Structured Perturbations of Semigroup Generators: Theory and Applications

DISSERTATION

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Introduction

Heat conduction (Jean Baptiste Joseph Fourier in [Fou09]), the propagation of waves in strings and membranes (Jean-Baptiste le Rond d'Alembert in [d'A47]), the population dynamics in biology (Alfred J. Lotka, Vito Volterra in [Vol26, Lot98]), the time evolution in quantum mechanics (Erwin Schrödinger in [Sch26a, Sch26b]) or the pricing of financial options (Fischer Black, Robert C. Merton, Myron Scholes in [BS73, Mer73]), all these processes can be described and studied by *partial differential equations* and there are many techniques to obtain the existence of unique solutions.

Using functional analytic concepts, each of the above equations (and many more) can be written as an *abstract Cauchy problem* on a Banach space X,

$$\text{(ACP)} \qquad \begin{cases} \dot{x}(t) = Gx(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

where G is an unbounded linear operator with domain $D(G) \subset X$.

The theory of strongly continuous operator semigroups provides a general and efficient method to deal with such (ACP). In particular, the properties of the corresponding semigroup directly yield properties of the solution of the differential equation.

In order to obtain the generator property of an operator G, we have, among others, the following tools:

(i) the Hille-Yosida theorem based on estimates of the resolvent;

(ii) the Lumer-Phillips theorem based on the notion of dissipative operators;

(iii) perturbation and approximation techniques using, under appropriate assumptions, the *robustness of properties*.

In this thesis we study additive perturbations. This dates back to R. Phillips who proved in his seminal work [Phi53]

- the generator property of $A + P$ for all bounded perturbations $P \in \mathcal{L}(X)$ and

- showed that certain qualitative properties of the unperturbed semigroup are preserved.

In such a perturbation approach to (ACP) we assume that the operator G is a certain sum of two operators A and P,

$$G = \text{“}A + P\text{”},$$

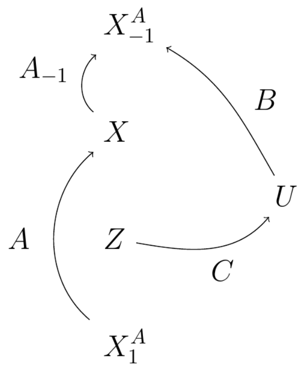
where the quotation marks indicate a yet unprecise formulation of the sum.

In this work we restrict ourselves to the study of so called *structured perturbations* $P = BC$ (cf. Definition III.3) studied by George Weiss in [Wei94a, Sects. 5-7] and Olof Staffans in [Sta05, Chap. 4 & 7] within their works on *closed-loop linear systems* (cf. [Sal87, Sect. 2.1] too). In such systems the operators B and C occur as *control* and *observation operators* of a system $\Sigma(A, B, C)$ given by

$$\text{(0.1)} \qquad \Sigma(A,B,C) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geq 0, \\ y(t) = Cx(t), & t \geq 0, \\ x(0) = x_0. \end{cases}$$

Formally, introducing the *feedback law* $y = u$ leads to a perturbation $P = BC$ and the abstract Cauchy problem (ACP) associated with $G = A + BC$.
In contrast to this system theoretic view we take a purely operator theoretic approach aiming to solve partial differential equations in a unified and systematic manner, while sometimes borrowing the control theoretic language.

At this point we describe the structured perturbation $P = BC$ for later reference (see [ABE14, ABE15, AE15] for the following notations). We first choose two Banach spaces X and U, called *state-* and *observation/control space*, respectively.



On these spaces we consider the operators

- $A : D(A) \subset X \to X$, called *state operator* and having non-empty resolvent set $\rho(A)$,
- $B \in \mathcal{L}(U, X_{-1}^A)$, called *control operator*,
- $C \in \mathcal{L}(Z, U)$, called *observation operator*.

Here, Z is a Banach space such that $X_1^A \hookrightarrow Z \hookrightarrow X$, where $\hookrightarrow$ denotes a continuous linear injection. Moreover, X_1^A and X_{-1}^A are the *inter-* and *extrapolation spaces* with respect to A, cf. [EN00, Sect. II.5].

Then we consider the operator

$$(0.2)\qquad \begin{aligned} A_{BC} &:= (A_{-1} + BC)|_X, \\ D(A_{BC}) &:= \{x \in Z : A_{-1}x + BCx \in X\}, \end{aligned}$$

where $A_{-1} : X \subset X_{-1}^A \to X_{-1}^A$ is the extension of A to X. Here, the sum $A_{-1} + BC$ is defined in X_{-1}^A and its restriction to the state space X yields A_{BC}.

Definition ([ABE14, Def. 5])**.** *Choose operators $A : D(A) \subset X \to X$ with $\rho(A) \neq \emptyset$, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(Z, U)$. The triple (A, B, C) is called* compatible *if for some (hence all) $\lambda \in \rho(A)$*

$$(0.3)\qquad \operatorname{rg}(R(\lambda, A_{-1})B) \subset Z = D(C).$$

The *compatibility condition* (0.3) expresses that the perturbation $P = BC \in \mathcal{L}(Z, X_{-1}^A)$ does not surmount the unboundedness of the operator A and was already used in, e.g., [Hel76, Sect. II.A], [Wei94b, Thm. 5.8] and [Sta05, Def. 5.1.1].

In the following we deal with various tasks.

1) Identify a large class of operators G that can be written as A_{BC};
2) Characterize the spectrum of $G = A_{BC}$ in terms of A, B and C;
3) Find conditions on the operators A, B and C implying that $G = A_{BC}$ generates a strongly continuous semigroup $(T_{BC}(t))_{t \geq 0}$;
4) Give conditions on the operators A, B and C such that $G = A_{BC}$ generates a strongly continuous cosine family $(C_{BC}(t))_{t \in \mathbb{R}}$;
5) On the basis of the established generator property of G, investigate the asymptotic properties of the strongly continuous semigroup $(T_{BC}(t))_{t \geq 0}$ in terms of A, B and C.

Beside the above questions, there is extensive literature covering, e.g.,

- unbounded perturbations preserving the exponential dichotomy and the Fredholm property (cf. [CL96, MS08]),
- robustness of maximal regularity under perturbations (cf. [KW01, HHK06]),
- regularity properties and positivity[1] of perturbed semigroups (cf. [Voi89, AR91, NP98, BMR02, Mát04, BA06, Mát08, DEHR09]), and

[1] We recommend [Nag86] for a detailed investigation of positive operators on Banach lattices and [Eng97] for a characterization of positive semigroup useful in applications.

- an in-depth analysis of spectrum related topics, i.e., a perturbation theorem for the essential spectral radius of a strongly continuous semigroup (cf. [Voi80, NR95] and [Voi94] as well) and a spectral mapping theorem for perturbed semigroups (cf. [BNP00]).

We now sketch the content of the chapters in this thesis.

In Chapter I we deal with perturbations of linear (e.g., differential) operators in the spirit of Günther Greiner. His approach in [Gre87] towards boundary perturbations is here generalized to cover unbounded perturbations as well. We complete this chapter with two examples showing what might occur when perturbing the boundary condition of a linear operator. This abstract approach is published in [ABE14, Sect. 4.3] (cf. also [Bom15, Chap. 3]) and in the manuscripts [ABE15, Sect. 3.1] (joint work with Miriam Bombieri and Klaus-Jochen Engel) and [AE15, Sect. 3.1] (joint work with Klaus-Jochen Engel).

In Chapter II we characterize the spectrum $\sigma(G)$ of the operator $G = A_{BC}$ in terms of the operators A, B and C. Beside the spectrum we investigate its finer subdivisions as well. The results are contained in the manuscript [AE15] (joint work with Klaus-Jochen Engel).

In Chapter III we cover the abstract Weiss-Staffans perturbation result (cf. Theorem III.8). We recall the definitions of *admissibility* and introduce the class of *structured* or *Weiss-Staffans perturbations* (cf. Definition III.3). The main result of this chapter is published in [ABE14] (cf. also [Bom15, Chap. 1]) and generalizes well-established perturbation theorems due to

(i) W. Desch, I. Lasiecka and W. Schappacher in [DLS85] and [DS89] (see Theorem III.1),

(ii) I. Miyadera and J. Voigt in [Miy66] and [Voi77] (see Theorem III.2), and

(iii) G. Greiner in [Gre87] (see Chapter I and Corollary III.13).

In Chapter IV we investigate structured perturbations in case the unperturbed semigroup is *analytic*. It is our goal to replace the admissibility conditions by assumptions on the range and domain of the operators B and C. This result (cf. Theorem IV.1) is contained in the forthcoming paper [ABE15] (joint work with Miriam Bombieri and Klaus-Jochen Engel) and [Bom15, Chap. 4].

In Chapter V we study perturbations of generators of *strongly continuous cosine families*. Our perturbation result in Theorem V.3 is applied to second order Cauchy problems associated with second order differential operators on an interval with perturbed Neumann- and Wentzell boundary conditions. The results are joint work with Klaus-Jochen Engel.

In Chapter VI we deal with asymptotic properties of the perturbed semigroup $(T_{BC}(t))_{t\geq 0}$ and give conditions on the operators B and C such that the asymptotic properties of the unperturbed semigroup $(T(t))_{t\geq 0}$ are preserved. This result in contained in the preprint [Adl15].

In Chapter VII we sketch an approach towards the wellposedness of heat and wave equations on networks.

In each chapter we give various applications of our results.

In the Appendix we study spectral properties of operators (and their parts), the invertibility of operator matrices using Schur complements (the results are contained in [AE15] which is joint work with Klaus-Jochen Engel) and parts of generators of strongly continuous cosine families (joint work with Klaus-Jochen Engel).

The content in this dissertation is in some parts verbatim consistent with the above mentioned manuscripts while we adapt the notations and further elaborate on some examples.

As general reference for the basic concepts of strongly continuous operator semigroups and the theory of perturbations we recommend the monographs [HP57, Kat95, Paz83, Gol85, EN00].

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I. Boundary Perturbations

In this chapter we introduce an abstract setting in order to study *boundary perturbations* of a (differential) operator as proposed by G. Greiner in [Gre87] (see also [DLS85, Sect. 3]). While Greiner's perturbation acts on the domain of the operator, it is the aim of this chapter to rewrite them as additive perturbations of the form (0.2). We point out that many concrete examples fit into this abstract framework, so we shall return throughout this thesis to the setup presented in this chapter.

This chapter summarizes results published in [ABE14, Sect. 4.3] (cf. [Bom15, Chap. 3] as well) and two forthcoming articles from which the presentation is taken:

- *Perturbation of analytic semigroups and applications to partial differential equations* which is joint work with Miriam Bombieri and Klaus-Jochen Engel (cf. [ABE15, Sect. 3.1]), and

- *Spectral theory for structured perturbations of linear operators* which is joint work with Klaus-Jochen Engel (cf. [AE15, Sect. 3.1]).

General Setting I.1. Consider

- a *Banach state space* X and a *Banach space of boundary conditions* ∂X;

- a *maximal operator*[1] $A_m : D(A_m) \subseteq X \to X$.

In order to single out a restriction A of A_m we take a *boundary operator* $L : D(A_m) \subset X \to \partial X$ and define

$$(1.1) \qquad A \subseteq A_m, \quad D(A) := \Big\{x \in D(A_m) : Lx = 0\Big\} = \ker(L).$$

This operator shall be perturbed in the following way. For a Banach space Z satisfying $D(A_m) \subseteq Z$ and $X_1^A \hookrightarrow Z \hookrightarrow X$, and operators $P \in \mathcal{L}(Z, X)$, $\Phi \in \mathcal{L}(Z, \partial X)$ we consider $G := A_P^\Phi$ given by

$$(1.2) \qquad A_P^\Phi := A_m + P, \quad D(A_P^\Phi) := \Big\{x \in D(A_m) : Lx = \Phi x\Big\} = \ker(L - \Phi),$$

$$
\begin{tikzcd}
X \supseteq Z \supseteq D(A_m) \arrow[r, "A_m"] \arrow[r, bend left, "P"] \arrow[dr, "L"] \arrow[dr, bend right, "\Phi"'] & X \\
& \partial X
\end{tikzcd}
$$

Diagram 1: The operators defining A_P^Φ in (1.2).

cf. Diagram 1.
Hence, A_P^Φ can be considered as a perturbation of A

- by the operator P to change its action, and

- by the operator Φ to change its domain.

The concrete cases we have in mind are differential operators A_m on a function space X over some domain Ω and ∂X as a space of functions over the boundary $\partial\Omega$ of the domain. In this context, the operator L maps a “differentiable” function, i.e., $f \in D(A_m)$, to certain boundary values, thus fixing the boundary condition.

Here, the main difference to Greiner’s setting is the possible degree of unboundedness of the perturbation, since P and Φ, being bounded on Z, are unbounded on X. Next we want to identify operators A, B and C such that $G = A_P^\Phi$ is given by A_{BC} as in (0.2). To this end we make the following

Assumptions I.2. (a) The operator A has non-empty resolvent set $\rho(A)$.

(b) For some $\mu \in \mathbb{C}$ the restriction

$$L|_{\ker(\mu - A_m)} : \ker(\mu - A_m) \to \partial X$$

is invertible with bounded inverse

$$L_\mu := \left(L|_{\ker(\mu - A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X).$$

Remark I.3. We call the operator L_μ the abstract *Dirichlet operator for the pair* (A_m, L) since it solves the abstract Dirichlet problem, i.e., for $g \in \partial X$ the element

[1]“maximal” in the sense of a “big” domain, e.g., a differential operator without boundary conditions.

$f := L_\mu g \in D(A_m)$ solves

$$\begin{cases} (\mu - A_m)f = 0, \\ Lf = g. \end{cases}$$

In [Gre87, Equ. (1.13)] and [CENN03, Ass. 2.1] there are sufficient conditions implying Assumption I.2.(b). This is summarized in the following lemma.

Lemma I.4. *If L is surjective and either*

- *A_m is closed and $L \in \mathcal{L}([D(A_m)], \partial X)$,[2] or*
- *$\binom{A_m-\mu}{L} : D(A_m) \subset X \to X \times \partial X$ is closed for some (hence all) $\mu \in \mathbb{C}$,[3]*

then for every $\lambda \in \rho(A)$, Assumption I.2.(b) is satisfied.

We mimic the proof in [CENN03, Lem. 2.2].

Proof. For $\lambda \in \rho(A)$ the maximal domain $D(A_m)$ can be decomposed into the direct sum

$$D(A_m) = D(A) \oplus \ker(\lambda - A_m),$$

cf. [Gre87, Lem. 1.2]. Since $A := A_m|_{\ker(L)}$ and L is surjective, the operator

$$L : \ker(\lambda - A_m) \to \partial X \quad \text{is bijective}$$

with inverse $L_\lambda : \partial X \to \ker(\lambda - A_m) \subset X$. To obtain boundedness of $L_\lambda \in \mathcal{L}(\partial X, X)$, it suffices to show that $L : \ker(\lambda - A_m) \subset X \to \partial X$ is closed by the closed graph theorem. Take $(f_n)_{n\in\mathbb{N}} \subset \ker(\lambda - A_m)$ such that

$$f_n \to f \in X \qquad \text{and} \qquad Lf_n \to x \in \partial X \qquad \text{as } n \to \infty.$$

First, assume that A_m is closed. The first convergence yields $A_m f_n = \lambda f_n \to \lambda f$. Thus, $f \in \ker(\lambda - A_m)$ by the closedness of A_m, and the convergence holds in $[D(A_m)]$. By the continuity assumption on L we obtain $Lf = x$.
Second, if $\binom{A_m-\mu}{L}$ is closed, then

$$\binom{A_m-\mu}{L} f_n = \binom{A_m f_n - \mu f_n}{L f_n} \to \binom{\lambda f - \mu f}{x} \in X \times \partial X \qquad \text{as } n \to \infty.$$

[2] For a linear operator T we write $[D(T)] := (D(T), \|\bullet\|_T)$ with the graph norm $\|\bullet\|_T$ given by $\|x\|_T := \|x\| + \|Tx\|$ for $x \in D(T)$.

[3] In [CENN03, Rem. 3.3], the authors give an example showing that $\binom{A_m}{L}$ is closed while A_m is not closed.

Thus, $f \in D(A_m)$ and $\binom{A_m f - \mu f}{Lf} = \binom{\lambda f - \mu f}{x}$, i.e., $f \in \ker(\lambda - A_m)$ and $Lf = x$. $\square$

Next we elaborate on the *Dirichlet operator* L_μ playing a crucial role in this approach.

Proposition I.5. *Let Assumption I.2.(b) be satisfied. Then for all* $\lambda \in \rho(A)$

$$L|_{\ker(\lambda - A_m)} : \ker(\lambda - A_m) \to \partial X$$

is invertible with bounded inverse given by

$$L_\lambda = (\mu - A)R(\lambda, A)L_\mu \in \mathcal{L}(\partial X, X). \tag{1.3}$$

Proof. Let $\tilde{L}_\lambda \in \mathcal{L}(\partial X, X)$ be the operator defined by the right-hand side of (1.3). Then the identity

$$\tilde{L}_\lambda = (\mu - A)R(\lambda, A)L_\mu = L_\mu + (\mu - \lambda)R(\lambda, A)L_\mu$$

implies that $\operatorname{rg}(\tilde{L}_\lambda) \subseteq \ker(\mu - A_m) + D(A) \subseteq D(A_m)$ and $L\tilde{L}_\lambda = \mathrm{Id}_{\partial X}$. Moreover, for $x \in \partial X$

$$\begin{aligned}(\lambda - A_m)\tilde{L}_\lambda x &= (\lambda - A_m)L_\mu x + (\mu - \lambda)(\lambda - A_m)R(\lambda, A)L_\mu x \\ &= (\lambda - \mu)L_\mu x + (\mu - \lambda)L_\mu x = 0,\end{aligned}$$

i.e., $\operatorname{rg}(\tilde{L}_\lambda) \subseteq \ker(\lambda - A_m)$. Summing up this proves that $L : \ker(\lambda - A_m) \to \partial X$ is surjective with right-inverse $\tilde{L}_\lambda$. To show injectivity assume that $x \in \ker(\lambda - A_m) \cap \ker(L)$. Then $x \in D(A)$ and $(\lambda - A)x = 0$ which implies $x = 0$ since $\lambda \in \rho(A)$. $\square$

Note that, by the previous result, the identity $L_\lambda = R(\lambda, A_{-1})(\mu - A_{-1})L_\mu$ holds. Hence, the operator

$$L_A := (\mu - A_{-1})L_\mu = (\lambda - A_{-1})L_\lambda \in \mathcal{L}(\partial X, X_{-1}^A) \tag{1.4}$$

is independent of $\lambda \in \rho(A)$.
The following result characterizes resolvent points of A in terms of the existence of L_λ.

Lemma I.6. *Let Assumption I.2.(b) hold. For A given by* (1.1) *we have $\lambda \in \rho(A)$ if and only if*

(i) A is closed,

(ii) $\lambda - A_m : D(A_m) \to X$ is surjective,

(iii) $L_\lambda : \partial X \to X$ *exists, i.e., for every* $x \in \partial X$ *there exists a unique* $f = L_\lambda x \in \ker(\lambda - A_m)$ *such that* $Lf = x$*.*

Proof. If $\lambda \in \rho(A)$, then clearly A is closed, $\lambda - A_m$ is surjective and L_λ exists by Proposition I.5. Now assume that (i)–(iii) hold. First we show that $\lambda - A$ is surjective. Let $g \in X$, then by (ii) there exist $\tilde{h} \in D(A_m)$ such that $(\lambda - A_m)\tilde{h} = g$. Define $h := (\mathrm{Id}_X - L_\lambda L)\tilde{h}$. Then $Lh = 0$, i.e., $h \in D(A)$ and

$$(\lambda - A)h = (\lambda - A_m)(\mathrm{Id}_X - L_\lambda L)\tilde{h} = g$$

which shows surjectivity. To show injectivity, assume $(\lambda - A)f = 0$ for some $f \in D(A) = \ker(L)$. Then $f \in \ker(\lambda - A_m)$ and $Lf = 0$. By the uniqueness in (iii) we conclude $f = 0$. Summing up, this shows that $\lambda - A$ is bijective, and since A is closed by (i), the closed graph theorem implies $\lambda \in \rho(A)$ as claimed. $\square$

We now wish to rewrite the operator A_P^Φ in (1.2) as A_{BC} in (0.2) for suitable operators B and C, as visualized in Diagram 2.

$$\begin{tikzcd}
Z \arrow[r, "P"] \arrow[dr, "\Phi"'] & X \arrow[r, "A_{-1}"] & X_{-1}^A \\
& \partial X \arrow[u, "L_\lambda"] \arrow[ur, "L_A = (\mu - A_{-1})L_\mu"'] &
\end{tikzcd}$$

Diagram 2: The operators appearing in the representation $A_P^\Phi = (A_{-1}+P+L_A\cdot\Phi)|_X$.

Lemma I.7. *We have*

$$G = \Big(A_{-1} + P + L_A \cdot \Phi\Big)\Big|_X. \tag{1.5}$$

Proof. Denote by $\tilde{G}$ the operator defined by the right-hand side of (1.5) and fix $\lambda \in \rho(A)$. Then for $x \in Z$ we have

$$\begin{aligned}
x \in D(\tilde{G}) \quad &\Longleftrightarrow \quad A_{-1}x + Px + (\lambda - A_{-1})L_\lambda\Phi x \in X \\
&\Longleftrightarrow \quad (\lambda - A_{-1})\Big(L_\lambda\Phi - \mathrm{Id}\Big)x + (P + \lambda)x \in X \\
&\Longleftrightarrow \quad \Big(L_\lambda\Phi - \mathrm{Id}\Big)x \in D(A) = \ker(L) \\
&\Longleftrightarrow \quad Lx = \Phi x \qquad\qquad (1.6)\\
&\Longleftrightarrow \quad x \in D(G),
\end{aligned}$$

where in (1.6) we used that $x = \Big(\mathrm{Id} - L_\lambda \Phi\Big)x + L_\lambda \Phi x \in D(A) + \ker(\lambda - A_m) \subseteq D(A_m)$ so that

$$0 = L(L_\lambda \Phi x - x) = \Phi x - Lx.$$

Moreover, for $x \in D(G)$ we obtain

$$\begin{aligned}\tilde{G}x &= (\lambda - A)\Big(L_\lambda \Phi - \mathrm{Id}\Big)x + (P + \lambda)x \\ &= (\lambda - A_m)L_\lambda \Phi x - (\lambda - A_m)x + (P + \lambda)x \\ &= A_m x + Px = Gx,\end{aligned}$$

hence $G = \tilde{G}$ as claimed. $\square$

In order to represent G given in (1.5) as A_{BC} as in (0.2), we define the product space

$$U := X \times \partial X \tag{1.7}$$

and the operators

$$B := \Big(\mathrm{Id}_X, L_A\Big) \in \mathcal{L}(U, X_{-1}^A) \quad \text{and} \quad C := \binom{P}{\Phi} \in \mathcal{L}(Z, U). \tag{1.8}$$

Then a simple computation shows the following.

Lemma I.8. *The triple (A, B, C) given by (1.1) and (1.8) is compatible. Moreover, $G = A_P^\Phi$ in (1.2) can be written as A_{BC} in (0.2), i.e., the operators $A_P^\Phi = A_m + P$ with domain $D(A_P^\Phi) = \ker(L - \Phi)$ and $A_{BC} = (A_{-1} + BC)|_X$ with domain $D(A_{BC}) := \{x \in Z : A_{-1}x + BCx \in X\}$ coincide.*

Proof. The compatibility condition (0.3) follows from (1.4) since for $\lambda \in \rho(A)$

$$\mathrm{rg}\Big(R(\lambda, A_{-1})B\Big) = \mathrm{rg}\Big(R(\lambda, A), R(\lambda, A_{-1})L_A\Big) \subset D(A) + \ker(\lambda - A_m) \subseteq D(A_m) \subseteq Z.$$

By Lemma I.7 we obtain $G = (A_{-1} + P + L_A \cdot \Phi)|_X = (A_{-1} + BC)|_X = A_{BC}$ which yields the assertion. $\square$

For additional literature concerning perturbations of the boundary conditions of an operator, we refer to [DLS85, Sect. 3], [Nic04], [HMR15, Sect. 4]. Moreover, there are many results dealing with *boundary control problems.*[4] For literature thereon we recommend [Fat68], [Sal87, Sect. 2.2], [TW09, Chap. 10] and references therein.

[4]Boundary control problems play a role in Lemma III.14 and Lemma V.11 in this thesis.

We close this chapter by presenting two simple situations illustrating some anomalies in the context of boundary perturbations. Even elementary properties such as the density of the domain or the existence of resolvent points fail to be preserved under perturbations of the boundary conditions.

Example I.9. Perturbing the domain of a densely defined operator does not necessarily yield a densely defined operator. On the space $X := \mathrm{C}[0,1]$ consider the first derivative $A_m := \frac{d}{ds}$ with domain $D(A_m) := \mathrm{C}^1[0,1] =: Z$. As boundary space choose $\partial X := \mathbb{C}$ and as boundary operator $L := \delta_1' : D(A_m) \subset X \to \mathbb{C}$, i.e., $Lf = f'(1)$. This yields the densely defined operator

$$\begin{aligned} Af &= f', \\ D(A) &= \{f \in \mathrm{C}^1[0,1] : f'(1) = 0\} = \ker(L). \end{aligned}$$

If we perturb the boundary condition by $\Phi := \delta_1 + \delta_1' \in (\mathrm{C}^1[0,1])'$ we obtain the operator

$$\begin{aligned} A^\Phi f &:= A_0^\Phi f = f', \\ D(A^\Phi) &= \{f \in \mathrm{C}^1[0,1] : f'(1) = \Phi f\} = \{f \in \mathrm{C}^1[0,1] : f(1) = 0\} = \ker(L - \Phi), \end{aligned}$$

which is not densely defined on $\mathrm{C}[0,1]$.

Example I.10. Perturbing the boundary conditions of an operator might result in a large spectrum. On the space $X := \mathrm{L}^p[0,1]$ introduce the first derivative $A_m := \frac{d}{ds}$ with domain $D(A_m) := \mathrm{W}^{1,p}[0,1] =: Z$. We choose as boundary space $\partial X := \mathbb{C}$, as boundary operator the point evaluation $L := \delta_1 : D(A_m) \subset X \to \mathbb{C}$ and as perturbation some $\Phi \in (\mathrm{W}^{1,p}[0,1])'$. This gives rise to the differential operators $A, A^\Phi \subset \frac{d}{ds}$ with domains

$$\begin{aligned} D(A) &= \{f \in \mathrm{W}^{1,p}[0,1] : f(1) = 0\} = \ker(L), \\ D(A^\Phi) &= \{f \in \mathrm{W}^{1,p}[0,1] : f(1) = \Phi f\} = \ker(L - \Phi). \end{aligned}$$

Then A satisfies $\rho(A) = \mathbb{C}$. However, choosing $\Phi := \delta_1$ yields $A^\Phi = A_m$, hence $\sigma_p(A^\Phi) = \mathbb{C}$.

II. Spectral Theory under Perturbations

The *spectrum*

$$\sigma(G) := \{\lambda \in \mathbb{C} : \lambda - G \text{ is not invertible in } \mathcal{L}(X)\}$$

as a subset of $\mathbb{C}$, and its finer subdivisions

$$\begin{aligned}
\sigma_p(G) &:= \Big\{\lambda \in \mathbb{C} : \lambda - G \text{ is not injective}\Big\} && \text{the } \textit{point spectrum}, \\
\sigma_a(G) &:= \left\{\lambda \in \mathbb{C} : \begin{array}{l} \lambda - G \text{ is not injective or} \\ \text{has non-closed range} \end{array}\right\} && \text{the } \textit{approximative point spectrum}, \\
\sigma_c(G) &:= \left\{\lambda \in \mathbb{C} : \begin{array}{l} \lambda - G \text{ is injective with} \\ \text{dense, non-closed range} \end{array}\right\} && \text{the } \textit{continuous spectrum}, \\
\sigma_r(G) &:= \left\{\lambda \in \mathbb{C} : \begin{array}{l} \lambda - G \text{ is injective with} \\ \text{non-dense range} \end{array}\right\} && \text{the } \textit{residual spectrum}, \\
\sigma_{\text{ess}}(G) &:= \left\{\lambda \in \mathbb{C} : \begin{array}{l} \dim(\ker(\lambda - G)) = \infty \text{ or} \\ \operatorname{codim}(\operatorname{rg}(\lambda - G)) = \infty \end{array}\right\} && \text{the } \textit{essential spectrum},
\end{aligned}$$

reflect much information about a (possibly unbounded) linear operator $G : D(G) \subset X \to X$ on a Banach space X. Here we only mention that for generators G of strongly continuous semigroups the location of $\sigma(G)$ in the complex plane influences the asymptotic behaviour of the solutions of the associated abstract Cauchy problem (for details see [EN00, Chap. V]).

However, in many applications it is difficult to determine $\sigma(G)$ by direct computations. The philosophy in this chapter is to represent a given operator G as A_{BC} as in (0.2) such that the spectrum $\sigma(A)$ of the unperturbed operator A is "small" and easy to compute. Then the theorems below lead to a description of $\sigma(G)$.

The results below will be published in the forthcoming paper *Spectral theory for structured perturbations of linear operators* which is joint work with Klaus-Jochen Engel (cf. [AE15]). The presentation is taken from that manuscript.

II.1. Characterizing the Spectral Values of A_{BC}

We investigate the spectrum of the operator $G = A_{BC}$ assuming the following.

Assumption II.1. We assume that the triple (A, B, C) with $A : D(A) \subset X \to X$, $\rho(A) \neq \emptyset$, $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(Z, U)$ is compatible (see Introduction).

To start our investigations, we fix $\mu \in \rho(A)$ and define the Banach space

$$Z_{-1} := (\mu - A_{-1})Z, \quad \|\bullet\|_{Z_{-1}} := \|R(\mu, A_{-1})\bullet\|_Z$$

which satisfies $X \hookrightarrow Z_{-1} \hookrightarrow X_{-1}$. Note that (0.3) implies $\mathrm{rg}(B) \subseteq Z_{-1}$, hence, by the closed graph theorem, $B \in \mathcal{L}(U, Z_{-1})$. We now define the operators

$$\begin{aligned} &A^Z_{BC} : Z \subseteq Z_{-1} \to Z_{-1}, \quad &&A^Z_{BC}x := A_{-1}x + BCx, \\ &A^Z : Z \subseteq Z_{-1} \to Z_{-1}, &&A^Z x := A_{-1}x, \end{aligned}$$

for which the following holds.

Lemma II.2. *We have $A^Z_{BC}, A^Z \in \mathcal{L}(Z, Z_{-1})$. Moreover, if A^Z_{BC} is closed, e.g., $\rho(A^Z_{BC}) \neq \emptyset$, then the norm of Z and the graph norm of A^Z_{BC} are equivalent on Z, i.e.,*

$$\|\bullet\|_Z \simeq \|\bullet\|_{A^Z_{BC}}, \tag{2.1}$$

where $\|x\|_{A^Z_{BC}} := \|x\|_{Z_{-1}} + \|A^Z_{BC}x\|_{Z_{-1}}$ for $x \in Z$. In other words, $Z \simeq (Z_{-1})_1^{A^Z_{BC}}$.

Proof. As already mentioned, $B \in \mathcal{L}(U, Z_{-1})$. Since $C \in \mathcal{L}(Z, U)$ and $A^Z \in \mathcal{L}(Z, Z_{-1})$ by the definition of the norm in Z_{-1}, we conclude that $A^Z_{BC} \in \mathcal{L}(Z, Z_{-1})$. If A^Z_{BC} is closed, then $(Z, \|\bullet\|_{A^Z_{BC}})$ is a Banach space. Moreover, $Z \hookrightarrow Z_{-1}$ and therefore $\|\bullet\|_Z$ is finer than $\|\bullet\|_{A^Z_{BC}}$, i.e.,

$$\|z\|_{A^Z_{BC}} = \|z\|_{Z_{-1}} + \|A^Z_{BC}z\|_{Z_{-1}} \leq C\|z\|_Z \quad \text{for all } z \in Z \text{ and some } C \geq 0$$

and the equivalence in (2.1) follows from the open mapping theorem. $\square$

The following operators will be a main tool in the sequel.

Definition II.3. For $\lambda \in \rho(A)$ define the operators

$$\Delta_U(\lambda) := CR(\lambda, A_{-1})B \in \mathcal{L}(U) \quad \text{and} \quad \Delta_Z(\lambda) := R(\lambda, A_{-1})BC \in \mathcal{L}(Z).$$

Note that the boundedness of $\Delta_U(\lambda)$ and $\Delta_Z(\lambda)$ follows from Assumption II.1, the closed graph theorem and the resolvent equation. Using these operators the spectral values of A_{BC} can be characterized in the following way.

Theorem II.4. *Let $\lambda \in \rho(A)$.*

(a) The following spectral characterizations hold.

$$\begin{aligned} \lambda \in \sigma(A_{BC}^Z) \quad &\Longleftrightarrow \quad 1 \in \sigma\Big(\Delta_U(\lambda)\Big), \\ \lambda \in \sigma_p(A_{BC}) \quad &\Longleftrightarrow \quad 1 \in \sigma_p\Big(\Delta_U(\lambda)\Big). \end{aligned}$$

(b) If $\Delta_U(\lambda) \in \mathcal{L}(U)$ is compact, then

$$\lambda \in \sigma(A_{BC}) \quad \Longleftrightarrow \quad \lambda \in \sigma_p(A_{BC}) \quad \Longleftrightarrow \quad 1 \in \sigma_p\Big(\Delta_U(\lambda)\Big).$$

In particular, if $\dim(U) < \infty$, *then*

$$\lambda \in \sigma(A_{BC}) \quad \Longleftrightarrow \quad \lambda \in \sigma_p(A_{BC}) \quad \Longleftrightarrow \quad \det\Big(\mathrm{Id}_U - \Delta_U(\lambda)\Big) = 0, \tag{2.2}$$

i.e., the spectral values in $\sigma(A_{BC}) \cap \rho(A)$ are given as the zeros of a (nonlinear) characteristic equation.

(c) If the condition

$$1 \in \rho\Big(\Delta_U(\nu)\Big) \text{ for some } \nu \in \rho(A) \quad \text{or, equivalently,} \quad \rho(A) \cap \rho(A_{BC}^Z) \neq \emptyset \tag{2.3}$$

holds, then

$$\lambda \in \sigma(A_{BC}) \quad \Longleftrightarrow \quad 1 \in \sigma\Big(\Delta_U(\lambda)\Big). \tag{2.4}$$

In particular, we obtain

$$\lambda \in \sigma_*(A_{BC}) \quad \Longleftrightarrow \quad 1 \in \sigma_*\Big(\Delta_U(\lambda)\Big) \tag{2.5}$$

for all $* \in \{p, a, r, c, ess\}$.

(d) Finally, if $1 \in \rho\Big(\Delta_U(\lambda)\Big)$, then $\lambda \in \rho(A_{BC})$ and the resolvent is given by

$$R(\lambda, A_{BC}) = R(\lambda, A) + R(\lambda, A_{-1})B\Big(Id_U - \Delta_U(\lambda)\Big)^{-1}CR(\lambda, A). \tag{2.6}$$

Proof. The proof is based on three ingredients: the extrapolated operator A_{BC}^Z, spectral properties of the part of an operator in a subspace (cf. Section A in the

Appendix) and *Schur complements* for operator matrices (cf. Section B in the Appendix). We start by showing some spectral inclusions. From Lemma A.3 the inclusions

$$\begin{aligned} \sigma(A_{BC}) &\subseteq \sigma(A^Z_{BC}), & \sigma_p(A_{BC}) &= \sigma_p(A^Z_{BC}), \\ \sigma_a(A_{BC}) &\subseteq \sigma_a(A^Z_{BC}), & \sigma_{\mathrm{ess}}(A_{BC}) &\subseteq \sigma_{\mathrm{ess}}(A^Z_{BC}) \end{aligned} \tag{2.7}$$

follow by choosing the spaces $F := Z_{-1}$, $E := X$ and operators $T := A^Z_{BC}$, $T_1 := T|_E = A_{BC}$, while we need the extra assumption that

$$X + \mathrm{rg}(A^Z_{BC}) \qquad \text{is dense in } Z_{-1} \tag{2.8}$$

to obtain the inclusions

$$\sigma_c(A_{BC}) \subseteq \sigma_c(A^Z_{BC}), \qquad \sigma_r(A_{BC}) \supseteq \sigma_r(A^Z_{BC}).$$

Since

$$X + \mathrm{rg}(A^Z_{BC}) = X + (\nu - A^Z_{BC})Z = (\nu - A_{-1})\Big[D(A) + (\mathrm{Id}_Z - R(\nu, A_{-1})BC)Z\Big]$$

holds for $\nu \in \rho(A)$, the condition (2.8) is satisfied if and only if

$$D(A) + \mathrm{rg}(\mathrm{Id}_Z - \Delta_Z(\nu)) \qquad \text{is dense in } Z. \tag{2.9}$$

To prove (a) we define for $\lambda \in \rho(A)$ the operator matrix

$$\mathfrak{T} := \begin{pmatrix} \lambda - A^Z & B \\ C & \mathrm{Id}_U \end{pmatrix} \in \mathcal{L}(Z \times U, Z_{-1} \times U). \tag{2.10}$$

Then the Schur complements of $\mathfrak{T}$ in Lemma B.1 are given by

$$\begin{aligned} \Delta_1 &= \lambda - A^Z_{BC} \in \mathcal{L}(Z, Z_{-1}), \\ \Delta_2 &= \mathrm{Id}_U - \Delta_U(\lambda) \in \mathcal{L}(U). \end{aligned}$$

Hence, from Lemma B.1.(iv)–(vi) it follows that the operator $\lambda - A^Z_{BC}$

- is injective,
- has dense range,
- has range with finite co-dimension,
- has closed range,
- has finite dimensional kernel,
- is invertible,

if and only if $\mathrm{Id}_U - \Delta_U(\lambda)$ has the same property, respectively. These properties characterize in particular the (point) spectrum and we obtain the equivalences

$$\begin{aligned}
\lambda \in \sigma(A^Z_{BC}) \qquad &\iff \qquad 1 \in \sigma\Big(\Delta_U(\lambda)\Big),\\
\lambda \in \sigma_p(A^Z_{BC}) = \sigma_p(A_{BC}) \qquad &\iff \qquad 1 \in \sigma_p\Big(\Delta_U(\lambda)\Big).
\end{aligned}$$

For (b) assume that $\Delta_U(\lambda)$ is compact. Then by (2.7) and (a) we conclude

$$\begin{aligned}
\lambda \in \sigma(A_{BC}) \implies \lambda \in \sigma(A^Z_{BC}) \iff 1 \in \sigma\Big(\Delta_U(\lambda)\Big) \iff 1 \in \sigma_p\Big(\Delta_U(\lambda)\Big)\\
\iff \lambda \in \sigma_p(A^Z_{BC}) \iff \lambda \in \sigma_p(A_{BC}) \implies \lambda \in \sigma(A_{BC}).
\end{aligned}$$

Therefore, all conditions are equivalent and we proved the first chain of equivalences. If U is finite dimensional, then $\Delta_U(\lambda)$ is compact for all $\lambda \in \rho(A)$ and (2.2) follows from the above.

For (c) assume that there exists $\nu \in \rho(A)$ such that $1 \in \rho(\Delta_U(\nu))$ which, by the first equivalence in (a), is equivalent to the existence of some $\nu \in \rho(A) \cap \rho(A^Z_{BC})$. Then A^Z_{BC} is closed and by Lemma II.2 we conclude

$$(Z_{-1})_1^{A^Z_{BC}} \simeq Z \hookrightarrow X \hookrightarrow Z_{-1}.$$

Hence, for $F := Z_{-1}$, $E := X$, $T := A^Z_{BC}$, $T_1 := T|_E = A_{BC}$ we have $F_1^T = (Z_{-1})_1^{A^Z_{BC}} \hookrightarrow E = X$. By Corollary A.3.(vii) this implies all the equivalences for the various parts of the spectra.

For (d) we now assume that $1 \in \rho(\Delta_U(\lambda))$, i.e., $\lambda \in \rho(A^Z_{BC})$. Then Lemma A.1.(vi) yields $R(\lambda, A_{BC}) = \Delta_1^{-1}|_X$ which is the restriction of $\Delta_1^{-1} \in \mathcal{L}(Z_{-1})$ to X. The formula for Δ_1^{-1} in Lemma B.1.(vi) then gives (2.6). $\square$

Corollary II.5. *Let $\lambda \in \rho(A)$. All assertions in Theorem II.4 hold with $\Delta_U(\lambda)$ replaced by $\Delta_Z(\lambda)$. If $1 \in \rho(\Delta_Z(\lambda))$, then $\lambda \in \rho(A_{BC})$ and the resolvent is given by*

$$R(\lambda, A_{BC}) = \Big(\mathrm{Id}_Z - \Delta_Z(\lambda)\Big)^{-1} R(\lambda, A). \tag{2.11}$$

Proof. The assertion follows from Corollary B.3 applied to $E = U$, $F = Z$ and the operators $R := R(\lambda, A_{-1})B \in \mathcal{L}(U, Z)$ and $Q := C \in \mathcal{L}(Z, U)$. $\square$

Our approach includes, as special cases, the spectra of delay equations (cf. [BP05, Lem. 3.20]), of flows on networks (cf. [KS05, Prop. 3.3]), and more, cf. [KVL92, Sect. II], [Nag97, Sect. 3] and [Eng99, Thm. 2.5.(a)]. We further point out that the above problem has already been studied by, e.g., Salamon, Weiss–Xu, Curtain–Jacob and Hadd–Manzo–Rhandi in the context of closed loop systems in control theory, cf. [Sal87, Lem. 4.4], [WX05, Thms. 1.1 & 1.2], [CJ09, Thm. 6.2] and [HMR15, Thm. 4.1].

Remark II.6. (i) The previous result establishes that for the point spectrum we always have

$$\lambda \in \sigma_p(A_{BC}) \quad \iff \quad 1 \in \sigma_p\big(\Delta_U(\lambda)\big) \quad \iff \quad 1 \in \sigma_p\big(\Delta_Z(\lambda)\big).$$

For the whole spectrum and its other parts only one implication holds in general (see the spectral inclusions proved in Theorem II.4), i.e.,

$$\begin{aligned} \lambda \in \sigma(A_{BC}) &\quad\Longrightarrow\quad 1 \in \sigma\big(\Delta_U(\lambda)\big) &&\iff\quad 1 \in \sigma\big(\Delta_Z(\lambda)\big),\\ \lambda \in \sigma_a(A_{BC}) &\quad\Longrightarrow\quad 1 \in \sigma_a\big(\Delta_U(\lambda)\big) &&\iff\quad 1 \in \sigma_a\big(\Delta_Z(\lambda)\big),\\ \lambda \in \sigma_{\mathrm{ess}}(A_{BC}) &\quad\Longrightarrow\quad 1 \in \sigma_{\mathrm{ess}}\big(\Delta_U(\lambda)\big) &&\iff\quad 1 \in \sigma_{\mathrm{ess}}\big(\Delta_Z(\lambda)\big). \end{aligned}$$

If $D(A) + \operatorname{rg}(\mathrm{Id}_Z - \Delta_Z(\nu))$ is dense in Z for some (hence all) $\nu \in \rho(A)$, then also

$$\begin{aligned} \lambda \in \sigma_c(A_{BC}) &\quad\Longrightarrow\quad 1 \in \sigma_c\big(\Delta_U(\lambda)\big) &&\iff\quad 1 \in \sigma_c\big(\Delta_Z(\lambda)\big),\\ \lambda \in \sigma_r(A_{BC}) &\quad\Longleftarrow\quad 1 \in \sigma_r\big(\Delta_U(\lambda)\big) &&\iff\quad 1 \in \sigma_r\big(\Delta_Z(\lambda)\big). \end{aligned}$$

To obtain equivalence, an additional assumption is needed, e.g., that $\Delta_U(\lambda) \in \mathcal{L}(U)$ is compact or that (2.3) is satisfied, see Example II.7.

(ii) We also have

$$\begin{aligned} \lambda - A_{BC}^Z \text{ surjective } &\Longrightarrow \lambda - A_{BC} \text{ surjective for all } \lambda \in \mathbb{C},\\ \lambda - A_{BC}^Z \text{ surjective } &\iff \mathrm{Id}_U - \Delta_U(\lambda) \text{ surjective for all } \lambda \in \rho(A). \end{aligned}$$

(iii) Note that, by the previous result, A_{BC} is closed if $1 \in \rho(\Delta_U(\nu))$ for some $\nu \in \rho(A)$. This condition is in particular satisfied if $P = BC$ is a Weiss–Staffans perturbation for A (cf. Definition III.3) or $\|CR(\nu, A_{-1})B\| < 1$ for some $\nu \in \rho(A)$.

(iv) In [SW02] the *system operator* $S_\Sigma(\lambda)$, similar to the operator matrix $\mathcal{T}$ in (2.10), is used to characterize spectral values of the generator of the *Lax-Phillips semigroup* in some half plane.

Equivalence in (2.4) or (2.5) does not hold in general without the extra assumption (2.3) or the compactness of $\Delta_U(\lambda)$. In fact, there are operators A, B and C such that A_{BC}^Z is not closed, hence $\sigma(A_{BC}^Z) = \mathbb{C}$, whereas $\sigma(A_{BC})$ might be rather small.

Example II.7. For an invertible, unbounded operator $(A, D(A))$ on a Banach space

X define on $\mathcal{X} := X \times X$ the operator matrix

$$\mathcal{G} := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_X & \mathrm{Id}_X \\ \mathrm{Id}_X & \mathrm{Id}_X \end{pmatrix}, \quad D(\mathcal{G}) := \left\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in X \times X : x + y \in D(A) \right\}.$$

The similarity of the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

implies that the operator $\mathcal{G}$ is similar to

$$\mathcal{D} := \begin{pmatrix} 2A & 0 \\ 0 & 0 \end{pmatrix}, \quad D(\mathcal{D}) := D(A) \times X.$$

In particular, this implies that $\mathcal{G}$ is closed and $\sigma(\mathcal{G}) = \sigma(\mathcal{D}) = \sigma(2A) \cup \{0\}$.
The most natural attempt to represent $\mathcal{G}$ as $\mathcal{A}_{\mathcal{BC}}$ is to choose the spaces $\mathcal{X} = X \times X =: \mathcal{Z} =: \mathcal{U}$ and the operators

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : D(\mathcal{A}) := D(A) \times D(A) \subset \mathcal{X} \to \mathcal{X},$$

$$\mathcal{B} := \begin{pmatrix} 0 & A_{-1} \\ A_{-1} & 0 \end{pmatrix} : \mathcal{U} \to X_{-1}^A \times X_{-1}^A, \qquad \text{and} \qquad \mathcal{C} := \mathrm{Id}_{\mathcal{X}} \in \mathcal{L}(\mathcal{Z}, \mathcal{U}).$$

For this choice we obtain $\mathcal{X}_{-1}^{\mathcal{A}} = X_{-1}^A \times X_{-1}^A = \mathcal{Z}_{-1}$. Hence, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}^{\mathcal{A}})$ and a simple computation shows $\mathcal{G} = \mathcal{A}_{\mathcal{BC}}$. However, the operator

$$\mathcal{A}_{\mathcal{BC}}^{\mathcal{Z}} = \begin{pmatrix} A_{-1} & A_{-1} \\ A_{-1} & A_{-1} \end{pmatrix} : \mathcal{Z} \subset \mathcal{Z}_{-1} \to \mathcal{Z}_{-1}$$

is not closed, hence $\sigma(\mathcal{A}_{\mathcal{BC}}^{\mathcal{Z}}) = \mathbb{C}$. Thus, for this choice of operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the condition (2.3) is not satisfied and Theorem II.4.(c) cannot be applied. In particular, the spectra of $\mathcal{A}_{\mathcal{BC}}^{\mathcal{Z}}$ and $\mathcal{G} = \mathcal{A}_{\mathcal{BC}}$ do not coincide.
However, the matrix $\mathcal{G}$ can still be treated within our framework. To this end choose the spaces $\mathcal{Z} := \mathcal{U} := \mathcal{X} = X \times X$. However, this time we consider the operators

$$\mathcal{A} := \begin{pmatrix} 2A & 0 \\ 2A & 0 \end{pmatrix} : D(\mathcal{A}) := D(A) \times X \subset \mathcal{X} \to \mathcal{X},$$

$$\mathcal{B} := \begin{pmatrix} A_{-1} & 0 \\ A_{-1} & 0 \end{pmatrix} : \mathcal{U} \to X_{-1}^A \times X_{-1}^A \qquad \text{and} \qquad \mathcal{C} := \begin{pmatrix} -\mathrm{Id}_X & \mathrm{Id}_X \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{U}).$$

For this choice of $\mathcal{A}$ we obtain

$$\mathfrak{X}_{-1}^{\mathcal{A}} = \left\{ \binom{x}{y} \in X_{-1}^{A} \times X_{-1}^{A} : x - y \in X \right\} = \mathcal{Z}_{-1}.^{1}$$

Hence, $\operatorname{rg}(\mathcal{B}) \subseteq \mathfrak{X}_{-1}^{\mathcal{A}}$ which implies $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathfrak{X}_{-1}^{\mathcal{A}})$. Again, $\mathcal{G} = \mathcal{A}_{\mathcal{B}\mathcal{C}}$ and

$$\mathcal{A}_{\mathcal{B}\mathcal{C}}^{\mathcal{Z}} = \begin{pmatrix} A_{-1} & A_{-1} \\ A_{-1} & A_{-1} \end{pmatrix} : \mathcal{Z} \subset \mathfrak{X}_{-1}^{\mathcal{A}} \to \mathfrak{X}_{-1}^{\mathcal{A}}.$$

The operator $\mathrm{Id}_{\mathcal{U}} - \mathcal{C}R(\lambda, \mathcal{A}_{-1})\mathcal{B} = \mathrm{Id}_{\mathcal{U}}$ is invertible for all $\lambda \in \rho(\mathcal{A})$. Thus, condition (2.3) is satisfied for all $\lambda \in \rho(\mathcal{A})$. Hence, we can apply Theorem II.4.(c) and conclude that $\sigma(\mathcal{G}) \subseteq \sigma(2A) \cup \{0\}$.

By applying Lemma B.1.(iii) to $\mathcal{T}$ in (2.10) we obtain, using Corollary B.3, the following representation of the eigenspaces which generalizes [WX05, Thm. 1.1].

Corollary II.8. *For $\lambda \in \rho(A)$ we have*

$$\begin{aligned} \ker(\lambda - A_{BC}) &= R(\lambda, A_{-1})B \ker\Big(\mathrm{Id}_U - \Delta_U(\lambda)\Big) = \ker\Big(\mathrm{Id}_Z - \Delta_Z(\lambda)\Big), \\ \ker\Big(\mathrm{Id}_U - \Delta_U(\lambda)\Big) &= C \ker(\lambda - A_{BC}). \end{aligned}$$

One drawback of Theorem II.4 is that it can be applied only to points $\lambda \in \rho(A)$. If one wants to determine the spectrum of a given operator G it is therefore important to represent G as A_{BC} with an operator A having "small" spectrum. In many cases this is possible due to the great freedom in the choices of B and C.

In the following sections we apply our abstract results to the following situations, see [AE15, Sect. 3].

(i) Boundary perturbations as presented in Chapter I,

(ii) a first derivative on $\mathrm{L}^p[0,1]$ with general boundary conditions,

(iii) a second derivative on $\mathrm{C}[0,1]$ with general boundary conditions,

(iv) a second derivative on $\mathrm{L}^p[0,1]$ with delay in the Neumann boundary conditions,

(v) a second order differential operator on $\mathrm{L}^p[0,1]$ with point delay in the Neumann boundary conditions,

(vi) a reduction matrix with damping and general boundary conditions.

[1] For $\lambda \in \rho(\mathcal{A}) = \rho(2A) \setminus \{0\}$ the resolvent is given by $R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, 2A) & 0 \\ R(\lambda, 2A) - \frac{1}{\lambda} & \frac{1}{\lambda} \end{pmatrix}$.

II.2. The Spectrum under Boundary Perturbations

By applying Theorem II.4 to the boundary perturbations studied in Chapter I, i.e., the operator A_P^Φ as in (1.2), A as in (1.1) and $P = BC$ defined in (1.8), we easily obtain the following result.

Corollary II.9. *For $\lambda \in \rho(A)$ define on $U := X \times \partial X$ and Z the operators*

$$\Delta_U(\lambda) := CR(\lambda, A_{-1})B = \begin{pmatrix} PR(\lambda, A) & PL_\lambda \\ \Phi R(\lambda, A) & \Phi L_\lambda \end{pmatrix} \in \mathcal{L}(U),$$
$$\Delta_Z(\lambda) := R(\lambda, A_{-1})BC = R(\lambda, A)P + L_\lambda \Phi \in \mathcal{L}(Z).$$

Then the following holds.

(a) If either $\Delta_U(\lambda) \in \mathcal{L}(U)$ or $\Delta_Z(\lambda) \in \mathcal{L}(Z)$ is compact, then

$$\lambda \in \sigma(A_P^\Phi) \iff \lambda \in \sigma_p(A_P^\Phi) \iff 1 \in \sigma_p\Big(\Delta_U(\lambda)\Big) \overset{\text{Lem. B.1.(iv)}}{\iff} 1 \in \sigma_p\Big(\Delta_Z(\lambda)\Big).$$

(b) If there exist $\nu \in \rho(A)$ such that $1 \in \rho\Big(\Delta_U(\nu)\Big)$, or equivalently $1 \in \rho\Big(\Delta_Z(\nu)\Big)$, then

$$\begin{aligned} \lambda \in \sigma(A_P^\Phi) \quad &\iff \quad 1 \in \sigma\Big(\Delta_U(\lambda)\Big) \quad \iff \quad 1 \in \sigma\Big(\Delta_Z(\lambda)\Big), \\ \lambda \in \sigma_*(A_P^\Phi) \quad &\iff \quad 1 \in \sigma_*\Big(\Delta_U(\lambda)\Big) \quad \iff \quad 1 \in \sigma_*\Big(\Delta_Z(\lambda)\Big) \end{aligned}$$

for all $* \in \{p, a, r, c, ess\}$. Moreover, if $1 \in \rho\Big(\Delta_U(\lambda)\Big)$, then $\lambda \in \rho(A_P^\Phi)$ and the resolvent is given by

$$\begin{aligned} R(\lambda, A_P^\Phi) &= R(\lambda, A) + \Big(R(\lambda, A), L_\lambda\Big) \cdot \Big(\mathrm{Id}_U - \Delta_U(\lambda)\Big)^{-1} \cdot \left(\begin{smallmatrix} PR(\lambda,A) \\ \Phi R(\lambda,A) \end{smallmatrix}\right) \\ &= \Big(\mathrm{Id}_Z - R(\lambda, A)P - L_\lambda \Phi\Big)^{-1} \cdot R(\lambda, A). \end{aligned}$$

If $P = 0$, we can cancel out the unnecessary terms and consider $U = \partial X$, $B = L_A$ and $C = \Phi$. Then $A^\Phi := A_0^\Phi = A_{BC}$ and the above result simplifies to the following.

Corollary II.10. *For $\lambda \in \rho(A)$ define the operators*

$$\Delta_{\partial X}(\lambda) = CR(\lambda, A_{-1})B = \Phi L_\lambda \in \mathcal{L}(\partial X),$$
$$\Delta_Z(\lambda) = R(\lambda, A_{-1})BC = L_\lambda \Phi \in \mathcal{L}(Z).$$

Then the following holds.

(a) *If either $\Delta_{\partial X}(\lambda) \in \mathcal{L}(U)$ or $\Delta_Z(\lambda) \in \mathcal{L}(Z)$ is compact, then*

$$\lambda \in \sigma(A^\Phi) \iff \lambda \in \sigma_p(A^\Phi) \iff 1 \in \sigma_p\Big(\Delta_{\partial X}(\lambda)\Big) \overset{\text{Lem. B.1.(iv)}}{\iff} 1 \in \sigma_p\Big(\Delta_Z(\lambda)\Big).$$

In particular, if $\dim(\partial X) < \infty$, *then*

$$\lambda \in \sigma(A^\Phi) \iff \lambda \in \sigma_p(A^\Phi) \iff \det\Big(\mathrm{Id}_{\partial X} - \Delta_{\partial X}(\lambda)\Big) = 0. \tag{2.12}$$

(b) *If there exist $\nu \in \rho(A)$ such that $1 \in \rho\Big(\Delta_{\partial X}(\nu)\Big)$, or equivalently $1 \in \rho\Big(\Delta_Z(\nu)\Big)$, then*

$$\begin{aligned} \lambda \in \sigma(A^\Phi) &\iff 1 \in \sigma\Big(\Delta_{\partial X}(\lambda)\Big) &&\iff 1 \in \sigma\Big(\Delta_Z(\lambda)\Big), \\ \lambda \in \sigma_*(A^\Phi) &\iff 1 \in \sigma_*\Big(\Delta_{\partial X}(\lambda)\Big) &&\iff 1 \in \sigma_*\Big(\Delta_Z(\lambda)\Big) \end{aligned}$$

for all $* \in \{p, a, r, c, ess\}$. Moreover, if $1 \in \rho\Big(\Delta_{\partial X}(\lambda)\Big)$, then $\lambda \in \rho(A^\Phi)$ and the resolvent is given by

$$\begin{aligned} R(\lambda, A^\Phi) &= R(\lambda, A) + L_\lambda \cdot \Big(\mathrm{Id}_{\partial X} - \Phi L_\lambda\Big)^{-1} \cdot \Phi R(\lambda, A) \\ &= \Big(\mathrm{Id}_Z - L_\lambda \Phi\Big)^{-1} \cdot R(\lambda, A). \end{aligned}$$

We mention that the spectrum of the operator A^Φ for bounded $\Phi \in \mathcal{L}(X, \partial X)$ has been studied by G. Greiner. If $\dim(\partial X) < \infty$, [Gre87, Prop. 3.1] characterizes the spectral values of A^Φ lying in the component of $\rho(A)$ which is unbounded to the right as the zeros of the function

$$F(\lambda) := \det(\mathrm{Id}_{\partial X} - \Delta_{\partial X}(\lambda)),$$

cf. (2.12) above. Further, Schappacher characterizes in [Sch91, Thm. 7] the point, residual and continuous spectrum of operators which are obtained through perturbations of, e.g., the boundary conditions (see also [DS85, Sect. 3] where the authors study spectral decompositions). Moreover, Nagel studied boundary perturbations for operator matrices and arrives in [Nag90, Thm. 2.7] at conditions similar to (2.12) assuming compactness of operators corresponding to our $\Delta_Z(\lambda) \in \mathcal{L}(Z)$.

II.3. Spectrum of a First Derivative with General Boundary Conditions.

The aim of this example is to illustrate our results in a very simple but typical context. Let $X := \mathrm{L}^p[0,1]$ and $\Psi \in (\mathrm{W}^{1,p}[0,1])'$ for some $1 \le p < \infty$. We show how the operator

$$G := \tfrac{d}{ds} \quad \text{with domain } D(G) := \Big\{ f \in \mathrm{W}^{1,p}[0,1] : \Psi f = 0 \Big\} \tag{2.13}$$

fits into the framework of the previous section. In fact, it suffices to choose the maximal operator $A_m := \frac{d}{ds}$ with domain $D(A_m) := \mathrm{W}^{1,p}[0,1]$, the boundary space $\partial X := \mathbb{C}$ and $L := \delta_0 : D(A_m) \subset X \to \partial X$ with $\delta_0 f := f(0)$. This yields

$$A = \tfrac{d}{ds} \quad \text{with domain } D(A) = \Big\{ f \in \mathrm{W}^{1,p}[0,1] : f(0) = 0 \Big\} = \ker(L).$$

Moreover, if we choose $Z := [D(A_m)] = \mathrm{W}^{1,p}[0,1]$ and $\Phi := L - \Psi \in \mathcal{L}(Z, \partial X)$, then $G = A^{\Phi}$.

Since $\sigma(A) = \emptyset$ and the Dirichlet operators $L_\lambda \in \mathcal{L}(\mathbb{C}, X)$, $\lambda \in \mathbb{C}$, for the pair (A_m, L) are given by

$$(L_\lambda z)(s) = z\, e^{\lambda s} \qquad \text{for } z \in \mathbb{C} \text{ and } s \in [0,1].$$

Corollary II.10 implies the following.

Corollary II.11. *The spectrum of G in* (2.13) *is characterized by*

$$\lambda \in \sigma(G) \quad \Longleftrightarrow \quad \lambda \in \sigma_p(G) \quad \Longleftrightarrow \quad \Psi\Big(e^{\lambda \bullet}\Big) = 0.$$

If, e.g., $\Psi = \delta_0 - \delta_1$, then $\lambda \in \sigma(G) \Longleftrightarrow e^\lambda = 1$, i.e., $\sigma(G) = \sigma_p(G) = 2\pi i \mathbb{Z}$.

Remark II.12. (a) We note that the choice of the unperturbed operator $A \subset A_m$ with domain $D(A) = \ker(L)$ in the example above (as well as in the following ones) is rather arbitrary. As already mentioned, due to the freedom of the perturbation $\Phi \in (\mathrm{W}^{1,p}[0,1])'$ it is convenient to choose A having large resolvent set.

(b) The point spectrum of G in (2.13) (as well as in Section II.4) can also be determined by solving the *eigenvalue equation*, i.e., find a solution $f \neq 0$ to the ordinary differential equation $(\lambda - \frac{d}{ds})f = 0$ subject to the boundary condition $\Psi f = 0$.

(c) The wellposedness of the (ACP) associated with the operator G in (2.13) on $\mathrm{L}^p[0,1]$ with appropriate boundary conditions is studied in [ABE14, Sect. 4.3] using results from Chapter III.

II.4. Spectrum of a Second Derivative with General Boundary Conditions.

On the state space $X := \mathrm{C}[0,1]$ we consider for some $\psi_1, \psi_2 \in (\mathrm{C}^2[0,1])'$ the second derivative

$$G := \tfrac{d^2}{ds^2} \quad \text{with domain } D(G) := \Big\{ f \in \mathrm{C}^2[0,1] : \psi_1 f = 0 = \psi_2 f \Big\}. \tag{2.14}$$

To compute $\sigma(G)$ we consider the maximal operator $A_m := \frac{d^2}{ds^2}$ with domain $D(A_m) := \mathrm{C}^2[0,1]$, the boundary space $\partial X := \mathbb{C}^2$ and $L := \binom{\delta_0}{\delta_0'} : D(A_m) \subset X \to \partial X$ where $\delta_0' f := f'(0)$. This gives the second derivative

$$A = \tfrac{d^2}{ds^2} \quad \text{with domain } D(A) = \Big\{ f \in \mathrm{C}^2[0,1] : Lf = 0 \Big\} = \ker(L). \tag{2.15}$$

Moreover, for $Z := [D(A_m)] = \mathrm{C}^2[0,1]$ we have $\Phi = \binom{\varphi_1}{\varphi_2} := L - \binom{\psi_1}{\psi_2} \in \mathcal{L}(Z, \partial X)$ and it follows that $G = A^\Phi$.

Since, by the Arzelà–Ascoli theorem (cf. [Ada75, Thm. 1.30]), $X_1^A \overset{c}{\hookrightarrow} X$,[2] the operator A has compact resolvent. Hence, $\sigma(A) = \sigma_p(A)$. Now a simple computations shows that

$$\sigma_p(A) = \emptyset.$$

Next, by solving for $\lambda \in \mathbb{C}$ and $z := \binom{z_1}{z_2} \in \partial X = \mathbb{C}^2$ the Dirichlet problem

$$\begin{cases} \big(\lambda - A_m\big) f = 0, \\ Lf = z, \end{cases} \qquad \text{i.e.,} \qquad \begin{cases} \big(\lambda - \frac{d^2}{ds^2}\big) f = 0, \\ f(0) = z_1, \ f'(0) = z_2, \end{cases}$$

we obtain the Dirichlet operators $L_\lambda \in \mathcal{L}(\partial X, X) = \mathcal{L}(\mathbb{C}^2, \mathrm{C}[0,1])$ for the pair (A_m, L) given by

$$\Big(L_\lambda \tbinom{z_1}{z_2}\Big)(s) = \begin{cases} z_1 \cdot \cosh(\sqrt{\lambda} s) + \frac{z_2 \cdot \sinh(\sqrt{\lambda} s)}{\sqrt{\lambda}} & \text{if } \lambda \neq 0, \\ z_1 + z_2 \cdot s & \text{if } \lambda = 0, \end{cases}$$

for $\binom{z_1}{z_2} \in \partial X$ and $s \in [0,1]$.

[2] Here "$\overset{c}{\hookrightarrow}$" denotes a compact embedding.

Now Corollary II.10 applied to this situation gives the following.

Corollary II.13. *For G given by (2.14) we have $\lambda \in \sigma(G) = \sigma_p(G)$ if and only if*

$$\begin{cases} \det \begin{pmatrix} \psi_1\Big(\cosh\big(\sqrt{\lambda}\,\bullet\big)\Big), & \psi_1\Big(\sinh\big(\sqrt{\lambda}\,\bullet\big)\Big) \\ \psi_2\Big(\cosh\big(\sqrt{\lambda}\,\bullet\big)\Big), & \psi_2\Big(\sinh\big(\sqrt{\lambda}\,\bullet\big)\Big) \end{pmatrix} = 0 & \text{for } \lambda \neq 0, \\ \det \begin{pmatrix} \psi_1(\mathbb{1}) & \psi_1(\mathbb{s}) \\ \psi_2(\mathbb{1}) & \psi_2(\mathbb{s}) \end{pmatrix} = 0 & \text{for } \lambda = 0, \end{cases} \tag{2.16}$$

where $\mathbb{1}(s) = 1$ and $\mathbb{s}(s) = s$ for all $s \in [0,1]$.

For particular choices of the boundary functionals ψ_1, ψ_2 the characteristic equation (2.16) simplifies considerably. For example, if we consider the second derivative $G = \frac{d^2}{ds^2}$ with Wentzell-type boundary conditions $f''(j) = f'(j)$, $j = 0, 1$, we obtain the following.

Corollary II.14. *For $\binom{\psi_1}{\psi_2} = \binom{\delta_0'' - \delta_0'}{\delta_1'' - \delta_1'}$ we obtain*

$$\sigma(G) = \sigma_p(G) = \{-\pi^2 \cdot n^2 : n \in \mathbb{N}_0\} \cup \{1\}.$$

Proof. By (2.16) we have $\lambda \in \sigma(G)$ if and only if

$$\lambda \cdot (\lambda - 1) \cdot \sinh\big(\sqrt{\lambda}\big) = 0. \qquad \square$$

The generator property of a system of operators as in (2.14) on $X = (\mathrm{L}^p[0,1])^n$ with appropriate boundary conditions is studied in Section IV.3.

II.5. Spectrum of a Second Derivative with Unbounded Delay at the Boundary.

In this example we investigate the spectrum of an operator associated with a heat equation with distributed unbounded delay at the boundary, cf. [HMR15, Expl. 5.2] where this operator appears for $p = 2$.

For $1 \leq p < \infty$ choose $X := \mathrm{L}^p[0,1]$ and $Y := \mathrm{L}^p([-1,0], X)$, which is isometrically isomorphic to $\mathrm{L}^p([-1,0] \times [0,1])$. For this reason we use the notation $v(r,s) := (v(r))(s)$ for $v \in Y$ and $r \in [-1,0]$, $s \in [0,1]$ in the sequel. On the product space

$\mathfrak{X} := X \times Y$ we consider the operator matrix

$$\text{(2.17)} \qquad \mathcal{G} := \begin{pmatrix} \frac{d^2}{ds^2} & 0 \\ 0 & \frac{d}{dr} \end{pmatrix},$$
$$D(\mathcal{G}) := \Big\{ \tbinom{f}{v} \in \mathrm{W}^{2,p}[0,1] \times \mathrm{W}^{1,p}([-1,0],X) : v(0) = f,\ f(1) = 0,$$
$$f'(0) = \int_0^1 \int_{-1}^0 v(r,s)\, d\nu(r)\, ds \Big\},$$

where $\nu : [-1,0] \to \mathbb{R}$ is a function of bounded variation.

We now characterize the spectrum of $\mathcal{G}$. In order to represent $\mathcal{G}$ as $\mathcal{A}^\Phi$ we first introduce the following operators and spaces.

- $A_m := \frac{d^2}{ds^2}$ with domain $D(A_m) := \{f \in \mathrm{W}^{2,p}[0,1] : f(1) = 0\}$ on X,
- $L := \delta_1' : D(A_m) \subset X \to \partial X := \mathbb{C}$, i.e., $Lf = f'(1)$,
- $D_m := \frac{d}{dr}$ with domain $D(D_m) := \mathrm{W}^{1,p}([-1,0],X)$ on Y,
- $K := \delta_0 : D(D_m) \subset Y \to \partial Y := X$, i.e., $Kv = v(0)$,
- $A = A_m|_{\ker(L)}$, $D := D_m|_{\ker(K)}$.

Next we define the maximal operator matrix

$$\mathcal{A}_m := \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix} \quad \text{with domain } D(\mathcal{A}_m) := D(A_m) \times D(D_m),$$

the boundary space $\partial\mathfrak{X} := \partial X \times \partial Y$,

$$\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & K \end{pmatrix} : D(\mathcal{A}_m) \to \partial\mathfrak{X},$$

and $\mathcal{A} \subset \mathcal{A}_m$ with domain $D(\mathcal{A}) := \ker(\mathcal{L}) = D(A) \times D(D)$. Finally, we take $\mathcal{Z} := X \times [D(D_m)]$ and define

$$\Phi := \begin{pmatrix} \delta_1' - \delta_0' & \varphi \\ \mathrm{Id}_X & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \partial\mathfrak{X}) \qquad \text{where} \quad \varphi(v) := \int_0^1 \int_{-1}^0 v(r,s)\, d\nu(r)\, ds.$$

Then, by definition, we obtain $\mathcal{G} = \mathcal{A}^\Phi$. In order to characterize the spectrum of $\mathcal{A}^\Phi$ we first note that $\sigma(A) = \sigma(D) = \emptyset$. Moreover, for $\lambda \in \mathbb{C}$ the Dirichlet operators

$L_\lambda \in \mathcal{L}(\partial X, X)$ and $K_\lambda \in \mathcal{L}(\partial Y, Y)$ for the pairs (A_m, L) and (D_m, K) are given by

$$
\begin{aligned}
(L_\lambda z)(s) &= \begin{cases} z \cdot \frac{\sinh(\sqrt{\lambda}(s-1))}{\sqrt{\lambda}} & \text{if } \lambda \neq 0, \\ z \cdot (s-1) & \text{if } \lambda = 0, \end{cases} && z \in \partial X,\ s \in [0,1], \\
(K_\lambda f)(r) &= e^{\lambda r} \cdot f, && f \in \partial Y,\ r \in [-1,0].
\end{aligned}
$$

Thus for $\lambda \in \mathbb{C}$ we obtain the Dirichlet operator for the pair $(\mathcal{A}_m, \mathcal{L})$ as

$$\mathcal{L}_\lambda := \begin{pmatrix} L_\lambda & 0 \\ 0 & K_\lambda \end{pmatrix} \in \mathcal{L}(\partial \mathcal{X}, \mathcal{X}).$$

We now are in the position to apply Corollary II.10 and obtain the following characterization of the spectral values of $\mathcal{G} = \mathcal{A}^\Phi$.

Corollary II.15. *Let $\lambda \in \mathbb{C}$ and $l_\lambda := L_\lambda 1$. Then*

$$\lambda \in \sigma(\mathcal{A}^\Phi) = \sigma_p(\mathcal{A}^\Phi) \quad \Longleftrightarrow \quad \int_0^1 \int_{-1}^0 e^{\lambda r} \cdot l_\lambda(s)\, d\nu(r)\, ds = \cosh\Big(\sqrt{\lambda}\Big). \tag{2.18}$$

In particular, if $\nu = \delta_{-1}$, then

$$\lambda \in \sigma_p(\mathcal{A}^\Phi) \quad \Longleftrightarrow \quad \Big(\lambda \cdot e^\lambda + 1\Big) \cdot \cosh\Big(\sqrt{\lambda}\Big) = 1.$$

Proof. For $\lambda \in \mathbb{C}$ we have

$$\Phi \mathcal{L}_\lambda = \begin{pmatrix} 1 - \cosh(\sqrt{\lambda}) & \varphi K_\lambda \\ L_\lambda & 0 \end{pmatrix} \in \mathcal{L}(\partial \mathcal{X}) = \mathcal{L}\Big(\mathbb{C} \times \mathrm{L}^p[0,1]\Big).$$

By Corollary II.10 and Lemma B.1(i) this implies that

$$
\begin{aligned}
\lambda \in \sigma(\mathcal{A}^\Phi) \quad &\Longrightarrow \quad 1 \in \sigma(\Phi \mathcal{L}_\lambda) && \Longleftrightarrow \quad 0 \in \sigma\Big(\cosh(\sqrt{\lambda}) - \varphi K_\lambda L_\lambda\Big) \\
&\Longleftrightarrow \quad 0 \in \sigma_p\Big(\cosh(\sqrt{\lambda}) - \varphi K_\lambda L_\lambda\Big) && \Longleftrightarrow \quad 1 \in \sigma_p(\Phi \mathcal{L}_\lambda) \\
&\Longleftrightarrow \quad \lambda \in \sigma_p(\mathcal{A}^\Phi) && \Longrightarrow \quad \lambda \in \sigma(\mathcal{A}^\Phi),
\end{aligned}
$$

where we used that ∂X is one-dimensional. The assertions follow by computing $\varphi K_\lambda L_\lambda : \mathbb{C} \to \mathbb{C}$. $\square$

Remark II.16. We note that the characteristic equations (2.18) holds, even if $\partial \mathcal{X}$ is only a product space with the finite dimensional factor $\partial X = \mathbb{C}$. (Use Schur complements (cf. Lemma B.1) and see Section II.6 as well).

The wellposedness and stability of the (ACP) associated with the operator $\mathcal{G}$ in (2.17) with appropriate boundary conditions is studied in Section VI.3.

II.6. Spectrum of a Second Order Differential Operator with Point Delay at the Boundary.

In this section we study the spectrum of an operator corresponding to a one-dimensional reaction-diffusion equation modeling a delayed chemical reaction, see Corollary IV.11.

Let $X := \mathrm{L}^p[0,1]$ and $Y := \mathrm{L}^p([-1,0], \partial Y)$ with $\partial Y := \mathrm{W}^{1,p}[0,1]$. On the product space $\mathcal{X} := X \times Y$ we consider for some fixed $c, k \in \mathbb{C}$ the operator matrix

$$\mathcal{G} := \begin{pmatrix} \frac{d^2}{ds^2} - 2c \cdot \frac{d}{ds} + k \cdot \mathrm{Id}_X & 0 \\ 0 & \frac{d}{dr} \end{pmatrix}, \tag{2.19}$$

$$D(\mathcal{G}) := \left\{ \binom{f}{v} \in \mathrm{W}^{2,p}[0,1] \times \mathrm{W}^{1,p}([-1,0], \partial Y) : \begin{array}{c} \binom{f'(0)}{f'(1)} = \binom{f(0)}{0} - \binom{v(-1,1)}{0} \\ v(0) = f \end{array} \right\}.$$

In order to compute $\sigma(\mathcal{G})$ we introduce the following operators and spaces.

- $A_m := \frac{d^2}{ds^2} - 2c \cdot \frac{d}{ds} + k \cdot \mathrm{Id}_X$ with domain $D(A_m) := \{f \in \mathrm{W}^{2,p}[0,1] : f'(1) = 0\}$ on X,
- $L := \delta_1 : D(A_m) \subset X \to \partial X := \mathbb{C}$, i.e., $Lf = f(1)$,
- $D_m := \frac{d}{dr}$ with domain $D(D_m) := \mathrm{W}^{1,p}([-1,0], \partial Y)$ on Y,
- $K := \delta_0 : D(D_m) \subset Y \to \partial Y$, i.e., $Kv = v(0)$,
- $A = A_m|_{\ker(L)}$, $D := D_m|_{\ker(K)}$.

Next we define the maximal operator matrix

$$\mathcal{A}_m := \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix} \quad \text{with domain } D(\mathcal{A}_m) := D(A_m) \times D(D_m).$$

Moreover, let $\mathcal{U} := \partial\mathcal{X} := \partial X \times \partial Y$,

$$\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & K \end{pmatrix} : D(\mathcal{A}_m) \to \partial\mathcal{X}$$

and $\mathcal{A} \subset \mathcal{A}_m$ with domain $D(\mathcal{A}) := \ker(\mathcal{L}) = D(A) \times D(D)$.

Finally, we define the spaces $Z_X := \mathrm{W}^{1,p}[0,1]$, $Z_Y := \mathrm{W}^{1,p}([-1,0],\partial Y)$, $\mathcal{Z} := Z_X \times Z_Y$ and consider

$$\begin{aligned}
\Phi &:= \begin{pmatrix} \varphi & \psi \\ \mathrm{Id}_{Z_X} & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z},\partial\mathcal{X}), && \text{where} \\
\varphi &:= \delta_0 + \delta_1 - \delta_0' \in \mathcal{L}(Z_X,\partial X), && \text{i.e., } \varphi(f) = f(0) + f(1) - f'(0), \\
\psi &:= -\delta_{-1} \otimes \delta_1 \in \mathcal{L}(Z_Y,\partial X), && \text{i.e., } \psi(v) = -v(-1,1).
\end{aligned}$$

Then, by definition, we obtain $\mathcal{G} = \mathcal{A}^\Phi$. In order to characterize the spectrum of $\mathcal{A}^\Phi$ we first note that $\sigma(A) = \sigma(D) = \emptyset$. Moreover, the Dirichlet operators $L_\lambda \in \mathcal{L}(\partial X, X)$ and $K_\lambda \in \mathcal{L}(\partial Y, Y)$ for the pairs (A_m, L) and (D_m, K) are explicitly given by

$$\begin{aligned}
(L_\lambda z)(s) &= \begin{cases} e^{c(s-1)} \cdot \Big(\cosh\Big((s-1)\cdot\sqrt{\lambda + c^2 - k}\Big) - \frac{c\sinh((s-1)\cdot\sqrt{\lambda+c^2-k})}{\sqrt{\lambda+c^2-k}}\Big) \cdot z \\ \hfill \text{if } \lambda \neq k - c^2, \\ e^{c(s-1)} \cdot (1 + c - cs) \cdot z \hfill \text{if } \lambda = k - c^2, \end{cases} \\
(K_\lambda f)(r) &= e^{\lambda r} \cdot f,
\end{aligned}$$

where $z \in \partial X = \mathbb{C}$, $s \in [0,1]$, $f \in \partial Y = \mathrm{W}^{1,p}[0,1]$ and $r \in [-1,0]$. Thus, for $\lambda \in \mathbb{C}$ we obtain the Dirichlet operator for the pair $(\mathcal{A}_m, \mathcal{L})$ as

$$\mathcal{L}_\lambda := \begin{pmatrix} L_\lambda & 0 \\ 0 & K_\lambda \end{pmatrix} \in \mathcal{L}(\partial\mathcal{X},\mathcal{X}).$$

Corollary II.10 yields the following characterization of the spectral values of $\mathcal{G} = \mathcal{A}^\Phi$.

Corollary II.17. *For $\lambda \in \mathbb{C}$, $\lambda \in \sigma(\mathcal{G}) = \sigma_p(\mathcal{G})$ if and only if*

$$e^{-\lambda} - l_\lambda(0) + l_\lambda'(0) = 0, \tag{2.20}$$

where $l_\lambda(s) := (L_\lambda 1)(s)$, $s \in [0,1]$.

Proof. For arbitrary $\lambda \in \mathbb{C}$ we have

$$\mathrm{Id}_{\partial\mathcal{X}} - \Phi\mathcal{L}_\lambda = \begin{pmatrix} l_\lambda'(0) - l_\lambda(0) & e^{-\lambda}\cdot\delta_1 \\ -L_\lambda & \mathrm{Id}_{\partial Y} \end{pmatrix} \in \mathcal{L}(\partial\mathcal{X}).$$

Using Schur complements (cf. Lemma B.1(i)), this matrix is not invertible if and only if (2.20) holds. The assertion then follows from Corollary II.10. $\square$

The generator property of $\mathcal{G}$ in (2.19) with appropriate boundary conditions is studied in Section IV.4.

II.7. Spectrum of a Reduction Matrix with Damping and General Boundary Conditions.

Let $X := \mathrm{L}^p[0,1]$, $\psi_1, \psi_2 \in (\mathrm{W}^{2,p}[0,1])'$ and $r \in \mathbb{C}$. On the space $\mathcal{X} := X \times X$ we consider the operator matrix

$$\tag{2.21} \begin{aligned} \mathcal{G} &:= \begin{pmatrix} 0 & \mathrm{Id}_X \\ \frac{d^2}{ds^2} & -r \cdot \mathrm{Id}_X \end{pmatrix}, \\ D(\mathcal{G}) &:= \left\{ \binom{f}{g} \in \mathrm{W}^{2,p}[0,1] \times \mathrm{L}^p[0,1] : \psi_1 f = \psi_2 f = 0 \right\}. \end{aligned}$$

In order to obtain a generator from $\mathcal{G}$ we have to restrict $\mathcal{G}$ to a smaller space $V \times X$ (cf. Proposition V.1). This *phase space* exists if and only if

$$G := \tfrac{d^2}{ds^2} \quad \text{with domain} \quad D(G) := \left\{ f \in \mathrm{W}^{2,p}[0,1] : \psi_1 f = \psi_2 f = 0 \right\}$$

generates a cosine family on X, cf. [ABHN11, Thm. 3.14.11]. In any case, by Lemma A.1.(vii) the spectrum $\sigma(\mathcal{G}|_{V\times X})$ coincides with $\sigma(\mathcal{G})$ provided that $\rho(\mathcal{G}) \neq \emptyset$.

We introduce the following operators and spaces.

- $A_m := \frac{d^2}{ds^2}$ with domain $D(A_m) := \mathrm{W}^{2,p}[0,1]$ on X,
- $L := \binom{\delta_0}{\delta_0'} : D(A_m) \subset X \to \partial X := \mathbb{C}^2$, i.e., $Lf = \binom{f(0)}{f'(0)}$,
- $A := A_m|_{\ker(L)}$ which satisfies $\rho(A) = \mathbb{C}$.

Then the Dirichlet operators $L_\lambda \in \mathcal{L}(\partial X, X)$ for the pair (A_m, L) are given by

$$\tag{2.22} \left(L_\lambda \binom{z_1}{z_2}\right)(s) = \begin{cases} z_1 \cosh(\sqrt{\lambda} s) + \frac{z_2 \sinh(\sqrt{\lambda} s)}{\sqrt{\lambda}} & \text{if } \lambda \neq 0, \\ z_1 + z_2 s & \text{if } \lambda = 0, \end{cases}$$

where $\binom{z_1}{z_2} \in \partial X$ and $s \in [0,1]$. Next we define on $\mathcal{X}$ the operator matrix

$$\mathcal{A} := \begin{pmatrix} 0 & \mathrm{Id}_X \\ A & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times X$$

with the extrapolation space $\mathcal{X}_{-1}^{\mathcal{A}} = X \times X_{-1}^{A}$ (cf. Proposition V.2). Finally, let

$Z := [D(A_m)]$, $\mathcal{Z} := Z \times X$, $\mathcal{U} := X \times \partial X$ and

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_X & L_A \end{pmatrix} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}^{\mathcal{A}}), \qquad \mathcal{C} := \begin{pmatrix} 0 & P \\ \Phi & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{U}) \quad \text{where}$$

- $L_A \in \mathcal{L}(\partial X, X_{-1}^A)$ is given by (1.4),
- $\Phi := \binom{\delta_0 - \psi_1}{\delta_0' - \psi_2} \in \mathcal{L}(Z, \mathbb{C}^2)$, and
- $P := -r \cdot \mathrm{Id}_X \in \mathcal{L}(X)$.

Then one easily verifies that indeed $\mathcal{G} = \mathcal{A}_{\mathcal{BC}}$.

Using Theorem II.4 we obtain the following characterization of the spectral values of $\mathcal{G}$.

Corollary II.18. *For $\mathcal{G}$ given by (2.21) we have $\lambda \in \sigma(\mathcal{G}) = \sigma_p(\mathcal{G})$ if and only if*

$$\begin{cases} \det \begin{pmatrix} \psi_1\Big(\cosh\big(\sqrt{\lambda^2 + r\lambda}\,\bullet\big)\Big), & \psi_1\Big(\sinh\big(\sqrt{\lambda^2 + r\lambda}\,\bullet\big)\Big) \\ \psi_2\Big(\cosh\big(\sqrt{\lambda^2 + r\lambda}\,\bullet\big)\Big), & \psi_2\Big(\sinh\big(\sqrt{\lambda^2 + r\lambda}\,\bullet\big)\Big) \end{pmatrix} = 0 & \text{for } \lambda \neq 0, -r, \\ \det \begin{pmatrix} \psi_1(\mathbb{1}) & \psi_1(\mathbb{s}) \\ \psi_2(\mathbb{1}) & \psi_2(\mathbb{s}) \end{pmatrix} = 0 & \text{for } \lambda = 0, -r. \end{cases}$$

Proof. We have $\rho(A) = \rho(\mathcal{A}) = \mathbb{C}$ and for $\lambda \in \mathbb{C}$ the resolvent is given by

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix}.$$

Thus, for $\lambda \in \mathbb{C}$ we obtain

$$\Delta_{\mathcal{U}}(\lambda) = \mathcal{C}R(\lambda, \mathcal{A}_{-1})\mathcal{B} = \begin{pmatrix} -r\lambda R(\lambda^2, A) & -r\lambda L_{\lambda^2} \\ \Phi R(\lambda^2, A) & \Phi L_{\lambda^2} \end{pmatrix} \in \mathcal{L}(\mathcal{U}).$$

By Theorem II.4 and Lemma B.1.(ii) this implies that

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}_{\mathcal{BC}}) \implies\ & 1 \in \sigma\Big(\Delta_{\mathcal{U}}(\lambda)\Big) \\ \iff\ & 1 \in \sigma\Big(\Phi L_{\lambda^2} - r\lambda\Phi \cdot R(\lambda^2, A)\Big(\mathrm{Id}_X + r\lambda R(\lambda^2, A)\Big)^{-1} \cdot L_{\lambda^2}\Big) \\ \iff\ & 1 \in \sigma_p\Big(\Phi L_{\lambda^2} - r\lambda\Phi \cdot R(\lambda^2 + r\lambda, A) \cdot L_{\lambda^2}\Big) \\ \iff\ & 1 \in \sigma_p\Big(\Delta_{\mathcal{U}}(\lambda)\Big) \iff \lambda \in \sigma_p(\mathcal{A}_{\mathcal{BC}}) \implies \lambda \in \sigma(\mathcal{A}_{\mathcal{BC}}). \end{aligned}$$

The assertions then follows from (2.22) by computing

$$\begin{aligned}\Phi L_{\lambda^2} - r\lambda\Phi \cdot R(\lambda^2 + r\lambda, A) \cdot L_{\lambda^2} &= \Phi \cdot (\lambda^2 - A) R(\lambda^2 + r\lambda, A) \cdot L_{\lambda^2} \\ &= \Phi \cdot R(\lambda^2 + r\lambda, A) \cdot L_A \\ &= \Phi \cdot L_{\lambda^2 + r\lambda} : \mathbb{C}^2 \to \mathbb{C}^2. \qquad \square\end{aligned}$$

The question whether $\mathcal{G}$ in (2.21) with appropriate boundary conditions generates a cosine family is treated in Chapter V.

III. The Weiss-Staffans Perturbation Theorem

In this chapter we present results guaranteeing that the perturbed operator $G = A_{BC}$ in (0.2) generates a C_0-semigroup $(T_{BC}(t))_{t\geq 0}$ on the Banach space X if the unperturbed operator A is already the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. As an application of Theorem III.8 we obtain the wellposedness of neutral differential equations (cf. Section III.3).

The main results of Chapter III are published in [ABE14] (cf. also [Bom15, Chap. 1]). Section III.3 is included in the manuscript *Asymptotic properties of C_0-semigroups under perturbations* (cf. [Adl15, Sect. 4]).

In the sequel $\omega_0(A)$ denotes the *growth bound* of the semigroup generated by A, cf. [EN00, Def. I.5.6], and Fav_α^A the *Favard space* of A of order $\alpha \in \mathbb{R}$, see [EN00, Sect. II.5.b].

III.1. A Survey on the Weiss-Staffans Perturbation Theorem

We recall two perturbation results as a conceptual motivation for the assumptions appearing in our main result Theorem III.8. First, the perturbation theorem in [DLS85, Thm. 2.1] due to Desch, Lasiecka and Schappacher investigates *range unboundedness*, i.e., deals with perturbations $P = A_{-1}B \in \mathcal{L}(X, X_{-1}^A)$. We formulate [DLS85, Thm. 2.1] using Proposition 8 from [DS89], see [EN00, Cor. III.3.4] as well.

Theorem III.1 (Desch-Lasiecka-Schappacher). *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with generator A. Take $B \in \mathcal{L}(X, Z)$ for a Banach space $Z \hookrightarrow X$ such that for some $t > 0$ and some $1 \leq p < \infty$ the following condition is satisfied:*

$$\begin{aligned} & \int_0^t T(t-s)Bu(s)\,ds \in D(A) && \forall\, u \in \mathrm{L}^p([0,t], X), \\ (3.1) \qquad \iff \quad & \int_0^t T_{-1}(t-s)A_{-1}Bu(s)\,ds \in X && \forall\, u \in \mathrm{L}^p([0,t], X). \end{aligned}$$

Then

$$A(\mathrm{Id}+B) = (A_{-1} + A_{-1}B)|_X \quad \textit{with domain } \{x \in X : x + Bx \in D(A)\}$$

is the generator of a C_0-semigroup on X.

On the contrary, the perturbation theorem due to Miyadera and Voigt investigates *domain unboundedness*, i.e., perturbations $P = C \in \mathcal{L}(X_1^A, X)$.

Theorem III.2 (Miyadera-Voigt). *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with generator A. Assume that there exists a constant $t > 0$ and $\gamma < 1$ such that*

$$\int_0^t \|CT(s)x\| \, ds \leq \gamma \|x\| \tag{3.2}$$

for all $x \in D(A)$. Then $A + C$ with domain $D(A)$ generates a C_0-semigroup on X.

We refer the reader to [Voi77, Thm. 1] for the proof, see [Miy66] and [EN00, Cor. III.3.16] as well. In particular, the proof in [Voi77] guarantees that the operator C is A-bounded with A-bounded at most γ.

Our goal is to combine the above assumptions to obtain a more general class of perturbations. The conditions (3.1) and (3.2) reappear in Definition III.3 as the *admissibility conditions* (ii) and (iii). The following class of perturbations and the related perturbation result was first studied by G. Weiss in [Wei94a, Thms. 6.1, 7.2] and O. Staffans in [Sta05, Sects. 7.1, 7.4] within the context of *linear systems*, while the *admissibility conditions* are studied, e.g., in [Wei89a, Def. 4.1], [Wei89b, Def. 6.1] and [GC96, Eng98] and [TW09, Chap. 4].

Definition III.3. Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Moreover, let Z be a Banach space satisfying $X_1^A \hookrightarrow Z \hookrightarrow X$. Then $P \in \mathcal{L}(Z, X_{-1}^A)$ is called a *Weiss-Staffans perturbation* for A if there exists $1 \leq p < \infty$, a Banach space U and operators $B \in \mathcal{L}(U, X_{-1}^A)$, $C \in \mathcal{L}(Z, U)$ such that

(i) the triple (A, B, C) is *compatible* (cf. (0.3));

(ii) the operator $B \in \mathcal{L}(U, X_{-1}^A)$ is *p-admissible*, i.e., there exists $t > 0$ such that for all $u \in \mathrm{L}^p([0,t], U)$

$$\int_0^t T_{-1}(t-s)Bu(s)\, ds \in X;$$

(iii) the operator $C \in \mathcal{L}(Z, U)$ is *p-admissible*, i.e., there exists $t > 0$ and $M_C \geq 0$ such that

$$\int_0^t \|CT(s)x\|^p \, ds \leq M_C^p \, \|x\|^p \quad \forall x \in D(A);$$

(iv) the pair $(B, C) \in \mathcal{L}(U, X_{-1}^A) \times \mathcal{L}(Z, U)$ is *p-admissible*, i.e., there exists $t > 0$ and $M_{BC} \geq 0$ such that

$$\int_0^t \left\| C \int_0^r T_{-1}(r-s)Bu(s)\, ds \right\|^p dr \leq M_{BC}^p \|u\|_p^p$$

holds for all $u \in \mathrm{W}_0^{2,p}([0,t], U) := \{f \in \mathrm{W}^{2,p}([0,t], U) : f(0) = f'(0) = 0\}$;

(v) the identity Id_U is a *p-admissible feedback* for (A, B, C), i.e., there exists $t > 0$ such that the operator $\mathrm{Id} - \mathcal{F}_t^{(A,B,C)}$ is invertible on $\mathrm{L}^p([0,t], U)$, where the operator $\mathcal{F}_t^{(A,B,C)} : \mathrm{L}^p([0,t], U) \to \mathrm{L}^p([0,t], U)$ is given by

$$\mathcal{F}_t^{(A,B,C)} :\ u \mapsto C \int_0^{\cdot} T_{-1}(\cdot - s)Bu(s)\, ds \qquad \text{for } u \in \mathrm{W}_0^{2,p}([0,t], U). \tag{3.3}$$

Remark III.4. (a) Condition (ii) in Definition III.3 is equivalent to the estimate

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\, ds \right\| \leq M_B \|u\|_p \qquad \forall u \in \mathrm{W}^{1,p}([0,t], U) \tag{3.4}$$

for some $t > 0$ and $M_B \geq 0$ by the closed graph theorem and integration by parts (cf. [ABE14, Rem. 2]).

(b) If the admissibility conditions (ii)-(v) hold for some $t > 0$, then the conditions already hold for all $t > 0$, see [Wei89a, Prop. 2.5], [Wei89b, Prop. 2.3], [Wei89c, Prop. 2.1] and [Sal87, Lem. 4.1]. The constants M_B, M_C and M_{BC} can be chosen independently of t if the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable (cf. Lemma III.7).

Definition III.5. We call the pair $(B, C) \in \mathcal{L}(U, X_{-1}^A) \times \mathcal{L}(Z, U)$ *jointly p-admissible* for A if the conditions (i) - (iv) in Definition III.3 hold.

Using the notions of admissibility above we can construct families of operators also studied in system theory, see [Wei94a, Def. 5.1].

Definition III.6. Let the pair $(B, C) \in \mathcal{L}(U, X_{-1}^A) \times \mathcal{L}(Z, U)$ be jointly p-admissible for A and take $t > 0$.

(i) The operator $\mathcal{B}_t : \mathrm{L}^p([0,t], U) \to X$ given by

$$\mathcal{B}_t :\ u \mapsto \int_0^t T_{-1}(t-s)Bu(s)\, ds$$

is called the *controllability* (or *input*) *map*. The operator $\mathcal{B}_t$ takes a control (or input) function u and issues the solution of the control system $\Sigma(A, B, 0)$ in (0.1) at time t with homogeneous initial data.

(ii) The operator $\mathcal{C}_t : X \to \mathrm{L}^p([0,t],X)$ given by

$$\mathcal{C}_t : \ x \mapsto CT(\cdot)x \qquad \text{for } x \in D(A)$$

is called the *observability* (or *output*) *map*. The function $\mathcal{C}_t x$ is the "observed" orbit of the Cauchy problem $\Sigma(A,0,C)$ in (0.1) with initial data $x \in X$. The *extended observability map* $\mathcal{C}_\infty : X \to \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+,X)$ is given by

$$(\mathcal{C}_\infty x)(s) = CT(s)x \qquad \text{for all } \ x \in D(A),\ s \in \mathbb{R}_+.$$

(iii) The operator $\mathcal{F}_t^{(A,B,C)} : \mathrm{L}^p([0,t],U) \to \mathrm{L}^p([0,t],U)$ given in (3.3) is called the *input-output map*, since $\mathcal{F}_t^{(A,B,C)}$ issues the "observed" output of the system $\Sigma(A,B,C)$ in (0.1) with input function u and homogeneous initial data. The *extended input-output map* $\mathcal{F}_\infty^{(A,B,C)} : \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+,U) \to \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+,U)$ is given by

$$(\mathcal{F}_\infty u)(\bullet) = C\int_0^\bullet T_{-1}(\bullet - s)Bu(s)\,ds \qquad \text{for all } \ u \in \mathrm{W}^{2,p}_{0,\mathrm{loc}}(\mathbb{R}_+,U).$$

The next lemma shows that we can neglect the time dependence of the constants M_B, M_C and M_{BC} for $\mathcal{B}_t$, $\mathcal{C}_t$ and $\mathcal{F}_t^{(A,B,C)}$, respectively, if the semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable semigroup, i.e., $\omega_0(A) < 0$. In particular, in this case we do not have to consider spaces of locally integrable functions in the above definitions of $\mathcal{C}_\infty$ and $\mathcal{F}_\infty^{(A,B,C)}$.

Lemma III.7. *Assume that $\omega_0(A) < 0$. Let (A,B,C) be compatible and (B,C) be jointly p-admissible for some $1 \leq p < \infty$. Then*

(i) the controllability maps $(\mathcal{B}_t)_{t\geq 0} \subset \mathcal{L}(\mathrm{L}^p(\mathbb{R}_+,U),X)$ form a strongly continuous, uniformly bounded family of operators,

(ii) the extended observability map $\mathcal{C}_\infty \in \mathcal{L}(X,\mathrm{L}^p(\mathbb{R}_+,U))$ is bounded, and

(iii) the extended input-output map $\mathcal{F}_\infty^{(A,B,C)} \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}_+,U))$ is bounded.

The proof of Lemma III.7 can be found in [BE14, Lem. 3.9, Lem. 3.15, Lem. 3.22].

The following theorem is the main result of this chapter and its operator theoretic proof is given in [ABE14, Thm. 10] (see [Sal87, Thms. 4.2 and 4.3] as well which, however, does not yield the form of the generator A_{BC} in (0.2)). System theoretic formulations and proofs are published in [Wei94a, Thms. 6.1, 7.2], [Sta05, Sects. 7.1, 7.4] and [Had05, HMR15].

Theorem III.8 (Weiss-Staffans)**.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with generator A. If $P = BC \in \mathcal{L}(Z, X_{-1}^A)$ is a Weiss-Staffans perturbation for A, then A_{BC} is the generator of a C_0-semigroup $(T_{BC}(t))_{t\geq 0}$ on X given by*

$$T_{BC}(t)x = T(t)x + \mathcal{B}_t\Big(\mathrm{Id} - \mathcal{F}_\infty^{(A,B,C)}\Big)^{-1}\mathcal{C}_\infty x \qquad \text{for } x \in X. \tag{3.5}$$

Moreover, we obtain the variation of parameter formula

$$T_{BC}(t)x = T(t)x + \int_0^t T_{-1}(t-s)\cdot BC \cdot T_{BC}(s)x\, ds \qquad \text{for } x \in D(A_{BC}).$$

Remark III.9. This theorem somehow interpolates Theorem III.1 and Theorem III.2. Due to this intermediate character we need to couple the admissibility conditions (ii) and (iii) in Definition III.3 which results in the admissibility condition (iv).

Due to the similarity of the assumptions in Theorems III.1, III.2 and III.8, one expects that the Desch-Lasiecka-Schappacher theorem and the Miyadera-Voigt theorem are special cases of Weiss-Staffans theorem. This is established in the following corollary. The proof can be found in [ABE14, Thms. 16, 18], see [TW09, Thm. 5.4.2, Cor. 5.5.1] as well.

Corollary III.10 ([ABE15, Lem. A.3])**.** *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and let $p \geq 1$.*

(a) If $B \in \mathcal{L}(U, X_{-1}^A)$ is a p-admissible control operator and $Z = X$, i.e., $C \in \mathcal{L}(X, U)$, then (A, B, C) is compatible, (B, C) is jointly p-admissible and there exists $M \geq 0$ and $t_0 > 0$ such that

$$\Big\|\mathcal{F}_t^{(A,B,C)}\Big\| \leq M \cdot t^{\frac{1}{p}} \quad \text{for all } 0 < t \leq t_0.$$

In particular, Id_U is a p-admissible feedback and, hence, A_{BC} is the generator of a C_0-semigroup on X.

(b) If $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(Z, U)$ is a p-admissible observation operator, then (A, B, C) is compatible, (B, C) is jointly p-admissible and there exists $M \geq 0$ and $t_0 > 0$ such that

$$\Big\|\mathcal{F}_t^{(A,B,C)}\Big\| \leq M \cdot t^{1-\frac{1}{p}} \quad \text{for all } 0 < t \leq t_0.$$

In particular, Id_U is a p-admissible feedback if $p > 1$ and, in this case, A_{BC} is the generator of a C_0-semigroup on X.

Remark III.11. Case (b) covers also the perturbation results [LT85, Thm. 2.1] and [LT00, Thm. 7.5.1.(a)].

III.2. Wellposedness under Boundary Perturbations

We now return to the abstract setting introduced in Chapter I. We show how Theorem III.8 can be used to generalize the approach by Greiner in [Gre87] to unbounded perturbations Φ of the boundary conditions of a generator. We strengthen Assumption I.2.(a) by assuming the following additional

Assumption III.12. Assume that $A := A_m|_{\ker(L)}$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.

We briefly recall the setting. The Banach space Z satisfies $X_1^A \hookrightarrow Z \hookrightarrow X$ and $D(A_m) \subset Z$. Further, we define the space $U := X \times \partial X$ and the operators

$$B := \Big(\mathrm{Id}_X, L_A\Big) \in \mathcal{L}(U, X_{-1}^A) \quad \text{and} \quad C := \begin{pmatrix} P \\ \Phi \end{pmatrix} \in \mathcal{L}(Z, U).$$

Then A_P^Φ in (1.2) can be written as $A_P^\Phi = A_{BC}$ by Lemma I.8. By applying Theorem III.8 one can now deal with unbounded $\Phi \in \mathcal{L}(Z, \partial X)$.

Corollary III.13 ([ABE15, Cor. 3.6])**.** *Assume that for some $1 \leq p < \infty$ the pairs (L_A, P) and (L_A, Φ) are jointly p-admissible for A and that there exists $t > 0$ such that $1 \in \rho\Big(\mathcal{F}_t^{(A,B,C)}\Big)$, where the input-output map is given by*

$$\mathcal{F}_t^{(A,B,C)} = \begin{pmatrix} \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} & \mathcal{F}_t^{(A,L_A,P)} \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\Phi)} & \mathcal{F}_t^{(A,L_A,\Phi)} \end{pmatrix} \in \mathcal{L}\Big(\mathrm{L}^p([0,t], X \times \partial X)\Big).$$

Then A_P^Φ given by (1.2) generates a C_0-semigroup on X. Here, the condition $1 \in \rho\Big(\mathcal{F}_t^{(A,B,C)}\Big)$ is in particular satisfied if $p > 1$ and $1 \in \rho\Big(\mathcal{F}_t^{(A,L_A,\Phi)}\Big)$ for some $t > 0$.

Proof. In order to guarantee that $P = BC \in \mathcal{L}(Z, X_{-1}^A)$ is a Weiss-Staffans perturbation for A, it suffices to show that for some $1 \leq p < \infty$ the pairs (Id_X, P), (Id_X, Φ), (L_A, P) and (L_A, Φ) are jointly p-admissible for A and $1 \in \rho\Big(\mathcal{F}_t^{(A,B,C)}\Big)$. All this follows directly by the assumptions and Corollary III.10.(b).

Finally, for $p > 1$, we have to verify the invertibility of $\mathrm{Id} - \mathcal{F}_t^{(A,B,C)}$ for some $t > 0$. The operator $\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}$ is invertible for all sufficiently small $t > 0$ with uniformly bounded inverses $\Big(\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}\Big)^{-1}$ by Corollary III.10.(b) and the Neumann series. Thus, for small $t > 0$, we have $1 \in \rho\Big(\mathcal{F}_t^{(A,B,C)}\Big)$ if and only if

$$\underbrace{\mathrm{Id} - \mathcal{F}_t^{(A,L_A,\Phi)}}_{\text{invertible}} - \underbrace{\mathcal{F}_t^{(A,\mathrm{Id}_X,\Phi)}\Big(\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}\Big)^{-1}\mathcal{F}_t^{(A,L_A,P)}}_{=:\Delta_t}$$

is invertible (use Schur complements in Lemma B.1.(ii)). The assertion follows since the set of invertible operators is open and

$$\|\Delta_t\| \to 0 \qquad \text{as } t \to 0^+$$

by Corollary III.10.(b). $\square$

In the remaining section we present two criteria for testing the admissibility of the control operator B. More criteria for checking the admissibility of the control operator B can be found in [TW09, Chap. 5].

The following lemma provides a criterion which is based on *boundary control problems* (BCP) on a Banach space X, which can be imagined as inhomogeneous Cauchy problems prior to closing a feedback. The proof uses ideas about the existence and representation of classical solutions of (BCP) on X as proposed in [EKFK$^+$10, Props. 2.7, 2.8].

Lemma III.14. *Let $A \subset A_m$ be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$. For some fixed $1 \leq p < \infty$ the following are equivalent.*

(i) The operator $B = L_A \in \mathcal{L}(\partial X, X_{-1}^A)$ in (1.4) is p-admissible for A.

(ii) There exists $t_0 > 0$ and a strongly continuous family of operators $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],\partial X), X)$ such that for every $u \in \mathrm{W}_0^{2,p}([0,t_0],\partial X)$ the function

$$x : [0,t_0] \to X, \quad x(t) := \mathbb{B}_t u \tag{3.6}$$

is the classical solution of the boundary control problem

$$\text{(BCP)} \qquad \begin{cases} \dot{x}(t) = A_m x(t), & 0 \leq t \leq t_0, \\ Lx(t) = u(t), & 0 \leq t \leq t_0, \\ x(0) = 0. \end{cases}$$

Moreover, in this case the controllability map is given by $\mathcal{B}_t = \mathbb{B}_t$ for $t \in [0,t_0]$.

Proof. (i)$\Longrightarrow$(ii). By the definition of p-admissibility, there exists some $t_0 > 0$ such that

$$\mathcal{B}_t u = \int_0^t T_{-1}(t-s)Bu(s)\,ds \in X \quad \text{for all } t \in [0,t_0] \text{ and } u \in \mathrm{L}^p([0,t_0],\partial X).$$

Define $\mathbb{B}_t := \mathcal{B}_t$ for $t \in [0,t_0]$. Remark III.4.(a) and Lemma III.7.(i) yield that $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],\partial X), X)$ is bounded and strongly continuous. It remains to show that the function $x := \mathbb{B}_{\bullet} u : [0, t_0] \to X$ in (3.6) is a *classical solution* of the boundary control problem (BCP) on X for $u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X)$, i.e.,

- $x \in \mathrm{C}^1([0, t_0], X)$ with $x(t) \in D(A_m)$ for every $0 \le t \le t_0$,
- $\dot{x}(t) = A_m x(t)$ and $Lx(t) = u(t)$ for every $0 \le t \le t_0$.

The following steps are performed in [EKFK+10, Prop. 2.8]. By the regularity of the function $u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X)$ we can apply integration by parts and obtain

$$x(t) = \int_0^t T_{-1}(t-s) B u(s)\, ds = L_\lambda u(t) + \int_0^t T(t-s) L_\lambda(\lambda u(s) - \dot{u}(s))\, ds.$$

Since $\lambda u(\bullet) - \dot{u}(\bullet) \in \mathrm{W}_0^{1,p}([0, t_0], \partial X)$, we obtain $\int_0^t T(t-s) L_\lambda(\lambda u(s) - \dot{u}(s))\, ds \in D(A) = \ker(L)$ (cf. [EN00, Cor. VI.7.6]) and thus $x(t) \in \ker(\lambda - A_m) + D(A) \subset D(A_m)$. Further, the function x is continuously differentiable with derivative

$$\begin{aligned}
\dot{x}(t) &= L_\lambda \dot{u}(t) + A \int_0^t T(t-s) L_\lambda(\lambda u(s) - \dot{u}(s))\, ds + \lambda L_\lambda u(t) - L_\lambda \dot{u}(t) \\
&= A \int_0^t T(t-s) L_\lambda(\lambda u(s) - \dot{u}(s))\, ds + A_m L_\lambda u(t) \\
&= A_m x(t).
\end{aligned}$$

Moreover, $Lx(t) = LL_\lambda u(t) + L \int_0^t T(t-s) L_\lambda(\lambda u(s) - \dot{u}(s))\, ds = u(t)$. Thus, x is a classical solution of (BCP) on X.

(ii)$\Longrightarrow$(i). Assume that the function $[0, t_0] \ni t \mapsto x(t) := \mathbb{B}_t u$ is a classical solution of the boundary control problem (BCP) on X for $u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X)$. Then (BCP) can be written as the inhomogeneous Cauchy problem

$$\text{(iACP)} \qquad \begin{cases} \dot{x}(t) = A_{-1} x(t) + B u(t) & \text{in } X_{-1}^A, \\ x(0) = 0. \end{cases}$$

The unique solution of this inhomogeneous Cauchy problem is given by

$$x(t) = \int_0^t T_{-1}(t-s) B u(s)\, ds \qquad \text{for } t \in [0, t_0]$$

(cf. [EN00, Sect. VI.7.a] or [EKFK+10, Prop. 2.7]) and by $\mathbb{B}_{t_0} \in \mathcal{L}(\mathrm{L}^p([0, t_0], \partial X), X)$ in assumption (ii) we obtain

$$\text{(3.7)} \quad \left\| \int_0^{t_0} T_{-1}(t_0 - s) B u(s)\, ds \right\|_X = \|\mathbb{B}_{t_0} u\|_X \le M \|u\|_p \quad \forall\, u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X).$$

Thus, the operator B is p-admissible by Remark III.4.(a). $\square$

The following lemma relates the range of the Dirichlet operators L_λ for the pair (A_m, L) to the Favard spaces Fav_α^A of A of order $\alpha \in \mathbb{R}$. This is helpful to check the admissibility of the control operator $B = L_A \in \mathcal{L}(\partial X, X_{-1}^A)$ in applications, see Lemma III.15.(d) or Corollary IV.6.(i).

Lemma III.15 ([ABE15, Lem. A.1])**.** *For $\alpha \in (0,1]$ the following are equivalent.*

(a) There exists $\lambda_0 > \omega_0(A)$ such that $\sup_{\lambda>\lambda_0} \|\lambda^\alpha L_\lambda x\| < \infty$ for all $x \in \partial X$.

(b) There exist $\lambda_0 > \omega_0(A)$ and $M > 0$ such that $\|Lx\| \geq \lambda^\alpha M \cdot \|x\|$ for all $\lambda \geq \lambda_0$ and $x \in \ker(\lambda - A_m)$.

(c) $\mathrm{rg}(L_\mu) = \ker(\mu - A_m) \subset \mathrm{Fav}_\alpha^A$ for some $\mu \in \rho(A)$.

Moreover, if $\alpha = 1$, then (a)–(c) are also equivalent to

(d) L_A is a 1-admissible control operator for A.

Proof. The equivalence of (a) and (b) follows immediately from the definition of L_λ as the inverse of $L : \ker(\lambda - A) \to \partial X$. To show the equivalence of (a) and (c), note that from (1.3) we obtain for $x \in \partial X$ and fixed $\mu \in \rho(A)$

$$\begin{aligned}\sup_{\lambda>\lambda_0}\Big\|\lambda^\alpha AR(\lambda,A)L_\mu x\Big\| &\leq \sup_{\lambda>\lambda_0}\Big\|\lambda^\alpha(\mu - A)R(\lambda,A)L_\mu x\Big\| + \sup_{\lambda>\lambda_0}\Big\|\mu\lambda^\alpha R(\lambda,A)L_\mu x\Big\| \\ &= \sup_{\lambda>\lambda_0}\Big\|\lambda^\alpha L_\lambda x\Big\| + \sup_{\lambda>\lambda_0}\Big\|\mu\lambda^\alpha R(\lambda,A)L_\mu x\Big\|.\end{aligned}$$

Since, by the Hille–Yosida theorem, we have

$$\sup_{\lambda>\lambda_0}\Big\|\mu\lambda^\alpha R(\lambda,A)L_\mu x\Big\| < \infty,$$

we conclude that

$$\sup_{\lambda>\lambda_0}\Big\|\lambda^\alpha AR(\lambda,A)L_\mu x\Big\| < \infty \quad \Longleftrightarrow \quad \sup_{\lambda>\lambda_0}\Big\|\lambda^\alpha L_\lambda x\Big\| < \infty.$$

By [EN00, Prop. II.5.12] the condition on the left-hand side is equivalent to $L_\mu x \in \mathrm{Fav}_\alpha^A$ and therefore we obtain (a)$\Longleftrightarrow$(c).

For the implication (c)$\Longrightarrow$(d) in case $\alpha = 1$, we refer to [NS94, Prop. 3.3] where the authors show that the operator $B = L_A : \partial X \to \mathrm{Fav}_0^A$ is 1-admissible (see [EN00, Proof of Cor. III.3.6] as well). For the converse statement we assume that L_A is

1-admissible. Hence, for $\mu \in \rho(A)$ and $t > 0$ we have

$$\begin{aligned} &(\mu - A_{-1}) \int_0^t T(t-s) L_\mu u(s)\, ds = \int_0^t T_{-1}(t-s) L_A u(s)\, ds \in X \\ \iff \quad & \int_0^t T(t-s) L_\mu u(s)\, ds \in D(A) \end{aligned}$$

for all $u \in \mathrm{L}^1([0,t], \partial X)$. Thus, $\mathrm{rg}(L_\mu)$ satisfies the condition (Z_1) in [DS89]. By [DS89, Thm. 9.(ii)] this yields $\mathrm{rg}(L_\mu) = \ker(\mu - A_m) \subset \mathrm{Fav}_1^A$ since for all $z \in \partial X$ we estimate

$$\begin{aligned} \|T(t)L_\mu z - L_\mu z\| &= \left\| A \int_0^t T(t-s) L_\mu z\, ds \right\| \\ &\leq \left\| \mu \int_0^t T(t-s) L_\mu z\, ds \right\| + \left\| \int_0^t T_{-1}(t-s) L_A z\, ds \right\| \\ &\leq |\mu| t \sup_{s \in [0,t]} \|T(s)\| \|L_\mu z\| + M \|L_A z\|_1 \\ &\leq Ct \|z\|_{\partial X} \end{aligned}$$

for some $C \geq 0$. The assertion follows by definition, see [EN00, Def. II.5.10]. $\square$

III.3. A Semigroup Approach to Wellposedness of Neutral Differential Equations

We now illustrate Theorem III.8 in the context of *Neutral Differential Equations*. It is our aim to present an operator theoretic formulation and proof of the wellposedness result for neutral differential equations in [HR08], see also [Adl15, Sect. 4].
To this end, let A generate a C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space X. For linear operators P, Q consider the system

$$\text{(NDE)} \qquad \begin{cases} \frac{d}{dt}[Qx_t] = A[Qx_t] + Px_t, & t \geq 0, \\ x_0 = f, \\ Qx_0 = x, \end{cases}$$

where the history segments x_t are defined by $x_t(r) := x(t+r)$ for $r \in [-1, 0]$.

The equation (NDE) can be transformed into an abstract Cauchy problem (ACP) on the space $\mathcal{X} := X \times Y := X \times \mathrm{L}^p([-1,0], X)$ for the operator matrix

$$\text{(3.8)} \qquad \mathcal{G} := \begin{pmatrix} A & P \\ 0 & \frac{d}{ds} \end{pmatrix}, \quad D(\mathcal{G}) := \left\{ \binom{x}{f} \in D(A) \times \mathrm{W}^{1,p}([-1,0], X) : x = Qf \right\}.$$

We assume that the operator Q is given by $Q = \delta_0 - K$ for some operator $K \in \mathcal{L}(\mathrm{C}([-1,0],X),X)$ and $P \in \mathcal{L}(\mathrm{C}([-1,0],X),X)$. In [HR08, Thm. 19, Prop. 21], Hadd and Rhandi show that $\mathcal{G}$ generates a C_0-semigroup on $\mathcal{X}$ (under appropriate assumptions) and that the neutral differential equation is *wellposed in a weak sense*, i.e., (NDE) has a unique *generalized solution*[1] for any initial value $\binom{x}{f} \in \mathcal{X}$. For further information, see [BHS83, HR08, HVL93, KZ86, NH03] and references therein.

We rewrite the operator $\mathcal{G}$ in (3.8) to meet the abstract framework from Chapter I and introduce the following operators and spaces:

- $D_m := \frac{d}{dr}$ with domain $D(D_m) := \mathrm{W}^{1,p}([-1,0],X)$ on Y;
- $L := \delta_0 : D(D_m) \subset Y \to \partial Y := X$, i.e., $Lv = v(0)$.

Then $D := D_m|_{\ker(L)}$ generates the nilpotent left shift semigroup $(S(t))_{t\geq 0}$ on Y. Moreover, the associated Dirichlet operator for the pair (D_m, L) exists for $\mu = 0$ and is given by $L_0 \in \mathcal{L}(\partial Y, Y)$, $(L_0 f)(r) := f$ for $r \in [-1,0]$.

Next, we define the spaces $\mathcal{U} := X \times X$ and $\mathcal{Z} := X \times \mathrm{C}([-1,0],X)$ and introduce the operator matrices

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : D(\mathcal{A}) := D(A) \times D(D) \subset \mathcal{X} \to \mathcal{X},$$

$$\mathcal{B} := \begin{pmatrix} \mathrm{Id}_X & 0 \\ 0 & L_D \end{pmatrix} : \mathcal{U} \to \mathcal{X}_{-1}^{\mathcal{A}} \qquad \mathcal{C} := \begin{pmatrix} 0 & P \\ \mathrm{Id}_X & K \end{pmatrix} : \mathcal{Z} \to \mathcal{U},$$

where $L_D := -D_{-1}L_0 \in \mathcal{L}(\partial Y, Y_{-1}^D)$. Then we obtain $\mathcal{G} = \mathcal{A}_{\mathcal{BC}}$ as in (0.2). We make the following

Assumptions III.16. Take μ and $\nu : [-1,0] \to \mathcal{L}(X)$ of bounded variation and assume that the operators K, $P \in \mathcal{L}(\mathrm{C}([-1,0],X),X)$ are given by

- $Pf = \int_{-1}^0 f(r)\, d\mu(r)$,
- $Kg = \int_{-1}^0 g(r)\, d\nu(r)$

for f, $g \in \mathrm{C}([-1,0],X)$. Moreover, we assume that the operator K has *no mass in zero*, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Kg\|_X \leq \varepsilon \|g\|_\infty$ for every $g \in \mathrm{C}([-1,0],X)$ satisfying $\operatorname{supp}(g) \subset [-\delta, 0]$.

Using Theorem III.8 we obtain the following generation result (cf. [HR08, Thm. 19]).

Theorem III.17. *For every P and $K \in \mathcal{L}(\mathrm{C}([-1,0],X),X)$ satisfing Assumption III.16, the operator $\mathcal{G}$ in* (3.8) *generates a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{X}$ which we call the* neutral semigroup.

[1] See [HR08, Def. 17] for the definition of a generalized solution of (NDE).

Proof. In order to guarantee that $\mathcal{P} = \mathcal{B}\mathcal{C} \in \mathcal{L}(\mathcal{Z}, \mathcal{X}_{-1}^{\mathcal{A}})$ is a Weiss-Staffans perturbation for $\mathcal{A}$, it suffices to show that

(i) (L_D, P) and (L_D, K) are jointly p-admissible for D, and

(ii) $1 \in \rho\Big(\mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})}\Big)$ for some $t > 0$, where the input-output map of the system is given by

$$\mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})} = \begin{pmatrix} 0 & \mathcal{F}_t^{(D,L_D,P)} \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_X)} & \mathcal{F}_t^{(D,L_D,K)} \end{pmatrix} \in \mathcal{L}(\mathrm{L}^p([0,t],\mathcal{U})).$$

The assertion (i) follows as in [ABE14, Cor. 25], where it is also shown that the operator $\mathrm{Id} - \mathcal{F}_t^{(D,L_D,K)}$ is invertible with uniformly bounded inverses $\Big(\mathrm{Id} - \mathcal{F}_t^{(D,L_D,K)}\Big)^{-1}$ for sufficiently small $t > 0$ since $\Big\|\mathcal{F}_t^{(D,L_D,K)}\Big\| \leq |\nu|[-t,0]$ (we use the "no mass in zero" assumption on K). Using Schur complements (cf. Lemma B.1(i)), the invertibility of $\mathrm{Id} - \mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})}$ is equivalent to the invertiblity of

$$\mathrm{Id} - \mathcal{F}_t^{(D,L_D,P)}\Big(\mathrm{Id} - \mathcal{F}_t^{(D,L_D,K)}\Big)^{-1}\mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_X)} =: \mathrm{Id} - \Delta_t.$$

Since $\Big\|\mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_X)}\Big\| \to 0$ as $t \to 0^+$ by Corollary III.10.(a) we obtain $\|\Delta_t\| \to 0$ as $t \to 0^+$. Thus, Theorem III.8 yields the assertion. ☐

Thus we obtain the following (cf. [HR08, Prop. 20.(ii)] as well).

Corollary III.18. *Let the linear operators P, K satisfy Assumption III.16. The neutral differential equation*

$$\text{(NDE)} \qquad \begin{cases} \frac{d}{dt}[x(t) - Kx_t] = A[x(t) - Kx_t] + Px_t, & t \geq 0, \\ x_0 = f, \\ x(0) - Kx_0 = x, \end{cases}$$

has a unique classical solution[2] *for all $f \in \mathrm{W}^{1,p}([-1,0],X)$ and $x \in D(A)$ satisfying $Qf = f(0) - Kf = x$, i.e., $\binom{x}{f} \in D(\mathcal{G})$.*

[2] See [HR08, Def. 15] for the definition of a classical solution of (NDE).

IV. Perturbations of Generators of Analytic Semigroups

In this chapter we present an approach towards perturbations of generators of analytic semigroups. In this case we are able to substitute the compatibility and admissibility conditions in Definition III.3 by

- an inclusion of the range of B in a certain intermediate space and
- the inclusion of some domain of a fractional power of A in the domain of C.

This yields, in particular, that A_{BC} generates an analytic semigroup of the same angle of analyticity as the unperturbed semigroup. The proof is based on Theorem III.8. We point out that the result presented in Theorem IV.1

(i) allows perturbations $P = BC$ which are not relatively A-bounded (compare, e.g., in [EN00, Sect. III.2]),

(ii) always give the angle of analyticity of the perturbed semigroup (compare [GK91, Thm. 2.6] in the situation of boundary perturbations),

(iii) are applicable also to coupled systems which are only in part governed by an analytic semigroup, cf. Section IV.4.

For the basic results on *analytic semigroups*, we refer to [Lun95, Chap. 2] and [EN00, Sect. II.4.(a)].

The results in Chapter IV will appear in the joint paper *Perturbation of analytic semigroups and applications to partial differential equations* together with Miriam Bombieri and Klaus-Jochen Engel (cf. [ABE15]). Since the presentation is taken from that manuscript, parts of Section IV.1 are included in [Bom15, Chap. 4] as well.

IV.1. The Abstract Perturbation Result

Here is our main perturbation result (cf. [Bom15, Thm. 4.2.3] too).

Theorem IV.1. *Let A generate an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on X. Moreover, assume that for $B \in \mathcal{L}(U, X_{-1}^A)$ and $C \in \mathcal{L}(Z, U)$ there exist $\lambda > \omega_0(A)$, $\beta \geq 0$ and $\gamma > 0$ such that*

(i) $\operatorname{rg}(R(\lambda, A_{-1})B) \subseteq \operatorname{Fav}_{1-\beta}^A$,

(ii) $[D((\lambda - A)^\gamma)] \hookrightarrow Z$,[1]

(iii) $\beta + \gamma < 1$.

Then the following holds.

(a) (A, B, C) *is compatible.*

(b) B *is a p-admissible control operator for all* $p > \frac{1}{1-\beta}$, *and, if* $\beta = 0$, *then also for* $p = 1$.

(c) C *is a p-admissible observation operator for all* $p < \frac{1}{\gamma}$.

(d) (B, C) *is jointly p-admissible for all* $\frac{1}{1-\beta} < p < \frac{1}{\gamma}$, *and, if* $\beta = 0$, *then also for* $p = 1$.

(e) for every $0 < \varepsilon < 1 - (\beta + \gamma)$ *and* $\frac{1}{1-\beta} \leq p < \frac{1}{\gamma}$ *there exists* $M \geq 0$ *such that*

$$\left\| \mathcal{F}_t^{(A,B,C)} \right\|_p \leq M \cdot t^\varepsilon \quad \text{for all } 0 < t \leq 1.$$

In particular, Id_U *is a p-admissible feedback operator.*

Finally, the operator A_{BC} *generates an analytic* C_0*-semigroup of angle* θ *on* X.

In order to prove part (b) and (d) of the above result we need the following version of Young's inequality. Here, for two functions K and v on $(0, t_0]$ for some $t_0 > 0$ we define their convolution by

$$\begin{aligned} (K * v)(0) &:= 0, \\ (K * v)(t) &:= \int_0^t K(t-s)v(s)\, ds \quad \text{for } t \in (0, t_0]. \end{aligned}$$

Lemma IV.2 and its proof are also contained in [Bom15, Lem. 4.2.1].

[1] The operator $(\lambda - A)^\gamma$ denotes the *fractional power* of order $\gamma \in \mathbb{R}$ of A for some $\lambda > \omega_0(A)$, see [EN00, Sect. II.5.(c)].

Lemma IV.2. *Let X, Y be Banach spaces and $K : (0,1] \to \mathcal{L}(Y,X)$ be strongly continuous. Moreover, assume that $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $k(\bullet) := ||K(\bullet)||_{\mathcal{L}(Y,X)} \in \mathrm{L}^q[0,1]$ and $v \in \mathrm{C}([0,1], Y)$, then $K * v \in \mathrm{L}^r([0,1], X)$ and*

$$\|K * v\|_r \leq \|k\|_q \cdot \|v\|_p. \tag{4.1}$$

Proof. We adapt the proof from [ABHN11, Prop. 1.3.5] where convolutions on $\mathbb{R}_+$ are considered and K is assumed to be strongly continuous on $\mathbb{R}_+$. Fix $0 < t \leq 1$ and $v \in \mathrm{C}([0,1], Y)$. Then $s \mapsto b(s) := K(t-s)v(s)$ is continuous on $(0,t)$, hence measurable. Note that $k(t - \bullet) \in \mathrm{L}^q[0,t] \subseteq \mathrm{L}^1[0,t]$, thus the estimate

$$\|b(s)\| = \|K(t-s)v(s)\| \leq k(t-s) \cdot \|v\|_\infty$$

implies that $||b(\bullet)||$ is integrable on $[0,t]$. By Bochner's theorem (see [ABHN11, Thm. 1.1.4]) this shows that b is integrable and hence $(K*v)(t)$ exists for all $t \in [0,1]$. Next we show that $t \mapsto (K * v)(t)$ is continuous on $[0,1]$. First, take $t \in [0,1)$ and choose $h > 0$ such that $t+h \in [0,1]$. Since $k \in \mathrm{L}^q[0,1] \subseteq \mathrm{L}^1[0,1]$ and $v \in \mathrm{C}([0,1], Y)$ is uniformly continuous we conclude

$$\begin{aligned}
&\Big\|(K * v)(t+h) - (K * v)(t)\Big\| \\
&\qquad \leq \int_0^t k(s) \cdot \Big\|v(t+h-s) - v(t-s)\Big\|\, ds + \int_t^{t+h} k(s) \cdot \Big\|v(t+h-s)\Big\|\, ds \\
&\qquad \leq \|k\|_1 \cdot \sup_{s, s+h \in [0,1]} \Big\|v(s+h) - v(s)\Big\| + \int_t^{t+h} k(s)\, ds \cdot \|v\|_\infty \\
&\qquad \to 0 \qquad \text{as } h \to 0^+.
\end{aligned}$$

For $t \in (0,1]$ and $h > 0$ such that $t - h \in [0,1]$, an analogous estimate finally shows the continuity of $t \mapsto (K * v)(t)$ on $[0,1]$. This shows that $K * v \in \mathrm{C}([0,1], X) \subset \mathrm{L}^r([0,1], X)$ and by the scalar-valued version of Young's inequality (see [RS75, Sect. IX.4, Expl. 1]) we finally obtain

$$\|K * v\|_r \leq \Big\|k * \|v(\bullet)\|_Y\Big\|_r \leq \|k\|_q \cdot \|v\|_p. \qquad \square$$

We now prove Theorem IV.1.

Proof. Note that for every $\lambda > \omega_0(A)$ we have $\mathrm{Fav}^A_{1-\beta} = \mathrm{Fav}^{A-\lambda}_{1-\beta}$. Hence, replacing, if necessary, A by $A - \lambda$ we can assume that $\omega_0(A) < 0$ and $\lambda = 0$.

(a) By [EN00, Props. II.5.14 & II.5.33] we have

$$D\Big((-A)^\alpha\Big) \subset \mathrm{Fav}^A_\alpha \subset D\Big((-A)^\delta\Big) \quad \text{for all } 1 > \alpha > \delta > 0. \tag{4.2}$$

Since by assumption (iii) we have $1-\beta>\gamma$, (4.2) and (ii) imply

$$\operatorname{rg}\Big(A_{-1}^{-1}B\Big) \subseteq \mathrm{Fav}^A_{1-\beta} \subseteq D\Big((-A)^\gamma\Big) \subseteq Z = D(C),$$

i.e., the triple (A,B,C) is compatible.

(b) Since $A_{-1}^{-1}B \in \mathcal{L}(U,X)$ and $\mathrm{Fav}^A_{1-\beta} \hookrightarrow X$, assumption (i) and the closed graph theorem imply that $A_{-1}^{-1}B \in \mathcal{L}(U,\mathrm{Fav}^A_{1-\beta})$. Hence, for all $u \in \mathrm{C}([0,1],U) \subset \mathrm{L}^p([0,1],U)$

$$v := A_{-1}^{-1}Bu \in \mathrm{C}\Big([0,1],\mathrm{Fav}^A_{1-\beta}\Big) \subset \mathrm{L}^p\Big([0,1],\mathrm{Fav}^A_{1-\beta}\Big). \tag{4.3}$$

Since $\operatorname{rg}(T(t)) \subseteq D(A^\infty)$ for all $t>0$, we can define

$$K:(0,1]\to\mathcal{L}\Big(\mathrm{Fav}^A_{1-\beta},X\Big), \quad K(t):=AT(t).$$

Then K is strongly continuous on $(0,1]$ and by [EN00, Prop. II.5.13] there exists $M>0$ such that

$$\Big\|t^\beta K(t)x\Big\|_X \le \sup_{s\in(0,1]}\Big\|s^\beta AT(s)x\Big\|_X \le M\cdot\|x\|_{\mathrm{Fav}^A_{1-\beta}} \qquad \text{for all } x\in\mathrm{Fav}^A_{1-\beta}.$$

This implies that

$$k(t) := \|K(t)\|_{\mathcal{L}(\mathrm{Fav}^A_{1-\beta},X)} \le M\cdot t^{-\beta} \quad \text{for all } t\in(0,1]. \tag{4.4}$$

Hence, $k\in\mathrm{L}^q[0,1]$ if $\beta\cdot q<1$, i.e.,

$$k\in\mathrm{L}^q[0,1] \quad \text{if} \quad \begin{cases} q<\frac{1}{\beta} & \text{and } \beta>0, \text{ or}\\ q\ge 1 & \text{and } \beta=0.\end{cases}$$

Now we choose $r=\infty$ in Young's inequality from Lemma IV.2. Then $q=\frac{p}{p-1}$ and from (4.1) it follows that there exists $M\ge0$ such that for all $u\in\mathrm{C}([0,1],U)$

$$\left\|\int_0^1 T_{-1}(1-s)Bu(s)\,ds\right\|_X = \Big\|(K*v)(1)\Big\|_X \le \|K*v\|_\infty \le M\cdot\|k\|_q\cdot\|u\|_p$$

provided

$$\begin{cases} \dfrac{p}{p-1}=q<\dfrac{1}{\beta} \text{ and } \beta>0 & \iff \quad p>\dfrac{1}{1-\beta} \text{ and } \beta>0, \text{ or}\\ \dfrac{p}{p-1}=q\ge1 \text{ and } \beta=0 & \iff \quad p\ge1 \text{ and } \beta=0.\end{cases}$$

Since $\mathrm{W}^{1,p}([0,1],U) \subset \mathrm{C}([0,1],U)$, the assertion follows from Remark III.4.(a).
(c) For all $t > 0$ we have by (ii)

$$\left\|CT(t)\right\|_{\mathcal{L}(X,U)} \le \left\|C(-A)^{-\gamma}\right\|_{\mathcal{L}(X,U)} \cdot \left\|(-A)^{\gamma}T(t)\right\|_{\mathcal{L}(X)}.$$

By [RR93, Lem. 11.36] there exists $M \ge 0$ such that

$$\left\|(-A)^{\gamma}T(t)\right\|_{\mathcal{L}(X)} \le M \cdot t^{-\gamma} \quad \text{for all } t \in (0,1], \tag{4.5}$$

and therefore C is a p-admissible observation operator for all $p < \frac{1}{\gamma}$.
(d) Since $\mathrm{rg}(T(t)) \subseteq D(A^{\infty})$, we can define

$$L : (0,1] \to \mathcal{L}\Big(\mathrm{Fav}^A_{1-\beta}, X\Big), \quad L(t) := (-A)^{1+\gamma}T(t) \quad \text{for } t \in (0,1].$$

Then L is strongly continuous on $(0,1]$. Using (4.4) and (4.5) we obtain for all $0 < t \le 1$ and suitable $M \ge 0$

$$\begin{aligned} l(t) := \|L(t)\|_{\mathcal{L}(\mathrm{Fav}^A_{1-\beta},X)} &\le \left\|(-A)^{\gamma}T(\tfrac{t}{2})\right\|_{\mathcal{L}(X)} \cdot \left\|AT(\tfrac{t}{2})\right\|_{\mathcal{L}(\mathrm{Fav}^A_{1-\beta},X)} \\ &\le M \cdot t^{-(\beta+\gamma)}. \end{aligned} \tag{4.6}$$

Now choose in Young's inequality from Lemma IV.2 $p = \frac{1}{1-\beta} \le r < \frac{1}{\gamma}$. Then we obtain $\frac{1}{q} = \beta + \frac{1}{r} > \beta + \gamma$ and hence

$$q \cdot (\beta + \gamma) < 1,$$

which implies that $l \in \mathrm{L}^q[0,1]$ by (4.6). Thus, by (4.1) there exists $M \ge 0$ such that the input-output map $\mathcal{F}_t^{(A,B,C)}$ for all $0 < t \le 1$ and $u \in \mathrm{C}([0,1],U)$ satisfies

$$\begin{aligned} \left\|\mathcal{F}_t^{(A,B,C)}u\right\|_r &\le \left(\int_0^1 \left\|C\int_0^t T_{-1}(t-s)Bu(s)\,ds\right\|_U^r dt\right)^{\frac{1}{r}} \\ &\le \left\|C(-A)^{-\gamma}\right\| \cdot \left(\int_0^1 \left\|\int_0^t (-A)^{1+\gamma}T_{-1}(t-s)\cdot A_{-1}^{-1}Bu(s)\,ds\right\|_X^r dt\right)^{\frac{1}{r}} \\ &\le M \cdot \|L * v\|_r \\ &\le M \cdot \|l\|_q \cdot \|u\|_{\frac{1}{1-\beta}} \end{aligned}$$

where $v \in \mathrm{C}\Big([0,1], \mathrm{Fav}^A_{1-\beta}\Big)$ is given by (4.3). This shows that for every $\frac{1}{1-\beta} \le r < \frac{1}{\gamma}$ and $0 < t \le 1$ the input-output map has a unique bounded extension

$$\mathcal{F}_t^{(A,B,C)} : \mathrm{L}^{\frac{1}{1-\beta}}\Big([0,t],U\Big) \to \mathrm{L}^r\Big([0,t],U\Big). \tag{4.7}$$

Since $\mathrm{L}^r\big([0,t],U\big) \hookrightarrow \mathrm{L}^{\frac{1}{1-\beta}}\big([0,t],U\big)$ for $r \ge \frac{1}{1-\beta}$, this together with (b) and (c) proves that the pair (B,C) is jointly p-admissible for all $p \in (\frac{1}{1-\beta},\frac{1}{\gamma})$ and in case $\beta = 0$ also for $p = 1$.

(e) By Jensen's inequality we have for all $1 \le p \le r < \infty$ and $u \in \mathrm{L}^r([0,t],U) \subseteq \mathrm{L}^p([0,t],U)$

$$\|u\|_p \le t^{\frac{1}{p}-\frac{1}{r}} \cdot \|u\|_r.$$

This combined with (4.7) gives for all $\frac{1}{1-\beta} \le p \le r < \frac{1}{\gamma}$ and $u \in \mathrm{L}^r([0,t],U)$ that

$$t^{-\frac{1}{p}+\frac{1}{r}} \cdot \|\mathcal{F}_t^{(A,B,C)}u\|_p \le \|\mathcal{F}_t^{(A,B,C)}u\|_r \le M \cdot \|u\|_{\frac{1}{1-\beta}} \le M \cdot t^{1-\beta-\frac{1}{p}} \cdot \|u\|_p.$$

For given $0 < \varepsilon < 1-(\beta+\gamma)$ we take $r := \frac{1}{1-\beta-\varepsilon} \in \Big(\frac{1}{1-\beta},\frac{1}{\gamma}\Big)$ and obtain by density of $\mathrm{L}^r([0,t],U)$ in $\mathrm{L}^p([0,t],U)$ that

$$\|\mathcal{F}_t^{(A,B,C)}u\|_p \le M \cdot t^{1-\beta-\frac{1}{r}}\|u\|_p \le M \cdot t^{\varepsilon}\|u\|_p$$

for all $u \in \mathrm{L}^p([0,t],U)$ as claimed. Clearly (d) and (e) combined with Theorem III.8 imply that A_{BC} generates a C_0-semigroup.

This semigroup is in fact analytic of angle θ by the following two lemmas. More precisely, Lemma IV.4 shows that the domains of fractional powers and the Favard spaces are invariant under rotations of the operator A. This allows us to repeat the above reasoning for A, B, C replaced by $e^{i\varphi}A$, $e^{i\varphi}B$, C. Hence, we obtain that also $e^{i\varphi}A_{BC}$ is a generator of a C_0-semigroup on X for all $\varphi \in (-\theta,\theta)$. By Lemma IV.3 this implies the assertion. $\square$

In the following for an operator A and $\varphi \in \mathbb{R}$ we use the notation

$$A_\varphi := e^{i\varphi}A.$$

Lemma IV.3. *Let $0 < \theta \le \frac{\pi}{2}$. Then A generates an analytic semigroup of angle θ if and only if A_φ generates a C_0-semigroup for every $\varphi \in (-\theta,\theta)$.*

Proof. Assume first that A generates an analytic semigroup $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ of angle θ where Σ_θ denotes the open sector

$$\Sigma_\theta := \Big\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \theta\Big\}.$$

Then it is clear that for every $\varphi \in (-\theta,\theta)$ the operators $T_\varphi(t) := T(e^{i\varphi}t)$ define a strongly continuous semigroup $(T_\varphi(t))_{t\ge 0}$ with generator A_φ, cf. [ABHN11, Prop. 3.7.2.(c)].

Conversely, assume that A_φ generates a C_0-semigroup $(T_\varphi(t))_{t\geq 0}$ for every $\varphi \in (-\theta, \theta)$. Then by [EN00, Thm. II.4.6.(b)] the operator A generates an analytic semigroup $(T(z))_{z\in\Sigma_{\theta'}\cup\{0\}}$ of some angle $\theta' > 0$. If $\theta' \geq \theta$ we are done, hence assume that $\theta' < \theta$. Then we have to show that the map $z \mapsto T(z)$ can be extended analytically from $\Sigma_{\theta'}$ to the open sector Σ_θ.

To this end we fix some $\varphi \in (\theta', \theta)$ and consider the two projections on the complex plane $P_{\pm\varphi} : \mathbb{C} \to \mathbb{C}$ onto $e^{\pm i\varphi} \cdot \mathbb{R}$ along $e^{\mp\varphi i} \cdot \mathbb{R}$. Then for $z \in \bar{\Sigma}_\varphi$ we put $r_\pm(z) := e^{\mp i\varphi} \cdot P_{\pm\varphi} z \geq 0$. Since $P_{\pm\varphi} = 1 - P_{\mp\varphi}$, this implies $z = r_+(z) \cdot e^{i\varphi} + r_-(z) \cdot e^{-i\varphi}$. Using this representation of z we define

$$\tilde{T} : \bar{\Sigma}_\varphi \to \mathcal{L}(X), \quad \tilde{T}(z) := T_\varphi\Big(r_+(z)\Big) \cdot T_{-\varphi}\Big(r_-(z)\Big). \tag{4.8}$$

Since the resolvents of $A_{\pm\varphi}$ commute, also the semigroups $(T_{\pm\varphi}(t))_{t\geq 0}$ commute. Using this fact and the equations $r_\pm(z + w) = r_\pm(z) + r_\pm(w)$ it follows that

$$\tilde{T}(z) \cdot \tilde{T}(w) = \tilde{T}(z + w) \quad \text{for all } z, w \in \bar{\Sigma}_\varphi.$$

Next we show that $(\tilde{T}(z))_{z\in\bar{\Sigma}_\varphi}$ is strongly continuous on the closed sector $\bar{\Sigma}_\varphi$. To this end choose $M, \omega > 0$ such that $\|T_{\pm\varphi}(t)\| \leq M \cdot e^{\omega t}$ for all $t \geq 0$. Then from the continuity of $r_\pm(\bullet)$ and the fact that $r_\pm(z) \leq \|P_{\pm\varphi}\| \cdot |z|$ we obtain for $x \in X$ and $z, w \in \bar{\Sigma}_\varphi$

$$\begin{aligned}
\Big\|\tilde{T}(z)x - \tilde{T}(w)x\Big\| &\leq \Big\|T_\varphi\Big(r_+(z)\Big) \cdot \Big[T_{-\varphi}\Big(r_-(z)\Big) - T_{-\varphi}\Big(r_-(w)\Big)\Big]x\Big\| \\
&\qquad + \Big\|\Big[T_\varphi\Big(r_+(z)\Big) - T_\varphi\Big(r_+(w)\Big)\Big] \cdot T_{-\varphi}\Big(r_-(w)\Big)x\Big\| \\
&\leq Me^{\omega\left(\|P_\varphi\|+\|P_{-\varphi}\|\right)\cdot|z|} \cdot \Big(\Big\|T_{-\varphi}\Big(r_-(z)\Big)x - T_{-\varphi}\Big(r_-(w)\Big)x\Big\| \\
&\qquad + \Big\|T_\varphi\Big(r_+(z)\Big)x - T_\varphi\Big(r_+(w)\Big)x\Big\|\Big) \\
&\to 0 \text{ as } w \to z \text{ in } \bar{\Sigma}_\varphi.
\end{aligned}$$

Hence, $(\tilde{T}(z))_{z\in\bar{\Sigma}_\varphi}$ is strongly continuous as claimed. This implies in particular that for every $\psi \in [-\varphi, \varphi]$ the restriction

$$\tilde{T}_\psi(t) := \tilde{T}\Big(e^{i\psi}t\Big), \quad t \geq 0,$$

defines a C_0-semigroup on X. Next we compute its generator $\tilde{A}_\psi$. Let

$$r_\pm := r_\pm\Big(e^{i\psi}\Big), \quad \text{i.e.,} \quad e^{i\psi} = r_+ \cdot e^{i\varphi} + r_- \cdot e^{-i\varphi}.$$

Then by definition of $\tilde{T}(z)$ in (4.8) we have $\tilde{T}_\psi(t) = T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)$. Hence, for $x \in D(A)$ we obtain

$$\begin{aligned}\frac{d}{dt}\tilde{T}_\psi(t)x &= r_+A_\varphi \cdot T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)x + r_-A_{-\varphi} \cdot T_\varphi(r_+t) \cdot T_{-\varphi}(r_-t)x \\ &= \Big(r_+e^{i\varphi}A + r_-e^{-i\varphi}A\Big) \cdot \tilde{T}(z)x = e^{i\psi}A \cdot \tilde{T}(z)x.\end{aligned}$$

This implies $e^{i\psi}A \subseteq \tilde{A}_\psi$ and since $\rho(e^{i\psi}A) \cap \rho(\tilde{A}_\psi) \neq \emptyset$, we obtain $\tilde{A}_\psi = e^{i\psi}A = A_\psi$. Since a generator uniquely determines the generated semigroup, we conclude that

$$T(z) = \tilde{T}(z) \quad \text{for all } z \in \Sigma_{\theta'},$$

i.e., $(\tilde{T}(z))_{z\in\bar{\Sigma}_\varphi}$ is a strongly continuous extension of $(T(z))_{z\in\Sigma_{\theta'}\cup\{0\}}$. For this reason from now on we can drop the tilde and write $T(z) = \tilde{T}(z)$ for all $z \in \bar{\Sigma}_\varphi$.

Summing up, we showed that A generates a semigroup $(T(z))_{z\in\bar{\Sigma}_\varphi}$ which is strongly continuous on $\bar{\Sigma}_\varphi$ and analytic on $\Sigma_{\theta'}$. It remains to show that $(T(z))_{z\in\bar{\Sigma}_\varphi}$ is analytic on Σ_φ. To this end note that for each $r > 0$

$$\begin{aligned}&z \mapsto T\Big(re^{\pm i\varphi}\Big) \cdot T(z) = T\Big(re^{\pm i\varphi} + z\Big) \quad \text{is analytic on } \Sigma_{\theta'} \quad \Longrightarrow \\ &z \mapsto T(z) \quad \text{is analytic on } re^{\pm i\varphi} + \Sigma_{\theta'}.\end{aligned}$$

Since

$$\Sigma_\varphi = \bigcup_{r>0}\Big(re^{\pm i\varphi} + \Sigma_{\theta'}\Big),$$

this implies that $(T(z))_{z\in\bar{\Sigma}_\varphi}$ is analytic on the whole open sector Σ_φ as claimed. Recall that $\varphi \in (\theta', \theta)$ was arbitrary. Thus, from

$$\Sigma_\theta = \bigcup_{\varphi\in(-\theta,\theta)} \Sigma_\varphi$$

we finally conclude that the semigroup $(T(t))_{t\geq 0}$ can be extended to an analytic semigroup $(T(z))_{z\in\bar{\Sigma}_\theta\cup\{0\}}$, i.e., is analytic of angle (at least) θ. □

Lemma IV.4 is also contained in [Bom15, Lem. 4.2.2].

Lemma IV.4. *Let A generate an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2})$. Moreover, let $\varphi \in (-\theta, \theta)$ and $\lambda > 0$ such that $\omega_0(A-\lambda), \omega_0(A_\varphi-\lambda) < 0$. Then for all $\alpha \in (0, 1]$ one has*

$$D\Big((\lambda - A)^\alpha\Big) = D\Big((\lambda - A_\varphi)^\alpha\Big) \quad \textit{and} \quad \mathrm{Fav}_\alpha^A = \mathrm{Fav}_\alpha^{A_\varphi}. \tag{4.9}$$

Proof. Note that by the previous result A_φ generates an analytic semigroup. Without loss of generality we assume that $\lambda = 0$.

To show the first equality in (4.9) fix some $\alpha \in (0,1)$. Then by the definition of $(-A)^{-\alpha}$ (see, e.g. [EN00, Def. II.5.25]), the equality

$$R(\lambda, A_\varphi) = e^{-i\varphi} R\Big(e^{-i\varphi}\lambda, A\Big) \tag{4.10}$$

and by Cauchy's integral theorem it follows that $(-A_\varphi)^{-\alpha} = e^{-i\varphi\alpha} \cdot (-A)^{-\alpha}$. This implies

$$D((-A)^\alpha) = \mathrm{rg}((-A)^{-\alpha}) = \mathrm{rg}((-A_\varphi)^{-\alpha}) = D((-A_\varphi)^\alpha)$$

for $\alpha \in (0,1)$ while for $\alpha = 1$ the assertion is obviously satisfied.

Next we show that $\mathrm{Fav}_\alpha^A \subseteq \mathrm{Fav}_\alpha^{A_\varphi}$ for $\alpha \in (0,1]$. Let $x \in \mathrm{Fav}_\alpha^A$. Then by (4.10), the resolvent equation, the Hille–Yosida theorem for A_φ and [EN00, Prop. II.5.12] we conclude that

$$\begin{aligned}
\sup_{\lambda>0}\Big\|\lambda^\alpha A_\varphi R(\lambda, A_\varphi)x\Big\| &= \sup_{\lambda>0}\Big\|\lambda^\alpha A\Big(R(e^{-i\varphi}\lambda, A) - R(\lambda, A)\Big)x + \lambda^\alpha AR(\lambda, A)x\Big\| \\
&\le \sup_{\lambda>0}\Big\|(1-e^{-i\varphi})\lambda R(e^{-i\varphi}\lambda, A)\lambda^\alpha AR(\lambda, A)x\Big\| + \sup_{\lambda>0}\Big\|\lambda^\alpha AR(\lambda, A)x\Big\| \\
&\le \left(1 + \sup_{\lambda>0}\Big\|(1-e^{-i\varphi})\lambda R(\lambda, A_\varphi)\Big\|\right) \cdot \sup_{\lambda>0}\Big\|\lambda^\alpha AR(\lambda, A)x\Big\| < \infty.
\end{aligned}$$

Again by [EN00, Prop. II.5.12] this implies that $x \in \mathrm{Fav}_\alpha^{A_\varphi}$, hence $\mathrm{Fav}_\alpha^A \subseteq \mathrm{Fav}_\alpha^{A_\varphi}$. In order to show the converse inclusion note that $A = e^{-i\varphi}A_\varphi$. The assertion then follows as above by interchanging the roles of A and A_φ and substituting φ by $-\varphi$. □

Remark IV.5. In Theorem IV.1, the implications (i) $\Longrightarrow$ (b), (ii) $\Longrightarrow$ (c) and (i) & (ii) $\Longrightarrow$ (a), (d), (e), e.g.,

- $\mathrm{rg}(R(\lambda, A_{-1})B) \subseteq \mathrm{Fav}_{1-\beta}^A \implies B$ is p-admissible for $p > \frac{1}{1-\beta}$, and
- $[D((\lambda - A)^\gamma)] \hookrightarrow Z \implies C$ is p-admissible for $1 \le p < \frac{1}{\gamma}$,

are the main results. These conclusions help to establish the wellposedness of systems of equations which are only partly governed by an analytic semigroup (cf. Section IV.4).

For further literature on perturbations of generators of analytic semigroups we refer to [GK91, KW01, HHK06] (cf. also [MS08, Sect. 3]). The authors are concerned with (R-)sectoriality[2] of the perturbed operators. The proofs rely on direct estimates of the resolvent operators while Theorem IV.1 is based on the Weiss-Staffans perturbation theorem.

IV.2. Boundary Perturbations of Generators of Analytic Semigroups

We now apply Theorem IV.1 to the operator $G = A_P^\Phi$ in (1.2) under the additional assumption that the operator A in Assumption III.12 generates an analytic semigroup. Note that condition (i) in the following corollary can be verified using the equivalences in Lemma III.15.

Corollary IV.6. *Let $A \subset A_m$ generate an analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on X, $P \in \mathcal{L}(Z, X)$ and $\Phi \in \mathcal{L}(Z, \partial X)$. If there exist $\lambda > \omega_0(A)$, $\beta \geq 0$ and $\gamma > 0$ such that*

(i) $\operatorname{rg}(L_\lambda) \subseteq \mathrm{Fav}_{1-\beta}^A$,

(ii) $[D((\lambda - A)^\gamma)] \hookrightarrow Z$,

(iii) $\beta + \gamma < 1$,

then $G = A_P^\Phi$ generates an analytic semigroup of angle θ on X.

Proof. For $B = (\mathrm{Id}_X,\ L_A) \in \mathcal{L}(X \times \partial X, U)$ as in (1.8) the assumption (i) in Theorem IV.1 requires

$$\operatorname{rg}(R(\lambda, A_{-1})B) = \operatorname{rg}((R(\lambda, A), R(\lambda, A_{-1})L_A) = \operatorname{rg}((R(\lambda, A), L_\lambda) \subseteq \mathrm{Fav}_{1-\beta}^A$$

and the assertion follows. ☐

Remark IV.7. The previous corollary improves [GK91, Thm. 2.6.(c)] where, by means of resolvent estimates, a similar result in the context of abstract Hölder spaces is proved. In contrast to our approach, the result in the above reference is not applicable to problems discussed in Section IV.4 where only part of the system is governed by an analytic semigroup, see Example IV.4 (cf. Remark IV.5).

In the following sections (see [ABE15, Sect. 3] as well) we apply Corollary IV.6 and Corollary III.13 to

[2] We refer to [Wei01, Def. 2.1] for the definition of R-boundedness.

(i) a system of second order differential operators with perturbed Neumann boundary conditions on $(\mathrm{L}^p[0,1])^n$,

(ii) a second order differential operator with delay in the Neumann boundary condition on $\mathrm{L}^p[0,1]$, and

(iii) the Neumann Laplacian with perturbed Neumann boundary conditions on $\mathrm{L}^2(\Omega)$.

We first embed the associated operators into the framework of Section IV.2 (see also Chapter I). The choice of the operators A_m, L and Φ differs from Chapter II since we are now interested in the generator property of the associated operators while Chapter II studies their spectrum.[3]

IV.3. Wellposedness of a System of Second Order Differential Operators with perturbed Neumann Boundary Conditions.

We take $P_1, ..., P_n \in \mathcal{L}(\mathrm{W}^{1,p}[0,1], \mathrm{L}^p[0,1])$ and $\Phi_1, \Phi_2 \in \mathcal{L}((\mathrm{W}^{1,p}[0,1])^n, \mathbb{C}^n)$ and consider on the space $\mathcal{X} := (\mathrm{L}^p[0,1])^n$ the operator

$$\mathcal{G} \subset \begin{pmatrix} c_1 \cdot \frac{d^2}{ds^2} + P_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & c_n \cdot \frac{d^2}{ds^2} + P_n \end{pmatrix} \tag{4.11}$$

with constants $c_1, ..., c_n > 0$ and domain $D(\mathcal{G}) := \{f \in (\mathrm{W}^{2,p}[0,1])^n : f'(0) = \Phi_1 f,\ f'(1) = \Phi_2 f\}$. Then the following holds.

Corollary IV.8. *For every $\Phi_1, \Phi_2 \in \mathcal{L}((\mathrm{W}^{1,p}[0,1])^n, \mathbb{C}^n)$ and every $P_1, ..., P_n \in \mathcal{L}(\mathrm{W}^{1,p}[0,1], \mathrm{L}^p[0,1])$, the operator $\mathcal{G}$ defined in* (4.11) *generates an analytic semigroup of angle $\frac{\pi}{2}$ on $\mathcal{X}$.*

A similar example is also included in [Bom15, Sect. 4.3].

Proof. We first introduce the maximal operator

$$\mathcal{A}_m := \operatorname{diag}\Big(c_j \cdot \tfrac{d^2}{ds^2}\Big)_{j=1,...,n} \quad \text{with domain } D(\mathcal{A}_m) := (\mathrm{W}^{2,p}[0,1])^n,$$

[3] In Chapter II, we choose the operator A such that $\sigma(A)$ is "small". However, the operators A in the applications of Chapter II do not generate strongly continuous semigroups on X in general, see, e.g., (2.15).

the boundary space $\partial\mathcal{X} := \mathbb{C}^{2n}$ and the boundary operator $\mathcal{L} := \binom{\delta_0'}{\delta_1'} : D(\mathcal{A}_m) \subset \mathcal{X} \to \partial\mathcal{X}$, i.e., $\mathcal{L}f = \binom{f'(0)}{f'(1)}$. Then $\mathcal{A} \subset \mathcal{A}_m$ with domain $D(\mathcal{A}) := \ker(\mathcal{L})$ is the uncoupled system of one-dimensional Laplacians with Neumann boundary conditions which generates an analytic semigroup of angle $\frac{\pi}{2}$ on $\mathcal{X}$ (use [CKW08, Thm. 2.2.(a)] and [ABHN11, Thm. 3.14.17]). Finally, we take $\mathcal{Z} := (\mathrm{W}^{1,p}[0,1])^n$. Then $\Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^n)$,

$$\mathcal{P} := \begin{pmatrix} P_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & P_n \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$$

and $\mathcal{G} = \mathcal{A}_{\mathcal{P}}^{\Phi}$. Next we verify the conditions (i)–(iii) in Corollary IV.6.

(i) The Dirichlet operators $\mathcal{L}_\lambda$, $\lambda > 0$, for the pair $(\mathcal{A}_m, \mathcal{L})$ are given by

$$(\mathcal{L}_\lambda z)(s) = \left(\frac{z_{j+n} \cosh\Big(\sqrt{\frac{\lambda}{c_j}} \cdot s\Big) - z_j \cosh\Big(\sqrt{\frac{\lambda}{c_j}}(1-s)\Big)}{\sqrt{\frac{\lambda}{c_j}} \sinh\Big(\sqrt{\frac{\lambda}{c_j}}\Big)} \right)_{j=1,\dots,n} \quad \text{for } z = \begin{pmatrix} z_1 \\ \vdots \\ z_{2n} \end{pmatrix} \in \mathbb{C}^{2n}$$

and $s \in [0,1]$. To check the inclusion $\mathrm{rg}(\mathcal{L}_\lambda) \subset \mathrm{Fav}_\alpha^{\mathcal{A}}$ for the Dirichlet operator for some $\lambda > \omega_0(\mathcal{A})$ and $\alpha \in (0,1)$ we use Lemma III.15.(a)$\Rightarrow$(c). It suffices to show $\sup_{\lambda > \lambda_0} \|\lambda^\alpha \varphi_\lambda\|_p < \infty$ for some $\lambda_0 > 0$, where

$$\varphi_\lambda(s) := \frac{\cosh\Big(\sqrt{\frac{\lambda}{c}} \cdot s\Big)}{\sqrt{\frac{\lambda}{c}} \sinh\Big(\sqrt{\frac{\lambda}{c}}\Big)} \quad \text{for } s \in [0,1] \text{ and some } 0 \neq c \in \mathbb{C}.$$

Using $\cosh\Big(\sqrt{\frac{\lambda}{c}} \cdot s\Big) \leq e^{\sqrt{\frac{\lambda}{c}} \cdot s} \leq 2\sinh\Big(\sqrt{\frac{\lambda}{c}} \cdot s\Big)$ for all $s \in [0,1]$, we obtain for $\lambda_0 > 0$ and $\alpha = \frac{p+1}{2p}$

$$\begin{aligned} \sup_{\lambda > \lambda_0} \Big\| \lambda^{\frac{p+1}{2p}} \varphi_\lambda \Big\|_p &= \sup_{\lambda > \lambda_0} \lambda^{\frac{1}{2p}} \sqrt{c} \cdot \left(\int_0^1 \left(\frac{\cosh\Big(\sqrt{\frac{\lambda}{c}} \cdot s\Big)}{\sinh\Big(\sqrt{\frac{\lambda}{c}}\Big)} \right)^p ds \right)^{\frac{1}{p}} \\ &\leq \sup_{\lambda > \lambda_0} \frac{c^{\frac{p+1}{2p}} e^{\sqrt{\frac{\lambda}{c}}}}{p^{\frac{1}{p}} \cdot \sinh\Big(\sqrt{\frac{\lambda}{c}}\Big)} < \infty. \end{aligned}$$

By Lemma III.15.(a)$\Rightarrow$(c), this implies the inclusion $\mathrm{rg}(\mathcal{L}_\lambda) \subset \mathrm{Fav}_{\frac{p+1}{2p}}^{\mathcal{A}}$, that is $\mathrm{rg}(\mathcal{L}_\lambda) \subset \mathrm{Fav}_{1-\beta}^{\mathcal{A}}$ for $\beta = \frac{p-1}{2p} < \frac{1}{2}$ for all $p \geq 1$.

(ii) To prove the inclusion $[D((\lambda - \mathcal{A})^\gamma)] \hookrightarrow \mathcal{Z}$ for some $\lambda > \omega_0(\mathcal{A})$, we note that the operator $\mathcal{K} = \mathrm{diag}\Big(\frac{d}{ds}\Big)_{j=1,\dots,n}$ with $[D(\mathcal{K})] = \mathcal{Z}$ is closed. Moreover, by [EN00, Expl. III.2.2] and Minkowski's inequality we have for every $\varepsilon > 0$ that

$$\|f'\|_p := \left(\sum_{j=1}^n \|f_j'\|_p^p\right)^{\frac{1}{p}} \leq \tfrac{9}{\varepsilon}\left(\sum_{j=1}^n \|f_j\|_p^p\right)^{\frac{1}{p}} + \varepsilon\left(\sum_{j=1}^n \|f_j''\|_p^p\right)^{\frac{1}{p}} = \tfrac{9}{\varepsilon}\cdot\|f\|_p + \varepsilon\cdot\|f''\|_p$$

for all $f \in (\mathrm{W}^{2,p}[0,1])^n$. For $\rho := \varepsilon^{-2}$ this gives

$$\|\mathcal{K}f\|_p \leq 9\rho^{\frac{1}{2}}\|f\|_p + C\rho^{-\frac{1}{2}}\|\mathcal{A}f\|_p \qquad \text{for all } f \in D(\mathcal{A}) \subset (\mathrm{W}^{2,p}[0,1])^n \tag{4.12}$$

with $C := \max_{j=1,\dots,n} c_j^{-1}$. The assertion follows from [RR93, Lem. 11.39] with $\alpha = \frac{1}{2}$, i.e., $[D((\lambda - \mathcal{A})^\gamma)] \hookrightarrow \mathcal{Z}$ for all $\gamma > \frac{1}{2}$ and $\lambda > \omega_0(\mathcal{A})$.
(iii) To finish the proof it suffices to choose in part (ii) some $\gamma \in (\frac{1}{2}, 1-\beta) = (\frac{1}{2}, \frac{p+1}{2p}) \neq \emptyset$ which then satisfies $\beta + \gamma < 1$. $\square$

As a concrete example, this approach allows to treat diffusion processes on networks studied, e.g., in [Bob12] and [BFN15, Equ. (5)]. The boundary conditions therein already appear in [Fel52, Equ. (2.5), p. 473] as *"elastic barrier" condition* in the context of diffusion processes on a finite interval.
Let us denote by $u \in (\mathrm{L}^p[0,1])^n$ the tupel $u = (u_1, ..., u_n)$. For $n \times n$-matrices $\mathbb{K}^{ij}$ $(i, j \in \{0,1\})$ with entries as in [BFN15, Equ. (4)], take $P_1 = ... = P_n = 0$, $\Phi_1 := \mathbb{K}^{00}\delta_0 + \mathbb{K}^{01}\delta_1$ and $\Phi_2 := \mathbb{K}^{10}\delta_0 + \mathbb{K}^{11}\delta_1 \in \mathcal{L}((\mathrm{W}^{1,p}[0,1])^n, \mathbb{C}^n)$. Corollary IV.8 yields the wellposedness of the diffusion process

$$\begin{cases} \dfrac{du_j}{dt}(t,s) = c_j\dfrac{d^2u_j}{ds^2}(t,s), & 0 < s < 1,\ j = 1,...,n,\ t \geq 0, \\ \dfrac{du}{ds}(t,0) = \mathbb{K}^{00}u(t,0) + \mathbb{K}^{01}u(t,1), & t \geq 0, \\ \dfrac{du}{ds}(t,1) = \mathbb{K}^{10}u(t,0) + \mathbb{K}^{11}u(t,1), & t \geq 0, \\ u(0,s) = (f_j(s))_{j=1,\dots,n}, & 0 < s < 1, \end{cases}$$

on $(\mathrm{L}^p[0,1])^n$, which proves [BFN15, Thm. 2.4]. Here, the entries of the matrices $\mathbb{K}^{ij}$ describe *"the possibility of passing [...] from the ith edge to the edges incident in the left and right endpoints"* [Bob12, p. 1503].

Please compare this situation to the heat and wave equations on networks studied in Chapter VII.

IV.4. Wellposedness of a Second Order Differential Operator with Unbounded Delay at the Boundary.

Here, we return to the operator $\mathcal{G}$ studied (in a slight variation) in Section II.6. This operator corresponds to a reaction-diffusion equation with unbounded delay in the Neumann boundary condition. Chapter II investigates the spectrum of the operator $\mathcal{G}$ while we are now interested in the generator property, e.g., the wellposedness of the system of equations in Corollary IV.11.

Further, this section partly generalizes the generation result in [BP02], where the authors study delay equations with unbounded operators in the delay term.[4]

For $1 \leq p < \infty$ define $X := \mathrm{L}^p[0,1]$. The operator $A \subset \frac{d^2}{ds^2}$ with domain

$$D(A) := \left\{ f \in \mathrm{W}^{2,p}[0,1] : f'(0) = 0 = f'(1) \right\}$$

generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ of angle $\frac{\pi}{2}$ on X. Choose

$$p' \in \begin{cases} \{p\} & \text{for } 1 \leq p < 2, \\ \left(\frac{2p}{p+1}, 2\right) & \text{for } 2 \leq p < \infty. \end{cases} \tag{4.13}$$

Then $p' \leq p$ and $\mathrm{L}^p[0,1] \hookrightarrow \mathrm{L}^{p'}[0,1]$. In particular, $p' \in \left[\frac{2p}{p+1}, 2\right)$ for all $1 \leq p < \infty$. Define $\partial Y := \mathrm{W}^{1,p}[0,1]$ and $Y := \mathrm{L}^{p'}([-1,0], \partial Y)$ where we use the notation $v(r,s) := (v(r))(s)$ for $v \in Y$ and $s \in [0,1]$, $r \in [-1,0]$. Then the following holds.

Theorem IV.9. *Let $1 \leq p < \infty$, fix p' as in (4.13) and choose an operator $\Phi \in \mathcal{L}(\mathrm{C}([-1,0], \partial Y), X)$ with representation*

$$\Phi f = \int_{-1}^{0} f(r)\, d\eta(r)$$

for $f \in \mathrm{C}([-1,0], \partial Y)$ and some $\eta : [-1,0] \to \mathcal{L}(\partial Y, X)$ of bounded variation. Then for every $P \in \mathcal{L}(\mathrm{W}^{1,p}[0,1], X)$, $\Psi \in \left(\mathrm{W}^{1,p}[0,1]\right)'$ and all functions $\mu : [-1,0] \to \mathbb{R}$

[4] In [BP02], the authors consider the abstract setup of generators of analytic semigroups instead of a second order differential operator.

and $\nu : [0,1] \to \mathbb{R}$ of bounded variation the operator

$$\mathcal{G} := \begin{pmatrix} \frac{\partial^2}{\partial s^2} + P & \Phi \\ 0 & \frac{\partial}{\partial r} \end{pmatrix},$$

$$D(\mathcal{G}) := \Big\{ \tbinom{f}{v} \in \mathrm{W}^{2,p}[0,1] \times \mathrm{W}^{1,p'}([-1,0],\partial Y) : v(0) = f,\ f'(1) = 0,$$
$$f'(0) = \Psi f + \int_0^1 \int_{-1}^0 v(r,s)\, d\mu(r)\, d\nu(s) \Big\}$$

generates a C_0-semigroup on $\mathcal{X} := X \times Y = \mathrm{L}^p[0,1] \times \mathrm{L}^{p'}([-1,0],\partial Y)$.

Remark IV.10. If we choose $P := -2c \cdot \frac{d}{ds} + k \cdot \mathrm{Id}_X$, $\Phi = 0$, $\Psi = \delta_0$ and $\mu = -\delta_{-1}$, $\nu = \delta_1$, then the operator $\mathcal{G}$ coincides with the operator matrix considered in Section II.6.

Proof. We first define operators such that the operator $\mathcal{G}$ can be studied in the context of Chapter I:

- $A_m := \frac{d^2}{ds^2}$ with domain $D(A_m) := \{f \in \mathrm{W}^{2,p}[0,1] : f'(1) = 0\}$ on X,
- $L := \delta_0' : D(A_m) \subset X \to \partial X := \mathbb{C}$, i.e., $Lf = f'(0)$,
- $D_m := \frac{d}{dr}$ with domain $D(D_m) := \mathrm{W}^{1,p'}([-1,0],\partial Y)$ on Y,
- $K := \delta_0 : D(D_m) \subset Y \to \partial Y = \mathrm{W}^{1,p}[0,1]$, i.e., $Kv = v(0)$.

Then, as mentioned above, $A := A_m|_{\ker(L)}$ is the generator of an analytic semigroup $(T(z))_{z \in \Sigma_{\frac{\pi}{2}}}$ on X while $D := D_m|_{\ker(K)}$ generates the nilpotent left shift semigroup $(S(t))_{t \geq 0}$ on Y. Moreover, the associated Dirichlet operators for the pairs (A_m, L) and (D_m, K) exist for $\mu > 0$ and are given by

- $L_\mu \in \mathcal{L}(\partial X, X)$ where $(L_\mu z)(s) = -z \cdot \frac{\cosh(\sqrt{\mu}(s-1))}{\sqrt{\mu} \cdot \sinh(\sqrt{\mu})}$ for $s \in [0,1]$, $z \in \mathbb{C}$,
- $K_\mu \in \mathcal{L}(\partial Y, Y)$ where $(K_\mu f)(r) := e^{\mu r} \cdot f$ for $r \in [-1,0]$, $f \in \partial Y$.

Next, we define the spaces $\partial\mathcal{X} := \partial X \times \partial Y = \mathbb{C} \times \mathrm{W}^{1,p}[0,1]$ and $\mathcal{Z} := \mathrm{W}^{1,p}[0,1] \times [D(D_m)] = \mathrm{W}^{1,p}[0,1] \times \mathrm{W}^{1,p'}([-1,0],\partial Y)$ and introduce the operator matrices

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : D(\mathcal{A}) := D(A) \times D(D) \subset \mathcal{X} \to \mathcal{X}, \quad \mathcal{L}_\mathcal{A} := \begin{pmatrix} L_A & 0 \\ 0 & K_D \end{pmatrix} : \mathcal{X} \to \mathcal{X}_{-1}^{\mathcal{A}},$$

$$\mathcal{P} := \begin{pmatrix} P & \Phi \\ 0 & 0 \end{pmatrix} : \mathcal{Z} \to \mathcal{X} \qquad \Theta := \begin{pmatrix} \Psi & \varphi \\ \mathrm{Id}_{\partial Y} & 0 \end{pmatrix} : \mathcal{Z} \to \partial\mathcal{X},$$

where $L_A := (\mu - A_{-1})L_\mu \in \mathcal{L}(\partial X, X_{-1}^A)$, $K_D := (\mu - D_{-1})K_\mu \in \mathcal{L}(\partial Y, Y_{-1}^D)$ as in (1.4) and

$$\varphi(v) := \int_0^1 \int_{-1}^0 v(r,s)\, d\mu(r)\, d\nu(s).$$

Then, as in Lemma I.7, we can write $\mathcal{G}$ as a perturbation of the form

$$\mathcal{G} = (\mathcal{A}_{-1} + \mathcal{P} + \mathcal{L}_{\mathcal{A}}\Theta)|_{\mathcal{X}}$$

and obtain $\mathcal{G} = \mathcal{A}_{\mathcal{P}}^{\Theta}$. We proceed by verifying the conditions of Corollary III.13. Since $\mathcal{A}$ is diagonal with diagonal domain, we can split the problem into several parts. We show that

(a) (L_A, P), (L_A, Ψ) and $(L_A, \mathrm{Id}_{\partial Y})$ are jointly q-admissible for A and $\frac{2p}{p+1} < q < 2$ (and if $p = 1$ also for $1 \le q < 2$),

(b) (K_D, φ), (K_D, Φ) are jointly p'-admissible for D,

(c) $1 \in \rho\Big(\mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})}\Big)$ for some $t > 0$ where $\mathcal{B}$ and $\mathcal{C}$ are defined analogously to (1.8) and the input-output map of the system is given by

$$\mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})} = \begin{pmatrix} \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} & \mathcal{F}_t^{(D,\mathrm{Id}_Y,\Phi)} & \mathcal{F}_t^{(A,L_A,P)} & \mathcal{F}_t^{(D,K_D,\Phi)} \\ 0 & 0 & 0 & 0 \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\Psi)} & \mathcal{F}_t^{(D,\mathrm{Id}_Y,\varphi)} & \mathcal{F}_t^{(A,L_A,\Psi)} & \mathcal{F}_t^{(D,K_D,\varphi)} \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_{\partial Y})} & 0 & \mathcal{F}_t^{(A,L_A,\mathrm{Id}_{\partial Y})} & 0 \end{pmatrix}. \tag{4.14}$$

Note that due to the non-analytic part stemming from the left shift semigroup $(S(t))_{t\ge 0}$ generated by D on Y, the operator matrix $\mathcal{G}$ will *not* generate an analytic semigroup on $\mathcal{X}$. Nevertheless, we will use Corollary IV.6 to treat the analytic part (a) and also to prove (c).

(a) First we use Lemma III.15.(a)$\Rightarrow$(c) and the estimate $\cosh(s) \le e^s \le 2\sinh(s)$ for $s \in [0,1]$ to show that $\mathrm{rg}(L_\mu) \subseteq \mathrm{Fav}^A_{\frac{p+1}{2p}}$ (cf. also Section IV.3). For $\lambda_0 > 0$ we compute

$$\begin{aligned} \sup_{\lambda > \lambda_0} \Big\| \lambda^{\frac{p+1}{2p}} L_\lambda \Big\| &\le \sup_{\lambda > \lambda_0} \frac{\lambda^{\frac{1}{2p}}}{\sinh(\sqrt{\lambda})} \cdot \left(\int_0^1 e^{p\sqrt{\lambda}(s-1)}\, ds \right)^{\frac{1}{p}} \\ &\le \sup_{\lambda > \lambda_0} \frac{e^{\sqrt{\lambda}}}{p^{\frac{1}{p}} \cdot \sinh(\sqrt{\lambda})} < \infty. \end{aligned}$$

This implies assumption (i) of Theorem IV.1 with $\beta = 1 - \frac{p+1}{2p} = \frac{p-1}{2p} < \frac{1}{2}$ for all $p \ge 1$. Hence, L_A is q-admissible for all $q > \frac{1}{1-\beta} = \frac{2p}{p+1}$ (and $q \ge 1$ if $p = 1$).

An estimate as in (4.12), together with [RR93, Lem. 11.39], implies that

$$[D((\lambda - A)^{\gamma})] \hookrightarrow \mathrm{W}^{1,p}[0,1] \quad \text{for all } \gamma > \tfrac{1}{2} \text{ and } \lambda > \omega_0(\mathcal{A})$$

and thus Theorem IV.1.(c) implies that P, Φ and $\mathrm{Id}_{\partial Y}$ are q-admissible for all $q < \frac{1}{\gamma} < 2$. Especially, we can choose $\gamma \in \left(\frac{1}{2}, 1-\beta\right) \neq \emptyset$ such that $\beta + \gamma < 1$. Further, we conclude that (L_A, P), (L_A, Φ) and $(L_A, \mathrm{Id}_{\partial Y})$ are jointly q-admissible for A for all $q \in (\frac{2p}{p+1}, 2) \neq \emptyset$ by Theorem IV.1.(d).

(b) We first show that the functional φ is a p'-admissible observation operator for D. In fact, for $v \in D(D)$ we have

$$\begin{aligned}
\int_0^1 \left|\varphi\, S(t) v\right|^{p'} dt &= \int_0^1 \left| \int_0^1 \int_{-1}^{-t} v(r+t,s)\, d\mu(r)\, d\nu(s) \right|^{p'} dt \\
&\leq \int_0^1 \left(\int_{-1}^{-t} \left| \int_0^1 v(t+r,s)\, d\nu(s) \right| d|\mu|(r) \right)^{p'} dt \qquad (4.15)\\
&\leq \int_0^1 \Big(|\mu|[-1,-t]\Big)^{p'-1} \cdot \int_{-1}^{-t} \left| \int_0^1 v(t+r,s)\, d\nu(s) \right|^{p'} d|\mu|(r)\, dt \qquad (4.16)\\
&\leq M^{p'} \Big(|\mu|[-1,0]\Big)^{p'-1} \cdot \int_0^1 \int_{-1}^{-t} \|v(t+r)\|_{\partial Y}^{p'}\, d|\mu|(r)\, dt \text{ [5]} \\
&= M^{p'} \Big(|\mu|[-1,0]\Big)^{p'-1} \cdot \int_{-1}^{0} \int_{0}^{-r} \|v(t+r)\|_{\partial Y}^{p'}\, dt\, d|\mu|(r) \qquad (4.17)\\
&\leq M^{p'} \Big(|\mu|[-1,0]\Big)^{p'} \cdot \|v\|_{\mathrm{L}^{p'}([-1,0],\partial Y)}^{p'},
\end{aligned}$$

where in (4.16) we used Hölder's inequality and the Fubini–Tonelli theorem in (4.15) and (4.17).

Next, by Lemma III.14, we obtain

$$\int_0^t S_{-1}(t-\sigma) K_D u(\sigma)\, d\sigma = \tilde{u}(t+\bullet) \text{ [6]} \qquad \text{for } t \in [0,1] \qquad (4.18)$$

and $u \in \mathrm{W}_0^{2,p'}([0,1],\partial Y)$ since $[0,1] \ni t \mapsto x(t) := \tilde{u}(t+\bullet)$ is a classical solution of the boundary control problem on $\mathrm{L}^{p'}([-1,0],\partial Y)$

$$\begin{cases} \dot{x}(t) = \frac{d}{dr} x(t), & 0 \leq t \leq 1, \\ x(t)(0) = u(t), & 0 \leq t \leq 1, \\ x(0) = 0. \end{cases}$$

[5] We choose $M := \|\psi\|$, where $\psi \in \mathcal{L}(\partial Y, \mathbb{C})$ is given by $\psi(f) := \int_0^1 f(s)\, d\nu(s)$.

[6] For the function u defined on the interval $[0,1]$ we denote by $\tilde{u}$ its extension to $[-1,1]$ by the value 0.

Hence, the operator K_D is a p'-admissible control operator for D since

$$\left\|\int_0^t S_{-1}(t-\sigma)K_D u(\sigma)\,d\sigma\right\|_Y = \left(\int_{-t}^0 \|u(t+r)\|_{\partial Y}^{p'}\,dr\right)^{\frac{1}{p'}} = \|u\|_{\mathrm{L}^{p'}([0,t],\partial Y)}.$$

Now we show that the pair (K_D,φ) is p'-admissible. In fact, using (4.18) we obtain for $u \in \mathrm{W}_0^{2,p'}([0,1],\partial Y)$ by essentially the same computations as above

$$\begin{aligned}
\int_0^1 \left|\varphi \int_0^t S_{-1}(t-\sigma)K_D u(\sigma)\,d\sigma\right|^{p'} dt &= \int_0^1 \left|\int_{-1}^0\int_0^1 \tilde{u}(t+r,s)\,d\nu(s)\,d\mu(r)\right|^{p'} dt \\
&\le \int_0^1 \left(\int_{-t}^0 \left|\int_0^1 u(t+r,s)\,d\nu(s)\right| d|\mu|(r)\right)^{p'} dt \\
&\le M^{p'}\Big(|\mu|[-1,0]\Big)^{p'-1} \cdot \int_0^1\int_{-t}^0 \|u(t+r)\|_{\partial Y}^{p'}\;d|\mu|(r)\,dt \\
&\le M^{p'}\Big(|\mu|[-1,0]\Big)^{p'-1} \cdot \int_{-1}^0\int_{-r}^1 \|u(t+r)\|_{\partial Y}^{p'}\;dt\,d|\mu|(r) \\
&\le M^{p'}\Big(|\mu|[-1,0]\Big)^{p'} \cdot \|u\|_{\mathrm{L}^{p'}([0,1],\partial Y)}^{p'}.
\end{aligned}$$

with the aid of Hölder's inequality and the Fubini–Tonelli theorem. This gives the joint p'-admissibility of the pair (K_D,Φ) (cf. [ABE14, Cor. 25] as well).

(c) Note that by Theorem IV.1.(e) we have

$$\left\|\mathcal{F}_t^{(A,*,\star)}\right\|_q \to 0 \quad \text{as } t \to 0^+ \text{ for all } \frac{2p}{p+1} \le q < 2, \tag{4.19}$$

where "$*,\star$" indicates one of the pairs "Id_X, P", "L_A, P", "Id_X, Ψ", "L_A, Ψ", "$\mathrm{Id}_X, \mathrm{Id}_{\partial Y}$" or "$L_A, \mathrm{Id}_{\partial Y}$". Take $q = p' \in \left[\frac{2p}{p+1}, 2\right)$ in (4.19) and split the matrix $\mathrm{Id} - \mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})} \in \mathcal{L}(\mathrm{L}^{p'}([0,t],\mathcal{X}\times\partial\mathcal{X}))$ as follows

$$\mathrm{Id} - \mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})} = \left(\begin{array}{c:cc|c}
\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} & -\mathcal{F}_t^{(D,\mathrm{Id}_Y,\Phi)} & -\mathcal{F}_t^{(A,L_A,P)} & -\mathcal{F}_t^{(D,K_D,\Phi)} \\ \hdashline
0 & \mathrm{Id} & 0 & 0 \\
-\mathcal{F}_t^{(A,\mathrm{Id}_X,\Psi)} & -\mathcal{F}_t^{(D,\mathrm{Id}_Y,\varphi)} & \mathrm{Id} - \mathcal{F}_t^{(A,L_A,\Psi)} & -\mathcal{F}_t^{(D,K_D,\varphi)} \\ \hline
-\mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_{\partial Y})} & 0 & -\mathcal{F}_t^{(A,L_A,\mathrm{Id}_{\partial Y})} & \mathrm{Id}
\end{array}\right).$$

Using Schur complements (cf. Lemma B.1.(i)), the upper left 3×3-matrix $\mathrm{Id} - \mathbb{F}_t$

is invertible for all sufficiently small $t > 0$ since

$$\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} - \begin{pmatrix} \mathcal{F}_t^{(D,\mathrm{Id}_Y,\Phi)} & \mathcal{F}_t^{(A,L_A,P)} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \mathcal{F}_t^{(D,\mathrm{Id}_Y,\varphi)} & \left(\mathrm{Id} - \mathcal{F}_t^{(A,L_A,\Psi)}\right)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\Psi)} \end{pmatrix}$$

$$= \mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} - \mathcal{F}_t^{(A,L_A,P)}\left(\mathrm{Id} - \mathcal{F}_t^{(A,L_A,\Psi)}\right)^{-1}\mathcal{F}_t^{(A,\mathrm{Id}_X,\Psi)} \tag{4.20}$$

is invertible for sufficiently small $t > 0$ by (4.19) and the Neumann series. Further, (4.20) has uniformly bounded inverses as well. Thus, by the representation $(B.2)$ for $(\mathrm{Id} - \mathbb{F}_t)^{-1}$, the inverses are uniformly bounded for all sufficiently small $t > 0$ since all occuring operators are uniformly bounded.

Using Schur complements again, the invertibility of $\mathrm{Id} - \mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})}$ is therefore equivalent to the invertibility of

$$\mathrm{Id} - \begin{pmatrix} \mathcal{F}_t^{(A,\mathrm{Id}_X,\mathrm{Id}_{\partial Y})} & 0 & \mathcal{F}_t^{(A,L_A,\mathrm{Id}_{\partial Y})} \end{pmatrix} \cdot (\mathrm{Id} - \mathbb{F}_t)^{-1} \cdot \begin{pmatrix} \mathcal{F}_t^{(D,K_D,\Phi)} \\ 0 \\ \mathcal{F}_t^{(D,K_D,\varphi)} \end{pmatrix} =: \mathrm{Id} - \Delta_t,$$

where $\mathcal{F}_t^{(D,K_D,\Phi)}$ and $\mathcal{F}_t^{(D,K_D,\varphi)}$ are uniformly bounded in t (cf. Lemma III.7.(iii)). Since by (4.19) (for $q = p' \in \left[\frac{2p}{p+1}, 2\right)$)

$$\|\Delta_t\|_{\mathcal{L}(\mathrm{L}^{p'}([0,t],\partial Y))} \to 0 \quad \text{as } t \to 0^+,$$

this yields $1 \in \rho\left(\mathcal{F}_t^{(\mathcal{A},\mathcal{B},\mathcal{C})}\right)$ for $t > 0$ sufficiently small by the Neumann series. Summing up (a)–(c), the matrix $\mathcal{G}$ generates a C_0-semigroup on $\mathcal{X}$ by Corollary III.13. $\square$

If we choose $P := b(\bullet)\frac{d}{ds} + c(\bullet)$ and $\Phi = 0$ with domain $Z := \mathrm{W}^{1,p}[0,1]$ and $b, c \in \mathrm{L}^\infty[0,1]$. By the previous result we obtain the following.

Corollary IV.11. *The reaction-diffusion equation subject to Neumann boundary conditions with distributed unbounded delay given by*

$$\text{(RDE)}\quad \begin{cases} \dfrac{du}{dt}(t,s) = \dfrac{d^2u}{ds^2}(t,s) + b(s)\,\dfrac{du}{ds}(t,s) + c(s)\,u(t,s), & 0 < s < 1,\ t \geq 0, \\ \dfrac{du}{ds}(t,0) = \Psi[u(t,\bullet)] + \displaystyle\int_0^1\int_{-1}^0 u(t+r,s)\,d\mu(r)\,d\nu(s), & t \geq 0, \\ \dfrac{du}{ds}(t,1) = 0, & t \geq 0, \\ u(r,s) = u_0(r,s), & 0 < s < 1,\ r \in [-1,0], \\ u(0,s) = f_0(s), & 0 < s < 1 \end{cases}$$

is wellposed on $L^p[0,1]$ *for all* $1 \leq p < \infty$ *and* $\Psi \in \left(\mathrm{W}^{1,p}[0,1]\right)'$, $b, c \in \mathrm{L}^\infty[0,1]$ *and* μ, ν *of bounded variation.*

Remark IV.12. Theorem IV.9 generalizes [HMR15, Expl. 5.2] where only the case $p = 2$ and $P = \Phi = \Psi = 0$ is considered. In fact, the perturbation P of the dynamic is nontrivial. Let us propose two different approaches and show where difficulties arise to highlight the strength of Theorem IV.1.

(a) The perturbation P is A-bounded with A-bound zero such that $A+P$ generates an analytic semigroup. Yet, it is a hassle to continue the analysis with the generator $A + P$ since there is no explicit representation of the corresponding resolvent or the Dirichlet operators in general.

(b) If one first performs the boundary perturbation and wishes to apply the perturbation P afterwards, one runs into difficulties since the delayed system does not originate from an analytic semigroup.

As a final remark, we note that the strategy proposed in the proof of Theorem IV.9 can be modified in order to study the generator property of the operators $\mathcal{G}$ in (2.17) of Section II.5 and (4.11) of Section IV.3 as well. In particular, the same procedure can be used to study the wellposedness of diffusion processes (as in [Bob12, BFN15]) with delay terms in both

- the equation determining the dynamics, and
- the Neumann boundary conditions.

IV.5. Wellposedness of the Neumann Laplacian with Perturbed Neumann Boundary Conditions.

For some open, bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ [7] we consider on the Hilbert space $X := \mathrm{L}^2(\Omega)$ the maximal operator

$$A_m f := \Delta f \qquad \text{with domain } D(A_m) := \left\{ f \in \mathrm{H}^{\frac{3}{2}}(\Omega) : \Delta f \in \mathrm{L}^2(\Omega) \right\}.$$

Next we choose the boundary space $\partial X := \mathrm{L}^2(\partial\Omega)$ and the boundary operator $L : D(A_m) \subset X \to \partial X$, $Lf := \frac{\partial f}{\partial \nu}|_{\partial\Omega}$ [8] which is well-defined by [LM72, Chap. 2,

[7] The boundary $\partial\Omega$ is an $(n-1)$-dimensional infinitely differentiable variety Ω being locally on one side of $\partial\Omega$, see [LM72, Chap. 2, Sect. 5.(i)]. For considerations of elliptic problems on nonsmooth domains, we refer to [Gri85].

[8] $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative.

Thm. 7.3]. Define $A := A_m|_{\ker(L)}$ whose domain is given by

$$D(A) := \left\{ f \in \mathrm{H}^{\frac{3}{2}}(\Omega) : \Delta f \in \mathrm{L}^2(\Omega),\ \tfrac{\partial f}{\partial \nu}|_{\partial\Omega} = 0 \right\}$$

which coincides with $\left\{ f \in \mathrm{H}^2(\Omega) : \frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0 \right\}$. In fact, for $f \in D(A)$ we have

$$\begin{cases} \Delta f \in \mathrm{L}^2(\Omega), \\ \frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0. \end{cases}$$

Then elliptic regularity (cf. [LM72, p.188-189]) gives $f \in \mathrm{H}^2(\Omega)$. On the other side, take $f \in \mathrm{H}^2(\Omega)$. Then $\Delta f \in \mathrm{L}^2(\Omega)$ by [LM72, Chap. 2, Thm. 5.4].
We call A the *Neumann Laplacian* on $\mathrm{L}^2(\Omega)$ which generates an analytic semigroup by [LT83, p. 248].
To verify the existence of the abstract Dirichlet operator $L_\mu : \partial X \to \ker(\mu - A_m)$ for some $\mu > 0$ we use the fact that the operator $\mathcal{P} := \binom{\Delta - \mu}{L}$ is an algebraic and topological isomorphism between appropriate spaces as shown in [LM72, Chap. 2, Thm. 7.4 for $s = \frac{3}{2}$]. This implies that for $g \in \mathrm{L}^2(\partial\Omega)$ there exists a unique $f \in \mathrm{H}^{\frac{3}{2}}(\Omega)$ such that

$$\mathcal{P}f = \begin{pmatrix} \Delta f - \mu f \\ L f \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad \text{i.e.,} \quad \begin{cases} \Delta f = \mu f \in \mathrm{L}^2(\Omega), \\ L f = g. \end{cases}$$

Therefore, $f \in D(A_m)$. For $g \in \partial X$ we define $L_\mu g := f$. This yields a bounded operator $L_\mu \in \mathcal{L}(\partial X, X)$ by the continuity of the inverse $\mathcal{P}^{-1}$. Hence, the Assumption I.2.(b) is satisfied.

In the following result we investigate the generator property of certain perturbations of A.

Theorem IV.13. *Let $\alpha \in (0, \frac{3}{2})$. Then for every operator $\Phi \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\partial\Omega))$ and $P \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\Omega))$ the operator*

$$G := \Delta + P,$$
$$D(G) := \{ f \in \mathrm{H}^{\frac{3}{2}}(\Omega) : \Delta f \in \mathrm{L}^2(\Omega),\ \tfrac{\partial f}{\partial \nu}|_{\partial\Omega} = \Phi f \} = \ker(L - \Phi),$$

generates a compact, analytic semigroup on $\mathrm{L}^2(\Omega)$.

Proof. We have $G = A_P^\Phi$ as in (1.2). In order to show that G generates an analytic semigroup we verify the conditions (i)-(iii) of Corollary IV.6. To this end we first choose $\delta \in (\alpha, \frac{3}{2})$ and $\lambda > 0$. By [Fuj67, Thm. 2] we have $D((\lambda - A)^{\frac{\theta}{2}}) = H^\theta(\Omega)$ for all $\theta \in (0, \frac{3}{2})$. Thus, the Dirichlet operators satisfy

$$\mathrm{rg}(L_\lambda) \subset D(A_m) \subset \mathrm{H}^{\frac{3}{2}}(\Omega) \subset \mathrm{H}^\delta(\Omega) = D((\lambda - A)^{\frac{\delta}{2}}) \subset \mathrm{Fav}^A_{\frac{\delta}{2}}$$

where the last inclusion follows from [EN00, Prop. II.5.33 & Prop. II.5.14]. This gives condition (i) for $\beta := 1 - \frac{\delta}{2}$. Using again [Fuj67, Thm. 2] we conclude

$$Z := H^\alpha(\Omega) = D((\lambda - A)^{\frac{\alpha}{2}})$$

which shows condition (ii) for $\gamma := \frac{\alpha}{2}$. Moreover, since $\delta > \alpha$ we obtain

$$\beta + \gamma = 1 - \tfrac{\delta}{2} + \tfrac{\alpha}{2} < 1$$

which shows (iii). Summing up, this implies that G generates an analytic semigroup. To prove compactness of this semigroup first note that by [Ada75, Thm. 6.2] we have the injections $[D(G)] \hookrightarrow \mathrm{H}^1(\Omega) \overset{c}{\hookrightarrow} \mathrm{L}^2(\Omega)$. Hence, [EN00, Prop. II.4.25] implies that G has compact resolvent and the assertion follows from [EN00, Thm. II.4.29]. $\square$

Remark IV.14. We note that the above result could be adapted to cover uniformly elliptic operators as studied in [LT83, Thm. 1.1] by completely different methods. In fact, Lasiecka and Triggiani extend the unperturbed operator A to the larger space $[D(A^{\frac{1}{4}+\varepsilon})]'$, $\varepsilon > 0$, where the perturbation becomes A-bounded with A-bound zero and then use a standard perturbation argument to obtain the result.

We give a concrete application of Theorem IV.13.

Corollary IV.15. *Let $1 < \alpha < \frac{3}{2}$. The second order differential equation with perturbed Neumann boundary conditions given by*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,s) = \Delta u(t,s) + \displaystyle\sum_{i=1}^n b_i(s)\,\frac{\partial u}{\partial x_i}(t,s) + c(s)\,u(t,s), & s \in \Omega,\ t \geq 0, \\ \dfrac{\partial u}{\partial \nu}(t,z) = \displaystyle\sum_{j=1}^m \langle [\varphi_j u](t,\bullet),\, \omega_j\rangle_{\mathrm{L}^2(\partial\Omega)} \cdot g_j(z) + \beta(z) u(t,z), & z \in \partial\Omega,\ t \geq 0, \\ u(0,s) = f_0(s), & s \in \Omega, \end{cases}$$

is wellposed on $\mathrm{L}^2(\Omega)$ for all $\varphi_1, ..., \varphi_m \in \mathcal{L}(H^\alpha(\Omega), \mathrm{L}^2(\partial\Omega))$, $b_1, ..., b_n, c \in \mathrm{L}^\infty(\Omega)$, $\omega_1, ..., \omega_m \in \mathrm{L}^2(\partial\Omega)$, $g_1, ..., g_m \in \mathrm{L}^2(\partial\Omega)$ and $\beta \in \mathrm{L}^2(\partial\Omega)$.

Proof. The assertion follows immediately from Theorem IV.13 by choosing $P := \sum_{i=1}^n b_i(\bullet)\,\frac{\partial}{\partial x_i} + c(\bullet) \in \mathcal{L}(Z, X)$ and $\Phi := \sum_{j=1}^m \langle \varphi_j \bullet,\, \omega_j\rangle_{\mathrm{L}^2(\partial\Omega)} \cdot g_j + \beta \cdot \mathrm{tr}(\bullet) \in \mathcal{L}(Z, \partial X)$. $\square$

V. Perturbations for Second Order Cauchy Problems

In this chapter we are concerned with *second order Cauchy problems* of the form

$$(ACP_2) \qquad \begin{cases} \ddot{x}(t) = Ax(t), \quad t \geq 0, \\ x(0) = x_0,\ \dot{x}(0) = x_1, \end{cases}$$

for some operator $A : D(A) \subset X \to X$ on a Banach space X.

This problem leads to the theory of *strongly continuous cosine families* $(C(t))_{t\in\mathbb{R}}$[1] since (ACP_2) is wellposed if and only if the operator A is the generator of a strongly continuous cosine family (cf. [Gol85, Chap. 2, Thm. 8.2]). In this chapter we prove a perturbation result for generators of strongly continuous cosine families. The results are subject to ongoing research with Klaus-Jochen Engel.

To this end we recall the following result which is crucial for our approach to (ACP_2). For a proof we refer to [ABHN11, Thm. 3.14.11].

Proposition V.1. *For an operator A on a Banach space X the following assertions are equivalent.*

(a) A generates a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X.

(b) There exists a Banach space V satisfying $X_1^A \hookrightarrow V \hookrightarrow X$ such that

$$(5.1) \qquad \mathcal{A} := \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times V,$$

generates a C_0-group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\mathcal{X} := V \times X$.

In this case, the space V is uniquely determined and $V \times X$ is called the phase space *(associated with A). Moreover, if we define the* sine family $(S(t))_{t\in\mathbb{R}}$ *associated with A by*

$$S(t)x := \int_0^t C(s)x\, ds \qquad \text{for } x \in X,$$

[1] As general references we recommend [Fat85, Gol85, ABHN11].

then

(i) $S(\cdot)x \in \mathrm{C}(\mathbb{R}, V)$ *for all* $x \in X$,

(ii) $S(\cdot)v \in \mathrm{C}(\mathbb{R}, X_1^A)$ *for all* $v \in V$,

(iii) $C(\cdot)v \in \mathrm{C}(\mathbb{R}, V) \cap \mathrm{C}^1(\mathbb{R}, X)$ *with derivative* $\frac{d}{dt}C(t)v = AS(t)v$ *for all* $v \in V$.

Finally, the C_0*-group* $(\mathcal{T}(t))_{t\in\mathbb{R}}$ *is given by*

$$\mathcal{T}(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix} \qquad \textit{for } t \in \mathbb{R}. \tag{5.2}$$

For $\lambda = \mu^2 \in \rho(A)$ we have $\mu \in \rho(\mathcal{A})$ (cf. [ABHN11, Equ. (3.97)]) and the resolvent is given by

$$R(\mu, \mathcal{A}) = \begin{pmatrix} \mu R(\lambda, A) & R(\lambda, A) \\ AR(\lambda, A) & \mu R(\lambda, A) \end{pmatrix}.$$

In order to obtain perturbation results for second order Cauchy problems we perturb the operator matrix in (5.1) using Theorem III.8. Let A generate a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ and fix $\lambda = \mu^2 \in \rho(A)$. Using the extrapolated operator $A_{-1} : X \to X_{-1}^A$, we define the Banach space

$$V_{-1} := (\lambda - A_{-1})V, \quad \|\bullet\|_{V_{-1}} := \left\|R(\lambda, A_{-1})\bullet\right\|_V.$$

Proposition V.2. *Let* A *generate a strongly continuous cosine family* $(C(t))_{t\in\mathbb{R}}$ *with phase space* $V \times X$, *i.e., let* $\mathcal{A}$ *generates a* C_0*-group* $(\mathcal{T}(t))_{t\in\mathbb{R}}$ *on* $\mathcal{X} = V \times X$. *The extrapolation space* $\mathcal{X}_{-1}^{\mathcal{A}}$ *with respect to* $\mathcal{A}$ *coincides with* $X \times V_{-1}$. *In particular,* $A_{-1} : V \subset V_{-1} \to V_{-1}$, *the extension of* A *to* V, *generates a strongly continuous cosine family* $(C_{-1}(t))_{t\in\mathbb{R}}$ *on* V_{-1} *with phase space* $X \times V_{-1}$.

Proof. Without loss of generality we assume that A is invertible.[2] Then $\mathcal{A}$ is invertible with inverse

$$\mathcal{A}^{-1} = \begin{pmatrix} 0 & A^{-1} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

The extrapolation space with respect to $\mathcal{A}$ is $\mathcal{X}_{-1}^{\mathcal{A}} := (\mathcal{X}, \|\bullet\|_{-1})^\sim$, where

$$\left\|\begin{pmatrix} v \\ x \end{pmatrix}\right\|_{-1} := \left\|\begin{pmatrix} A^{-1}x \\ v \end{pmatrix}\right\|_{\mathcal{X}} \quad \text{for } \begin{pmatrix} v \\ x \end{pmatrix} \in \mathcal{X}.$$

[2]Extrapolation spaces are invariant under bounded perturbations, i.e., rescaling of the operator does not affect the extrapolation space.

Since the norm $\|\bullet\|_{-1}$ respects the product structure of $\mathcal{X} = V \times X$ we obtain

$$\mathcal{X}_{-1}^{\mathcal{A}} = (\mathcal{X}, \|\bullet\|_{-1})^{\tilde{}} = X \times V_{-1}$$

and the operator $\mathcal{A}$ extends to

$$\mathcal{A}_{-1} = \begin{pmatrix} 0 & \mathrm{Id} \\ A_{-1} & 0 \end{pmatrix} \quad \text{with domain } D(\mathcal{A}_{-1}) = V \times X. \qquad \square$$

To proceed within our general framework we consider

- intermediate Banach spaces Z_V, Z_X satisfying

  (5.3) $$X_1^A \hookrightarrow Z_V \hookrightarrow V \hookrightarrow Z_X \hookrightarrow X$$

  and the product space $\mathcal{Z} := Z_V \times Z_X$,
- a Banach space U,
- two bounded linear operators $B_1 : U \to X$, $B_2 : U \to V_{-1}$ and the operator $\mathcal{B} \in \mathcal{L}(U, \mathcal{X}_{-1}^{\mathcal{A}})$ given by

  (5.4) $$\mathcal{B} := \binom{B_1}{B_2} : U \to \mathcal{X}_{-1}^{\mathcal{A}} = X \times V_{-1},$$
- two bounded linear operators $C_1 : Z_V \to U$, $C_2 : Z_X \to U$ and the operator $\mathcal{C} \in \mathcal{L}(\mathcal{Z}, U)$ given by

  (5.5) $$\mathcal{C} := (C_1, C_2) : \mathcal{Z} = Z_V \times Z_X \to U.$$

Note that for this choice of operators $\mathcal{B}$ and $\mathcal{C}$, the perturbation $\mathcal{P} = \mathcal{B}\mathcal{C}$ takes the form

$$\mathcal{P} = \begin{pmatrix} B_1C_1 & B_1C_2 \\ B_2C_1 & B_2C_2 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{X}_{-1}^{\mathcal{A}}).$$

V.1. Perturbations of Generators of Cosine Families

Applying Theorem III.8 to study the generator property of $\mathcal{A}_{\mathcal{B}\mathcal{C}}$ with operators $\mathcal{B}$ and $\mathcal{C}$ given by (5.4) and (5.5) would yield to a quite cumbersome list of hypotheses. Therefore, we only present a perturbation result for generators of strongly continuous cosine families by choosing $Z_X := X$ in (5.3) and putting $B_1 := 0$, $C_2 := 0$.

Let A be the generator of a cosine family $(C(t))_{t\in\mathbb{R}}$ on X. If $B := B_2 \in \mathcal{L}(U, V_{-1})$ and $C := C_1 \in \mathcal{L}(Z_V, U)$ in (5.4), (5.5), then we obtain

$$\mathcal{A}_{\mathcal{BC}} = \begin{pmatrix} 0 & \mathrm{Id} \\ A_{-1} + BC & 0 \end{pmatrix}, \quad D(\mathcal{A}_{\mathcal{BC}}) = \Big\{ \tbinom{v}{x} \in Z_V \times V : (A_{-1} + BC)v \in X \Big\}.$$

We now apply Theorem III.8 and Proposition V.1 to this situation and obtain the following result.

Theorem V.3. *Let A generate a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ with phase space $V \times X$, i.e., the operator $\mathcal{A}$ generates a C_0-group $(\mathcal{T}(t))_{t\geq 0}$ on $V \times X$. Assume that for $B \in \mathcal{L}(U, V_{-1})$ and $C \in \mathcal{L}(Z_V, U)$ there exist $1 \leq p < \infty$, $t > 0$ and $M \geq 0$ such that*

(i) $\mathrm{rg}(R(\lambda, A_{-1})B) \subset Z_V$ *for some* $\lambda \in \rho(A)$,

(ii) $\displaystyle\int_0^t S_{-1}(t-s)Bu(s)\,ds \in V$ *for all* $u \in \mathrm{L}^p([0,t], U)$,

(ii') $\displaystyle\int_0^t C_{-1}(t-s)Bu(s)\,ds \in X$ *for all* $u \in \mathrm{L}^p([0,t], U)$,

(iii) $\displaystyle\int_0^t \Big\| C\,C(s)v \Big\|_U^p ds \leq M \cdot \|v\|_V^p$ *for all* $v \in D(A)$,

(iii') $\displaystyle\int_0^t \Big\| C\,S(s)x \Big\|_U^p ds \leq M \cdot \|x\|_X^p$ *for all* $x \in V$,

(iv) $\displaystyle\int_0^t \Big\| C \int_0^r S_{-1}(r-s)Bu(s)\,ds \Big\|_U^p dr \leq M \cdot \|u\|_p^p$ *for all* $u \in \mathrm{W}_0^{2,p}([0,t], U)$,

(v) $1 \in \rho\Big(\mathcal{F}_t^{(A,B,C)}\Big)$,

where $\mathcal{F}_t^{(A,B,C)} \in \mathcal{L}(\mathrm{L}^p([0,t], U))$ is the unique bounded extension of the operator given by

$$\Big(\mathcal{F}_t^{(A,B,C)} u\Big)(r) = C \int_0^r S_{-1}(r-s)Bu(s)\,ds \tag{5.6}$$

for $u \in \mathrm{W}_0^{2,p}([0,t], U)$ and $r \in [0,t]$. Then the operator

$$A_{BC} := (A_{-1} + BC)|_X, \quad D(A_{BC}) := \Big\{ x \in Z_V : (A_{-1} + BC)x \in X \Big\},$$

generates a strongly continuous cosine family with phase space $V \times X$.

In the sequel we use for $t > 0$ and $u \in \mathrm{L}^p([0,t], U)$ the notations

$$\mathcal{B}_t^S u := \int_0^t S_{-1}(t-s)Bu(s)\,ds \qquad \text{and} \qquad \mathcal{B}_t^C u := \int_0^t C_{-1}(t-s)Bu(s)\,ds.$$

Since $(S_{-1}(t))_{t\in\mathbb{R}} \subset \mathcal{L}(V_{-1}, X)$ is strongly continuous, [ABHN11, Prop. 1.3.4] implies that $\mathcal{B}_t^S \in \mathcal{L}(\mathrm{L}^p([0,t],U), X)$. Similarly, the strong continuity of $(C_{-1}(t))_{t\in\mathbb{R}}$ on V_{-1} implies that $\mathcal{B}_t^C \in \mathcal{L}(\mathrm{L}^p([0,t],U), V_{-1})$.

Remark V.4. The identity

$$\int_0^{t_0} \mathcal{T}_{-1}(t_0 - s)\binom{0}{B}u(s)\,ds = \begin{pmatrix} \mathcal{B}_{t_0}^S u \\ \mathcal{B}_{t_0}^C u \end{pmatrix} \tag{5.7}$$

holds by (5.2) for all $u \in \mathrm{L}^p([0,t_0],U)$. Thus, the conditions (ii) and (ii') are equivalent to the p-admissibility of $\binom{0}{B}$ for $\mathcal{A}$. By [BE14, Lem. 3.15] we then conclude that $\mathcal{B}_t^S u \in V$ and $\mathcal{B}_t^C u \in X$ for all $t > 0$ and $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+, U)$ and, by [ABE14, Rem. 2], the conditions (ii) and (ii') are equivalent to the estimates

$$\left\| \int_0^{t_0} S_{-1}(t_0 - s)Bu(s)\,ds \right\|_V \le M \left\|u\right\|_p, \tag{5.8}$$

$$\left\| \int_0^{t_0} C_{-1}(t_0 - s)Bu(s)\,ds \right\|_X \le M \left\|u\right\|_p, \tag{5.9}$$

for some $M \ge 0$ and all $u \in \mathrm{W}^{1,p}([0,t_0],U)$.

Lemma V.5. *For $t_0 > 0$ the following are equivalent.*

(a) $\mathcal{B}_{t_0}^S u \in V$ *for all* $u \in \mathrm{L}^p([0,t_0],U)$.

(b) $\mathcal{B}_{t_0}^C u \in X$ *for all* $u \in \mathrm{L}^p([0,t_0],U)$.

Proof. (a)$\Longrightarrow$(b). By assumption (a) and (5.8) we have $\mathcal{B}_{t_0}^S \in \mathcal{L}(\mathrm{L}^p([0,t_0],U), V)$. We show that

$$(\mathcal{B}_t^S)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],U), V) \quad \text{is strongly continuous.} \tag{5.10}$$

To see this, we first define $u_t \in \mathrm{L}^p([0,t_0],U)$ as the right translation of u by $t_0 - t$,

$$u_t(s) := \begin{cases} 0 & \text{if } 0 \le s < t_0 - t, \\ u(s + t - t_0) & \text{if } t_0 - t \le s \le t_0. \end{cases}$$

By this choice of u and u_t we then have $\mathcal{B}_t^S u = \mathcal{B}_{t_0}^S u_t \in V$ and, if $t, t+h \in [0,t_0]$,

$$\begin{aligned} \left\|\mathcal{B}_{t+h}^S u - \mathcal{B}_t^S u\right\|_V &= \left\|\mathcal{B}_{t_0}^S (u_{t+h} - u_t)\right\|_V \\ &\le \left\|\mathcal{B}_{t_0}^S\right\|_{\mathcal{L}(\mathrm{L}^p([0,t_0],U),V)} \cdot \left\|u_{t+h} - u_t\right\|_p \to 0 \quad \text{as } h \to 0 \end{aligned}$$

by the strong continuity of the right shift semigroup on $\mathrm{L}^p([0,t_0],U)$. This shows (5.10).

In order to show (b) we use the fact that the phase space of the extrapolated cosine family $(C_{-1}(t))_{t\in\mathbb{R}}$ is given by $X \times V_{-1}$ and the domain of the extrapolated generator A_{-1} is $D(A_{-1}) = V$ (cf. Proposition V.2). Hence, by [ABHN11, Proof of Thm. 3.14.11, p. 211] applied to the extrapolated cosine family we have

$$X = \{x \in V_{-1} : S_{-1}(\bullet)x \in \mathrm{C}([0,r_0],V)\} \tag{5.11}$$

for each $r_0 > 0$ fixed. Choose now $r_0 := \frac{t_0}{2}$ and $r \in [0,r_0]$. Then, using the functional equation

$$2\,S_{-1}(r)\,C_{-1}(r_0-s) = S_{-1}(r_0+r-s) - S_{-1}(r_0-r-s)$$

(see [ABHN11, p. 206]), we obtain for $v := Bu \in \mathrm{L}^p([0,t_0],V_{-1})$ and $\bar v := \mathbb{1}_{[0,r_0]}\cdot v$

$$\begin{aligned}
2\,S_{-1}(r)\,\mathcal{B}^C_{r_0}u &= \int_0^{r_0} S_{-1}(r_0+r-s)v(s)\,ds - \int_0^{r_0} S_{-1}(r_0-r-s)v(s)\,ds\\
&= \int_0^{r_0+r} S_{-1}(r_0+r-s)\bar v(s)\,ds\\
&\qquad - \int_0^{r_0-r} S_{-1}(r_0-r-s)v(s)\,ds - \int_{r_0-r}^{r_0} S_{-1}(r_0-r-s)v(s)\,ds\\
&=: I_1(r) - I_2(r) - I_3(r).
\end{aligned}$$

We consider the three integrals separately.

First note that $I_1(r) = \mathcal{B}^S_{r_0+r}\bar v$, hence (5.10) implies $I_1(\bullet) \in \mathrm{C}([0,r_0],V)$. Similarly, $I_2(r) = \mathcal{B}^S_{r_0-r}v$ and therefore $I_2(\bullet) \in \mathrm{C}([0,r_0],V)$. Finally, using that $S_{-1}(\bullet)$ is an odd function on $\mathbb{R}$, we obtain $I_3(r) = \mathcal{B}^S_r\hat v$ for the third term where $\hat v(\bullet) := -v(r_0-\bullet)$. Again this implies $I_3(\bullet) \in \mathrm{C}([0,r_0],V)$. Summing up, this shows that $S_{-1}(\bullet)\mathcal{B}^C_{r_0}u \in \mathrm{C}([0,r_0],V)$. By (5.11) we conclude $\mathcal{B}^C_{r_0}u \in X$ for all $u \in \mathrm{L}^p([0,t_0],U)$. Hence, $\mathcal{B}^C_{t_0}u \in X$ for all $u \in \mathrm{L}^p([0,t_0],U)$ by Remark V.4.

(b)$\Longrightarrow$(a). By assumption (b) and (5.9) we have $\mathcal{B}^C_{t_0} \in \mathcal{L}(\mathrm{L}^p([0,t_0],U),X)$. Again, using the translation argument, we see that $(\mathcal{B}^C_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],U),X)$ is strongly continuous. Using the functional equation

$$AS(r)S(r_0-s)x = \tfrac12 C(r_0+r-s)x - \tfrac12 C(r_0-r-s)x \qquad t,s\in\mathbb{R},\ x\in X,$$

in [ABHN11, Prop. 3.14.5.(f)], we obtain for $v := Bu \in \mathrm{L}^p([0,t_0],V_{-1})$ and $r \in [0,r_0]$, $r_0 := \frac{t_0}{2}$, $\bar v := \mathbb{1}_{[0,r_0]}\cdot v$

$$A_{-1}\,S_{-1}(r)\,\mathcal{B}^S_{r_0}u = \tfrac12\int_0^{r_0} C_{-1}(r_0+r-s)v(s)\,ds - \tfrac12\int_0^{r_0} C_{-1}(r_0-r-s)v(s)\,ds$$

$$
\begin{aligned}
&= \tfrac{1}{2}\int_0^{r_0+r} C_{-1}(r_0+r-s)\bar{v}(s)\,ds \\
&\quad - \tfrac{1}{2}\int_0^{r_0-r} C_{-1}(r_0-r-s)v(s)\,ds - \tfrac{1}{2}\int_{r_0-r}^{r_0} C_{-1}(r_0-r-s)v(s)\,ds \\
&=: J_1(r) - J_2(r) - J_3(r).
\end{aligned}
$$

Thus, by a similar argument as in the first part of the proof,[3] we conclude that

$$A_{-1}S(\bullet)\mathcal{B}_{r_0}^S u \in \mathrm{C}([0,r_0],X),$$

i.e., $S(r)\mathcal{B}_{r_0}^S u \in D(A)$ for every $r \in [0,r_0]$ and $AS(\bullet)\mathcal{B}_{r_0}^S u \in \mathrm{C}([0,r_0],X)$. By the definition of the phase space (cf. [ABHN11, Proof of Thm. 3.14.11, p. 211]), this yields $\mathcal{B}_{r_0}^S u \in V$. Hence, $\mathcal{B}_{t_0}^S u \in V$ for all $u \in \mathrm{L}^p([0,t_0],U)$ by Remark V.4. □

Thus, Theorem V.3 can be stated with a reduced number of assumptions.

Corollary V.6. *Let A generate a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ with phase space $V \times X$. Assume that for $B \in \mathcal{L}(U,V_{-1})$ and $C \in \mathcal{L}(Z_V,U)$ there exist $1 \le p < \infty$, $t > 0$ and $M \ge 0$ such that*

- *(i)* $\operatorname{rg}(R(\lambda,A_{-1})B) \subset Z_V$ *for some* $\lambda \in \rho(A)$,
- *(ii)* $\int_0^t S_{-1}(t-s)Bu(s)\,ds \in V$ *for all* $u \in \mathrm{L}^p([0,t],U)$,
- *(iii)* $\int_0^t \left\|C\,C(s)v\right\|_U^p ds \le M \cdot \|v\|_V^p$ *for all* $v \in D(A)$,
- *(iii')* $\int_0^t \left\|C\,S(s)x\right\|_U^p ds \le M \cdot \|x\|_X^p$ *for all* $x \in V$,
- *(iv)* $\int_0^t \left\|C\int_0^r S_{-1}(r-s)Bu(s)\,ds\right\|_U^p dr \le M \cdot \|u\|_p^p$ *for all* $u \in \mathrm{W}_0^{2,p}([0,t],U)$,
- *(v)* $1 \in \rho\big(\mathcal{F}_t^{(A,B,C)}\big)$ *with* $\mathcal{F}_t^{(A,B,C)}$ *defined in* (5.6).

Then A_{BC} generates a strongly continuous cosine family with phase space $V \times X$.

We state the analogue of Corollary III.10 which follows from Corollary V.6 and Proposition V.1.

Corollary V.7. *Let A be the generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ with phase space $V \times X$, i.e., $\mathcal{A}$ in* (5.1) *generates a C_0-group $(\mathcal{T}(t))_{t\ge 0}$ on $\mathcal{X} = V \times X$, and let $p \ge 1$.*

(a) If $B \in \mathcal{L}(U,V_{-1})$ satisfies condition (ii) in Corollary V.6 and $Z_V = V$, i.e., $C \in \mathcal{L}(V,U)$, then conditions (i)-(v) in Corollary V.6 hold and there exists

[3] Use the strong continuity of $(\mathcal{B}_t^C)_{t\in[0,t_0]}$ and the fact that $C_{-1}(\bullet)$ is an even function on $\mathbb{R}$, together with the assumption (b) that $\mathcal{B}_{r_0+r}^C \bar{v}$, $\mathcal{B}_{r_0-r}^C v$, $\mathcal{B}_r^C \hat{v} \in \mathrm{C}([0,t_0],X)$, where $\hat{v} := v(r_0 - \bullet)$.

$M \geq 0$ and $t_0 > 0$ such that

$$\left\|\mathcal{F}_t^{(A,B,C)}\right\| \leq M \cdot t^{\frac{1}{p}} \quad \textit{for all } 0 < t \leq t_0.$$

Hence, A_{BC} is the generator of a strongly continuous cosine family with phase space $V \times X$.

(b) If $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(Z_V, U)$ satisfies condition (iii) and (iii') in Corollary V.6, then conditions (i)-(iv) in Corollary V.6 hold and there exists $M \geq 0$ and $t_0 > 0$ such that

$$\left\|\mathcal{F}_t^{(A,B,C)}\right\| \leq M \cdot t^{1-\frac{1}{p}} \quad \textit{for all } 0 < t \leq t_0.$$

In particular, if $p > 1$, then condition (v) in Corollary V.6 holds. In this case, A_{BC} is the generator of a strongly continuous cosine family with phase space $V \times X$.

V.2. Boundary Perturbations for Second Order Equations

In order to investigate second order equations under boundary perturbations, we proceed as in Chapter I and take a linear operator $A_m : D(A_m) \subseteq X \to X$ on a Banach spaces X, a Banach space ∂X and a boundary operator $L : D(A_m) \subset X \to \partial X$ to define

$$A \subseteq A_m \quad \text{with domain } D(A) := \ker(L). \tag{5.12}$$

We now assume that the operator A generates a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$. In particular, Assumption I.2 (a) is then satisfied. We perturb the matrix in (5.1) associated with A in the following way. Let Z_V be a Banach space satisfying

$$D(A_m) \subset Z_V \hookrightarrow V \hookrightarrow X.$$

Moreover, choose operators $P \in \mathcal{L}(Z_V, X)$ and $\Phi \in \mathcal{L}(Z_V, \partial X)$.

Then we define on $\mathcal{X} = V \times X$ the operator matrix

$$\begin{aligned} \mathcal{G} &:= \begin{pmatrix} 0 & \mathrm{Id} \\ A_m + P & 0 \end{pmatrix}, \\ D(\mathcal{G}) &:= \left\{ \binom{v}{x} \in D(A_m) \times V : Lv = \Phi v \right\}. \end{aligned} \tag{5.13}$$

The abstract Cauchy problem (ACP_2) associated with $\mathcal{G}$ corresponds to the second order equation

$$\begin{cases} \ddot{x}(t) = A_m x(t) + Px(t), & t \geq 0, \\ Lx(t) = \Phi x(t), & t \geq 0, \\ x(0) = x_0,\ \dot{x}(0) = x_1. \end{cases}$$

Note that similarly to (1.2) we then have

$$\mathcal{G} = \mathcal{A}_{\mathcal{P}}^{\Phi} := \mathcal{A}_m + \mathcal{P} \quad \text{with domain } D(\mathcal{A}_{\mathcal{P}}^{\Phi}) := \left\{ \binom{v}{x} \in D(A_m) \times V : Lv = \Phi v \right\} \tag{5.14}$$

for the space $\mathcal{Z} := Z_V \times X$ and the operators

$$\begin{aligned} \mathcal{A}_m &:= \begin{pmatrix} 0 & \mathrm{Id} \\ A_m & 0 \end{pmatrix} \quad \text{with domain } D(\mathcal{A}_m) := D(A_m) \times V, \\ \mathcal{P} &:= \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{X}). \end{aligned}$$

Hence, if Assumption I.2 (b) is satisfied, the following representation holds as in (1.5).

Proposition V.8. *Assume that Assumption I.2 (b) holds and let A be defined as in* (5.12). *Then*

$$\mathcal{G} = \begin{pmatrix} 0 & \mathrm{Id} \\ A_{-1} + P + L_A \cdot \Phi & 0 \end{pmatrix}\Bigg|_{\mathcal{X}}. \tag{5.15}$$

Proof. Denote by $\tilde{\mathcal{G}}$ the operator defined by the right-hand side of (5.15) and fix $\lambda \in \rho(A)$. Then for $\binom{v}{x} \in Z_V \times X$ we have

$$\begin{aligned} \binom{v}{x} \in D(\tilde{\mathcal{G}}) \quad &\iff \quad x \in V \ \text{ and } \ A_{-1}v + Pv + (\lambda - A_{-1})L_\lambda \Phi v \in X \\ &\iff \quad x \in V \ \text{ and } \ (\lambda - A_{-1})\Big(L_\lambda \Phi - \mathrm{Id}\Big)v + (P + \lambda)v \in X \\ &\iff \quad x \in V \ \text{ and } \ \Big(L_\lambda \Phi - \mathrm{Id}\Big)v \in D(A) = \ker(L) \\ &\iff \quad x \in V \ \text{ and } \ Lv = \Phi v \qquad (5.16) \\ &\iff \quad \binom{v}{x} \in D(\mathcal{G}), \end{aligned}$$

where in (5.16) we used that $v = \Big(\mathrm{Id} - L_\lambda \Phi\Big)v + L_\lambda \Phi v \in D(A) + \ker(\lambda - A_m) \subseteq D(A_m)$ so that

$$0 = L(L_\lambda \Phi x - x) = \Phi x - Lx.$$

Moreover, for $\binom{v}{x} \in D(\mathcal{G})$ we obtain

$$\begin{aligned}
\tilde{\mathcal{G}}\binom{v}{x} &= \begin{pmatrix} x \\ (\lambda - A)\Big(L_\lambda \Phi - \mathrm{Id}\Big)v + (P+\lambda)v \end{pmatrix} \\
&= \begin{pmatrix} x \\ (\lambda - A_m)L_\lambda \Phi v - (\lambda - A_m)v + (P+\lambda)v \end{pmatrix} \\
&= \begin{pmatrix} x \\ A_m v + Pv \end{pmatrix} = \mathcal{G}\binom{v}{x}.
\end{aligned}$$

□

In order to represent $\mathcal{G}$ as in (0.2) we define the space

$$\mathcal{U} := X \times \partial X$$

and the operators $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}^{\mathcal{A}})$ and $\mathcal{C} \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ by

$$\mathcal{B} := \begin{pmatrix} 0 \\ B \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_X & L_A \end{pmatrix}, \qquad \mathcal{C} := \begin{pmatrix} C & 0 \end{pmatrix} := \begin{pmatrix} P & 0 \\ \Phi & 0 \end{pmatrix}. \tag{5.17}$$

As in Lemma I.8, we then obtain the following representation of $\mathcal{G}$.

Corollary V.9. *The triple* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *given by* $\mathcal{A}$ *as in* (5.1) *with operator* A *as in* (5.12) *and* $\mathcal{B}$, $\mathcal{C}$ *as in* (5.17) *is compatible. Moreover,* $\mathcal{G} = \mathcal{A}_{\mathcal{B}\mathcal{C}}$.

In this setting our perturbation result in Corollary V.6 reads as follows. Here we consider the situation $P \in \mathcal{L}(V, X)$ which applies to Section V.3 and V.4 below.[4]

Corollary V.10. *Let* $A \subset A_m$ *generate a strongly continuous cosine family* $(C(t))_{t\in\mathbb{R}}$ *with phase space* $V \times X$ *and assume that* $P \in \mathcal{L}(V, X)$. *Assume that there exist* $1 \le p < \infty$, $t > 0$ *and* $M \ge 0$ *such that*

(a) $\displaystyle\int_0^t S_{-1}(t-s)L_A u(s)\,ds \in V$ *for all* $u \in \mathrm{L}^p([0,t], \partial X)$,

(b) $\displaystyle\int_0^t \Big\|\Phi\, C(s)v\Big\|_{\partial X}^p ds \le M \cdot \|v\|_V^p$ *for all* $v \in D(A)$,

(c) $\displaystyle\int_0^t \Big\|\Phi\, S(s)x\Big\|_{\partial X}^p ds \le M \cdot \|x\|_X^p$ *for all* $x \in V$,

(d) $\displaystyle\int_0^t \Big\|\Phi \int_0^r S_{-1}(r-s)L_A u(s)\,ds\Big\|_{\partial X}^p dr \le M \cdot \|u\|_p^p$ *for all* $u \in \mathrm{W}_0^{2,p}([0,t], \partial X)$,

(e) $1 \in \rho\Big(\mathcal{F}_t^{(A,L_A,\Phi)}\Big)$,

[4] Likewise assumptions yield the generator property of A_P^Φ in case $P \in \mathcal{L}(Z_V, X)$ (cf. Corollary III.13).

where $\mathcal{F}_t^{(A,L_A,\Phi)} \in \mathcal{L}(\mathrm{L}^p([0,t],\partial X))$ *is the unique bounded extension of the operator given by*

$$\left(\mathcal{F}_t^{(A,L_A,\Phi)}u\right)(r) = \Phi \int_0^r S_{-1}(r-s)L_A u(s)\,ds \tag{5.18}$$

for $u \in \mathrm{W}_0^{2,p}([0,t],\partial X)$ *and* $r \in [0,t]$. *Then* A_P^Φ *generates a strongly continuous cosine family with phase space* $V \times X$.

Proof. By Corollary V.6,[5] the boundedness of $P \in \mathcal{L}(V,X)$ and Proposition V.1, it suffices to show that $1 \in \rho\left(\mathcal{F}_t^{(A,B,C)}\right)$ for some $t > 0$, with B and C as in (5.17) and

$$\mathcal{F}_t^{(A,B,C)} = \begin{pmatrix} \mathcal{F}_t^{(A,\mathrm{Id}_X,P)} & \mathcal{F}_t^{(A,L_A,P)} \\ \mathcal{F}_t^{(A,\mathrm{Id}_X,\Phi)} & \mathcal{F}_t^{(A,L_A,\Phi)} \end{pmatrix} \in \mathcal{L}(\mathrm{L}^p([0,t],\mathcal{U})).^6$$

The operator $\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}$ is invertible for all sufficiently small $t > 0$ with uniformly bounded inverses $\left(\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}\right)^{-1}$ by Corollary V.7.(a) and the Neumann series. Thus, for small $t > 0$, the operator $\mathrm{Id} - \mathcal{F}_t^{(A,B,C)}$ is invertible if and only if

$$\mathrm{Id} - \mathcal{F}_t^{(A,L_A,\Phi)} - \underbrace{\mathcal{F}_t^{(A,\mathrm{Id}_X,\Phi)}\left(\mathrm{Id} - \mathcal{F}_t^{(A,\mathrm{Id}_X,P)}\right)^{-1}\mathcal{F}_t^{(A,L_A,P)}}_{=:\Delta_t}$$

is invertible (use Schur complements in Lemma B.1.(ii)). The assertion follows by hypothesis (e), the fact that the set of invertible operators is open and

$$\|\Delta_t\| \to 0 \qquad \text{as } t \to 0^+$$

by Corollary V.7.(a). $\square$

Analogously to Lemma III.14, we now present a criterion in order to obtain condition (a) in Corollary V.10 for control operators $B = L_A$ as in (1.4) and the generator A of a cosine family.

Lemma V.11. *Let* $A \subset A_m$ *generate the strongly continuous cosine family* $(C(t))_{t\in\mathbb{R}}$ *and let* $B := L_A = (\lambda - A_{-1})L_\lambda \in \mathcal{L}(\partial X, V_{-1})$ *for some* $\lambda \in \rho(A)$. *Then for some fixed* $p \geq 1$ *the following are equivalent.*

(a) B *satisfies condition (a) in Corollary V.10.*

[5] The hypotheses (a)–(d) imply (ii), (iii), (iii') and (iv), respectively, while $\mathrm{rg}(R(\lambda,A_{-1})B) = \mathrm{rg}((R(\lambda,A),L_\lambda)) \subset D(A_m) \subset Z_V$ is a direct consequence of our standing assumptions.

[6] The operators $\mathcal{F}_t^{(A,\bullet,\bullet)}$ are defined analogously to $\mathcal{F}_t^{(A,L_A,\Phi)}$ in (5.18).

(b) *There exists* $t_0 > 0$ *and a strongly continuous family of operators* $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],\partial X),V)$ *such that for every* $u \in \mathrm{W}_0^{2,p}([0,t_0],\partial X)$ *the function*

$$x : [0,t_0] \to V, \quad x(t) := \mathbb{B}_t u \tag{5.19}$$

is the classical solution of the boundary control problem

$$(BCP_2) \qquad \begin{cases} \ddot{x}(t) = A_m x(t), & 0 \le t \le t_0, \\ Lx(t) = u(t), & 0 \le t \le t_0, \\ x(0) = 0,\ \dot{x}(0) = 0. \end{cases}$$

Moreover, in this case, for $t \in [0,t_0]$ *we obtain*

$$\mathbb{B}_t u = \int_0^t S_{-1}(t-s)Bu(s)\,ds \quad \textit{for all } u \in \mathrm{L}^p([0,t_0],\partial X).$$

Proof. (a)$\Longrightarrow$(b). By Corollary V.10.(a) there exists some $t_0 > 0$ such that

$$\mathcal{B}_t^S u = \int_0^t S_{-1}(t-s)Bu(s)\,ds \in V \quad \text{for all } t \in [0,t_0] \text{ and } u \in \mathrm{L}^p([0,t_0],\partial X).$$

Define $\mathbb{B}_t := \mathcal{B}_t^S$ for $t \in [0,t_0]$. Remark V.4 yields $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0],\partial X),V)$. It remains to show that for $u \in \mathrm{W}_0^{2,p}([0,t_0],\partial X)$ the function $x := \mathbb{B}_\bullet u : [0,t_0] \to V$ in (5.19) is a *classical solution* of the boundary control problem (BCP_2) on X, i.e.,

- $x \in \mathrm{C}^2([0,t_0],X)$ with $x(t) \in D(A_m)$ for every $0 \le t \le t_0$,
- $\ddot{x}(t) = A_m x(t)$ and $Lx(t) = u(t)$ for every $0 \le t \le t_0$.

For $u \in \mathrm{W}^{2,p}([0,t_0],\partial X)$ and $t \in [0,t_0]$ we define

$$y(t) := \mathcal{B}_t^C u = \int_0^t C_{-1}(t-s)Bu(s)\,ds \in X \qquad \text{(by Lemma V.5).}$$

Then $z := \binom{x}{y}$ satisfies

$$z(t) = \int_0^t \mathcal{T}_{-1}(t-s)\tbinom{0}{B}u(s)\,ds \in \mathcal{X},$$

i.e., $\binom{0}{B}$ is p-admissible for $\mathcal{A}$ and, using Lemma III.14.(i)⇒(ii), the function $z \in \mathrm{C}^1([0,t_0],\mathcal{X})$ is a classical solution of the inhomogeneous Cauchy problem

$$\text{(iACP)} \qquad \begin{cases} \dot{z}(t) = \mathcal{A}_{-1}z(t) + \tbinom{0}{B}u(t), & 0 \le t \le t_0, \\ z(0) = 0. \end{cases}$$

However, (iACP) can be rewritten as

$$\begin{cases} \dot{x}(t) = y(t), & 0 \le t \le t_0, \\ \ddot{x}(t) = \dot{y}(t) = A_{-1}x(t) + Bu(t), & 0 \le t \le t_0, \\ x(0) = 0,\ \dot{x}(0) = 0. \end{cases}$$

Thus, x is twice continuously differentiable and solves the equations

$$\begin{cases} \ddot{x}(t) = A_m x(t), & 0 \le t \le t_0, \\ Lx(t) = u(t), & 0 \le t \le t_0, \\ x(0) = 0,\ \dot{x}(0) = 0, \end{cases}$$

i.e., x is a classical solution of (BCP_2) on X.

(b)$\Longrightarrow$(a). Assume that the function $[0, t_0] \ni t \mapsto x(t) := \mathbb{B}_t u$ is a classical solution of the boundary control problem (BCP_2) on X for all $u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X)$, i.e., of the inhomogeneous Cauchy problem

$$\begin{cases} \ddot{x}(t) = A_{-1}x(t) + Bu(t) & \text{in } V_{-1}, \\ x(0) = 0,\ \dot{x}(0) = 0. \end{cases}$$

The unique solution of the inhomogeneous Cauchy problem is given by

$$x(t) = \int_0^t S_{-1}(t-s)Bu(s)\, ds$$

for $t \in [0, t_0]$ (cf. [LT81, Thm. 3.1] whose proof can be adapted to this situation, or use the first order reduction of (iACP)). Since $\mathbb{B}_{t_0} \in \mathcal{L}(\mathrm{L}^p([0, t_0], \partial X), V)$ by assumption (b) we obtain

$$\left\| \int_0^{t_0} S_{-1}(t_0 - s)Bu(s)\, ds \right\|_V = \|\mathbb{B}_{t_0} u\|_V \le M\|u\|_p \tag{5.20}$$

for all $u \in \mathrm{W}_0^{2,p}([0, t_0], \partial X)$ which yields Corollary V.10.(a) by Remark V.4. $\square$

To illustrate our results we next show the wellposedness of

- a second order Cauchy problem associated with a second order differential operator on $\mathrm{L}^p[0,1]$ and perturbed Neumann boundary conditions, and
- a second order Cauchy problem associated with a second order differential operator on $\mathrm{C}[0,1]$ and perturbed Wentzell boundary conditions.

V.3. Wellposedness of a Second Order Cauchy Problem associated with a Second Order Differential Operator with perturbed Neumann Boundary Conditions

On $X := \mathrm{L}^p[0,1]$ we consider the second derivative with Neumann boundary conditions

$$A := \Delta_N \subset \tfrac{d^2}{ds^2}, \quad D(A) := \Big\{ f \in \mathrm{W}^{2,p}[0,1] : f'(0) = f'(1) = 0 \Big\}. \tag{5.21}$$

It is a well-known result that Δ_N generates a strongly continuous cosine family $(C_N(t))_{t\in\mathbb{R}}$ on $\mathrm{L}^p[0,1]$ with phase space $\mathrm{W}^{1,p}[0,1] \times \mathrm{L}^p[0,1]$. Below, we give our own proof based on the restriction of cosine families to closed invariant subspaces (cf. Section C in the Appendix). Also in [CKW08] an extension technique has been used to show the generator property of second order differential operators.

Proposition V.12. *The operator $A = \Delta_N$ generates a strongly continuous cosine family $(C_N(t))_{t\in\mathbb{R}}$ with phase space* $\mathrm{W}^{1,p}[0,1] \times \mathrm{L}^p[0,1]$.

Proof. To show that $A = \Delta_N$ generates a strongly continuous cosine family we use Lemma C.1. Therefore, define $\hat{X} := \mathrm{L}^p[-1,1]$ and consider on $\hat{X}$ the left shift group $(\hat{G}(t))_{t\in\mathbb{R}}$ generated by

$$\hat{W} = \tfrac{d}{ds} \quad \text{with domain } D(\hat{W}) = \{\hat{f} \in \mathrm{W}^{1,p}[-1,1] : \hat{f}(-1) = \hat{f}(1)\}.$$

Hence, for $t \in \mathbb{R}$ the operators

$$\hat{C}(t) := \tfrac{1}{2}\Big(\hat{G}(t) + \hat{G}(-t)\Big) \in \mathcal{L}(\hat{X})$$

yield a strongly continuous cosine family $(\hat{C}(t))_{t\in\mathbb{R}}$ on $\hat{X}$ with generator $\hat{A} = \hat{W}^2$ (see Section C in the Appendix), i.e., the operator $\hat{A}$ is given by

$$\hat{A} = \tfrac{d^2}{ds^2} \quad \text{with domain } D(\hat{A}) = \{\hat{f} \in \mathrm{W}^{2,p}[-1,1] : \hat{f}(-1) = \hat{f}(1),\ \hat{f}'(-1) = \hat{f}'(1)\}.$$

Define the closed subspace $X \subset \hat{X}$ by

$$X :=\Big\{\hat{f} \in \hat{X} : \hat{f} \text{ is even}\Big\}.$$ [7]

The space X is $\hat{C}(t)$-invariant since even functions are $\hat{C}(t)$-invariant (but *not* $\hat{G}(t)$-invariant). Thus, we can apply Lemma C.1 to conclude that $\left(\hat{C}(t)|_X\right)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ defines a strongly continuous cosine family on X with generator $\hat{A}|_X$ and phase space $\mathcal{X} = V \times X$ where $V := D(\hat{W}) \cap X$ is endowed with the graph norm of $\hat{W}$.

It remains to determine $D(\hat{A}|_X)$ and V. Note that an even function $\hat{f} \in \mathrm{W}^{2,p}[-1,1]$ always satisfies $\hat{f}'(0) = 0$ and $\hat{f}'(-r) = -\hat{f}'(r)$ for $r \in [0,1]$. Identifying $\hat{f} \in X$ with its restriction $f = \hat{f}|_{[0,1]} \in \mathrm{L}^p[0,1]$ we obtain

$$\hat{f} \in D(\hat{A}|_X) \quad :\Longleftrightarrow \quad \hat{f} \in \mathrm{W}^{2,p}[-1,1] \cap X,\ \hat{f}'(-1) = \hat{f}'(1) \text{ and } \hat{f}'' \in X.$$

In particular, the function $\hat{f}$ satisfies $-\hat{f}'(1) = \hat{f}'(-1) = \hat{f}'(1)$ and we obtain

$$\hat{f} \in D(\hat{A}|_X) \quad \Longrightarrow \quad f \in \mathrm{W}^{2,p}[0,1] \text{ and } f'(0) = f'(1) = 0,$$

i.e., $f \in D(A)$. On the contrary, take $f \in D(A)$. Then it is an easy consequence that the even extension $\hat{f}$ satisfies $\hat{f} \in D(\hat{A}|_X)$ and hence $A = \Delta_N$ is similar to $\hat{A}|_X$. Further,

$$\begin{aligned} \hat{f} \in V \quad &:\Longleftrightarrow \quad \hat{f} \in \mathrm{W}^{1,p}[-1,1] \cap X \\ &\Longleftrightarrow \quad f \in \mathrm{W}^{1,p}[0,1]. \end{aligned}$$

Summing up, this proves that $A = \Delta_N$ generates a strongly continuous cosine family with phase space $\mathrm{W}^{1,p}[0,1] \times \mathrm{L}^p[0,1]$. □

Remark V.13. The strategy of extending functions in a certain way and restricting a given cosine family to an invariant subspace has also been proposed in, e.g., [Bob10], therein called *Lord Kelvin's method of images*.

We now perturb, for simplicity, only the boundary condition $f'(0) = 0$. For this purpose we introduce the maximal operator

$$A_m := \tfrac{d^2}{ds^2} \qquad \text{with domain } D(A_m) := \{f \in \mathrm{W}^{2,p}[0,1] : f'(1) = 0\}$$

and define the boundary space $U := \partial X := \mathbb{C}$ and the boundary operator

$$L : D(A_m) \subset X \to \partial X, \quad Lf := f'(0).$$

Then one easily verifies that the associated Dirichlet operators $L_\lambda : \partial X \to \ker(\lambda -$

[7] Here, we use the same symbol for the state space $\mathrm{L}^p[0,1]$ and the closed subspace $X \subset \hat{X}$ since the two spaces are isomorphic.

$A_m) \subset X$ for the pair (A_m, L) and $\lambda > 0$ exist and are given by

$$(L_\lambda z)(s) = -z \cdot \frac{\cosh(\sqrt{\lambda}(s-1))}{\sqrt{\lambda} \cdot \sinh(\sqrt{\lambda})} \qquad \text{for } z \in \partial X = \mathbb{C},\ s \in [0,1].$$

Using Corollary V.10 we obtain the following generation result.

Corollary V.14. *Assume that* $\Phi \in \mathcal{L}(\mathrm{C}^1[0,1], \mathbb{C})$ *has the representation*

$$\Phi f = \int_0^1 f'(s)\, d\mu(s) \qquad \textit{for } f \in \mathrm{C}^1[0,1]$$

and $\mu : [0,1] \to \mathbb{R}$ *of bounded variation and choose* $P \in \mathcal{L}(\mathrm{W}^{1,p}[0,1], \mathrm{L}^p[0,1])$*. If* μ *has no mass in 0 (cf. Assumption III.16), then the second order differential operator* $A_P^\Phi \subset \frac{d^2}{ds^2} + P$ *with domain*

$$D(A_P^\Phi) = \left\{ f \in \mathrm{W}^{2,p}[0,1] : \begin{pmatrix} f'(0) \\ f'(1) \end{pmatrix} = \begin{pmatrix} \Phi f \\ 0 \end{pmatrix} \right\}$$

generates a strongly continuous cosine family with phase space $\mathrm{W}^{1,p}[0,1] \times \mathrm{L}^p[0,1]$.

Proof. By Proposition V.12, the cosine family $(C_N(t))_{t\in\mathbb{R}}$ is given by

$$[C_N(t)f](s) = \frac{\hat{f}(s+t) + \hat{f}(s-t)}{2} \qquad \text{for } t \in \mathbb{R},\ s \in [0,1], \text{[8]}$$

while the sine family $(S_N(t))_{t\in\mathbb{R}}$ is given by

$$[S_N(t)f](s) = \frac{1}{2} \int_{s-t}^{s+t} \hat{f}(r)\, dr \qquad \text{for } t \in \mathbb{R},\ s \in [0,1].$$

We have $P \in \mathcal{L}(V, X)$ and choose $Z_V := \mathrm{C}^1[0,1]$. We check the conditions (a)-(e) in Corollary V.10. To this end, we first use Lemma V.11 and define for $0 < t_0 \le 1$ the strongly continuous operator family $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p[0,t_0], \mathrm{W}^{1,p}[0,1])$ by

$$[\mathbb{B}_t u](s) := -\int_0^{\max\{0, t-s\}} u(r)\, dr. \tag{5.22}$$

The function $x(t) := \mathbb{B}_t u$ solves the boundary control problem (BCP_2) on $\mathrm{L}^p[0,1]$ in Lemma V.11 for $u \in \mathrm{W}_0^{2,p}[0,t_0]$. Thus, $\mathcal{B}_t^S = \mathbb{B}_t$ for $t \in [0,t_0]$ which yields (a).

[8]For $f \in \mathrm{L}^p[0,1]$ we denote by $\hat{f}$ its even and 2-periodic extension to $\mathbb{R}$.

For $f \in D(A)$ we obtain

$$
\begin{aligned}
\int_0^1 |\Phi C(t) f|^p \, dt &= \int_0^1 \Big| \int_0^1 \frac{\hat{f}'(s+t) + \hat{f}'(s-t)}{2} \, d\mu(s) \Big|^p \, dt \\
&\leq 2^{-p} (|\mu|[0,1])^{p-1} \int_0^1 \int_0^1 |\hat{f}'(s+t) + \hat{f}'(s-t)|^p \, d|\mu|(s) \, dt \qquad (5.23)\\
&\leq \tfrac{1}{2} (|\mu|[0,1])^{p-1} \int_0^1 \int_0^1 |\hat{f}'(s+t)|^p + |\hat{f}'(s-t)|^p \, dt \, d|\mu|(s) \qquad (5.24)\\
&\leq \tfrac{1}{2} (|\mu|[0,1])^{p-1} \cdot \Big(\int_0^1 d|\mu|(s) \Big) \cdot \Big(\int_0^2 |\hat{f}'(t)|^p \, dt + \int_{-1}^1 |\hat{f}'(t)|^p \, dt \Big) \\
&\leq \tfrac{1}{2} (|\mu|[0,1])^p \cdot (2\|f'\|_p^p + 2\|f'\|_p^p) \\
&\leq 2(|\mu|[0,1])^p \cdot \|f\|^p_{\mathrm{W}^{1,p}[0,1]}.
\end{aligned}
$$

where we used Hölder's inequality in (5.23) and the Fubini–Tonelli theorem in (5.24). For $f \in V$ we obtain as above

$$
\begin{aligned}
\int_0^1 |\Phi S(t) f|^p \, dt &= 2^{-p} \int_0^1 \Big| \Phi \int_{\bullet - t}^{\bullet + t} \hat{f}(r) \, dr \Big|^p \, dt \\
&\leq 2^{-p} \int_0^1 \Big| \int_0^1 \hat{f}(s+t) - \hat{f}(s-t) \, d\mu(s) \Big|^p \, dt \\
&\leq 2^{-p} (|\mu|[0,1])^{p-1} \int_0^1 \int_0^1 |\hat{f}(s+t) - \hat{f}(s-t)|^p \, d|\mu|(s) \, dt \\
&\leq \tfrac{1}{2} (|\mu|[0,1])^{p-1} \int_0^1 \int_0^1 |\hat{f}(s+t)|^p + |\hat{f}(s-t)|^p \, dt \, d|\mu|(s) \\
&\leq \tfrac{1}{2} (|\mu|[0,1])^{p-1} \cdot \Big(\int_0^1 d|\mu|(s) \Big) \cdot \Big(\int_0^2 |\hat{f}(t)|^p \, dt + \int_{-1}^1 |\hat{f}(t)|^p \, dt \Big) \\
&\leq 2(|\mu|[0,1])^p \cdot \|f\|_p^p.
\end{aligned}
$$

Thus, conditions (b) and (c) hold.

Finally, for $t_0 \leq 1$ and $u \in \mathrm{W}_0^{2,p}([0,t_0],U)$ we estimate (using (5.22), Hölder's inequality and the Fubini–Tonelli theorem)

$$
\begin{aligned}
\int_0^{t_0} \Big| \Phi \int_0^t S_{-1}(t-r) L_A u(r) \, dr \Big|^p \, dt &= \int_0^{t_0} \Big| \int_0^t u(t-s) \, d\mu(s) \Big|^p \, dt \\
&\leq \int_0^{t_0} (|\mu|[0,t])^{p-1} \int_0^t |u(t-s)|^p \, d|\mu|(s) \, dt \\
&\leq (|\mu|[0,t_0])^{p-1} \int_0^{t_0} \int_s^{t_0} |u(t-s)|^p \, dt \, d|\mu|(s) \\
&\leq (|\mu|[0,t_0])^p \cdot \|u\|_p^p.
\end{aligned}
$$

This yields conditions (d) and (e) since μ is assumed to have no mass in zero. □

Remark V.15. Corollary V.14 can also be used to establish the result in [APW11, Thm. 3.4.(a)] and the example in [AP12, Expl. 5.3].

V.4. Wellposedness of a Second Order Cauchy Problem associated with a Second Order Differential Operator with perturbed Wentzell Boundary Conditions

On $X := \mathrm{C}[0,1]$ we consider the second derivative with Wentzell boundary conditions

$$A := \Delta_W \subset \tfrac{d^2}{ds^2}, \quad D(A) := \Big\{ f \in \mathrm{C}^2[0,1] : f''(0) = f''(1) = 0 \Big\}. \tag{5.25}$$

Applying a similar technique of proof as in Proposition V.12 yields the generator property of Δ_W. It is then our aim to perturb the boundary conditions and the action of the dynamics using Corollary V.10.

Proposition V.16. *The operator $A = \Delta_W$ generates a strongly continuous cosine family $(C_W(t))_{t\in\mathbb{R}}$ with phase space $\mathrm{C}^1[0,1] \times \mathrm{C}[0,1]$.*

Proof. To show that $A = \Delta_W$ generates a strongly continuous cosine family we use Lemma C.1. Therefore, define the function space

$$\hat{X} := \mathrm{C}_{ub}(\mathbb{R}) \oplus \langle \mathbb{s} \rangle \,,$$

where $\mathrm{C}_{ub}(\mathbb{R})$ denotes the space of all bounded, uniformly continuous functions on $\mathbb{R}$ and $\mathbb{s} \in \mathrm{C}(\mathbb{R})$ is defined by $\mathbb{s}(s) := s$ for all $s \in \mathbb{R}$, i.e., all functions $\hat{f} \in \hat{X}$ are of the form $\hat{f} = f + \alpha \cdot \mathbb{s}$ for some function $f \in \mathrm{C}_{ub}(\mathbb{R})$ and $\alpha \in \mathbb{C}$ and this representation is unique. On $\hat{X}$ we introduce the norm

$$\left\| \hat{f} \right\| := \|f\|_\infty + |\alpha| \quad \text{for } \hat{f} = f + \alpha \cdot \mathbb{s} \in \hat{X} \quad (\alpha \in \mathbb{C})$$

making it into a Banach space. Next we consider on $\hat{X}$ the left shift group $(\hat{G}(t))_{t\in\mathbb{R}}$ given by

$$\Big(\hat{G}(t)\hat{f}\Big)(s) := \hat{f}(s+t), \quad s,t \in \mathbb{R}.$$

By [EN00, II.2.10, Prop. 1] it follows that $\hat{G}(t)$ is strongly continuous with generator

$$\hat{W} = \tfrac{d}{ds} \quad \text{and domain } D(\hat{W}) = \{\hat{f} \in \hat{X} : \hat{f} \text{ is differentiable and } \hat{f}' \in \mathrm{C}_{ub}(\mathbb{R})\}.$$

Hence, for $t \in \mathbb{R}$ the operators

$$\hat{C}(t) := \tfrac{1}{2}\Big(\hat{G}(t) + \hat{G}(-t)\Big) \in \mathcal{L}(\hat{X})$$

yield a strongly continuous cosine family $(\hat{C}(t))_{t\in\mathbb{R}}$ on $\hat{X}$ with generator $\hat{A} = \hat{W}^2 = \frac{d^2}{ds^2}$ and domain

$$D(\hat{A}) = \{\hat{f} \in D(\hat{W}) : \hat{f}' \text{ is differentiable and } \hat{f}'' \in \mathrm{C}_{ub}(\mathbb{R})\}.$$

Define now the closed subspace $X \subset \hat{X}$ by

$$X := \Big\{\hat{f} + \alpha \cdot \mathbb{1} + \beta \cdot \mathbb{s} \in \hat{X} : \hat{f} \text{ is continuous, odd and 2-periodic on } \mathbb{R},\ \alpha,\ \beta \in \mathbb{C}\Big\}.$$

Using that continuous, odd, 2-periodic functions are $\hat{C}(t)$-invariant, as well as $\hat{C}(t)\mathbb{1} = \mathbb{1}$ and $\hat{C}(t)\mathbb{s} = \mathbb{s}$, it follows that X is $\hat{C}(t)$-invariant (but *not* $\hat{G}(t)$-invariant) since

$$\begin{aligned}\hat{C}(t)[\hat{f} + \alpha\mathbb{1} + \beta\mathbb{s}] &= \hat{C}(t)\hat{f} + \alpha \cdot \hat{C}(t)\mathbb{1} + \beta \cdot \hat{C}(t)\mathbb{s} \\ &= \hat{C}(t)\hat{f} + \alpha \cdot \mathbb{1} + \beta \cdot \mathbb{s} \in X.\end{aligned}$$

Thus, we can apply Lemma C.1 to conclude that $\Big(\hat{C}(t)|_X\Big)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ defines a strongly continuous cosine family on X with generator $\hat{A}|_X$ and phase space $\mathfrak{X} = V \times X$ where $V := D(\hat{W}) \cap X$.

Next observe that every $f \in \mathrm{C}[0,1]$ has a unique extension $\hat{f} \in X$ such that $\hat{f}|_{[0,1]} = f$, cf. Figure V.1, with continuous extension and restriction operators.

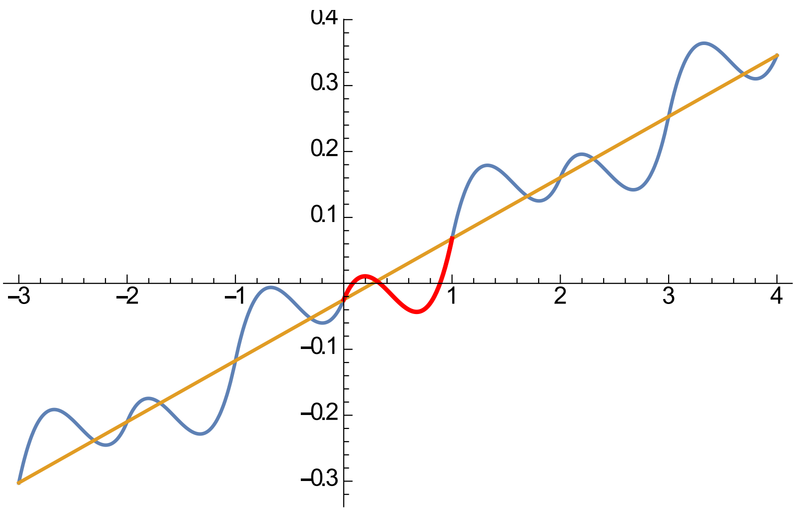

Figure V.1.: $\hat{f}$ and $f(0)\cdot\mathbb{1}+(f(1)-f(0))\cdot\mathbb{s}$ for $f(s) = (x-1/12)\cdot(x-1/3)\cdot(x-8/9)$.

To show this, note first that the odd and 2-periodic extension of a function $f \in \mathrm{C}[0,1]$

is continuous on $\mathbb{R}$ if and only if $f(0) = f(1) = 0$. Hence, if p_f denotes the odd and 2-periodic extension of $f - f(0)\cdot \mathbb{1} - \big(f(1) - f(0)\big)\cdot \mathbb{s} \in \mathrm{C}[0,1]$, then $p_f \in \mathrm{C}_{ub}(\mathbb{R})$ and

$$\hat{f} := p_f + f(0)\cdot \mathbb{1} + \big(f(1) - f(0)\big)\cdot \mathbb{s} \in X. \tag{5.26}$$

Moreover, for the extension in (5.26) we have $\hat{f}|_{[0,1]} = f$ and

$$\begin{aligned}
\|f\|_\infty = \sup_{s\in[0,1]} |\hat{f}(s)| &\le \sup_{s\in[0,1]} |\hat{f}(s) - \big(\hat{f}(1) - \hat{f}(0)\big)\cdot s| + |\hat{f}(1) - \hat{f}(0)| \\
&\le \sup_{s\in\mathbb{R}} |\hat{f}(s) - \big(\hat{f}(1) - \hat{f}(0)\big)\cdot s| + |\hat{f}(1) - \hat{f}(0)| = \big\|\hat{f}\big\| \\
&\le \|p_f\|_\infty + |f(0)| + |f(1) - f(0)| \le \sup_{s\in[0,1]} |p_f(s)| + 3\|f\|_\infty \le 7\|f\|_\infty.
\end{aligned}$$

Thus, we can identify X and $\mathrm{C}[0,1]$ by extending and restricting functions as indicated above (cf. Figure V.1).

It remains to determine $D(\hat{A}|_X)$ and V. By definition we have

$$\hat{f} \in D(\hat{A}|_X) \quad :\Longleftrightarrow \quad \hat{f} \in D(\hat{A}) \cap X \text{ and } \hat{f}'' \in X.$$

In particular, $\hat{f}''$ is continuous, odd and 2-periodic and we obtain

$$\hat{f} \in D(\hat{A}|_X) \quad \Longrightarrow \quad f \in \mathrm{C}^2[0,1] \text{ and } f''(0) = f''(1) = 0,$$

i.e., $f \in D(A)$. On the contrary, take $f \in D(A)$ and let p_f be the continuous, odd and 2-periodic extension of $f - f(0)\cdot \mathbb{1} - \big(f(1) - f(0)\big)\cdot \mathbb{s} \in \mathrm{C}[0,1]$. Then

- p_f is twice differentiable, hence $\hat{f} = p_f + f(0)\cdot \mathbb{1} + \big(f(1) - f(0)\big)\cdot \mathbb{s} \in X$ is twice differentiable,
- $\hat{f}'$, $\hat{f}'' \in \mathrm{C}_{ub}(\mathbb{R})$, i.e., $\hat{f} \in D(\hat{A})$,
- $\hat{f}'' = p_f''$ is odd and 2-periodic, i.e., $\hat{f} \in D(\hat{A}|_X)$.

It is an easy consequence that $A = \Delta_W$ is similar to $\hat{A}|_X$. Further,

$$\begin{aligned}
\hat{f} \in V \quad &:\Longleftrightarrow \quad \hat{f} \in D(\hat{W}) \cap X \\
&\Longleftrightarrow \quad f \in \mathrm{C}^1[0,1].
\end{aligned}$$

Summing up, this proves that $A = \Delta_W$ generates a strongly continuous cosine family with phase space $\mathrm{C}^1[0,1] \times \mathrm{C}[0,1]$. $\square$

In the second step we perturb the operator $A = \Delta_W$. To this end we consider the maximal operator

$$A_m := \tfrac{d^2}{ds^2} \qquad \text{with domain } D(A_m) := \mathrm{C}^2[0,1]$$

and define the boundary space $U := \partial X := \mathbb{C}^2$ and the boundary operator

$$L : D(A_m) \subset X \to \partial X, \quad Lf := \binom{f''(0)}{f''(1)}.$$

Then one easily verifies that the associated Dirichlet operators $L_\lambda : \partial X \to \ker(\lambda - A_m) \subset X$ for the pair (A_m, L) and $\lambda > 0$ exist and are given by

$$(L_\lambda z)(s) = \frac{z_1 \sinh(\sqrt{\lambda}(1-s)) + z_2 \sinh(\sqrt{\lambda}\, s)}{\lambda \cdot \sinh(\sqrt{\lambda})} \qquad \text{for } z = \binom{z_1}{z_2} \in \partial X = \mathbb{C}^2,\ s \in [0,1].$$

Using Corollary V.7.(a) and Corollary V.10 we obtain the following generation result.

Corollary V.17. *For $\Phi \in \mathcal{L}(\mathrm{C}^1[0,1], \mathbb{C}^2)$ and $P \in \mathcal{L}(\mathrm{C}^1[0,1], \mathrm{C}[0,1])$ the operator $A_P^\Phi \subset \frac{d^2}{ds^2} + P$ with domain*

$$D(A_P^\Phi) = \left\{ f \in \mathrm{C}^2[0,1] : \binom{f''(0)}{f''(1)} = \Phi f \right\}$$

generates a strongly continuous cosine family with phase space $\mathrm{C}^1[0,1] \times \mathrm{C}[0,1]$.

Proof. By the choice of $P \in \mathcal{L}(V, X)$ and $\Phi \in \mathcal{L}(V, \partial X)$, we are in the situation of Corollary V.7.(a) with $\mathcal{C} = (C, 0)$ defined in (5.17) for $C \in \mathcal{L}(V, X \times \partial X)$. It thus suffices to check condition (a) in Corollary V.10. To this end, we use Lemma V.11 and define for $0 \le t_0 \le 1$ and arbitrary $1 \le p < \infty$ the strongly continuous operator family $(\mathbb{B}_t)_{t\in[0,t_0]} \subset \mathcal{L}(\mathrm{L}^p([0,t_0], \mathbb{C}^2), \mathrm{C}^1[0,1])$,

$$\left[\mathbb{B}_t \binom{u_1}{u_2}\right](s) := \int_0^{\max\{0,t-s\}} (t-s-r) u_1(r)\, dr + \int_0^{\max\{0,t+s-1\}} (t+s-1-r) u_2(r)\, dr.$$

The function $x(t) := \mathbb{B}_t u$ solves the boundary control problem (BCP_2) on $\mathrm{C}[0,1]$ in Lemma V.11 for $u \in \mathrm{W}_0^{2,p}([0,t_0], \mathbb{C}^2)$. Thus, $\mathcal{B}_t^S = \mathbb{B}_t$ for $t \in [0, t_0]$, which yields the assertion. $\square$

Remark V.18. The technique in Section V.4 can be used to treat the example [AP12, Expl. 5.1]. Similar situations have been studied in [BE04, XL08, Bob10].

VI. Robustness of Asymptotic Properties

In this chapter we study the robustness of asymptotic properties under perturbations and follow the presentation in the manuscript *Asymptotic properties of C_0-semigroups under perturbations* (cf. [Adl15]). We identify conditions on the perturbing operators giving rise to a semigroup, such that the orbits of the perturbed semigroup inherit the same asymptotic behavior as the original semigroup (cf. Theorem VI.7). We apply our main result to study the strong stability of a neutral semigroup (cf. Section VI.2) and a semigroup associated with a delayed heat equation (cf. Section VI.3).

VI.1. A Perturbation Result Preserving Asymptotic Properties

For a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, the orbits $\mathcal{T}x := [t \mapsto T(t)x]$ belong to $\mathrm{C}_{ub}(\mathbb{R}_+, X)$ for all $x \in X$, where $\mathrm{C}_{ub}(\mathbb{R}_+, X)$ is the space of all bounded, uniformly continuous functions from $\mathbb{R}_+$ to X. In order to formalize the *asymptotic behavior* of the semigroup, we consider closed function subspaces $\mathcal{E} \subset \mathrm{C}_{ub}(\mathbb{R}_+, X)$ such that the functions $f \in \mathcal{E}$ possess a certain characteristic asymptotic property. Then $(T(t))_{t\geq 0}$ is said to have this characteristic asymptotic property if $\mathcal{T}x \in \mathcal{E}$ for all $x \in X$.

We first look for subspaces $\mathcal{E} \subset \mathrm{C}_{ub}(\mathbb{R}_+, X)$ appropriate for our purpose.

Definition VI.1. Let $\mathcal{E} \subset \mathrm{C}_{ub}(\mathbb{R}_+, X)$ be a closed subspace. We call $\mathcal{E}$ an *asymptotic subspace* (or *translation-biinvariant* in the terminology of [BC99, Sect. 7]) if for all $t \geq 0$ and $f \in \mathrm{C}_{ub}(\mathbb{R}_+, X)$

$$S(t)f \in \mathcal{E} \quad \Longrightarrow \quad f \in \mathcal{E}, \tag{6.1}$$

where $(S(t))_{t\geq 0}$ denotes the left shift semigroup on $\mathrm{C}_{ub}(\mathbb{R}_+, X)$.

We present a list of asymptotic subspaces and strongly continuous semigroups whose orbits belong to these asymptotic subspaces, see [BC99, Sect. 7] and [CP01, Sect. 3].

(i) The space $\mathrm{C}_{ub}(\mathbb{R}_+, X)$ is an asymptotic subspace and the orbits of a *bounded* semigroup belong to $\mathrm{C}_{ub}(\mathbb{R}_+, X)$.

(ii) The space $\mathrm{C}_0(\mathbb{R}_+, X) := \{f \in \mathrm{C}_{ub}(\mathbb{R}_+, X) : \lim_{t\to\infty} \|f(t)\| = 0\}$ is an asymptotic subspace. The orbits of a *strongly stable* semigroup belong to $\mathrm{C}_0(\mathbb{R}_+, X)$.

(iii) The space $\mathrm{C}_{0,\mathrm{w}}(\mathbb{R}_+, X) := \{f \in \mathrm{C}_{ub}(\mathbb{R}_+, X) : \lim_{t\to\infty} |\langle f(t), x'\rangle| = 0$ for all $x' \in X'\}$ is an asymptotic subspace. The orbits of a *weakly stable* semigroup belong to $\mathrm{C}_{0,\mathrm{w}}(\mathbb{R}_+, X)$.

(iv) The space $AAP(\mathbb{R}_+, X) := C_0(\mathbb{R}_+, X) \oplus AP(\mathbb{R}_+, X)$ is an asymptotic subspace by [AB99], where $AP(\mathbb{R}_+, X) := \overline{\operatorname{span}}\{\varepsilon_{i\alpha} \otimes x : \alpha \in \mathbb{R}, x \in X\}$ with $\varepsilon_{i\alpha}(s) = e^{i\alpha s}$ for $s \in \mathbb{R}_+$. We call a semigroup $(T(t))_{t\geq 0}$ *asymptotically almost periodic* if all orbits are relatively compact in X. Then, the orbits of an asymptotically almost periodic semigroup belong to $AAP(\mathbb{R}_+, X)$.

Now let $\mathcal{E} \subset \mathrm{C}_{ub}(\mathbb{R}_+, X)$ be an asymptotic subspace and let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup such that all orbits $\mathcal{T}x$ belong to $\mathcal{E}$. Further, assume that the perturbation P gives rise to a C_0-semigroup. Under what assumptions on the perturbation P do the orbits of the perturbed semigroup remain in $\mathcal{E}$?

Robust asymptotic properties have been investigated for *Miyadera-Voigt perturbations* in [CP01] and for *Desch-Schappacher perturbations* in [Man05]. In this chapter we turn to the investigation of robustness of asymptotic properties of C_0-semigroups under Weiss-Staffans perturbations.

We introduce robust asymptotic subspaces for Weiss-Staffans perturbations.

Definition VI.2. Let $\mathcal{E}$ be an asymptotic subspace and take

(i) a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X with generator A such that $\mathcal{T}x \in \mathcal{E}$ for all $x \in X$, and

(ii) a Weiss-Staffans perturbation $P = BC$ for A.

The space $\mathcal{E}$ is called a *robust asymptotic subspace* for P if the semigroup $(T_{BC}(t))_{t\geq 0}$ generated by A_{BC} has its orbits in $\mathcal{E}$ too.

In order to find appropriate Weiss-Staffans perturbations, we strengthen the assumptions in Definition III.3 (see also Definition III.6 for the operators $\mathcal{C}_t$, $\mathcal{C}_\infty$, $\mathcal{F}_t^{(A,B,C)}$ and $\mathcal{F}_\infty^{(A,B,C)}$).

Definition VI.3. Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. We call a Weiss-Staffans perturbation $P = BC \in \mathcal{L}(Z, X_{-1}^A)$ (with given $1 \leq p < \infty$) an *infinite-time Weiss-Staffans perturbation* for A if

(i) B is *infinite-time p-admissible*, i.e., there exists $M_B \geq 0$ such that for all $t > 0$ and $u \in \mathrm{L}^p([0,t], U)$ we have

$$\int_0^t T_{-1}(t-s)Bu(s)\,ds \in X$$

and

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,ds \right\| \leq M_B \|u\|_p,$$

(ii) $\sup_{t>0} \left\| \left(I - \mathcal{F}_t^{(A,B,C)} \right)^{-1} \mathcal{C}_t \right\|_{\mathcal{L}(X, \mathrm{L}^p([0,t],U))} < \infty.$

Remark VI.4. Condition (ii) in Definition VI.3 implies

$$\left(\mathrm{Id} - \mathcal{F}_\infty^{(A,B,C)} \right)^{-1} \mathcal{C}_\infty \in \mathcal{L}(X, \mathrm{L}^p(\mathbb{R}_+, U)).$$

Under these additional assumptions, our perturbation result in Theorem III.8 reads as follows.

Proposition VI.5. *Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ and assume that $P = BC$ is an infinite-time Weiss-Staffans perturbation for A. Then A_{BC} generates a bounded C_0-semigroup $(T_{BC}(t))_{t\geq 0}$ on X.*

Proof. By Theorem III.8, the perturbed semigroup $(T_{BC}(t))_{t\geq 0}$ is given by

$$T_{BC}(t)x = T(t)x + \mathcal{B}_t \left(I - \mathcal{F}_\infty^{(A,B,C)} \right)^{-1} \mathcal{C}_\infty x \quad \text{for } x \in X. \tag{6.2}$$

The boundedness of $(T_{BC}(t))_{t\geq 0}$ follows since $(T(t))_{t\geq 0}$ is bounded and

$$\sup_{t>0} \left\| \mathcal{B}_t \left(I - \mathcal{F}_\infty^{(A,B,C)} \right)^{-1} \mathcal{C}_\infty \right\| \leq M_B \left\| \left(I - \mathcal{F}_\infty^{(A,B,C)} \right)^{-1} \mathcal{C}_\infty \right\|_{\mathcal{L}(X, \mathrm{L}^p(\mathbb{R}_+,U))} < \infty$$

by Definition VI.3.(i) and Remark VI.4. □

Remark VI.6. Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X. Proposition VI.5 proves that $\mathcal{E} = \mathrm{C}_{ub}(\mathbb{R}_+, X)$ is a robust asymptotic subspace for all infinite-time Weiss-Staffans perturbations for A.

The following theorem is the main result of this chapter.

Theorem VI.7. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a bounded C_0-semigroup on X with generator A and let $\mathcal{E}$ be an asymptotic subspace such that $\mathcal{T}x \in \mathcal{E}$ for all $x \in X$. If P is an infinite-time Weiss-Staffans perturbation for A, then $\mathcal{E}$ is a robust asymptotic subspace for P.*

In the proof we shall use techniques proposed in [BC99, CP01, Man05].

Proof. We notice that $\left(I - \mathcal{F}_{\infty}^{(A,B,C)}\right)^{-1}\mathcal{C}_{\infty}x \in \mathrm{L}^p(\mathbb{R}_+, U)$ for all $x \in X$ by Definition VI.3.(ii) where $1 \leq p < \infty$ is as in Definition III.3. By Formula (6.2) it suffices to show that $[t \mapsto \mathcal{B}_t u] \in \mathcal{E}$ for all $u \in \mathrm{L}^p(\mathbb{R}_+, U)$. The operator

$$\begin{aligned}\mathbb{B} : \mathrm{L}^p(\mathbb{R}_+, U) &\to \mathrm{C}_b(\mathbb{R}_+, X),\\ u &\mapsto (\mathbb{B}u)(t) := \mathcal{B}_t u \qquad \text{for } t \in \mathbb{R}_+\end{aligned}$$

is bounded by the assumption (i) in Definition VI.3 on B. Here, the strong continuity of $(\mathcal{B}_t)_{t\geq 0}$ (see Lemma III.7.(i)) yields the continuity of $\mathbb{B}u$ while the boundedness of $\mathbb{B}u$ follows from Definition VI.3.(i), i.e.,

$$\|\mathbb{B}u\|_{\mathrm{C}_b(\mathbb{R}_+,X)} = \sup_{t>0} \|\mathcal{B}_t u\| \leq M_B \|u\|_p .$$

We show that $f := \mathbb{B}\tilde{u} \in \mathcal{E}$ for all $\tilde{u} = \mathbb{1}_{(a,b)} \otimes u$, $u \in U$ and $0 \leq a < b < \infty$, where

$$(\mathbb{1}_{(a,b)} \otimes u)(s) = \begin{cases} u & \text{for } s \in (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

For all $t > 0$ the identity [1]

$$\begin{aligned}(S(b)f)(t) = \mathcal{B}_{t+b}\tilde{u} &= \int_0^{t+b} T_{-1}(t+b-s)B\tilde{u}(s)\,ds \\ &= \int_a^b T_{-1}(t+b-s)Bu\,ds \\ &= T(t)\int_0^{b-a} T_{-1}(b-a-s)Bu\,ds \\ &= T(t)\mathcal{B}_{b-a}\left(\mathbb{1}_{(0,b-a)} \otimes u\right)\end{aligned}$$

holds. By the admissibility of B we have $\mathcal{B}_{b-a}\left(\mathbb{1}_{(0,b-a)} \otimes u\right) \in X$. Since $\mathcal{T}x \in \mathcal{E}$ for all $x \in X$, we obtain $S(b)f \in \mathcal{E}$. Thus, $\mathbb{B}\tilde{u} \in \mathcal{E}$ since $\mathcal{E}$ is an asymptotic subspace.

Finally, we obtain $\mathbb{B}u \in \mathcal{E}$ for all $u \in \mathrm{L}^p(\mathbb{R}_+, U)$ since the step functions are dense in $\mathrm{L}^p(\mathbb{R}_+, X)$ and $\mathcal{E}$ is closed. □

[1]The left shift semigroup on $\mathrm{C}_{ub}(\mathbb{R}_+, X)$ is denoted by $(S(t))_{t\geq 0}$.

In the following we focus on stability results for a neutral semigroup (cf. Section III.3 for the wellposedness) and a semigroup associated with a delayed heat equation.

VI.2. Strong Stability of a Neutral Semigroup

We return to the example studied in Section III.3 where we showed that the operator $\mathcal{G}$ in (3.8) can be written as $\mathcal{A}_{\mathcal{B}\mathcal{C}}$ as in (0.2) and generates a C_0-semigroup under appropriate assumptions.

We want to apply Theorem VI.7 to study the asymptotic properties of this semigroup. Therefore, let A be the generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, i.e., $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$. For $\mathbb{k}$, $\mathbb{p} \in \mathcal{L}(X)$ we consider the neutral equation

$$\frac{d}{dt}[x(t) - \mathbb{k}x(t-1)] = A[x(t) - \mathbb{k}x(t-1)] + \alpha \cdot \mathbb{p}x(t-1), \quad t \geq 0, \quad \alpha > 0,$$

with initial data $x_0 = f$ and $x(0) = y$ and it is our goal to study the asymptotic properties of the semigroup on $\mathcal{X} := X \times Y := X \times \mathrm{L}^1([-1,0], X)$ generated by

$$\mathcal{G} := \begin{pmatrix} A & \alpha\mathbb{p}\delta_{-1} \\ 0 & \frac{d}{dr} \end{pmatrix},$$
$$D(\mathcal{G}) := \left\{ \binom{x}{f} \in D(A) \times \mathrm{W}^{1,1}([-1,0], X) : x = f(0) - \mathbb{k}f(-1) \right\} \subset \mathcal{X}.$$

Here, the operators $P := \alpha\mathbb{p}\delta_{-1}$, $K := \mathbb{k}\delta_{-1} \in \mathcal{L}(\mathrm{C}([-1,0], X), X)$ both have no mass in zero. Hence, $\mathcal{G}$ indeed generates a C_0-semigroup on $\mathcal{X}$ by Theorem III.17. For $\alpha > 0$, we use the isomorphism $\mathfrak{S}_\alpha \binom{x}{f} = \binom{x}{\alpha f}$ on $\mathcal{X}$ in order to consider the similar operator matrix

$$\tilde{\mathcal{G}} := \begin{pmatrix} A & \mathbb{p}\delta_{-1} \\ 0 & \frac{d}{dr} \end{pmatrix}$$
$$D(\tilde{\mathcal{G}}) := \left\{ \binom{x}{f} \in D(A) \times \mathrm{W}^{1,1}([-1,0], X) : \alpha x = f(0) - \mathbb{k}f(-1) \right\}$$

given by $\tilde{\mathcal{G}} = \mathfrak{S}_\alpha \mathcal{G} \mathfrak{S}_\alpha^{-1}$. The operator $\tilde{\mathcal{G}}$ can be written as $\mathcal{A}_{\tilde{\mathcal{B}}\tilde{\mathcal{C}}}$, where

- $\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{dr} \end{pmatrix}$

with domain $D(\mathcal{A}) := D(A) \times \{f \in \mathrm{W}^{1,1}([-1,0], X) : f(0) = 0\}$,

- $\tilde{\mathcal{B}} := \begin{pmatrix} \mathrm{Id}_X & 0 \\ 0 & L_D \end{pmatrix} : \mathcal{U} := X \times X \to \mathcal{X}_{-1}^{\mathcal{A}}$, where $L_D := -D_{-1}L_0 \in \mathcal{L}(X, Y_{-1}^D)$ as in Section III.3 and

- $\tilde{\mathcal{C}} := \begin{pmatrix} 0 & P \\ \alpha \cdot \mathrm{Id}_X & K \end{pmatrix} : \mathcal{Z} := X \times \mathrm{C}([-1,0], X) \to \mathfrak{U}.$

Then we obtain the following stability result.

Corollary VI.8. *For all* $\mathbb{p}$, $\mathbb{k} \in \mathcal{L}(X)$ *satisfying* $\|\mathbb{p}\| + \|\mathbb{k}\| < 1$ *and* $\alpha > 0$ *such that* $M\alpha < \omega$ *the neutral semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *generated by* $\mathcal{G}$ *is strongly stable on* $\mathcal{X}$.

Proof. It suffices to show that the operator $\tilde{\mathcal{G}}$ generates a strongly stable semigroup. The unperturbed semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is strongly stable since A generates the uniformly exponentially stable semigroup $(T(t))_{t\geq 0}$ and D generates the nilpotent left shift semigroup $(S(t))_{t\geq 0}$. Thus, by Theorem VI.7, it suffices to show that $P := \tilde{\mathcal{B}}\tilde{\mathcal{C}}$ is an infinite-time Weiss-Staffans perturbation for $\mathcal{A}$ with $p = 1$. We recall that for all $t > 0$ and $u \in \mathrm{W}_0^{2,1}([0,t], X)$ [2]

$$\int_0^t S_{-1}(t-s)L_D u(s)\, ds = \tilde{u}(t + \bullet)$$

(cf. (4.18)) and that for all $t > 0$

$$\int_0^t \|\alpha T(s)x\|\, ds \leq q\|x\| \qquad \text{for all } x \in X$$

and $q := \frac{M\alpha}{\omega} < 1$ by assumption.

Condition (i) in Definition VI.3 is satisfied since for all $t > 0$ and u_1, $u_2 \in \mathrm{W}_0^{1,1}([0,t], X)$ we obtain

$$\begin{aligned}
\left\| \int_0^t \mathcal{T}_{-1}(t-s)\tilde{\mathcal{B}}\tbinom{u_1(s)}{u_2(s)}\, ds \right\|_{\mathcal{X}} &= \left\| \int_0^t T(t-s)u_1(s)\, ds \right\| + \left\| \int_0^t S_{-1}(t-s)L_D u_2(s)\, ds \right\|_Y \\
&= \int_0^t \|T(t-s)\|\|u_1(s)\|\, ds + \|\tilde{u_2}(t+\bullet)\|_Y \\
&\leq M\, \|u_1\|_1 + \int_{-1}^0 \|\tilde{u_2}(t+s)\|\, ds \\
&\leq M \left\| \tbinom{u_1}{u_2} \right\|_{\mathrm{L}^1([0,t],\mathfrak{U})}.
\end{aligned}$$

In order to verify condition (ii) in Definition VI.3 we first show that $1 \in \rho\Big(\mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})}\Big)$.

[2] For the function u defined on the interval $[0,t]$ we denote by $\tilde{u}$ its extension to $[-1,t]$ by the value 0.

For u_1, $u_2 \in \mathrm{W}_0^{2,1}(\mathbb{R}_+, X)$ we obtain

$$
\begin{aligned}
\int_0^\infty \left\| \tilde{\mathcal{C}} \int_0^r \mathcal{T}_{-1}(r-s)\tilde{\mathcal{B}}\binom{u_1(s)}{u_2(s)}\, ds \right\|_{\mathcal{U}} dr &\le \int_0^\infty \left\| P \int_0^r S_{-1}(r-s) L_D u_2(s)\, ds \right\| dr \\
&\quad + \int_0^\infty \left\| K \int_0^r S_{-1}(r-s) L_D u_2(s)\, ds \right\| dr \\
&\quad + \int_0^\infty \left\| \int_0^r \alpha T(r-s) u_1(s)\, ds \right\| dr \qquad (6.3)\\
&\le (\|\mathbb{p}\| + \|\mathbb{k}\|) \int_0^\infty \|\tilde{u_2}(r-1)\|\; dr + q\,\|u_1\|_1 \\
&\le (\|\mathbb{p}\| + \|\mathbb{k}\|)\, \|u_2\|_1 + q\,\|u_1\|_1
\end{aligned}
$$

where we estimate the integral in (6.3) as in the proof of [ABE14, Thm. 18] using the denseness of the step functions in $\mathrm{L}^1([0,1], X)$. Hence, $\left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\| < 1$ and $1 \in \rho\Big(\mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})}\Big)$. Hence, we obtain

$$
\begin{aligned}
\left\| \Big(1 - \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})}\Big)^{-1} \mathcal{C}_\infty \binom{x}{f} \right\| &\le \Big(1 - \left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\|\Big)^{-1} \int_0^\infty \left\| \tilde{\mathcal{C}} \mathcal{T}(s) \binom{x}{f} \right\| ds \\
&\le \Big(1 - \left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\|\Big)^{-1} \left(\int_0^\infty \|\alpha T(s) x\|\; ds + (\|\mathbb{k}\| + \|\mathbb{p}\|) \int_0^\infty \|\delta_{-1} S(s) f\|\; ds \right) \\
&\le \Big(1 - \left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\|\Big)^{-1} \left(q\,\|x\| + (\|\mathbb{k}\| + \|\mathbb{p}\|) \cdot \int_0^1 \|f(s-1)\|\; ds \right) \\
&= \Big(1 - \left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\|\Big)^{-1} (q\,\|x\| + (\|\mathbb{k}\| + \|\mathbb{p}\|)\, \|f\|_1) \\
&\le \Big(1 - \left\| \mathcal{F}_\infty^{(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})} \right\|\Big)^{-1} \left\| \binom{x}{f} \right\|_{\mathcal{X}}
\end{aligned}
$$

for all $\binom{x}{f} \in D(\mathcal{A})$, i.e., the condition (ii) in Definition VI.3 holds. Therefore $\tilde{\mathcal{G}}$ generates a C_0-semigroup whose orbits belong to $\mathrm{C}_0(\mathbb{R}_+, \mathcal{X})$ for all $\binom{x}{f} \in \mathcal{X}$. $\square$

VI.3. Stability of a Heat Equation with Unbounded Distributed Delay at the Boundary.

In this section we prove a stability result for a heat equation with distributed delay in the Neumann boundary condition given by the equations

$$
\text{(DHE)} \qquad \begin{cases} \dfrac{du}{dt}(t,s) = \dfrac{d^2u}{ds^2}(t,s), & 0 < s < 1,\ t \geq 0, \\ \dfrac{du}{ds}(t,0) = \displaystyle\int_0^1 \int_{-1}^0 u(t+r,s)\, d\nu(r)\, ds, & t \geq 0, \\ u(t,1) = 0, & t \geq 0, \\ u(0,s) = f_0(s), & 0 < s < 1, \end{cases}
$$

where $\nu : [-1,0] \to \mathbb{R}$ is of bounded variation. This system of equations can be reformulated as an (ACP) associated with the operator $\mathcal{G}$ in (2.17) of Section II.5.

Lemma VI.9. *The operator* $\mathcal{G}$ *in* (2.17) *is the generator of a* C_0*-semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $\mathcal{X} := X \times Y := \mathrm{L}^p[0,1] \times \mathrm{L}^p([-1,0],X)$*, i.e., (DHE) is wellposed on* $\mathrm{L}^p[0,1]$.

The wellposedness of (DHE) on $\mathrm{L}^2[0,\pi]$ was proven in [Bom15, Sect. 3.2, Expl. 2] using Hilbert space methods (cf. [HMR15, Expl. 5.2] as well).

Proof. The generator property of $\mathcal{G}$ can be shown as in Section IV.4. For this purpose we represent $\mathcal{G}$ as $\mathcal{A}^\Phi$ and introduce the following operators and spaces. Consider

- $A_m := \frac{d^2}{ds^2}$ with domain $D(A_m) := \{f \in \mathrm{W}^{2,p}[0,1] : f(1) = 0\}$ on X,
- $L := \delta_0' : D(A_m) \subset X \to \partial X := \mathbb{C}$, i.e., $Lf = f'(0)$,
- $D_m := \frac{d}{dr}$ with domain $D(D_m) := \mathrm{W}^{1,p}([-1,0],X)$ on Y,
- $K := \delta_0 : D(D_m) \subset Y \to \partial Y := X$, i.e., $Kv = v(0)$,
- $A := A_m|_{\ker L}$, $D := D_m|_{\ker K}$.

Next we define the maximal operator matrix

$$
\mathcal{A}_m := \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix} \qquad \text{with domain } D(\mathcal{A}_m) := D(A_m) \times D(D_m),
$$

the boundary operator

$$\mathcal{L} := \begin{pmatrix} L & 0 \\ 0 & K \end{pmatrix} : D(\mathcal{A}_m) \subset \mathcal{X} \to \partial\mathcal{X} := \partial X \times \partial Y,$$

the operator $\mathcal{A} \subset \mathcal{A}_m$ with domain $D(\mathcal{A}) := \ker(\mathcal{L}) = D(A) \times D(D)$ and $\mathcal{Z} := X \times [D(D_m)]$. Define

$$\mathcal{L}_{\mathcal{A}} := \begin{pmatrix} L_A & 0 \\ 0 & K_D \end{pmatrix} := \begin{pmatrix} -A_{-1}L_0 & 0 \\ 0 & -D_{-1}K_0 \end{pmatrix} : \partial\mathcal{X} \to \mathcal{X}_{-1}^{\mathcal{A}},$$

with operators

$$\begin{aligned} (L_0 z)(s) &:= z \cdot (s-1) && \text{for } z \in \mathbb{C} \text{ and } s \in [0,1], \\ (K_0 x)(r) &:= x && \text{for } x \in X \text{ and } r \in [-1,0], \text{ and} \\ \Phi &:= \begin{pmatrix} 0 & \varphi \\ \mathrm{Id}_X & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{Z}, \partial\mathcal{X}) && \text{where } \varphi(v) := \int_0^1 \int_{-1}^0 v(r,s)\, d\nu(r)\, ds. \end{aligned}$$

With these definitions we obtain $\mathcal{G} = \mathcal{A}^\Phi$. As in Section IV.4 one shows that $P := \mathcal{L}_{\mathcal{A}}\Phi$ is a Weiss-Staffans perturbation for $\mathcal{A}$, i.e.,

(a) (L_A, Id_X) is jointly q-admissible for A and for all $q > \frac{2p}{p+1}$ (and for all $q \geq 1$ if $p = 1$), and

(b) (K_D, φ) is jointly p-admissible for D. □

Then the following stability result holds.

Proposition VI.10. *There exists $\varepsilon_0 > 0$ such that the semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{X}$ generated by $\mathcal{G}$ is strongly stable if ν satisfies $|\nu|[-1,0] < \varepsilon_0$.*

Proof. Using the spaces and operators defined above it suffices to show that $P = \mathcal{L}_{\mathcal{A}}\Phi$ is an infinite-time Weiss-Staffans perturbation for $\mathcal{A}$, i.e.,

(a) $\mathcal{L}_{\mathcal{A}}$ is infinite-time p-admissible for $\mathcal{A}$, and

(b) $\sup_{t>0} \left\| \left(\mathrm{Id} - \mathcal{F}_t^{(\mathcal{A},\mathcal{L}_{\mathcal{A}},\Phi)}\right)^{-1} \mathcal{C}_t \right\| < \infty$,

where

$$\mathcal{F}_t^{(\mathcal{A},\mathcal{L}_{\mathcal{A}},\Phi)} = \begin{pmatrix} 0 & \mathcal{F}_t^{(D,K_D,\varphi)} \\ \mathcal{F}_t^{(A,L_A,\mathrm{Id}_X)} & 0 \end{pmatrix} \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}_+, \partial\mathcal{X})).$$

Condition (a) is a consequence of Lemma VI.9 and the fact that A and D both generate uniformly exponentially stable semigroups on X and Y respectively (we can neglect the time dependence by Lemma III.7). Further, this yields the existence of the operators $\mathcal{C}_\infty \in \mathcal{L}(X, \mathrm{L}^p(\mathbb{R}_+, \partial\mathcal{X}))$ and $\mathcal{F}_\infty^{(\mathcal{A},\mathcal{L}_\mathcal{A},\Phi)} \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}_+, \partial\mathcal{X}))$.

For (b) it is sufficient to show that $1 \in \rho(\mathcal{F}_\infty^{(\mathcal{A},\mathcal{L}_\mathcal{A},\Phi)})$. Using Schur complements (cf. Lemma B.1(i)) the invertibility of $\mathrm{Id} - \mathcal{F}_\infty^{(\mathcal{A},\mathcal{L}_\mathcal{A},\Phi)}$ is equivalent to the invertibility of $\mathrm{Id} - \mathcal{F}_\infty^{(D,K_D,\varphi)}\mathcal{F}_\infty^{(A,L_A,\mathrm{Id}_X)} \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}_+))$, where $\|\mathcal{F}_\infty^{(D,K_D,\varphi)}\| \le |\nu|[-1,0]$ (cf. [ABE14, Cor. 25.(iv)]). Thus, choosing

$$\varepsilon_0 := \|\mathcal{F}_\infty^{(A,L_A,\mathrm{Id}_X)}\|^{-1},$$

we obtain $1 \in \rho(\mathcal{F}_\infty^{(\mathcal{A},\mathcal{L}_\mathcal{A},\Phi)})$ and the assertion follows by Theorem VI.7. $\square$

As a consequence of the semigroup result we obtain stability for the solutions of (DHE).

Corollary VI.11. *There exists $\varepsilon_0 > 0$ such that the solutions of the delayed heat equation (DHE) are stable for all $\nu : [-1,0] \to \mathbb{R}$ of bounded variation with $|\nu|[-1,0] < \varepsilon_0$, i.e., for all $f_0 \in \mathrm{L}^p[0,1]$ the solution $u \in \mathrm{C}(\mathbb{R}_+, \mathrm{L}^p[0,1])$ of (DHE) satisfies*

$$\int_0^1 |u(t,s)|^p \, ds \to 0 \qquad \text{as } t \to \infty.$$

VII. A Final Application

Among the many perturbation problems to which our abstract theory might be applied, we sketch an approach towards the wellposedness of *heat and wave equations on networks* studied, e.g., by Marjeta Kramar Fijavž, Delio Mugnolo and Eszter Sikolya in [KFMS07] and Arendt et al. in [ADKF14] using *form methods* on Hilbert spaces (cf. [Kat95, Chap. 6], [Ouh05] and [AtE12]) and [Klö12] using a flow approach (see below). Given are a graph and two systems of partial differential equations

$$
\text{(HE)} \qquad \begin{cases} \frac{du_j}{dt}(t,s) = \frac{d}{ds}\Big(c_j \frac{du_j}{ds}\Big)(t,s), & t \geq 0,\ 0 < s < 1,\ j = 1,...,m, \\ u_j(t,v_i) = u_l(t,v_i), & t \geq 0,\ j,l \in \Gamma(v_i),\ i = 1,...,n, \\ \qquad 0 = \displaystyle\sum_{j=1}^{m} \phi_{ij}\mu_j c_j(v_i) \frac{du}{ds}(t,v_i), & t \geq 0,\ i = 1,...,n, \\ u_j(0,s) = f_j(s), & 0 < s < 1,\ j = 1,...,m, \end{cases}
$$

$$
\text{(WE)} \qquad \begin{cases} \frac{d^2u_j}{dt^2}(t,s) = \frac{d}{ds}\Big(c_j \frac{du_j}{ds}\Big)(t,s), & t \geq 0,\ 0 < s < 1,\ j = 1,...,m, \\ u_j(t,v_i) = u_l(t,v_i), & t \geq 0,\ j,l \in \Gamma(v_i),\ i = 1,...,n, \\ \qquad 0 = \displaystyle\sum_{j=1}^{m} \phi_{ij}\mu_j c_j(v_i) \frac{du}{ds}(t,v_i), & t \geq 0,\ i = 1,...,n, \\ u_j(0,s) = f_j(s), & 0 < s < 1,\ j = 1,...,m, \\ \frac{du_j}{dt}(0,s) = g_j(s), & 0 < s < 1,\ j = 1,...,m, \end{cases}
$$

(cf. [KFMS07, Sects. 2 & 3]) where

- m is the number of edges e_j of the graph,
- n is the number of vertices v_i of the graph,
- $\Gamma(v_i) := \{j \in \{1,...,m\} : e_j(0) = v_i \text{ or } e_j(1) = v_i\}$ is the set of indices of the edges connected to the vertex v_i $(i = 1,...,n)$,
- $c_j : [0,1] \to \mathbb{R}_+$ determine the *speed of propagation* on each edge $(j = 1,...,m)$,

- $\mu_j > 0$ are *distribution coefficients* depending on the edges e_j $(j = 1, ..., m)$, and

- $\Phi = (\phi_{ij})_{i,j} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ is the incidence matrix of the graph and represents its structure, see [KS05]. Further, we denote by Φ_ω^+ and Φ_ω^- the *weighted incoming and outgoing incidence matrices.*

To show the wellposedness of (HE) and (WE) on $(\mathrm{L}^p[0,1])^m$ for all $1 \le p < \infty$ we want to use Theorem III.8. To this end, we need an operator $\mathcal{G}$ such that (HE) and (WE) can be written as a first or second order abstract Cauchy problem, respectively. For simplicity of notations we only consider constant velocities $(c_j)_{j=1,\dots,m}$ equal to 1.

For $1 \le p < \infty$ we define the state space $\mathcal{X} := (\mathrm{L}^p[0,1])^m$ and consider the operator

$$\mathcal{G} := \begin{pmatrix} \frac{d^2}{ds^2} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{d^2}{ds^2} \end{pmatrix},$$

$$\begin{aligned} D(\mathcal{G}) := \{f \in (\mathrm{W}^{2,p}[0,1])^m : \Phi_\omega^+ f'(0) = \Phi_\omega^- f'(1) \text{ and} \\ \exists\, \delta \in \mathbb{C}^n \text{ such that } f(0) = (\Phi^+)^T\delta \text{ and } f(1) = (\Phi^-)^T\delta\}, \end{aligned}$$

see [KFMS07, Equ. (2.4)]. Then (ACP) and (ACP_2) associated with the operator $\mathcal{G}$ are equivalent to (HE) and (WE), respectively. In particular, the boundary conditions

$$\begin{cases} u_j(t, v_i) = u_l(t, v_i), & \forall\, t \ge 0,\ j, l \in \Gamma(v_i),\ i = 1, ..., n, \\ 0 = \sum_{j=1}^m \phi_{ij}\mu_j \frac{du}{ds}(t, v_i), & \forall\, t \ge 0,\ i = 1, ..., n, \end{cases}$$

in (HE) and (WE) are equivalent to

$$\begin{cases} \forall\, t \ge 0 \quad \exists\, \delta(t) \in \mathbb{C}^n \text{ s.t.} & \binom{u(t,0)}{u(t,1)} = (\Phi^+|\Phi^-)^T\delta(t),\ ^1 \\ \forall\, t \ge 0 & \Phi_\omega^+ \frac{du}{ds}(t, 0) = \Phi_\omega^- \frac{du}{ds}(t, 1). \end{cases}$$

It is our aim to show that $\mathcal{G}$ generates a strongly continuous cosine family on $\mathcal{X}$ with phase space $\mathcal{V} \times \mathcal{X}$, where

$$\mathcal{V} := \{f \in (\mathrm{W}^{1,p}[0,1])^m :\ \exists\, \delta \in \mathbb{C}^n \text{ such that } f(0) = (\Phi^+)^T\delta \text{ and } f(1) = (\Phi^-)^T\delta\}.$$

[1] Here, we use the notation $u = (u_1, ..., u_m)$.

Then, $\mathcal{G}$ is the generator of an analytic semigroup of angle $\frac{\pi}{2}$, cf. [ABHN11, Thm. 3.14.17].

By Proposition V.1, the operator $\mathcal{G}$ generates a strongly continuous cosine family on $\mathcal{X}$ with phase space $\mathcal{V} \times \mathcal{X}$ if and only if the *reduction matrix*

$$\begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{G} & 0 \end{pmatrix} \qquad \text{with domain } D(\mathcal{G}) \times \mathcal{V}$$

generates a strongly continuous semigroup on $\mathcal{V} \times \mathcal{X}$. This, however, is equivalent to

(i) $\rho(\mathcal{G}) \neq \emptyset$ (e.g., use results from Chapter II), and

(ii) the existence of a unique solution $u = (u_1, ..., u_m)$ of (WE) for all initial values $\binom{f}{g} \in D(\mathcal{G}) \times \mathcal{V}$, see [EN00, Thm. II.6.7].

We propose a strategy to show (ii) and use the flow approach in [Klö12] for the wave equation (WE).
(a) Doubling the graph and introducing *energy variables*, the wave system in [Klö12, Equ. (2.6)],[2]

$$(*) \qquad \begin{cases} \frac{d^2u_j}{dt^2}(t,s) = \frac{d^2u_j}{ds^2}(t,s), & t \geq 0,\ 0 < s < 1,\ j = 1, ..., m, \\ \frac{du_j}{dt}(t,v_i) = \frac{du_l}{dt}(t,v_i), & t \geq 0,\ j,l \in \Gamma(v_i),\ i = 1, ..., n, \\ \qquad 0 = \sum_{j=1}^{m} \phi_{ij}\mu_j \frac{du}{ds}(t,v_i), & t \geq 0,\ i = 1, ..., n, \\ u_j(0,s) = f_j(s), & 0 < s < 1,\ j = 1, ..., m, \\ \frac{du_j}{dt}(0,s) = g_j(s), & 0 < s < 1,\ j = 1, ..., m, \end{cases}$$

can be studied as a transport system on $(\mathrm{L}^p[0,1])^{2m}$ with system operator

$$\tilde{\mathcal{G}} := \mathrm{diag}\left(\tfrac{d}{ds}\right)_{j=1,\dots,2m} \qquad \text{and domain } D(\tilde{\mathcal{G}}) := \{f \in (\mathrm{W}^{1,p}[0,1])^{2m} : f(1) = \mathbb{B}f(0)\}$$

where $\mathbb{B} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{R})$ (cf. [Klö12, Sects. 2.3 & 2.4]).

(b) The difference operator $\tilde{\mathcal{G}}$ is the generator of a strongly continuous semigroup on $(\mathrm{L}^p[0,1])^{2m}$ by [ABE14, Cor. 25] (use Theorem III.8 or see [Klö10, Thm. 2.6] for $p = 2$).

[2]Here, we additionally introduce the coefficients μ_j to meet the boundary conditions of the system (WE). Further, we set $\Psi^+ = \Psi^- = 0$.

(c) The element $\binom{f}{g} \in D(\mathcal{G}) \times \mathcal{V}$ satisfies the boundary conditions

$$\begin{aligned} g_j(v_i) &= g_l(v_i), && t \geq 0,\ j,l \in \Gamma(v_i),\ i = 1,...,n, && \text{see [Klö12, Equ. (2.3)]}, \\ \Phi_\omega^+ f'(0) &= \Phi_\omega^- f'(1), && t \geq 0, && \text{see [Klö12, Equ. (2.5)]}. \end{aligned}$$

Hence, [Klö12, Thm. 2.12] yields the unique classical solution $u \in \mathrm{C}^2(\mathbb{R}_+, (\mathrm{L}^p[0,1])^m)$ of (*) for given initial value $\binom{f}{g} \in D(\mathcal{G}) \times \mathcal{V}$.

(d) It remains to verify that this classical solution $u \in \mathrm{C}^2(\mathbb{R}_+, (\mathrm{L}^p[0,1])^m)$ of (*) verifies the remaining boundary conditions

$$u_j(t, v_i) = u_l(t, v_i), \qquad t \geq 0,\ j,l \in \Gamma(v_i),\ i = 1,...,n,$$

in (WE). However, *"if the displacement of two adjacent edges in a common vertex is initially equal, it will remain equal in time"* [Klö12, p. 113], which yields the desired boundary conditions.

Once that we established the wellposedness of (HE) and (WE) on $(\mathrm{L}^p[0,1])^m$ using the strategy (a)-(d) proposed above, it is of interest to

- achieve the wellposedness directly using perturbation results for analytic semigroups and strongly continuous cosine families as presented in Chapter IV and Chapter V,
- investigate the wellposedness of heat equations on networks with delay terms in both the dynamics and the boundary condition. Mixed dynamics on networks have been studied in, e.g., [HM13].

Appendix

In this appendix we provide some results concerning spectral properties of operators (and their parts) as well as the invertibility of operator matrices by means of so-called Schur complements. Sections A and B are contained in the forthcoming paper *Spectral theory for structured perturbations of linear operators* which is joint work with Klaus-Jochen Engel (we follow the presentation in [AE15]). In Section C we study parts of generators of strongly continuous cosine families (joint work with Klaus-Jochen Engel).

A. Spectral Theory for Parts of Operators

Lemma A.1 generalizes [EN00, Lem. IV.1.15, Prop. IV.2.17] and connects some spectral properties of an operator T on a Banach space F to those of its part $T|_E$ in a subspace E of F.

Lemma A.1. *Let $T : D(T) \subset F \to F$ be a linear operator on a Banach space F, let E be a Banach space satisfying $D(T) \subseteq E \hookrightarrow F$ and define the part of T in E as*

$$T_1 := T|_E : D(T_1) \subseteq E \to E \quad \textit{with domain} \quad D(T_1) := \{x \in D(T) : Tx \in E\}.$$

Then the following holds.

(i) $\ker(T) = \ker(T_1)$*; in particular, T is injective $\iff$ T_1 is injective.*

(ii) $\mathrm{rg}(T_1) = \mathrm{rg}(T) \cap E$*; in particular, T is surjective $\implies$ T_1 is surjective.*

(iii) $\mathrm{rg}(T)$ *is closed in F* $\implies$ $\mathrm{rg}(T_1)$ *is closed in E.*

(iv) $\mathrm{codim}(\mathrm{rg}(T)) < \infty \implies \mathrm{codim}(\mathrm{rg}(T_1)) < \infty$.

(v) Assume that $E + \mathrm{rg}(T)$ is dense in F. If $\mathrm{rg}(T_1)$ is dense in E, then $\mathrm{rg}(T)$ is dense in F.

(vi) If T is closed, then T_1 is closed. Moreover,

$$\rho(T) \subseteq \rho(T_1) \qquad \textit{and} \qquad R(\lambda, T_1) = R(\lambda, T)|_E \quad \textit{for all } \lambda \in \rho(T).$$

(vii) If $\rho(T) \neq \emptyset$ and $F_1^T := [D(T)] \hookrightarrow E$, then in (ii)–(v) equivalences hold.

Proof. The equalities in (i) and (ii) follow easily from the definition of $D(T_1)$.
To show (iii), take $(y_n)_{n\in\mathbb{N}} \subset \operatorname{rg}(T_1)$ such that $y_n \to y \in E$ as $n \to \infty$. Since $E \hookrightarrow F$ and $\operatorname{rg}(T)$ is closed in F, this implies $y \in \operatorname{rg}(T) \cap E = \operatorname{rg}(T_1)$, i.e., $\operatorname{rg}(T_1)$ is closed in E.
For (iv) assume that $\operatorname{codim}(\operatorname{rg}(T_1)) = \infty$. Then there exists an infinite, linearly independent subset $S \subset E \setminus \operatorname{rg}(T_1)$. Since by (ii), $\operatorname{rg}(T_1) = \operatorname{rg}(T) \cap E$ we conclude $S \subset F \setminus \operatorname{rg}(T)$, i.e., $\operatorname{codim}(\operatorname{rg}(T)) = \infty$.
To show (v) we assume that $\operatorname{rg}(T)$ is not dense in F. Then there exists $0 \neq \psi \in F'$ such that $\psi|_{\operatorname{rg}(T)} = 0$. Let $\varphi := \psi|_E \in E'$. If $\varphi = 0$, then $\psi|_{E+\operatorname{rg}(T)} = 0$ and by the denseness assumption it follows that $\psi = 0$, contradicting the choice of ψ. Hence, $\varphi \neq 0$ and $\varphi|_{\operatorname{rg}(T_1)} = 0$ which implies that $\operatorname{rg}(T_1)$ is not dense in E either.
For (vi), take $(x_n)_{n\in\mathbb{N}} \subset D(T_1)$ such that $x_n \to x \in E$ and $T_1 x_n \to y \in E$ as $n \to \infty$. Since $E \hookrightarrow F$, this implies $x_n \to x$ in F and $Tx_n \to y$ in F as $n \to \infty$. By the closedness of T this gives $x \in D(T)$ and $Tx = y$. From $y \in E$ it follows that $x \in D(T_1)$ and $T_1 x = y$, i.e., T_1 is closed. Now take $\lambda \in \rho(T)$. Then the restriction $R := R(\lambda, T)|_E$ is a closed algebraic inverse of $\lambda - T_1$ defined on all of E and having range in E since $D(T) \subseteq E$. By the closed graph theorem this implies $R \in \mathcal{L}(E)$, i.e., $\lambda \in \rho(T_1)$ and $R = R(\lambda, T_1)$. This shows (i)–(vi).
To verify (vii) we first define the part of T_1 in $F_1^T = [D(T)]$, i.e.,

$$T_2 := T_1|_{F_1^T} : D(T_2) \subseteq F_1^T \to F_1^T \quad \text{with domain} \quad D(T_2) := \{x \in D(T_1) : T_1 x \in F_1^T\}.$$

Then the pair T_2, T_1 satisfies the assumptions made for T_1, T, hence we can repeat the reasoning in (ii)-(v) with T_1, T replaced by T_2, T_1, respectively. Thus, for (v) we need the additional assumption that $F_1^T + \operatorname{rg}(T_1)$ is dense in E. Note that for $\mu \in \rho(T) \subseteq \rho(T_1)$, we always have $E = \operatorname{rg}(\mu - T_1) \subseteq F_1^T + \operatorname{rg} T_1$. Hence, the denseness assumption is automatically satisfied. Moreover, for such μ the operator $\mu - T_1 \in \mathcal{L}(F_1^T, F)$ is an isomorphism which induces a similarity transformation between T_2 and T. This implies that T_2 is surjective/has closed range/has range with finite co-dimension/has dense range, respectively, if and only if T has. Summing up, this shows the equivalences in (ii)–(v) if $\rho(T) \neq \emptyset$. $\square$

Remark A.2. Without the denseness assumption on $E + \operatorname{rg}(T)$ the assertion in Lemma A.1.(v) does not hold. To see this take an operator $S : D(S) \subset X \to X$ with dense range. Then for a Banach space $Y \neq \{0\}$, define $E := X \times \{0\}$, $F := X \times Y$ and the operator $T : D(T) \subseteq F \to F$ by $T\binom{x}{y} := \binom{Sx}{0}$ for $\binom{x}{y} \in D(T) := D(S) \times Y$. Then $T_1 := T|_E$ has dense range in E while $\operatorname{rg}(T) = \operatorname{rg}(S) \times \{0\} \subseteq E$ is not dense in F. Clearly, in this case $E + \mathrm{rg}(T) = E$ is not dense in F.

The following is the main result of Section A.

Corollary A.3. *In the situation of Lemma A.1 the following relations hold.*

(i) $\sigma_p(T_1) = \sigma_p(T)$.

(ii) $\sigma(T_1) \subseteq \sigma(T)$.

(iii) $\sigma_a(T_1) \subseteq \sigma_a(T)$.

(iv) $\sigma_c(T_1) \subseteq \sigma_c(T)$ *if* $E + \mathrm{rg}(T)$ *is dense in* F.

(v) $\sigma_r(T_1) \supseteq \sigma_r(T)$ *if* $E + \mathrm{rg}(T)$ *is dense in* F.

(vi) $\sigma_{ess}(T_1) \subseteq \sigma_{ess}(T)$.

(vii) If $\rho(T) \neq \emptyset$ *and* $F_1^T \hookrightarrow E$, *then in (ii)–(vi) equality holds.*

Proof. All assertions follow easily from the definitions and Lemma A.1 applied to $\lambda - T$ for $\lambda \in \mathbb{C}$ instead of T. For (iv) and (v), note that $E + \mathrm{rg}(\lambda - T)$ is independent of $\lambda \in \mathbb{C}$. $\square$

B. Schur Complements for Operator Matrices.

In Section B we give conditions characterizing various spectral properties of an operator matrix. This leads to the notion of *Schur complement* which, in a certain sense, generalizes the concept of the determinant for scalar matrices to matrices with non-commuting entries. For a more systematic treatment we refer to [Eng95] or [Zha05].

Lemma B.1. *For Banach spaces* E, F, G, H *and linear operators* $P \in \mathcal{L}(E, G)$, $Q \in \mathcal{L}(F, G)$, $R \in \mathcal{L}(E, H)$, $S \in \mathcal{L}(F, H)$ *define the (linear) operator matrix*

$$\mathcal{T} := \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathcal{L}(E \times F, G \times H).$$

Then the following holds.

(i) If $S \in \mathcal{L}(F, H)$ *is invertible, then for* $\Delta_1 := P - QS^{-1}R \in \mathcal{L}(E, G)$ *we have*

$$\mathcal{T} = \begin{pmatrix} \mathrm{Id}_G & QS^{-1} \\ 0 & \mathrm{Id}_H \end{pmatrix} \cdot \begin{pmatrix} \Delta_1 & 0 \\ 0 & S \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_E & 0 \\ S^{-1}R & \mathrm{Id}_F \end{pmatrix}. \tag{B.1}$$

Hence,

$$\begin{aligned} &\mathcal{T} \in \mathcal{L}(E \times F, G \times H) \text{ is injective/surjective/has closed range/} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{has dense range, resp.} \\ \iff\quad &\Delta_1 \in \mathcal{L}(E, G) \qquad\qquad \text{is injective/surjective/has closed range/} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{has dense range, resp.} \end{aligned}$$

In particular, $\mathcal{T}$ *is invertible if and only if* Δ_1 *is invertible and in this case*

$(B.2)$
$$\mathcal{T}^{-1} = \begin{pmatrix} \Delta_1^{-1} & -\Delta_1^{-1} \cdot QS^{-1} \\ -S^{-1}R \cdot \Delta_1^{-1} & S^{-1} + S^{-1}R \cdot \Delta_1^{-1} \cdot QS^{-1} \end{pmatrix} \in \mathcal{L}(G \times H, E \times F).$$

Moreover, $\dim(\ker(\mathcal{T})) = \dim(\ker(\Delta_1))$ *and* $\operatorname{codim}(\operatorname{rg}(\mathcal{T})) = \operatorname{codim}(\operatorname{rg}(\Delta_1))$.

(ii) *If* $P \in \mathcal{L}(E, G)$ *is invertible, then for* $\Delta_2 := S - RP^{-1}Q \in \mathcal{L}(F, H)$ *we have*

$$(B.3) \qquad \mathcal{T} = \begin{pmatrix} \mathrm{Id}_G & 0 \\ RP^{-1} & \mathrm{Id}_H \end{pmatrix} \cdot \begin{pmatrix} P & 0 \\ 0 & \Delta_2 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_E & P^{-1}Q \\ 0 & \mathrm{Id}_F \end{pmatrix}.$$

Hence,

$$\begin{aligned} &\mathcal{T} \in \mathcal{L}(E \times F, G \times H) \text{ is injective/surjective/has closed range/} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{has dense range, resp.} \\ \iff\quad &\Delta_2 \in \mathcal{L}(F, H) \qquad\qquad \text{is injective/surjective/has closed range/} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{has dense range, resp.} \end{aligned}$$

In particular, $\mathcal{T}$ *is invertible if and only if* Δ_2 *is invertible and in this case*

$(B.4)$
$$\mathcal{T}^{-1} = \begin{pmatrix} P^{-1} + P^{-1}Q \cdot \Delta_2^{-1} \cdot RP^{-1} & -P^{-1}Q \cdot \Delta_2^{-1} \\ -\Delta_2^{-1} \cdot RP^{-1} & \Delta_2^{-1} \end{pmatrix} \in \mathcal{L}(G \times H, E \times F).$$

Moreover, $\dim(\ker(\mathcal{T})) = \dim(\ker(\Delta_2))$ *and* $\operatorname{codim}(\operatorname{rg}(\mathcal{T})) = \operatorname{codim}(\operatorname{rg}(\Delta_2))$.

If P *and* S *are both invertible, then the following holds.*

(iii) $\ker(\Delta_1) = P^{-1}Q \ker(\Delta_2)$ *and* $\ker(\Delta_2) = S^{-1}R \ker(\Delta_1)$.

(iv) Δ_1 *is injective/surjective/has closed range/has dense range* $\iff$ Δ_2 *is injective/surjective/has closed range/has dense range, resp.*

(v) $\dim(\ker(\Delta_1)) = \dim(\ker(\Delta_2))$ *and* $\operatorname{codim}(\operatorname{rg}(\Delta_1)) = \operatorname{codim}(\operatorname{rg}(\Delta_2))$.

(vi) Δ_1 *is invertible if and only if* Δ_2 *is invertible and in this case*

$$\begin{aligned}\Delta_1^{-1} &= P^{-1} + P^{-1}Q \cdot \Delta_2^{-1} \cdot RP^{-1} \in \mathcal{L}(G, E),\\ \Delta_2^{-1} &= S^{-1} + S^{-1}R \cdot \Delta_1^{-1} \cdot QS^{-1} \in \mathcal{L}(H, F).\end{aligned}$$

Proof. All assertions in (i) follow as a consequence of the factorization of $\mathcal{T}$ given in $(B.1)$ using the fact that the upper/lower triangular matrices involved are all isomorphisms while the boundedness of the inverses of $\mathcal{T}$, Δ_1 and Δ_2 follows from the closed graph theorem. In fact, the identities

$$\begin{aligned}\begin{pmatrix}\mathrm{Id}_G & -QS^{-1}\\ 0 & \mathrm{Id}_H\end{pmatrix} \cdot \mathcal{T} &= \begin{pmatrix}\Delta_1 & 0\\ 0 & S\end{pmatrix} \cdot \begin{pmatrix}\mathrm{Id}_E & 0\\ S^{-1}R & \mathrm{Id}_F\end{pmatrix}, \quad \text{and}\\ \mathcal{T} \cdot \begin{pmatrix}\mathrm{Id}_E & 0\\ -S^{-1}R & \mathrm{Id}_F\end{pmatrix} &= \begin{pmatrix}\mathrm{Id}_G & QS^{-1}\\ 0 & \mathrm{Id}_H\end{pmatrix} \cdot \begin{pmatrix}\Delta_1 & 0\\ 0 & S\end{pmatrix}\end{aligned}$$

yield that the (co)dimensions of the kernels (and ranges, respectively) coincide since $\binom{x}{y} \in \ker(\mathcal{T})$ if and only if $x \in \ker(\Delta_1)$ and $y = -S^{-1}Rx$.
The assertions in (ii) follow analogously. Here, $\binom{x}{y} \in \ker(\mathcal{T})$ if and only if $y \in \ker(\Delta_2)$ and $x = -P^{-1}Qy$. Thus, (iii) follows immediately.

The assertions in (iv)-(vi) follow from (i) and (ii) since the properties hold for Δ_1 if and only if they hold for $\mathcal{T}$ if and only if they hold for Δ_2. The representations of Δ_1^{-1} and Δ_2^{-1} follow by comparing the diagonal entries of the representations of $\mathcal{T}^{-1}$. $\square$

Remark B.2. The operators $\Delta_1 = P - QS^{-1}R : E \to G$ and $\Delta_2 = S - RP^{-1}Q : F \to H$ appearing above are frequently called *characteristic operator functions* or *Schur complements* of the matrix $\mathcal{T}$, cf. [Nag89, Def. 2.3], [Tre08, Def. 1.6.1, Def. 2.2.12].

The previous result has the following useful application.

Corollary B.3. *Let* E, F *be Banach spaces and* $Q \in \mathcal{L}(F, E)$, $R \in \mathcal{L}(E, F)$. *Then*

$$1 \in \sigma(QR) \Longleftrightarrow 1 \in \sigma(RQ), \qquad 1 \in \sigma_*(QR) \Longleftrightarrow 1 \in \sigma_*(RQ)$$

for all $* \in \{p, a, r, c, ess\}$. *Moreover,* $\ker(\mathrm{Id}_E - QR) = Q\ker(\mathrm{Id}_F - RQ)$ *and* $\ker(\mathrm{Id}_F - RQ) = R\ker(\mathrm{Id}_E - QR)$. *Finally, if* $1 \in \rho(RQ)$ *or, equivalently,* $1 \in$

$\rho(QR)$, *then*

$$(B.5)\qquad \begin{aligned}(\mathrm{Id}_E - QR)^{-1} &= \mathrm{Id}_E + Q(\mathrm{Id}_F - RQ)^{-1}R,\\ (\mathrm{Id}_F - RQ)^{-1} &= \mathrm{Id}_F + R(\mathrm{Id}_E - QR)^{-1}Q.\end{aligned}$$

Proof. In the situation of Lemma B.1 choose $G = E$, $H = F$, $P = \mathrm{Id}_E$ and $S = \mathrm{Id}_F$. Then $\Delta_1 = \mathrm{Id}_E - QR$ and $\Delta_2 = \mathrm{Id}_F - RQ$. Hence, all assertions concerning the spectra follow easily from the characterizations of the corresponding spectral properties of $\mathcal{T}$, Δ_1 and Δ_2 in Lemma B.1.(iii)–(v). Finally, $(B.5)$ follows from Lemma B.1.(vi). □

C. Parts of Generators of Cosine Families

An important class of generators of strongly continuous cosine families is given by squares of group generators. More precisely, assume that $A = W^2$ for the generator $(W, D(W))$ of a C_0-group $(G(t))_{t\in\mathbb{R}}$ on a Banach space X. Then by [ABHN11, Expl. 3.14.15] A generates a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X given by

$$(\mathrm{C.1})\qquad C(t) = \tfrac{1}{2}\Big(G(t) + G(-t)\Big).$$

Moreover, in this case the phase space is $V\times X := [D(W)]\times X$ and if W is invertible, then

$$\begin{aligned}WS(t)x &:= \tfrac{1}{2}W\int_0^t G(s)x\,ds - \tfrac{1}{2}(-W)\int_0^t G(-s)x\,ds\\ &= \tfrac{1}{2}(G(t)x - G(-t)x)\quad \text{for } x\in X.\end{aligned}$$

In particular,

$$(\mathrm{C.2})\qquad S(t) = \tfrac{1}{2}W^{-1}(G(t) - G(-t)).$$

The following is a version of [EN00, II.2.3, Prop.] for strongly continuous cosine families.

Lemma C.1. *Let $(\hat{C}(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family on a Banach space $\hat{X}$ with generator $\hat{A}$ and phase space $\hat{V}\times\hat{X}$. Moreover, assume that $X\subset\hat{X}$ is a closed, $\hat{C}(t)$-invariant subspace of $\hat{X}$. Then*

$$C(t) := \hat{C}(t)|_X \in \mathcal{L}(X)$$

defines a strongly continuous cosine family on X with generator $A = \hat{A}|_X$ and phase space $V \times X$ where $V := \hat{V} \cap X$ is equipped with the norm of $\hat{V}$.

Proof. By Proposition V.1,

$$\hat{\mathcal{A}} := \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{A} & 0 \end{pmatrix}, \quad D(\hat{\mathcal{A}}) := D(\hat{A}) \times \hat{V}$$

generates a C_0-group $(\hat{\mathcal{T}}(t))_{t\in\mathbb{R}}$ on $\hat{\mathcal{X}} := \hat{V} \times \hat{X}$ given by

$$\hat{\mathcal{T}}(t) = \begin{pmatrix} \hat{C}(t) & \hat{S}(t) \\ \hat{A}\hat{S}(t) & \hat{C}(t) \end{pmatrix}.$$

Define $\mathcal{X} := V \times X$, where $V := \hat{V} \cap X$ is equipped with the norm of $\hat{V}$. Take $(v_n)_{n\in\mathbb{N}} \subset V$ such that $v_n \to v \in \hat{V}$ as $n \to \infty$. Then $v \in V$ since $\hat{V} \hookrightarrow \hat{X}$ and $X \subset \hat{X}$ is closed, i.e., V is closed in $\hat{V}$. Hence, $\mathcal{X}$ is a closed subspace of $\hat{\mathcal{X}}$. We show next that $\mathcal{X}$ is $\hat{\mathcal{T}}(t)$-invariant, i.e., for every $t \in \mathbb{R}$ we have

(i) $\hat{C}(t)V \subset V$, (ii) $\hat{S}(t)X \subset V$,

(iii) $\hat{A}\hat{S}(t)V \subset X$, (iv) $\hat{C}(t)X \subset X$.

Here (i) and (iv) follow immediately from $\hat{C}(t)\hat{V} \subset \hat{V}$ (cf. Proposition V.1.(iii)) and the assumption $\hat{C}(t)X \subset X$. Moreover, (ii) follows since $\hat{S}(t)\hat{X} \subset \hat{V}$ and by the closedness of X in $\hat{X}$, i.e., for all $x \in X$ we obtain

$$\hat{S}(t)x = \int_0^t \hat{C}(s)x\,ds \in X \cap \hat{V} = V.$$

Finally, to show (iii) note that for $v \in V = \hat{V} \cap X$ we have $\hat{C}(\bullet)v \in \mathrm{C}^1(\mathbb{R}, \hat{X}) \cap \mathrm{C}(\mathbb{R}, X)$ by Proposition V.1.(iii) and the assumption $\hat{C}(t)X \subset X$. This implies

$$\hat{A}\hat{S}(t)v = \tfrac{d}{dt}\hat{C}(t)v \in X$$

since $X \subset \hat{X}$ is closed. Next we apply [EN00, II.2.3, Prop.] to obtain that

$$\mathcal{T}(t) := \hat{\mathcal{T}}(t)|_{\mathcal{X}} \in \mathcal{L}(\mathcal{X})$$

defines a C_0-group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $\mathcal{X}$ with generator $\mathcal{A} := \hat{\mathcal{A}}|_{\mathcal{X}}$. Take $A := \hat{A}|_X$. Then $D(\hat{A}) \subseteq \hat{V}$ implies

$$D(A) := \left\{x \in D(\hat{A}) \cap X : \hat{A}x \in X\right\} = \left\{x \in D(\hat{A}) \cap V : \hat{A}x \in X\right\} \tag{C.3}$$

and therefore

$$\begin{aligned}\binom{v}{x} \in D(\mathcal{A}) \quad &:\Longleftrightarrow \quad \binom{v}{x} \in D(\hat{\mathcal{A}}) \cap \mathfrak{X} \text{ and } \hat{\mathcal{A}}\binom{v}{x} \in \mathfrak{X} \\ &\Longleftrightarrow \quad \binom{v}{x} \in \Big(D(\hat{A}) \cap V\Big) \times (\hat{V} \cap X) \text{ and } \binom{x}{\hat{A}v} \in V \times X \\ &\Longleftrightarrow \quad \binom{v}{x} \in D(A) \times V\end{aligned}$$

by (C.3). The assertion now follows from Proposition V.1. □

In Sections V.3 and V.4 we use Lemma C.1 to show that the operator Δ_N in (5.21) and Δ_W in (5.25) are generators of strongly continuous cosine families $(C_N(t))_{t\in\mathbb{R}}$ and $(C_W(t))_{t\in\mathbb{R}}$ on $\mathrm{L}^p[0,1]$ and $\mathrm{C}[0,1]$, respectively. Let us make this more precise. For $1 \le p < \infty$ we show that, e.g., $(C_N(t))_{t\in\mathbb{R}} \subset \mathcal{L}(\mathrm{L}^p[0,1])$ is similar to the restriction $(\hat{C}(t)|_X)_{t\in\mathbb{R}}$ on a closed $\hat{C}(t)$-invariant subspace X of a strongly continuous cosine family $(\hat{C}(t))_{t\in\mathbb{R}}$ on a Banach space $\hat{X}$ satisfying

$$\hat{C}(t) = \tfrac{1}{2}\Big(\hat{G}(t) + \hat{G}(-t)\Big) \tag{C.4}$$

for a C_0-group $(\hat{G}(t))_{t\in\mathbb{R}}$ on $\hat{X}$ with generator $\hat{W}$.
We point out that we use the invariance of X under $(\hat{C}(t))_{t\in\mathbb{R}}$ but do *not* assume that X is $\hat{G}(t)$-invariant (which is, in fact, not the case).

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Zusammenfassung in deutscher Sprache

In der vorliegenden Arbeit verwenden wir einen Störungsansatz, um lineare Operatoren und deren Eigenschaften zu untersuchen. Dabei nehmen wir an, dass der Operator G als eine Summe von zwei Operatoren A und P geschrieben werden kann, wobei der Operator A „schöne" Eigenschaften hat, während die Störung P „schlimm" sein kann. Genauer betrachten wir strukturierte Störungen $P = BC$, welche von George Weiss [Wei94a, Sects. 5-7] und Olof Staffans [Sta05, Chap. 4 & 7] im Kontext *linearer Systeme mit geschlossenem Feedback* betrachtet wurden (siehe auch [Sal87]).

Dazu betrachten wir zwei Banachräume X und U und Operatoren

- $A : D(A) \subset X \to X$ mit $\rho(A) \neq \emptyset$,
- $B \in \mathcal{L}(U, X_{-1}^A)$, und
- $C \in \mathcal{L}(Z, U)$,

wobei Z ein Banachraum ist, welcher $X_1^A \hookrightarrow Z \hookrightarrow X$ erfüllt. Dann definieren wir den Operator

$$A_{BC} := (A_{-1} + BC)|_X, \qquad D(A_{BC}) := \{x \in Z : A_{-1}x + BCx \in X\}$$

und beschäftigen uns mit folgenden Aufgaben.

1) Identifiziere eine Klasse von Operatoren G, die als A_{BC} geschrieben werden können;

2) Charakterisiere das Spektrum des Operators $G = A_{BC}$ mittels A, B und C;

3) Finde Bedingungen an die Operatoren A, B und C, sodass $G = A_{BC}$ eine stark stetige Halbgruppe $(T_{BC}(t))_{t\geq 0}$ erzeugt;

4) Finde Bedingungen an die Operatoren A, B und C, sodass $G = A_{BC}$ eine stark stetige Kosinusfamilie $(C_{BC}(t))_{t\in\mathbb{R}}$ erzeugt;

5) Beschreibe die Asymptotik der stark stetigen Halbgruppe $(T_{BC}(t))_{t\geq 0}$ mittels A, B und C.

In Kapitel I betrachten wir zunächst unbeschränkte Randstörungen von (Differential)Operatoren. Dies verallgemeinert den Zugang von Günther Greiner in [Gre87]. Die Analyse der abstrakten Situation findet man in [ABE14, Sect. 4.3] (vgl. auch [Bom15, Chap. 3]) und den Manuskripten [ABE15, Sect. 3.1] (eine gemeinsame Arbeit mit Miriam Bombieri und Klaus-Jochen Engel) und [AE15, Sect. 3.1] (gemeinsam mit Klaus-Jochen Engel).

In Kapitel II wird das (Fein-)Spektrum $\sigma(G)$ und $\sigma_*(G)$, $* \in \{p, ap, c, r, ess\}$ des Operators $G = A_{BC}$ mittels der Operatoren A, B und C charakterisiert. Die Resultate werden im Artikel [AE15] (gemeinsam mit Klaus-Jochen Engel) veröffentlicht.

In Kapitel III wird das Weiss-Staffans Störungsresultat präsentiert (siehe Theorem III.8). Die benötigten *Zulässigkeitsbedingungen* werden eingeführt und die Klasse der *Weiss-Staffans Störungen* definiert (siehe Definition III.3). Das Hauptresultat ist in [ABE14] veröffentlicht (vgl. auch [Bom15, Chap. 1]) und verallgemeinert Störungsresultate u.a. von

(i) W. Desch, I. Lasiecka und W. Schappacher in [DLS85] und [DS89] (siehe Theorem III.1),

(ii) I. Miyadera und J. Voigt in [Miy66] und [Voi77] (siehe Theorem III.2), und

(iii) G. Greiner in [Gre87] (siehe Kapitel I und Korollar III.13).

In Kapitel IV wird das Weiss-Staffans Störungsresultat auf Generatoren *analytischer Halbgruppen* angewandt, indem die Zulässigkeitsbedingungen durch Bedingungen an das Bild und den Definitionsbereich der Operatoren B und C ersetzt werden. Dieses Störungsresultat (siehe Theorem IV.1) ist in der Arbeit [ABE15] (gemeinsam mit Miriam Bombieri und Klaus-Jochen Engel) und [Bom15, Chap. 4] enthalten.

In Kapitel V werden Störungen von Generatoren von *stark stetigen Kosinusfamilien* untersucht. Das Störungsresultat (siehe Theorem V.3) wird auf abstrakte Cauchyprobleme zweiter Ordnung mit zweiter Ableitung und gestörten Neumann- und Wentzell-Randbedingungen angewandt. Diese Resultate sind eine gemeinsame Arbeit mit Klaus-Jochen Engel.

In Kapitel VI wird das asymptotische Verhalten der gestörten Halbgruppe untersucht. Unter Bedingungen an die Operatoren B und C bleibt das asymptotische Verhalten der ungestörten Halbgruppe erhalten. Diese Resultate sind in meiner Arbeit [Adl15] enthalten.

In Kapitel VII wird ein Zugang zur Wohlgestelltheit von Wärmeleitungs- und Wellengleichungen auf Netzwerken skizziert.

Im Appendix werden das (Fein-)Spektrum des Teils eines Operators, die Invertierbarkeit von Operatormatrizen mittels Schurkomplementen (die Resultate sind in der gemeinsamen Arbeit [AE15] mit Klaus-Jochen Engel enthalten) und die Generatoreigenschaft von Einschränkungen von Generatoren von Kosinusfamilie untersucht.