N_{B(\mathbf{x})}(\mathbf{x}) \wedge \overline{B(\mathbf{x})}) \supset \bot$ for the negated subformulae $\neg B(\mathbf{x})$ looked at in the analysis of A.

Antonius' Translation, Contd

Theorem [Antonius]. A theory $\mathcal{T}$ proves a sequent $B \Rightarrow B'$ (implicitly universally quantified) iff $\overline{\mathcal{T}}$ proves the sequent $\overline{B} \Rightarrow \overline{B'}$, where the latter form of proof is in a sequent calculus restricted to coherent sequents.

We can strengthen this as follows:

Theorem [RD & SN]. Let A be a sentence, axiomatising the theory $\mathcal{T}$; then the theory $\overline{\mathcal{T}}$, axiomatised by $\top \supset \overline{A}$ and the CIs generated from A by Antonius' method, is a coherent conservative relational extension of $\mathcal{T}$.

Note that there are, in comparison to the atomisation techniques, relatively few new relation symbols.

A variant on Skolem (1920)

Another approach is to construct the PNF of the axiom, and put the body into CNF or DNF. Skolem (1920) showed (in effect) that every first-order sentence can be replaced by a $\forall\exists$-sentence (its "normal form") giving a conservative relational extension.

Pairs of quantifier blocks are eliminated by use of one trick; negation symbols are eliminated by another. Without bothering with PNF or CNF, we illustrate the main idea with examples:

- For the McKinsey axiom $\forall x \exists y.\ xRy \wedge (\forall z.\ yRz \supset y = z)$, introduce a new unary predicate symbol M (for *Maximal*, with $M(y)$ "meaning" $\forall z.\ yRz \supset y = z$), and two SCIs:

$$\forall yz.\ (M(y) \wedge yRz) \supset y = z$$

$$\forall x.\ \top \supset (\exists y.\ xRy \wedge M(y))$$

- For the strict seriality axiom, $\forall x.\exists y.\ xRy \wedge \neg(yRx)$ introduce a new binary predicate symbol S, with xSy "meaning" $\neg xRy$ and two SCIs:

$$\forall xy.\ (xRy \wedge xSy) \supset \bot$$

$$\forall x.\top \supset \exists y.xRy \wedge ySx$$

New Coherentisation Algorithm

Existing published algorithms to convert a sentence into a finite set of SCIs may not just generate lots of new relation symbols, but also fail to be *idempotent*, i.e. to leave the SCIs unchanged, since conversion to PNF and then to CNF or DNF, or to NNF, can destroy too much of the sentence's structure. Can we do any better?

Definition [RD & SN]. A formula is *weakly positive* iff the only occurrences of $\forall$, $\supset$ and $\neg$ are strictly positive, i.e. not within a negation or the antecedent of an implication.

Proposition [RD & SN]. A formula is weakly positive iff the only occurrences of $\forall$, $\supset$ and $\neg$ are positive, i.e. within the scope of an even number of negations and antecedents of implications.

Examples. Positive formulae (i.e. having **no** such occurrences); CIs; Negation normal formulae; the McKinsey condition; the strict seriality condition; the frame condition for Kreisel-Putnam logic (see later).

New Coherentisation Algorithm: Weak Positivisation

Every first-order formula can be converted to a classically equivalent weakly positive formula as follows. Note that conversion to NNF would suffice, but may change too much of the formula.

First, we treat negations as implications. Second, if the formula is an implication with positive antecedent we leave the antecedent unchanged (and, recursively, we convert the succedent). Then (with x not free in B) we can use the classical equivalences

$$
\begin{aligned}
(C \supset \bot) \supset B &\equiv C \vee B && (1)\\
(C \supset D) \supset B &\equiv (C \wedge \neg D) \vee B && (2)\\
\forall x A \supset B &\equiv \exists x.\ A \supset B && (3)
\end{aligned}
$$

and the intuitionistic equivalences

$$
\begin{aligned}
(C \vee D) \supset B &\equiv (C \supset B) \wedge (D \supset B) && (4)\\
(C \wedge D) \supset B &\equiv C \supset (D \supset B) && (5)\\
\exists x A \supset B &\equiv \forall x. A \supset B && (6)
\end{aligned}
$$

New Coherentisation Algorithm: Analysis of W.P. Formula

Proposition [RD & SN]. Every weakly positive formula A **is** either

- an atom or
- a universally quantified implication $\forall \mathbf{x}.\ C \supset D$ (with C positive) and D a disjunction of zero or more existentially quantified conjunctions of zero or more of the following:
  - atoms
  - weakly positive formulae A_i simpler than A.

We allow empty quantification; we consider negations to be implications and D to be the same as $\top \supset D$.

Proof. By analysis of the structure of A. We also allow trivial disjunctions (at most one disjunct) and trivial conjunctions (at most one conjunct)—but, to avoid an infinite recursion, some step of analysing A, if non-atomic, must be non-trivial. □

New Coherentisation Algorithm: Constructing CIs

Corollary [RD & SN]. Given a weakly positive formula A, we can introduce fresh relation symbols and "semi-definitional implications" so that A is simplified to a CI and the new implications are CIs, making a coherent theory conservative over A.

Proof. By induction on the structure of A. If, when analysing a conjunction, we meet a subformula $C(\mathbf{x})$ other than an an atom or a conjunction $A_1 \wedge A_2$, we introduce in its place a fresh relation symbol N_i (with appropriate arguments) and a "semi-definitional implication" $\forall \mathbf{x}.\, N_i(\mathbf{x}) \supset C(\mathbf{x})$ (which we may need to analyse further).

Any universal quantifiers or implications (with positive antecedents) at the front of $C(\mathbf{x})$ can easily be shifted, so we get a formula of the form $\forall \mathbf{xy}.\, N_i(\mathbf{x}) \wedge P(\mathbf{x}, \mathbf{y}) \supset B(\mathbf{x}, \mathbf{y})$, where $P(\mathbf{x}, \mathbf{y})$ is positive and $B(\mathbf{x}, \mathbf{y})$ is weakly positive and smaller than A.

After a finite number of steps we have replaced A by a finite number of CIs axiomatising a theory conservative over the theory with axiom A. $\square$

New Coherentisation Algorithm: Constructing SCIs

We can (intuitionistically) convert the coherent implications (the CIs) to SCIs by methods discussed earlier.

We can reduce the number of steps that we go round the loop (and thus reduce the number of fresh relation symbols and of SCIs generated) by applying (as transformations) any of the intuitionistic "permutations":

```
exists x. A v B      ==>  exists x A  v  exists x B
A & (B v C)          ==>  (A & B) v (A & C)
(B v C) & A          ==>  (B & A) v (C & A)
C & exists x D       ==>  exists x. C & D  (x chosen not free in C)
(exists x C) & D     ==>  exists x. C & D  (x chosen not free in D)
A => forall x B      ==>  forall x. A => B (x chosen not free in A)
A => B => C          ==>  (A & B) => C
~A                   ==>  A => \bot
```

Theorem (RD & SN) With or without these permutations, we have an **idempotent** translation, i.e. any SCI is transformed by this process to itself. (NB: no part of an SCI matches the LHS of any of these permutations. We allow trivial simplifications, e.g. $\bot \vee B \equiv B$.)

Remark Examples such as the McKinsey condition are translated exactly as we have illustrated. [Otherwise, trouble!]

Example: Frame Condition for Kreisel-Putnam Logic

A suitable frame condition for the Kreisel-Putnam (intermediate) logic **KP**, axiomatised by $(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))$ is

$$\forall xyz.\ (x \leqslant y \ \wedge \ x \leqslant z) \ \supset \ (y \leqslant z \ \vee \ z \leqslant y \ \vee \ \exists u.(x \leqslant u \ \wedge \ u \leqslant y \ \wedge \ u \leqslant z \ \wedge \ F(u, y, z)))$$

where $F(u, y, z)$ abbreviates $\forall v.\ u \leqslant v \supset \exists w.\ (v \leqslant w \wedge (y \leqslant w \vee z \leqslant w))$;

By changing F from a (bi-directional) abbreviation to a new predicate symbol with an associated SCI (in just one direction)

$$\forall uvyz.\ (F(u, y, z) \ \wedge \ u \leqslant v) \supset (\exists w(v \leqslant w \ \wedge \ y \leqslant w) \vee \exists w(v \leqslant w \ \wedge \ z \leqslant w))$$

we achieve our goal of making the condition for **KP** an SCI.

Our coherentisation algorithm does just this.

A Glivenko-style Theorem

Theorem If $\mathcal{T}$ is a theory whose axioms are weakly positive sentences, and A is a positive sentence provable in $\mathcal{T}$, then A has an intuitionistic proof from $\mathcal{T}$.

This is just an extension of what we have called "Barr's theorem"; a proof of $\mathcal{T} \Rightarrow A$ in **G3c** is (because of the syntactic restrictions) already a proof in **m-G3i** (multi-succedent intuitionistic calculus.

Result can be strengthened by allowing A to be an SCI.

Result is due to Negri: see her paper "From rule systems to systems of rules" in the JLC 2014 on "generalised geometric implications".

Implementation

The new coherentisation algorithm is implemented in *OCaml* and available on my website. It requires the loading of John Harrison's excellent *OCaml* library for manipulating first-order terms, as documented in his wonderful book.

I don't have a *Coq* or *Isabelle* proof of its correctness. (There isn't even one for Harrison's code.)

Work is in progress on **YAPE**, Yet Another Proof Engine, in *Prolog*, which allows natural expression of rules for root-first search in sequent calculi for propositional logics such as **Int**, and LaTeX output of proofs. The 2014 version is on my website, but doesn't properly cover labelled calculi; the 2015 version (to be completed REAL SOON) will be, once termination conditions are properly implemented.

Once that is done, incorporation of ideas from coherent logic automation can be started. The two kinds of system need to interact; it's not just a matter of using the work of (e.g.) Bezem & Coquand as an oracle. Termination will of course continue to be an issue.

Concept Formation

One idea not yet explored is that the replacement of an axiom by a finite conjunction of coherent implications (in an extended language) is a simple form of concept formation, i.e. the fresh relation symbols stand for what might turn out to be concepts of interest in their own right, whose inclusion helps clarify and simplify the structure of the theory.

Kant's Logic

Achourioti & van Lambalgen (2011) argue that coherent/geometric logic is the "general logic" underlying Kant's Table of Judgments, with a semantically defined notion (for sentences) of "objective validity" such that a sentence is objectively valid iff equivalent to a conjunction of finitely many coherent implications.

See also `https://cast.itunes.uni-muenchen.de/vod/clips/jSeSzC9jG0/quicktime.mp4`

Caution

It may appear that all we are doing is constructing a definitional extension of the language.

Here is a counter-argument. Let us first distinguish **three** ways of extending a theory with (for example) a new unary symbol M, informally with $M(y)$ "meaning" $\forall z. y \leqslant z \supset y = z$:

- Adding an abbreviative definition (aka "[definitional] abbreviation"). The new symbol is a "defined symbol", so the formula $M(y)$ is indistinguishable from the formula $\forall z. y \leqslant z \supset y = z$; in particular, it is not atomic, and not even quantifier-free.
- Making a definitional extension (aka a "extension by definitions"), i.e. adding a fresh primitive relation symbol M and a new axiom $\forall y.\ M(y) \equiv (\forall z.\ y \leqslant z \supset y = z)$. Well-known to give a conservative extension [Shoenfield (1967), van Dalen's book].
- Making a semi-definitional extension, i.e. adding a fresh primitive relation symbol M and a new axiom: $\forall y. M(y) \supset \forall z. y \leqslant z \supset y = z$ [or the equivalent SCI $\forall yz. M(y) \wedge y \leqslant z \supset y = z$]. This suffices for our needs. Note that $\forall y.\ (\forall z. y \leqslant z \supset y = z) \supset M(y)$ is not a CI.

Caution, 2

The theory of posets in which **every element is below a maximum element** is axiomatised by easy conditions including the McKinsey Condition: $\forall x.\, \exists y.\, x \leqslant y \wedge \forall z.y \leqslant z \supset y = z$.

Model-theoretic considerations show the theory to have no coherent axiomatisation. (Its models (with order-preserving morphisms) are not closed under filtered direct limits in **Set**.)

It doesn't help to make the **abbreviations** $M(y) \equiv \forall z(y \leqslant z \supset y = z)$ and $N(x, y) \equiv \neg(x = y)$; this doesn't change the classes of models and of their morphisms. [See next slide for why N is useful.]

But the other techniques described construct a conservative extension that **is** coherent. (Having a new **primitive** M **changes** the class of morphisms.)

There is thus a subtle difference between the other techniques described and the use of abbreviations; the latter can't change an incoherent theory into a coherent one. For details of the argument, based on Johnstone (2002), see our paper.

Caution, 3

But how does what we do differ from Shoenfield's "extension by definitions"?

We could add both halves of the equivalence, i.e. **both** $\forall yz.M(y) \wedge y \leqslant z \supset y = z)$ **and** $\forall y.(\forall z.y \leqslant z \supset y = z) \supset M(y)$; that gives an extension by definitions.

The first formula is an SCI; the second formula is not even a CI. Rearranging it classically, we get $\forall y.\ M(y) \vee \exists z.y \leqslant z \wedge y \neq z$.

A fresh relation symbol N and the axiom $\forall yz.\ N(y, z) \equiv y \neq z$ allows the second formula to be turned into an SCI ; the new axiom is equivalent to the conjunction of $\forall yz.\ (N(y, z) \wedge y = z) \supset \bot$ and $\forall yz.\ y = z \vee N(y, z)$.

Two fresh relation symbols and three SCIs in total, and effectively a definitional extension.

But, we don't need the second formula, provided that we are only replacing a **positive** occurrence of an instance of $\forall z.y \leqslant z \supset y = z)$.

We have therefore introduced the notion of a semi-definitional extension.

Summary

So, what have we done? We have

1. tried to stick to our (unstated) principles of following Gentzen by (i) limiting pre-processing and (ii) retaining naturality
2. retrieved from obscurity a result that uses a technique of Skolem (1920), "Relational Skolemisation" (a better name here than "Atomisation"): a result best formulated as "Every f.-o. theory has a coherent conservative [relational] extension".
3. introduced the notion of "weakly positive formula", and shown that any f.-o. formula can easily be put (equivalently) into this form, in a way that leaves w.p. formulae unchanged.
4. given (and implemented) an idempotent algorithm for converting weakly positive sentences to conjunctions of SCIs—not necessarily equivalent but at least giving a conservative extension.
5. continued implementation of generic framework for exploiting coherent axioms for intermediate and modal logics.

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Gentzen's justification of inferences*

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When Peter Schroeder-Heister invited me to give a talk at this conference, on the occasion of the 50th anniversary of the appearance of my thesis, as he said kindly, he gave me the option to present something scientific in the narrower sense or to reflect more personally or historically on my thesis and what came afterwards in the last half-century. I have chosen to try to do both. I shall start with some personal and historical remarks[1] and at the end I shall present some new things. Gentzen's ideas about how inferences may be justified will be the central theme throughout the talk.

1 Prelude to my "Natural Deduction"

In the summer of 1961, I started to work towards a doctor's degree at the Philosophy Department of Stockholm University. I had got a scholarship for four years to write a doctoral dissertation. In Sweden at that time, at least at my department, a thesis for a doctorate was written essentially without supervision. The thesis should be the result of independent research and should be published. It was presupposed that you had already written a less demanding licentiate's dissertation. Mine had been about automated deduction. Now I wanted to work on something more philosophical.

My aim was to study the concept of proof. The vague question I had was: What is it that makes something a deductive proof? I took for granted that the answer must somehow take into account the meaning of the sentences involved in the proof. I started to read works by Lorenzen and Curry. They related in different ways the meanings of compound sentences to inferential matters and saw logic as a super-structure over deductive systems for atomic sentences, but they did not really try to explain inferences or proofs. Then I read Gentzen's "Untersuchungen über das logische Schließen". I had read that work earlier in the way most people at that time read it, putting most attention

*I am indebted to professor Cesare Cozzo and professor Per Martin-Löf for comments on an earlier draft. The text is an edited version of the manuscript for my talk at the conference, at which I used the weaker condition of fn. 5 in the definition of analytical validity.

[1]Some of them overlap with a part of "A short scientific autobiography", in *Dag Prawitz on Proofs and Meaning*, (Outstanding Contributions to Logic, vol. 7), H. Wansing (ed.), Springer 2015, pp. 33–64.

to Gentzen's Sequenzenkalkül. But now I realized that what he called "Kalkül des natürlichen Schließens" gave a kind of answer to my question what it is that makes something a proof.

Gentzen drew attention to some noteworthy features of his system of natural deduction. The first one that he mentioned was the affinity with actual reasoning, which was, he said, the fundamental aim in setting up the system. The affinity consists of course first of all in the deductions being free to start not only from axioms but also from assumptions that could be discharged in the course of the reasoning.

Secondly, he emphasized another aspect that he described by saying: "[. . .] daß hier eine beachtenswerte Systematik vorliegt." (ibid., p. 189; transl.: "that a noteworthy *systematics* is present here"). This was an understatement in my view. It is indeed true that his system is remarkably systematic in containing introduction rules and elimination rules for each logical constant, but the really remarkable thing is the different statuses given to these two kinds of inference rules. What especially attracted me at least was Gentzen's now well-known idea that the introduction rules for a logical constant could be seen as constituting a definition of the constant in question and that the corresponding elimination rules are "no more than a consequence of this definition", as he put it.

This I saw as a way of justifying the inference rules: the introductions stated sufficient conditions for inferring compound sentences of different logical forms in accordance with the constructive meanings of the logical constants and could be seen as specifying these meanings, and the eliminations were justified by the meanings given to the logical constants by the introductions. What justified an elimination rule was more precisely the fact that a deduction of the conclusion could be obtained directly (without using the elimination) from deductions of the premisses provided that the condition to infer the major premiss by introduction was satisfied.

Having seen this it was immediately obvious that nothing new was obtained by inferring a conclusion by elimination if its major premiss had been proved by introduction. Clearly, all such eliminations could be removed by simple transformations or *reductions*, which were exemplified by Gentzen, and which I now formulated explicitly. One could then not but expect that these reductions should make it possible to remove all such eliminations from a deduction. To prove that the reductions terminated in what I called a *normal deduction* required only a suitable induction measure and a suitable order of reductions.

I recognized that the indispensable use of eliminations consisted in inferring conclusions from assumptions or what had already been inferred by successive eliminations from an initial assumption. In a normal deduction, the eliminations were restricted to this use, and as a result of this, the normal deductions had a perspicuous form that struck me: each of its threads consisted of two consecutive parts, an initial part containing only eliminations, followed by a second part containing only introductions (disregarding here the constant for falsehood). I also realized that a normal intuitionistic natural deduction could easily be transformed to a cut-free proof in the sequent calculus. In this way the normalization theorem for natural deduction gave Gentzen's Hauptsatz as a corollary. This theorem now got a deeper meaning for me.

I saw these things in the summer of 1961, and it made me quite happy – I felt like having hit, almost by chance, on important secrets. In the autumn I told about my discoveries at two seminars jointly arranged by the philosophy departments at Stockholm and Uppsala.

After having worked out the results in more detail, I presented them (in German) at a colloquium at the Institut für Mathematische Logik at Münster in the summer of 1962, when I was spending a term there. The essential part of what would become my dissertation three years later as far as intuitionistic logic was concerned was ready at this time. But I wanted my thesis to contain many other things, and in any case, I wanted to use the three years that remained of my scholarship to get other results.

I experimented with several different natural deduction systems for classical logic that could be suitable for proving a normalization theorem. One version used the axiom of the excluded third instead of the classical form of reductio ad absurdum that I finally chose in the thesis. It was presented at a meeting of the Association for Symbolic Logic in New York in 1964. Another discovery that I made was the parallelism of natural deduction systems and extended lambda-calculi. I presented it in Stockholm in 1963, and lectured on it in seminars that I gave as visiting assistant professor at UCLA in the spring of 1964. I also spent much energy in trying to extend the normalization theorem to 2nd order logic but in vain.

2 The publication of "Natural Deduction"

In the spring of 1965 when my scholarship was soon to end, I got in a hurry to put in order what I had ready in publishable form so that I could give it to the printer, and on the last day of my scholarship period, there was the public defence of my doctor's thesis. It then lay ready in the form of a printed book as required, titled *Natural Deduction. A Proof-Theoretical Study*.

One cannot say that the justification of inferences was an explicit theme in the book. Nor did I write there about my question what it is that makes something a proof. I did not know how to speak about such things in a stringent way.

The closest I came to speaking of justifications was in an indirect way, pointing out that an elimination inference is the inverse of the corresponding introduction inference in the sense that a deduction of the conclusion of an elimination is already "contained" in the deductions of its premisses when the major premiss is inferred by introduction; since my presentation at Münster, I had referred to this as the *inversion principle*, using a term from Lorenzen. I also emphasized that the inversion principle was the essential intuitive idea behind not only the normalization theorem but also behind Gentzen's Hauptsatz, whose essence was best seen in this way, I claimed. Otherwise, the thesis gave most attention to the reductions, to the precise normal form of deductions obtained by them, and to corollaries of the existence of this normal form.

The reception of the dissertation was quite varied. Many people did not at all agree with my perspective on Gentzen's work. When I presented my results at Münster, three senior logicians were present: Ackermann, Hermes and Hasenjaeger. I do not remember the comments from the first two, but they were not negative. Hasenjaeger wondered however why I bothered with natural deduction, because, as far as he knew, Gentzen had been so happy to leave the troublesome system of natural deduction when he had found his calculus of sequents and its Hauptsatz.

At the public defence of my thesis, Stig Kanger was the faculty opponent. His main criticism of the thesis was that I should have derived the normalization theorem for natural deduction from the Hauptsatz for the sequent calculus. Thus, in effect, he rejected totally my perspective. In private he also told me that my dissertation was far too long – I could preferably have written it in half of the space that I had used, he said.

In view of this, it was comforting that the reviewer in the *Journal of Symbolic Logic*, who was Richmond Thomason, wrote: "this very compressed book will require the reader to fill in many details". Thomason granted the naturalness of the meta-theory for natural deduction but was of the opinion that the meta-theory for the sequent calculus lent itself better for rigorous proofs. Mints, who also reviewed the book, appreciated the normalization theorem for classical logic, but meant that my approach became complicated in the case of intuitionistic logic with its commutative reductions.

There were also more positive remarks. At a conference in Hannover in 1966, I met in a hallway a person who said very briefly something like this: "Hello. I am Robin Gandy. I liked your book. This is the way to present Gentzen's stuff."

3 Gentzen's view of natural deduction

Allow me here an interlude to speculate about what Gentzen's own attitude could have been. In the summer of 1971, I was very surprised that Gentzen had not presented the normalization theorem for natural deduction. It seemed clear to me that he must have seen the possibility of such a theorem. In his "Untersuchungen über das logische Schließen", he says in fact that natural deduction for intuitionistic logic contains the properties essential for something like the Hauptsatz, but that natural deduction for classical logic does not.

As is now known, thanks to Jan von Plato's investigations of Gentzen's Nachlass, Gentzen actually finished a manuscript where he proved the normalization theorem for intuitionistic logic more or less in the way I did in my thesis. But he never published it. Perhaps Hasenjaeger was right: Gentzen preferred later not to think about natural deduction.

For two reasons, I think that this is not the truth. Firstly, it is clear already from Gentzen's published writings that the normalization theorem for natural deduction played an essential role in his own intuitive thinking. When he is to describe the underlying idea of his second consistency proof, what he actually describes, informally, but in some detail, is the normalization theorem for natural deduction. Secondly, we now know from his Nachlass that he even planned to write a book about the foundations of mathematics where the starting point would be the normalization theorem for natural deduction, called "der Gipfelsatz" or "der Hügelsatz" in his notes, which was to be assimilated with the proof of the consistency of arithmetic. It is true that he was disturbed by the fact that classical logic has an inference rule or an axiom that falls outside the introduction-elimination pattern. But I think it is likely that we would have seen publications by Gentzen about the normalization theorem for natural deduction at the end of the forties, if he had not died shortly after the war.

4 My return to the theme of "Natural Deduction"

In the years after the publication of my thesis, I turned to other things, among them to extending the Hauptsatz to higher order logic by model theoretic means. Only four years later I returned to the theme of the dissertation. There were two particular stimuli for that.

One came from Per Martin-Löf who was in Chicago in 1968-69 where he was introduced to Bill Howard's ideas about "formulae-as-types". Howard did not know about my work on natural deduction and tried first to connect the terms in his extended lambda-calculus with proofs in sequent calculus. Per, who knew about my thesis but had so far shown little interest in it, saw that it made more sense to connect the terms with natural deductions. From that time Per has been a strong supporter of my perspective on Gentzen's work and he developed it further in several respects. Particularly stimulating was his extension of Gentzen's introduction-elimination pattern to arithmetic and, more generally, to inductively defined predicates. Furthermore, I was fascinated by the very powerful method for establishing normalization theorems that he obtained by carrying over to natural deductions Tait's notion of convertibility for terms in the lambda-calculus, calling it computability. He presented these things at the 2nd Scandinavian Logic Symposium at Oslo in 1970.

A second stimulus came from Kreisel, with whom I had joint seminars at Stanford in the academic year 1969-70. He too became convinced of my perspective on Gentzen's work and got especially interested in the reductions by which I had transformed deductions into normal form and their relation to the identity of proofs. In a review (*Journal of Philosophy* 68 (1971), 238–265), he wrote: "As I see it now, [. . .] [footnote: Guided by D. Prawitz's reading of Gentzen] the single most striking element of Gentzen's work occurs already in his doctoral dissertation", which "provided the *germs for a theory of proofs*". This theory would concern "the process of reasoning, not only its result". Kreisel was as we see an early advocate of general proof theory. He too presented a paper, "A survey of proof theory II", at the Oslo symposium, where he further developed his ideas about these things.

One thing that he and I discussed at Stanford was the hope that the normal form theorem for 2nd order logic obtained by model-theoretic means could be strengthened by showing that every 2nd order deduction reduces to a normal one; the term "normalization theorem" was suggested at this time by Kreisel to name this stronger result. In the spring of 1970 when he was one of the supervisors of Girard's doctoral work, he suggested that an idea that Girard was working on could be used to solve this problem. This was also discussed at the Oslo symposium and turned out to be right; in the proceedings of the symposium, Girard, Martin-Löf, and myself presented solutions based on the idea of Girard.

Several of the papers of these proceedings were concerned in this way with developing themes of my thesis; I think one could say that its perspective became generally accepted at this time as a fruitful approach in proof-theory. In the decade that was to come, there was also Martin-Löf's type theory where introductions and eliminations of Gentzen's kind and reductions of my kind, now in the form of definitional equality rules, became cornerstones.

5 The notion of valid deduction

My contribution to the Oslo symposium was a survey paper where among other things I brought up to date themes that I had been dealing with in my thesis. The notion of computability that Per had defined for natural deductions now seemed to me to offer a way to make explicit Gentzen's ideas about the justification of inferences, which I had not known how to state in a general and rigorous way in my thesis.

Somewhat modifying the notion of computability, now calling it validity, I stated in effect the following two principles for intuitionistic natural deductions:

(I) A closed deduction is valid if and only if it reduces to a deduction ending with an introduction – what a little later I started to call a deduction in *canonical form* – whose immediate subdeductions are valid.

(II) An open deduction is valid if and only if the result of substituting, first, closed terms for the free variables in the deduction, and then, closed valid deductions for the free assumptions, is always valid.

The validity was to be relative to a base of valid deductions of atomic sentences.

Principle (II) is an expression of the idea that an open deduction is seen as a schema for closed deductions and is therefore valid if and only if its closed instances obtained by replacing parameters with constants and open assumptions with valid deductions are valid; this can be said to be implicit in Gentzen.

Principle (I) is an expression of the idea that the meaning of a sentence is given by its introduction rule viewed as the canonical way of proving the sentence; dubbing it canonical is to say that proofs in other forms can be valid if and only if they can be rewritten in that form. The principle can be broken up into two principles (since "reduce" is a reflexive relation):

(Ia) A closed deduction in canonical form is valid if and only if its immediate subdeductions are valid; and

(Ib) A closed deduction in non-canonical form is valid if and only if it reduces to a valid deduction in canonical form.

Since the premisses and the assumptions closed by the application of an introduction rule are subformulas of the conclusion, the subdeductions referred to in clause (Ia) are of lower complexity than the given deduction. It is therefore possible to give by recursion a definition of validity relative to a base that satisfies principles (I) and (II). This definition of validity seemed to me at that time to make precise Gentzen's idea about how inferences may be justified.

To see that the intuitionistic natural deductions really are valid in the defined sense, we have to verify that the inference rules preserve validity. For the introductions this is immediate by (Ia), and should be so since the meanings of the conclusions are supposed to be given by them. For the eliminations this is seen by essentially spelling out the inversion principle: when $\mathcal{D}^*$ is the reduction of a deduction $\mathcal{D}$ that ends with an elimination whose major premiss is inferred by introduction, $\mathcal{D}^*$ is contained in the immediate subdeductions of $\mathcal{D}$ and is valid if they are.

6 The notion of valid argument

However, the definitional domain of this notion of validity consists of deductions that all turn out to have the property defined. One would like a notion of validity to be defined, not for deductions already expected to be valid, but for reasoning or argumentation in general, some of which is clearly not valid. Shortly after the Oslo symposium, I therefore extended the notion of validity to what I called *arguments*.

A collection of arbitrary inferences, not just applications of some given inference rules, arranged in tree-form like a natural deduction, was called an *argument*. Some inference rules, referred to as introductions, were to be taken as meaning explanatory, and an argument whose last inference was an instance of such a rule was referred to as *canonical* (so called in my second presentation of these ideas, published in 1974). To all other inference rules were to be assigned (alleged) justifications in the form of reduction operations, and the notion of reduction of natural deduction was generalized to them, so that one could speak of one argument reducing to another *relative to* a set $\mathcal{R}$ of reductions.

An argument $\mathcal{A}$ together with a set of reductions $\mathcal{R}$ was called a justified argument. Validity of a justified argument $(\mathcal{A}, \mathcal{R})$ was now defined in essentially the same way as I had defined validity for deductions. A natural deduction $\mathcal{D}$ paired with the set $\mathcal{R}$ of standard reductions assigned to elimination rules constituted an example of a justified argument, and $(\mathcal{D}, \mathcal{R})$ would come out as valid. But a justified argument in general could be built up of any kinds of inferences and reductions, only some of which would be valid.

Introduction rules were assumed to satisfy the same complexity condition as Gentzen's introduction rules, and a base that determined what counted as valid arguments for atomic sentences was presupposed as before. Then, what it is for a justified argument $(\mathcal{A}, \mathcal{R})$ to be valid – or for an argument $\mathcal{A}$ to be *valid relative to* a set $\mathcal{R}$ of reductions, as I also said synonymously – was possible to define by recursion by clauses similar to the ones that defined the validity of deductions:

1. A closed argument $\mathcal{A}$ in canonical form is valid relative to $\mathcal{R}$, if and only if its immediate subarguments are.

2. A closed argument $\mathcal{A}$ in non-canonical form is valid relative to $\mathcal{R}$, if and only if it reduces relative to $\mathcal{R}$ to a closed argument in canonical form that is valid relative to $\mathcal{R}$.

3. An open argument $\mathcal{A}$ is valid relative to $\mathcal{R}$ if and only if the result obtained from $\mathcal{A}$ by first substituting closed terms for the free variables in $\mathcal{A}$, and then substituting for the free assumptions closed arguments valid relative to an extension $\mathcal{R}^*$ of $\mathcal{R}$, is always valid relative to $\mathcal{R}^*$.

The notion of validity obtained in this way came to be discussed by others as time went on. In his book *The Logical Basis of Metaphysics* from 1991, Michael Dummett devoted a couple of chapters to a discussion of my notion, which he slightly modified, calling it a proof-theoretical justification of logical laws. In the volume *Proof-Theoretic Semantics*, which grew out of a conference with the same name held here in Tübingen in 1999, Peter Schroeder-Heister discussed and modified my notion in some essential respects, in particular with respect to what is to be counted as a set of reductions paired with an argument.[2]

[2] "Validity concepts in proof-theoretic semantics", in *Proof-Theoretic Semantics*, R. Kahle and P. Schroeder-Heister (eds.), *Synthese* 148 (2006), 525–571.

7 Valid and legitimate inference

It is to be noted that this notion of valid argument (in its different versions) is defined by recursion over the complexity of the sentences they are arguments for, as BHK-proofs are defined by recursion over the sentences they are proofs of, whereas intuitively we think usually of a proof as arising inductively by making accepted inferences. As we have understood the notion of proof since ancient Greece, a proof is built up stepwise by making inferences, one after another, by which conclusions are demonstrated categorically or shown to hold under certain assumptions, given that the premisses have been established in the same way. Similarly, a proof in a formal system is defined inductively over its length, each step being an application of one of the given inference rules.

In contrast, the notion of valid argument, like the notion of a BHK-proof, does not presuppose a notion of accepted inference. But given the notion of valid argument, we can define an inference rule as valid relative to a set $\mathcal{R}$ of reductions when any application of the rule to a valid argument relative to $\mathcal{R}$ yields a new argument that is valid relative to $\mathcal{R}$. However, we should be aware that this is to turn the usual conceptual order upside-down.

If the intuitive notion of proof is to be explicated, we need to make precise what it is for an inference to be acceptable in a proof. It cannot be enough that the conclusion in fact follows from the premisses, whatever is meant by that. In particular, it is clearly not sufficient that the inference is valid in the traditional sense of necessarily preserving truth or preserving truth under all variations of the meanings of the non-logical terms. Unless this is evident, the inference is not accepted in a deductive proof but is seen as a gap in the reasoning.

However, to get an objective notion of proof we do not want to rely on a primitive notion of evidence or an explanation of that notion in psychological terms. I have suggested elsewhere[3] that we make precise what it is to have objectively a binding *ground* for an assertion or judgement, so that to be in possession of such a ground amounts to the assertion or the judgement being warranted or justified. The notion of ground, which must of course not presuppose the notion of proof if it is to work in an explication of that concept, may then be used to define what it is for an inference to be acceptable or *legitimate*, as I have called it: To be legitimate the inference should deliver a ground for the conclusion given grounds for the premisses.

For the same reason that an inference may be valid in the traditional sense without being legitimate, an inference may be valid as defined above in terms of valid arguments without being legitimate; it may need an elaborate proof to establish that an argument is valid or that an inference preserves validity.

[3] "Explaining deductive inference" in *Dag Prawitz on Proofs and Meaning* (Outstanding Contributions to Logic, vol. 7), ed. H. Wansing, Springer 2015, pp. 65–100.

More generally speaking, for an inference to be legitimate it is not enough that the conclusion *can be* justified, in other words that the conclusion is *justifiable*, given that the premisses have been justified. What is required is that the conclusion becomes *justified* by the inference without further reasoning; in other words, it must be the very inference that delivers a ground for the conclusion.

Note that I am here speaking of *acts* of inference; in this I am following Per Martin-Löf and Göran Sundholm, who have for a long time emphasised that an inference is primarily an act. It is only by performing an act that we can hope to justify a judgement. In particular, it is an inference act that can bring us in possession of a ground for its conclusion – a mere inference figure cannot bring about anything. However, the act must consist of something more than a mere transition of assertions or judgements, if it is to make the conclusion justified. I have suggested that we should see an inference as an operation on grounds. Then a legitimate inference delivers literally a ground for the conclusion when applied to grounds for the premisses. In this reconceptualization of the notion of inference, I have taken the grounds to be objects denoted by terms in an extended typed lambda-calculus, isomorphic to Gentzen's natural deduction; in other words, what Per Martin-Löf in his talk yesterday called proof-objects and took to be the modern interpretation of natural deduction.

This I see today as the most promising analysis of our intuitive notions of inference and proof. Ideas from Gentzen are involved here in two ways. Firstly, the meaning of a proposition is explained by telling how a ground for asserting the proposition is formed, and the grounds are formed by applying an operation structurally similar to Gentzen's corresponding introduction rule. Secondly, inferences other than introductions, when understood as essentially operations on grounds, become structurally of the same kind as the operations by which the reductions that justify Gentzen's elimination rules are obtained.

8 A new notion of valid argument

It is possible however to describe Gentzen's ideas about the justification of inferences in precise, general terms that stay closer to how he saw it and to how inferences are commonly seen. But it requires another notion than the one of valid argument that I have been speaking of here. That notion has other undesirable features than the one just noted above.

The problem is, roughly stated, that the reduction assigned to an inference of an argument may not have much to do with that inference; it may come "just out of the blue", as Peter has put it in discussions of this issue. As an extreme case, the validity of an argument relative to a set $\mathcal{R}$ of reductions may depend totally on $\mathcal{R}$ and not at all on its inferences. In contrast, the standard reductions associated with Gentzen's elimination rules depend essentially on the deductions to which they are applied. They are obtained by operations of three very simple kinds applied to these deductions: (a) extracting subdeductions, (b) substituting terms for free variables in subdeductions, and (c) composing two subdedcutions, that is, substituting one subdeduction for free assumptions in another subdeduction.

At the second conference on proof-theoretic semantics held here in Tübingen two years ago, I investigated the relation between valid arguments and BHK-proofs. But as I suggested in the paper[4], a notion of valid argument where the reductions were restricted to operations of the kind (a)-(c) may be in much better concordance with the intuition behind Gentzen's justification of elimination rules than the notion studied in the paper.

In my thesis, I tried to catch that intuition by the informally stated inversion principle, which said, as we recall, that a deduction of the conclusion of an elimination inference is already "contained" in the deductions of the premisses when the major premiss is inferred by introduction, but I did not try to give a general definition of the term "contained". However, such a definition is easily given, since for the inversion principle to hold for Gentzen's eliminations it is sufficient and necessary that what is obtained by a reduction is counted as contained in the redex, and as just noted there are just three kinds of simple operations that are used in order to get the reductions.

Let us say that the argument $\mathcal{A}$ is *immediately extracted* from the set Σ of arguments if and only if either

(a) $\mathcal{A}$ is an argument in Σ or a sub-argument of some argument in Σ, or

(b) $\mathcal{A}$ is the result of substituting a term for the occurrences of a free variable in an argument in Σ, or

(c) $\mathcal{A}$ is the result of composing two arguments $\mathcal{B}$ and $\mathcal{C}$ in Σ, or more precisely,

$$\mathcal{A} = \begin{array}{c}\mathcal{B}\\ [B]\\ \mathcal{C}\end{array},$$

that is, $\mathcal{A}$ is the result of replacing some free assumptions B in $\mathcal{C}$ by $\mathcal{B}$.

We can then define an argument $\mathcal{A}$ to be *contained* in a set Σ of arguments, if there is a sequence of arguments $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ where $\mathcal{A}_n = \mathcal{A}$ and for each $i < n$, $\mathcal{A}_i$ is immediately extracted from $\Sigma \cup \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{i-1}\}$. I shall say that the argument $\mathcal{A}$ is contained in the argument $\mathcal{B}$ or that $\mathcal{B}$ contains $\mathcal{A}$, if $\mathcal{A}$ is contained in $\{\mathcal{B}\}$.

[4]"On the relation between Heyting's and Gentzen's approach to meaning" in *Advances in Proof-Theoretic Semantics*, T. Piecha and P. Schroeder-Heister (eds.), Springer 2016, pp. 5–25.

In the entry "Proof-Theoretic Semantic" in *Stanford Encyclopedia of Philosophy* (Summer 2014 Edition), Peter Schroeder-Heister also notes that the standard reductions for the connectives are elementary in the sense that they are obtained by composing given subderivations.

Given this notion of containment, we can consider a new notion of valid argument. To distinguish it from the previously defined notions we may call it *analytical validity*[5]; I shall sometimes drop the prefix "analytical" in contexts where I only speak of this new notion of validity. Most of the terminology and assumptions in connection with valid arguments can be used as before, in particular concerning introductions, canonical arguments, and the base for atomic sentences. The important difference is that analytical validity will not be relative to a set of reductions.

Of the three recursive clauses in the previous definition of validity, the first and the third can essentially be kept while the second is different:

1) A closed argument in canonical form is *analytically valid*, if and only if its immediate sub-arguments are.

2) A closed non-canonical argument for a sentence A is *analytically valid*, if and only if it contains an *analytically valid* closed canonical argument for A.[6]

3) An open argument is *analytically valid* if and only if all results of substituting, first, closed terms for the free variables in the argument, and then, closed *analytically valid* argument for the free assumptions, are valid.

Note that a closed non-canonical argument $\mathcal{A}$ for a sentence A that contains a closed (analytically) valid argument $\mathcal{A}^*$ for A is itself valid. This is seen by noting that $\mathcal{A}^*$ must in its turn contain a valid closed canonical argument for A (by clause 2) which is also contained in $\mathcal{A}$ (because of the transitivity of the containment relation), and that therefore $\mathcal{A}$ is valid by clause 2.

The definition of this new notion of analytical validity is still using recursion over the complexity of sentences and is not presupposing a notion of validity for inferences. Such a notion, analytical validity of inferences, may be defined again as preservation of the property in question. Then, introductions are analytically valid in virtue of clause 1. To demonstrate for any other inference rule that it is analytically valid, it is sufficient to show in view of the remark just made that for any closed argument obtained by applying the rule, an analytically valid, closed argument for its conclusion is contained in the set of arguments for its premisses, given that these arguments are analytically valid.

Analytical validity is a much more demanding notion of validity than we had before. Derivable inferences that are not essentially variations or iterations of introductions or eliminations seem usually not to be analytically valid. But given that the usual introductions determine what is counted as canonical

[5] The terminology may be appropriate in view of the emphasis that is here put on an argument containing another, although, as Göran Sundholm pointed out in the discussion after my talk, as far as Kant's notion of analytical truth is concerned, the containment is a relation between predicates, not between arguments.

[6] A slightly weaker condition would be to require that it contain the immediate sub-arguments of an *analytically valid* closed canonical argument for A.

arguments, it is easy to see that all the usual eliminations are valid in this strong sense.

To verify (analytical) validity for $\rightarrow$E, let $\mathcal{A}$ be a valid argument for A and let $\mathcal{B}$ be a valid argument for $A \rightarrow B$. We have to show that the argument

$$\mathcal{C} = \frac{\begin{array}{cc} \mathcal{A} & \mathcal{B} \\ A & A \rightarrow B \end{array}}{B}$$

is valid. In case $\mathcal{C}$ is an open argument, this amounts to showing that any closed instance $\mathcal{C}^*$ of $\mathcal{C}$ (in the sense of clause 3), which we may write

$$\frac{\begin{array}{cc} \mathcal{A}^* & \mathcal{B}^* \\ A^* & A^* \rightarrow B^* \end{array}}{B^*}$$

is valid (if $\mathcal{C}$ is already closed, let $\mathcal{C}^* = \mathcal{C}$). Since $\mathcal{A}$ and $\mathcal{B}$ are valid, so are the closed arguments $\mathcal{A}^*$ and $\mathcal{B}^*$ (by clause 3). Hence, $\mathcal{B}^*$ contains a closed, canonical, and valid argument for $A^* \rightarrow B^*$, which in turn must contain, in fact, must have as immediate sub-argument, a valid argument $\mathcal{B}_1$ for $\mathcal{B}^*$ from $\mathcal{A}^*$ (by clause 2). Let $\mathcal{B}_2$ be

$$\begin{array}{c} \mathcal{A}^* \\ [A^*] \\ \mathcal{B}_1 \end{array}$$

that is, the closed argument for B^* obtained by substituting $\mathcal{A}^*$ for all free assumptions A^* in $\mathcal{B}_1$. By clause 3, $\mathcal{B}_2$ is valid, and by clause (c), $\mathcal{B}_2$ is contained in $\{\mathcal{A}^*, \mathcal{B}_1\}$ and hence also in $\mathcal{C}^*$ (by the transitivity of containment). It follows by the remark following the definition of analytical validity that $\mathcal{C}^*$ is valid.

This kind of result can be extended beyond the elimination rules for logical constants. If we follow Per Martin-Löf in seeing Peano's first and second axioms (reformulated as inference rules) as introduction rules for the predicate of being a natural number, then it is not difficult to see that the rule of induction (which is to be seen as the corresponding elimination rule) is analytically valid.

For an example of a simple inference that is intuitionistically derivable but is not analytically valid, consider the following inference taken from the paper by Peter Schroeder-Heister referred to earlier:

$$\frac{A \rightarrow (B \rightarrow C)}{B \rightarrow (A \rightarrow C)}$$

It is easily seen to be valid in the previously defined sense relative to a set $\mathcal{R}$ containing the standard reduction assigned to $\rightarrow$E and an obvious reduction that transforms any argument for the premiss valid relative to an extension of $\mathcal{R}$ to an argument for the conclusion valid relative to that extension. The transformed argument contains however inferences that need not be a part of an analytically valid argument for the premiss. That the inference is not analytically valid for arbitrary A, B, and C is seen by letting A be $B \to C$. An example of an analytically valid, closed argument for the premiss is the one-step argument that assumes $B \to C$ and then applies $\to$I. Clearly it does not contain an argument for the conclusion, not even an argument for $(B \to C) \to C$ from B; thus, when weakening clause 2 as described in footnote 5, the inference remains invalid.

Not even the trivial inference

$$\frac{A \vee B}{B \vee A}$$

is analytically valid. An analytically valid closed argument for $A \vee B$ does not need to contain a canonical argument for $B \vee A$. However, it does contain an analytically valid closed argument for A or for B, which would make the inference valid if we change the notion in the way described in footnote 5.

Let me close with a few final remarks, in particular concerning Gentzen's idea that in an elimination inference we are using the major premiss only "in the sense afforded it by the [corresponding] introduction". It is this idea that becomes better reflected by the new, stronger condition for the analytical validity of a closed non-canonical argument. For validity it was sufficient (and necessary) that the argument could be rewritten in canonical form by applying a reduction and that the result was valid; how the reduction operation looked was left essentially open, which meant that the meanings of the premisses of the last inference of the argument did not necessarily matter. When we now require for analytical validity that the argument contain a closed, analytically valid argument for the sentence in question, we get a condition that can be satisfied only thanks to what the major premiss means, that is, because of the nature of a canonical argument for it. The inversion principle formulated in my thesis indicates the general feature that Gentzen's elimination rules must possess, and in fact do possess, in order to satisfy this stronger condition.

It is also noteworthy that given that the introductions take the usual Gentzen form, essentially only they and Gentzen's eliminations (including inferences that compress a number of iterated eliminations) seem to be analytically valid. The analytically valid inferences may in this way correspond to what may be called essentially gap-free inferences.

To the extent that there are also other analytically valid inferences expressible in a first order language, they can be expected to be derivable in the intuitionistic system for natural deduction. This conjecture seems to make precise the plausible idea that Gentzen's eliminations are the strongest inferences expressible in the usual first order language that can be justified in terms of his introductions. It may have a greater chance to be proved than the similar and more doubtful conjecture formulated more than 40 years ago in terms of my previous notion of validity[7].

[7] "Towards a foundation of general proof theory", in *Logic, Methodology and Philosophy of Science IV*, P. Suppes et al. (eds.), North Holland, 1973, pp. 225–250.

On Proof Compressions in Sequent Calculi and Natural Deductions

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Tuebingen, January 26, 2016
[unable to attend the conference]

Abstract

Theorem

A natural embedding of Hudelmaier's sequent calculus for purely implicational logic into analogous Prawitz-style tree-like natural deduction calculus followed by appropriate dag-like horizontal compression allows to obtain polynomial-size dag-like natural deductions d' of arbitrary tree-like inputs d. ('dag' = directed acyclic graph).

A suitable formalization of the theorem should prove the conjecture **NP = PSPACE**.

§1. Overview of the procedure -1

- Formalize purely implicational minimal propositional logic as fragment $\mathrm{LM}_{\to}$ of Hudelmaier's tree-like cutfree intuitionistic sequent calculus. Note that for any $\mathrm{LM}_{\to}$ proof d of a given formula α we have:
  1. the height $h(d)$ of d is polynomial (actually linear) in the length $|\alpha|$,
  2. the total number $\phi(d)$ of pairwise distinct formulas occurring in d is also polynomial (actually quadratic) in $|\alpha|$.
- Embed $\mathrm{LM}_{\to}$ into Prawitz's tree-like natural deduction formalism for minimal logic, $\mathrm{NM}_{\to}$. Observe that this translation preserves polynomial estimates (1) and (2).

§1. Overview of the procedure -2

- Elaborate the dag-like deducibility in $\mathrm{NM}_{\to}$.
- Elaborate and apply *horizontal tree-to-dag proof compression* in $\mathrm{NM}_{\to}$. Note that for any given tree-like input d, the size of the resulting dag-like output d' is bounded by the product of $h(d)$ and $\phi(d)$. Hence, in the dag-like version of $\mathrm{NM}_{\to}$, the size of the compressed embedded tree-like $\mathrm{LM}_{\to}$ proof of α is polynomially bounded in $|\alpha|$.
- Since purely implicational minimal propositional logic is known to be PSPACE-complete, conclude: **NP = PSPACE**.

§2. Hudelmaier's sequent calculus for minimal logic -1-

Definition (Sequent calculus $\mathrm{LM}_{\to}$)

$\mathrm{LM}_{\to}$ includes the following axioms $(\mathrm{M}A)$ and inference rules $(\mathrm{M}I1 \to)$, $(\mathrm{M}I2 \to)$, $(\mathrm{M}E \to P)$, $(\mathrm{M}E \to\to)$ in standard intuitionistic sequent formalism[a] of one connective $\to$.
(α, β, γ formulas; p, q distinct propositional variables; in $(\mathrm{M}I1 \to)$, no $(\alpha \to \beta) \to \gamma$ occurs in Γ, while in $(\mathrm{M}E \to P)$ and $(\mathrm{M}E \to\to)$, q occurs in Γ, γ).[b]

[a] The antecedents, Γ, of our sequents $\Gamma \Longrightarrow \alpha$ are viewed as multisets of formulas. Sequents $\Longrightarrow \alpha$, i.e. $\emptyset \Longrightarrow \alpha$, are identified with formulas α.

[b] This slight modification is equivalent to the corresponding subsystem of Hudelmaier's original calculus LG. The constraints $q \in VAR(\Gamma, \gamma)$ are added just for the sake of transparency.

§2. Hudelmaier's sequent calculus $\mathrm{LM}_{\to}$ -2-

$$(\mathrm{M}A): \quad \Gamma, p \Longrightarrow p$$

$$(\mathrm{M}I1 \to): \quad \frac{\Gamma, \alpha \Longrightarrow \beta}{\Gamma \Longrightarrow \alpha \to \beta} \; [(\nexists\gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]$$

$$(\mathrm{M}I2 \to): \quad \frac{\Gamma, \alpha, \beta \to \gamma \Longrightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Longrightarrow \alpha \to \beta}$$

$$(\mathrm{M}E \to P): \quad \frac{\Gamma, p, \gamma \Longrightarrow q}{\Gamma, p, p \to \gamma \Longrightarrow q} \; [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q]$$

$$(\mathrm{M}E \to\to): \quad \frac{\Gamma, \alpha, \beta \to \gamma \Longrightarrow \beta \qquad \Gamma, \gamma \Longrightarrow q}{\Gamma, (\alpha \to \beta) \to \gamma \Longrightarrow q} \; [q \in \mathrm{VAR}(\Gamma, \gamma)]$$

§2. Hudelmaier's sequent calculus $\mathrm{LM}_{\to}$ -3-

Theorem (Hudelmaier)

$\mathrm{LM}_{\to}$ is sound and complete with respect to minimal propositional logic and tree-like deducibility. So any given formula α is valid in the minimal logic iff sequent $\Longrightarrow \alpha$ is tree-like deducible in $\mathrm{LM}_{\to}$.

For any (tree-like or dag-like) deduction d denote by $h(d)$ and $\phi(d)$ its *height* (:= maximal thread length) and *foundation* (:= the total number of pairwise distinct formulas), respectively.
For any sequent (in particular, formula) s denote by $|s|$ the total number of '$\to$'-occurrences in s and define its complexity degree $\deg(s)$:

1. $\deg(\Gamma, \alpha \to \beta \Longrightarrow \alpha) := |\alpha \to \beta| + \sum_{\xi \in \Gamma} |\xi|$,
2. $\deg(\Gamma \Longrightarrow \alpha) := |\alpha| + \sum_{\xi \in \Gamma} |\xi|$, *if* $(\nexists \beta) : \alpha \to \beta \in \Gamma$.

§2. Hudelmaier's sequent calculus $\mathrm{LM}_{\to}$ -4-

Lemma (Hudelmaier)

1. *Tree-like $\mathrm{LM}_{\to}$ deductions have the semi-subformula property, where semi-subformulas of $(\alpha \to \beta) \to \gamma$ include $\beta \to \gamma$ along with proper subformulas $\alpha \to \beta$, α, β, γ. That is, any β occurring in a given tree-like $\mathrm{LM}_{\to}$ deduction of α is a semi-subformula of α.*
2. *If s' occurs strictly above s in a given tree-like $\mathrm{LM}_{\to}$ deduction d, then $\deg(s') < \deg(s)$.*
3. *The height of any tree-like $\mathrm{LM}_{\to}$ deduction d of s is linear in $|s|$. In particular if $s = \alpha$, then $h(d) \leq 3|\alpha|$.*
4. *The foundation of any tree-like $\mathrm{LM}_{\to}$ deduction d of s is at most quadratic in $|s|$. In particular if $s = \alpha$, then $\phi(d) \leq (|\alpha| + 1)^2$.*

§3. Basic Prawitz-style formalism $\mathrm{NM}_{\to}$

- We consider Prawitz's purely implicational system for minimal propositional logic that contains just two rules

$$\boxed{(\to I) : \dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}}{\alpha \to \beta}} \qquad \boxed{(\to E) : \dfrac{\alpha \quad \alpha \to \beta}{\beta}}$$

where $\alpha, \beta, \gamma, \cdots$ denote arbitrary formulas over propositionl variables $p, q, r, \cdots$ and one propositional connective $\to$.

Theorem (Prawitz)

$\mathrm{NM}_{\to}$ *is sound and complete with respect to minimal propositional logic and tree-like deducibility.*

§4. Embedding $\mathrm{LM}_{\to}$ into $\mathrm{NM}_{\to}$ -1-

$\mathrm{M}_{\to}$ consists of natural deductions for minimal logic with rules $(\to \mathrm{I})$, $(\to \mathrm{E})$. We embed $\mathrm{LM}_{\to}$ into $\mathrm{M}_{\to}$ following standard *sequent deduction* $\hookrightarrow$ *natural deduction* pattern, where sequent deduction of $\Gamma \Longrightarrow \alpha$ is interpreted as a natural deduction of α from assumptions in Γ (we don't expose minor assumptions).

$$\boxed{(\mathrm{M}A) : \Gamma, p \Rightarrow p} \hookrightarrow \boxed{p}$$

$$\boxed{(\mathrm{M}I1 \to) : \dfrac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}} \hookrightarrow \boxed{\dfrac{\begin{array}{c}[\alpha]_1 \\ \Downarrow \\ \beta\end{array}}{\alpha \to \beta_{[1]}} (\to \mathrm{I})}$$

(*discharging premise-assumption* α; $[(\nexists\gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]$)

§4. Embedding $\mathrm{LM}_{\to}$ into $\mathrm{NM}_{\to}$ -2-

$$\boxed{(\mathrm{M}I2 \to) : \dfrac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow \alpha \to \beta}} \hookrightarrow$$

$$\boxed{\dfrac{\dfrac{\begin{matrix}[\alpha]_1\\ \Downarrow\end{matrix}\qquad \begin{matrix}\dfrac{\dfrac{\dfrac{[\beta]_2}{\alpha \to \beta}(\to \mathrm{I}) \quad (\alpha \to \beta) \to \gamma}{\gamma}(\to \mathrm{E})}{\beta \to \gamma_{[2]}}(\to \mathrm{I})\\ \Downarrow\end{matrix}}{\beta}}{\alpha \to \beta_{[1]}}(\to \mathrm{I})}$$

(*discharging/deducing premise-assumptions* $\alpha, \beta \to \gamma$)

§4. Embedding $\mathrm{LM}_{\to}$ into $\mathrm{NM}_{\to}$ -3-

$$\boxed{(\mathrm{M}E \to P) : \dfrac{\Gamma, p, \gamma \Longrightarrow q}{\Gamma, p, p \to \gamma \Longrightarrow q}} \hookrightarrow$$

$$\boxed{\dfrac{\begin{matrix}p\\ \Downarrow\end{matrix}\qquad \begin{matrix}\dfrac{p \quad p \to \gamma}{\gamma}(\to \mathrm{E})\\ \Downarrow\end{matrix}}{q}}$$

$$\left(\begin{matrix}\textit{deducing premise-assumption } \gamma;\\ [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q]\end{matrix}\right)$$

§4. Embedding $\mathrm{LM}_{\to}$ into $\mathrm{NM}_{\to}$ -4-

$$\boxed{(\mathrm{M}E \to\to) : \dfrac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta \qquad \Gamma, \gamma \Rightarrow q}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow q}} \hookrightarrow$$

$$\boxed{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\alpha]_1}{\Downarrow} \quad \dfrac{\dfrac{\dfrac{\dfrac{[\beta]_2}{\alpha \to \beta} \quad (\alpha \to \beta) \to \gamma}{\gamma}}{\beta \to \gamma}\,[2]}{\Downarrow}}{\beta}}{\alpha \to \beta}\,[1] \quad (\alpha \to \beta) \to \gamma}{\gamma} \quad \dfrac{\dfrac{\dfrac{[\gamma]_3}{\Downarrow}}{q}}{\gamma \to q}\,[3]}{q}}$$

$$\left(\begin{matrix} \textit{discharging/deducing premise-assumptions } \alpha, \beta \to \gamma, \gamma; \\ [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q] \end{matrix} \right)$$

§4. Summary

Lemma (embedding)

There is a recursive operator $\mathcal{F}$ transforming any given tree-like $\mathrm{LM}_{\to}$ deduction d of $\Gamma \Longrightarrow \alpha$ into a tree-like $\mathrm{NM}_{\to}$ deduction $\mathcal{F}(d)$ with endformula α and assumptions occurring in Γ. Moreover d and $\mathcal{F}(d)$ share the semi-subformula property, linearity of the height and polynomial upper bounds on the foundation. In particular if $\Gamma = \emptyset$, then $\mathcal{F}(d)$ is a $\mathrm{NM}_{\to}$ proof of α such that $h(\mathcal{F}(d)) \leq 18|\alpha|$ and $\phi(\mathcal{F}(d)) < (|\alpha|+1)^2(|\alpha|+2)$.

§5. Dag-like natural deductions -1-

- We wish to formalize dag-like deducibility in Prawitz's world. Recall that *'dag'* stands for *directed acyclic graph* (edges directed upwards).
- The main difference between tree-like and dag-like natural deductions is caused by the art of discharging, as the following examples show.

§5. Dag-like natural deductions -2-

Example

Consider a dag-like natural deduction *d* =

$$(\to \mathrm{E})\,\dfrac{(\to \mathrm{E})\,\dfrac{\dfrac{\overbrace{\vdots}^{\Gamma}}{\beta\to\alpha} \qquad \dfrac{[\alpha]^1 \quad \alpha\to\beta}{\beta}\,(\to \mathrm{E})}{\alpha \qquad\qquad \alpha\to\beta\;[1]}\,(\to \mathrm{I})}{\beta}$$

in which the right-hand side premise of second $(\to \mathrm{E})$ coincides with $(\to \mathrm{I})$ premise β. Note that the assumption α above β is discharged by this $(\to \mathrm{I})$. However, we can only infer that *d* deduces β from $\Gamma \cup \{\alpha, \alpha \to \beta\}$, instead of expected $\Gamma \cup \{\alpha \to \beta\}$, which leaves the option $\Gamma \cup \{\alpha \to \beta\} \nvdash \beta$ open, if $\alpha \notin \Gamma$.

§5. Dag-like natural deductions -3-

Example (*continued*)

This becomes obvious if we replace *d* by its "unfolded" tree-like version d_{U} =

$$\dfrac{\dfrac{\dfrac{\alpha \quad \alpha \to \beta}{\beta} \qquad \dfrac{\overbrace{\quad\vdots\quad}^{\Gamma}}{\beta \to \alpha}}{\alpha} \qquad \dfrac{\dfrac{[\alpha]^1 \quad \alpha \to \beta}{\beta}}{\alpha \to \beta}\,[1]}{\beta}$$

Clearly d_{U} deduces β from $\Gamma \cup \{\alpha, \alpha \to \beta\}$, instead of $\Gamma \cup \{\alpha \to \beta\}$, which leaves the option $\Gamma \cup \{\alpha \to \beta\} \nvdash \beta$ open, if $\alpha \notin \Gamma$.

§6. Global dag-like proof correctness

- Keeping this in mind we'll say that in a dag-like natural deduction *d*, a given leaf *u* labeled with formula α is an *open* (or *undischarged*) *assumption-node*, and α is an *open* (or *undischarged*) *assumption*, iff there exists a thread θ connecting *u* with the root such that no $w \in \theta$ is the $(\to \mathrm{I})$ conclusion labeled with $\alpha \to \beta$. Other leaves are called *closed* (or *discharged*) *assumption-nodes*.
- A natural deduction *d* is called a *proof* (of its root-formula) iff the set of open assumptions is empty.
- Note that the corresponding condition ' *u is open* (resp. *closed*) *in d* ' belongs merely to **NP** (resp. **coNP**), unless *d* is a tree-like deduction, in which case both conditions are in **P**, as desired. (Thus tree-like deducibility is no problem.)

§7. Local dag-like proof correctness -1-

- We overcome this trouble by a suitable notion of *local correctness* that includes special edge-labeling function $\ell^{\mathrm{d}} : \mathrm{E}(d) \times \mathrm{F}(d) \to \{0,1\}$, where $\mathrm{E}(d)$ and $\mathrm{F}(d)$ are respectively the edges and formulas of d.

Definition

Local correctness conditions for ℓ^{d} are as follows, where $\mathrm{S}(u,d)$ contains τ as u's children ($\mathrm{C}(u,d)$) or pairs of children and $\ell^{\mathrm{G}}(\tau)$ are corresponding grandparents.

- Suppose $u \neq root(d)$ and $\tau \in \mathrm{S}(u,d)$. Then:
  1. If $\tau = x \in \mathrm{C}(u,d)$, then $\ell^{\mathrm{d}}(\langle u,x\rangle,\alpha) =$
  $$\begin{cases} 1\,, & \textit{if } \ell^{\mathrm{F}}(u) = \alpha \to \ell^{\mathrm{F}}(x)\,, \\ \prod_{v\in\ell^{\mathrm{G}}(\tau)} \ell^{\mathrm{d}}(\langle v,u\rangle,\alpha)\,, & \textit{else.} \end{cases}$$
  2. If $\tau = \langle y,z\rangle$ with $y,z \in \mathrm{C}(u,d)$, then $\ell^{\mathrm{d}}(\langle u,y\rangle,\alpha) = \ell^{\mathrm{d}}(\langle u,z\rangle,\alpha) = \prod_{v\in\ell^{\mathrm{G}}(\tau)} \ell^{\mathrm{d}}(\langle v,u\rangle,\alpha)$.

§7. Local dag-like proof correctness -2-

Definition

Denote by $\mathbb{D}_{\to}$ the set of locally correct dag-like $\mathrm{NM}_{\to}$ deductions. A given assumption α is called *discharged* (or *closed*) in $\mathcal{D} = \left\langle d,\ell^{\mathrm{F}},\ell^{\mathrm{G}},\ell^{\mathrm{d}}\right\rangle \in \mathbb{D}_{\to}$ iff for every leaf u with $\ell^{\mathrm{F}}(u) = \alpha$ we have $\ell^{\mathrm{d}}(\langle x,u\rangle,\alpha) = 1$, $\forall x \in \mathrm{P}(u,D)$. Otherwise α is called *open*. Denote by $\Gamma_{\mathcal{D}}$ the set of open assumptions, in $\mathcal{D}$. $\mathcal{D}$ is called an encoded dag-like $\mathrm{NM}_{\to}$ *proof of* $\ell^{\mathrm{F}}(root(d))$ iff $\Gamma_{\mathcal{D}} = \emptyset$.

Lemma (global = local)

There is an isomorphism between global (i.e. unencoded) and encoded dag-like natural proofs of $\ell^{\mathrm{F}}(root(d))$.

Proof.

Easy bottom-up induction on the height of d. □

§8. Local dag-like proof correctness -2-

Lemma (soundness and completeness)

1. *For any given quadruple $\mathcal{D} = \left\langle d, \ell^{\mathrm{F}}, \ell^{\mathrm{G}}, \ell^{\mathrm{d}} \right\rangle \in \mathbb{D}_{\rightarrow}$, the condition '$\mathcal{D}$ is an encoded dag-like proof of root (D)' is decidable in $|\mathcal{D}|$-polynomial time.*
2. *Dag-like version of $\mathrm{NM}_{\rightarrow}$ (whether global or encoded) is sound and complete with respect to minimal propositional logic.*

Proof.

1. Straightforward.
2. Completeness trivial ($\because$ trees are dags). Soundness proved via unfolding (cf. Example).

□

§9. Horizontal compression -1-

Horizontal dag-like compression of a tree-like deduction is an ultimate inversion of the unfolding. It is obtained by iteration of *horizontal collapsing* of distinct vertices labeled by equal formulas.

Definition (*horizontal collapsing*)

Let $\mathcal{D} = \left\langle d, \ell^{\mathrm{F}}, \ell^{\mathrm{G}}, \ell^{\mathrm{d}} \right\rangle \in \mathbb{D}_{\rightarrow}$, $k \in [h(D)]$, $\alpha \in \mathrm{F}$ and $S_{\alpha} = \{x \in L_k(D) : \ell^{\mathrm{F}}(x) = \alpha\}$. [a] Moreover we assume that $(d)_{\succeq x}$ are pairwise disjoint (sub)trees, for all $x \in S_{\alpha}$. Let $u \in S_{\alpha}$ be fixed. A required collapsed dag-like deduction $\mathcal{D}^{\mathrm{C}}_{k,\alpha} = \left\langle d_{k,\alpha}, \ell^{\mathrm{F}}_{k,\alpha}, \ell^{\mathrm{G}}_{k,\alpha}, \ell^{\mathrm{d}}_{k,\alpha} \right\rangle \in \mathbb{D}_{\rightarrow}$ is stipulated as follows, where $R_{\alpha} = \bigcup_{x \in S_{\alpha}} \mathrm{S}(x, d)$ and $\left[(d)_{R_{\alpha}}\right]_u$ is a dag that extends upper subdags $\biguplus_{\tau \in R_{\alpha}} (d)_{\succeq \tau}$ by a new root u.

[a] $L_k(d)$ is the set of nodes of the height (level) k.

§9. Horizontal compression -2-

Definition (*horizontal collapsing*)

1. $d_{k,\alpha}$ arises from d by substituting $\left[(d)_{R_\alpha}\right]_u$ for $(d)_{\succeq u}$, while deleting all $(d)_{\succeq x}$, $u \neq x \in S_\alpha$.
   Note that $\mathrm{V}(d_{k,\alpha}) = \{u\} \cup (\mathrm{V}(d) \setminus S_\alpha)$. The edges are given by $\mathrm{E}\,(d_{k,\alpha}) = \mathrm{E}\,(d) \downarrow_{\mathrm{V}(d_{k,\alpha})^2} \cup \left\{ \langle w, u \rangle : w \in \bigcup_{x \in S_\alpha} \mathrm{P}\,(x, d) \right\}$.
2. $\ell^{\mathrm{F}}_{k,\alpha}$ and $\ell^{\mathrm{G}}_{k,\alpha}$ are naturally inherited from ℓ^{F} and ℓ^{G}.${}^{\mathrm{F}}_{k,\alpha}$
3. Recall that (by global = local) $\ell^{\mathrm{d}}_{k,\alpha}$ is determined by $\ell^{\mathrm{F}}_{k,\alpha}$, $\ell^{\mathrm{G}}_{k,\alpha}$ and/but also explicitly definable (omitted for brevity).

$\mathcal{D} \hookrightarrow \mathcal{D}^{\mathrm{C}}_{k,\alpha}$ is called dag-like *horizontal collapsing*, in $\mathrm{NM}_{\rightarrow}$.

Lemma

$\mathcal{D}^{\mathrm{C}}_{k,\alpha} \in \mathbb{D}_{\rightarrow}$. $\mathcal{D}$ and $\mathcal{D}^{\mathrm{C}}_{k,\alpha}$ have the same root formulas and assumptions, while $\Gamma_{\mathcal{D}^{\mathrm{C}}_{k,\alpha}} = \Gamma_{\mathcal{D}}$. Besides, $\left|\mathcal{D}^{\mathrm{C}}_{k,\alpha}\right| < |\mathcal{D}|$, if $|S_\alpha| > 1$.

§9. Horizontal compression -3-

Definition (*horizontal tree-to-dag compression*)

Let $\mathbb{T}_{\rightarrow}$ be the set of tree-like dags $\in \mathbb{D}_{\rightarrow}$. Our *compressing operator* $\mathcal{C} : \mathbb{T}_{\rightarrow} \rightarrow \mathbb{D}_{\rightarrow}$ is obtained by bottom-up iteration of the horizontal collapsing so long as possible, starting with $\mathcal{T}_{\rightarrow}$.

Theorem

For any $\mathrm{NM}_{\rightarrow}$ deduction $\mathcal{T} = \left\langle d, \ell^{\mathrm{F}}, \ell^{\mathrm{G}}, \ell^{d} \right\rangle \in \mathbb{T}_{\rightarrow}$, $h\,(\mathcal{T}) > 2$, and compressed encoded dag-like $\mathrm{NM}_{\rightarrow}$ deduction $\mathcal{C}\,(\mathcal{T}) \in \mathbb{D}_{\rightarrow}$,

$$\boxed{|\mathcal{C}\,(\mathcal{T})| < h\,(d) \cdot \phi\,(d)}$$

Moreover $\mathcal{T}$ and $\mathcal{C}\,(\mathcal{T})$ both have the same root formulas and assumptions, while $\Gamma_{\mathcal{C}(\mathcal{T})} = \Gamma_{\mathcal{T}}$. In particular, if $\mathcal{T}$ is an encoded tree-like $\mathrm{NM}_{\rightarrow}$ proof of any given α, then $\mathcal{C}\,(\mathcal{T})$ is an encoded dag-like $\mathrm{NM}_{\rightarrow}$ proof of α, whose size, $|\mathcal{C}\,(\mathcal{T})|$, is polynomial in $|\alpha|$, provided that so are both $h\,(\mathcal{T})$ and $\phi\,(\mathcal{T})$.

§10. Summary -1-

Proof.

By the definition we have

$$|\mathcal{C}(\mathcal{T})| = \bigcup_{n=0}^{h(T)} |\ell_n^{\mathrm{F}}(L_n(d))| \leq$$

$$1 + 2 + \bigcup_{n=2}^{h(T)} |\ell_n^{\mathrm{F}}(L_n(d))| \leq 3 + (h(d) - 1) \cdot \phi(d) < h(d) \cdot \phi(d).$$

The rest follows from Lemma by induction on $h(T)$. □

§10. Summary -2-

Corollary

Let d be any given tree-like $\mathrm{LM}_{\rightarrow}$ deduction of α and $\mathcal{T}$ be the encoded tree-like $\mathrm{NM}_{\rightarrow}$ proof of α that corresponds to $F(d)$. Then $|\mathcal{C}(\mathcal{T})| < 18(|\alpha| + 1)^4$.

Proof.

By previous estimates

$$\begin{aligned} |\mathcal{C}(\mathcal{T})| &\leq h(F(d)) \cdot \phi(F(d)) < 18|\alpha|(|\alpha| + 1)^2(|\alpha| + 2) \\ &< 18(|\alpha| + 1)^4. \end{aligned}$$

□

Corollary

NP = **PSPACE**, *and hence*
NP = **coNP** = **PSPACE** = **NPSPACE**.

§10. Summary -3-

Proof.

Recall that the validity problem for minimal propositional logic is PSPACE-complete. Now by standard arguments, the Corollary shows that it is a NP problem. Indeed, consider any given purely implicational formula α. By Hudelmaier's result, α is valid in the minimal logic iff there exists a tree-like $\mathrm{LM}_{\rightarrow}$ deduction *d* of α. Hence, by the embedding lemma and soundness and completeness of dag-like $\mathrm{NM}_{\rightarrow}$, α is valid in the minimal logic iff we can "guess" a dag-like $\mathrm{NM}_{\rightarrow}$ proof $\mathcal{C}(\mathcal{T})$ of α, whose size is polynomial in $|\alpha|$. Moreover, we know that the assertion ' $\mathcal{C}(\mathcal{T})$ *is an encoded dag-like* $\mathrm{NM}_{\rightarrow}$ *proof of* α ' is decidable in polynomial time with respect to $|\mathcal{C}(\mathcal{T})|$, and hence also $|\alpha|$. Thus the existence of an encoded dag-like $\mathrm{NM}_{\rightarrow}$ proof of α is verifiable in polynomial time by a non-deterministic algorithm, and hence so is the problem of minimal validity of α, Q.E.D. $\square$