The Adiabatic Limit of the Connection Laplacian with Applications to Quantum Waveguides

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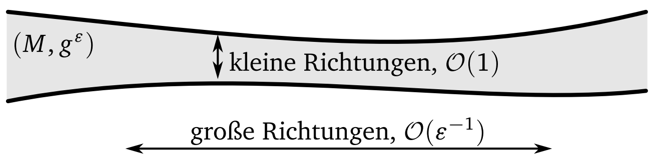
Zusammenfassung

Im Rahmen dieser Dissertation werden wir uns mit Schrödinger-Operatoren von der Gestalt

$$H^{\mathcal{E}} = -\Delta^{\mathcal{E}}_{g^\varepsilon} + V^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}$$

auf einem hermiteschen Vektorbündel (mit Totalraum) $\mathcal{E}$ über einer ε-dünnen Basismannigfaltigkeit M beschäftigen. Hierbei wird die kinetische Energie durch den Zusammenhangs-Laplace-Operator $-\Delta^{\mathcal{E}}_{g^\varepsilon}$ bezüglich eines metrischen Zusammenhangs $\nabla^{\mathcal{E}}$ auf $\mathcal{E}$ mit Dirichlet-Randbedingungen beschrieben. Darüber hinaus repräsentiert $V^{\mathcal{E}}$ ein $\mathrm{End}(\mathcal{E})$-wertiges Potential und der Differentialoperator $\varepsilon H_1^{\mathcal{E}}$ eine kleine Störung. Die Eigenschaft von M ε-dünn zu sein bedeutet intuitiv, dass die Größenordnung einiger (vertikaler) Richtungen mit einem kleinen Faktor $\varepsilon \ll 1$ gegenüber den übrigen (horizontalen) Richtungen skaliert. Wir hingegen werden den gleichwertigen Standpunkt einnehmen, dass wir die vertikale Längenskala festhalten und die horizontalen Richtungen mit dem Faktor ε^{-1} groß skalieren.



Wir werden dies mittels einer riemannschen Metrik $g = g_{\mathsf{H}} + g_{\mathsf{V}}$ auf M und einer geeigneten Reskalierung

$$g^\varepsilon = \varepsilon^{-2} g_{\mathsf{H}} + g_{\mathsf{V}} = \varepsilon^{-2}(g_{\mathsf{H}} + \varepsilon^2 g_{\mathsf{V}})$$

umsetzen, was schließlich zu einer Aufspaltung

$$-\Delta^{\mathcal{E}}_{g^\varepsilon} = -\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}} - \Delta^{\mathcal{E}}_{\mathsf{V}} \tag{1}$$

des zugehörigen Laplace-Operators führt. Die Untersuchung solcher skalierten Metriken wird häufig als der adiabatische Limes bezeichnet. Wir werden eine Komplexitätsreduktion des Operators $H^{\mathcal{E}}$ durchführen, indem wir ausnutzen, dass der Einfluss der vergleichsweise kleinen vertikalen Richtungen immer weiter vernachlässigt werden kann. Genauer gesagt werden wir es uns zum Ziel setzen, einen effektiven Operator $H^{\mathcal{P}}_{\mathrm{eff}}$ abzuleiten, welcher lediglich auf einem Vektorbündel $\mathcal{P}$ über den horizontalen Richtungen wirkt und wesentliche Eigenschaften von $H^{\mathcal{E}}$ wie die erzeugte Dynamik oder das Spektrum approximiert.

Als Hauptanwendung für diese Fragestellung dient uns die geometrische Situation, bei welcher M von einer Familie von ε-dünnen Tubenumgebungen $\mathcal{T}^\varepsilon$ um eine Untermannigfaltigkeit eines höherdimensionalen euklidischen Raums herrührt. Die Untersuchung des Laplace-Beltrami-Operators (d.h. des Zusammenhangs-Laplace-Operators auf dem trivialen Linienbündel $\mathcal{T}^\varepsilon \times \mathbb{C}$ bezüglich des flachen Zusammenhangs d) mit Dirichlet-Randbedingungen entspricht der quantenmechanischen Bewegung von ungeladenen, spinlosen Teilchen, welche innerhalb der Tube $\mathcal{T}^\varepsilon$ lokalisiert sind. Aus diesem Grund nennt man solche Tuben in der Literatur auch Quantenwellenleiter. Die Erweiterung auf Zusammenhänge von der Form $\mathrm{d} + \mathrm{i}\mathcal{A}$ erlaubt es uns geladene Quantenteilchen zu betrachten, die an ein externes magnetisches Potential $\mathcal{A}$ koppeln. Zusätzlich ermöglicht unser geometrischer Rahmen eine Verallgemeinerung zu Spin behafteten, geladenen Teilchen, welche an ein möglicherweise nicht-abelsches Eichfeld koppeln.

Die Betrachtung einer dünnen Mannigfaltigkeit mit wohl separierten Längenskalen (große horizontale Richtungen und kleine vertikale Richtungen) kann tatsächlich als ein adiabatisches Problem aufgefasst werden. Diese Skalentrennung bedeutet, dass sich zu den vertikalen Richtungen gehörende Eigenschaften sehr langsam entlang der horizontalen Richtungen verändern und somit in einer festen Konfiguration verharren. In der

Tat erinnert die Struktur (1) des Laplace-Operators stark an das bekannteste Beispiel des zuvor genannten Prinzips: Bei der Born-Oppenheimer-Näherung in der Moleküldynamik wird die Annahme ausgenutzt, dass die leichten (schnellen, „vertikalen“) Elektronen ihren schweren (langsamen, „horizontalen“) Kernen adiabatisch folgen, d.h. dass sich der Eigenzustand der Elektronen augenblicklich der momentanen Position der Kerne anpasst. Daraus folgt, dass die Dynamik der Kerne durch eine effektive Gleichung auf ihrem eigenen Konfigurationsraum von niedrigerer Dimension geregelt wird, wobei der Einfluss der Elektronen durch ein effektives elektronisches Potential zum Ausdruck kommt. Dieser Reduktionsprozess (das „Einfrieren der elektronischen Freiheitsgrade“) nennt man adiabatisches Entkoppeln.

Aufbau der Arbeit und Übersicht der Ergebnisse

In Kapitel 2 werden wir ausführlich die Geometrie eines hermiteschen Vektorbündels $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ über einer ε-dünnen Mannigfaltigkeit (M, g^{ε}) beleuchten. Dabei werden wir M selbst durch ein Faserbündel $M \xrightarrow{\pi_M} B$ mit kompakten Fasern $M_x = \pi_M^{-1}(x)$, $x \in B$, modellieren. Es wird sich herausstellen, dass die Verknüpfung der beteiligten Bündel als ein Faserbündel $\mathcal{E}$ über B mit Projektion $\Pi_{\mathcal{E}} = \pi_M \circ \pi_{\mathcal{E}}$ betrachtet werden kann, wobei die Fasern $\mathcal{E}_x = \Pi_{\mathcal{E}}^{-1}(x)$ hermitesche Vektorbündel über M_x für alle $x \in B$ sind (siehe Proposition 2.1). Die Aufspaltung (1) legt es nahe, den Schrödinger-Operator $H^{\mathcal{E}}$ gemäß

$$H^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + H^{\mathcal{F}}$$

auf $\mathcal{H} = L^2(\mathcal{E})$ aufzuteilen, wobei $H^{\mathcal{F}}$ den ε-unabhängigen, $\Pi_{\mathcal{E}}$-faserweisen, selbstadjungierten Operator

$$H^{\mathcal{F}}(x) = -\Delta_{g_{M_x}}^{\mathcal{E}_x} + V^{\mathcal{E}}\big|_{M_x}$$

mit Dirichlet-Randbedingungen bezeichnet. Da die Fasern M_x kompakt sind, besteht das Spektrum von $H^{\mathcal{F}}(x)$ lediglich aus Eigenwerten mit endlicher Vielfachheit. Deshalb werden wir uns mit Eigenbändern $\lambda : B \to \mathbb{R}$

befassen, für die $\lambda(x) \in \sigma(H^{\mathcal{F}}(x))$ und die zugehörige faserweise spektrale Projektion $P_0(x)$ der Gleichung $H^{\mathcal{F}}(x)P_0(x) = \lambda(x)P_0(x)$ für alle $x \in B$ genügen. Das Bild von $P_0(x)$ beinhaltet gerade die $\lambda(x)$-Eigenschnitte und spannt einen faserweisen, endlich dimensionalen Unterraum von $L^2(\mathcal{E}_x)$ auf. Diese Unterräume können zu einem Vektorbündel $\mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} B$ von endlichem Rang zusammengefügt werden, sofern das zugrunde liegende Eigenband λ vom Rest des Spektrums durch eine gleichmäßige Lücke getrennt ist.

Ein natürlicher Kandidat zur Approximation von $H^{\mathcal{E}}$ ist der adiabatische Operator $H_{\mathrm{a}}^{\mathcal{P}} = P_0 H^{\mathcal{E}} P_0$ auf $L^2(\mathcal{P})$. Mit der Maßgabe, dass $[H^{\mathcal{E}}, P_0] = [-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0] = \mathcal{O}(\varepsilon)$ in einem geeigneten Sinne gilt, ist der adiabatische Operator ε-nahe am anfänglichen Operator auf dem adiabatischen Unterraum $P_0\mathcal{H}$, d.h.

$$\begin{aligned}
\left(H^{\mathcal{E}} - H_{\mathrm{a}}^{\mathcal{P}}\right)P_0 &= \left(-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\right)P_0 - P_0\left(-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\right)P_0 \\
&= \underbrace{\left[-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0\right]}_{=\mathcal{O}(\varepsilon)} P_0 \\
&= \mathcal{O}(\varepsilon).
\end{aligned}$$

Dies lässt sich auf eine näherungsweise Invarianz des Unterraums $P_0\mathcal{H}$ unter der Wirkung von $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ für endliche Zeiten $t = \mathcal{O}(1)$ übertragen. Sobald man jedoch eine Invarianz für größere Zeiten beweisen will, muss man die spektrale Projektion P_0 durch eine superadiabatische Projektion $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$ ersetzen. Deren Konstruktion wird in Kapitel 3 durchgeführt. Wir werden dabei in Proposition 3.17 zeigen, dass für alle $n \in \mathbb{N}$ eine orthogonale Projektion P_ε existiert, sodass $[H^{\mathcal{E}}, P_\varepsilon] = \mathcal{O}(\varepsilon^{n+1})$ in $\mathcal{L}(\mathcal{H})$ für beschränkte Energien gilt.

In Kapitel 4 werden wir uns der Aufstellung der (abstrakten) Hauptergebnisse dieser Arbeit widmen. Hierbei werden wir stets

(i) angemessene Beschränktheitseigenschaften für die Geometrie der beteiligten Bündel $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ und $M \xrightarrow{\pi_M} B$ (siehe Bedingung 2.2),

(ii) angemessene Beschränktheitseigenschaften für die Störung $\varepsilon H_1^{\mathcal{E}}$

und das Potential $V^{\mathcal{E}}$ (siehe Bedingung 2.16)

(iii) und eine gleichmäßige spektrale Lücke für das Eigenband λ (siehe Bedingung 2.26)

annehmen. Die superadiabatische Projektion P_ε dient als Ausgangspunkt für den letztendlichen effektiven Operator. Man erhält ihn, indem man $H^{\mathcal{E}}$ zunächst auf den superadiabatischen Unterraum $P_\varepsilon \mathcal{H}$ einschränkt und dann mittels eines unitären Operators U_ε, welcher P_ε und P_0 miteinander verflechtet (siehe Lemma 4.2), zurück auf $L^2(\mathcal{P})$ abbildet:

Theorem 1 (Theorem 4.3) *Es gelten die drei oben genannten Bedingungen. Dann existieren für alle $n \in \mathbb{N}$ und $\Lambda > 0$ ein effektiver Operator $H_{\mathrm{eff}}^{\mathcal{P}} = U_\varepsilon^\dagger P_\varepsilon H^{\mathcal{E}} P_\varepsilon U_\varepsilon$ auf $L^2(\mathcal{P})$ sowie Konstanten $C > 0$ und $\varepsilon_0 > 0$, sodass*

$$\left\| \left(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}} t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}^{\mathcal{P}} t} U_\varepsilon^\dagger \right) P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C \varepsilon^{n+1} |t|$$

für alle $0 < \varepsilon < \varepsilon_0$ erfüllt ist.

Es sei darauf hingewiesen, dass der ursprüngliche Operator $H^{\mathcal{E}}$ auf L^2-Schnitten von $\mathcal{E}$ (einem Vektorbündel über M von endlichem Rang) wirkt, während $H_{\mathrm{eff}}^{\mathcal{P}}$ auf L^2-Schnitten von $\mathcal{P}$ (einem Vektorbündel über B von endlichem Rang) operiert. Die Approximation der von $H^{\mathcal{E}}$ erzeugten Dynamik durch die von $H_{\mathrm{eff}}^{\mathcal{P}}$ generierte Dynamik markiert somit einen Prozess der Dimensionsreduktion. Neben den jeweiligen Dynamiken besteht auch ein gewisser Zusammenhang zwischen den Spektren:

Theorem 2 (Theorem 4.4) *Sei $\delta > 0$ beliebig und es gelten die Voraussetzungen von Theorem 1. Dann gibt es Konstanten $C > 0$ und $\varepsilon_0 > 0$, sodass $\mu \in \sigma(H_{\mathrm{eff}}^{\mathcal{P}})$ mit $\mu \leqslant \Lambda - \delta$*

$$\mathrm{dist}\big(\mu, \sigma(H^{\mathcal{E}})\big) \leqslant C \varepsilon^{n+1}$$

für alle $0 < \varepsilon < \varepsilon_0$ impliziert.

Offensichtlich ist die umgekehrte Richtung – d.h. die Approximation von $\sigma(H^{\mathcal{P}}_{\mathrm{eff}})$ durch $\sigma(H^{\mathcal{E}})$ – genau dann möglich, wenn lediglich das Grundzustandsband $\lambda_0(x) = \min(\sigma(H^{\mathcal{F}}(x)))$ den signifikanten Beitrag zum Erwartungswert von $H^{\mathcal{E}}$ liefert. Mit anderen Worten: Die einzige Möglichkeit, um eine gegenseitige Näherung der beiden betreffenden Operatoren zu erreichen, ist die Einschränkung von $\sigma(H^{\mathcal{E}})$ auf den zu λ_0 gehörenden spektralen Unterraum. Dies kann dadurch erzielt werden, dass man Energien oberhalb von $\Lambda_1 = \inf_{x\in B}(\sigma(H^{\mathcal{F}}(x))\setminus\lambda_0(x))$ mittels einer geeigneten Abschneidefunktion $\chi \in C^{\infty}_{\mathrm{b}}((-\infty,\Lambda_1))$ unterdrückt:

Theorem 3 (Theorem 4.5) *Es gelten die Voraussetzungen von Theorem 1 für das Grundzustandsband. Sei darüber hinaus $-\varepsilon^2\Delta^{\mathcal{E}}_{\mathrm{H}} + \varepsilon H^{\mathcal{E}}_1$ nach unten beschränkt durch $-C\varepsilon\mathbf{1}_{\mathcal{H}}$ für eine Konstante $C > 0$. Dann ist $H^{\mathcal{P}}_{\mathrm{eff}}\chi(H^{\mathcal{P}}_{\mathrm{eff}})$ unitär äquivalent zu $H^{\mathcal{E}}\chi(H^{\mathcal{E}})$ bis auf Fehler der Ordnung ε^{n+1} in $\mathcal{L}(\mathcal{H})$ für $\varepsilon > 0$ klein genug.*

Hieraufhin werden wir die Diskussion über den effektiven Operator mit einer genaueren Inspizierung seiner semiklassischen Entwicklung beenden. Hierbei werden wir die konkreten Entwicklungen $U_\varepsilon = \mathbf{1}_{\mathcal{H}} + \mathcal{O}(\varepsilon)$ und $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$ verwenden und im Wesentlichen

$$H^{\mathcal{P}}_{\mathrm{eff}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}} + \mathcal{O}(\varepsilon^3)$$

mit dem $\mathcal{O}(\varepsilon^2)$-term $\mathcal{M}^{\mathcal{P}}$ als erste superadiabatische Korrektur erhalten (vergleiche Proposition 4.10). Im letzten Abschnitt dieses Kapitels werden wir uns mit der Untersuchung von niedrigen Energien der Ordnung ε^α, $\alpha \in (0,2]$, über dem unteren Rand von $\sigma(H^{\mathcal{E}})$ befassen. In diesem Fall werden wir sehen, dass der (bedeutend simplere) adiabatische Operator eine genauere Näherung liefert, als man von $H^{\mathcal{P}}_{\mathrm{eff}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{O}(\varepsilon^2)$ zunächst erwartet hätte. Genauer gesagt werden wir in Proposition 4.14 zeigen, dass sich die unteren Teile von $\sigma(H^{\mathcal{E}})$ und $\sigma(H^{\mathcal{P}}_{\mathrm{a}})$ gegenseitig bis auf Fehler der Ordnung $\varepsilon^{2+\alpha/2}$ approximieren. Besteht der untere Teil von $\sigma(H^{\mathcal{P}}_{\mathrm{a}})$ – beziehungsweise der untere Teil vom Spektrum des positiven Operators $H^{\mathcal{P}}_{\mathrm{a}} - \Lambda_0\mathbf{1}_{L^2(\mathcal{P})}$ für $\Lambda_0 = \inf_{x\in B}(\lambda_0(x))$ – lediglich aus Eigenwerten, so erhält man eine noch bessere Näherung:

Theorem 4 (Theorem 4.15) *Es gelten die Voraussetzungen von Theorem 1 für das Grundzustandsband und die Störung $H_1^{\mathcal{E}}$ erfülle zusätzlich Bedingung 4.11. Falls nun für ein $\alpha \in (0,2]$ positive Konstanten C, δ und ε_0 existieren, sodass $\sigma(H_{\mathrm{a}}^{\mathcal{P}} - \Lambda_0 \mathbf{1}_{L^2(\mathcal{P})}) \cap (-\infty, (C+\delta)\varepsilon^{\alpha})$ aus $K+1$ Eigenwerten $\nu_0 \leqslant \cdots \leqslant \nu_K$ unter seinem wesentlichen Spektrum für alle $0 < \varepsilon < \varepsilon_0$ besteht, so gilt:*

(i) *$H^{\mathcal{E}}$ hat $K+1$ Eigenwerte $\upsilon_0 \leqslant \cdots \leqslant \upsilon_K$ unter seinem wesentlichen Spektrum und $|\upsilon_j - \nu_j| = \mathcal{O}(\varepsilon^{2+\alpha})$ für alle $j \in 0,\dots,K$.*

(ii) *Falls zusätzlich ein $\nu \in \{\nu_0,\dots,\nu_K\}$ einfach ist und vom Rest von $\sigma(H_{\mathrm{a}}^{\mathcal{P}} - \Lambda_0 \mathbf{1}_{L^2(\mathcal{P})})$ mindestens durch $C_\nu \varepsilon^{\alpha}$ für ein $C_\nu > 0$ getrennt ist, so ist der zugehörige Eigenwert υ von $H^{\mathcal{E}}$ ebenfalls einfach und es gibt eine Konstante $C_\upsilon > 0$, sodass $\operatorname{dist}(\upsilon, \sigma(H^{\mathcal{E}} \backslash \{\upsilon\})) \geqslant C_\upsilon \varepsilon^{\alpha}$ gilt.*

Wir werden diese Resultate in Kapitel 5 auf den geometrischen Rahmen von verallgemeinerten Quantenwellenleiter anwenden. Letztere werden durch eine Familie von ε-dünnen Tubenumgebungen $\mathcal{T}^{\varepsilon}$ um eine glatt eingebettete, b-dimensionale Untermannigfaltigkeit $B \hookrightarrow (\mathbb{R}^{b+f}, \delta^{b+f})$ modelliert. Die Untersuchung des Dirichlet-Laplace-Operators $-\varepsilon^2 \Delta_{\delta^{b+f}}^{\varepsilon^{-\beta}\mathcal{A}}$ auf $\mathcal{T}^{\varepsilon} \times \mathbb{C}^N$ bezüglich eines Zusammenhangs $\nabla^{\varepsilon^{-\beta}\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon^{-\beta}\mathcal{A}$ entspricht einer Eichtheorie, bei welcher geladene, nichtrelativistische Quantenteilchen (charakterisiert durch eine lokalisierte Wellenfunktion $\psi : \mathbb{R} \to L^2(\mathcal{T}^{\varepsilon} \times \mathbb{C}^N)$, welche die zugehörige Schrödinger-Gleichung löst) an ein möglicherweise nicht-abelsches, $\mathbb{C}^{N\times N}_{\mathrm{Herm}}$-wertiges Eichfeld $\mathcal{A}$ der Stärke $\varepsilon^{-\beta}$, $\beta \in \{0,1\}$, koppeln. Wir werden zunächst einen Diffeomorphismus konstruieren, um die ε-dünnen Tuben $(\mathcal{T}^{\varepsilon}, \delta^{b+f})$ isometrisch auf eine ε-unabhängige Mannigfaltigkeit (M, G^{ε}) abzubilden, welche die zusätzliche Struktur eines Faserbündels $M \xrightarrow{\pi_M} B$ mit einer gestörten riemannschen Metrik $G^{\varepsilon} = g^{\varepsilon} + \mathcal{O}(\varepsilon)$ aufweist. Ferner ist das induzierte Eichfeld $\mathcal{A}_{\varepsilon} = \mathcal{A}_0 + \mathcal{O}(\varepsilon)$ ebenfalls eine zulässige Störung. Wir werden somit folgern, dass $-\varepsilon^2 \Delta_{\delta^{b+f}}^{\varepsilon^{-\beta}\mathcal{A}}$ unitär äquivalent zum Operator

$$H^{\mathcal{E}} = -\Delta_{G^{\varepsilon}}^{\varepsilon^{-\beta}\mathcal{A}_{\varepsilon}}$$

ist.

Schwache Eichfelder ($\beta = 0$, $\nabla^{\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}$)

Wir werden die Ergebnisse zu den niedrigen Energien für verallgemeinerte Wellenleiter in der Anwesenheit von schwachen $\mathbb{C}^{N\times N}$-wertigen Eichfeldern aufgreifen:

Theorem 5 (Theorem 5.12) *Es erzeuge $H^{\mathrm{w}}_{\mathrm{tube}} = -\varepsilon^2\Delta^{\mathcal{A}}_{\delta^{b+f}} + V$ die Dynamik eines nichtrelativistischen Quantenteilchens mit Spin N, welches in einem Quantenwellenleiter lokalisiert ist und an ein schwaches $\mathbb{C}^{N\times N}$-wertiges Eichfeld $\mathcal{A}$ koppelt sowie unter dem Einfluss eines $\mathbb{C}^{N\times N}$-wertigen Potentials V steht. Sind dann die Voraussetzungen von Theorem 4 für das Grundzustandsband für ein $\alpha \in (0,2]$ erfüllt und ist $\nu^\varepsilon < C\varepsilon^\alpha$ ein Eigenwert von $H^{\mathrm{w}}_{\mathrm{QWG}} = H^{\mathcal{P}}_{\mathrm{a}} - \Lambda_0\mathbf{1}_{L^2(\mathcal{P})}$, so existiert ein Eigenwert υ^ε von $H^{\mathrm{w}}_{\mathrm{tube}}$ unter seinem wesentlichen Spektrum mit der asymptotischen Entwicklung*

$$\upsilon^\varepsilon = \Lambda_0 + \nu^\varepsilon + \mathcal{O}\big(\varepsilon^{2+\alpha}\big).$$

Anschließend werden wir die auftretenden Operatoren $H^{\mathrm{w}}_{\mathrm{QWG}}$ für ein spinloses Teilchen in Quantenröhren um eine unendlich ausgedehnte Kurve $c : \mathbb{R} \to \mathbb{R}^3$ für folgende zwei geometrische Konfigurationen exemplarisch ausrechnen:

- die Röhrenquerschnitte sind zentrierte, sich um die Kurve drehende Ellipsen mit variierenden Halbachsen (Beispiel für einen massiven Wellenleiter), siehe Korollar 5.15,

- die Röhrenquerschnitte sind zentrierte Kreise mit variierendem Radius entlang der Kurve (Beispiel für einen hohlen Wellenleiter), siehe Korollar 5.18.

Diese Rechnungen können einfach auf andere geometrische Situationen übertragen werden und stellen daher eine große Verallgemeinerung der vorhandenen Ergebnisse bereit. Insbesondere können die Resultate für massive, sich drehende Röhren im $\mathbb{R}^3$ mit festem Querschnitt direkt reproduziert werden.

Starke Eichfelder ($\beta = 1$, $\nabla^{\varepsilon^{-1}\mathcal{A}} = \mathbf{d1}_{\mathbb{C}^N} + \mathbf{i}\varepsilon^{-1}\mathcal{A}$)
Wir werden schließlich noch den Fall von starken abelschen Eichfeldern der Form $\varepsilon^{-1}\mathcal{A}\mathbf{1}_{\mathbb{C}^N}$ für verallgemeinerte Quantenwellenleiter untersuchen, wobei $\mathcal{A}$ wieder einem reellwertigen magnetischen Potential entspricht:

Theorem 6 (Theorem 5.21) *Es erzeuge $H^{\mathrm{s}}_{\mathrm{tube}} = -\varepsilon^2\Delta^{\varepsilon^{-1}\mathcal{A}}_{\delta^{b+f}}\mathbf{1}_{\mathbb{C}^N} + V$ die Dynamik eines nichtrelativistischen Quantenteilchens mit Spin N, welches in einem Quantenwellenleiter lokalisiert ist und an ein starkes magnetisches Potential $\varepsilon^{-1}\mathcal{A}$ koppelt sowie unter dem Einfluss eines $\mathbb{C}^{N\times N}$-wertigen Potentials V steht. Sind dann die Voraussetzungen von Theorem 3 für das Grundzustandsband erfüllt und ist $\chi \in C^\infty_{\mathrm{b}}((-\infty, \Lambda_1))$ eine geeignete Abschneidefunktion, so ist $\chi(H^{\mathcal{P}}_{\mathrm{eff}})\, H^{\mathrm{s}}_{\mathrm{QWG}}\, \chi(H^{\mathcal{P}}_{\mathrm{eff}})$ für $H^{\mathrm{s}}_{\mathrm{QWG}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}}$ unitär äquivalent zu $\chi(H^{\mathrm{s}}_{\mathrm{tube}})\, H^{\mathrm{s}}_{\mathrm{tube}}\, \chi(H^{\mathrm{s}}_{\mathrm{tube}})$ bis auf Fehler der Ordnung ε^3 in $\mathcal{L}(\mathcal{H})$ für $\varepsilon > 0$ klein genug.*

Wir werden auch hier die reduzierten Hamilton-Operatoren $H^{\mathrm{s}}_{\mathrm{QWG}}$ für die beiden beispielhaften Geometrien eines massiven Wellenleiters (vergleiche Korollar 5.23) und eines hohlen Wellenleiters (vergleiche Korollar 5.18) bestimmen. Während die Resultate für den massiven Fall wiederum mit den bereits bekannten Ergebnissen in Verbindung gebracht werden können, werden wir für $H^{\mathrm{s}}_{\mathrm{QWG}}$ im hohlen Fall einen vollständig neuen Ausdruck erhalten.

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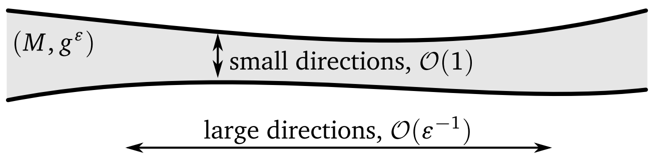
Chapter 1

Introduction

In this thesis, we will analyse Schrödinger-type operators of the form

$$H^{\mathcal{E}} = -\Delta^{\mathcal{E}}_{g^\varepsilon} + V^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} \tag{1.1}$$

on a Hermitian vector bundle (with total space) $\mathcal{E}$ over an ε-thin base manifold M, where the kinetic energy operator $-\Delta^{\mathcal{E}}_{g^\varepsilon}$ is the connection Laplacian associated with some metric connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ with Dirichlet boundary conditions, $V^{\mathcal{E}}$ stands for an $\mathrm{End}(\mathcal{E})$-valued potential, and the differential operator $\varepsilon H_1^{\mathcal{E}}$ denotes a small perturbation. The aforementioned ε-thinness of M intuitively means that the magnitude of some compact (vertical) directions scales by a small factor $\varepsilon \ll 1$ compared to the remaining (horizontal) directions. We will adopt the equivalent approach of keeping a fixed vertical length scale and scaling the horizontal directions by a factor ε^{-1} instead.



This can be implemented by introducing a Riemannian metric $g = g_{\mathsf{H}} + g_{\mathsf{V}}$ on M and an appropriate rescaling

$$g^\varepsilon = \varepsilon^{-2} g_{\mathsf{H}} + g_{\mathsf{V}} = \varepsilon^{-2}(g_{\mathsf{H}} + \varepsilon^2 g_{\mathsf{V}}),$$

which consequently leads to an adapted splitting

$$-\Delta_{g^\varepsilon}^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} - \Delta_{\mathsf{V}}^{\mathcal{E}} \tag{1.2}$$

of the associated Laplacian. The examination of such rescaled metrics is often referred to as the adiabatic limit. We will reduce the complexity of the operator $H^{\mathcal{E}}$, taking advantage of the increasingly negligible influence of the small vertical directions. More precisely, we will derive effective operators $H_{\mathrm{eff}}^{\mathcal{P}}$, acting on a vector bundle $\mathcal{P}$ over the large horizontal directions alone, which approximate essential features of $H^{\mathcal{E}}$ such as the generated dynamics or the spectrum.

Our main application, and therefore motivation, is the geometric framework where M arises from a family of ε-thin tubular neighbourhoods $\mathcal{T}^\varepsilon$ around a submanifold of some higher dimensional ambient Euclidean space. Physically speaking, the study of the Laplace-Beltrami operator (which is the connection Laplacian on the trivial line bundle $\mathcal{T}^\varepsilon \times \mathbb{C}$ associated with the flat connection d) with Dirichlet boundary conditions in such tubes corresponds to the quantum mechanical motion of uncharged, spinless particles that are localised within the tube $\mathcal{T}^\varepsilon$. That is why these tubes are often referred to as quantum waveguides in the literature. Such waveguides have been studied for more than twenty-five years, ranging from strips around infinite curves in $\mathbb{R}^2$ [EŠ89] to so-called generalised quantum waveguides [HLT15], where $\mathbb{R}^2$ is replaced by $\mathbb{R}^{b+f}$ for $b, f \in \mathbb{N}$ and the role of the curve is taken by an embedded, complete submanifold of dimension b and codimension f. For example, the latter submanifold could represent a vibrational equilibrium configuration of nuclei which form a molecule, whereas the surrounding tubular neighbourhood models the region in which the nuclei carry out their small vibrations. The extension to metric connections of the form $\mathrm{d} + \mathrm{i}\mathcal{A}$ enables us to consider charged quantum particles coupled to some external magnetic potential $\mathcal{A}$. Moreover, our geometric framework allows for the generalisation to charged particles that carry a spin and couple to a possibly non-Abelian gauge field.

There is a vast amount of literature dealing with the spectral analysis of the Laplacian in quantum waveguides. The two main problems that are considered are:

(i) What are natural conditions on the geometry of the tube $\mathcal{T}^{\varepsilon=1}$ (in particular on that of the submanifold) which yield the existence of eigenvalues below the essential spectrum ("bound states")?

(ii) Assume that there are eigenvalues below the essential spectrum. What is the asymptotic expansion of these eigenvalues in the shrinking tube $\mathcal{T}^{\varepsilon\to 0}$? Can they be approximated by means of a limiting operator?

We will give an overview of the extensive literature addressing these questions and discuss the related results in Section 1.2 after the establishment of the necessary geometric and analytic language. Let us finally mention some of the related problems that will not be examined any further in this thesis:

- Apart from quantum mechanical dynamics governed by the Schrödinger equation, the Dirichlet Laplacian is also the generator of the heat semi-group describing the conduction of heat in such thin tubes with fixed temperature on the outside. Some of the relevant results (effective equations, decay rate of the solutions) can be found in [Wit07, KZ10, GKP14, KK14].

- Other interesting questions arising from the Laplacian in thin tubes concern the consideration of Neumann boundary conditions, the location of nodal domains of the eigenfunctions, and the analysis of thin neighbourhoods of embedded graphs. A nice survey of the corresponding questions and methods is given by Grieser [Gri08].

- Another conceivable application for this framework is the vector bundle $\mathcal{E} = \Lambda^k M$ of alternating k-Forms over (M, g^ε). Weitzenböck's formula implies that the Hodge Laplacian $\Delta^{\mathrm{Hodge}}_{g^\varepsilon,k}$ on k-forms differs from the connection Laplacian, whose associated connection is induced by the Levi-Civita connection ∇^{g^ε}, merely by an

End$(\Lambda^k M)$-valued potential [BGV92]. If M is a compact manifold, de Rham's theorem [BGV92, Theorem 3.54] then states that the kernel of $\Delta^{\mathrm{Hodge}}_{g^\varepsilon,k}$ is isomorphic to the k-th de Rham cohomology group, providing insights into the differential topological structure of M. Consequently, it should be possible to associate $\ker(\Delta^{\mathrm{Hodge}}_{g^\varepsilon,k})$ with the (less complicated) kernel of an effective operator and link this to already existing results concerning the adiabatic limit of the Hodge Laplacian [MM90, For95, LK00, Lot02].

The treatment of a thin manifold, where the length scales of large horizontal directions and small vertical directions are well separated, can actually be considered as an adiabatic problem. This separation means that properties related to the vertical directions vary slowly along the horizontal directions and thus remain in a fixed configuration. The structure of (1.2) is in fact evocative of the most famous example of the aforementioned principle: The Born-Oppenheimer approximation within molecular dynamics exploits the assumption that the light (fast, "vertical") electrons follow their heavy (slow, "horizontal") nuclei adiabatically, i.e., the eigenstate of the electrons instantaneously adjusts to the momentary position of the nuclei [BO27]. It follows that the dynamics of the nuclei are governed by an effective equation on their own lower dimensional configuration space, whereas the effects of the electrons are expressed by an effective electronic potential. This reduction procedure ("freezing of the electronic degrees of freedom") is called adiabatic decoupling.

For this reason we use the techniques of higher order space-adiabatic perturbation theory which were developed in the context of Born-Oppenheimer approximation [MS02, Sor03, Teu03, NS04, PST07, MS09] for flat geometries. These ideas have been extended to constrained quantum systems for a great variety of geometries, where the localisation procedure was implemented through either a strongly confining potential for non-compact vertical directions [WT14] or Dirichlet boundary conditions for the compact case [Lam14]. In this thesis, we will adopt the geometric framework introduced in the last-mentioned work and broaden the viewpoint from operators acting on complex-valued functions on M to operators acting on sections of a $\mathbb{C}^N$-vector bundle $\mathcal{E}$ over M that can locally be represented by $\mathbb{C}^N$-valued functions ($N \in \mathbb{N}$).

1.1 Derivation of Effective Operators

We will explore the geometry of a Hermitian vector bundle $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ over an ε-thin manifold (M, g^ε) in great detail throughout Chapter 2. The latter manifold itself will be modelled by a fibre bundle $M \xrightarrow{\pi_M} B$ with compact fibres $M_x = \pi_M^{-1}(x)$, $x \in B$. It will turn out that the composition of the involved bundles can be viewed as a fibre bundle $\mathcal{E}$ over B with projection $\Pi_{\mathcal{E}} = \pi_M \circ \pi_{\mathcal{E}}$, where the fibres $\mathcal{E}_x = \Pi_{\mathcal{E}}^{-1}(x)$ are Hermitian vector bundles over M_x for all $x \in B$ (see Proposition 2.1). The decomposition (1.2) suggests to split

$$H^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + H^{\mathcal{F}}$$

on $\mathcal{H} = L^2(\mathcal{E})$, where $H^{\mathcal{F}}$ is the ε-independent, $\Pi_{\mathcal{E}}$-fibrewise, self-adjoint operator

$$H^{\mathcal{F}}(x) = -\Delta_{g_{M_x}}^{\mathcal{E}_x} + V^{\mathcal{E}}\big|_{M_x}$$

with Dirichlet boundary conditions. The compactness of the fibres M_x implies that the spectrum of $H^{\mathcal{F}}(x)$ consists solely of eigenvalues of finite multiplicity. We will therefore deal with eigenbands $\lambda : B \to \mathbb{R}$ such that $\lambda(x) \in \sigma(H^{\mathcal{F}}(x))$ and its associated fibrewise spectral projection $P_0(x)$ satisfy $H^{\mathcal{F}}(x)P_0(x) = \lambda(x)P_0(x)$ for all $x \in B$. The image of $P_0(x)$ exactly contains the $\lambda(x)$-eigensections and spans a fibrewise finite-dimensional subspace of $L^2(\mathcal{E}_x)$. These subspaces may be merged into a finite-rank eigenspace bundle $\mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} B$ if the primary eigenband is separated from the rest of the spectrum by a uniform gap.

A natural candidate for the approximation of $H^{\mathcal{E}}$ is given by the adiabatic operator $H_{\mathrm{a}}^{\mathcal{P}} = P_0 H^{\mathcal{E}} P_0$ acting on $L^2(\mathcal{P})$. Provided that $[H^{\mathcal{E}}, P_0] = [-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0]$ is of order ε in a suitable sense, the adiabatic operator is ε-close to the initial operator on the adiabatic subspace $P_0\mathcal{H}$ by virtue of the fact that

$$\begin{aligned}\big(H^{\mathcal{E}} - H_{\mathrm{a}}^{\mathcal{P}}\big)P_0 &= \big(-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\big)P_0 - P_0\big(-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\big)P_0 \\ &= \underbrace{\big[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0\big]}_{=\mathcal{O}(\varepsilon)} P_0 \\ &= \mathcal{O}(\varepsilon).\end{aligned}$$

This translates into an approximate invariance of the subspace $P_0\mathcal{H}$ under $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ for finite times $t = \mathcal{O}(1)$. If one wants to prove invariance for larger time scales, one has to replace the spectral projection P_0 by a super-adiabatic projection $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$. Its construction will be carried out in Chapter 3. We will demonstrate in Proposition 3.17 that for all $n \in \mathbb{N}$ there exists an orthogonal projection P_ε such that $[H^{\mathcal{E}}, P_\varepsilon] = \mathcal{O}(\varepsilon^{n+1})$ holds in $\mathcal{L}(\mathcal{H})$ for bounded energies.

Chapter 4 will be dedicated to stating the main (abstract) results of this thesis. Here, we will always assume

(i) suitable boundedness properties for the geometry of the involved bundles $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ and $M \xrightarrow{\pi_M} B$ (cf. Condition 2.2),

(ii) suitable boundedness properties for the perturbation $\varepsilon H_1^{\mathcal{E}}$ and the potential $V^{\mathcal{E}}$ (cf. Condition 2.16),

(iii) and a uniform spectral gap of the eigenband λ (cf. Condition 2.26).

The super-adiabatic projection P_ε serves as the starting point for the ultimate effective operator. This operator is obtained by first restricting $H^{\mathcal{E}}$ to the super-adiabatic subspace $P_\varepsilon\mathcal{H}$ and then mapping it back to $L^2(\mathcal{P})$ via a unitary operator U_ε which intertwines P_ε and P_0. We will prove in Theorem 4.3 that the effective operator $H_{\mathrm{eff}}^{\mathcal{P}} = U_\varepsilon^\dagger P_\varepsilon H^{\mathcal{E}} P_\varepsilon U_\varepsilon$ is self-adjoint on $L^2(\mathcal{P})$ and satisfies

$$\big(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}^{\mathcal{P}}t} U_\varepsilon^\dagger\big)P_\varepsilon = \mathcal{O}\big(\varepsilon^{n+1}\,|t|\big)$$

in $\mathcal{L}(\mathcal{H})$ for bounded energies. We remark that while the initial operator acts on L^2-sections of $\mathcal{E}$ (which is a finite rank vector bundle over M), the effective operator acts on L^2-sections of $\mathcal{P}$ (which is a finite rank vector bundle over the lower dimensional manifold B). Hence the approximation of $H^{\mathcal{E}}$ by $H^{\mathcal{P}}_{\text{eff}}$ represents a dimensional reduction procedure. Apart from the dynamics, there also exists a certain relationship between the respective spectra. More precisely, we will show in Theorem 4.4 that $\operatorname{dist}(\mu, \sigma(H^{\mathcal{E}})) = \mathcal{O}(\varepsilon^{n+1})$ for any $\mu \in \sigma(H^{\mathcal{P}}_{\text{eff}})$. It is clear that the converse direction – namely the approximation of $\sigma(H^{\mathcal{P}}_{\text{eff}})$ by $\sigma(H^{\mathcal{E}})$ – is possible if and only if the ground state band $\lambda_0(x) = \min(\sigma(H^{\mathcal{F}}(x)))$ contributes significantly to the expectation value of $H^{\mathcal{E}}$. Put differently, the only possible way to obtain a mutual approximation of the two operators in question is the restriction of $\sigma(H^{\mathcal{E}})$ to the spectral subspace which is associated with λ_0. This can be implemented by the suppression of energies above $\Lambda_1 = \inf_{x \in B}(\sigma(H^{\mathcal{F}}(x)) \setminus \lambda_0(x))$ via some appropriate cut-off function $\chi \in C_0^\infty((-\infty, \Lambda_1))$. Theorem 4.5 will then yield that $H^{\mathcal{E}}\chi(H^{\mathcal{E}})$ is unitarily equivalent to $H^{\mathcal{P}}_{\text{eff}}\chi(H^{\mathcal{P}}_{\text{eff}})$ up to errors of order ε^{n+1} in $\mathcal{L}(\mathcal{H})$. We will then end the discussion of the effective operator by taking a closer look at its semi-classical expansion. To do so, we will use the concrete expansions $U_\varepsilon = \mathbf{1}_{\mathcal{H}} + \mathcal{O}(\varepsilon)$ and $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$ and essentially obtain

$$H^{\mathcal{P}}_{\text{eff}} = H^{\mathcal{P}}_{\text{a}} + \mathcal{M}^{\mathcal{P}} + \mathcal{O}(\varepsilon^3)$$

in Proposition 4.10, including the $\mathcal{O}(\varepsilon^2)$-term $\mathcal{M}^{\mathcal{P}}$ as first super-adiabatic correction. Section 4.4 will deal with the examination of low energies, which are of order ε^α, $\alpha \in (0,2]$, above the bottom of $\sigma(H^{\mathcal{E}})$. In this case we will see that the (much simpler) adiabatic operator provides a more accurate approximation for such energies than one would expect from $H^{\mathcal{P}}_{\text{eff}} = H^{\mathcal{P}}_{\text{a}} + \mathcal{O}(\varepsilon^2)$. More precisely, we will prove that the low-lying parts of $\sigma(H^{\mathcal{E}})$ and $\sigma(H^{\mathcal{P}}_{\text{a}})$ approximate each other up to errors of order $\varepsilon^{2+\alpha/2}$ (see Proposition 4.14). Moreover, if the low-lying part of $\sigma(H^{\mathcal{P}}_{\text{a}})$ solely consists of eigenvalues, Theorem 4.15 will give that the mutual approximation of the eigenvalues is valid even up to errors of order $\varepsilon^{2+\alpha}$.

1.2 Application to Quantum Waveguides

We will apply the aforementioned results to the geometric framework of generalised quantum waveguides [HLT15]. These are a family of ε-thin tubular neighbourhoods $\mathcal{T}^\varepsilon$ around a smoothly embedded, b-dimensional submanifold $B \hookrightarrow (\mathbb{R}^{b+f}, \delta^{b+f})$. The analysis of the Dirichlet Laplacian $-\varepsilon^2 \Delta^{\mathcal{A}}_{\delta^{b+f}}$ on $\mathcal{T}^\varepsilon \times \mathbb{C}^N$ associated with a connection $\nabla^{\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}$ corresponds to a gauge theory where charged, non-relativistic quantum particles (described by a localised wave function $\psi : \mathbb{R} \to L^2(\mathcal{T}^\varepsilon \times \mathbb{C}^N)$ that solves the corresponding Schrödinger equation) couple to a possibly non-Abelian, $\mathbb{C}^{N\times N}_{\mathrm{Herm}}$-valued gauge field $\mathcal{A}$. In Section 5.1, we will specify a diffeomorphism in order to map the ε-thin tubes $(\mathcal{T}^\varepsilon, \delta^{b+f})$ isometrically to an ε-independent Riemannian manifold (M, G^ε) that has the additional structure of a fibre bundle $M \xrightarrow{\pi_M} B$ with a Riemannian metric $G^\varepsilon = g^\varepsilon + \mathcal{O}(\varepsilon)$ that is perturbed in an admissible manner. Moreover, the induced gauge field likewise has the structure of an admissible perturbation $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \mathcal{O}(\varepsilon)$. Thus, we will conclude that $-\varepsilon^2 \Delta^{\mathcal{A}}_{\delta^{b+f}}$ is unitarily equivalent to the Born-Oppenheimer-like Laplacian

$$H^{\mathcal{E}} = -\Delta^{\mathcal{A}_\varepsilon}_{G^\varepsilon} = -\Delta^{\mathcal{A}_0}_{g^\varepsilon} + \mathcal{O}(\varepsilon) = -\varepsilon^2 \Delta^{\mathcal{A}_0}_{\mathsf{H}} - \Delta^{\mathcal{A}_0}_{\mathsf{V}} + \mathcal{O}(\varepsilon).$$

In the course of this section, we will try to cover at least a small part of the vast amount of literature concerning quantum waveguides (with and without gauge field). Moreover, we will relate the results for $\mathcal{A} \neq 0$ to those obtained in Section 5.2 and Section 5.3.

1.2.1 Absence of Gauge Fields

Let us first survey the existing literature for the case of a vanishing gauge field $\mathcal{A}$ (or equivalently of uncharged quantum particles). The components of the wave function ψ then decouple (the Laplacian $\Delta^{\mathcal{A}=0}_{\delta^{b+f}} = \Delta^{\mathrm{L.B.}}_{\delta^{b+f}} \mathbf{1}_{\mathbb{C}^N}$ is obviously diagonal with respect to the spin degrees of freedom) and each component may be treated separately. We thus restrict ourselves to the case $N = 1$ of spinless particles without loss of generality. The object of interest is the Laplace-Beltrami operator $\Delta^{\mathrm{L.B.}}_{\delta^{b+f}}$ in such waveguides with Dirichlet boundary conditions. Most of the relevant literature deals with quantum strips ($b = 1$ and $f = 1$), quantum tubes ($b = 1$ and $f \geqslant 2$), and quantum layers ($b \geqslant 2$ and $f = 1$). We refer to Figure 5.1 for an illustration of their lower dimensional realisations.

Existence of Bound States

Quantum strips were initially considered as planar tubular neighbourhoods around bent curves with respective cross-sections being an interval of constant length. This corresponds to a fibre bundle $M = \mathbb{R} \times [-1, 1]$ with isometric fibres in our picture. In this context, the authors of [EŠ89, GJ92] proved the existence of bound states for asymptotically flat strips, while Duclos and Exner obtained the same result for more general non-straight strips [DE95]. Their results imply that the bending of the curve always has an attractive character. Later the range of geometry was broadened to varying strips. From our viewpoint this is implemented by an x-dependent function $h > 0$ within the unscaled metric

$$g = \underbrace{\mathrm{d}x \otimes \mathrm{d}x}_{=g_{\mathsf{H}}} + h(x)\, \underbrace{\mathrm{d}y \otimes \mathrm{d}y}_{=g_{\mathsf{V}}}.$$

For instance the authors of [BGRB97, FS08b] analysed straight strips with a local bump and with periodic cross-sections, respectively.

The existence of curvature-induced bound states for quantum tubes with $f = 2$ was first proved by Goldstone and Jaffe as well as by Duclos and Exner for disc-shaped cross-sections [GJ92, DE95]. Beyond that, the addition of a further codimension significantly enriches the geometric variety of the waveguide: Another interesting effect besides the attractive bending arises if one considers quantum tubes $\mathcal{T}^{\varepsilon=1}$ where the cross-sections are all isomorphic to a fixed, not rotationally invariant domain $F \subset \mathbb{R}^f$ and are allowed to twist around the curve with respect to a "reference framing" induced by some x-dependent $\mathrm{SO}(f)$-transformation for the transversal directions. The existence of bound states for bent (but asymptotically straight) and twisted tubes was proven for example in

[CDFD05]. It is shown that twisting has a repulsive character [EKK08, Kre08] for $f = 2$. Particularly, twisting can destroy bound states if the curve is only mildly bent.

In the case of quantum layers, which are built over complete, oriented and non-compact[1] hypersurfaces with a constant interval as cross-section at each point, the effect of bending is not necessarily attractive. It was pointed out in [Haa12] that bending is attractive for $b = 1$, non-repulsive for $b = 2$ and indefinite for $b \geqslant 3$, irrespective of the codimension f. Hence, further conditions have to be imposed on the hypersurface in order to guarantee the existence of bound states. This was done for instance in [DEK01, CEK04, LR12] for asymptotically flat hypersurfaces in the case $b = 2$. Lin and Lu eventually introduced the notion of asymptotically flat, parabolic hypersurfaces and showed the existence of eigenvalues below the essential spectrum for arbitrary $b \geqslant 2$ [LL06b].

Let us also mention some of the less studied geometric generalisations. Krejčiřík considered strips which are embedded into an asymptotically flat Riemannian manifold [Kre03]. In this situation the effects of bending consists of both a purely attractive part induced by the extrinsic curvature of the curve and an indefinite part induced by the intrinsic curvature of the ambient surface. Lin and Lu studied the existence of discrete spectrum for tubular neighbourhoods of fixed radius around complete, oriented and non-compact submanifolds with arbitrary dimensions $b, f \geqslant 1$ [LL06a]. Wittich investigated tubular neighbourhoods of closed submanifolds of ambient Riemannian manifolds in the context of the heat equation [Wit07]. Finally, the authors of [HLT15] generalised the framework of [LL06a] to allow for varying cross-sections. Moreover, they introduced an entirely new class of conceivable geometries: In contrast to the previous ("massive") waveguides, whose cross-sections are given by the closure of an open and bounded domain, "hollow" waveguides are modelled on the basis of the latter by restricting to their boundary in each cross-section.

[1] We can obviously exclude the compact case since then the associated Laplacian always admits a purely discrete spectrum [Dav95, Theorem 6.2.3].

Asymptotic Expansion of the Eigenvalues

Once the existence of eigenvalues of the Dirichlet Laplacian $-\varepsilon^2\Delta^{\mathrm{L.B.}}_{\delta^{b+f}}$ on $\mathcal{T}^\varepsilon \times \mathbb{C}$ is assumed, one might ask for their asymptotic expansion as ε tends to zero. Most of the existing literature deals with the low-lying eigenvalues that are associated with the non-degenerate, positive ground state ϕ_0 of the vertical operator $-\Delta^{\mathrm{L.B.}}_{\mathrm{V}}$ with associated eigenband λ_0. The general strategy is to specify an ε-independent limiting operator H_0 on the lower dimensional submanifold B and to show

$$-\Delta^{\mathrm{L.B.}}_{\delta^{b+f}} - \tfrac{\Lambda_0}{\varepsilon^2}\mathbf{1}_{\mathcal{H}} \xrightarrow{\varepsilon\to 0} H_0 \otimes \mathbf{1}, \quad H_0 := -\Delta^{\mathrm{L.B.}}_{g_B} + \mathcal{V} \tag{1.3}$$

on the subspace $L^2(B, g_B) \otimes \mathrm{span}(\phi_0) \subset \mathcal{H}$, where $\Lambda_0 = \inf_{x\in B} \lambda_0(x)$ denotes the bottom of the vertical mode and $\mathcal{V}$ is some appropriate potential on B.

The majority of the literature is restricted to a constant ground state band $\lambda_0 = \Lambda_0$. This is for example satisfied for twisted waveguides with isometric cross-sections F. The convergence in (1.3) has to be understood in the sense of resolvents [BMT07, Gru09, KŠ12] or in the sense of quadratic forms [deO11], which only implies strong resolvent convergence. The potential $\mathcal{V} = V_{\mathrm{bend}} + V_{\mathrm{twist}}$ in the limiting operator H_0 encodes the effects of bending and twisting. Particularly for the case $(b, f) = (1, 2)$ of a bent and twisted tube around a smoothly embedded curve $c : \mathbb{R} \to \mathbb{R}^3$, where the twisting with respect to the "reference framing" can be characterised by means of a smooth angle function $\vartheta : \mathbb{R} \to \mathbb{R}$, the two respective potentials are given by

$$V_{\mathrm{bend}} = -\frac{\|c''\|^2_{\mathbb{R}^3}}{4} \quad , \quad V_{\mathrm{twist}} = (\vartheta')^2 \int_F \left|(y \times \nabla_y)\phi_0(y^1, y^2)\right|^2 \,\mathrm{d}y,$$

where ϕ_0 is the x-independent ground state of the Dirichlet Laplacian on $F \subset \mathbb{R}^2$. These expressions were derived in [BMT07] for a finite curve and by the authors of [Gru09, deO11, KŠ12] for unbounded curves. It was ultimately shown in [HLT15] that the corresponding potentials for generalised (massive) quantum waveguides read in our notation

$$V_{\text{bend}} = \lim_{\varepsilon\to 0} \varepsilon^{-2} V_{\rho_\varepsilon} \quad , \quad V_{\text{twist}} = \int_{M_x} \left\| \mathrm{P}^{\mathrm{H}M} \operatorname{grad}_g \phi_0 \right\|_g^2 \operatorname{vol}_{g_{\mathrm{V}}}$$

with geometric potential V_{ρ_ε} (2.18) depending on the embedding of the curve. The convergence of resolvents in norm yields an expansion

$$v_i^\varepsilon = \Lambda_0 + \varepsilon^2 \nu_i + o(\varepsilon^2), \quad i \in \mathcal{I} \tag{1.4}$$

for the eigenvalues of $-\varepsilon^2 \Delta^{\text{L.B.}}_{\delta^{b+f}}$, where $\{\nu_i\}_{i\in\mathcal{I}}$ are the eigenvalues of H_0 if they exist. The limiting operator $H_0 \otimes \mathbf{1}$ coincides with the leading order of the rescaled adiabatic operator $\varepsilon^{-2}(H_{\mathrm{a}}^{\mathcal{P}} - \lambda_0 \mathbf{1}_{\mathcal{H}})$ that is related to the ground state band. If one takes into account the entire ε-dependent adiabatic operator with eigenvalues $\{\nu_i^\varepsilon\}_{i\in\mathcal{I}}$, [HLT15, Theorem 3.3] states that the low-lying eigenvalues of $-\varepsilon^2 \Delta^{\text{L.B.}}_{\delta^{b+f}}$ have the even more accurate expansion

$$v_i^\varepsilon = \Lambda_0 + \varepsilon^2 \nu_i^\varepsilon + \mathcal{O}(\varepsilon^4), \quad i \in \mathcal{I}.$$

While the level spacing of the low-lying eigenvalues $\{v_i^\varepsilon\}_{i\in\mathcal{I}}$ is of order ε^2 for a constant ground state band, the situation changes if one looks at a varying ground state band $x \mapsto \lambda_0(x)$. In this situation one expects a spacing of order ε^α for $\alpha \in (0,2)$, which is in particular the case if λ_0 has a unique minimum. Let us consider for example a one-dimensional base ($b = 1$) and $\lambda_0(x) = \Lambda_0 + cx^{2k} + \mathcal{O}(x^{2k+1})$ for some $c > 0$ and $k \in \mathbb{N}$. Then the first eigenvalues of $-\varepsilon^2 \Delta^{\text{L.B.}}_{\delta^{b+f}}$ asymptotically behave as

$$v_i^\varepsilon = \Lambda_0 + \varepsilon^{\frac{2k}{k+1}} \nu_i + o\left(\varepsilon^{\frac{2k}{k+1}}\right), \quad i \in \mathcal{I},$$

where $\{\nu_i\}_{i\in\mathcal{I}}$ are the eigenvalues of the limiting operator H_0 with potential $\mathcal{V}(x) = cx^{2k}$. Friedlander and Solomyak considered a similar situation for straight strips [FS08a], while de Oliveira and Verri investigated the case $k = 1$ ($\alpha = 1$) for bent and twisted tubes in $\mathbb{R}^3$ [deOV11]. They found that the effects of bending and twisting are no longer apparent on this energy scale. Given that the level spacing of

$$\left(-\varepsilon^2 \Delta^{\text{L.B.}}_{g_B} + \left(\lambda_0(x) - \Lambda_0\right)\right) \otimes \mathbf{1}$$

is of order ε^α, the low-lying eigenvalues of $-\varepsilon^2\Delta^{\mathrm{L.B.}}_{\delta^{b+f}}$ may be expanded as [HLT15, Theorem 3.3]

$$\upsilon_i^\varepsilon = \Lambda_0 + \varepsilon^\alpha \nu_i^\varepsilon + \mathcal{O}(\varepsilon^{\alpha+2}), \quad i \in \mathcal{I},$$

where $\{\nu_i^\varepsilon\}_{i\in\mathcal{I}}$ are the ε-dependent eigenvalues of $\varepsilon^{-\alpha}(H_{\mathrm{a}}^{\mathcal{P}} - \Lambda_0 \mathbf{1}_{\mathcal{H}})$.

1.2.2 Presence of Gauge Fields

The behaviour of the discrete spectrum associated with the Dirichlet Laplacian

$$-\varepsilon^2\Delta^{\varepsilon^{-\beta}\mathcal{A}}_{\delta^{b+f}} = (\varepsilon \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon^{1-\beta}\mathcal{A})^*(\varepsilon \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon^{1-\beta}\mathcal{A})$$

in tubular neighbourhoods with a gauge field $\mathcal{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*\mathcal{T}^\varepsilon \times \mathbb{C}^{N\times N}_{\mathrm{Herm}})$ has been far less explored. The parameter $\beta \in \{0,1\}$ in the Laplacian reflects the strength of the coupling of the gauge field. We will refer to $\beta = 0$ as weak gauge fields and to $\beta = 1$ as strong gauge fields.

Weak Gauge Fields ($\beta = 0$, $\nabla^{\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}$)
Most of the literature deals with the case of a real-valued gauge field $\mathcal{A}$ and charged spinless particles. This corresponds to the line bundle $\mathcal{T}^\varepsilon \times \mathbb{C}$, where $\mathcal{A}$ represents an ordinary Abelian magnetic potential. Therefore, we will always assume $N = 1$, unless otherwise indicated.

The authors of [EJK01, EK05, BEK05] considered quantum strips $\mathcal{T}^{\varepsilon=1}$ and showed that the magnetic effects have a repulsive nature. Exner et al. confined the motion of a quantum particle to a straight line by means of a strong harmonic oscillator potential in a homogeneous magnetic field, and allowed a periodic perturbation in the longitudinal direction. They proved that the spectrum of the magnetic Laplacian is absolutely continuous at the bottom. Ekholm and Kovařík examined a straight strip with either a L^∞_{loc}- or Aharonov-Bohm field and proved that the discrete spectrum for both a local bump and a mildly curved strip is empty, although both configurations induce bound states in the absence of the magnetic field [DE95, BGRB97]. Borisov et al. demonstrated that eigenvalues below the essential spectrum in a straight strip in the presence of a C_0^1-magnetic potential can be produced by the inclusion of a sufficiently large Neumann window on the boundary. By gauging away the vertical component of the magnetic potential, however, the authors of [KR14] showed that the magnetic effects are not apparent to leading order for shrinking strips $\mathcal{T}^{\varepsilon\to 0}$. More precisely, they proved a convergence result

$$-\Delta^{\mathcal{A}}_{\delta^2} - \tfrac{\lambda_0}{\varepsilon^2}\mathbf{1}_{\mathcal{H}} \xrightarrow{\varepsilon\to 0} H_0 \otimes \mathbf{1}$$

on $L^2(\mathbb{R}, \mathrm{d}x \otimes \mathrm{d}x) \otimes \operatorname{span}(\phi_0)$ in the norm resolvent sense for curved equidistant strips with limiting operator $H_0 = -\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + V_{\mathrm{bend}}$. Here, λ_0 stands for the (once again constant) ground state band of $-\Delta^{\mathrm{L.B.}}_{\mathrm{V}}$. Moreover, they approximated the eigenvalues below the essential spectrum according to (1.4).

Shrinking quantum tubes with magnetic potentials in $\mathbb{R}^3$ were investigated in [Gru08, BdeOV13] for $B = \mathbb{S}^1$ and in [KR14] for $B = \mathbb{R}$. In both cases, the leading order of the vertical magnetic potential can be gauged away completely, which results in $|\lambda_0^{\mathrm{m}} - \lambda_0| = \mathcal{O}(\varepsilon^4)$ for the difference of the ground state bands with and without magnetic potential. Consequently, they obtained

$$-\Delta^{\mathcal{A}}_{\delta^3} - \tfrac{\lambda_0}{\varepsilon^2}\mathbf{1}_{\mathcal{H}} \xrightarrow{\varepsilon\to 0} H_0^{(\mathcal{A}_B)} \otimes \mathbf{1}$$

in the norm resolvent sense for a twisting but fixed, simply connected cross-section. Since it also possible to simultaneously gauge away the leading order of the horizontal component for an unbounded curve $B = \mathbb{R}$, Krejčiřík and Raymond obtained $H_0 = -\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + V_{\mathrm{bend}} + V_{\mathrm{twist}}$ for the limiting operator, which again gives an approximation of the eigenvalues (1.4) [KR14]. In contrast to this, the horizontal contribution of $\mathcal{A}$ cannot be gauged away in general for a closed curve $B = \mathbb{S}^1$, and Bedoya et al. thus derived the limiting operator

$$H_0^{\mathcal{A}_B} = -\Delta^{\mathcal{A}_B}_{\mathrm{d}x\otimes\mathrm{d}x} + V_{\mathrm{bend}} + V_{\mathrm{twist}}, \quad \nabla^{\mathcal{A}_B} = \mathrm{d} + \mathrm{i}\mathcal{A}_B$$

with magnetic potential $\mathcal{A}_B = c^*\mathcal{A}$ (i.e., $\mathcal{A}_B(\partial_x)|_x = \mathcal{A}_{c(x)}(c'(x))$ for all $x \in \mathbb{S}^1$), where $c : \mathbb{S}^1 \to \mathbb{R}^3$ is the embedding of the closed curve

[BdeOV13]. Grushin considered more general Schrödinger operators with an additional potential of the form $V(x, y^1, y^2) = V_1(x) + \varepsilon^{-2} V_2(y^1, y^2)$. Then $\Lambda_0 = \lambda_0$ is the ground state of the vertical operator $-\Delta_{\mathrm{V}}^{\mathrm{L.B.}} + V_2$ and V_1 is simply added to the limiting operator H_0.

As far as quantum layers are concerned, it is always possible to gauge away the single vertical component of the magnetic potential (just as is done for quantum strips). Ferrari and Cuoghi, as well as de Oliveira, considered the quantum mechanical motion of a particle that is confined to the vicinity of a hypersurface in $\mathbb{R}^3$ by a large constraining potential that does not change its shape along the surface [FC08, deO14]. In the limit where the strength of that potential tends to infinity, the dynamics converge to a product of dynamics on the surface generated by a limiting operator $H_0^{\mathcal{A}_B} = -\Delta_{g_B}^{\mathcal{A}_B} + V_{\mathrm{bend}}$ and highly oscillating dynamics for the transversal mode. Here, $\mathcal{A}_B$ denotes the restriction of the horizontal magnetic potential to the surface. Krejčiřík et al. proved the norm resolvent convergence result

$$-\Delta_{\delta^3}^{\mathcal{A}} - \tfrac{\lambda_0}{\varepsilon^2} \mathbf{1}_{\mathcal{H}} \xrightarrow{\varepsilon \to 0} H_0^{\mathcal{A}_B} \otimes \mathbf{1}$$

for equidistant layers with Dirichlet boundary conditions [KRT15]. Moreover, they determined the corresponding magnetic field $\mathcal{B}_B = \star \mathrm{d}\mathcal{A}_B$ on B to be the component $\mathcal{B}^{\perp}$ of the initial magnetic field perpendicular to the surface.

We will apply the results of the low energy asymptotics to generalised quantum waveguides in the presence of $\mathbb{C}^{N\times N}_{\mathrm{Herm}}$-valued gauge fields in Section 5.2. More precisely, we will show in Theorem 5.12 that the low-lying eigenvalues of $-\varepsilon^2 \Delta_{\delta^{b+f}}^{\mathcal{A}}$ can be approximated by those of the appropriate adiabatic operator linked to the ground state band of the vertical operator up to errors of order $\varepsilon^{2+\alpha}$. We will calculate the operators for quantum tubes around an infinite curve $c : \mathbb{R} \to \mathbb{R}^3$ with two geometric configurations:

- the cross-sections are centred ellipses with varying semi-axes twisting around the curve (massive waveguide),

- the cross-sections are centred circles with varying radius along the curve (hollow waveguide).

These calculations can be easily transferred to other geometric realisations and therefore provide a great generalisation of the existing results. In particular, the results for massive twisted tubes in $\mathbb{R}^3$ with fixed cross-section [BdeOV13, KR14] can be recovered immediately.

Strong Gauge Fields ($\boldsymbol{\beta = 1}$, $\boldsymbol{\nabla^{\varepsilon^{-1}\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon^{-1}\mathcal{A}}$)
Brüning et al. examined a charged spin-1/2-particle ($N = 2$) which is constrained around a surface $B \hookrightarrow \mathbb{R}^3$ using a steep constraining potential $V_{\mathrm{c}}^{\varepsilon}$ [BDNT08, BDN09]. The latter is modelled on B's normal bundle by means of the scaling behaviour $V_{\mathrm{c}}^{\varepsilon}(\nu) = V_{\mathrm{c}}(\varepsilon^{-1}\nu)$ (see also [WT14]). Moreover, the particle is influenced by an external electric potential φ and a magnetic potential $\mathcal{A}$ which induces a homogeneous magnetic field $\mathcal{B}$. In the "thin layer limit" of very strong confinement $\varepsilon \to 0$, they derived, under appropriate conditions, an effective scalar Schrödinger equation

$$\mathrm{i}\varepsilon \tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \left(-\varepsilon^2 \Delta_{g_B}^{\varepsilon^{-1}\mathcal{A}_B} + \lambda + \varphi_B + \varepsilon^2 V_{\mathrm{bend}}\right)\psi(t) \tag{1.5}$$

on the submanifold B with restricted electric potential φ_B and magnetic potential $\mathcal{A}_B$, bending potential V_{bend} as before and eigenband λ of the vertical $\mathbb{C}^{2\times 2}$-valued operator

$$H^{\mathcal{F}} = \underbrace{\left(-\Delta_{\mathrm{V}}^{\mathrm{L.B.}} + V_{\mathrm{c}}\right)}_{\substack{\text{operator}\\ \text{without spin}}} \mathbf{1}_{\mathbb{C}^2} + \underbrace{\begin{pmatrix} \mathcal{B}^3 & \mathcal{B}^1 - \mathrm{i}\mathcal{B}^2 \\ \mathcal{B}^1 + \mathrm{i}\mathcal{B}^2 & -\mathcal{B}^3 \end{pmatrix}}_{\substack{\text{coupling of the spin}\\ \text{with magnetic field}}}.$$

Here, the time t is measured in macroscopic units (whereas the longitudinal scale of the layer is of order one). Hence, solutions of (1.5) are expected to have non-trivial effects, i.e., to propagate macroscopic distances in finite times $t = \mathcal{O}(1)$.

Krejčiřík and Raymond examined the effects of a strong external magnetic field for the geometric framework of shrinking quantum tubes in $\mathbb{R}^3$ [KR14] which twist around infinite curves with fixed, simply connected cross-section $F \subset \mathbb{R}^2$. They again derived a limiting operator H_0 in the norm resolvent sense, given by

$$H_0 = \left\langle \phi_0, -\Delta_{\mathrm{H}}^{\mathcal{B}_\perp \times y}(\cdot \phi_0) \right\rangle_{L^2(F,\mathrm{d}y)} + V_{\mathrm{bend}} + V_{\mathrm{eff}}(\mathcal{B}_\|, \phi_0)$$

Here, $\mathcal{B}_\|(x) \in \mathbb{R}$ and $\mathcal{B}_\perp(x) \in \mathbb{R}^2$ denote the components of the magnetic field parallel and perpendicular to the curve, respectively, and ϕ_0 is the ground state of the vertical Laplacian $-\Delta^{\mathrm{L.B.}}_{\mathrm{d}y \otimes \mathrm{d}y}$ with Dirichlet boundary conditions on $L^2(F, \mathrm{d}y)$.

We will consider the initial operator $-\varepsilon^2 \Delta^{\varepsilon^{-1}\mathcal{A}}_{\delta^{b+f}} \mathbf{1}_{\mathbb{C}^N} + V$ on $\mathcal{T}^\varepsilon \times \mathbb{C}^N$ for generalised quantum waveguides with an Abelian magnetic potential $\mathcal{A}$ and $\mathbb{C}^{N\times N}$-valued potential V in Section 5.3. To do so, we will approximate parts of the initial spectrum by those of an appropriate effective operator $H^{\mathcal{P}}_{\mathrm{eff}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}} + \mathcal{O}(\varepsilon^3)$ associated with the ground state band up to errors of order ε^3 (see Theorem 5.21). We will again examine the two geometric configurations of infinite massive and hollow quantum tubes in $\mathbb{R}^3$. While the results for the massive case may be related to those of [KR14], we will obtain a completely new expression for the effective operator in the hollow case.

Chapter 2

2 The Framework

This chapter gives a detailed presentation of the framework within which we will develop the general super-adiabatic perturbation theory. We will begin with an explanation of the double bundle structure that was mentioned in the introduction. Hereupon, we will turn to the connection Laplacian and thoroughly examine its decomposition into a horizontal and vertical differential operator. We will then introduce the conditions on the ultimate Schrödinger operator (1.1) and discuss two possible situations leading to an admissible perturbation. Finally, we will explain the fundamentals of adiabatic perturbation theory.

2.1 Geometric Setting

An adequate decomposition of M into horizontal and vertical directions is naturally obtained by the imposition that M has the additional structure of a fibre bundle with compact fibres. More precisely, we assume that there exist

- a smooth, connected manifold M with or without boundary (total space),
- a smooth, connected manifold B without boundary (base manifold),
- a smooth, compact manifold F with or without boundary (typical fibre),

- and a smooth surjective map $\pi_M : M \to B$,

such that for all $x \in B$ there exists an open neighbourhood $U \subset B$ of x together with a diffeomorphism $\Phi : \pi_M^{-1}(U) \to U \times F$ for which the diagram

$$\begin{tikzcd} \pi_M^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi_M"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes.

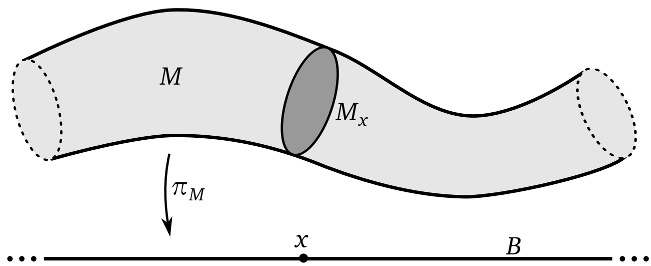


Figure 2.1: Illustration of a manifold M that has the additional structure of a fibre bundle $M \xrightarrow{\pi_M} B$ with base manifold $B = \mathbb{R}$ and typical fibre $F = \mathbb{B}_1^2(0) \subset \mathbb{R}^2$.

We set $b := \dim(B)$, $f := \dim(F)$, and consequently $b + f = \dim(M)$, for the respective dimensions of the involved manifolds. Moreover, we denote by $M_x := \pi_M^{-1}(x)$ the fibres of M, all being diffeomorphic to the typical fibre F. It follows that $\partial M = \bigcup_{x \in B} \partial M_x$, and thus M has a boundary if and only if F does. Here, one should always keep in mind the concrete example of a "massive tube" as depicted in Figure 2.1, where the base B is given by the real line and the typical fibre F by a two-dimensional disc.

We also introduce a $\mathbb{C}^N$-vector bundle over M for some $N \in \mathbb{N}$ (see Definition A.1), i.e., there additionally exist

- a smooth manifold $\mathcal{E}$,
- and a smooth surjective map $\pi_{\mathcal{E}} : \mathcal{E} \to M$,

such that for all $p \in M$

(i) there exists an open neighbourhood $W \subset M$ of p together with a diffeomorphism $\Lambda : \pi_{\mathcal{E}}^{-1}(W) \to W \times \mathbb{C}^N$ for which the diagram

$$\begin{tikzcd} \pi_{\mathcal{E}}^{-1}(W) \arrow[rr, "\Lambda"] \arrow[dr, "\pi_{\mathcal{E}}"'] & & W \times \mathbb{C}^N \arrow[dl, "\mathrm{pr}_1"] \\ & W & \end{tikzcd}$$

commutes,

(ii) and the fibre $\mathcal{E}_p := \pi_{\mathcal{E}}^{-1}(p)$ is a complex N-dimensional vector space and $\Lambda|_{\mathcal{E}_p} : \mathcal{E}_p \to \{p\} \times \mathbb{C}^N$ is a vector space isomorphism.

2.1.1 Double Fibre Bundle

The composition of the involved fibre bundles allows us to view $\mathcal{E}$ as the total space of a fibre bundle over B (see Figure 2.2). The following proposition makes this precise:

Proposition 2.1 *Let $\pi_M : M \to B$ be a smooth fibre bundle with compact typical fibre F and $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be a smooth $\mathbb{C}^N$-vector bundle. Then $\Pi_{\mathcal{E}} : \mathcal{E} \to B$ is a smooth fibre bundle with projection $\Pi_{\mathcal{E}} := \pi_M \circ \pi_{\mathcal{E}}$, where its typical fibre $\mathcal{F}$ carries a unique (up to isomorphism) structure of a $\mathbb{C}^N$-vector bundle $\pi_{\mathcal{F}} : \mathcal{F} \to F$.*

PROOF. Let $x_0 \in B$ be arbitrary and $U \subset B$ be an open, contractible, x_0-centred neighbourhood[1] that comes along with a local trivialisation

[1] We will introduce a Riemannian metric g_B on B in Subsection 2.1.3 such that we can choose U to be for example a geodesic ball $\mathsf{B}_r(x_0)$ with radius $r < r_{\mathrm{inj}}(B, g_B)$.

$\Phi : \pi_M^{-1}(U) \to U \times F$. Instead of considering $\Pi_{\mathcal{E}}^{-1}(U) = \mathcal{E}|_{\pi_M^{-1}(U)}$, we rather restrict our attention to the associated pullback bundle

$$\begin{aligned}\bar{\mathcal{E}} &:= (\Phi^{-1})^* \mathcal{E}|_{\pi_M^{-1}(U)} \\ &= \left\{ \big((x,y),\psi\big) \in (U \times F) \times \mathcal{E}|_{\pi_M^{-1}(U)} \text{ such that } \psi \in \mathcal{E}_{\Phi^{-1}(x,y)} \right\}\end{aligned}$$

over $U \times F$, whose fibre over (x,y) coincides with $\{(x,y)\} \times \mathcal{E}_{\Phi^{-1}(x,y)}$, and define $\pi_{\mathcal{F}} : \mathcal{F} \to F$ to be the $\mathbb{C}^N$-vector bundle ("the model") with total space

$$\mathcal{F} := \left\{ (y,\psi) \in \mathcal{F} \times \bar{\mathcal{E}} \text{ such that } \psi \in \bar{\mathcal{E}}_{(x_0,y)} = \big\{(x_0,y)\big\} \times \mathcal{E}_{\Phi^{-1}(x_0,y)} \right\}$$

and projection $\pi_{\mathcal{F}} := \mathrm{pr}_1 : (y,\psi) \mapsto y$. The contractibility of U now induces a diffeotopy $h : (U \times F) \times [0,1] \to U \times F$ between $\mathbf{1}_{U \times F} = h(\cdot,1)$ and $\{x_0\} \times \mathbf{1}_F = h(\cdot,0)$, and [Hat09, Theorem 1.6], which is also valid in the smooth case, states that $\mathbf{1}_{U\times F}^* \bar{\mathcal{E}} = \bar{\mathcal{E}}$ and

$$\begin{aligned}(\{x_0\} \times \mathbf{1}_F)^* \bar{\mathcal{E}} &= \left\{ \big((x,y),\psi\big) \in (U \times F) \times \bar{\mathcal{E}} \text{ such that } \psi \in \bar{\mathcal{E}}_{(x_0,y)} \right\} \\ &= U \times \underbrace{\big\{ (y,\psi) \in F \times \bar{\mathcal{E}} \text{ such that } \psi \in \bar{\mathcal{E}}_{(x_0,y)} \big\}}_{=\mathcal{F}}\end{aligned}$$

are diffeomorphic vector bundles over U. In summary, we constructed a local trivialisation $\Psi : \Pi_{\mathcal{E}}^{-1}(U) \to U \times \mathcal{F}$, for which the diagram

$$\begin{tikzcd} \Pi_{\mathcal{E}}^{-1}(U) \arrow[rr, "\Psi"] \arrow[dr, "\Pi_{\mathcal{E}}"'] & & U \times \mathcal{F} \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes.

We finally cover B with a set $\{U_\nu\}_{\nu \in \mathcal{I}}$ of such contractible neighbourhoods, apply the described construction of the corresponding (model) bundles for each U_ν and observe that, due to the fact that B is a connected manifold, these models are pairwise isomorphic. $\square$

Given a local trivialisation $\Phi : \pi_M^{-1}(U) \to U \times F$ of $M \xrightarrow{\pi_M} B$ and a point $x_0 \in U$, the above proof shows that the model $\mathcal{F}$ is isomorphic to the pullback bundle $(\Phi|_{M_{x_0}}^{-1})^* \mathcal{E}$.

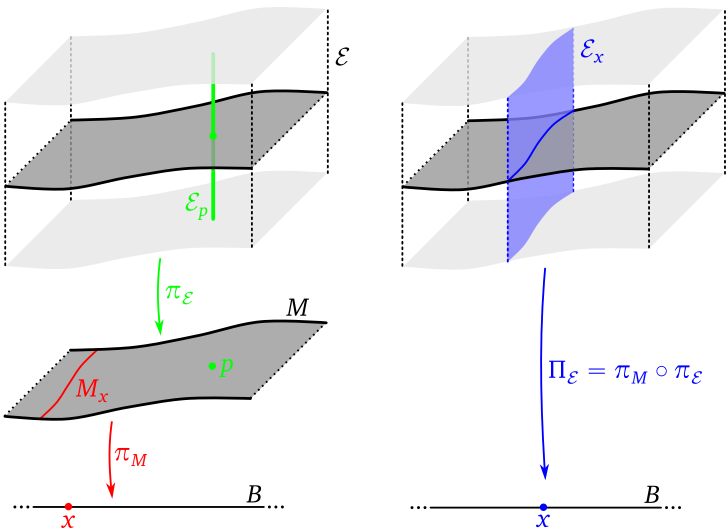


Figure 2.2: Due to the double bundle structure $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ and $M \xrightarrow{\pi_M} B$, we may consider $\mathcal{E}$ as the total space of a fibre bundle over the base B, where the typical fibre itself is a $\mathbb{C}^N$-vector bundle over M's typical fibre F.

2.1.2 Metrics and Connections

We now introduce Riemannian metrics g on M and g_B on B, which turn π_M into a Riemannian submersion [ONe66]. This means that the differential $\mathsf{T}\pi_M : \mathsf{T}M \to \mathsf{T}B$ induces an isometry $\mathsf{T}M/\ker(\mathsf{T}\pi_M) \to \mathsf{T}B$, which yields an orthogonal decomposition of M's tangent bundle into a horizontal and vertical subbundle with respect to g:

$$\mathsf{T}M = \ker(\mathsf{T}\pi_M)^{\perp} \oplus \ker(\mathsf{T}\pi_M) =: \mathsf{H}M \oplus \mathsf{V}M. \tag{2.1}$$

Note that the vectors in the kernel of $\mathsf{T}\pi_M$ are tangent to the fibres, i.e., $\mathsf{V}_\xi M = \mathsf{T}_\xi M_x$ for all $\xi \in M_x$. Consequently, g may be written as

$$g = \pi_M^* g_B + g_{\mathsf{V}},$$

where $g_{\mathsf{V}} \in C^\infty(\Sigma^2 \mathsf{V}M)$ is the restriction of g to the vertical subbundle, i.e., $g_{\mathsf{V}}(\mathfrak{h}, \cdot)$ for all $\mathfrak{h} \in \mathsf{H}M$. We finally note that the horizontal subbundle $\mathsf{H}M$ is isomorphic to $\pi_M^* \mathsf{T}B$ for any metric g. Given a smooth vector field $X \in C^\infty(\mathsf{T}B)$, there exists a unique horizontal vector field on M that is π_M-related to X. This vector field is referred to as the *horizontal lift* $X^{\mathsf{H}} \in C^\infty(\mathsf{H}M)$ and therefore satisfies

$$g(X^{\mathsf{H}}, V) = 0 \text{ for all } V \in C^\infty(\mathsf{V}M) \quad \rightsquigarrow X^{\mathsf{H}} \text{ is horizontal,}$$

and

$$\mathsf{T}\pi_M \circ X^{\mathsf{H}} = X \circ \pi_M \qquad \rightsquigarrow X^{\mathsf{H}} \text{ is a lift.}$$

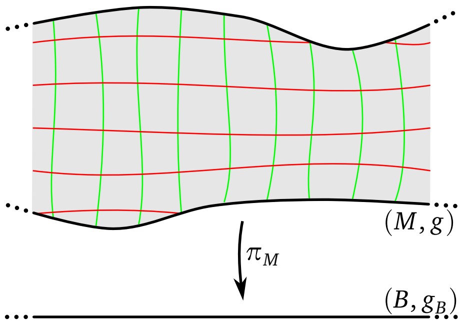


Figure 2.3: The differential $\mathsf{T}\pi_M$ induces the decomposition (2.1): While the integral curves of horizontal vector fields (red) are lifts of integral curves in B, the integral curves of vertical vector fields (green) are tangent to the fibres in M.

We introduce the *integrability tensor* $\Omega_{\mathsf{H}} \in C^\infty(\Lambda^2 \mathsf{H}M \otimes \mathsf{V}M)$ of the horizontal distribution and the *second fundamental form* $\mathrm{II}_{\mathsf{V}} \in C^\infty(\Sigma^2 \mathsf{V}M \otimes \mathsf{H}M)$

of the fibres to be

$$\Omega_{\mathsf{H}}(X^{\mathsf{H}}, Y^{\mathsf{H}}) := \mathrm{P}^{\mathsf{V}M}\big([X^{\mathsf{H}}, Y^{\mathsf{H}}]\big) = [X^{\mathsf{H}}, Y^{\mathsf{H}}] - [X, Y]^{\mathsf{H}}$$

and

$$\mathrm{II}_{\mathsf{V}}(V, W) := \mathrm{P}^{\mathsf{H}M}\big(\nabla^{g}_{V} W\big),$$

respectively. The g_{V}-trace of the latter is the horizontal *mean curvature vector*

$$\eta_{\mathsf{V}} := \mathrm{tr}_{g_{\mathsf{V}}}\big(\mathrm{II}_{\mathsf{V}}(\cdot, \cdot)\big). \tag{2.2}$$

Note that $\eta_{\mathsf{V}}|_{M_x}$ coincides with the mean curvature of the submanifold $M_x \hookrightarrow M$ (see Definition A.11(iii)) for all $x \in B$.

As far as the vector bundle $\mathcal{E}$ is concerned, we introduce a Hermitian bundle metric $h \in C^{\infty}(\mathcal{E}^{*\otimes 2})$ and a metric connection $\nabla^{\mathcal{E}} : C^{\infty}(\mathcal{E}) \to C^{\infty}(\mathsf{T}^{*}M \otimes \mathcal{E})$.

The fibres $\mathcal{E}_x = \Pi_{\mathcal{E}}^{-1}(x)$ of the composed bundle $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$ coincide with the pullback bundles $\iota_x^{*}\mathcal{E}$ for all $x \in B$, where $M_x \xhookrightarrow{\iota_x} M$ is the embedding of the fibre M_x into M. In this context, we equip each of these bundles $\mathcal{E}_x$ with the pullback bundle metric $h_x = \iota_x^{*}h \in C^{\infty}(\mathcal{E}_x^{*\otimes 2})$ and the pullback connection $\nabla^{\mathcal{E}_x} = \iota_x^{*}\nabla^{\mathcal{E}} : C^{\infty}(\mathcal{E}_x) \to C^{\infty}(\mathsf{T}^{*}M_x \otimes \mathcal{E}_x)$, as well as each of the related base manifolds M_x with the pullback Riemannian metric $g_{M_x} = \iota_x^{*} g_{\mathsf{V}} \in C^{\infty}(\Sigma^2 \mathsf{T} M_x)$, and obtain fibrewise Hermitian vector bundles $\pi_{\mathcal{E}_x} : (\mathcal{E}_x, h_x, \nabla^{\mathcal{E}_x}) \to (M_x, g_{M_x})$.

2.1.3 Boundedness Properties

In order to study global properties of differential operators on a vector bundle over a non-compact base, we need to specify additional uniformity conditions on the geometric objects.

We first require suitable boundedness properties for the Riemannian manifold (M, g) with an additional fibre bundle structure. Therefore, we transfer the notion of a vector bundle of bounded geometry to a fibre bundle with compact fibres, in that we require the existence of a suitable set of local trivialisations with uniform bounds. This concept of so-called *uniformly locally trivial fibre bundles* was introduced in [Lam14, Section A.1]. In particular, Lampart proved that

- (M, g) is a ∂-manifold of bounded geometry in the sense of Definition A.15 if $\partial F \neq \emptyset$,
- and (M, g) is manifold of bounded geometry in the sense of Definition A.14 if $\partial F = \emptyset$.

Thus, we can finally impose uniform boundedness properties for the $\mathbb{C}^N$-vector bundle over M. Let us state the precise requirements on the underlying geometry that will be used throughout this thesis:

Condition 2.2 With the notation previously introduced in this section, we require that

(i) $\pi_M : (M, g) \to (B, g_B)$ be a uniformly locally trivial fibre bundle (see [Lam14, Definition A.3]) with compact typical fibre F, i.e.,
   - (B, g_B) is a manifold of bounded geometry,
   - and there exists a Riemannian metric g_F on F such that for all $r < r_{\mathrm{inj}}(B, g_B)$ and $x \in B$ there is a local trivialisation
   $$\Phi : \left(\pi_M^{-1}(U), g\right) \to (U \times F, g_B \times g_F)$$
   on the geodesic ball $U = \mathsf{B}_r(x)$ for which $\mathsf{T}\Phi$ and Φ^* are bounded tensors together with all their covariant derivatives uniformly in r and x,

(ii) and $\pi_{\mathcal{E}} : \mathcal{E} \to M$ together with the set of local synchronous trivialisations (see Definition A.18) be a $\mathbb{C}^N$-vector bundle of bounded geometry. ◊

Remark 2.3 The bounded geometry of the manifold (B, g_B) implies the existence of an atlas $\{(U_\nu, \kappa_\nu)\}_{\nu \in \mathbb{N}_0}$ of geodesic balls $U_\nu = \mathsf{B}_r(x_\nu)$ with centres x_ν and fixed radius $r < r_{\mathrm{inj}}(B, g_B)$, see Lemma A.16. In this context, we introduce ν-dependent models $\mathcal{F}_\nu$ as Hermitian $\mathbb{C}^N$-vector bundles $(\mathcal{F}, h_\nu, \nabla_\nu) \xrightarrow{\pi_{\mathcal{F}}} (F, g_F)$, each of them endowed with bundle metric $h_\nu = (\Psi_\nu|_{\mathcal{E}_{x_\nu}}^{-1})^* h_{x_\nu}$ and metric connection $\nabla_\nu = (\Psi_\nu|_{\mathcal{E}_{x_\nu}}^{-1})^* \nabla^{\mathcal{E}_{x_\nu}}$ for local trivialisations $\Phi_\nu : \pi_M^{-1}(U_\nu) \to U_\nu \times F$ and $\Psi_\nu : \Pi_{\mathcal{E}}^{-1}(U_\nu) \to U_\nu \times \mathcal{F}$. Note that Remark A.13 yields $W^k(\mathcal{F}_\nu) = W^k(\mathcal{F}_{\nu'})$ for all $k, \nu, \nu' \in \mathbb{N}_0$ as topological vector spaces. $\Diamond$

One might now ask for the boundedness properties of the composed fibre bundle $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$. If one could show that $\mathcal{F}$ as a Riemannian manifold endowed with the (most natural) Sasaki metric $g_{\mathcal{F}}$, see Definition A.9, is a manifold of bounded geometry, one could adapt the definition of a uniformly locally trivial fibre bundle by replacing the compact manifold with a manifold of bounded geometry and obtain local trivialisations that are bounded with all their derivatives. $(\mathcal{F}, g_{\mathcal{F}})$, however, is not a manifold of bounded geometry since the associated curvature $\mathrm{R}^{g_{\mathcal{F}}}$ at $\psi \in \mathcal{F}_y$, $y \in F$, grows with the length $\|\psi\|$ of the vector ψ (see [Bla10, Section 9.1]). Nevertheless, we will show in the remainder of this subsection that the local trivialisations of $\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} F$ constructed in Proposition 2.1 are in fact uniformly bounded. To start with, let $\Phi : \pi_M^{-1}(U) \to U \times F$ be the local trivialisation of M that is provided for an geodesic ball $U \subset B$. Then for any $x, x' \in U$ and $y, y' \in F$ there is a length-minimising curve $\gamma : [0,1] \to U \times F$ with end points (x, y) and (x', y'). The Riemannian distance between the points $\Phi^{-1}(x, y)$ and $\Phi^{-1}(x', y')$ may thus be estimated from above by the length of the curve $\Phi^{-1} \circ \gamma : [0,1] \to \pi_M^{-1}(U)$, i.e.,

$$\begin{aligned}
&\mathrm{dist}_g\big(\Phi^{-1}(x,y), \Phi^{-1}(x',y')\big)\\
&\leqslant \int_0^1 \sqrt{g\big(\Phi^* \dot\gamma, \Phi^* \dot\gamma\big)}\,\mathrm{d}t\\
&\leqslant \int_0^1 C(\Phi)(g_B \times g_F)(\dot\gamma, \dot\gamma)\,\mathrm{d}t\\
&\leqslant C(\Phi)\big(\mathrm{dist}_{g_B}(x,x') + \mathrm{dist}_{g_F}(y,y')\big).
\end{aligned} \tag{2.3}$$

Note that C can be chosen independently of U (i.e., independently of Φ) due to the uniform local triviality of $(M, g) \xrightarrow{\pi_M} (B, g_B)$.

Lemma 2.4 *There exist $r_B < r_{\mathrm{inj}}(B, g_B)$ and a covering of (F, g_F) by normal charts $\{(V_\lambda, \varpi_\lambda)\}_{\lambda \in \mathbb{Z}}$ such that each set $\Phi^{-1}(U \times V_\lambda)$ is contained in a normal chart of (M, g) for all geodesic balls $U = \mathsf{B}_r(x) \subset B$ with radius $r < r_B$, where $\Phi : \pi_M^{-1}(U) \to U \times F$ is the local trivialisation associated with U.*

Proof ($\partial M = \emptyset$). We choose $r_B < r_{\mathrm{inj}}(B, g_B)$ as well as $r_F < r_{\mathrm{inj}}(F, g_F)$ small enough such that $r_B + r_F < \frac{r_{\mathrm{inj}}(M,g)}{C}$ with constant $C > 0$ as in (2.1.3). This gives

$$\mathrm{dist}_g\left(\Phi^{-1}(x, y), \Phi^{-1}(x', y')\right) < r_{\mathrm{inj}}(M, g)$$

for all $x' \in \mathsf{B}_{r_B}(x)$, $y' \in \mathsf{B}_{r_F}(y)$ and $(x, y) \in B \times F$, and it follows that $\Phi^{-1}(\mathsf{B}_{r_B}(x) \times \mathsf{B}_{r_F}(y))$ is entirely contained in an (M, g)-geodesic ball around $\Phi^{-1}(x, y)$ with radius $r_{\mathrm{inj}}(M, g)$. We conclude by choosing points $\{y_\lambda\}_{\lambda \in \mathbb{N}_0}$ in F such that

$$\bigcup_{\lambda \in \mathbb{N}_0} \underbrace{\mathsf{B}_{r_F}(y_\lambda)}_{=:V_\lambda} = F.$$

□

Proof ($\partial M \neq \emptyset$). We have data $\{r_{\mathrm{C},M}, r_{\mathrm{inj}}(\partial M, g|_{\partial M}), r_{\mathrm{I},M}\}$ for (M, g), $r_{\mathrm{inj}}(B, g_B)$ for (B, g_B) and $\{r_{\mathrm{C},F}, r_{\mathrm{inj}}(\partial F, g_F|_{\partial F}), r_{\mathrm{I},F}\}$ for (F, g_F). We want to construct an atlas of (F, g_F) as in Lemma A.16. Therefore, we reduce the collar width of F in such a way that $r_{\mathrm{C},F} < \frac{r_{\mathrm{C},M}}{C}$ with $C > 0$ derived in (2.1.3). This implies

$$\Phi^{-1}\left(\mathsf{B}_r(x) \times \mathsf{NC}_F(r_{\mathrm{C},F})\right) \subset \mathsf{NC}_M(r_{\mathrm{C},M}) \tag{2.4}$$

for all $0 < r < r_{\mathrm{inj}}(B, g_B)$ and $x \in B$. We treat the cases of boundary collar charts $\{(V_\lambda, \varpi_\lambda)\}_{\lambda < 0}$ and interior charts $\{(V_\lambda, \varpi_\lambda)\}_{\lambda \in \mathbb{N}_0}$ separately:

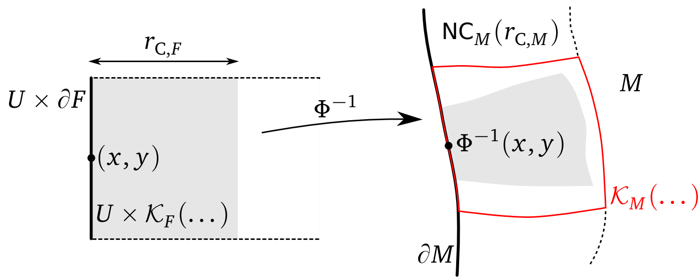
Boundary Collar Charts:
Let $y \in \partial F$ be arbitrary. Choose $r_B < r_{\text{inj}}(B, g_B)$ and $r_{\partial,F} < r_{\text{inj}}(\partial F, g_F|_{\partial F})$ small enough such that $r_B + r_{\partial,F} < \frac{r_{\text{inj}}(\partial M, g|_{\partial M})}{C}$, which means

$$\Phi^{-1}\big(\mathsf{B}_{r_B}(x) \times \mathsf{B}_{r_{\partial,F}}(y)\big) \subset \mathsf{B}_{r_{\text{inj}}(\partial M, g|_{\partial M})}\big(\Phi^{-1}(x,y)\big)$$

for all $(x,y) \in B \times \partial F$. Moreover, we apply (2.4) to further scale down r_B and $r_{\partial,F}$ so that

$$\begin{aligned}&\Phi^{-1}\Big(\mathsf{B}_{r_B}(x) \times \mathcal{K}_F\big(\mathsf{B}_{r_{\partial,F}}(y) \times [0, r_{\mathrm{C},F})\big)\Big)\\ &\subset \mathcal{K}_M\Big(\mathsf{B}_{r_{\text{inj}}(\partial M, g|_{\partial M})}\big(\Phi^{-1}(x,y)\big) \times [0, r_{\mathrm{C},M})\Big)\end{aligned}$$

(cf. Definition A.15 for the precise meaning of the collar maps $\mathcal{K}_F$ and $\mathcal{K}_M$).

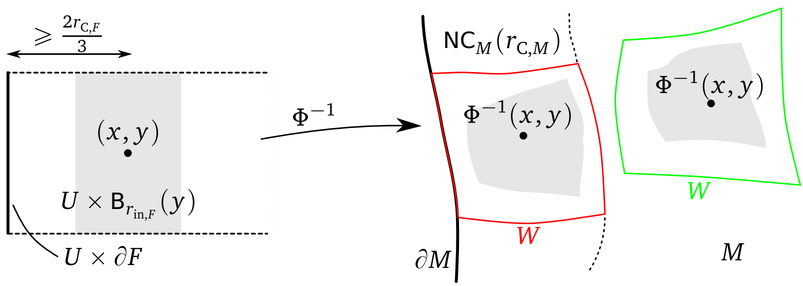


Note that this downsizing can always be carried out uniformly in $x \in B$ and $y \in \partial F$, which can be seen as follows: By the triangle inequality, the set $\mathsf{B}_{r_B}(x) \times \mathcal{K}_F(\mathsf{B}_{r_{\partial,F}}(y) \times [0, r_{\mathrm{C},F}))$ is completely contained in a $(g_B \times g_F)$-metric ball around (x,y) with radius $r_B + r_{\partial,F} + r_{\mathrm{C},F}$ for all $(x,y) \in B \times \partial F$. The image of the latter ball under Φ^{-1} is in turn contained in a g-metric ball $\mathcal{B}$ around $\Phi^{-1}(x,y)$ with radius $C(r_B + r_{\partial,F} + r_{\mathrm{C},F})$. We then scale down r_B, $r_{\partial,F}$ and $r_{\mathrm{C},F}$ (if necessary) until the former radius goes below some x-independent constant $r = r(r_{\mathrm{C},M}, r_{\text{inj}}(\partial M, g|_{\partial M})$ and [Sch96, Lemma 3.19] is applicable, asserting that $\mathcal{B}$ is contained in

$\mathcal{K}_M(\mathsf{B}_{r_{\mathrm{inj}}(\partial M, g|_{\partial M})}(\Phi^{-1}(x,y)) \times [0, r_{\mathrm{C},M}))$. We then choose points $\{y_\lambda\}_{\lambda<0}$ in ∂F such that the set $\{\mathsf{B}_{r_{\partial,F}}(y_\lambda)\}_{\lambda<0}$ is covering of ∂F, i.e.,

$$\bigcup_{\lambda<0} \underbrace{\mathcal{K}_F\left(\mathsf{B}_{r_{\partial,F}}(y_\lambda) \times [0, r_{\mathrm{C},F})\right)}_{=:V_\lambda} = \mathsf{NC}_F(r_{\mathrm{C},F}).$$

Interior Charts:
For any point $y \in F \backslash \mathsf{NC}_F(\frac{2r_{\mathrm{C},F}}{3})$, we consider geodesic balls $\mathsf{B}_{r_{\mathrm{in},F}}(y)$ with radius $r_{\mathrm{in},F} < r_{\mathrm{I},F}$ small enough such that $\mathsf{B}_{r_{\mathrm{in},F}}(y) \cap F \backslash \mathsf{NC}_F(\frac{r_{\mathrm{C},F}}{3}) = \emptyset$. Then $\mathsf{B}_{r_B}(x) \times \mathsf{B}_{r_{\mathrm{in},F}}(y)$ is contained in a $(g_B \times g_F)$-metric ball around (x, y) with radius $r_B + r_{\mathrm{in},F}$ for all $x \in B$. Consequently, the image of the ball under Φ^{-1} is contained in a g-metric ball $\mathcal{B}$ around $\Phi^{-1}(x, y)$ with radius $C(r_B + r_{\mathrm{in},F})$. By scaling down r_B and $r_{\mathrm{in},F}$ appropriately, we can make this radius smaller than some constant $r = r(r_{\mathrm{C},M}, r_{\mathrm{inj}}(\partial M, g|_{\partial M}), r_{\mathrm{I},M})$ and [Sch96, Lemma 3.19] yields again an (M, g)-normal coordinate chart (W, τ) – either a boundary collar chart (red) or an interior chart (green) – such that $\mathcal{B} \subset W$.



Finally, we again choose points $\{y_\lambda\}_{\lambda \in \mathbb{N}_0}$ in F such that

$$\bigcup_{\lambda \in \mathbb{N}_0} \underbrace{\mathsf{B}_{r_{\mathrm{in},F}}(y_\lambda)}_{=:V_\lambda} = F \backslash \mathsf{NC}_F\left(\tfrac{2r_{\mathrm{C},F}}{3}\right).$$

□

Due to the compactness of F, the covering provided by the previous lemma always admits (after a possible renaming of the indices) a finite subcovering $\{(V_\lambda, \varpi_\lambda)\}_{\lambda=1}^K$ with normal coordinates.

We may now link the boundedness properties of the two involved bundles in the following sense:

Lemma 2.5 *Let $r_B < r_{\mathrm{inj}}(B, g_B)$ and $\{(V_\lambda, \varpi_\lambda)\}_{\lambda=1}^K$ be the quantities of Lemma 2.4. Cover (B, g_B) with any set of geodesic balls $\{U_\nu = \mathsf{B}_r(x_\nu)\}_{\nu\in\mathbb{N}_0}$ with radius $r < r_B$ and obtain local trivialisations $\Phi_\nu^{-1}(U_\nu) \to U_\nu \times F$. Then there exist local trivialisations*

$$\Lambda_{\nu\lambda} : \mathcal{E}|_{\Phi_\nu^{-1}(U_\nu \times V_\lambda)} \to \Phi_\nu^{-1}(U_\nu \times V_\lambda) \times \mathbb{C}^N, \quad (\nu, \lambda) \in \mathbb{N}_0 \times \{1, \ldots, K\}$$

of $\pi_{\mathcal{E}} : \mathcal{E} \to M$ which are bounded with all their derivatives uniformly in ν and λ.

Proof. Lemma 2.4 states that every set $\Phi_\nu^{-1}(U_\nu \times V_\lambda)$ is completely contained in an (M, g)-normal chart (W, τ), for which the bounded geometry of $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ provides a local synchronous trivialisation $\Lambda : \pi_{\mathcal{E}}^{-1}(W) \to W \times \mathbb{C}^N$. Therefore, the restrictions

$$\Lambda_{\nu\lambda} := \Lambda|_{\Phi_\nu^{-1}(U_\nu \times V_\lambda)}$$

have all the desired boundedness properties uniformly in ν and λ. □

We are now in a position to give a result for the boundedness properties of the fibre bundle $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$. We will therefore revise the construction of the local trivialisations $\{\Psi_\nu : \pi_{\mathcal{E}}^{-1}(U_\nu) \to U \times \mathcal{F}\}_{\nu\in\mathbb{N}_0}$ obtained in Proposition 2.1 by thoroughly following the steps in the proof of [Hat09, Theorem 1.6] and specifying all involved objects explicitly.

Proposition 2.6 *Let $\pi_M : (M, g) \to (B, g_B)$ be a uniformly locally trivial fibre bundle with compact typical fibre (F, g_F) and $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be a $\mathbb{C}^N$-vector bundle of bounded geometry. Cover (B, g_B) with geodesic balls $\{U_\nu\}_{\nu\in\mathbb{N}_0}$ with radius $r < r_B$ and obtain local trivialisations $\Phi_\nu^{-1}(U_\nu) \to U_\nu \times F$. Then the corresponding local trivialisations $\Psi_\nu : \Pi_{\mathcal{E}}^{-1}(U_\nu) \to U_\nu \times \mathcal{F}_\nu$ of*

Proposition 2.1 are bounded with all their derivatives uniformly in ν in the following sense: For all synchronous trivialisations $\Lambda_{\mathcal{E}} : \pi_{\mathcal{E}}^{-1}(W) \to W \times \mathbb{C}^N$ and $\Lambda_{\mathcal{F}_\nu} : \pi_{\mathcal{F}}^{-1}(V) \to V \times \mathbb{C}^N$ of $\mathcal{E}$ and $\mathcal{F}_\nu$ (cf. Remark 2.3), respectively, the mappings

$$\left(\mathbf{1}_{U_\nu}, \Lambda_{\mathcal{F}_\nu}\right) \circ \Psi_\nu \circ \Lambda_{\mathcal{E}}^{-1} \circ \left(\Phi_\nu^{-1}, \mathbf{1}_{\mathbb{C}^N}\right) \tag{2.5}$$

and their inverses are homomorphisms on $\mathbb{C}^N$, bounded with all their derivatives on $\Phi_\nu(W) \cap (U_\nu \times F)$ uniformly in ν.

PROOF. The first ingredient is the diffeotopy

$$\begin{aligned} h_\nu : (U_\nu \times F) \times [0,1] &\to U_\nu \times F, \\ (x, y, t) &\mapsto \left(\kappa_\nu^{-1}\big(t\kappa_\nu(x)\big), y\right) =: (tx, y), \end{aligned}$$

which describes the shrinking of the geodesic ball U_ν with constant speed to the point $x_\nu = \kappa_\nu^{-1}(0)$ using the normal chart $\kappa_\nu : U_\nu \to \mathbb{B}_r^b(0)$. Hence, it holds that

$$\left\{(tx, y) \text{ such that } (x, y) \in U_\nu \times F\right\} \subset U_\nu \times F$$

for all $t \in [0,1]$ and

$$h(\cdot, 1) = \mathbf{1}_{U_\nu \times F} \quad , \quad h(\cdot, 0) = \{x_\nu\} \times \mathbf{1}_F .$$

This induces pullback bundles according to the following diagram (with the shortcut $\mathcal{E}_\nu := \Pi_{\mathcal{E}}^{-1}(U_\nu)$):

$$\begin{tikzcd} \mathcal{E}_\nu \arrow[r] \arrow[d, "\pi_{\mathcal{E}}"'] & \overbrace{(\Phi_\nu^{-1})^* \mathcal{E}_\nu}^{=:\bar{\mathcal{E}}_\nu} \arrow[r] \arrow[d, "\bar{\pi}_{\mathcal{E}}"] & \overbrace{h_\nu^* (\Phi_\nu^{-1})^* \mathcal{E}_\nu}^{=:\hat{\mathcal{E}}_\nu} \arrow[d, "\hat{\pi}_{\mathcal{E}}"] \\ \pi_M^{-1}(U_\nu) & U_\nu \times F \arrow[l, "\Phi_\nu^{-1}"] & (U_\nu \times F) \times [0,1] \arrow[l, "h_\nu"] \end{tikzcd}$$

The utilised covering $\{(V_\lambda, \varpi_\lambda)\}_{\lambda=1}^K$ of the fibre (F, g_F) comes along with a smooth partition of unity $\{\chi_\lambda\}_{\lambda=1}^K$, which also induces a partition of unity of $U_\nu \times F$ via $\chi_{\nu\lambda}(x,y) := \chi_\lambda(y)$. Then the family $\{\zeta_{\nu\sigma}(x,\cdot)\}_{\sigma=0}^{K}$ defined by the relations

$$\zeta_{\nu 0} \equiv 1 \quad , \quad \zeta_{\nu\sigma}(x,y) := \sum_{\lambda=1}^{\sigma} \chi_{\nu\lambda}(x,y) = \sum_{\lambda=1}^{\sigma} \chi_\lambda(y) \text{ for } \sigma \geqslant 1$$

is monotonically increasing and approaches the constant function $\zeta_{\nu K} \equiv 1$ pointwise. Two neighbouring mappings $\zeta_{\nu\sigma}$ and $\zeta_{\nu,\sigma-1}$ obviously differ only on the set $U_\nu \times V_\sigma$. Then for all $\sigma \in \{0,\dots,K\}$ we set

$$\Xi_{\nu\sigma} := \mathrm{graph}(\zeta_{\nu\sigma}) \subset (U_\nu \times F) \times [0,1], \quad \sigma \in \{0,\dots,K\}$$

and consider the respective restriction

$$\hat{\mathcal{E}}_\nu^\sigma := \hat{\mathcal{E}}_\nu\big|_{\Xi_{\nu\sigma}} = \Big\{ \Big(\big(x,y,\zeta_{\nu\sigma}(x,y)\big), \psi \Big) \in \Xi_{\nu\sigma} \times \bar{\mathcal{E}}_\nu \\ \text{such that } \psi \in \big(\hat{\mathcal{E}}_\nu\big)_{(\zeta_{\nu\sigma}(x,y)x,y)} \Big\}$$

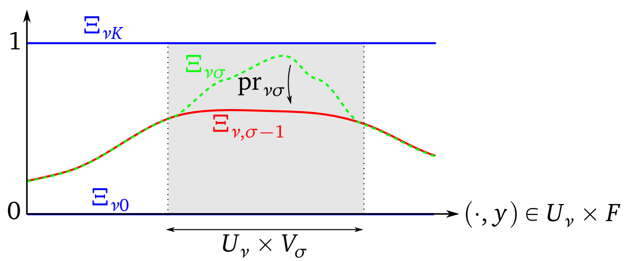
as a vector bundle over $U_\nu \times F$, whose fibre at (x,y) is given by that of $\mathcal{E}$ at $\Phi_\nu^{-1}(\zeta_\sigma(x,y)x,y) \subset \pi_M^{-1}(U_\nu)$. Consequently, we get (see the proof of Proposition 2.1)

$$\hat{\mathcal{E}}_\nu^K = \hat{\mathcal{E}}_\nu\big|_{(U_\nu\times F)\times\{1\}} = h(\cdot,1)^*\bar{\mathcal{E}}_\nu = \bar{\mathcal{E}}_\nu$$

and

$$\hat{\mathcal{E}}_\nu^0 = \hat{\mathcal{E}}_\nu\big|_{(U_\nu\times F)\times\{0\}} = h(\cdot,0)^*\bar{\mathcal{E}}_\nu = U_\nu \times \mathcal{F}$$

at the endpoints $\sigma = K$ and $\sigma = 0$, respectively.



We note that these bundles come along with local trivialisations

$$\begin{aligned} \hat{\Lambda}^{\sigma}_{\nu|\lambda} : \hat{\mathcal{E}}^{\sigma}_{\nu}\big|_{U_\nu \times V_\lambda} &\to (U_\nu \times V_\lambda) \times \mathbb{C}^N, \\ \Big(\big(x, y, \zeta_{\nu\sigma}(x,y)\big), \psi\Big) &\mapsto \big(x, y, \mathrm{pr}_2\, \Lambda_{\nu\lambda}(\psi)\big) \end{aligned} \tag{2.6}$$

for all $\lambda \in \{1, \dots, K\}$, where $\{\Lambda_{\nu\lambda}\}_{\lambda=1}^{K}$ are the local trivialisations of $\mathcal{E}_\nu$ constructed in Lemma 2.5.

$$\begin{tikzcd}
& & & \hat{\mathcal{E}}^K_\nu = \bar{\mathcal{E}}_\nu \arrow[dd, "\Upsilon_\nu"] \\
\mathcal{E}_\nu \arrow[r, "(\Phi_\nu^{-1})^*"] \arrow[drrr, leftrightarrow, red, "\Psi_\nu"'] & \bar{\mathcal{E}}_\nu \arrow[urr, "\mathbf{1}_{\bar{\mathcal{E}}_\nu}"] \arrow[r, dotted] & \hat{\mathcal{E}}_\nu \arrow[ur, dotted] \arrow[dr, dotted] & \\
& & & \hat{\mathcal{E}}^0_\nu = U_\nu \times \mathcal{F}
\end{tikzcd}$$

It remains to specify the diffeomorphism $\Upsilon_\nu : \bar{\mathcal{E}}_\nu \to U_\nu \times \mathcal{F}$, whose construction is actually carried out in the proof of [Hat09, Theorem 1.6]. To do so, we first need to lift each projection $\mathrm{pr}_{\nu\sigma} : \Xi_{\nu\sigma} \to \Xi_{\nu,\sigma-1}$ to an isomorphism $\mathrm{PR}_{\nu\sigma} : \hat{\mathcal{E}}^{\sigma}_{\nu} \to \hat{\mathcal{E}}^{\sigma-1}_{\nu}$. Therefore, we note that $\zeta_{\nu\sigma} = \zeta_{\nu,\sigma-1}$ on $U_\nu \times \complement V_\sigma$ implies

$$\hat{\mathcal{E}}^{\sigma}_{\nu}\big|_{U_\nu \times \complement V_\sigma} = \hat{\mathcal{E}}^{\sigma-1}_{\nu}\big|_{U_\nu \times \complement V_\sigma}.$$

Moreover, on $U_\nu \times V_\sigma$ we identify vectors of $(\hat{\mathcal{E}}^{\sigma}_{\nu})_{(\cdot, \zeta_{\nu\sigma}(\cdot))}$ with those of $(\hat{\mathcal{E}}^{\sigma-1}_{\nu})_{(\cdot, \zeta_{\nu,\sigma-1}(\cdot))}$ by means of the local trivialisations (2.6). This eventually leads to

$$\mathrm{PR}_{\nu\sigma} := \begin{cases} \mathbf{1}_{\hat{\mathcal{E}}^{\sigma}_{\nu}}, & \text{on } \hat{\mathcal{E}}^{\sigma}_{\nu}\big|_{U_\nu \times \complement V_\sigma} \\ \big(\hat{\Lambda}^{\sigma-1}_{\nu|\sigma}\big)^{-1} \circ \hat{\Lambda}^{\sigma}_{\nu|\sigma}, & \text{on } \hat{\mathcal{E}}^{\sigma}_{\nu}\big|_{U_\nu \times V_\sigma}, \end{cases} \qquad \sigma \in \{1, \dots, K\}.$$

Due to the fact that $\chi_\sigma \equiv 0$ on $\complement V_\sigma$, the transition function $(\hat{\Lambda}^{\sigma-1}_{\nu|\sigma})^{-1} \circ \hat{\Lambda}^{\sigma}_{\nu|\sigma}$ is the identity over all $(x,y) \in U_\nu \times \complement V_\sigma$, in particular near $U_\nu \times \partial V_\sigma$, so there is no discontinuity there. The desired diffeomorphism is finally obtained by the finite composition

$$\Upsilon_\nu = \mathrm{PR}_{\nu 1} \circ \dots \circ \mathrm{PR}_{\nu K}\,.$$

Now take trivialisations $\Lambda_{\mathcal{E}}$ and $\Lambda_{\mathcal{F}_\nu}$ as in the statement and let $\bar{\Lambda}_{\mathcal{E}}$ be the induced local trivialisation of $\bar{\mathcal{E}}_\nu$ over $\Phi_\nu(W)$. Suppose without loss of generality that $\Phi_\nu(W) \cap (U_\nu \times F) \neq \emptyset$ (otherwise the statement would be trivial) and take any point (x, y) of this intersection. Denote the elements of the set

$$\{\lambda \in \{1, \ldots, K\} \text{ such that } y \in V_\lambda\}$$

by $\theta_1 < \cdots < \theta_L$. Then there is an open neighbourhood of (x, y) over which we have $\Upsilon_\nu = \mathrm{PR}_{\nu\theta_1} \circ \cdots \circ \mathrm{PR}_{\nu\theta_L}$. Using the fact that $\hat{\Lambda}^{\theta_s - 1}_{\nu|\theta_s} = \hat{\Lambda}^{\theta_{s-1}}_{\nu|\theta_s}$ for all $s \in \{1, \ldots, L\}$ (with $\theta_0 = 0$), we consequently find that (2.5) turns into

$$\left((\mathbf{1}_{U_\nu}, \Lambda_{\mathcal{F}_\nu}) \circ \left(\hat{\Lambda}^0_{\nu|\theta_1}\right)^{-1} \right) \circ \left(\hat{\Lambda}^{\theta_1}_{\nu|\theta_1} \circ \left(\hat{\Lambda}^{\theta_1}_{\nu|\theta_2}\right)^{-1} \right) \circ \cdots \circ \left(\hat{\Lambda}^{\theta_L}_{\nu|\theta_L} \circ \bar{\Lambda}^{-1}_{\mathcal{E}} \right)$$

near (x, y).

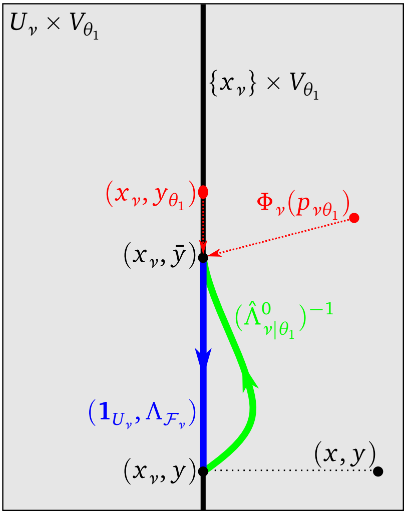
- As far as the last term is concerned, it follows that (since $\Xi_{\nu\theta_L} \equiv 1$ near (x, y)) $\mathcal{E}^{\theta_L}_\nu = \hat{\mathcal{E}}^K_\nu = \bar{\mathcal{E}}_\nu$ near (x, y), and hence $\hat{\Lambda}^{\theta_L}_{\nu|\theta_L} \circ \bar{\Lambda}^{-1}_{\mathcal{E}}$ is essentially a transition function between synchronous trivialisations of $\bar{\mathcal{E}}_\nu$.

- Each intermediate term

  $$\hat{\Lambda}^{\theta_s}_{\nu|\theta_s} \circ \left(\hat{\Lambda}^{\theta_s}_{\nu|\theta_{s+1}}\right)^{-1}, \quad s \in \{1, \ldots, L-1\}$$

  is a transition function of the bundle $\hat{\mathcal{E}}^{\theta_s}_\nu$ over $U_\nu \times (V_{\theta_s} \cap V_{\theta_{s+1}})$, smooth and its bounds depend on those of $\{\hat{\Lambda}_{\nu\lambda}\}^K_{\lambda=1}$, $\{\chi_\lambda\}^K_{\lambda=1}$ and κ_ν.

- We note that $\hat{\Lambda}^0_{\nu|\theta_1}$ – although it is the restriction of the local trivialisation $(\Phi_\nu, \mathbf{1}_{\mathbb{C}^N}) \circ \Lambda_{\nu\theta_1}$ to $\{x_\nu\} \times V_{\theta_1}$ – does not give an synchronous trivialisation of $\mathcal{F}_\nu$ since it is associated with normal coordinates on (M, g) which do in general not restrict to normal coordinates on (M_x, g_{M_x}), for example if $M_x \hookrightarrow M$ is not totally geodesic. At the expense of introducing further transition functions on $\mathcal{E}$ and $\mathcal{F}_\nu$, however, we may assume that both $(\Phi, \mathbf{1}_{\mathbb{C}^N}) \circ \Lambda_{\nu\theta_1}$ and $(\mathbf{1}_{U_\nu}, \Lambda_{\mathcal{F}_\nu})$ stem from parallel transport along normal coordinates that are centred around the same point $(x_\nu, \bar{y})$. Then the transition function $(\mathbf{1}_{U_\nu}, \Lambda_{\mathcal{F}_\nu}) \circ (\hat{\Lambda}^0_{\nu|\theta_1})^{-1}$ is obtained by parallel transport along a closed curve in $U_\nu \times V_{\theta_1}$, starting/ending at (x_ν, y) and passing through $(x_\nu, \bar{y})$, and therefore given by the holonomy of $(\Phi_\nu^{-1})^* \nabla^{\mathcal{E}}$. In view of [GS13, Lemma 5.13], this can be bounded in terms of the bounds on Φ_ν and $\mathrm{R}^{\mathcal{E}}$ by writing it as the solution of a differential equation.



Since all bounds are independent of (x, y) and uniform in ν, the statement is proven. $\square$

Due to the compactness of F, another choice of bundle metric and metric connection on $\mathcal{F}$ still gives bounded trivialisations. These bounds, however, may then depend on x, which occurs for example if one scales the bundle metric h_ν with an x-dependent factor.

We close this subsection by collecting the characteristics which reflect the uniform boundedness of the underlying geometry:

Definition 2.7 We establish the following properties associated with the bundles $\pi_{\mathcal{E}} : \mathcal{E} \to M$ and $\pi_M : M \to B$:

- Since (B, g_B) is a manifold of bounded geometry without boundary, it comes along with an atlas $\mathfrak{U} := \{(U_\nu, \kappa_\nu)\}_{\nu \in \mathbb{N}_0}$ of geodesic balls $U_\nu = \mathsf{B}_r(x_\nu)$ with centres x_ν and uniform radius $r < r_B < r_{\mathrm{inj}}(B, g_B)$, and a subordinate partition of unity $\{\chi_\nu\}_{\nu \in \mathbb{N}_0}$ with finite multiplicity $N_{\mathfrak{U}}$ as in Lemma A.16.
- We equip each ball U_ν with a g_B-orthonormal frame $\{X_i^\nu\}_{i=1}^b$ of $\mathsf{T}B|_{U_\nu} = \mathsf{T}U_\nu$, obtained by parallel transport along radial geodesics starting at x_ν with respect to the Levi-Civita connection ∇^{g_B}. These form a set of smooth sections, uniformly bounded in i and ν because of the bounded geometry.
- The uniformly locally trivial fibre bundle $\pi_M : M \to B$ provides local trivialisations $\Phi_\nu : (\pi_M^{-1}(U_\nu), g) \to (U_\nu \times F, g_B \times g_F)$ for $\nu \in \mathbb{N}_0$, bounded uniformly in ν with all their derivatives.
- Proposition 2.1 states that each local trivialization Φ_ν induces a local trivialisation $\Psi_\nu : \Pi_{\mathcal{E}}^{-1}(U_\nu) \to U_\nu \times \mathcal{F}_\nu$ of $\Pi_{\mathcal{E}} : \mathcal{E} \to B$. These trivialisations are bounded uniformly in ν with all their derivatives due to Proposition 2.6. $\Diamond$

Note that $\{\chi_\nu^M := \chi_\nu \circ \pi_M\}_{\nu \in \mathbb{N}_0}$ is a smooth partition of unity of M that is subordinate to the covering $\{\pi_M^{-1}(U_\nu)\}_{\nu \in \mathbb{N}_0}$ and is bounded with all its derivatives.

2.1.4 The Adiabatic Limit

Our aim now is to consider a family of rescaled Riemannian submersion metrics

$$g^\varepsilon = \varepsilon^{-2} \pi_M^* g_B + g_{\mathsf{V}} \tag{2.7}$$

on M, which blows up the volume of the base B (the volume of the horizontal directions in M). This is also referred to as the *adiabatic limit* and was introduced by Witten to investigate the limit of the η-invariant of the Dirac operator [Wit85]. We remark that the horizontal distribution $\mathsf{H}M$ was fixed for $g^{\varepsilon=1}$ and hence remains ε-independent.

Definition 2.8 Let $\{X_i\}_{i=1}^b$ and $\{V_j\}_{j=1}^f$ be local orthonormal frames of the Euclidean vector bundles $(\mathsf{T}B, g_B)$ and $(\mathsf{V}M, g_{\mathsf{V}})$, respectively. We call the local orthonormal frame $(\{\varepsilon X_i^{\mathsf{H}}\}_{i=1}^b, \{V_j\}_{j=1}^f)$ of $(\mathsf{T}M, g^\varepsilon)$ *adapted* to the decomposition (2.1). $\Diamond$

Let us consider the $\varepsilon^{-2} g_B$-geodesic ball around any $x \in B$ with radius

$$r < r_{\mathrm{inj}}(B, g_B) = \varepsilon r_{\mathrm{inj}}(B, \varepsilon^{-2} g_B) < r_{\mathrm{inj}}(B, \varepsilon^{-2} g_B), \quad 0 < \varepsilon < 1.$$

This geodesic ball $\mathsf{B}_r^{\varepsilon^{-2} g_B}(x) = \mathsf{B}_{\varepsilon r}^{g_B}(x)$ is contained in the ball $\mathsf{B}_r^{g_B}(x)$ for all $0 < \varepsilon < 1$. Thus, we may cover $(B, \varepsilon^{-2} g_B)$ with a possibly greater number of balls $\{U_\nu^\varepsilon = \mathsf{B}_r^{\varepsilon^{-2} g_B}(x_\nu^\varepsilon)\}_{\nu \in \mathbb{N}_0}$, all of them being contained in balls $\mathsf{B}_r^{g_B}(x_\nu^\varepsilon)$ which depend on ε only via their centres. We can hence use the same (suitably restricted) local trivialisations of $(M, g) \xrightarrow{\pi_M} (B, g_B)$ for all $0 < \varepsilon < 1$ and conclude that $\pi_M : (M, g^\varepsilon) \to (B, \varepsilon^{-2} g_B)$ is a uniformly locally trivial fibre bundle with ε-independent bounds on the trivialisations. Moreover, (M, g^ε) is a (∂-)manifold of bounded geometry and the constants and bounds appearing in Definition A.15 can be chosen to be those of (M, g) [Lam14, Proposition A.9].

Remark 2.9 Lemma A.16 allows us to equip (M, g^ε) with a countable atlas $\{(W_\mu^\varepsilon, \tau_\mu^\varepsilon)\}_{\mu \in \mathbb{Z}}$ consisting of geodesic cylinders ($\mu < 0$) and geodesic balls ($\mu \in \mathbb{N}_0$) with radius $r < \frac{1}{3} \min\{r_{\mathrm{inj}}(\partial M, g|_{\partial M}), r_{\mathrm{I}}\}$ and centres p_μ^ε, and a subordinate partition of unity $\{\chi_\mu^\varepsilon\}_{\nu \in \mathbb{Z}}$.

(i) In view of [Sch96, Theorem A.1], the bounds on the metric coefficients $(g_\mu^\varepsilon)_{\alpha\beta}$ and their inverse $(g_\mu^\varepsilon)^{\alpha\beta}$ in these normal coordinates can be chosen independently of ε.

(ii) As can be seen in the construction in [Sch96, Lemma 3.22], both the bounds on $\{\mathrm{D}^{\mathfrak{a}}(\tau^{\varepsilon}_{\mu *}\chi^{\varepsilon}_{\mu}) \in C^{\infty}_{\mathrm{b}}(\mathbb{R}^{b+f})\}_{\mu\in\mathbb{Z}}$ for all $\mathfrak{a} \in \mathbb{N}_0^{b+f}$ and the multiplicity N_M can be chosen independently of ε. ◊

Passing over to the vector bundle $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g^{\varepsilon})$, one can similarly show that the bounds on the ε-dependent curvature $\mathrm{R}^{\mathcal{E}}$ of $\nabla^{\mathcal{E}}$ improve with decreasing parameter ε. More precisely, if $X, Y \in C^{\infty}_{\mathrm{b}}(\mathsf{T}B)$ are of g_B-length one, the related horizontal lifts $\varepsilon X^{\mathsf{H}}, \varepsilon Y^{\mathsf{H}} \in C^{\infty}_{\mathrm{b}}(\mathsf{H}M)$ are of g^{ε}-length one and we obtain for any $V, W \in C^{\infty}_{\mathrm{b}}(\mathsf{V}M)$ and $\psi \in C^{\infty}_{\mathrm{b}}(\mathcal{E})$:

$$\begin{aligned}
\left\|\mathrm{R}^{\mathcal{E}}\left(\varepsilon X^{\mathsf{H}}, \varepsilon Y^{\mathsf{H}}\right)\psi\right\|_h &= \varepsilon^2 \left\|\mathrm{R}^{\mathcal{E}}\left(X^{\mathsf{H}}, Y^{\mathsf{H}}\right)\psi\right\|_h, \\
\left\|\mathrm{R}^{\mathcal{E}}\left(\varepsilon X^{\mathsf{H}}, V\right)\psi\right\|_h &= \varepsilon \left\|\mathrm{R}^{\mathcal{E}}\left(X^{\mathsf{H}}, V\right)\psi\right\|_h, \\
\left\|\mathrm{R}^{\mathcal{E}}(V, W)\psi\right\|_h &= \left\|\mathrm{R}^{\mathcal{E}}(V, W)\psi\right\|_h.
\end{aligned}$$

As far as the local synchronous trivialisations (see Definition A.18) associated with the normal coordinates introduced in Remark 2.9 are concerned, we can argue as before to see $W^{\varepsilon}_{\mu} \subset W^{\varepsilon=1}_{\mu}$ for all $0 < \varepsilon < 1$. Hence, we may again use the same (suitably restricted) trivialisations as for $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$.

Remark 2.10 Let $(\Gamma^{\mathcal{E}}_{\mu})^A_{\alpha B}$ for $\alpha \in \{1, \ldots, b+f\}$ and $A, B \in \{1, \ldots, N\}$ be the Christoffel symbols of $\nabla^{\mathcal{E}}$ with respect to the synchronous trivialisations that arise from the atlas $\{(W^{\varepsilon}_{\mu}, \tau^{\varepsilon}_{\mu})\}_{\mu\in\mathbb{Z}}$ introduced in Remark 2.9. Then [Eic91, Theorem B], which can be extended to the case of ∂-base manifolds of bounded geometry by means of Definition A.18 (in the spirit of [Sch96, Theorem A.1]), asserts that all derivatives of the Christoffel symbols can be bounded uniformly in $\mu \in \mathbb{Z}$, i.e., for all $k \in \mathbb{N}_0$ there is a constant $C(k) > 0$ such that

$$\left\|\mathrm{D}^{\mathfrak{a}}\left(\tau^{\varepsilon}_{\mu *}(\Gamma^{\mathcal{E}}_{\mu})^A_{\alpha B}\right)\right\|_{\infty} \leqslant C(k)$$

for all $\mu \in \mathbb{Z}$ and all multi-indices $\mathfrak{a} \in \mathbb{N}_0^{b+f}$ with $|\mathfrak{a}| \leqslant k$. Moreover, these bounds $C(k)$ depend only on those of R^{ε}, and in particular, can be chosen to be those for $\varepsilon = 1$. ◊

2.2 The Laplacian

Denote by $\mathcal{H} := L^2(\mathcal{E}, \mathrm{vol}_g)$ the ε-independent Hilbert space of square-integrable sections of the vector bundle $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$, which is endowed with the scalar product

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \int_M h(\psi, \phi)\ \mathrm{vol}_g\,.$$

We will deal with Schrödinger operators on $\mathcal{H}$ whose kinetic part is essentially the (negative of the) *connection Laplacian* $-\Delta^{\mathcal{E}}_{g^\varepsilon}$ with Dirichlet boundary conditions. As is mentioned in Section A.3, the latter is initially defined as a differential operator on $C_0^\infty(\mathcal{E}^\circ)$ by the relation

$$-\Delta^{\mathcal{E}}_{g^\varepsilon}\psi = -\operatorname{tr}_{g^\varepsilon}\big((\nabla^{\mathcal{E}})^2\psi\big)$$

and extends to a positive self-adjoint operator on $\mathcal{H}$ with domain

$$\mathrm{dom}\big(-\Delta^{\mathcal{E}}_{g^\varepsilon}\big) = W^2\big(\mathcal{E}, \mathrm{vol}_{g^\varepsilon}\big) \cap W^1_0\big(\mathcal{E}, \mathrm{vol}_{g^\varepsilon}\big).$$

If we take an adapted local orthonormal frame of $(\mathsf{T}M, g^\varepsilon)$ as in Definition 2.8, we observe, using Einstein's sum convention in the case of repeated indices (which will always be employed throughout the thesis), that this Laplacian resembles the kinetic energy operator of the molecular Born-Oppenheimer Hamiltonian [PST07]:

$$\begin{aligned}
-\Delta^{\mathcal{E}}_{g^\varepsilon} &\overset{(A.14)}{=} -\Big(\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}} - \nabla^{\mathcal{E}}_{\nabla^{g^\varepsilon}_{\varepsilon X_i^{\mathsf{H}}}\varepsilon X_i^{\mathsf{H}}}\Big) - \Big(\nabla^{\mathcal{E}}_{V_j}\nabla^{\mathcal{E}}_{V_j} - \nabla^{\mathcal{E}}_{\nabla^{g^\varepsilon}_{V_j}V_j}\Big) \\
&= -\varepsilon^2\Big(\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}} - \nabla^{\mathcal{E}}_{(\nabla^{g_B}_{X_i}X_i)^{\mathsf{H}}}\Big) \\
&\quad - \Big(\nabla^{\mathcal{E}}_{V_j}\nabla^{\mathcal{E}}_{V_j} - \nabla^{\mathcal{E}}_{\nabla^{\mathsf{V}}_{V_j}V_j} - \nabla^{\mathcal{E}}_{\mathrm{II}^\varepsilon_{\mathsf{V}}(V_j,V_j)}\Big) \\
&= -\varepsilon^2\Big(\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}} - \nabla^{\mathcal{E}}_{(\nabla^{g_B}_{X_i}X_i)^{\mathsf{H}}}\Big) + \nabla^{\mathcal{E}}_{\eta^\varepsilon_{\mathsf{V}}} - \Big(\nabla^{\mathcal{E}}_{V_j}\nabla^{\mathcal{E}}_{V_j} - \nabla^{\mathcal{E}}_{\nabla^{\mathsf{V}}_{V_j}V_j}\Big) \\
&= -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} - \Delta^{\mathcal{E}}_{\mathsf{V}},
\end{aligned} \tag{2.8}$$

where we abbreviated

$$\nabla^{\mathsf{V}}_V W := \mathrm{P}^{\mathsf{V}M}\,\nabla^{g^\varepsilon}_V W \overset{(A.3)}{=} \mathrm{P}^{\mathsf{V}M}\,\nabla^{g}_V W$$

for vertical vector fields $V, W \in C^\infty(\mathsf{V}M)$. The *horizontal Laplacian*

$$\Delta_{\mathsf{H}}^{\mathcal{E}} = \operatorname{tr}_{\pi_M^* g_B}\big((\nabla^{\mathcal{E}})^2 \cdot\big) - \nabla^{\mathcal{E}}_{\varepsilon^{-2}\eta_{\mathsf{V}}^{\varepsilon}} = \operatorname{tr}_{\pi_M^* g_B}\big((\nabla^{\mathcal{E}})^2 \cdot\big) - \nabla^{\mathcal{E}}_{\eta_{\mathsf{V}}}$$

incorporates all derivatives in the horizontal directions including the mean curvature vector η_{V} of the fibres in (M, g^ε), which equals ε^2 times the respective vector η_{V} of the fibres in (M, g) [Lam14, Lemma 1.6]. The remainder $\Delta_{\mathsf{V}}^{\mathcal{E}}$ incorporates only vertical derivatives and is therefore called the *vertical Laplace operator*. The latter will be the starting point for the analysis of fibrewise acting operators of the following subsection.

2.2.1 Function Space Bundles

We begin with the observation that the vertical Laplacian is compatible with the embedding $M_x \xhookrightarrow{\iota_x} M$ of the fibres for all $x \in B$ in the sense that

$$\iota_x^*\big(\Delta_{\mathsf{V}}^{\mathcal{E}}\psi\big) = \Delta_{g_{M_x}}^{\mathcal{E}_x}\iota_x^*\psi, \quad \iota_x^*\psi \in C^\infty(\mathcal{E}_x)$$

holds for $\psi \in C^\infty(\mathcal{E})$. Put differently, the vertical Laplace operator $\Delta_{\mathsf{V}}^{\mathcal{E}}(x)$ at $x \in B$ coincides with the connection Laplacian

$$\Delta_{g_{M_x}}^{\mathcal{E}_x} = \operatorname{tr}_{g_{M_x}}\big(\nabla^{\mathsf{T}^* M_x \otimes \mathcal{E}_x}\nabla^{\mathcal{E}_x}\cdot\big), \quad \nabla^{g_{M_x}} = \iota_x^*\nabla^{\mathsf{V}}$$

on the vector bundle $(\mathcal{E}_x, h_x, \nabla^{\mathcal{E}_x}) \xrightarrow{\pi_{\mathcal{E}_x}} (M_x, g_{M_x})$. This can easily be seen as the covariant derivative $\nabla_{\mathsf{V}}^{\mathcal{E}}\psi$ along the vertical vector fields $V \in \{V_j, \nabla^{\mathsf{V}}_{V_j}V_j\}$ at $\xi \in M_x$ merely depends on the values of ψ along a curve in M_x that is tangent to $V(\xi) \in \mathsf{T}_\xi M_x$ and thus

$$\iota_x^*\big(\nabla_V^{\mathcal{E}}\psi\big) = \nabla^{\mathcal{E}_x}_{V|_{M_x}}\iota_x^*\psi.$$

It follows that the vertical Laplacian operates for each $x \in B$ on the fibrewise Hilbert space

$$L^2(\mathcal{E}_x) = \Big\{\text{mappings } \phi : M_x \to \mathcal{E}_x \text{ with } \phi(\xi) \in (\mathcal{E}_x)_\xi \text{ such}$$
$$\text{that } \|\phi\|^2_{L^2(\mathcal{E}_x)} \overset{\text{(A.11)}}{=} \int_{M_x} \|\phi(\xi)\|^2_{(h_x)_\xi} \operatorname{vol}_{g_{M_x}}(\xi) < \infty\Big\}.$$

We saw at the end of Subsection 2.1.2 that these spaces $L^2(\mathcal{E}_x)$ are all isomorphic to the topological vector space $L^2(\mathcal{F})$. Hence, we may think of these as the fibres of an infinite-dimensional vector bundle over B with typical fibre $L^2(\mathcal{F})$. More generally, we aim to construct vector bundles over B with various function spaces $X(\mathcal{F})$ as typical fibre, for instance certain Sobolev spaces and therefore the completions of $C_0^\infty(\mathcal{F})$ and $C_0^\infty(\mathcal{F}^\circ)$. A detailed exposition of the construction of continuous vector bundles $\pi_X : X(\mathcal{F};\Pi_{\mathcal{E}}) \to B$ with typical fibre $X(\mathcal{F})$ is given in [Lam14, Appendix B.1] for the case of a trivial line bundle $\mathcal{E} = M \times \mathbb{C}$ over M. But all of the arguments occurring there may be transferred immediately to our situation, so we will only briefly give the necessary modifications.

We take initial data from Definition 2.7 and define $C^\infty(\mathcal{F};\Pi_{\mathcal{E}})$ in terms of transition functions (using the abbreviation $U_{\nu\nu'} := U_\nu \cap U_{\nu'}$)

$$t_{\nu\nu'} : U_{\nu\nu'} \times C^\infty(\mathcal{F}) \to U_{\nu\nu'} \times C^\infty(\mathcal{F}), \quad (x,\phi) \mapsto \big(x, \mathsf{g}_{\nu\nu'}(x)\phi\big),$$

where

$$\begin{aligned} \mathsf{g}_{\nu\nu'}(x) : C^\infty(\mathcal{F}) &\to C^\infty(\mathcal{F}), \\ \phi &\mapsto \Psi_\nu|_{\mathcal{E}_x} \circ \Psi_{\nu'}|_{\mathcal{E}_x}^{-1} \circ \phi \circ \Phi_{\nu'}|_{M_x} \circ \Phi_\nu|_{M_x}^{-1}. \end{aligned} \tag{2.9}$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\Psi_{\nu'}|_{\mathcal{E}_x}^{-1}"] & \mathcal{E}_x \arrow[r, "\Psi_\nu|_{\mathcal{E}_x}"] & \mathcal{F} \\
F \arrow[u, "\phi"] & M_x \arrow[l, "\Phi_{\nu'}|_{M_x}"'] & F \arrow[l, "\Phi_\nu|_{M_x}^{-1}"'] \arrow[u, "\mathsf{g}_{\nu\nu'}(x)\phi"']
\end{tikzcd}$$

The completion of $C^\infty(\mathcal{F})$ with respect to the $\|\cdot\|_{W^k(\mathcal{F})}$-norm, which is independent of the choice of bundle metric and metric connection on $\mathcal{F}$ by Remark A.13, uniquely defines the structure of a continuous vector bundle $\pi_{W_k} : W^k(\mathcal{F};\Pi_{\mathcal{E}}) \to B$ with fibres $W^k(\mathcal{F};\Pi_{\mathcal{E}})\big|_x = W^k(\mathcal{E}_x)$ if the transition functions $t_{\nu\nu'}$ are continuous mappings from $U_{\nu\nu'} \times W^k(\mathcal{F})$ to $U_{\nu\nu'} \times W^k(\mathcal{F})$ for all $\nu, \nu' \in \mathbb{N}_0$ [Lam14, Lemma B.4]. By the uniform boundedness principle, this requirement is equivalent to the strong continuity of $\mathsf{g}_{\nu\nu'} : U_{\nu\nu'} \to \mathcal{L}(W^k(\mathcal{F}))$ for all $\nu, \nu' \in \mathbb{N}_0$. For this purpose, the uniform local triviality of $M \xrightarrow{\pi_M} B$ and Proposition 2.6 yield that the local trivialisations used for the "transition matrices" (2.9) are bounded uniformly with all their derivatives and the family (cf. Remark 2.3)

$$\Big\{\mathsf{g}_{\nu\nu'} : U_{\nu\nu'} \to \mathcal{L}\big(W^k(\mathcal{F}_\nu)\big) \text{ with } \nu, \nu' \in \mathbb{N}_0 \text{ such that } U_{\nu\nu'} \neq \varnothing\Big\}$$

is actually strongly equicontinuous. Thus, the infinite-dimensional vector bundles $W^k(\mathcal{F}; \Pi_{\mathcal{E}})$ are well-defined and their associated local trivialisations $\Theta_\nu : W^k(\mathcal{F}; \Pi_{\mathcal{E}})\big|_{U_\nu} \to U_\nu \times W^k(\mathcal{F}_\nu)$ are given in each fibre by the homeomorphisms

$$\Theta_\nu|_{W^k(\mathcal{E}_x)} : W^k(\mathcal{E}_x) \to W^k(\mathcal{F}_\nu), \quad \phi \mapsto \Psi_\nu|_{\mathcal{E}_x} \circ \phi \circ \Phi_\nu|_{M_x}^{-1}$$

for $x \in U_\nu$.

$$\begin{array}{ccc} \mathcal{E}_x & \xrightarrow{\Psi_\nu|_{\mathcal{E}_x}} & \mathcal{F}_\nu \\ \phi \Big\uparrow & & \Big\uparrow \Theta_\nu|_{W^k(\mathcal{E}_x)}\phi \\ M_x & \xleftarrow[\Phi_\nu|_{M_x}^{-1}]{} & F \end{array}$$

Consequently, the latter are bounded uniformly in the following sense: For all $k \in \mathbb{N}_0$ there exists a constant $C(k) > 0$ such that

$$\left\| \Theta_\nu|_{W^k(\mathcal{E}_x)} \right\|_{\mathcal{L}(W^k(\mathcal{E}_x), W^k(\mathcal{F}_\nu))} \leqslant C(k) \tag{2.10}$$

for all $x \in U_\nu$ and $\nu \in \mathbb{N}_0$.

Remark 2.11 $L^2(\mathcal{F})$-derivatives of the mapping $x \mapsto \mathsf{g}_{\nu\nu'}(x)\phi$ incorporate vertical derivatives of $\phi \in W^k(\mathcal{F})$ due to the chain rule, and so in general $\partial_x \mathsf{g}_{\nu\nu'}(x)$ is expected to be continuous only from $W^{k-1}(\mathcal{F})$ to $W^k(\mathcal{F})$ and consequently $\partial_x \mathsf{g}_{\nu\nu'}(x) \notin \mathcal{L}(W^k(\mathcal{F}))$. Hence, differentiability is too strong a requirement for these infinite-dimensional vector bundles. $\Diamond$

One can similarly construct spaces of bounded bundle maps between two vector bundles $W^k(\mathcal{F};\Pi_{\mathcal{E}})$ and $W^l(\mathcal{F};\Pi_{\mathcal{E}})$ for $k,l \in \mathbb{N}_0$ [Lam14, Lemma B.5]. For example, any fibrewise operator T with

$$T(x) \in \mathcal{L}\big(W^k(\mathcal{E}_x), W^l(\mathcal{E}_x)\big)$$

is a section of the bundle $\mathcal{L}(W^k(\mathcal{F};\Pi_{\mathcal{E}}), W^l(\mathcal{F};\Pi_{\mathcal{E}}))$.

We close the discussion by returning to the initial motivation for this subsection and therefore introduce the vector bundles

$$\mathcal{H}_{\mathcal{F}} := W^0(\mathcal{F};\Pi_{\mathcal{E}}) = L^2(\mathcal{F};\Pi_{\mathcal{E}}) \tag{2.11a}$$

and

$$\mathcal{D}_{\mathcal{F}} := W^2(\mathcal{F};\Pi_{\mathcal{E}}) \cap W^1_0(\mathcal{F};\Pi_{\mathcal{E}}). \tag{2.11b}$$

The fact that the fibrewise vertical Laplacian $-\Delta^{\mathcal{E}}_{\mathsf{V}}(x)$ defines a self-adjoint operator on $L^2(\mathcal{E}_x) = \mathcal{H}_{\mathcal{F}}|_x$ with Dirichlet domain $\operatorname{dom}(-\Delta^{\mathcal{E}}_{\mathsf{V}}(x)) = W^2(\mathcal{E}_x) \cap W^1_0(\mathcal{E}_x) = \mathcal{D}_{\mathcal{F}}|_x$ for all $x \in B$ can be merged into the statement $-\Delta^{\mathcal{E}}_{\mathsf{V}} \in L^\infty(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}))$.

2.2.2 Elliptic Regularity

The structure of the composed fibre bundle $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$ suggests that we introduce adapted Sobolev spaces on $\mathcal{E}$ which take into account the ε-dependent scaling of the horizontal and vertical directions of M via the rescaled submersion metric g^ε (2.7), which causes the different scaling of horizontal and vertical derivatives in $-\Delta^{\mathcal{E}}_{g^\varepsilon} = -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} - \Delta^{\mathcal{E}}_{\mathsf{V}}$.

Definition 2.12 With the notation of Definition 2.7, we denote by $W^k_\varepsilon(\mathcal{E})$ the completion of $C^\infty_0(\mathcal{E})$ with respect to the norm

$$\|\psi\|^2_{W^k_\varepsilon(\mathcal{E})} := \sum_{\nu\in\mathbb{N}_0} \sum_{|\alpha|\leqslant k} \int_{U_\nu} \underbrace{\Bigg\| \prod_{i=1}^{b} \big(\nabla^{\mathcal{E}}_{\varepsilon\Phi^*_\nu X^\nu_i}\big)^{\alpha_i} \chi^M_\nu \psi \Bigg\|^2_{W^{k-|\alpha|}(\mathcal{E}_x)}}_{\substack{\text{includes } |\alpha|\ \partial\text{-horizontal and}\\ k-|\alpha| \text{ vertical derivatives}}} \operatorname{vol}_{g_B}(x).$$

Moreover, the spaces $W^k_{0,\varepsilon}(\mathcal{E})$ stand for the completion of $C_0^\infty(\mathcal{E}^\circ)$ with respect to the same norm. $\Diamond$

We will give a more detailed discussion of the so-called ∂-horizontal vector fields $\Phi_\nu^* X_i^\nu \in C_{\mathrm{b}}^\infty(\mathsf{T}M|_{\pi_M^{-1}(U_\nu)})$ at the beginning of Section 3.1 and content ourselves at this point with the intuition that these vector fields are tangent to the boundary ∂M and are obtained by slightly tilting the respective horizontal lifts $(X_i^\nu)^{\mathsf{H}} \in C_{\mathrm{b}}^\infty(\mathsf{H}M|_{\pi_M^{-1}(U_\nu)})$.

Remark 2.13 Let $\psi \in C_0^\infty(\mathcal{E})$ be arbitrary. We then obtain on the one hand

$$
\begin{aligned}
\|\psi\|^2_{W^0_\varepsilon(\mathcal{E})} &= \sum_{\nu\in\mathbb{N}_0} \int_{U_\nu} \left\|\chi_\nu^M \psi\right\|^2_{L^2(\mathcal{E}_x)} \operatorname{vol}_{g_B}(x) \\
&= \sum_{\nu\in\mathbb{N}_0} \int_{U_\nu} \left(\int_{M_x} \left\|(\chi_\nu^M \psi)(\xi)\right\|^2_{(h_x)_\xi} \operatorname{vol}_{g_{M_x}}(\xi) \right) \operatorname{vol}_{g_B}(x) \\
&= \sum_{\nu\in\mathbb{N}_0} \int_{\pi_M^{-1}(U_\nu)} {\chi_\nu^M}^2(p) \left\|\psi(p)\right\|^2_{h_p} \operatorname{vol}_g(p) \\
&\leqslant \sum_{\nu'\in\mathbb{N}_0} \sum_{\nu\in\mathbb{N}_0} \int_M \chi_\nu^M(p) \chi_{\nu'}^M(p) \left\|\psi(p)\right\|^2_{h_p} \operatorname{vol}_g(p) \\
&= \sum_{\nu\in\mathbb{N}_0} \int_M \chi_\nu^M(p) \left\|\psi(p)\right\|^2_{h_p} \operatorname{vol}_g(p) \\
&= \|\psi\|^2_{\mathcal{H}},
\end{aligned}
$$

while on the other hand it holds that

$$
\begin{aligned}
\|\psi\|^2_{\mathcal{H}} &= \sum_{\nu\in\mathbb{N}_0} \sum_{\nu'\in\mathbb{N}_0} \int_M \chi_\nu^M \chi_{\nu'}^M \|\psi\|_h^2 \operatorname{vol}_g \\
&\leqslant \sum_{\substack{\nu,\nu'\in\mathbb{N}_0 \\ U_{\nu\nu'}\neq\varnothing}} \int_M \tfrac{1}{2} \left({\chi_\nu^M}^2 + {\chi_{\nu'}^M}^2 \right) \|\psi\|_h^2 \operatorname{vol}_g \\
&\leqslant \sum_{\nu\in\mathbb{N}_0} N_{\mathfrak{U}} \int_M \tfrac{1}{2} \left({\chi_\nu^M}^2 + {\chi_\nu^M}^2 \right) \|\psi\|_h^2 \operatorname{vol}_g
\end{aligned}
$$

$$= N_{\mathfrak{U}} \sum_{\nu \in \mathbb{N}_0} \int_{\pi_M^{-1}(U_\nu)} \chi_\nu^{M\,2} \left\|\psi(p)\right\|_h^2 \,\mathrm{vol}_g$$
$$= N_{\mathfrak{U}} \left\|\psi\right\|_{W_\varepsilon^0(\mathcal{E})}^2 .$$

The second inequality is valid because there are at most $N_{\mathfrak{U}}$ partition functions $\chi_{\nu'}^M \leqslant 1$ with $U_{\nu\nu'} \neq \varnothing$ for all $\nu \in \mathbb{N}_0$ and similarly for the second term. Thus, $W_\varepsilon^0(\mathcal{E}) = \mathcal{H}$ as topological vector spaces with ε-independent, equivalent norms

$$\left\|\psi\right\|_{W_\varepsilon^0(\mathcal{E})} \leqslant \left\|\psi\right\|_{\mathcal{H}} \leqslant N_{\mathfrak{U}}^{1/2} \left\|\psi\right\|_{W_\varepsilon^0(\mathcal{E})} . \tag{2.12}$$

◊

We now show that the two norms $\|\cdot\|_{W_\varepsilon^k(\mathcal{E})}$ and $\|\cdot\|_{W^k(\mathcal{E},\mathrm{vol}_{g^\varepsilon})}$ are equivalent. The required constants are ε-independent up to a global factor ε^b, which comes from the fact that we used vol_{g_B} for the volume measure in Definition 2.12 instead of $\mathrm{vol}_{\varepsilon^{-2}g_B}$.

Proposition 2.14 *For every $k \in \mathbb{N}_0$ there are constants $0 < c(k) \leqslant C(k)$ such that*

$$c(k) \left\|\psi\right\|_{W_\varepsilon^k(\mathcal{E})} \leqslant \varepsilon^b \left\|\psi\right\|_{W^k(\mathcal{E},\mathrm{vol}_{g^\varepsilon})} \leqslant C(k) \left\|\psi\right\|_{W_\varepsilon^k(\mathcal{E})}$$

for all $\psi \in W_\varepsilon^k(\mathcal{E})$.

PROOF. We choose

- a covering $\{(U_\nu^\varepsilon, \kappa_\nu^\varepsilon)\}_{\nu \in \mathbb{N}_0}$ of $(B, \varepsilon^{-2} g_B)$ with local trivialisations $\Phi_\nu : \pi_M^{-1}(U_\nu^\varepsilon) \to U_\nu^\varepsilon \times F$ of $M \xrightarrow{\pi_M} B$,
- a covering $\{(V_\lambda, \varpi_\lambda)\}_{\lambda \in \mathbb{Z}}$ of (F, g_F) with synchronous trivialisations $\Lambda_\lambda^{\mathcal{F}_\nu} : \pi_{\mathcal{F}}^{-1}(V_\lambda) \to V_\lambda \times \mathbb{C}^N$ of the typical bundle $\mathcal{F}_\nu \xrightarrow{\pi_{\mathcal{F}}} F$ (see Remark 2.3) for all $\nu \in \mathbb{N}_0$,
- a covering $\{(W_\mu^\varepsilon, \tau_\mu^\varepsilon)\}_{\mu \in \mathbb{Z}}$ of (M, g^ε) with synchronous trivialisations $\Lambda_\mu^{\mathcal{E}} : \pi_{\mathcal{E}}^{-1}(W_\mu^\varepsilon) \to W_\mu^\varepsilon \times \mathbb{C}^N$ of $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$,
- and local trivialisations $\Psi_\nu : \Pi_{\mathcal{E}}^{-1}(U_\nu^\varepsilon) \to U_\nu^\varepsilon \times \mathcal{F}$ of $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$.

We saw in Subsection 2.1.4 that all constants and curvature bounds, that belong to (M, g^ε), can be chosen independently of ε. In particular, there is a partition of unity $\{\chi_\mu^\varepsilon\}_{\mu\in\mathbb{Z}}$ subordinate to the cover introduced above with derivatives and multiplicity N_M uniformly bounded in ε (cf. Remark 2.9(ii)).

The coordinate changes $t_{\lambda\mu\nu}$ in virtue of the diagram

$$\begin{tikzcd}
\mathbb{C}^N \arrow[rr, "\Lambda_\mu^{\mathcal{E}} \circ \Psi_\nu^{-1} \circ (\mathbf{1}_{U_\nu}, \Lambda_\lambda^{\mathcal{F}_\nu})^{-1}"] & & \mathbb{C}^N \\
\mathbb{R}^{b+f} \arrow[u, "\phi"] & & \mathbb{R}^{b+f} \arrow[ll, "(\kappa_\nu^\varepsilon, \varpi_\lambda) \circ \Phi_\nu \circ (\tau_\mu^\varepsilon)^{-1}"] \arrow[u, "t_{\lambda\nu\mu}\phi"']
\end{tikzcd}$$

are continuous maps from $W^k(\mathbb{R}^{b+f}, \mathbb{C}^N)$ to $W^k(\mathbb{R}^{b+f}, \mathbb{C}^N)$ with bounds uniform in λ, μ and ν due to Proposition 2.6, [Lam14, Proposition A.9] and the fact that both $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ and $\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} F$ are vector bundles of bounded geometry (having the base manifolds (M, g^ε) and (F, g_F) of bounded geometry). Moreover, the latter bounds are independent of ε because $\{\varepsilon\Phi_\nu^* X_i^\nu\}_{i=1}^b$ extends to a g^ε-orthonormal basis at $p_\nu \in \pi_M^{-1}(U_\nu)$. Hence, the coordinate changes are orthogonal maps to first order while the higher derivatives are bounded by the very construction of the coordinates. Consequently, if we expand both norms in their local expressions and rescale the volume measure properly, they are related by globally bounded maps. These expressions can be patched together using Lemma A.16. □

We are finally able to formulate an elliptic regularity statement for the connection Laplacian $\Delta_{g^\varepsilon}^{\mathcal{E}}$ with Dirichlet boundary conditions:

Proposition 2.15 *Let $\psi \in W_\varepsilon^2(\mathcal{E}) \cap W_{0,\varepsilon}^1(\mathcal{E})$ and $k \in \mathbb{N}_0$ such that $\Delta_{g^\varepsilon}^{\mathcal{E}}\psi \in W_\varepsilon^k(\mathcal{E})$. Then $\psi \in W_\varepsilon^{k+2}(\mathcal{E})$ and there is a constant $C(k) > 0$ such that*

$$\|\psi\|^2_{W_\varepsilon^{k+2}(\mathcal{E})} \leqslant C(k) \left(\left\|\Delta_{g^\varepsilon}^{\mathcal{E}}\psi\right\|^2_{W_\varepsilon^k(\mathcal{E})} + \|\psi\|^2_{\mathcal{H}} \right).$$

Due to the fact that the coefficients of the Laplacian are smooth, this proposition implies in particular that solutions of the Dirichlet boundary value problem

$$-\Delta^{\mathcal{E}}_{g^\varepsilon}\psi = \phi, \quad \psi|_{\partial M} = 0 \tag{2.13}$$

have infinite (classical) derivatives if $\phi \in W^\infty(\mathcal{E})$. Moreover, an iterated application of this proposition and appropriate interpolation inequalities for Sobolev spaces [GT98, Theorem 7.28] provide the estimate

$$\|\psi\|^2_{W^{2k}(\mathcal{E})} \leqslant \tilde{C}(k)\left(\left\|\left(\Delta^{\mathcal{E}}_{g^\varepsilon}\right)^k\psi\right\|^2_{\mathcal{H}} + \|\psi\|^2_{\mathcal{H}}\right) \tag{2.14}$$

for all $k \in \mathbb{N}_0$.

Outline of Proof (of Proposition 2.15). Since Proposition A.20 shows that Sobolev norms are first calculated locally on subsets of $\mathbb{R}^{b+f} \times \mathbb{C}^N$ and then patched together accordingly, the proof consists of considering local boundary value problems that are induced by normal coordinates and synchronous trivialisations associated with the vector bundle $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g^\varepsilon)$ of bounded geometry. More precisely, let $\{(W^\varepsilon_\mu, \tau^\varepsilon_\mu)\}_{\mu\in\mathbb{Z}}$ be a cover of M by normal coordinates (with respective centres p^ε_μ) and $\Lambda^{\mathcal{E}}_\mu : \pi^{-1}_{\mathcal{E}}(W^\varepsilon_\mu) \to W^\varepsilon_\mu \times \mathbb{C}^N$ the corresponding synchronous trivialisations. For the sake of clarity, we use $q = \tau^\varepsilon_\mu(p)$ to distinguish between points $p \in W^\varepsilon_\mu \subset M$ and coordinates $q \in \tau^\varepsilon_\mu(W^\varepsilon_\mu) \subset \mathbb{R}^{b+f}$ (in particular, we have $\tau^\varepsilon_\mu(p^\varepsilon_\mu) = 0$ for all $\mu \in \mathbb{Z}$). We then denote by $q \mapsto \psi(q) = (\psi^1(q), \dots, \psi^N(q))$ a section of the trivial vector bundle $\tau^\varepsilon_\mu(W^\varepsilon_\mu) \times \mathbb{C}^N \subset \mathbb{R}^{b+f} \times \mathbb{C}^N$ in these coordinates. Consequently, the boundary value problem (2.13) is transferred to a countable system of local boundary value problems

$$\begin{cases} (\mathcal{L}_{\mu,\varepsilon}\psi)^A = \phi^A & \text{in } \Omega^{\mathrm{bc}} := \mathbb{B}^{b+(f-1)}_{r_{\partial,M}}(0) \times [0, r_{\mathrm{C},M}) \\ \psi^A = 0 & \text{on } \partial\Omega^{\mathrm{bc}} \end{cases}$$

on boundary collar charts ($\mu < 0$) and

$$(\mathcal{L}_{\mu,\varepsilon}\psi)^A = \phi^A \quad \text{in } \Omega^{\mathrm{in}} := \mathbb{B}^{b+f}_{r_{\mathrm{in},M}}(0)$$

on interior charts ($\mu \in \mathbb{N}_0$) for all $A \in \{1, \ldots, N\}$. Note that these subsets $\Omega^{\text{bc,in}} \subset \mathbb{R}^{b+f}$ are the same for all $\mu \in \mathbb{Z}$ and all $0 < \varepsilon < 1$, while the coefficients of the differential operator

$$\mathcal{L}_{\mu,\varepsilon} = -(g^{\varepsilon}_{\mu})^{\alpha\beta}(q)\mathbf{1}_{\mathbb{C}^N}\partial_{q^\alpha}\partial_{q^\beta} + \mathfrak{A}^{\alpha}_{\mu,\varepsilon}(q)\partial_{q^\alpha} + \mathfrak{B}_{\mu,\varepsilon}(q)$$

depend on the metric coefficients $(g^{\varepsilon}_{\mu})^{\alpha\beta}$, the Christoffel symbols $(\Gamma^{\mathcal{E}}_{\mu})^{A}_{\alpha B}$ of $\nabla^{\mathcal{E}}$ and their respective derivatives up to second order. Remark 2.9 and Remark 2.10 reveal that $\mathfrak{A}^{\alpha}_{\mu,\varepsilon}$ and $\mathfrak{B}_{\mu,\varepsilon}$ can be bounded uniformly in $\mu \in \mathbb{Z}$ and $0 < \varepsilon < 1$.

Following the exposition of [Sch96, Chapter 4], we need (2.13) to be a uniformly elliptic boundary value problem, i.e., we have to check that

(i) the local boundary value problems are elliptic in the classical sense (cf. [Hör76, Definition 10.6.2]),

(ii) $\mathcal{L}_{\mu,\varepsilon}$ is a uniformly elliptic differential operator with ellipticity constants that are uniform in μ (and ε),

(iii) and the local problems admit (after a possible contraction of Ω^{bc} and Ω^{in}) local fundamental solutions

$$\mathcal{R}^{\text{bc}}_{\mu,\varepsilon} : L^2(\Omega^{\text{bc}}, \mathbb{C}^N) \oplus W^{3/2}(\partial\Omega^{\text{bc}}, \mathbb{C}^N) \to W^2(\Omega^{\text{bc}}, \mathbb{C}^N), \quad \mu < 0$$

and

$$\mathcal{R}^{\text{in}}_{\mu,\varepsilon} : L^2(\Omega^{\text{in}}, \mathbb{C}^N) \to W^2(\Omega^{\text{in}}, \mathbb{C}^N), \quad \mu \in \mathbb{N}_0$$

that are bounded uniformly in μ (and ε).

Sufficient conditions for a boundary value problem to be uniformly elliptic are derived in [Sch96, Proposition 5.13]. In particular, this criterion merely depends on the principal parts

$$\mathcal{L}^{\text{P}}_{\mu,\varepsilon}(q,\zeta) = -g^{\varepsilon}_{\mu}(\zeta,\zeta)\big|_q \mathbf{1}_{\mathbb{C}^N}, \quad \zeta \in \mathsf{T}^*_q\big(\tau^{\varepsilon}_{\mu}(W^{\varepsilon}_{\mu})\big) \cong \mathbb{R}^{b+f}$$

of the involved differential operators and their respective derivatives. These requirements are verified in [Sch96, Proposition 5.14] for the Hodge Laplacian with Dirichlet boundary conditions, which clearly transfers to our case. Moreover, this procedure explicitly preserves the ε-uniformity (cf. Remark 2.9(i)). Finally, [Sch96, Theorem 4.15] immediately implies the desired inequality for the $\|\cdot\|_{W^k(\mathcal{E},\mathrm{vol}_{g^\varepsilon})}$-norm, which is equivalent to the $\|\cdot\|_{W^k_\varepsilon(\mathcal{E})}$-norm up to a global factor ε^b due to Proposition 2.14. □

2.3 The Schrödinger Operator

We will consider specific "perturbations" of the connection Laplacian introduced in the previous section. More precisely, we will analyse Schrödinger operators satisfying the following prerequisites:

Condition 2.16 Let

$$H^{\mathcal{E}} := -\Delta^{\mathcal{E}}_{g^\varepsilon} + \varepsilon H_1^{\mathcal{E}} + V^{\mathcal{E}} \tag{2.15}$$

be the densely defined operator with Dirichlet boundary conditions, where

(i) the perturbation $H_1^{\mathcal{E}}$
   - is symmetric on $\mathrm{dom}(-\Delta^{\mathcal{E}}_{g^\varepsilon}) \subset \mathcal{H}$ and bounded independently of ε as a map from $W^{k+2}_\varepsilon(\mathcal{E})$ to $W^k_\varepsilon(\mathcal{E})$ for all $k \in \mathbb{N}_0$,
   - and carries additional ∂-horizontal smoothness, i.e., $H_1^{\mathcal{E}} T \in \mathcal{A}^{p+2,q}$ for every $T \in \mathcal{A}^{p,q}_H$ (see Definition 3.5),

(ii) and the ε-independent potential $V^{\mathcal{E}}$ is an element of $C^\infty_{\mathrm{b}}(\mathrm{Herm}(\mathcal{E}))$, where $\mathrm{Herm}(\mathcal{E}) \subset \mathrm{End}(\mathcal{E})$ denotes the subbundle of self-adjoint (Hermitian) vector bundle endomorphisms of $\mathcal{E}$. ◊

Under these conditions, $H^{\mathcal{E}}$ is self-adjoint on $\mathcal{H}$ with domain

$$\mathrm{dom}(H^{\mathcal{E}}) = \mathrm{dom}\big(-\Delta^{\mathcal{E}}_{g^\varepsilon}\big) \overset{(A.19)}{=} W^2_\varepsilon(\mathcal{E}) \cap W^1_{0,\varepsilon}(\mathcal{E})$$

and bounded from below by the Katō-Rellich theorem [RS75, Theorem X.12]. From now on $H^{\mathcal{E}}$ always denotes this self-adjoint operator, while expressions like the Laplacian $\Delta_{g^\varepsilon}^{\mathcal{E}}$ or $H_1^{\mathcal{E}}$ may also stand for a differential operator without reference to a specific domain. The smoothness conditions on the perturbation $H_1^{\mathcal{E}}$ and the potential $V^{\mathcal{E}}$ allow us to extend the elliptic estimates for the Laplacian $-\Delta_{g^\varepsilon}^{\mathcal{E}}$ to the Schrödinger operator $H^{\mathcal{E}}$:

Corollary 2.17 *Let $\psi \in \operatorname{dom}(H^{\mathcal{E}})$ and $k \in \mathbb{N}$ be such that $(H^{\mathcal{E}})^k\psi \in \mathcal{H}$ for $H^{\mathcal{E}}$ satisfying Condition 2.16. Then $\psi \in W_\varepsilon^{2k}(\mathcal{E})$ and there are constants $\varepsilon_0(k), C(k) > 0$ such that*

$$\|\psi\|^2_{W_\varepsilon^{2k}(\mathcal{E})} \leqslant C(k)\left(\left\|(H^{\mathcal{E}})^k\psi\right\|^2_{\mathcal{H}} + \|\psi\|^2_{\mathcal{H}}\right)$$

for $0 < \varepsilon < \varepsilon_0(k)$, and in particular $\operatorname{dom}((H^{\mathcal{E}})^k) \subset W_\varepsilon^{2k}(\mathcal{E})$ for $\varepsilon > 0$ small enough.

Proof. Condition 2.16 yields that the operator $H^{\mathcal{E}}$ defines a bounded map from $W_\varepsilon^{2l+2}(\mathcal{E})$ to $W_\varepsilon^{2l}(\mathcal{E})$ for all $l \in \mathbb{N}_0$ with bounds independent of ε. The aspired estimate then follows using iterated elliptic regularity of the Laplacian (see Proposition 2.15):

$$\begin{aligned}
&\|\psi\|^2_{W_\varepsilon^{2k}(\mathcal{E})} \\
&\overset{(2.14)}{\leqslant} C(k)\left(\left\|\big(\underbrace{-H^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + V^{\mathcal{E}}}_{=\Delta_{g^\varepsilon}^{\mathcal{E}}}\big)^k\psi\right\|^2_{\mathcal{H}} + \|\psi\|^2_{\mathcal{H}}\right) \\
&\leqslant \ C(k)\Bigg(2^{3^k-1}\sum_{n,m=0}^{n+m=k} C_{nm}\Big\| \underbrace{(H^{\mathcal{E}})^{k-(n+m)}\big(\varepsilon H_1^{\mathcal{E}}\big)^n(V^{\mathcal{E}})^m}_{\substack{\text{this ordering is symbolical as the mixed}\\ \text{terms are clearly not in this order!}}}\ \psi\Big\|^2_{\mathcal{H}} \\
&\qquad\qquad + \|\psi\|^2_{\mathcal{H}}\Bigg)
\end{aligned}$$

$$\leqslant \; C(k)\Big(2^{3^k-1}\left\|(H^{\mathcal{E}})^k\psi\right\|^2_{\mathcal{H}} + \Big(1+2^{3^k-1}\left\|V^{\mathcal{E}}\right\|^{2k}_{\mathcal{L}(\mathcal{H})}\Big)\,\|\psi\|^2_{\mathcal{H}}\Big)$$
$$+\,C(k)\,2^{3^k-1}\sum_{n=1,m=0}^{n+m=k-1} C_{nm}\underbrace{\Big\|\underbrace{(H^{\mathcal{E}})^{k-(n+m)}\big(\varepsilon H_1^{\mathcal{E}}\big)^n(V^{\mathcal{E}})^m}_{\substack{\text{this ordering is symbolical as the mixed}\\ \text{terms are clearly not in this order!}}}\,\psi\Big\|^2_{\mathcal{H}}}_{\leqslant c_{nm}\varepsilon^{2n}\|\psi\|^2_{W_\varepsilon^{2(k-m)}(\mathcal{E})}}.$$

The remaining sum in the last line is of order ε. Thus, it can be moved to the left hand side and be absorbed into $\|\psi\|^2_{W_\varepsilon^{2k}(\mathcal{E})}$ for $\varepsilon=\varepsilon(k)$ small enough. $\square$

We will now discuss two types of perturbations which occur in the context of generalised quantum waveguides. As will become clear from Section 5.1, such a waveguide is modelled by a family of ε-thin tubular neighbourhoods around a smoothly embedded submanifold $B\hookrightarrow\mathbb{R}^{b+f}$ and may be mapped diffeomorphically to the total space M of an appropriate fibre bundle $M\xrightarrow{\pi_M}B$. The initial tube operator (without potential) will turn out to be unitarily equivalent to the Dirichlet Laplacian $-\Delta^{\mathcal{E},\varepsilon}_{G^\varepsilon}$ associated with some Riemannian metric $G^\varepsilon=g^\varepsilon+\mathcal{O}(\varepsilon)$ on M and metric connection $\nabla^{\mathcal{E},\mathcal{A}_\varepsilon}=\nabla^{\mathcal{E}}+\mathcal{O}(\varepsilon)$ on a vector bundle $\mathcal{E}\xrightarrow{\pi_{\mathcal{E}}}M$ in a suitable sense. We will analyse the effects due to these perturbations separately.

2.3.1 Perturbation of the Riemannian Submersion

Let us consider the connection Laplacian

$$-\Delta^{\mathcal{E}}_{G^\varepsilon}=-\operatorname{tr}_{G^\varepsilon}\big((\nabla^{\mathcal{E}})^2\cdot\big)$$

with Dirichlet boundary conditions on $L^2(\mathcal{E},\mathrm{vol}_{G^\varepsilon})$ associated with some perturbed Riemannian metric

$$G^\varepsilon=g^\varepsilon+\mathcal{O}(\varepsilon)\tag{2.16}$$

on M. We will see that $-\Delta^{\mathcal{E}}_{G^\varepsilon}$ is unitarily equivalent to an operator of the form (2.15). More precisely, let

$$\rho_\varepsilon := \frac{\mathrm{vol}_{G^\varepsilon}}{\mathrm{vol}_{g^\varepsilon}} = 1 + \mathcal{O}(\varepsilon)$$

be the Radon-Nikodym density of the two volume measures. In view of the fact that $\mathrm{vol}_{g^\varepsilon} = \varepsilon^{-b}\,\mathrm{vol}_g$, this induces a unitary map $\widehat{U}_{\rho_\varepsilon} := (\varepsilon^{-b}\rho_\varepsilon)^{1/2}\mathbf{1}_{\mathcal{E}}$ from $L^2(\mathcal{E}, \mathrm{vol}_{G^\varepsilon})$ to $L^2(\mathcal{E}, \mathrm{vol}_g) = \mathcal{H}$. Equation (2.16) suggests that the unitarily transformed operator

$$H^{\mathcal{E}} = \widehat{U}_{\rho_\varepsilon}\big(-\Delta^{\mathcal{E}}_{G^\varepsilon}\big)\widehat{U}^\dagger_{\rho_\varepsilon}$$

equals $-\Delta^{\mathcal{E}}_{g^\varepsilon}$ to leading order, whereas the subsequent orders encode the difference between the two metrics. Therefore, we first observe that

$$\begin{aligned}\mathrm{div}_{G^\varepsilon}(K)\,\mathrm{vol}_{G^\varepsilon} &= \mathcal{L}_K\,\mathrm{vol}_{G^\varepsilon} = \mathcal{L}_K\big(\rho_\varepsilon\,\mathrm{vol}_{g^\varepsilon}\big)\\ &= \mathrm{d}\rho_\varepsilon(K)\,\mathrm{vol}_{g^\varepsilon} + \rho_\varepsilon\mathcal{L}_K\,\mathrm{vol}_{g^\varepsilon}\\ &= \big(\mathrm{d}\ln\rho_\varepsilon(K) + \mathrm{div}_{g^\varepsilon}(K)\big)\,\mathrm{vol}_{G^\varepsilon}\end{aligned}$$

holds true for arbitrary $K \in C^\infty(\mathsf{T}M)$, and hence

$$\mathrm{div}_{G^\varepsilon} = \mathrm{div}_{g^\varepsilon} + \mathrm{d}\ln\rho_\varepsilon. \tag{2.17a}$$

We now need to express the $\sharp$-isomorphism in terms of both metrics. In this context, we introduce the tensor $s^\varepsilon := \widetilde{G}^\varepsilon - \widetilde{g}^\varepsilon \in C^\infty(\Sigma^2\mathsf{T}^*M)$ and get

$$\Xi^{\sharp, G^\varepsilon} = \Xi^{\sharp, g^\varepsilon} + s^\varepsilon(\Xi, \cdot) \tag{2.17b}$$

for all $\Xi \in C^\infty(\mathsf{T}^*M)$.

Lemma 2.18 *It holds that*

$$\widehat{U}_{\rho_\varepsilon}\left(-\Delta^{\mathcal{E}}_{G^\varepsilon}\right)\widehat{U}^\dagger_{\rho_\varepsilon} = -\Delta^{\mathcal{E}}_{g^\varepsilon} + \nabla^{\mathcal{E},*,g^\varepsilon} \circ S^{\mathcal{E}} + V_{\rho_\varepsilon}\mathbf{1}_{\mathcal{E}}$$

on $C_0^\infty(\mathcal{E}^\circ) \overset{dense}{\subset} \mathcal{H}$. *Here,* $S^{\mathcal{E}} \in \mathrm{Diff}_1(\mathcal{E}, \mathsf{T}^*M \otimes \mathcal{E})$ *is given by*

$$S^{\mathcal{E}} : \psi \mapsto \big((\flat, g^\varepsilon) \otimes \mathbf{1}_{\mathcal{E}}\big) \circ \Big(\mathrm{tr}_{\mathsf{T}M,(13)}\big(s^\varepsilon \otimes \nabla^{\mathcal{E}}\psi\big)\Big)$$

and the geometric potential by

$$V_{\rho_\varepsilon} := \tfrac{1}{2}\Delta^{\mathrm{L.B.}}_{G^\varepsilon}(\ln\rho_\varepsilon) - \tfrac{1}{4}\left\|\operatorname{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)\right\|^2_{G^\varepsilon},$$

where $\Delta^{\mathrm{L.B.}}_{G^\varepsilon} = \operatorname{div}_{G^\varepsilon}\circ\operatorname{grad}_{G^\varepsilon}$ *is the Laplace-Beltrami operator on functions introduced in Example A.24.*

An alternative expression for the geometric potential is given by

$$V_{\rho_\varepsilon} \overset{(2.17\mathrm{a})}{=} \tfrac{1}{2}\operatorname{div}_{g^\varepsilon}\circ\operatorname{grad}_{G^\varepsilon}(\ln\rho_\varepsilon) + \tfrac{1}{4}\left\|\operatorname{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)\right\|^2_{G^\varepsilon}. \tag{2.18}$$

In order to avoid ambiguity when dealing with the adjoints ($*$ and $\dagger$) or the musical isomorphisms ($\sharp$ and $\flat$), we indicate the underlying Riemannian metric in the notation for these operations.

Example 2.19 Let us examine the special case of a trivial line bundle $\mathcal{E} = M\times\mathbb{C}$ equipped with the flat connection $\nabla^{M\times\mathbb{C}} = \mathrm{d}$. We then obtain for the action of $S^{M\times\mathbb{C}}$ on any $\psi\in C^\infty_0(M\backslash\partial M\times\mathbb{C})\cong C^\infty(M\backslash\partial M,\mathbb{C})$:

$$\begin{aligned}
S^{M\times\mathbb{C}}\psi &= \big((\flat,g^\varepsilon)\otimes\mathbf{1}_{\mathbb{C}}\big)\circ\Big(\operatorname{tr}_{\mathrm{T}M,(13)}\big(s^\varepsilon\otimes\mathrm{d}\psi\big)\Big)\\
&= \Big(\big(s^\varepsilon\otimes\mathrm{d}\psi\big)(\omega^\alpha,\cdot,v_\alpha)\Big)^{\flat,g^\varepsilon} = \Big(s^\varepsilon(\omega^\alpha,\cdot)\otimes\mathrm{d}\psi(v_\alpha)\Big)^{\flat,g^\varepsilon}\\
&= s^\varepsilon\big(\underbrace{\omega^\alpha\otimes\mathrm{d}\psi(v_\alpha)}_{=\mathrm{d}\psi},\cdot\big)^{\flat,g^\varepsilon},
\end{aligned}$$

and so

$$\mathrm{d}^{*,g^\varepsilon}\big(S^{M\times\mathbb{C}}\psi\big) = \mathrm{d}^{*,g^\varepsilon}\big(s^\varepsilon(\mathrm{d}\psi,\cdot)^{\flat,g^\varepsilon}\big) \overset{(\mathrm{A}.17)}{=} -\operatorname{div}_{g^\varepsilon}\big(s^\varepsilon(\mathrm{d}\psi,\cdot)\big).$$

Consequently, the Laplace-Beltrami operator $-\Delta^{\mathrm{L.B.}}_{G^\varepsilon}$ is unitarily equivalent to an operator whose action on smooth functions ψ vanishing on ∂M reads

$$-\Delta^{\mathrm{L.B.}}_{g^\varepsilon}\psi - \operatorname{div}_{g^\varepsilon}\big(s^\varepsilon(\mathrm{d}\psi,\cdot)\big) + V_{\rho^\varepsilon}\psi,$$

in accordance with [Lam14, Equation (1.7)]. $\Diamond$

Proof (of Lemma 2.18). We use Lemma A.22 and calculate for arbitrary $\psi \in C_0^\infty(\mathcal{E}^\circ)$:

$$\begin{aligned}\widehat{U}_{\rho_\varepsilon}\big(-\Delta^{\mathcal{E}}_{G^\varepsilon}\big)\widehat{U}^\dagger_{\rho_\varepsilon}\psi &= \rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\nabla^{\mathcal{E}}\big(\rho_\varepsilon^{-1/2}\psi\big)\\ &= \rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\big(\rho_\varepsilon^{-1/2}\nabla^{\mathcal{E}}\psi + \mathrm{d}\rho_\varepsilon^{-1/2}\otimes\psi\big)\\ &= \rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\rho_\varepsilon^{-1/2}\big(\nabla^{\mathcal{E}}\psi - \tfrac{1}{2}\mathrm{d}\ln\rho_\varepsilon\otimes\psi\big).\end{aligned}$$

Denote by $\{v_\alpha\}_{\alpha=1}^{b+f}$ and $\{\omega^\alpha\}_{\alpha=1}^{b+f}$ local dual frames of $\mathsf{T}M$ and T^*M, respectively. In virtue of (A.16) and Lemma A.21, the first term then evaluates as

$$\begin{aligned}&\rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\rho_\varepsilon^{-1/2}\nabla^{\mathcal{E}}\psi\\ &= \rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\big(\omega^\alpha\otimes\rho_\varepsilon^{-1/2}\nabla^{\mathcal{E}}_{v_\alpha}\psi\big)\\ &= \rho_\varepsilon^{1/2}\Big(-\nabla^{\mathcal{E}}_{(\omega^\alpha)^{\sharp,G^\varepsilon}}\rho_\varepsilon^{-1/2}\nabla^{\mathcal{E}}_{v_\alpha}\psi - \mathrm{div}_{G^\varepsilon}\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big)\rho_\varepsilon^{-1/2}\nabla^{\mathcal{E}}_{v_\alpha}\psi\Big)\\ &= -\nabla^{\mathcal{E}}_{(\omega^\alpha)^{\sharp,G^\varepsilon}}\nabla^{\mathcal{E}}_{v_\alpha}\psi + \tfrac{1}{2}\mathrm{d}\ln\rho_\varepsilon\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big)\nabla^{\mathcal{E}}_{v_\alpha}\psi\\ &\quad - \mathrm{div}_{G^\varepsilon}\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big)\nabla^{\mathcal{E}}_{v_\alpha}\psi\\ &\overset{(2.17a)}{=} -\nabla^{\mathcal{E}}_{(\omega^\alpha)^{\sharp,G^\varepsilon}}\nabla^{\mathcal{E}}_{v_\alpha}\psi + \tfrac{1}{2}\mathrm{d}\ln\rho_\varepsilon\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big)\nabla^{\mathcal{E}}_{v_\alpha}\psi\\ &\quad - \Big(\mathrm{div}_{g^\varepsilon}\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big) + \mathrm{d}\ln\rho_\varepsilon\big((\omega^\alpha)^{\sharp,G^\varepsilon}\big)\Big)\nabla^{\mathcal{E}}_{v_\alpha}\psi\\ &\overset{(2.17b)}{=} \underbrace{\Big(-\nabla^{\mathcal{E}}_{(\omega^\alpha)^{\sharp,g^\varepsilon}}\nabla^{\mathcal{E}}_{v_\alpha} - \mathrm{div}_{g^\varepsilon}\big((\omega^\alpha)^{\sharp,g^\varepsilon}\big)\nabla^{\mathcal{E}}_{v_\alpha}\Big)}_{=-\Delta^{\mathcal{E}}_{g^\varepsilon},\ \text{cf. the proof of Lemma A.22}}\psi - \tfrac{1}{2}\nabla^{\mathcal{E}}_{\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)}\psi\\ &\quad + \underbrace{\Big(-\nabla^{\mathcal{E}}_{s^\varepsilon(\omega^\alpha,\cdot)} - \mathrm{div}_{g^\varepsilon}\big(s^\varepsilon(\omega^\alpha,\cdot)\big)\Big)\nabla^{\mathcal{E}}_{v_\alpha}\psi}_{=(*)},\end{aligned}$$

where we used the expansion $\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon) = \mathrm{d}\ln\rho_\varepsilon((\omega^\alpha)^{\sharp,G^\varepsilon})v_\alpha$. As far as the second term is concerned, we proceed similarly and obtain

$$\begin{aligned}&-\tfrac{1}{2}\rho_\varepsilon^{1/2}\nabla^{\mathcal{E},*,G^\varepsilon}\rho_\varepsilon^{-1/2}\mathrm{d}\ln\rho_\varepsilon\otimes\psi\\ &= -\tfrac{1}{2}\rho_\varepsilon^{1/2}\Big(-\nabla^{\mathcal{E}}_{(\rho_\varepsilon^{-1/2}\mathrm{d}\ln\rho_\varepsilon)^{\sharp,G^\varepsilon}}\psi - \mathrm{div}_{G^\varepsilon}\big((\rho_\varepsilon^{-1/2}\mathrm{d}\ln\rho_\varepsilon)^{\sharp,G^\varepsilon}\big)\psi\Big)\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\nabla^{\mathcal{E}}_{(\mathrm{d}\ln\rho_\varepsilon)^{\sharp,G^\varepsilon}}\psi + \tfrac{1}{2}\,\mathrm{div}_{G^\varepsilon}\big((\mathrm{d}\ln\rho_\varepsilon)^{\sharp,G^\varepsilon}\big)\psi \\
&\quad + \tfrac{1}{2}\rho_\varepsilon^{1/2}G^\varepsilon\Big(\underbrace{\mathrm{grad}_{G^\varepsilon}(\rho_\varepsilon^{-1/2})}_{=-\frac{1}{2}\rho_\varepsilon^{-1/2}\,\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)}, (\mathrm{d}\ln\rho_\varepsilon)^{\sharp,G^\varepsilon}\Big)\psi \\
&= \tfrac{1}{2}\nabla^{\mathcal{E}}_{\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)}\psi \\
&\quad + \underbrace{\Big(\tfrac{1}{2}\,\mathrm{div}_{G^\varepsilon}\big(\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)\big) - \tfrac{1}{4}\left\|\mathrm{grad}_{G^\varepsilon}(\ln\rho_\varepsilon)\right\|^2_{G^\varepsilon}\Big)}_{=V_{\rho_\varepsilon}}\psi.
\end{aligned}
$$

The addition of both terms finally yields the statement, since

$$
(*) = \big(\nabla^{\mathcal{E}}_{s^\varepsilon(\omega^\alpha,\cdot)}\big)^{\dagger,g^\varepsilon}\nabla^{\mathcal{E}}_{v_\alpha}\psi \overset{(A.16)}{=} \nabla^{\mathcal{E},*,g^\varepsilon}\underbrace{\Big(\big(s^\varepsilon(\omega^\alpha,\cdot)\big)^{\flat,g^\varepsilon}\otimes\nabla^{\mathcal{E}}_{v_\alpha}\psi\Big)}_{=S^{\mathcal{E}}\psi},
$$

again due to Lemma A.21. □

Remark 2.20 Again let $\{v_\alpha\}_{\alpha=1}^{b+f}$ be a local frame of $\mathsf{T}M$ and $\{\omega^\alpha\}_{\alpha=1}^{b+f}$ its local dual frame. Then note that

$$
\begin{aligned}
&\int_M \underbrace{\big((G^\varepsilon)^{\alpha\beta} - (g^\varepsilon)^{\alpha\beta}\big)}_{=s^\varepsilon(\omega^\alpha,\omega^\beta)} h\big(\nabla^{\mathcal{E}}_{v_\alpha}\phi, \nabla^{\mathcal{E}}_{v_\beta}\psi\big)\ \mathrm{vol}_{g^\varepsilon} \\
&= \int_M h\big(\nabla^{\mathcal{E}}_{v_\alpha}\phi, s^\varepsilon(\omega^\alpha,\omega^\beta)\nabla^{\mathcal{E}}_{v_\beta}\psi\big)\ \mathrm{vol}_{g^\varepsilon} \\
&= \int_M h\Big(\phi, \big(\nabla^{\mathcal{E}}_{v_\alpha}\big)^{\dagger,g^\varepsilon} s^\varepsilon(\omega^\alpha,\omega^\beta)\nabla^{\mathcal{E}}_{v_\beta}\psi\Big)\ \mathrm{vol}_{g^\varepsilon} \\
&= \int_M h\Big(\phi, \nabla^{\mathcal{E},*,g^\varepsilon}\underbrace{\big((v_\alpha)^{\flat,g^\varepsilon}\otimes s^\varepsilon(\omega^\alpha,\omega^\beta)\nabla^{\mathcal{E}}_{v_\beta}\big)\psi}_{=S^{\mathcal{E}}\psi}\Big)\ \mathrm{vol}_{g^\varepsilon}
\end{aligned}
$$

for all $\phi,\psi \in C_0^\infty(\mathcal{E}^\circ)$. Multiplication of the latter equality by a constant factor ε^b (which corresponds to replacing the volume measure $\mathrm{vol}_{g^\varepsilon}$ by $\mathrm{vol}_g = \varepsilon^b\,\mathrm{vol}_{g^\varepsilon}$) shows that the operator

$$
-\Delta^{\mathcal{E}}_{g^\varepsilon} + \nabla^{\mathcal{E},*,g^\varepsilon}\circ S^{\mathcal{E}}
$$

is given by the quadratic form

$$\left\langle \psi, (-\Delta^{\mathcal{E}}_{g^\varepsilon} + \nabla^{\mathcal{E},*,g^\varepsilon} \circ S^{\mathcal{E}})\psi \right\rangle_{\mathcal{H}} = \int_M \mathrm{tr}_{G^\varepsilon}\Big(h\big(\nabla^{\mathcal{E}}_{\cdot}\psi, \nabla^{\mathcal{E}}_{\cdot}\psi\big)\Big)\ \mathrm{vol}_g$$

and thus defines a positive operator. ◊

The following definition gives precise meaning to the vague notation (2.16) of G^ε being ε-close to g^ε:

Definition 2.21 We call G^ε an *admissible perturbation of* g^ε, denoted by $G^\varepsilon = g^\varepsilon + \mathcal{O}(\varepsilon)$, if the matrix representing the difference $G^\varepsilon - g^\varepsilon$ with respect to any adapted local orthonormal frame (cf. Definition 2.8) is bounded and of order ε with all its derivatives, i.e., for all $\mathfrak{a} \in \mathbb{N}_0^{b+f}$ there exists an ε-independent constant $C(|\mathfrak{a}|) > 0$ such that

$$\left\| \mathrm{D}^{\mathfrak{a}}(G^\varepsilon - \mathbf{1}_{m\times m}) \right\|_{\infty,\mathrm{Mat}} \leqslant C\big(|\mathfrak{a}|\big)\varepsilon,$$

independently of the chosen local frames of $\mathsf{T}B$ and $\mathsf{V}M$. ◊

If we apply the musical isomorphism $(\flat, g^\varepsilon) : \mathsf{T}M \to \mathsf{T}^*M$ to some adapted local orthonormal frame, we obtain a local orthonormal frame of T^*M, which we will also call adapted since it still respects the orthogonal decomposition $\mathsf{T}^*M = \mathsf{H}^*M \oplus \mathsf{V}^*M$ with respect to $\widetilde{g}^\varepsilon$.

Lemma 2.22 *Let G^ε be an admissible perturbation of g^ε. Then for all $\mathfrak{a} \in \mathbb{N}_0^{b+f}$ there exists a constant $c(|\mathfrak{a}|)$ such that the matrix representation of s^ε with respect to any adapted local orthonormal frame of $(\mathsf{T}^*M, \widetilde{g}^\varepsilon)$ satisfies*

$$\|\mathrm{D}^{\mathfrak{a}} s^\varepsilon\|_{\infty,\mathrm{Mat}} \leqslant c(|\mathfrak{a}|)\varepsilon$$

for all $0 < \varepsilon < 1$, independently of the chosen frame.

Proof. The matrix representation $\widetilde{G}^\varepsilon$ with respect to an adapted local orthonormal frame of $(\mathsf{T}^*M, \widetilde{g}^\varepsilon)$ can be expressed by a convergent Neumann series

$$\widetilde{G}^\varepsilon = \big(\underbrace{\mathbf{1}_{m\times m} - (\mathbf{1}_{m\times m} - G^\varepsilon)}_{=G^\varepsilon}\big)^{-1} = \sum_{k=0}^{\infty} (\mathbf{1}_{m\times m} - G^\varepsilon)^k,$$

where G^ε is the matrix representation with respect to the dual adapted local orthonormal frame of $(\mathsf{T}M, g^\varepsilon)$. Thus, the matrix norm of

$$s^\varepsilon = \widetilde{G}^\varepsilon - \mathbf{1}_{m\times m} = \sum_{k=1}^{\infty} (\mathbf{1}_{m\times m} - G^\varepsilon)^k$$

can be estimated as

$$\begin{aligned} \|s^\varepsilon\|_{\infty,\mathrm{Mat}} &\leqslant \frac{1}{1 - \left\|\mathbf{1}_{m\times m} - G^\varepsilon\right\|_{\infty,\mathrm{Mat}}} - 1 \\ &= \underbrace{\left\|\mathbf{1}_{m\times m} - G^\varepsilon\right\|_{\infty,\mathrm{Mat}}}_{\leqslant C(|0|)\varepsilon} + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{O}(\varepsilon). \end{aligned}$$

Any derivative

$$\mathrm{D}^{\mathfrak{a}} s^\varepsilon = \mathrm{D}^{\mathfrak{a}}\big(\widetilde{G}^\varepsilon - \mathbf{1}_{m\times m}\big) = \mathrm{D}^{\mathfrak{a}} \widetilde{G}^\varepsilon, \quad |\mathfrak{a}| \geqslant 1$$

is a polynomial of degree $|\mathfrak{a}|$, where the monomials are given by the composition of at least one matrix of the form $\mathrm{D}^{\mathfrak{b}} G^\varepsilon$ for $1 \leqslant |\mathfrak{b}| \leqslant |\mathfrak{a}|$ with norm

$$\begin{aligned} \left\|\mathrm{D}^{\mathfrak{b}} G^\varepsilon\right\|_{\infty,\mathrm{Mat}} &\leqslant \underbrace{\left\|\mathrm{D}^{\mathfrak{b}} \mathbf{1}_{m\times m}\right\|_{\infty,\mathrm{Mat}}}_{=0} + \left\|\mathrm{D}^{\mathfrak{b}}(G^\varepsilon - \mathbf{1}_{m\times m})\right\|_{\infty,\mathrm{Mat}} \\ &= \varepsilon C\big(|\mathfrak{b}|\big) \end{aligned}$$

and the matrix $\widetilde{G}^\varepsilon$ with norm of order one. Consequently, the norms $\|\mathrm{D}^{\mathfrak{a}} s^\varepsilon\|_{\infty,\mathrm{Mat}} = \mathcal{O}(\varepsilon)$ depend only on $|\mathfrak{a}|$. □

Proposition 2.23 *Let G^ε be an admissible perturbation of g^ε. Then the Dirichlet Laplacian $-\Delta^{\mathcal{E}}_{G^\varepsilon}$ is unitarily equivalent to an operator which satisfies Condition 2.16.*

Proof. We already know that

$$\widehat{U}_{\rho_\varepsilon}\left(-\Delta^{\mathcal{E}}_{G^\varepsilon}\right)\widehat{U}^\dagger_{\rho_\varepsilon} = -\Delta^{\mathcal{E}}_{g^\varepsilon} + \varepsilon H^{\mathcal{E}}_1$$

holds for the perturbation

$$H_1^{\mathcal{E}} := \varepsilon^{-1}\big(\nabla^{\mathcal{E},*,g^\varepsilon} \circ S^{\mathcal{E}} + V_{\rho_\varepsilon}\mathbf{1}_{\mathcal{E}}\big)$$

in virtue of Lemma 2.18. As far as the volume density ρ_ε is concerned, one obtains with the aid of an adapted local orthonormal frame of $(\mathsf{T}M, g^\varepsilon)$:

$$\begin{aligned}
\ln \rho_\varepsilon &= \ln \sqrt{\frac{\det(G^\varepsilon)}{\det(\mathbf{1}_{m\times m})}} \\
&= \tfrac{1}{2}\ln\Big(\det\big((G^\varepsilon - \mathbf{1}_{m\times m}) + \mathbf{1}_{m\times m}\big)\Big) \\
&= \tfrac{1}{2}\operatorname{tr}\Big(\ln\big((G^\varepsilon - \mathbf{1}_{m\times m}) + \mathbf{1}_{m\times m}\big)\Big) \\
&= \tfrac{1}{2}\operatorname{tr}\Big(\underbrace{(G^\varepsilon - \mathbf{1}_{m\times m})}_{\|\ldots\|_{\mathrm{Mat},\infty} \leqslant C(0)\varepsilon} + \mathcal{O}\big(\|G^\varepsilon - g^\varepsilon\|^2_{\mathrm{Mat},\infty}\big)\Big). \qquad (2.19) \\
&= \mathcal{O}(\varepsilon)
\end{aligned}$$

Thus, the geometric potential (2.18) is of order ε in $C_{\mathrm{b}}^\infty(M)$. If we finally express $H_1^{\mathcal{E}}$ over $\pi_M^{-1}(U)$ with $U \in \mathfrak{U}$ from Definition 2.7 and use an adapted local orthonormal frame, we arrive at a second-order differential operator whose coefficients are given in terms of the matrix entries of $\varepsilon^{-1}s^\varepsilon$ (smooth and bounded with all their derivatives), and are of order one by Lemma 2.22. This implies that $H_1^{\mathcal{E}}$ is a continuous map from $W_\varepsilon^{k+2}(\mathcal{E})$ to $W_\varepsilon^k(\mathcal{E})$ for all $k \in \mathbb{N}_0$ with ε-independent bounds and $H_1^{\mathcal{E}} T \in \mathcal{A}^{p+2,q}$ for all $T \in \mathcal{A}_H^{p,q}$ (see Definition 3.5). The proof is completed by the remark that the resulting operator is clearly bounded from below, since it is unitarily equivalent to the positive operator $-\Delta^{\mathcal{E}}_{G^\varepsilon}$. □

2.3.2 Perturbation of the Connection

Since two connections on $\mathcal{E}$ always differ by an $\mathrm{End}(\mathcal{E})$-valued one-form, we may obtain a new connection by an appropriate extension of the initial ("reference") connection $\nabla^{\mathcal{E}}$:

$$\nabla^{\mathcal{E},\omega} := \nabla^{\mathcal{E}} + \omega, \quad \omega \in C^\infty\big(\mathsf{T}^*M \otimes \mathrm{End}(\mathcal{E})\big).$$

Provided that the reference connection $\nabla^{\mathcal{E}}$ is metric with respect to a given bundle metric h, we want the extended connection $\nabla^{\mathcal{E},\omega}$ to be metric (with respect to the same bundle metric) as well. One can easily verify that $\nabla^{\mathcal{E},\omega}$ is metric if and only if ω is anti-self-adjoint everywhere, i.e.,

$$h\big(\psi, \omega(K)\phi\big) = -h\big(\omega(K)\psi, \phi\big)$$

holds for all $K \in C^\infty(\mathsf{T}M)$ and $\psi, \phi \in C^\infty(\mathcal{E})$. Thus, we restrict to connections of the form $\nabla^{\mathcal{E}} + \mathrm{i}\mathcal{A}$ with $\mathcal{A} \in C^\infty(\mathsf{T}^*M \otimes \mathrm{Herm}(\mathcal{E}))$:

Definition 2.24 Let $\pi_M : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ be a $\mathbb{C}^N$-vector bundle. We call the family

$$\nabla^{\mathcal{E},\mathcal{A}^\varepsilon} := \nabla^{\mathcal{E}} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon, \quad 0 \leqslant \varepsilon < 1,$$

an *admissible perturbation of* $\nabla^{\mathcal{E}}$, denoted by $\nabla^{\mathcal{E},\mathcal{A}^\varepsilon} = \nabla^{\mathcal{E}} + \mathcal{O}(\varepsilon)$, if $\mathcal{A}^\varepsilon \in C^\infty_{\mathrm{b}}(\mathsf{T}^*M \otimes \mathrm{Herm}(\mathcal{E}))$ has bounds independent of ε. $\lozenge$

The smallness of the difference between the involved connections implies that the respective Laplacians are also ε-close to each other, i.e.,

$$-\Delta^{\mathcal{E},\mathcal{A}^\varepsilon}_{g^\varepsilon} = -\Delta^{\mathcal{E}}_{g^\varepsilon} + \mathcal{O}(\varepsilon)$$

with errors in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}), \mathcal{H})$. More specifically, it will turn out to be beneficial for the applications to expand only the horizontal deviation of these operators and to keep the unaffected vertical operator

$$-\Delta^{\mathcal{E},\mathcal{A}^\varepsilon}_{\mathsf{V}} = -\Delta^{\mathcal{E}}_{\mathsf{V}} + \mathcal{O}(\varepsilon)$$

with errors in $\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})$.

Proposition 2.25 *Let $\nabla^{\mathcal{E},\mathcal{A}^\varepsilon}$ be an admissible perturbation of $\nabla^{\mathcal{E}}$. Then*

$$-\Delta^{\mathcal{E},\mathcal{A}^\varepsilon}_{g^\varepsilon} = -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} - \Delta^{\mathcal{E},\mathcal{A}^\varepsilon}_{\mathsf{V}} + \varepsilon H^{\mathcal{E}}_1,$$

where the perturbation $H^{\mathcal{E}}_1$ satisfies Condition 2.16.

PROOF. We take an adapted local orthonormal frame of $(\mathsf{T}M, g^\varepsilon)$ from Definition 2.8 and perform similar calculations as for (2.8). In this context, the entire horizontal Laplacian

$$
\begin{aligned}
-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E},\mathcal{A}^\varepsilon} = -\varepsilon^2\Big[&\Big(\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\Big)\Big(\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\Big) \\
&- \Big(\nabla^{\mathcal{E}}_{(\nabla^{g_B}_{X_i}X_i)^{\mathsf{H}}} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon\big((\nabla^{g_B}_{X_i}X_i)^{\mathsf{H}}\big)\Big) \\
&- \Big(\nabla^{\mathcal{E}}_{\eta_{\mathsf{V}}} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon(\eta_{\mathsf{V}})\Big)\Big]
\end{aligned}
$$

splits into the sum of the unperturbed horizontal Laplacian $-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}$ and the symmetric perturbation

$$
\begin{aligned}
H_1^{\mathcal{E}} &= -\mathrm{i}\varepsilon\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}} \\
&\quad \underbrace{-\mathrm{i}\varepsilon\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}\big(\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\big) + \mathrm{i}\varepsilon^2\mathcal{A}^\varepsilon\Big(\big(\nabla^{g_B}_{X_i}X_i\big)^{\mathsf{H}} + \eta_{\mathsf{V}}\Big)}_{=\left[-\mathrm{i}\varepsilon\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}\right]^\dagger \text{ with Lemma A.21}} \\
&\quad + \varepsilon^3\mathcal{A}^\varepsilon(X_i^{\mathsf{H}})\mathcal{A}^\varepsilon(X_i^{\mathsf{H}}) \\
&= -2\mathrm{i}\varepsilon\,\mathrm{tr}_{\pi_M^* g_B}\big(\mathcal{A}^\varepsilon\otimes\nabla^{\mathcal{E}}_{\varepsilon\cdot}\big) \\
&\quad + \mathrm{i}\varepsilon^2\Big(\mathcal{A}^\varepsilon(\eta_{\mathsf{V}}) - \mathrm{tr}_{\pi_M^* g_B}\big(\nabla^{\mathsf{T}^*M\otimes\mathrm{End}(\mathcal{E})}\mathcal{A}^\varepsilon\big)\Big) \\
&\quad + \varepsilon^3\,\mathrm{tr}_{\pi_M^* g_B}\big(\mathcal{A}^\varepsilon\otimes\mathcal{A}^\varepsilon\big),
\end{aligned}
$$

where $\nabla^{\mathrm{End}(\mathcal{E})} := [\nabla^{\mathcal{E}},\cdot]$ is the connection on $\mathrm{End}(\mathcal{E}) = \mathcal{E}^*\otimes\mathcal{E}$ associated with the reference connection $\nabla^{\mathcal{E}}$ and $\nabla^{\mathsf{T}^*M\otimes\mathrm{End}(\mathcal{E})}$ stands for the induced tensor product connection (A.5). The C^∞-boundedness of $\mathcal{A}^\varepsilon$ implies that $H_1^{\mathcal{E}}$ defines a first-order horizontal differential operator, which is a bounded map from $W^{k+1}_\varepsilon(\mathcal{E})$ to $W^k_\varepsilon(\mathcal{E})$ with ε-independent constants for all $k\in\mathbb{N}_0$. In particular, it has the desired properties discussed in Remark 3.8. □

2.4 Adiabatic Perturbation Theory

We split the Schrödinger operator (2.15) according to

$$H^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + H^{\mathcal{F}}$$

with fibrewise acting *vertical operator* $H^{\mathcal{F}}$

$$H^{\mathcal{F}}(x) := -\Delta_{\mathsf{V}}^{\mathcal{E}}(x) + V^{\mathcal{E}}\big|_{M_x} = -\Delta_{g_{M_x}}^{\mathcal{E}_x} + V^{\mathcal{E}}\big|_{M_x}.$$

The latter defines a self-adjoint operator on $L^2(\mathcal{E}_x) = \mathcal{H}_{\mathcal{F}}|_x$ (2.11a) with Dirichlet domain $W^2(\mathcal{E}_x) \cap W_0^1(\mathcal{E}_x) = \mathcal{D}_{\mathcal{F}}|_x$ (2.11b) due to the Katō-Rellich theorem. Hence we treat it as an essentially bounded section of $\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})$. The compactness of the fibres (M_x, g_{M_x}) yields that the spectrum of the elliptic operator $H^{\mathcal{F}}(x)$ is the discrete set of eigenvalues

$$-\big\| V^{\mathcal{E}}\big|_{M_x} \big\|_{L^\infty(\mathrm{End}(\mathcal{E}_x))} \leqslant \lambda_0(x) \leqslant \lambda_1(x) \leqslant \lambda_2(x) \leqslant \dots$$

of finite multiplicity accumulating at infinity [Nic96, Theorem 10.4.19]. In this context, an *eigenband* is a function $\lambda : B \to \mathbb{R}$ with the property $\lambda(x) \in \sigma(H^{\mathcal{F}}(x))$ for all $x \in B$. Throughout this thesis, we will exclusively be interested in eigenbands with the following specific property:

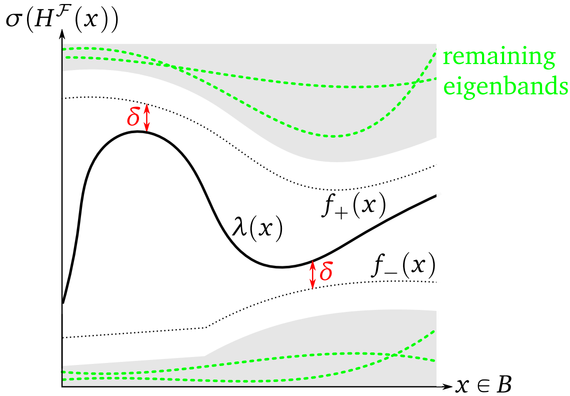


Figure 2.4: The eigenband $\lambda : B \to \mathbb{R}$ satisfies the subsequent gap condition and is separated from the rest of $H^{\mathcal{F}}$'s spectrum by at least 2δ, i.e., it holds that $\inf_{x\in B} \operatorname{dist}(\lambda, \sigma(H^{\mathcal{F}})\backslash\lambda) \geqslant 2\delta$.

Condition 2.26 We say that an eigenband $\lambda : B \to \mathbb{R}$ has a *spectral gap*, if there exist $\delta > 0$ and $f_\pm \in C_\mathrm{b}(B)$ with $\operatorname{dist}(f_\pm(x), \sigma(H^\mathcal{F}(x))) \geqslant \delta$ such that

$$\big[f_-(x), f_+(x)\big] \cap \sigma\big(H^\mathcal{F}(x)\big) = \lambda(x)$$

for all $x \in B$. ◊

Given any eigenband λ, we can assign a fibrewise *spectral projection* P_0 that satisfies $H^\mathcal{F} P_0 = \lambda P_0$. One clearly has the estimates

$$\|P_0\|_{\mathcal{L}(\mathcal{H}_\mathcal{F})|_x} \leqslant 1 \quad , \quad \|P_0\|_{\mathcal{L}(\mathcal{D}_\mathcal{F})|_x} \leqslant \big|\lambda(x)\big| + 1$$

for all $x \in B$, and so P_0 is an essentially bounded section of both $\mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{L}(\mathcal{D}_\mathcal{F})$ provided that the eigenband λ is a bounded function. The finite multiplicity of the eigenvalues of $H^\mathcal{F}(x)$ immediately implies the finite rank of the projection $P_0(x)$ for all $x \in B$. If the eigenband λ is additionally separated from the rest of $\sigma(H^\mathcal{F})$ by a spectral gap, then P_0 as a section of $\mathcal{L}(\mathcal{H}_\mathcal{F})$ is continuous (see Proposition 3.14), and $q = \operatorname{rank}(P_0) = \operatorname{tr}(P_0)$ must be constant and the *eigenspace bundle* $\mathcal{P} := P_0\mathcal{H}_\mathcal{F}$ is a well-defined subbundle of $\mathcal{H}_\mathcal{F}$ of finite rank q. Via the identification $\mathcal{H} \cong L^2(\mathcal{H}_\mathcal{F})$ (cf. [Lam14, Corollary B.6]), the operator P_0 defines a bounded operator on $\mathcal{H}$, whose image $P_0\mathcal{H}$ is isomorphic to $L^2(\mathcal{P})$, the L^2-sections of the $\mathbb{C}^q$-vector bundle $\pi_\mathcal{P} : \mathcal{P} \to B$.

Remark 2.27 The considerations of Subsection 2.3.2 show that an admissible perturbation $\nabla^{\mathcal{E},\mathcal{A}^\varepsilon} = \nabla^\mathcal{E} + \mathrm{i}\varepsilon\mathcal{A}^\varepsilon$ of the connection leads to an additional perturbation

$$\begin{aligned} H_1^\mathcal{F} &= \nabla^{\mathcal{E},*} \circ (\mathrm{i}\mathcal{A}^\varepsilon) + (\mathrm{i}\mathcal{A}^\varepsilon)^* + \varepsilon(\mathrm{i}\mathcal{A}^\varepsilon)^* \circ (\mathrm{i}\mathcal{A}^\varepsilon) \\ &= \operatorname{tr}_{g_\mathsf{V}}\big(-2\mathrm{i}\mathcal{A}^\varepsilon \otimes \nabla^\mathcal{E} - \nabla^{\mathsf{T}^*M\otimes\mathrm{End}(\mathcal{E})}\mathcal{A}^\varepsilon + \varepsilon\mathcal{A}^\varepsilon \otimes \mathcal{A}^\varepsilon\big) \end{aligned}$$

within the (by now ε-dependent) vertical operator

$$H^{\mathcal{F},\varepsilon} = -\Delta_\mathsf{V}^\mathcal{E} + \varepsilon H_1^\mathcal{F} + V_\varepsilon^\mathcal{E},$$

where the potential $V_\varepsilon^\mathcal{E}$ is allowed to depend on ε as well. As long as Condition 2.16 and Condition 2.26 are still fulfilled, i.e.,

- the perturbation $H_1^{\mathcal{F}} \in L^\infty(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}))$ is symmetric on $\mathcal{D}_{\mathcal{F}}$ and carries additional ∂-horizontal smoothness (which will be stated more precisely in the discussion directly after Lemma 3.12) with bounds that are independent of ε,

- the bounds of the potential $V_\varepsilon^{\mathcal{E}} \in C_{\mathrm{b}}^\infty(\mathrm{Herm}(\mathcal{E}))$ can be chosen independently of ε,

- and the spectral gap condition is maintained uniformly in ε,

the methods presented in the following chapter are also applicable. All objects derived from $H^{\mathcal{F},\varepsilon}$ (like eigensections, eigenbands λ^ε with associated spectral projections P_0^ε and eigenspace bundles $\mathcal{P}^\varepsilon$), however, then depend on ε, making the notation somewhat laborious. Therefore, we will drop the superscript ε and will not treat such ε-dependences explicitly, but bear in mind that the uniformity conditions on the perturbation $H_1^{\mathcal{F}}$, the potential $V_\varepsilon^{\mathcal{E}}$ and the spectral gap δ are sufficient in order to obtain the main results. $\Diamond$

The adiabatic approximation consists of comparing certain features of $H^{\mathcal{E}}$ with those of the much simpler *adiabatic operator*

$$H_{\mathrm{a}}^{\mathcal{P}} := P_0 H^{\mathcal{E}} P_0 = P_0\big(-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\big) P_0 + \lambda P_0.$$

Due to the fact that $H_{\mathrm{a}}^{\mathcal{P}}(\mathbf{1}_{\mathcal{H}} - P_0) = \mathbf{0}$, a mutual approximation of $H^{\mathcal{E}}$ and $H_{\mathrm{a}}^{\mathcal{P}}$ is possible only on the adiabatic subspace $P_0\mathcal{H}$. If one initially starts on the image of P_0, Duhamel's principle allows for the comparison of the respective generated dynamics

$$\begin{aligned}\big(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - \mathrm{e}^{-\mathrm{i}H_{\mathrm{a}}^{\mathcal{P}}t}\big)P_0 &= -\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\int_0^t \tfrac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{\mathrm{i}H^{\mathcal{E}}s}\mathrm{e}^{-\mathrm{i}H_{\mathrm{a}}^{\mathcal{P}}s}\big)P_0 \,\mathrm{d}s\\ &= -\mathrm{i}\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\int_0^t \mathrm{e}^{\mathrm{i}H^{\mathcal{E}}s}(H^{\mathcal{E}} - H_{\mathrm{a}}^{\mathcal{P}})P_0\mathrm{e}^{-\mathrm{i}H_{\mathrm{a}}^{\mathcal{P}}s}P_0 \,\mathrm{d}s\\ &= -\mathrm{i}\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\int_0^t \mathrm{e}^{\mathrm{i}H^{\mathcal{E}}s}[H^{\mathcal{E}}, P_0]P_0\mathrm{e}^{-\mathrm{i}H_{\mathrm{a}}^{\mathcal{P}}s}P_0 \,\mathrm{d}s\end{aligned}$$

by means of a commutator estimate

$$[H^{\mathcal{E}}, P_0]P_0 = \big[-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0\big]P_0 = \mathcal{O}(\varepsilon)$$

in a suitable sense. Therefore, one has approximate invariance of the subspace $P_0\mathcal{H}$ under $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ only for finite times t of order one. The growth of the basis $(B, \varepsilon^{-2}g_B)$ by a factor ε^{-1}, however, yields that non-trivial dynamical effects do not occur until times $t = \mathcal{O}(\varepsilon^{-1})$. This deficiency is the actual motivation for the considerations of the following chapter: For all $n \in \mathbb{N}$ we will construct a super-adiabatic projection $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$ that satisfies $[H^{\mathcal{E}}, P_\varepsilon] = \mathcal{O}(\varepsilon^{n+1})$ in a suitable sense.

Chapter 3

Super-Adiabatic Perturbation Theory

In this chapter, we will introduce the tools required to derive effective operators which give an accurate approximation of Schrödinger operators (2.15) in the adiabatic limit $\varepsilon \ll 1$. Therefore, we will use the techniques of super-adiabatic perturbation theory and extend the ideas developed by Lampart in [Lam14], where he considered operators acting on complex-valued functions on an ε-thin fibre bundle $M \xrightarrow{\pi_M} B$ with compact typical fibre F (sections of the trivial line bundle $M \times \mathbb{C} \xrightarrow{\mathrm{pr}_1} M$). More precisely, we will investigate operators that act on sections of a Hermitian, possibly non-trivial $\mathbb{C}^N$-vector bundle $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$.

The rigorous assumptions on the underlying geometry are listed in Condition 2.2. We saw in Section 2.1 that these bundles give rise to a double fibre bundle structure, i.e., we may treat $\Pi_{\mathcal{E}} : \mathcal{E} \to B$ as a fibre bundle, where the typical fibre is itself a $\mathbb{C}^N$-vector bundle $\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} F$. The boundedness properties of the individual bundles allow for local trivialisations of the composed fibre bundle that are bounded in a suitable manner (see Proposition 2.6).

Let $\mathcal{H}$ be the ε-independent Hilbert space of square-integrable sections of $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$ associated with the unscaled Riemannian submersion metric g. We will consider Schrödinger operators

$$H^{\mathcal{E}} = -\Delta^{\mathcal{E}}_{g^{\varepsilon}} + \varepsilon H^{\mathcal{E}}_1 + V^{\mathcal{E}} = -\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}} + \varepsilon H^{\mathcal{E}}_1 + H^{\mathcal{F}}$$

with Dirichlet boundary conditions, which mark suitable extensions of the connection Laplacian $-\Delta^{\mathcal{E}}_{g^\varepsilon}$ in the sense of Condition 2.16. We mentioned in Section 2.3 that $H^{\mathcal{E}}$ is self-adjoint on $\mathcal{H}$ with domain $\mathrm{dom}(H^{\mathcal{E}}) = W^2_\varepsilon(\mathcal{E}) \cap W^1_{0,\varepsilon}(\mathcal{E})$ and satisfies reasonable elliptic regularity estimates (see Corollary 2.17). The vertical operator

$$H^{\mathcal{F}} = -\Delta^{\mathcal{E}}_{\mathsf{V}} + V^{\mathcal{E}}$$

acts fibrewise with respect to the fibres of $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$. It defines an operator on the Hilbert space $\mathcal{H}_{\mathcal{F}}$, which is a vector bundle over the base manifold B with infinite-dimensional typical fibre $L^2(\mathcal{F})$. Moreover, it was shown in Section 2.4 that $H^{\mathcal{F}}$ is a self-adjoint operator with Dirichlet domain $\mathcal{D}_{\mathcal{F}} \subset \mathcal{H}_{\mathcal{F}}$ and possesses a spectrum consisting only of eigenbands of finite multiplicity. In fact, we will deal with eigenbands that obey a spectral gap condition (see Condition 2.26). This allows us to associate a spectral projection P_0 with λ (such that $H^{\mathcal{F}} P_0 = \lambda P_0$) and to construct the eigenspace bundle $\mathcal{P} = P_0 \mathcal{H}_{\mathcal{F}}$, which is a finite-rank vector bundle over B.

3.1 Introduction of Suitable Algebras

We will see that the construction of the super-adiabatic projection $P_\varepsilon = P_0 + \mathcal{O}(\varepsilon)$ relies heavily on the fact that the commutator $[H^{\mathcal{E}}, P_0]$ is small in a suitable sense. Inasmuch as P_0 commutes with $H^{\mathcal{F}}$ and the perturbation $\varepsilon H^{\mathcal{E}}_1$ itself is small, this reduces to locally proving

$$\left[-\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right] = \mathcal{O}(\varepsilon). \tag{3.1}$$

If λ is separated from the rest of $\sigma(H^{\mathcal{F}})$ by a gap δ, P_0 is defined locally by Riesz's formula in terms of a contour integral of the resolvent of $H^{\mathcal{F}}$ so that

$$P_0(x) = \mathbf{1}_{(\lambda(x)-\delta, \lambda(x)+\delta)}\big(H^{\mathcal{F}}(x)\big).$$

Thus, it seems natural to first consider the commutator $[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}, H^{\mathcal{F}}]$, which basically means we need to calculate commutators of the form $[\varepsilon\nabla_{X^{\mathsf{H}}}^{\mathcal{E}}, H^{\mathcal{F}}]$ for $X \in C^\infty(\mathsf{T}B)$, and then to use the functional calculus to analyse (3.1). However, the latter commutator is delicate in the presence of a boundary ∂M: Since X^{H} is generally not tangent to the boundary (cf. Figure 2.3), the derivative $\nabla_{\varepsilon X^{\mathsf{H}}}^{\mathcal{E}}\phi$ (for $\phi \in C^\infty(\mathcal{E})$ with $\phi|_{\partial M} = 0$) need not vanish on ∂M in general. Consequently, the object $[\nabla_{X^{\mathsf{H}}}^{\mathcal{E}}, H^{\mathcal{F}}]$ has no sensible meaning on $\mathcal{D}_{\mathcal{F}}$. In contrast, one would ultimately calculate derivatives of $-\Delta_{\mathsf{V}}^{\mathcal{E}} + V^{\mathcal{E}}$ on $C^\infty(\mathcal{E})$ rather than on $C_0^\infty(\mathcal{E}^\circ)$.

In order to circumvent these difficulties, we want to work with vector fields on M that are adapted to the boundary. These are naturally given in terms of local trivialisations $\Phi : \pi_M^{-1}(U) \to U \times F$ of M because they come along with trivialisations of the boundary $\pi_M^{-1}(U) \cap \partial M \cong U \times \partial F$. More precisely, for any $X \in C^\infty(\mathsf{T}U)$, whose product lift onto $U \times F$ is tangent to $U \times \partial F$, the vector field $\Phi^*X \in C^\infty(\mathsf{T}M|_{\pi_M^{-1}(U)})$ is tangent to ∂M. We will refer to such fields as *∂-horizontal vector fields*. Note that $\mathsf{T}\pi_M \circ (\Phi^*X) = X$, so Φ^*X is a lift of X and we may locally decompose $X^{\mathsf{H}} = \Phi^*X + V$, where V is a smooth section of $\ker(\mathsf{T}\pi_M) = \mathsf{V}M$. Consequently, we obtain

$$\nabla_{\varepsilon X^{\mathsf{H}}}^{\mathcal{E}} = \varepsilon\nabla_{\mathsf{V}}^{\mathcal{E}} + \varepsilon\nabla_{\Phi^*X}^{\mathcal{E}} = T_0\big(\varepsilon\nabla_{\Phi^*X}^{\mathcal{E}}\big)^0 + T_1\big(\varepsilon\nabla_{\Phi^*X}^{\mathcal{E}}\big)^1,$$

and so we consider $\nabla_{\varepsilon X^{\mathsf{H}}}^{\mathcal{E}}$ locally as a polynomial of degree one in $\varepsilon\nabla_{\Phi^*X}^{\mathcal{E}}$ with fibrewise (vertical) operators as coefficients T_0 and T_1 which carry an additional ∂-horizontal smoothness. This section will deal with making this idea more precise.

We recall that Definition 2.7 yields g_B-orthonormal, uniformly C^∞-bounded frames $\{X_i^\nu\}_{i=1}^b$ of $\mathsf{T}U_\nu$ for all $\nu \in \mathbb{N}_0$ and that each of the respective horizontal lifts has the decomposition

$$(X_i^\nu)^{\mathsf{H}} = \Phi_\nu^* X_i^\nu + V_i^\nu \tag{3.2}$$

with $V_i^\nu \in C^\infty(\mathsf{V}M|_{\pi_M^{-1}(U_\nu)})$.

Lemma 3.1 *The vector fields V_i^ν defined by* (3.2) *are C^∞-bounded uniformly in $i \in \{1, \ldots, b\}$ and $\nu \in \mathbb{N}_0$.*

PROOF. The horizontal lifts $(X_i^\nu)^{\mathsf{H}}$ are uniformly bounded by [Lam14, Corollary A.6 (5)] due to $X_i^\nu \in C_{\mathrm{b}}^\infty(\mathsf{T}U_\nu)$. The uniform local triviality of $M \xrightarrow{\pi_M} B$ further implies $\Phi_\nu^* X_i^\nu \in C_{\mathrm{b}}^\infty(\mathsf{T}M|_{\pi_M^{-1}(U_\nu)})$ in a uniform manner. Thus, the difference $V_i^\nu = (X_i^\nu)^{\mathsf{H}} - \Phi_\nu^* X_i^\nu$ itself is uniformly bounded with all its derivatives. □

Note that the g^ε-length of V_i^ν is of order ε compared to that of $(X_i^\nu)^{\mathsf{H}}$. Hence, the choice of ∂-horizontal vector fields corresponds to locally assigning ε-tilted horizontal directions and is suitable for dealing with operators on $\mathcal{E}$

- which are locally given by polynomials in $\nabla^{\mathcal{E}}_{\varepsilon\Phi_\nu^* X_i^\nu} = \varepsilon\nabla^{\mathcal{E}}_{\Phi_\nu^* X_i^\nu}$,
- and whose coefficients are L^∞-sections of $\mathcal{L}(\mathcal{H}_{\mathcal{F}})$ (i.e., the fibrewise acting operators in $\mathcal{L}(\mathcal{H})$) or $\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})$ and carry an additional ∂-horizontal smoothness.

This is stated precisely within the next two definitions:

Definition 3.2 Let $\{X_i^\nu\}_{i=1}^b$ be the g_B-orthonormal, C^∞-bounded frame of $\mathsf{T}U_\nu$ for all $\nu \in \mathbb{N}_0$ provided by Definition 2.7.

(i) We denote by $\mathcal{C}^\nu \subset L^\infty(\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu})$ the space of those linear operators T for which

$$\left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_1}^\nu}, \left[\ldots, \left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_k}^\nu}, T\right]\ldots\right]\right] \in L^\infty\big(\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu}\big)$$

holds for all $k \in \mathbb{N}_0$ and $i_1, \ldots, i_k \in \{1, \ldots, b\}$.

(ii) We similarly define $\mathcal{C}_H^\nu \subset L^\infty(\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_{U_\nu})$ to be the space of those linear operators T for which

$$\left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_1}^\nu}, \left[\ldots, \left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_k}^\nu}, T\right]\ldots\right]\right] \in L^\infty\big(\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_{U_\nu}\big)$$

holds for all $k \in \mathbb{N}_0$ and $i_1, \ldots, i_k \in \{1, \ldots, b\}$. ◊

Equivalently, the subspace $\mathcal{C}^\nu_H$ consists of those linear operators $T \in \mathcal{C}^\nu$ that additionally fulfil $H^\mathcal{F} T \in \mathcal{C}^\nu$.

Definition 3.3 Let $\{X_i^\nu\}_{i=1}^b$ be the g_B-orthonormal, C^∞-bounded frame of $\mathsf{T}U_\nu$ for all $\nu \in \mathbb{N}_0$ provided by Definition 2.7.

(i) Denote by $\mathcal{A} \subset \mathcal{L}(W^\infty(\mathcal{E}), \mathcal{H})$ the set of those operators T that satisfy

- $\pi_M(\operatorname{supp} T\psi) \subset \pi_M(\operatorname{supp}\psi)$ for all $\psi \in W^\infty(\mathcal{E})$,
- and T is locally given by the polynomial

$$\begin{aligned} T|_{\pi_M^{-1}(U_\nu)} &= \sum_{\mathfrak{a}\in\mathbb{N}_0^b} T_\mathfrak{a}^\nu (\varepsilon\nabla^\mathcal{E}_{\Phi_\nu^* X_1^\nu})^{\mathfrak{a}_1} \dots (\varepsilon\nabla^\mathcal{E}_{\Phi_\nu^* X_b^\nu})^{\mathfrak{a}_b} \\ &= \sum_{\mathfrak{a}\in\mathbb{N}_0^b} T_\mathfrak{a}^\nu \varepsilon^{|\mathfrak{a}|} (\nabla^\mathcal{E}_{\Phi_\nu^* X_1^\nu})^{\mathfrak{a}_1} \dots (\nabla^\mathcal{E}_{\Phi_\nu^* X_b^\nu})^{\mathfrak{a}_b} \end{aligned} \tag{3.3}$$

with $T_\mathfrak{a}^\nu \in \mathcal{C}^\nu$ such that

$$\left\| \left[\nabla^\mathcal{E}_{\Phi_\nu^* X_{i_1}^\nu}, \left[\dots, \left[\nabla^\mathcal{E}_{\Phi_\nu^* X_{i_k}^\nu}, T_\mathfrak{a}^\nu \right] \dots \right] \right] \right\|_{\mathcal{L}(\mathcal{H}_\mathcal{F})|_{U_\nu}} < C(\mathfrak{a}, k)$$

holds for all $\nu \in \mathbb{N}$ and there exists $l \in \mathbb{N}_0$ such that $T_\mathfrak{a}^\nu = \mathbf{0}$ for $|\mathfrak{a}| > l$.

(ii) $\mathcal{A}_H$ is defined similarly, with the modifications that $T_\mathfrak{a}^\nu \in \mathcal{C}_H^\nu$ and that the commutators are bounded in $\mathcal{L}(\mathcal{H}_\mathcal{F}, \mathcal{D}_\mathcal{F})|_{U_\nu}$. $\Diamond$

We will write $\mathcal{C}^\nu_\bullet$ if any statement holds for $\mathcal{C}^\nu$ as well as for $\mathcal{C}^\nu_H$, and introduce $\mathcal{A}_\bullet$ analogously. Moreover, we will frequently use the self-explanatory notations

$$\left(\varepsilon\nabla^\mathcal{E}_{\Phi_\nu^* X^\nu}\right)^\mathfrak{a} = \prod_{i=1}^b \left(\varepsilon\nabla^\mathcal{E}_{\Phi_\nu^* X_i^\nu}\right)^{\mathfrak{a}_i} = \varepsilon^{|\mathfrak{a}|} \prod_{i=1}^b \left(\nabla^\mathcal{E}_{\Phi_\nu^* X_i^\nu}\right)^{\mathfrak{a}_i}.$$

Lemma 3.4 *Each of the two spaces $\mathcal{A}$ and $\mathcal{A}_H$, equipped with the composition, forms an algebra.*

Proof. Take two elements $S, T \in \mathcal{A}_\bullet$. One clearly has $\pi_M(\operatorname{supp} ST\psi) \subset \pi_M(\operatorname{supp} T\psi) \subset \pi_M(\operatorname{supp}\psi)$ by definition. In order to see that ST is locally of the form (3.3) over some $U \in \mathfrak{U}$ (see Definition 2.7), one has to commute all ∂-horizontal derivatives $\nabla^{\mathcal{E}}_{\Phi * X_j}$ of S to the right appropriately. Hence, one gets commutators of the form

$$\left[\nabla^{\mathcal{E}}_{\Phi * X_j}, T_{\mathfrak{a}}\right] \in \mathcal{C}_\bullet$$

and

$$\left[\nabla^{\mathcal{E}}_{\Phi * X_j}, \nabla^{\mathcal{E}}_{\Phi * X_l}\right] = \nabla^{\mathcal{E}}_{\Phi * [X_j, X_l]} + \mathrm{R}^{\mathcal{E}}\left(\Phi^* X_j, \Phi^* X_l\right) \quad \text{for } j > l.$$

The latter curvature term restricted to M_x defines a C^∞-bounded section of $\operatorname{End}(\mathcal{E}_x)$ for all $x \in U$ by Definition A.19. Thus, iterated commutators

$$\left[\nabla^{\mathcal{E}}_{\Phi * X_{i_1}}, \left[\ldots, \left[\nabla^{\mathcal{E}}_{\Phi * X_{i_k}}, \mathrm{R}^{\mathcal{E}}\left(\Phi^* X_j, \Phi^* X_l\right)\right] \ldots\right]\right] \tag{3.4}$$

are bounded both in $\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_U$ as well as in $\mathcal{L}(\mathcal{D}_{\mathcal{F}})|_U$ (more precisely, the latter are elements of $L^\infty(W^\infty(\mathcal{F}; \Pi_{\mathcal{E}})|_U)$) and preserve the regularity. Consequently, if one commutes all terms (3.4) to the left, one has

$$ST|_{\pi_M^{-1}(U)} = \sum_{\mathfrak{a}, \mathfrak{b}} S_{\mathfrak{a}} T_{\mathfrak{b}} \left(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\right)^{\mathfrak{a}+\mathfrak{b}} + \text{lower order polynomials},$$

where the lower order polynomials are of degree at most $|\mathfrak{a}| + |\mathfrak{b}| - 1$ and the coefficients

$$S_{\mathfrak{a}}\left[\ldots, \left[\ldots, T_{\mathfrak{b}}\right] \ldots\right]\left[\ldots, \left[\ldots, \mathrm{R}^{\mathcal{E}}\left(\Phi^* X_j, \Phi^* X_l\right)\right] \ldots\right]$$

are bounded in $\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_U$ and $\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_U$. □

We now grade these algebras by the degree of the polynomials (3.3) and the ε-order of the coefficients $T^\nu_{\mathfrak{a}}$:

Definition 3.5 We introduce the filtrations

$$\mathcal{A}^p_\bullet := \left\{T \in \mathcal{A}_\bullet \text{ such that } \left(|\mathfrak{a}| > p \Rightarrow T^\nu_{\mathfrak{a}} = \mathbf{0}\right) \text{ for all } \nu \in \mathbb{N}_0\right\}$$

and

$$\mathcal{A}_\bullet^{p,q} := \big\{T \in \mathcal{A}_\bullet^p \text{ such that the constants } C(\mathfrak{a},k) \\ \text{introduced in Definition 3.3 are of order } \varepsilon^q\big\}$$

for $p,q \in \mathbb{N}_0$. ◊

A p-th-order differential operator is also one of order $p+1$, which gives $\mathcal{A}_\bullet^p \subset \mathcal{A}_\bullet^{p+1}$. Furthermore, if the above constants $C(\mathfrak{a},k)$ are of order ε^{q+1}, they clearly also are of order ε^q, hence $\mathcal{A}_\bullet^{p,q+1} \subset \mathcal{A}_\bullet^{p,q}$.

We end this section with some comments on the algebras $\mathcal{A}_\bullet$ and their relationship to the $W_\varepsilon^p(\mathcal{E})$-spaces.

Remark 3.6 (i) The condition $\pi_M(\operatorname{supp} T\psi) \subset \pi_M(\operatorname{supp}\psi)$ for any $T \in \mathcal{A}^p$ implies

$$\begin{aligned}
&\|T\psi\|_{\mathcal{H}}^2 \\
&\overset{(2.12)}{\leqslant} N_{\mathfrak{U}} \|T\psi\|_{W_\varepsilon^0(\mathcal{E})}^2 \\
&= N_{\mathfrak{U}} \sum_{\nu'\in\mathbb{N}_0} \left\|\chi_{\nu'}^M T\psi\right\|_{\mathcal{H}}^2 = N_{\mathfrak{U}} \sum_{\nu'\in\mathbb{N}_0} \Bigg\|\chi_{\nu'}^M \sum_{\nu\in\mathbb{N}_0} \underbrace{T\big(\chi_\nu^M\psi\big)}_{\substack{\text{support in}\\ \pi_M^{-1}(U_\nu)}}\Bigg\|_{\mathcal{H}}^2 \\
&= N_{\mathfrak{U}} \sum_{\nu'\in\mathbb{N}_0} \Bigg\|\chi_{\nu'}^M \sum_{\nu\in\mathbb{N}_0} \underbrace{\sum_{|\mathfrak{a}|\leqslant p} T_{\mathfrak{a}}^\nu \big(\varepsilon\nabla_{\Phi_\nu^* X^\nu}^{\mathcal{E}}\big)^{\mathfrak{a}} \big(\chi_\nu^M\psi\big)}_{\mathcal{N}_{b,p} := \sum_{k=0}^p b^k \text{ terms}}\Bigg\|_{\mathcal{H}}^2 \\
&\leqslant N_{\mathfrak{U}}^2 \mathcal{N}_{b,p} \sum_{\nu,\nu',|\mathfrak{a}|} \left\|\chi_{\nu'}^M T_{\mathfrak{a}}^\nu \big(\varepsilon\nabla_{\Phi_\nu^* X^\nu}^{\mathcal{E}}\big)^{\mathfrak{a}} \big(\chi_\nu^M\psi\big)\right\|_{\mathcal{H}}^2 \\
&\leqslant N_{\mathfrak{U}}^2 \mathcal{N}_{b,p} \sum_{\nu,\nu',|\mathfrak{a}|} \int_{U_\nu} \left\|\chi_{\nu'}^M T_{\mathfrak{a}}^\nu\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_x}^2 \\
&\qquad\qquad \times \underbrace{\left\|\big(\varepsilon\nabla_{\Phi_\nu^* X^\nu}^{\mathcal{E}}\big)^{\mathfrak{a}} \big(\chi_\nu^M\psi\big)\right\|_{L^2(\mathcal{E}_x)}^2}_{\leqslant \|\dots\|_{W^{p-|\mathfrak{a}|}(\mathcal{E}_x)}^2} \operatorname{vol}_{g_B}(x)
\end{aligned}$$

$$\begin{aligned}
&\leqslant N_{\mathfrak{U}}^2 \mathcal{N}_{b,p} \sum_{\nu\in\mathbb{N}_0} \Bigg(N_{\mathfrak{U}} \underbrace{\sup_{\nu'\in\mathbb{N}_0} \max_{|\mathfrak{a}|\leqslant p} \left\| \chi_{\nu'}^M T_{\mathfrak{a}}^{\nu} \right\|^2_{L^\infty(\,\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu}}}_{\leqslant \max_{|\mathfrak{a}|\leqslant p} \left\| T_{\mathfrak{a}}^{\nu}\right\|^2_{L^\infty(\,\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu}}} \Bigg) \\
&\qquad \times \sum_{|\mathfrak{a}|\leqslant p} \int_{U_\nu} \left\| \left(\varepsilon \nabla^{\mathcal{E}}_{\Phi_\nu^* X^\nu}\right)^{\mathfrak{a}} (\chi_\nu^M \psi) \right\|^2_{W^{p-|\mathfrak{a}|}(\mathcal{E}_x)} \operatorname{vol}_{g_B}(x) \\
&\leqslant N_{\mathfrak{U}}^3 \mathcal{N}_{b,p} \left(\sup_{\nu\in\mathbb{N}_0} \max_{|\mathfrak{a}|\leqslant p} \left\| T_{\mathfrak{a}}^{\nu} \right\|^2_{L^\infty(\,\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu}} \right) \\
&\qquad \times \underbrace{\sum_{\nu\in\mathbb{N}_0} \sum_{|\mathfrak{a}|\leqslant p} \int_{U_\nu} \left\| \left(\varepsilon \nabla^{\mathcal{E}}_{\Phi_\nu^* X^\nu}\right)^{\mathfrak{a}} (\chi_\nu^M \psi) \right\|^2_{W^{p-|\mathfrak{a}|}(\mathcal{E}_x)} \operatorname{vol}_{g_B}(x)}_{= \|\psi\|^2_{W^p_\varepsilon(\mathcal{E})}, \text{ see Definition 2.12}}
\end{aligned}$$

for all $\psi \in W^p_\varepsilon(\mathcal{E})$. Thus, $\mathcal{A}^p \subset \mathcal{L}(W^p_\varepsilon(\mathcal{E}), \mathcal{H})$ with the norm

$$\|T\|_p := \|T\|_{\mathcal{L}(W^p_\varepsilon(\mathcal{E}),\mathcal{H})} \leqslant N_{\mathfrak{U}}^{3/2} \mathcal{N}_{b,p}^{1/2} \sup_{\nu\in\mathbb{N}_0} \max_{|\mathfrak{a}|\leqslant p} \left\| T_{\mathfrak{a}}^{\nu} \right\|_{L^\infty(\,\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_{U_\nu}} .$$

In particular, we may calculate these norms locally with respect to the covering $\{\pi_M^{-1}(U_\nu)\}_{\nu\in\mathbb{N}_0}$ of M.

(ii) Let $p_1 \geqslant p_2$. Then $\|\psi\|_{W^{p_1}_\varepsilon(\mathcal{E})} \geqslant \|\psi\|_{W^{p_2}_\varepsilon(\mathcal{E})}$ for all $\psi \in W^{p_1}_\varepsilon(\mathcal{E}) \overset{\text{dense}}{\subset} W^{p_2}_\varepsilon(\mathcal{E})$ implies

$$\frac{\|T\psi\|_{\mathcal{H}}}{\|\psi\|_{W^{p_1}_\varepsilon(\mathcal{E})}} \leqslant \frac{\|T\psi\|_{\mathcal{H}}}{\|\psi\|_{W^{p_2}_\varepsilon(\mathcal{E})}}$$

for all $T \in \mathcal{A}^{p_2} \subset \mathcal{A}^{p_1}$. Moreover, the inequality still holds after passing to the supremum over all $\psi \in W^{p_1}_\varepsilon(\mathcal{E})$ on the left hand side and over all $\psi \in W^{p_2}_\varepsilon(\mathcal{E})$ on the right hand side. We thus have $\|T\|_{p_1} \leqslant \|T\|_{p_2}$ if $p_1 \geqslant p_2$. In addition, $T \in \mathcal{A}^{p,q}$ yields $\|T\|_p = \mathcal{O}(\varepsilon^q)$.

(iii) Now let $T \in \mathcal{A}^p_H$. Then

$$H^{\mathcal{E}} T = -\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}} T \underbrace{+ \varepsilon H_1^{\mathcal{E}} T + H^{\mathcal{F}} T}_{\in \mathcal{A}^{p+2}}$$

holds true and $-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}T\big|_{\pi_M^{-1}(U)}$ for $U \in \mathfrak{U}$ from Definition 2.7 consists of $p+2$ ∂-horizontal derivatives of the form $\varepsilon\nabla^{\mathcal{E}}_{\Phi * X}$. If we commute all of these derivatives to the right (similar calculations will be carried out in Lemma 3.7), we finally observe that $-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}T \in \mathcal{A}^{p+2}$. Thus, sections in the image of T satisfy Dirichlet boundary conditions and $H^{\mathcal{E}}T \in \mathcal{A}^{p+2} \subset \mathcal{L}(W_\varepsilon^{p+2}(\mathcal{E}), \mathcal{H})$. This already implies that T, and therefore $\mathcal{A}_H^p$, is contained in $\mathcal{L}(W_\varepsilon^{p+2}(\mathcal{E}), \mathrm{dom}(H^{\mathcal{E}}))$. $\Diamond$

3.2 Classification of the Constituents

Now that we have introduced the algebras $\mathcal{A}_\bullet$ with their respective filtrations, we want to classify accordingly the essential building blocks that are needed for the construction of the super-adiabatic projection P_ε. These building blocks will turn out to be the connection Laplacian (more precisely the commutator of Δ_{g^ε} with elements of $\mathcal{A}_H$), the resolvent of the vertical operator $H^{\mathcal{F}}$ and the spectral projection P_0 associated with an eigenband λ with spectral gap.

3.2.1 The Connection Laplacian

Let us start with the connection Laplacian and its horizontal contribution. The following lemma states that the commutation with those operators raises the order of the resulting ∂-horizontal differential operator by one, while the commutator with $\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}$ additionally gives a "free" ε:

Lemma 3.7 *Let $S, T \in \mathcal{A}_H$ with $ST \in \mathcal{A}_H^{p,q}$. It then holds that*

(i) $[\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}, S]T \in \mathcal{A}^{p+1,q+1}$,

(ii) $[\Delta_{g^\varepsilon}^{\mathcal{E}}, S]T \in \mathcal{A}^{p+1,q}$

Proof. In view of Remark 3.6(i), the spaces $\mathcal{A}^p$ together with their norms $\|\cdot\|_p$ are characterised by means of local quantities induced by the cover $\mathfrak{U}$ of B introduced in Definition 2.7. Thus, it suffices to show the claims for one (and thus all) $U \in \mathfrak{U}$.

(i) We write

$$\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}}\big|_{\pi_M^{-1}(U)} = \sum_{i=1}^{b} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big)\big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big) + \varepsilon^2 D,$$

where D contains first-order differential operators and second-order parts with at least one vertical derivative. Let us first consider the commutator

$$\Big[\nabla^{\mathcal{E}}_{\Phi * X_i}, S\Big]\Big|_{\pi_M^{-1}(U)} = \sum_{\mathfrak{a} \in \mathbb{N}_0^b} \varepsilon^{|\mathfrak{a}|} \bigg(\underbrace{\big[\nabla^{\mathcal{E}}_{\Phi * X^i}, S_{\mathfrak{a}}\big]}_{\in \mathcal{C}_H} \big(\nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}} + S_{\mathfrak{a}} \Big[\nabla^{\mathcal{E}}_{\Phi * X_i}, \big(\nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}}\Big]\bigg)$$

for arbitrary $i \in \{1, \dots, b\}$. Here, the second commutator may be evaluated as a finite linear combination of monomials of the form $(\nabla^{\mathcal{E}}_{\Phi * X})^{\mathfrak{b}}$ with $|\mathfrak{b}| = |\mathfrak{a}|$ and lower order polynomials, which involve iterated commutators (3.4) that are – in combination with $S_{\mathfrak{a}}$ – bounded in $\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_U$. The commutator $[\nabla^{\mathcal{E}}_{\Phi * X_i}, S]$ over $\pi_M^{-1}(U)$ then is of the same order in $\mathcal{A}_H$ as S (and analogously for T) and it locally holds with Einstein's sum convention that

$$\begin{aligned}
&\Big[\big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big)\big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big), S\Big] T \\
&= \varepsilon \bigg(\underbrace{\big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big) \underbrace{\big[\nabla^{\mathcal{E}}_{\Phi * X_i}, S\big] T}_{\in \mathcal{A}_H^{p,q}}}_{\in \mathcal{A}_H^{p+1,q}} + \underbrace{\varepsilon \underbrace{\big[\nabla^{\mathcal{E}}_{\Phi * X_i}, S\big]\big[\nabla^{\mathcal{E}}_{\Phi * X_i}, T\big]}_{\in \mathcal{A}_H^{p,q}}}_{\in \mathcal{A}_H^{p,q+1} \subset \mathcal{A}_H^{p+1,q}} \\
&\qquad + \underbrace{\varepsilon \underbrace{\big[\nabla^{\mathcal{E}}_{\Phi * X_i}, S\big] T}_{\in \mathcal{A}_H^{p,q}} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big)}_{\in \mathcal{A}_H^{p+1,q}}\bigg) \\
&\in \mathcal{A}_H^{p+1,q+1} \subset \mathcal{A}^{p+1,q+1}.
\end{aligned}$$

As far as the remaining commutator with D is concerned, observe that all vertical covariant derivatives contained in D are along C^∞-bounded vertical vector fields, since $V_i = X_i^{\mathsf{H}} - \Phi^* X_i$ is bounded by Lemma 3.1 and

$$\begin{aligned}\eta_{\mathsf{V}} &= g\big(X_i^{\mathsf{H}}, \eta_{\mathsf{V}}\big) X_i^{\mathsf{H}} \\ &= g_B\big(X_i, \mathsf{T}\pi_M(\eta_{\mathsf{V}})\big)\Phi^* X_i + g_B\big(X_i, \mathsf{T}\pi_M(\eta_{\mathsf{V}})\big) V_i\end{aligned}$$

is bounded by [Lam14, Corollary A.6 (4)]. Hence, one locally has

$$\varepsilon^2 D = \varepsilon D_i \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big) + \varepsilon^2 D_0,$$

where $D_i \in L^\infty\big(\mathcal{L}(W^1(\mathcal{F}; \Pi_{\mathcal{E}}), \mathcal{H}_{\mathcal{F}})\big|_U\big)$ is a first-order vertical differential operator and $D_0 \in L^\infty\big(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})|_U\big)$ is a second-order vertical differential operator. Thus, we may evaluate the commutator $[\varepsilon^2 D, S]$ over $\pi_M^{-1}(U)$ as:

$$\begin{aligned}[\varepsilon^2 D, S] = \varepsilon &\underbrace{D_i S_{\mathfrak{a}}}_{\in \mathcal{C}} \Big[\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}, \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}}\Big] \\ &+ \varepsilon \underbrace{D_i \underbrace{\Big[\nabla^{\mathcal{E}}_{\Phi * X_i}, S_{\mathfrak{a}}\Big]}_{\in \mathcal{C}_H}}_{\in \mathcal{C}} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}} \\ &+ \varepsilon S_{\mathfrak{a}} \underbrace{\Big[D_i, \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}}\Big]}_{=(*)} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big) \\ &+ \varepsilon \underbrace{\Big[D_i, S_{\mathfrak{a}}\Big]}_{\in \mathcal{C}} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X_i}\big) \\ &+ \varepsilon^2 S_{\mathfrak{a}} \underbrace{\Big[D_0, \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}}\Big]}_{=(*)} + \varepsilon^2 \underbrace{\Big[D_0, S_{\mathfrak{a}}\Big]}_{\in \mathcal{C}} \big(\varepsilon \nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}}.\end{aligned}$$

It suffices to treat the terms $(*)$, since the other four terms are clearly elements of $\mathcal{A}^{\tilde{p}+1, \tilde{q}+1}$ if $S \in \mathcal{A}_H^{\tilde{p}, \tilde{q}}$. To do so, if one commutates all the derivatives $\varepsilon\nabla^{\mathcal{E}}_{\Phi * X_j}$ to the right, one ends up calculating iterated commutators of the form

$$\left[\nabla^{\mathcal{E}}_{V}, \varepsilon\nabla^{\mathcal{E}}_{\Phi * X_j}\right] = \varepsilon\Big(\nabla^{\mathcal{E}}_{[V,\Phi * X_j]} + \mathrm{R}^{\mathcal{E}}\big(V, \Phi^* X_j\big)\Big),$$

where $V \in C^{\infty}_{\mathrm{b}}(\mathsf{V}M|_{\pi_M^{-1}(U)})$ and $[\Phi^* X_j, Y]$ is again a C^{∞}-bounded vertical field because $[X_j^{\mathsf{H}}, V]$ is vertical [Lam14, Lemma 1.4 (4)]. Consequently, both $S_{\mathfrak{a}}[D_i, (\varepsilon\Phi^* X)^{\mathfrak{a}}]$ and $S_{\mathfrak{a}}[D_0, (\varepsilon\Phi^* X)^{\mathfrak{a}}]$ gain a "free" ε and hence are elements of $\mathcal{A}^{\tilde{p}-1,\tilde{q}+1} \subset \mathcal{A}^{\tilde{p}+1,\tilde{q}+1}$ if $S \in \mathcal{A}_H^{\tilde{p},\tilde{q}}$.

(ii) This claim immediately follows from (i) due to the fact that

$$\left[\Delta^{\mathcal{E}}_{\mathsf{V}}, T\right]S = \underbrace{\Delta^{\mathcal{E}}_{\mathsf{V}} T}_{\in\mathcal{A}}\, S - T\, \underbrace{\Delta^{\mathcal{E}}_{\mathsf{V}} S}_{\in\mathcal{A}} \in \mathcal{A}^{p,q},$$

i.e.,

$$\left[\Delta^{\mathcal{E}}_{g^{\varepsilon}}, S\right]T = \underbrace{\left[\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, S\right]T}_{\in\mathcal{A}^{p+1,q+1}\subset\mathcal{A}^{p+1,q}} + \underbrace{\left[\Delta^{\mathcal{E}}_{\mathsf{V}}, S\right]T}_{\in\mathcal{A}^{p,q}\subset\mathcal{A}^{p+1,q}} \in \mathcal{A}^{p+1,q}.$$

□

In view of Remark 3.6(ii), the calculations of the previous lemma show in particular that (if we again commute all ∂-horizontal derivatives to the right)

$$\left\|\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, T\right]\right\|_{p+1+l} \leqslant \left\|\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, T\right]\right\|_{p+1} = \mathcal{O}(\varepsilon^{q+1}) \tag{3.5}$$

for all $l \in \mathbb{N}_0$ if $T \in \mathcal{A}_H^{p,q}$.

Remark 3.8 The requirement that $H_1^{\mathcal{E}} T \in \mathcal{A}^{p+2,q}$ for every $T \in \mathcal{A}_H^{p,q}$ in the perturbation within Condition 2.16 is fulfilled if $H_1^{\mathcal{E}}$ takes the local form

$$H_1^{\mathcal{E}}\big|_{\pi_M^{-1}(U)} = \sum_{|\mathfrak{a}|=2} A_{\mathfrak{a}}\big(\varepsilon\nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{a}} + \sum_{|\mathfrak{b}|=1} B_{\mathfrak{b}}\big(\varepsilon\nabla^{\mathcal{E}}_{\Phi * X}\big)^{\mathfrak{b}} + C,$$

where the coefficients satisfy $A_{\mathfrak{a}} \in \mathcal{C}$, $B_{\mathfrak{b}} \in L^{\infty}\big(\mathcal{L}(W^1(\mathcal{F};\Pi_{\mathcal{E}}), \mathcal{H}_{\mathcal{F}})\big|_U\big)$ and $C \in L^{\infty}\big(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})|_U\big)$ as well as commutator conditions analogous to those of Definition 3.2. ◊

3.2.2 The Vertical Resolvent

We now want to classify the resolvent of the vertical operator in view of Definition 3.5:

Lemma 3.9 *Let* $z \in C_{\mathrm{b}}^{\infty}(B, \mathbb{C})$ *with* $\operatorname{dist}(z(x), \sigma(H^{\mathcal{F}}(x))) \geqslant C > 0$. *Then*

$$R^{\mathcal{F}}(z) := \left(H^{\mathcal{F}} - z\mathbf{1}_{\mathcal{H}}\right)^{-1} \in \mathcal{A}_H^{0,0}.$$

The main part of the proof consists of showing the boundedness of the commutator $[\nabla^{\mathcal{E}}_{\Phi_\nu^* X}, R^{\mathcal{F}}(z)]$ in $\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_{U_\nu}$ for $X \in C_{\mathrm{b}}^{\infty}(\mathsf{T}U_\nu)$ for some (and thus all) $\nu \in \mathbb{N}_0$. In order to make sense of this expression ($H^{\mathcal{F}}$ has an x-dependent domain!), it is more convenient to consider the corresponding operators on the local product $U_\nu \times \mathcal{F}$ rather than on $\mathcal{E}|_{\pi_M^{-1}(U_\nu)} = \Pi_{\mathcal{E}}^{-1}(U_\nu)$. Therefore, as we saw in Subsection 2.2.1, the completion of $C^{\infty}(\mathcal{F})$ with respect to $\|\cdot\|_{L^2(\mathcal{F})}$ yields a vector bundle isomorphism (cf. Remark 2.3)

$$\begin{aligned} \Theta_\nu : \mathcal{H}_{\mathcal{F}}|_{U_\nu} &\to U_\nu \times L^2(\mathcal{F}_\nu), \\ L^2(\mathcal{E}_x) \ni \phi &\mapsto \Big((\Theta_\nu \phi)(x) \in L^2(\mathcal{F}_\nu) : y \mapsto \Psi_\nu \circ \phi \circ \Phi_\nu^{-1}(x,y) \Big) \end{aligned} \tag{3.6}$$

by means of the local trivialisations $\Phi_\nu : \pi_M^{-1}(U_\nu) \to U_\nu \times F$ of $M \xrightarrow{\pi_M} B$ and $\Psi_\nu : \Pi_{\mathcal{E}}^{-1}(U_\nu) \to U_\nu \times \mathcal{F}_\nu$ of $\mathcal{E} \xrightarrow{\Pi_{\mathcal{E}}} B$. This induces a (weak) covariant derivative

$$\nabla_X^{\Theta_\nu} : L^2\big(U_\nu, W^1(\mathcal{F}_\nu)\big) \cap W^1\big(U_\nu, L^2(\mathcal{F}_\nu)\big) \to L^2\big(U_\nu, L^2(\mathcal{F}_\nu)\big)$$

along $X \in C^{\infty}(\mathsf{T}U_\nu)$ given by the extension of

$$C^{\infty}\big(U_\nu, C^{\infty}(\mathcal{F})\big) \ni \phi \mapsto \nabla_X^{\Theta_\nu}\phi := \big(\Theta_\nu \circ \nabla^{\mathcal{E}}_{\Phi_\nu^* X} \circ \Theta_\nu^{-1}\big)\phi.$$

This mapping in fact satisfies

$$\nabla_{fX}^{\Theta_\nu} = f\nabla_X^{\Theta_\nu}\phi \quad , \quad \nabla_X^{\Theta_\nu}(f\phi) = \mathrm{d}f(X) + f\nabla_X^{\Theta_\nu}\phi$$

for all $\phi \in C^{\infty}(U_\nu, C^{\infty}(\mathcal{F}))$, $X \in C^{\infty}(\mathsf{T}U_\nu)$ and $f \in C^{\infty}(U_\nu)$. In order to not overburden the notation, we will drop the index ν for the rest of this subsection.

Lemma 3.10 *There exists a "connection one-form" A^Θ such that for all $X \in C^\infty_{\mathrm{b}}(\mathsf{T}U)$ the mapping $A^\Theta(X) : U \to \mathcal{L}(W^k(\mathcal{F}))$ is smooth and bounded with all its derivatives for all $k \in \mathbb{N}_0$, and satisfies*

$$\nabla^\Theta_X \phi = X \cdot \phi + A^\Theta(X)\phi$$

for all $\phi \in L^2(U, W^1(\mathcal{F})) \cap W^1(U, L^2(\mathcal{F}))$. Here, $X \cdot \phi$ denotes the pointwise $L^2(\mathcal{F})$-limit

$$(X \cdot \phi)(x) := \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \phi\big(\gamma^X_x(t)\big),$$

for the integral curve $\gamma^X_x : (-\delta, \delta) \to U$ of X through $\gamma^X_x(0) = x$ with $\delta = \delta(x) > 0$ small enough.

Proof. Let us calculate the covariant derivative ∇^Θ_X along $X \in C^\infty_{\mathrm{b}}(\mathsf{T}U)$ at any $x \in U$: The corresponding integral curve γ^X_x induces a ∂-horizontal curve $\alpha^X_{x,y} : (-\delta, \delta) \to \pi_M^{-1}(U)$, $t \mapsto \Phi^{-1}(\gamma^X_x(t), y)$ for all $y \in F$, which coincides with the integral curve of $\Phi^* X$ through $\alpha^X_{x,y}(0) = \Phi^{-1}(x, y)$. Moreover, we denote by $\mathfrak{p}^X_{x,y}(t) : \mathcal{E}_{\alpha^X_{x,y}(0)} \to \mathcal{E}_{\alpha^X_{x,y}(t)}$ the parallel transport along $\alpha^X_{x,y}$ with respect to the connection $\nabla^{\mathcal{E}}$, i.e., $\mathfrak{p}^X_{x,y}(t)$ satisfies the differential equation (A.2)

$$\nabla^{\mathcal{E}}_{\dot\alpha^X_{x,y}(t)}\big(\mathfrak{p}^X_{x,y}(t) w\big) = 0$$

for all $w \in \mathcal{E}_{\Phi^{-1}(x,y)}$ and $t \in (-\delta, \delta)$. This finally yields the smooth mapping $\mathfrak{t}^X_{x,y} : (-\delta, \delta) \to \mathrm{Aut}(\mathcal{F}_y)$ defined by

$$\mathfrak{t}^X_{x,y}(t) := \Psi|_{M_{\tilde x}} \circ \mathfrak{p}^X_{x,y}(t) \circ \Psi|^{-1}_{M_x}, \quad \tilde x = \gamma^X_x(t).$$

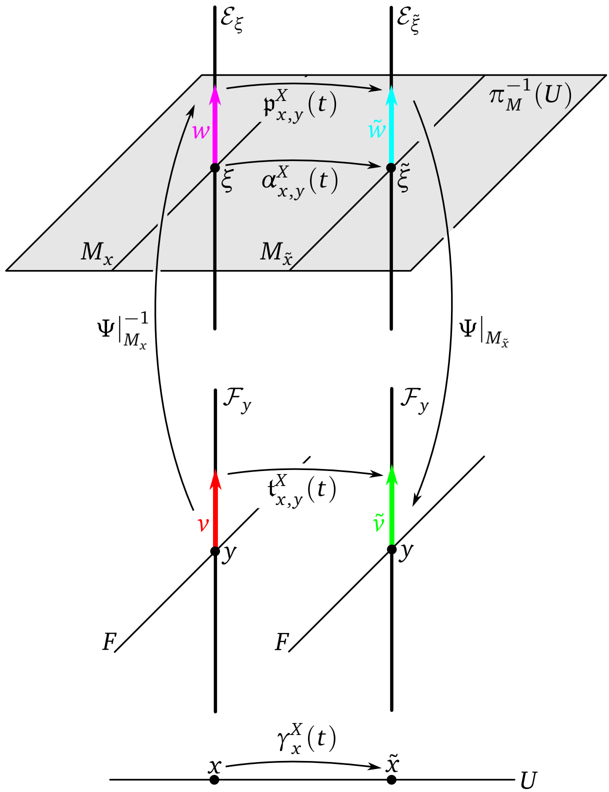
Thus, we may think of ∇^Θ_X in terms of the induced parallel transport $\mathfrak{t}^X_{\cdot,\cdot}$ in view of [KN63, Section III.1], i.e.,

$$\begin{aligned}
\big(\nabla^\Theta_X \phi\big)(x) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \big(\mathfrak{t}^X_{x,\cdot}(t)\big)^{-1} \circ \phi \circ \gamma^X_x(t) \\
&= \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathfrak{t}^X_{x,\cdot}(t)\Big)^{-1} \underbrace{\phi\big(\gamma^X_x(0)\big)}_{=\phi(x)} \\
&\quad + \underbrace{\big(\mathfrak{t}^X_{x,\cdot}(0)\big)^{-1}}_{=\mathbf{1}_{\mathcal{F}}} \underbrace{\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \phi\big(\gamma^X_x(t)\big)}_{=(X\cdot\phi)(x)}
\end{aligned}$$

for all $\phi \in C^\infty(U, C^\infty(\mathcal{F})) \overset{\text{dense}}{\subset} \in L^2(U, W^1(\mathcal{F})) \cap W^1(U, L^2(\mathcal{F}))$, and identify $(A^\Theta(X))(x) \in C_b^\infty(\mathrm{End}(\mathcal{F}))$ as the mapping

$$y \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{t}^X_{x,y}(t) \right)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{t}^X_{x,y}(-t) \in \mathrm{End}(\mathcal{F}_y)$$

if $X \in C_b^\infty(\mathsf{T}U)$.



The boundedness of $A^\Theta(X)$ is clearly implied by the uniform boundedness properties of the local trivialisations Φ (uniform local triviality of $M \xrightarrow{\pi_M} B$) and Ψ (see Proposition 2.6) as well as the C^∞-boundedness of the parallel transport map $\mathfrak{p}^X_{\cdot,\cdot}$ which stems from the bounded geometry of the vector bundle $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$. In particular, $A^{\Theta}(X)$ defines a smooth and uniformly bounded mapping from U to $\mathcal{L}(W^k(\mathcal{F}))$ for all $k \in \mathbb{N}_0$. $\square$

The vector bundle isomorphism (3.6) induces a bounded map

$$\mathcal{W} : L^2\big(\mathcal{H}_{\mathcal{F}}|_U\big) \to L^2\big(U, L^2(\mathcal{F})\big), \quad \phi \mapsto \Theta\phi$$

between Hilbert spaces. This allows us to transfer various operators to the local product with identical fibres:

Definition 3.11 Let $\mathcal{D}(\mathcal{F}) := W^2(\mathcal{F}) \cap W^1_0(\mathcal{F}) \subset L^2(\mathcal{F})$ be the x-independent Dirichlet domain of the Laplacian.

(i) The operator

$$\begin{aligned} H^{\mathcal{F}(\cdot)} := \mathcal{W} \circ H^{\mathcal{F}} \circ \mathcal{W}^{-1} : U \to \mathcal{L}\big(\mathcal{D}(\mathcal{F}), L^2(\mathcal{F})\big),& \\ x \mapsto -\Delta^{\mathcal{F}(x)} + V^{\mathcal{F}(x)}& \end{aligned}$$

is fibrewise self-adjoint with $L^2(\mathcal{F})$-domain $\mathrm{dom}(H^{\mathcal{F}(x)}) = \mathcal{D}(\mathcal{F})$ for all $x \in U$. Here,

- $\Delta^{\mathcal{F}(x)}$ stands for the connection Laplacian associated with the connection
  $$\nabla^{\mathcal{F}(x)} := \big(\Psi|_{\mathcal{E}_X}^{-1}\big)^* \nabla^{\mathcal{E}_x} : C^\infty(\mathcal{F}) \to C^\infty(\mathsf{T}^*F \otimes \mathcal{F})$$
  on $\mathcal{F}$ and the Riemannian metric
  $$g_F^{(x)} := \big(\Phi|_{M_x}^{-1}\big)^* g_{M_x} \in C^\infty(\Sigma^2 \mathsf{T}F)$$
  on F,
- and $V^{\mathcal{F}(\cdot)} : U \to C_{\mathrm{b}}^\infty(\mathrm{End}(\mathcal{F}))$ is the potential given by
  $$V^{\mathcal{F}(x)} := (\Psi|_{\mathcal{E}_x}^{-1})^* V^{\mathcal{E}}\big|_{\mathcal{E}_x} = \Psi|_{\mathcal{E}_x} \circ V^{\mathcal{E}} \circ \Phi|_{M_x}^{-1}.$$

(ii) The associated resolvent

$$R^{\mathcal{F}(\cdot)}(z) := \mathcal{W} \circ R^{\mathcal{F}}(z) \circ \mathcal{W}^{-1} : U \to \mathcal{L}\big(L^2(\mathcal{F}), \mathcal{D}(\mathcal{F})\big)$$

is the mapping

$$
\begin{aligned}
x \mapsto \big(H^{\mathcal{F}(x)} - z(x)\mathbf{1}_{L^2(\mathcal{F})}\big)^{-1} \\
= \big(-\Delta^{\mathcal{F}(x)} + V^{\mathcal{F}(x)} - z(x)\mathbf{1}_{L^2(\mathcal{F})}\big)^{-1}.
\end{aligned}
$$

(iii) Let $\lambda : B \to \mathbb{R}$ be an eigenband of $H^{\mathcal{F}}$. We let

$$
P_0^{\mathcal{F}(\cdot)} := \mathcal{W} \circ P_0 \circ \mathcal{W}^{-1} : U \to \mathcal{L}\big(L^2(\mathcal{F})\big) \cap \mathcal{L}\big(\mathcal{D}(\mathcal{F})\big)
$$

denote the spectral projection of $H^{\mathcal{F}(\cdot)}$ related to $\lambda|_U$. $\Diamond$

Note that $\sigma(H^{\mathcal{F}}(x))$ coincides with $\sigma(H^{\mathcal{F}(x)})$ and one has the equivalence

$$
H^{\mathcal{F}} P_0 = \lambda P_0 \quad \Leftrightarrow \quad H^{\mathcal{F}(x)} P_0^{\mathcal{F}(x)} = \lambda(x) P_0^{\mathcal{F}(x)} \text{ for all } x \in U.
$$

Furthermore, Condition 2.26 on the spectral gap immediately carries over with the same parameter δ. It is important to assure that the x-dependence of the operators introduced in the previous definition is appropriate:

Lemma 3.12 *Let $U \in \mathfrak{U}$ be a geodesic ball in B from Definition 2.7. Moreover, let $\lambda : U \to \mathbb{R}$ be an eigenband of $H^{\mathcal{F}(\cdot)}$ with a spectral gap and $z \in C_{\mathrm{b}}^{\infty}(U, \mathbb{C})$ with $z(x) \in \mathbb{C} \backslash \{\sigma(H^{\mathcal{F}(x)})\}$ for all $x \in U$. Then the mappings*

(i) $H^{\mathcal{F}(\cdot)} : U \to \mathcal{L}(\mathcal{D}(\mathcal{F}), L^2(\mathcal{F}))$,

(ii) $R^{\mathcal{F}(\cdot)}(z) : U \to \mathcal{L}(L^2(\mathcal{F}), \mathcal{D}(\mathcal{F}))$,

(iii) $P_0^{\mathcal{F}(\cdot)} : U \to \mathcal{L}(L^2(\mathcal{F}), \mathcal{D}(\mathcal{F})) \subset \mathcal{L}(L^2(\mathcal{F}))$

are smooth and bounded with all their derivatives.

Proof. (i) The Laplacian $\Delta^{\mathcal{F}(\cdot)}$ is a second-order vertical differential operator whose coefficients are smooth mappings from U to $\mathrm{End}(\mathcal{F})$ with uniform bounds due to the bounded geometry of the involved bundles $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ and $M \xrightarrow{\pi_M} B$ (see again Proposition 2.6). To make the latter precise, note that we need to check differentiability only locally thanks to the compactness of F. Therefore, we take any normal coordinate chart $V \subset F$ and a C^∞-bounded g_F-orthonormal frame $\{V_j\}_{j=1}^f$ of $\mathsf{T}F|_V$. Then

$$\Delta^{\mathcal{F}(x)}\Big|_{U\times\,\mathcal{F}|_V} = a^{jj'}(x)\nabla^{\mathcal{F}}_{V_j}\nabla^{\mathcal{F}}_{V_{j'}} + b^j(x)\nabla^{\mathcal{F}}_{V_j} + c(x),$$

where $a^{jj'} : U \to C^\infty(V)$ and $b^j, c : U \to \mathrm{End}(\mathcal{F}|_V)$ depend on the coefficients of $g_F^{(\cdot)}$ (and its inverse), the Christoffel symbols of the associated Levi-Civita connection $\nabla^{g_F^{(\cdot)}}$ and the Christoffel symbols of $\nabla^{\mathcal{F}^{(\cdot)}}$. But these functions are bounded uniformly with all their derivatives, since they arise from the respective coefficients associated with g_{V} and $\nabla^{\mathcal{E}}$ (transported back via the smooth and bounded maps Φ and Ψ). Moreover, $V^{\mathcal{F}^{(\cdot)}} = \mathcal{W}\circ V^{\mathcal{E}}\circ\mathcal{W}^{-1}$ is clearly bounded with all its derivatives.

(ii) We first show that $R^{\mathcal{F}^{(\cdot)}} : U \to \mathcal{L}(L^2(\mathcal{F}), \mathcal{D}(\mathcal{F}))$ is differentiable with bounded derivatives. Therefore, again let $\gamma_x^X : (-\delta,\delta) \to U$ be the integral curve of $X \in C_{\mathrm{b}}^\infty(\mathsf{T}U)$ through $\gamma_x^X(0) = x$. Writing $R(t,z(t)) := R^{\mathcal{F}(\gamma_x^X(t))}(z(\gamma_x^X(t)))$ and $H(t) := H^{\mathcal{F}(\gamma_x^X(t))}$, we obtain for the difference quotient with the aid of the resolvent identities:

$$\begin{aligned}
&\big(X\cdot R^{\mathcal{F}^{(\cdot)}}(z)\big)(x)\\
&= \lim_{t\to 0}\frac{R(t,z(t)) - R(0,z(0))}{t}\\
&= \lim_{t\to 0}\left(\frac{R(t,z(t)) - R(0,z(t))}{t} + \frac{R(0,z(t)) - R(0,z(0))}{t}\right)\\
&= -\lim_{t\to 0}\Bigg(R\big(t,z(t)\big)\frac{H(t)-H(0)}{t}R\big(0,z(t)\big)\\
&\qquad\qquad - R\big(0,z(t)\big)\frac{z(t)-z(0)}{t}R\big(0,z(0)\big)\Bigg)\\
&= \Big(-R^{\mathcal{F}^{(\cdot)}}(z)\big(X\cdot H^{\mathcal{F}^{(\cdot)}} - \mathrm{d}z(X)\mathbf{1}_{L^2(\mathcal{F})}\big)R^{\mathcal{F}^{(\cdot)}}(z)\Big)(x),
\end{aligned}$$

where we used part (i) of the lemma, $z \in C_{\mathrm{b}}^{\infty}(U, \mathbb{C})$ and the continuity of the mapping $x \mapsto R^{\mathcal{F}(x)}(z(x))$ (which can easily be checked by a similar calculation) in order to take the limit. Consequently,

$$\begin{aligned} & X \cdot R^{\mathcal{F}(\cdot)}(z) \\ & = -R^{\mathcal{F}(\cdot)}(z)\big(X \cdot H^{\mathcal{F}(\cdot)} - \mathrm{d}z(X)\mathbf{1}_{L^2(\mathcal{F})}\big)R^{\mathcal{F}(\cdot)}(z) \end{aligned}$$

defines a bounded map from U to $\mathcal{L}(\mathcal{D}(\mathcal{F}), L^2(\mathcal{F}))$. Iteration of the argument proves the statement.

(iii) Let $x_0 \in U$ be arbitrary and γ be the circle of radius δ around $\lambda(x_0)$ in $\mathbb{C}$. The rest of the spectrum $\sigma(H^{\mathcal{F}}(x_0)) \setminus \{\lambda(x_0)\}$ is entirely contained outside of the circle around $\lambda(x_0)$ of radius 2δ due to the spectral gap condition (cf. Figure 2.4). Hence, there is a neighbourhood $\tilde{U} \subset U$ of x_0 such that $\sigma(H^{\mathcal{F}(x)}) \setminus \{\lambda(x)\} \subset \complement(\mathbb{B}_{3\delta/2}(x_0))$ and thus $\operatorname{dist}(\gamma, \sigma(H^{\mathcal{F}(x)})) > \delta/2$ for all $x \in \tilde{U}$. The spectral projection $P_0^{\mathcal{F}(\cdot)}$ on $\tilde{U}$ is finally given by Riesz's projection formula [Kat80, Theorem III–6.17]

$$P_0^{\mathcal{F}(\cdot)} = \frac{\mathrm{i}}{2\pi} \int_{\gamma} R^{\mathcal{F}(\cdot)}(z)\, \mathrm{d}z.$$

Now (ii) and the compactness of the contour γ allow us to prove the claim by using the dominated convergence theorem applied to the Bochner integral. For instance, one has

$$X \cdot P_0^{\mathcal{F}(\cdot)} = \frac{\mathrm{i}}{2\pi} \int_{\gamma} X \cdot R^{\mathcal{F}(\cdot)}(z)\, \mathrm{d}z.$$

□

As pointed out in Remark 2.27, one often treats perturbed vertical operators of the form

$$\begin{aligned} H^{\mathcal{F}(\cdot),\varepsilon} &= \mathcal{W} \circ \big(-\Delta_{\mathsf{V}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{F}} + V_{\varepsilon}^{\mathcal{E}}\big) \circ \mathcal{W}^{-1} \\ &= -\Delta^{\mathcal{F}(\cdot)} + \varepsilon H_1^{\mathcal{F}(\cdot)} + V_{\varepsilon}^{\mathcal{F}(\cdot)}. \end{aligned}$$

The Katō-Rellich theorem ensures that this is a smooth and bounded mapping from U to $\mathcal{L}(\mathcal{D}(\mathcal{F}), L^2(\mathcal{F}))$ if the potential $V_\varepsilon^{\mathcal{F}(\cdot)} : U \to \mathcal{L}(L^2(\mathcal{F}))$ and the perturbation $H_1^{\mathcal{F}(\cdot)} : U \to \mathcal{L}(\mathcal{D}(\mathcal{F}), L^2(\mathcal{F}))$ are also smooth and bounded. This is satisfied if the initial perturbation $H_1^{\mathcal{F}} \in L^\infty(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}))$ carries additional ∂-horizontal smoothness in the sense of Definition 3.3: Take C^∞-bounded orthonormal frames $\{X_i^\nu\}_{i=1}^b$ of $(\mathsf{T}U_\nu, g_B)$ for all $\nu \in \mathbb{N}_0$ according to Definition 2.7 and require that

$$\left\| \left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_1}^\nu}, \left[\dots, \left[\nabla^{\mathcal{E}}_{\Phi_\nu^* X_{i_k}^\nu}, H_1^{\mathcal{F}}\big|_{\pi_M^{-1}(U_\nu)} \right] \dots \right] \right] \right\|_{\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})|_{U_\nu}} < C(k)$$

for all $k \in \mathbb{N}_0$ and $\{i_1, \dots, i_k\} \in \{1, \dots, b\}$.

We now have all the necessary tools at hand in order to prove the property claimed for the vertical resolvent.

Proof (of Lemma 3.9). The operator $R^{\mathcal{F}}(z)$ acts fibrewise with norms

$$\begin{aligned} \left\| \left(R^{\mathcal{F}}(z) \right)(x) \right\|^2_{\mathcal{L}(\mathcal{H}_{\mathcal{F}})|_x} &\leqslant \tfrac{1}{C^2}, \\ \left\| \left(R^{\mathcal{F}}(z) \right)(x) \right\|^2_{\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_x} &\leqslant \left(1 + \tfrac{|z(x)|}{C} \right)^2 + \tfrac{1}{C^2} \leqslant 2 + \tfrac{1 + 2|z(x)|^2}{C^2} \end{aligned}$$

for all $x \in B$, and hence $R^{\mathcal{F}}(z) \in L^\infty(\mathcal{L}(\mathcal{H}_{\mathcal{F}})) \cap L^\infty(\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}))$. Therefore, it remains to check the commutator condition

$$\left\| \left[\nabla^{\mathcal{E}}_{\Phi * X_{i_1}}, \left[\dots, \left[\nabla^{\mathcal{E}}_{\Phi * X_{i_k}}, R^{\mathcal{F}}(z) \right] \dots \right] \right] \right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_U} < C(k) \tag{3.7}$$

for some $U \in \mathfrak{U}$ from Definition 2.7.

Let us first consider a single commutator $[\nabla^{\mathcal{E}}_{\Phi * X_i}, R^{\mathcal{F}}(z)]$. By virtue of (2.10), the boundedness of this commutator in $\mathcal{L}(\mathcal{H}_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}})|_U$ (uniformly in $x \in U$) amounts to proving uniform boundedness of

$$\left[\nabla^{\Theta}_{X_i}, R^{\mathcal{F}(x)}(z) \right] = \mathcal{W} \circ \left[\nabla^{\mathcal{E}}_{\Phi * X_i}, R^{\mathcal{F}}(z) \right] \circ \mathcal{W}^{-1}$$

in $\mathcal{L}(L^2(\mathcal{F}),\mathcal{D}(\mathcal{F}))$. To do so, we compute for any $\phi \in C^\infty(U, C^\infty(\mathcal{F})) \overset{\text{dense}}{\subset} L^2(U, L^2(\mathcal{F}))$:

$$\begin{aligned}
\left[\nabla^\Theta_{X_i}, R^{\mathcal{F}^{(\cdot)}}(z)\right]\phi &= \nabla^\Theta_{X_i}\left(R^{\mathcal{F}^{(\cdot)}}(z)\phi\right) - R^{\mathcal{F}^{(\cdot)}}(z)\nabla^\Theta_{X_i}\phi \\
&= X^i \cdot \left(R^{\mathcal{F}^{(\cdot)}}(z)\phi\right) + A^\Theta(X_i)\left(R^{\mathcal{F}^{(\cdot)}}(z)\phi\right) \\
&\quad - R^{\mathcal{F}^{(\cdot)}}(z)\left(X_i \cdot \phi\right) - R^{\mathcal{F}^{(\cdot)}}(z) A^\Theta(X_i)\phi \\
&= \Big(\left(X_i \cdot R^{\mathcal{F}^{(\cdot)}}(z)\right) + \left[A^\Theta(X_i), R^{\mathcal{F}^{(\cdot)}}(z)\right]\Big)\phi .
\end{aligned}$$

Thus, the fibrewise operator

$$\left[\nabla^\Theta_{X_i}, R^{\mathcal{F}^{(x)}}(z)\right] = X_i \cdot R^{\mathcal{F}^{(x)}}(z) + \left[A^\Theta(X_i), R^{\mathcal{F}^{(x)}}(z)\right]$$

is an element of $\mathcal{L}(L^2(\mathcal{F}),\mathcal{D}(\mathcal{F}))$ with bounds uniform in $x \in U$ and $i \in \{1,\dots,b\}$ because both the resolvent $R^{\mathcal{F}^{(\cdot)}}(z) : U \to \mathcal{L}(L^2(\mathcal{F}),\mathcal{D}(\mathcal{F}))$ and the "connection one-form" $A^\Theta(X) : U \to \mathcal{L}(L^2(\mathcal{F})) \cap \mathcal{L}(\mathcal{D}(\mathcal{F}))$ are smooth and bounded with all their derivatives by virtue of Lemma 3.12(ii) and Lemma 3.10.

The same arguments apply to iterated commutators (3.7) and the commutator condition follows. □

3.2.3 The Spectral Projection

We continue with the classification of the spectral projection P_0 associated with an eigenband $\lambda : B \to \mathbb{R}$ that has a spectral gap:

Lemma 3.13 *Let $\lambda : B \to \mathbb{R}$ be an eigenband of $H^{\mathcal{F}}$ with a spectral gap, i.e., λ satisfies Condition 2.26. Then the associated spectral projection satisfies $P_0 \in \mathcal{A}^{0,0}_H$.*

Proof. The spectral gap condition yields that each $x_0 \in B$ possesses an open neighbourhood $\tilde{U} \subset B$ such that $\operatorname{dist}(\gamma, \sigma(H^{\mathcal{F}^{(x)}})) > \delta/2$ for all $x \in \tilde{U}$, where γ is the circle of radius δ around $\lambda(x_0)$ just as in the proof of Lemma 3.12(iii). Thus,

$$P_0 = \frac{\mathrm{i}}{2\pi}\int_\gamma R^{\mathcal{F}}(z)\,\mathrm{d}z$$

for all $x \in \tilde{U}$ and $P_0 \in \mathcal{A}_H^{0,0}$ follows from Lemma 3.9. $\square$

Next we show that the corresponding eigenspace bundle $\mathcal{P} = P_0\mathcal{H}_\mathcal{F}$ is a well-defined subbundle of $\mathcal{H}_\mathcal{F}$ and that the associated eigenband is a smooth function that is bounded together with all its derivatives.

Proposition 3.14 *Let $\lambda : B \to \mathbb{R}$ be an eigenband of $H^\mathcal{F}$ with a spectral gap. Then the corresponding eigenspace bundle $\mathcal{P} \subset \mathcal{H}_\mathcal{F}$ is a smooth, finite-rank subbundle and $\lambda \in C_\mathrm{b}^\infty(B)$.*

PROOF. We first note that the smoothness of the projections

$$P_0^{\mathcal{F}_\nu^{(\cdot)}} : U_\nu \to \mathcal{L}\big(\big(L^2(\mathcal{F}_\nu)\big), \quad x \mapsto P_0^{\mathcal{F}_\nu^{(x)}}$$

for all $\nu \in \mathbb{N}_0$ (cf. Remark 2.3), which was shown in Lemma 3.12(iii), implies that $q(x) := \operatorname{rank}(P_0(x)) < \infty$ is continuous and therefore constant. Moreover, for every $x_0 \in B$ we may show the existence of an open neighbourhood $U_{x_0} \subset B$ around x_0 and an associated, smooth local trivialisation $\Lambda_{x_0} : \pi_\mathcal{P}^{-1}(U_{x_0}) \to U_{x_0} \times \mathbb{C}^q$ as follows: Choose $\nu \in \mathbb{N}_0$ with $x_0 \in U_\nu$ and let $\mathcal{P}_{x_0} = \operatorname{im}(P_0(x_0))$ be spanned by the vectors $\phi_1^{x_0}, \ldots, \phi_q^{x_0}$, which induce vectors

$$\psi_j^{x_0} := \frac{\mathcal{W}_\nu|_{L^2(\mathcal{E}_{x_0})}\phi_j^{x_0}}{\big\|\mathcal{W}_\nu|_{L^2(\mathcal{E}_{x_0})}\phi_j^{x_0}\big\|_{L^2(\mathcal{F}_\nu)}} \in L^2(\mathcal{F}_\nu), \quad j \in \{1, \ldots, q\}.$$

Again due to Lemma 3.12(iii), there exists a constant $r_{x_0} < r_\mathrm{inj}(B, g_B)$ such that

$$\left\|\left(P_0^{\mathcal{F}_\nu^{(x)}} - P_0^{\mathcal{F}_\nu^{(x_0)}}\right)\psi_j^{x_0}\right\|_{L^2(\mathcal{F}_\nu)} \leqslant \tfrac{1}{2}$$

for all $j = 1, \ldots, q$ and $x \in \mathsf{B}_{r_{x_0}}(x_0) =: U_{x_0}$. Then the mappings

$$x \mapsto \psi_j^{x_0}(x) := P_0^{\mathcal{F}_\nu^{(x)}}\psi_j^{x_0}, \quad j \in \{1, \ldots, q\}$$

are smooth, form a fibrewise basis of $(\mathcal{W}_\nu\mathcal{P})|_{U_{x_0}} \subset L^2(U_{x_0}, L^2(\mathcal{F}_\nu))$ and provide a smooth local trivialisation

$$\begin{aligned}\Lambda_{x_0} : \pi_{\mathcal{P}}^{-1}(U_{x_0}) &\to U_{x_0} \times \mathbb{C}^q,\\ \phi(x) = \mathcal{W}|_{\mathcal{P}_x}^{-1}\Big(\textstyle\sum_{j=1}^q c^j \psi_j^{x_0}(x)\Big) &\mapsto \big(x, (c^1, \dots, c^q)\big).\end{aligned}$$

Let us now turn to the boundedness of the eigenband λ. Its restriction to U_ν for some $\nu \in \mathbb{N}_0$ satisfies the formula

$$\lambda|_{U_\nu} = \tfrac{1}{q}\operatorname{tr}\Big(H^{\mathcal{F}_\nu^{(\cdot)}} P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big).$$

The directional derivative of $\lambda|_{U_\nu}$ along some $X \in C_{\mathrm{b}}^\infty(\mathsf{T}U_\nu)$ then reads

$$\begin{aligned}X \cdot \lambda &= [X, \lambda]\\ &= \tfrac{1}{q}\operatorname{tr}\Big(\Big[X, H^{\mathcal{F}_\nu^{(\cdot)}} P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big]\Big)\\ &= \tfrac{1}{q}\operatorname{tr}\Big(H^{\mathcal{F}_\nu^{(\cdot)}} P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big[X, P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big] + H^{\mathcal{F}_\nu^{(\cdot)}}\Big[X, P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big] P_0^{\mathcal{F}_\nu^{(\cdot)}}\\ &\qquad + \Big[X, H^{\mathcal{F}_\nu^{(\cdot)}}\Big] P_0^{\mathcal{F}_\nu^{(\cdot)}}\Big)\end{aligned}$$

by means of the projection property $P_0^{\mathcal{F}_\nu^{(\cdot)}} \circ P_0^{\mathcal{F}_\nu^{(\cdot)}} = P_0^{\mathcal{F}_\nu^{(\cdot)}}$. Each of those three resulting terms is trace class, since each has finite rank of at most q, and defines a continuous and bounded mapping from U_ν to $\mathcal{L}(L^2(\mathcal{F}_\nu))$ because of Lemma 3.12(i),(iii). Thus, $(X \cdot \lambda)|_{U_\nu}$ is continuous and bounded for all $X \in C_{\mathrm{b}}^\infty(\mathsf{T}U_\nu)$ and $\nu \in \mathbb{N}_0$, which proves $\lambda \in C_{\mathrm{b}}^1(B)$. Iterated application of these arguments yields iterated commutators, which are again bounded and continuous for the same reasons. □

We end this subsection with the classification of the *reduced resolvent*

$$R^{\mathcal{F}}(\lambda) := (H_{\mathcal{F}} - \lambda \mathbf{1}_{\mathcal{H}})^{-1}(\mathbf{1}_{\mathcal{H}} - P_0)$$

related to an eigenband λ:

Corollary 3.15 *Let $\lambda : B \to \mathbb{R}$ be an eigenband of $H^{\mathcal{F}}$ with a spectral gap and P_0 its associated spectral projection. Then the reduced resolvent satisfies $R^{\mathcal{F}}(\lambda) \in \mathcal{A}_H^{0,0}$.*

PROOF. This immediately follows from the formula

$$R^{\mathcal{F}}(\lambda) = (\mathbf{1}_{\mathcal{H}} - P_0)\left(\frac{\mathrm{i}}{2\pi}\int_\gamma \frac{1}{\lambda - z} R^{\mathcal{F}}(z)\,\mathrm{d}z\right)(\mathbf{1}_{\mathcal{H}} - P_0)$$

together with Lemma 3.9 ($R^{\mathcal{F}}(z) \in \mathcal{A}_H^{0,0}$), Lemma 3.13 ($P_0 \in \mathcal{A}_H^{0,0}$) and Proposition 3.14 ($\lambda \in C_{\mathrm{b}}^\infty(B)$). □

3.3 Construction of Super-Adiabatic Projections

We start this section with the construction of an "almost-projection" P^n for $n \in \mathbb{N}_0$, which will be the basis for the super-adiabatic projection P^ε:

Lemma 3.16 *For every $n \in \mathbb{N}_0$ there exist $P_k \in \mathcal{A}_H^{2^k,0}$ for $k \in \{0, 1, \ldots, n\}$, such that*

$$P^n := \sum_{k=0}^n \varepsilon^k P_k$$

satisfies

(i) $(P^n)^2 - P^n \in \mathcal{A}_H^{2^{n+1},n+1}$,

(ii) $\left\|[H^{\mathcal{E}}, P^n]\right\|_{2^n+2} = \mathcal{O}(\varepsilon^{n+1})$ *on* $\mathrm{dom}(H^{\mathcal{E}})$.

The proof is based on [Teu03, Lemma 3.8]. In contrast to using the methods of pseudo-differential calculus, we instead explicitly construct the expansion in terms of commutators. This is possible since we consider (isolated) eigenbands and not more complicated subsets of $\sigma(H^{\mathcal{F}})$.

Let us first introduce the notation $P_0^\perp := \mathbf{1}_{\mathcal{H}} - P_0$. The decomposition $\mathcal{H} = P_0\mathcal{H} \oplus P_0^\perp\mathcal{H}$ allows for the splitting of an operator

$$T = \begin{pmatrix} P_0 T P_0 & P_0 T P_0^\perp \\ P_0^\perp T P_0 & P_0^\perp T P_0^\perp \end{pmatrix}$$

into a diagonal block

$$T^{\mathrm{D}} := \begin{pmatrix} P_0 T P_0 & \mathbf{0} \\ \mathbf{0} & P_0^\perp T P_0^\perp \end{pmatrix} = P_0 T P_0 + P_0^\perp T P_0^\perp$$

and an off-diagonal block

$$T^{\mathrm{O}} := \begin{pmatrix} \mathbf{0} & P_0 T P_0^\perp \\ P_0^\perp T P_0 & \mathbf{0} \end{pmatrix} = P_0 T P_0^\perp + P_0^\perp T P_0.$$

Moreover, we remark that

$$[T, P_0] = \left[T, P_0^2\right] = P_0[T, P_0] + [T, P_0]P_0,$$

and so

$$\begin{aligned} & P_0[T, P_0]P_0 = P_0^2[T, P_0]P_0 + P_0[T, P_0]P_0^2 = 2P_0[T, P_0]P_0 \\ \Leftrightarrow \quad & P_0[T, P_0]P_0 = \mathbf{0}, \end{aligned}$$

i.e., any commutator with P_0 is always off-diagonal.

PROOF (of Lemma 3.16). We prove the statement by induction.

Base Case ($n = 0$):
$P^0 := P_0$ is an element of $\mathcal{A}_H^{0,0} \subset \mathcal{A}_H^{2^0,0}$ by Lemma 3.13.

(i) It holds that $P_0^2 - P_0 = 0 \in \mathcal{A}_H^{2^{0+1},0+1}$ since P_0 is a spectral projection.

(ii) Using the fact that $[H^{\mathcal{F}}, P_0] = \mathbf{0}$, we get

$$\left\|[H^{\mathcal{E}}, P_0]\right\|_{2^0+2} \leqslant \underbrace{\left\|[-\varepsilon^2 \Delta_{\mathrm{H}}^{\mathcal{E}}, P_0]\right\|_3}_{=\mathcal{O}(\varepsilon)\text{ by (3.5)}} + \varepsilon \underbrace{\left\|[H_1, P_0]\right\|_3}_{=\mathcal{O}(1)} = \mathcal{O}(\varepsilon^{0+1})$$

since both $H_1 P_0$ and $P_0 H_1$ are elements of $\mathcal{A}^{2,0} \subset \mathcal{A}^{3,0}$ due to the fact that $P_0 \in \mathcal{A}_H^{0,0}$.

Inductive Step ($n \to n+1$):
Let $P^n = \sum_{k=0}^{n}$ be given We will frequently use the fact that

$$P^n = P_0 + \mathcal{A}_H^{2^n,1}. \tag{3.8}$$

We then define the additional term within $P^{n+1} = P^n + \varepsilon^{n+1} P_{n+1}$ by

$$\begin{aligned}\varepsilon^{n+1} P_{n+1} := &\underbrace{-P_0\big((P^n)^2 - P^n\big)P_0 + P_0^\perp\big((P^n)^2 - P^n\big)P_0^\perp}_{=\varepsilon^{n+1}P_{n+1}^{\mathrm{D}}} \\ &\underbrace{-P_0^\perp\big(R^{\mathcal{F}}(\lambda)[H^{\mathcal{E}}, P^n]\big)P_0 + P_0\big([H^{\mathcal{E}}, P^n]R^{\mathcal{F}}(\lambda)\big)P_0^\perp}_{=\varepsilon^{n+1}P_{n+1}^{\mathrm{O}}}.\end{aligned}$$

This is an element of $\mathcal{A}_H^{2^{n+1},n+1}$ (i.e., $P_{n+1} \in \mathcal{A}_H^{2^{n+1},0}$) because

- $P_0, P_0^\perp \in \mathcal{A}_H^{0,0}$ and $(P^n)^2 - P^n \in \mathcal{A}_H^{2^{n+1},n+1}$ by the inductive hypothesis for the diagonal term,
- and $P_0, P_0^\perp \in \mathcal{A}_H^{0,0}$, $R^{\mathcal{F}}(\lambda) \in \mathcal{A}_H^{0,0}$ as well as $[H^{\mathcal{E}}, P^n] \in \mathcal{A}_H^{2^n+2,n+1} \subset \mathcal{A}_H^{2^{n+1},n+1}$ by the inductive hypothesis, since $2^{n+1} \geqslant 2^n + 2$ for $n \geqslant 1$, for the off-diagonal term.

We will verify the two properties by separately examining the diagonal and off-diagonal blocks.

(i) For the P_0-P_0-block we get:

$$\begin{aligned}&P_0\big((P^{n+1})^2 - P^{n+1}\big)P_0 \\ &\quad= P_0\big((P^n)^2 - P^n\big)P_0 \\ &\qquad + \varepsilon^{n+1}P_0\big(P^n P_{n+1} + P_{n+1}P^n - P_{n+1}\big)P_0 \\ &\qquad + \underbrace{\varepsilon^{2n+2}P_0 P_{n+1}^2 P_0}_{\in \mathcal{A}_H^{2^{n+1}+2^{n+1},2n+2}}\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(3.8)}{=} \underbrace{P_0\big((P^n)^2 - P^n\big)P_0}_{=-\varepsilon^{n+1}P_0P^{\mathrm{D}}_{n+1}P_0} \\
&\quad + \varepsilon^{n+1}\underbrace{P_0\big(P_0P_{n+1} + P_{n+1}P_0 - P_{n+1}\big)P_0}_{=P_0P_{n+1}P_0=P_0P^{\mathrm{D}}_{n+1}P_0} \\
&\quad + \mathcal{A}_H^{2^{n+2},n+2} \\
&\in \mathcal{A}_H^{2^{n+2},n+2}.
\end{aligned}
$$

The considerations for the $P_0^\perp$-$P_0^\perp$-block are essentially the same. As far as the $P_0^\perp$-P_0-block is concerned, we first observe that

$$
P_0^\perp\big((P^{n+1})^2 - P^{n+1}\big)P_0
$$

equals

$$
\begin{aligned}
&P_0^\perp\big((P^n)^2 - P^n\big)P_0 + \varepsilon^{n+1}P_0^\perp\big(P_{n+1}P^n - P_{n+1}\big)P_0 \\
&+\varepsilon^{n+1}P_0^\perp P^nP_{n+1}P_0 + \underbrace{\varepsilon^{2n+2}P_0^\perp P_{n+1}^2P_0}_{\in\mathcal{A}_H^{2^{n+1}+2^{n+1},2n+2}}.
\end{aligned}
$$

The second and the third term are in $\mathcal{A}_H^{2^{n+1}+2^n,n+2} \subset \mathcal{A}_H^{2^{n+2},n+2}$:

$$
\begin{aligned}
&\varepsilon^{n+1}P_0^\perp\big(P_{n+1}P^n - P_{n+1}\big)P_0 \\
&\stackrel{(3.8)}{=} \varepsilon^{n+1}P_0^\perp\Big(P_{n+1}\big(P_0 + \mathcal{A}_H^{2^n,1}\big) - P_{n+1}\Big)P_0 \\
&\;=\; \varepsilon^{n+1}\underbrace{\big(P_0^\perp P_{n+1}P_0 - P_0^\perp P_{n+1}P_0\big)}_{=\mathbf{0}} + \mathcal{A}_H^{2^{n+1}+2^n,n+2} \\
&\in \mathcal{A}_H^{2^{n+1}+2^n,n+2}
\end{aligned}
$$

and

$$
\varepsilon^{n+1}P_0^\perp P^nP_{n+1}P_0 \stackrel{(3.8)}{=} \varepsilon^{n+1}P_0^\perp\big(P_0 + \mathcal{A}_H^{2^n,1}\big)P_{n+1}P_0
$$

$$
\begin{aligned}
&= \varepsilon^{n+1} \underbrace{P_0^\perp P_0}_{=\mathbf{0}} P_{n+1} P_0 + \mathcal{A}_H^{2^n+2^{n+1},n+2} \\
&\in \mathcal{A}_H^{2^n+2^{n+1},n+2}.
\end{aligned}
$$

Hence, $P_0^\perp((P^{n+1})^2 - P^{n+1})P_0$ simplifies to

$$
\begin{aligned}
&P_0^\perp \big((P^n)^2 - P^n\big) \underbrace{(P^n + P_0 - P^n)P_0}_{=P_0} + \mathcal{A}_H^{2^{n+2},n+2} \\
&= P_0^\perp \big((P^n)^2 - P^n\big) P^n P_0 \\
&\quad + P_0^\perp \underbrace{\underbrace{\big((P^n)^2 - P^n\big)}_{\in \mathcal{A}_H^{2^{n+1},n+1}} \underbrace{(P_0 - P^n)}_{\in \mathcal{A}_H^{2^n,1}}}_{\in \mathcal{A}^{2^{n+1}+2^n,n+2} \subset \mathcal{A}_H^{2^{n+2},n+2}} P_0 + \mathcal{A}_H^{2^{2n+2},n+2} \\
&= P_0^\perp P^n \big((P^n)^2 - P^n\big) P_0 + P_0^\perp \underbrace{\big[(P^n)^2 - P_n, P_n\big]}_{=\mathbf{0}} P_0 \\
&\quad + \mathcal{A}_H^{2^{n+2},n+2} \\
&\overset{(3.8)}{=} P_0^\perp (P_0 + \mathcal{A}_H^{2^n,1}) \underbrace{\big((P^n)^2 - P_n\big)}_{\in \mathcal{A}_H^{2^{n+1},n+1}} P_0 + \mathcal{A}_H^{2^{n+2},n+2} \\
&= \underbrace{P_0^\perp P_0}_{=0} \big(P^n)^2 - P_n\big) P_0 + \mathcal{A}_H^{2^{n+2},n+2} \\
&\in \mathcal{A}_H^{2^{n+2},n+2}.
\end{aligned}
$$

The computations for the P_0-$P_0^\perp$-block are similar.

(ii) Once again we begin with the diagonal block. Therefore, we first consider

$$
P_0[H^{\mathcal{E}}, \varepsilon^{n+1} P_{n+1}^{\mathrm{O}}]P_0 = \varepsilon^{n+1} P_0 (H^{\mathcal{E}} P_0^\perp P_{n+1}^{\mathrm{O}} - P_{n+1}^{\mathrm{O}} P_0^\perp H^{\mathcal{E}}) P_0.
$$

Adding

$$\begin{aligned}\mathbf{0} &= \varepsilon^{n+1} P_0 \underbrace{\left(-[H^{\mathcal{E}}, P_0] + H^{\mathcal{E}} P_0 - P_0 H^{\mathcal{E}}\right)}_{=\mathbf{0}} P^{\mathrm{O}}_{n+1} P_0 \\ &\quad + \varepsilon^{n+1} P_0 P^{\mathrm{O}}_{n+1} P_0^{\perp} \underbrace{\left(-[H^{\mathcal{E}}, P_0] + H^{\mathcal{E}} P_0 - P_0 H^{\mathcal{E}}\right)}_{=\mathbf{0}} P_0 \\ &= -\varepsilon^{n+1}\left(P_0 [H^{\mathcal{E}}, P_0] P^{\mathrm{O}}_{n+1} P_0 + P_0 P^{\mathrm{O}}_{n+1} P_0^{\perp} [H^{\mathcal{E}}, P_0] P_0\right) \\ &\quad - \varepsilon^{n+1}\left(P_0 H^{\mathcal{E}} P^{\mathrm{O}}_{n+1} P_0 - P_0 P^{\mathrm{O}}_{n+1} P_0^{\perp} H^{\mathcal{E}} P_0\right)\end{aligned}$$

to the equality, a rearrangement of the terms yields that

$$P_0 [H^{\mathcal{E}}, \varepsilon^{n+1} P^{\mathrm{O}}_{n+1}] P_0$$

is equal to

$$\begin{aligned}&\varepsilon^{n+1}\Big(-P_0 [H^{\mathcal{E}}, P_0] P^{\mathrm{O}}_{n+1} P_0 - P_0 P_{n+1} P_0^{\perp} [H^{\mathcal{E}}, P_0] P_0\big) \\ &\quad + \varepsilon^{n+1}\big(P_0 H^{\mathcal{E}} \underbrace{\underbrace{\left(P_0^{\perp} - \mathbf{1}_{\mathcal{H}}\right)}_{=-P_0} P^{\mathrm{O}}_{n+1} P_0}_{=\mathbf{0}} \\ &\qquad \underbrace{-P_0 P^{\mathrm{O}}_{n+1} P_0^{\perp} H^{\mathcal{E}} P_0 + P_0 P^{\mathrm{O}}_{n+1} P_0^{\perp} H^{\mathcal{E}} P_0}_{=\mathbf{0}}\Big) \\ &= \varepsilon^{n+1}\Big(-P_0 \underbrace{[H^{\mathcal{E}}, P_0] P^{\mathrm{O}}_{n+1}}_{\substack{\in \mathcal{A}^{2^{n+1}+2,1} \text{ by} \\ \text{Lemma 3.7(i)}}} P_0 \underbrace{-P_0 P_{n+1} P_0^{\perp} \underbrace{[H^{\mathcal{E}}, P_0] P_0}_{\substack{\in \mathcal{A}^{1,1} \text{ by} \\ \text{Lemma 3.7(i)}}}}_{\in \mathcal{A}^{2^{n+1}+1,1} \subset \mathcal{A}^{2^{n+1}+2,1}}\Big) \\ &\in \mathcal{A}^{2^{n+1}+2,n+2}.\end{aligned}$$

This implies

$$\left\| P_0 \big[H^{\mathcal{E}}, \varepsilon^{n+1} P^{\mathrm{O}}_{n+1}\big] P_0 \right\|_{2^{n+1}+2} = \mathcal{O}(\varepsilon^{n+2})$$

by Remark 3.6(ii). It remains to prove a similar estimate for the term $P^{n+1} - \varepsilon^{n+1} P^{\mathrm{O}}_{n+1} = P^n + \varepsilon^{n+1} P^{\mathrm{D}}_{n+1}$:

$$
\begin{aligned}
&P_0[H^{\mathcal{E}}, P^n + \varepsilon^{n+1} P^{\mathrm{D}}_{n+1}]P_0 \\
&= P_0[H^{\mathcal{E}}, P^n + \underbrace{\varepsilon^{n+1} P_0 P^{\mathrm{D}}_{n+1} P_0}_{=-P_0((P^n)^2 - P^n)P_0}]P_0 \\
&= P_0[H^{\mathcal{E}}, P^n]P_0 \\
&\quad - \Big(\underbrace{P_0 H^{\mathcal{E}} P_0\big((P^n)^2 - P_n\big)P_0}_{\substack{=P_0 H^{\mathcal{E}}((P^n)^2 - P^n)P_0 \\ +P_0[H^{\mathcal{E}}, P_0]((P^n)^2 - P^n)P_0}} - \underbrace{P_0\big((P^n)^2 - P^n\big)P_0 H^{\mathcal{E}} P_0}_{\substack{=P_0((P^n)^2 - P^n)H^{\mathcal{E}} P_0 \\ +P_0((P^n)^2 - P^n)[P_0, H^{\mathcal{E}}]P_0}}\Big) \\
&= P_0[H^{\mathcal{E}}, P^n]P_0 - P_0\big[H^{\mathcal{E}}, (P^n)^2 - P^n\big]P_0 \\
&\quad - P_0 \underbrace{[H^{\mathcal{E}}, P_0] \underbrace{\big((P^n)^2 - P^n\big)}_{\in \mathcal{A}_H^{2^{n+1}, n+1}}}_{\in \mathcal{A}^{2^{n+1}+1, n+2} \text{ by Lemma 3.7(i)}} P_0 \\
&\quad + P_0 \underbrace{\big((P^n)^2 - P^n\big)}_{\in \mathcal{A}^{2^{n+1}, n+1}} \underbrace{[P_0, H^{\mathcal{E}}]P_0}_{\substack{\in \mathcal{A}^{1,1} \text{ by} \\ \text{Lemma 3.7(i)}}} \\
&= 2P_0[H^{\mathcal{E}}, P^n]P_0 - P_0\big[H^{\mathcal{E}}, (P^n)^2\big]P_0 + \mathcal{A}^{2^{n+1}+2, n+2} \\
&= P_0 \underbrace{(P_0 - P^n)}_{\in \mathcal{A}_H^{2^n, 1} \text{ by (3.8)}} \underbrace{[H^{\mathcal{E}}, P^n]}_{=\mathcal{O}(\varepsilon^{n+1})} P_0 + P_0 \underbrace{[H^{\mathcal{E}}, P^n]}_{=\mathcal{O}(\varepsilon^{n+1})} \underbrace{(P_0 - P^n)}_{\in \mathcal{A}_H^{2^n, 1} \text{ by (3.8)}} P_0 \\
&\quad + \mathcal{A}^{2^{n+1}+2, n+2}
\end{aligned}
$$

and thus

$$
\left\| P_0[H^{\mathcal{E}}, P^n + \varepsilon^{n+1} P^{\mathrm{D}}_{n+1}]P_0 \right\|_{2^{n+1}+2} = \mathcal{O}(\varepsilon^{n+2})
$$

using Remark 3.6(ii), which is what we wished to prove. The remaining $P_0^{\perp}$-$P_0^{\perp}$-block is again similar.

Dealing with the off-diagonal block, we already know that

$$
\big[-\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}} + \lambda \mathbf{1}_{\mathcal{H}} + \varepsilon H^{\mathcal{E}}_1, P_{n+1}\big]P_0 \in \mathcal{A}^{2^{n+1}+2, 1}
$$

by Lemma 3.7(i), Proposition 3.14 and the fact that $[H_1^{\mathcal{E}}, P_{n+1}] \in \mathcal{A}^{2^{n+1}+2,0}$. It then follows that

$$
\begin{aligned}
&P_0^\perp [H^{\mathcal{E}}, P^{n+1}] P_0 \\
&= P_0^\perp [H^{\mathcal{E}}, P^n] P_0 + \varepsilon^{n+1} P_0^\perp [H^{\mathcal{F}} - \lambda \mathbf{1}_{\mathcal{H}}, P_{n+1}] P_0 \\
&\quad + \underbrace{\varepsilon^{n+1} P_0^\perp [-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \lambda \mathbf{1}_{\mathcal{H}} + \varepsilon H_1^{\mathcal{E}}, P_{n+1}] P_0}_{\in \mathcal{A}^{2^{n+1}+2,n+2}} \\
&= P_0^\perp [H^{\mathcal{E}}, P^n] P_0 + P_0^\perp [H^{\mathcal{F}} - \lambda \mathbf{1}_{\mathcal{H}}, \ \underbrace{\varepsilon^{n+1} P_0^\perp P_{n+1} P_0}_{= -P_0^\perp R^{\mathcal{F}}(\lambda) [H^{\mathcal{E}}, P^n] P_0} \ ] P_0 \\
&\quad + \mathcal{A}^{2^{n+1}+2,n+2} \\
&= \underbrace{P_0^\perp [H^{\mathcal{E}}, P^n] P_0 - P_0^\perp \underbrace{(H^{\mathcal{F}} - \lambda \mathbf{1}_{\mathcal{H}}) R^{\mathcal{F}}(\lambda)}_{= P_0^\perp} [H^{\mathcal{E}}, P^n] P_0}_{= \mathbf{0}} \\
&\quad + P_0^\perp R^{\mathcal{F}}(\lambda) [H^{\mathcal{E}}, P^n] P_0 \underbrace{(H_F - \lambda \mathbf{1}_{\mathcal{H}}) P_0}_{= \mathbf{0}} + \mathcal{A}^{2^{n+1}+2,n+2},
\end{aligned}
$$

so finally

$$
\left\| P_0^\perp [H^{\mathcal{E}}, P^{n+1}] P_0 \right\|_{2^{n+1}+2} = \mathcal{O}(\varepsilon^{n+2})
$$

and in the same manner for the P_0-$P_0^\perp$-block.

This completes the proof. □

This "almost-projection" $P^n = \sum_{k=0}^n \varepsilon^k P_k$ leads to the ultimate super-adiabatic projection P_ε having the same asymptotic expansion in ε. The construction basically relies on the methods of [Sor03, Theorem 2.1] and is adapted to the situation at hand in [Lam14, Proposition 2.13]: Since P_k includes differential operators of order 2^k for $k \geqslant 1$, P^n generally does not define an element of $\mathcal{L}(\mathcal{H})$. Therefore, P_ε is extracted from P^n by a cut-off procedure.

Proposition 3.17 *For all $n \in \mathbb{N}$ and $\Lambda > 0$ there exists an orthogonal projection $P_\varepsilon \in \mathcal{L}(\mathcal{H}) \cap \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$, satisfying $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ in $\mathcal{L}(\mathcal{H})$ as well as in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$, such that*

$$\left\| [H^{\mathcal{E}}, P_\varepsilon] \varrho(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^{n+1})$$

for every Borel function $\varrho : \mathbb{R} \to [0,1]$ with support in $(-\infty, \Lambda]$.

We will need the notion of regular cut-off functions for the proof of this proposition. These are functions $f : \mathbb{R} \to [0,1]$ such that $f^s \in C_0^\infty(\mathbb{R})$ for all $s > 0$, i.e., f does not possess any zero of finite order.

Lemma 3.18 *Let A be a self-adjoint operator on some Hilbert space H with domain $\mathrm{dom}(A)$. Moreover, let $T \in \mathcal{L}(\mathsf{H}) \cap \mathcal{L}(\mathrm{dom}(A))$ be a self-adjoint operator on H.*

(i) If χ is a regular cut-off function and both conditions

$$\begin{aligned} \left\| [T, A] \right\|_{\mathcal{L}(\mathrm{dom}(A), \mathsf{H})} &= \mathcal{O}(\varepsilon), \\ \left\| [T, A] \chi^s(A) \right\|_{\mathcal{L}(\mathsf{H})} &= \mathcal{O}(\varepsilon^k) \textit{ for some } k \in \mathbb{N}_0 \textit{ and all } s > 0 \end{aligned}$$

are satisfied, then

$$\left\| \left[T, \chi(A) \right] \right\|_{\mathcal{L}(\mathsf{H}, \mathrm{dom}(A))} = \mathcal{O}(\varepsilon^k).$$

(ii) If in addition T is a projection, one has

$$\left\| \chi(TAT) T - T \chi(A) T \right\|_{\mathcal{L}(\mathsf{H}, \mathrm{dom}(A))} = \mathcal{O}(\varepsilon^k).$$

Proof. See [Lam14, Lemma C.2]. The proof shows that one actually has to add a further projection T to $\chi(TAT)$, which can be placed either to the left or to the right due to the fact that

$$[T, TAT] = \mathbf{0} \quad \Rightarrow \quad \left[T, \chi(TAT) \right] = \mathbf{0},$$

in order to obtain the correct second statement. □

We are now in a position to prove the above proposition:

PROOF (of Proposition 3.17). Let $n \in \mathbb{N}$ as well as $\Lambda > 0$ be arbitrary but fixed. Take P^n from Lemma 3.16 and a regular cut-off function χ which is equal to one on $[\inf \sigma(H^{\mathcal{E}}) - 1, \Lambda + 1]$ and equal to zero outside $(\inf \sigma(H^{\mathcal{E}}) - 2, \Lambda + 2)$. Set

$$\tilde{P} := P^n - P_0 = \sum_{k=1}^{n} \varepsilon^k P_k \in \mathcal{A}_H^{2^n,1} \subset \mathcal{A}^{2^n,1} \underset{3.6(\mathrm{i})}{\overset{\text{Remark}}{\subset}} \mathcal{L}\big(W_\varepsilon^{2^n}(\mathcal{E}), \mathcal{H}\big)$$

and define

$$P^\chi := P_0 + \tilde{P}\chi(H^{\mathcal{E}}) + \chi(H^{\mathcal{E}})\tilde{P}\big(\mathbf{1}_{\mathcal{H}} - \chi(H^{\mathcal{E}})\big) = P_0 + \mathcal{O}(\varepsilon).$$

This means that we exclude the diagonal $\chi^\perp$-$\chi^\perp$-block from $\tilde{P}$, i.e., the diagonal block with "energies" larger than $\Lambda + 2$. We now treat P^χ acting on $\mathcal{H}$ and $\mathrm{dom}(H^{\mathcal{E}})$ separately:

- We have $\chi(H^{\mathcal{E}}) \in \mathcal{L}(\mathcal{H}, \mathrm{dom}((H^{\mathcal{E}})^k))$ for every $k \in \mathbb{N}_0$ with norm

$$\big\|\chi(H^{\mathcal{E}})\big\|^2_{\mathcal{L}(\mathcal{H},\mathrm{dom}((H^{\mathcal{E}})^k))} \leqslant \underbrace{\big(\sup_t \big|\chi(t)\big|\big)^2}_{=1} + \underbrace{\big(\sup_t \big|t^k\chi(t)\big|\big)^2}_{<\infty}.$$

  Thanks to the regularity result $\mathrm{dom}((H_\varepsilon^{\mathcal{E}})^k) \subset W_\varepsilon^{2k}(\mathcal{E})$ of Corollary 2.17, $\tilde{P}\chi(H^{\mathcal{E}})$ defines a bounded operator on $\mathcal{H}$. Therefore, its adjoint operator is also bounded and one infers

$$\chi(H^{\mathcal{E}})\tilde{P} = \big(\tilde{P}\chi(H^{\mathcal{E}})\big)^\dagger$$

  on $W_\varepsilon^{2^k}(\mathcal{E}) \overset{\text{dense}}{\subset} \mathcal{H}$ due to the construction of P_k for $k \geqslant 1$ (cf. the inductive step within the proof of Lemma 3.16). Hence, its extension is also bounded in $\mathcal{L}(\mathcal{H})$ with norm

$$\begin{aligned}\big\|\chi(H^{\mathcal{E}})\tilde{P}\big\|_{\mathcal{L}(\mathcal{H})} &= \Big\|\big(\tilde{P}\chi(H^{\mathcal{E}})\big)^\dagger\Big\|_{\mathcal{L}(\mathcal{H})} = \big\|\tilde{P}\chi(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} \\ &= \mathcal{O}(\varepsilon)\end{aligned}$$

  since $\tilde{P} \in \mathcal{A}_H^{2^n,1} \subset \mathcal{A}^{2^n,1}$. Finally, $P^\chi \in \mathcal{L}(\mathcal{H})$ is self-adjoint on $\mathcal{H}$ by the very construction and $P^\chi - P_0 = \mathcal{O}(\varepsilon)$ holds in $\mathcal{L}(\mathcal{H})$.

- By virtue of the fact that $[H^{\mathcal{E}}, \chi(H^{\mathcal{E}})] = \mathbf{0}$, we obtain for all $k \in \mathbb{N}$ and arbitrary $\psi \in \mathrm{dom}(H^{\mathcal{E}})$:

$$\begin{aligned}
&\left\|\chi(H^{\mathcal{E}})\psi\right\|^2_{\mathrm{dom}((H^{\mathcal{E}})^k)}\\
&= \left\|\chi(H^{\mathcal{E}})\psi\right\|^2_{\mathcal{H}} + \left\|(H^{\mathcal{E}})^k\chi(H^{\mathcal{E}})\psi\right\|^2_{\mathcal{H}}\\
&= \left\|\chi(H^{\mathcal{E}})\psi\right\|^2_{\mathcal{H}} + \left\|(H^{\mathcal{E}})^{k-1}\chi(H^{\mathcal{E}})H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}}\\
&\leqslant \left\|\chi(H^{\mathcal{E}})\right\|^2_{\mathcal{L}(\mathcal{H})}\|\psi\|^2_{\mathcal{H}} + \left\|(H^{\mathcal{E}})^{k-1}\chi(H^{\mathcal{E}})\right\|^2_{\mathcal{L}(\mathcal{H})}\left\|H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}}\\
&\leqslant \underbrace{\max\left\{1, \sup_t\left|t^{k-1}\chi(t)\right|\right\}}_{<\infty}\underbrace{\left(\|\psi\|^2_{\mathcal{H}} + \left\|H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}}\right)}_{=\|\psi\|^2_{\mathrm{dom}(H^{\mathcal{E}})}}.
\end{aligned}$$

This together with

$$\tilde{P} \in \mathcal{A}_H^{2^n,1} \underset{3.6(\mathrm{iii})}{\overset{\text{Remark}}{\subset}} \mathcal{L}\left(W_\varepsilon^{2^n+2}(\mathcal{E}), \mathrm{dom}(H^{\mathcal{E}})\right)$$

yields $\tilde{P}\chi(H^{\mathcal{E}}) \in \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$. Consequently, it remains to show that $\chi(H^{\mathcal{E}})\tilde{P} \in \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$. In view of

$$\begin{aligned}
&\left\|\chi(H^{\mathcal{E}})\tilde{P}\psi\right\|^2_{\mathrm{dom}(H^{\mathcal{E}})}\\
&= \left\|\chi(H^{\mathcal{E}})\tilde{P}\psi\right\|^2_{\mathcal{H}} + \left\|H^{\mathcal{E}}\chi(H^{\mathcal{E}})\tilde{P}\psi\right\|^2_{\mathcal{H}}\\
&= \left\|\chi(H^{\mathcal{E}})\tilde{P}\psi\right\|^2_{\mathcal{H}} + \left\|\chi(H^{\mathcal{E}})H^{\mathcal{E}}\tilde{P}\psi\right\|^2_{\mathcal{H}}\\
&\leqslant \left\|\chi(H^{\mathcal{E}})\tilde{P}\psi\right\|^2_{\mathcal{H}} + \left\|\chi(H^{\mathcal{E}})\tilde{P}H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}} + \left\|\chi(H^{\mathcal{E}})\left[H^{\mathcal{E}}, \tilde{P}\right]\psi\right\|^2_{\mathcal{H}}\\
&\leqslant \left\|\chi(H^{\mathcal{E}})\tilde{P}\right\|^2_{\mathcal{L}(\mathcal{H})}\underbrace{\left(\|\psi\|^2_{\mathcal{H}} + \left\|H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}}\right)}_{=\|\psi\|^2_{\mathrm{dom}(H^{\mathcal{E}})}}\\
&\quad + \left\|\chi(H^{\mathcal{E}})\left[H^{\mathcal{E}}, \tilde{P}\right]\right\|^2_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}\|\psi\|^2_{\mathrm{dom}(H^{\mathcal{E}})}\\
&= \left(\underbrace{\left\|\chi(H^{\mathcal{E}})\tilde{P}\right\|^2_{\mathcal{L}(\mathcal{H})}}_{=\mathcal{O}(\varepsilon^2)} + \left\|\chi(H^{\mathcal{E}})\left[H^{\mathcal{E}}, \tilde{P}\right]\right\|^2_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}\right)\\
&\quad \times \|\psi\|^2_{\mathrm{dom}(H^{\mathcal{E}})}
\end{aligned}$$

for arbitrary $\psi \in \mathrm{dom}(H^{\mathcal{E}})$, it remains to prove that $\chi(H^{\mathcal{E}})[H^{\mathcal{E}}, \tilde{P}]$ is an element of $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}), \mathcal{H})$. But actually by the same argument as above,

$$\chi(H^{\mathcal{E}})\big[H^{\mathcal{E}}, \tilde{P}\big] = \big(-[H^{\mathcal{E}}, \tilde{P}]\chi(H^{\mathcal{E}})\big)^{\dagger}$$

on $W_{\varepsilon}^{2^n+2}(\mathcal{E}) \cap \mathrm{dom}(H^{\mathcal{E}}) \overset{\text{dense}}{\subset} \mathcal{H}$, and its extension is again bounded in $\mathcal{L}(\mathcal{H})$ and one has

$$\big\|\chi(H^{\mathcal{E}})\big[H^{\mathcal{E}}, \tilde{P}\big]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})} \leqslant \big\|[H^{\mathcal{E}}, \tilde{P}]\chi(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon)$$

by means of $\tilde{P} \in \mathcal{A}_H^{2^n,1}$. We thus conclude $P^{\chi} \in \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$ with $\|P^{\chi} - P_0\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon)$.

We finally connect the intermediate results of the previous two bullet points and obtain the estimate

$$\begin{aligned}
&\big\|[H^{\mathcal{E}}, P^{\chi}]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})} \\
&= \big\|[H^{\mathcal{E}}, P_0]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})} + \mathcal{O}(\varepsilon) \\
&= \underbrace{\big\|[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}, P_0]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon)\text{ by (3.5)}} + \varepsilon \underbrace{\big\|[H_1^{\mathcal{E}}, P_0]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(1)\text{, cf. proof of Lemma 3.16}} + \mathcal{O}(\varepsilon) \\
&= \mathcal{O}(\varepsilon).
\end{aligned} \tag{3.9}$$

Now let $\bar{\chi}$ be another regular cut-off function, which is equal to one on $[\inf\sigma(H^{\mathcal{E}}), \Lambda]$ and equal to zero where $\chi \not\equiv 1$. This implies $\chi\bar{\chi} = \bar{\chi}$ as well as $(1-\chi)\bar{\chi} = 0$ and consequently $P^{\chi}\bar{\chi}(H^{\mathcal{E}}) = P^n\bar{\chi}(H^{\mathcal{E}})$ by the functional calculus. Thus, we obtain the estimate

$$\begin{aligned}
\big\|[H^{\mathcal{E}}, P^{\chi}]\bar{\chi}(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} &= \big\|[H^{\mathcal{E}}, P^n]\bar{\chi}(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} \\
&\leqslant \underbrace{\big\|[H^{\mathcal{E}}, P^n]\big\|_{2^n+2}}_{\substack{=\mathcal{O}(\varepsilon^{n+1})\text{ by}\\ \text{Lemma 3.16(ii)}}} \underbrace{\big\|\bar{\chi}(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H}, W_{\varepsilon}^{2^n+2}(\mathcal{E}))}}_{<\infty} \\
&= \mathcal{O}(\varepsilon^{n+1}).
\end{aligned} \tag{3.10}$$

Since P^χ is ε-close to the projection P_0, it holds that

$$\left\|(P^\chi)^2 - P^\chi\right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon) \quad , \quad \left\|(P^\chi)^2 - P^\chi\right\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon).$$

The spectral mapping theorem then implies the existence of a constant $C > 0$ for which

$$\sigma(P^\chi) \subset [-C\varepsilon, C\varepsilon] \cup [1 - C\varepsilon, 1 + C\varepsilon]$$

as an operator both in $\mathcal{L}(\mathcal{H})$ and in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$. This suggests that we now define the super-adiabatic projection P_ε for $\varepsilon < \frac{1}{4C}$ by means of the integral

$$P_\varepsilon := \frac{\mathrm{i}}{2\pi} \int_{|z-1|=1/2} (P^\chi - z\mathbf{1}_{\mathcal{H}})^{-1}\, \mathrm{d}z.$$

Then $\mathrm{dist}(\sigma(P^\chi), z) \geqslant \frac{1}{4}$ for all $\{z \in \mathbb{C}$ such that $|z-1| = \frac{1}{2}\}$, and P_ε is well-defined and bounded by two in both $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$. Moreover, it is an orthogonal projection by the functional calculus and the second resolvent identity yields

$$\begin{aligned} P_\varepsilon - P_0 &= \frac{\mathrm{i}}{2\pi} \int_{|z-1|=1/2} (P^\chi - z\mathbf{1}_{\mathcal{H}})^{-1} - (P_0 - z\mathbf{1}_{\mathcal{H}})^{-1}\, \mathrm{d}z \\ &= \frac{\mathrm{i}}{2\pi} \int_{|z-1|=1/2} \underbrace{(P^\chi - z\mathbf{1}_{\mathcal{H}})^{-1}}_{\|\ldots\| \leqslant 4} \underbrace{(P_0 - P^\chi)}_{=\mathcal{O}(\varepsilon)} \underbrace{(P_0 - z\mathbf{1}_{\mathcal{H}})^{-1}}_{\|\ldots\| \leqslant 4}\, \mathrm{d}z \\ &= \mathcal{O}(\varepsilon) \end{aligned}$$

in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$.

Now that we have constructed the projection P_ε, it remains to prove the estimate in the proposition. Therefore, we first note that $\chi\bar{\chi}^s = \bar{\chi}^s$ and $(1-\chi)\bar{\chi}^s = 0$ for all $s > 0$, and so (3.10) stays valid for any positive power of $\bar{\chi}(H^{\mathcal{E}})$. This together with (3.9) allows us to apply Lemma 3.18(i) in order to get

$$\left\|\left[P^\chi, \bar{\chi}(H^{\mathcal{E}})\right]\right\|_{\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{n+1}),$$

and we ultimately arrive at

$$\begin{aligned}
&\left\|\left[R^\chi(z), \bar{\chi}(H^{\mathcal{E}})\right]\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\
&\quad = \left\|R^\chi(z)\left[P^\chi, \bar{\chi}(H^{\mathcal{E}})\right]R^\chi(z)\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{n+1}),
\end{aligned} \tag{3.11}$$

where $R^\chi(z) := (P^\chi - z\mathbf{1}_{\mathcal{H}})^{-1}$. Note that the operator in (3.11) makes sense since $R^\chi(z)$ is an element both of $\mathcal{L}(\mathcal{H})$ and of $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$ due to the open mapping theorem.

Finally, if we take into account that $\bar{\chi}(H^{\mathcal{E}})\varrho(H^{\mathcal{E}}) = \varrho(H^{\mathcal{E}})$, we can estimate

$$\begin{aligned}
&\left\|[H^{\mathcal{E}}, P_\varepsilon]\varrho(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H})} \\
&= \frac{1}{2\pi}\left\|\int_\gamma \left[H^{\mathcal{E}}, R^\chi(z)\right]\varrho(H^{\mathcal{E}})\,\mathrm{d}z\right\|_{\mathcal{L}(\mathcal{H})} \\
&= \frac{1}{2\pi}\left\|\int_\gamma R^\chi(z)[H^{\mathcal{E}}, P^\chi]R^\chi(z)\bar{\chi}(H^{\mathcal{E}})\varrho(H^{\mathcal{E}})\,\mathrm{d}z\right\|_{\mathcal{L}(\mathcal{H})} \\
&= \frac{1}{2\pi}\left\|\int_\gamma R^\chi(z)[H^{\mathcal{E}}, P^\chi] \right. \\
&\qquad\qquad \left. \times \Big(\bar{\chi}(H^{\mathcal{E}})R^\chi(z) + \left[R^\chi(z), \bar{\chi}(H^{\mathcal{E}})\right]\Big)\varrho(H^{\mathcal{E}})\,\mathrm{d}z\right\|_{\mathcal{L}(\mathcal{H})} \\
&\leqslant \frac{1}{2\pi}\left\|\int_\gamma R^\chi(z)\underbrace{[H^{\mathcal{E}}, P^\chi]\bar{\chi}(H^{\mathcal{E}})}_{=\mathcal{O}(\varepsilon^{n+1})\text{ by (3.10)}}R^\chi(z)\varrho(H^{\mathcal{E}})\,\mathrm{d}z\right\|_{\mathcal{L}(\mathcal{H})} \\
&\quad + \frac{1}{2\pi}\left\|\int_\gamma R^\chi(z)\underbrace{[H^{\mathcal{E}}, P^\chi]}_{\substack{=\mathcal{O}(\varepsilon)\\ \text{by (3.9)}}}\underbrace{\left[R^\chi(z), \bar{\chi}(H^{\mathcal{E}})\right]}_{=\mathcal{O}(\varepsilon^{n+1})\text{ by (3.11)}}\varrho(H^{\mathcal{E}})\,\mathrm{d}z\right\|_{\mathcal{L}(\mathcal{H})} \\
&= \mathcal{O}(\varepsilon^{n+1}),
\end{aligned}$$

and the proof is completed. □

We end this section with the comment that an analogous argumentation gives

$$\begin{aligned}
[H^{\mathcal{E}}, P_{\varepsilon}] &= \frac{\mathrm{i}}{2\pi} \int_{\gamma} \left[H^{\mathcal{E}}, R^{\chi}(z)\right] \mathrm{d}z \\
&= -\frac{\mathrm{i}}{2\pi} \int_{\gamma} R^{\chi}(z) \underbrace{\left[H^{\mathcal{E}}, P^{\chi}\right]}_{= \mathcal{O}(\varepsilon) \text{ by } (3.9)} R^{\chi}(z)\, \mathrm{d}z \\
&= \mathcal{O}(\varepsilon)
\end{aligned} \tag{3.12}$$

in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}), \mathcal{H})$.

Chapter 4
4 Main Results

We are now in a position to harvest the fruits of what has been sown in the course of the previous chapters. Therefore, we will assume throughout this chapter that Condition 2.2 on the geometry, Condition 2.16 on the Schrödinger operator $H^{\mathcal{E}}$ and Condition 2.26 on the eigenband λ are satisfied. Then Proposition 3.17 is applicable, i.e., we fix $n \in \mathbb{N}$ as well as $\Lambda > 0$ and construct the associated family of orthogonal projections P_ε for $0 < \varepsilon < \varepsilon_0$. We will exploit the specific properties of P_ε and define a self-adjoint operator $H_{\mathrm{eff}}^{\mathcal{P}}$ on $L^2(\mathcal{P})$ that approximates essential features of the full operator $H^{\mathcal{E}}$. The proofs basically rely on the standard methods which are used in the context of perturbation theory (cf. for example the detailed exposition in [Teu03]).

4.1 Dynamical Properties

We first show that the image of P_ε is almost invariant under the unitary group $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ generated by $H^{\mathcal{E}}$ in the following sense:

Lemma 4.1 *There exists a constant $C > 0$ such that*

$$\left\| \left[\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}, P_\varepsilon\right] \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C\varepsilon^{n+1}\,|t|$$

for every $0 < \varepsilon < \varepsilon_0$.

PROOF. This is an immediate consequence of Duhamel's principle. More precisely, we calculate

$$\begin{aligned}
&\big[\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}, P_\varepsilon\big]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \\
&= \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\big(P_\varepsilon - \mathrm{e}^{\mathrm{i}H^{\mathcal{E}}t}P_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\big)\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \\
&= \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\int_0^t \tfrac{\mathrm{d}}{\mathrm{d}s}\big(-\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}P_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\big)\,\mathrm{d}s\,\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \\
&= -\mathrm{i}\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\int_0^t \mathrm{e}^{\mathrm{i}H^{\mathcal{E}}s}[H^{\mathcal{E}}, P_\varepsilon]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\,\mathrm{d}s
\end{aligned}$$

on $\mathrm{dom}(H^{\mathcal{E}})$, where we used $[\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}, \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})] = \mathbf{0}$ for all $s \in [0,t]$, which follows from the spectral theorem. The $\mathcal{L}(\mathcal{H})$-norm of the latter equality then yields

$$\begin{aligned}
&\Big\|\big[\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}, P_\varepsilon\big]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})\Big\|_{\mathcal{L}(\mathcal{H})} \\
&\leqslant \underbrace{\big\|-\mathrm{i}\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}\big\|_{\mathcal{L}(\mathcal{H})}}_{=1} \\
&\quad\times \int_0^t \underbrace{\big\|\mathrm{e}^{\mathrm{i}H^{\mathcal{E}}s}\big\|_{\mathcal{L}(\mathcal{H})}}_{=1}\,\underbrace{\big\|[H^{\mathcal{E}}, P_\varepsilon]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{n+1})\text{ by Proposition 3.17}}\,\underbrace{\big\|\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\big\|_{\mathcal{L}(\mathcal{H})}}_{=1}\,\mathrm{d}s \\
&\leqslant C\varepsilon^{n+1}\,|t|
\end{aligned}$$

for some constant $C > 0$. □

This suggests that $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ may be well approximated on the image of P_ε by the unitary group $\mathrm{e}^{-\mathrm{i}P_\varepsilon H^{\mathcal{E}}P_\varepsilon}$. Therefore, let us first look at the diagonal block $H^{\mathcal{E},\mathrm{D}} = P_\varepsilon H^{\mathcal{E}}P_\varepsilon + P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp$. The difference

$$\begin{aligned}
H^{\mathcal{E},\mathrm{D}} - H^{\mathcal{E}} &= -(P_\varepsilon H^{\mathcal{E}}P_\varepsilon^\perp + P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon) \\
&= -P_\varepsilon H^{\mathcal{E}} + P_\varepsilon H^{\mathcal{E}}P_\varepsilon - H^{\mathcal{E}}P_\varepsilon + P_\varepsilon H^{\mathcal{E}}P_\varepsilon \\
&= -(\mathbf{1}_{\mathcal{H}} - 2P_\varepsilon)[H^{\mathcal{E}}, P_\varepsilon]
\end{aligned}$$

is symmetric and of order ε in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})$ by means of (3.12), i.e., it is $H^{\mathcal{E}}$-bounded with relative bound smaller than one for ε small enough (which causes a possible reduction of ε_0). Consequently, the Katō-Rellich theorem yields that $H^{\mathcal{E},\mathrm{D}}$ is self-adjoint on $\mathcal{H}$ with domain

$$\begin{aligned}\mathrm{dom}(H^{\mathcal{E},\mathrm{D}}) &= \mathrm{dom}(H^{\mathcal{E}})\\ &= \big(P_\varepsilon\,\mathrm{dom}(H^{\mathcal{E}})\oplus P_\varepsilon^{\perp}\,\mathrm{dom}(H^{\mathcal{E}})\big)\cap\big(P_\varepsilon\mathcal{H}\oplus P_\varepsilon^{\perp}\mathcal{H}\big)\\ &= \underbrace{\big(P_\varepsilon\,\mathrm{dom}(H^{\mathcal{E}})\oplus P_\varepsilon^{\perp}\mathcal{H}\big)}_{=\mathrm{dom}(P_\varepsilon H^{\mathcal{E}}P_\varepsilon)}\cap\underbrace{\big(P_\varepsilon^{\perp}\,\mathrm{dom}(H^{\mathcal{E}})\oplus P_\varepsilon\mathcal{H}\big)}_{=\mathrm{dom}(P_\varepsilon^{\perp}H^{\mathcal{E}}P_\varepsilon^{\perp})}.\end{aligned}$$

Although the so-called *super-adiabatic subspace* $P_\varepsilon\mathcal{H}$ may be rather complicated to characterise, it is still ε-close to $\mathcal{H}_{\mathcal{P}} := P_0\mathcal{H} = L^2(\mathcal{P})$, so we therefore aim to consider a unitarily equivalent, *effective operator*

$$H_{\mathrm{eff}}^{\mathcal{P}} := U_\varepsilon^{\dagger}P_\varepsilon H^{\mathcal{E}}P_\varepsilon U_\varepsilon \tag{4.1}$$

with domain $\mathcal{D}_{\mathcal{P}}^{\mathrm{eff}} := U_\varepsilon^{\dagger}P_\varepsilon\,\mathrm{dom}(H^{\mathcal{E}}) \subset \mathcal{H}_{\mathcal{P}}$, where $U_\varepsilon = \mathbf{1}_{\mathcal{H}} + \mathcal{O}(\varepsilon)$ is a unitary operator from $P_0\mathcal{H}$ to $P_\varepsilon\mathcal{H}$ constructed as follows:

Lemma 4.2 *For all sufficiently small $\varepsilon > 0$ there exists a unitary map $U_\varepsilon \in \mathcal{L}(\mathcal{H})\cap\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$ that intertwines P_ε and P_0, i.e., $U_\varepsilon P_0 = P_\varepsilon U_\varepsilon$.*

PROOF. We follow the exposition of [Kat80, Section I – § 4.6]. The operator

$$\tilde{U}_\varepsilon := P_\varepsilon P_0 + P_\varepsilon^{\perp}P_0^{\perp}$$

clearly maps $P_0\mathcal{H}$ to $P_\varepsilon\mathcal{H}$ and its adjoint $\tilde{U}_\varepsilon^{\dagger} = P_0P_\varepsilon + P_0^{\perp}P_\varepsilon^{\perp}$ does the reverse. The operator

$$S := \tilde{U}_\varepsilon\tilde{U}_\varepsilon^{\dagger} = \tilde{U}_\varepsilon^{\dagger}\tilde{U}_\varepsilon = \mathbf{1}_{\mathcal{H}} - (P_\varepsilon - P_0)^2$$

is positive and invertible for $\varepsilon > 0$ small enough due to the fact that $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ both in $\mathcal{L}(\mathcal{H})$ and in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$, which was obtained within the proof of Proposition 3.17. Thus, the desired unitary map is defined by the Sz.-Nagy formula

$$U_\varepsilon := \big(P_\varepsilon P_0 + P_\varepsilon^{\perp}P_0^{\perp}\big)S^{-1/2} = S^{-1/2}\big(P_\varepsilon P_0 + P_\varepsilon^{\perp}P_0^{\perp}\big),$$

where we took advantage of the fact that S commutes with both P_ε and P_0. We conclude that U_ε actually defines a unitary mapping from $P_\varepsilon\mathcal{H}$ to $P_0\mathcal{H}$ and that

$$U_\varepsilon P_0 = S^{-1/2} P_\varepsilon P_0 = P_\varepsilon P_0 S^{-1/2} = P_\varepsilon U_\varepsilon$$

holds true. □

We proceed with the first main theorem:

Theorem 4.3 *There exists* $\varepsilon_0 > 0$ *such that the effective operator* $(H^{\mathcal{P}}_{\text{eff}}, \mathcal{D}^{\text{eff}}_{\mathcal{P}})$ *defined by* (4.1) *and Lemma 4.2 is self-adjoint on* $\mathcal{H}_{\mathcal{P}}$ *and there exists a constant* $C > 0$ *with*

$$\left\| \left(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\text{eff}}t} U^\dagger_\varepsilon \right) P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C\varepsilon^{n+1}\,|t|$$

for all $0 < \varepsilon < \varepsilon_0$.

This theorem states that – after excluding energies larger than Λ – if we initially start in the super-adiabatic subspace $P_\varepsilon\mathcal{H} \subset \mathcal{H}$, the dynamics of $H^{\mathcal{E}}$ (i.e., the evolution under the unitary group $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$) may be approximated by those of $H^{\mathcal{P}}_{\text{eff}}$ up to errors of order $\varepsilon^{N+1}\,|t|$. Put differently, the approximation of $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ by $\mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\text{eff}}t}$ is accurate for very long times of order ε^{-n}. Moreover, $P^\perp_\varepsilon U_\varepsilon P_0 = P^\perp_\varepsilon P_\varepsilon U_\varepsilon = \mathbf{0}$ implies that

$$\begin{aligned}
&\left\| P^\perp_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} \\
&= \left\| P^\perp_\varepsilon \left(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\text{eff}}t} U^\dagger_\varepsilon \right) P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})} \\
&\leqslant \left\| \left(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\text{eff}}t} U^\dagger_\varepsilon \right) P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \right\|_{\mathcal{L}(\mathcal{H})},
\end{aligned}$$

and hence the subspace $P_\varepsilon\mathcal{H}$ is invariant under the dynamics of $H^{\mathcal{E}}$ for energies below Λ up to the same error.

PROOF (of Theorem 4.3). We already know that $(H^{\mathcal{P}}_{\text{eff}}, \mathcal{D}^{\text{eff}}_{\mathcal{P}})$ is unitarily equivalent to the self-adjoint operator $(P_\varepsilon H^{\mathcal{E}} P_\varepsilon, P_\varepsilon\,\mathrm{dom}(H^{\mathcal{E}}))$ on $P_\varepsilon\mathcal{H}$, so it only remains to check the estimate. We therefore again use Duhamel's principle and obtain

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}t} U^\dagger_\varepsilon \\
&= \int_0^t \tfrac{\mathrm{d}}{\mathrm{d}s}\big(U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)} U^\dagger_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\big)\,\mathrm{d}s \\
&= -\mathrm{i}\int_0^t U_\varepsilon \underbrace{(-H^{\mathcal{P}}_{\mathrm{eff}})\mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)}}_{=-\mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)}U^\dagger_\varepsilon U_\varepsilon H^{\mathcal{P}}_{\mathrm{eff}}} U^\dagger_\varepsilon \mathrm{e}^{-iH^{\mathcal{E}}s} \\
&\qquad + U_\varepsilon \mathrm{e}^{-iH^{\mathcal{P}}_{\mathrm{eff}}(t-s)} U^\dagger_\varepsilon H^{\mathcal{E}} \mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\,\mathrm{d}s \\
&= -\mathrm{i}\int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)} U^\dagger_\varepsilon \big(H^{\mathcal{E}} - \underbrace{U_\varepsilon H^{\mathcal{P}}_{\mathrm{eff}} U^\dagger_\varepsilon}_{=P_\varepsilon H^{\mathcal{E}} P_\varepsilon}\big)\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\,\mathrm{d}s
\end{aligned} \tag{4.2}$$

on $\mathrm{dom}(H^{\mathcal{E}})$. Thanks to the fact that

$$\big[U_\varepsilon H^{\mathcal{P}}_{\mathrm{eff}} U^\dagger_\varepsilon, P_\varepsilon\big] = [P_\varepsilon H^{\mathcal{E}} P_\varepsilon, P_\varepsilon] = \mathbf{0},$$

we finally infer

$$\begin{aligned}
&\big(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}t} U^\dagger_\varepsilon\big) P_\varepsilon \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \\
&\;=\; P_\varepsilon\big(\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t} - U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}t} U^\dagger_\varepsilon\big)\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}}) \\
&\qquad + \underbrace{\big[\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}, P_\varepsilon\big]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})}_{\substack{=\mathcal{O}(\varepsilon^{n+1}|t|)\text{ in }\mathcal{L}(\mathcal{H})\\ \text{by Lemma 4.1}}} \\
&\overset{(4.2)}{=} -\mathrm{i}P_\varepsilon \int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)} U^\dagger_\varepsilon \big(H^{\mathcal{E}} - P_\varepsilon H^{\mathcal{E}} P_\varepsilon\big)\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})\,\mathrm{d}s \\
&\qquad + \mathcal{O}\big(\varepsilon^{n+1}|t|\big) \\
&\;=\; -\mathrm{i}\int_0^t U_\varepsilon \mathrm{e}^{-\mathrm{i}H^{\mathcal{P}}_{\mathrm{eff}}(t-s)} U^\dagger_\varepsilon \underbrace{\big(P_\varepsilon H^{\mathcal{E}} - P_\varepsilon H^{\mathcal{E}} P_\varepsilon\big)}_{=-P_\varepsilon[H^{\mathcal{E}},P_\varepsilon]} \mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}\,\mathrm{d}s \\
&\qquad + \mathcal{O}\big(\varepsilon^{n+1}|t|\big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{i}\int_0^t \underbrace{U_\varepsilon \mathrm{e}^{-\mathrm{i}H_{\text{eff}}^{\mathcal{P}}(t-s)}U_\varepsilon^\dagger P_\varepsilon}_{\|\ldots\|_{\mathcal{L}(\mathcal{H})}\leqslant 1} \underbrace{[H^{\mathcal{E}}, P_\varepsilon]\mathbf{1}_{(-\infty,\Lambda]}(H^{\mathcal{E}})}_{\substack{=\mathcal{O}(\varepsilon^{n+1}) \text{ in } \mathcal{L}(\mathcal{H}) \\ \text{by Proposition 3.17}}} \underbrace{\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}s}}_{\|\ldots\|_{\mathcal{L}(\mathcal{H})}\leqslant 1} \,\mathrm{d}s \\
&\quad + \mathcal{O}\big(\varepsilon^{n+1}|t|\big) \\
&\leqslant C\varepsilon^{n+1}|t|
\end{aligned}
$$

in $\mathcal{L}(\mathcal{H})$ for some constant $C > 0$. □

4.2 Spectral Properties

Now that we know that the full dynamics $\mathrm{e}^{-\mathrm{i}H^{\mathcal{E}}t}$ are well approximated by the effective dynamics $\mathrm{e}^{-\mathrm{i}H_{\text{eff}}^{\mathcal{P}}t}$, we can pass to their respective spectra and ask for their mutual approximation:

Theorem 4.4 *Let $H_{\text{eff}}^{\mathcal{P}}$ be the effective operator from Theorem 4.3. Then for every $\delta > 0$ there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $\mu \in \sigma(H_{\text{eff}}^{\mathcal{P}})$ with $\mu \leqslant \Lambda - \delta$ one has*

$$\operatorname{dist}\big(\mu, \sigma(H^{\mathcal{E}})\big) \leqslant C\varepsilon^{n+1}$$

for all $0 < \varepsilon < \varepsilon_0$.

PROOF. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be a Weyl sequence in $\mathcal{H}_{\mathcal{P}}$ for μ, i.e., $\|\psi_k\|_{\mathcal{H}_{\mathcal{P}}} = 1$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big\|\big(H_{\text{eff}}^{\mathcal{P}} - \mu\big)\psi_k\big\|_{\mathcal{H}_{\mathcal{P}}} = 0.$$

The requirement $\mu \leqslant \Lambda - \delta$ enables us to choose this sequence to be in the image of $\mathbf{1}_{(-\infty,\Lambda-\delta/2]}(H_{\text{eff}}^{\mathcal{P}})$.

It seems natural to consider the normalised sequence $\{\varphi_k := U_\varepsilon\psi_k\}_{k\in\mathbb{N}}$ in $P_\varepsilon\mathcal{H}$ and to examine the behaviour of $\big(H^{\mathcal{E}} - \mu\big)\varphi_k$. Therefore,

$$
\begin{aligned}
&\big\|(H^{\mathcal{E}} - \mu)U_\varepsilon\psi_k\big\|_{\mathcal{H}} \\
&= \big\|\underbrace{(P_\varepsilon + P_\varepsilon^\perp)}_{=\mathbf{1}_{\mathcal{H}}}(H^{\mathcal{E}} - \mu)P_\varepsilon U_\varepsilon\psi_k\big\|_{\mathcal{H}}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \Big\| U_\varepsilon \underbrace{U_\varepsilon^\dagger P_\varepsilon (H^{\mathcal{E}} - \mu) P_\varepsilon U_\varepsilon}_{= P_0 (H^{\mathcal{P}}_{\mathrm{eff}} - \mu) P_0} \psi_k \Big\|_{\mathcal{H}} \\
&\quad + \Big\| P_\varepsilon^\perp (H^{\mathcal{E}} - \mu) P_\varepsilon U_\varepsilon \underbrace{\mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 \psi_k}_{= \psi_k} \Big\|_{\mathcal{H}} \\
&= \underbrace{\Big\| U_\varepsilon (H^{\mathcal{P}}_{\mathrm{eff}} - \mu) \psi_k \Big\|_{\mathcal{H}}}_{= \| (H^{\mathcal{P}}_{\mathrm{eff}} - \mu) \psi_k \|_{\mathcal{H}_{\mathcal{P}}}} \\
&\quad + \Big\| P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon U_\varepsilon \mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 \psi_k \Big\|_{\mathcal{H}} .
\end{aligned} \tag{4.3}$$

We now show that the second term merely contributes an $\mathcal{O}(\varepsilon^{n+1})$-error. To do so, let ϱ be a regular cut-off function with support in $(-\infty, \Lambda]$ such that ϱ restricted to $(-\infty, \Lambda - \frac{\delta}{2}] \cap \sigma(H^{\mathcal{P}}_{\mathrm{eff}})$ equals one. Then

$$\begin{aligned}
\varrho(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 &= \varrho\big(U_\varepsilon^\dagger (P_\varepsilon H^{\mathcal{E}} P_\varepsilon) U_\varepsilon \big) P_0 \\
&= U_\varepsilon^\dagger \varrho\big(P_\varepsilon H^{\mathcal{E}} P_\varepsilon \big) U_\varepsilon P_0 = U_\varepsilon^\dagger \varrho\big(P_\varepsilon H^{\mathcal{E}} P_\varepsilon \big) P_\varepsilon U_\varepsilon \\
&= U_\varepsilon^\dagger \big(P_\varepsilon \varrho(H^{\mathcal{E}}) P_\varepsilon + \mathcal{O}(\varepsilon^{n+1}) \big) U_\varepsilon
\end{aligned}$$

with errors in $\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))$ by Lemma 3.18(ii) for $T = P_\varepsilon$. This gives (with $[P_0, H^{\mathcal{P}}_{\mathrm{eff}}] = \mathbf{0}$)

$$\begin{aligned}
&\mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 \\
&= \varrho(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 \mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) \\
&= U_\varepsilon^\dagger P_\varepsilon \varrho(H^{\mathcal{E}}) P_\varepsilon U_\varepsilon \mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) + \mathcal{O}(\varepsilon^{n+1})
\end{aligned}$$

with errors in $\mathcal{L}(\mathcal{H}_{\mathcal{P}}, \mathcal{D}^{\mathrm{eff}}_{\mathcal{P}})$. Consequently, we may rewrite the second term of (4.3) as

$$\begin{aligned}
&P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon U_\varepsilon \mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) P_0 \\
&= \underbrace{P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon}_{= P_\varepsilon^\perp [H^{\mathcal{E}}, P_\varepsilon]} \varrho(H^{\mathcal{E}}) P_\varepsilon U_\varepsilon \mathbf{1}_{(-\infty, \Lambda - \delta/2]}(H^{\mathcal{P}}_{\mathrm{eff}}) + \mathcal{O}(\varepsilon^{n+1}) \\
&= \mathcal{O}(\varepsilon^{n+1})
\end{aligned}$$

in $\mathcal{L}(\mathcal{H}_{\mathcal{P}},\mathcal{H})$, because $\left\|[H^{\mathcal{E}},P_\varepsilon]\varrho(H^\varepsilon)\right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^{n+1})$ with the aid of Proposition 3.17.

Finally, we choose $K \in \mathbb{N}$ large enough so that the first term of (4.3) falls below the value of the second. This shows the existence of a constant $C > 0$ such that $\left\|(H^{\mathcal{E}} - \mu)\varphi\right\|_{\mathcal{H}} \leqslant C\varepsilon^{n+1}$ for $\varphi = U_\varepsilon \psi_K$. Now either it holds that $(H^{\mathcal{E}} - \mu)\varphi = 0$ and μ is an eigenvalue of $H^{\mathcal{E}}$, or the vector $\frac{(H^{\mathcal{E}}-\mu)\varphi}{\left\|(H^{\mathcal{E}}-\mu)\varphi\right\|_{\mathcal{H}}}$ is normalised and

$$\begin{aligned}
\frac{1}{\operatorname{dist}(\mu,(\sigma(H^{\mathcal{E}})))} &= \left\|(H^{\mathcal{E}} - \mu)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \\
&\geqslant \left\|(H^{\mathcal{E}} - \mu)^{-1}\frac{(H^{\mathcal{E}}-\mu)\varphi}{\left\|(H^{\mathcal{E}}-\mu)\varphi\right\|_{\mathcal{H}}}\right\|_{\mathcal{H}} \\
&= \frac{1}{\underbrace{\left\|(H^{\mathcal{E}} - \mu)\varphi\right\|_{\mathcal{H}}}_{\geqslant 1/(C\varepsilon^{n+1})}} \underbrace{\left\|(H^{\mathcal{E}} - \mu)^{-1}(H^{\mathcal{E}} - \mu)\varphi\right\|_{\mathcal{H}}}_{=\|\varphi\|_{\mathcal{H}}=1} \\
&\geqslant \frac{1}{C\varepsilon^{n+1}}.
\end{aligned}$$

□

We can adapt the proof to the other direction in the following way: Choose $\upsilon \in \sigma(H^{\mathcal{E}})$ with associated normalised Weyl sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in the image of $\mathbf{1}_{(-\infty,\Lambda-\delta/2]}(H^{\mathcal{E}})$. Then looking at the sequence of quasi-modes $\{\psi_k = U_\varepsilon^\dagger P_\varepsilon \varphi_k\}_{k\in\mathbb{N}}$, we get

$$\begin{aligned}
&\left\|(H_{\mathrm{eff}}^{\mathcal{P}} - \upsilon)U_\varepsilon^\dagger P_\varepsilon \varphi_k\right\|_{\mathcal{H}_{\mathcal{P}}} \\
&= \left\|U_\varepsilon^\dagger P_\varepsilon (H^{\mathcal{E}} - \upsilon) P_\varepsilon \varphi_k\right\|_{\mathcal{H}_{\mathcal{P}}} \\
&\leqslant \underbrace{\left\|U_\varepsilon^\dagger P_\varepsilon\right\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_{\mathcal{P}})}}_{\leqslant 2} \\
&\quad \times \Big(\underbrace{\left\|(H^{\mathcal{E}} - \upsilon)\varphi_k\right\|_{\mathcal{H}}}_{\to 0 \text{ as } k\to\infty} + \underbrace{\left\|[H^{\mathcal{E}},P_\varepsilon]\mathbf{1}_{(-\infty,\Lambda-\delta/2]}(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{n+1}) \text{ by Proposition 3.17}}\Big),
\end{aligned}$$

and so again $\left\|(H_{\mathrm{eff}}^{\mathcal{P}} - \upsilon)\psi_K\right\|_{\mathcal{H}_{\mathcal{P}}} \leqslant C\varepsilon^{n+1}$ for $K \in \mathbb{N}$ large enough. If the sequence $\{\psi_k\}_{k\in\mathbb{N}}$ is bounded from below, then either υ is an eigenvalue of $H^{\mathcal{P}}_{\text{eff}}$ or $\operatorname{dist}(\upsilon, \sigma(H^{\mathcal{P}}_{\text{eff}})) \leqslant C\varepsilon^{n+1}\|\psi_K\|^{-1}_{\mathcal{H}_{\mathcal{P}}}$. The sequence $\{P_\varepsilon \varphi_k\}_{k\in\mathbb{N}}$, however, is not bounded from below if υ is associated with an eigenband λ' of $H^{\mathcal{F}}$ other than the one used for the construction of P_ε.

Suppose for a moment that the spectrum of $H^{\mathcal{F}}$ consists solely of separated bands $\{\lambda_j\}_{j\in\mathbb{N}_0}$ with according spectral projections P_0^j, which yields the orthogonal decomposition $\mathbf{1}_{\mathcal{H}} = \bigoplus_{j\in\mathbb{N}_0} P_0^j$. If $\psi \in \operatorname{dom}(H^{\mathcal{E}})$ has energy $\langle \psi, H^{\mathcal{E}}\psi\rangle_{\mathcal{H}} \leqslant \Lambda$ for some $\Lambda \in \mathbb{R}$, then only finitely many spectral projections P_0^j, namely those associated with eigenbands with $\inf_{x\in B}\lambda_j(x) < \Lambda$, contribute significantly to ψ because

$$\Lambda \geqslant \langle \psi, H^{\mathcal{E}}\psi\rangle_{\mathcal{H}} = \sum_{j\in\mathbb{N}} \langle \psi, (\underbrace{-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} + \varepsilon H^{\mathcal{E}}_1}_{\gtrsim -C\varepsilon} + \lambda_j) P_0^j \psi\rangle_{\mathcal{H}}$$

under the condition that $-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} + \varepsilon H^{\mathcal{E}}_1 \geqslant -C\varepsilon\mathbf{1}_{\mathcal{H}}$ (note that the level spacing of the eigenbands λ_j is of order one). A successive lowering of the threshold Λ reduces the number of substantially participating eigenbands to the point where only the *ground state band*

$$\lambda_0(x) := \min \sigma\big(H^{\mathcal{F}}(x)\big) \tag{4.4}$$

is involved. In this case, i.e., for energies below $\Lambda_1 := \inf_{x\in B}(\sigma(H^{\mathcal{F}})\backslash\lambda_0)$, we indeed expect a mutual approximation of $\sigma(H^{\mathcal{E}})$ and $\sigma(H^{\mathcal{P}}_{\text{eff}})$.

Theorem 4.5 *Let $-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} + \varepsilon H^{\mathcal{E}}_1$ be bounded from below by $-C\varepsilon\mathbf{1}_{\mathcal{H}}$ for some constant $C > 0$ and $H^{\mathcal{P}}_{\text{eff}}$ be the effective operator from Theorem 4.3 associated with the ground state band λ_0* (4.4). *Moreover, let χ be a regular cut-off function with support in $(-\infty, \Lambda_1)$. Then $H^{\mathcal{P}}_{\text{eff}}\chi(H^{\mathcal{P}}_{\text{eff}})$ is unitarily equivalent to $H^{\mathcal{E}}\chi(H^{\mathcal{E}})$ up to errors of order ε^{n+1} in $\mathcal{L}(\mathcal{H})$ for $\varepsilon > 0$ small enough.*

If we use Weyl sequences as in Theorem 4.4 and the discussion afterwards, the latter statement implies that for every $\delta > 0$ both

$$\sup_{\substack{\upsilon\in\sigma(H^{\mathcal{E}}),\\ \upsilon\leqslant\Lambda_1-\delta}} \inf_{\substack{\mu\in\sigma(H^{\mathcal{P}}_{\text{eff}}),\\ \mu\leqslant\Lambda_1-\delta}} |\upsilon-\mu| = \sup_{\substack{\upsilon\in\sigma(H^{\mathcal{E}}),\\ \upsilon\leqslant\Lambda_1-\delta}} \underbrace{\operatorname{dist}\big(\upsilon, \sigma(H^{\mathcal{P}}_{\text{eff}}) \cap (-\infty, \Lambda_1-\delta]\big)}_{\leqslant C_1\varepsilon^{n+1}}$$

and

$$\sup_{\substack{\mu\in\sigma(H_{\text{eff}}^{\mathcal{P}}),\\ \mu\leqslant\Lambda_1-\delta}} \inf_{\substack{\upsilon\in\sigma(H^{\mathcal{E}}),\\ \upsilon\leqslant\Lambda_1-\delta}} |\mu-\upsilon| = \sup_{\substack{\mu\in\sigma(H_{\text{eff}}^{\mathcal{P}}),\\ \mu\leqslant\Lambda_1-\delta}} \underbrace{\operatorname{dist}\bigl(\mu,\sigma(H^{\mathcal{E}})\cap(-\infty,\Lambda_1-\delta]\bigr)}_{\leqslant C_2\varepsilon^{n+1}}$$

are of order ε^{n+1} for $\varepsilon>0$ small enough. Thus, one has

$$\operatorname{dist}_{\mathrm{H}}\bigl(\sigma(H^{\mathcal{E}})\cap(-\infty,\Lambda_1-\delta],\sigma(H_{\text{eff}}^{\mathcal{P}})\cap(-\infty,\Lambda_1-\delta]\bigr)=\mathcal{O}(\varepsilon^{n+1})$$

for all $\delta>0$, where

$$\operatorname{dist}_{\mathrm{H}}(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|\,,\sup_{b\in B}\inf_{a\in A}|b-a|\right\} \tag{4.5}$$

denotes the Hausdorff distance between compact sets $A,B\subset\mathbb{R}$, if one chooses $\varepsilon=\varepsilon(\delta)>0$ sufficiently small.

The following two lemmas give the key consequences of the condition on the operator $-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}+\varepsilon H_1^{\mathcal{E}}$ that it be bounded from below by $-C\varepsilon\mathbf{1}_{\mathcal{H}}$:

Lemma 4.6 *Under the conditions of Theorem 4.5, it holds that*

$$\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)P_\varepsilon^\perp=\mathbf{0}.$$

Proof. The operator $P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp$ is self-adjoint on the Hilbert space $\mathcal{H}$ with domain $P_\varepsilon\mathcal{H}\oplus P_\varepsilon^\perp\operatorname{dom}(H^{\mathcal{E}})$. We will show that $\sigma(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)=\{0\}\cup[\Lambda_1,\infty)$. Using the fact that $\operatorname{supp}(\chi)\cap[\Lambda_1,\infty)=\varnothing$, we then get

$$\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)=\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)P_\varepsilon+\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)P_\varepsilon^\perp=\chi(0)P_\varepsilon+\mathbf{0}P_\varepsilon^\perp$$

by the spectral theorem, and thus $\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)P_\varepsilon^\perp=\mathbf{0}$.

We first note that $P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp\big|_{P_\varepsilon\mathcal{H}}=\mathbf{0}$ implies $\sigma(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp\big|_{P_\varepsilon\mathcal{H}})=\{0\}$. As far as the contribution of $P_\varepsilon^\perp\operatorname{dom}(H^{\mathcal{E}})$ is concerned, we start by introducing a regular cut-off function $\varrho_1\in C_{\mathrm{b}}^\infty(\mathbb{R},[0,1])$, equal to one on $[\inf\sigma(H^{\mathcal{E}})-1,\Lambda_1+1]$ and equal to zero outside of $(\inf\sigma(H^{\mathcal{E}})-2,\Lambda_1+2)$, and set

$$\varrho_2: t\mapsto\begin{cases}0, & t<\Lambda_1+1\\ 1-\varrho_1(t), & t\geqslant\Lambda_1+1\end{cases}.$$

Then by construction

$$(\varrho_1 + \varrho_2)\big|_{\sigma(H^{\mathcal{E}})} \equiv 1 \quad \Rightarrow \quad \underbrace{\varrho_1(H^{\mathcal{E}})}_{=:\widehat{\varrho}_1} + \underbrace{\varrho_2(H^{\mathcal{E}})}_{=:\widehat{\varrho}_2} = \mathbf{1}_{\mathcal{H}},$$

and we get for any $\psi \in P_\varepsilon^\perp \operatorname{dom}(H^{\mathcal{E}})$:

$$\begin{aligned}
&\big\langle \psi, P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon^\perp \psi \big\rangle_{\mathcal{H}} \\
&= \big\langle \psi, P_\varepsilon^\perp H^{\mathcal{E}} (\widehat{\varrho}_1 + \widehat{\varrho}_2) P_\varepsilon^\perp \psi \big\rangle_{\mathcal{H}} \\
&= \big\langle \psi, P_\varepsilon^\perp \widehat{\varrho}_1^{1/2} H^{\mathcal{E}} \widehat{\varrho}_1^{1/2} P_\varepsilon^\perp \psi \big\rangle_{\mathcal{H}} + \underbrace{\big\langle \widehat{\varrho}_2^{1/2} P_\varepsilon^\perp \psi, H^{\mathcal{E}} \widehat{\varrho}_2^{1/2} P_\varepsilon^\perp \psi \big\rangle_{\mathcal{H}}}_{\substack{\geqslant \Lambda_1 \|\widehat{\varrho}_2^{1/2} P_\varepsilon^\perp \psi\|_{\mathcal{H}}^2 = \Lambda_1 \|\widehat{\varrho}_2^{1/2} \psi\|_{\mathcal{H}}^2, \\ \text{since } \operatorname{supp}(\widehat{\varrho}_2) \subset [\Lambda_1 + 1, \infty)}} \\
&\overset{\text{(i)}}{\geqslant} \big\langle \psi, P_0^\perp \widehat{\varrho}_1^{1/2} H^{\mathcal{E}} \widehat{\varrho}_1^{1/2} P_0^\perp \psi \big\rangle_{\mathcal{H}} + \Lambda_1 \big\| \widehat{\varrho}_2^{1/2} \psi \big\|_{\mathcal{H}}^2 + \mathcal{O}\big(\varepsilon \|\psi\|_{\mathcal{H}}^2\big) \\
&\overset{\text{(ii)}}{\geqslant} \big\langle \widehat{\varrho}_1^{1/2} \psi, P_0^\perp H^{\mathcal{E}} P_0^\perp \widehat{\varrho}_1^{1/2} \psi \big\rangle_{\mathcal{H}} + \Lambda_1 \big\| \widehat{\varrho}_2^{1/2} \psi \big\|_{\mathcal{H}}^2 + \mathcal{O}\big(\varepsilon \|\psi\|_{\mathcal{H}}^2\big) \\
&= \underbrace{\big\langle P_0^\perp \widehat{\varrho}_1^{1/2} \psi, P_0^\perp \big(-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}\big) P_0^\perp \widehat{\varrho}_1^{1/2} \psi \big\rangle_{\mathcal{H}}}_{\geqslant -C\varepsilon \|P_0^\perp \widehat{\varrho}_1^{1/2} \psi\|_{\mathcal{H}}^2 \geqslant -C\varepsilon \|\psi\|_{\mathcal{H}}^2 = \mathcal{O}(\varepsilon \|\psi\|_{\mathcal{H}}^2)} \\
&\quad + \underbrace{\big\langle P_0^\perp \widehat{\varrho}_1^{1/2} \psi, P_0^\perp H^{\mathcal{F}} P_0^\perp \widehat{\varrho}_1^{1/2} \psi \big\rangle_{\mathcal{H}}}_{\substack{\geqslant \Lambda_1 \|P_0^\perp \widehat{\varrho}_1^{1/2} \psi\|_{\mathcal{H}}^2, \text{ since } P_0^\perp \text{ projects} \\ \text{onto states with energy } \sigma(H^{\mathcal{F}}) \setminus \lambda \geqslant \Lambda_1}} + \Lambda_1 \big\| \widehat{\varrho}_2^{1/2} \psi \big\|_{\mathcal{H}}^2 + \mathcal{O}\big(\varepsilon \|\psi\|_{\mathcal{H}}^2\big) \\
&\geqslant \Lambda_1 \Big(\big\| P_0^\perp \widehat{\varrho}_1^{1/2} \psi \big\|_{\mathcal{H}}^2 + \big\| \widehat{\varrho}_2^{1/2} \psi \big\|_{\mathcal{H}}^2 \Big) + \mathcal{O}\big(\varepsilon \|\psi\|_{\mathcal{H}}^2\big) \\
&\overset{\text{(iii)}}{\geqslant} \Lambda_1 \underbrace{\Big(\big\| \widehat{\varrho}_1^{1/2} \psi \big\|_{\mathcal{H}}^2 + \big\| \widehat{\varrho}_2^{1/2} \psi \big\|_{\mathcal{H}}^2 \Big)}_{= \langle \psi, (\widehat{\varrho}_1 + \widehat{\varrho}_2) \psi \rangle_{\mathcal{H}} = \|\psi\|_{\mathcal{H}}^2} + \mathcal{O}\big(\varepsilon \|\psi\|_{\mathcal{H}}^2\big).
\end{aligned}$$

Thus, we can find for all $c > 0$ a constant $\varepsilon_0 > 0$ small enough such that

$$\big\langle \psi, P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon^\perp \psi \big\rangle_{\mathcal{H}} \geqslant \big(\Lambda_1 - \tilde{C}\varepsilon\big) \|\psi\|_{\mathcal{H}}^2 \geqslant (\Lambda_1 - c) \|\psi\|_{\mathcal{H}}^2$$

holds true for all $0 < \varepsilon < \varepsilon_0$. This shows that $\sigma\big(P_\varepsilon^\perp H^{\mathcal{E}} P_\varepsilon^\perp \big|_{P_\varepsilon^\perp \operatorname{dom}(H^{\mathcal{E}})}\big)$ is contained in $[\Lambda_1, \infty)$ and we only need to justify the estimates (i), (ii) and (iii):

(i) The absolute value of

$$\left\langle \psi, P_\varepsilon^\perp \widehat{\varrho}_1^{1/2} H^\mathcal{E} \widehat{\varrho}_1^{1/2} P_\varepsilon^\perp \psi \right\rangle_\mathcal{H} - \left\langle \psi, P_0^\perp \widehat{\varrho}_1^{1/2} H^\mathcal{E} \widehat{\varrho}_1^{1/2} P_0^\perp \psi \right\rangle_\mathcal{H}$$

can be estimated from above by

$$2 \underbrace{\left\| P_\varepsilon^\perp - P_0^\perp \right\|_{\mathcal{L}(\mathcal{H})}}_{= \mathcal{O}(\varepsilon) \text{ by Prop. 3.17}} \underbrace{\left\| \widehat{\varrho}_1^{1/2} H^\mathcal{E} \widehat{\varrho}_1^{1/2} \right\|_{\mathcal{L}(\mathcal{H})}}_{<\infty} \|\psi\|_\mathcal{H}^2 = \mathcal{O}\big(\varepsilon \|\psi\|_\mathcal{H}^2\big).$$

(ii) A short calculation shows that

$$\begin{aligned} &\left| \left\langle \psi, P_0^\perp \widehat{\varrho}_1^{1/2} H^\mathcal{E} \widehat{\varrho}_1^{1/2} P_0^\perp \psi \right\rangle_\mathcal{H} - \left\langle \psi, \widehat{\varrho}_1^{1/2} P_0^\perp H^\mathcal{E} P_0^\perp \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} \right| \\ &\leqslant 2 \underbrace{\left\| \left[P_0, \widehat{\varrho}_1^{1/2} \right] \right\|_{\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^\mathcal{E}))}}_{\substack{= \mathcal{O}(\varepsilon) \text{ by Lemma 4.6(i)} \\ \text{with the aid of (3.9)}}} \underbrace{\left\| \widehat{\varrho}_1^{1/2} \right\|_{\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^\mathcal{E}))}}_{\substack{< \infty \text{ (cf. the proof} \\ \text{of Proposition 3.17)}}} \|\psi\|_\mathcal{H}^2 \\ &= \mathcal{O}\big(\varepsilon \|\psi\|_\mathcal{H}^2\big). \end{aligned}$$

(iii) Observe that

$$\begin{aligned} &\left\langle P_0^\perp \widehat{\varrho}_1^{1/2} \psi, P_0^\perp \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} \\ &= \left\langle \widehat{\varrho}_1^{1/2} \psi, P_0^\perp \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} \\ &= \left\langle \widehat{\varrho}_1^{1/2} \psi, \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} - \left\langle \widehat{\varrho}_1^{1/2} \psi, P_0 \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} \\ &= \left\langle \widehat{\varrho}_1^{1/2} \psi, \widehat{\varrho}_1^{1/2} \psi \right\rangle_\mathcal{H} - \left\langle \widehat{\varrho}_1^{1/2} \psi, P_0 \widehat{\varrho}_1^{1/2} P_\varepsilon^\perp \psi \right\rangle_\mathcal{H} \end{aligned}$$

for all $\psi \in P_\varepsilon^\perp \operatorname{dom}(H^\mathcal{E})$, and hence

$$\begin{aligned} &\left| \left\| P_0^\perp \widehat{\varrho}_1^{1/2} \psi \right\|_\mathcal{H}^2 - \left\| \widehat{\varrho}_1^{1/2} \psi \right\|_\mathcal{H}^2 \right| \\ &\leqslant \underbrace{\left\| \widehat{\varrho}_1^{1/2} \right\|_{\mathcal{L}(\mathcal{H})}}_{\leqslant 1} \underbrace{\left\| P_0 \widehat{\varrho}_1^{1/2} P_\varepsilon^\perp \right\|_{\mathcal{L}(\mathcal{H})}}_{= \mathcal{O}(\varepsilon)} \|\psi\|_\mathcal{H}^2, \end{aligned}$$

because

$$P_0\widehat{\varrho}_1^{1/2}P_\varepsilon^\perp = \widehat{\varrho}_1^{1/2}\underbrace{P_\varepsilon P_\varepsilon^\perp}_{=\mathbf{0}} + \widehat{\varrho}_1^{1/2}(P_0 - P_\varepsilon)P_\varepsilon^\perp + \big[P_0, \widehat{\varrho}_1^{1/2}\big]P_\varepsilon^\perp$$

finally gives

$$\begin{aligned}\big\|P_0\widehat{\varrho}_1^{1/2}P_\varepsilon^\perp\big\|_{\mathcal{L}(\mathcal{H})} &\leqslant \underbrace{\|P_\varepsilon - P_0\|_{\mathcal{L}(\mathcal{H})}}_{=\mathcal{O}(\varepsilon)\text{ by Prop. 3.17}} + \underbrace{\Big\|\big[P_0, \widehat{\varrho}_1^{1/2}\big]\Big\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))}}_{=\mathcal{O}(\varepsilon)\text{ similarly as for (ii)}} \\ &= \mathcal{O}(\varepsilon).\end{aligned}$$

This completes the proof. □

Lemma 4.7 *Under the conditions of Theorem 4.5, it holds that*

$$P_\varepsilon^\perp\chi(H^{\mathcal{E}}) = \mathcal{O}(\varepsilon^{n+1}) \quad , \quad \chi(H^{\mathcal{E}})P_\varepsilon^\perp = \mathcal{O}(\varepsilon^{n+1})$$

in $\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))$.

PROOF. We already know that

$$\big\|[P_\varepsilon^\perp, H^{\mathcal{E}}]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})} = \big\|[P_\varepsilon, H^{\mathcal{E}}]\big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})} \overset{(3.12)}{=} \mathcal{O}(\varepsilon)$$

and

$$\big\|[P_\varepsilon^\perp, H^{\mathcal{E}}]\chi^s(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} = \big\|[P_\varepsilon, H^{\mathcal{E}}]\chi^s(H^{\mathcal{E}})\big\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^{n+1})$$

for all $s > 0$ by Proposition 3.17. Thus Lemma 3.18(ii) with $T = P_\varepsilon^\perp$ together with the previous lemma implies

$$\begin{aligned}&\big\|P_\varepsilon^\perp\chi(H^{\mathcal{E}})P_\varepsilon^\perp\big\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\ &= \big\|P_\varepsilon^\perp\chi(H^{\mathcal{E}})P_\varepsilon^\perp - \underbrace{\chi(P_\varepsilon^\perp H^{\mathcal{E}}P_\varepsilon^\perp)P_\varepsilon^\perp}_{=\mathbf{0}}\big\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\ &= \mathcal{O}(\varepsilon^{n+1}).\end{aligned} \tag{4.6}$$

Next we apply Lemma 3.18(i) with $T = P_\varepsilon$ in order to deduce

$$\begin{aligned}\left\|P_\varepsilon^\perp \chi(H^\mathcal{E})P_\varepsilon\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))} &= \left\|\left[\chi(H^\mathcal{E}), P_\varepsilon\right]P_\varepsilon\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))} \\ &\leqslant \underbrace{\left\|\left[\chi(H^\mathcal{E}), P_\varepsilon\right]\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\mathcal{O}(\varepsilon^{n+1})} \underbrace{\|P_\varepsilon\|_{\mathcal{L}(\mathcal{H})}}_{\leqslant 2} \\ &= \mathcal{O}(\varepsilon^{n+1}). \end{aligned} \tag{4.7}$$

Consequently, we arrive at the first estimate

$$\begin{aligned}&\left\|P_\varepsilon^\perp \chi(H^\mathcal{E})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))} \\ &\leqslant \underbrace{\left\|P_\varepsilon^\perp \chi(H^\mathcal{E})P_\varepsilon^\perp\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\mathcal{O}(\varepsilon^{n+1}) \text{ by (4.6)}} + \underbrace{\left\|P_\varepsilon^\perp \chi(H^\mathcal{E})P_\varepsilon\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\mathcal{O}(\varepsilon^{n+1}) \text{ by (4.7)}} \\ &= \mathcal{O}(\varepsilon^{n+1})\end{aligned}$$

of the statement. The second one is obtained from the latter by another application of Lemma 3.18(i) with $T = P_\varepsilon^\perp$. □

We finally turn to the proof of the theorem:

PROOF (of Theorem 4.5). Let us first observe that

$$\begin{aligned}&\left\|P_\varepsilon \chi(H^\mathcal{E}) - P_\varepsilon U_\varepsilon \chi(H_{\mathrm{eff}}^\mathcal{P})U_\varepsilon^\dagger\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))} \\ &\leqslant \underbrace{\left\|P_\varepsilon \chi(H^\mathcal{E})P_\varepsilon - P_\varepsilon \chi(P_\varepsilon H^\mathcal{E} P_\varepsilon)\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\mathcal{O}(\varepsilon^{n+1}) \text{ by Lemma 3.18(ii) for } T = P_\varepsilon} \\ &\quad + \underbrace{\|P_\varepsilon\|_{\mathcal{L}(\mathrm{dom}(H^\mathcal{E}))}}_{\leqslant 2} \underbrace{\left\|\chi(H^\mathcal{E}) - \chi(H^\mathcal{E})P_\varepsilon\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\left\|\chi(H^\mathcal{E})P_\varepsilon^\perp\right\| = \mathcal{O}(\varepsilon^{n+1}) \text{ by Lemma 4.7}} \\ &\quad + \underbrace{\|P_\varepsilon\|_{\mathcal{L}(\mathrm{dom}(H^\mathcal{E}))}}_{\leqslant 2} \underbrace{\left\|\underbrace{U_\varepsilon \chi(H_{\mathrm{eff}}^\mathcal{P})U_\varepsilon^\dagger}_{=\chi(U_\varepsilon H_{\mathrm{eff}}^\mathcal{P} U_\varepsilon^\dagger)} - \chi(P_\varepsilon H^\mathcal{E} P_\varepsilon)\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^\mathcal{E}))}}_{=\mathbf{0}} \\ &= \mathcal{O}(\varepsilon^{n+1}).\end{aligned} \tag{4.8}$$

Then an easy calculation shows

$$
\begin{aligned}
& H^{\mathcal{E}}\chi(H^{\mathcal{E}}) \\
&= \left(H^{\mathcal{E},\mathrm{D}} + (\mathbf{1}_{\mathcal{H}} - 2P_\varepsilon)[H^{\mathcal{E}}, P_\varepsilon]\right)\chi(H^{\mathcal{E}}) \\
&= U_\varepsilon H^{\mathcal{P}}_{\text{eff}} U_\varepsilon^\dagger P_\varepsilon \chi(H^{\mathcal{E}}) + P_\varepsilon^\perp H^{\mathcal{E}} \underbrace{P_\varepsilon^\perp \chi(H^{\mathcal{E}})}_{\substack{= \mathcal{O}(\varepsilon^{n+1}) \text{ in} \\ \mathcal{L}(\mathcal{H}) \text{ by} \\ \text{Lemma 4.7}}} \\
&\qquad + (\mathbf{1}_{\mathcal{H}} - 2P_\varepsilon)\underbrace{[H^{\mathcal{E}}, P_\varepsilon]\chi(H^{\mathcal{E}})}_{\substack{= \mathcal{O}(\varepsilon^{n+1}) \text{ in } \mathcal{L}(\mathcal{H}) \\ \text{by Proposition 3.17}}} \\
&\overset{(4.8)}{=} U_\varepsilon H^{\mathcal{P}}_{\text{eff}} U_\varepsilon^\dagger \underbrace{P_\varepsilon U_\varepsilon}_{\substack{=U_\varepsilon \\ \text{on } P_0\mathcal{H}}} \chi(H^{\mathcal{P}}_{\text{eff}}) U_\varepsilon^\dagger + \mathcal{O}(\varepsilon^{n+1}) \\
&= U_\varepsilon H^{\mathcal{P}}_{\text{eff}} \chi(H^{\mathcal{P}}_{\text{eff}}) U_\varepsilon^\dagger + \mathcal{O}(\varepsilon^{n+1})
\end{aligned}
$$

with errors in $\mathcal{L}(\mathcal{H})$, where we used the fact that

$$
P_\varepsilon H^{\mathcal{E}} P_\varepsilon = U_\varepsilon U_\varepsilon^\dagger P_\varepsilon H^{\mathcal{E}} U_\varepsilon U_\varepsilon^\dagger P_\varepsilon = U_\varepsilon H^{\mathcal{P}}_{\text{eff}} U_\varepsilon^\dagger P_\varepsilon
$$

in the second equality. □

4.3 The Effective Operator

We will follow the exposition of [Lam14, Subsection 2.2.1] in order to gain a better understanding of the leading-order terms that arise from the asymptotic expansion of the effective operator

$$
H^{\mathcal{P}}_{\text{eff}} = U_\varepsilon^\dagger P_\varepsilon H^{\mathcal{E}} P_\varepsilon U_\varepsilon.
$$

Therefore, we use the concrete expression of U_ε from Lemma 4.2, i.e., we expand

$$
\left(\mathbf{1}_{\mathcal{H}} - (P_\varepsilon - P_0)\right)^{-1/2} = \mathbf{1}_{\mathcal{H}} + \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k-1}k!(k-1)!}(P_\varepsilon - P_0)^{2k},
$$

and obtain

$$\begin{aligned} P_\varepsilon U_\varepsilon &= P_\varepsilon P_0\big(\mathbf{1}_{\mathcal{H}} - (P_\varepsilon - P_0)\big)^{-1/2} \\ &= P_0 + P_0^\perp (P_\varepsilon - P_0) P_0 - \tfrac{1}{2} P_0 (P_\varepsilon - P_0)^2 P_0 + \mathcal{O}(\varepsilon^3) \\ &=: P_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{4.9}$$

with errors in $\mathcal{L}(\mathcal{H})$ and $U_1, U_2 \in \mathcal{L}(\mathcal{H}) \cap \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$ since $P_\varepsilon - P_0 = \mathcal{O}(\varepsilon)$ in both $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$ as shown in the proof of Proposition 3.17. This leads to the decomposition of

$$H^{\mathcal{P}}_{\mathrm{eff}} = P_0 H^{\mathcal{E}} P_0 + H^{\mathcal{P}}_{\mathrm{sa}}$$

into an *adiabatic operator* $H^{\mathcal{P}}_{\mathrm{a}} := P_0 H^{\mathcal{E}} P_0$ and a remainder $H^{\mathcal{P}}_{\mathrm{sa}}$ that incorporates the super-adiabatic corrections.

4.3.1 The Adiabatic Operator

The starting point of this subsection is the following lemma which provides an additional smoothness property for the eigenspace bundle $\mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} B$ that is associated with some eigenband $\lambda : B \to \mathbb{R}$ and spectral projection P_0:

Proposition 4.8 *The eigenspace bundle $\pi_{\mathcal{P}} : \mathcal{P} \to B$ associated with an eigenband λ with a spectral gap has a differential structure such that $C^\infty(\mathcal{P}) \subset C^\infty(\mathcal{E})$.*

The intuition underlying this statement is that one may choose λ-eigensections of $H^{\mathcal{F}}$ locally over some $U \subset B$ which are smooth sections of $\mathcal{E}|_{\pi_M^{-1}(U)}$ at the same time. While the smoothness in the vertical directions results from the elliptic regularity of $H^{\mathcal{F}}$, i.e., any such λ-eigensection ϕ satisfies $\phi(x) \in C^\infty(\mathcal{E}_x)$ for all $x \in U$, the horizontal differentiability is established by the ∂-horizontal smoothness of $P_0 \in \mathcal{A}^{0,0}_H$ (see Lemma 3.13).

Outline of Proof (of Proposition 4.8). Let $x_0 \in B$ be arbitrary but fixed. Then the proof of Proposition 3.14 reveals that any vector $\phi^{x_0} \in \mathcal{P}_{x_0}$ induces a mapping $\psi^{x_0} \in C^\infty(U_{x_0}, L^2(\mathcal{F}))$ with geodesic ball $U_{x_0} = \mathsf{B}_{r_{x_0}}(x_0)$

for $r_{x_0} < r_{\mathrm{inj}}(B, g_B)$ small enough, such that $\psi^{x_0}(x) \in \mathrm{im}(P_0^{\mathcal{F}(x)}) \subset L^2(\mathcal{F})$ is a λ-eigensection of $H^{\mathcal{F}(x)}$ for all $x \in U_{x_0}$. Next we choose normal coordinates around any $(x, y) \in U_{x_0} \times F$ with synchronous trivialisation of $\mathcal{F}$ in order to map ψ^{x_0} diffeomorphically to some function $\psi^{(x,y)} : \mathbb{R}^{b+f} \to \mathbb{C}^N$ after a possible extension past the boundary (if y lies in a boundary collar chart) and a regularisation outside of $\mathbb{B}^b_{r/2}(0) \times \mathbb{B}^f_{r/2}(0)$ (see the proof of [Lam14, Lemma B.8] for the details, with obvious modifications for $N \geqslant 2$). Then one uses Wachsmuth's lemma [Lam14, Lemma B.8] to inductively show $\psi^{(x,y)} \in C^\infty(\mathbb{R}^{b+f}, \mathbb{C}^N)$, which implies the smoothness of ψ^{x_0} at (x, y) and consequently $\psi^{x_0} \in C^\infty(U_{x_0} \times F, \mathcal{F})$. The pullback of ψ^{x_0} to $\mathcal{E}|_{\pi_M^{-1}(U_{x_0})}$ via the local trivialisations $\Phi_{x_0} : \pi_M^{-1}(U_{x_0}) \to U_{x_0} \times F$ and $\Psi_{x_0} : \Pi_{\mathcal{E}}^{-1}(U_{x_0}) \to U_{x_0} \times \mathcal{F}$ finally preserves the smoothness, i.e., it holds that

$$\Psi_{x_0}^{-1} \circ \psi^{x_0} \circ \Phi_{x_0} \in C^\infty\left(\mathcal{E}|_{\pi_M^{-1}(U_{x_0})}\right)$$

and

$$\left(\Psi_{x_0}^{-1} \circ \psi^{x_0}\right)(x, \cdot) \in \mathcal{P}_x$$

for all $x \in U_{x_0}$. In the end, we apply this procedure to an entire basis $\{\phi_1^{x_0}, \ldots, \phi_q^{x_0}\}$ of $\mathcal{P}_{x_0}$, $q = \mathrm{rank}(P_0)$, and obtain mappings $\psi_1^{x_0}, \ldots, \psi_q^{x_0} \in C^\infty(U_{x_0} \times F, \mathcal{F})$ such that

$$\mathrm{span}\Big(\left(\Psi_{x_0}^{-1} \circ \psi_1^{x_0}\right)(x, \cdot), \ldots, \left(\Psi_{x_0}^{-1} \circ \psi_q^{x_0}\right)(x, \cdot)\Big) = \mathcal{P}_x$$

for all $x \in U_{x_0}$. $\square$

We endow $\mathcal{P}$ with the bundle metric

$$\langle \cdot, \cdot \rangle_{\mathcal{P}} \in C^\infty(\mathcal{P}^* \otimes \mathcal{P}^*), \quad \langle \phi, \psi \rangle_{\mathcal{P}_x} := \int_{M_x} h_x(P_0\phi, P_0\psi) \ \mathrm{vol}_{g_{M_x}}$$

and the so-called *Berry connection*

$$\nabla^{\mathrm{B}} : C^\infty(\mathcal{P}) \to C^\infty(\mathsf{T}^*B \otimes \mathcal{P}), \quad \nabla^{\mathcal{P}}_X \phi := P_0 \nabla^{\mathcal{E}}_{X^{\mathrm{H}}} P_0 \phi. \tag{4.10}$$

As a matter of fact, the previous proposition ensures that an application of the horizontal derivative $\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}$ on $P_0\phi_0 \in C^\infty(\mathcal{P}) \subset C^\infty(\mathcal{E})$ is well-defined. Since the sections of $\mathcal{P}$ vanish on ∂M_x for all $x \in B$, it holds for arbitrary $X \in C^\infty(\mathsf{T}B)$ and $\phi, \psi \in C^\infty(\mathcal{P})$ that

$$
\begin{aligned}
&X \cdot \langle \phi, \psi \rangle_{\mathcal{P}_x} \\
&= \mathcal{L}_X \int_{M_x} h_x(P_0\phi, P_0\psi)\ \mathrm{vol}_{g_{M_x}} \\
&= \int_{M_x} \mathcal{L}_{X^{\mathsf{H}}}\Big(h_x(P_0\phi, P_0\psi)\ \mathrm{vol}_{g_{M_x}}\Big) \\
&= \int_{M_x} h_x\big(P_0\nabla^{\mathsf{B}}_X\phi, P_0\psi\big) + h_x\big(P_0\phi, P_0\nabla^{\mathsf{B}}_X\psi\big)\ \mathrm{vol}_{g_{M_x}} \\
&\quad + \int_{M_x} h_x(P_0\phi, P_0\psi)\ \mathcal{L}_{X^{\mathsf{H}}}\mathrm{vol}_{g_{M_x}} \\
&= \big\langle \nabla^{\mathsf{B}}_X\phi, \psi \big\rangle_{\mathcal{P}_x} + \big\langle \phi, \nabla^{\mathsf{B}}_X\psi \big\rangle_{\mathcal{P}_x} - \langle \phi, \overline{\eta}_{\mathsf{V}}(X)\psi \rangle_{\mathcal{P}_x}.
\end{aligned}
\tag{4.11}
$$

Here, the variation of area formula [Lan99, Theorem XV.6.6(a)]

$$
\mathcal{L}_{X^{\mathsf{H}}}\mathrm{vol}_{g_{M_x}}\Big|_{\xi} = -g_\xi\big(X^{\mathsf{H}}, \eta_{\mathsf{V}}\big)\,\mathrm{vol}_{g_{M_x}}, \quad \xi \in M_x
$$

induces the tensor $\overline{\eta}_{\mathsf{V}} \in C^\infty(\mathsf{T}^*B \otimes \mathrm{End}(\mathcal{P}))$ given by

$$
\overline{\eta}_{\mathsf{V}}(X) := P_0\, g\big(X^{\mathsf{H}}, \eta_{\mathsf{V}}\big)P_0 = P_0\, g_B\big(X, \mathsf{T}\pi_M(\eta_{\mathsf{V}})\big)P_0,
$$

which is basically the mean curvature of the fibres $M_x \hookrightarrow M$ averaged over the $\lambda(x)$-eigensections. Moreover, the latter calculation shows in particular that the connection

$$
\nabla^{\mathcal{P}} : C^\infty(\mathcal{P}) \to C^\infty(\mathsf{T}^*B \otimes \mathcal{P}), \quad \nabla^{\mathcal{P}} := \nabla^{\mathsf{B}} - \tfrac{1}{2}\overline{\eta}_{\mathsf{V}}
$$

is metric with respect to the bundle metric $\langle\cdot,\cdot\rangle_{\mathcal{P}}$. In summary, we will treat $(\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{P}}, \nabla^{\mathcal{P}}) \xrightarrow{\pi_{\mathcal{P}}} (B, g_B)$ as a Hermitian vector bundle of finite rank $q = \mathrm{rank}(P_0)$. In addition, a straightforward calculation gives the formula

$$
\begin{aligned}
\mathrm{R}^{\mathcal{P}}(X,Y) = P_0\Big(&\mathrm{R}^{\mathcal{E}}(X^{\mathsf{H}},Y^{\mathsf{H}}) - \nabla^{\mathcal{E}}_{\Omega_{\mathsf{H}}(X^{\mathsf{H}},Y^{\mathsf{H}})}\Big)P_0 \\
&+ P_0\Big[\big[P_0,\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\big],\big[P_0,\nabla^{\mathcal{E}}_{Y^{\mathsf{H}}}\big]\Big]P_0 \\
&- \tfrac{1}{2}\big(\nabla^{\mathsf{T}^*B\otimes\mathrm{End}(\mathcal{P})}_X\overline{\eta}_{\mathsf{V}}\big)(Y) + \tfrac{1}{2}\big(\nabla^{\mathsf{T}^*B\otimes\mathrm{End}(\mathcal{P})}_Y\overline{\eta}_{\mathsf{V}}\big)(X) \\
&- \tfrac{1}{4}\big[\overline{\eta}_{\mathsf{V}}(X),\overline{\eta}_{\mathsf{V}}(Y)\big]
\end{aligned}
$$

for the curvature of $\nabla^{\mathcal{P}}$, where

$$
\begin{aligned}
\big(\nabla^{\mathsf{T}^*B\otimes\mathrm{End}(\mathcal{P})}_X\overline{\eta}_{\mathsf{V}}\big)(Y)\phi := \nabla^{\mathcal{P}}_X\big(\overline{\eta}_{\mathsf{V}}(Y)\phi\big)& \\
- \overline{\eta}_{\mathsf{V}}\big(\nabla^{g_B}_X Y\big)\phi - \overline{\eta}_{\mathsf{V}}(Y)\nabla^{\mathcal{P}}_X\phi&
\end{aligned}
$$

denotes the tensor product connection on $\mathsf{T}^*B\otimes\mathrm{End}(\mathcal{P})$. In view of the C^∞-boundedness of the horizontal lift, Ω_{H} and η_{V} (see [Lam14, Corollary A.6]), the ∂-horizontal smoothness of $P_0\in\mathcal{A}^{0,0}_H$ by Lemma 3.13 and the bounded geometry of $\mathcal{E}\xrightarrow{\pi_{\mathcal{E}}}M$, we deduce $\mathrm{R}^{\mathcal{P}}\in C^\infty_{\mathrm{b}}(\Lambda^2\mathsf{T}B\otimes\mathrm{End}(\mathcal{P}))$ and the eigenspace bundle $\mathcal{P}$ turns out to be a vector bundle of bounded geometry by Definition A.19.

Let $\phi\in C^\infty_0(\mathcal{P})$ and $X,Y\in C^\infty(\mathsf{T}B)$ be arbitrary. Then Stokes' theorem yields

$$
\begin{aligned}
0 &= \int_B \mathcal{L}_X\left(\left\langle\phi,\nabla^{\mathrm{B}}_Y\phi\right\rangle_{\mathcal{P}}\ \mathrm{vol}_{g_B}\right) \\
&= \int_B\left(\mathcal{L}_X\left\langle\phi,\nabla^{\mathrm{B}}_Y\phi\right\rangle_{\mathcal{P}}\right)\ \mathrm{vol}_{g_B} + \int_B\left\langle\phi,\nabla^{\mathrm{B}}_Y\phi\right\rangle_{\mathcal{P}}\quad\underbrace{\mathcal{L}_X\,\mathrm{vol}_{g_B}}_{=\mathrm{div}_{g_B}(X)\,\mathrm{vol}_{g_B}} \\
&\overset{(4.11)}{=} \int_B\left\langle\nabla^{\mathrm{B}}_X\phi,\nabla^{\mathrm{B}}_Y\phi\right\rangle_{\mathcal{P}}\mathrm{vol}_{g_B} - \int_B\left\langle\phi,\overline{\eta}_{\mathsf{V}}(X)\nabla^{\mathrm{B}}_Y\phi\right\rangle_{\mathcal{P}}\ \mathrm{vol}_{g_B} \\
&\quad + \int_B\Big\langle\phi,\big(\underbrace{\nabla^{\mathrm{B}}_X\nabla^{\mathrm{B}}_Y + \mathrm{div}_{g_B}(X)\nabla^{\mathrm{B}}_Y}_{=(\nabla^{\mathrm{B}})^2(X,Y),\text{ cf. Lemma A.22}}\big)\phi\Big\rangle_{\mathcal{P}}\ \mathrm{vol}_{g_B},
\end{aligned}
$$

which suggests that we define the negative of the so-called *Berry Lapla-*

cian $-\Delta^{\mathrm{B}}$ by means of the quadratic form

$$\begin{aligned}\left\langle \phi, -\Delta^{\mathrm{B}}\phi \right\rangle_{\mathcal{H}_{\mathcal{P}}} &:= \int_B \left\langle \nabla^{\mathrm{B}}\phi, \nabla^{\mathrm{B}}\phi \right\rangle_{\mathsf{T}^*B\otimes\mathcal{P}} \ \mathrm{vol}_{g_B} \\ &= \int_B \left\langle \phi, -\operatorname{tr}_{g_B}\big((\nabla^{\mathrm{B}})^2 - \overline{\eta}_{\mathsf{V}}(\cdot)\nabla^{\mathrm{B}}_{\cdot}\big)\phi \right\rangle_{\mathcal{P}} \ \mathrm{vol}_{g_B} .\end{aligned}$$

It differs from the negative of the connection Laplacian

$$-\Delta^{\mathcal{P}}_{g_B} = -\operatorname{tr}_{g_B}\big((\nabla^{\mathcal{P}})^2\cdot\big)$$

merely by the potential

$$V^{\mathcal{P}}_{\eta_{\mathsf{V}}} := \Delta^{\mathcal{P}}_{g_B} - \Delta^{\mathrm{B}} = \operatorname{tr}_{g_B}\Big(-\tfrac{1}{2}\big(\nabla^{\mathsf{T}^*B\otimes\mathrm{End}(\mathcal{P})}_{\cdot}\overline{\eta}_{\mathsf{V}}\big)(\cdot) + \tfrac{1}{4}\overline{\eta}_{\mathsf{V}}(\cdot)\overline{\eta}_{\mathsf{V}}(\cdot)\Big).$$

On the other hand, the difference between the Berry Laplacian and the projected horizontal Laplacian is the generalisation of the so-called *Born-Huang potential* $V^{\mathcal{P}}_{\mathrm{BH}}$, which is well-known from the Born-Oppenheimer approximation [PST07]. It may be computed easily with the aid of a g_B-orthonormal frame $\{X_i\}_{i=1}^b$ of $\mathsf{T}U$ with $U \in \mathfrak{U}$ (cf. Definition 2.7):

$$\begin{aligned}V^{\mathcal{P}}_{\mathrm{BH}} &:= \Delta^{\mathrm{B}} - P_0\Delta^{\mathcal{E}}_{\mathrm{hor}}P_0 \\ &= \Big(\nabla^{\mathrm{B}}_{X_i}\nabla^{\mathrm{B}}_{X_i} - \nabla^{\mathrm{B}}_{\nabla^{g_B}_{X_i}X_i} - \overline{\eta}_{\mathsf{V}}(X_i)\nabla^{\mathrm{B}}_{X_i}\Big) \\ &= -\Big(P_0\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}P_0^{\perp}\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}P_0 - g(X_i^{\mathsf{H}},\eta_{\mathsf{V}})P_0^{\perp}\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}P_0\Big) \\ &= \operatorname{tr}_{g_B}\Bigg(P_0\big[\nabla^{\mathcal{E}}_{(\cdot)^{\mathsf{H}}},P_0\big]\big[\nabla^{\mathcal{E}}_{(\cdot)^{\mathsf{H}}},P_0\big]P_0 \\ &\qquad\qquad - P_0\Big[g_B\big(\cdot,\mathsf{T}\pi_M(\eta_{\mathsf{V}})\big)\mathbf{1}_{\mathcal{H}},P_0\Big]\big[\nabla^{\mathcal{E}}_{(\cdot)^{\mathsf{H}}},P_0\big]P_0\Bigg).\end{aligned}$$

To sum up, the calculations show that

$$-P_0\Delta^{\mathcal{E}}_{\mathsf{H}}P_0 = -\Delta^{\mathrm{B}} + V^{\mathcal{P}}_{\mathrm{BH}} = -\Delta^{\mathcal{P}}_{g_B} + V^{\mathcal{P}}_{\mathrm{BH}} + V^{\mathcal{P}}_{\eta_{\mathsf{V}}}$$

and the adiabatic operator takes the form

$$H^{\mathcal{P}}_{\mathrm{a}} = -\varepsilon^2\Delta^{\mathrm{B}} + \lambda\mathbf{1}_{\mathcal{H}_{\mathcal{P}}} + \varepsilon P_0H^{\mathcal{E}}_1P_0 + \varepsilon^2V_{\mathrm{BH}}. \tag{4.12}$$

We finally note that the bounded geometry of the eigenspace bundle allows for elliptic regularity estimates for the Berry Laplacian and the adiabatic operator in the spirit of Proposition 2.15 and Corollary 2.17:

Proposition 4.9 *Denote by $W^k_\varepsilon(\mathcal{P})$ the rescaled Sobolev norms* (A.12) *associated with the $\mathbb{C}^q$-vector bundle $\pi_{\mathcal{P}} : (\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{P}}, \nabla^{\mathcal{P}}) \to (B, \varepsilon^{-2} g_B)$.*

(i) Let $\phi \in W^2_\varepsilon(\mathcal{P})$ and $k \in \mathbb{N}_0$ be such that $\varepsilon^2 \Delta^{\mathrm{B}} \phi \in W^k_\varepsilon(\mathcal{P})$. Then $\phi \in W^{k+2}_\varepsilon(\mathcal{P})$ and there is a constant $C(k) > 0$ such that

$$\|\phi\|^2_{W^{k+2}_\varepsilon(\mathcal{P})} \leqslant C(k) \left(\left\| \varepsilon^2 \Delta^{\mathrm{B}} \phi \right\|^2_{W^k_\varepsilon(\mathcal{P})} + \|\phi\|^2_{\mathcal{H}_{\mathcal{P}}} \right).$$

(ii) Let $\phi \in \mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}})$ and $k \in \mathbb{N}$ such that $(H^{\mathcal{P}}_{\mathrm{a}})^k \phi \in \mathcal{H}_{\mathcal{P}}$. Then there are constants $\varepsilon_0(k), C(k) > 0$ such that

$$\|\phi\|^2_{W^{2k}_\varepsilon(\mathcal{P})} \leqslant C(k) \left(\left\| (H^{\mathcal{P}}_{\mathrm{a}})^k \phi \right\|^2_{\mathcal{H}_{\mathcal{P}}} + \|\phi\|^2_{\mathcal{H}_{\mathcal{P}}} \right)$$

for $0 < \varepsilon < \varepsilon_0(k)$.

This implies in particular that

$$\mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}}) = W^2_\varepsilon(\mathcal{P}) = P_0\, \mathrm{dom}(H^{\mathcal{E}}),$$

where the second equality is valid because of Proposition 4.8. Moreover, Proposition 4.9(ii) basically relies on the fact that $V^{\mathcal{P}}_{\mathrm{BH}} \in C^\infty_{\mathrm{b}}(\mathrm{End}(\mathcal{P}))$, which follows from the facts that $P_0 \in \mathcal{A}^{0,0}_H$, $\overline{\eta}_{\mathsf{V}} \in C^\infty_{\mathrm{b}}(\mathsf{T}^*B \otimes \mathrm{End}(\mathcal{P}))$ and $\lambda \in C^\infty_{\mathrm{b}}(B)$.

4.3.2 Super-Adiabatic Corrections

The super-adiabatic corrections are the difference between the effective operator and adiabatic operator:

$$\begin{aligned} H^{\mathcal{P}}_{\mathrm{sa}} &= H^{\mathcal{P}}_{\mathrm{eff}} - H^{\mathcal{P}}_{\mathrm{a}} \\ &\overset{(4.9)}{=} \varepsilon \big(U_1^\dagger H^{\mathcal{E}} P_0 + P_0 H^{\mathcal{E}} U_1 \big) \\ &\quad + \varepsilon^2 \big(U_1^\dagger H^{\mathcal{E}} U_1 + P_0 H^{\mathcal{E}} U_2 + U_2 H^{\mathcal{E}} P_0 \big) \\ &\quad + \mathcal{O}(\varepsilon^3) \end{aligned}$$

$$
\begin{aligned}
&= P_0 \underbrace{\Big((P_\varepsilon - P_0)[H^{\mathcal{E}}, P_0] + [P_0, H^{\mathcal{E}}](P_\varepsilon - P_0) \Big)}_{= \mathcal{O}(\varepsilon^2) \text{ in } \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}), \mathcal{H})} P_0 \\
&\quad + \varepsilon^2 \big(U_1^\dagger H^{\mathcal{E}} U_1 + P_0 H^{\mathcal{E}} U_2 + U_2 H^{\mathcal{E}} P_0 \big) \\
&\quad + \mathcal{O}(\varepsilon^3)
\end{aligned} \tag{4.13}
$$

with errors in $\mathcal{L}(\mathcal{D}^{\mathrm{eff}}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}})$. This implies $\|H^{\mathcal{P}}_{\mathrm{sa}}\|_{\mathcal{L}(\mathcal{D}^{\mathrm{eff}}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}})} = \mathcal{O}(\varepsilon^2)$ and hence $\mathrm{dom}(H^{\mathcal{P}}_{\mathrm{eff}}) = \mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}})$ by the Katō-Rellich theorem. The first super-adiabatic correction to $H^{\mathcal{P}}_{\mathrm{a}}$ (i.e., the $\mathcal{O}(\varepsilon^2)$-contribution within $H^{\mathcal{P}}_{\mathrm{sa}}$) turns out to be [PST07]

$$
\mathcal{M}^{\mathcal{P}} := P_0 [H^{\mathcal{E}}, P_0] R^{\mathcal{F}}(\lambda) [H^{\mathcal{E}}, P_0] P_0. \tag{4.14}
$$

However, this is a fourth-order differential operator if $H^{\mathcal{E}}_1$ is of second order, and therefore does not define a bounded operator from $\mathcal{D}^{\mathrm{eff}}_{\mathcal{P}}$ to $\mathcal{H}_{\mathcal{P}}$. This inadequacy is circumvented by the addition of energy cut-offs $\chi(H^{\mathcal{P}}_{\mathrm{eff}})$ – just as we did in the construction of P_ε – in the formal expansion

$$
H^{\mathcal{P}}_{\mathrm{eff}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}} + \mathcal{O}(\varepsilon^3). \tag{4.15}
$$

The precise statement is the following:

Proposition 4.10 *Let $H^{\mathcal{P}}_{\mathrm{eff}}$ be the effective operator of Theorem 4.3 and χ be a regular cut-off function with* $\mathrm{supp}(\chi) \subset (-\infty, \Lambda]$. *Then*

$$
\Big\| H^{\mathcal{P}}_{\mathrm{eff}} \chi^2(H^{\mathcal{P}}_{\mathrm{eff}}) - \chi(H^{\mathcal{P}}_{\mathrm{eff}}) \big(H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}} \big) \chi(H^{\mathcal{P}}_{\mathrm{eff}}) \Big\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})} = \mathcal{O}(\varepsilon^3).
$$

PROOF. The statement is proven in [Lam14, Proposition 2.23], but we mention the most important arguments for the sake of transparency.

First observe that

$$
\begin{aligned}
H^{\mathcal{P}}_{\mathrm{eff}} \chi^2(H^{\mathcal{P}}_{\mathrm{eff}}) - \chi(H^{\mathcal{P}}_{\mathrm{eff}}) H^{\mathcal{P}}_{\mathrm{a}} \chi(H^{\mathcal{P}}_{\mathrm{eff}}) &= \chi(H^{\mathcal{P}}_{\mathrm{eff}}) \underbrace{H^{\mathcal{P}}_{\mathrm{sa}} \chi(H^{\mathcal{P}}_{\mathrm{eff}})}_{= \mathcal{O}(\varepsilon^2)} \\
&= \chi(H^{\mathcal{E}}) H^{\mathcal{P}}_{\mathrm{sa}} \chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^3)
\end{aligned}
$$

with errors in $\mathcal{L}(\mathcal{H}_{\mathcal{P}})$. Here, we utilised that the interchange of $\chi(H_{\mathrm{eff}}^{\mathcal{P}})$ and $\chi(H^{\mathcal{E}})$ on $\mathcal{P}$ yields error terms only of order ε in $\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))$ for the last step because

$$H_{\mathrm{eff}}^{\mathcal{P}} = U_\varepsilon^\dagger P_\varepsilon H^{\mathcal{E}} P_\varepsilon U_\varepsilon \overset{(4.9)}{=} P_0 H^{\mathcal{E}} P_0 + \mathcal{O}(\varepsilon)$$

in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}), \mathcal{H})$ and $\left\|\chi(P_0 H^{\mathcal{E}} P_0) P_0 - P_0 \chi(H^{\mathcal{E}}) P_0\right\|_{\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon)$ by Lemma 3.18(ii). Thus, it remains to show

$$\left\|\chi(H^{\mathcal{E}})\big(H_{\mathrm{sa}}^{\mathcal{P}} - \mathcal{M}^{\mathcal{P}}\big)\chi(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^3).$$

In order to verify that the first super-adiabatic correction (4.13) – once sandwiched with cut-offs $\chi(H^{\mathcal{E}})$ – is given by the expression (4.14) plus higher orders, we start by noting that

$$(P_\varepsilon - P_0)\chi(H^{\mathcal{E}}) = \varepsilon P_1 \chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^2)$$

in $\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))$, as shown in [Lam14, Lemma 2.25], where

$$P_1 = -P_0^\perp R^{\mathcal{F}}(\lambda)[H^{\mathcal{E}}, P_0]P_0 + P_0[H^{\mathcal{E}}, P_0]R^{\mathcal{F}}(\lambda)P_0^\perp$$

is given by the explicit construction of Lemma 3.16. This immediately gives

$$\begin{aligned}\varepsilon U_1 \chi(H^{\mathcal{E}}) &= P_0^\perp(\varepsilon P_1)\chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^2)\\ &= -P_0^\perp R^{\mathcal{F}}(\lambda)[H^{\mathcal{E}}, P_0]P_0 \chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

and

$$\begin{aligned}\varepsilon^2 U_2 \chi(H^{\mathcal{E}}) &= -\tfrac{1}{2}P_0(\varepsilon P_1)(\varepsilon P_1)P_0\chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^3)\\ &= \tfrac{1}{2}P_0[H^{\mathcal{E}}, P_0]R^{\mathcal{F}}(\lambda)P_0^\perp R^{\mathcal{F}}(z)[H^{\mathcal{E}}, P_0]P_0\chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

with errors in $\mathcal{L}(\mathcal{H}, \mathrm{dom}(H^{\mathcal{E}}))$. Consequently, the leading order (4.13) of the super-adiabatic correction may be expressed in terms of reduced resolvents $R^{\mathcal{F}}(\lambda)$ as well as commutators $[H^{\mathcal{E}}, P_0]$ up to higher order errors and the calculations of [Lam14, Lemma 2.26] ultimately show that the leading part of $\chi(H^{\mathcal{E}})H_{\mathrm{sa}}^{\mathcal{P}}\chi(H^{\mathcal{E}})$ in $\mathcal{L}(\mathcal{H})$ coincides with $\chi(H^{\mathcal{E}})\mathcal{M}^{\mathcal{P}}\chi(H^{\mathcal{E}})$. □

4.4 Low Energy Asymptotics

In this section, we will analyse the low-lying part of the spectrum for Schrödinger operators of the form

$$-\Delta_{g^\varepsilon}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + V^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + \Delta_{\mathsf{V}}^{\mathcal{E}} + V^{\mathcal{E}}$$

with C^∞-bounded, $\mathrm{Herm}(\mathcal{E})$-valued potential $V^{\mathcal{E}}$ and a second-order horizontal differential operator $\varepsilon H_1^{\mathcal{E}}$ as a small perturbation. More precisely, we are interested in the study of small energies with a distance of order ε^α, with parameter $0 < \alpha \leqslant 2$, above the bottom

$$\Lambda_0 := \inf_{x \in B} \min \sigma \left(-\Delta^{\mathcal{E}_x} + V^{\mathcal{E}}\big|_{M_x} \right)$$

of the vertical operator. In order to relate the low-lying part of the spectrum of the (shifted) operator

$$H^{\mathcal{E}} := -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + \underbrace{\Delta_{\mathsf{V}}^{\mathcal{E}} + V^{\mathcal{E}} - \Lambda_0}_{=:H^{\mathcal{F}}} \tag{4.16}$$

with that of some effective operator, the only eigenband in question is the ground state band λ_0 (4.4). Hence, we fix P_ε and U_ε constructed for λ_0 with $n \geqslant 3$ and any $\Lambda > 0$ provided by Theorem 4.3. Finally, the study of the low energy asymptotics corresponds to the consideration of $H^{\mathcal{E}}$ on the image of $\varrho_\alpha(H^{\mathcal{E}}) := \mathbf{1}_{(-\infty, \varepsilon^\alpha \Lambda]}(H^{\mathcal{E}})$ for $\alpha \in (0, 2]$.

Under the assumptions that $-\varepsilon^2 \Delta_{\mathsf{H}} + \varepsilon H_1^{\mathcal{E}}$ is bounded from below by $-C\varepsilon \mathbf{1}_{\mathcal{H}}$ for some constant $C > 0$ and that $\varepsilon^\alpha \Lambda < \Lambda_1$, we saw in Theorem 4.5 that $H_{\mathrm{eff}}^{\mathcal{P}}$ is unitarily equivalent to $H^{\mathcal{E}}$ in this low energy regime up to errors of order ε^4 in $\mathcal{L}(\mathcal{H})$. In the further course of this section, we will show that the (much simpler) adiabatic operator $H_{\mathrm{a}}^{\mathcal{P}}$ yields an even better approximation for $H^{\mathcal{E}}$ than one would expect from the fact that

$$H_{\mathrm{eff}}^{\mathcal{P}} \overset{(4.13)}{=} H_{\mathrm{a}}^{\mathcal{P}} + \mathcal{O}(\varepsilon^2) \tag{4.17}$$

in $\mathcal{L}(\mathcal{D}^{\mathrm{eff}}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}})$. Put differently, the mere existence of P_ε and U_ε provides an improvement of the adiabatic approximation, even though $H^{\mathcal{P}}_{\mathrm{a}} = P_0 H^{\mathcal{E}} P_0$ does not involve any super-adiabatic corrections.

The choice of α represents the typical energy scale for the eigenvalues of the adiabatic operator below its essential spectrum. The parameter range $\alpha \in (0,2]$ incorporates the two most relevant scales:

- $\alpha = 1$:
  If $x \mapsto \lambda_0(x)$ has a unique non-degenerate minimum on B, the results obtained in [Sim83] suggest that the leading part of the adiabatic operator behaves like a b-dimensional harmonic oscillator $H^{\mathcal{P}}_{\mathrm{a}} = -\varepsilon^2 \Delta^{\mathrm{B}} + \lambda_0(x)$ with eigenvalue spacing of order ε.

- $\alpha = 2$:
  If $\lambda_0 \equiv 0$, which for example occurs in the case of unitarily equivalent operators $(H^{\mathcal{F}}(x), \mathcal{D}_{\mathcal{F}}(x))$ for all $x \in B$, the adiabatic operator is given by $H^{\mathcal{P}}_{\mathrm{a}} = \varepsilon^2(-\Delta^{\mathrm{B}} + \mathcal{O}(\varepsilon))$ and its eigenvalues, if they exist, scale as ε^2. We will see that the latter approximate those of $H^{\mathcal{E}}$ up to errors of order ε^4.

We now state the precise requirements for the perturbation $H^{\mathcal{E}}_1$ which strengthen the prerequisites of Theorem 4.5:

Condition 4.11 The perturbation within the Schrödinger operator (4.16) is symmetric on $W^2_\varepsilon(\mathcal{E}) \cap W^1_{0,\varepsilon}(\mathcal{E})$ and may locally be expanded as

$$H^{\mathcal{E}}_1\big|_{\pi_M^{-1}(U_\nu)} = \sum_{i,j=1}^{b} \nabla^{\mathcal{E}}_{\varepsilon(X^\nu_i)^{\mathrm{H}}} A^{ij}_\nu \nabla^{\mathcal{E}}_{\varepsilon(X^\nu_j)^{\mathrm{H}}} + \sum_{i=1}^{b} B^i_\nu \nabla^{\mathcal{E}}_{\varepsilon(X^\nu_i)^{\mathrm{H}}} + \varepsilon C_\nu$$

for any $U_\nu \in \mathfrak{U}$ of Definition 2.7, where we require that

(i) the according coefficients $A^{ij}_\nu, B^i_\nu, C_\nu \in C^\infty_{\mathrm{b}}(\mathrm{End}(\mathcal{E})|_{\pi_M^{-1}(U_\nu)})$ have bounds uniform in $i,j \in \{1,\dots,b\}$ as well as $\nu \in \mathbb{N}_0$, and independent of ε,

(ii) and $-\varepsilon^2 \Delta^{\mathcal{E}}_{\mathrm{H}} + \varepsilon H^{\mathcal{E}}_1$ is bounded from below by $-C\varepsilon^2 \mathbf{1}_{\mathcal{H}}$ for an ε-independent constant $C > 0$. ◊

This condition implies that the entire operator $H^{\mathcal{E}}$ is bounded from below by $-C\varepsilon^2 \mathbf{1}_{\mathcal{H}}$, so in particular $\|H^{\mathcal{E}} \varrho_\alpha(H^{\mathcal{E}})\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^\alpha)$.

We will show that the low-lying part of $\sigma(H^{\mathcal{E}})$ is adequately reflected by that of the adiabatic operator $H_{\mathrm{a}}^{\mathcal{P}}$ up to errors $\mathcal{O}(\varepsilon^{2+\alpha/2})$, i.e., the adiabatic approximation is more accurate by at least a factor $\varepsilon^{\alpha/2}$ than (4.17) for the general case. The reason for this improvement can be seen by the following heuristic argument: The super-adiabatic corrections (with main contribution $P_\varepsilon - P_0 \approx \varepsilon P_1$) essentially consist of horizontal differential operators. But such derivatives $\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ for $X \in C^\infty_{\mathrm{b}}(\mathsf{T}B)$ are of order $\varepsilon^{\alpha/2}$ and are therefore small on this ε-dependent energy scale, since $\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}} = \mathcal{O}(\varepsilon^\alpha)$ on the image of $\varrho_\alpha(H^{\mathcal{E}})$.

To be more specific, note that small energies of order ε^α of $H^{\mathcal{E}}$ correspond to bounded energies of the rescaled operator $\varepsilon^{-\alpha} H^{\mathcal{E}}$. In this context, we denote by $\mathrm{dom}^2_\alpha(T)$, $T \in \{H^{\mathcal{E}}, H^{\mathcal{P}}_{\mathrm{a}}, H^{\mathcal{P}}_{\mathrm{eff}}\}$, the domain of $(\varepsilon^{-\alpha} T)^2$ with its graph-norm. The desired control of horizontal derivatives in this low energy regime, i.e., on the image of $\varrho_\alpha(T) \in \mathcal{L}(\mathcal{H}, \mathrm{dom}^2_\alpha(T))$, is then essentially established by the estimate

$$\left\| P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(T), \mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{\alpha/2})$$

for arbitrary $X \in C^\infty_{\mathrm{b}}(\mathsf{T}B)$ whose proof is given in Lemma B.1. A first consequence of this is the estimate (with $\|P_0\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))} \leqslant 2$)

$$\begin{aligned} &\left\| \nabla^{\mathrm{B}}_{\varepsilon X} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(T), \mathrm{dom}(H^{\mathcal{E}}))} \\ &\quad \leqslant \left\| P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(T), \mathrm{dom}(H^{\mathcal{E}}))} + 2\varepsilon \underbrace{\left\| \left[\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}, P_0 \right] \right\|_{\mathcal{L}(\mathrm{dom}(T), \mathrm{dom}(H^{\mathcal{E}}))}}_{= \mathcal{O}(1) \text{ due to } P_0 \in \mathcal{A}^{0,0}_H} \\ &\quad \overset{\alpha \leqq 2}{=} \mathcal{O}(\varepsilon^{\alpha/2}) \end{aligned} \tag{4.18}$$

for $X \in C^\infty_{\mathrm{b}}(\mathsf{T}B)$. From this we deduce the following assertion:

Lemma 4.12 *Let $\alpha \in (0, 2]$ and Condition 4.11 be satisfied. Then one has*

$$\left\| \left[-\varepsilon^2 \Delta^{\mathcal{E}}_{\mathsf{H}}, P_0 \right] P_0 \varrho_\alpha(T) \right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^{1+\alpha/2})$$

for $T \in \{H^{\mathcal{E}}, H^{\mathcal{P}}_{\mathrm{a}}, H^{\mathcal{P}}_{\mathrm{eff}}\}$.

PROOF. We adjust the proof of [Lam14, Lemma 3.9] appropriately: Using local calculations on $\pi_M^{-1}(U)$ for $U \in \mathfrak{U}$ (see Definition 2.7), one can show that

$$\big[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}, P_0\big]P_0\big|_{\pi_M^{-1}(U)} = -\operatorname{tr}_{g_B}\big(\varepsilon T_1 \otimes \nabla^{\mathsf{B}}_{\varepsilon\cdot} + \varepsilon^2 T_2\big),$$

where the mappings

$$\begin{aligned} T_1 : \qquad & X \mapsto \big[-2\nabla^{\mathcal{E}}_{X^{\mathsf{H}}} + g(\eta_{\mathsf{V}}, X^{\mathsf{H}}), P_0\big], \\ T_2 : (X,Y) & \mapsto P_0^{\perp}\Bigg(-\Big[\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}, \big[\nabla^{\mathcal{E}}_{Y^{\mathsf{H}}}, P_0\big]\Big] \\ & \qquad + \Big[\nabla^{\mathcal{E}}_{(\nabla_X Y)^{\mathsf{H}}}, P_0\Big] + g(\eta_{\mathsf{V}}, X)\big[\nabla_{Y^{\mathsf{H}}}, P_0\big]\Bigg)P_0 \end{aligned}$$

define sections of $\mathsf{T}^*B \otimes \mathcal{L}(\mathcal{H}_{\mathcal{F}})$ and $\mathsf{T}^*B^{\otimes 2} \otimes \mathcal{L}(\mathcal{H}_{\mathcal{F}})$, respectively. Consequently as a result of Remark 3.6 and (4.18), the above expression is of order $\varepsilon^{1+\alpha/2}$ on the image of $\operatorname{dom}^2_\alpha(T)$ and finally $\varrho_\alpha(T) \in \mathcal{L}(\mathcal{H}, \operatorname{dom}^2_\alpha(T))$ with ε-independent bound yields the statement. $\square$

By means of this lemma, we can specify our qualitative discussion from before and derive refined estimates for the operator $P_\varepsilon - P_0$ on the image of $\varrho_\alpha(T)$ for $T \in \{H^{\mathcal{E}}, H^{\mathcal{P}}_{\mathrm{a}}, H^{\mathcal{P}}_{\mathrm{eff}}\}$:

Lemma 4.13 *Let $\alpha \in (0,2]$ and Condition 4.11 be satisfied. Then it holds that*

$$\big\|P_0^{\perp}(P_\varepsilon - P_0)P_0\varrho_\alpha(T)\big\|_{\mathcal{L}(\mathcal{H},\operatorname{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{1+\alpha/2})$$

for $T \in \{H^{\mathcal{E}}, H^{\mathcal{P}}_{\mathrm{a}}, H^{\mathcal{P}}_{\mathrm{eff}}\}$.

PROOF. Let χ be a regular cut-off function with support in $(-\infty, \Lambda]$ such that $\chi = 1$ on $\operatorname{supp}(\varrho_\alpha)$, which means $\varrho_\alpha(T) = \chi(T)\varrho_\alpha(T)$ by the functional calculus. We first show

$$\begin{aligned} \big\|P_0^{\perp}(P_\varepsilon - P_0)\varrho_\alpha(T)\big\|_{\mathcal{L}(\mathcal{H},\operatorname{dom}(H^{\mathcal{E}}))} \leqslant{} & \big\|P_0^{\perp}\varepsilon P_1 P_0 \varrho_\alpha(T)\big\|_{\mathcal{L}(\mathcal{H},\operatorname{dom}(H^{\mathcal{E}}))} \\ & + \mathcal{O}(\varepsilon^2) \end{aligned}$$

for $T \in \{H^{\mathcal{E}}, H_{\mathrm{a}}^{\mathcal{P}}, H_{\mathrm{eff}}^{\mathcal{P}}\}$ with the aid of the estimate

$$\left\|(P_\varepsilon - P_0)\chi(H^{\mathcal{E}}) - \varepsilon P_1 \chi(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^2) \tag{4.19}$$

proven in [Lam14, Lemma 2.25]:

- $T = H^{\mathcal{E}}$:
  A short calculation shows

$$\begin{aligned}
&\left\|P_0^{\perp}(P_\varepsilon - P_0)\varrho_\alpha(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\
&\quad = \left\|P_0^{\perp}(P_\varepsilon - P_0)\chi(H^{\mathcal{E}})\varrho_\alpha(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\
&\quad \overset{(4.19)}{\leqslant} \left\|P_0^{\perp}\varepsilon P_1 P_0 \chi(H^{\mathcal{E}})\varrho_\alpha(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2) \\
&\quad = \left\|P_0^{\perp}\varepsilon P_1 P_0 \varrho_\alpha(H^{\mathcal{E}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

  Here, we used that P_1 is off-diagonal (i.e., $P_0^{\perp}P_1 = P_0^{\perp}P_1P_0$) in the third line.

- $T = H_{\mathrm{a}}^{\mathcal{P}}$:
  We take advantage of the two facts

$$\left\|\left[P_0, \chi(H^{\mathcal{E}})\right]\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon), \tag{4.20a}$$

$$\left\|P_0\chi(H^{\mathcal{E}})P_0 - P_0\chi(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon), \tag{4.20b}$$

  which follow from Lemma 3.18. In analogy to [Lam14, Equation (2.19)], we then get

$$\begin{aligned}
P_0^{\perp}(P_\varepsilon - P_0) &= P_0^{\perp}(P_\varepsilon - P_0)P_0 + P_0^{\perp}(P_\varepsilon - P_0)P_0^{\perp} \\
&= P_0^{\perp}(P_\varepsilon - P_0)P_0 + P_0^{\perp}\underbrace{(P_\varepsilon - P_0)^2}_{=\mathcal{O}(\varepsilon^2)} \\
&= P_0^{\perp}(P_\varepsilon - P_0)P_0 + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{4.21}$$

in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))$. Thus,

$$\begin{aligned}
&\left\| P_0^{\perp}(P_\varepsilon - P_0)\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} \\
&\overset{(4.21)}{\leqslant} \left\| P_0^{\perp}(P_\varepsilon - P_0)P_0\chi(H_{\mathrm{a}}^{\mathcal{P}})\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2) \\
&\overset{(4.20b)}{\leqslant} \Big\| P_0^{\perp} \underbrace{(P_\varepsilon - P_0)P_0\chi(H^{\mathcal{E}})}_{\substack{= \varepsilon P_1 P_0 \chi(H^{\mathcal{E}}) + \mathcal{O}(\varepsilon^2) \\ \text{by (4.19) and (4.20a)}}} P_0\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\Big\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2) \\
&\overset{(4.20b)}{\leqslant} \left\| P_0^{\perp}\varepsilon P_1 P_0\chi(H_{\mathrm{a}}^{\mathcal{P}})\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2) \\
&= \left\| P_0^{\perp}\varepsilon P_1 P_0\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H},\mathrm{dom}(H^{\mathcal{E}}))} + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

- $T = H_{\mathrm{eff}}^{\mathcal{P}}$:
  This immediately follows from the latter case, using

  $$\left\|\chi(H_{\mathrm{eff}}^{\mathcal{P}}) - \chi(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}},\mathcal{D}_{\mathcal{P}}^{\mathrm{eff}})} \leqslant C \underbrace{\left\|H_{\mathrm{eff}}^{\mathcal{P}} - H_{\mathrm{a}}^{\mathcal{P}}\right\|_{\mathcal{L}(\mathcal{D}_{\mathcal{P}}^{\mathrm{eff}},\mathcal{H}_{\mathcal{P}})}}_{=\mathcal{O}(\varepsilon^2)}$$

  by the Helffer-Sjöstrand formula.

In view of the fact that $\varrho_\alpha(T) \in \mathcal{L}(\mathcal{H}, \mathrm{dom}_\alpha^2(T))$, it remains to estimate

$$P_0^{\perp}\varepsilon P_1 P_0 = -R^{\mathcal{F}}(\lambda_0)[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}}, P_0]P_0$$

in $\mathcal{L}(\mathrm{dom}_\alpha^2(T), \mathrm{dom}(H^{\mathcal{E}}))$, which results from the explicit form of P_1^{O} constructed in Lemma 3.16. To do so, we first observe that the claim for the $\mathcal{L}(\mathrm{dom}_\alpha^2(T), \mathcal{H})$-norm is implied by Lemma 4.12 and the fact that $\|\varepsilon H_1^{\mathcal{E}} P_0\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(T),\mathcal{H})} = \mathcal{O}(\varepsilon^{1+\alpha/2})$ due to Lemma B.1. In order to extend this to a claim with respect to the desired norm, we note that the image of $P_1 \in \mathcal{A}_H^{2,0}$ satisfies Dirichlet boundary conditions, and hence Proposition 2.15 is applicable for any $P_0^{\perp}\varepsilon P_1 P_0 \varrho_\alpha(T)\psi$ with normalised $\psi \in \mathcal{H}$, which leads to

$$
\begin{aligned}
&\left\|P_0^{\perp}\varepsilon P_1 P_0 \varrho_\alpha(T)\psi\right\|^2_{W^2_\varepsilon(\mathcal{E})} \\
&\leqslant C\left(\left\|\Delta^{\mathcal{E}}_{g^\varepsilon} P_0^{\perp}\varepsilon P_1 P_0\varrho_\alpha(T)\psi\right\|^2_{\mathcal{H}} + \underbrace{\left\|P_0^{\perp}\varepsilon P_1 P_0\varrho_\alpha(T)\psi\right\|^2_{\mathcal{H}}}_{=\mathcal{O}(\varepsilon^{2(1+\alpha/2)}\|\psi\|^2_{\mathcal{H}})}\right) \\
&\leqslant C\left(\left\|\Delta^{\mathcal{E}}_{g^\varepsilon} R^{\mathcal{F}}(\lambda_0)\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0\varrho_\alpha(T)\psi\right\|^2_{\mathcal{H}}\right. \\
&\qquad\left. + \left\|\Delta^{\mathcal{E}}_{g^\varepsilon} R^{\mathcal{F}}(\lambda_0)\left[\varepsilon H^{\mathcal{E}}_1, P_0\right]P_0\varrho_\alpha(T)\psi\right\|^2_{\mathcal{H}}\right) \\
&\quad + \mathcal{O}\left(\varepsilon^{2+\alpha}\|\psi\|^2_{\mathcal{H}}\right).
\end{aligned}
$$

As far as the $-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}$-term is concerned, the fact that $R^{\mathcal{F}}(\lambda_0) \in \mathcal{A}^{0,0}_H$ due to Corollary 3.15 shows that all vertical derivatives of $\Delta^{\mathcal{E}}_{g^\varepsilon}$ give bounded operators, whereas all ∂-horizontal derivatives may be commuted to the right, and thus

$$
\begin{aligned}
&\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}} R^{\mathcal{F}}(\lambda_0)\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0 \\
&= \varepsilon^2 \underbrace{\left[\varepsilon\Delta^{\mathcal{E}}_{\mathsf{H}}, R^{\mathcal{F}}(\lambda_0)\right]\left[-\varepsilon\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0}_{\in\mathcal{L}(\mathrm{dom}^2_\alpha(T),\mathcal{H}),\ \text{cf. Lemma 3.7(i)}} + R^{\mathcal{F}}(\lambda_0)\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0
\end{aligned}
$$

on the image of $\varrho_\alpha(T) \in \mathcal{L}(\mathcal{H}, \mathrm{dom}^2_\alpha(T))$. One is then faced with similar calculations as in the proof of Lemma 4.12 for the second term, so formally

$$
R^{\mathcal{F}}(\lambda_0)\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0\Big|_{\pi_M^{-1}(U)} = \mathrm{tr}_{g_B}\left(\varepsilon\tilde{T}_1 \otimes \nabla^{\mathsf{B}}_{\varepsilon\cdot} + \varepsilon^2\tilde{T}_2\right)
$$

with coefficients $S(X), T(X,Y) \in \mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})\big|_{\pi_M^{-1}(U)}$ bounded independently of ε for $X, Y \in C^\infty_{\mathrm{b}}(\mathsf{T}U)$ with $U \in \mathfrak{U}$ (see Definition 2.7). Then (4.18) results in

$$
\left\|\Delta^{\mathcal{E}}_{g^\varepsilon} R^{\mathcal{F}}(\lambda_0)\left[-\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}, P_0\right]P_0\varrho_\alpha(T)\psi\right\|_{\mathcal{H}} = \mathcal{O}\left(\varepsilon^{1+\alpha/2}\|\psi\|_{\mathrm{dom}^2_\alpha(T)}\right).
$$

We complete the proof by using the estimate

$$
\begin{aligned}
&\Big\| \Delta^{\mathcal{E}}_{g^\varepsilon} R^{\mathcal{F}}(\lambda_0) \big[\varepsilon H_1^{\mathcal{E}}, P_0\big] P_0 \Big\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(T),\mathcal{H})} \\
&\leqslant \varepsilon \underbrace{\Big\| \Delta^{\mathcal{E}}_{g^\varepsilon} R^{\mathcal{F}}(\lambda_0) \Big\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}}_{<\infty \text{ due to } R^{\mathcal{F}}(\lambda_0)\in \mathcal{A}^{0,0}_H} \underbrace{\big\| \big[H_1^{\mathcal{E}}, P_0\big] P_0 \big\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(T),\mathrm{dom}(H^{\mathcal{E}}))}}_{=\mathcal{O}(\varepsilon^{\alpha/2}) \text{ by Lemma B.1}} \\
&= \mathcal{O}(\varepsilon^{1+\alpha/2})
\end{aligned}
$$

for the remaining $H_1^{\mathcal{E}}$-term. □

We are now in a position to state a refined comparison result for the respective low-lying parts of $\sigma(H^{\mathcal{E}})$ and $\sigma(H_{\mathrm{a}}^{\mathcal{P}})$:

Proposition 4.14 *Let* $\alpha \in (0,2]$ *and Condition 4.11 be satisfied. Then for all* $C > 0$ *it holds that*

$$
\mathrm{dist}_{\mathrm{H}}\big(\sigma(H^{\mathcal{E}}) \cap (-\infty, C\varepsilon^\alpha], \sigma(H_{\mathrm{a}}^{\mathcal{P}}) \cap (-\infty, C\varepsilon^\alpha]\big) = \mathcal{O}(\varepsilon^{2+\alpha/2}),
$$

where $\mathrm{dist}_{\mathrm{H}}$ *is the Hausdorff distance* (4.5) *between compact subsets of* $\mathbb{R}$.

Proof. It suffices to show the mutual $\varepsilon^{2+\alpha/2}$-closeness between the sets $\sigma(H_{\mathrm{eff}}^{\mathcal{P}}) \cap (-\infty, C\varepsilon^\alpha]$ and $\sigma(H_{\mathrm{a}}^{\mathcal{P}}) \cap (-\infty, C\varepsilon^\alpha]$ by means of the unitary equivalence of $H^{\mathcal{E}}$ and $H_{\mathrm{eff}}^{\mathcal{P}}$ up to errors of order ε^4 in this low energy regime. In view of Theorem 4.4 and the discussion right after Theorem 4.5, we prove the statement using a Weyl sequence argument. Therefore, first let μ be an element of $\sigma(H_{\mathrm{eff}}^{\mathcal{P}})$ with normalised Weyl sequence $\{\psi_k\}_{k\in\mathbb{N}}$ in the image of $\varrho_\alpha(H_{\mathrm{eff}}^{\mathcal{P}})$. Then we easily calculate

$$
\begin{aligned}
&\big\| (H_{\mathrm{a}}^{\mathcal{P}} - \mu)\psi_k \big\|_{\mathcal{H}_{\mathcal{P}}} \\
&\leqslant \big\| (H_{\mathrm{a}}^{\mathcal{P}} - H_{\mathrm{eff}}^{\mathcal{P}})\psi_k \big\|_{\mathcal{H}_{\mathcal{P}}} + \underbrace{\big\| (H_{\mathrm{eff}}^{\mathcal{P}} - \mu)\psi_k \big\|_{\mathcal{H}_{\mathcal{P}}}}_{\text{arbitrarily small}} \\
&= \big\| (H_{\mathrm{a}}^{\mathcal{P}} - H_{\mathrm{eff}}^{\mathcal{P}})\varrho_\alpha(H_{\mathrm{eff}}^{\mathcal{P}})\psi_k \big\|_{\mathcal{H}_{\mathcal{P}}} + \mathcal{O}(\varepsilon^{2+\alpha/2})
\end{aligned}
$$

for $k \in \mathbb{N}$ large enough. The concrete form (4.13) of $H_{\mathrm{a}}^{\mathcal{P}} - H_{\mathrm{eff}}^{\mathcal{P}}$ then shows

$$\left\|(H_{\mathrm{a}}^{\mathcal{P}} - H_{\mathrm{eff}}^{\mathcal{P}})\varrho_\alpha(H_{\mathrm{eff}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})} = \mathcal{O}(\varepsilon^{2+\alpha/2})$$

due to Proposition 3.17, the previous two lemmas and Lemma B.1, just as in the proof of [Lam14, Proposition 3.11]. Hence, $\{\psi_k\}_{k\in\mathbb{N}}$ is an approximate Weyl sequence for μ, i.e., $\left\|(H_{\mathrm{a}}^{\mathcal{P}} - \mu)\psi_k\right\|_{\mathcal{H}_{\mathcal{P}}} = \mathcal{O}(\varepsilon^{2+\alpha/2})$ if $k \in \mathbb{N}$ is chosen sufficiently large, which immediately implies the fact that $\operatorname{dist}(\mu, \sigma(H_{\mathrm{a}}^{\mathcal{P}})) = \mathcal{O}(\varepsilon^{2+\alpha/2})$.

In order to verify $\operatorname{dist}(\nu, \sigma(H_{\mathrm{eff}}^{\mathcal{P}})) = \mathcal{O}(\varepsilon^{2+\alpha})$ for $\nu \in \sigma(H_{\mathrm{a}}^{\mathcal{P}})$, one ends up proving a similar estimate

$$\left\|(H_{\mathrm{eff}}^{\mathcal{P}} - H_{\mathrm{a}}^{\mathcal{P}})\varrho_\alpha(H_{\mathrm{a}}^{\mathcal{P}})\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})} = \mathcal{O}(\varepsilon^{2+\alpha/2}),$$

once again using Lemma 4.12, Lemma 4.13 and Lemma B.1. □

The following theorem gives an even better approximation for the case of low-lying eigenvalues.

Theorem 4.15 *Let $\alpha \in (0,2]$ and Condition 4.11 be satisfied. Moreover, assume that there exist constants $C, \delta, \varepsilon_0 > 0$ such that $\sigma(H_{\mathrm{a}}^{\mathcal{P}}) \cap (-\infty, C\varepsilon^\alpha)$ consists of $K+1$ eigenvalues $\nu_0 \leqslant \cdots \leqslant \nu_K$ (repeated according to their multiplicity) and $\operatorname{rank}(\mathbf{1}_{(-\infty,(C+\delta)\varepsilon^\alpha)}(H_{\mathrm{a}}^{\mathcal{P}}))$ is finite for all $0 < \varepsilon < \varepsilon_0$. Then the following hold:*

(i) $H^{\mathcal{E}}$ has $K+1$ eigenvalues $\upsilon_0 \leqslant \cdots \leqslant \upsilon_K$ below its essential spectrum and $|\upsilon_j - \nu_j| = \mathcal{O}(\varepsilon^{2+\alpha})$ for all $j = 0, \ldots, K$.

(ii) If additionally any $\nu \in \{\nu_0, \ldots, \nu_K\}$ is a simple eigenvalue separated from the rest of $\sigma(H_{\mathrm{a}}^{\mathcal{P}})$ by at least $C_\nu\varepsilon^\alpha$ for some constant $C_\nu > 0$, the corresponding eigenvalue υ of $H^{\mathcal{E}}$ is also simple and there exists a constant $C_\upsilon > 0$ such that $\operatorname{dist}(\upsilon, \sigma(H^{\mathcal{E}})\setminus\{\upsilon\}) \geqslant C_\upsilon\varepsilon^\alpha$.

PROOF. We again utilise the approximate unitary equivalence of Theorem 4.5 for small energies of order ε^α, and hence the claims may be reduced to those between the eigenvalues of $H_{\mathrm{a}}^{\mathcal{P}}$ and $H_{\mathrm{eff}}^{\mathcal{P}}$.

(i) First note that

$$\left|\left\langle\psi,(H_{\mathrm{a}}^{\mathcal{P}}-H_{\mathrm{eff}}^{\mathcal{P}})\psi\right\rangle_{\mathcal{H}_{\mathcal{P}}}\right|\leqslant\left\|\varrho_\alpha(T)(H_{\mathrm{a}}^{\mathcal{P}}-H_{\mathrm{eff}}^{\mathcal{P}})\varrho_\alpha(T)\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}$$
$$=\mathcal{O}(\varepsilon^{2+\alpha}) \tag{4.22}$$

for all normalised $\psi\in\varrho_\alpha(T)\mathcal{H}_{\mathcal{P}}$ with $T\in\{H_{\mathrm{a}}^{\mathcal{P}},H_{\mathrm{eff}}^{\mathcal{P}}\}$. Here, the additional factor $\varepsilon^{\alpha/2}$ compared to $\left\|(H_{\mathrm{a}}^{\mathcal{P}}-H_{\mathrm{eff}}^{\mathcal{P}})\varrho_\alpha(T)\right\|_{\mathcal{L}(H_{\mathrm{eff}}^{\mathcal{P}})}=\mathcal{O}(\varepsilon^{2+\alpha/2})$ comes from the second cut-off $\varrho_\alpha(T)$ acting on the left. For example, we obtain for the first term in the expansion (4.13):

$$\begin{aligned}
&\left\|\varrho_\alpha(T)P_0(P_\varepsilon-P_0)P_0^\perp[H^{\mathcal{E}},P_0]P_0\varrho_\alpha(T)\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}\\
&=\left\|\left(P_0^\perp(P_\varepsilon-P_0)P_0\varrho_\alpha(T)\right)^\dagger\left([H^{\mathcal{E}},P_0]P_0\varrho_\alpha(T)\right)\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}\\
&\leqslant\underbrace{\left\|P_0^\perp(P_\varepsilon-P_0)P_0\varrho_\alpha(T)\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}}_{=\mathcal{O}(\varepsilon^{1+\alpha/2})\text{ by Lemma 4.13}}\underbrace{\left\|[H^{\mathcal{E}},P_0]P_0\varrho_\alpha(T)\right\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}}_{\substack{=\mathcal{O}(\varepsilon^{1+\alpha/2})\text{ by Lemma B.1}\\ \text{and Lemma 4.12}}}\\
&=\mathcal{O}(\varepsilon^{2+\alpha}).
\end{aligned}$$

Now suppose for a moment that $\operatorname{rank}(\mathbf{1}_{(-\infty,(C+\delta/2)\varepsilon^\alpha)}(H_{\mathrm{eff}}^{\mathcal{P}}))$ was infinite. But then (4.22) implies that the dimension of

$$\left\{\psi\in\operatorname{dom}(H_{\mathrm{a}}^{\mathcal{P}})\text{ such that }\left\langle\psi,H_{\mathrm{a}}^{\mathcal{P}}\psi\right\rangle_{\mathcal{H}_{\mathcal{P}}}\leqslant(C+\delta)\varepsilon^\alpha\left\|\psi\right\|_{\mathcal{H}_{\mathcal{P}}}^2\right\}$$

would also be infinite, which clearly contradicts the assumption of the theorem. Hence, the space

$$\left\{\psi\in\operatorname{dom}(H_{\mathrm{eff}}^{\mathcal{P}})\text{ s.t. }\left\langle\psi,H_{\mathrm{eff}}^{\mathcal{P}}\psi\right\rangle_{\mathcal{H}_{\mathcal{P}}}\leqslant\left(C+\tfrac{\delta}{2}\right)\varepsilon^\alpha\left\|\psi\right\|_{\mathcal{H}_{\mathcal{P}}}^2\right\}$$

is finite-dimensional and $\sigma(H_{\mathrm{eff}}^{\mathcal{P}})\cap(-\infty,(C+\frac{\delta}{2})\varepsilon^\alpha)$ consists solely of finitely many degenerate eigenvalues $\mu_0\leqslant\mu_1\leqslant\dots$.

The eigenvalues of $H_{\mathrm{a}}^{\mathcal{P}}$ (and in the same way those of $H_{\mathrm{eff}}^{\mathcal{P}}$) are characterised by the max-min principle

$$\nu_j=\min_{W_j}\max\left\{\left\langle\psi,H_{\mathrm{a}}^{\mathcal{P}}\psi\right\rangle_{\mathcal{H}_{\mathcal{P}}}\text{ s.t. }\psi\in W_j\text{ and }\|\psi\|_{\mathcal{H}_{\mathcal{P}}}=1\right\},$$

where W_j runs over all $(j+1)$-dimensional subspaces of $\mathrm{dom}(H_{\mathrm{a}}^{\mathcal{P}})$. In particular for the special choice $\widetilde{W}_j = \bigoplus_{k=0}^{j} \ker(H_{\mathrm{a}}^{\mathcal{P}} - \nu_k)$, which is clearly the minimising subspace for ν_j, it follows that (using the fact that $\mathrm{dom}(H_{\mathrm{a}}^{\mathcal{P}}) = \mathrm{dom}(H_{\mathrm{eff}}^{\mathcal{P}})$)

$$\begin{aligned}
\mu_j &\leqslant \max\Big\{\big\langle \psi, H_{\mathrm{eff}}^{\mathcal{P}}\psi\big\rangle_{\mathcal{H}_{\mathcal{P}}} \text{ such that } \psi \in \widetilde{W}_j \text{ and } \|\psi\|_{\mathcal{H}_{\mathcal{P}}} = 1\Big\} \\
&\overset{(4.22)}{\leqslant} \max\Big\{\big\langle \psi, H_{\mathrm{a}}^{\mathcal{P}}\psi\big\rangle_{\mathcal{H}_{\mathcal{P}}} \text{ such that } \psi \in \widetilde{W}_j \text{ and } \|\psi\|_{\mathcal{H}_{\mathcal{P}}} = 1\Big\} \\
&\quad + \mathcal{O}(\varepsilon^{2+\alpha}) \\
&= \nu_j + \mathcal{O}(\varepsilon^{2+\alpha}).
\end{aligned}$$

The latter shows that $H_{\mathrm{eff}}^{\mathcal{P}}$ has $K+1$ eigenvalues $\mu_0 \leqslant \cdots \leqslant \mu_K$ below $(C + \frac{\delta}{2})\varepsilon^{\alpha}$. Reversing the roles of $H_{\mathrm{a}}^{\mathcal{P}}$ and $H_{\mathrm{eff}}^{\mathcal{P}}$ analogously shows $\nu_j \leqslant \mu_j + \mathcal{O}(\varepsilon^{2+\alpha})$ and the statement is proven.

(ii) Denote by P_ν the projection onto $\ker(H_{\mathrm{a}}^{\mathcal{P}} - \nu)$ and by μ the corresponding eigenvalue of $H_{\mathrm{eff}}^{\mathcal{P}}$. Suppose that there were another eigenvalue $\mu' \neq \mu$ of $H_{\mathrm{eff}}^{\mathcal{P}}$ for which $|\nu - \mu'| = \mathcal{O}(\varepsilon^{2+\alpha/4})$. Let $\{\psi_k\}_{k\in\mathbb{N}}$ and $\{\psi'_k\}_{k\in\mathbb{N}}$ be normalised Weyl sequences of μ and μ' such that $\big\langle \psi_k, \psi'_l\big\rangle_{\mathcal{H}_{\mathcal{P}}} = 0$ for all $k, l \in \mathbb{N}$ (this can be achieved by choosing the respective sequences in the image of the spectral projections on disjoint intervals around μ and μ'). Moreover, set $\psi = \psi_{k(\varepsilon)}$ and $\psi' = \psi'_{k(\varepsilon)}$ so that

$$\begin{aligned}
\big\|(H_{\mathrm{a}}^{\mathcal{P}} - \nu)\psi^{(\prime)}\big\|_{\mathcal{H}_{\mathcal{P}}} &\leqslant \underbrace{\big\|(H_{\mathrm{a}}^{\mathcal{P}} - \mu^{(\prime)})\psi^{(\prime)}\big\|_{\mathcal{H}_{\mathcal{P}}}}_{= \mathcal{O}(\varepsilon^{2+\alpha/2}) \text{ by Prop. 4.14}} + \underbrace{\big|\nu - \mu^{(\prime)}\big|}_{=\mathcal{O}(\varepsilon^{2+\alpha/4})} \\
&= \mathcal{O}(\varepsilon^{2+\alpha/4}),
\end{aligned}$$

and consequently

$$\begin{aligned}
&\big\|(\mathbf{1}_{\mathcal{H}_{\mathcal{P}}} - \mathrm{P}_\nu)\psi^{(\prime)}\big\|_{\mathcal{H}_{\mathcal{P}}} \\
&\leqslant \underbrace{\big\|(H_{\mathrm{a}}^{\mathcal{P}} - \nu)^{-1}(\mathbf{1}_{\mathcal{H}_{\mathcal{P}}} - \mathrm{P}_\nu)\big\|_{\mathcal{L}(\mathcal{H}_{\mathcal{P}})}}_{\leqslant 1/(C_\nu \varepsilon^{\alpha})} \underbrace{\big\|(H_{\mathrm{a}}^{\mathcal{P}} - \nu)\psi^{(\prime)}\big\|_{\mathcal{H}_{\mathcal{P}}}}_{=\mathcal{O}(\varepsilon^{2+\alpha/4})}
\end{aligned}$$

is of order $\varepsilon^{2-3\alpha/4}$. But then

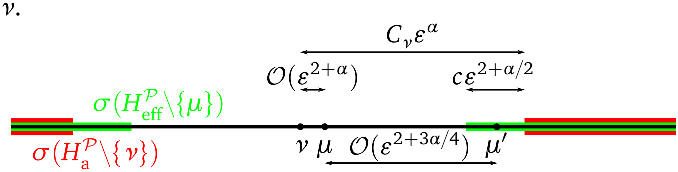
$$\begin{aligned}
&\left|\langle \mathrm{P}_\nu \psi, \mathrm{P}_\nu \psi' \rangle_{\mathcal{H}_\mathcal{P}}\right| \\
&\leqslant \Big|\underbrace{\langle \psi, \psi' \rangle_{\mathcal{H}_\mathcal{P}}}_{=0}\Big| + \left|\left\langle (\mathbf{1}_{\mathcal{H}_\mathcal{P}} - \mathrm{P}_\nu)\psi, (\mathbf{1}_{\mathcal{H}_\mathcal{P}} - \mathrm{P}_\nu)\psi' \right\rangle_{\mathcal{H}_\mathcal{P}}\right| \\
&\leqslant \left\|(\mathbf{1}_{\mathcal{H}_\mathcal{P}} - \mathrm{P}_\nu)\psi\right\|_{\mathcal{H}_\mathcal{P}} \left\|(\mathbf{1}_{\mathcal{H}_\mathcal{P}} - \mathrm{P}_\nu)\psi'\right\|_{\mathcal{H}_\mathcal{P}} \\
&= \mathcal{O}(\varepsilon^{4-3\alpha/2})
\end{aligned}$$

and

$$\left|1 - \left\|\mathrm{P}_\nu \psi^{(\prime)}\right\|_{\mathcal{H}_\mathcal{P}}\right| \leqslant \left\|(\mathbf{1}_{\mathcal{H}_\mathcal{P}} - \mathrm{P}_\nu)\psi^{(\prime)}\right\|_{\mathcal{H}_\mathcal{P}} = \mathcal{O}(\varepsilon^{2-3\alpha/4})$$

imply that $\mathrm{P}_\nu \psi$ and $\mathrm{P}_\nu \psi'$ are two almost orthonormal vectors in $\mathrm{P}_\nu \mathcal{H}_\mathcal{P}$, which contradicts the simplicity of ν. Therefore, μ is an isolated eigenvalue of $H^\mathcal{P}_{\mathrm{eff}}$.

The eigenvalue must also be simple: If there were two orthogonal eigensections ψ_1 and ψ_2 of μ, the above argumentation with the choice $\psi = \psi_1$ and $\psi' = \psi_2$ would again contradict the simplicity of ν.



Finally, μ is separated from the rest of $\sigma(H^\mathcal{P}_{\mathrm{eff}})$ by at least the quantity

$$C_\nu \varepsilon^\alpha - c\varepsilon^{2+\alpha/2} = \underbrace{\left(C_\nu - c\varepsilon^{2-\alpha/2}\right)}_{=:C_\mu} \varepsilon^\alpha,$$

where the constant $c > 0$ stems from Proposition 4.14. $\square$

5 Chapter 5

Quantum Waveguides with Gauge Fields

There has been a considerable progress within mesoscopic physics throughout the last decades. In particular, the usage of semiconductors composed of aluminium, gallium and arsenic nowadays allows for an enormous variety of shapes in the fabrication of microscopic structures, which confine an electron gas into quasi two-dimensional “films” or quasi one-dimensional “channels” (see [LCM99, Hun00] and references therein for a more detailed physical background). The characteristic properties of such devices are itemised as follows [DE95]:

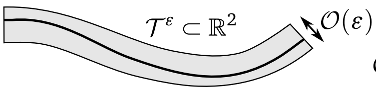
(i) small size with typical length scale of 10 nm to 100 nm,

(ii) high purity: the electron mean free path can be a few μm or even larger,

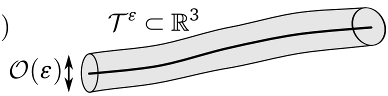
(iii) crystallic structure,

(iv) the wave function representing the particles inside the structure are usually suppressed at the boundaries between different semiconductor materials.

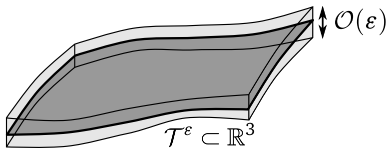
The motion of particles in such thin devices is quantised in the transversal directions and the corresponding transverse energy levels form a set of discrete values. This motivates referring to these microstructures as *quantum waveguides*. They are modelled mathematically by a family of ε-thin tubes $\mathcal{T}^\varepsilon$ around a smoothly embedded curve in $\mathbb{R}^2$ or around a smoothly embedded curve/hypersurface in $\mathbb{R}^3$.



(a) quantum strips



(b) conventional quantum tubes



(c) conventional quantum layers

Figure 5.1: The most prominent and most investigated examples are (see for example [DE95, DEK01]) tubular neighbourhoods around (a) an embedded curve in $\mathbb{R}^2$ with possibly varying ε-intervals as cross-sections, (b) an embedded curve in $\mathbb{R}^3$ with possibly varying ε-discs as cross-sections and (c) an embedded hypersurface in $\mathbb{R}^3$ with possibly varying ε-intervals as cross-sections.

Although the behaviour of an electron is governed by a non-relativistic many-body Schrödinger equation with a Hamiltonian modelling the interaction with the lattice atoms and impurities, the properties (i) – (iii) allow us to neglect all scattering effects ("ballistic regime") and to approximate the motion of the electron by a free one-particle equation with some effective mass m_{eff} (e.g., $m_{\text{eff}} = 0.067 m_{\text{e}}$ for GaAs [Vre68]). There is a sizeable amount of literature that implements property (iv) by means of a steep confining potential in the transversal directions (see for instance [JK71, daC82, Tol88, Mar95, FH01, Mit01, WT14]). We, in contrast, impose Dirichlet boundary conditions on $\partial\mathcal{T}^\varepsilon$ in order to localise the quantum particle to the interior of the tube. Thus, a reasonable model for an electronic motion inside the device $\mathcal{T}^\varepsilon \subset \mathbb{R}^{2,3}$ are solutions $\psi : \mathbb{R} \to L^2(\mathcal{T}^\varepsilon, \mathbb{C})$ of the free Schrödinger equation

$$\mathrm{i}\hbar \tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -\tfrac{\hbar^2}{2m_{\mathrm{eff}}}\Delta^{\mathrm{L.B.}}_{\delta^{2,3}}\psi(t), \quad \psi(0)|_{\partial\mathcal{T}^\varepsilon} = 0$$

with the Dirichlet Laplace-Beltrami operator $-\Delta^{\mathrm{L.B.}}_{\delta^{2,3}} = \mathrm{d}^*\mathrm{d}$.

The described geometric framework can be extended to so-called *generalised* quantum waveguides, i.e., to a family of ε-thin tubes $\mathcal{T}^\varepsilon \subset \mathbb{R}^{b+f}$ around a smoothly embedded, complete, b-dimensional submanifold $B \hookrightarrow \mathbb{R}^{b+f}$ [HLT15]. In this context, we initially think of $(\mathbb{R}^{b+f}, \delta^{b+f})$ as the "laboratory", where external forces like electromagnetic fields described by Maxwell's classical electrodynamics or gravitational effects described by Einstein's general relativity are well-understood, and restrict the configuration space of multiple quantum particles to the tube $\mathcal{T}^\varepsilon$. This situation typically arises in molecular physics, where a formation of chemically bonded atoms vibrates slightly around their equilibrium configuration B. These deviations are quantised and the resulting experimentally observable spectrum corresponds to the study of eigenvalues of some appropriate Schrödinger operator on $\mathcal{T}^\varepsilon$ below the essential spectrum.

If we switch on an external electromagnetic field (in terms of an electric potential $\varphi \in C^\infty_{\mathrm{b}}(\mathbb{R}^{b+f})$ and a magnetic potential $\mathcal{A} \in C^\infty_{\mathrm{b}}(\mathsf{T}^*\mathbb{R}^{b+f})$), the Hamiltonian has to be modified according to the principle of minimal coupling [Gre01, Section 9.1], i.e., by the addition of the potential $q_{\mathrm{eff}}\varphi$ and the replacement of d by $\mathrm{d}^{\mathcal{A}} := \mathrm{d} + \frac{\mathrm{i}q_{\mathrm{eff}}}{\hbar}\mathcal{A}$, where $q_{\mathrm{eff}} \in \mathbb{R}$ denotes the effective charge of the quantum particles. Moreover, the particles may possess a total intrinsic angular momentum (*spin*) expressed by a spin quantum number $\frac{N-1}{2}$ ($N \in \mathbb{N}$) and are therefore characterised by a $\mathbb{C}^N$-valued wave function. In this context, we will refer to particles with spin quantum number 0 ($N = 1$) as spinless particles which are represented by a scalar wave function. The possible interaction of the spin with the magnetic forces (and, of course, the interaction with the remaining forces) is modelled by an additional potential $V \in C^\infty(\mathbb{R}^{b+f}, \mathbb{C}^{N\times N}_{\mathrm{Herm}})$ which takes its values in the Hermitian $N \times N$-matrices and therefore mixes the components of ψ. Eliminating the physical dimensions by setting $\hbar = 1 = q_{\mathrm{eff}}$ as well as $m_{\mathrm{eff}} = 1/2$, we obtain the non-relativistic, time-dependent Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H^{\mathrm{EM}}\psi(t), \quad \psi(0)|_{\partial\mathcal{T}^{\varepsilon}} = 0 \tag{5.1}$$

with Hamiltonian

$$H^{\mathrm{EM}} = \nabla^{\mathcal{A},*}\nabla^{\mathcal{A}}\,\mathbf{1}_{\mathbb{C}^N} + V, \quad \nabla^{\mathcal{A}} = (\mathrm{d} + \mathrm{i}\mathcal{A})\mathbf{1}_{\mathbb{C}^N} \tag{5.2}$$

in the presence of external electromagnetic fields, where we absorbed the electric potential $\varphi\,\mathbf{1}_{\mathbb{C}^N}$ into V.

Example 5.1 Let us consider a non-relativistic spin-1/2-particle (for example an electron) in three-dimensional Euclidean space with coordinates (z^1, z^2, z^3) in the presence of electromagnetic fields $\mathcal{A} \in C_{\mathrm{b}}^{\infty}(\mathsf{T}^*\mathbb{R}^3)$ and $\varphi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^3)$. Then the possible states $\psi = (\psi_\uparrow, \psi_\downarrow)^{\mathrm{t}}$ of such a particle, where $\psi_\uparrow$ ("spin-up") and $\psi_\downarrow$ ("spin-down") represent the eigenstates of the spin along the z^3-axis, are the solutions of the famous Pauli equation [Gre01, Section 12.5]

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H^{\mathrm{EM}}_{\mathrm{Pauli}}\psi(t)$$

with Hamiltonian

$$H^{\mathrm{EM}}_{\mathrm{Pauli}} = \underbrace{\left(-\Delta^{\mathcal{A}}_{\delta^3} + \varphi\right)}_{\substack{\text{Hamiltonian}\\ \text{without spin}}}\mathbf{1}_{\mathbb{C}^2} + \underbrace{\begin{pmatrix} \mathcal{B}^3 & \mathcal{B}^1 - \mathrm{i}\mathcal{B}^2 \\ \mathcal{B}^1 + \mathrm{i}\mathcal{B}^2 & -\mathcal{B}^3 \end{pmatrix}}_{\text{coupling of spinor components}}.$$

While the first diagonal term describes the usual kinetic and potential energy of the particle due to the minimal coupling, the second term models the interaction of the particle's spin with the components

$$\mathcal{B}^l = \sum_{m<n}^{3} \varepsilon^{lmn}\,\mathrm{d}\mathcal{A}(\partial_{z^m}, \partial_{z^n}) \quad \text{for } l \in \{1,2,3\}$$

of the magnetic field $\mathcal{B} = (\star \mathrm{d}\mathcal{A})^\sharp \in C^\infty_\mathrm{b}(\mathsf{T}\mathbb{R}^3)$. Consequently, the latter operator coincides with H^{EM} (5.2) for the connection $\nabla^{\mathcal{A}} = (\mathrm{d} + \mathrm{i}\mathcal{A})\mathbf{1}_{\mathbb{C}^2}$ and the $\mathbb{C}^{2\times 2}_{\mathrm{Herm}}$-valued potential

$$V = \begin{pmatrix} \varphi + \mathcal{B}^3 & \mathcal{B}^1 - \mathrm{i}\mathcal{B}^2 \\ \mathcal{B}^1 + \mathrm{i}\mathcal{B}^2 & \varphi - \mathcal{B}^3 \end{pmatrix}.$$

◊

A solution of the related time-dependent Schrödinger equation (5.1) is a priori a mapping $\psi : \mathbb{R} \to L^2(\mathbb{R}^{b+f}, \mathbb{C}^N)$, i.e., $\psi(t)$ can be identified with an L^2-section of the trivial vector bundle $\mathbb{R}^{b+f} \times \mathbb{C}^N$ for all $t \in \mathbb{R}$. From this geometric point of view, $\nabla^{\mathcal{A}}$ takes the role of a metric connection with respect to the Hermitian bundle metric $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ and $\nabla^{\mathcal{A},*}\nabla^{\mathcal{A}} = -\Delta^{\mathcal{A}}_{\delta^{b+f}}$ is the associated connection Laplacian. From now on we will consider the more general class of metric connections

$$\nabla^{\mathcal{A}} := \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}$$

on $\mathbb{R}^{b+f} \times \mathbb{C}^N$ with connection one-form $\mathcal{A} \in C^\infty_\mathrm{b}(\mathsf{T}^*\mathbb{R}^{b+f}) \otimes \mathbb{C}^{N\times N}_{\mathrm{Herm}}$.

Remark 5.2 A common aspect seen in applications is the transformation behaviour of physical quantities under smooth mappings $\mathsf{g} : \mathbb{R}^{b+f} \to G$ for some Lie group $G \subset \mathrm{GL}(N, \mathbb{C})$, which describes fibrewise transformations $\psi \mapsto \mathsf{g}\psi$ of wave functions $\psi : \mathbb{R}^{b+f} \to \mathbb{C}^N$ (sections of $\mathbb{R}^{b+f} \times \mathbb{C}^N$). The connection $\nabla^{\mathcal{A}}$ transforms as

$$\mathsf{g}\,\nabla^{\mathcal{A}}\,\mathsf{g}^{-1} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\,\underbrace{\left(\mathsf{g}\,\mathcal{A}\,\mathsf{g}^{-1} - \mathrm{i}\mathsf{g}\,\mathrm{d}\mathsf{g}^{-1}\right)}_{=:\mathcal{A}_\mathsf{g}} = \nabla^{\mathcal{A}_\mathsf{g}}.$$

Consequently, if $\psi(t)$ is a solution of the time-dependent Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -\Delta^{\mathcal{A}}_{\delta^{b+f}}\psi(t),$$

it follows that $\psi_\mathsf{g}(t) := \mathsf{g}\psi(t)$ is a solution of the transformed equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi_\mathsf{g}(t) = -\Delta^{\mathcal{A}_\mathsf{g}}_{\delta^{b+f}}\psi_\mathsf{g}(t)$$

and the examination of the two systems $(\psi, \mathcal{A})$ and $(\mathsf{g}\psi, \mathcal{A}_{\mathsf{g}})$ is equivalent, as they describe the same *(G-)gauge theory*. This additional degree of freedom in terms of g is often referred to as gauge invariance and $\mathcal{A}$ is the corresponding non-Abelian *gauge field*. $\Diamond$

The localisation of the particles within the ε-thin tube $\mathcal{T}^\varepsilon \subset \mathbb{R}^{b+f}$ results in solutions of the Schrödinger equation with oscillations of order ε^{-1} in the transversal (vertical) directions. Thus, the associated transversal kinetic energy is expected to be proportional to ε^{-2} and the limit $\varepsilon \ll 1$ seems to be rather ill-defined. This is why we multiply the Laplacian in (5.2) with a factor ε^2, so that both the kinetic energy $-\varepsilon^2 \Delta^{\mathcal{A}}_{\delta^{b+f}}$ and the potential energy V are of order one. We observe, however, that this additional factor ε^2 leads to a rescaling of the gauge field by a factor ε,

$$-\varepsilon^2 \Delta^{\mathcal{A}}_{\delta^{b+f}} = (\varepsilon \mathrm{d} \mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon \mathcal{A})^* (\varepsilon \mathrm{d} \mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon \mathcal{A}),$$

In this context, we refer to this scaling as the coupling to *weak* gauge fields $\varepsilon\mathcal{A}$. Finally, this leads to the study of the operator

$$H^{\mathrm{w}}_{\mathrm{tube}} := -\varepsilon^2 \Delta^{\mathcal{A}}_{\delta^{b+f}} + V = -\Delta^{\mathcal{A}}_{\varepsilon^{-2}\delta^{b+f}} + V \tag{5.3}$$

on $L^2(\mathcal{T}^\varepsilon \times \mathbb{C}^N, \mathrm{vol}_{\delta^{b+f}}) = L^2\big((\mathbb{R}^{b+f} \times \mathbb{C}^N)\big|_{\mathcal{T}^\varepsilon}\big)$ with Dirichlet boundary conditions, where we used the same symbols $\mathcal{A}$ and V for the respective restrictions from $\mathbb{R}^{b+f}$ to $\mathcal{T}^\varepsilon$. Physically speaking, $H^{\mathrm{w}}_{\mathrm{tube}}$ implements the dynamics of particles with total spin number $\frac{N-1}{2}$, which are coupled to a weak gauge field $\varepsilon\mathcal{A}$ and are localised in the tube $\mathcal{T}^\varepsilon$.

We will begin the next section with the construction of a diffeomorphism $\Psi_\varepsilon : M \to \mathcal{T}^\varepsilon$ in order to identify the family $\{\mathcal{T}^\varepsilon\}_{0<\varepsilon\leqslant 1}$ (the "tube") with an ε-independent manifold M (the "waveguide") that has the additional structure of a fibre bundle over B and whose typical fibre F is given by a compact subset of $\mathbb{R}^f$. This diffeomorphism then lifts to a unitary operator

$$\begin{aligned}\widehat{\Psi}_\varepsilon : L^2(\mathcal{T}^\varepsilon \times \mathbb{C}^N, \mathrm{vol}_{\delta^{b+f}}) &\to L^2(M \times \mathbb{C}^N, \mathrm{vol}_{G^\varepsilon}),\\ \psi \mapsto \widehat{\Psi}_\varepsilon \psi &= \varepsilon^{\frac{b+f}{2}} \psi \circ \Psi_\varepsilon\end{aligned}$$

and, by means of the relations

$$\widehat{\Psi}_\varepsilon\Big(\mathrm{d}\big(\widehat{\Psi}_\varepsilon^\dagger\psi\big)(Z)\Big) = \mathrm{d}\psi\big(\mathsf{T}\Psi_\varepsilon^{-1}Z\big) \quad , \quad \widehat{\Psi}_\varepsilon\mathcal{A}(Z)\widehat{\Psi}_\varepsilon^\dagger = \big(\Psi_\varepsilon^*\mathcal{A}\big)\big(\mathsf{T}\Psi_\varepsilon^{-1}Z\big)$$

for any $\psi \in C^\infty(M \times \mathbb{C}^N)$ and vector field Z on $\mathcal{T}^\varepsilon$, one concludes that the Schrödinger operator (5.3) is unitarily equivalent to

$$\widehat{\Psi}_\varepsilon H_{\text{tube}}^{\mathrm{w}} \widehat{\Psi}_\varepsilon^\dagger = -\Delta_{G^\varepsilon}^{\mathcal{A}_\varepsilon} + V_\varepsilon$$

on $L^2(\mathcal{E}, \mathrm{vol}_{G^\varepsilon})$ with $\mathbb{C}^N$-vector bundle $\mathcal{E} = M \times \mathbb{C}^N$, where we introduced

- the induced Riemannian metric

  $$G^\varepsilon := \Psi_\varepsilon^*(\varepsilon^{-2}\delta^{b+f}) = g^\varepsilon + \mathcal{O}(\varepsilon),$$

  which is an admissible perturbation of a rescaled submersion metric (2.7) in the sense of Definition 2.21,

- the induced gauge field

  $$\mathcal{A}_\varepsilon := \Psi_\varepsilon^*\mathcal{A} = \mathcal{A}_0 + \varepsilon\mathcal{A}_1^\varepsilon \in C_{\mathrm{b}}^\infty(\mathsf{T}^*M) \otimes \mathbb{C}_{\text{Herm}}^{N\times N},$$

  so that the connection $\nabla^{\mathcal{A}_\varepsilon} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}_\varepsilon$ is an admissible perturbation of $\nabla^{\mathcal{A}_0} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}_0$ (see Definition 2.24),

- and the induced potential

  $$V_\varepsilon := V \circ \Psi_\varepsilon \in C_{\mathrm{b}}^\infty\big(M, \mathbb{C}_{\text{Herm}}^{N\times N}\big).$$

Hence, this fits in the framework discussed in Section 2.3, provided that the tube $\mathcal{T}^\varepsilon$ satisfies adequate boundedness properties (see Definition 5.3 below).

5.1 Embedded Tubular Neighbourhoods

Let $c : B \to \mathbb{R}^{b+f}$ be a smooth embedding of a complete, b-dimensional manifold B into $\mathbb{R}^{b+f}$. This induces an orthogonal decomposition

$$c^*\mathsf{T}\mathbb{R}^{b+f} = \mathsf{T}B \oplus \mathsf{N}B, \tag{5.4}$$

and there is a one-to-one correspondence

$$\tau + \nu \in \mathsf{T}_xB \oplus \mathsf{N}_xB \quad \Leftrightarrow \quad \overline{\tau + \nu} \in \mathsf{T}_{c(x)}\mathbb{R}^{b+f} \cong \mathbb{R}^{b+f}$$

such that $\overline{\tau}$ is tangent to $c(B)$ and $\overline{\nu}$ is normal to $c(B)$, cf. Subsection A.1.3.

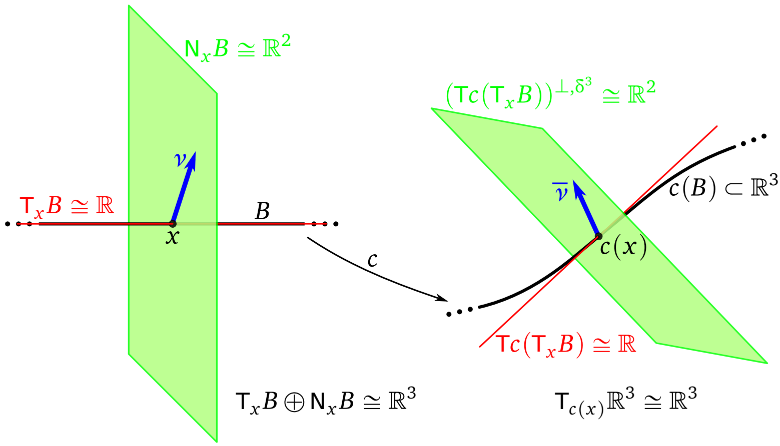


Figure 5.2: Illustration of the identification (5.4) for the case of an embedded curve in three-dimensional Euclidean space ($b = 1$ and $f = 2$).

In view of Definition A.10, we equip $\mathsf{T}B \oplus \mathsf{N}B$ with

- a bundle metric $\delta^{\mathsf{T}B\oplus\mathsf{N}B}$ that splits into a Riemannian metric g_B on B and a bundle metric $\delta^{\mathsf{N}B}$ on $\mathsf{N}B$,
- and a pullback connection $\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$ that splits into the Levi-Civita connection ∇^{g_B} on $\mathsf{T}B$ and the metric normal connection $\nabla^{\mathsf{N}B}$ on $\mathsf{N}B$.

Suppose that there exists a tubular neighbourhood $\mathcal{T}^r \subset \mathbb{R}^{b+f}$ of $c(B)$ with globally fixed radius, i.e., there is $r > 0$ such that normals to $c(B)$ of length r do not intersect. This is equivalent to the requirement that the map

$$\Phi : \mathsf{N}B \to \mathbb{R}^{b+f}, \quad \mathsf{N}_x B \ni \nu \mapsto c(x) + \overline{\nu}$$

restricted to

$$\mathsf{N}B^r := \left\{ \nu \in \mathsf{N}B \text{ such that } \|\nu\|_{\delta^{\mathsf{N}B}} < r \right\}$$

be a diffeomorphism onto $\mathcal{T}^r$ and it follows that the entire analysis can be carried out on (a subset of) the normal bundle.

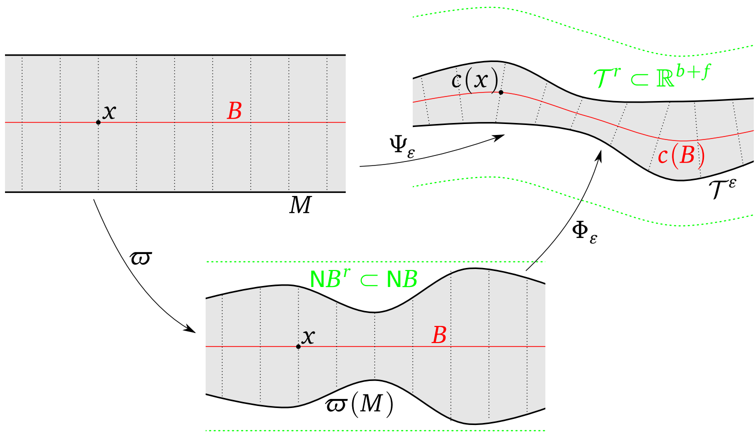


Figure 5.3: The two embeddings $c : B \to \mathbb{R}^{b+f}$ and $\varpi : M \to \mathsf{N}B^r \subset \mathsf{N}B$ allow for the identification of the fixed waveguide $M \xrightarrow{\pi_M} B$ with the family of ε-thin tubes $\mathcal{T}^\varepsilon \subset \mathbb{R}^{b+f}$.

This suggests that we view the initial tube as an ε-independent, fibrewise subset $M \subset \mathsf{N}B^r$, which is then mapped back into $\mathbb{R}^{b+f}$ by means of a rescaled diffeomorphism

$$\Phi_\varepsilon : \nu \mapsto c(x) + \varepsilon\overline{\nu}, \quad 0 < \varepsilon \leqslant 1.$$

More precisely, we think of M as a smooth fibre bundle $\pi_M : M \to B$ with compact typical fibre F with (possibly empty) smooth boundary that is embedded into $\mathsf{N}B^r$ via some smooth mapping $\varpi : M \to \mathsf{N}B^r$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\varpi"] \arrow[d, "\pi_M"'] & \mathsf{N}B^r \arrow[d, "\pi_{\mathsf{N}B}"] \\ B \arrow[r, "\mathbf{1}_B"'] & B \end{tikzcd}$$

commutes, or equivalently that $\varpi(M_x) \subset \mathsf{N}_x B^r := \mathsf{N}_x B \cap \mathsf{N}B^r$ for all $x \in B$. The composition

$$\Psi_\varepsilon := \Phi_\varepsilon \circ \varpi, \quad 0 < \varepsilon \leqslant 1 \tag{5.5}$$

finally yields the desired change of perspective from the family of ε-thin tubes $\{\mathcal{T}^\varepsilon\}_{0<\varepsilon\leqslant 1}$ to the ε-independent waveguide M.

In order to apply the results of Chapter 4, we need to ensure that Condition 2.2 on the geometry is satisfied. Therefore, we require the following uniformity properties for the family of ε-thin embedded tubes $\mathcal{T}^\varepsilon \subset \mathbb{R}^{b+f}$ (cf. [HLT15, Definition 3.1]):

Definition 5.3 We call M a *quantum waveguide of bounded geometry* if the associated family of diffeomorphisms $\{\Psi_\varepsilon : M \to \mathcal{T}^\varepsilon\}_{0<\varepsilon\leqslant 1}$ (5.5) satisfies the following:

(i) tubular neighbourhood:

- (B, g_B) is a manifold of bounded geometry,
- $c : (B, g_B) \to (\mathbb{R}^{b+f}, \delta^{b+f})$ is smooth and bounded with all its derivatives,
- and there exists $r > 0$ such that $\Phi : \mathsf{N}B^r \to \mathcal{T}^r$ is a diffeomorphism,

(ii) modelling of the quantum waveguide:
- $\pi_M : (M, G) \to (B, g_B)$ is a uniformly locally trivial fibre bundle,
- and $\varpi : (M, G) \to (\mathsf{N}B^r, \mathcal{G})$ is smooth and bounded with all its derivatives,

where $\mathcal{G} := \Phi^* \delta^{b+f}$ and $G := (\Psi_{\varepsilon=1})^* \delta^{b+f}$ stand for the ε-independent Riemannian metrics on $\mathsf{N}B^r$ and M, respectively. $\Diamond$

Condition (i) is fulfilled if B is a uniformly embedded submanifold [Eld13, Definition 2.21], i.e., the embedding $c : B \to \mathbb{R}^{b+f}$ is smooth and bounded with all its derivatives, and there exists $d > 0$ such that for all $x \in B$ the set $\mathbb{B}^{b+f}_d(c(x)) \cap c(B) \subset \mathbb{R}^{b+f}$ is given by the graph of a function $h_x \in C^\infty_{\mathrm{b}}(\mathbb{R}^b, \mathbb{R}^f)$ with bounds uniform in x. It follows that a uniformly embedded submanifold (B, g_B) is of bounded geometry [Eld13, Lemma 2.27] and admits a tubular radius $r > 0$ [Eld13, Theorem 2.33]. One concludes that both the Weingarten map $\mathcal{W}$ and the second fundamental form II are C^∞-bounded tensors, and hence the curvature of $\nabla^{\mathsf{N}B}$ (A.10) satisfies $\mathrm{R}^{\mathsf{N}B} \in C^\infty_{\mathrm{b}}(\Lambda^2 \mathsf{T}B \otimes \mathrm{End}(\mathsf{N}B))$ and $(\mathsf{N}B, \delta^{\mathsf{N}B}, \nabla^{\mathsf{N}B}) \xrightarrow{\pi_{\mathsf{N}B}} (B, g_B)$ is a vector bundle of bounded geometry by Definition A.19.

The pullback of $\mathcal{T}^\varepsilon \times \mathbb{C}^N$ via the map Ψ_ε gives a family $\Psi_\varepsilon^*(\mathcal{T}^\varepsilon \times \mathbb{C}^N)$ of vector bundles over M, all of which are isomorphic to the same trivial vector bundle $\mathcal{E} = M \times \mathbb{C}^N \xrightarrow{\mathrm{pr}_1} M$. Moreover, (M, G) is a manifold of bounded geometry [Lam14, Proposition A.4] and thus $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathbb{C}^N}, \nabla^{\mathcal{A}_0}) \xrightarrow{\pi_{\mathcal{E}}} (M, G)$ is a vector bundle of bounded geometry [Sch96, Example 3.13] with C^∞-bounded curvature $\mathrm{R}^{\mathcal{A}_0} = \mathrm{d}\mathcal{A}_0 + [\mathcal{A}_0(\cdot), \mathcal{A}_0(\cdot)]$. We finally mention that Ψ_ε is smooth and bounded with all its derivatives independently of ε, because its constituents ϖ and c are as well, which immediately carries over to the induced potential $V_\varepsilon = V \circ \Psi_\varepsilon$.

5.1.1 The Induced Riemannian Metric

We first turn $\Psi_\varepsilon : (M, G^\varepsilon) \to (\mathcal{T}^\varepsilon, \varepsilon^{-2}\delta^{b+f})$ into an isometry by means of the rescaled pullback metric $G^\varepsilon := \Psi_\varepsilon^*(\varepsilon^{-2}\delta^{b+f})$. We will see that the resulting metric $G^\varepsilon = g^\varepsilon + \mathcal{O}(\varepsilon)$ is an admissible perturbation of the rescaled Riemannian submersion metric (2.7) in the sense of Definition 2.21.

We start with the examination of the intermediate pullback metric $\mathcal{G}^\varepsilon := \Phi_\varepsilon^*(\varepsilon^{-2}\delta^{b+f})$ on $\mathsf{N}B^r$. Therefore, we first recall the considerations of Subsection A.1.2, which show that the tangent bundle of $\mathsf{N}B$ may be decomposed as

$$\mathsf{T}(\mathsf{N}B) = \mathsf{H}(\mathsf{N}B) \oplus \mathsf{V}(\mathsf{N}B) = \ker(\mathcal{K}_{\mathsf{N}B}) \oplus \ker(\mathsf{T}\pi_{\mathsf{N}B}), \tag{5.6}$$

where $\mathcal{K}_{\mathsf{N}B} : \mathsf{T}(\mathsf{N}B) \to \mathsf{N}B$ stands for the connection map induced by $\nabla^{\mathsf{N}B}$. Moreover, the restricted maps

$$\mathsf{T}\pi_{\mathsf{N}B}|_{\mathsf{H}_\nu \mathsf{N}B} : \mathsf{H}_\nu \mathsf{N}B \to \mathsf{T}_x B \quad , \quad \mathcal{K}_{\mathsf{N}B}|_{\mathsf{V}_\nu \mathsf{N}B} : \mathsf{V}_\nu \mathsf{N}B \to \mathsf{N}_x B$$

are vector space isomorphisms for all $\nu \in \mathsf{N}_x B$ and $x \in B$.

Let $\{x^i\}_{i=1}^b$ denote local coordinates on some $U \subset B$ and $\{e_j\}_{j=1}^f$ be a local orthonormal frame that trivialises $\mathsf{N}B|_U$. This yields bundle coordinates

$$\begin{aligned} &q^i := \pi_{\mathsf{N}B} \circ x^i, \quad && i = 1, \dots, b \\ &n^j, && j = 1, \dots, f \end{aligned}$$

on $\mathsf{N}B^r|_U$ in such a way that every $\nu = n^j e_j(x) \in \mathsf{N}_x B^r$ corresponds to a point $(x, n) \in U \times \mathbb{B}_r^f(0)$. Moreover, Remark A.8 states that the associated coordinate vector fields on $\mathsf{N}B^r|_U$ induce horizontal and vertical lifts

$$\begin{aligned} \widehat{\partial}_{x^i}^{\mathsf{H}} &= \partial_{q^i} - n^j \,\delta^{\mathsf{N}B}\big(\omega^{\mathsf{N}B}(\partial_{x^i}) e_j, e_{j'}\big)\, \partial_{n^{j'}}, \\ e_j^{\mathsf{V}} &= \partial_{n^j}, \end{aligned} \tag{5.7}$$

where $\omega^{\mathsf{N}B} \in C_{\mathrm{b}}^\infty\big((\mathsf{T}^*B \otimes \mathrm{End}(\mathsf{N}B))|_U\big)$ represents the connection one-form of $\nabla^{\mathsf{N}B}$ over U with respect to the local frame $\{e_j\}_{j=1}^f$. Since we will later have to deal with two different horizontal lifts (one with respect to $\mathsf{N}B^r \subset \mathsf{N}B \xrightarrow{\pi_{\mathsf{N}B}} B$ and the other with respect to $M \xrightarrow{\pi_M} B$), we indicate the former with an additional hat for the sake of a clarity.

Lemma 5.4 *The decomposition* (5.6) *is orthogonal with respect to* $\mathcal{G}^\varepsilon$ *for all* $0 < \varepsilon \leqslant 1$.

PROOF. Since $\mathsf{V}(\mathsf{N}B|_U) = \ker(\mathsf{T}\pi_{\mathsf{N}B}|_U)$ is spanned by $\{\partial_{n^j}\}_{j=1}^f$ (independently of $\mathcal{G}^\varepsilon$), it remains to check that the horizontal lift (5.7) is $\mathcal{G}^\varepsilon$-orthonormal to ∂_{n^j} for all $0 < \varepsilon \leqslant 1$. Therefore, we start by evaluating the differential $\mathsf{T}\Phi_\varepsilon$ on tangent vectors $w \in \mathsf{T}_\nu(\mathsf{N}B^r|_U)$ for $\nu = n^j e_j(x) \in \mathsf{N}_x B^r$. To do so, let $I \subset \mathbb{R}$ be a neighbourhood of zero, $b : I \to U$ be a smooth curve with $b(0) = x$ and

$$\gamma : I \to \mathsf{N}B^r|_U, \quad s \mapsto \mathfrak{n}^j(s) e_j\big(b(s)\big)$$

be a smooth curve with initial data $\mathfrak{n}^j(0) = n^j$ and $\dot\gamma(0) = w$. We then obtain

$$\begin{aligned}
\mathsf{T}\Phi_\varepsilon(w) &= \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{ \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \Phi_\varepsilon\big(\gamma(s)\big)\right\} \\
&= \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{ \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} c\big(b(s)\big) + \varepsilon\, \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \overline{\mathfrak{n}^j(s) e_j\big(b(s)\big)}\right\} \\
&= \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{ \overline{\dot b(0) + \varepsilon \dot{\mathfrak{n}}^j(0) e_j(x) + \varepsilon n^j \nabla_{\dot b(0)}^{\mathsf{T}B \oplus \mathsf{N}B} e_j(x)}\right\},
\end{aligned}$$

where $\mathfrak{p}_\nu^{\Phi_\varepsilon} : \mathsf{T}_{c(x)}\mathbb{R}^{b+f} \to \mathsf{T}_{\Phi_\varepsilon(\nu)}\mathbb{R}^{b+f}$ denotes the parallel transport of any tangent vector $\overline{w} \in \mathbb{R}^{b+f}$ from $c(x)$ to $\Phi_\varepsilon(\nu)$. Let us consider the computations for curves associated to ∂_{q^i} and ∂_{n^j} separately:

- Let $\gamma : I \to \mathsf{N}B^r|_U$ be the curve

  $$\gamma : s \mapsto n^j e_j\big(b(s)\big) \quad , \quad \dot b(0) = \partial_{x^i}.$$

  Then $\dot\gamma(0) = \partial_{q^i}$ and

  $$\begin{aligned}
  \mathsf{T}\Phi_\varepsilon(\partial_{q^i}) \;&=\; \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{\overline{\partial_{x^i} + \varepsilon n^j \nabla_{\partial_{x^i}}^{\mathsf{T}B \oplus \mathsf{N}B} e_\alpha(x)}\right\} \\
  &\overset{(A.9)}{=} \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{\overline{\partial_{x^i} + \varepsilon n^j \Big(-\mathcal{W}\big(\mathrm{e}_\alpha(x)\big)\partial_{x^i} + \nabla_{\partial_{x^i}}^{\mathsf{N}B} e_\alpha(x)\Big)}\right\}.
  \end{aligned}$$

- Denote by $\gamma : I \to \mathsf{N}B^r|_U$ the curve

$$\gamma : s \mapsto \left(n^{j'} + s\delta_j^{j'}\right) e_{j'}(x).$$

Thus, $\dot{\gamma}(0) = \partial_{n^j}$ and

$$\mathsf{T}\Phi_\varepsilon(\partial_{n^j}) = \mathfrak{p}_\nu^{\Phi_\varepsilon}\left\{\overline{\varepsilon e_j(x)}\right\}.$$

Now note that the parallel transport $\mathfrak{p}_\nu^{\Phi_\varepsilon}$ trivially preserves both lengths and angles (the mapping $\mathfrak{p}_\nu^{\Phi_\varepsilon}$ obviously coincides with the identity $\mathbf{1}_{\mathbb{R}^{b+f}}$ as an endomorphism of $\mathbb{R}^{b+f}$ and the metric δ^{b+f} is the the same on each fibre of $\mathsf{T}\mathbb{R}^{b+f}$). So we may simplify

$$\delta^{b+f}_{\Phi_\varepsilon(\nu)}\left(\mathfrak{p}_\nu^{\Phi_\varepsilon}\{\overline{w_1}\}, \mathfrak{p}_\nu^{\Phi_\varepsilon}\{\overline{w_2}\}\right) = \delta^{b+f}_{c(x)}\left(\overline{w_1}, \overline{w_2}\right) = \delta_x^{\mathsf{T}B\oplus\mathsf{N}B}(w_1, w_2)$$

for all $w_1, w_2 \in \mathsf{T}_x B \oplus \mathsf{N}_x B$ and $\nu \in \mathsf{N}_x B^r$. This implies

$$\begin{aligned}\mathcal{G}^\varepsilon\left(\partial_{q^i}, \partial_{n^j}\right) &= \varepsilon^{-2}\delta^{\mathsf{T}B\oplus\mathsf{N}B}\left(\partial_{x^i} - \varepsilon\mathcal{W}(\nu)\partial_{x^i}, \varepsilon\nabla^{\mathsf{N}B}_{\partial_{x^i}}\nu, \varepsilon e_j\right)\\ &= \delta^{\mathsf{N}B}\left(\nabla^{\mathsf{N}B}_{\partial_{x^i}}\nu, e_j\right)\end{aligned}$$

and

$$\mathcal{G}^\varepsilon\left(\partial_{n^{j'}}, \partial_{n^j}\right) = \varepsilon^{-2}\delta^{\mathsf{T}B\oplus\mathsf{N}B}(\varepsilon e_{j'}, \varepsilon e_j) = \delta^{\mathsf{N}B}(e_{j'}, e_j) = \delta_{j'j}.$$

Thus in view of (5.7):

$$\begin{aligned}\mathcal{G}^\varepsilon\left(\widehat{\partial}^{\mathsf{H}}_{x^i}, \partial_{n^j}\right) &= \delta^{\mathsf{N}B}\left(\nabla^{\mathsf{N}B}_{\partial_{x^i}}\nu, e_j\right) - \sum_{j'=1}^{f}\delta^{\mathsf{N}B}\left(\omega^{\mathsf{N}B}(\partial_{x^i})\nu, e_{j'}\right)\delta_{j'j}\\ &= \left(\omega^{\mathsf{N}B}(\partial_{x^i})\nu, e_j\right) - \left(\omega^{\mathsf{N}B}(\partial_{x^i})\nu, e_j\right)\\ &= 0.\end{aligned}$$

□

This directly leads to an explicit expression for the metric $\mathcal{G}^\varepsilon$:

Lemma 5.5 *The intermediate pullback metric $\mathcal{G}^\varepsilon$ on $\mathsf{N}B^r$ is given by*

$$\mathcal{G}^\varepsilon = \varepsilon^{-2}g_B\Big(\left(\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W}\right)\mathsf{T}\pi_{\mathsf{N}B}\cdot, \left(\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W}\right)\mathsf{T}\pi_{\mathsf{N}B}\cdot\Big)\circ\pi_{\mathsf{N}B} + \delta^{\mathsf{V}(\mathsf{N}B)}$$

with vertical bundle metric $\delta^{\mathsf{V}(\mathsf{N}B)} := \delta^{\mathsf{N}B}(\mathcal{K}_{\mathsf{N}B}\cdot, \mathcal{K}_{\mathsf{N}B}\cdot)$.

PROOF. It suffices to evaluate $\mathcal{G}^\varepsilon$ merely for the horizontal and vertical diagonal blocks due to the previous lemma. The computations there imply

$$\mathsf{T}\Phi_\varepsilon\big(\widehat{\partial}^{\mathsf{H}}_{x^i}\big) = \mathfrak{p}^{\Phi_\varepsilon}_\nu\left\{\overline{\big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W}(\nu)\big)\,\partial_{x^i}}\right\},$$

which shows that the horizontal part of $\mathcal{G}^\varepsilon$ is given by

$$\mathcal{G}^\varepsilon\big(\widehat{X}^{\mathsf{H}}, \widehat{Y}^{\mathsf{H}}\big) = \varepsilon^{-2} g_B\big((\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W})X, \big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W})Y\big)$$

for all $X, Y \in C^\infty(\mathsf{T}B)$. As far as the vertical part is concerned, we simply have $\mathcal{G}^\varepsilon(V, W) = \delta^{\mathsf{N}B}(\mathcal{K}_{\mathsf{N}B}V, \mathcal{K}_{\mathsf{N}B}W)$ for arbitrary $V, W \in C^\infty(\mathsf{V}(\mathsf{N}B^r))$. $\square$

We observe that the leading term of $\mathcal{G}^\varepsilon$ equals the rescaled Sasaki metric (cf. Definition A.9)

$$\begin{aligned} g^\varepsilon_{\mathsf{N}B} &= \varepsilon^{-2} g_B\big(\mathsf{T}\pi_{\mathsf{N}B}\cdot, \mathsf{T}\pi_{\mathsf{N}B}\cdot\big) \circ \pi_{\mathsf{N}B} + \delta^{\mathsf{N}B}\big(\mathcal{K}_{\mathsf{N}B}\cdot, \mathcal{K}_{\mathsf{N}B}\cdot\big) \\ &= \varepsilon^{-2}\pi^*_{\mathsf{N}B} g_B + \delta^{\mathsf{V}(\mathsf{N}B)}, \end{aligned}$$

which turns $(\mathsf{N}B, g^\varepsilon_{\mathsf{N}B}) \xrightarrow{\pi_{\mathsf{N}B}} (B, \varepsilon^{-2} g_B)$ into a Riemannian submersion with totally geodesic fibres.

Example 5.6 Let us elaborate on the situation for conventional quantum tubes as depicted in Figure 5.2: Let $(B, g_B) = (\mathbb{R}, \mathrm{d}x \otimes \mathrm{d}x)$ and $c : \mathbb{R} \to \mathbb{R}^3$ be a smoothly embedded curve in $(\mathbb{R}^3, \delta^3)$ that is bounded with all its derivatives and parametrised by arc length. We pick an orthonormal basis $\{\overline{\tau_0}, \overline{e_{1,0}}, \overline{e_{2,0}}\}$ of $\mathsf{T}_{c(0)}\mathbb{R}^3 \cong \mathbb{R}^3$ such that $\overline{\tau_0} = c'(0)$ is tangent and $\{\overline{e_{1,0}}, \overline{e_{2,0}}\}$ are normal to the curve at $c(0)$. We then obtain the so-called relatively parallel adapted frame $\{\overline{\tau}, \overline{e_1}, \overline{e_2}\}$ as the solution of the coupled system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \overline{\tau(x)} \\ \overline{e_1(x)} \\ \overline{e_2(x)} \end{pmatrix} = \begin{pmatrix} 0 & \kappa^1(x) & \kappa^2(x) \\ -\kappa^1(x) & 0 & 0 \\ -\kappa^2(x) & 0 & 0 \end{pmatrix}\begin{pmatrix} \overline{\tau(x)} \\ \overline{e_1(x)} \\ \overline{e_2(x)} \end{pmatrix}$$

with initial data $\{\overline{\tau(0)}, \overline{e_1(0)}, \overline{e_2(0)}\} = \{\overline{\tau_0}, \overline{e_{1,0}}, \overline{e_{2,0}}\}$, where

$$\kappa^j : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \kappa^j(x) := \delta^3\big(c''(x), \overline{e_j(x)}\big) \text{ for } j \in \{1, 2\}$$

are the mean curvatures of the curve [Bis75]. This is an orthonormal frame of $(\mathsf{T}\mathbb{R}^3|_{c(\mathbb{R})}, \delta^3)$ with $\overline{\tau(x)} = c'(x)$ for all $x \in \mathbb{R}$ and corresponds via (5.4) to an orthonormal frame $\{\tau, e_1, e_2\}$ of $(\mathsf{T}\mathbb{R} \oplus \mathsf{N}\mathbb{R}, \delta^{\mathsf{T}\mathbb{R}\oplus\mathsf{N}\mathbb{R}})$ obtained by the parallel transport of the orthonormal basis $\{\tau_0, e_{1,0}, e_{2,0}\}$ of $\mathsf{T}_0\mathbb{R} \oplus \mathsf{N}_0\mathbb{R}$ along $\mathbb{R}$ with respect to the induced connection $c^*\nabla^{\delta^3} = \nabla^{\mathsf{T}\mathbb{R}\oplus\mathsf{N}\mathbb{R}}$. Consequently, $\tau(x) = \partial_x$ for all $x \in \mathbb{R}$ trivialises $\mathbb{R}$'s tangent bundle, whereas $e_1, e_2 : \mathbb{R} \to \mathbb{R}^2$ form an orthonormal frame of $\mathsf{N}\mathbb{R} \cong \mathbb{R} \times \mathbb{R}^2$, so that

$$\begin{aligned}\mathsf{N}\mathbb{R}^r &= \Big\{\underbrace{n^1 e_1(x) + n^2 e_2(x)}_{=\nu\in\mathsf{N}_x\mathbb{R}} \in \mathsf{N}\mathbb{R} \text{ such that } \underbrace{\sqrt{(n^1)^2 + (n^2)^2}}_{=\|\nu\|_{\delta^{\mathsf{N}\mathbb{R}}}} < r\Big\}\\ &\cong \mathbb{R} \times \mathbb{B}^2_r(0)\end{aligned}$$

yields bundle coordinates (x, n^1, n^2). We then identify $\mathsf{T}_{(x,n)}\mathbb{R}^r = \mathsf{T}_{(x,n)}\mathbb{R}$ with $\mathsf{N}\mathbb{R} \cong \mathbb{R}^3$ by means of (A.7):

$$\partial_q\big|_{(x,n)} = (1,0,0)^{\mathrm{t}} \quad , \quad \partial_{n^1}\big|_{(x,n)} = (0,1,0)^{\mathrm{t}} \quad , \quad \partial_{n^2}\big|_{(x,n)} = (0,0,1)^{\mathrm{t}}.$$

Here, we associated $x \in \mathbb{R}$ with $q(x) = (x,0,0) \in \mathsf{N}\mathbb{R}$. The normal connection $\nabla^{\mathsf{N}\mathbb{R}}$ is flat due to the fact that $\dim(\mathbb{R}) = 1$ ($\omega^{\mathsf{N}\mathbb{R}} = 0$), and hence the horizontal lift $\widehat{\partial}^{\mathsf{H}}_x|_{(x,n)}$ of $\mathsf{T}_x\mathbb{R} \ni \partial_x|_x = 1 \in \mathbb{R}$ coincides with $\partial_q\big|_{(x,n)}$ and we obtain the vector space isomorphisms

$$\begin{aligned}\mathsf{T}\pi_{\mathsf{N}\mathbb{R}}|_{\mathsf{H}_{(x,n)}\mathsf{N}\mathbb{R}} &: (1,0,0)^{\mathrm{t}} \mapsto 1,\\ \mathcal{K}_{\mathsf{N}\mathbb{R}}|_{\mathsf{V}_{(x,n)}\mathsf{N}\mathbb{R}} &: (0,1,0)^{\mathrm{t}} \mapsto (1,0)^{\mathrm{t}} \text{ and } (0,0,1)^{\mathrm{t}} \mapsto (0,1)^{\mathrm{t}}.\end{aligned}$$

Apart from that, the action of the rescaled diffeomorphism Φ_ε and the Weingarten map $\mathcal{W}$ on $\nu = n^j e_j(x) \in \mathsf{N}_x B^r$ are given by

$$\Phi_\varepsilon : \nu \mapsto c(x) + \varepsilon\Big(n^1\overline{e_1(x)} + n^2\overline{e_2(x)}\Big)$$

and

$$\mathcal{W} : \nu \mapsto -\langle n, \kappa(x)\rangle_{\mathbb{R}^2}\, \mathrm{d}x \otimes \partial_x,$$

where $\kappa(x) = \kappa^j(x)e_j(x) \in \mathsf{N}_x\mathbb{R}$ is the mean curvature vector. Finally, the induced metric according to Lemma 5.5 is given by

$$\begin{aligned}\mathcal{G}^\varepsilon|_{(x,n)} &= \varepsilon^{-2}\big(1-\varepsilon\langle n,\kappa(x)\rangle_{\mathbb{R}^2}\big)^2\, \mathrm{d}\hat{x}^{\mathsf{H}} \otimes \mathrm{d}\hat{x}^{\mathsf{H}} \\ &\quad + \underbrace{\mathrm{d}n^1 \otimes \mathrm{d}n^1 + \mathrm{d}n^2 \otimes \mathrm{d}n^2}_{=:\delta^{\mathsf{V}(\mathsf{N}\mathbb{R})}},\end{aligned}$$

where we used the notation $\mathrm{d}\hat{x}^{\mathsf{H}} := \pi^*_{\mathsf{N}B}\mathrm{d}x = \mathrm{d}q$.

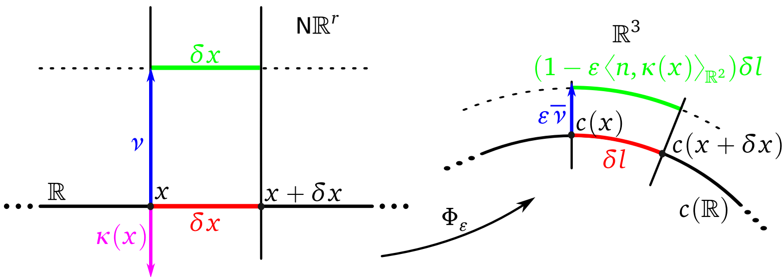


Figure 5.4: The images of horizontal, equally long curves in $\mathsf{N}\mathbb{R}^r$ under Φ_ε yield variably long curves in $\mathbb{R}^3$.

The additional factor $1-\varepsilon\langle n,\kappa\rangle_{\mathbb{R}^2}$ in the horizontal block of $\mathcal{G}^\varepsilon$ reflects the fact that the tube in $\mathbb{R}^3$ is either stretched or compressed due to the curvature of the curve c (see Figure 5.4). $\Diamond$

Now that we have gained a better understanding of the intermediate Riemannian metric $\mathcal{G}^\varepsilon$ on $\mathsf{N}B^r$, we may incorporate the embedding $\varpi : M \to \mathsf{N}B^r$ and determine the ultimate metric $G^\varepsilon = \varpi^*\mathcal{G}^\varepsilon$. Therefore, we distinguish between two different situations depending on the codimension $\operatorname{codim}(\varpi)$ of the submanifold $\varpi(M)$ in $\mathsf{N}B^r$:

Vanishing Codimension

If the codimension of the initial ε-tubes $\mathcal{T}^\varepsilon$ in $\mathbb{R}^{b+f}$ is zero, the typical fibre F of M is given by the closure of an open, bounded and connected subset of $\mathbb{B}^f_r(0) \subset \mathbb{R}^f$ with smooth boundary and the waveguide M is called *massive* [HLT15, Definition 2.2(1)]). In this case, the restriction $G^\varepsilon = \varpi^* \mathcal{G}^\varepsilon$ is straightforward and its leading term

$$g^\varepsilon = \varepsilon^{-2} \pi_M^* g_B + \varpi^* \delta^{\mathsf{V}(\mathsf{N}B)}$$

turns out to be a rescaled Riemannian submersion with totally geodesic fibres.

Example 5.7 We apply the preliminary work of Example 5.6 to a special class of massive conventional quantum tubes. More precisely, we consider a family of tubes $\{\mathcal{T}^\varepsilon\}_{0<\varepsilon\leqslant 1}$ around an embedded curve $c \in C^\infty_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^3)$ parametrised by arc length, where the respective cross-sections at $c(x)$ are given by elliptical discs with semi-major axis $\varepsilon\mathfrak{a}(x)$ and semi-minor axis $\varepsilon\mathfrak{b}(x)$ that twist relatively to the normal frame $\{\overline{e_1}, \overline{e_2}\}$ by an angle $\vartheta(x)$. Here, we assume that $\mathfrak{a}, \mathfrak{b} : \mathbb{R} \to [r_-, r_+]$ with $0 < r_- < r_+ < r$ and $\vartheta : \mathbb{R} \to [0, 2\pi)$ are smooth and bounded with all their derivatives. Then the associated waveguide $M = \mathbb{R} \times \mathbb{B}^2_1(0) \xrightarrow{\mathrm{pr}_1} \mathbb{R}$ is of bounded geometry (see Definition 5.3) and smoothly embedded into $\mathsf{N}\mathbb{R}^r$ via the fibrewise mapping

$$\varpi : M \to \mathsf{N}\mathbb{R}^r, \quad (x, y^1, y^2) \mapsto \big(\mathfrak{r}(x) y\big)^1 e_1(x) + \big(\mathfrak{r}(x) y\big)^2 e_2(x),$$

where we introduced the rotation-dilation matrix $\mathfrak{r} : \mathbb{R} \to \mathbb{R}^{2\times 2}$ given by

$$x \mapsto \begin{pmatrix} \mathfrak{r}_{11}(x) & \mathfrak{r}_{12}(x) \\ \mathfrak{r}_{21}(x) & \mathfrak{r}_{22}(x) \end{pmatrix} := \begin{pmatrix} \mathfrak{a}(x) \cos\big(\vartheta(x)\big) & -\mathfrak{b}(x) \sin\big(\vartheta(x)\big) \\ \mathfrak{a}(x) \sin\big(\vartheta(x)\big) & \mathfrak{b}(x) \cos\big(\vartheta(x)\big) \end{pmatrix}.$$

We start with the vertical part $g_{\mathsf{V}} = \varpi^* \delta^{\mathsf{V}(\mathsf{N}\mathbb{R})}$ of the metric. A small calculation shows

$$\begin{aligned} \mathsf{T}\varpi(\partial_{y^1}) &= \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \varpi(x, y^1 + s, y^2) = \mathfrak{r}_{11}(x)\, \partial_{n^1} + \mathfrak{r}_{21}(x)\, \partial_{n^2}, \\ \mathsf{T}\varpi(\partial_{y^2}) &= \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \varpi(x, y^1, y^2 + s) = \mathfrak{r}_{12}(x)\, \partial_{n^1} + \mathfrak{r}_{22}(x)\, \partial_{n^2} \end{aligned}$$

for the respective vectors at $\varpi(x, y) \in \mathsf{N}\mathbb{R}^r$. Hence, we get

$$G^\varepsilon\big(\partial_{y^1}, \partial_{y^1}\big) = \mathcal{G}^\varepsilon\big(\mathsf{T}\varpi(\partial_{y^1}), \mathsf{T}\varpi(\partial_{y^1})\big) = \mathfrak{r}_{11}^2 + \mathfrak{r}_{21}^2 = \mathfrak{a}^2$$

using $\mathcal{G}^\varepsilon(\partial_{n^j}, \partial_{n^{j'}}) = \delta_{jj'}$ and similarly

$$G^\varepsilon\big(\partial_{y^2}, \partial_{y^2}\big) = \mathfrak{b}^2 \quad , \quad G^\varepsilon\big(\partial_{y^1}, \partial_{y^2}\big) = 0 = G^\varepsilon\big(\partial_{y^2}, \partial_{y^1}\big).$$

As far as the horizontal part is concerned, we first apply the differential $\mathsf{T}\varpi$ on the product lift $\partial_x^{\mathrm{pr}}\big|_{(x,y)} = (1,0)^{\mathrm{t}} \in \mathsf{T}_{(x,y)}M$ of $\partial_x|_x = 1 \in \mathsf{T}_x\mathbb{R}$:

$$\begin{aligned}\mathsf{T}\varpi(\partial_x^{\mathrm{pr}}) &= \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\varpi(x+s, y^1, y^2) \\ &= \widehat{\partial}_x^{\mathsf{H}} + \big(\mathfrak{r}'(x)y\big)^1 \partial_{n^1} + \big(\mathfrak{r}'(x)y\big)^2 \partial_{n^2},\end{aligned}$$

where the prime denotes differentiation with respect to x. This in fact shows that ∂_x^{pr} does not coincide with the horizontal lift $\widehat{\partial}_x^{\mathsf{H}}$ of ∂_x.

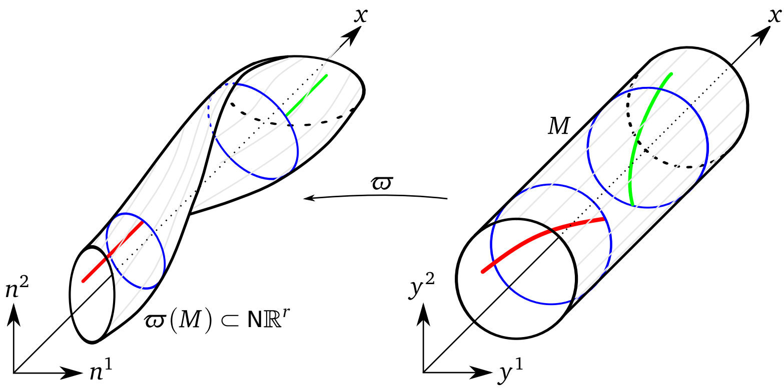


Figure 5.5: Relation between horizontal curves in $\mathsf{N}\mathbb{R}^r$ and M. The curves in the fixed waveguide M are twisted clockwise because one considers the straight lines in $\mathsf{N}\mathbb{R}^r$ from the perspective of the anti-clockwise twisted set $\varpi(M)$.

Yet we can construct the ε-independent horizontal lift ∂_x^{H} from ∂_x^{pr} by subtracting the G-vertical component:

$$\begin{aligned}\partial_x^{\mathsf{H}} &= \partial_x^{\mathrm{pr}} - G^{jj'}\, G\big(\partial_x^{\mathrm{pr}}, \partial_{y^j}\big)\, \partial_{y^{j'}} \\ &= \partial_x^{\mathrm{pr}} - \mathfrak{a}^{-2}\, \mathcal{G}\big(\mathsf{T}\varpi(\partial_x^{\mathrm{pr}}), \mathsf{T}\varpi(\partial_{y^1})\big)\, \partial_{y^1} \\ &\qquad - \mathfrak{b}^{-2}\, \mathcal{G}\big(\mathsf{T}\varpi(\partial_x^{\mathrm{pr}}), \mathsf{T}\varpi(\partial_{y^2})\big)\, \partial_{y^2}\end{aligned}$$

$$
\begin{aligned}
&= \partial_x^{\mathrm{pr}} - \tfrac{1}{\mathfrak{a}}\Big((\mathfrak{r}'y)^1 \cos\vartheta + (\mathfrak{r}'y)^2 \sin\vartheta\Big)\partial_{y^1} \\
&\qquad - \tfrac{1}{\mathfrak{b}}\Big(-(\mathfrak{r}'y)^1 \sin\vartheta + (\mathfrak{r}'y)^2 \cos\vartheta\Big)\partial_{y^2} \\
&= \partial_x^{\mathrm{pr}} - \underbrace{\begin{pmatrix}(\ln a)'y^1 \\ (\ln b)'y^2\end{pmatrix} \cdot \nabla_y}_{\substack{\text{variation of the}\\ \text{cross-section}}} - \underbrace{\vartheta' \begin{pmatrix}\frac{a}{b}y^1 \\ \frac{b}{a}y^2\end{pmatrix} \times \nabla_y}_{\substack{\text{variation of the}\\ \text{twisting angle}}},
\end{aligned}
\tag{5.8}
$$

where we introduced the notations for the inner product

$$
\begin{pmatrix}v^1 \\ v^2\end{pmatrix} \cdot \begin{pmatrix}w^1 \\ w^2\end{pmatrix} = v^1 w^1 + v^2 w^2
$$

and the cross product

$$
\begin{pmatrix}v^1 \\ v^2\end{pmatrix} \times \begin{pmatrix}w^1 \\ w^2\end{pmatrix} = v^1 w^2 - v^2 w^1
\tag{5.9}
$$

for vectors $v, w \in \mathbb{R}^2$. Equation (5.8) yields a separation of the effects induced by the variation of the ellipses (in terms of $\mathfrak{a}'$ and $\mathfrak{b}'$) and of the twist (in terms of ϑ') along the curve. Finally, it follows from the construction that $G^\varepsilon(\partial_x^{\mathsf{H}}, \partial_{n^j}) = 0$ for $j \in \{1, 2\}$, and also

$$
\begin{aligned}
G^\varepsilon\left(\partial_x^{\mathsf{H}}, \partial_x^{\mathsf{H}}\right)\big|_{(x,y)} &= \mathcal{G}^\varepsilon\left(\widehat{\partial}_x^{\mathsf{H}}, \widehat{\partial}_x^{\mathsf{H}}\right)\Big|_{(x, n = \mathfrak{r}(x)y)} \\
&= \varepsilon^{-2}\Big(1 - \varepsilon\big\langle \mathfrak{r}(x)y, \kappa(x)\big\rangle_{\mathbb{R}^2}\Big)^2,
\end{aligned}
$$

and the rescaled pullback metric reads

$$
\begin{aligned}
G^\varepsilon|_{(x,y)} &= \varepsilon^{-2}\Big(1 - \varepsilon\big\langle \mathfrak{r}(x)y, \kappa(x)\big\rangle_{\mathbb{R}^2}\Big)^2 \mathrm{d}x^{\mathsf{H}} \otimes \mathrm{d}x^{\mathsf{H}} \\
&\quad + \underbrace{\mathfrak{a}^2(x)\,\mathrm{d}y^1 \otimes \mathrm{d}y^1 + \mathfrak{b}^2(x)\,\mathrm{d}y^2 \otimes \mathrm{d}y^2}_{=: g_{M_x}}
\end{aligned}
\tag{5.10}
$$

for $(x, y) \in \mathbb{R} \times \mathbb{B}_1^2(0)$, where $\mathrm{d}x^{\mathsf{H}} := \mathrm{pr}_1^* \,\mathrm{d}x$. We observe that the fibres $M_x = \mathbb{B}_1^2(0)$ are x-dependent in terms of the respective Riemannian metrics $g_{M_x} = g_{\mathsf{V}}|_{M_x}$. ◊

Codimension greater than or equal to one

If the codimension of $\varpi(M)$ in $\mathsf{N}B^r$ does not vanish, we have to introduce further decompositions apart from (5.4):

- We denote by

$$\mathsf{T}(\mathsf{N}B^r)|_{\varpi(M)} = \mathsf{T}\varpi(\mathsf{T}M) \oplus \mathsf{N}\big(\varpi(M)\big) \tag{5.11}$$

  the orthogonal decomposition of the tangent bundle of $\mathsf{N}B^r$ restricted to $\varpi(M)$ with respect to $\mathcal{G} := \mathcal{G}^{\varepsilon=1}$.

- We decompose the tangent bundle of M orthogonally into

$$\mathsf{T}M = \mathsf{H}M \oplus \mathsf{V}M$$

  with respect to G^ε. This decomposition (more precisely, the horizontal subbundle $\mathsf{H}M$) does not depend on ε [Lam14, Lemma 3.3].

- Note that $\mathsf{T}\varpi(\mathsf{V}M) \subset \mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)}$ since $\pi_M = \pi_{\mathsf{N}B} \circ \varpi$ (i.e., it holds that $\varpi(M_x) \subset \mathsf{N}_x B^r$ for all $x \in B$). This suggests the introduction of the corresponding orthogonal complement with respect to the bundle metric $\delta^{\mathsf{V}(\mathsf{N}B)}$, so that

$$\mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)} = \mathsf{T}\varpi(\mathsf{V}M) \oplus \underbrace{\big(\mathsf{T}\varpi(\mathsf{V}M)\big)^{\perp,\delta^{\mathsf{V}(\mathsf{N}B)}}}_{=:\mathsf{V}M^\perp}. \tag{5.12}$$

By virtue of the fact that

$$\mathsf{T}\pi_{\mathsf{N}B}\Big(\mathsf{T}\varpi(X^\mathsf{H}) - \widehat{X}^\mathsf{H}\big|_{\varpi(M)}\Big) = \mathsf{T}\pi_M(X^\mathsf{H}) - \mathsf{T}\pi_{\mathsf{N}B}\big(\widehat{X}^\mathsf{H}\big) = X - X = 0$$

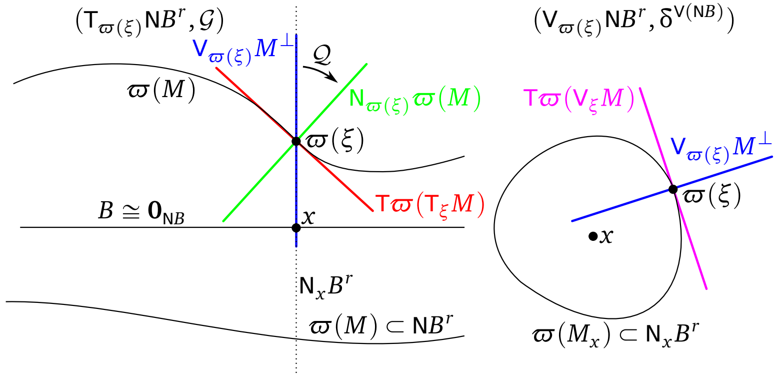
for all $X \in C^\infty(\mathsf{T}B)$, the difference $\mathsf{T}\varpi(X^\mathsf{H}) - \widehat{X}^\mathsf{H}\big|_{\varpi(M)}$ between the horizontal lifts is an element of the kernel of $\mathsf{T}\pi_{\mathsf{N}B}$ and hence a vertical field:

$$\mathsf{T}\varpi(X^\mathsf{H}) = \widehat{X}^\mathsf{H}\big|_{\varpi(M)} + \daleth(X), \quad \daleth(X) \in C^\infty\big(\mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)}\big). \tag{5.13}$$

Moreover,

$$
\begin{aligned}
0 = G^{\varepsilon}(X^{\mathsf{H}}, V) &= \mathcal{G}^{\varepsilon}\left(\mathsf{T}\varpi(X^{\mathsf{H}}), \mathsf{T}\varpi(V)\right) \\
&= \mathcal{G}^{\varepsilon}\left(\widehat{X}^{\mathsf{H}}\big|_{\varpi(M)} + \daleth(X), \mathsf{T}\varpi(V)\right) = \delta^{\mathsf{V}(\mathsf{N}B)}\left(\daleth(X), \mathsf{T}\varpi(V)\right)
\end{aligned}
$$

for all $V \in C^{\infty}(\mathsf{V}M)$, which shows that $\daleth(X)$ is already a smooth section of $\mathsf{V}M^{\perp}$.



(a) Decomposition (5.11) (b) Decomposition (5.12)

Figure 5.6: Sketch of the aforementioned orthogonal decompositions at the point $\varpi(\xi) \in \varpi(M_x) \subset \mathsf{N}_x B^r$. The isomorphism (A.7) allows for the identifications (a) of $\mathsf{N}B^r$ with its tangent space $\mathsf{T}_{\varpi(\xi)}\mathsf{N}B^r$ and (b) of $\mathsf{N}_x B^r$ with its tangent space $\mathsf{T}_{\varpi(\xi)}\mathsf{N}_x B^r = \mathsf{V}_{\varpi(\xi)}\mathsf{N}B^r$.

Next we use [Lam14, Lemma 3.2], which asserts that the fibrewise projection $\mathcal{Q} : \mathsf{T}(\mathsf{N}B^r)|_{\varpi(M)} \to \mathsf{N}(\varpi(M))$ restricted to $\mathsf{V}M^{\perp}$ is a bundle isomorphism, in order to obtain a formula for $\daleth \in C^{\infty}(\mathsf{T}^*B) \otimes C^{\infty}(\mathsf{V}M^{\perp})$. We first note that $\mathsf{T}\varpi(X^{\mathsf{H}}) \in C^{\infty}(\mathsf{T}\varpi(\mathsf{T}M))$ implies

$$
0 \overset{(5.11)}{=} \mathcal{Q}\left(\mathsf{T}\varpi(X^{\mathsf{H}})\right) \overset{(5.13)}{=} \mathcal{Q}\left(\widehat{X}^{\mathsf{H}}\big|_{\varpi(M)}\right) + \mathcal{Q}\left(\daleth(X)\right)
$$

and thus

$$\daleth(X) = -\mathcal{Q}^{-1}\mathcal{Q}\Big(\widehat{X}^{\mathsf{H}}\big|_{\varpi(M)}\Big). \tag{5.14}$$

Finally, the uniform local triviality of M implies that $\daleth(X) \in C^{\infty}_{\mathrm{b}}(\mathsf{V}M^{\perp})$ if X is a C^{∞}-bounded vector field on B.

Now that we have thoroughly examined the relation between the respective horizontal lifts X^{H} and $\widehat{X}^{\mathsf{H}}$, we may sum up our new insights as follows:

Proposition 5.8 *The ultimate pullback metric G^{ε} on M is given by*

$$\begin{aligned} G^{\varepsilon} = \varepsilon^{-2}\Big[& g_B\Big(\big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\varpi^*\mathcal{W}\big)\mathsf{T}\pi_M\cdot, \big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\varpi^*\mathcal{W}\big)\mathsf{T}\pi_M\cdot\Big) \circ \pi_M \\ & + \varepsilon^2 \delta^{\mathsf{V}(\mathsf{N}B)}\big(\daleth \circ \mathsf{T}\pi_M\cdot, \daleth \circ \mathsf{T}\pi_M\cdot\big)\Big] \\ & + \varpi^*\delta^{\mathsf{V}(\mathsf{N}B)}, \end{aligned}$$

where $\daleth$ is defined by expression (5.14).

Proof. While the horizontal block of G^{ε} is seen to be

$$\begin{aligned} G^{\varepsilon}\big(X^{\mathsf{H}}, Y^{\mathsf{H}}\big) &= \mathcal{G}^{\varepsilon}\big(\mathsf{T}\varpi(X^{\mathsf{H}}), \mathsf{T}\varpi(Y^{\mathsf{H}})\big) \\ &\overset{(5.13)}{=} \mathcal{G}^{\varepsilon}\Big(\widehat{X}^{\mathsf{H}}\big|_{\varpi(M)} + \daleth(X), \widehat{Y}^{\mathsf{H}}\big|_{\varpi(M)} + \daleth(Y)\Big) \\ &= \varepsilon^{-2} g_B\big((\mathbf{1}_{\mathsf{T}B} - \varepsilon\varpi^*\mathcal{W})X, \big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\varpi^*\mathcal{W})Y\big) \\ &\quad + \delta^{\mathsf{V}(\mathsf{N}B)}\big(\daleth(X), \daleth(Y)\big) \end{aligned}$$

for all $X, Y \in C^{\infty}(\mathsf{T}B)$, the vertical metric is simply obtained by the fibrewise restriction of $\delta^{\mathsf{V}(\mathsf{N}B)}$ to $\mathsf{T}\varpi(\mathsf{V}M) \subset \mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)}$. □

This proposition shows that leading term of G^{ε} is given by the rescaled Riemannian submersion metric (2.7) with $g_{\mathsf{V}} := \varpi^*\delta^{\mathsf{V}(\mathsf{N}B)}$ as vertical bundle metric. The remainder is non-zero on the $\mathsf{H}M$-$\mathsf{H}M$-block alone, where it is formally given by

$$\big((\mathbf{1}_{b\times b} - \varepsilon\mathcal{W})^2 + \varepsilon^2\daleth\daleth^{\mathrm{t}}\big) - \mathbf{1}_{b\times b} = \varepsilon\big(-2\mathcal{W} + \varepsilon\mathcal{W}^2 + \varepsilon\daleth\daleth^{\mathrm{t}}\big) \tag{5.15}$$

with respect to an adapted local orthonormal frame (cf. Definition 2.8). Here, $\mathcal{W} \in \mathbb{R}^{b \times b}$ and $\daleth \in \mathbb{R}^{b \times \operatorname{codim}(\varpi)}$ denote the respective matrix representations (with respect to an g_B-orthonormal frame) with smooth and uniformly bounded coefficients. Thus, $G^\varepsilon = g^\varepsilon + \mathcal{O}(\varepsilon)$ is an admissible perturbation of g^ε in the sense of Definition 2.21 and the Radon-Nikodym density $\rho_\varepsilon = \mathrm{vol}_{G^\varepsilon} / \mathrm{vol}_{g^\varepsilon}$ formally reads

$$\begin{aligned}\rho_\varepsilon &= \sqrt{\det\Big(\mathbf{1}_{b\times b} + \varepsilon\big(-2\mathcal{W} + \varepsilon\mathcal{W}^2 + \varepsilon\daleth\daleth^{\mathrm{t}}\big)\Big)} \\ &= \sqrt{1 + \varepsilon \operatorname{tr}\big(-2\mathcal{W} + \varepsilon\mathcal{W}^2 + \varepsilon\daleth\daleth^{\mathrm{t}}\big) + \mathcal{O}(\varepsilon^2)} \\ &= 1 + \mathcal{O}(\varepsilon)\end{aligned} \tag{5.16}$$

with errors in $C^\infty_{\mathrm{b}}(M)$.

Relation (5.14) can be made concrete in the case where the codimension of $\varpi(M)$ in $\mathsf{N}B^r$ equals to one. This situation occurs for example if one considers *hollow* quantum waveguides [HLT15, Definition 2.2(2)], where $\operatorname{rank}(\mathsf{N}B) \geqslant 2$ and the bundle $M \xrightarrow{\pi_M} B$ is given by the fibrewise boundary of a massive waveguide. In this case, $\mathsf{N}(\varpi(M))$ is a line bundle and trivialised by a unit (outer-pointing) vector field N normal to $\varpi(M)$. A decomposition $N = N_{\mathsf{H}} + N_{\mathsf{V}}$ into a horizontal field $N_{\mathsf{H}} \in C^\infty(\mathsf{H}(\mathsf{N}B^r)|_{\varpi(M)})$ and a vertical field $N_{\mathsf{V}} \in C^\infty(\mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)})$ then gives

$$\delta^{\mathsf{V}(\mathsf{N}B)}\big(\mathsf{T}\varpi(W), N_{\mathsf{V}}\big) = \mathcal{G}\big(\mathsf{T}\varpi(W), N_{\mathsf{V}}\big) \overset{(5.12)}{=} \mathcal{G}\big(\mathsf{T}\varpi(W), N\big) \overset{(5.11)}{=} 0$$

for all $W \in C^\infty(\mathsf{V}M)$. Hence, $N_{\mathsf{V}} = \mathcal{Q}^{-1}N \in C^\infty(\mathsf{V}M^\perp)$ is a non-vanishing vector field by [Lam14, Lemma 3.2] and

$$\daleth(X) = \frac{\mathcal{G}\big(\daleth(X), N_{\mathsf{V}}\big)}{\mathcal{G}(N_{\mathsf{V}}, N_{\mathsf{V}})} N_{\mathsf{V}} \overset{(5.13)}{=} -\frac{\mathcal{G}\big(\widehat{X}^{\mathsf{H}}|_{\varpi(M)}, N_{\mathsf{H}}\big)}{\mathcal{G}(N_{\mathsf{V}}, N_{\mathsf{V}})} N_{\mathsf{V}} \tag{5.17}$$

for all $X \in C^\infty(\mathsf{T}B)$.

Example 5.9 Let us return to the setting of Example 5.6 and consider the trivial bundle $M = \mathbb{R} \times \mathbb{S}^1 \xrightarrow{\mathrm{pr}_1} \mathbb{R}$ over $B = \mathbb{R}$ with circle $\mathbb{S}^1$ as (typical) fibre.

We then introduce polar coordinates $(x, \rho, \varphi) \in \mathbb{R} \times (0, r) \times (0, 2\pi)$ for $\mathsf{N}\mathbb{R}^r = \mathbb{R} \times \mathbb{B}^2_r(0)$. This choice yields coordinate vector fields $\{\partial_q, \partial_\rho, \partial_\varphi\}$ on $\mathsf{N}\mathbb{R}^r$, where we again set $q(x) = (x, 0, 0)$ for $x \in \mathbb{R}$. The identification (A.7) of $\mathsf{T}_{(x,\rho,\varphi)}\mathsf{N}\mathbb{R}$ with $\mathsf{N}\mathbb{R} \cong \mathbb{R}^3$ for each $(x, \rho, \varphi) \in \mathsf{N}\mathbb{R}^r$ provides an orthogonal basis

$$\begin{aligned}
\partial_q\big|_{(x,\rho,\varphi)} &= (1, 0, 0)^{\mathrm{t}}, \\
\partial_\rho\big|_{(x,\rho,\varphi)} &= (0, \cos\varphi, \sin\varphi)^{\mathrm{t}}, \\
\partial_\varphi\big|_{(x,\rho,\varphi)} &= (0, -\rho\sin\varphi, \rho\cos\varphi)^{\mathrm{t}}
\end{aligned}$$

such that $\mathsf{H}(\mathsf{N}\mathbb{R}^r) = \operatorname{span}(\partial_q)$ and $\mathsf{V}(\mathsf{N}\mathbb{R}^r) = \operatorname{span}(\partial_\rho, \rho^{-1}\partial_\varphi)$. Consequently, the unscaled intermediate metric $\mathcal{G}$ on $\mathsf{N}\mathbb{R}^r$ reads

$$\mathcal{G}|_{(x,\rho,\varphi)} = \big(1 - \rho\kappa^\rho(x, \varphi)\big)^2 \, \mathrm{d}\hat{x}^{\mathsf{H}} \otimes \mathrm{d}\hat{x}^{\mathsf{H}} + \underbrace{\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi}_{=\delta^{\mathsf{V}(\mathsf{N}\mathbb{R})}},$$

where

$$\kappa^\rho = \big\langle \kappa, e_\rho \big\rangle_{\mathbb{R}^2} = \left\langle \begin{pmatrix} \kappa^1 \\ \kappa^2 \end{pmatrix}, \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \right\rangle_{\mathbb{R}^2} = \kappa^1 \cos\varphi + \kappa^2 \sin\varphi$$

is the radial component of the mean curvature vector. Next we embed M into $\mathsf{N}\mathbb{R}^r$ according to

$$\varpi : \mathbb{R} \times \mathbb{S}^1 \to \mathsf{N}\mathbb{R}^r, \quad (x, y) \mapsto \ell(x, y)\big(\cos y \, e_1(x) + \sin y \, e_2(x)\big),$$

where $\ell : \mathbb{R} \times \mathbb{S}^1 \to [l_-, l_+]$ with $0 < l_- < l_+ < r$ is smooth and bounded with all its derivatives. This represents the boundary of a (generally untwisted) massive waveguide whose typical fibre is $\mathbb{D}^2 \subset \mathbb{R}^2$. It is easy to verify that

$$\mathsf{T}\varpi\big(\mathsf{T}_{(x,y)}M\big) = \operatorname{span}\Big(\underbrace{\hat{\partial}^{\mathsf{H}}_x + \tfrac{\partial\ell}{\partial x} \, \partial_\rho\big|_{\varphi=y}}_{=\mathsf{T}\varpi(\partial^{\mathrm{pr}}_x)}, \underbrace{\tfrac{\partial\ell}{\partial y} \, \partial_\rho\big|_{\varphi=y} + \partial_\varphi\big|_{(\rho,\varphi)=(\ell,y)}}_{=\mathsf{T}\varpi(\partial_y)}\Big)$$

and a small calculation shows that its $\mathcal{G}$-orthogonal complement in the tangent space $\mathsf{T}_{(x,\rho=\ell(x,y),\varphi=y)}\mathsf{N}\mathbb{R}^r$ is spanned by the vector

$$-\tfrac{\ell}{(1-\ell\varkappa)^2} \tfrac{\partial\ell}{\partial x} \, \hat{\partial}^{\mathsf{H}}_x + \ell \, \partial_\rho - \tfrac{\partial \ln\ell}{\partial y} \, \partial_\varphi\Big|_{(\rho,\varphi)=(\ell(x,y),y)}.$$

Here, we introduced

$$\varkappa(x,y) := \kappa^{\rho}(x,\varphi = y) = \kappa^1(x)\cos y + \kappa^2(x)\sin y.$$

After an adequate normalisation, we obtain a unit normal field

$$N = \underbrace{\frac{-\frac{\ell}{(1-\ell\varkappa)^2}\frac{\partial \ell}{\partial x}\,\widehat{\partial}^{\mathsf{H}}_x}{\ell\sqrt{\left(\frac{\partial \ell}{\partial x}\right)^2 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2 + 1}}}_{=N_{\mathsf{H}}} + \underbrace{\frac{\ell\,\partial_\rho - \frac{\partial \ln \ell}{\partial y}\,\partial_\varphi}{\ell\sqrt{\left(\frac{\partial \ell}{\partial x}\right)^2 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2 + 1}}}_{=N_{\mathsf{V}}},$$

which consequently gives

$$\begin{aligned}
\daleth(\partial_x) &\overset{(5.17)}{=} \frac{\frac{\partial \ell}{\partial x}\sqrt{\left(\frac{\partial \ell}{\partial x}\right)^2 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2 + 1}}{1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2} N_{\mathsf{V}} \\
&= \frac{\frac{\partial \ell}{\partial x}}{1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2}\left(\partial_\rho - \frac{1}{\ell}\frac{\partial \ln \ell}{\partial y}\,\partial_\varphi\right),
\end{aligned}$$

again evaluated at the point $(\rho,\varphi) = (\ell(x,y),y)$. As far as the vertical part of the metric is concerned, we easily calculate

$$\begin{aligned}
&\big(\varpi^*(\mathrm{d}\varphi \otimes \mathrm{d}\varphi + \rho^2\,\mathrm{d}\rho \otimes \mathrm{d}\rho)\big)(\partial_y,\partial_y) \\
&= \big(\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi\big)\big(\mathsf{T}\varpi(\partial_y),\mathsf{T}\varpi(\partial_y)\big)\big|_{(\rho,\varphi)=(\ell(x,y),y)} \\
&= \left(\frac{\partial \ell}{\partial y}\right)^2 + \ell^2.
\end{aligned}$$

Finally, we insert these results into the expression from Proposition 5.8 and get

$$\begin{aligned}
G^{\varepsilon} = \varepsilon^{-2}&\left[\big(1 - \varepsilon\ell\varkappa\big)^2 + \varepsilon^2\frac{\left(\frac{\partial \ell}{\partial x}\right)^2}{1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2}\right]\mathrm{d}x^{\mathsf{H}} \otimes \mathrm{d}x^{\mathsf{H}} \\
&+ \left[1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2\right]\ell^2\,\mathrm{d}y \otimes \mathrm{d}y.
\end{aligned} \tag{5.18}$$

We continue with the explicit computation of the horizontal lift of ∂_x:

$$\begin{aligned}
\partial_x^{\mathsf{H}} &= \partial_x^{\mathrm{pr}} - G^{yy} G\big(\partial_x^{\mathrm{pr}}, \partial_y\big)\,\partial_y \\
&= \partial_x^{\mathrm{pr}} - \frac{1}{\left[1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2\right]\ell^2}\, \mathcal{G}\big(\mathsf{T}\varpi(\partial_x^{\mathrm{pr}}), \mathsf{T}\varpi(\partial_y)\big)\,\partial_y \\
&= \partial_x^{\mathrm{pr}} - \frac{1}{\left[1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2\right]\ell^2}\, \mathcal{G}\left(\widehat{\partial}_x^{\mathsf{H}} + \frac{\partial \ell}{\partial x}\partial_\rho, \frac{\partial \ell}{\partial y}\partial_\rho + \partial_\varphi\right)\partial_y \\
&= \partial_x^{\mathrm{pr}} - \frac{\frac{\partial \ln \ell}{\partial x}\frac{\partial \ln \ell}{\partial y}}{1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2}\,\partial_y .
\end{aligned} \tag{5.19}$$

Here, we encounter a similar phenomenon as in Example 5.7: Reducing the complexity of the set $M = \mathbb{R} \times \mathbb{S}^1$ is accompanied by a more complicated horizontal lift ∂_x^{H}. In contrast to the aforementioned example, the deviation from ∂_x^{H} to ∂_x^{pr} does not include any twisting effects, but reflects only the variation of the waveguide (in terms of $\frac{\partial \ell}{\partial x}$ and $\frac{\partial \ell}{\partial y}$). We close with the remark that the rescaled pullback metric G^ε is ε-close to the rescaled Riemannian submersion

$$g^\varepsilon|_{(x,y)} = \varepsilon^{-2}\,\mathrm{d}x^{\mathsf{H}} \otimes \mathrm{d}x^{\mathsf{H}} + \underbrace{\left[1 + \left(\frac{\partial \ln \ell(x,y)}{\partial y}\right)^2\right]\ell^2(x,y)\,\mathrm{d}y \otimes \mathrm{d}y}_{=: g_{M_x}}$$

in the sense of Definition 2.21: If we use the adapted local orthonormal frame

$$\varepsilon \partial_x^{\mathsf{H}} \quad , \quad \left[1 + \left(\frac{\partial \ln \ell}{\partial y}\right)^2\right]^{-1/2} \ell^{-1}\partial_y$$

of $(\mathsf{T}M, g^\varepsilon)$, one has the matrix representation

$$G^\varepsilon - g^\varepsilon = \begin{pmatrix} \varepsilon h_\varepsilon & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$h_\varepsilon = -2\ell\varkappa + \varepsilon\left[(\ell\varkappa)^2 + \frac{\left(\frac{\partial\ell}{\partial x}\right)^2}{1+\left(\frac{\partial\ln\ell}{\partial y}\right)^2}\right]$$

is of order one in $C_{\mathrm{b}}^\infty(M)$. ◊

5.1.2 The Induced Connection

Once again we first consider an intermediate quantity, to be specific the gauge field $\mathfrak{A}_\varepsilon := \Phi_\varepsilon^*\mathcal{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*(\mathsf{N}B^r))\otimes\mathbb{C}_{\mathrm{Herm}}^{N\times N}$. It is determined by means of

$$\mathsf{T}\Phi_\varepsilon\big(\widehat{X}^{\mathsf{H}}\big) = \overline{\big(\mathbf{1}_{\mathsf{T}B} - \varepsilon\mathcal{W}(\cdot)\big)X} \quad , \quad \mathsf{T}\Phi_\varepsilon(V) = \varepsilon\overline{\mathcal{K}_{\mathsf{N}B}V}$$

for $X \in C^\infty(\mathsf{T}B)$ and $V \in C^\infty(\mathsf{V}(\mathsf{N}B^r))$ as was shown in Lemma 5.4 and Lemma 5.5. Here, we omitted the parallel transport map $\mathfrak{p}_{\cdot}^{\Psi_\varepsilon} = \mathbf{1}_{\mathbb{R}^{b+f}}$ from $\mathsf{T}_{\Psi_0(\cdot)}\mathbb{R}^{b+f} = \mathsf{T}_{c(\cdot)}\mathbb{R}^{b+f}$ to $\mathsf{T}_{\Psi_\varepsilon(\cdot)}\mathbb{R}^{b+f}$, i.e., it suffices to evaluate the vectors in the image of $\mathsf{T}\Phi_\varepsilon$ merely on the submanifold $c(B) \subset (\mathbb{R}^{b+f}, \delta^{b+f})$. We then get

$$\mathfrak{A}_\varepsilon = \mathfrak{A}_{\mathsf{H}}^0 + \varepsilon\big(\mathfrak{A}_{\mathsf{H}}^{1,\varepsilon} + \mathfrak{A}_{\mathsf{V}}^\varepsilon\big), \tag{5.20}$$

where

$$\begin{aligned}
\mathfrak{A}_{\mathsf{H}}^0\big(\widehat{X}^{\mathsf{H}}\big) &= \mathcal{A}_{\Psi_0(\cdot)}\big(\overline{X}\big) = \mathcal{A}_{c(\cdot)}\big(\overline{X}\big) = \big(\pi_{\mathsf{N}B}^*\mathcal{A}_B\big)\big(\widehat{X}^{\mathsf{H}}\big),\\
\mathfrak{A}_{\mathsf{H}}^{1,\varepsilon}\big(\widehat{X}^{\mathsf{H}}\big) &= \varepsilon^{-1}\Big(\mathcal{A}_{\Psi_\varepsilon(\cdot)}\big(\overline{X}\big) - \mathcal{A}_{\Psi_0(\cdot)}\big(\overline{X}\big)\Big) - \mathcal{A}_{\Psi_\varepsilon(\cdot)}\big(\overline{\mathcal{W}(\cdot)X}\big),\\
\mathfrak{A}_{\mathsf{V}}^\varepsilon(V) &= \mathcal{A}_{\Psi_\varepsilon(\cdot)}\big(\overline{\mathcal{K}_{\mathsf{N}B}V}\big) = \mathcal{A}_{c(\cdot)}\big(\overline{\mathcal{K}_{\mathsf{N}B}V}\big) + \mathcal{O}(\varepsilon).
\end{aligned}$$

The gauge field $\mathcal{A}_B := c^*\mathcal{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*B)\otimes\mathbb{C}_{\mathrm{Herm}}^{N\times N}$ stands for the pullback of the original gauge field $\mathcal{A}$ to the submanifold B.

The pullback of $\mathfrak{A}_\varepsilon$ to M via the mapping ϖ ultimately yields the induced gauge field $\mathcal{A}_\varepsilon := \varpi^*\mathfrak{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*M)\otimes\mathbb{C}_{\mathrm{Herm}}^{N\times N}$. Therefore, we note that

$$\mathsf{T}\varpi(X^{\mathsf{H}}) = \widehat{X}^{\mathsf{H}}\big|_{\varpi(M)} + \daleth(X), \quad \daleth(X) \in C_{\mathrm{b}}^\infty\big(\mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)}\big)$$

and $\mathsf{T}\varpi(V) \in C^{\infty}(\mathsf{V}(\mathsf{N}B^r)|_{\varpi(M)})$ for $X \in C^{\infty}(\mathsf{T}B)$ and $V \in C^{\infty}(\mathsf{V}M)$. Thus, we obtain from (5.20) the following expression:

$$\mathcal{A}_{\varepsilon} = \pi_M^* \mathcal{A}_B + \varepsilon \mathcal{A}_{\mathsf{H}}^{\varepsilon} + \varepsilon \mathcal{A}_{\mathsf{V}}^{\varepsilon},$$

where $\mathcal{A}_{\mathsf{H}}^{\varepsilon} \in C_{\mathrm{b}}^{\infty}(\mathsf{H}^*M) \otimes \mathbb{C}_{\mathrm{Herm}}^{N \times N}$ and $\mathcal{A}_{\mathsf{V}}^{\varepsilon} \in C_{\mathrm{b}}^{\infty}(\mathsf{V}^*M) \otimes \mathbb{C}_{\mathrm{Herm}}^{N \times N}$ are given by

$$\mathcal{A}_{\mathsf{H}}^{\varepsilon}(X^{\mathsf{H}}) = \mathfrak{A}_{\mathsf{H}}^{\varepsilon}\big(\widehat{X}^{\mathsf{H}}\big) + \varepsilon \mathfrak{A}_{\mathsf{V}}^{\varepsilon}\big(\daleth(X)\big) \quad , \quad \mathcal{A}_{\mathsf{V}}^{\varepsilon}(V) = \mathfrak{A}_{\mathsf{V}}^{\varepsilon}\big(\mathsf{T}\varpi(V)\big). \tag{5.21}$$

We conclude that

$$\nabla^{\mathcal{A}_{\varepsilon}} = \nabla^{\mathcal{A}_0} + \mathrm{i}\varepsilon \mathcal{A}_1^{\varepsilon}, \quad \mathcal{A}_1^{\varepsilon} := \mathcal{A}_{\mathsf{H}}^{\varepsilon} + \mathcal{A}_{\mathsf{V}}^{\varepsilon} \in C_{\mathrm{b}}^{\infty}(\mathsf{T}^*M) \otimes \mathbb{C}_{\mathrm{Herm}}^{N \times N}$$

is an admissible perturbation of

$$\nabla^{\mathcal{A}_0} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}_0, \quad \mathcal{A}_0 := \pi_M^* \mathcal{A}_B \in C_{\mathrm{b}}^{\infty}(\mathsf{H}^*M) \otimes \mathbb{C}_{\mathrm{Herm}}^{N \times N}$$

in the sense of Definition 2.24.

As far as the vertical contribution of the corresponding connection Laplacian $-\Delta_{g^{\varepsilon}}^{\mathcal{A}_{\varepsilon}}$ is concerned, a short calculation shows

$$\begin{aligned} -\Delta_{\mathsf{V}}^{\mathcal{A}_{\varepsilon}} &= (\mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}_{\varepsilon})^{*,g_{\mathsf{V}}}(\mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}_{\varepsilon}) \\ &= (\mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon\mathcal{A}_{\mathsf{V}}^{\varepsilon})^{*,g_{\mathsf{V}}}(\mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon\mathcal{A}_{\mathsf{V}}^{\varepsilon}) \\ &= -\Delta_{\mathsf{V}}^{\varepsilon\mathcal{A}_{\mathsf{V}}^{\varepsilon}} = -\Delta_{\mathsf{V}}^{\mathrm{L.B.}}\mathbf{1}_{\mathbb{C}^N} + \mathcal{O}(\varepsilon) \end{aligned}$$

with errors in $L^{\infty}(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}))$. Thus, the vertical part of the gauge field transforms into a weak perturbation of order ε, although we initially started with the connection $\nabla^{\mathcal{A}} = \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\mathcal{A}$ on $\mathcal{T}^{\varepsilon} \times \mathbb{C}^N$. Physically speaking, this effect occurs because the shrinking of the initial tube $\mathcal{T}^{\varepsilon}$ in the transversal directions leaves the magnitude of the gauge field unchanged, while the influence of the vertical differential operator increases.

The Case of a Trivial Line Bundle ($N = 1$)

The main example for an Abelian gauge theory are spinless quantum particles that are coupled to a classical electromagnetic field, in which case, $\mathcal{A} \in C_{\mathrm{b}}^{\infty}(\mathsf{T}^*\mathbb{R}^{b+f}|_{\mathcal{T}^{\varepsilon}})$ represents the magnetic potential. This corresponds to a U(1)-gauge theory, where the system $(\psi, \mathcal{A})$ is invariant under transformations of the form $\mathsf{g} = \mathrm{e}^{\mathrm{i}\Omega}$ for some real-valued function $\Omega \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^{b+f}\big|_{\mathcal{T}^{\varepsilon}})$. This induces the transformation behaviour

$$\psi \mapsto \mathsf{g}\psi = \mathrm{e}^{\mathrm{i}\Omega}\phi \quad , \quad \mathcal{A} \mapsto \mathcal{A}_{\mathsf{g}} = \mathcal{A} - \mathrm{d}\Omega$$

for the wave function ψ and the magnetic potential $\mathcal{A}$ (see Remark 5.2). Moreover, we note that the curvature $\mathrm{R}^{\mathcal{A}} = \mathrm{d}\mathcal{A}$ is gauge invariant, i.e.,

$$\mathrm{R}^{\mathcal{A}_{\mathsf{g}}} = \mathrm{d}(\mathcal{A} - \mathrm{d}\Omega) = \mathrm{d}\mathcal{A} - \underbrace{\mathrm{d}(\mathrm{d}\Omega)}_{=0} = \mathrm{R}^{\mathcal{A}}.$$

This two-form is referred to as the *magnetic field* $\mathcal{B} = \mathrm{d}\mathcal{A}$, which is in fact the experimentally observable (and therefore relevant) quantity. Put differently, the magnetic field is invariant under gauge transformations and the respective connection Laplacians $\Delta^{\mathcal{A}}$ and $\Delta^{\mathcal{A}-\mathrm{d}\Omega}$ are unitarily equivalent. This additional degree of freedom (in terms of Ω) may be used to simplify the calculations. In this context, it is always possible to gauge away the leading order of the intermediate vertical gauge field $\mathfrak{A}_{\mathsf{V}}^{\varepsilon}$ using

$$\Omega_{\varepsilon} : \mathsf{N}B^{r} \to \mathbb{R}, \quad \mathsf{N}_{x}B^{r} \ni \nu \mapsto \mathcal{A}_{c(x)}\big(\varepsilon\overline{\nu}\big).$$

As a matter of fact, if we take a smooth curve $\gamma : I \to \mathsf{N}_{x}B^{r}$ with $\gamma(0) = \nu$ and $\dot{\gamma}(0) = w \in \mathsf{T}_{\nu}(\mathsf{N}_{x}B^{r}) = \mathsf{V}_{\nu}\mathsf{N}B^{r}$, we easily calculate

$$\begin{aligned} \mathrm{d}\Omega_{\varepsilon}(w)\big|_{\nu} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\Omega_{\varepsilon}\circ\gamma)(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{A}_{c(x)}\Big(\varepsilon\overline{\gamma(t)}\Big) \\ &= \mathcal{A}_{c(x)}\Big(\varepsilon\overline{\mathcal{K}_{\mathsf{N}B}\dot{\gamma}(0)}\Big) \\ &= \mathfrak{A}_{\mathsf{V}}^{\varepsilon=0}(w)\big|_{\nu}, \end{aligned}$$

where we used the connection map $\mathcal{K}_{\mathsf{N}B}$ to identify $\mathsf{V}_{\nu}\mathsf{N}B^{r}$ with $\mathsf{N}_{x}B^{r}$ for $\nu \in \mathsf{N}_{x}B^{r}$. This in turn means that the induced magnetic potential on M may be assumed a priori to be of the form

$$\mathcal{A}_{\varepsilon} = \pi_{M}^{*}\mathcal{A}_{B} + \varepsilon\mathcal{A}_{\mathsf{H}}^{\varepsilon} + \varepsilon^{2}\mathcal{A}_{\mathsf{V}}^{\varepsilon}.$$

Example 5.10 Let us consider the intermediate magnetic potential $\mathfrak{A}_\varepsilon$ for the geometric framework of conventional quantum tubes introduced in Example 5.6, i.e., tubular neighbourhoods around smoothly embedded curves $c : \mathbb{R} \to \mathbb{R}^3$. It is shown in [KR14, Section 4.1] that this situation admits an extremely convenient gauge on $\mathsf{N}\mathbb{R}^r = \mathbb{R} \times \mathbb{B}^2_r(0)$: The magnetic potential $\mathcal{A}$ restricted to the curve $c(\mathbb{R})$ can be gauged away completely in both the horizontal and vertical directions. This implies that $\mathfrak{A}_\varepsilon$ vanishes identically when evaluated on the zero section of $\mathsf{N}\mathbb{R}$, which implies the vanishing of both $\mathcal{A}_B$ and $\mathfrak{A}_\mathsf{V}^{\varepsilon=0}$. The subsequent order in the Taylor expansion of $\mathfrak{A}_\varepsilon$ around zero (incorporating derivatives of $\mathcal{A}$) can be expressed in terms of the physically relevant magnetic field

$$\mathcal{B} = \mathrm{d}\mathcal{A} \in C^\infty_\mathrm{b}\left(\Lambda^2 \mathsf{T}\mathbb{R}^3\big|_{\mathcal{T}^\varepsilon}\right)$$

evaluated on the curve, which consists of one parallel component

$$\mathcal{B}_\parallel(x) = \mathcal{B}_{c(x)}\left(\overline{e_1(x)}, \overline{e_2(x)}\right)$$

and two perpendicular components

$$\mathcal{B}^1_\perp(x) = \mathcal{B}_{c(x)}\left(\overline{e_2(x)}, c'(x)\right) \quad , \quad \mathcal{B}^2_\perp(x) = \mathcal{B}_{c(x)}\left(c'(x), \overline{e_1(x)}\right)$$

with respect to the curve $c(\mathbb{R}) \subset \mathbb{R}^3$. More precisely, it can be deduced from [KR14, Equation (4.2)] that the intermediate magnetic potential has the expansion

$$\mathfrak{A}_\varepsilon\big|_{(x,n)} = \varepsilon\big(\mathcal{B}_\perp(x) \times n + \mathcal{O}(\varepsilon)\big)\, \mathrm{d}\hat{x}^\mathsf{H} + \varepsilon^2\left(\tfrac{1}{2}\mathcal{B}_\parallel(x) n + \mathcal{O}(\varepsilon)\right) \times \mathrm{d}n$$

in the notation of (5.9). This allows for the examination of the induced magnetic potential $\mathcal{A}_\varepsilon$ for the massive waveguide of Example 5.7 and the hollow waveguide of Example 5.9:

(i) The massiveness of the waveguide yields $\daleth = 0$. Hence, the horizontal part of $\mathcal{A}_\varepsilon$ reduces to

$$\mathcal{A}^\varepsilon_\mathsf{H}(\partial^\mathsf{H}_x) = \mathcal{B}_\perp \times (\mathfrak{r} y) + \mathcal{O}(\varepsilon), \tag{5.22}$$

whereas its vertical contribution is given by

$$\begin{aligned}
\mathcal{A}_{\mathsf{V}}^{\varepsilon}(\partial_{y^1}) &= \mathfrak{A}_{\varepsilon}(\mathfrak{r}_{11}\,\partial_{n^1} + \mathfrak{r}_{21}\,\partial_{n^2}) + \mathcal{O}(\varepsilon) \\
&= \tfrac{1}{2}\mathcal{B}_{\|}\big(-\mathfrak{r}_{11}(\mathfrak{r}y)^2 + \mathfrak{r}_{21}(\mathfrak{r}y)^2\big) + \mathcal{O}(\varepsilon) \\
&= -\tfrac{1}{2}\mathcal{B}_{\|}\mathfrak{a}\mathfrak{b}y^2 + \mathcal{O}(\varepsilon), \\
\mathcal{A}_{\mathsf{V}}^{\varepsilon}(\partial_{y^2}) &= \cdots = \tfrac{1}{2}\mathcal{B}_{\|}\mathfrak{a}\mathfrak{b}y^1 + \mathcal{O}(\varepsilon).
\end{aligned}$$

Consequently, the magnetic potential of this massive quantum waveguide reads

$$\mathcal{A}_{\varepsilon} = \varepsilon\big(\mathcal{B}_{\perp} \times (\mathfrak{r}y) + \mathcal{O}(\varepsilon)\big)\,\mathrm{d}x^{\mathsf{H}} + \varepsilon^2\Big(\tfrac{1}{2}\mathfrak{a}\mathfrak{b}\mathcal{B}_{\|}y + \mathcal{O}(\varepsilon)\Big) \times \mathrm{d}y.$$

(ii) Although $\beth(\partial_x)$ does not vanish in the hollow case, it is of lower order in the horizontal part $\mathcal{A}_{\mathsf{H}}^{\varepsilon}$ within (5.21) and may thus be neglected, i.e.,

$$\mathcal{A}_{\mathsf{H}}^{\varepsilon}(\partial_x^{\mathsf{H}}) = \mathcal{B}_{\perp} \times \begin{pmatrix} \ell\cos y \\ \ell\sin y \end{pmatrix} + \mathcal{O}(\varepsilon). \tag{5.23}$$

As far as the vertical contribution is concerned, we use

$$\begin{aligned}
\mathsf{T}\varpi(\partial_y) &= \tfrac{\partial\ell}{\partial y}\,\partial_\rho\big|_{\varphi=y} + \partial_\varphi\big|_{(\rho,\varphi)=(\ell,y)} \\
&= \tfrac{\partial\ell}{\partial y}(\cos y\,\partial_{n^1} + \sin y\,\partial_{n^2}) + \ell(-\sin y\,\partial_{n^1} + \cos y\,\partial_{n^2})
\end{aligned}$$

in order to obtain

$$\begin{aligned}
\mathcal{A}_{\mathsf{V}}^{\varepsilon}(\partial_y) &= \tfrac{1}{2}\mathcal{B}_{\|}\begin{pmatrix} \ell\cos y \\ \ell\sin y \end{pmatrix} \times \begin{pmatrix} (\partial_y\ell)\cos y - \ell\sin y \\ (\partial_y\ell)\sin y + \ell\cos y \end{pmatrix} + \mathcal{O}(\varepsilon) \\
&= \tfrac{1}{2}\mathcal{B}_{\|}\ell^2 + \mathcal{O}(\varepsilon).
\end{aligned}$$

Finally, we see that

$$\mathcal{A}_{\varepsilon} = \varepsilon\Big(\mathcal{B}_{\perp} \times \begin{pmatrix} \ell\cos y \\ \ell\sin y \end{pmatrix} + \mathcal{O}(\varepsilon)\Big)\,\mathrm{d}x^{\mathsf{H}} + \varepsilon^2\Big(\tfrac{1}{2}\ell^2\mathcal{B}_{\|} + \mathcal{O}(\varepsilon)\Big)\,\mathrm{d}y$$

for the magnetic potential. $\Diamond$

5.1.3 The Induced Operator

The previous subsections showed that the initial tube operator $H^{\mathrm{w}}_{\mathrm{tube}}$ (5.3) is unitarily equivalent to

$$-\Delta^{\mathcal{A}_\varepsilon}_{G^\varepsilon} + V_\varepsilon \tag{5.24}$$

on $L^2(\mathcal{E}, \mathrm{vol}_{G^\varepsilon})$ with $\mathcal{E} = M \times \mathbb{C}^N$, i.e., we treat a Riemannian metric $G^\varepsilon = g^\varepsilon + \mathcal{O}(\varepsilon)$ and a metric connection $\nabla^{\mathcal{A}_\varepsilon} = \nabla^{\mathcal{A}_0} + \mathcal{O}(\varepsilon)$, both of which are perturbed in an admissible manner. Therefore, we have to combine the results of Subsection 2.3.1 and Subsection 2.3.2 for the connection Laplacian $-\Delta^{\mathcal{A}_\varepsilon}_{G^\varepsilon}$. We will see that $H^{\mathrm{w}}_{\mathrm{tube}}$ is unitarily equivalent to an operator of the form

$$H^{\mathcal{E},\mathrm{w}} = -\varepsilon^2 \Delta^{\mathcal{A}_0}_{\mathsf{H}} + \varepsilon H^{\mathcal{E},\mathrm{w}}_1 + H^{\mathcal{F},\varepsilon,\mathrm{w}}$$

on $\mathcal{H} = L^2(\mathcal{E}, \mathrm{vol}_g)$ that is suitable for the low energy analysis developed in Section 4.4. Put differently, we will conclude that the perturbation $H^{\mathcal{E},\mathrm{w}}_1$ satisfies Condition 4.11 and that the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ is a perturbation of some suitable operator $H^{\mathcal{F},\varepsilon=0,\mathrm{w}}$ in the sense of Remark 2.27.

To start with, let $\widehat{U}_{\rho_\varepsilon} = (\varepsilon^{-b}\rho_\varepsilon)^{1/2}\mathbf{1}_{\mathbb{C}^N}$ for $\rho_\varepsilon = \mathrm{vol}_{G^\varepsilon}/\mathrm{vol}_{g^\varepsilon}$ be the unitary operator from $L^2(\mathcal{E}, \mathrm{vol}_{G^\varepsilon})$ to $\mathcal{H} = L^2(\mathcal{E}, \mathrm{vol}_g)$. Then Lemma 2.18 asserts

$$\widehat{U}_{\rho_\varepsilon}\big(-\Delta^{\mathcal{A}_\varepsilon}_{G^\varepsilon}\big)\widehat{U}^\dagger_{\rho^\varepsilon} = -\Delta^{\mathcal{A}_\varepsilon}_{g^\varepsilon} + \nabla^{\mathcal{A}_\varepsilon,*,g^\varepsilon} \circ S^{\mathcal{A}_\varepsilon} + V_{\rho_\varepsilon}\mathbf{1}_{\mathbb{C}^N},$$

where the perturbed Laplacian $-\Delta^{\mathcal{A}_\varepsilon}_{g^\varepsilon}$ is expanded with the aid of Proposition 2.25 as

$$\begin{aligned}
-\Delta^{\mathcal{A}_\varepsilon}_{g^\varepsilon} = -\varepsilon^2 \Delta^{\mathcal{A}_0}_{\mathsf{H}} - \Delta^{\varepsilon^2 \mathcal{A}^\varepsilon_{\mathsf{V}}}_{\mathsf{V}}& \\
- \mathrm{i}\varepsilon^3 \operatorname{tr}_{\pi_M^* g_B}&\big(2\mathcal{A}^\varepsilon_{\mathsf{H}} \otimes \nabla^{\mathcal{A}_0} + \nabla^{\mathsf{T}^*M \otimes \mathrm{End}(\mathcal{E})}\mathcal{A}^\varepsilon_{\mathsf{H}}\big) \\
+ \mathrm{i}\varepsilon^3 \mathcal{A}^\varepsilon_{\mathsf{H}}&(\eta_{\mathsf{V}}) + \varepsilon^4 \operatorname{tr}_{\pi_M^* g_B}\big(\mathcal{A}^\varepsilon_{\mathsf{H}} \otimes \mathcal{A}^\varepsilon_{\mathsf{H}}\big).
\end{aligned}$$

The $S^{\mathcal{A}_\varepsilon}$-term within the transformed Laplacian is evaluated by means of a local frame $\{v_\alpha\}_{\alpha=1}^m$ of $\mathsf{T}M$ and its dual frame $\{\omega^\alpha\}_{\alpha=1}^m$ as well as (A.16) and Lemma A.21 to be:

$$
\begin{aligned}
&\nabla^{\mathcal{A}_\varepsilon,*,g^\varepsilon} \circ S^{\mathcal{A}_\varepsilon} \\
&= -\nabla^{\mathcal{A}_\varepsilon}_{v_\beta}\big(s^\varepsilon(\omega^\alpha,\omega^\beta)\nabla^{\mathcal{A}_\varepsilon}_{v_\alpha}\cdot\big) - s^\varepsilon\big(\omega^\alpha, \underbrace{\operatorname{div}_{g^\varepsilon}(v_\beta)\omega^\beta}_{=-\widetilde{\nabla}^{g^\varepsilon}_{v_\beta}\omega^\beta}\big)\nabla^{\mathcal{A}_\varepsilon}_{v_\alpha} \\
&= -\nabla^{\mathcal{A}_0}_{v_\beta}\big(s^\varepsilon(\omega^\alpha,\omega^\beta)\nabla^{\mathcal{A}_0}_{v_\alpha}\cdot\big) \\
&\quad + \Big(-2\mathrm{i}\varepsilon\, s^\varepsilon(\omega^\alpha,\omega^\beta)\mathcal{A}_1^\varepsilon(v_\beta) + s^\varepsilon\big(\omega^\alpha, \widetilde{\nabla}^{g^\varepsilon}_{v_\beta}\omega^\beta\big)\Big)\nabla^{\mathcal{A}_0}_{v_\alpha} \\
&\quad - \mathrm{i}\varepsilon\Big(\big(\nabla^{\mathsf{T}M^{\otimes 2}}_{v_\beta}s^\varepsilon\big)(\omega^\alpha,\omega^\beta) + s^\varepsilon\big(\widetilde{\nabla}^{g^\varepsilon}_{v_\beta}\omega^\alpha,\omega^\beta\big)\Big)\mathcal{A}_1^\varepsilon(v_\alpha) \\
&\quad - \mathrm{i}\varepsilon\, s^\varepsilon(\omega^\alpha,\omega^\beta)\Big(\big(\nabla^{\mathsf{T}^*M\otimes\mathrm{End}(\mathcal{E})}_{v_\beta}\mathcal{A}_1^\varepsilon\big)(v_\alpha) + \mathcal{A}_1^\varepsilon\big(\nabla^{g^\varepsilon}_{v_\beta}v_\alpha\big)\Big) \\
&\quad + \varepsilon^2 s^\varepsilon(\omega^\alpha,\omega^\beta)\mathcal{A}_1^\varepsilon(v_\beta)\mathcal{A}_1^\varepsilon(v_\alpha).
\end{aligned}
$$

Next we introduce the tensor

$$\sigma^\varepsilon := \varepsilon^{-3}s^\varepsilon \in C_{\mathrm{b}}^\infty(\Sigma^2\mathsf{H}^*M)$$

so as to illustrate (and compare) the different orders of ε properly. Actually the bounds of σ^ε can be chosen to be of order one independently of ε, since

- Lemma 2.22 implies the existence of a constant $c > 0$ such that

  $$\varepsilon^3\big|\sigma^\varepsilon(\omega,\omega)\big| = \big|s^\varepsilon(\omega,\omega)\big| \leqslant c\varepsilon\widetilde{g}^\varepsilon(\omega,\omega)$$

  for all $\omega \in \mathsf{T}^*M$,

- and Proposition 5.8 implies that $(G^\varepsilon - g^\varepsilon)(\mathfrak{v},\cdot) = 0$ for all $\mathfrak{v} \in \mathsf{V}M$, and hence s^ε is a section of $\Sigma^2\mathsf{H}^*M$ and the latter equation simplifies to

  $$\big|\sigma^\varepsilon(\pi_M^*\zeta,\pi_M^*\zeta)\big| \leqslant c\varepsilon^{-2}\widetilde{g}(\pi_M^*\zeta,\pi_M^*\zeta) = c\widetilde{g}_B(\zeta,\zeta)$$

  for all $\zeta \in \mathsf{T}^*B$.

In summary, the operator (5.24) and therefore the initial Schrödinger operator $H^{\mathrm{w}}_{\mathrm{tube}}$ is unitarily equivalent to

$$H^{\mathcal{E},\mathrm{w}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{A}_0} - \Delta_{\mathsf{V}}^{\varepsilon \mathcal{A}_{\mathsf{V}}^{\varepsilon}} + V_\varepsilon + V_{\rho_\varepsilon} \mathbf{1}_{\mathbb{C}^N} + \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}},$$

where the deviation $\tilde{H}_1^{\mathcal{E},\mathrm{w}}$ from the horizontal part of $-\Delta_{G^\varepsilon}^{\mathcal{A}_\varepsilon}$ to $-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{A}_0}$ equals

$$\begin{aligned}
&\mathrm{tr}_{\mathsf{H}M,(12),(34)} \left(\nabla_{\varepsilon \cdot}^{\mathcal{A}_0} \left(-\sigma^\varepsilon(\cdot,\cdot) \nabla_{\varepsilon \cdot}^{\mathcal{A}_0} \cdot \right) \right) \\
&-2\mathrm{i}\varepsilon\, \mathrm{tr}_{\pi_M^* g_B} \left(\mathcal{A}_{\mathsf{H}}^\varepsilon \otimes \nabla_{\varepsilon \cdot}^{\mathcal{A}_0} \right) \\
&+\varepsilon\, \mathrm{tr}_{\mathsf{H}M} \left(\mathrm{tr}_{\mathsf{H}M,(23)} \left(\sigma^\varepsilon \left(\cdot, \widetilde{\nabla}_{\cdot}^{g^\varepsilon} \cdot \right) - 2\mathrm{i}\varepsilon \sigma^\varepsilon \otimes \mathcal{A}_{\mathsf{H}}^\varepsilon \right) \nabla_{\varepsilon \cdot}^{\mathcal{A}_0} \cdot \right) \\
&+\varepsilon \Bigg[-\mathrm{i}\varepsilon\, \mathrm{tr}_{\pi_M^* g_B} \left(\nabla^{\mathsf{T}^* M \otimes \mathrm{End}(\mathcal{E})} \mathcal{A}_{\mathsf{H}}^\varepsilon \right) + \mathrm{i}\varepsilon \mathcal{A}_{\mathsf{H}}^\varepsilon(\eta_{\mathsf{V}}) + \varepsilon^2\, \mathrm{tr}_{\pi_M^* g_B} \left(\mathcal{A}_{\mathsf{H}}^\varepsilon \otimes \mathcal{A}_{\mathsf{H}}^\varepsilon \right) \\
&\qquad - \mathrm{i}\varepsilon^2\, \mathrm{tr}_{\mathsf{H}M,(13),(24)} \left(\left(\nabla^{\mathsf{H}M^{\otimes 2}} \sigma^\varepsilon + \sigma^\varepsilon \left(\widetilde{\nabla}_{\cdot}^{g^\varepsilon} \cdot, \cdot \right) \right) \otimes \mathcal{A}_{\mathsf{H}}^\varepsilon \right) \\
&\qquad - \mathrm{i}\varepsilon^2\, \mathrm{tr}_{\mathsf{H}M,(13),(24)} \left(\sigma^\varepsilon \otimes \left(\nabla^{\mathsf{T}^* M \otimes \mathrm{End}(\mathcal{E})} \mathcal{A}_{\mathsf{H}}^\varepsilon + \mathcal{A}_{\mathsf{H}}^\varepsilon \left(\nabla_{\cdot}^{g^\varepsilon} \cdot \right) \right) \right) \\
&\qquad + \varepsilon^4\, \mathrm{tr}_{\mathsf{H}M,(13),(24)} \left(\sigma^\varepsilon \otimes \mathcal{A}_{\mathsf{H}}^\varepsilon \otimes \mathcal{A}_{\mathsf{H}}^\varepsilon \right) \Bigg].
\end{aligned}$$

Remark 5.11 (i) The tensor product connections $\nabla^{\mathsf{T}^* M \otimes \mathrm{End}(\mathcal{E})}$ and $\nabla^{\mathsf{H}M^{\otimes 2}}$ depend implicitly on ε via the Levi-Civita connection ∇^{g^ε} involved in their respective definitions. It turns out, however, that we only need their ε-independent components (Christoffel symbols):

- Koszul's formula (A.3) yields

$$\nabla_{X^{\mathsf{H}}}^{g_\varepsilon} Y^{\mathsf{H}} = \left(\nabla_X^{g_B} Y \right)^{\mathsf{H}} + \Omega_{\mathsf{H}}(X^{\mathsf{H}}, Y^{\mathsf{H}}) = \nabla_{X^{\mathsf{H}}}^{g} Y^{\mathsf{H}} \tag{5.25}$$

  for $X, Y \in C^\infty(\mathsf{T}B)$. Hence,

$$\begin{aligned}
\left(\nabla_{X^{\mathsf{H}}}^{\mathsf{T}^* M \otimes \mathrm{End}(\mathcal{E})} \mathcal{A}_{\mathsf{H}}^\varepsilon \right)(Y^{\mathsf{H}}) &= X^{\mathsf{H}} \cdot \mathcal{A}_{\mathsf{H}}^\varepsilon(Y^{\mathsf{H}}) - \mathcal{A}_{\mathsf{H}}^\varepsilon \left(\nabla_{X^{\mathsf{H}}}^{g^\varepsilon} Y^{\mathsf{H}} \right) \\
&= X^{\mathsf{H}} \cdot \mathcal{A}_{\mathsf{H}}^\varepsilon(Y^{\mathsf{H}}) - \mathcal{A}_{\mathsf{H}}^\varepsilon \left(\nabla_{X^{\mathsf{H}}}^{g} Y^{\mathsf{H}} \right)
\end{aligned}$$

is the same as for the unscaled Levi-Civita connection $\nabla^{g^{\varepsilon=1}}$.

- Let $\{v_\alpha\}_{\alpha=1}^m$ be a local frame of $\mathsf{T}M$ and $\{\omega^\alpha\}_{\alpha=1}^m$ its dual frame such that $\{v_i\}_{i=1}^b$ and $\{\omega^i\}_{i=1}^b$ are local frames of $\mathsf{H}M$ and H^*M, respectively. Then

$$\begin{aligned}
\mathrm{P}^{\mathsf{H}^*M}\left(\widetilde{\nabla}^{g^\varepsilon}_{v_i}\omega^j\right) &= \left(\widetilde{\Gamma}^{g^\varepsilon}\right)^j_{i\alpha}\omega^\alpha \\
&\overset{(A.4)}{=} -\left(\Gamma^{g^\varepsilon}\right)^j_{i\alpha}\omega^\alpha \\
&\overset{(5.25)}{=} -\left(\Gamma^{g}\right)^j_{i\alpha}\omega^\alpha \\
&= \mathrm{P}^{\mathsf{H}^*M}\left(\widetilde{\nabla}^{g}_{v_i}\omega^j\right)
\end{aligned}$$

does not depend on ε, and likewise for

$$\begin{aligned}
&\left(\nabla^{\mathsf{H}M^{\otimes 2}}_{X^{\mathsf{H}}}\sigma^\varepsilon\right)(\Xi,\Upsilon) \\
&= X^{\mathsf{H}}\cdot\sigma^\varepsilon(\Xi,\Upsilon) - \sigma^\varepsilon\left(\widetilde{\nabla}^{g^\varepsilon}_{X^{\mathsf{H}}}\Xi,\Upsilon\right) - \sigma^\varepsilon\left(\Xi,\widetilde{\nabla}^{g^\varepsilon}_{X^{\mathsf{H}}}\Upsilon\right) \\
&= X^{\mathsf{H}}\cdot\sigma^\varepsilon(\Xi,\Upsilon) - \sigma^\varepsilon\left(\widetilde{\nabla}^{g}_{X^{\mathsf{H}}}\Xi,\Upsilon\right) - \sigma^\varepsilon\left(\Xi,\widetilde{\nabla}^{g}_{X^{\mathsf{H}}}\Upsilon\right)
\end{aligned}$$

for $X \in C^\infty(\mathsf{T}B)$ and $\Xi,\Upsilon \in C^\infty(\mathsf{H}^*M)$.

(ii) Due to the fact that $\ln\rho_\varepsilon = \mathcal{O}(\varepsilon)$ as shown in the proof of Proposition 2.23, the geometric potential (2.18) equals

$$V_{\rho_\varepsilon} = V^{\mathsf{V}}_{\rho_\varepsilon} + \underbrace{\tfrac{1}{2}\varepsilon^2\Delta^{\mathrm{L.B.}}_{\mathsf{H}}\ln\rho_\varepsilon}_{=\mathcal{O}(\varepsilon^3)} + \mathcal{O}(\varepsilon^4) \tag{5.26}$$

with leading order

$$V^{\mathsf{V}}_{\rho_\varepsilon} = \underbrace{\tfrac{1}{2}\Delta^{\mathrm{L.B.}}_{\mathsf{V}}\ln\rho_\varepsilon}_{=\mathcal{O}(\varepsilon)} + \underbrace{\tfrac{1}{4}\widetilde{g}_{\mathsf{V}}(\mathrm{d}\ln\rho_\varepsilon,\mathrm{d}\ln\rho_\varepsilon)}_{=\mathcal{O}(\varepsilon^2)}.$$

Let us examine this potential in the two most relevant situations:

- If M is a massive quantum waveguide ($\operatorname{codim}(\varpi) = 0$), the leading order of

$$\begin{aligned} \ln \rho_\varepsilon &\overset{(2.19)}{\underset{(5.15)}{=}} \tfrac{1}{2} \operatorname{tr}_{TB}\Big(-2\varepsilon\varpi^*\mathcal{W} \\ &\qquad + \varepsilon^2\big(\varpi^*\mathcal{W}\circ\varpi^*\mathcal{W} + \delta^{\mathsf{V}(NB)}\big(\daleth(\cdot),\daleth(\cdot)\big)\big)\Big) \\ &= -\varepsilon \operatorname{tr}_{TB}(\varpi^*\mathcal{W}) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

  is linear with respect to the fibre coordinate $\xi \in M_x$. Consequently, $\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ vanishes on the leading term of $\ln\rho_\varepsilon$ and $V^{\mathsf{V}}_{\rho_\varepsilon} = \mathcal{O}(\varepsilon^2)$.
- If $\partial M = \emptyset$, the induced Weingarten map $\varpi^*\mathcal{W}$ is generally not linear with respect to y and thus $\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ is not zero when applied to the leading order $\varpi^*\mathcal{W}$. As a consequence, one expects an $\mathcal{O}(\varepsilon)$-contribution within $V^{\mathsf{V}}_{\rho_\varepsilon}$. ◊

Depending on the particular geometric situation at hand, we can take advantage of the freedom to distribute the potential $V_{\rho_\varepsilon} = V^{(1)}_{\rho_\varepsilon} + V^{(2)}_{\rho_\varepsilon}$ to the vertical operator and the perturbation so that

$$H^{\mathcal{F},\varepsilon,\mathrm{w}} = -\Delta_{\mathsf{V}}^{\varepsilon\mathcal{A}^\varepsilon_{\mathsf{V}}} + V_\varepsilon + V^{(1)}_{\rho_\varepsilon}\mathbf{1}_{\mathbb{C}^N} \quad , \quad \varepsilon H_1^{\mathcal{E},\mathrm{w}} = \varepsilon\tilde{H}_1^{\mathcal{E},\mathrm{w}} + V^{(2)}_{\rho_\varepsilon}\mathbf{1}_{\mathbb{C}^N}.$$

In virtue of Remark 2.20, the operator $-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{A}_0} + \varepsilon\tilde{H}_1^{\mathcal{E},\mathrm{w}}$ is the horizontal part of the Laplacian $-\Delta_{G^\varepsilon}^{\mathcal{A}_\varepsilon}$ and hence defines a positive operator. It then follows that $-\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{A}_0} + \varepsilon H_1^{\mathcal{E},\mathrm{w}}$ is bounded from below by $-\big\|V^{(2)}_{\rho_\varepsilon}\big\|_{L^\infty(M)}\mathbf{1}_{\mathcal{H}}$. Thus, the perturbation satisfies all the requirements of Condition 4.11 if we split the geometric potential in such a way that $V^{(2)}_{\rho_\varepsilon}$ is of order ε^2.

5.2 Application to Weak Magnetic Fields

The considerations of the previous section showed that the initial tube operator $H^{\mathrm{w}}_{\mathrm{tube}}$ (5.3) is unitarily equivalent to an operator of the form

$$H^{\mathcal{E},\mathrm{w}} = -\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{A}_0} + H^{\mathcal{F},\varepsilon,\mathrm{w}} + \varepsilon H_1^{\mathcal{E},\mathrm{w}}$$

with a perturbation $H_1^{\mathcal{E},\mathrm{w}}$ satisfying Condition 4.11. Hence, we may apply Theorem 4.15 and obtain the following approximation result for the low-lying eigenvalues of $H^{\mathcal{E},\mathrm{w}}$ (and consequently for those of $H^{\mathrm{w}}_{\mathrm{tube}}$):

Theorem 5.12 *Let $H^{\mathrm{w}}_{\mathrm{tube}}$ be the Hamiltonian generating the dynamics of non-interacting, non-relativistic quantum particles with total spin quantum number N, which are localised within a quantum waveguide of bounded geometry and are coupled to a weak external $\mathbb{C}^{N\times N}_{\mathrm{Herm}}$-valued, C^∞-bounded gauge field $\mathcal{A}$. Assume that*

(i) the ground state band λ_0^{m} of $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ (with associated eigenspace bundle $\mathcal{P}^{\mathrm{m}}$) has a spectral gap, cf. Condition 2.26, for $\varepsilon > 0$ small enough,

(ii) and there is a constant $C > 0$ such that $C\varepsilon^\alpha$ is strictly below the essential spectrum of $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}} - \Lambda_0 \mathbf{1}_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}}$ in the sense of Theorem 4.15 for some $\alpha \in (0,2]$, where $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}}$ is the adiabatic operator and Λ_0 denotes the bottom of $H^{\mathcal{F},\varepsilon,\mathrm{w}}$'s spectrum (see (5.32) below).

Then if $\nu^\varepsilon < C\varepsilon^\alpha$ is an eigenvalue of $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}} - \Lambda_0 \mathbf{1}_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}}$, there exists an eigenvalue υ^ε of $H^{\mathrm{w}}_{\mathrm{tube}}$ below its essential spectrum with the asymptotic expansion

$$\upsilon^\varepsilon = \Lambda_0 + \nu^\varepsilon + \mathcal{O}(\varepsilon^{2+\alpha}).$$

We will apply this theorem to quantum waveguides arising from conventional quantum tubes for a non-relativistic, charged, spinless quantum particle in the presence of a weak external magnetic field (described by means of some magnetic potential $\mathcal{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*\mathbb{R}^3|_{\mathcal{T}^\varepsilon})$) but without any other forces ($V = 0$). Such a particle is characterised by a solution $\psi : \mathbb{R} \to L^2(\mathcal{T}^\varepsilon \times \mathbb{C})$ of the time-dependent Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H^{\mathrm{w}}_{\mathrm{tube}}\psi(t), \quad \psi(t)|_{\partial\mathcal{T}^\varepsilon} = 0$$

for some tube $\mathcal{T}^\varepsilon \subset \mathbb{R}^3$ around a smoothly embedded curve $c : \mathbb{R} \to \mathbb{R}^3$. Recalling the specific gauge for $\mathfrak{A}_\varepsilon$ of Example 5.10 (which implied both $\mathcal{A}_B = 0$ and $\mathfrak{A}_{\mathsf{V}}^{\varepsilon=0} = 0$), we conclude that the associated tube operator $H^{\mathrm{w}}_{\mathrm{tube}}$ is unitarily equivalent to

$$H^{\mathcal{E},\mathrm{w}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathrm{L.B.}} - \Delta_{\mathsf{V}}^{\varepsilon^2\mathcal{A}_{\mathsf{V}}^\varepsilon} + \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}} + V_{\rho_\varepsilon} \tag{5.27}$$

on $\mathcal{H} = L^2(\mathcal{E}, \mathrm{vol}_g)$ with trivial line bundle $\mathcal{E} = M \times \mathbb{C}$ over the waveguide M. In view of Remark 5.11(ii) and the subsequent discussion, we set

$$H^{\mathcal{F},\varepsilon,\mathrm{w}} := -\Delta_{\mathsf{V}}^{\varepsilon^2 \mathcal{A}_{\mathsf{V}}^{\varepsilon}} \tag{5.28a}$$

for $\mathrm{codim}(\varpi) = 0$ and

$$\begin{aligned} H^{\mathcal{F},\varepsilon,\mathrm{w}} &:= -\Delta_{\mathsf{V}}^{\varepsilon^2 \mathcal{A}_{\mathsf{V}}^{\varepsilon}} + V_{\rho_\varepsilon}^{\mathsf{V}} \\ &= \widehat{U}_{\rho_\varepsilon} \underbrace{\left(-\operatorname{tr}_{g_{\mathsf{V}}}\left((\nabla^{\varepsilon^2 \mathcal{A}_{\mathsf{V}}^{\varepsilon}})^2 \right) - \nabla^{\varepsilon^2 \mathcal{A}_{\mathsf{V}}^{\varepsilon}}_{\mathrm{grad}_{g_{\mathsf{V}}} \ln \rho_\varepsilon} \right)}_{= -\widetilde{\Delta}_{\mathsf{V}}^{\varepsilon^2 \mathcal{A}_{\mathsf{V}}^{\varepsilon}} \text{ by (A.20)}} \widehat{U}_{\rho_\varepsilon}^{\dagger} \end{aligned} \tag{5.28b}$$

for $\mathrm{codim}(\varpi) \geqslant 1$ on $\mathcal{H}_{\mathcal{F}}|_x := L^2(M_x, \mathrm{vol}_{g_{M_x}})$ with appropriate Dirichlet domains. Here, the latter transformation via the fibrewise unitary map $\widehat{U}_{\rho_\varepsilon}(x) : L^2(M_x, \rho_\varepsilon \, \mathrm{vol}_{g_{M_x}}) \to \mathcal{H}_{\mathcal{F}}|_x$ is obtained by calculations analogous to those in Subsection 2.3.1. It then immediately follows that the perturbation

$$\varepsilon H_1^{\mathcal{E},\mathrm{w}} = \begin{cases} \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}} + V_{\rho_\varepsilon}, & \mathrm{codim}(\varpi) = 0 \\ \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}} + \frac{1}{2}\varepsilon^2 \Delta_{\mathsf{H}}^{\mathrm{L.B.}} \ln \rho_\varepsilon, & \mathrm{codim}(\varpi) \geqslant 1 \end{cases}$$

satisfies Condition 4.11 as desired.

Let us analyse the asymptotic expansion of the magnetic ground state band $\lambda_0^{\mathrm{m}}(x) := \min \sigma(H^{\mathcal{F},\varepsilon,\mathrm{w}}(x))$. To do this, we start with the examination of the ε-independent ground state

$$\begin{aligned} \lambda_0(x) &:= \min \sigma\left(H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_{\mathsf{V}}^{\varepsilon}=0}(x) \right) \\ &= \begin{cases} \min \sigma\left(-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}(x) \right), & \mathrm{codim}(\varpi) = 0 \\ \min \sigma\left(-\widetilde{\Delta}_{\mathsf{V}}^{\mathrm{L.B.}}(x) \right), & \mathrm{codim}(\varpi) \geqslant 1 \end{cases} \end{aligned}$$

of the unperturbed vertical operator (in the absence of any magnetic potential) for the two most relevant geometric configurations:

- *Massive quantum waveguides* ($\operatorname{codim}(\varpi) = 0$):
  Denote by $\phi_0(x)$ the fibrewise, uniquely defined, positive ground state with corresponding eigenband $\lambda_0(x) > 0$. We will assume that $x \mapsto \lambda_0(x)$ has a unique non-degenerate minimum on $\mathbb{R}$. Moreover, a natural condition on ϕ_0 is the requirement that its barycentre lie on the curve:
  $$\left\langle \phi_0, y^1 \phi_0 \right\rangle_{\mathcal{H}_\mathcal{F}} = 0 = \left\langle \phi_0, y^2 \phi_0 \right\rangle_{\mathcal{H}_\mathcal{F}} . \tag{5.29}$$
  This corresponds to the "correct" parametrisation of the waveguide, i.e., to the proper positioning of $c(\mathbb{R})$ within the tube $\mathcal{T}^\varepsilon \subset \mathbb{R}^3$.

- *Hollow quantum waveguides* ($\operatorname{codim}(\varpi) = 1$):
  In view of the discussion right at the end of Section A.3, the Laplacian $-\widetilde{\Delta}_{\mathsf{V}}^{\text{L.B.}}$ is symmetric with quadratic form
  $$\left\langle \psi, -\widetilde{\Delta}_{\mathsf{V}}^{\text{L.B.}} \psi \right\rangle_{L^2(M_x, \rho_\varepsilon \operatorname{vol}_{g_{M_x}})} = \int_{M_x} \widetilde{g}_{M_x} \left(\overline{\mathrm{d}\psi}, \mathrm{d}\psi \right) \rho_\varepsilon \operatorname{vol}_{g_{M_x}}$$
  and hence defines a positive operator. Its ground state is a fibrewise constant function, say some $\tilde{\phi}(x) \circ \pi_M$, with eigenband $\lambda_0 \equiv 0$. The corresponding normalised ground state of $H^{\mathcal{F},\varepsilon,\mathrm{w}}(x)\big|_{\mathcal{A}_{\mathsf{V}}^\varepsilon = 0}$ is then
  $$\begin{aligned} \phi_0(x) &= \frac{\widehat{U}_{\rho_\varepsilon} \left(\tilde{\phi}_0(x) \circ \pi_M \right)}{\left\| \widehat{U}_{\rho_\varepsilon} \left(\tilde{\phi}_0(x) \circ \pi_M \right) \right\|_{\mathcal{H}_\mathcal{F}|_x}} = \frac{\rho_\varepsilon^{1/2}}{\int_{M_x} \rho_\varepsilon \ \operatorname{vol}_{g_{M_x}}} \\ &= \operatorname{Vol}_{g_{M_x}}(M_x)^{-1/2} \circ \pi_M + \mathcal{O}(\varepsilon), \end{aligned}$$
  where we took advantage of the fact that $\rho_\varepsilon = 1 + \mathcal{O}(\varepsilon)$ in $C_{\mathrm{b}}^\infty(M)$ due to (5.16).

As far as the corrections to λ_0 within λ_0^{m} are concerned, we use the methods of finite-dimensional perturbation theory developed in [Kat80, Section II – § 2]). The entire vertical operator has the expansion

$$
\begin{aligned}
H^{\mathcal{F},\varepsilon,\mathrm{w}} = H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}^{\varepsilon}_{\mathsf{V}}=0} &+ \varepsilon^2 \underbrace{\Big(\mathrm{d}^*\big(\mathrm{i}\mathcal{A}^{\varepsilon=0}_{\mathsf{V}}\big) + \big(\mathrm{i}\mathcal{A}^{\varepsilon=0}_{\mathsf{V}}\big)^*\mathrm{d}\Big)}_{=:H^{\mathcal{F},\mathrm{w}}_2} \\
&+ \varepsilon^3 \underbrace{\Big(\mathrm{d}^*\big(\mathrm{i}\mathcal{A}^{1,\varepsilon=0}_{\mathsf{V}}\big) + \big(\mathrm{i}\mathcal{A}^{1,\varepsilon=0}_{\mathsf{V}}\big)^*\mathrm{d}\Big)}_{=:H^{\mathcal{F},\mathrm{w}}_3} \\
&+ \mathcal{O}(\varepsilon^4)
\end{aligned}
$$

with errors in $L^\infty(\mathcal{L}(\mathcal{D}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}))$, where $\mathcal{A}^{\varepsilon}_{\mathsf{V}} = \mathcal{A}^{\varepsilon=0}_{\mathsf{V}} + \varepsilon\mathcal{A}^{1,\varepsilon=0}_{\mathsf{V}} + \mathcal{O}(\varepsilon^2)$. But then

$$
\begin{aligned}
&\int_{M_x} \phi_0\big(\mathrm{d}^*(\mathrm{i}\mathcal{A}_{\mathsf{V}}) + (\mathrm{i}\mathcal{A}_{\mathsf{V}})^*\mathrm{d}\big)\phi_0 \ \mathrm{vol}_{g_{M_x}} \\
&= 2\,\mathrm{Re}\Bigg(\mathrm{i}\underbrace{\int_{M_x} \widetilde{g}_{M_x}(\mathrm{d}\phi_0, \mathcal{A}_{\mathsf{V}}\phi_0) \ \mathrm{vol}_{g_{M_x}}}_{\text{real-valued}}\Bigg) \\
&= 0
\end{aligned}
\tag{5.30}
$$

for any $\mathcal{A}_{\mathsf{V}} \in C^\infty(\mathsf{V}^*M)$, which gives

$$
\big\langle \phi_0, H^{\mathcal{F},\mathrm{w}}_2\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}|_x} = 0 = \big\langle \phi_0, H^{\mathcal{F},\mathrm{w}}_3\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}|_x}
$$

for all $x \in \mathbb{R}$. This, together with the fact that the $\mathcal{O}(\varepsilon)$-correction to the unperturbed ground state ϕ_0 vanishes, shows that the first non-zero correction to λ_0 is actually of order ε^4. The diamagnetic inequality [FH10, Section 2.1] finally asserts that the infimum of $\sigma(H^{\mathcal{F},\varepsilon,\mathrm{w}})$ increases when a magnetic field is switched on. Thus, the ground state band λ^{m}_0 of the entire magnetic vertical operator is always pointwise greater than or equal to λ_0 and we end up with the asymptotic expansion

$$
\lambda^{\mathrm{m}}_0(x) = \begin{cases} \lambda_0(x) + \varepsilon^4\lambda_{0,4}(x) + \mathcal{O}(\varepsilon^5), & \mathrm{codim}(\varpi) = 0 \\ 0 + \varepsilon^4\lambda_{0,4}(x) + \mathcal{O}(\varepsilon^5), & \mathrm{codim}(\varpi) = 1 \end{cases}
$$

with $\lambda_{0,4} \geqslant 0$ and errors in $C^\infty_{\mathrm{b}}(\mathbb{R})$.

Lemma 5.13 *Let the typical fibre $F \subset \mathbb{R}^2$ of $M = \mathbb{R} \times F$ be connected. Then there exists $\varepsilon_0 > 0$ such that λ_0^{m} satisfies Condition 2.26 for all $0 < \varepsilon < \varepsilon_0$.*

PROOF. We first note that the ground state band λ_0 of the unperturbed operator $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ satisfies Condition 2.26 by [Lam14, Lemma 3.7], i.e., there is a constant $d > 0$ such that

$$\operatorname{dist}\Big(\lambda_0(x), \sigma\big(-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}(x)\big)\backslash\big\{\lambda_0(x)\big\}\Big) \geqslant d$$

for all $x \in \mathbb{R}$. Moreover, $\bar{\lambda}_0 := \|\lambda_0\|_{L^\infty(\mathbb{R})} < \infty$ by Proposition 3.14. As far as the perturbation

$$T_\varepsilon = -2\mathrm{i}\mathcal{A}_{\mathsf{V}}^\varepsilon\big(\operatorname{div}_{g_{\mathsf{V}}} \cdot\big) + \mathrm{i}\big(\mathrm{d}^*\mathcal{A}_{\mathsf{V}}^\varepsilon\big) + \varepsilon^2 \operatorname{tr}_{g_{\mathsf{V}}}\big(\mathcal{A}_{\mathsf{V}}^\varepsilon \otimes \mathcal{A}_{\mathsf{V}}^\varepsilon\big)$$

in $H^{\mathcal{F},\varepsilon,\mathrm{w}} = H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_{\mathsf{V}}^\varepsilon=0} + \varepsilon^2 T_\varepsilon$ is concerned, we obtain the estimate

$$\begin{aligned}
&\Big\|\varepsilon^2 T_\varepsilon \psi\Big\|_{L^2(M_x)} \\
&\leqslant 2\varepsilon^2\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(\mathsf{T}^*M)}\Big\|\operatorname{grad}_{g_{M_x}} \psi\Big\|_{L^2(M_x)} \\
&\quad + \bigg(\varepsilon^2\Big\|\mathrm{d}^*\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(M)} + \underbrace{\varepsilon^4\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|^2_{L^\infty(\mathsf{T}^*M)}}_{\substack{\leqslant \varepsilon^2\|\mathcal{A}_{\mathsf{V}}^\varepsilon\|_{L^\infty(\mathsf{T}^*M)} \\ \text{for } \varepsilon \text{ small enough}}}\bigg)\|\psi\|_{L^2(M_x)} \\
&\leqslant 2\varepsilon^2\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(\mathsf{T}^*M)}\bigg(\Big\|H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_{\mathsf{V}}^\varepsilon=0}(x)\psi\Big\|_{L^2(M_x)} + \tfrac{1}{2}\|\psi\|_{L^2(M_x)}\bigg) \\
&\quad + \bigg(\varepsilon^2\Big\|\mathrm{d}^*\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(M)} + \varepsilon^2\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(\mathsf{T}^*M)}\bigg)\|\psi\|_{L^2(M_x)} \\
&= a_\varepsilon\|\psi\|_{L^2(M_x)} + b_\varepsilon\Big\|H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_{\mathsf{V}}^\varepsilon=0}(x)\psi\Big\|_{L^2(M_x)}
\end{aligned}$$

for all $\psi \in \mathcal{D}_{\mathcal{F}}|_x$ with x-uniform bounds

$$\begin{aligned}
a_\varepsilon &= 2\varepsilon^2\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(\mathsf{T}^*M)} + \varepsilon^2\Big\|\mathrm{d}^*\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(M)}, \\
b_\varepsilon &= 2\varepsilon^2\Big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\Big\|_{L^\infty(\mathsf{T}^*M)}.
\end{aligned}$$

We infer from the C^∞-boundedness of $\mathcal{A}_{\mathsf{V}}^\varepsilon$ that the quantity

$$\varepsilon_0 := \sqrt{\frac{d/2}{\big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\big\| + 2\big\|\mathrm{d}^*\mathcal{A}_{\mathsf{V}}^\varepsilon\big\| + \big\|\mathcal{A}_{\mathsf{V}}^\varepsilon\big\|\big(\bar{\lambda}_0 + d\big)}} > 0$$

is well-defined and one easily verifies the inequality

$$a_\varepsilon + b_\varepsilon\big(\lambda_0(x) + d\big) < \tfrac{d}{2}$$

for all $x \in \mathbb{R}$ and all $0 < \varepsilon < \varepsilon_0$. Consequently, [Kat80, Theorem IV−3.18] yields

$$\sigma\big(H^{\mathcal{F},\varepsilon,\mathrm{w}}(x)\big) \cap \Big(\lambda_0(x) - \tfrac{d}{2}, \lambda_0(x) + \tfrac{d}{2}\Big) = \big\{\lambda_0^{\mathrm{m}}(x)\big\}$$

for all $x \in \mathbb{R}$ and all $0 < \varepsilon < \varepsilon_0$. Thus, Condition 2.26 is satisfied with spectral gap $\delta = d/4$ and separation functions $f_\pm(x) := \lambda_0(x) \pm \delta$. □

This lemma allows for the definition of the spectral projection

$$P_0^{\mathrm{m}}(x) = \frac{\mathrm{i}}{2\pi} \int_{|z-\lambda_0(x)|=\delta} \big(H^{\mathcal{F},\varepsilon,\mathrm{w}}(x) - z\mathbf{1}_{\mathcal{H}_\mathcal{F}}\big)^{-1}\,\mathrm{d}z$$

associated with the magnetic ground state band λ_0^{m}, which gives rise to the corresponding smooth normalised ground state

$$\phi_0^{\mathrm{m}}(x) = \frac{P_0^{\mathrm{m}}(x)\phi_0(x)}{\big\|P_0^{\mathrm{m}}\phi_0\big\|_{\mathcal{H}_\mathcal{F}|_x}} \quad \rightsquigarrow \quad P_0^{\mathrm{m}} = \big\langle \phi_0^{\mathrm{m}}, \cdot \big\rangle_{\mathcal{H}_\mathcal{F}} \phi_0^{\mathrm{m}}.$$

The bounded geometry of the waveguide M (cf. Definition 5.3) and the C^∞-boundedness of the vertical magnetic potential imply that

$$\partial_x^k\Big(H^{\mathcal{F},\varepsilon,\mathrm{w}}(x) - H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_\mathrm{V}^\varepsilon=0}(x)\Big) \leqslant C(k)\varepsilon^2$$

in the norm-resolvent sense for all $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Consequently in view of

$$P_0^{\mathrm{m}} - P_0 = \frac{1}{2\pi\mathrm{i}} \int_{|z-\lambda_0|=\delta} R^{\mathcal{F},\varepsilon,\mathrm{w}}(z)\Big(H^{\mathcal{F},\varepsilon,\mathrm{w}} - H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_\mathrm{V}^\varepsilon=0}\Big)R^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_\mathrm{V}^\varepsilon=0}(z)\,\mathrm{d}z,$$

the magnetic ground state ϕ_0^{m} is very close to the unperturbed state ϕ_0, i.e., it holds for all $k \in \mathbb{N}_0$ that

$$\sup_{x\in\mathbb{R}} \Big\| \big(\partial_x^{\mathrm{H}}\big)^k (\phi_0^{\mathrm{m}} - \phi_0) \Big\|_{\mathcal{H}_\mathcal{F}|_x} = \mathcal{O}(\varepsilon^2). \tag{5.31}$$

Theorem 5.12 states that the low-lying eigenvalues of the initial tube operator $H^{\mathrm{w}}_{\mathrm{tube}}$ may be approximated by those of the adiabatic operator

$$\begin{aligned} H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}} &= P_0^{\mathrm{m}} H^{\mathcal{E},\mathrm{w}} P_0^{\mathrm{m}} \\ &\overset{(4.12)}{=} \underbrace{-\varepsilon^2 \Delta^{\mathrm{B}} + \varepsilon^2 V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}}}_{\text{Berry terms}} + \lambda_0^{\mathrm{m}} \mathbf{1}_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}} + \underbrace{\varepsilon P_0^{\mathrm{m}} H_1^{\mathcal{E},\mathrm{w}} P_0^{\mathrm{m}}}_{\text{perturbation}} \end{aligned}$$

acting on $\mathcal{H}_{\mathcal{P}^{\mathrm{m}}} = L^2(\mathcal{P}^{\mathrm{m}})$, i.e., on the L^2-sections of the eigenspace bundle $\mathcal{P}^{\mathrm{m}} = P_0^{\mathrm{m}} \mathcal{H}_{\mathcal{F}}$. To be more precise, with "low-lying" we mean those eigenvalues of $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}}$ with a distance of order ε^{α} to the bottom

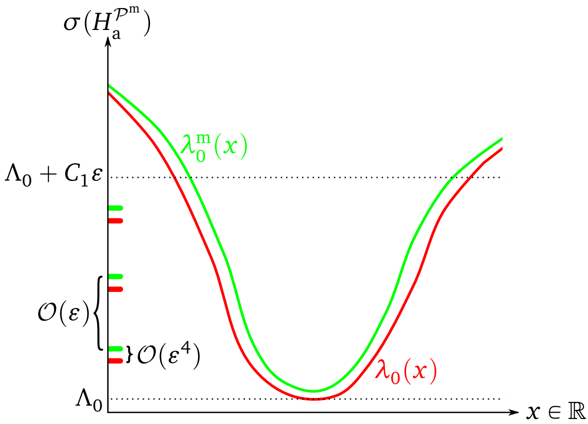
$$\Lambda_0 := \min_{0 \leqslant \varepsilon \leqslant 1} \inf_{x \in \mathbb{R}} \lambda_0^{\mathrm{m}}(x) = \inf_{x \in \mathbb{R}} \lambda_0(x) \tag{5.32}$$

of $\sigma(H^{\mathcal{F},\varepsilon,\mathrm{w}})$. The correct scaling ε^{α} of the eigenvalues can be read off the renormalised leading term of the adiabatic operator, which is given by

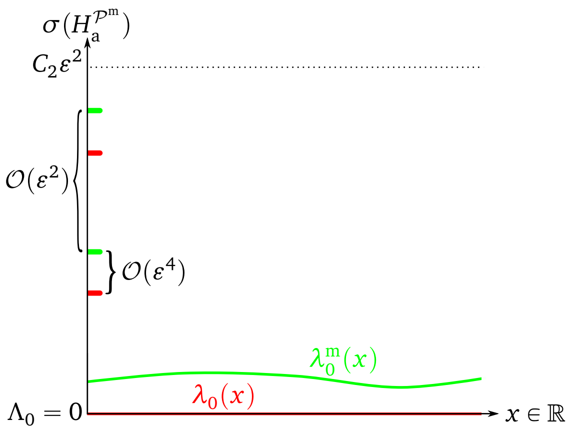
$$-\varepsilon^2 \Delta^{\mathrm{B}} + \big(\lambda_0^{\mathrm{m}} - \Lambda_0\big) \mathbf{1}_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}} = -\varepsilon^2 \Delta^{\mathrm{B}} + (\lambda_0 - \Lambda_0) \mathbf{1}_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}} + \mathcal{O}(\varepsilon^4).$$

We observe that this is essentially determined by the behaviour $x \mapsto \lambda_0(x)$ of the ground state band associated with the unperturbed vertical operator. Consequently, we retrieve $\alpha = 1$ for massive waveguides (with λ_0 having a unique non-degenerate minimum) and $\alpha = 2$ for hollow waveguides.

We saw above that the switching on of the magnetic field in general leads to an $\mathcal{O}(\varepsilon^4)$-increase of the ground state band ($\lambda_0 \to \lambda_0^{\mathrm{m}}$) and consequently to an increase of the same order for the eigenvalues of the adiabatic operator $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}}$ (cf. Figure 5.7). In both cases the shift of the ground state due to the magnetic field is much smaller than the level spacing of the low-lying eigenvalues of $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}}$ (three orders in ε for the massive waveguide and two orders in ε for the hollow waveguide). Hence, these eigenvalues stay ε^{α}-close to Λ_0 and the constants C_1 and C_2 can be chosen independently of the vertical magnetic potential $\mathcal{A}_{\mathrm{V}}^{\varepsilon}$ for $\varepsilon > 0$ small enough.



(a) massive quantum waveguides ($\alpha = 1$)



(b) hollow quantum waveguides ($\alpha = 2$)

Figure 5.7: The low-lying eigenvalues of the adiabatic operator $H_a^{\mathcal{P}^m}$ associated with the ground state band below $\Lambda_0 + C_\alpha\varepsilon^\alpha$ (a) for massive quantum waveguides, where λ_0 has a unique and non-degenerate minimum, and (b) for hollow quantum waveguides are depicted on the vertical axis in the absence (red, $\mathcal{A}_V^\varepsilon = 0$) and presence (green, $\mathcal{A}_V^\varepsilon \neq 0$) of the magnetic potential.

The global trivialisation of the eigenspace bundle $\mathcal{P}^{\mathrm{m}}$ via the ground state ϕ_0^{m} yields an identification of $\mathcal{H}_{\mathcal{P}^{\mathrm{m}}} \subset L^2(M)$ with $L^2(\mathbb{R}) \otimes (\phi_0^{\mathrm{m}})$ and the associated adiabatic operator may be treated as an operator H_{a} acting on $L^2(\mathbb{R})$. In the next two subsections, we will calculate the adiabatic operators for the waveguides introduced in Example 5.7 and Example 5.9 up to errors of order $\varepsilon^{2+\alpha}$ in $\mathcal{L}(\mathrm{dom}(\varepsilon^{-\alpha} H_{\mathrm{a},0}), L^2(\mathbb{R}))$, where $H_{\mathrm{a},0} = -\varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathrm{d}x \otimes \mathrm{d}x} + (\lambda_0 - \Lambda_0)$, which is the exact accuracy provided by Theorem 5.12.

5.2.1 Example for Massive Quantum Waveguides

We consider a single non-relativistic, spinless, charged quantum particle, which is localised within a family of massive tubes $\{\mathcal{T}^\varepsilon \subset \mathbb{R}^3\}_{0<\varepsilon\leqslant 1}$ around a smoothly embedded curve $c : \mathbb{R} \to \mathbb{R}^3$ with twisted elliptical discs as respective cross-sections (see Example 5.7 for the details), in the presence of an external magnetic field represented by some potential $\mathcal{A} \in C_{\mathrm{b}}^\infty(\mathsf{T}^*\mathbb{R}^3|_{\mathcal{T}^\varepsilon})$. The quantum waveguide turns out to be the trivial fibre bundle $M = \mathbb{R} \times \mathbb{B}_1(0) \xrightarrow{\mathrm{pr}_1} \mathbb{R}$ with coordinates (x, y^1, y^2) and rescaled pullback metric (5.10)

$$G^\varepsilon = \varepsilon^{-2}\big(1 - \varepsilon \langle \mathfrak{r} y, \kappa \rangle_{\mathbb{R}^2}\big)^2 \, \mathrm{d}x^{\mathsf{H}} \otimes \mathrm{d}x^{\mathsf{H}} + g_{\mathsf{V}},$$

where the vertical contribution reads

$$g_{\mathsf{V}}|_{M_x} = g_{M_x} = \mathfrak{a}^2(x)\, \mathrm{d}y^1 \otimes \mathrm{d}y^1 + \mathfrak{b}^2(x)\, \mathrm{d}y^2 \otimes \mathrm{d}y^2.$$

The corresponding Schrödinger operator (5.27) on $L^2(M, \mathrm{vol}_g)$ incorporates the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ (5.28a) and the perturbation given by $\varepsilon H_1^{\mathcal{E},\mathrm{w}} = \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}} + V_{\rho_\varepsilon}$.

Vertical Operator

We briefly discuss the ground state associated with the unperturbed operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}^\varepsilon_{\mathsf{V}}=0} = -\Delta^{\mathrm{L.B.}}_{\mathsf{V}}$. The fibrewise unitary map $\widehat{V}(x)$ from $\mathcal{H}_{\mathcal{F}}|_x$ to $L^2(\mathbb{E}^2_{\mathfrak{a}(x),\mathfrak{b}(x)}, \mathrm{d}n)$, given by

$$\phi \mapsto \big(\widehat{V}(x)\phi\big)(y^1, y^2) := \phi\left(\tfrac{y^1}{\mathfrak{a}(x)}, \tfrac{y^2}{\mathfrak{b}(x)}\right),$$

induces a unitarily equivalent operator $-\Delta^{\mathrm{L.B.}}_{\delta^2}$ with Dirichlet boundary conditions on the x-dependent elliptical domain

$$\mathbb{E}^2_{\mathfrak{a}(x),\mathfrak{b}(x)} := \left\{ (y^1, y^2) \in \mathbb{R}^2 : \ \left(\tfrac{y^1}{\mathfrak{a}(x)}\right)^2 + \left(\tfrac{y^2}{\mathfrak{b}(x)}\right)^2 \leqslant 1 \right\}.$$

Introducing elliptical coordinates

$$\begin{aligned} \mathfrak{E}(x) : \big[0, \xi_0(x)\big] \times [0, 2\pi) &\to \mathbb{E}^2_{\mathfrak{a}(x),\mathfrak{b}(x)}, \\ (\xi, \eta) &\mapsto \big(\mathfrak{f}(x) \cosh \xi \cos \eta, \mathfrak{f}(x) \sinh \xi, \sin \eta\big) \end{aligned}$$

for all $x \in \mathbb{R}$ with focal length function $\mathfrak{f} = \sqrt{\mathfrak{a}^2 - \mathfrak{b}^2}$ and elliptical boundary $\xi_0 = \operatorname{artanh}\left(\frac{\mathfrak{b}}{\mathfrak{a}}\right)$, we see that the non-degenerate, positive and normalised ground state of $-\Delta^{\mathrm{L.B.}}_{\delta^2}$ is given by [GRM03]:

$$\Phi_0(x) = \frac{1}{\sqrt{C(x)}} \varphi_0(x) \circ \mathfrak{E}(x)^{-1},$$

with

$$\big(\varphi_0(x)\big)(\xi, \eta) := \mathrm{Je}_0\big(\xi; q_{01}(x)\big)\, \mathrm{ce}_0\big(\eta; q_{01}(x)\big)$$

and smooth normalisation function

$$C(x) = \int_0^{\xi_0(x)} \int_0^{2\pi} \Big(\big(\varphi_0(x)\big)(\xi, \eta)\Big)^2 \big(\cosh(2\xi) - \cos(2\eta)\big)\, \mathrm{d}\eta\, \mathrm{d}\xi.$$

Here, Je_0 denotes the zeroth radial Mathieu function and ce_0 the zeroth angular Mathieu function. By reason of the Dirichlet boundary conditions, both functions depend implicitly on the x-dependent first zero $q_{01}(x)$ of the radial function Je_0 (which takes the role of the radial Bessel function in the case of disc-shaped domains) at the elliptical boundary $\xi_0(x)$, i.e., it is the solution of $\mathrm{Je}_0(\xi_0(x); q) = 0$. The ground state associated with $-\Delta^{\mathrm{L.B.}}_{\mathsf{V}}$ finally reads

$$\begin{aligned} \big(\phi_0(x)\big)(y^1, y^2) &:= \big(\widehat{V}^\dagger(x) \Phi_0(x)\big)(y^1, y^2) \\ &= \big(\Phi_0(x)\big)\big(\mathfrak{a}(x) y^1, \mathfrak{b}(x) y^2\big). \end{aligned}$$

The boundedness of the semi-axes with all their derivatives directly carries over to the ground state $\phi_0 \in C_{\mathrm{b}}^{\infty}(M)$. In the end, we remark that the elliptical cross-section is positioned symmetrically with respect to the curve $c(\mathbb{R})$, and thus ϕ_0 is centred and satisfies (5.29).

The corresponding ground state band is given by $\lambda_0(x) = \frac{4q_{01}(x)}{\mathfrak{f}(x)}$. Hence, the requirement that $\lambda_0(x)$ have a unique non-degenerate minimum is an implicit condition on the semi-axis functions $\mathfrak{a}(x)$ and $\mathfrak{b}(x)$. Moreover, the ground state band is separated from the rest of the spectrum by a gap $\frac{4}{\mathfrak{f}(x)}(q_{11}(x) - q_{01}(x))$, where $q_{11}(x)$ is first zero of the first (even) radial Mathieu function. The uniformity properties of the semi-axis functions $r_- \leqslant \mathfrak{b}(x) < \mathfrak{a}(x) \leqslant r_+$ imply the uniformity of the spectral gap, i.e., there is an x-independent constant $d > 0$ such that

$$\inf_{x\in\mathbb{R}} \operatorname{dist}\Big(\lambda_0(x), \sigma\big(-\Delta^{\mathrm{L.B.}}_{g_{M_x}}\big)\backslash\{\lambda_0(x)\}\Big) \geqslant d.$$

Moreover, it is shown in [Mak59] that $\|\lambda_0\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{3}\frac{L}{A}$ with perimeter $L \leqslant 4r_+$ and area $A \geqslant \pi r_-^2$ of the respective elliptical domains.

Berry Terms

The Berry connection (4.10) is given by $\nabla^{\mathrm{B}}_{\partial_x} = P_0^{\mathrm{m}} \partial_x^{\mathsf{H}} P_0^{\mathrm{m}}$ for $\partial_x \in C^\infty(\mathsf{T}\mathbb{R})$. The massiveness of the waveguide yields $\eta_{\mathsf{V}} = 0$, and hence this defines a metric connection (i.e., ∇^{B} coincides with $\nabla^{\mathcal{P}^{\mathrm{m}}}$, see Subsection 4.3.1). Thus, it holds that

$$\begin{aligned} 0 &= \partial_x \cdot \big\langle \phi_0^{\mathrm{m}}, \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{P}^{\mathrm{m}}} = \big\langle \nabla^{\mathrm{B}}_{\partial_x} \phi_0^{\mathrm{m}}, \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{P}^{\mathrm{m}}} + \big\langle \phi_0^{\mathrm{m}}, \nabla^{\mathrm{B}}_{\partial_x} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{P}^{\mathrm{m}}} \\ &= 2\,\mathrm{Re}\left(\big\langle \phi_0^{\mathrm{m}}, \nabla^{\mathrm{B}}_{\partial_x} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{P}^{\mathrm{m}}}\right) = 2\,\mathrm{Re}\left(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}}\right) \end{aligned}$$

and $\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}}$ is purely imaginary. The application of ∇^{B} to $\psi\phi_0^{\mathrm{m}}$ for $\psi \in C^\infty(\mathbb{R})$ is then calculated to be

$$\nabla^{\mathrm{B}}_{\partial_x}(\psi\phi_0^{\mathrm{m}}) = \mathrm{d}\psi(\partial_x)\phi_0^{\mathrm{m}} + \psi P_0^{\mathrm{m}} \nabla^{\mathrm{B}}_{\partial_x} \phi_0^{\mathrm{m}} = \Big(\underbrace{(\mathrm{d} + \mathrm{i}\mathcal{A}^{\mathrm{B}})}_{=:\nabla^{\mathcal{A}^{\mathrm{B}}}}(\partial_x)\psi\Big)\phi_0^{\mathrm{m}},$$

where the "effective magnetic potential" reads

$$\begin{aligned}\mathcal{A}^{\mathrm{B}}(\partial_x) &= \operatorname{Im}\Big(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big) \\ &\overset{(5.31)}{=} \underbrace{\operatorname{Im}\Big(\big\langle \phi_0, \partial_x^{\mathsf{H}} \phi_0 \big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)}_{=0 \text{ since } \phi_0 \text{ is real-valued}} + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{5.33}$$

Finally, the Berry Laplacian applied on $\psi\phi_0^{\mathrm{m}}$ equals

$$\Delta^{\mathrm{B}}(\psi\phi_0^{\mathrm{m}}) = \big(\Delta^{\mathcal{A}^{\mathrm{B}}}_{\mathrm{d}x\otimes\mathrm{d}x}\psi\big)\phi_0^{\mathrm{m}} = \Big(\big(\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + \mathcal{O}(\varepsilon^{3/2})\big)\psi\Big)\phi_0^{\mathrm{m}}.$$

The difference between the Berry Laplacian and the projected horizontal Laplace operator equals the Born-Huang potential

$$\begin{aligned} V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}}\phi_0^{\mathrm{m}} &= P_0^{\mathrm{m}}\big(\partial_x^{\mathsf{H}} P_0^{\mathrm{m}} \partial_x^{\mathsf{H}} - \Delta_{\mathsf{H}}^{\mathrm{L.B.}}\big)\phi_0^{\mathrm{m}} \\ &= \Big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}}\Big(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}}\Big)\Big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}} \\ &\quad - \big\langle \phi_0^{\mathrm{m}}, \Delta_{\mathsf{H}}^{\mathrm{L.B.}}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}} \\ &= \Big(\big\|\partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\|^2_{\mathcal{H}_{\mathcal{F}}} - \underbrace{\big|\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\big|^2}_{=\mathcal{O}(\varepsilon^4)}\Big)\phi_0^{\mathrm{m}} \\ &\overset{(5.31)}{=} \Big(\big\|\partial_x^{\mathsf{H}}\phi_0\big\|^2_{\mathcal{H}_{\mathcal{F}}} + \mathcal{O}(\varepsilon^2)\Big)\phi_0^{\mathrm{m}}. \end{aligned} \tag{5.34}$$

Its leading term is non-negative and has been mainly studied in the context of "twisted quantum waveguides" with isometric cross-sections, see for example [CB96, EKK08].

Example 5.14 Let us consider the special case of a fixed elliptical domain with constant semi-axes $\mathfrak{a}$ and $\mathfrak{b}$. This implies that

- the horizontal lift (5.8) simplifies to

$$\partial_x^{\mathsf{H}} = \partial_x^{\mathrm{pr}} - \vartheta'\Big(\tfrac{\mathfrak{a}}{\mathfrak{b}}y^1\,\partial_{y^2} - \tfrac{\mathfrak{b}}{\mathfrak{a}}y^2\,\partial_{y^1}\Big) =: \partial_x^{\mathrm{pr}} - \mathrm{i}\vartheta' L_y^{\mathfrak{a},\mathfrak{b}}, \tag{5.35}$$

- and the leading order ϕ_0 of the vertical ground state becomes (up to the x-dependent twist) independent of the base coordinate x.

Consequently, the main contribution of the Born-Huang potential is rather simple, to be specific, one has

$$V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}}(x) = \big(\vartheta'(x)\big)^2 \big\| L_y^{\mathfrak{a},\mathfrak{b}} \phi_0 \big\|^2_{\mathcal{H}_{\mathcal{F}}|_x} + \mathcal{O}(\varepsilon^2).$$

This expression is in agreement with the result in [KŠ12], i.e., $V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}}$ coincides with the well-known twisting potential plus some very small ε-corrections due to the presence of the magnetic field. ◊

Perturbation

In order to calculate $\varepsilon P_0^{\mathrm{m}} H_1^{\mathcal{E},\mathrm{w}} P_0^{\mathrm{m}}$ up to errors of order ε^3 with respect to the norm $\mathcal{L}(\mathrm{dom}(\varepsilon^{-1}H_{\mathrm{a},0}), L^2(\mathbb{R}))$, first note that any $\psi \in \mathrm{dom}(\varepsilon^{-1}H_{\mathrm{a},0})$ of order one oscillates on a length scale of order $\varepsilon^{-1/2}$, i.e.,

$$\big\| \mathrm{d}\psi(\varepsilon\partial_x)\big\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{1/2}) \quad \Leftrightarrow \quad \big\|\varepsilon\partial_x^{\mathsf{H}}(\psi\phi_0^{\mathrm{m}})\big\|_{\mathcal{H}_{\mathcal{P}^{\mathrm{m}}}} = \mathcal{O}(\varepsilon^{1/2}).$$

Moreover, the potential term appearing in $\tilde{H}_1^{\mathcal{E},\mathrm{w}}$ is already of order ε^3 and may a priori be neglected in the following considerations.

We need an explicit formula for the tensor σ^ε which encodes the difference between the rescaled pullback metric G^ε and the rescaled submersion metric g^ε due to the extrinsic curvature of the curve. A small calculation shows

$$\begin{aligned}
\sigma^\varepsilon(\mathrm{d}x^{\mathsf{H}}, \mathrm{d}x^{\mathsf{H}}) &= \varepsilon^{-3}\Big(\widetilde{G}^\varepsilon(\mathrm{d}x^{\mathsf{H}}, \mathrm{d}x^{\mathsf{H}}) - \widetilde{g}^\varepsilon(\mathrm{d}x^{\mathsf{H}}, \mathrm{d}x^{\mathsf{H}})\Big)\\
&= \varepsilon^{-1}\left(\frac{1}{(1-\varepsilon\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2})^2} - 1\right)\\
&= \sum_{l=0}^{\infty} \varepsilon^l (l+2)\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2}^{l+1}\\
&= 2\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2} + 3\varepsilon\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2}^2 + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{5.36}$$

with errors in $C_{\mathrm{b}}^\infty(M)$. As far as the geometric potential V_{ρ_ε} (5.26) is concerned, we first determine the associated Radon-Nikodym density to be

$$\rho_\varepsilon = \sqrt{\frac{\det(G^\varepsilon)}{\det(g^\varepsilon)}} = \sqrt{\frac{\varepsilon^{-2}(1-\varepsilon\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2})^2 \mathfrak{a}^2\mathfrak{b}^2}{\varepsilon^{-2}\mathfrak{a}^2\mathfrak{b}^2}} = 1 - \varepsilon\langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2},$$

which gives

$$\ln\rho_\varepsilon = -\varepsilon\,\langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2} - \tfrac{1}{2}\varepsilon^2\,\langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}^2 - \tfrac{1}{3}\varepsilon^3\,\langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}^3 + \mathcal{O}(\varepsilon^4). \qquad (5.37)$$

We may now treat the three terms appearing in V_{ρ_ε} separately:

- Using the facts that

  $$\mathrm{d}\ln\rho_\varepsilon = -\varepsilon\Big(\big(\partial_x^{\mathsf{H}}\cdot\langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\big)\,\mathrm{d}x^{\mathsf{H}} + \langle \mathfrak{r}^{\mathrm{t}}\kappa,\mathrm{d}y\rangle_{\mathbb{R}^2}\Big) + \mathcal{O}(\varepsilon^2)$$

  and $\widetilde{g}_{\mathsf{V}} = \mathfrak{a}^{-2}\,\partial_{y^1}\otimes\partial_{y^1} + \mathfrak{b}^{-2}\,\partial_{y^2}\otimes\partial_{y^2}$, it immediately follows that

  $$\begin{aligned}\tfrac{1}{4}\widetilde{g}_{\mathsf{V}}(\mathrm{d}\ln\rho_\varepsilon,\mathrm{d}\ln\rho_\varepsilon) &= \tfrac{1}{4}\varepsilon^2\left[\big(\tfrac{1}{\mathfrak{a}}(\mathfrak{r}^{\mathrm{t}}\kappa)^1\big)^2 + \big(\tfrac{1}{\mathfrak{b}}(\mathfrak{r}^{\mathrm{t}}\kappa)^2\big)^2\right] + \mathcal{O}(\varepsilon^3)\\ &= \tfrac{1}{4}\varepsilon^2\,\|\kappa\|_{\mathbb{R}^2}^2 + \mathcal{O}(\varepsilon^3)\end{aligned}$$

- A similar computation shows

  $$\begin{aligned}\tfrac{1}{2}\Delta_{\mathsf{V}}^{\mathrm{L.B.}}\ln\rho_\varepsilon &= -\tfrac{1}{2}\varepsilon^2\big(\mathfrak{a}^{-2}\,\partial_{y^1}^2 + \mathfrak{b}^{-2}\,\partial_{y^2}^2\big)\frac{\langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}^2}{2} + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{2}\varepsilon^2\Big(\tfrac{1}{\mathfrak{a}^2}(\mathfrak{r}^{\mathrm{t}}\kappa)^1(\mathfrak{r}^{\mathrm{t}}\kappa)^1 + \tfrac{1}{\mathfrak{b}^2}(\mathfrak{r}^{\mathrm{t}}\kappa)^2(\mathfrak{r}^{\mathrm{t}}\kappa)^2\Big) + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{2}\varepsilon^2\,\|\kappa\|_{\mathbb{R}^2}^2 + \mathcal{O}(\varepsilon^3).\end{aligned}$$

- The contribution of $\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}\ln\rho_\varepsilon$ is of order ε^3 and hence negligible for the regime $\alpha = 1$.

Thus, the leading-order contribution of the geometric potential V_{ρ_ε} depends only on the base coordinate x and is given by

$$V_{\rho_\varepsilon}(x,y) = -\tfrac{1}{4}\varepsilon^2\,\big\|\kappa(x)\big\|_{\mathbb{R}^2}^2 + \mathcal{O}(\varepsilon^3). \qquad (5.38)$$

This attractive leading-order contribution has been widely discussed in the literature in the context of “bent quantum waveguides”, see for instance [DE95, EKK08]. Its effect due to the extrinsic curvature has been known in fact for a long time [Tol88]. In particular, the sign of this potential is indefinite if one considers higher dimensional base manifolds [Haa12].

We are left with the calculation of the differential operators within $\tilde{H}_1^{\mathcal{E}}$. Many terms vanish by virtue of the facts that $\nabla^g_{\partial_x^{\mathsf{H}}}\partial_x^{\mathsf{H}} = 0 = \widetilde{\nabla}^g_{\partial_x^{\mathsf{H}}}\mathrm{d}x^{\mathsf{H}}$ and $\eta_{\mathsf{V}} = 0$, which results in

$$\big(\nabla^{\mathsf{H}M^{\otimes 2}}_{\partial_x^{\mathsf{H}}}\sigma^\varepsilon\big)(\mathrm{d}x^{\mathsf{H}},\mathrm{d}x^{\mathsf{H}}) = \partial_x^{\mathsf{H}}\cdot\sigma^\varepsilon(\mathrm{d}x^{\mathsf{H}},\mathrm{d}x^{\mathsf{H}}).$$

Recalling that $\varepsilon\partial_x^{\mathsf{H}} = \mathcal{O}(\varepsilon^{1/2})$ on $\mathrm{dom}(\varepsilon^{-1}H_{\mathrm{a},0})\otimes\mathrm{span}(\phi_0^{\mathrm{m}})$, we get

$$\begin{aligned}
\varepsilon\tilde{H}_1^{\mathcal{E},\mathrm{w}} &= -\varepsilon^2\partial_x^{\mathsf{H}}\big(\sigma^\varepsilon(\mathrm{d}x^{\mathsf{H}},\mathrm{d}x^{\mathsf{H}})\varepsilon\partial_x^{\mathsf{H}}\cdot\big) - 2\mathrm{i}\varepsilon^2\mathcal{A}^\varepsilon_{\mathsf{H}}(\partial_x^{\mathsf{H}})\varepsilon\partial_x^{\mathsf{H}} + \mathcal{O}(\varepsilon^3)\\
&\overset{(5.22)}{\underset{(5.36)}{=}} \underbrace{-2\varepsilon^2\partial_x^{\mathsf{H}}\big(\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\varepsilon\partial_x^{\mathsf{H}}\cdot\big)}_{=\mathcal{O}(\varepsilon^2)}\underbrace{-2\mathrm{i}\varepsilon^2\big(\mathcal{B}_\perp\times(\mathfrak{r}y)\big)\varepsilon\partial_x^{\mathsf{H}}}_{=\mathcal{O}(\varepsilon^{5/2})} + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

We then apply $P_0^{\mathrm{m}} = \langle\phi_0^{\mathrm{m}},\cdot\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}}$ on both sides and evaluate for any function $\psi\in C^\infty(\mathbb{R})$:

$$\begin{aligned}
&\varepsilon P_0^{\mathrm{m}}\tilde{H}_1^{\mathcal{E},\mathrm{w}}P_0^{\mathrm{m}}(\psi\phi_0^{\mathrm{m}})\\
&\overset{(5.31)}{=} \varepsilon\big\langle\phi_0,\tilde{H}_1^{\mathcal{E}}(\psi\phi_0)\big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}} + \mathcal{O}(\varepsilon^3)\\
&= -2\varepsilon\big\langle\phi_0,\varepsilon\partial_x^{\mathsf{H}}\big(\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\varepsilon\partial_x^{\mathsf{H}}(\psi\phi_0)\big)\big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}} && (5.39a)\\
&\quad - 2\mathrm{i}\varepsilon^2\big\langle\phi_0,\big(\mathcal{B}_\perp\times(\mathfrak{r}y)\big)\varepsilon\partial_x^{\mathsf{H}}(\psi\phi_0)\big\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}} && (5.39b)\\
&\quad + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Because of the fact that $\phi_0(x)$ is real-valued and vanishes on $\partial\mathbb{B}_1^2(0)$ for all $x\in\mathbb{R}$, we can rewrite (5.39a) as

$$\begin{aligned}
&-2\varepsilon\big\langle\phi_0,\big(\varepsilon\partial_x^{\mathsf{H}}\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\big)\phi_0 + 2\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\varepsilon\partial_x^{\mathsf{H}}\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}\mathrm{d}\psi(\varepsilon\partial_x)\phi_0^{\mathrm{m}}\\
&-2\varepsilon\langle\phi_0,\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\phi_0\rangle_{\mathcal{H}_{\mathcal{F}}}\big(\varepsilon^2\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x}\psi\big)\phi_0^{\mathrm{m}}\\
&+\mathcal{O}(\varepsilon^3)\\
&= \Big(\varepsilon\partial_x\Big(-2\varepsilon\langle\phi_0,\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\phi_0\rangle_{\mathcal{H}_{\mathcal{F}}}\,\mathrm{d}\psi(\varepsilon\partial_x)\Big)\Big)\phi_0^{\mathrm{m}} + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

But this terms vanishes since $\langle\mathfrak{r}(x)y,\kappa\rangle_{\mathbb{R}^2}$ is linear in the fibre coordinate y and the barycentre of the unperturbed ground state ϕ_0 lies on the curve (see (5.29)). One likewise concludes that the second term (5.39b)

$$-2\mathrm{i}\varepsilon^2 \underbrace{\big\langle \phi_0, \big(\mathcal{B}_\perp \times (\mathfrak{r}y)\big)\phi_0 \big\rangle_{\mathcal{H}_\mathcal{F}}}_{=0 \text{ since } \phi_0 \text{ is centred (5.29)}} \mathrm{d}\psi(\varepsilon\partial_x)\phi_0^{\mathrm{m}} + \mathcal{O}(\varepsilon^3)$$

is of lower order.

Conclusion

We are now in a position to gather all the relevant terms in the adiabatic operator up to errors of order ε^3 in $\mathcal{L}(\mathrm{dom}(\varepsilon^{-1}H_{\mathrm{a},0}), L^2(\mathbb{R}))$. We then use the low-lying eigenvalues of the resulting operator to approximate those of the initial tube operator by means of Theorem 5.12:

Corollary 5.15 *Let $H^{\mathrm{w,mas}}_{\mathrm{tube}}$ be the Hamiltonian generating the dynamics of a non-relativistic, charged, spinless quantum particle, which is localised within the massive quantum waveguide introduced in Example 5.7, in the presence of a weak, C^∞-bounded external magnetic potential $\mathcal{A}$. Assume that*

(i) the associated unperturbed ground state band λ_0 admits a unique non-degenerate minimum,

(ii) and there is a constant $C > 0$ such that $C\varepsilon$ is strictly below the essential spectrum of

$$\begin{aligned} H^{\mathrm{w,mas}}_{\mathrm{QWG}} := -\varepsilon^2\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + (\lambda_0 - \Lambda_0) \\ + \varepsilon^2\Big(\big\|\mathrm{d}\phi_0(\partial_x^{\mathsf{H}})\big\|^2_{\mathcal{H}_\mathcal{F}} - \tfrac{1}{4}\,\|\kappa\|^2_{\mathbb{R}^2}\Big) \end{aligned}$$

in the sense of Theorem 4.15.

Then if $\nu^\varepsilon < C\varepsilon$ is an eigenvalue of $H^{\mathrm{w,mas}}_{\mathrm{QWG}}$, there exists an eigenvalue υ^ε of $H^{\mathrm{w,mas}}_{\mathrm{tube}}$ below its essential spectrum with the asymptotic expansion

$$\upsilon^\varepsilon = \Lambda_0 + \nu^\varepsilon + \mathcal{O}(\varepsilon^3).$$

We observe that the magnetic effects are not apparent up to this accuracy.

We close this subsection with a discussion of the situation for a constant ground state band λ_0. This can be attained by

(i) smoothly varying the semi-axes $\mathfrak{a}(x), \mathfrak{b}(x)$ in such a way that the ratio of $q_{01}(x)$ and $\mathfrak{f}(x)$ remains constant,

(ii) or fixing the elliptical cross-section which is allowed to twist around the curve.

We consequently obtain the regime $\alpha = 2$ with eigenvalues of order ε^2 above the minimum $\Lambda_0 = \lambda_0$. In this case, the related eigenfunctions are of order one with respect to the graph-norm of $\varepsilon^{-2} H_{\mathrm{a},0} = -\Delta^{\mathrm{L.B.}}_{\mathrm{d}x \otimes \mathrm{d}x}$ (i.e., with respect to the $W^2(\mathbb{R})$-norm) and oscillate on a length scale of order one:

$$\left\| \mathrm{d}\psi(\partial_x) \right\|_{L^2(\mathbb{R})} = \mathcal{O}(1) \quad \Leftrightarrow \quad \left\| \partial^{\mathsf{H}}_x (\psi \phi^{\mathrm{m}}_0) \right\|_{\mathcal{H}_{\mathcal{P}\mathrm{m}}} = \mathcal{O}(1).$$

Since Theorem 4.15 now gives access to an accuracy of order ε^4 for the approximation of the eigenvalues, we briefly itemize the modified and additional terms:

- The Berry one-form $\mathcal{A}^{\mathrm{B}}$ (5.33) of the Berry connection $\nabla^{\mathrm{B}} = \mathrm{d} + \mathrm{i}\mathcal{A}^{\mathrm{B}}$ is of order ε^2, and hence

$$-\varepsilon^2 \Delta^{\mathcal{A}^{\mathrm{B}}}_{\mathrm{d}x \otimes \mathrm{d}x} = -\varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathrm{d}x \otimes \mathrm{d}x} + \mathcal{O}(\varepsilon^4).$$

- The geometric potential V_{ρ_ε} is determined up to order ε^3 by means of the expansion (5.37). The subsequent order of both $\Delta^{\mathrm{L.B.}}_{\mathsf{V}} \ln \rho_\varepsilon$ and $\widetilde{g}_{\mathsf{V}}(\mathrm{d} \ln \rho_\varepsilon, \mathrm{d} \ln \rho_\varepsilon)$ may be calculated analogously. As far as the remaining term $\varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathsf{H}} \ln \rho_\varepsilon$ is concerned, we note the intermediate result $\partial^{\mathsf{H}}_x \cdot \langle \mathfrak{r} y, \kappa \rangle_{\mathbb{R}^2} = \langle \mathfrak{r} y, \kappa' \rangle_{\mathbb{R}^2}$, which follows from a straightforward calculation. In summary, the geometric potential $V_{\rho_\varepsilon}(x, y)$ equals

$$-\tfrac{1}{4}\varepsilon^2 \left\| \kappa(x) \right\|^2_{\mathbb{R}^2} - \tfrac{1}{2}\varepsilon^3 \Big(\left\| \kappa(x) \right\|^2_{\mathbb{R}^2} \langle \mathfrak{r} y, \kappa \rangle_{\mathbb{R}^2} + \langle \mathfrak{r} y, \kappa'' \rangle_{\mathbb{R}^2} \Big)$$

plus errors of order ε^4. The $\mathcal{O}(\varepsilon^3)$-term, however, is linear in y and vanishes when it is integrated against the centred ground state ϕ_0.

- The only additional term arising from the perturbation $\varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}}$ is the leading term $-\mathrm{i}\varepsilon^3 \varepsilon \partial_x^{\mathsf{H}} (\mathcal{B}_\perp \times (\mathfrak{r}y))$ of the potential. But this term clearly vanishes because ϕ_0 is centred. Consequently, we repeat the same calculations as for (5.39) without dropping the $\mathcal{O}(\varepsilon^3)$-terms and get

$$\begin{aligned}
&\varepsilon P_0^{\mathrm{m}} \tilde{H}_1^{\mathcal{E},\mathrm{w}} P_0^{\mathrm{m}} (\psi \phi_0^{\mathrm{m}}) \\
&= -2\varepsilon^3 \left(\partial_x \left(\underbrace{\langle \phi_0, \langle \mathfrak{r}y, \kappa \rangle_{\mathbb{R}^2} \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \mathrm{d}\psi(\partial_x) \right) \right) \phi_0^{\mathrm{m}} \\
&\quad - 2\varepsilon^3 \left\langle \phi_0, \partial_x^{\mathsf{H}} \big(\langle \mathfrak{r}y, \kappa \rangle_{\mathbb{R}^2} \partial_x^{\mathsf{H}} \phi_0 \big) \right\rangle_{\mathcal{H}_{\mathcal{F}}} \psi \phi_0^{\mathrm{m}} \\
&\quad - 2\mathrm{i}\varepsilon^3 \left(\partial_x \left(\underbrace{\left\langle \phi_0, \big(\mathcal{B}_\perp \times (\mathfrak{r}y)\big) \phi_0 \right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \psi \right) \right) \phi_0^{\mathrm{m}} \\
&\quad - 2\mathrm{i}\varepsilon^3 \underbrace{\left\langle \phi_0, \big(\mathcal{B}_\perp \times (\mathfrak{r}y)\big) \phi_0 \right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \mathrm{d}\psi(\partial_x) \phi_0^{\mathrm{m}} \\
&\quad + \mathcal{O}(\varepsilon^4),
\end{aligned}$$

  once again because ϕ_0 is centred (5.29).

We already know that all terms contained in the adiabatic operator are of order ε^2 in $\mathcal{L}(W^2(\mathbb{R}), L^2(\mathbb{R}))$ in the regime $\alpha = 2$. Therefore, we may divide the latter operator by ε^2 and define

$$\begin{aligned}
H_{\mathrm{QWG},\alpha=2}^{\mathrm{w,mas}} &:= \varepsilon^{-2} P_0^{\mathrm{m}} \big(H^{\mathcal{E},\mathrm{w}} - \Lambda_0 \mathbf{1}_{\mathcal{H}} \big) P_0^{\mathrm{m}} \\
&= -\Delta_{\mathrm{d}x \otimes \mathrm{d}x}^{\mathrm{L.B.}} - \tfrac{1}{4} \|\kappa\|_{\mathbb{R}^2}^2 + \left\| \partial_x^{\mathsf{H}} \phi_0 \right\|_{\mathcal{H}_{\mathcal{F}}}^2 \\
&\quad - 2\varepsilon \left\langle \phi_0, \partial_x^{\mathsf{H}} \big(\langle \mathfrak{r}y, \kappa \rangle_{\mathbb{R}^2} \partial_x^{\mathsf{H}} \phi_0 \big) \right\rangle_{\mathcal{H}_{\mathcal{F}}} \\
&\quad + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

It then follows that if ν^ε is an eigenvalue of $H_{\mathrm{QWG},\alpha=2}^{\mathrm{w,mas}}$ below its essential spectrum in the sense of Theorem 4.15, there exists an eigenvalue υ^ε of $H_{\mathrm{tube}}^{\mathrm{w,mas}}$ below its essential spectrum with the asymptotic expansion

$$\upsilon^\varepsilon = \Lambda_0 + \varepsilon^2 \nu^\varepsilon + \mathcal{O}(\varepsilon^4).$$

We conclude that the low-lying eigenvalues of $H^{\mathrm{w}}_{\mathrm{tube}}$ are again entirely independent of the magnetic potential up to this order.

Remark 5.16 The leading part of $H^{\mathrm{w,mas}}_{\mathrm{QWG},\alpha=2}$ was already derived in [KR14, Definition 2.5 ($\delta = 0$ in their notation)] for an arbitrary but constant, possibly twisting, simply connected cross-section $F \subset \mathbb{B}^2_r(0)$. This corresponds to the embedding

$$\begin{aligned}\varpi : \mathbb{R} \times F &\to \mathsf{N}\mathbb{R}^r,\\ (x, y^1, y^2) &\mapsto \Big(\cos\big(\vartheta(x)\big)y^1 - \sin\big(\vartheta(x)\big)y^2\Big)e_1(x)\\ &\qquad + \Big(\sin\big(\vartheta(x)\big)y^1 + \cos\big(\vartheta(x)\big)y^2\Big)e_2(x)\end{aligned}$$

of the waveguide into the fibrewise subset $\mathsf{N}\mathbb{R}^r$ of the normal bundle in our notation, where $\vartheta \in C^\infty_{\mathrm{b}}(\mathbb{R})$ is the twisting angle. It is then an easy calculation to see $\partial^{\mathsf{H}}_x = \partial^{\mathrm{pr}}_x - \vartheta'(y \times \nabla_y)$ and finally to obtain

$$\begin{aligned}H^{\mathrm{w,mas}}_{\mathrm{QWG},\alpha=2} = &-\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} - \tfrac{1}{4}\,\|\kappa\|^2_{\mathbb{R}^2} + (\vartheta')^2\,\Big\|(y \times \nabla_y)\phi_0\Big\|^2_{L^2(F,\mathrm{d}y)}\\ &+ \mathcal{O}(\varepsilon)\end{aligned}$$

in this case. ◊

5.2.2 Example for Hollow Quantum Waveguides

Let us consider a single, non-relativistic, spinless, charged quantum particle in the presence of an external magnetic field in the geometric situation of hollow quantum waveguides $M = \mathbb{R} \times \mathbb{S}^1 \xrightarrow{\mathrm{pr}_1} \mathbb{R}$ as introduced in Example 5.9, where additionally the radius function ℓ is assumed to depend on the base coordinate x alone (and not also on the fibre coordinate y). In view of (5.18), the resulting rescaled pullback metric on M is then given by

$$G^\varepsilon = \varepsilon^{-2}\underbrace{\left[(1 - \varepsilon\ell\varkappa)^2 + \varepsilon^2\left(\tfrac{\partial\ell}{\partial x}\right)^2\right]}_{=\rho^2_\varepsilon}\mathrm{d}x^{\mathsf{H}} \otimes \mathrm{d}x^{\mathsf{H}} + g_{\mathsf{V}}$$

with vertical bundle metric $g_{\mathsf{V}}|_{M_x} = g_{M_x} = \ell^2(x)\,\mathrm{d}y \otimes \mathrm{d}y$. This hollow waveguide corresponds to the boundary of the massive quantum waveguide of the former subsection with $\vartheta(x) = 0$ and $\mathfrak{a}(x) = \mathfrak{b}(x) = \ell(x)$ for all $x \in \mathbb{R}$. The Schrödinger operator (5.27) for this setting incorporates the perturbation $\varepsilon H_1^{\mathcal{E},\mathrm{w}} = \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{w}} + \frac{1}{2}\varepsilon^2 \Delta_{\mathsf{H}}^{\mathrm{L.B.}} \ln \rho_\varepsilon$ and the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ (5.28b).

Vertical Operator

We already know that the ground state of the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ is given by $\phi_0^{\mathrm{m}} = \phi_0 + \mathcal{O}(\varepsilon^2)$, where

$$\phi_0(x) = \mathrm{Vol}_{g_{M_x}}(\mathbb{S}^1)^{-1/2} \circ \pi_M + \mathcal{O}(\varepsilon) = \tfrac{1}{\sqrt{2\pi\ell(x)}} \circ \pi_M + \mathcal{O}(\varepsilon)$$

is the ground state of

$$H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}^\varepsilon_{\mathsf{V}}=0} = -\Delta_{\mathsf{V}}^{\mathrm{L.B.}} \underbrace{+\tfrac{1}{2}\Delta_{\mathsf{V}}^{\mathrm{L.B.}} \ln \rho_\varepsilon + \tfrac{1}{4}\tilde{g}_{\mathsf{V}}(\mathrm{d}\ln\rho_\varepsilon, \mathrm{d}\ln\rho_\varepsilon)}_{=V^{\mathsf{V}}_{\rho_\varepsilon}}$$

with constant eigenband $\lambda_0(x) = 0$. Let us explicitly calculate the $\mathcal{O}(\varepsilon)$-contribution of ϕ_0^{m} using finite-dimensional perturbation theory. To do this, we expand $H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}^\varepsilon_{\mathsf{V}}=0}$ up to an accuracy of order ε. Using

$$\rho_\varepsilon = \sqrt{(1-\varepsilon\ell\varkappa)^2 + \varepsilon^2\left(\tfrac{\partial\ell}{\partial x}\right)^2},$$

we compute

$$\begin{aligned}\ln\rho_\varepsilon &= \tfrac{1}{2}\ln\big((1-\varepsilon\ell\varkappa)^2 + \varepsilon^2(\ell')^2\big)\\ &= -\varepsilon\ell\varkappa - \tfrac{1}{2}\varepsilon^2\big((\ell\varkappa)^2 - (\ell')^2\big) + \mathcal{O}(\varepsilon^3)\end{aligned} \tag{5.40}$$

and thus

$$V^{\mathsf{V}}_{\rho_\varepsilon} = \tfrac{1}{2}\varepsilon\tfrac{\varkappa}{\ell} + \mathcal{O}(\varepsilon^2) \quad \Rightarrow \quad H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}^\varepsilon_{\mathsf{V}}=0} = -\Delta_{\mathsf{V}}^{\mathrm{L.B.}} + \tfrac{1}{2}\varepsilon\tfrac{\varkappa}{\ell} + \mathcal{O}(\varepsilon^2).$$

A set of orthonormal eigenfunctions of $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}} = -\ell^{-2}\,\partial_y^2$ with Dirichlet boundary conditions on $\mathcal{H}_{\mathcal{F}} = L^2(\mathbb{S}^1, \ell\,\mathrm{d}y)$ is given by

$$\Big\{\phi_0^{\mathrm{e}}, \{\phi_k^{\mathrm{e}}, \phi_k^{\mathrm{o}}\}_{k>0}\Big\} \tag{5.41}$$

with eigenvalues $\lambda_k^\Delta = k^2/\ell^2$. The ground state ($k = 0$) is the fibrewise constant function $\phi_0^{\mathrm{e}} = \frac{1}{\sqrt{2\pi\ell}} \circ \pi_M$. The excited states ($k > 0$) are doubly degenerate with eigenfunctions

$$\phi_k^{\mathrm{e}} = \tfrac{1}{\sqrt{\pi\ell}} \cos(ky) \quad , \quad \phi_k^{\mathrm{o}} = \tfrac{1}{\sqrt{\pi\ell}} \sin(ky).$$

Consequently, the first-order correction to the unperturbed ground state ϕ_0 is given by (with $\varkappa = \kappa^1 \cos y + \kappa^2 \sin y$)

$$\begin{aligned}
\phi_{0,1} &= -\sum_{k>0} \sum_{\bullet \in \{\mathrm{e},\mathrm{o}\}} \frac{1}{\lambda_k^\Delta - \lambda_0^\Delta} \left\langle \phi_k^\bullet, \tfrac{1}{2} \tfrac{\varkappa}{\ell} \phi_0^{\mathrm{e}} \right\rangle_{\mathcal{H}_{\mathcal{F}}} \phi_k^\bullet \\
&= -\sum_{k>0} \sum_{\bullet \in \{\mathrm{e},\mathrm{o}\}} \frac{1}{k^2/\ell^2} \left\langle \phi_k^\bullet, \tfrac{1}{2} \tfrac{1}{\sqrt{2}} \left(\tfrac{\kappa^1}{\ell} \phi_1^{\mathrm{e}} + \tfrac{\kappa^2}{\ell} \phi_1^{\mathrm{o}} \right) \right\rangle_{\mathcal{H}_{\mathcal{F}}} \phi_k^\bullet \\
&= -\tfrac{1}{\sqrt{8}} \ell \left(\kappa^1 \phi_1^{\mathrm{e}} + \kappa^2 \phi_1^{\mathrm{o}} \right).
\end{aligned}$$

and the asymptotic expansion of the entire magnetic ground state reads

$$\phi_0^{\mathrm{m}} = \tfrac{1}{\sqrt{2\pi\ell}} - \tfrac{1}{\sqrt{8}} \varepsilon \ell \left(\kappa^1 \phi_1^{\mathrm{e}} + \kappa^2 \phi_1^{\mathrm{o}} \right) + \mathcal{O}(\varepsilon^2), \tag{5.42}$$

where we dropped the composition with π_M in the leading term of ϕ_0^{m} for the sake of clarity. Once again, the magnetic effects do not appear until the subsequent order proportional to ε^2.

Berry Terms

In view of Subsection 4.3.1, we start with the calculation of the mean curvature vector η_{V} of $M_x \hookrightarrow (M, g)$. Equation (5.19) shows that the horizontal lift ∂_x^{H} reduces to the product lift ∂_x^{pr} in this case, and hence we obtain

$$\begin{aligned}
\eta_{\mathsf{V}} &\overset{(2.2)}{=} g^{yy} g\left(\nabla^g_{\partial_y} \partial_y, \partial_x^{\mathsf{H}} \right) \partial_x^{\mathsf{H}} \\
&\overset{(A.3)}{=} g^{yy} \left(-\tfrac{1}{2} \partial_x^{\mathrm{pr}} \cdot g(\partial_y, \partial_y) \right) \partial_x^{\mathsf{H}} \\
&= -(\ln \ell)' \, \partial_x^{\mathsf{H}}.
\end{aligned}$$

Moreover, it holds that

$$\begin{aligned}
0 &= \partial_x \cdot \underbrace{\big\langle \phi_0^{\mathrm{m}}, \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=1} \\
&= 2\,\mathrm{Re}\left(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \right) - \underbrace{\big\langle \phi_0^{\mathrm{m}}, \underbrace{g\left(\partial_x^{\mathsf{H}}, \eta_{\mathsf{V}}\right)}_{=-(\ln \ell)'} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=\overline{\eta}_{\mathsf{V}}(\partial_x)},
\end{aligned}$$

or equivalently

$$\mathrm{Re}\left(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \right) = \tfrac{1}{2} \overline{\eta}_{\mathsf{V}}(\partial_x) = -\tfrac{1}{2} (\ln \ell)'.$$

It follows that the metric connection $\nabla^{\mathcal{P}^{\mathrm{m}}} := \nabla^{\mathrm{B}} - \frac{1}{2}\overline{\eta}_{\mathsf{V}}$ coincides with $\nabla^{\mathcal{A}^{\mathrm{B}}} = \mathrm{d} + \mathrm{i}\mathcal{A}^{\mathrm{B}}$ with Berry one-form $\mathcal{A}^{\mathrm{B}} = \mathcal{O}(\varepsilon^2)$ as in (5.33). The Berry terms are summarised as

$$\begin{aligned}
&-\varepsilon^2 \Delta^{\mathcal{A}^{\mathrm{B}}}_{\mathrm{d}x \otimes \mathrm{d}x} + \varepsilon^2 \big(V^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{BH}} + V^{\mathcal{P}^{\mathrm{m}}}_{\eta_{\mathsf{V}}} \big) \\
&\quad = -\varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathrm{d}x \otimes \mathrm{d}x} + \varepsilon^2 \big(V^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{BH}} + V^{\mathcal{P}^{\mathrm{m}}}_{\eta_{\mathsf{V}}} \big) + \mathcal{O}(\varepsilon^4)
\end{aligned}$$

with potential

$$\begin{aligned}
V^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{BH}} + V^{\mathcal{P}^{\mathrm{m}}}_{\eta_{\mathsf{V}}} &= \left\| \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \right\|^2_{\mathcal{H}_{\mathcal{F}}} - \left| \big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \right|^2 \\
&\quad - \tfrac{1}{2} \partial_x \cdot \overline{\eta}_{\mathsf{V}}(\partial_x) + \tfrac{1}{4} \left| \overline{\eta}_{\mathsf{V}}(\partial_x) \right|^2 \\
&\overset{(5.42)}{=} \underbrace{\left\| \partial_x^{\mathsf{H}} \tfrac{1}{\sqrt{2\pi\ell}} \right\|^2_{\mathcal{H}_{\mathcal{F}}} - \left| \left\langle \tfrac{1}{\sqrt{2\pi\ell}}, \partial_x^{\mathsf{H}} \tfrac{1}{\sqrt{2\pi\ell}} \right\rangle_{\mathcal{H}_{\mathcal{F}}} \right|^2}_{=0} + \mathcal{O}(\varepsilon^2) \\
&\quad + \tfrac{1}{2} (\ln \ell)'' + \tfrac{1}{4} \left| (\ln \ell)' \right|^2 \\
&= \tfrac{1}{2} \tfrac{\ell''}{\ell} - \tfrac{1}{4} \left(\tfrac{\ell'}{\ell} \right)^2 + \mathcal{O}(\varepsilon^2). \qquad (5.43)
\end{aligned}$$

Remark 5.17 This potential was already derived in [HLT15] for arbitrary hollow quantum waveguides. More precisely, it generally holds that

$$\begin{aligned}\overline{\eta}_{\mathsf{V}}(X) &= 2\,\mathrm{Re}\Big(\big\langle \phi_0^{\mathrm{m}}, \mathrm{d}\phi_0^{\mathrm{m}}(X^{\mathsf{H}})\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)\\ &= 2\Big\langle \mathrm{Vol}_{g_{M_x}}(M_x)^{-1/2}, X^{\mathsf{H}}\cdot \mathrm{Vol}_{g_{M_x}}(M_x)^{-1}\Big\rangle_{\mathcal{H}_{\mathcal{F}}} + \mathcal{O}(\varepsilon)\\ &= -X\cdot \ln\big(\mathrm{Vol}_{g_{M_x}}(M_x)\big) + \mathcal{O}(\varepsilon)\end{aligned}$$

for any $X \in C^{\infty}(\mathsf{T}B)$ and the sum $V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}} + V_{\eta_{\mathsf{V}}}^{\mathcal{P}^{\mathrm{m}}}$ equals

$$\tfrac{1}{2}\Delta_{g_B}^{\mathrm{L.B.}} \ln\big(\mathrm{Vol}_{g_{M_x}}(M_x)\big) + \tfrac{1}{4}\Big\|\mathrm{grad}_{g_B} \ln\big(\mathrm{Vol}_{g_{M_x}}(M_x)\big)\Big\|_{g_B}^2 + \mathcal{O}(\varepsilon).$$

The higher precision of (5.43) stems from the fact that the mean curvature vector $\eta_{\mathsf{V}} = (-(\ln \ell)'\,\partial_x)^{\mathsf{H}}$ is a horizontal lift and hence does not carry any y-dependency, i.e., $\overline{\eta}_{\mathsf{V}}(\partial_x) = -(\ln \ell)'$ holds exactly without any ε-corrections. ◊

Perturbation

The relevant geometric quantities arising from the deviation $G^{\varepsilon} - g^{\varepsilon}$ are

$$\begin{aligned}\sigma^{\varepsilon}(\mathrm{d}x^{\mathsf{H}}, \mathrm{d}x^{\mathsf{H}}) &= \varepsilon^{-1}\left(\frac{1}{(1-\varepsilon\ell\varkappa)^2 + \varepsilon^2(\ell')^2} - 1\right)\\ &= 2\ell\varkappa + \varepsilon\big(3(\ell\varkappa)^2 - (\ell')^2\big) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

and $\ln\rho_{\varepsilon}$ (5.40). The latter gives rise to the remainder of the original geometric potential (5.26):

$$\begin{aligned}\tfrac{1}{2}\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}\ln\rho_{\varepsilon} &= \tfrac{1}{2}\varepsilon^2\big(\partial_x^{\mathsf{H}}\,\partial_x^{\mathsf{H}} + \underbrace{(\ln\ell)'\partial_x^{\mathsf{H}}}_{=-\eta_{\mathsf{V}}}\big)\ln\rho_{\varepsilon}\\ &= -\tfrac{1}{2}\varepsilon^3\big((\ell\varkappa)'' + (\ln\ell)'(\ell\varkappa)'\big) + \mathcal{O}(\varepsilon^4).\end{aligned}$$

The function $\varkappa(x,\cdot)$, and hence the leading term of $\Delta_{\mathsf{H}}^{\mathrm{L.B.}}\ln\rho_{\varepsilon}(x,\cdot)$, is a linear combination of $\cos y$ and $\sin y$ for all $x \in \mathbb{R}$, which means

$$P_0^{\mathrm{m}}\tfrac{1}{2}\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}\ln\rho_{\varepsilon}P_0^{\mathrm{m}} = -\tfrac{1}{2}\varepsilon^3\underbrace{\Big\langle \tfrac{1}{\sqrt{2\pi\ell}}, \big((\ell\varkappa)'' + (\ln\ell)'(\ell\varkappa)'\big)\tfrac{1}{\sqrt{2\pi\ell}}\Big\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}$$

plus errors of order ε^4. The remainder $\varepsilon\tilde{H}_1^{\mathcal{E},\mathrm{w}}$ of the perturbation can be handled as in the massive case of the previous subsection (plus the leading-order contribution of the potential):

$$\begin{aligned}
\varepsilon\tilde{H}_1^{\mathcal{E},\mathrm{w}} &\overset{(5.23)}{=} -2\varepsilon^3\partial_x^{\mathsf{H}}\big(\ell\varkappa\partial_x^{\mathsf{H}}\cdot\big) - 2\mathrm{i}\varepsilon^3\left(\mathcal{B}_\perp\times\begin{pmatrix}\ell\cos y\\ \ell\sin y\end{pmatrix}\right)\partial_x^{\mathsf{H}}\\
&\quad + \mathrm{i}\varepsilon^3\underbrace{\left[-\partial_x^{\mathsf{H}}\left(\mathcal{B}_\perp\times\begin{pmatrix}\ell\cos y\\ \ell\sin y\end{pmatrix}\right) - (\ln\ell)'\left(\mathcal{B}_\perp\times\begin{pmatrix}\ell\cos y\\ \ell\sin y\end{pmatrix}\right)\right]}_{=f_1(x)\cos y + f_2(x)\sin y \text{ for smooth functions } f_{1,2} \text{ to leading order}}\\
&\quad + \mathcal{O}(\varepsilon^4).
\end{aligned}$$

Consequently, we get (keeping in mind the mean curvature term (4.11))

$$\begin{aligned}
&\varepsilon P_0^{\mathrm{m}}\tilde{H}_1^{\mathcal{E},\mathrm{w}}P_0^{\mathrm{m}}(\psi\phi_0^{\mathrm{m}})\\
&= -2\varepsilon^3\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\partial_x^{\mathsf{H}}\left(\ell\varkappa\,\partial_x^{\mathsf{H}}\big(\psi\tfrac{1}{\sqrt{2\pi\ell}}\big)\right)\right\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}}\\
&\quad - 2\mathrm{i}\varepsilon^3\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\left(\mathcal{B}_\perp\times\begin{pmatrix}\ell\cos y\\ \ell\sin y\end{pmatrix}\right)\partial_x^{\mathsf{H}}\big(\psi\tfrac{1}{\sqrt{2\pi\ell}}\big)\right\rangle_{\mathcal{H}_{\mathcal{F}}}\phi_0^{\mathrm{m}}\\
&\quad + \mathcal{O}(\varepsilon^4)\\
&= -2\varepsilon^3\Bigg(\partial_x\Big(\underbrace{\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\ell\varkappa\tfrac{1}{\sqrt{2\pi\ell}}\right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}\mathrm{d}\psi(\partial_x)\Big)\\
&\qquad\qquad + \overline{\eta}_{\mathsf{V}}(\partial_x)\underbrace{\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\ell\varkappa\tfrac{1}{\sqrt{2\pi\ell}}\right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}\mathrm{d}\psi(\partial_x)\\
&\qquad\qquad + \underbrace{\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\ \underbrace{\partial_x^{\mathsf{H}}\left(\ell\varkappa\,\partial_x^{\mathsf{H}}\tfrac{1}{\sqrt{2\pi\ell}}\right)}_{\substack{=g_1(x)\cos y+g_2(x)\sin y\\ \text{for smooth functions } g_{1,2}}}\ \right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}\psi\Bigg)\phi_0^{\mathrm{m}}\\
&\quad - 2\mathrm{i}\varepsilon^3\underbrace{\left\langle\tfrac{1}{\sqrt{2\pi\ell}},\left(\mathcal{B}_\perp\times\begin{pmatrix}\ell\cos y\\ \ell\sin y\end{pmatrix}\right)\tfrac{1}{\sqrt{2\pi\ell}}\right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}\mathrm{d}\psi(\partial_x)\,\phi_0^{\mathrm{m}}
\end{aligned}$$

$$
\begin{aligned}
&- 2\mathrm{i}\varepsilon^3 \underbrace{\left\langle \frac{1}{\sqrt{2\pi\ell}}, \underbrace{\left(\mathcal{B}_\perp \times \begin{pmatrix} \ell \cos y \\ \ell \sin y \end{pmatrix}\right)}_{\substack{=h_1(x)\cos y + h_2(x)\sin y \\ \text{for smooth functions } h_{1,2}}} \partial_x^{\mathsf{H}} \frac{1}{\sqrt{2\pi\ell}} \right\rangle_{\mathcal{H}_\mathcal{F}}}_{=0} \psi \phi_0^{\mathrm{m}} \\
&+ \mathcal{O}(\varepsilon^4) \\
= {} & \mathcal{O}(\varepsilon^4).
\end{aligned}
$$

To sum up, the very specific form of the leading order ϕ_0 of the ground state ϕ_0^{m} causes the disappearance of the $\mathcal{O}(\varepsilon^3)$-contribution within the perturbation.

Conclusion

We finally collect all the relevant terms of the adiabatic operator up to errors of order ε^4 in $\mathcal{L}(W^2(\mathbb{R}), L^2(\mathbb{R}))$ and apply Theorem 5.12 for the mutual approximation of the low-lying eigenvalues:

Corollary 5.18 *Let $H_{\mathrm{tube}}^{\mathrm{w,hol}}$ be the Hamiltonian generating the dynamics of a non-relativistic, charged, spinless quantum particle, which is localised within the hollow quantum waveguide introduced in Example 5.9 with $\ell(x, y) = \ell(x)$, in the presence of a weak, C^∞-bounded external magnetic potential $\mathcal{A}$. Assume that there is a constant $C > 0$ strictly below the essential spectrum of*

$$
H_{\mathrm{QWG}}^{\mathrm{w,hol}} := -\Delta_{\mathrm{d}x\otimes\mathrm{d}x}^{\mathrm{L.B.}} + \frac{1}{2}\frac{\ell''}{\ell} - \frac{1}{4}\left(\frac{\ell'}{\ell}\right)^2
$$

in the sense of Theorem 4.15. Then if $\nu < C$ is an eigenvalue of $H_{\mathrm{QWG}}^{\mathrm{w,hol}}$, there exists an eigenvalue υ^ε of $H_{\mathrm{tube}}^{\mathrm{w,hol}}$ below its essential spectrum with the asymptotic expansion

$$
\upsilon^\varepsilon = \varepsilon^2 \nu + \mathcal{O}(\varepsilon^4).
$$

We immediately see that the potential terms in $H_{\mathrm{QWG}}^{\mathrm{w,hol}}$ do not depend on the extrinsic curvature induced by the embedding of the curve $c : \mathbb{R} \to \mathbb{R}^3$ (in terms κ), but merely on the deformation of the waveguide along the curve (in terms of ℓ). This was already stressed in [HLT15]. In this context, a radius function with a constricting profile (e.g., $\ell(x) = 2 - \frac{1}{1+x^2}$) can lead to a potential with wells and can thus support the formation of bound states.

5.3 Application to Strong Magnetic Fields

The discussion at the beginning of this chapter showed that an initial gauge field $\mathcal{A}$ of order one in the "macroscopic" laboratory $(\mathbb{R}^{b+f}, \delta^{b+f})$ leads to a "miscroscopic" Schrödinger operator (5.3)

$$H^{\mathrm{w}}_{\mathrm{tube}} = (\varepsilon \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon\mathcal{A})^*(\varepsilon \mathrm{d}\mathbf{1}_{\mathbb{C}^N} + \mathrm{i}\varepsilon\mathcal{A}) + V$$

on $\mathcal{T}^\varepsilon \times \mathbb{C}^N$, where the gauge field is weakly coupled by means of the additional ε in front of $\mathcal{A}$. In contrast to this, this section will deal with so-called *strong* Abelian gauge fields, i.e., we initially start with the field $\varepsilon^{-1}\mathcal{A}\mathbf{1}_{\mathbb{C}^N}$ related to some magnetic potential $\mathcal{A} \in C^\infty_{\mathrm{b}}(\mathsf{T}^*\mathbb{R}^{b+f}|_{\mathcal{T}^\varepsilon})$ and end up with the operator

$$H^{\mathrm{s}}_{\mathrm{tube}} := (\varepsilon\mathrm{d} + \mathrm{i}\mathcal{A})^*(\varepsilon\mathrm{d} + \mathrm{i}\mathcal{A})\mathbf{1}_{\mathbb{C}^N} + V, \quad V \in C^\infty_{\mathrm{b}}\big(\mathbb{R}^{b+f}|_{\mathcal{T}^\varepsilon}, \mathbb{C}^{N\times N}_{\mathrm{Herm}}\big)$$

on $\mathcal{T}^\varepsilon \times \mathbb{C}^N$ with Dirichlet boundary conditions. We again assume the family of ε-thin tubes $\mathcal{T}^\varepsilon$ (or likewise the associated waveguide M) to be of bounded geometry in the sense of Definition 5.3. The diffeomorphism $\Psi_\varepsilon : M \to \mathcal{T}^\varepsilon$ (5.5) again induces a unitarily equivalent operator

$$\widehat{\Psi}_\varepsilon H^{\mathrm{s}}_{\mathrm{tube}} \widehat{\Psi}^\dagger_\varepsilon = -\Delta^{\varepsilon^{-1}\mathcal{A}_\varepsilon}_{G^\varepsilon}\mathbf{1}_{\mathbb{C}^N} + V_\varepsilon \tag{5.44}$$

on $L^2(\mathcal{E}, \mathrm{vol}_{G^\varepsilon})$ with trivial vector bundle $\mathcal{E} = M \times \mathbb{C}^N$. Here, the induced magnetic potential reads

$$\varepsilon^{-1}\mathcal{A}_\varepsilon = \varepsilon^{-1}\pi^*_M\mathcal{A}_B + \mathsf{A}_\varepsilon,$$

where $\mathsf{A}_\varepsilon = \mathsf{A}_0 + \varepsilon\mathsf{A}^\varepsilon_1$ is defined by

$$\mathsf{A}_0 = \mathcal{A}^{\varepsilon=0}_{\mathsf{H}} \quad , \quad \mathsf{A}^\varepsilon_1 = \varepsilon^{-1}\big(\mathcal{A}^\varepsilon_{\mathsf{H}} - \mathcal{A}^{\varepsilon=0}_{\mathsf{H}}\big) + \mathcal{A}^\varepsilon_{\mathsf{V}} =: \mathsf{A}^\varepsilon_{\mathsf{H}} + \mathsf{A}^\varepsilon_{\mathsf{V}}.$$

This suggests that we consider A_ε as our "new" weak magnetic potential and $\nabla^{\mathsf{A}_\varepsilon} = \nabla^{\mathsf{A}_0} + \varepsilon \mathsf{A}_1^\varepsilon$ as a suitable perturbation of $\nabla^{\mathsf{A}_0} = \mathrm{d} + \mathrm{i}\mathsf{A}_0$ in the sense of Definition 2.24 plus a "strong part" $\mathrm{i}\varepsilon^{-1}\pi_M^* \mathcal{A}_B$.

In view of the unitary map $\widehat{U}_{\rho_\varepsilon} = (\varepsilon^{-b}\rho_\varepsilon)^{1/2}\mathbf{1}_{\mathbb{C}^N}$, the operator (5.44) is unitarily equivalent to (see Lemma 2.18)

$$
\begin{aligned}
H^{\mathcal{E},\mathrm{s}} &:= \widehat{U}_{\rho_\varepsilon} \widehat{\Psi}_\varepsilon H^{\mathrm{s}}_{\mathrm{tube}} \widehat{\Psi}_\varepsilon^\dagger \widehat{U}_{\rho_\varepsilon}^\dagger \\
&= \bigl(-\Delta_{g^\varepsilon}^{\varepsilon^{-1}\mathcal{A}_\varepsilon} + \nabla^{\varepsilon^{-1}\mathcal{A}_\varepsilon,*,g^\varepsilon} \circ \mathcal{S}^{\varepsilon^{-1}\mathcal{A}_\varepsilon} + V_{\rho_\varepsilon}\bigr)\mathbf{1}_{\mathbb{C}^N} + V_\varepsilon
\end{aligned}
$$

on $\mathcal{H} = L^2(\mathcal{E}, \mathrm{vol}_g)$. Similar calculations as for Proposition 2.25 show that the involved Laplacian splits as

$$
\begin{aligned}
&-\Delta_{g^\varepsilon}^{\varepsilon^{-1}\mathcal{A}_\varepsilon} \\
&= -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathsf{A}_0} - \Delta_{\mathsf{V}}^{\varepsilon \mathsf{A}_{\mathsf{V}}^\varepsilon} + \mathcal{D}_B \\
&\quad + \varepsilon^2 \Bigl[-2\mathrm{i}\,\mathrm{tr}_{\pi_M^* g_B}\bigl(\mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \nabla_{\varepsilon\cdot}^{\mathsf{A}_0}\bigr) + \mathrm{tr}_{\pi_M^* g_B}\bigl(\pi_M^*\mathcal{A}_B \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon + \mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \pi_M^* \mathcal{A}_B\bigr) \\
&\qquad\quad + \mathrm{i}\varepsilon \Bigl(\mathsf{A}_{\mathsf{H}}^\varepsilon(\eta_{\mathsf{V}}) - \mathrm{tr}_{\pi_M^* g_B}\bigl(\widetilde{\nabla}^{g^\varepsilon} \mathsf{A}_{\mathsf{H}}^\varepsilon\bigr)\Bigr) + \varepsilon^2\,\mathrm{tr}_{\pi_M^* g_B}\bigl(\mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon\bigr)\Bigr]
\end{aligned}
$$

with first-order horizontal differential operator

$$
\begin{aligned}
\mathcal{D}_B &:= -2\mathrm{i}\,\mathrm{tr}_{\pi_M^* g_B}\bigl(\pi_M^*\mathcal{A}_B \otimes \nabla_{\varepsilon\cdot}^{\mathsf{A}_0}\bigr) \\
&\qquad + \mathrm{i}\varepsilon \mathcal{A}_B\bigl(\mathsf{T}\pi_M(\eta_{\mathsf{V}})\bigr) + \underbrace{\Bigl[\mathrm{tr}_{g_B}\Bigl(\mathcal{A}_B \otimes \mathcal{A}_B - \mathrm{i}\varepsilon\bigl(\widetilde{\nabla}^{g_B}\mathcal{A}_B\bigr)\Bigr)\Bigr]}_{\in C_{\mathrm{b}}^\infty(B)} \circ \pi_M,
\end{aligned}
$$

incorporating the main contribution of the strong part $\mathcal{A}_B$. Moreover, an adaptation of the calculations in Subsection 5.1.3 yields that the second term $\nabla^{\varepsilon^{-1}\mathcal{A}_\varepsilon,*,g^\varepsilon} \circ S^{\varepsilon^{-1}\mathcal{A}_\varepsilon}$ may be rewritten as ε times the expression

$$
\begin{aligned}
&-\nabla_{\varepsilon v_j}^{\mathsf{A}_0}\bigl(\sigma^\varepsilon(\omega^i, \omega^j)\nabla_{\varepsilon v_i}^{\mathsf{A}_0}\cdot\bigr) \\
&-\Bigl(2\mathrm{i}\sigma^\varepsilon(\omega^i,\omega^j)\bigl(\pi_M^*\mathcal{A}_B + \varepsilon^2 \mathsf{A}_{\mathsf{H}}^\varepsilon\bigr)(v_j) - \varepsilon\sigma^\varepsilon\bigl(\omega^i, \widetilde{\nabla}_{v_j}^{g^\varepsilon}\omega^j\bigr)\Bigr)\nabla_{\varepsilon v_i}^{\mathsf{A}_0}
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma^{\varepsilon}(\omega^i,\omega^j)\pi_M^*\mathcal{A}_B(v_i)\pi_M^*\mathcal{A}_B(v_j)\\
&-\mathrm{i}\varepsilon\Big(\big(\nabla^{\mathsf{H}M^{\otimes 2}}_{v_j}\sigma^{\varepsilon}\big)(\omega^i,\omega^j)+\sigma^{\varepsilon}\big(\widetilde{\nabla}^{g^\varepsilon}_{v_j}\omega^i,\omega^j\big)\Big)\big(\pi_M^*\mathcal{A}_B+\varepsilon^2\mathsf{A}^{\varepsilon}_{\mathsf{H}}\big)(v_i)\\
&-\mathrm{i}\varepsilon\sigma^{\varepsilon}(\omega^i,\omega^j)\Big[\Big(\widetilde{\nabla}^{g^\varepsilon}_{v_j}\big(\pi_M^*\mathcal{A}_B+\varepsilon^2\mathsf{A}^{\varepsilon}_{\mathsf{H}}\big)\Big)(v_i)\\
&\qquad\qquad+\big(\pi_M^*\mathcal{A}_B+\varepsilon^2\mathsf{A}^{\varepsilon}_{\mathsf{H}}\big)\big(\nabla^{g^\varepsilon}_{v_j}v_i\big)\Big]\\
&+\varepsilon^2\sigma^{\varepsilon}(\omega^i,\omega^j)\big(\pi_M^*\mathcal{A}_B(v_i)\mathsf{A}^{\varepsilon}_{\mathsf{H}}(v_j)+\mathsf{A}^{\varepsilon}_{\mathsf{H}}(v_i)\pi_M^*\mathcal{A}_B(v_j)\big)\\
&+\varepsilon^4\sigma^{\varepsilon}(\omega^i,\omega^j)\mathsf{A}^{\varepsilon}_{\mathsf{H}}(v_i)\mathsf{A}^{\varepsilon}_{\mathsf{H}}(v_j),
\end{aligned}
$$

where $\{v_i\}_{i=1}^b$ and $\{\omega^i\}_{i=1}^b$ are local frames of $\mathsf{H}M$ and H^*M, respectively. In view of Remark 5.11(i), we may again drop the ε within ∇^{g^ε} and $\widetilde{\nabla}^{g^\varepsilon}$ and consider $\nabla^{\mathsf{H}M^{\otimes 2}}$ to be ε-independent (since we only differentiate along horizontal directions). We finally obtain

$$
H^{\mathcal{E},\mathrm{s}}=\big(-\varepsilon^2\Delta^{\mathsf{A}_0}_{\mathsf{H}}+\mathcal{D}_B\big)\mathbf{1}_{\mathbb{C}^N}+H^{\mathcal{F},\varepsilon,\mathrm{s}}+\underbrace{\varepsilon\tilde{H}^{\mathcal{E},\mathrm{s}}_1\mathbf{1}_{\mathbb{C}^N}+\mathfrak{V}_\varepsilon}_{=:\varepsilon H^{\mathcal{E},\mathrm{s}}_1}.
$$

with vertical operator

$$
H^{\mathcal{F},\varepsilon,\mathrm{s}}=-\Delta^{\varepsilon\mathsf{A}^{\varepsilon}_{\mathsf{V}}}_{\mathsf{V}}\mathbf{1}_{\mathbb{C}^N}+V_c,\quad V_c:=c^*V\in C^\infty_{\mathrm{b}}(B,\mathbb{C}^{N\times N}_{\mathrm{Herm}}).
$$

The perturbation is made up of the differential operator

$$
\begin{aligned}
&\tilde{H}^{\mathcal{E},\mathrm{s}}_1\\
&=-\operatorname{tr}_{\mathsf{H}M,(12),(34)}\Big(\nabla^{\mathsf{A}_0}_{\varepsilon\cdot}\big(\sigma^{\varepsilon}(\cdot,\cdot)\nabla^{\mathsf{A}_0}_{\varepsilon\cdot}\cdot\big)\Big)\\
&\quad-\operatorname{tr}_{\mathsf{H}M}\Big(\operatorname{tr}_{\mathsf{H}M,(23)}\Big(2\mathrm{i}\sigma^{\varepsilon}\otimes\big(\pi_M^*\mathcal{A}_B+\varepsilon^2\mathsf{A}^{\varepsilon}_{\mathsf{H}}\big)-\varepsilon\sigma^{\varepsilon}\big(\cdot,\widetilde{\nabla}^{g}_{\cdot}\cdot\big)\Big)\nabla^{\mathsf{A}_0}_{\varepsilon\cdot}\cdot\Big)\\
&\quad-2\mathrm{i}\varepsilon\operatorname{tr}_{\pi_M^*g_B}\big(\mathsf{A}^{\varepsilon}_{\mathsf{H}}\otimes\nabla^{\mathsf{A}_0}_{\varepsilon\cdot}\big)\\
&\quad+\operatorname{tr}_{\mathsf{H}M,(13),(24)}\big(\sigma^{\varepsilon}\otimes\pi_M^*\mathcal{A}_B\otimes\pi_M^*\mathcal{A}_B\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon\Big[\operatorname{tr}_{\pi_M^* g_B}\big(\pi_M^*\mathcal{A}_B \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon + \mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \pi_M^*\mathcal{A}_B\big) \\
&\qquad - \mathrm{i}\operatorname{tr}_{\mathsf{H}M,(13),(24)}\Big(\Big(\nabla^{\mathsf{H}M^{\otimes 2}}\sigma^\varepsilon + \sigma^\varepsilon\big(\widetilde{\nabla}^g_{\cdot}\cdot,\cdot\big)\Big) \otimes \pi_M^*\mathcal{A}_B\Big) \\
&\qquad - \mathrm{i}\operatorname{tr}_{\mathsf{H}M,(13),(24)}\Big(\sigma^\varepsilon \otimes \Big(\big(\widetilde{\nabla}^g \pi_M^*\mathcal{A}_B\big) + \pi_M^*\mathcal{A}_B\big(\nabla^g_{\cdot}\cdot\big)\Big)\Big)\Big] \\
&+ \varepsilon^2\Big[\mathrm{i}\Big(\mathsf{A}_{\mathsf{H}}^\varepsilon(\eta_{\mathsf{V}}) - \operatorname{tr}_{\pi_M^* g_B}\big(\widetilde{\nabla}^g \mathsf{A}_{\mathsf{H}}^\varepsilon\big)\Big) \\
&\qquad + \operatorname{tr}_{\mathsf{H}M,(13),(24)}\Big(\sigma^\varepsilon \otimes \big(\pi_M^*\mathcal{A}_B \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon + \mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \pi_M^*\mathcal{A}_B\big)\Big)\Big] \\
&+ \varepsilon^3\Big[\operatorname{tr}_{\pi_M^* g_B}\big(\mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon\big) \\
&\qquad - \mathrm{i}\operatorname{tr}_{\mathsf{H}M,(13),(24)}\Big(\Big(\nabla^{\mathsf{H}M^{\otimes 2}}\sigma^\varepsilon + \sigma^\varepsilon\big(\widetilde{\nabla}^g_{\cdot}\cdot,\cdot\big)\Big) \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon\Big) \\
&\qquad - \mathrm{i}\operatorname{tr}_{\mathsf{H}M,(13),(24)}\Big(\sigma^\varepsilon \otimes \Big(\big(\widetilde{\nabla}^g \mathsf{A}_{\mathsf{H}}^\varepsilon\big) + \mathsf{A}_{\mathsf{H}}^\varepsilon\big(\nabla^g_{\cdot}\cdot\big)\Big)\Big)\Big] \\
&+ \varepsilon^4 \operatorname{tr}_{\mathsf{H}M,(13),(24)}\big(\sigma^\varepsilon \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon \otimes \mathsf{A}_{\mathsf{H}}^\varepsilon\big)
\end{aligned}
$$

and the $\mathcal{O}(\varepsilon)$-potential

$$\mathfrak{V}_\varepsilon = V_{\rho_\varepsilon}\mathbf{1}_{\mathbb{C}^N} + (V_\varepsilon - V_c).$$

In order to apply the results of Section 4.2 to generalised quantum waveguides in the presence of strong magnetic fields, we essentially need to adjust Lemma 3.7 by the addition of $\mathcal{D}_B$ to the initial horizontal Laplacian, i.e., we must show the following:

Lemma 5.19 (Extension of Lemma 3.7) *Let $S, T \in \mathcal{A}_H$ with $ST \in \mathcal{A}_H^{p,q}$. It then holds that*

$$\big[\mathcal{D}_B, S\big]T \in \mathcal{A}^{p+1,q+1}.$$

PROOF. It suffices to consider the case $N = 1$ since all relevant operators are a multiple of the identity $\mathbf{1}_{\mathbb{C}^N}$. The claim is once again a local issue, i.e., we only need to show it over $\pi_M^{-1}(U) \subset M$ for some $U \in \mathfrak{U}$ (cf. Definition 2.7). Therefore, we take a g_B-orthonormal frame $\{X_i\}_{i=1}^b$ of $\mathsf{T}U$ and take advantage of the fact that $V_i = X_i^{\mathsf{H}} - \Phi^* X_i \in C_{\mathrm{b}}^\infty(\mathsf{V}M|_{\pi_M^{-1}(U)})$ by Lemma 3.1 to get:

$$\mathcal{D}_B|_{\pi_M^{-1}(U)} = \left(\pi_M^* A_i\right)\left(\varepsilon \nabla^{\mathsf{A}_0}_{\Phi^* X_i} + \varepsilon \nabla^{\mathsf{A}_0}_{V_i}\right) + \pi_M^* B + \varepsilon C,$$

where $A_i, B \in C_{\mathrm{b}}^\infty(U)$ and $C \in C_{\mathrm{b}}^\infty(\pi_M^{-1}(U))$ are smooth functions given by

$$\begin{aligned}
A_i &:= -2\mathrm{i}\mathcal{A}_B(X_i),\\
B &:= \mathcal{A}_B(X_i)\mathcal{A}_B(X_i) - \mathrm{i}\varepsilon\left(\widetilde{\nabla}^{g_B}_{X_i}\mathcal{A}_B\right)(X_i),\\
C &:= \mathrm{i}\mathcal{A}_B\left(\mathsf{T}\pi_M(\eta_{\mathsf{V}})\right).
\end{aligned}$$

If we expand S locally over $\pi_M^{-1}(U)$ as

$$S|_{\pi_M^{-1}(U)} = \sum_{\mathfrak{a} \in \mathbb{N}_0^b} S_{\mathfrak{a}} \varepsilon^{|\mathfrak{a}|} \left(\nabla^{\mathsf{A}_0}_{\Phi^* X}\right)^{\mathfrak{a}},$$

the crucial fact is

$$[\pi_M^* A_i, S^{\mathfrak{a}}] = \mathbf{0} = [\pi_M^* B, S_{\mathfrak{a}}] \quad \text{for all } \mathfrak{a} \in \mathbb{N}_0^b,$$

because $\pi_M^* A_i$ and $\pi_M^* B$ are fibrewise constant. Thus, the term $[\mathcal{D}_B, S]T$ over $\pi_M^{-1}(U)$ merely consists of these four terms:

$$\begin{aligned}
\left[\left(\pi_M^* A^i\right)\left(\varepsilon \nabla^{\mathsf{A}_0}_{\Phi^* X_i}\right), S\right] T &= \underbrace{\varepsilon\left(\pi_M^* A_i\right) \underbrace{\left[\nabla^{\mathsf{A}_0}_{\Phi^* X_i}, S\right] T}_{\in \mathcal{A}_H^{p,q}}}_{\in \mathcal{A}_H^{p,q+1} \subset \mathcal{A}^{p+1,q+1}} \\
&\quad + \underbrace{\sum_{\mathfrak{a} \in \mathbb{N}_0^b} S_{\mathfrak{a}} \underbrace{\left[\pi_M^* A^i, \left(\varepsilon \nabla^{\mathsf{A}_0}_{\Phi^* X}\right)^{\mathfrak{a}}\right]}_{= \varepsilon \sum_{|\mathfrak{b}| \leqslant |\mathfrak{a}|-1} C_{\mathfrak{b}} \left(\varepsilon \nabla^{\mathsf{A}_0}_{\Phi^* X}\right)^{\mathfrak{b}}} \left(\varepsilon \nabla^{\mathsf{A}_0}_{\Phi^* X_i}\right) T}_{\in \mathcal{A}_H^{p,q+1} \subset \mathcal{A}^{p+1,q+1}} \\
&\in \mathcal{A}^{p+1,q+1},
\end{aligned}$$

$$
\begin{aligned}
\Big[\big(\pi_M^* A^i\big)\big(\varepsilon\nabla^{\mathsf{A}_0}_{V_i}\big), S\Big] T = {} & \underbrace{\varepsilon\big(\pi_M^* A_i\big)\,\underbrace{\big[\nabla^{\mathsf{A}_0}_{V_i}, S\big] T}_{\in \mathcal{A}^{p,q}}}_{\in \mathcal{A}^{p,q+1} \subset \mathcal{A}^{p+1,q+1}} \\
& \underbrace{+\, \varepsilon \sum_{\mathfrak{a}\in\mathbb{N}_0^b} S_{\mathfrak{a}} \underbrace{\Big[\pi_M^* A^i, \big(\varepsilon\nabla^{\mathsf{A}_0}_{\Phi * X}\big)^{\mathfrak{a}}\Big]}_{=\varepsilon \sum_{|\mathfrak{b}|\leqslant|\mathfrak{a}|-1} C_{\mathfrak{b}} (\varepsilon\nabla^{\mathsf{A}_0}_{\Phi * X})^{\mathfrak{b}}} \big(\nabla^{\mathsf{A}_0}_{V_i}\big) T}_{\in \mathcal{A}^{p-1,q+2} \subset \mathcal{A}^{p+1,q+1}} \\
& \in \mathcal{A}^{p+1,q+1},
\end{aligned}
$$

$$
\begin{aligned}
\big[\pi_M^* B, S\big] T = {} & \sum_{\mathfrak{a}\in\mathbb{N}_0^b} S^{\mathfrak{a}} \underbrace{\Big[\pi_M^* B, \big(\varepsilon\nabla^{\mathsf{A}_0}_{\Phi * X}\big)^{\mathfrak{a}}\Big]}_{=\varepsilon \sum_{|\mathfrak{b}|\leqslant|\mathfrak{a}|-1} C_{\mathfrak{b}} (\varepsilon\nabla^{\mathsf{A}_0}_{\Phi * X})^{\mathfrak{b}}} \; T \\
& \in \mathcal{A}_H^{p-1,q+1} \subset \mathcal{A}^{p+1,q+1}
\end{aligned}
$$

and

$$
[\varepsilon C, S] T = \varepsilon [C, S] T \in \mathcal{A}_H^{p,q+1} \subset \mathcal{A}^{p+1,q+1}.
$$

This completes the proof. □

In order to ensure all requirements for the application of Theorem 4.5 are met, we first observe that

$$
\big(-\varepsilon^2 \Delta_{\mathsf{H}}^{\mathsf{A}_0} + \mathcal{D}_B\big) + \varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{s}}
$$

is the horizontal Laplacian of $-\Delta_{G^\varepsilon}^{\varepsilon^{-1}\mathcal{A}_\varepsilon}$ in the sense of Remark 2.20 and hence defines a positive operator. It then follows that the entire horizontal operator is bounded from below by $-\|\mathfrak{V}_\varepsilon\|_{L^\infty(M,\mathbb{C}^{N\times N})} = \mathcal{O}(\varepsilon)$. The (possibly q-fold degenerate) ground state $\{\phi_0 \zeta_0^j\}_{j=1}^q$ of the unperturbed vertical operator

$$
H^{\mathcal{F},\varepsilon=0,\mathrm{s}} = -\Delta_{\mathsf{V}}^{\mathrm{L.B.}} \mathbf{1}_{\mathbb{C}^N} + V_c
$$

with eigenband $\lambda_0 = \lambda_0^{\Delta} + \lambda_0^{V_c}$ is composed of the positive ground state ϕ_0 of the scalar Dirichlet Laplacian $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ with eigenband λ_0^{Δ} and the smooth functions $\zeta_0^j : B \to \mathbb{C}^N$, $j \in \{1,\dots,q\}$, such that $\{\zeta_0^j(x)\}_{j=1}^q$ is an orthonormal basis of $V_c(x)$'s eigenspace associated with its lowest eigenvalue $\lambda_0^{V_c}(x)$ for all $x \in B$. We will assume that λ_0 is separated from the rest of the spectrum of $H^{\mathcal{F},\varepsilon=0,\mathrm{s}}$ by a uniform gap, i.e., λ_0 satisfies Condition 2.26. Then a straightforward modification of Lemma 5.13 shows that the magnetic ground state band $\lambda_0^{\mathrm{m}}(x) = \min \sigma(H^{\mathcal{F},\varepsilon,\mathrm{s}}(x))$ also satisfies a gap condition and the corresponding spectral projection

$$P_0^{\mathrm{m}} = \frac{\mathrm{i}}{2\pi} \int_{|z-\lambda_0|=\delta} \left(H^{\mathcal{F},\varepsilon,\mathrm{s}} - z\mathbf{1}_{\mathcal{H}}\right)^{-1} \mathrm{d}z$$

is well-defined. In view of [Lam14, Lemma 3.7], the spectral gap condition on λ_0 is satisfied for $V_c = \mathbf{0}$ and connected typical fibre F of $M \xrightarrow{\pi_M} B$. Since all corrections to $H^{\mathcal{F},\varepsilon=0,\mathrm{s}}$ within the operator

$$H^{\mathcal{F},\varepsilon,\mathrm{s}} = H^{\mathcal{F},\varepsilon=0,\mathrm{s}} + \varepsilon H_1^{\mathcal{F},\mathrm{s}} \mathbf{1}_{\mathbb{C}^N} + \varepsilon^2 H_2^{\mathcal{F},\mathrm{s}} \mathbf{1}_{\mathbb{C}^N} + \dots$$

are obviously diagonal with respect to the spin degrees of freedom (recall that we started with an Abelian gauge field $\mathcal{A}\mathbf{1}_{\mathbb{C}^N}$), the finite-dimensional perturbation theory once again gives $\lambda_0^{\mathrm{m}} = \lambda_0 + \varepsilon^2 \lambda_{0,2} + \mathcal{O}(\varepsilon^3)$ for the asymptotic expansion of the magnetic ground state band with spin-independent second-order correction

$$\begin{aligned} \lambda_{0,2}(x) &= \left\langle \phi_0, H_2^{\mathcal{F},\mathrm{s}} \phi_0 \right\rangle_{L^2(M_x, \mathrm{vol}_{g_{M_x}})} \\ &\quad - \left\langle \phi_0, H_1^{\mathcal{F},\mathrm{s}} (-\Delta_{\mathrm{V}}^{\mathrm{L.B.}} - \lambda_0^{\Delta})^{-1} \tilde{P}_0^{\perp} H_1^{\mathcal{F},\mathrm{s}} \phi_0 \right\rangle_{L^2(M_x, \mathrm{vol}_{g_{M_x}})} . \end{aligned}$$

Here, $\tilde{P}_0^{\perp}$ denotes the projection onto the orthogonal complement of $\mathrm{span}(\phi_0)$ in $L^2(M_x, \mathrm{vol}_{g_{M_x}})$ and the scalar corrections are given by

$$\begin{aligned} H_1^{\mathcal{F},\mathrm{s}} &= \mathrm{d}^*\left(\mathrm{i}\mathcal{A}_{\mathrm{V}}^{\varepsilon=0}\right) + \left(\mathrm{i}\mathcal{A}_{\mathrm{V}}^{\varepsilon=0}\right)^* \mathrm{d}, \\ H_2^{\mathcal{F},\mathrm{s}} &= \left(\mathcal{A}_{\mathrm{V}}^{\varepsilon=0}\right)^* \left(\mathcal{A}_{\mathrm{V}}^{\varepsilon=0}\right) + \underbrace{\mathrm{d}^*\left(\mathrm{i}\mathcal{A}_{\mathrm{V}}^{1,\varepsilon=0}\right) + \left(\mathrm{i}\mathcal{A}_{\mathrm{V}}^{1,\varepsilon=0}\right)^* \mathrm{d}}_{\langle \phi_0, \dots \phi_0 \rangle_{L^2(M_x)} = 0 \text{ due to } (5.30)} . \end{aligned}$$

Remark 5.20 Let us mention two possible ways to make the expression for $\lambda_{0,2}$ more concrete:

(i) Since $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ is a compact operator, the Fredholm alternative provides a unique, fibrewise map $\theta : x \mapsto \theta(x) \in \tilde{P}_0^{\perp} L^2(M_x, \mathrm{vol}_{g_{M_x}})$ such that

$$\int_B \left\|\theta(x)\right\|^2_{L^2(M_x,\mathrm{vol}_{g_{M_x}})} \ \mathrm{vol}_{g_B}(x) < \infty$$

and

$$\left(-\Delta_{\mathsf{V}}^{\mathrm{L.B.}} - \lambda_0^{\Delta}\right)\theta = \tilde{P}_0^{\perp} H_1^{\mathcal{F},\mathrm{s}} \phi_0.$$

With this, the second-order correction reads

$$\lambda_{0,2}(x) = \left\langle \phi_0, H_2^{\mathcal{F},\mathrm{s}} \phi_0 \right\rangle_{L^2(M_x,\mathrm{vol}_{g_{M_x}})} - \left\langle \phi_0, H_1^{\mathcal{F},\mathrm{s}} \theta \right\rangle_{L^2(M_x,\mathrm{vol}_{g_{M_x}})}.$$

(ii) Let $\{\phi_k(x)\}_{k\geqslant 0}$ be a complete set of (local) real-valued eigensections of $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}(x) = -\Delta_{g_{M_x}}^{\mathrm{L.B.}}$ on $L^2(M_x, \mathrm{vol}_{g_{M_x}})$ with respective Dirichlet eigenvalues $\lambda_k^{\Delta}(x)$. Then $\lambda_{0,2}(x)$ equals

$$\int_{M_x} \widetilde{g}_{M_x}\left(\mathcal{A}_{\mathsf{V}}^{\varepsilon=0}\phi_0, \mathcal{A}_{\mathsf{V}}^{\varepsilon=0}\phi_0\right) \ \mathrm{vol}_{g_{M_x}}$$
$$- \sum_{k>0} \frac{\left| \int_{M_x} \widetilde{g}_{M_x}(\mathrm{d}\phi_k, \mathcal{A}_{\mathsf{V}}^{\varepsilon=0}\phi_0) - \widetilde{g}_{M_x}(\mathcal{A}_{\mathsf{V}}^{\varepsilon=0}\phi_k, \mathrm{d}\phi_0) \ \mathrm{vol}_{g_{M_x}} \right|^2}{\lambda_k - \lambda_0}. \ \Diamond$$

The switching on of the magnetic field yields an $\mathcal{O}(\varepsilon^2)$-displacement of the ground state band λ_0^{m} with respect to λ_0 (and analogously for the rest of the spectrum). Thus, it holds that

$$\Lambda_1 := \inf_{x\in B}\Big(\sigma\left(H^{\mathcal{F},\varepsilon=0,\mathrm{s}}(x)\right)\backslash\lambda_0(x)\Big) < \inf_{x\in B}\Big(\sigma\left(H^{\mathcal{F},\varepsilon,\mathrm{s}}(x)\right))\backslash\lambda_0^{\mathrm{m}}(x)\Big) + \delta$$

for all $\delta > 0$ if $\varepsilon = \varepsilon(\delta) > 0$ is chosen sufficiently small.

We formally gather the leading terms ("reduced Hamiltonian")

$$\begin{aligned} H^{\mathrm{s}}_{\mathrm{QWG}} &:= H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{a}} + \mathcal{M}^{\mathcal{P}^{\mathrm{m}}} \\ &= -\varepsilon^2 \Delta^{\mathrm{B}} + \lambda^{\Delta}_0 + \lambda^{V_c}_0 + \varepsilon^2 \lambda_{0,2} + \varepsilon^2 V^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{BH}} \\ &\quad + P^{\mathrm{m}}_0 \Big(\mathcal{D}_B + \varepsilon H^{\mathcal{E},\mathrm{s}}_1 + \big[H^{\mathcal{E},\mathrm{s}}, P^{\mathrm{m}}_0 \big] \mathcal{R}^{\mathcal{F},\varepsilon,\mathrm{s}}(\lambda^{\mathrm{m}}_0) \big[H^{\mathcal{E},\mathrm{s}}, P^{\mathrm{m}}_0 \big] \Big) P^{\mathrm{m}}_0 \\ &\quad + \mathcal{O}(\varepsilon^3) \end{aligned}$$

of the effective operator (4.15) and combine Theorem 4.5 and Proposition 4.10 as follows:

Theorem 5.21 *Let $H^{\mathrm{s}}_{\mathrm{tube}}$ be the Hamiltonian generating the dynamics of non-interacting, non-relativistic, charged quantum particles with total spin quantum number N, which are localised within a quantum waveguide of bounded geometry, in the presence of a strong, C^∞-bounded external magnetic potential $\varepsilon^{-1}\mathcal{A}$ and a C^∞-bounded, $\mathbb{C}^{N\times N}_{\mathrm{Herm}}$-valued external potential V. Assume that the ground state band λ_0 of $-\Delta^{\mathrm{L.B.}}_{\mathrm{V}} \mathbf{1}_{\mathbb{C}^N} + V_c$ has a spectral gap, cf. Condition 2.26. Moreover, let χ be a regular cut-off function with support in $(-\infty, \Lambda_1)$. Then $\chi(H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{eff}})\, H^{\mathrm{s}}_{\mathrm{QWG}}\, \chi(H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{eff}})$ is unitarily equivalent to $\chi(H^{\mathrm{s}}_{\mathrm{tube}})\, H^{\mathrm{s}}_{\mathrm{tube}}\, \chi(H^{\mathrm{s}}_{\mathrm{tube}})$ up to errors of order ε^3 for $\varepsilon > 0$ small enough, where $H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{eff}}$ is the effective operator* (4.1) *associated with λ^{m}_0.*

This theorem allows for the approximation of the part of the spectrum of $H^{\mathrm{s}}_{\mathrm{tube}}$ that is "related" to the ground state band λ^{m}_0, by that of the reduced scalar operator $H^{\mathrm{s}}_{\mathrm{QWG}}$ up to errors of order ε^3. The fact that $H^{\mathrm{s}}_{\mathrm{QWG}}$ now may contain fourth-order differential operators via the $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$-term does not create new problems because we have to consider $H^{\mathrm{s}}_{\mathrm{QWG}}$ on the image of $\chi(H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{eff}})$, i.e., on $W^\infty(\mathcal{P}^{\mathrm{m}})$, rather than on $\mathrm{dom}(H^{\mathcal{P}^{\mathrm{m}}}_{\mathrm{a}}) = W^2(\mathcal{P}^{\mathrm{m}})$.

As in the previous section, we will examine the structure of $H^{\mathrm{s}}_{\mathrm{QWG}}$ for a single spinless particle ($N = 1$) that is localised within the conventional quantum tubes arising from Example 5.7 (massive waveguide) and Example 5.9 (hollow waveguide). The absence of the spin implies that the external potential V is real-valued and trivially has a single "eigenvalue" alone. Note that in both geometric settings we may completely gauge away the strong part $\pi_M^* \mathcal{A}_B$ of the magnetic potential (cf. Example 5.10), and hence the computations simplify tremendously since $\mathcal{D}_B = \mathbf{0}$.

5.3.1 Example for Massive Quantum Waveguides

Let us calculate the operator $H^{\mathrm{s}}_{\mathrm{QWG}}$ for a single non-relativistic, spinless, charged quantum particle which is localised within an infinite tube in $\mathbb{R}^3$, whose cross-sections are given by elliptical discs with varying semi-axes twisting around a smoothly embedded curve $c : \mathbb{R} \to \mathbb{R}^3$. This setting was already established in Example 5.7 and elaborated for weak magnetic potentials in Subsection 5.2.1 (we will no longer assume the existence of a unique, non-degenerate minimum for λ_0). Consequently, the strong magnetic potential for this purpose takes the form (cf. Example 5.10(i))

$$\varepsilon^{-1}\mathcal{A}_\varepsilon = \mathsf{A}_\varepsilon = \underbrace{\big(\mathcal{B}_\perp \times (\mathfrak{r}y)\big)\,\mathrm{d}x^{\mathsf{H}}}_{=\mathsf{A}_0} + \varepsilon\left(\mathsf{A}^\varepsilon_{\mathsf{H}} + \underbrace{\left(\tfrac{1}{2}\mathfrak{a}\mathfrak{b}\mathcal{B}_\| y + \mathcal{O}(\varepsilon)\right) \times \mathrm{d}y}_{=\mathsf{A}^\varepsilon_{\mathsf{V}}}\right).$$

Since the particle does not carry a spin ($N = 1$), the ground state of the unperturbed vertical operator is non-degenerate ($q = 1$) and $\zeta_0^1(x) = 1$ for all $x \in \mathbb{R}$. The fibre Hilbert space then reduces to $\mathcal{H}_{\mathcal{F}}|_x = L^2(M_x, \mathrm{vol}_{g_{M_x}})$ and the contribution of the potential V_c is just $\lambda_0^{V_c}(x) = V_c(x) \in \mathbb{R}$. We remark that the corresponding unperturbed part ϕ_0 of the entire magnetic ground state $\phi_0^{\mathrm{m}} = \phi_0 + \varepsilon\delta\phi_0^{\mathrm{m}}$ is positive and centred, since it has these properties for the Laplacian $-\Delta_{\mathsf{V}}^{\mathrm{L.B.}}$ and V_c is fibrewise constant. In the end, the global trivialisation $W^\infty(\mathcal{P}^{\mathrm{m}}) \cong W^\infty(\mathbb{R}) \otimes \mathrm{span}(\phi_0^{\mathrm{m}})$ induced by ϕ_0^{m} allows us to view $H^{\mathrm{s}}_{\mathrm{QWG}}$ as an operator acting on $W^\infty(\mathbb{R})$.

Ground State Band

A short calculation shows that the corrections to the unperturbed vertical operator $H^{\mathcal{F},\varepsilon=0,\mathrm{s}}$ are given by

$$H_1^{\mathcal{F},\mathrm{s}} = -\mathrm{i}\mathcal{B}_\| \begin{pmatrix} \frac{a}{b} y^1 \\ \frac{b}{a} y^2 \end{pmatrix} \times \nabla_y \overset{(5.35)}{=} \mathcal{B}_\| L_y^{\mathfrak{a},\mathfrak{b}} \quad , \quad H_2^{\mathcal{F},\mathrm{s}} = -\tfrac{1}{4}\mathfrak{a}\mathfrak{b}\mathcal{B}_\|^2\, \|y\|^2_{\mathbb{R}^2} \,.$$

So in view of Remark 5.20(i), the magnetic ground state band becomes

$$\lambda_0^{\mathrm{m}} = \lambda_0^{\Delta} + V_c + \varepsilon^2 \mathcal{B}_{\parallel}^2 \Big\langle \phi_0, \tfrac{1}{4}\big((\mathfrak{a}y^1)^2 + (\mathfrak{b}y^2)^2\big)\phi_0 - L_y^{\mathfrak{a},\mathfrak{b}}\theta \Big\rangle_{\mathcal{H}_{\mathcal{F}}},$$

where θ is the unique solution of

$$\big(-\Delta_{\mathsf{V}}^{\mathrm{L.B.}} - \lambda_0\big)\theta = P_0^{\perp} L_y^{\mathfrak{a},\mathfrak{b}} \phi_0.$$

Berry Terms

The Berry connection is given by

$$\nabla_{\partial_x}^{\mathsf{B}}(\psi\phi_0^{\mathrm{m}}) = \big(\nabla_{\partial_x}^{\mathsf{A}^{\mathsf{B}}}\psi\big)\phi_0^{\mathrm{m}}, \quad \nabla^{\mathsf{A}^{\mathsf{B}}} = \mathrm{d} + \mathrm{i}\mathsf{A}^{\mathsf{B}}$$

with one-form

$$\begin{aligned}
\mathsf{A}^{\mathsf{B}}(\partial_x) &= \big\langle \phi_0^{\mathrm{m}}, \mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}} + \mathrm{Im}\Big(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big) \\
&= \underbrace{\big\langle \phi_0, \big(\mathcal{B}_{\perp} \times (\mathfrak{r}y)\big)\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0 \text{ since } \phi_0 \text{ is centred (5.29)}} \\
&\quad + 2\varepsilon\Big[\mathrm{Re}\Big(\big\langle \phi_0, \big(\mathcal{B}_{\perp} \times (\mathfrak{r}y)\big)\delta\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big) \\
&\qquad + \mathrm{Im}\Big(\big\langle \phi_0, \partial_x^{\mathsf{H}}\delta\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)\Big].
\end{aligned}$$

Thus, the "effective magnetic potential" is actually of order ε. The Born-Huang potential $V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}}$ may be calculated similarly as in (5.34):

$$\begin{aligned}
V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}} &= \big\|\nabla_{\partial_x^{\mathsf{H}}}^{\mathsf{A}_0}\phi_0^{\mathrm{m}}\big\|_{\mathcal{H}_{\mathcal{F}}}^2 - \Big|\big\langle \phi_0^{\mathrm{m}}, \nabla_{\partial_x^{\mathsf{H}}}^{\mathsf{A}_0}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big|^2 \\
&= \big\|\partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\|_{\mathcal{H}_{\mathcal{F}}}^2 \big\|\mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0^{\mathrm{m}}\big\|_{\mathcal{H}_{\mathcal{F}}}^2 + 2\underbrace{\mathrm{Im}\Big(\big\langle \partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}, \mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)}_{=\mathcal{O}(\varepsilon) \text{ since } \phi_0 \text{ is real-valued}} \\
&\quad - \Big|\big\langle \phi_0, \partial_x^{\mathsf{H}}\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}} + \mathrm{i}\big\langle \phi_0, \mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}} + \mathcal{O}(\varepsilon)\Big|^2
\end{aligned}$$

$$= \left\|\partial_x^{\mathsf{H}}\phi_0\right\|^2_{\mathcal{H}_{\mathcal{F}}} - \Big|\underbrace{\left\langle\phi_0, \partial_x^{\mathsf{H}}\phi_0\right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0}\Big|^2$$
$$+ \left\|\left(\mathcal{B}_\perp \times (\mathfrak{r}y)\right)\phi_0\right\|^2_{\mathcal{H}_{\mathcal{F}}} - \Big|\underbrace{\left\langle\phi_0, \left(\mathcal{B}_\perp \times (\mathfrak{r}y)\right)\phi_0\right\rangle_{\mathcal{H}_{\mathcal{F}}}}_{= 0 \text{ since } \phi_0 \text{ is centred (5.29)}}\Big|^2$$
$$+ \mathcal{O}(\varepsilon).$$

If we add up the Berry Laplacian and the Born-Hung potential, the Berry terms as an operator on $W^\infty(\mathbb{R})$ finally read

$$-\varepsilon^2 \Delta^{\mathsf{A}^{\mathsf{B}}}_{\mathrm{d}x\otimes\mathrm{d}x} + \varepsilon^2 \left(\left\|\partial_x^{\mathsf{H}}\phi_0\right\|^2_{\mathcal{H}_{\mathcal{F}}} + \left\|\left(\mathcal{B}_\perp \times (\mathfrak{r}y)\right)\phi_0\right\|^2_{\mathcal{H}_{\mathcal{F}}}\right) + \mathcal{O}(\varepsilon^3).$$

Perturbation

The differential operator within the perturbation is (with (5.36))

$$\varepsilon \tilde{H}_1^{\mathcal{E},\mathrm{s}} = -2\varepsilon\Big[\varepsilon\partial_x^{\mathsf{H}}\left(\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\varepsilon\partial_x^{\mathsf{H}}\cdot\right) + 2\mathrm{i}\varepsilon\,\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\mathsf{A}_0(\partial_x^{\mathsf{H}})\varepsilon\partial_x^{\mathsf{H}}\Big]$$
$$- 3\varepsilon^2\,\langle\mathfrak{r},y\rangle^2_{\mathbb{R}^2}\,\varepsilon^2\Delta^{\mathrm{L.B.}}_{\mathsf{H}} - 2\mathrm{i}\varepsilon^2\mathsf{A}^{\varepsilon=0}_{\mathsf{H}}(\partial_x^{\mathsf{H}})\varepsilon\partial_x^{\mathsf{H}}$$
$$+ \mathcal{O}(\varepsilon^3)$$

with errors in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E},\mathrm{s}}),\mathcal{H})$ (in which case $\varepsilon\partial_x^{\mathsf{H}} = \mathcal{O}(1)$). Consequently, we obtain for the projected operator (in analogy to the steps carried out in Subsection 5.2.1 for the weak magnetic fields):

$$\varepsilon P_0^{\mathrm{m}}\tilde{H}_1^{\mathcal{E},\mathrm{s}}P_0^{\mathrm{m}}(\psi\phi_0^{\mathrm{m}})$$
$$= -2\varepsilon\Bigg[\varepsilon\partial_x\Big(\big\langle\phi_0^{\mathrm{m}}, \underbrace{\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}}_{\text{linear in } y}\,\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\mathrm{d}\psi(\varepsilon\partial_x)\Big)$$
$$+ \varepsilon\Bigg(\mathrm{Im}\Big(\big\langle\phi_0^{\mathrm{m}}, \langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\partial_x^{\mathsf{H}}\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)$$
$$+ 2\mathrm{i}\big\langle\phi_0^{\mathrm{m}}, \underbrace{\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\mathsf{A}_0(\partial_x^{\mathsf{H}})}_{\text{quadratic in } y}\,\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Bigg)\mathrm{d}\psi(\varepsilon\partial_x)\Bigg]\phi_0^{\mathrm{m}}$$
$$- 3\varepsilon^2\big\langle\phi_0^{\mathrm{m}}, \underbrace{\langle\mathfrak{r}y,\kappa\rangle^2_{\mathbb{R}^2}}_{\substack{\text{quadratic}\\ \text{in } y}}\,\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\left(\varepsilon^2\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x}\psi\right)\phi_0^{\mathrm{m}}$$

$$- 2\mathrm{i}\varepsilon^2 \big\langle \phi_0^{\mathrm{m}}, \underbrace{\mathsf{A}_{\mathsf{H}}^{\varepsilon=0}(\partial_x^{\mathsf{H}})}_{\text{quadratic in } y}\, \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \mathrm{d}\psi(\varepsilon\partial_x)\phi_0^{\mathrm{m}}$$
$$+ \mathcal{O}(\varepsilon^3).$$

The projected potential within the perturbation reads (with (5.38))

$$\begin{aligned} P_0^{\mathrm{m}} \mathfrak{V}_\varepsilon P_0^{\mathrm{m}}(\psi\phi_0^{\mathrm{m}}) = \Bigg(&-\tfrac{1}{4}\varepsilon^2 \|\kappa\|^2 + \varepsilon \Big\langle \phi_0^{\mathrm{m}}, \underbrace{\langle \mathrm{grad}_{\mathsf{V}} V, \mathfrak{r}y\rangle_{\mathbb{R}^2}}_{\text{linear in } y} \phi_0^{\mathrm{m}} \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \\ &+ \varepsilon^2 \Big\langle \phi_0^{\mathrm{m}}, \underbrace{\langle \mathfrak{r}y, (\mathrm{Hess}_{\mathsf{V}} V)(\mathfrak{r}y)\rangle_{\mathbb{R}^2}}_{\text{quadratic in } y} \phi_0^{\mathrm{m}} \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \Bigg) \psi\phi_0^{\mathrm{m}} \\ &+ \mathcal{O}(\varepsilon^3). \end{aligned}$$

Here, the mappings $\mathrm{grad}_{\mathsf{V}} V : \mathbb{R} \to \mathbb{R}^2$ and $\mathrm{Hess}_{\mathsf{V}} V : \mathbb{R} \to \mathbb{R}^{2\times 2}_{\mathrm{Sym}}$ are the first vertical coefficients of the Taylor expansion of $V_\varepsilon - V_c$ around the curve, i.e.,

$$\begin{aligned} (\mathrm{grad}_{\mathsf{V}} V)^j(x) &= \Big\langle \mathrm{grad}\, V\big(c(x)\big), \overline{e_j(x)} \Big\rangle_{\mathbb{R}^3}, \\ (\mathrm{Hess}_{\mathsf{V}} V)^j_{j'}(x) &= \tfrac{1}{2} \Big\langle \overline{e_j(x)}, \mathrm{Hess}\, V\big(c(x)\big)\overline{e_{j'}(x)} \Big\rangle_{\mathbb{R}^3}. \end{aligned}$$

We finally use that the leading part ϕ_0 of $\phi_0^{\mathrm{m}} = \phi_0 + \varepsilon\delta\phi_0^{\mathrm{m}}$ is real-valued and centred (5.29), as well as the fact that all derivatives not acting on ψ yield an additional ε, and arrive at the following expression for the entire projected perturbation as an operator on $W^\infty(\mathbb{R})$:

$$\begin{aligned} \varepsilon P_0^{\mathrm{m}} H_1^{\mathcal{E},\mathrm{s}} P_0^{\mathrm{m}} = -\varepsilon^2 \Bigg[& 3 \Big\langle \phi_0, \langle \mathfrak{r}y, \kappa\rangle^2_{\mathbb{R}^2} \phi_0 \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \\ &+ 4\,\mathrm{Re}\Big(\big\langle \phi_0, \langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2} \delta\phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big) \Bigg] \varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} \\ - 2\varepsilon^2 \Big[& \big\langle \phi_0, \mathsf{A}_{\mathsf{H}}^{\varepsilon=0}(\partial_x^{\mathsf{H}})\phi_0 \big\rangle_{\mathcal{H}_{\mathcal{F}}} \\ &+ 2 \big\langle \phi_0, \langle \mathfrak{r}y, \kappa\rangle_{\mathbb{R}^2} \mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0 \big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big] \mathrm{i}\varepsilon\partial_x \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^2\Big[-\tfrac{1}{4}\,\|\kappa\|^2_{\mathbb{R}^2}+2\,\mathrm{Re}\,\Big(\big\langle\phi_0,\langle\mathrm{grad}_{\mathsf{V}}\,V,\mathfrak{r}y\rangle_{\mathbb{R}^2}\,\delta\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)\\
&\qquad\qquad\big\langle\phi_0,\langle\mathfrak{r}y,\mathrm{Hess}_{\mathsf{V}}\,V(\mathfrak{r}y)\rangle_{\mathbb{R}^2}\,\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big]\\
&+\mathcal{O}(\varepsilon^3).
\end{aligned}
$$

M-Term

We start with $[H^{\mathcal{E},\mathrm{s}},P_0^{\mathrm{m}}]=[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathsf{A}_0},P_0^{\mathrm{m}}]+[\varepsilon H_1^{\mathcal{E},\mathrm{s}},P_0^{\mathrm{m}}]$. The first term may be evaluated as

$$
\begin{aligned}
\big[-\varepsilon^2\Delta_{\mathsf{H}}^{\mathsf{A}_0},P_0^{\mathrm{m}}\big]&=\big[-\nabla^{\mathsf{A}_0}_{\varepsilon\partial_x^{\mathsf{H}}}\nabla^{\mathsf{A}_0}_{\varepsilon\partial_x^{\mathsf{H}}},P_0^{\mathrm{m}}\big]\\
&=-\varepsilon\nabla^{\mathsf{A}_0}_{\varepsilon\partial_x^{\mathsf{H}}}\big[\nabla^{\mathsf{A}_0}_{\partial_x^{\mathsf{H}}},P_0^{\mathrm{m}}\big]-\varepsilon\big[\nabla^{\mathsf{A}_0}_{\partial_x^{\mathsf{H}}},P_0^{\mathrm{m}}\big]\nabla^{\mathsf{A}_0}_{\varepsilon\partial_x^{\mathsf{H}}}\\
&=\underbrace{-2\varepsilon\big[\nabla^{\mathsf{A}_0}_{\partial_x^{\mathsf{H}}},P_0^{\mathrm{m}}\big]}_{=:\varepsilon B}\,\varepsilon\partial_x^{\mathsf{H}}+\mathcal{O}(\varepsilon^2).
\end{aligned}
$$

Using the fact that

$$
\varepsilon H_1^{\mathcal{E},\mathrm{s}}=-2\varepsilon\,\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}+\varepsilon\,\langle\mathrm{grad}_{\mathsf{V}}\,V,\mathfrak{r}y\rangle_{\mathbb{R}^2}+\mathcal{O}(\varepsilon^2)\tag{5.45}
$$

with errors in $\mathcal{L}(\mathrm{dom}(H^{\mathcal{E},\mathrm{s}}),\mathcal{H})$, the second term is calculated to be

$$
\big[\varepsilon H_1^{\mathcal{E},\mathrm{s}},P_0^{\mathrm{m}}\big]=\underbrace{-2\varepsilon\big[\langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2},P_0^{\mathrm{m}}\big]}_{=:\varepsilon A}\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}+\underbrace{\varepsilon\big[\langle\mathrm{grad}_{\mathsf{V}}\,V,\mathfrak{r}y\rangle_{\mathbb{R}^2},P_0^{\mathrm{m}}\big]}_{=:\varepsilon C}
$$

plus errors of order ε^2. If we insert these expressions into (4.14), the operator $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$ is formally equal to

$$
\begin{aligned}
&\varepsilon^2P_0^{\mathrm{m}}\Big(A\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}+B\,\varepsilon\partial_x^{\mathsf{H}}+C\Big)R^{\mathcal{F},\varepsilon,\mathrm{s}}(\lambda_0^{\mathrm{m}})\Big(A\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}+B\,\varepsilon\partial_x^{\mathsf{H}}+C\Big)P_0^{\mathrm{m}}\\
&+\mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Let us consider the term

$$
\begin{aligned}
&\varepsilon^2P_0^{\mathrm{m}}A\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}\,R^{\mathcal{F},\varepsilon,\mathrm{s}}(\lambda_0^{\mathrm{m}})\,B\,\varepsilon\partial_x^{\mathsf{H}}P_0^{\mathrm{m}}\\
&\quad=\varepsilon^2P_0A\,\varepsilon^2\Delta_{\mathsf{H}}^{\mathrm{L.B.}}R^{\mathcal{F},\varepsilon=0,\mathrm{s}}(\lambda_0)\,B\,\varepsilon\partial_x^{\mathsf{H}}P_0+\mathcal{O}(\varepsilon^3)
\end{aligned}\tag{5.46}
$$

as an example. Again let $\{\phi_k(x)\}_{k\geqslant 0}$ be a complete set of real-valued eigensections of $H^{\mathcal{F},\varepsilon=0,\mathrm{s}}(x)$ with associated eigenvalues $\lambda_k(x)$. Straightforward computations then yield

$$\begin{aligned}
B(\psi\phi_0) &= -2\sum_{k>0}\big\langle \phi_k, \nabla^{\mathsf{A}_0}_{\partial^{\mathsf{H}}_x}\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}\psi\phi_k + \mathcal{O}(\varepsilon),\\
R^{\mathcal{F},\varepsilon=0,\mathrm{s}}(\lambda_0)(\psi\phi_k) &= \frac{1}{\lambda^{\Delta}_k-\lambda^{\Delta}_0}\psi\phi_k \quad (k>0),\\
A(\psi\phi_k) &= 2\big\langle \phi_0, \langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\phi_k\big\rangle_{\mathcal{H}_{\mathcal{F}}}\psi\phi_0 + \mathcal{O}(\varepsilon) \quad (k>0)
\end{aligned}$$

for $\psi\in W^{\infty}(\mathbb{R})$. Furthermore, we notice that we may commute all the derivatives to the right and act with them on ψ alone, producing only higher order errors. This ultimately gives

$$\begin{aligned}
&-4\varepsilon^2\sum_{k>0}\frac{1}{\lambda^{\Delta}_k-\lambda^{\Delta}_0}\big\langle \phi_0, \langle \mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\phi_k\big\rangle_{\mathcal{H}_{\mathcal{F}}}\big\langle \phi_k, \nabla^{\mathsf{A}_0}_{\partial^{\mathsf{H}}_x}\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}(\varepsilon\partial_x)^3\\
&+\mathcal{O}(\varepsilon^3)
\end{aligned}$$

as an operator on $W^{\infty}(\mathbb{R})$ for (5.46). It follows that the entire $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$-term formally resembles a fourth-order differential operator

$$\mathcal{M}^{\mathcal{P}^{\mathrm{m}}} = \varepsilon^2\sum_{k>0}\frac{1}{\lambda^{\Delta}_k-\lambda^{\Delta}_0}\sum_{n=0}^{4}\mathcal{M}^{(k)}_n(\varepsilon\partial_x)^n + \mathcal{O}(\varepsilon^3)$$

with appropriate operators $\mathcal{M}^{(k)}_n\in\mathcal{L}(L^2(\mathbb{R}))$.

Remark 5.22 If one again considers low energies, i.e., the action of $H^{\mathrm{s}}_{\mathrm{QWG}}$ on states $\psi\in W^{\infty}(\mathbb{R})$ with $\|\varepsilon\partial_x\psi\|_{L^2(\mathbb{R})}=\mathcal{O}(\varepsilon^{\alpha})$ for $\alpha>0$, the leading contribution of $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$ is given by

$$\begin{aligned}
&\varepsilon^2\sum_{k>0}\frac{1}{\lambda^{\Delta}_k-\lambda^{\Delta}_0}\mathcal{M}^{(k)}_0\\
&= -\varepsilon^2\sum_{k>0}\frac{1}{\lambda^{\Delta}_k-\lambda^{\Delta}_0}\Big|\big\langle \phi_k, \langle \operatorname{grad}_{\mathsf{V}} V,\mathfrak{r}y\rangle_{\mathbb{R}^2}\,\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big|^2.
\end{aligned}$$

The appearance of $\operatorname{grad}_{\mathsf{V}} V$ within the super-adiabatic correction is not very surprising because the corresponding potential term $\varepsilon \langle \operatorname{grad}_{\mathsf{V}} V, \mathfrak{r}y\rangle_{\mathbb{R}^2}$ in the perturbation (5.45) prevents $H_1^{\mathcal{E},\mathrm{s}}$ from satisfying Condition 4.11. Put differently, if $\operatorname{grad}_{\mathsf{V}} V$ was zero, $H_1^{\mathcal{E},\mathrm{s}}$ would satisfy the requirements for applying the results for the low energy asymptotics (cf. Section 4.4), which in particular yield an approximation of the initial operator $H^{\mathcal{E},\mathrm{s}}$ solely in terms of the adiabatic operator without the $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$-term. ◊

Conclusion

Let us finally gather all terms within the reduced Hamiltonian up to errors of order ε^3 when applied on states $\psi \in W^{\infty}(\mathbb{R})$ with $\|\varepsilon\partial_x\psi\|_{L^2(\mathbb{R})} = \mathcal{O}(1)$. This allows for an accurate approximation of points in the spectrum of the initial tube operator by means of Theorem 5.21:

Corollary 5.23 *Let $H_{\mathrm{tube}}^{\mathrm{s,mas}}$ be the Hamiltonian generating the dynamics of a non-relativistic, charged, spinless quantum particle, which is localised within the massive quantum waveguide introduced in Example 5.7, in the presence of a strong, C^{∞}-bounded external magnetic potential $\varepsilon^{-1}\mathcal{A}$ and a C^{∞}-bounded external potential V. Then the reduced Hamiltonian reads*

$$
\begin{aligned}
H_{\mathrm{QWG}}^{\mathrm{s,mas}} = & -\varepsilon^2\Delta_{\mathrm{d}x\otimes\mathrm{d}x}^{\mathsf{A}^{\mathrm{B}}} + \lambda_0^{\Delta} + V_{\mathrm{c}} \\
& + \varepsilon^2\mathcal{B}_{\|}^2\Big\langle \phi_0, \tfrac{1}{4}\big((\mathfrak{a}y^1)^2 + (\mathfrak{b}y^2)^2\big)\phi_0 - L_y^{\mathfrak{a},\mathfrak{b}}\theta\Big\rangle_{\mathcal{H}_{\mathcal{F}}} \\
& + \varepsilon^2\left\|\partial_x^{\mathsf{H}}\phi_0\right\|_{\mathcal{H}_{\mathcal{F}}}^2 + \varepsilon^2\left\|\big(\mathcal{B}_{\perp}\times(\mathfrak{r}y)\big)\phi_0\right\|_{\mathcal{H}_{\mathcal{F}}}^2 - \tfrac{1}{4}\varepsilon^2\|\kappa\|_{\mathbb{R}^2}^2 \\
& - \varepsilon^2\Big[3\Big\langle\phi_0, \langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}^2\,\phi_0\Big\rangle_{\mathcal{H}_{\mathcal{F}}} \\
& \qquad + 4\,\mathrm{Re}\Big(\big\langle\phi_0, \langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\delta\phi_0^{\mathrm{m}}\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big)\Big]\varepsilon^2\Delta_{\mathrm{d}x\otimes\mathrm{d}x}^{\mathrm{L.B.}} \\
& - 2\varepsilon^2\Big[\big\langle\phi_0, \mathsf{A}_{\mathsf{H}}^{\varepsilon=0}(\partial_x^{\mathsf{H}})\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}} \\
& \qquad + 2\big\langle\phi_0, \langle\mathfrak{r}y,\kappa\rangle_{\mathbb{R}^2}\,\mathsf{A}_0(\partial_x^{\mathsf{H}})\phi_0\big\rangle_{\mathcal{H}_{\mathcal{F}}}\Big]\mathrm{i}\varepsilon\partial_x
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon^2 \Big[2\,\mathrm{Re}\Big(\big\langle \phi_0, \langle \mathrm{grad}_{\mathsf{V}} V, \mathfrak{r}y \rangle_{\mathbb{R}^2}\, \delta\phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big) \\
&\qquad + \big\langle \phi_0, \langle \mathfrak{r}y, \mathrm{Hess}_{\mathsf{V}} V(\mathfrak{r}y) \rangle_{\mathbb{R}^2}\, \phi_0 \big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big] \\
&+ \varepsilon^2 \sum_{k>0} \frac{1}{\lambda_k^{\Delta} - \lambda_0^{\Delta}} \sum_{n=0}^{4} \mathcal{M}_n^{(k)} (\varepsilon \partial_x)^n .
\end{aligned}
$$

We close the discussion with a look at the case of low energies, and so we assume that λ_0^{Δ} is constant and that the external potential $\varepsilon^2 V$ is very weak. This implies $\lambda_0^{\mathrm{m}}(x) - \lambda_0^{\Delta} = \mathcal{O}(\varepsilon^2)$ with errors uniform in x. If we subtract λ_0^{Δ} from the operator, all remaining potentials are of order ε^2 and thus so is the kinetic energy operator $-\varepsilon^2 \Delta^{\mathsf{A}_{\mathrm{B}}}_{\mathrm{d}x\otimes\mathrm{d}x} = -\varepsilon^2 \Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + \mathcal{O}(\varepsilon^3)$, which is due to the fact that $\mathsf{A}_{\mathrm{B}}(\partial_x) = \mathcal{O}(\varepsilon)$ for a centred unperturbed ground state ϕ_0. Consequently, if we consider $H^{\mathrm{s,mas}}_{\mathrm{QWG}}$ on states $\psi \in W^{\infty}(\mathbb{R})$ with $\|\varepsilon\partial_x \psi\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon)$, we get

$$
\begin{aligned}
H^{\mathrm{s,mas}}_{\mathrm{QWG},\alpha=2} &:= \varepsilon^{-2} \big(H^{\mathrm{s,mas}}_{\mathrm{QWG}} - \lambda_0^{\Delta} \mathbf{1}_{L^2(\mathbb{R})} \big) \\
&= -\Delta^{\mathrm{L.B.}}_{\mathrm{d}x\otimes\mathrm{d}x} + V_c + \big\| \partial_x^{\mathsf{H}} \phi_0 \big\|^2_{\mathcal{H}_{\mathcal{F}}} + \big\| \big(\mathcal{B}_{\perp} \times (\mathfrak{r}y) \big) \phi_0 \big\|^2_{\mathcal{H}_{\mathcal{F}}} \\
&\quad + \mathcal{B}_{\|}^2 \Big\langle \phi_0, \tfrac{1}{4} \big((\mathfrak{a}y^1)^2 + (\mathfrak{b}y^2)^2 \big) \phi_0 - L_y^{\mathfrak{a},\mathfrak{b}} \theta \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \\
&\quad - \tfrac{1}{4} \|\kappa\|^2_{\mathbb{R}^2} \\
&\quad + \mathcal{O}(\varepsilon).
\end{aligned}
$$

Any $\nu \in \sigma(H^{\mathrm{s,mas}}_{\mathrm{QWG},\alpha=2})$ in the spectrum of this operator corresponds to some point υ^{ε} in the spectrum of the initial tube operator $H^{\mathrm{s,mas}}_{\mathrm{tube}}$ with the asymptotic expansion $\upsilon^{\varepsilon} = \lambda_0^{\Delta} + \varepsilon^2 \nu + \mathcal{O}(\varepsilon^3)$. We can finally relate this result to the effective operator derived in [KR14, Definition 2.5 ($\delta = 1$ in their notation)] for their geometric framework, which was sketched in Remark 5.16. The correction to the unperturbed ground state band is then given by

$$
\lambda_{0,2} = \mathcal{B}_{\|}^2 \Big(\tfrac{1}{4} \big\langle \phi_0, \|y\|^2_{\mathbb{R}^2}\, \phi_0 \big\rangle_{L^2(F,\mathrm{d}y)} - \big\langle \phi_0, L_y \theta \big\rangle_{L^2(F,\mathrm{d}y)} \Big)
$$

with vertical angular momentum operator $L_y := -\mathrm{i}(y \times \nabla_y)$, where θ is the unique solution of

$$(-\Delta^{\mathrm{L.B.}}_{\mathrm{d}y\otimes\mathrm{d}y} - \lambda_0^{\Delta})\theta = P_0^{\perp} L_y \phi_0.$$

If one drops the property for ϕ_0 to be centred, it is easy to verify on the basis of the previous calculations that the modified reduced operator takes the form

$$\begin{aligned} H^{\mathrm{s,mas}}_{\mathrm{QWG},\alpha=2} = {} & \Big\langle \phi_0, -\Delta_{\mathsf{H}}^{\mathcal{B}_\perp \times y}(\cdot\phi_0)\Big\rangle_{L^2(F,\mathrm{d}y)} + V_c - \tfrac{1}{4}\,\|\kappa\|^2_{\mathbb{R}^2} \\ & + \mathcal{B}_\parallel^2 \Big(\tfrac{1}{4}\big\langle \phi_0, \|y\|^2_{\mathbb{R}^2}\,\phi_0\big\rangle_{L^2(F,\mathrm{d}y)} - \big\langle \phi_0, L_y\theta\big\rangle_{L^2(F,\mathrm{d}y)}\Big) \\ & + \mathcal{O}(\varepsilon). \end{aligned}$$

5.3.2 Example for Hollow Quantum Waveguides

This subsection is dedicated to the computation of the reduced Hamiltonian $H^{\mathrm{s}}_{\mathrm{QWG}}$ for a single non-relativistic, spinless, charged quantum particle localised within hollow quantum waveguides as introduced in Example 5.9, where one considers the boundary of a cylindrical tube with varying radius around a smoothly embedded curve $c : \mathbb{R} \to \mathbb{R}^3$. We deduce from Example 5.10(ii) that the corresponding strong magnetic field is given by

$$\begin{aligned} \varepsilon^{-1}\mathcal{A}_\varepsilon &= \mathsf{A}_\varepsilon \\ &= \underbrace{\left(\mathcal{B}_\perp \times \begin{pmatrix} \ell\cos y \\ \ell \sin y\end{pmatrix}\right)\mathrm{d}x^{\mathsf{H}}}_{=\mathsf{A}_0} + \varepsilon\left(\mathsf{A}^\varepsilon_{\mathsf{H}} + \underbrace{\left(\tfrac{1}{2}\ell^2\mathcal{B}_\parallel + \mathcal{O}(\varepsilon)\right)\times \mathrm{d}y}_{=\mathsf{A}^\varepsilon_{\mathsf{V}}}\right). \end{aligned}$$

Moreover, the spin degrees of freedom again vanish (i.e., $q = 1$ with $\zeta_0^1(x) = 1$ for all $x \in \mathbb{R}$) and the ground state of the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{s}} = -\Delta_{\mathsf{V}}^{\varepsilon\mathsf{A}^\varepsilon_{\mathsf{V}}} + V_c$ (we no longer need to insert $V^{\mathsf{V}}_{\rho_\varepsilon}$ into $H^{\mathcal{F},\varepsilon,\mathrm{s}}$) reads

$$\phi_0^{\mathrm{m}} = \underbrace{\tfrac{1}{\sqrt{2\pi\ell}} \circ \pi_M}_{\phi_0} + \mathcal{O}(\varepsilon^2) \quad , \quad \lambda_0^{\mathrm{m}} = V_c + \varepsilon^2\lambda_{0,2} + \mathcal{O}(\varepsilon^3).$$

We will again drop the composition with π_M for the sake of clarity. In the end, this ground state ϕ_0^{m} provides a global trivialisation of $W^\infty(\mathcal{P}^{\mathrm{m}})$ in terms of $W^\infty(\mathbb{R}) \otimes \mathrm{span}(\phi_0^{\mathrm{m}})$ and $H^{\mathrm{s}}_{\mathrm{QWG}}$ may be regarded as an operator acting on $W^\infty(\mathbb{R})$.

Since the kinetic energy $-\varepsilon^2 \Delta^{\mathrm{A^B}}_{\mathrm{d}x\otimes\mathrm{d}x}$ is of the same order as the leading-order potential V_c, one a priori expects highly oscillating states $\psi \in W^\infty(\mathbb{R})$ such that $\|\varepsilon \partial_x \psi\|_{L^2(\mathbb{R})} = \mathcal{O}(1)$ and one has to calculate a rather high number of terms in the same manner as in the massive case of the previous subsection. In order to not overburden this example, we will furthermore assume that the external potential $\varepsilon^2 V$ is very weak. This implies that λ_0^{m} and all other potentials in $H^{\mathrm{s}}_{\mathrm{QWG}}$ are of order ε^2. Hence, we will again retrieve the low energy regime $\alpha = 2$, where the reduced Hamiltonian $H^{\mathrm{s}}_{\mathrm{QWG}}$ is considered on states $\psi \in W^\infty(\mathbb{R})$ for low-lying eigenvalues with $\|\varepsilon \partial_x \psi\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon)$.

Ground State Band

In view of Remark 5.20(ii), the two terms contributing to $\lambda_{0,2}$ are easily computed with the aid of the orthonormal eigenfunctions (5.41) of $-\Delta^{\mathrm{L.B.}}_{\mathrm{V}}$. To be specific, one obtains the expressions

$$\int_{\mathbb{S}^1} \tfrac{1}{\ell^2} \left(\tfrac{1}{2}\ell^2 \mathcal{B}_\| \phi_0\right) \left(\tfrac{1}{2}\ell^2 \mathcal{B}_\| \phi_0\right) \mathrm{vol}_{g_{\mathrm{V}}} = \tfrac{1}{4}\ell^2 \mathcal{B}_\|^2 \underbrace{\langle \phi_0, \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}}}_{=1}$$

and

$$\begin{aligned}
&\sum_{k>0} \sum_{\bullet \in \{\mathrm{e},\mathrm{o}\}} \frac{\left| \int_{\mathbb{S}^1} \frac{1}{\ell^2} (\partial_y \phi_k^\bullet) \left(\frac{1}{2}\ell^2 \mathcal{B}_\| \phi_0\right) - \frac{1}{\ell^2} \left(\frac{1}{2}\ell^2 \mathcal{B}_\| \phi_k^\bullet\right) \overbrace{(\partial_y \phi_0)}^{=0} \mathrm{vol}_{g_{\mathrm{V}}} \right|^2}{k^2/\ell^2} \\
&= \sum_{k>0} \tfrac{1}{2} \tfrac{\ell^2}{k^2} \mathcal{B}_\|^2 \left(\left| \langle \partial_y \phi_k^{\mathrm{e}}, \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}} \right|^2 - \left| \langle \partial_y \phi_k^{\mathrm{o}}, \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}} \right|^2 \right) \\
&= \sum_{k>0} \tfrac{1}{2} \ell^2 \mathcal{B}_\|^2 \left(\Big| \underbrace{\langle \phi_k^{\mathrm{o}}, \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \Big|^2 - \Big| \underbrace{\langle \phi_k^{\mathrm{e}}, \phi_0 \rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \Big|^2 \right) \\
&= 0
\end{aligned}$$

for them. The contribution of $\varepsilon^2 V_c$ is trivial and we conclude

$$\lambda_0^{\mathrm{m}} = \varepsilon^2 \Big(V_c + \tfrac{1}{4}\ell^2 \mathcal{B}_{\|}^2 \Big) + \mathcal{O}(\varepsilon^3).$$

Berry Terms

As in the case of a weak magnetic field (see Subsection 5.2.2), it holds that

$$-\varepsilon^2 \Delta^{\mathrm{B}} + \varepsilon^2 V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}} = -\varepsilon^2 \Delta^{\mathsf{A}^{\mathrm{B}}}_{\mathrm{d}x \otimes \mathrm{d}x} + \varepsilon^2 \big(V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}} + V_{\eta_{\mathsf{V}}}^{\mathcal{P}^{\mathrm{m}}} \big)$$

with Berry connection one-form (the “effective magnetic potential”)

$$\begin{aligned}
\mathsf{A}^{\mathrm{B}}(\partial_x) &= \big\langle \phi_0^{\mathrm{m}}, \mathsf{A}_0(\partial_x^{\mathsf{H}}) \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} + \operatorname{Im} \Big(\big\langle \phi_0^{\mathrm{m}}, \partial_x^{\mathsf{H}} \phi_0^{\mathrm{m}} \big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big) \\
&= \underbrace{\Big\langle \tfrac{1}{\sqrt{2\pi\ell}}, \Big(\mathcal{B}_{\perp} \times \begin{pmatrix} \ell \cos y \\ \ell \sin y \end{pmatrix} \Big) \tfrac{1}{\sqrt{2\pi\ell}} \Big\rangle_{\mathcal{H}_{\mathcal{F}}}}_{=0} \\
&\quad + \underbrace{\operatorname{Im} \Big(\Big\langle \tfrac{1}{\sqrt{2\pi\ell}}, \partial_x^{\mathsf{H}} \tfrac{1}{\sqrt{2\pi\ell}} \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big)}_{=0} \\
&\quad + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

Born-Huang potential

$$\begin{aligned}
V_{\mathrm{BH}}^{\mathcal{P}^{\mathrm{m}}} &= \Big\| \partial_x^{\mathsf{H}} \tfrac{1}{\sqrt{2\pi\ell}} \Big\|_{\mathcal{H}_{\mathcal{F}}}^2 + \Big| \Big\langle \tfrac{1}{\sqrt{2\pi\ell}}, \partial_x^{\mathsf{H}} \tfrac{1}{\sqrt{2\pi\ell}} \Big\rangle_{\mathcal{H}_{\mathcal{F}}} \Big|^2 \\
&\quad + \Big\| \Big(\mathcal{B}_{\perp} \times \begin{pmatrix} \ell \cos y \\ \ell \sin y \end{pmatrix} \Big) \tfrac{1}{\sqrt{2\pi\ell}} \Big\|_{\mathcal{H}_{\mathcal{F}}}^2 \\
&\quad + \mathcal{O}(\varepsilon^2) \\
&= \Big\| \mathcal{B}_{\perp} \times \tfrac{\ell}{\sqrt{2}} \begin{pmatrix} \phi_1^{\mathrm{e}} \\ \phi_1^{\mathrm{o}} \end{pmatrix} \Big\|_{\mathcal{H}_{\mathcal{F}}}^2 + \mathcal{O}(\varepsilon^2) \\
&= \tfrac{1}{2}\ell^2 \left\| \mathcal{B}_{\perp} \right\|_{\mathbb{R}^2}^2 + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

and η_{V}-potential

$$V_{\eta_{\mathsf{V}}}^{\mathcal{P}^{\mathrm{m}}} = -\tfrac{1}{2} \partial_x \cdot \overline{\eta}_{\mathsf{V}}(\partial_x) + \tfrac{1}{4} \left| \overline{\eta}_{\mathsf{V}}(\partial_x) \right|^2 = \tfrac{1}{2} \tfrac{\ell''}{\ell} - \tfrac{1}{4} \Big(\tfrac{\ell'}{\ell} \Big)^2 .$$

Perturbation

The only term of order ε^2 within $\varepsilon H_1^{\mathcal{E},\mathrm{s}}$ (for this low energy regime) is the potential

$$\begin{aligned}\mathfrak{V}_\varepsilon &= V^{\mathsf{V}}_{\rho_\varepsilon} + \mathcal{O}(\varepsilon^3)\\ &\overset{(5.40)}{=} \tfrac{1}{2}\varepsilon\tfrac{\varkappa}{\ell} + \tfrac{1}{4}\varepsilon^2\big(2\varkappa^2 - (\varkappa^\perp)^2\big) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

with $\varkappa^\perp = -\kappa^1 \sin y + \kappa^2 \cos y$. While the term proportional to ε is a linear combination of $\cos y$ and $\sin y$ and thus vanishes when it is integrated against the square of the unperturbed, fibrewise constant ground state $\frac{1}{\sqrt{2\pi\ell}}$, the quadratic term may be rewritten as

$$\tfrac{1}{4}\varepsilon^2\big[(\kappa^1)^2(2\cos^2 y - \sin^2 y) + (\kappa^2)^2(2\sin^2 y - \cos^2 y)\big].$$

If we now bear in mind the fact that

$$\big\langle\phi_0, \cos^2 y\,\phi_0\big\rangle_{\mathcal{H}_\mathcal{F}} = \tfrac{1}{2} = \big\langle\phi_0, \sin^2 y\,\phi_0\big\rangle_{\mathcal{H}_\mathcal{F}},$$

we arrive at the compact expression

$$\varepsilon P_0^{\mathrm{m}} H_1^{\mathcal{E},\mathrm{s}} P_0^{\mathrm{m}} = \tfrac{1}{8}\varepsilon^2\,\|\kappa\|^2_{\mathbb{R}^2} + \mathcal{O}(\varepsilon^3).$$

M-Term

In virtue of Remark 5.22, the leading-order contribution of the $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$-term is essentially determined by the $\mathcal{O}(\varepsilon)$-contribution of $\mathfrak{V}_\varepsilon$, i.e.,

$$\begin{aligned}\mathcal{M}^{\mathcal{P}^{\mathrm{m}}} &= \varepsilon^2 \sum_{k>0} \frac{1}{k^2/\ell^2} \sum_{\bullet\in\{\mathrm{e},\mathrm{o}\}} \mathcal{M}_0^{(k,\bullet)} + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{4}\varepsilon^2 \sum_{k>0} \frac{1}{k^2/\ell^2} \sum_{\bullet\in\{\mathrm{e},\mathrm{o}\}} \Big|\big\langle\phi_k^\bullet, \tfrac{\varkappa}{\ell}\phi_0\big\rangle_{\mathcal{H}_\mathcal{F}}\Big|^2 + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{4}\varepsilon^2 \sum_{k>0} \frac{1}{k^2/\ell^2} \sum_{\bullet\in\{\mathrm{e},\mathrm{o}\}} \Big|\Big\langle\phi_k^\bullet, \tfrac{\kappa^1}{\sqrt{2}\ell}\phi_1^{\mathrm{e}}\Big\rangle_{\mathcal{H}_\mathcal{F}} + \Big\langle\phi_k^\bullet, \tfrac{\kappa^2}{\sqrt{2}\ell}\phi_1^{\mathrm{o}}\Big\rangle_{\mathcal{H}_\mathcal{F}}\Big|^2\\ &\quad + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{4}\varepsilon^2\ell^2\left[\left(\tfrac{\kappa^1}{\sqrt{2}\ell}\right)^2 + \left(\tfrac{\kappa^2}{\sqrt{2}\ell}\right)^2\right] + \mathcal{O}(\varepsilon^3)\\ &= -\tfrac{1}{8}\varepsilon^2\,\|\kappa\|^2_{\mathbb{R}^2} + \mathcal{O}(\varepsilon^3).\end{aligned}$$

This term exactly cancels the remainder of the geometric potential. We have already observed in the weak case that V_{ρ_ε} does not contribute to the reduced Hamiltonian up to order ε^2. In this context, one may adopt two different perspectives concerning the geometric potential:

- One absorbs $V_{\rho_\varepsilon}^{\vee}$ into the vertical operator and shows that the ground state band remains unchanged. On the one hand, one has to calculate (formally) fewer terms in $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}} + \mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$, but on the other hand this absorption comes along with a more complicated ground state (and thus a more complicated spectral projection P_0^{m})

$$\phi_0^{\mathrm{m}} \overset{(5.42)}{=} \frac{1}{\sqrt{2\pi\ell}} - \frac{1}{\sqrt{8}}\varepsilon\ell(\kappa^1\cos y + \kappa^2 \sin y) + \mathcal{O}(\varepsilon^2).$$

- One incorporates $V_{\rho_\varepsilon}^{\vee}$ into the perturbation. Then the advantages and disadvantages are reversed: One has to evaluate more terms in $H_{\mathrm{a}}^{\mathcal{P}^{\mathrm{m}}} + \mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$ (which again leads to cancellations as above). But on the other hand, the finite-dimensional perturbation theory of the vertical ground state, which is reflected in $P_0^{\mathrm{m}} = P_0 + \mathcal{O}(\varepsilon^2)$, is much simpler.

Conclusion

We now collect all relevant terms in the sum of the adiabatic operator and the $\mathcal{M}^{\mathcal{P}^{\mathrm{m}}}$-term and approximate some parts of the spectrum associated with the initial tube operator:

Corollary 5.24 *Let $H_{\mathrm{tube}}^{\mathrm{s,hol}}$ be the Hamiltonian generating the dynamics of a non-relativistic, charged, spinless quantum particle, which is localised within the hollow waveguide introduced in Example 5.9 with $\ell(x,y) = \ell(x)$, in the presence of a strong, C^∞-bounded external magnetic field $\varepsilon^{-1}\mathcal{A}$ and very weak, C^∞-bounded external potential $\varepsilon^2 V$. Then the reduced Hamiltonian reads*

$$H_{\mathrm{QWG}}^{\mathrm{s,hol}} = -\Delta_{\mathrm{d}x\otimes\mathrm{d}x}^{\mathrm{L.B.}} + V_c + \frac{1}{2}\frac{\ell''}{\ell} - \frac{1}{4}\left(\frac{\ell'}{\ell}\right)^2 + \frac{\ell^2}{4}\left(\mathcal{B}_{\parallel}^2 + 2\,\|\mathcal{B}_\perp\|_{\mathbb{R}^2}^2\right).$$

Appendix A

A Analytic Aspects of the Connection Laplacian

In this appendix, we will collect the most important facts which build the basis for the analysis of the connection Laplacian in the main part of this thesis.

A.1 Sobolev Spaces on Vector Bundles

In the first section, we will introduce the geometric and analytic terminology that is needed for our purpose. Therefore, we will give a brief overview of the elementary definitions concerning finite-dimensional vector bundles and define Sobolev spaces on these bundles.

A vector bundle over a manifold M makes precise the idea of "attaching" an $\mathbb{F}$-vector space $\mathcal{E}_p$ ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) at every point $p \in M$ in such a way that these spaces fit together appropriately and form a total space $\mathcal{E}$.

Definition A.1 Let M be a smooth, real manifold with (possibly empty) boundary ∂M and $\mathcal{E}$ be a smooth, real/complex manifold. We call the smooth surjective map $\pi_{\mathcal{E}} : \mathcal{E} \to M$ a smooth $\mathbb{F}^N$*-vector bundle* over M if for all $p \in M$

(i) there exists an open neighbourhood $W \subset M$ of p together with a diffeomorphism $\Lambda : \pi_{\mathcal{E}}^{-1}(W) \to W \times \mathbb{F}^N$ for which the diagram

$$\begin{tikzcd} \pi_{\mathcal{E}}^{-1}(W) \arrow[rr, "\Lambda"] \arrow[dr, "\pi_{\mathcal{E}}"'] & & W \times \mathbb{F}^N \arrow[dl, "\mathrm{pr}_1"] \\ & W & \end{tikzcd}$$

commutes,

(ii) and the set $\mathcal{E}_p := \pi_{\mathcal{E}}^{-1}(p)$ has the structure of an $\mathbb{F}$-vector space and $\Lambda|_{\mathcal{E}_p} : \mathcal{E}_p \to \{p\} \times \mathbb{F}^N$ is a vector space isomorphism. $\Diamond$

On calls $\mathcal{E}$ the *total space*, M the *base space* and $\mathcal{E}_p = \pi_{\mathcal{E}}^{-1}(p)$ the *fibres*. We will often write $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$ or merely $\mathcal{E}$ for the vector bundle if there is no danger of confusion. Furthermore, we introduce the abbreviations

$$\mathcal{E}^\circ := \mathcal{E}|_{M\setminus\partial M} \quad , \quad \partial\mathcal{E} := \mathcal{E}|_{\partial M}.$$

The simplest example of a vector bundle $\mathcal{E}$ is the case where each fibre $\mathcal{E}_p$ is a copy of the same $\mathbb{F}$-vector space, i.e., $\mathcal{E}$ is globally diffeomorphic to the product manifold $M \times \mathbb{F}^N$. In this context, one calls $\mathrm{pr}_1 : M \times \mathbb{F}^N \to M$ the *trivial* $\mathbb{F}^N$-vector bundle over M. Moreover, one often refers to $\mathbb{F}^1$-vector bundles as real/complex *line bundles*.

Let $\{W_\mu\}_{\mu\in\mathcal{I}}$ be an open cover of M with an associated set of *local trivialisations* $\{\Lambda_\mu : \pi_{\mathcal{E}}^{-1}(W_\mu) \to W_\mu \times \mathbb{F}^N\}_{\mu\in\mathcal{I}}$. Then condition (ii) of the previous definition is equivalent to requiring smoothness of the transition functions

$$\begin{aligned} t_{\mu\mu'} := \Lambda_{\mu'} \circ \Lambda_\mu^{-1} : (W_\mu \cap W_{\mu'}) \times \mathbb{F}^N &\to (W_\mu \cap W_{\mu'}) \times \mathbb{F}^N, \\ (p, v) &\mapsto \big(x, \mathsf{g}_{\mu\mu'}(p)v\big), \end{aligned}$$

or equivalently to requiring smoothness of the induced transition matrices $\mathsf{g}_{\mu\mu'} : W_\mu \cap W_{\mu'} \to \mathrm{GL}(N, \mathbb{F})$. Conversely, an open cover $\{W_\mu\}_{\mu\in\mathcal{I}}$ of M together with a set of smooth mappings $\{\mathsf{g}_{\mu\mu'} : W_\mu \cap W_\nu \to \mathrm{GL}(N, \mathbb{F})\}_{\mu,\mu'\in\mathcal{I}}$ that satisfy $\mathsf{g}_{\mu\mu'} = \mathsf{g}_{\mu'\mu}^{-1}$ and the cocycle condition

$$\mathsf{g}_{\mu\mu'} \circ \mathsf{g}_{\mu'\mu''} = \mathsf{g}_{\mu\mu''}$$

for all $\mu, \mu', \mu'' \in \mathcal{I}$, uniquely define the structure of an $\mathbb{F}^N$-vector bundle $\mathcal{E} \xrightarrow{\pi_\mathcal{E}} M$ with total space

$$\mathcal{E} := \mathsf{E}/\sim, \quad \mathsf{E} = \bigcup_{\mu \in \mathcal{I}} W_\mu \times \mathbb{F}^N$$

and projection $\pi_\mathcal{E}[(\mu, p, v)] := p$. Here, two points (μ, p, v) and (μ', p', v') are said to be equivalent whenever $p = p'$ and $v' = \mathsf{g}_{\mu\mu'}(p)v$.

Given two vector bundles $\mathcal{E} \xrightarrow{\pi_\mathcal{E}} M$ and $\mathcal{F} \xrightarrow{\pi_\mathcal{F}} M$ over the same base manifold M, one may use several vector space operations in order to construct new vector bundles over M, for example

- direct sums $\mathcal{E} \oplus \mathcal{F}$ with fibres $\mathcal{E}_p \oplus \mathcal{F}_p$,
- the bundle of linear maps $\mathcal{L}(\mathcal{E}, \mathcal{F})$ with fibres $\mathcal{L}(\mathcal{E}_p, \mathcal{F}_p)$,
  - the bundle of endomorphisms $\mathrm{End}(\mathcal{E}) = \mathcal{L}(\mathcal{E}, \mathcal{E})$,
  - the dual bundle $\mathcal{E}^* = \mathcal{L}(\mathcal{E}, M \times \mathbb{F})$ with fibres $\mathcal{L}(\mathcal{E}_p, \mathbb{F})$,
- and tensor products $\mathcal{E} \otimes \mathcal{F}$ with fibres $\mathcal{E}_p \otimes \mathcal{F}_p$.

The k-fold application ($k \in \mathbb{N}$) of the tensor product yields the bundle $\mathcal{E}^{*\otimes k}$ of k-fold covariant tensors. We will denote by $\Sigma^k\mathcal{E}$ and $\Lambda^k\mathcal{E}$ its subbundles (with fibrewise subspaces) of symmetric and alternating k-fold covariant tensors, respectively.

Definition A.2 Let $\pi_\mathcal{E} : \mathcal{E} \to M$ be an $\mathbb{F}^N$-vector bundle. A *section* of $\mathcal{E}$ is a map $\psi : M \to \mathcal{E}$ with the property $\pi_\mathcal{E} \circ \psi = \mathbf{1}_M$, i.e., $\psi(p) \in \mathcal{E}_p$ for all $p \in M$.

(i) $C^\infty(\mathcal{E})$ denotes the space of smooth sections of $\mathcal{E}$.

(ii) $C_0^\infty(\mathcal{E}) \subset C^\infty(\mathcal{E})$ stands for the Fréchet space of smooth section of $\mathcal{E}$ which are compactly supported in M. $\Diamond$

A (local) *frame* of $\mathcal{E}$ is a set of N (local) sections $\{\psi_1, \ldots, \psi_N\}$ of $\mathcal{E}$ which form a basis of $\mathcal{E}_p$ at each point $p \in M$ (wherever they are defined).

A.1.1 Metrics and Connections

Any $\mathbb{F}$-vector space can be turned into a Euclidean/unitary vector space by means of an inner product. This idea can be transferred to vector bundles by the fibrewise assignment of such inner products:

Definition A.3 Let $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be an $\mathbb{F}^N$-vector bundle. A *bundle metric* is a map $h \in C^\infty(\mathcal{E}^{*\otimes 2})$, such that $h_p : \mathcal{E}_p \times \mathcal{E}_p \to \mathbb{F}$ defines a Euclidean/Hermitian scalar product for all $p \in M$. We refer to the pair $(\mathcal{E}, h)$ as a *Euclidean/Hermitian vector bundle*. ♢

We note that every smooth $\mathbb{F}^N$-vector bundle admits a smooth bundle metric

$$h_p(\mathfrak{v}, \mathfrak{w}) = \sum_{\mu \in \mathcal{I}} \chi_\mu(p) \langle \mathfrak{v}, \mathfrak{w} \rangle_{\mathbb{F}^N, \mu} \quad \text{for } \mathfrak{v}, \mathfrak{w} \in \mathcal{E}_p,$$

where $\{\chi_\mu\}_{\mu \in \mathcal{I}}$ is a smooth partition of unity of M subordinate to the cover $\{W_\mu\}_{\mu \in \mathcal{I}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{F}^N, \mu}$ denotes the bundle metric over W_μ, obtained by the pullback of the standard scalar product $\langle \cdot, \cdot \rangle_{\mathbb{F}^N}$ in $\mathbb{F}^N$ via the local trivialisation Λ_μ.

Furthermore, we want to enrich the geometric structure of vector bundles with the notion of connections, which play a crucial role throughout this thesis.

Definition A.4 Let $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be an $\mathbb{F}^N$-vector bundle.

(i) A *connection* on $\mathcal{E}$ is a map $\nabla^{\mathcal{E}} : C^\infty(\mathcal{E}) \to C^\infty(\mathsf{T}^* M \otimes \mathcal{E})$ such that the Leibniz rule

$$\nabla^{\mathcal{E}}(f\psi) = \mathrm{d}f \otimes \psi + f \nabla^{\mathcal{E}} \psi$$

holds for all $\psi \in C^\infty(\mathcal{E})$ and $f \in C^\infty(M)$.

(ii) If $(\mathcal{E}, h)$ is an Euclidean/Hermitian vector bundle, we call a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ *metric* if

$$\mathrm{d}\big(h(\phi, \psi)\big) = h(\nabla^{\mathcal{E}} \phi, \psi) + h(\phi, \nabla^{\mathcal{E}} \psi)$$

for all $\phi, \psi \in C^{\infty}(\mathcal{E})$. ◊

The condition that the connection be metric clearly is equivalent to requiring

$$X \cdot h(\phi, \psi) = h\big(\nabla^{\mathcal{E}}_X \phi, \psi\big) + h\big(\phi, \nabla^{\mathcal{E}}_X, \psi\big) \tag{A.1}$$

for all $X \in C^{\infty}(\mathsf{T}M)$. Moreover, the mapping

$$\nabla^{\mathcal{E}}_X : C^{\infty}(\mathcal{E}) \to C^{\infty}(\mathcal{E}), \quad \psi \mapsto \nabla^{\mathcal{E}}_X \psi$$

is called the *covariant derivative* along X. A connection allows us to identify (literally "connect") nearby fibres $\mathcal{E}_p$ and $\mathcal{E}_{p'}$ as follows: Let $\gamma : [0,1] \to M$ be a path joining $\gamma(0) = p$ with $\gamma(1) = p'$. A section ψ of $\mathcal{E}$ along γ (i.e., a mapping $\psi : [0,1] \to \mathcal{E}$ with $\psi(t) \in \mathcal{E}_{\gamma(t)}$ for all $t \in [0,1]$) is said to be parallel with respect to $\nabla^{\mathcal{E}}$ if

$$\nabla^{\mathcal{E}}_{\dot{\gamma}(t)} \psi(t) = 0 \quad \text{for all } t \in [0,1]. \tag{A.2}$$

This first-order ordinary differential equation with initial data $\psi(0) = \mathfrak{v} \in \mathcal{E}_p$ has a unique smooth solution $\psi^{\mathfrak{v}} : [0,1] \to \mathcal{E}$, which induces the isomorphism

$$\mathfrak{p}_{\gamma} : \mathcal{E}_p \to \mathcal{E}_{p'}, \quad \psi^{\mathfrak{v}}(0) = \mathfrak{v} \mapsto \mathfrak{p}_{\gamma}(\mathfrak{v}) := \psi^{\mathfrak{v}}(1),$$

the so-called *parallel transport map*.

Definition A.5 Let $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be an $\mathbb{F}^N$-vector bundle endowed with a connection $\nabla^{\mathcal{E}}$. The *curvature* $\mathrm{R}^{\mathcal{E}} \in C^{\infty}(\Lambda^2 \mathsf{T}M \otimes \mathrm{End}(\mathcal{E}))$ of the connection $\nabla^{\mathcal{E}}$ is given by

$$\mathrm{R}^{\mathcal{E}}(X, Y)\psi := \nabla^{\mathcal{E}}_X \nabla^{\mathcal{E}}_Y \psi - \nabla^{\mathcal{E}}_Y \nabla^{\mathcal{E}}_X \psi - \nabla^{\mathcal{E}}_{[X,Y]} \psi$$

for $X, Y \in C^{\infty}(\mathsf{T}M)$ and $\psi \in C^{\infty}(\mathcal{E})$. ◊

The curvature of a connection $\nabla^{\mathcal{E}}$ can be viewed as a measure of the lack of commutativity of two covariant derivatives $\nabla^{\mathcal{E}}_X$ and $\nabla^{\mathcal{E}}_Y$ provided that the vector fields X and Y commute. A connection is called *flat* if its associated curvature vanishes identically.

The Tangent Bundle

The tangent bundle $\pi_{\mathsf{T}M} : \mathsf{T}M \to M$ of a smooth, m-dimensional manifold is the collection of all tangent vectors ("velocity vectors of curves", see $\dot{\gamma}(t) \in \mathsf{T}_{\gamma(t)}M$ in (A.2)) in M, i.e., the total space $\mathsf{T}M$ is the disjoint union of the tangent spaces $\{\mathsf{T}_pM = \pi_{\mathsf{T}M}^{-1}(p)\}_{p\in M}$. One usually equips this $\mathbb{R}^m$-vector bundle with a bundle metric $g \in C^\infty(\Sigma^2\mathsf{T}M)$ and calls the pair (M, g) a *Riemannian manifold* (g is said to be a *Riemannian metric* on M). The fundamental theorem of Riemannian geometry [Lan99, Theorem VIII.4.1] asserts that the Riemannian metric g uniquely defines a connection (the so-called *Levi-Civita connection*) ∇^g on $\mathsf{T}M$ with the properties

$$\underbrace{X \cdot g(Y,Z) = g\big(\nabla^g_X Y, Z\big) + g\big(Y, \nabla^g_X Z\big)}_{\nabla^g \text{ is metric with respect to } g} \quad , \quad \underbrace{\nabla^g_X Y - \nabla^g_Y X = [X,Y]}_{\nabla^g \text{ is torsion-free}}$$

for all $X, Y, Z \in C^\infty(\mathsf{T}M)$. These features allow us to express the action of the Levi-Civita connection in terms of the metric and Lie brackets:

$$\begin{aligned} 2g\big(\nabla^g_X Y, Z\big) &= X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) \\ &\qquad - g\big(Y, [X,Z]\big) - g\big(Z, [Y,X]\big) + g\big(X, [Z,Y]\big). \end{aligned} \tag{A.3}$$

This is often referred to as *Koszul's formula*. Apart from that, bearing in mind that any curve $\gamma : [0,1] \to M$ itself induces a section $\dot{\gamma}$ of $\mathsf{T}M$ along γ, we will call such a curve *geodesic* if $\dot{\gamma}$ is parallel with respect to the Levi-Civita connection ∇^g, i.e., if γ is the solution of the differential equation $\nabla^g_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all $t \in [0,1]$.

The Riemannian metric g enables us to identify vectors in $\mathsf{T}M$ and covectors in $\mathsf{T}^*M := (\mathsf{T}M)^*$ by means of the (musical) bundle isomorphism

$$\flat : \mathsf{T}M \to \mathsf{T}^*M, \quad \mathsf{T}_pM \ni v \mapsto v^\flat := g_p(v, \cdot)$$

with inverse

$$\sharp : \mathsf{T}^*M \to \mathsf{T}M, \quad \mathsf{T}^*_pM \ni \omega \mapsto \omega^\sharp \text{ such that } g_p(\omega^\sharp, w) = \omega(w) \\ \text{for all } w \in \mathsf{T}_pM.$$

Example A.6 The *gradient* of a function $f \in C^\infty(M)$ is the uniquely defined vector field dual to $\mathrm{d}f$:

$$\operatorname{grad}_g f := (\mathrm{d}f)^\sharp.$$

This is equivalent to the condition

$$\mathrm{d}f(X) = g\big(\operatorname{grad}_g f, X\big)$$

for all $X \in C^\infty(\mathsf{T}M)$. ◊

The musical isomorphism furthermore allows us to endow the cotangent bundle T^*M with

- a symmetric bundle metric $\widetilde{g} \in C^\infty(\Sigma^2\mathsf{T}^*M)$ given by

  $$\widetilde{g}(\Xi, \Upsilon) := g(\Xi^\sharp, \Upsilon^\sharp),$$

- and a metric connection $\widetilde{\nabla}^g : C^\infty(\mathsf{T}^*M) \to C^\infty(\mathsf{T}^*M^{\otimes 2})$ defined by the relation

  $$\begin{aligned} &X \cdot \Xi(Y) = g\big(\nabla^g_X Y, \Xi^\sharp\big) + \widetilde{g}\big(Y^\flat, \widetilde{\nabla}^g_X \Xi\big) \\ \Leftrightarrow \quad &\big(\widetilde{\nabla}^g_X \Xi\big)(Y) := X \cdot \Xi(Y) - \Xi\big(\nabla^g_X Y\big) \end{aligned}$$

  for all $X, Y \in C^\infty(\mathsf{T}M)$ and $\Xi \in C^\infty(\mathsf{T}^*M)$.

If $\{v_\alpha\}_{\alpha=1}^m$ is a local frame of $\mathsf{T}M$ with dual local frame $\{\omega^\alpha\}_{\alpha=1}^m$ of T^*M, i.e., $\omega^\alpha(v_\beta) = \delta^\alpha_\beta$ for all $\alpha, \beta \in \{1, \ldots, m\}$, the corresponding Christoffel symbols

$$\nabla^g_{v_\alpha} v_\beta = \Gamma^\gamma_{\alpha\beta} v_\gamma \quad , \quad \widetilde{\nabla}^g_{v_\alpha} \omega^\beta = \widetilde{\Gamma}^\beta_{\alpha\gamma} \omega^\gamma$$

satisfy the symmetry relation

$$\widetilde{\Gamma}^\gamma_{\alpha\beta} = -\Gamma^\gamma_{\alpha\beta} \tag{A.4}$$

for all $\alpha, \beta, \gamma \in \{1, \ldots, m\}$. Moreover, let $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ be the matrix representations of g and $\widetilde{g}$, respectively:

$$g_{\alpha\beta} := g(v_\alpha, v_\beta) \quad , \quad g^{\alpha\beta} := \widetilde{g}(\omega^\alpha, \omega^\beta).$$

Then one easily verifies

$$(v_\alpha)^\flat = g_{\alpha\beta}\,\omega^\beta \quad , \quad (w^\alpha)^\sharp = g^{\alpha\beta} v_\beta$$

and the identity $\sharp \circ \flat = \mathbf{1}_{\mathsf{T}M}$ yields

$$w^\alpha v_\alpha = w = w^\alpha\big((v_\alpha)^\flat\big)^\sharp = w^\alpha\big(g^{\alpha\beta} g_{\beta\gamma} v_\gamma\big)$$

for all $w \in \mathsf{T}M$ (wherever the local frame $\{v_\alpha\}_{\alpha=1}^m$ is defined). This is equivalent to $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$ for all $\alpha, \gamma \in \{1, \dots, m\}$, and hence $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$.

C^∞-bounded Sections

Given any $\mathbb{F}^N$-vector bundle $(\mathcal{E}, h\nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$, we may define the tensor product connection $\nabla^{\mathsf{T}^*M\otimes\mathcal{E}}$ on the tensor product $\mathsf{T}^*M \otimes \mathcal{E}$ by means of the individual connections $\widetilde{\nabla}^g$ on T^*M and $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ by the relation

$$\nabla^{\mathsf{T}^*M\otimes\mathcal{E}} := \widetilde{\nabla}^g \otimes \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\mathsf{T}^*M} \otimes \nabla^{\mathcal{E}}. \tag{A.5}$$

We may then repeat this procedure and obtain connections $\nabla^{\mathsf{T}^*M^{\otimes j}\otimes\mathcal{E}}$ on $\mathsf{T}^*M^{\otimes j} \otimes \mathcal{E}$ for all $j \in \{1, \dots, k-1\}$, $k \in \mathbb{N}$. The composition of those connections finally leads to the smooth k-th-order differential operator

$$(\nabla^{\mathcal{E}})^k := \nabla^{\mathsf{T}^*M^{\otimes k-1}\otimes\mathcal{E}} \circ \dots \circ \nabla^{\mathsf{T}^*M\otimes\mathcal{E}} \circ \nabla^{\mathcal{E}}. \tag{A.6}$$

Likewise, we can merge the bundle metrics $\widetilde{g}$ of T^*M and h of $\mathcal{E}$ into a bundle metric on $\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E}$ via

$$\begin{aligned} &\big\langle(\Xi_1 \otimes \dots \otimes \Xi_k \otimes \psi), (\Upsilon_1 \otimes \dots \otimes \Upsilon_k \otimes \phi)\big\rangle_{\mathsf{T}^*M^{\otimes k}\otimes\mathcal{E}} \\ &\quad := \widetilde{g}(\Xi_1, \Upsilon_1) \dots \widetilde{g}(\Xi_k, \Upsilon_k)\, h(\psi, \phi). \end{aligned}$$

Definition A.7 We call $\psi \in C^\infty(\mathcal{E})$ C^∞*-bounded* if for all $k \in \mathbb{N}_0$ there is a constant $C(k) > 0$ such that

$$\sup_M \big\langle(\nabla^{\mathcal{E}})^k\psi, (\nabla^{\mathcal{E}})^k\psi\big\rangle_{\mathsf{T}^*M^{\otimes k}\otimes\mathcal{E}} \leqslant C(k)$$

with the convention $\langle\cdot,\cdot\rangle_{\mathsf{T}^*M^{\otimes 0}\otimes\mathcal{E}} = h$ and $(\nabla^{\mathcal{E}})^0 = \mathbf{1}_{\mathcal{E}}$. We denote the Fréchet space of C^∞-bounded sections by $C^\infty_{\mathrm{b}}(\mathcal{E})$. $\Diamond$

A.1.2 Excursus 1: The Sasaki Metric

We will give a brief explanation for the construction of a Riemannian metric $g_{\mathcal{E}} \in C^{\infty}(\Sigma^2 \mathsf{T}\mathcal{E})$ on the total space $\mathcal{E}$ of a Euclidean $\mathbb{R}^n$-vector bundle $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$, following [Sak96, Section II.4].

Due to the fact that each fibre $\mathcal{E}_p = \pi_{\mathcal{E}}^{-1}(p)$ of $\mathcal{E}$ is a vector space, it may be identified with its own tangent space $\mathsf{T}_{\mathfrak{v}}\mathcal{E}_p$ for $\mathfrak{v} \in \mathcal{E}_p$ by means of

$$\iota_{\mathfrak{v}} : \mathcal{E}_p \to \mathsf{T}_{\mathfrak{v}}\mathcal{E}_p, \quad \mathfrak{w} \mapsto \big[t \mapsto (\mathfrak{v} + t\mathfrak{w})\big]. \tag{A.7}$$

Moreover, each tangent space $\mathsf{T}_{\mathfrak{v}}\mathcal{E}_p$ is an N-dimensional subspace of $\mathsf{T}_{\mathfrak{v}}\mathcal{E}$ and coincides with $\ker(\mathsf{T}\pi_{\mathcal{E}}|_{\mathsf{T}_{\mathfrak{v}}\mathcal{E}})$. We will refer to it as the *vertical subspace* of $\mathsf{T}_{\mathfrak{v}}\mathcal{E}$ and denote it by $\mathsf{V}_{\mathfrak{v}}\mathcal{E}$. A complementary subspace $\mathsf{H}_{\mathfrak{v}}\mathcal{E} \subset \mathsf{T}_{\mathfrak{v}}\mathcal{E}$ may be assigned with the aid of the connection $\nabla^{\mathcal{E}}$ as follows: Take a curve $\gamma : I \to M$ through $\gamma(0) = p$ with $\dot{\gamma}(0) = v$ for some $v \in \mathsf{T}_p M$. In view of (A.2), the initial condition $\psi(0) = \mathfrak{v}$ uniquely defines a parallel section $\psi^{\mathfrak{v}} : I \to \mathcal{E}$ along γ with respect to $\nabla^{\mathcal{E}}$. Then $v^{\mathsf{H}}_{\mathfrak{v}} := \dot{\psi}^{\mathfrak{v}}(0) \in \mathsf{T}_{\mathfrak{v}}\mathcal{E}$ defines the horizontal lift of $v \in \mathsf{T}_p M$ at $\mathfrak{v} \in \mathcal{E}_p$, which is independent of the specific choice of the curve γ. The space

$$\mathsf{H}_{\mathfrak{v}}\mathcal{E} := \left\{ v^{\mathsf{H}}_{\mathfrak{v}} \in \mathsf{T}_{\mathfrak{v}}\mathcal{E} \text{ such that } v \in \mathsf{T}_{\pi_{\mathcal{E}}(\mathfrak{v})} M \right\}$$

forms an m-dimensional subspace of $\mathsf{T}_{\mathfrak{v}}\mathcal{E}$, $m = \dim(M)$, and is called the *horizontal subspace*. This gives a smooth, fibrewise decomposition

$$\mathsf{T}\mathcal{E} = \bigcup_{\mathfrak{v} \in \mathcal{E}} \mathsf{T}_{\mathfrak{v}}\mathcal{E} = \bigcup_{\mathfrak{v} \in \mathcal{E}} \mathsf{H}_{\mathfrak{v}}\mathcal{E} \oplus \mathsf{V}_{\mathfrak{v}}\mathcal{E} = \mathsf{H}\mathcal{E} \oplus \mathsf{V}\mathcal{E} \tag{A.8}$$

of $\mathcal{E}$'s tangent bundle into a horizontal and vertical subbundle, which leads to the definition of the *connection map* $\mathcal{K}_{\mathcal{E}} : \mathsf{T}\mathcal{E} \to \mathcal{E}$ by the fibrewise homomorphism

$$\mathcal{K}_{\mathcal{E}}|_{\mathsf{T}_{\mathfrak{v}}\mathcal{E}} : \mathsf{T}_{\mathfrak{v}}\mathcal{E} \to \mathcal{E}_p, \quad \mathfrak{w} \mapsto \iota_{\mathfrak{v}}^{-1} \circ \mathfrak{w}^{\mathsf{V}}$$

for all $\mathfrak{v} \in \mathcal{E}_p$, where $\mathfrak{w}^{\mathsf{V}}$ is the vertical part of $\mathfrak{w}$ according to the decomposition (A.8).

Let us summarise the previous considerations of this subsection: An $\mathbb{R}^N$-vector bundle $(\mathcal{E}, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} M$ admits the splitting

$$\mathsf{T}\mathcal{E} = \mathsf{H}\mathcal{E} \oplus \mathsf{V}\mathcal{E} = \ker(\mathcal{K}_{\mathcal{E}}) \oplus \ker(\mathsf{T}\pi_{\mathcal{E}}),$$

where each of the restricted maps

$$\mathsf{T}\pi_{\mathcal{E}}|_{\mathsf{H}_{\mathfrak{v}}\mathcal{E}} : \mathsf{H}_{\mathfrak{v}}\mathcal{E} \to \mathsf{T}_p M \quad , \quad \mathcal{K}_{\mathcal{E}}|_{\mathsf{V}_{\mathfrak{v}}\mathcal{E}} : \mathsf{V}_{\mathfrak{v}}\mathcal{E} \to \mathcal{E}_p$$

is a vector space isomorphism for all $\mathfrak{v} \in \mathcal{E}_p$ and $p \in M$. This gives rise to the introduction of horizontal and vertical lifts associated with $X \in C^\infty(\mathsf{T}M)$ and $\psi \in C^\infty(\mathcal{E})$, respectively, via

$$X^{\mathsf{H}}(\mathfrak{v}) := \left(\mathsf{T}\pi_{\mathcal{E}}|_{\mathsf{H}_{\mathfrak{v}}\mathcal{E}}\right)^{-1} \circ X(p) \quad , \quad \psi^{\mathsf{V}}(\mathfrak{v}) := \left(\mathcal{K}_{\mathcal{E}}|_{\mathsf{V}_{\mathfrak{v}}\mathcal{E}}\right)^{-1} \circ \psi(p).$$

Remark A.8 Let us express these lifts in terms of coordinate vector fields. Local coordinates $\{p^\alpha\}_{\alpha=1}^m$ on some subset $W \subset M$ and a frame $\{e_A\}_{A=1}^N$ of $\mathcal{E}|_W$ yield bundle coordinates

$$\begin{aligned} q^\alpha &:= \pi_{\mathcal{E}} \circ p^\alpha, \quad && \alpha = 1, \ldots, m \\ v^A, & && A = 1, \ldots, N \end{aligned}$$

on $\mathcal{E}|_W$ such that the vector $\mathfrak{v} = v^A e_A(p) \in \mathcal{E}_p$ corresponds to the point $(p, v) \in W \times \mathbb{R}^N$. Then the respective horizontal and vertical lift of the associated coordinate vector fields are given by [Bla10, Section 9.3]

$$\partial_{p^\alpha}^{\mathsf{H}} = \partial_{q^\alpha} - v^A \Gamma_{\alpha A}^B \partial_{v^B} \quad , \quad e_A^{\mathsf{V}} = \partial_{v^A}.$$

Here, the Christoffel symbols of $\nabla^{\mathcal{E}}$ that are associated with the local frame $\{e_A\}_{A=1}^N$ are defined by the relation $\nabla^{\mathcal{E}}_{\partial_{p^\alpha}} e_A = \Gamma_{\alpha A}^B e_B$. ◊

Now that we can split any vector field $\mathfrak{V}$ on $\mathcal{E}$ into

- a horizontal part $\mathfrak{V}^{\mathsf{H}} \overset{\mathsf{T}\pi_{\mathcal{E}}}{\rightsquigarrow} C^\infty(\mathsf{T}M)$,
- and a vertical part $\mathfrak{V}^{\mathsf{H}} \overset{\mathcal{K}_{\mathcal{E}}}{\rightsquigarrow} C^\infty(\mathcal{E})$,

the Riemannian metric g on M (i.e., the bundle metric on $\mathsf{T}M$) and the bundle metric h on $\mathcal{E}$ allow for the construction of a Riemannian metric on $\mathcal{E}$ as follows:

Definition A.9 Let $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ be an $\mathbb{R}^N$-vector bundle. The *Sasaki metric* is defined by

$$g_{\mathcal{E}} := g\big(\mathsf{T}\pi_{\mathcal{E}}\cdot, \mathsf{T}\pi_{\mathcal{E}}\cdot\big) \circ \pi_{\mathcal{E}} + h(\mathcal{K}_{\mathcal{E}}\cdot, \mathcal{K}_{\mathcal{E}}\cdot) = \pi_{\mathcal{E}}^* g + h(\mathcal{K}_{\mathcal{E}}\cdot, \mathcal{K}_{\mathcal{E}}\cdot). \qquad \lozenge$$

It was initially studied for the tangent bundle $\mathcal{E} = \mathsf{T}M$, where $\nabla^{\mathcal{E}} = \nabla^g$ is the Levi-Civita connection and the bundle metric h coincides with the Riemannian metric g [Sas58].

A.1.3 Excursus 2: Submanifolds

We survey the most important geometric objects that are involved in the treatment of submanifolds. We refer to [Lan99, Section XIV, § 1] for further details.

Let A be a smooth a-dimensional (ambient) manifold and $\iota : B \to A$ be a smooth embedding of a b-dimensional submanifold B into A. Since the differential $\mathsf{T}\iota : \mathsf{T}B \to \mathsf{T}A$ is a fibrewise monomorphism from $\mathsf{T}_x B \to \mathsf{T}_{\iota(x)} A$ for all $x \in B$, there exists a bundle monomorphism

$$\bar{\iota} : \mathsf{T}B \to \iota^* \mathsf{T}A = \big\{(x, v) \in B \times \mathsf{T}A \text{ such that } v \in \mathsf{T}_{\iota(x)} A\big\}$$

given by $\bar{\iota} = (\pi_{\mathsf{T}B}, \mathsf{T}\iota)$. In this context, the $\mathbb{R}^{a-b}$-vector bundle

$$\mathsf{N}B := \iota^* \mathsf{T}A / \operatorname{im}(\bar{\iota}) = \operatorname{coker}(\bar{\iota})$$

over B is called the *normal bundle* of B in A [Wal04, Definition 5.3].

If A is equipped with a Riemannian metric G, one may identify the fibres $\mathsf{N}_x B$ with the G-orthogonal complement of $\mathsf{T}\iota(\mathsf{T}_x B)$ in $\mathsf{T}_{\iota(x)} A$, i.e.,

$$\mathsf{T}_{\iota(x)} A = \mathsf{T}\iota(\mathsf{T}_x B) \oplus \big(\mathsf{T}\iota(\mathsf{T}_x B)\big)^{\perp, G} \cong \mathsf{T}_x B \oplus \mathsf{N}_x B,$$

and obtain the orthogonal decomposition $\iota^* \mathsf{T}A = \mathsf{T}B \oplus \mathsf{N}B$. Hence, every vector $t + n \in \mathsf{T}_x B \oplus \mathsf{N}_x B$ ($t \in \mathsf{T}_x B$ and $n \in \mathsf{N}_x B$) corresponds to a vector $\overline{t + n} \in \mathsf{T}_{\iota(x)} A$ with $G(\bar{t}, \bar{n}) = 0$ for all $x \in B$.

Definition A.10 We introduce

(i) a bundle metric $G^{\mathsf{T}B\oplus\mathsf{N}B}$ on $\mathsf{T}B\otimes\mathsf{N}B$ such that

$$\begin{aligned}G\big(\overline{t_1+n_1},\overline{t_2+n_2}\big) &= G^{\mathsf{T}B\oplus\mathsf{N}B}(t_1+n_1,t_2+n_2)\\ &= g_B(t_1,t_2)+G^{\mathsf{N}B}(n_1,n_2)\end{aligned}$$

holds for $t_1,t_2\in\mathsf{T}_xB$ and $n_1,n_2\in\mathsf{N}_xB$ with

- a Riemannian metric $g_B=\iota^*G$ on B,
- and a bundle metric $G^{\mathsf{N}B}=G|_{\mathsf{N}B}$ on $\mathsf{N}B$,

(ii) and a pullback connection

$$\nabla^{\mathsf{T}B\oplus\mathsf{N}B}:=\iota^*\nabla^G : C^\infty(\mathsf{T}B\oplus\mathsf{N}B)\to C^\infty\big(\mathsf{T}^*B\otimes(\mathsf{T}B\oplus\mathsf{N}B)\big)$$

which induces

- a connection $\nabla^{\mathsf{T}B}:=\mathrm{P}^{\mathsf{T}B}\,\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$ on $\mathsf{T}B$ coinciding with the Levi-Civita connection associated with g_B, i.e., $\nabla^{\mathsf{T}B}=\nabla^g$,
- and a metric connection $\nabla^{\mathsf{N}B}:=\mathrm{P}^{\mathsf{N}B}\,\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$ on $\mathsf{N}B$, the so-called *normal connection*. ◊

We similarly establish operators for the off-diagonal blocks of $\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$:

Definition A.11 Let $\tau,\sigma\in C^\infty(\mathsf{T}B)$ and $\nu\in C^\infty(\mathsf{N}B)$.

(i) The *Weingarten map* $\mathcal{W}\in C^\infty(\mathsf{N}^*B\otimes\mathrm{End}(\mathsf{T}B))$ is defined by

$$\nu\mapsto\mathcal{W}(\nu)\tau:=-\mathrm{P}^{\mathsf{T}B}\,\nabla_\tau^{\mathsf{T}B\oplus\mathsf{N}B}\,\nu.$$

(ii) The *second fundamental form* $\mathrm{II}\in C^\infty(\Sigma^2\mathsf{T}B\otimes\mathsf{N}B)$ is given by

$$(\tau,\sigma)\mapsto\mathrm{II}(\tau,\sigma):=\mathrm{P}^{\mathsf{N}B}\,\nabla_\tau^{\mathsf{T}B\oplus\mathsf{N}B}\sigma.$$

(iii) The *mean curvature* $\eta\in C^\infty(\mathsf{N}B)$ of the submanifold (B,g_B) is the unique normal field

$$\eta:=\mathrm{tr}_{g_B}\big(\mathrm{II}(\cdot,\cdot)\big).$$ ◊

The sign of the Weingarten map is chosen in such a way that

$$g_B\big(\mathcal{W}(\nu)\tau,\sigma\big) = G^{\mathsf{N}B}\big(\mathrm{II}(\tau,\sigma),\nu\big)$$

holds true. Apart from that, we may summarise the action of $\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$ by the Gauß formula

$$\nabla^{\mathsf{T}B\oplus\mathsf{N}B}_\tau\sigma = \nabla^{g_B}_\tau\sigma + \mathrm{II}(\tau,\sigma)$$

and the Weingarten equation

$$\nabla^{\mathsf{T}B\oplus\mathsf{N}B}_\tau\nu = -\mathcal{W}(\nu)\tau + \nabla^{\mathsf{N}B}_\tau\nu. \tag{A.9}$$

The latter relation paves the way to explicitly calculating the induced curvature $\mathrm{R}^{\mathsf{T}B\oplus\mathsf{N}B} = \iota^*\,\mathrm{R}^G$ of the pullback connection $\nabla^{\mathsf{T}B\oplus\mathsf{N}B}$. Starting with $\mathrm{R}^{\mathsf{T}B\oplus\mathsf{N}B}(\sigma_1,\sigma_2)\in C^\infty(\mathrm{End}(\mathsf{T}B\oplus\mathsf{N}B))$ for $\sigma_1,\sigma_2\in C^\infty(\mathsf{T}B)$, its application to another tangent vector field $\tau\in C^\infty(\mathsf{T}B)$ equals

$$\begin{aligned}
&\mathrm{R}^{\mathsf{T}B\oplus\mathsf{N}B}(\sigma_1,\sigma_2)\tau\\
&= \mathrm{R}^{g_B}(\sigma_1,\sigma_2)\tau - \mathcal{W}\big(\mathrm{II}(\sigma_2,\tau)\big)\sigma_1 + \mathcal{W}\big(\mathrm{II}(\sigma_1,\tau)\big)\sigma_2\\
&\quad + \big(\nabla^{\mathsf{T}^*B^{\otimes 2}\otimes\mathsf{N}B}_{\tau_1}\mathrm{II}\big)(\sigma_2,\tau) - \big(\nabla^{\mathsf{T}^*B^{\otimes 2}\otimes\mathsf{N}B}_{\tau_2}\mathrm{II}\big)(\sigma_1,\tau),
\end{aligned}$$

whereas its action on some normal field $\nu\in C^\infty(\mathsf{N}B)$ is given by

$$\begin{aligned}
&\mathrm{R}^{\mathsf{T}B\oplus\mathsf{N}B}(\sigma_1,\sigma_2)\nu\\
&= \mathrm{R}^{\mathsf{N}B}(\sigma_1,\sigma_2)\nu - \mathrm{II}\big(\tau_1,\mathcal{W}(\nu)\sigma_2\big) + \mathrm{II}\big(\mathcal{W}(\nu)\sigma_1,\sigma_2\big)\\
&\quad + \big(\nabla^{\mathsf{N}^*B\otimes\mathrm{End}(\mathsf{T}B)}_{\tau_2}\mathcal{W}\big)(\nu)\sigma_1 - \big(\nabla^{\mathsf{N}^*B\otimes\mathrm{End}(\mathsf{T}B)}_{\tau_1}\mathcal{W}\big)(\nu)\sigma_2.
\end{aligned}$$

Here, R^{g_B} and $\mathrm{R}^{\mathsf{N}B}$ are the curvatures of ∇^{g_B} and $\nabla^{\mathsf{N}B}$, respectively. If the ambient space (A,G) is flat, both tangent and normal components of the latter two equations vanish identically and one obtains rather simple expressions for the curvatures in terms of the extrinsic geometry:

$$\begin{aligned}
\mathrm{R}^{g_B}(\sigma_1,\sigma_2)\tau &= \mathcal{W}\big(\mathrm{II}(\sigma_2,\tau)\big)\sigma_1 - \mathcal{W}\big(\mathrm{II}(\sigma_1,\tau)\big)\sigma_2,\\
\mathrm{R}^{\mathsf{N}B}(\sigma_1,\sigma_2)\nu &= \mathrm{II}\big(\sigma_1,\mathcal{W}(\nu)\sigma_2\big) - \mathrm{II}\big(\mathcal{W}(\nu)\sigma_1,\sigma_2\big).
\end{aligned} \tag{A.10}$$

A.1.4 Functional Framework

We will assume that $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ is a $\mathbb{C}^N$-vector bundle with a metric connection over an m-dimensional Riemannian manifold. This subsection aims to introduce Sobolev spaces on such vector bundles, following the exposition of [Nic96, Subsection 10.2.4].

Let $\mathrm{vol}_g := \star 1 \in C^\infty(\Lambda^k M)$ be the volume form associated with the Riemannian metric g. We then set

$$\begin{aligned}\mathcal{L}^q(\mathcal{E}) := \big\{&\text{Borel measurable sections } \psi : M \to \mathcal{E} \\ &\text{such that } p \mapsto \big\|\psi(p)\big\|_{h_p}^q \text{ is } \mathrm{vol}_g\text{-integrable}\big\}\end{aligned}$$

for all $q \in [1, \infty]$ and identify sections ϕ, ψ thereof whenever they coincide pointwise vol_g-almost everywhere. The resulting quotient space

$$L^q(\mathcal{E}) := \mathcal{L}^q(\mathcal{E}) / \sim$$

together with the norm

$$\|\psi\|_{L^q(\mathcal{E})} := \begin{cases} \left(\int_M \big\|\psi(p)\big\|_{h_p}^q \ \mathrm{vol}_g(p)\right)^{1/q}, & 1 \leqslant q < \infty \\ \operatorname{ess\,sup}_{p \in M} \big\|\psi(p)\big\|_{h_p}, & q = \infty \end{cases} \tag{A.11}$$

turns out to be a Banach space [Nic96, Proposition 10.2.31]. These considerations may be extended in order to obtain Banach spaces $L^q(\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E})$ for $q \in [1, \infty]$ and $k \in \mathbb{N}$ with the norms

$$\|\psi\|_{L^q(\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E})} := \begin{cases} \left(\int_M \|\psi\|_{\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E}}^q \ \mathrm{vol}_g\right)^{1/q}, & 1 \leqslant q < \infty \\ \operatorname{ess\,sup}_M \|\psi\|_{\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E}}, & q = \infty \end{cases}.$$

In order to define Sobolev spaces $W^{k,q}(\mathcal{E}) \subset L^q(\mathcal{E})$ for $k \in \mathbb{N}_0$, we first need to introduce the weak derivative of a locally integrable section $\psi \in L^1_{\mathrm{loc}}(\mathcal{E})$. Therefore, we call $\phi \in L^1_{\mathrm{loc}}(\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E})$ the k-th weak derivative of ψ (i.e., $(\nabla^{\mathcal{E}})^k \psi = \phi$ weakly) for $k \in \mathbb{N}_0$ if

$$\int_M \langle \phi, \chi \rangle_{\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E}} \ \mathrm{vol}_g = \int_M h\big(\psi, (\nabla^{\mathcal{E}})^{k,*} \chi\big) \ \mathrm{vol}_g$$

holds for all $\chi \in C_0^\infty(\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E})$. Here, the operator $(\nabla^{\mathcal{E}})^{k,*}$ from $C^\infty(\mathsf{T}^*M^{\otimes k} \otimes \mathcal{E})$ to $C^\infty(\mathcal{E})$ denotes the adjoint of $(\nabla^{\mathcal{E}})^k$, cf. (A.15) below.

Definition A.12 Let $k \in \mathbb{N}_0$ and $q \in [1, \infty]$.

(i) We define the k-th L^q-Sobolev space $W^{k,q}(\mathcal{E}) \subset L^q(\mathcal{E})$ as the set

$$W^{k,q}(\mathcal{E}) = \big\{\psi \in L^q(\mathcal{E}) \text{ such that for all } 0 \leqslant j \leqslant k \text{ there exists} \\ \phi_j \in L^q(\mathsf{T}^*M^{\otimes j} \otimes \mathcal{E}) \text{ with } (\nabla^{\mathcal{E}})^j \psi = \phi_j \text{ weakly}\big\}$$

endowed with the norm

$$\|\psi\|_{W^{k,q}(\mathcal{E})} := \sum_{j=0}^{k} \left\|(\nabla^{\mathcal{E}})^j \psi\right\|_{L^q(\mathsf{T}^*M^{\otimes j} \otimes \mathcal{E})}. \tag{A.12}$$

(ii) The k-th L^q-Sobolev space with zero boundary conditions $W_0^{k,q}(\mathcal{E})$ is the subspace

$$W_0^{k,q}(\mathcal{E}) = \big\{\psi \in W^{k,q}(\mathcal{E}) \text{ such that } (\nabla^{\mathcal{E}})^j \psi(p) = 0_p \\ \text{for all } p \in \partial M \text{ and } 0 \leqslant j \leqslant k-1\big\}.$$ ◊

If one considers the special case of a line bundle $\mathcal{E} = \mathbb{R}^m \times \mathbb{C}$, which is equipped with the constant bundle metric $h = \langle \cdot, \cdot \rangle_{\mathbb{C}}$ and the flat connection $\nabla^{\mathbb{R}^m \times \mathbb{C}} = \mathrm{d}$, over the Euclidean space $(\mathbb{R}^m, \delta^m)$ as base manifold, the Sobolev norms (A.12) read

$$\|\psi\|_{W^{k,q}(\mathbb{R}^m \times \mathbb{C})} = \sum_{j=0}^{k} \left\| \left(\textstyle\sum_{|\alpha|=j} |\mathrm{D}^\alpha \psi|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^m)}, \quad \alpha \in \mathbb{N}_0^m.$$

These norms are equivalent to the usual Sobolev norms, and hence $W^{k,q}(\mathbb{R}^m \times \mathbb{C})$ coincides as a set with $W^{k,q}(\mathbb{R}^m)$.

Remark A.13 The Sobolev spaces $W^{k,q}(\mathcal{E})$ introduced in Definition A.12 depend on the choice of the bundle metric h, metric connection $\nabla^{\mathcal{E}}$ and the Riemannian metric g. A different choice of these quantities can alter the respective spaces significantly. It turns out, however, that $W^{k,q}(\mathcal{E})$ is independent of these choices if the base manifold M is compact [Nic96, Theorem 10.2.36]. ◊

Since we will solely work with the L^2-Sobolev spaces, we will drop the superscript q from now on and simply write $W^k(\mathcal{E}) := W^{k,2}(\mathcal{E})$ and $W_0^k(\mathcal{E}) := W_0^{k,2}(\mathcal{E})$. Moreover, when there is danger of confusion, we will sometimes write $W^k(\mathcal{E}, \mathrm{vol}_g)$ in order to emphasise the volume measure vol_g on M which was used in the construction of the Sobolev spaces.

A.2 Bounded Geometry

The utilisation of Sobolev spaces on vector bundles over non-compact manifolds as introduced in Definition A.12 is a very delicate issue if certain geometric features are unbounded. As a matter of fact, many classical results obtained for Sobolev spaces on Euclidean spaces (having "zero geometry") do not generally carry over to the setting of vector bundles over arbitrary Riemannian manifolds. In order to circumvent this inadequacy, we will restrict ourselves to vector bundles of so-called bounded geometry, where the relevant geometric quantities satisfy additional uniformity properties. These bundles are built over base manifolds of bounded geometry, where the curvature bounds are well-controlled, and are endowed with a set of uniformly bounded local trivialisations.

A.2.1 Manifolds of Bounded Geometry

The concept of bounded geometry for Riemannian manifolds (M, g) without boundary was introduced by [Shu92, Appendix A1]. In this context, we first introduce the injectivity radius $r_{\mathrm{inj}} : M \to [0, \infty]$ defined by

$$\begin{aligned} r_{\mathrm{inj}}(p) := \sup_{r>0} \Big\{ &\text{the exponential map } \exp_p^M \text{ restricted} \\ &\text{to } \mathbb{B}_r^m(0) \subset \mathsf{T}_p M \text{ is a diffeomorphism} \Big\}, \end{aligned}$$

where $m = \dim(M)$. This basically means that all points $p' \in M$ with distance $\mathrm{dist}_g(p, p') < r_{\mathrm{inj}}(p)$ may be joined with p by a unique geodesic.

Definition A.14 A connected Riemannian manifold (M, g) of dimension m without boundary is a *manifold of bounded geometry* if the following hold:

(i) *Injectivity radius:*
The injectivity radius $r_{\mathrm{inj}}(M, g) = \inf_{p \in M} r_{\mathrm{inj}}(p)$ of M is positive.

(ii) *Curvature bounds:*
The curvature of ∇^g satisfies $\mathrm{R}^g \in C_{\mathrm{b}}^{\infty}(\mathsf{T}^*M^{\otimes 3} \otimes \mathsf{T}M)$. ◊

If the manifold has a boundary ∂M, one has to adapt the former definition of bounded geometry, since $r_{\mathrm{inj}}(p) = 0$ for all $p \in \partial M$. In this case, one instead needs [Sch01, Definition 2.2]

- bounded geometry of the interior $M^{\circ} = M \backslash \partial M$,
- bounded geometry of the boundary ∂M,
- and the boundedness of the inclusion $\partial M \hookrightarrow M$.

In view of the orthogonal decomposition

$$\mathsf{T}M\big|_{\partial M} \cong \mathsf{T}\partial M \otimes \mathsf{N}\partial M$$

with respect to g, the last requirement is implemented by taking into account the second fundamental form $\mathrm{II}^g_{\partial M} \in C^{\infty}(\Sigma^2 \mathsf{T}\partial M \otimes \mathsf{N}\partial M)$ of the boundary $\partial M \hookrightarrow (M, g)$.

Definition A.15 A connected Riemannian manifold (M, g) with (possibly empty) boundary ∂M is said to be a *∂-manifold of bounded geometry* if the following requirements are satisfied:

(i) *Normal collar:*
Denote by $\nu \in C^{\infty}(\mathsf{N}\partial M)$ the inward-pointing unit normal field of the boundary ∂M. There exists $r_{\mathrm{C}} > 0$ such that the (collar) map

$$\mathcal{K} : \partial M \times [0, r_{\mathrm{C}}) \to M, \quad (p, s) \mapsto \exp_p(s\,\nu)$$

is a diffeomorphism onto its image. Therefore, we denote by

$$\begin{aligned}\mathsf{NC}(r) &:= \left\{\exp_p(s\nu) \in M \text{ such that } (p,s) \in \partial M \times [0,r)\right\}\\ &= \mathcal{K}\big(\partial M \times [0,r)\big)\end{aligned}$$

the normal collar of width $r < r_{\mathrm{C}}$.

(ii) *Injectivity radius of the boundary:*
The injectivity radius $r_{\mathrm{inj}}(\partial M,\, g|_{\partial M})$ of the boundary ∂M is positive.

(iii) *Injectivity radius of the interior:*
There is $r_{\mathrm{I}} > 0$ such that $r_{\mathrm{inj}}(p) \geqslant r_{\mathrm{I}}$ for all $p \in M \backslash \mathsf{NC}(\frac{r_{\mathrm{C}}}{3})$.

(iv) *Curvature bounds:*
The curvature of ∇^g and the second fundamental form of the boundary $\partial M \hookrightarrow (M,g)$ satisfy

$$\mathrm{R}^g\, C^\infty_{\mathrm{b}}\left(\Lambda^2\mathsf{T}M \otimes \mathrm{End}(\mathsf{T}M)\right)$$

and

$$\mathrm{II}^g_{\partial M} \in C^\infty_{\mathrm{b}}(\Sigma^2\mathsf{T}\partial M \otimes \mathsf{N}\partial M),$$

respectively. ◊

Typical examples for manifolds of bounded geometry are compact Riemannian manifolds and Lie groups with (left-)invariant metrics [Sch96, Example 3.13].

A ∂-manifold of bounded geometry provides a suitable set of coordinates via charts for the boundary collar and for the interior [Sch96, Definition 3.2], see Figure A.1:

- *Boundary collar charts:*
Let $r_\partial < r_{\mathrm{inj}}(\partial M,\, g|_{\partial M})$ and choose an orthonormal basis to identify $\mathsf{T}_p\partial M$ with $\mathbb{R}^{m-1}$ for some $p \in \partial M$. Then boundary collar coordinates are defined by

$$\lambda^{\mathrm{bc}}_p : \underbrace{\mathbb{B}^{m-1}_{r_\partial}(0) \times [0, r_{\mathrm{C}})}_{\subset \mathbb{R}^m_{\geqslant 0}} \to M, \quad (u,s) \mapsto \exp^M_{\exp^{\partial M}_p(u)}(s\nu).$$

- *Interior charts:*
  Let $r_{\text{in}} < r_{\mathrm{I}}$ and choose an orthonormal basis to identify $\mathsf{T}_{p'}M$ with $\mathbb{R}^m$ for some $p' \in M \backslash \mathsf{NC}(\frac{r_{\mathrm{C}}}{3})$. Then interior coordinates are given by

$$\lambda_{p'}^{\text{in}} : \underbrace{\mathbb{B}_{r_{\text{in}}}^m(0)}_{\subset \mathbb{R}^m} \to M, \quad w \mapsto \exp_{p'}^M(w).$$

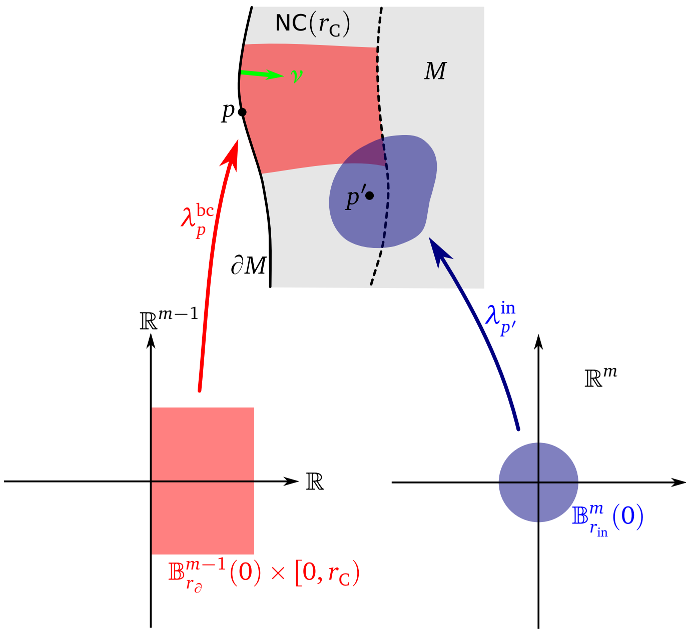


Figure A.1: Sketch of one boundary collar chart λ_p^{bc} ($p \in \partial M$) and one interior chart λ_p^{in} ($p' \in M \backslash \mathsf{NC}(\frac{r_{\mathrm{C}}}{3})$).

The coordinates associated with such charts are called *normal coordinates*. These charts yield a suitable atlas of M and a nice subordinate partition of unity:

Lemma A.16 *For all* $0 < r < \frac{1}{3}\min\{r_{\mathrm{inj}}(\partial M, g|_{\partial M}), r_{\mathrm{I}}\}$ *there exists an atlas* $\{(W_\mu, \tau_\mu)\}_{\mu\in\mathbb{Z}}$ *with either* $W_\mu = \emptyset$ *or*

(i) for $\mu < 0$ *there is* $p_\mu \in \partial M$ *such that*

$$W_\mu = \lambda^{\mathrm{bc}}_{p_\mu}\big(\mathbb{B}^{m-1}_r(0) \times [0, r_{\mathrm{C}})\big) \quad , \quad \tau_\mu = (\lambda^{\mathrm{bc}}_{p_\mu})^{-1}\big|_{W_\mu},$$

(ii) and for $\mu \in \mathbb{N}_0$ *there is* $p_\mu \in M\backslash \mathsf{NC}(\frac{2r_{\mathrm{C}}}{3})$ *such that*

$$W_\mu = \lambda^{\mathrm{in}}_{p_\mu}\big(\mathbb{B}^m_r(0)\big) \quad , \quad \tau_\mu = (\lambda^{\mathrm{in}}_{p_\mu})^{-1}\big|_{W_\mu}.$$

This covering is uniformly locally finite: There is a number $N_M \in \mathbb{N}$ *such that for all* $\mu \in \mathbb{Z}$ *the set* $\{\mu' \in \mathbb{Z}$ *such that* $W_\mu \cap W_{\mu'} \neq \emptyset\}$ *has at most* N_M *elements. Moreover, there is a smooth partition of unity* $\{\chi_\mu\}_{\mu\in\mathbb{Z}}$ *of* M *subordinate to the covering* $\{W_\mu\}_{\mu\in\mathbb{Z}}$ *such that* $\tau_{\mu*}\chi_\mu \subset C^\infty_{\mathrm{b}}(\mathbb{R}^m)$ *with bounds uniform in* μ*, i.e., for all* $k \in \mathbb{N}_0$ *there is a constant* $C(k) > 0$ *such that* $\big\|\mathrm{D}^{\mathfrak{a}}(\tau_{\mu*}\chi_\mu)\big\|_\infty \leqslant C(k)$ *for all* $\mu \in \mathbb{Z}$ *and all multi-indices* $\mathfrak{a} \in \mathbb{N}^m_0$ *with* $|\mathfrak{a}| \leqslant k$.

PROOF. See [Sch96, Lemma 3.22]. $\square$

A.2.2 Vector Bundles of Bounded Geometry

The basic purpose of vector bundles of bounded geometry is to avoid an unbounded variation of the respective fibres along the base manifold. This is encoded in terms of suitable local trivialisations [Shu92, Appendix A1]:

Definition A.17 Let $\pi_{\mathcal{E}} : \mathcal{E} \to M$ be an $\mathbb{F}^N$-vector bundle over an m-dimensional ∂-manifold M of bounded geometry. We call $\mathcal{E}$ a *vector bundle of bounded geometry* if for any pair of normal coordinate charts (W, τ) and (W', τ') of M with $W \cap W' \neq \emptyset$ there exist local trivialisations $\Lambda : \pi^{-1}_{\mathcal{E}}(W) \to W \times \mathbb{F}^N$ and $\Lambda' : \pi^{-1}_{\mathcal{E}}(W') \to W' \times \mathbb{F}^N$ for which the associated transition matrix $\mathsf{g} : W \cap W' \to \mathrm{GL}(N, \mathbb{F})$ is uniformly bounded, i.e., for all $k \in \mathbb{N}_0$ there is a constant $C(k) > 0$ (independent of τ and g) such that

$$\left\|\mathrm{D}^{\mathfrak{a}}(\tau_* \mathsf{g})\right\|_{\infty,\mathrm{Mat}} \leqslant C(k)$$

for all multi-indices $\mathfrak{a} \in \mathbb{N}^m$ with $|\mathfrak{a}| \leqslant k$. $\Diamond$

Typical examples for vector bundles of bounded geometry are the trivial bundle $\mathcal{E} = M \times \mathbb{F}^N$, the tangent bundle $\mathcal{E} = \mathsf{T}M$ and the cotangent bundle $\mathcal{E} = \mathsf{T}^*M$. Moreover, every vector bundle over a compact base manifold M is a vector bundle of bounded geometry [Sch96, Example 3.13].

The provision of an additional bundle metric h and metric connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ allows for the introduction of a particular set of local trivialisations [GS13, Definition 5.2]:

Definition A.18 Let $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ be an $\mathbb{F}^N$-vector bundle over an m-dimensional ∂-manifold M of bounded geometry, where the latter is equipped with an atlas $\{W_\mu, \tau_\mu)\}_{\mu \in \mathbb{Z}}$ of normal coordinate charts with centres $p_\mu = \tau_\mu^{-1}(0)$ as in Lemma A.16. Choose an orthonormal basis $\{e_A^\mu\}_{A=1}^N$ of $\mathcal{E}_{p_\mu}$ for all $\mu \in \mathbb{Z}$ and define a local frame $\{e_A^\mu(p)\}_{A=1}^N$ on W_μ by the parallel transport with respect of $\nabla^{\mathcal{E}}$

- for $\mu < 0$ (boundary collar charts) along the curve

$$\gamma_\mu^{(u,s)} : [0,2] \to W_\mu, \quad t \mapsto \begin{cases} \tau_\mu^{-1}(tu, 0), & t \leqslant 1 \\ \tau_\mu^{-1}(u, ts), & t > 1 \end{cases}$$

  with $(u,s) \in \mathbb{B}_r^{m-1}(0) \times [0, r_{\mathrm{C}})$,

- and for $\mu \in \mathbb{N}_0$ (interior charts) along the curve

$$\gamma^w : [0,1] \to W_\mu, \quad t \mapsto \tau_\mu^{-1}(tw)$$

  with $w \in \mathbb{B}_r^m(0)$.

We then denote by

$$\Lambda_\mu : \pi_{\mathcal{E}}^{-1}(W_\mu) \to W_\mu \times \mathbb{F}^N, \quad \mathcal{E}_p \ni w = w^A e_A^\mu(p) \mapsto \big(p, (w^1, \ldots, w^N)\big)$$

the corresponding *synchronous trivialisations*. ◊

A coordinate-free definition for vector bundles of bounded geometry with a metric connection can be established by means of curvature bounds [Eic07, Section 1.A.1]:

Definition A.19 Let $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ be an $\mathbb{F}^N$-vector bundle endowed with a metric connection. Then $\mathcal{E}$ is said to be of bounded geometry if

(i) (M, g) is a ∂-manifold of bounded geometry,

(ii) and the curvature of $\nabla^{\mathcal{E}}$ satisfies $\mathrm{R}^{\mathcal{E}} \in C_{\mathrm{b}}^{\infty}(\Lambda^2 \mathsf{T}M \otimes \mathrm{End}(\mathcal{E}))$. ◊

These two concepts for vector bundles of bounded geometry are consistent with each other. To be precise, [GS13, Theorem 5.4] – with straightforward modifications for the case of a ∂-base manifold – states that a vector bundle $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$ with a metric connection is of bounded geometry in the sense of Definition A.17 together with synchronous trivialisations if and only if it is of bounded geometry in the sense of Definition A.19. In this context, we will always assume that a vector bundle is of bounded geometry with respect to a set of synchronous trivialisations.

We finally mention that one may relate the global definition of the Sobolev norms (cf. Definition A.12) to a local definition in terms of synchronous trivialisations:

Proposition A.20 *Let (M, g) be a ∂-manifold of bounded geometry with normal charts $\{(W_\mu, \tau_\mu)\}_{\mu \in \mathbb{Z}}$ and subordinate partition of unity $\{\chi_\mu\}_{\mu \in \mathbb{Z}}$ as in Lemma A.16. Moreover, let $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ be an $\mathbb{F}^N$-vector bundle of bounded geometry over M with synchronous trivialisations $\{\Lambda_\mu : \pi_{\mathcal{E}}^{-1}(W_\mu) \to W_\mu \times \mathbb{F}^N\}_{\mu \in \mathbb{Z}}$ as in Definition A.18. Then the norm*

$$\|\psi\|_k := \left(\sum_{\mu \in \mathbb{Z}} \big\| \tau_{\mu *}\big(\Lambda_\mu \circ (\chi_\mu \psi)\big) \big\|^2_{W^k(\tau_\mu(W_\mu), \mathbb{F}^N)} \right)^{1/2},$$

is equivalent to (A.12) *and one has the equalities*

$$W^k(\mathcal{E}) = \overline{C_0^\infty(\mathcal{E})}^{\|\cdot\|_k} \quad , \quad W_0^k(\mathcal{E}) = \overline{C_0^\infty(\mathcal{E}^\circ)}^{\|\cdot\|_k}. \tag{A.13}$$

PROOF. See [GS13, Theorem 5.7], which verbatim extends to ∂-base manifolds. □

The spaces (A.13) are well-defined since another choice of normal coordinates satisfying Lemma A.16 and corresponding synchronous trivialisations yields equivalent norms and, therefore, the same Sobolev spaces as sets [Sch96, Lemma 3.24]. Moreover, these spaces possess the well-known properties [Sch96, Proposition 3.25]

- $W^k(\mathcal{E})$ is a Hilbert space for all $k \in \mathbb{N}_0$,
- the embedding $W^k(\mathcal{E}) \hookrightarrow C_\mathrm{b}^l(\mathcal{E})$ is bounded whenever $k > \frac{m}{2} + l$ for $m = \dim(M)$,
- the embedding $W^{k+1}(\mathcal{E}) \hookrightarrow W^k(\mathcal{E})$ is bounded with dense image for all $k \in \mathbb{N}_0$, and is compact if and only if M is compact,
- any differential operator $T \in \mathrm{Diff}_l(\mathcal{E}, \mathcal{F})$ with C^∞-bounded coefficients (with respect to adequate synchronous trivialisations) extends to a bounded map from $W^{k+l}(\mathcal{E})$ to $W^k(\mathcal{E})$ for all $k, l \in \mathbb{N}_0$,
- and the restriction map (trace operator) $\mathrm{res} : C^\infty(\mathcal{E}) \to C^\infty(\partial\mathcal{E})$ extends to a bounded map from $W^k(\mathcal{E})$ to $W^{k-1/2}(\mathcal{E})$ for all $k \in \mathbb{N}$.

A.3 The Connection Laplacian

The main object of interest throughout this thesis is the (negative of the) *connection Laplacian* with Dirichlet boundary conditions. Starting with a smooth $\mathbb{C}^N$-vector bundle $(\mathcal{E}, \nabla^\mathcal{E})$ over a smooth, m-dimensional Riemannian manifold (M, g) with (possibly empty) boundary, it is initially defined on $C_0^\infty(\mathcal{E}^\circ)$ as the second-order differential operator [BGV92, Definition 2.4]

$$\psi \mapsto -\operatorname{tr}_g(\nabla^{\mathsf{T}^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}}\psi) \overset{(A.6)}{=} -\operatorname{tr}_g\big((\nabla^{\mathcal{E}})^2\psi\big).$$

We want to evaluate this expression by means of local dual frames $\{v_\alpha\}_{\alpha=1}^m$ of $\mathsf{T}M$ and $\{\omega^\alpha\}_{\alpha=1}^m$ of T^*M. Therefore, the composition of $\nabla^{\mathcal{E}}$ and $\nabla^{\mathsf{T}^*M\otimes\mathcal{E}}$ (the *Hessian*) applied to $\psi \in C_0^\infty(\mathcal{E}^\circ)$ yields

$$\begin{aligned}
(\nabla^{\mathcal{E}})^2\psi &= \nabla^{\mathsf{T}^*M\otimes\mathcal{E}}\underbrace{\big(\omega^\beta\otimes\nabla^{\mathcal{E}}_{v_\beta}\psi\big)}_{=\nabla^{\mathcal{E}}\psi}\\
&= \widetilde{\nabla}^g\omega^\beta\otimes\nabla^{\mathcal{E}}_{v_\beta}\psi + \omega^\beta\otimes\nabla^{\mathcal{E}}(\nabla^{\mathcal{E}}_{v_\beta}\psi)\\
&= \omega^\alpha\otimes\big(\widetilde{\nabla}^g_{v_\alpha}\omega^\beta\otimes\nabla^{\mathcal{E}}_{v_\beta}\psi + \omega^\beta\otimes\nabla^{\mathcal{E}}_{v_\alpha}(\nabla^{\mathcal{E}}_{v_\beta}\psi)\big)\\
&= \omega^\alpha\otimes\big(\widetilde{\Gamma}^\beta_{\alpha\gamma}\omega^\gamma\otimes\nabla^{\mathcal{E}}_{v_\beta}\psi + \omega^\beta\otimes\nabla^{\mathcal{E}}_{v_\alpha}(\nabla^{\mathcal{E}}_{v_\beta}\psi)\big)\\
&= \omega^\alpha\otimes\omega^\beta\otimes\Big(\nabla^{\mathcal{E}}_{-\Gamma^\gamma_{\alpha\beta}v_\gamma}\psi + \nabla^{\mathcal{E}}_{v_\alpha}(\nabla^{\mathcal{E}}_{v_\beta}\psi)\Big)\\
&= \omega^\alpha\otimes\omega^\beta\otimes\Big(-\nabla^{\mathcal{E}}_{\nabla^g_{v_\alpha}v_\beta}\psi + \nabla^{\mathcal{E}}_{v_\alpha}(\nabla^{\mathcal{E}}_{v_\beta}\psi)\Big).
\end{aligned}$$

We then take the g-trace and arrive at

$$-\operatorname{tr}_g\big((\nabla^{\mathcal{E}})^2\psi\big) = -g^{\alpha\beta}\Big(\nabla^{\mathcal{E}}_{v_\alpha}\nabla^{\mathcal{E}}_{v_\beta}\psi - \nabla^{\mathcal{E}}_{\nabla^g_{v_\alpha}v_\beta}\psi\Big). \tag{A.14}$$

Now that we have understood the action of the Laplacian as a differential operator on smooth sections, we want to embed the Dirichlet Laplacian into an appropriate functional analytic framework, i.e., we introduce some bundle metric h on $\mathcal{E}$ and view the Laplacian as an operator acting on the Hilbert space $\mathcal{H} := L^2(\mathcal{E})$ with scalar product

$$\langle\phi,\psi\rangle_{\mathcal{H}} = \int_M h(\phi,\psi)\ \mathrm{vol}_g\,.$$

As outlined in the former section, the use of Sobolev spaces necessitates the imposition of further boundedness properties on the underlying vector bundle $\mathcal{E}$ (over a non-compact manifold M). Therefore, we will assume that $(\mathcal{E}, h, \nabla^{\mathcal{E}}) \xrightarrow{\pi_{\mathcal{E}}} (M, g)$ is a $\mathbb{C}^N$-vector bundle of bounded geometry (cf. Definition A.17).

The connection $\nabla^{\mathcal{E}}$ induces an adjoint operator $\nabla^{\mathcal{E},*}$ from $C^\infty(\mathsf{T}^*M \otimes \mathcal{E})$ to $C^\infty(\mathcal{E})$ defined by the relation

$$\int_M \left\langle \nabla^{\mathcal{E}} \phi, \chi \right\rangle_{\mathsf{T}^*M\otimes\mathcal{E}} \; \mathrm{vol}_g = \int_M h\big(\phi, \nabla^{\mathcal{E},*} \chi\big) \; \mathrm{vol}_g \tag{A.15}$$

for all $\phi \in C_0^\infty(\mathcal{E}^\circ)$ and $\chi \in C^\infty(\mathsf{T}^*M \otimes \mathcal{E})$. An easy calculation using local frames proves the formula

$$\nabla^{\mathcal{E},*}(\Xi \otimes \psi) = \big(\nabla^{\mathcal{E}}_{\Xi^\sharp}\big)^\dagger \psi \tag{A.16}$$

for $\Xi \in C^\infty(\mathsf{T}^*M)$ and $\psi \in C^\infty(\mathcal{E})$, where the latter differential operator is the formal adjoint of $\nabla^{\mathcal{E}}_{\Xi^\sharp}$, i.e.,

$$\int_M h\big(\nabla^{\mathcal{E}}_{\Xi^\sharp} \phi, \psi\big) \; \mathrm{vol}_g = \int_M h\Big(\phi, \big(\nabla^{\mathcal{E}}_{\Xi^\sharp}\big)^\dagger \psi\Big) \; \mathrm{vol}_g$$

for all $\phi \in C_0^\infty(\mathcal{E}^\circ)$ and $\psi \in C^\infty(\mathcal{E})$.

Lemma A.21 *If $\nabla^{\mathcal{E}}$ is a metric connection on $\mathcal{E}$ with respect to h, one has*

$$\big(\nabla^{\mathcal{E}}_X\big)^\dagger = -\nabla^{\mathcal{E}}_X - \mathrm{div}_g(X) \mathbf{1}_{\mathcal{E}}$$

for all $X \in C^\infty(\mathsf{T}M)$, where the divergence $\mathrm{div}_g \in C^\infty(\mathsf{T}^*M)$ *is uniquely defined by*

$$\mathcal{L}_X \, \mathrm{vol}_g = \mathrm{div}_g(X) \, \mathrm{vol}_g \,.$$

If we apply this lemma to the trivial line bundle $\mathcal{E} = M \times \mathbb{C}$ with flat connection $\nabla^{M\times\mathbb{C}} = \mathrm{d}$, we obtain

$$\begin{aligned} \mathrm{d}^*(X^\flat) = \mathrm{d}^*(X^\flat \otimes 1) \overset{(\mathrm{A}.16)}{=} \mathrm{d}_X^\dagger 1 &= -\underbrace{\mathrm{d}1(X)}_{=0} - \mathrm{div}_g(X) 1 \\ &= -\mathrm{div}_g(X) \end{aligned} \tag{A.17}$$

for all $X \in C^\infty(\mathsf{T}M)$. The adjoint of d on the trivial line bundle is hence given by the negative of the divergence.

PROOF (of Lemma A.21). First of all note that for any $f \in C_0^\infty(M \backslash \partial M)$ we have the equality

$$\begin{aligned} 0 &= \int_{\partial M} i_X(f \ \mathrm{vol}_g) = \int_M \mathcal{L}_X(f \ \mathrm{vol}_g) \\ &= \int_M X \cdot f \ \ \mathrm{vol}_g + \int_M f \, \mathrm{div}_g(X) \ \mathrm{vol}_g \, . \end{aligned}$$

We then integrate (A.1), which holds true since $\nabla^{\mathcal{E}}$ is metric, over M and apply the above observation for $f = h(\phi, \psi)$:

$$\begin{aligned} &\int_M h\big(\nabla^{\mathcal{E}}_X \phi, \psi\big) \ \mathrm{vol}_g \\ &= -\int_M h\big(\phi, \nabla^{\mathcal{E}}_X \psi\big) \ \mathrm{vol}_g + \int_M X \cdot h(\phi, \psi) \ \mathrm{vol}_g \\ &= -\int_M h\big(\phi, \nabla^{\mathcal{E}}_X \psi\big) \ \mathrm{vol}_g - \int_M h(\phi, \psi) \, \mathrm{div}_g(X) \ \mathrm{vol}_g \\ &= \int_M h\Big(\phi, \big(-\nabla^{\mathcal{E}}_X - \mathrm{div}_g(X) \mathbf{1}_{\mathcal{E}}\big)\psi\Big) \ \mathrm{vol}_g, \end{aligned}$$

which proves the statement. □

We will need an alternative expression for the divergence applied to some vector field $X \in C^\infty(\mathsf{T}M)$. Therefore, let $\nabla^{\mathsf{T}^*M^{\otimes k}}$, $k \in \{1, \dots, m\}$, be the tensor product connection

$$\begin{aligned} \big(\nabla^{\mathsf{T}^*M^{\otimes k}}_X \omega\big)(Y_1, \dots, Y_k) := X \cdot \big(\omega(Y_1, \dots, Y_k)\big) \\ - \sum_{j=1}^k \omega(Y_1, \dots, \nabla^g_X Y_j, \dots, Y_k) \end{aligned}$$

on $\mathsf{T}^*M^{\otimes k}$. Then [Lan99, Theorem XV.2.2] yields $\nabla^{\mathsf{T}^*M^{\otimes m}} \mathrm{vol}_g = 0$, and so we obtain for any local frame $\{v_\alpha\}_{\alpha=1}^m$ of $\mathsf{T}M$:

$$\begin{aligned} 0 &= X \cdot \big(\mathrm{vol}_g(v_1, \dots, v_m)\big) - \sum_{\alpha=1}^m \mathrm{vol}_g(v_1, \dots, \nabla^g_X v_\alpha, \dots, v_m) \\ &= (\mathcal{L}_X \mathrm{vol}_g)(v_1, \dots, v_m) + \sum_{\alpha=1}^m \mathrm{vol}_g\big(v_1, \dots, [X, v_\alpha] - \nabla^g_X v_\alpha, \dots, v_m\big) \end{aligned}$$

$$= \operatorname{div}_g(X)\operatorname{vol}_g(v_1,\ldots,v_m) - \sum_{\alpha=1}^{m}\operatorname{vol}_g\big(v_1,\ldots,\nabla^g_{v_\alpha}X,\ldots,v_m\big),$$

where we took advantage of the torsion-freeness of the Levi-Civita connection ∇^g for the last transformation. Thus, the latter equation is equivalent to

$$\operatorname{div}_g(X) = g^{\alpha\beta} g\big(\nabla^g_{v_\alpha}X, v_\beta\big) = \operatorname{tr}_g\big(\nabla^g X\big). \tag{A.18}$$

Lemma A.22 *If $\nabla^{\mathcal{E}}$ is a metric connection on $\mathcal{E}$ with respect to h, it holds that*

$$-\operatorname{tr}_g\big((\nabla^{\mathcal{E}})^2\cdot\big) = \nabla^{\mathcal{E},*}\nabla^{\mathcal{E}}$$

on $C_0^\infty(\mathcal{E}^\circ)$.

PROOF. This is a straightforward computation once again using dual local frames $\{v_\alpha\}_{\alpha=1}^m$ of $\mathsf{T}M$ and $\{\omega^\alpha\}_{\alpha=1}^m$ of T^*M. More precisely, we obtain for arbitrary $\psi \in C_0^\infty(\mathcal{E}^\circ)$ in virtue of Lemma A.21:

$$\begin{aligned}
\nabla^{\mathcal{E},*}\nabla^{\mathcal{E}}\psi &= \nabla^{\mathcal{E},*}\big(\omega^\beta \otimes \nabla_{v_\beta}\psi\big)\\
&= -\nabla^{\mathcal{E}}_{(\omega^\beta)^\sharp}\nabla^{\mathcal{E}}_{v_\beta}\psi - \operatorname{div}_g\big((\omega^\beta)^\sharp\big)\nabla^{\mathcal{E}}_{v_\beta}\psi\\
&= -\nabla^{\mathcal{E}}_{g^{\beta\alpha}v_\alpha}\nabla^{\mathcal{E}}_{v_\beta}\psi + \nabla^{\mathcal{E}}_{-\operatorname{div}_g(g^{\beta\alpha}v_\alpha)v_\beta}\psi\\
&= -g^{\alpha\beta}\Big(\nabla^{\mathcal{E}}_{v_\alpha}\nabla^{\mathcal{E}}_{v_\beta}\psi - \nabla^{\mathcal{E}}_{\nabla^g_{v_\alpha}v_\beta}\psi\Big)\\
&\overset{(A.14)}{=} -\operatorname{tr}_g\big((\nabla^{\mathcal{E}})^2\psi\big).
\end{aligned}$$

Here, we utilised

$$\begin{aligned}
-\operatorname{div}_g(g^{\beta\alpha}v_\alpha)v_\beta &\overset{(A.18)}{=} -\operatorname{tr}_g\big(\nabla^g(g^{\beta\alpha}v_\alpha)\big)v_\beta\\
&= -g^{\kappa\lambda}g\big(\nabla^g_{v_\kappa}(g^{\beta\alpha}v_\alpha), v_\lambda\big)v_\beta\\
&\overset{(A.1)}{=} -g^{\kappa\lambda}\big[\underbrace{v_\kappa\cdot\underbrace{g(g^{\beta\alpha}v_\alpha, v_\lambda)}_{=g^{\beta\alpha}g_{\alpha\lambda}=\delta^\beta_\lambda}}_{=0} - g(g^{\beta\alpha}v_\alpha, \nabla^g_{v_\kappa}v_\lambda)\big]v_\beta
\end{aligned}$$

$$= \quad g^{\kappa\lambda}\underbrace{\left(g^{\beta\alpha}g(\nu_\alpha, \nabla^g_{\nu_\kappa}\nu_\lambda)\nu_\beta\right)}_{=\nabla^g_{\nu_\kappa}\nu_\lambda}$$

for the next-to-last line. □

This lemma allows us to easily prove the necessary facts so as to extend the densely defined connection Laplacian to a self-adjoint operator:

Proposition A.23 *Let* $\pi_{\mathcal{E}} : (\mathcal{E}, h, \nabla^{\mathcal{E}}) \to (M, g)$ *be a* $\mathbb{C}^N$*-vector bundle with a metric connection. Then the associated connection Laplacian* (A.14) *defines a symmetric and positive operator on* $C_0^\infty(\mathcal{E}^\circ)$.

PROOF. The Laplacian is clearly symmetric due to the previous lemma, since

$$\begin{aligned}
\left\langle -\operatorname{tr}_g\left((\nabla^{\mathcal{E}})^2\phi\right), \psi\right\rangle_{\mathcal{H}} &= \int_M h(\nabla^{\mathcal{E},*}\nabla^{\mathcal{E}}\phi, \psi)\ \mathrm{vol}_g \\
&= \int_M \left\langle \nabla^{\mathcal{E}}\phi, \nabla^{\mathcal{E}}\psi\right\rangle_{\mathsf{T}^*M\otimes\mathcal{E}}\ \mathrm{vol}_g \\
&= \int_M h(\phi, \nabla^{\mathcal{E},*}\nabla^{\mathcal{E}}\psi)\ \mathrm{vol}_g \\
&= \left\langle \phi, -\operatorname{tr}_g\left((\nabla^{\mathcal{E}})^2\psi\right)\right\rangle_{\mathcal{H}}
\end{aligned}$$

for all $\phi, \psi \in C_0^\infty(\mathcal{E}^\circ)$. Moreover, the corresponding quadratic form is seen to be

$$\mathcal{Q}[\psi] := \int_M \left\langle \nabla^{\mathcal{E}}\psi, \nabla^{\mathcal{E}}\psi\right\rangle_{\mathsf{T}^*M\otimes\mathcal{E}}\ \mathrm{vol}_g \geqslant 0.$$

□

Hence, $\mathcal{Q}$ is closable and its Friedrichs extension is the quadratic form of the Dirichlet realisation of the Laplacian which we will denote by $-\Delta_g^{\mathcal{E}}$. This operator is self-adjoint on $\mathcal{H}$ with domain

$$\operatorname{dom}\left(-\Delta_g^{\mathcal{E}}\right) = \overline{C_0^\infty(\mathcal{E}^\circ)}^{(\mathcal{Q}[\cdot]+\|\cdot\|^2_{\mathcal{H}})^{1/2}} = W^2(\mathcal{E}) \cap W_0^1(\mathcal{E}) \tag{A.19}$$

by virtue of Proposition A.20.

Example A.24 Let us consider the case where $\mathcal{E} = M \times \mathbb{C}$ is the complex, trivial line bundle endowed with the flat connection $\nabla^{M\times\mathbb{C}} = \mathrm{d}$. We will refer to the associated connection Laplacian

$$-\Delta_g^{\text{L.B.}} := -\operatorname{tr}_g(\mathrm{d}^2\cdot)$$

as the (negative of the) *Laplace Beltrami operator*. Its action as a differential operator on $C^\infty(M\times\mathbb{C}) \cong C^\infty(M,\mathbb{C})$ is given by

$$-\Delta_g^{\text{L.B.}}\psi = -g^{\alpha\beta}\Big(v_\beta\cdot(v_\alpha\cdot\psi) - \nabla^g_{v_\alpha}v_\beta\cdot\psi\Big)$$

with respect to a local frame $\{v_\alpha\}_{\alpha=1}^m$ of $\mathsf{T}M$. Alternatively, Lemma A.22 together with Example A.6 yield

$$-\Delta_g^{\text{L.B.}}\psi = \mathrm{d}^*(\mathrm{d}\psi) = \mathrm{d}^*\big((\operatorname{grad}_g\psi)^\flat\big) \overset{\text{(A.17)}}{=} -\operatorname{div}_g\big(\operatorname{grad}_g\psi\big)$$

for all $\psi \in C^\infty(M,\mathbb{C})$, so $\Delta_g^{\text{L.B.}} = \operatorname{div}_g\circ\operatorname{grad}_g$ on smooth functions. ◊

We close this subsection with the brief discussion of a modified Laplacian $-\widetilde{\Delta}_g^{\mathcal{E}}$ with Dirichlet boundary conditions, given by the quadratic form

$$\big\langle\psi, -\widetilde{\Delta}_g^{\mathcal{E}}\psi\big\rangle_{L^2(\mathcal{E},\mathrm{vol}_G)} := \int_M \operatorname{tr}_g\Big(h\big(\nabla^{\mathcal{E}}_{\cdot}\psi,\nabla^{\mathcal{E}}_{\cdot}\psi\big)\Big)\ \mathrm{vol}_G,$$

where $G := \rho^{m/2}g$, $\rho\in C_{\mathrm{b}}^\infty(M,[C_1,C_2])$ with $0<C_1<C_2<\infty$, is a conformally equivalent Riemannian metric (for which $\mathrm{vol}_G = \rho\,\mathrm{vol}_g$). To do so, we start with $\psi\in C_0^\infty(\mathcal{E}^\circ)$ as well as $X,Y\in C^\infty(\mathsf{T}M)$ and calculate similarly as in the proof of Lemma A.21:

$$\begin{aligned}
0 \;&=\; \int_{\partial M} i_X\Big(h\big(\psi,\nabla^{\mathcal{E}}_Y\psi\big)\ \mathrm{vol}_G\Big)\\
&=\; \int_M X\cdot h\big(\psi,\nabla^{\mathcal{E}}_Y\psi\big)\ \mathrm{vol}_G + \int_M h\big(\psi,\nabla^{\mathcal{E}}_Y\psi\big)\operatorname{div}_G(X)\ \mathrm{vol}_G\\
&\underset{\text{(2.17b)}}{\overset{\text{(A.1)}}{=}} \int_M h\big(\nabla^{\mathcal{E}}_X,\nabla^{\mathcal{E}}_Y\psi\big) + h\big(\psi,\nabla^{\mathcal{E}}_X\nabla^{\mathcal{E}}_Y\psi\big)\ \mathrm{vol}_G\\
&\qquad + \int_M h\big(\psi,\nabla^{\mathcal{E}}_Y\psi\big)\big(\operatorname{div}_g(X) + \mathrm{d}\ln\rho(X)\big)\ \mathrm{vol}_G,
\end{aligned}$$

or equivalently

$$\int_M h\left(\nabla^{\mathcal{E}}_X\psi, \nabla^{\mathcal{E}}_Y\psi\right)\ \mathrm{vol}_G = \int_M h\Big(\psi, \big(-\nabla^{\mathcal{E}}_X\nabla^{\mathcal{E}}_Y - \mathrm{div}_g(X)\nabla^{\mathcal{E}}_Y\big)\psi\Big)\ \mathrm{vol}_G$$
$$+ \int_M h\left(\psi, -\nabla^{\mathcal{E}}_{\mathrm{d}\ln\rho(X)Y}\psi\right)\ \mathrm{vol}_G\,.$$

If we finally take the g-trace of the respective integrands, we may ultimately identify

$$-\widetilde{\Delta}^{\mathcal{E}}_g = -\,\mathrm{tr}_g\big((\nabla^{\mathcal{E}})^2\cdot\big) - \nabla^{\mathcal{E}}_{\mathrm{grad}_g \ln\rho} \tag{A.20}$$

as an operator on $L^2(\mathcal{E}, \mathrm{vol}_G)$.

Appendix B

B Technical Estimates

The following lemma provides the key ingredients that are needed to prove the results for the low energy regime discussed in Section 4.4. There, we consider energies of the order ε^α $(0 < \alpha \leqslant 2)$ above the bottom of the spectrum of the vertical operator $H^{\mathcal{F}} = -\Delta_{\mathsf{V}}^{\mathcal{E}} + V^{\mathcal{E}}$ in

$$H^{\mathcal{E}} = -\varepsilon^2 \Delta_{\mathsf{H}}^{\mathcal{E}} + \varepsilon H_1^{\mathcal{E}} + H^{\mathcal{F}}.$$

The perturbation $H_1^{\mathcal{E}}$ is assumed to be a second-order horizontal differential operator with C^∞-bounded coefficients. We recall that we introduced $\mathrm{dom}_\alpha^k(T)$ as the domain of the operator $(\varepsilon^{-\alpha}T)^k$ equipped with its graph-norm for $k \in \{1,2\}$.

Lemma B.1 *Let $\alpha \in (0,2]$ and $T \in \{H^{\mathcal{E}}, H_{\mathrm{a}}^{\mathcal{P}}, H_{\mathrm{eff}}^{\mathcal{P}}\}$. If $H_1^{\mathcal{E}}$ satisfies Condition 4.11, it holds that*

$$\left\| P_0 \nabla_{\varepsilon X^{\mathsf{H}}}^{\mathcal{E}} \right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(T), \mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{\alpha/2})$$

for all $X \in C_{\mathrm{b}}^\infty(\mathsf{T}B)$ and

$$\left\| H_1^{\mathcal{E}} P_0 \right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(T), \mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(\varepsilon^{\alpha/2}).$$

We will prove this technical lemma in great detail for $T = H^{\mathcal{E}}$ and sketch the necessary adjustments for the remaining cases afterwards.

PROOF (of Lemma B.1, $T = H^{\mathcal{E}}$). The actual proof consists of four steps:

(i) $P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{E}})$ to $\mathcal{H}$:
We show an even stronger bound, namely we consider the operator $\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ as a mapping from $\mathrm{dom}_\alpha(H^{\mathcal{E}})$ to $\mathcal{H}$. To start with, we first observe that

$$\begin{aligned}\left\|\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\right\|^2_{\mathcal{H}} &= \int_M h\left(\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi, \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\right)\ \mathrm{vol}_g \\ &\leqslant \int_M \pi^*_M g_B(X^{\mathsf{H}}, X^{\mathsf{H}}) \left\langle \nabla^{\mathcal{E}}_{\varepsilon\cdot}\psi, \nabla^{\mathcal{E}}_{\varepsilon\cdot}\psi\right\rangle_{\mathsf{H}*M\otimes\mathcal{E}}\ \mathrm{vol}_g \\ &\leqslant \underbrace{\left\|g_B(X,X)\right\|_\infty}_{<\infty} \underbrace{\int_M h\left(\psi, -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\psi\right)\mathrm{vol}_g}_{=\left\langle\psi, -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\psi\right\rangle_{\mathcal{H}}}\end{aligned}$$

holds true for $\psi \in \mathrm{dom}(H^{\mathcal{E}})$. Then, since

$$\begin{aligned}\left\langle\psi, -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\psi\right\rangle_{\mathcal{H}} &\leqslant \left\langle\psi, -\varepsilon^2\Delta^{\mathcal{E}}_{\mathsf{H}}\psi\right\rangle_{\mathcal{H}} + \underbrace{\left\langle\psi, H^{\mathcal{F}}\psi\right\rangle_{\mathcal{H}}}_{\geqslant 0} \\ &= \left\langle\psi, (H^{\mathcal{E}} - \varepsilon H^{\mathcal{E}}_1)\psi\right\rangle_{\mathcal{H}} \\ &\leqslant \varepsilon^\alpha \underbrace{\|\psi\|_{\mathcal{H}} \left\|\varepsilon^{-\alpha}H^{\mathcal{E}}\right\|_{\mathcal{H}}}_{\leqslant\|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{E}})}} + \varepsilon\left|\left\langle\psi, H^{\mathcal{E}}_1\psi\right\rangle_{\mathcal{H}}\right|,\end{aligned}$$

the first estimate leads to

$$\left\|\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\right\|^2_{\mathcal{H}} \leqslant C\left(\varepsilon^\alpha\|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{E}})} + \varepsilon\left|\left\langle\psi, H^{\mathcal{E}}_1\psi\right\rangle_{\mathcal{H}}\right|\right). \tag{B.1}$$

We now take a closer look at the term in (B.1) which incorporates the perturbation. Using the local form of $H^{\mathcal{E}}_1$ from Condition 4.11, this term,

$$\begin{aligned}\left|\left\langle\psi, H^{\mathcal{E}}_1\psi\right\rangle_{\mathcal{H}}\right| &= \left|\left\langle\psi, H^{\mathcal{E}}_1\textstyle\sum_{\nu\in\mathbb{N}_0}\chi^M_\nu\psi\right\rangle_{\mathcal{H}}\right| \\ &\leqslant \left|\left\langle\psi, \textstyle\sum_{\nu\in\mathbb{N}_0}\nabla^{\mathcal{E}}_{\varepsilon(X^\nu_i)^{\mathsf{H}}} A^{ij}_\nu\, \nabla^{\mathcal{E}}_{\varepsilon(X^\nu_j)^{\mathsf{H}}}\chi^M_\nu\psi\right\rangle_{\mathcal{H}}\right| \\ &\quad + \left|\left\langle\psi, \textstyle\sum_{\nu\in\mathbb{N}_0}B^i_\nu\,\nabla^{\mathcal{E}}_{\varepsilon(X^\nu_i)^{\mathsf{H}}}\chi^M_\nu\psi\right\rangle_{\mathcal{H}}\right| \\ &\quad + \varepsilon\left|\left\langle\psi, \textstyle\sum_{\nu\in\mathbb{N}_0}C_\nu\chi^M_\nu\psi\right\rangle_{\mathcal{H}}\right|,\end{aligned}$$

may be split into three terms, which we now consider separately:

- The second-order term may be estimated as

$$\begin{aligned}
&\left|\sum_{\nu'\in\mathbb{N}_0}\Big\langle\psi,\textstyle\sum_{\nu\in\mathbb{N}_0}\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu})^{\mathsf{H}}}\chi^M_{\nu'}A^{ij}_{\nu}\,\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu})^{\mathsf{H}}}\chi^M_{\nu}\psi\Big\rangle_{\mathcal{H}}\right|\\
&=\left|\sum_{\nu'\in\mathbb{N}_0}\Big\langle\psi,\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\chi^M_{\nu'}A^{ij}_{\nu'}\,\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\underbrace{\textstyle\sum_{\nu\in\mathbb{N}_0}\chi^M_{\nu}}_{=1}\psi\Big\rangle_{\mathcal{H}}\right|\\
&\leqslant\left|\sum_{\nu'\in\mathbb{N}_0}\Big\langle\chi^M_{\nu'}\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\psi,A^{ij}_{\nu'}\,\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\psi\Big\rangle_{\mathcal{H}}\right|\\
&\qquad+\varepsilon\left|\sum_{\nu'\in\mathbb{N}_0}\Big\langle\psi,\chi^M_{\nu'}\operatorname{div}_g\big((X_i^{\nu'})^{\mathsf{H}}\big)A^{ij}_{\nu'}\,\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\psi\Big\rangle_{\mathcal{H}}\right|
\end{aligned}$$

with the aid of Lemma A.21 for the last step. Considering the first term, we insert the identity $\sum_{\nu''\in\mathbb{N}_0}\chi^M_{\nu''}=1$ appropriately into the second argument of the scalar product and obtain the upper bound

$$\begin{aligned}
&\sup_{\nu,k,l}\big\|A^{kl}_{\nu}\big\|_{\infty}\sum_{\substack{\nu',\nu''\in\mathbb{N}_0\\ U_{\nu'\nu''}\neq\emptyset}}\sum_{i,j=1}^{b}\Big\|\chi^M_{\nu'}\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\psi\Big\|_{\mathcal{H}}\Big\|\chi^M_{\nu''}\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\psi\Big\|_{\mathcal{H}}\\
&\leqslant\tfrac{1}{2}\sup_{\nu,k,l}\big\|A^{kl}_{\nu}\big\|_{\infty}\sum_{\substack{\nu',\nu''\in\mathbb{N}_0\\ U_{\nu'\nu''}\neq\emptyset}}\sum_{i,j=1}^{b}\bigg(\Big\|\chi^M_{\nu'}\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\psi\Big\|^2_{\mathcal{H}}\\
&\qquad\qquad+\Big\|\chi^M_{\nu''}\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\psi\Big\|^2_{\mathcal{H}}\bigg)\\
&\leqslant\underbrace{\sup_{\nu',k,l}\big\|A^{kl}_{\nu'}\big\|_{\infty}N_{\mathfrak{U}}\,b^2\,C}_{=:C_1}\Big(\varepsilon^{\alpha}\,\|\psi\|^2_{\mathrm{dom}_{\alpha}(H^{\varepsilon})}+\varepsilon\,\big|\langle\psi,H_1^{\mathcal{E}}\psi\rangle_{\mathcal{H}}\big|\Big).
\end{aligned}$$

Here, we used the facts that

$$\sum_{\nu'\in\mathbb{N}_0}\big\|\chi^M_{\nu'}\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\big\|^2_{\mathcal{H}}=\int_M\underbrace{\sum_{\nu'\in\mathbb{N}_0}\chi^{M\,2}_{\nu'}}_{\leqslant 1\text{ pointwise}}\,h\big(\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi,\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\big)\ \mathrm{vol}_g$$

$$\leqslant \left\|g_B(X,X)\right\|_\infty \left\langle\psi, -\varepsilon^2\Delta_{\mathsf{H}}^{\mathcal{E}}\psi\right\rangle_{\mathcal{H}} \tag{B.2}$$

and that the vector fields $\{X_i^{\nu'}\}_{\nu',i}$ are uniformly bounded in order to establish a similar estimate to (B.1) for the last transformation. As far as the second term is concerned, we proceed similarly and bound it from above by

$$\begin{aligned}
&\varepsilon\,\|\psi\|_{\mathcal{H}} \sum_{\nu'\in\mathbb{N}_0}\sum_{j=1}^{b} \left\|\chi_{\nu'}^{M}\,\mathrm{div}_g\left((X_i^{\nu'})^{\mathsf{H}}\right)A_{\nu'}^{ij}\,\nabla^{\mathcal{E}}_{\varepsilon(X_j^{\nu'})^{\mathsf{H}}}\psi\right\|_{\mathcal{H}}\\
&\overset{(\mathrm{B.2})}{\leqslant}\ \varepsilon\,\|\psi\|_{\mathcal{H}}\, b\sup_{\nu,k,l}\left(\left\|\mathrm{div}_g\left((X_k^{\nu})^{\mathsf{H}}\right)A_{\nu}^{kl}\right\|_\infty \underbrace{\left\|\nabla^{\mathcal{E}}_{\varepsilon(X_l^{\nu})^{\mathsf{H}}}\psi\right\|_{\mathcal{H}}}_{\leqslant c\|\psi\|_{\mathrm{dom}(H^{\mathcal{E}})}}\right)\\
&\leqslant\ \underbrace{\varepsilon\, b\, c\sup_{\nu,k,l}\left\|\mathrm{div}_g\left((X_k^{\nu})^{\mathsf{H}}\right)A_{\nu}^{kl}\right\|_\infty}_{=:C_2} \|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{E}})}\,.
\end{aligned}$$

In summary, the second-order term can be bounded by

$$(C_1\varepsilon^\alpha + C_2\varepsilon)\,\|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{E}})} + C_1\varepsilon\left|\left\langle\psi, H_1^{\mathcal{E}}\psi\right\rangle_{\mathcal{H}}\right|.$$

- The first-order term gives the following contribution:

$$\begin{aligned}
&\left|\left\langle\psi, \textstyle\sum_{\nu\in\mathbb{N}_0} B_\nu^i\,\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu})^{\mathsf{H}}}\chi_\nu^M\psi\right\rangle_{\mathcal{H}}\right|\\
&\leqslant\ \|\psi\|_{\mathcal{H}}\underbrace{\left\|\textstyle\sum_{\nu\in\mathbb{N}_0} B_\nu^i\,\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu})^{\mathsf{H}}}\chi_\nu^M\psi\right\|_{\mathcal{H}}}_{\leqslant N_{\mathfrak{U}}^{1/2}\|\ldots\|_{W^0_\varepsilon(\mathcal{E})}\text{ by Remark 2.13}}\\
&\leqslant\ \|\psi\|_{\mathcal{H}} N_{\mathfrak{U}}^{1/2}\left(\sum_{\nu'\in\mathbb{N}_0}\left\|\chi_{\nu'}\,B_{\nu'}^i\,\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\underbrace{\textstyle\sum_{\nu\in\mathbb{N}_0}\chi_\nu^M}_{=1}\psi\right\|_{\mathcal{H}}\right)^{1/2}\\
&\leqslant\ \sup_{\nu,k}\left\|B_\nu^k\right\|_\infty N_{\mathfrak{U}}^{1/2}\,\|\psi\|_{\mathcal{H}}\left(\sum_{\nu'\in\mathbb{N}_0}\left\|\chi_{\nu'}\,B_{\nu'}^i\,\nabla^{\mathcal{E}}_{\varepsilon(X_i^{\nu'})^{\mathsf{H}}}\psi\right\|_{\mathcal{H}}\right)^{1/2}
\end{aligned}$$

$$\overset{(B.1)}{\underset{(B.2)}{\leqslant}} \sup_{\nu,k} \left\| B_\nu^k \right\|_\infty N_{\mathfrak{U}}^{1/2} \, \|\psi\|_{\mathcal{H}} \, b^{1/2} \, C^{1/2}$$

$$\times \underbrace{\left(\varepsilon^\alpha \, \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} + \varepsilon \left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right| \right)^{1/2}}_{\leqslant \varepsilon^{\alpha/2} \|\psi\|_{\mathrm{dom}_\alpha(H^\mathcal{E})} + \varepsilon^{1/2} \left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right|^{1/2}}$$

$$\leqslant \underbrace{\sup_{\nu,k} \left\| B_\nu^k \right\|_\infty (N_{\mathfrak{U}} \, b \, C)^{1/2}}_{=:C_3} \varepsilon^{\alpha/2} \, \|\psi\|_{\mathcal{H}} \, \|\psi\|_{\mathrm{dom}_\alpha(H^\mathcal{E})}$$

$$+ \Bigg(\underbrace{\sup_{\nu,k} \left\| B_\nu^k \right\|_\infty (N_{\mathfrak{U}} \, b \, C)^{1/2}}_{=:(2C_4)^{1/2}} \varepsilon^{1/2} \, \|\psi\|_{\mathcal{H}} \Bigg) \left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right|^{1/2}$$

$$\leqslant \left(C_3 \varepsilon^{\alpha/2} + C_4 \varepsilon \right) \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} + \tfrac{1}{2} \left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right| .$$

- The remaining potential term is easily estimated to be

$$\varepsilon \left| \left\langle \psi, \textstyle\sum_{\nu \in \mathbb{N}_0} C_\nu \chi_\nu^M \psi \right\rangle_{\mathcal{H}} \right| \leqslant \underbrace{\sup_{\nu'} \|C_{\nu'}\|_\infty}_{=:C_5} \varepsilon \, \|\psi\|_{\mathcal{H}} \left\| \underbrace{\textstyle\sum_{\nu \in \mathbb{N}_0} \chi_\nu^M}_{=1} \psi \right\|_{\mathcal{H}}$$

$$= C_5 \varepsilon \, \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} .$$

Thus, the addition of these three estimates implies

$$\left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right| \leqslant \left(C_1 \varepsilon^\alpha + (C_2 + C_4 + C_5) \varepsilon + C_3 \varepsilon^{\alpha/2} \right) \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} + \left(\tfrac{1}{2} + C_1 \varepsilon \right) \left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right| ,$$

and consequently

$$\left| \left\langle \psi, H_1^\mathcal{E} \psi \right\rangle_{\mathcal{H}} \right| \leqslant \tilde{C} \varepsilon^{\alpha/2} \, \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})}$$

for some constant $\tilde{C} > 0$ if $\varepsilon > 0$ is chosen sufficiently small. We finally insert this into (B.1) and obtain

$$\left\| \nabla^\mathcal{E}_{\varepsilon X^{\mathsf{H}}} \psi \right\|^2_{\mathcal{H}} \leqslant C \left(\varepsilon^\alpha + \tilde{C} \varepsilon^{1+\alpha/2} \right) \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} \leqslant c \varepsilon^\alpha \, \|\psi\|^2_{\mathrm{dom}_\alpha(H^\mathcal{E})} ,$$

where we used the fact that $\alpha \leqslant 1 + \alpha/2$ for $\alpha \in (0, 2]$.

(ii) $P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{E}})$ to $\mathrm{dom}(H^{\mathcal{E}})$:
We start with the estimate

$$\left\|H^{\mathcal{E}}P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\psi\right\|_{\mathcal{H}} \leqslant \underbrace{\left\|\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\right\|_{\mathcal{L}(\mathrm{dom}_\alpha(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{\alpha/2})\text{ by (i)}}\left\|H^{\mathcal{E}}\psi\right\|_{\mathrm{dom}_\alpha(H^{\mathcal{E}})} + \varepsilon\left\|\left(\left[H^{\mathcal{E}},\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]+[H^{\mathcal{E}},P_0]\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right)\psi\right\|_{\mathcal{H}}$$

for $\psi\in\mathrm{dom}^2_\alpha(H^{\mathcal{E}})\subset\mathrm{dom}((H^{\mathcal{E}})^2)$. We estimate the first term as

$$\begin{aligned}\left\|H^{\mathcal{E}}\psi\right\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{E}})} &= \left\|\varepsilon^{-\alpha}(H^{\mathcal{E}})^2\psi\right\|^2_{\mathcal{H}}+\left\|H^{\mathcal{E}}\psi\right\|^2_{\mathcal{H}}\\ &\leqslant \varepsilon^{2\alpha}\left(\left\|(\varepsilon^{-\alpha}H^{\mathcal{E}})^2\psi\right\|^2_{\mathcal{H}}+\|\psi\|_{\mathcal{H}}\left\|(\varepsilon^{-\alpha}H^{\mathcal{E}})^2\psi\right\|_{\mathcal{H}}\right)\\ &\leqslant 2\varepsilon^{2\alpha}\|\psi\|^2_{\mathrm{dom}^2_\alpha(H^{\mathcal{E}})}.\end{aligned}$$

Expanding the second term

$$\begin{aligned}\left[H^{\mathcal{E}},\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]+[H^{\mathcal{E}},P_0]\nabla^{\mathcal{E}}_{X^{\mathsf{H}}} &= \left[-\Delta^{\mathcal{E}}_{g^\varepsilon},\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]+\left[V^{\mathcal{E}},\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]+\varepsilon\left[H^{\mathcal{E}}_1,\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]\\ &\quad+\left[-\varepsilon\Delta^{\mathcal{E}}_{\mathsf{H}},P_0\right]\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}+\left[H^{\mathcal{E}}_1,P_0\right]\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\end{aligned}$$

locally over $\pi_M^{-1}(U)$ for some $U\in\mathfrak{U}$ of Definition 2.7, we see, in view of Remark 3.6(i), that this defines a bounded operator from $W^3_\varepsilon(\mathcal{E})$ to $\mathcal{H}$ with a bound that can be chosen independently of ε due to the fact that $P_0\in\mathcal{A}^{0,0}_H$ (see also (3.5)) as well as Condition 4.11. Thus, Corollary 2.17 implies that

$$\begin{aligned}\left\|\left(\left[H^{\mathcal{E}},\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right]+[H^{\mathcal{E}},P_0]\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right)\psi\right\|_{\mathcal{H}} &\leqslant c\,\|\psi\|_{\mathrm{dom}((H^{\mathcal{E}})^2)}\\ &\leqslant c\,\|\psi\|_{\mathrm{dom}^2_\alpha(H^{\mathcal{E}})}.\end{aligned}$$

In summary, this shows that

$$\begin{aligned}&\left\|P_0\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathrm{dom}(H^{\mathcal{E}}))}\\ &\leqslant \underbrace{\left\|H^{\mathcal{E}}P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{\min\{3\alpha/2,1\}})}+\underbrace{\left\|P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{\alpha/2})\text{ by (i)}}\\ &\overset{\alpha\leqq 2}{=}\mathcal{O}(\varepsilon^{\alpha/2}).\end{aligned}$$

(iii) $H_1^{\mathcal{E}} P_0$ from $\mathrm{dom}_\alpha^2(H^{\mathcal{E}})$ to $\mathcal{H}$:
We consider the operator $H_1^{\mathcal{E}} P_0$ again locally over $\pi_M^{-1}(U)$ for some $U \in \mathfrak{U}$ (cf. Definition 2.7) and get, thanks to Condition 4.11, that

$$H_1^{\mathcal{E}} P_0\big|_{\pi_M^{-1}(U)} = \underbrace{\Big(A^{ji}\nabla^{\mathcal{E}}_{\varepsilon X_j^{\mathsf{H}}} + B^i + \varepsilon\big[\nabla^{\mathcal{E}}_{X_j^{\mathsf{H}}}, A^{ji}\big]\Big)}_{=:\, S^i \text{ with } \|S^i\|_{\mathcal{L}(W^1_\varepsilon(\mathcal{E}),\mathcal{H})} = \mathcal{O}(1)} \nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}} P_0 + \underbrace{\varepsilon C P_0}_{\substack{=\mathcal{O}(\varepsilon)\\ \text{in } \mathcal{L}(\mathcal{H})}}$$

with the aid of Lemma 3.13. But then the fact that

$$\begin{aligned}
&\left\|\nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}} P_0\right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(H^{\mathcal{E}}),W^2_\varepsilon(\mathcal{E}))} \\
&\leqslant\ \varepsilon \underbrace{\left\|\big[\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}, P_0\big]\right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(H^{\mathcal{E}}),W^2_\varepsilon(\mathcal{E}))}}_{\substack{\leqslant c\|[\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}},P_0]\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}))} = \mathcal{O}(1)\\ \text{due to } P_0 \in \mathcal{A}_H^{0,0}}} + \underbrace{\left\|P_0 \nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}\right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(H^{\mathcal{E}}),W^2_\varepsilon(\mathcal{E}))}}_{=\,\mathcal{O}(\varepsilon^{\alpha/2}) \text{ by (ii)}} \\
&\overset{\alpha\leqq 2}{=} \mathcal{O}(\varepsilon^{\alpha/2})
\end{aligned}$$

immediately implies $\left\|H_1^{\mathcal{E}} P_0\right\|_{\mathcal{L}(\mathrm{dom}_\alpha^2(H^{\mathcal{E}}),\mathcal{H})} = \mathcal{O}(\varepsilon^{\alpha/2})$.

(iv) $H_1^{\mathcal{E}} P_0$ from $\mathrm{dom}_\alpha^2(H^{\mathcal{E}})$ to $\mathrm{dom}(H^{\mathcal{E}})$:
Let us first analyse the commutator of $H^{\mathcal{E}}$ with $H_1^{\mathcal{E}} P_0$. Expressing this over $\pi_M^{-1}(U)$ for some $U \in \mathfrak{U}$ from Definition 2.7, we see that

$$\begin{aligned}
\big[H^{\mathcal{E}}, H_1^{\mathcal{E}} P_0\big] &= H_1^{\mathcal{E}} \underbrace{\big[H^{\mathcal{E}}, P_0\big]}_{=\mathcal{O}(\varepsilon)} + \varepsilon S^i \underbrace{\big[H^{\mathcal{E}}, \nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}\big]}_{=\mathcal{O}(1)} P_0 \\
&\quad + \underbrace{\big[-\Delta^{\mathcal{E}}_{g^\varepsilon} + V^{\mathcal{E}}, S^i\big]}_{=\mathcal{O}(1)} \Big(\varepsilon \underbrace{\big[\nabla^{\mathcal{E}}_{X_i^{\mathsf{H}}}, P_0\big]}_{=\mathcal{O}(1)} + P_0 \nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}\Big) \\
&\quad + \varepsilon \underbrace{\big[H_1^{\mathcal{E}}, S^i\big]}_{=\mathcal{O}(1)} \nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}} P_0 + \varepsilon \underbrace{\big[H^{\mathcal{E}}, C\big]}_{=\mathcal{O}(1)} \\
&= \varepsilon T_1 + T_2^i P_0 \nabla^{\mathcal{E}}_{\varepsilon X_i^{\mathsf{H}}}
\end{aligned}$$

for some $T_1 \in \mathcal{L}(W^4_\varepsilon(\mathcal{E}),\mathcal{H})$ and $T_2^i \in \mathcal{L}(W^2_\varepsilon(\mathcal{E}),\mathcal{H})$, using $P_0 \in \mathcal{A}_H^{0,0}$ due to Lemma 3.13. While Corollary 2.17 implies that $\|T_1\psi\|_{\mathcal{H}}$ can be bounded by an ε-independent constant times $\|\psi\|_{\mathrm{dom}((H^{\mathcal{E}})^2)} \leqslant \|\psi\|_{\mathrm{dom}^2_\alpha(H^{\mathcal{E}})}$, the second term gives

$$\begin{aligned}
&\left\| T^i_2 P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}_i} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})} \\
&\leqslant c \underbrace{\left\| T^i_2 \right\|_{\mathcal{L}(W^2_\varepsilon(\mathcal{E}),\mathcal{H})}}_{<\infty} \underbrace{\left\| P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}_i} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathrm{dom}(H^{\mathcal{E}}))}}_{=\mathcal{O}(\varepsilon^{\alpha/2}) \text{ by (ii)}}.
\end{aligned}$$

Consequently, this leads to the estimate

$$\left\| \left[H^{\mathcal{E}}, H^{\mathcal{E}}_1 P_0 \right] \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})} \overset{\alpha\leqq 2}{=} \mathcal{O}(\varepsilon^{\alpha/2}),$$

and we proceed similarly as in step (ii), i.e.,

$$\begin{aligned}
&\left\| H^{\mathcal{E}}_1 P_0 \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathrm{dom}(H^{\mathcal{E}}))} \\
&\leqslant \underbrace{\left\| H^{\mathcal{E}}_1 P_0 \right\|_{\mathcal{L}(\mathrm{dom}(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(1)} \underbrace{\left\| H^{\mathcal{E}} \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathrm{dom}(H^{\mathcal{E}}))}}_{=\mathcal{O}(\varepsilon^\alpha)} \\
&\quad + \underbrace{\left\| \left[H^{\mathcal{E}}, H^{\mathcal{E}}_1 P_0 \right] \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{\alpha/2})} + \underbrace{\left\| H^{\mathcal{E}}_1 P_0 \right\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{E}}),\mathcal{H})}}_{=\mathcal{O}(\varepsilon^{\alpha/2}) \text{ by (iii)}} \\
&= \mathcal{O}(\varepsilon^{\alpha/2}).
\end{aligned}$$

□

PROOF (of Lemma B.1, $T = H^{\mathcal{P}}_{\mathrm{a}}$). The basic idea here is to repeat the four steps of the case $T = H^{\mathcal{E}}$ and thereby to bound the individual terms by $H^{\mathcal{P}}_{\mathrm{a}}$ instead of $H^{\mathcal{E}}$:

(i) $P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathcal{H}$:
Using $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}}) \subset \mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}}) = P_0 \,\mathrm{dom}(H^{\mathcal{E}})$, we arrive at the analogue of estimate (B.1), i.e.,

$$\left\| \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} \psi \right\|^2_{\mathcal{H}} \leqslant C \left(\varepsilon^\alpha \|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{P}}_{\mathrm{a}})} + \varepsilon \left| \left\langle \psi, H^{\mathcal{E}}_1 \psi \right\rangle_{\mathcal{H}_{\mathcal{P}}} \right| \right)$$

for $\psi \in \mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}})$. We then similarly bound $|\langle \psi, H^{\mathcal{E}}_1 \psi \rangle_{\mathcal{H}_{\mathcal{P}}}|$ by a constant times $\varepsilon^{\alpha/2} \|\psi\|^2_{\mathrm{dom}_\alpha(H^{\mathcal{P}}_{\mathrm{a}})}$ and obtain the desired estimate.

(ii) $P_0\nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathrm{dom}(H^{\mathcal{E}})$:
We consider

$$
\begin{aligned}
&H^{\mathcal{E}} P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} P_0 \\
&= P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} H^{\mathcal{E}} P_0 + \big[H^{\mathcal{E}}, P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}}\big] P_0 \\
&= \nabla^{\mathrm{B}}_{\varepsilon X} H^{\mathcal{P}}_{\mathrm{a}} + P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} [H^{\mathcal{E}}, P_0] P_0 \\
&\quad + [H^{\mathcal{E}}, P_0] \nabla^{\mathcal{E}}_{\varepsilon X^{\mathsf{H}}} P_0 + \varepsilon P_0 \big[H^{\mathcal{E}}, \nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\big] P_0 \\
&= \nabla^{\mathrm{B}}_{\varepsilon X} H^{\mathcal{P}}_{\mathrm{a}} + \underbrace{\nabla^{\mathrm{B}}_{\varepsilon X} \underbrace{[H^{\mathcal{E}}, P_0]}_{\substack{\text{off-diagonal} \\ \text{w.r.t. } P_0}} P_0}_{=\mathbf{0}} - \varepsilon P_0 \big[\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}, P_0\big] [H^{\mathcal{E}}, P_0] P_0 \\
&\quad + [H^{\mathcal{E}}, P_0] \nabla^{\mathrm{B}}_{\varepsilon X} + \varepsilon \underbrace{[H^{\mathcal{E}}, P_0]\big[\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}, P_0\big]}_{\text{diagonal w.r.t. } P_0} P_0 + \varepsilon P_0 \big[H^{\mathcal{E}}, \nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\big] P_0 \\
&= \nabla^{\mathrm{B}}_{\varepsilon X} H^{\mathcal{P}}_{\mathrm{a}} + \varepsilon [H^{\mathcal{E}}, P_0] \nabla^{\mathrm{B}}_{X} \\
&\quad + \underbrace{\varepsilon\, P_0 \underbrace{\Big(\Big[[H^{\mathcal{E}}, P_0], \big[\nabla^{\mathcal{E}}_{X^{\mathsf{H}}}, P_0\big]\Big] + \big[H^{\mathcal{E}}, \nabla^{\mathcal{E}}_{X^{\mathsf{H}}}\big]\Big)}_{=\,\mathcal{O}(1) \text{ in } \mathcal{L}(W^2_\varepsilon(\mathcal{E}), \mathcal{H})} P_0}_{=\,\mathcal{O}(1) \text{ in } \mathcal{L}(W^2_\varepsilon(\mathcal{P}), \mathcal{H}_{\mathcal{P}}) \text{ due to } P_0 \in \mathcal{A}^{0,0}_H}
\end{aligned}
$$

The first term can be estimated using $H^{\mathcal{P}}_{\mathrm{a}} = \mathcal{O}(\varepsilon^\alpha)$ as a bounded mapping from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}})$ as well as the regularity result of Proposition 4.9(ii):

$$
\begin{aligned}
&\big\|\nabla^{\mathrm{B}}_{\varepsilon X} H^{\mathcal{P}}_{\mathrm{a}}\big\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}}), \mathcal{H})} \\
&\leqslant c \underbrace{\big\|\nabla^{\mathrm{B}}_{\varepsilon X}\big\|_{\mathcal{L}(W^2_\varepsilon(\mathcal{P}), \mathcal{H}_{\mathcal{P}})}}_{=\mathcal{O}(1)} \underbrace{\big\|H^{\mathcal{P}}_{\mathrm{a}}\big\|_{\mathcal{L}(\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}}), \mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}}))}}_{=\mathcal{O}(\varepsilon^\alpha)} \\
&= \mathcal{O}(\varepsilon^\alpha).
\end{aligned}
$$

The last-mentioned proposition may also be used to show that the third term is of order ε as a map from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathcal{H}_{\mathcal{P}}$. As far as the second term is concerned, a local expansion shows that

$$
[H^{\mathcal{E}}, P_0] \nabla^{\mathrm{B}}_{X} = \big[-\varepsilon \Delta^{\mathcal{E}}_{\mathsf{H}} + H^{\mathcal{E}}_1, P_0\big] \nabla^{\mathrm{B}}_{\varepsilon X}
$$

is a third-order differential operator on $\mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} (B, \varepsilon^{-2} g_B)$ with coefficients in $\mathcal{L}(L^2(\mathcal{P}), \mathcal{H})$ that are off-diagonal with respect to P_0. Thus, another application of Proposition 4.9(ii) leads to the fact that

$$\left\| [H^{\mathcal{E}}, P_0] \nabla^{\mathrm{B}}_X \psi \right\|_{\mathcal{H}} \leqslant c \, \|\psi\|_{\mathrm{dom}((H^{\mathcal{P}}_{\mathrm{a}})^2)} \leqslant c \, \|\psi\|_{\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})}$$

for an ε-independent constant $c > 0$. To sum up, $H^{\mathcal{E}} P_0 \nabla^{\mathcal{E}}_{\varepsilon X^{\mathrm{H}}}$ is of order $\varepsilon^{\alpha/2}$ as a mapping from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathcal{H}$.

(iii) $H^{\mathcal{E}}_1 P_0$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathcal{H}$:
This is shown in exactly the same manner as in the case $T = H^{\mathcal{E}}$, using $\mathrm{dom}(H^{\mathcal{P}}_{\mathrm{a}}) \subset P_0 \, \mathrm{dom}(H^{\mathcal{E}})$.

(iv) $H^{\mathcal{E}}_1 P_0$ from $\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})$ to $\mathrm{dom}(H^{\mathcal{E}})$:
We proceed analogously in order to see that $[H^{\mathcal{E}}, H^{\mathcal{E}}_1 P_0] P_0$ is of order ε in $\mathcal{L}(W^4_\varepsilon(\mathcal{P}), \mathcal{H})$, i.e., $\left\| [H^{\mathcal{E}}, H^{\mathcal{E}}_1 P_0] \psi \right\|_{\mathcal{H}}$ can be bounded by an ε-independent constant times $\varepsilon \, \|\psi\|_{\mathrm{dom}((H^{\mathcal{P}}_{\mathrm{a}})^2)} \leqslant \varepsilon \, \|\psi\|_{\mathrm{dom}^2_\alpha(H^{\mathcal{P}}_{\mathrm{a}})}$ with the aid of Proposition 2.15(ii). $\square$

PROOF (of Lemma B.1, $T = H^{\mathcal{P}}_{\mathrm{eff}}$). This immediately follows from the previous case and the fact that $H^{\mathcal{P}}_{\mathrm{eff}} = H^{\mathcal{P}}_{\mathrm{a}} + \mathcal{O}(\varepsilon^2)$ in $\mathcal{L}(\mathcal{D}^{\mathrm{eff}}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}})$ due to the explicit formula (4.13). Put differently, the replacement of the adiabatic operator by the effective operator in the necessary estimates only yields higher order errors. $\square$

List of Symbols

1 General Notation

Symbol	Explanation
C^∞	space of smooth functions/sections
C_0^∞	Fréchet space of smooth functions/sections with compact support
C_b^∞	Fréchet space of C^∞-bounded functions/sections
L^2	Lebesgue space of square-integrable functions/sections
L^∞	Lebesgue space of essentially bounded functions/sections
W^k	Sobolev space of functions/sections with weak derivatives of order up to $k \in \mathbb{N}_0$ in L^2
W_0^k	subspace of W^k for $k \in \mathbb{N}$, where in addition all weak derivatives up to order $k-1$ vanish on the boundary
W^∞	intersection of all Sobolev spaces $\{W^k\}_{k\in\mathbb{N}_0}$, space of functions/sections with infinitely many weak derivatives
$\mathcal{L}(U,V)$	space of continuous linear maps between the vector spaces U and V
vol_G	volume measure associated with the Riemannian metric G
∇^G	Levi-Civita connection related to the Riemannian metric G
$r_\mathrm{inj}(N,G)$	injectivity radius of a Riemannian manifold (N,G)

$\mathsf{B}_r(p)$	geodesic ball around $p \in N$ with radius r, image of the metric ball $\mathbb{B}_r(0) \subset \mathsf{T}_pN$ under the exponential map
$\mathsf{NC}(r)$	normal collar of width r, image of the cylinder $\partial N \times [0, r)$ under the collar map $\mathcal{K}$
$\mathrm{R}^{\mathcal{V}}$	curvature of the connection $\nabla^{\mathcal{V}}$
$\Delta_G^{\mathcal{V}}$	connection Laplacian associated with the connection $\nabla^{\mathcal{V}}$ and the Riemannian metric G
$\mathcal{L}(\mathcal{V}, \mathcal{W})$	space of continuous bundle homomorphisms between the vector bundles $\mathcal{V}$ and $\mathcal{W}$ over the same base manifold
$\mathrm{End}(\mathcal{V})$	space of vector bundle endomorphisms of the vector bundle $\mathcal{V}$
$\mathrm{Herm}(\mathcal{V})$	subbundle of $\mathrm{End}(\mathcal{V})$, space of self-adjoint (Hermitian) vector bundle endomorphisms of the vector bundle $\mathcal{V}$
$\Delta_G^{\mathrm{L.B.}}$	Laplace-Beltrami operator, connection Laplacian associated with the flat connection d on the trivial line bundle and the Riemannian metric G
f^*	pullback map related to a smooth mapping $f : N_1 \to N_2$
$\mathsf{T}f$	differential (tangent map) of a smooth map $f : N_1 \to N_2$, mapping from $\mathsf{T}N_1$ to $\mathsf{T}N_2$
$\mathrm{D}f$	derivative (Jacobian matrix) of a smooth map $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$
$\mathcal{L}_Z$	Lie derivative along the vector field Z

2 The Framework

Symbol	**Explanation**	**Page**
(B, g_B)	complete, connected b-dimensional manifold of bounded geometry without boundary	19, 26
$\mathfrak{U}$	atlas of normal coordinate charts (geodesic balls) for B with finite multiplicity $N_{\mathfrak{U}}$	37

$\{\chi_\nu\}_{\nu\in\mathbb{N}_0}$	partition of unity of B which is subordinate to the cover $\mathfrak{U} = \{U_\nu\}_{\nu\in\mathbb{N}_0}$	37
(F, g_F)	compact, f-dimensional Riemannian manifold with (possibly empty) boundary	19, 26
$M \xrightarrow{\pi_M} B$	uniformly locally trivial fibre bundle over B with typical fibre F, M is a ∂-manifold of bounded geometry equipped with submersion metric g	19, 26
$\{\chi_\nu^M\}_{\nu\in\mathbb{N}_0}$	respective composition of χ_ν with π_M, partition of unity of M which is subordinate to the cover $\{\pi_M^{-1}(U_\nu)\}_{\nu\in\mathbb{N}_0}$	37
$\mathsf{H}M$	horizontal subbundle of $\mathsf{T}M$	23
$\mathsf{V}M$	vertical subbundle of $\mathsf{T}M$	23
X^{H}	horizontal lift of the vector field $X \in C^\infty(\mathsf{T}B)$	24
Φ^*X	∂-horizontal vector field associated with some local trivialisation $\Phi : \pi_M^{-1}(U) \to U \times F$ of M and a vector field $X \in C^\infty(\mathsf{T}U)$	69
Ω_{H}	integrability tensor of the horizontal subbundle $\mathsf{H}M$	24
II_{V}	second fundamental form of the fibres in M	24
η_{V}	mean curvature vector of the fibres in M	25
g^ε	rescaled Riemannian submersion metric on M	37
g_{V}	restriction of g^ε to the vertical subbundle $\mathsf{V}M$	24
$\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$	$\mathbb{C}^N$-vector bundle of bounded geometry over M which is endowed with a Hermitian bundle metric h and a metric connection $\nabla^{\mathcal{E}}$	21, 26
$\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} F$	the model, $\mathbb{C}^N$-vector bundle over F that is isomorphic to $\mathcal{E}_x = \mathcal{E}\|_{\pi_M^{-1}(x)}$ for all $x \in B$	21

$\mathcal{F}_\nu$	Hermitian vector bundle $(\mathcal{F}, h_\nu, \nabla_\nu) \xrightarrow{\pi_\mathcal{F}} (F, g_F)$ that is equipped with ν-dependent bundle metric h_ν and metric connection ∇_ν	27, 31
$\mathcal{H}$	Hilbert space of square-integrable sections of the vector bundle $(\mathcal{E}, h) \xrightarrow{\pi_\mathcal{E}} (M, g)$	40
$W^k_\varepsilon(\mathcal{E})$	weighted Sobolev spaces on the ε-dependent vector bundle $(\mathcal{E}, h, \nabla^\mathcal{E}) \xrightarrow{\pi_\mathcal{E}} (M, g^\varepsilon)$	44
$\mathcal{H}_\mathcal{F}$	infinite-dimensional, Hermitian vector bundle over B with typical fibre $L^2(\mathcal{F})$	44
$\mathcal{D}_\mathcal{F}$	infinite-dimensional, Hermitian vector bundle over B with typical fibre $\mathcal{D}(\mathcal{F}) = W^2(\mathcal{F}) \cap W^1_0(\mathcal{F})$	44
$\varepsilon^2 \Delta^\mathcal{E}_\mathsf{H}$	horizontal part of $\Delta^\mathcal{E}_{g^\varepsilon}$	41
$\Delta^\mathcal{E}_\mathsf{V}$	vertical part of $\Delta^\mathcal{E}_{g^\varepsilon}$	41
$H^\mathcal{E}$	self-adjoint operator on $\mathcal{H}$ with Dirichlet domain $\mathrm{dom}(H^\mathcal{E})$	50, 128
$H^\mathcal{F}$	self-adjoint vertical operator on $\mathcal{H}_\mathcal{F}$ with Dirichlet domain $\mathcal{D}_\mathcal{F}$	62, 128
$\varepsilon H^\mathcal{E}_1$	perturbation within the operator $H^\mathcal{E}$	50, 129
$\mathcal{A}$	algebra of ∂-horizontal differential operators on $\mathcal{E}$ with coefficients in $L^\infty(\mathcal{L}(\mathcal{H}_\mathcal{F}))$	71
$\mathcal{A}_H$	subalgebra of $\mathcal{A}$, ∂-horizontal differential operators on $\mathcal{E}$ with coefficients in $L^\infty(\mathcal{L}(\mathcal{H}_\mathcal{F}, \mathcal{D}_\mathcal{F}))$	71

3 (Super-)Adiabatic Perturbation Theory

Symbol	**Explanation**	**Page**
λ	eigenband of the vertical operator $H^\mathcal{F}$	62, 88

P_0	spectral projection associated with the eigenband λ	63, 87
$\mathcal{P}$	eigenspace bundle associated with λ, Hermitian vector bundle over B of finite rank $q = \operatorname{rank}(P_0)$	63, 120
$\mathcal{H}_{\mathcal{P}}$	Hilbert space of square-integrable sections of $\mathcal{P}$	107
$H_{\mathrm{a}}^{\mathcal{P}}$	adiabatic operator, self-adjoint operator on $\mathcal{H}_{\mathcal{P}}$	120
∇^{B}	Berry connection on $\mathcal{P}$	121
Δ^{B}	Berry Laplacian, second-order differential operator on $\mathcal{P}$	124
$V_{\mathrm{BH}}^{\mathcal{P}}$	Born-Huang potential, deviation from the Berry Laplacian Δ_{B} to the projected horizontal Laplacian $P_0 \Delta_{\mathsf{H}}^{\mathcal{E}} P_0$	124
$V_{\eta_{\mathsf{V}}}^{\mathcal{P}}$	potential induced by the mean curvature vector η_{V}	124
P_ε	super-adiabatic projection associated with the eigenband λ	98
U_ε	unitary operator on $\mathcal{H}$ which intertwines P_0 and P_ε	107
$H_{\mathrm{eff}}^{\mathcal{P}}$	effective operator, self-adjoint operator on $\mathcal{H}_{\mathcal{P}}$ with domain $\mathcal{D}_{\mathcal{P}}^{\mathrm{eff}} = U_\varepsilon^\dagger P_\varepsilon \operatorname{dom}(H^{\mathcal{E}})$	107
$H_{\mathrm{sa}}^{\mathcal{P}}$	super-adiabatic corrections, deviation from $H_{\mathrm{eff}}^{\mathcal{P}}$ to $H_{\mathrm{a}}^{\mathcal{P}}$	120
$\mathcal{M}^{\mathcal{P}}$	$\mathcal{M}^{\mathcal{P}}$-term, first super-adiabatic correction	126
$\operatorname{dom}_\alpha^k(T)$	domain of $(\varepsilon^{-\alpha} T)^k$ for $k \in \{1, 2\}$, parameter $\alpha \in (0, 2]$ and operators $T \in \{H^{\mathcal{E}}, H_{\mathrm{a}}^{\mathcal{P}}, H_{\mathrm{eff}}^{\mathcal{P}}\}$, which is equipped with the graph-norm	130

4 Quantum Waveguides

Symbol	Explanation	Page
$\mathcal{T}^\varepsilon$	ε-thin tubular neighbourhood around a smoothly embedded submanifold $(B, g_B) \overset{c}{\hookrightarrow} (\mathbb{R}^{b+f}, \delta^{b+f})$	150
Ψ_ε	diffeomorphism from the ε-independent wave-guide $M \subset \mathsf{N}B$ to the ε-thin tube $\mathcal{T}^\varepsilon \subset \mathbb{R}^{b+f}$	150
G^ε	pullback of the Riemannian metric $\varepsilon^{-2}\delta^{b+f}$ to M via Ψ_ε, admissible perturbation of g^ε	158, 163
$\mathcal{A}_\varepsilon$	pullback of $\mathcal{A} \in C^\infty_{\mathrm{b}}(\mathsf{T}^*\mathbb{R}^{b+f}\big\|_{\mathcal{T}^\varepsilon}) \otimes \mathbb{C}^{N\times N}_{\mathrm{Herm}}$ to M via Ψ_ε, admissible perturbation of $\pi_M^*\mathcal{A}_B$ with $\mathcal{A}_B = c^*\mathcal{A} \in C^\infty_{\mathrm{b}}(\mathsf{T}^*B) \otimes \mathbb{C}^{N\times N}_{\mathrm{Herm}}$	169
$\widehat{\Psi}_\varepsilon$	unitary operator from $L^2(\mathcal{T}^\varepsilon \times \mathbb{C}^N, \mathrm{vol}_{\delta^{b+f}})$ to $L^2(M \times \mathbb{C}^N, \mathrm{vol}_{G^\varepsilon})$ induced by Ψ_ε	146
$\widehat{U}_{\rho_\varepsilon}$	unitary operator from $L^2(M \times \mathbb{C}^N, \mathrm{vol}_{G^\varepsilon})$ to $\mathcal{H} = L^2(M \times \mathbb{C}^N, \mathrm{vol}_g)$ induced by the Radon-Nikodym density $\rho_\varepsilon = \mathrm{vol}_{G^\varepsilon}/\mathrm{vol}_{g^\varepsilon}$	173
V_{ρ_ε}	geometric potential induced by ρ_ε	54, 176
$H^{\mathcal{E},\mathrm{w}}$	Schrödinger operator that is associated with weak gauge fields	175, 177
$H^{\mathcal{F},\varepsilon,\mathrm{w}}$	magnetic vertical operator within $H^{\mathcal{E},\mathrm{w}}$, $\mathcal{O}(\varepsilon^2)$-perturbation of $H^{\mathcal{F},\varepsilon,\mathrm{w}}_{\mathcal{A}_\varepsilon=0}$	179
$H^{\mathcal{E},\mathrm{s}}$	Schrödinger operator that is associated with strong gauge fields	205
$H^{\mathcal{F},\varepsilon,\mathrm{s}}$	magnetic vertical operator within $H^{\mathcal{E},\mathrm{s}}$, $\mathcal{O}(\varepsilon)$-perturbation of $H^{\mathcal{F},\varepsilon=0,\mathrm{s}}$	205

λ_0^{m}	magnetic ground state band of the vertical operator $H^{\mathcal{F},\varepsilon,\mathrm{w}}$ respectively $H^{\mathcal{F},\varepsilon,\mathrm{s}}$, which comes along with a ground state ϕ_0^{m}, a spectral projection P_0^{m}, an eigenspace bundle $\mathcal{P}^{\mathrm{m}}$, etc.	179, 209

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