Strongly constrained maximal subgroups and Sims order bounds for finite almost simple linear and unitary groups

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Introduction

The main topics of the present thesis are the strongly constrained maximal subgroups of the finite almost simple linear and unitary groups, i.e. maximal subgroups whose generalized Fitting subgroup coincides with the largest normal p-subgroup for a prime p (see Definition 1.4.11). The motivation to consider this particular class of local maximal subgroups arises by the Sims conjecture, a theorem of Wielandt and the O'Nan-Scott theorem: In the middle of the 1960s, Sims conjectured that for a finite primitive permutation group G the order of a point stabilizer is bounded by a function f in terms of an arbitrary non-trivial subdegree d of G (see [Th, p. 135] or [CPSS, Theorem 1]). A function f which satisfies the conditions of the conjecture of Sims (for a collection $\mathscr{G}$ of finite primitive permutation groups) is called a *Sims order bound* (for $\mathscr{G}$). By a theorem of Wielandt (see [Kn, Theorems 2.1 and 4.2] and [Wie2, Theorem 6.7]), which is of more general nature, one can establish an explicit Sims order bound for the collection consisting of the finite primitive permutation groups whose point stabilizers are not strongly constrained. Furthermore, using the O'Nan-Scott theorem (see e.g. [LPS]), one may show that an explicit Sims order bound f can be determined if an explicit Sims order bound for the collection consisting of the finite almost simple primitive permutation groups is known (see [CPSS, Section 1]). So, to obtain an explicit Sims order bound f it is sufficient to investigate the case of finite almost simple primitive permutation groups which have a strongly constrained point stabilizer.
In the present thesis we consider finite almost simple groups with linear or unitary socle. We achieve the following two goals: First, we determine the pairs (G, M) where G is a finite almost simple linear or unitary group and M a strongly constrained maximal subgroup of G. In particular, we classify all strongly constrained maximal subgroups of the finite almost simple linear and unitary groups. Second, using this classification, we determine an explicit Sims order bound for the collection consisting of the finite almost simple primitive permutation groups whose socle is isomorphic to a projective special linear or unitary group and which have a strongly constrained point stabilizer. On the basis of the two intended goals, this thesis is divided into three chapters: After the preparation of basic facts and lemmas in Chapter 1, Chapter 2 is dedicated to the first intended goal and Chapter 3 to the second.

For the first intended goal it is important to know the maximal subgroups of the finite almost simple linear and unitary groups. In the fundamental paper of Aschbacher [As], published in 1984, a theorem is proved about the subgroup structure of the finite almost simple classical groups (with a certain exception in the orthogonal case). By this theorem, a maximal subgroup M of a finite almost simple classical group G belongs to one of eight collections of (geometrically defined) subgroups of G, which we denote by $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of G and call *Aschbacher classes* (or in short *A-classes*) of G, or is an almost simple subgroup fulfilling some specific conditions. In particular, a member of an A-class of G is

a candidate for a maximal subgroup of G. In the year 1990, the book [KL] of Kleidman and Liebeck was published where the authors studied the A-classes of G (with two exceptions in the symplectic and orthogonal case). The group theoretic structures of and the conjugacy amongst the members of the A-classes of G were determined, carrying on the work of Aschbacher. For this, Kleidman and Liebeck introduced a further division on the A-classes of G into subcollections which they denoted as *types*. Furthermore, for the finite almost simple classical groups G associated to vector spaces of dimension at least 13 the authors determined the exact conditions under which a member of an A-class of G is a maximal subgroup of G.
There is a long history about the study and classification of the maximal subgroups of the finite (almost) simple classical groups associated to vector spaces of low dimension. The most complete work is Kleidman's Ph.D. thesis in 1987 where according to reports a classification in the case of finite simple classical groups associated to vector spaces of dimension at most 12 ("simple classical groups of low dimension") without proof was presented. But, these results were not publicly available before the appearance of the book [BHR], by Bray, Holt and Roney-Dougal in 2013, considering the case of almost simple classical groups of low dimension. Many sources in the literature refer to a book by Kleidman on this topic announced to appear in the Longman Research Notes Series which, however, never appeared in print (for a short history of these facts see [BHR, p. viii-ix]). Unfortunately, the author had spent a considerable amount of time in the study of unitary groups before the appearance of [BHR] which made several of his previously obtained results obsolete. Now, the present thesis is based in an essential way on the results contained in both books [KL] and [BHR].
For the intended classification we have to analyze the members of the A-classes of the finite almost simple linear or unitary group G which are maximal subgroups of G for the condition of strong constraint. In particular, we have to use and work with the facts provided in the paper [As] and the books [KL] and [BHR] frequently. (Note: By the condition of strong constraint, we do not have to provide the full information contained in the three works. Furthermore, despite the fact that it is claimed in [KL, Corollary 1.2.4] that the local maximal subgroups of the finite classical groups are known, a huge amount of work is necessary to determine the class of specific local maximal subgroups we are interested in.) Unfortunately, some terminology and notation differ in [As], [KL] and [BHR] and some are also not appropriate for our purposes. Based on these works, we provide the terminology and notation which is more useful for our research. A reason to investigate those finite almost simple classical groups which are linear or unitary groups in this thesis is that in these two cases facts may be stated and assertions may be shown by analogous considerations and arguments. To work simultaneously in these two cases, we also introduce in Subsection 1.2.3 a specific notation, based on [KL], which we call *standard notation*.

In Chapter 2, we determine for a finite almost simple linear or unitary group G precise conditions under which a member of an A-class of G is a strongly constrained maximal subgroup of G. The division of the chapter is chosen, such that the members of A-class $\mathscr{C}_j$ are considered in Section $2.j$. The results are presented in main theorems where we usually consider the linear and unitary case separately for the sake of clearness. In particular, we present the results of these main theorems not using the standard notation. Sometimes, we also drop the condition of maximality and more generally determine precise conditions under which a member of an A-class is strongly constrained. Several mathematical disciplines are used to achieve the intended classification, such as finite (local) group theory or elementary number theory. E.g., in particular cases the condition of strong constraint leads to some interesting problems in elementary number theory displaying a nice interaction of finite group theory and number theory. As its name suggests, the condition of strong constraint is a strong property and has a significant effect on the structure of a group. There may arise various necessary conditions from the assumption that a member M of an A-class of G is a strongly constrained maximal subgroup of G. As an overview, in the following list we collect several of these conditions. For this, let n denote the dimension of the vector space associated to the action of the finite almost simple linear (or unitary) group G and q (or q^2) the order of the associated ground field.

There is/are

- conditions on the type of M, such as in Corollary 2.2.13.
- conditions on the position of G in the automorphism group of its socle, such as in Main Theorem 2.1.24.
- the condition that n or q is small and fixed, or fulfills certain other conditions, such as in Main Theorems 2.1.10, 2.2.26 or 2.5.25.
- no further condition, such as in Main Theorem 2.1.25.
- elementary number theoretic conditions with respect to both n and q, such as in Main Theorems 2.2.21 or 2.3.17.

Furthermore, there are cases where no member of an A-class $\mathscr{C}_j$ fulfills the demanded conditions of maximality and strong constraint, such as in Main Theorems 2.4.5 or 2.7.2.
In the following table, we list the mentioned main theorems of Chapter 2 with respect to the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ (see the definitions in Chapter 2).

A-class	Main Theorem(s) linear case	Main Theorem(s) unitary case	Notes
$\mathscr{C}_1$	2.1.9 and 2.1.24	2.1.10 and 2.1.25	
$\mathscr{C}_2$	2.2.21 and 2.2.25	2.2.22 and 2.2.26	see also Corollary 2.2.13
$\mathscr{C}_3$	2.3.16	2.3.17	
$\mathscr{C}_4$	2.4.5	2.4.5	
$\mathscr{C}_5$	2.5.24	2.5.25	
$\mathscr{C}_6$	2.6.28, 2.6.31 and 2.6.40	2.6.28 and 2.6.40	without demanding maximality of M in G
$\mathscr{C}_7$	2.7.2	2.7.2	
$\mathscr{C}_8$	2.8.11		A-class $\mathscr{C}_8$ is empty in the unitary case

Let G be a primitive permutation group on a finite set X and G_α the stabilizer in G of a point $\alpha \in$ X. An important requirement of the Sims conjecture, stated at the beginning of the introduction, is that the order of G_α is bounded by a function in terms of an *arbitrary* non-trivial subdegree of G: An explicit order bound for G_α in terms of the maximal subdegree of G may be determined by comparative elementary considerations, see [Kn6]. After the conjecture of Sims was formulated, much effort was invested to prove the conjecture. In [Wie2, Theorem 6.7], Wielandt showed that there is a prime p such that $|G_\alpha/\mathrm{O}_p(G_\alpha)|$ divides $d!((d-1)!)^d =: \mathrm{wdt}(d)$ where d denotes an arbitrary non-trivial subdegree of G. The previously defined function $\mathrm{wdt}(d)$ we call the *Wielandt order bound*, because of its importance. Furthermore, it may be shown that the Wielandt order bound $\mathrm{wdt}(d)$ is a Sims order bound for the collection consisting of the primitive permutation groups G for which the stabilizer of a point G_α is not strongly constrained, see [Kn, Proposition 4.1 and Theorem 4.2]. Several authors have obtained extensive partial results concerning the Sims conjecture, such as Thompson (see [Th]), Wielandt (see e.g. [Wie2] or [Kn, Theorem 4.2]) and Knapp (see e.g. [Kn] or [Kn5]). But, the full conjecture remained open until 1983. In this year, the paper [CPSS] of Cameron, Praeger, Saxl and Seitz was published where the authors proved the Sims conjecture completely. For the proof the authors used the classification of finite simple groups (which was recently announced at that time), the O'Nan-Scott theorem on the structure of primitive permutation groups, observations of Thompson and Wielandt and Lie theory. In particular, the four authors showed that an explicit Sims order bound may be determined if an explicit Sims order bound for the collection $\mathscr{H}$ consisting of the finite almost simple primitive permutation groups is known. But, in the paper [CPSS], the authors focused on proving the existence of a Sims order bound for $\mathscr{H}$ rather than on providing an explicit function. Therefore, it is a worthy endeavor to determine an explicit Sims order bound for $\mathscr{H}$.

Chapter 3 is dedicated to the determination of an explicit Sims order bound $h(d)$, in Main Theorem 3.1.19, for the collection consisting of the finite primitive permutation groups G which are almost simple linear or unitary groups and which have a strongly constrained point stabilizer G_α. For this, we use intensively the facts and results provided and obtained in Chapter 2. In particular, we use the method of Aschbacher, by considering separately the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$: We divide the problem of determining the order bound $h(d)$ into the partial problems of determining Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ (or $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$) for the collections $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ (or $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$) consisting of the finite primitive permutation groups G which are almost simple linear (or unitary) groups and which have a strongly constrained point stabilizer G_α belonging to A-class $\mathscr{C}_j$ of G. (Here, we note that w.l.o.g. we assume in the case of unitary groups that the dimension of the associated vector space is at least 3.)
Recalling that the Wielandt order bound $\mathrm{wdt}(d)$ is a Sims order bound for the collection consisting of the primitive permutation groups for which the stabilizer of a point is not strongly constrained, it is an reasonable goal to determine Sims

order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$ which are not greater than $\mathrm{wdt}(d)$. But, we do not restrict to show that $\mathrm{wdt}(d)$ bounds the order of a point stabilizer in the considered cases. Our aim is to determine for each A-class $\mathscr{C}_j$ more appropriate Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$. There is an enormous amount of cases of pairs (G, G_α) to consider ($G \in \mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j} \cup \mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$) and it is hard to determine the (possible) minimal non-trivial subdegree of G. Therefore, within the scope of this thesis one should not expect to determine sharp Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$. By using various results of finite permutation group theory (esp. observations and techniques by Wielandt), finite group theory, elementary number theory, linear algebra and the facts of Chapter 2, we determine Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$ which are smaller than $\mathrm{wdt}(d)$. Moreover, for large d these order bounds are considerably smaller than $\mathrm{wdt}(d)$; for small d we have to consider the possible cases which may occur to obtain our intended goal of order bounds not greater than $\mathrm{wdt}(d)$.
The following table describes roughly the shape of the determined Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$. The entry 0 is placed if the corresponding collection $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ or $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$ is empty.

A-class	$h^{\mathbf{L}}_{\mathscr{C}_j}(d)$	$h^{\mathbf{U}}_{\mathscr{C}_j}(d)$	Associated theorem(s)
$\mathscr{C}_1$	$e^{\mathcal{O}(\ln(d)^2)}$	$e^{\mathcal{O}(\ln(d)^2)}$	3.3.5 and 3.3.6
$\mathscr{C}_2$	$e^{\mathcal{O}(d\ln(d))}$	$e^{\mathcal{O}(d\ln(d))}$	3.4.6 and 3.4.7
$\mathscr{C}_3$	$\mathcal{O}(d\ln(d))$	$\mathcal{O}(d\ln(d))$	3.5.4 and 3.5.5
$\mathscr{C}_4$	0	0	3.2.1
$\mathscr{C}_5$	small constant integer	small constant integer	3.2.3 and 3.2.5
$\mathscr{C}_6$	$e^{\mathcal{O}(\ln(d)^2)}$	$e^{\mathcal{O}(\ln(d)^2)}$	3.6.6
$\mathscr{C}_7$	0	0	3.2.1
$\mathscr{C}_8$	small constant integer	0	3.2.6 and 3.2.1

As a direct conclusion of Main Theorem 3.1.19 and the observations of Wielandt, we may state in Main Theorem 3.1.20 that the Wielandt order bound $\mathrm{wdt}(d)$ is a Sims order bound for the collection consisting of all finite almost simple primitive permutation groups whose socle is isomorphic to a projective special linear or unitary group. This yields a partial answer to the question of an explicit Sims order bound for $\mathscr{H}$.

Finally, as an overview, we give a brief summary of each chapter.

In Chapter 1, we recall basic terminology, notation and facts and we provide lemmas for later use. Especially, we recall the needed terminology and notation of the finite classical groups for our later investigations. Differences in terminology and notation in [As], [KL] and [BHR] and also between these works and this thesis are pointed out, as well as the consequences arising from them. Furthermore, in Subsection 1.2.3, we introduce the so called *standard notation*. We note that in this thesis we also pay attention to the problem of well-definedness in the case of the finite unitary groups, as described in the paper [BHR2].

Chapter 2 is dedicated to the determination of the pairs (G, M) where G is a finite almost simple linear or unitary group and M a strongly constrained maximal subgroup of G. At the beginning of Section $2.j$, we give an exact definition of A-class $\mathscr{C}_j$ as it is appropriate for our purposes. These definitions we base on the definitions in [As], [KL] and [BHR]. As we take advantage of these three works, we indicate the differences between the definitions in [As], [KL] and [BHR], as well as the differences between the definitions in these works and this thesis. The needed facts of [As], [KL] and [BHR] we provide properly. We read off the facts from tables provided in [KL] and [BHR], give further notes and sometimes provide alternative references. In Section $2.j$, we give main theorems where the intended pairs (G, M) are determined with respect to an A-class $\mathscr{C}_j$. Often, we determine also additional facts in these sections, such as the determination of the (only non-trivial) largest normal p-subgroup $\mathrm{O}_p(M)$ of the strongly constrained maximal subgroup M of G (p the appropriate prime), or the determination of the centralizer of $\mathrm{O}_p(M)$ in G. We note that we also correct some mistakes in [As], [KL] and [BHR], as well as in other references, such as a wrong definition of A-class $\mathscr{C}_5$ in [BHR] (see Remark 2.5.3 (a)).

In Chapter 3, we determine an explicit Sims order bound $h(d)$ for the collection consisting of the finite primitive permutation groups which are almost simple linear or unitary groups and which have a strongly constrained point stabilizer. In Section 3.1, we first provide a detailed historical overview about the Sims conjecture and the background of the intended goal of this chapter. We recall previously obtained results of other authors about the Sims conjecture which are useful for our later investigations. Then, in Main Theorem 3.1.19, we state the intended order bound $h(d)$. Using this result, we deduce a Sims order bound for the collection consisting of the finite primitive permutation groups which are almost simple linear or unitary groups, in Main Theorem 3.1.20. Main Theorem 3.1.19 is stated with respect to Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$. The Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$ are determined in the following sections of this chapter. Further possibilities to get sharper results or more elementary proofs for less precise results are provided in several cases.

Chapter 1

Basic Tools

In this chapter, we recall basic terminology, notation and facts for our later investigations and we provide lemmas for later use.

1.1 Basic notation and some basic terminology

We start by introducing some general notation used in this thesis. If more specialized notation is needed we will provide it at the time it arises, such as for the layer of a finite group or Zsigmondy primes (see Definitions 1.4.10 and 1.5.1).
The greatest common divisor and the lowest common multiple of the positive integers $a_1, \ldots, a_n$ we denote by $\gcd(a_1, \ldots, a_n)$ and $\mathrm{lcm}(a_1, \ldots, a_n)$, or we simple write $(a_1, \ldots, a_n)$ and $[a_1, \ldots, a_n]$ as abbreviations. For a set π of prime numbers let π' be the set of primes complementary to π. If r is a real number the largest integer $\leq r$ is denoted by $\lfloor r \rfloor$ and the smallest integer $\geq r$ is denoted by $\lceil r \rceil$. As usual, we write e for Euler's number. For two positive real numbers r and b where $b \neq 1$ we denote by $\log_b(r)$ the logarithm of r to base b, and we write $\ln(r)$ for the *natural logarithm* of r, i.e. the logarithm of r to base e.
We recall the basic facts about finite fields, such as a finite field has always prime power order, that the multiplicative group (consisting of all non-zero elements) is cyclic and that the automorphism group is cyclic, generated by the Frobenius automorphism (see e.g. [BHR, p. 11-12] or [Wil2, p. 42-43]). Up to isomorphism, there is only one finite field of prime power order q, and by $\mathrm{GF}(q)$ we denote a finite field of order q. The multiplicative group of $\mathrm{GF}(q)$ we denote by $\mathrm{GF}(q)^*$, and for the characteristic of $\mathrm{GF}(q)$ we write $\mathrm{char}(\mathrm{GF}(q))$. If the characteristic of $\mathrm{GF}(q)$ is the prime p then we denote the *Frobenius* automorphism of $\mathrm{GF}(q)$ by φ_p, i.e. $\varphi_p : \mathrm{GF}(q) \to \mathrm{GF}(q), x \mapsto x^p$.
For positive integers n, m we define the following. By $\mathrm{Mat}_{m,n}(q)$, we denote the set of all $m \times n$-matrices $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} = (a_{ij})_{m \times n}$ with entries in the finite field $\mathrm{GF}(q)$ and we set $\mathrm{Mat}_n(q) = \mathrm{Mat}_{m,n}(q)$ if $n = m$. For the rank of a matrix $M \in \mathrm{Mat}_{m,n}(q)$ we write $\mathrm{rk}(M)$. The symbol $\mathbb{1}_n$ denotes the $n \times n$ identity matrix, and $0_{m,n}$ denotes the $m \times n$ matrix with zero in every entry. Of course,

we will only use the last notation if we want to describe the structure of a matrix more clearly and otherwise drop the subscript or omit writing the zero entries of a matrix. By $\mathrm{E}_{i,j}^{m,n}$, we denote the matrix in $\mathrm{Mat}_{m,n}(q)$ with 1 in position (i,j) and 0 elsewhere. For $\alpha_1,\dots,\alpha_n \in \mathrm{GF}(q)$ we write $\mathrm{diag}(\alpha_1,\dots,\alpha_n)$ for the diagonal matrix $(a_{ij})_{n\times n}$ where $a_{jj}=\alpha_j$ for $j\in\{1,\dots,n\}$ and $a_{ij}=0$ otherwise, and $\mathrm{antidiag}(\alpha_1,\dots,\alpha_n)$ for the anti-diagonal matrix $(a_{ij})_{n\times n}$ where $a_{ij}=\alpha_j$ if $j=n-i+1$ for $i\in\{1,\dots,n\}$ and $a_{ij}=0$ otherwise. We will also permit descriptions of matrices including matrices, such as $D_1=\begin{pmatrix}\lambda & 0_{1,n-1}\\ 0_{n-1,1} & \mathbb{1}_{n-1}\end{pmatrix}\in$ $\mathrm{Mat}_n(q)$ or $D_2=\begin{pmatrix}\lambda & 0_{1,n-1}\end{pmatrix}\in\mathrm{Mat}_{1,n}(q)$ for $\lambda\in\mathrm{GF}(q)$. Therefore, we generalize the definition of $\mathbb{1}$ and 0 naturally for having subscripts which are non-negative integers, such as $D_1=D_2=(\lambda)\in\mathrm{Mat}_1(q)$ for $n=1$. Furthermore, we generalize the notation for the diagonal and anti-diagonal matrices by writing $\mathrm{diag}(A_1,\dots,A_k)$ and $\mathrm{antidiag}(A_1,\dots,A_k)$ for the respective matrices

$$\begin{pmatrix} A_1 & 0_{n_1,m} & 0_{n_1,n_k}\\ 0_{m,n_1} & \ddots & 0_{m,n_k}\\ 0_{n_k,n_1} & 0_{n_k,m} & \mathrm{A}_k\end{pmatrix},\ \begin{pmatrix} 0_{n_k,n_1} & 0_{n_k,m} & \mathrm{A}_k\\ 0_{m,n_1} & \iddots & 0_{m,n_k}\\ A_1 & 0_{n_1,m} & 0_{n_1,n_k}\end{pmatrix}\in\mathrm{Mat}_n(q)$$

where $A_i\in\mathrm{Mat}_{n_i}(q)$ for $i\in\{1,\dots,k\}$ with $n=\sum_{i=1}^k n_i$ and $m=n-n_1-n_k$. (For consistence we define $\mathrm{Mat}_0(q)$ to be the empty set if A_i is chosen to be $\mathbb{1}_0$). The transpose of a matrix A is denoted by A^t. By δ_{ij}, we denote the *Kronecker delta*, i.e. $\delta_{jj}=1$ for all j and $\delta_{ij}=0$ otherwise.

For a vector space V over a field K we denote by $\dim_K(V)$ the dimension of V, and we will drop the subscript K and also write $\dim(V)$ if the role of K is clear by the situation. By V^*, we denote the dual space of V. For subspaces $V_1,V_2\le V$ we write $V=V_1\oplus V_2$ if and only if $V=V_1+V_2$ and $V_1\cap V_2=\{0\}$. We declare that linear maps act on the right in this thesis (if nothing else is assumed). So, with respect to a fixed ordered basis, we can identify the $\mathrm{GF}(q)$-linear maps from an n-dimensional vector space over $\mathrm{GF}(q)$ to an m-dimensional vector space over $\mathrm{GF}(q)$ with the matrices from $\mathrm{Mat}_{n,m}(q)$, and we have an action of matrices on row vectors by right multiplication.

Let N and M be sets. If N is a subset of M we write $N\subseteq M$, and we write $N\subset M$ if the inclusion is proper. For a group G and for a subset $H\subseteq G$ we write $\langle H\rangle$ for the subgroup of G generated by H, and we write $H\le G$ if H is a subgroup of G and $H<G$ if H is a proper subgroup of G. If H is a maximal, normal, subnormal, or characteristic subgroup of G we write $H\lessdot G$, $H\trianglelefteq G$, $H\trianglelefteq\trianglelefteq G$, or H char G, respectively. By $G\cong H$, we denote that two groups G and H are *isomorphic* (i.e. there is a group isomorphism from G to H); and if a group K is isomorphic to a subgroup of G we write $K\lesssim G$. By $\mathrm{Z}(G)$, we denote the centre of G, and for a subset H of G we write $\mathrm{C}_G(H)$ and $\mathrm{N}_G(H)$ for the centralizer and the normalizer of H in G, respectively. If $x,y\in G$ we write $[x,y]=x^{-1}y^{-1}xy$ for the *commutator* of x with y. For two subsets M_1 and M_2 of G we denote by $[M_1,M_2]$ the subgroup of G generated by the commutators $[m_1,m_2]$ for $m_i\in M_i$. The *commutator* (or *derived*) subgroup $[G,G]$ of G is denoted by G' or $G^{(1)}$. Recursively, we define $G^{(n)}=[G^{(n-1)},G^{(n-1)}]$ for a positive

integer n where $G^{(0)} = G$. The descending normal series $G \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \dots$ is called the *derived series* of G, and the last term of the derived series of G is denoted by G^{∞}.
Let p be a prime and let for the following G be a finite group. We denote by $\mathrm{O}_p(G)$ the *p-core* of G, i.e. the largest normal p-subgroup of G. For the smallest normal subgroup of G such that its quotient group is a p-group we write $\mathrm{O}^p(G)$. Evidently, the last two introduced subgroups of G are characteristic subgroups of G (see e.g. [DH, A. (8.5) Definition]). By $\mathrm{Syl}_p(G)$, we denote the set of all Sylow p-subgroups of G. For an element $g \in G$ we write $\mathrm{o}(g)$ for the order of g. The least common multiple of the orders of the elements of G is called the *exponent* of G, and denoted by $\exp(G)$. (Note, that the exponent of G is equal to the maximal order of all elements in G if G is a p-group). We will write $\Phi(G)$ for the *Frattini subgroup* of G, i.e. the intersection of all maximal subgroups of G where we set $\Phi(G) = 1$ for $G = 1$. By $\mathrm{soc}(G)$, we denote the *socle* of G, i.e. the subgroup of G generated by its minimal normal subgroups. If G is a p-group we define the characteristic subgroups $\Omega_j(G) = \langle g \mid g \in G, g^{p^j} = 1\rangle$ of G for positive integers j.
For the automorphism group of G we write $\mathrm{Aut}(G)$, and for the group of inner automorphisms of G we write $\mathrm{Inn}(G)$. Let g, h be elements of G. Then the conjugation of the element g by h is denoted by $g^h = h^{-1}gh$. By the homomorphism $G \to \mathrm{Inn}(G), h \mapsto \tilde{h}$ where $\tilde{h}$ denotes the conjugation map by h on G, we have $\mathrm{Inn}(G) \cong G/\mathrm{Z}(G)$. Since there is no danger of confusion, we will drop the symbol $\tilde{}$ and only write $h \in \mathrm{Inn}(G)$ instead of $\tilde{h}$ for an element $h \in G$. Moreover, in the case that G is *simple* (i.e. every normal subgroup of G is trivial) and non-abelian we have $G \cong \mathrm{Inn}(G)$, and we will identify these two groups when working with inner automorphisms of G. Clearly, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, and the factor group $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G.
The notation for describing group structures is based on the Atlas notation [At]. We write $G \times H$ for the direct product of the groups G and H. A split extension of G by H is denoted by $G : H$, or $G \rtimes H$. For a non-split extension of G by H we write $G{}^{\cdot}H$ and we write $G.H$ when it is not known or we do not want to specify whether the extension splits. We write $G \wr H$ for the wreath product of a group G by a permutation group H. A cyclic group of order m we denote by $\mathbf{Z}_m$, or as an abbreviation by m (e.g. when we describe group structures). Also as an abbreviation, an elementary abelian group of order p^m (p a prime) we denote by p^m when we describe group structures. For an elementary abelian group A we denote by A^{m+n} a group with an elementary abelian normal subgroup A^m such that the quotient group is isomorphic to A^n. We will write $[m]$ for a group of order m of unspecified structure.
For a finite set Ω we denote by $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$ the symmetric and alternating groups on the set Ω. If $\Omega = \{1, \dots, n\}$ we also write S_n and A_n for the symmetric and alternating groups of degree n.

Now, we introduce some basic group theoretic terminology for our thesis.

Definition 1.1.1. Let G be a finite group. We call G *almost simple* if $\operatorname{soc}(G)$ is a non-abelian simple group.

We provide an equivalent definition for this terminology which we will also use in this thesis.

Proposition 1.1.2. *Let G be a finite group and let S denote the socle of G. Then the following is equivalent.*

(i) G is almost simple.

(ii) There is a group $\tilde{G}$ with $G \cong \tilde{G}$ and $X \leq \tilde{G} \leq \operatorname{Aut}(X)$ for a non-abelian simple group X.

In assertion (ii) we have that $\operatorname{soc}(\tilde{G}) = X$, and hence we will identify the groups G and $\tilde{G}$, as well as S and X.

Proof. Let G be an almost simple group. We have that $\mathrm{C}_G(S)$ is a normal subgroup of G, and so $[S, \mathrm{C}_G(S)] \leq S \cap \mathrm{C}_G(S)$. Hence, we easily obtain that $\mathrm{C}_G(S) = 1$, since S is the unique minimal normal subgroup of G and also non-abelian. So, the implication (i) to (ii) easily follows. To complete the prove, assume that (ii) holds. We only have to show that $X = \operatorname{soc}(\tilde{G})$. Obviously, X is a minimal normal subgroup of $\tilde{G}$. Suppose that M is a minimal normal subgroup of $\tilde{G}$ with $M \neq X$. Since M and X intersect trivial, we obtain that M and X centralize each other, which contradicts the fact that $\tilde{G}$ acts faithfully on X. □

Remark. The terminology of an almost simple group, provided in part (ii) of the last proposition, coincides with that used in the book of Kleidman and Liebeck (cf. [KL, p. 1]) and the book of Bray, Holt and Roney-Dougal (cf. [BHR, p. 1]).

For an almost simple group G and a maximal subgroup M of G, it is useful to consider and analyze the intersection of M with the socle of G when observing M. Obviously, we have that $\operatorname{soc}(G) \cap M$ is a normal subgroup of M, and the following proposition provides that this intersection is always non-trivial.

Proposition 1.1.3. *For a finite almost simple group G and a maximal subgroup M of G we have that $\operatorname{soc}(G) \cap M > 1$.*

Proof. See [BHR, Theorem 1.3.6], [LPS, p. 395-396] or [Wil, proof of Lemma 2.1.]. □

Remark. We note that Schreier's conjecture (see e.g. [BHR, Theorem 1.3.2]), which is a direct consequence of the classification of finite simple groups, is used in the proof of the last proposition.

Next, we introduce a terminology related to the intersection of a maximal subgroup of an almost simple group G with the socle of G.

Definition 1.1.4. Let M be a maximal subgroup of an almost simple group G. We call M a *novelty* (of G) if $\mathrm{soc}(G) \cap M$ is not a maximal subgroup of $\mathrm{soc}(G)$.

Remark 1.1.5. (a) For the definition of a novelty we keep to the book [BHR, see Definition 1.3.8]. So, in this thesis novelties of a group G are always maximal subgroups of G. We note that the definition of a G-novelty in the book of Kleidman and Liebeck [KL, see p. 66] is different. There, a G-novelty has not to be a maximal subgroup of G, such as in the first row of [KL, Table 3.5.H].

(b) For a more detailed description of the occurrence of novelties (which can be separated into two types), see [BHR, p. 9-11].

(c) For typical examples of novelties, see Proposition 2.1.15 and Remark 2.1.16, below.

Finally, the terminology of a quasisimple group is introduced.

Definition 1.1.6. A group Q is called *quasisimple* if Q is perfect (i.e. $Q = Q'$) and $Q/\mathrm{Z}(Q)$ is a (non-abelian) simple group.

1.2 The finite classical groups

In this section, we recall the construction of the finite classical groups and provide elementary properties concerning them. We also introduce the finite classical simple groups, where two cases, denoted by $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$, will be of fundamental importance in this thesis. Then we will describe the automorphism groups of $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$ by introducing generators for them. Finally, we introduce the (associated) standard notation which we use in this thesis.
The books of Kleidman and Liebeck [KL] and Bray, Holt and Roney-Dougal [BHR], as well as the paper of Aschbacher [As], are main sources for our research. There, the maximal subgroups of the finite almost simple classical groups are determined (cf. the description at the beginning of Chapter 2). Unfortunately, some terminology and notation differ in these three works and for our purposes they are sometimes not (quite) appropriate. Hence, we will provide the needed terminology and notation, as it is advantageous for our research. But we will stay close to these three works. Differences between the three works and also between these works and our thesis will be pointed out, as well as the consequences arising from them.

1.2.1 Definition and properties of the finite classical groups

We start by providing the basic definitions, terminology, notation and properties concerning the finite classical groups, needed for our research. We keep our approach mainly to the book of Kleidman and Liebeck (see [KL, p. 9-47]) and the paper of Aschbacher (see [As, p. 470-472]), and in case of the definition of the classical groups we proceed as Kleidman and Liebeck.

General linear group, special linear group and general semilinear group of V

Let V denote a vector space of finite dimension $n > 0$ over the finite field $\mathrm{GF}(q)$ of prime power order q. For the *general linear group* of V (the group of all non-singular $\mathrm{GF}(q)$-linear transformations of V), we write $\mathrm{GL}(V)$, or $\mathrm{GL}(V, \mathrm{GF}(q))$ to emphasize the role of $\mathrm{GF}(q)$. The *special linear group* of V (i.e. the commutator group of $\mathrm{GL}(V)$ if $(n, q) \neq (2, 2)$ and $\mathrm{GL}(V)$ itself if $(n, q) = (2, 2)$, or equivalently the subgroup of $\mathrm{GL}(V)$ consisting of the elements with determinant 1) is denoted by $\mathrm{SL}(V)$.
We call a map g from V to V a *semilinear transformation* of V, if g preserves addition (i.e. $(v + w)g = vg + wg$ for all $v, w \in V$) and $(\lambda v)g = \lambda^{\sigma(g)}(vg)$ holds for all $v \in V$, $\lambda \in \mathrm{GF}(q)$ and an automorphism $\sigma(g)$ of $\mathrm{GF}(q)$ depending on g but not on λ and v. By $\Gamma\mathrm{L}(V)$ we denote the *general semilinear group* of V, which is the group consisting of all non-singular semilinear transformations g of V (i.e. g is a semilinear transformation of V and $\{v \in V \mid vg = 0\} = \{0\}$). We have a canonical surjective homomorphism

$$\sigma : \Gamma\mathrm{L}(V) \to \mathrm{Aut}(\mathrm{GF}(q)), \quad g \mapsto \sigma(g) \tag{1.2.1}$$

with kernel $\mathrm{GL}(V)$. Hence, $\mathrm{GL}(V)$ and also $\mathrm{SL}(V)$ are normal subgroups of $\Gamma\mathrm{L}(V)$.
Let $B = (b_1, \ldots, b_n)$ be an ordered basis of V. Each element g of $\mathrm{GL}(V)$ is determined by its action on B. So, we can identify g (with respect to B) with the $n \times n$ matrix $(g_{ij})_{1 \leq i,j \leq n}$ over $\mathrm{GF}(q)$ where $b_i g = \sum_{j=1}^{n} g_{ij} b_j$ (recall that linear maps act on the right). Hence, we obtain a canonical isomorphism from $\mathrm{GL}(V)$ to $\mathrm{GL}_n(q)$ where $\mathrm{GL}_n(q)$ denotes the group of all non-singular $n \times n$ matrices over the finite field $\mathrm{GF}(q)$. Analogously, we have that each element of $\Gamma\mathrm{L}(V)$ is determined by its action on B together with $\sigma(g)$, see (1.2.10) below. By $\Gamma\mathrm{L}_n(q)$, we denote $\Gamma\mathrm{L}(V)$ considered with respect to the basis B (and $\mathrm{Aut}(\mathrm{GF}(q))$). For the ordered basis B of V and an automorphism φ of $\mathrm{GF}(q)$ we can define $\overline{\varphi} : V \to V$ by $(\sum_{j=1}^{n} \lambda_j b_j)\overline{\varphi} = \sum_{j=1}^{n} \lambda_j^{\varphi} b_j$. So, we obtain that $\Gamma\mathrm{L}_n(q)$ is a semidirect product of $H = \{\overline{\varphi} \mid \varphi \in \mathrm{Aut}(\mathrm{GF}(q))\}$ with $\mathrm{GL}_n(q)$.

Forms on V

Here, we introduce the (finite) classical groups where we keep to the book of Kleidman and Liebeck [KL]. Loosely speaking, we obtain them as stabilizers in $\Gamma\mathrm{L}(V), \mathrm{GL}(V)$ and $\mathrm{SL}(V)$ of certain forms on a vector space V. Therefore, we need the following definitions concerning forms on V.

A map $f : V \times V \to \mathrm{GF}(q)$ is called a *left linear form on* $V \times V$, or in short a *left linear form* (*on* V), if it is $\mathrm{GF}(q)$-linear in the first argument, and it is called a *bilinear form* (*on* V) if it is $\mathrm{GF}(q)$-linear in both arguments. For a map $Q : V \to \mathrm{GF}(q)$, we define $f_Q : V \times V \to \mathrm{GF}(q)$ via $f_Q(v, w) = Q(v + w) - Q(v) - Q(w)$ for all $v, w \in V$. Q is called a *quadratic form* (*on* V) if $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in \mathrm{GF}(q)$, $v \in V$ and f_Q is a bilinear form on V. We call f_Q the *associated bilinear form* of Q if Q is a quadratic form. If V is equipped with a left linear form f or a quadratic form Q we will also write (V, f) or (V, Q) (instead of V) to emphasize this additional structure.
We provide some further definitions about forms on V. Let $f : V \times V \to \mathrm{GF}(q)$ be a map. We call f *non-degenerate* if for each $v \in V \backslash \{0\}$ there exist elements $w_1, w_2 \in V$ such that $f(w_1, v) \neq 0$ and $f(v, w_2) \neq 0$. Quadratic forms are called *non-degenerate* if their associated bilinear forms are non-degenerate. We call the map f *symmetric* if $f(v, w) = f(w, v)$ for all $v, w \in V$. f is called *skew-symmetric* if $f(v, w) = -f(w, v)$ for all $v, w \in V$. We note that if the order q of the finite field $\mathrm{GF}(q)$ is even, then the terms symmetric and skew-symmetric coincide. Further, we note that if Q is a quadratic form on V then the associated bilinear form f_Q is symmetric by definition.
Now, we introduce two important kinds of forms. We call f a *symplectic form* (on V) if f is a skew-symmetric bilinear form, and if $f(v, v) = 0$ for all $v \in V$. f is called a *unitary form* (on V) if $\mathrm{GF}(q)$ has a non-trivial involutory field automorphism φ (note that if such an automorphism exists then q is a square), f is a left linear form and $f(v, w) = f(w, v)^\varphi$ holds for all $v, w \in V$. We note that if f is a unitary form then f is additive in the second argument (i.e. $f(v, u + w) = f(v, u) + f(v, w)$ for all $u, v, w \in V$) and we have that $f(v, \lambda w) = \lambda^\varphi f(v, w)$ for all $\lambda \in GF(q)$ and $v, w \in V$.

Isometry groups

To introduce the classical groups, we need some additional terminology about isometry groups. Assume that (V_1, f_1) and (V_2, f_2) (or (V_1, Q_1) and (V_2, Q_2)) are two vector spaces of dimension n over the same finite field $\mathrm{GF}(q)$ equipped with left linear forms f_1 and f_2 (or quadratic forms Q_1 and Q_2). An invertible $\mathrm{GF}(q)$-linear homomorphism g from (V_1, f_1) to (V_2, f_2) (or (V_1, Q_1) to (V_2, Q_2)) is called an *isometry* if we have $f_1(v, w) = f_2(vg, wg)$ (or $Q_1(v) = Q_2(vg)$) for all $v, w \in V$. If there is such an isometry we call (V_1, f_1) and (V_2, f_2) (or (V_1, Q_1) and (V_2, Q_2)) *isometric* and we write $(V_1, f_1) \cong (V_2, f_2)$ (or $(V_1, Q_1) \cong (V_2, Q_2)$). A *similarity* we call an invertible $\mathrm{GF}(q)$-linear homomorphism g from (V_1, f_1) to (V_2, f_2) (or (V_1, Q_1) to (V_2, Q_2)) which satisfies $f_2(vg, wg) = \lambda(g) f_1(v, w)$ (or $Q_2(vg) = \lambda(g) Q_1(v)$) for all $v, w \in V$ and an element $\lambda(g)$ from $\mathrm{GF}(q)^*$ depending on g and not on v and w. If there is such an similarity we call (V_1, f_1) and (V_2, f_2) (or (V_1, Q_1) and (V_2, Q_2)) *similar*.
From now on we consider the case $V = V_1 = V_2$ and $f = f_1 = f_2$ (or $Q = Q_1 =$

Q_2). Here, the set of isometries of the vector space (V, f) (or (V, Q)) forms a subgroup of $\mathrm{GL}(V)$, which we call the *isometry* group of (V, f) (or (V, Q)) and denote by $\mathrm{I}(V, f)$ (or $\mathrm{I}(V, Q)$). By intersecting $\mathrm{I}(V, f)$ (or $\mathrm{I}(V, Q)$) with the special linear group $\mathrm{SL}(V)$, we obtain the group of all *special isometries* of (V, f) (or (V, Q)), which we denote by $\mathrm{S}(V, f)$ (or $\mathrm{S}(V, Q)$). The set of all similarities of the vector space (V, f) (or (V, Q)) forms also a group, which we call the *similarity* group of (V, f) (or (V, Q)) and denote by $\Delta(V, f)$ (or $\Delta(V, Q)$). We call an element g of $\Gamma\mathrm{L}(V)$ a *semisimilarity* of (V, f) (or (V, Q)) if

$$f(vg, wg) = \lambda(g) f(v, w)^{\sigma(g)} \qquad (\text{or } Q(vg) = \lambda(g) Q(v)^{\sigma(g)}) \tag{1.2.2}$$

holds for all $v, w \in V$ and elements $\sigma(g) \in \mathrm{Aut}(\mathrm{GF}(q))$, $\lambda(g) \in \mathrm{GF}(q)^*$ depending on g and not on v and w. By elementary considerations, we see that the set of all semisimilarities of (V, f) (or (V, Q)) forms a group, the *semisimilarity* group of (V, f) (or (V, Q)), which we denote by $\Gamma(V, f)$ (or $\Gamma(V, Q)$). We obtain that

$$\mathrm{S}(V, \kappa) \leq \mathrm{I}(V, \kappa) \leq \Delta(V, \kappa) \leq \Gamma(V, \kappa) \tag{1.2.3}$$

is a $\Gamma(V, \kappa)$-invariant sequence of groups where κ is either f or Q (see [As, p. 471] and [KL, p.12 Remark]).
For the groups defined above we sometimes emphasize that V is a $\mathrm{GF}(q)$-vector space by noting in addition $\mathrm{GF}(q)$, such as in $\mathrm{I}(V, \mathrm{GF}(q), \kappa)$.
Considering (1.2.2), we note that the element $\lambda(g)$ (and also $\sigma(g)$) is uniquely determined by g if κ is surjective, see [KL, Lemma 2.1.2 (i) and (iii)]. Hence, we can introduce the well-defined map

$$\lambda : \ \Gamma(V, \mathrm{GF}(q), \kappa) \to \mathrm{GF}(q)^*, \quad g \mapsto \lambda(g) \tag{1.2.4}$$

for surjective κ. If $\kappa = f$ is trivial (i.e. $f(v, w) = 0$ for all $v, w \in V$) we define the map as

$$\lambda : \ \Gamma\mathrm{L}(V) \to \mathrm{GF}(q)^*, \quad g \mapsto 1. \tag{1.2.5}$$

For surjective κ the restriction of the homomorphism σ from (1.2.1) to the subgroup $\Gamma(V, \mathrm{GF}(q), \kappa)$ yield a homomorphism

$$\sigma : \ \Gamma(V, \mathrm{GF}(q), \kappa) \to \mathrm{Aut}(\mathrm{GF}(q)) \tag{1.2.6}$$

with kernel $\Delta(V, \mathrm{GF}(q), \kappa)$, cf. [KL, Lemma 2.1.2 (iv)]. (Concerning the (chosen) accordance of the notation $\sigma(g)$ in (1.2.1) and (1.2.2) we refer to [KL, Lemma 2.1.2 (iii) and Table 2.1.A]).
We note that the above definitions for the map λ (in (1.2.4) and (1.2.5)) and the homomorphism σ (in (1.2.1) and (1.2.6)) coincide with the definitions of the map τ and the homomorphism σ in the book of Kleidman and Liebeck, see [KL, p. 12-13].

The classical groups

Here, we define the classical groups together with their notation and provide basic facts about them. For this, we introduce the following four cases **L**, **U**, **Sp** and **O** of forms f and Q on a vector space V.

Case	Form on V
L	f is trivial (i.e. $f(v,w)=0$ for all $v,w \in V$)
U	f is a non-degenerate unitary form
Sp	f is a non-degenerate symplectic form
O	Q is a non-degenerate quadratic form

If f or Q is one of the forms on V occurring in the previous table we call it a *classical form* on V.
For the following let κ be either a classical form f or Q on V. We call (V,κ) a *classical geometry*, and more specifically, we call (V,κ) a *linear*, *unitary*, *symplectic*, or *orthogonal geometry* in cases **L**, **U**, **Sp**, or **O**, respectively.
For each case of the previous table we obtain by (1.2.3) a sequence of groups. We extend this sequences by groups $\mathrm{A}(V,\kappa)$ and $\Omega(V,\kappa)$. For $n \geq 3$ in case **L** there is a non-trivial involutory automorphism τ of $\mathrm{S}(V,f)=\mathrm{SL}(V)$, called the graph automorphism (see Subsection 1.2.2, below). So, we set

$$\mathrm{A}(V,\kappa)=\begin{cases}\Gamma(V,f):\langle\tau\rangle & \text{in case } \mathbf{L} \text{ if } n\geq 3,\\ \Gamma(V,\kappa) & \text{otherwise.}\end{cases}$$

In case **O**, we have a certain normal subgroup K of $\mathrm{S}(V,Q)$ of index 2, described in [KL, § 2.5, esp. Proposition 2.5.7. and the following Descriptions 1.-4.]. We define $\Omega(V,Q)=K$ in case **O** and $\Omega(V,f)=\mathrm{S}(V,f)$ in the remaining three cases.
So, we obtain the following $\mathrm{A}(V,\kappa)$-invariant sequence of groups (see [KL, p. 14])

$$\Omega(V,\kappa)\leq \mathrm{S}(V,\kappa)\leq \mathrm{I}(V,\kappa)\leq \Delta(V,\kappa)\leq \Gamma(V,\kappa)\leq \mathrm{A}(V,\kappa). \tag{1.2.7}$$

By definition we have that $\mathrm{Z}(\mathrm{GL}(V))\trianglelefteq \mathrm{A}(V,\kappa)$, and we define the projective map P which we call *projection map* (cf. [As, p. 473]) by

$$\begin{aligned}\mathrm{P}:\mathrm{A}(V,\kappa)\to \mathrm{P}(\mathrm{A}(V,\kappa))=\mathrm{PA}(V,\kappa)=\mathrm{A}(V,\kappa)/\mathrm{Z}(\mathrm{GL}(V)),\\ a\mapsto \mathrm{P}(a)=\mathrm{P}a=a\cdot \mathrm{Z}(\mathrm{GL}(V))/\mathrm{Z}(\mathrm{GL}(V)).\end{aligned} \tag{1.2.8}$$

Hence, we obtain the $\mathrm{PA}(V,\kappa)$-invariant sequence of groups

$$\mathrm{P}\Omega(V,\kappa)\leq \mathrm{PS}(V,\kappa)\leq \mathrm{PI}(V,\kappa)\leq \mathrm{P}\Delta(V,\kappa)\leq \mathrm{P}\Gamma(V,\kappa)\leq \mathrm{PA}(V,\kappa), \tag{1.2.9}$$

as the projective version of (1.2.7).

Now, we are able to define the classical groups (in the sense of the book of Kleidman and Liebeck [KL, p. 13-14]). Recall that κ denotes a form on the vector space V for one of the four cases **L**, **U**, **Sp**, or **O**.

We call a group G a *(finite) classical group* if it satisfies

$$\Omega(V,\kappa) \leq G \leq \mathrm{A}(V,\kappa), \text{ or } \mathrm{P}\Omega(V,\kappa) \leq G \leq \mathrm{PA}(V,\kappa).$$

Let G be a classical group, then we call G a *linear group*, *unitary group*, *symplectic group*, or *orthogonal group* in the cases **L**, **U**, **Sp**, or **O**, respectively. Except for a few cases, the groups $\mathrm{P}\Omega(V,\kappa)$ are non-abelian simple. So, we call the collection of the non-abelian simple groups $\mathrm{P}\Omega(V,\kappa)$ the *(finite) classical simple groups*. For precise information about the simplicity see Proposition 1.2.12, below.

Remark 1.2.1. (a) Our definition for the finite classical groups coincides with the definition provided in the book by Kleidman and Liebeck [KL, p. 13-14], except that our notation **Sp** for the "symplectic case" differs from that in [KL], given by **S**, for avoiding confusion with other abbreviations in this thesis. Aschbacher's definition for the classical groups in [As, p. 470-471] is slightly different. Aschbacher denotes the isometry group of (V,κ) by $O(V,\kappa)$, whereas we have chosen to use the symbol $\mathrm{I}(V,\kappa)$. Furthermore, Aschbacher defines $\Omega(V,\kappa)$ as the commutator group of $O(V,\kappa)$, while our definition for $\Omega(V,\kappa)$ is different in few cases, such as for $\mathrm{I}(V,f) \cong \mathrm{GL}(2,2)$ (cf. [KL, p. 14 Remark]).

(b) Since we will use frequently the results from [BHR], we will remark the differences between the notation in our definition of the classical groups and the notation provided there. In [BHR], the notation for the "symplectic case" is **S**, whereas we use the notation **Sp** (recall also part (a)). We write $\mathrm{I}(V,\kappa)$ for the isometry group and $\Delta(V,\kappa)$ for the similarity group of (V,κ), whereas in the book [BHR] the symbols G and C are chosen.
Further, we remark that in [BHR] the classical groups are introduced related to a standard form on the vector space V, i.e. choosing a suitable standard basis for V for each of the different cases of forms on V (regarding additional conditions in case **O**), and defining the classical groups related to that basis (see [BHR, p. 27-31 and Table 1.1]). In [BHR], the choice of a fixed basis of V is advantageous for the analysis of inclusion among each other of certain subgroups of a classical group. Furthermore, it solves the problem of well-definedness when they present their results in [BHR, Chapter 8] (cf. Remark 1.2.18 (below) and [BHR2]). For our purposes it is more advantageous not to choose a fixed basis. Since we have to analyze the structure of certain subgroups of the classical groups deeply, we shall choose for each situation a suitable basis of V separately, to obtain an advantageous representation for the analysis (such as in Propositions 2.1.13 and 2.2.8).

(c) We note that it is sufficient to restrict our attention to the four forms on a vector space V occurring in the cases **L**, **U**, **Sp** and **O**, and call their stabilizers classical groups. This is justified in [BHR, p. 13-16, esp. Theorem 1.5.13].

(d) We have defined our projection map P by reduction modulo $\mathrm{Z}(\mathrm{GL}(V))$. For any subgroup H of $\mathrm{A}(V,\kappa)$ we have the isomorphism

$$\mathrm{P}H = H \cdot \mathrm{Z}(\mathrm{GL}(V))/\mathrm{Z}(\mathrm{GL}(V)) \cong H/(H \cap \mathrm{Z}(\mathrm{GL}(V))).$$

Hence, it is also possible to consider $\mathrm{P}H$ as the reduction of H modulo $H \cap \mathrm{Z}(\mathrm{GL}(V))$. More general, for $H \leq G \leq \mathrm{A}(V,\kappa)$ we can also consider $\mathrm{P}H$ as the reduction of H modulo $\mathrm{Z}(\mathrm{GL}(V)) \cap G$. (Important for our research are especially the cases $G = \Omega(V,\kappa)$). In this thesis we will use (also without further mention) that kind of reduction which is more advantageous in the particular situation. I.e. assume that $G = H = \mathrm{SL}(V)$. For the inclusion of $\mathrm{PSL}(V)$ in $\mathrm{PGL}(V)$ it is appropriate considering $\mathrm{PSL}(V)$ to be the quotient $\mathrm{SL}(V) \cdot \mathrm{Z}(\mathrm{GL}(V))/\mathrm{Z}(\mathrm{GL}(V))$. However, some authors define $\mathrm{PSL}(V) = \mathrm{SL}(V)/\mathrm{Z}(\mathrm{SL}(V))$, since this is advantageous in certain other situations.

Convention 1.2.2. Working with the projective cases, we introduce the following notations. The full preimage under P we denote by a hat, as in $\widehat{\mathrm{PGL}(V)} = \mathrm{GL}(V)$. For a matrix $A = (a_{ij})_{n\times n} \in \mathrm{GL}_n(q)$ we write for the image under the projection map $\mathrm{P}A = [a_{ij}]_{n\times n} = A \cdot \mathrm{Z}(\mathrm{GL}_n(q)) \in \mathrm{PGL}_n(q)$. In view of Remark 1.2.1 (d), the last notation is also generalized to a reduction modulo another (for the respective situation appropriate) subgroup of $\mathrm{Z}(\mathrm{GL}(V))$.

Next, we introduce our notation for the groups occurring in the sequences (1.2.7) and (1.2.9). For the purpose of well-definedness we provide the following proposition and lemma.

Proposition 1.2.3. *Let V be an n-dimensional vector space over the finite field $\mathrm{GF}(q)$ and let κ be a form on V belonging to the cases* **U**, **Sp** *or* **O**. *By δ_{ij} we denote the Kronecker delta. Then the following hold.*

(i) If $\kappa = f$ is a non-degenerate unitary form then (V,f) has

(a) an orthonormal basis, i.e. there is a basis $B = \{b_1,\dots,b_n\}$ of V with $f(b_i,b_j) = \delta_{ij}$,

(b) a basis

$$B = \begin{cases} \{b_1,\dots,b_{\frac{n}{2}},c_1,\dots,c_{\frac{n}{2}}\} & \text{if } n \text{ is even,} \\ \{b_1,\dots,b_{\frac{n-1}{2}},c_1,\dots,c_{\frac{n-1}{2}},x\} & \text{if } n \text{ is odd} \end{cases}$$

where $f(b_i,b_j) = f(c_i,c_j) = 0$, $f(b_i,c_j) = \delta_{ij}$, $f(b_i,x) = f(c_i,x) = 0$ for all i,j and $f(x,x) = 1$.

(ii) *If $\kappa = f$ is a non-degenerate symplectic form then the dimension n of V is even, and (V, f) has a basis $B = \{b_1, \ldots, b_{\frac{n}{2}}, c_1, \ldots, c_{\frac{n}{2}}\}$ such that $f(b_i, b_j) = f(c_i, c_j) = 0$ and $f(b_i, c_j) = \delta_{ij}$ for all i, j.*

(iii) *If $\kappa = Q$ is a non-degenerate quadratic form with associated bilinear form f_Q then (V, Q) has a basis B of one of the following types.*

   (a) *n is even and $B = \{b_1, \ldots, b_{\frac{n}{2}}, c_1, \ldots, c_{\frac{n}{2}}\}$ where $Q(b_i) = Q(c_i) = 0$ and $f_Q(b_i, c_j) = \delta_{ij}$ for all i, j.*

   (b) *n is even and $B = \{b_1, \ldots, b_{\frac{n}{2}-1}, c_1, \ldots, c_{\frac{n}{2}-1}, x, y\}$ where $Q(b_i) = Q(c_i) = 0$, $f_Q(b_i, c_j) = \delta_{ij}$ and $f_Q(b_i, x) = f_Q(b_i, y) = f_Q(c_i, x) = f_Q(c_i, y) = 0$ for all i, j and $Q(x) = 1, f_Q(x, y) = 1, Q(y) = \zeta$ where the polynomial $T^2 + T + \zeta$ is irreducible over $\mathrm{GF}(q)$.*

   (c) *n is odd and $B = \{b_1, \ldots, b_{\frac{n-1}{2}}, c_1, \ldots, c_{\frac{n-1}{2}}, x\}$ where $Q(b_i) = Q(c_i) = 0$, $f_Q(b_i, c_j) = \delta_{ij}$ and $f_Q(b_i, x) = f_Q(c_i, x) = 0$ for all i, j and x is non-singular (i.e. $Q(x) \neq 0$).*

Proof. For assertion (i) see [KL, Propositions 2.3.1. and 2.3.2.][1], or [BHR, Proposition 1.5.29]. Assertion (ii) follows by [KL, Proposition 2.4.1.], or [BHR, Proposition 1.5.26]. We obtain assertion (iii) by [KL, Proposition 2.5.3.]. □

Lemma 1.2.4. *Let V_1, V_2 be n-dimensional vector spaces over $\mathrm{GF}(q)$. Let κ_1 and κ_2 be either both left-linear or both quadratic forms. If (V_1, κ_1) and (V_2, κ_2) are similar then $X(V_1, \kappa_1) \cong X(V_2, \kappa_2)$ for $X \in \{\mathrm{S}, \mathrm{I}, \Delta, \Gamma, \mathrm{A}\}$.*

Proof. See [KL, Lemma 2.1.1. and p. 21]. □

By the previous proposition, we obtain that there is a unique non-degenerate unitary form and a unique non-degenerate symplectic form, up to isometry, on V. So, we may introduce a notation for the groups in the sequences (1.2.7) and (1.2.9) in cases **L**, **U** and **Sp**, because of the uniqueness of the groups, by Lemma 1.2.4. Furthermore, we obtain that case **O** will split up into three subcases, which can be identified by the occurrence of one of the specific bases for the vector space in Proposition 1.2.3 (iii). For more information about the case **O** see [KL, p. 26-28]. Following Kleidman and Liebeck, we denote the three subcases by $\mathbf{O}^\circ$ if n is odd, $\mathbf{O}^+$ if n is even and Proposition 1.2.3 (iii) (a) holds and $\mathbf{O}^-$ if n is even and Proposition 1.2.3 (iii) (b) holds (see [KL, (2.5.4)]). We also define for a quadratic form Q on V that $\mathrm{sgn}(Q) = \circ, +$, or $-$ in cases $\mathbf{O}^\circ$, $\mathbf{O}^+$, or $\mathbf{O}^-$, respectively.

By Table 1.2.1, we now introduce our notation for the groups in the sequence (1.2.7). It is adopted from the book of Kleidman and Liebeck (see [KL, Table 2.1.B]), except for a few changes.
The different cases for the forms on V occur in the first column. In the second column we list the groups from the sequence (1.2.7), and in the third column we

[1] We have to note that the proof of [KL, Proposition 2.3.2.] is not correct, cf. the footnote on p. 76. But, the assertion of the proposition is valid.

provide our notation for them. Recall that by κ we denote a form on the vector space V for one of the four cases **L**, **U**, **Sp**, or **O**. For many considerations and applications (e.g. calculations) it is advantageous to choose a (suitable) fixed ordered basis B for the vector space V and consider the groups $X(V,\kappa)$ for $X \in \{\Omega, \mathrm{S}, \mathrm{I}, \Gamma, \mathrm{A}\}$ with respect to that basis. So, in the fourth column we provide our notation for the groups with respect to a fixed ordered basis B. If we use the formulation from this column in our thesis, we will (usually) specify the related basis, or omit the specification if it is clear by the situation, or if it is not relevant. We recall the well-definedness of the groups given in Table 1.2.1, by our previous considerations (or see [KL, p.15 ff.]).

The projective groups appearing in the sequence (1.2.9) we denote analogously to sequence (1.2.7), by preceding each notation from the third column of Table 1.2.1 with the symbol P, such as $\mathrm{PGL}(V)$ denotes the projective general linear group. Equally, we choose our notation for these groups with respect to a fixed ordered basis, by preceding each notation from the fourth column with the symbol P.

Table 1.2.1

Case	Group from sequence (1.2.7)	Notation	Notation with resp. to an ordered basis
L	$\Omega(V,f) = \mathrm{S}(V,f)$	$\mathrm{SL}(V)$	$\mathrm{SL}_n(q)$
	$\mathrm{I}(V,f) = \Delta(V,f)$	$\mathrm{GL}(V)$	$\mathrm{GL}_n(q)$
	$\Gamma(V,f)$	$\Gamma\mathrm{L}(V)$	$\Gamma\mathrm{L}_n(q)$
	$\mathrm{A}(V,f)^\star$	$\mathrm{A}\Gamma\mathrm{L}(V)$	$\mathrm{A}\Gamma\mathrm{L}_n(q)$
U	$\Omega(V,f) = \mathrm{S}(V,f)$	$\mathrm{SU}(V)$	$\mathrm{SU}_n(q^2)$
	$\mathrm{I}(V,f)$	$\mathrm{GU}(V)$	$\mathrm{GU}_n(q^2)$
	$\Delta(V,f)$	$\Delta\mathrm{U}(V)$	$\Delta\mathrm{U}_n(q^2)$
	$\mathrm{A}(V,f) = \Gamma(V,f)$	$\Gamma\mathrm{U}(V)$	$\Gamma\mathrm{U}_n(q^2)$
Sp	$\Omega(V,f) = \mathrm{S}(V,f) = \mathrm{I}(V,f)$	$\mathrm{Sp}(V)$	$\mathrm{Sp}_n(q)$
	$\Delta(V,f)$	$\mathrm{GSp}(V)$	$\mathrm{GSp}_n(q)$
	$\mathrm{A}(V,f) = \Gamma(V,f)$	$\Gamma\mathrm{Sp}(V)$	$\Gamma\mathrm{Sp}_n(q)$
O$^\epsilon$, $\epsilon \in \{+,-,\circ\}$	$\Omega(V,Q)$	$\Omega^\epsilon(V)$	$\Omega^\epsilon_n(q)$
	$\mathrm{S}(V,Q)$	$\mathrm{SO}^\epsilon(V)$	$\mathrm{SO}^\epsilon_n(q)$
	$\mathrm{I}(V,Q)$	$\mathrm{O}^\epsilon(V)$	$\mathrm{O}^\epsilon_n(q)$
	$\Delta(V,Q)$	$\mathrm{GO}^\epsilon(V)$	$\mathrm{GO}^\epsilon_n(q)$
	$\Gamma(V,Q) = \mathrm{A}(V,Q)$	$\Gamma\mathrm{O}^\epsilon(V)$	$\Gamma\mathrm{O}^\epsilon_n(q)$

* $\mathrm{A}(V,f) = \Gamma(V,f) : \langle\tau\rangle$ for $n \geq 3$ and $\mathrm{A}(V,f) = \Gamma(V,f)$ otherwise.

Concerning Table 1.2.1, we provide the following remark.

Remark 1.2.5. (a) Except for a few changes (see part (b)), we note that in the book of Kleidman and Liebeck the notation for the groups of column two is equal to our notation from column four, see [KL, p. 15]. We have decided to introduce the two notations from column three and four to have a proper notation for the groups of column two if they are considered with respect to a fixed ordered basis. This notation is advantageous if several bases can be chosen to obtain suitable representations for different subgroups of a classical group. Also, it is makes sense when we indicate if a given classical group is well-defined (then we use the notation from column three), or only unique if the related basis is specified (then the notation from column four is used and the related basis is listed). Recall that we have also the freedom not to indicate the chosen basis in column four, such that we have also the advantage to work with this notation without specifying it. For more details concerning the problem of well-definedness see the paper [BHR2] and Remark 1.2.18, below.

(b) In case **U** the order of the finite field of the vector space is a square. Kleidman and Liebeck use the notations $\mathrm{SU}_n(q), \mathrm{GU}_n(q)$ and $\Gamma\mathrm{U}_n(q)$ in [KL], to indicate unitary groups over the finite field of order q^2. However, we have decided to follow the book of Huppert [Hu] and write in our notation in column four the full order q^2 of the finite field (such that $\mathrm{SU}_n(q^2) \leq \mathrm{SL}_n(q^2)$).

(c) As in [KL], the book [BHR] uses the root of the order of the finite field for the notation of the unitary groups. Furthermore, in [BHR] the notation of the groups of all semisimilarities $\Gamma(V, \kappa)$ are in cases **U**, **Sp** and **O** given by an additional symbol C at the beginning, to indicate that these groups are not the corresponding semilinear groups of V, see [BHR, Table 1.2 and p. 30]. We note that the notation for the orthogonal groups in [BHR, see p. 31] differs from that chosen in [KL, see Table 2.1.B], and also differs from our notation.

(d) In case **Sp** we have that $\mathrm{S}(V, f) = \mathrm{I}(V, f)$, i.e. all isometries have determinant 1. This fact is well-known, see e.g. [As2, (22.4)].

(e) For even q in case $\mathbf{O}^\epsilon$ we have that the associated bilinear form f_Q of Q is also a non-degenerate symplectic form on V (cf. page 13). By Proposition 1.2.3 (ii), we know that the dimension of V has to be even. Hence, we set that q is odd if case $\mathbf{O}^\circ$ occurs (i.e. if the dimension n of the vector space V is odd). For more details see [KL, p. 26] and [BHR, p. 23-24].

For representing subgroups of classical groups with respect to an ordered basis and calculating with the corresponding matrices, the following considerations are important.

Definition 1.2.6. Let f be a left-linear form on a vector space V of dimension $n \geq 1$. Let $B = (b_1, \ldots, b_n)$ be an ordered basis of V. We call $J_{f,B}$ the *matrix of the form* f on V (with respect to B) where $J_{f,B} = (f(b_i, b_j))_{1 \leq i,j \leq n}$. We will drop the subscripts and also write $J_{f,B} = J_f = J_B = J$ if the roles of f or B are clear by the situation or are not relevant.

Remark 1.2.7. (cf. [BHR, Lemma 1.5.20])

(a) It is not hard to see that if f is a unitary form on the $\mathrm{GF}(q^2)$-vector space V then $J_f^t = J_f^{\varphi_q}$ where t denotes the transpose map and φ_q denotes the map which replaces all matrix entries by their q-th powers. Conversely, every matrix $A \in \mathrm{Mat}_n(q^2)$ with $A^t = A^{\varphi_q}$ determines on V a unitary form.

(b) For a symplectic form f on the $\mathrm{GF}(q)$-vector space V we have $J_f^t = -J_f$ and all entries on the diagonal are zero; conversely, a matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathrm{Mat}_n(q)$ with $A^t = -A$ and $a_{ii} = 0$ for all i determines on V a symplectic form. (Note, that if the characteristic of $\mathrm{GF}(q)$ is odd then $a_{ii} = 0$ follows from the condition $A^t = -A$).

(c) If Q is a quadratic form on the $\mathrm{GF}(q)$-vector space V then for the associated symmetric bilinear form f_Q we have $J_{f_Q}^t = J_{f_Q}$. Conversely, if $A \in \mathrm{Mat}_n(q)$ with $A^t = A$ then A determines on V a symmetric bilinear form. Furthermore, we note that Q and f_Q determine one another if the characteristic of $\mathrm{GF}(q)$ is odd, see [KL, (2.5.1)], or more precisely [BHR, Proposition 1.5.15].

Lemma 1.2.8. *Let B be an ordered basis of the vector space V over $\mathrm{GF}(q^2)$ in case* **U** *and $\mathrm{GF}(q)$ in case* **Sp**. *Let $\mathrm{GU}_n(q^2)$ be a representation of $\mathrm{GU}(V)$ and $\mathrm{Sp}_n(q)$ be a representation of $\mathrm{Sp}(V)$ (both) with respect to B. By φ_q we denote the map described in the previous remark. Then we have that*

(a) $\mathrm{GU}_n(q^2) = \{g \in \mathrm{GL}_n(q^2) \mid gJ_Bg^{t\varphi_q} = J_B\}$ *and*

(b) $\mathrm{Sp}_n(q) = \{g \in \mathrm{GL}_n(q) \mid gJ_Bg^t = J_B\}$.

Proof. See [KL, Lemma 2.1.8. (i)]. □

Remark. For an assertion concerning the representation of the isometry group of (V, Q) with respect to an ordered basis for a non-degenerate quadratic form Q, see [KL, Lemma 2.1.8. (iii)], or more precisely [BHR, Lemma 1.5.21].

Next, we introduce further terminology important for our thesis. Let (V, κ) be a classical geometry, and let U be an m-dimensional subspace of V where $1 \leq m \leq \dim(V)$. By κ_U, we denote the restriction of the form κ to $U \times U$ in cases **L**, **U** and **Sp**, or U in case **O**. We obtain a *sub-geometry* (U, κ_U) of (V, κ), and note that in general (U, κ_U) has not to be a classical geometry. Sometimes, we omit writing the index U and simply write (U, κ). If κ_U is a non-degenerate form on U we call U *non-degenerate*. Obviously, if U is non-degenerate and (V, κ) is a unitary, symplectic, or orthogonal geometry then (U, κ_U) is a unitary, symplectic, or orthogonal geometry, respectively. If κ_U is the trivial form on U we call U *totally singular*. So, in the case **L** there are only totally singular subspaces of V. Clearly, if U is totally singular then (U, κ_U) is a linear geometry. The following terminology will be introduced only for the cases **L**, **U** and **Sp**.[2]

[2] For a more general definition see [BHR, Definition 1.5.10].

Let U be a subspace of V where (V, f) is a linear, unitary, or symplectic geometry. The *orthogonal complement* of U (in (V, f)) is defined by

$$U^{\perp}_{(V,f)} = U^{\perp} = \{v \mid v \in V, f(v, x) = 0 \text{ for all } x \in U\}.$$

Two subspaces U_1 and U_2 of V we call *orthogonal* if $U_1 \leq U_2^{\perp}$ or $U_2 \leq U_1^{\perp}$. We write $U_1 + U_2 = U_1 \perp U_2$ if U_1 and U_2 are orthogonal and $U_1 + U_2 = U_1 \oplus U_2$. Clearly, we have $U^{\perp} = V$ if case **L** holds. In cases **U** and **Sp** we obtain that $\dim(U) + \dim(U^{\perp}) = \dim(V)$ and $(U^{\perp})^{\perp} = U$. Furthermore, for these two cases we have that U is non-degenerate if and only if $V = U \perp U^{\perp}$, and U is totally singular if and only if $U \leq U^{\perp}$ (see [KL, Lemma 2.1.5.]).

We note an important lemma and a consequence arising from it.

Lemma 1.2.9. (Witt's Lemma) *Let (V_1, κ_1) and (V_2, κ_2) be two isometric classical geometries, and let $U_i \leq V_i$ for $i \in \{1, 2\}$. If there is an isometry g from (U_1, κ_1) to (U_2, κ_2) then g extends to an isometry from (V_1, κ_1) to (V_2, κ_2).*

Proof. See [As2, Section 20] or [KL, Proposition 2.1.6.]. □

Corollary 1.2.10. *(a) All maximal totally singular subspaces of a classical geometry (V, κ) have the same dimension which we call the Witt index of (V, κ).*

(b) If (V, κ) is a unitary geometry we have that the Witt index of (V, κ) is $\lfloor \frac{n}{2} \rfloor$.

Proof. The assertion follows from Lemma 1.2.9 together with our previous observations. □

In the rest of this subsection we provide basic facts about the classical groups which are members of the sequences (1.2.7) and (1.2.9).

Isomorphisms and simplicity of the finite classical simple groups

The results of the following two propositions are standard and we quote them from the book [BHR]. First, we provide the information about isomorphisms of the groups $\mathrm{P}\Omega(V, \kappa)$ in (1.2.9) for the different cases **L**, **U**, **Sp** and **O**, taken from [BHR, Proposition 1.10.1]. Recall, that except for a few cases the groups $\mathrm{P}\Omega(V, \kappa)$ are the finite classical simple groups.

Proposition 1.2.11. *Let M be the collection consisting of the groups $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q^2)$ and $\mathrm{PSp}_n(q)$ all for $n \geq 2$, and of $\mathrm{P}\Omega^{\epsilon}_n(q)$ for $n \geq 3$ where q is odd if $\epsilon = \circ$ (see Remark 1.2.5 (e)). The following provides a complete list of isomorphisms between two elements of M, and also a further isomorphism for $\mathrm{P}\Omega^{+}_4(q)$.*

$$\mathrm{PSL}_2(q) \cong \mathrm{PSU}_2(q^2) \cong \mathrm{PSp}_2(q) \cong \mathrm{P}\Omega_3^\circ(q),$$
$$\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5), \mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2), \mathrm{PSU}_4(2^2) \cong \mathrm{PSp}_4(3),$$
$$\mathrm{P}\Omega_4^+(q) \cong \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q), \mathrm{P}\Omega_4^-(q) \cong \mathrm{PSL}_2(q^2),$$
$$\mathrm{P}\Omega_5^\circ(q) \cong \mathrm{PSp}_4(q), \mathrm{P}\Omega_6^+(q) \cong \mathrm{PSL}_4(q), \mathrm{P}\Omega_6^-(q) \cong \mathrm{PSU}_4(q^2).$$

Next, we state an important fact about the simplicity of the groups $\mathrm{P}\Omega(V,\kappa)$ from sequence (1.2.9).

Proposition 1.2.12. *Let X be $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q^2)$, $\mathrm{PSp}_n(q)$, or $\mathrm{P}\Omega_n^\epsilon(q)$ for $n \geq 2$ and q odd if X is $\mathrm{P}\Omega_n^\circ(q)$ (recall Remark 1.2.5 (e)). Let Q_8 denote the quaternion group (for a definition see Lemma 2.6.2 (a), below). Then the following hold.*

(i) If X is soluble, then X is isomorphic to one of the following groups:

$$\mathrm{PSL}_2(2) \cong \mathrm{S}_3, \mathrm{PSL}_2(3) \cong \mathrm{A}_4, \mathrm{PSU}_3(2^2) \cong 3^2 : Q_8,$$
$$\mathrm{P}\Omega_2^+(q) \cong (q-1)/(2,q-1), \mathrm{P}\Omega_2^-(q) \cong (q+1)/(2,q-1),$$
$$\mathrm{P}\Omega_4^+(2) \cong \mathrm{S}_3 \times \mathrm{S}_3, \mathrm{P}\Omega_4^+(3) \cong \mathrm{A}_4 \times \mathrm{A}_4.$$

(ii) If X is not simple and not soluble, then X is isomorphic to $\mathrm{P}\Omega_4^+(q) \cong \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$ for $q \geq 4$, or to $\mathrm{PSp}_4(2) \cong \mathrm{S}_6$.

(iii) If X is not isomorphic to a group listed in (i) and (ii), then X is simple and the associated group $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q^2)$, $\mathrm{Sp}_n(q)$, or $\Omega_n^\epsilon(q)$ is quasisimple.

*(iv) Assume that $n \geq 2, 3, 4, 7$ in cases **L**, **U**, **Sp** and $\mathbf{O}^\epsilon$, respectively. Then X is non-abelian simple, except for the cases $\mathrm{PSL}_2(2)$, $\mathrm{PSL}_2(3)$, $\mathrm{PSU}_3(2^2)$ and $\mathrm{PSp}_4(2)$.*

Proof. For (i) - (iii) see [BHR, Proposition 1.10.3]. Assertion (iv) follows immediately from (i) and (ii) together with Proposition 1.2.11, or see [KL, Theorem 2.1.3]. □

Orders of the finite classical groups

Now, we provide the important facts about the orders of the classical groups occurring in the sequences (1.2.7) and (1.2.9) for the different cases **L**, **U**, **Sp** and **O**. These orders are well-known and we cite them from [KL, Tables 2.1.C and 2.1.D] and [BHR, p. 32-33]. The first proposition list the orders of the isometry groups.

Proposition 1.2.13. *(a) The order of $\mathrm{GL}_n(q)$ is $q^{\frac{n(n-1)}{2}} \prod_{i=1}^n (q^i - 1)$.*

(b) The order of $\mathrm{GU}_n(q^2)$ is $q^{\frac{n(n-1)}{2}} \prod_{i=1}^n (q^i - (-1)^i)$.

(c) *The order of* $\mathrm{Sp}_n(q)$ *is* $q^{\frac{n^2}{4}} \prod_{i=1}^{\frac{n}{2}}(q^{2i}-1)$ *(note that* n *is even, see Proposition 1.2.3 (ii)).*

(d) *The order of* $\mathrm{O}_n^{\circ}(q)$ *is* $2q^{\frac{(n-1)^2}{4}} \prod_{i=1}^{\frac{n-1}{2}}(q^{2i}-1)$ *(note that* nq *is odd).*

(e) *The order of* $\mathrm{O}_n^{\pm}(q)$ *is* $2q^{\frac{n(n-2)}{4}}(q^{\frac{n}{2}} \mp 1) \prod_{i=1}^{\frac{n}{2}-1}(q^{2i}-1)$ *(note that* n *is even).*

The following proposition provides two tables which give the indices of the groups in the sequences (1.2.7) and (1.2.9), together with the order of the intersection of the isometry group and the centre of the general linear group. So, in combination with Proposition 1.2.13, we have all information to calculate the order of any group occurring in the sequences (1.2.7) and (1.2.9). As above, let κ denote a form on V for one of the four cases.

Proposition 1.2.14. *Let* p *be a prime and* a *be a positive integer. Let* (V, κ) *be a vector space of dimension* $n > 0$ *over a finite field of order* $q = p^a$ *in case* **L**, **Sp**, *or* **O** *and of order* $q^2 = p^{2a}$ *in case* **U**. *If* n *is odd in case* **O** *let* q *be odd (see Remark 1.2.5 (e)). Here, as an abbreviation we write* $X = X(V, \kappa)$ *where* X *ranges over the symbols* Ω, S, I, Δ, Γ *and* A, *and we write* $\mathrm{Z} = \mathrm{Z}(\mathrm{GL}(V))$. *In view of the sequences (1.2.7) and (1.2.9) the following hold.*

Case	$\lvert\mathrm{S}:\Omega\rvert$	$\lvert\mathrm{I}:\mathrm{S}\rvert$	$\lvert\Delta:\mathrm{I}\rvert$	$\lvert\Gamma:\Delta\rvert$	$\lvert\mathrm{A}:\Gamma\rvert$
L	1	$q-1$	1	a	$2^{\dagger}$
U	1	$q+1$	$q-1$	$2a$	1
Sp	1	1	$q-1$	a	1
$\mathbf{O}^{\circ}$	$2^{\ddagger}$	2	$\frac{q-1}{2}$	a	1
$\mathbf{O}^{\pm}$	2	$(2,q-1)$	$q-1$	a	1

Case	$\lvert\mathrm{I}\cap\mathrm{Z}\rvert$	$\lvert\mathrm{PS}:\mathrm{P}\Omega\rvert$	$\lvert\mathrm{PI}:\mathrm{PS}\rvert$	$\lvert\mathrm{P}\Delta:\mathrm{PI}\rvert$	$\lvert\mathrm{P}\Gamma:\mathrm{P}\Delta\rvert$	$\lvert\mathrm{PA}:\mathrm{P}\Gamma\rvert$
L	$q-1$	1	$(q-1,n)$	1	a	$2^{\dagger}$
U	$q+1$	1	$(q+1,n)$	1	$2a$	1
Sp	$(2,q-1)$	1	1	$(2,q-1)$	a	1
$\mathbf{O}^{\circ}$	2	$2^{\ddagger}$	1	1	a	1
$\mathbf{O}^{\pm}$	$(2,q-1)$	$e_{\pm}^{\star}$	$(2,q-1)$	$(2,q-1)$	a	1

† If $n = 1, 2$ we have $\mathrm{A}(V,f) = \Gamma(V,f)$ and the value 2 should be replaced by 1.
‡ When $n = 1$ then $\Omega(V,Q) = \mathrm{S}(V,Q)$ and the value 2 should be replaced by 1.
⋆ We have $e_{\pm} \in \{1,2\}$, $e_+e_- = 2^{(2,q)}$, and if q is odd then $e_+ = 2$ if and only if $\frac{n(q-1)}{4}$ is even.

Remark. We note a slight mistake in [KL, Table 2.1.D] in the case $\mathbf{O}^{\circ}$ of dimension $n = 1$, where we have $\lvert\mathrm{I}(V,Q) \cap \mathrm{Z}(\mathrm{GL}(V))\rvert = 2$ and $\lvert\mathrm{PS}(V,Q) : \mathrm{P}\Omega(V,Q)\rvert = 1$. This mistake was also corrected by [BHR, in Table 1.3] without mentioning the correction.

1.2.2 The automorphism groups of $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$

In this subsection, we introduce the generators of the automorphism groups of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ for $n \geq 2$ and $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$ for $n \geq 3$.
We start by providing a general result concerning the automorphism groups of the finite classical simple groups. For this, let κ be a form on the n-dimensional vector space V for one of the cases **L**, **U**, **Sp**, or **O**, and recall the notation introduced in Table 1.2.1. As we have seen in Proposition 1.2.12, the groups $\mathrm{P\Omega}(V, \kappa)$ are non-abelian simple, except for a few cases. We also have that $\mathrm{P\Omega}(V, \kappa)$ is a normal subgroup of $\mathrm{PA}(V, \kappa)$, see (1.2.9). In fact, for the automorphism group of the classical simple groups $\mathrm{P\Omega}(V, \kappa)$ we obtain the following.

Proposition 1.2.15. *Let $\Omega(V, \kappa)$ be a group from Table 1.2.1 and let $n \geq 2$ in case **L**, $n \geq 3$ in case **U**, $n \geq 4$ in case **Sp** and $n \geq 7$ in case **O**. If $\mathrm{P\Omega}(V, \kappa)$ is simple, then we have $\mathrm{Aut}(\mathrm{P\Omega}(V, \kappa)) \cong \mathrm{PA}(V, \kappa)$, except when $\Omega(V, \kappa) \cong \mathrm{Sp}_4(q)$ and q is even, or $\Omega(V, \kappa) \cong \Omega_8^+(q)$. So, we may identify $\mathrm{PA}(V, \kappa)$ with $\mathrm{Aut}(\mathrm{P\Omega}(V, \kappa))$, in the not excluded cases.*

Proof. See [KL, Theorem 2.1.4.] or [Car, Chapter 12]. □

Remark 1.2.16. (a) Recall Proposition 1.1.2. In the situation of Proposition 1.2.15, we obtain that the classical groups G occurring in $\mathrm{P\Omega}(V, \kappa) \leq G \leq \mathrm{PA}(V, \kappa)$ are almost simple with socle $\mathrm{P\Omega}(V, \kappa)$.

(b) In view of Proposition 1.2.11, we note that the restriction on the dimension in the different cases of the last proposition is appropriate and does not exclude any classical simple group $\mathrm{P\Omega}(V, \kappa)$. Furthermore, we note that the dimension restriction of $n \geq 3$ in case **U** is not redundant: Consider the case where f is a non-degenerate unitary form on a 2-dimensional vector space V over the finite field of order $q^2 = p^{2a}$ for a prime p and a positive integer a. By Propositions 1.2.11, 1.2.13, 1.2.14 and 1.2.15, we obtain that $|\mathrm{PA}(V, f)| = 2aq(q^2-1) > aq(q^2-1) = |\mathrm{Aut}(\mathrm{PSL}_2(q))| = |\mathrm{Aut}(\mathrm{P\Omega}(V, f))|$. For more details concerning this case, see Remark 1.2.19 (c), below.

(c) For the excluded cases in the last proposition, we note the following. In case $\Omega(V, \kappa) \cong \mathrm{Sp}_4(q)$ where q is even, we have $|\mathrm{Aut}(\mathrm{PSp}_4(q)) : H| = 2$ where $H \cong \mathrm{P\Gamma Sp}_4(q)$. Here, there is a non-trivial involutory graph automorphism acting on $\mathrm{PSp}_4(q)$. In case $\Omega(V, \kappa) \cong \Omega_8^+(q)$, we have $|\mathrm{Aut}(\mathrm{P\Omega}_8^+(q)) : K| = 3$ where $K \cong \mathrm{P\Gamma O}_8^+(q)$, and there is a graph automorphism of order three. For more information about the mentioned graph automorphisms in the excluded cases see [Car, Chapter 12] (using Lie theory), or in case $\mathrm{PSp}_4(q)$ see also [BHR, p. 358-360].

In this thesis we consider the situation for almost simple groups G where the socle of G is isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$. Hence, we can assume that $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ or $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ where $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$ are non-abelian simple (recall Proposition 1.1.2). So, it is important to provide a detailed description of the structure of the (outer)

automorphism groups of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$. For this, we note that it is helpful to keep the assertion of Proposition 1.2.15 in mind.
It is well-known that the automorphism groups of the classical simple groups $\mathrm{P}\Omega(V,\kappa)$ have a description in terms of $\mathrm{Inn}(\mathrm{P}\Omega(V,\kappa)) \cong \mathrm{P}\Omega(V,\kappa)$ and so called diagonal, field and graph automorphisms. Here, we define for our intended groups $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ and $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$ these specific elements of their automorphism groups. We do so, by providing generators for a group from sequence (1.2.7) modulo the preceding group in the cases **L** and **U**. These generators induce automorphisms on $\Omega(V,f)$ (recall that $\Omega(V,f)$ is $\mathrm{A}(V,f)$-invariant), and on their projective versions. So, we will obtain our intended automorphisms of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ and $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$. We will list properties of these specific automorphisms, and summarize them in Corollaries 1.2.20 and 1.2.22, below.
As mentioned before, the books [KL] and [BHR] are main sources for our research. We note that in these books the description of the automorphism groups of $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$ and the chosen elements to generate them are not (quite) appropriate for our work (see e.g. Remark 1.2.18 for reasons). So, we will provide the needed terminology, as it is advantageous for our research, but we will stay close to these books. We will also note concordances and differences of our terminology and notation to the terminology and notation used in these two books. For the differences between the books [KL] and [BHR] concerning forms and generators of the outer automorphism groups, see [BHR, p. 57-58].
We introduce the following notation for the rest of this subsection. Let $u \in \{1,2\}$ where $u = 1$ in case **L** and $u = 2$ in case **U**. Let V be an n-dimensional vector space over a finite field of order q^u where $q = p^a$ for a prime p and a positive integer a. Let $n \geq 2$ in case **L** and $n \geq 3$ in case **U**. Recall from page 7 that φ_p denotes the Frobenius automorphism of $\mathrm{GF}(q^u)$, hence $\langle\varphi_p\rangle = \mathrm{Aut}(\mathrm{GF}(q^u))$ and $\mathrm{o}(\varphi_p) = a$ in case **L** and $\mathrm{o}(\varphi_p) = 2a$ in case **U**. By ω, we denote a primitive element of $\mathrm{GF}(q^u)^*$, so $\mathrm{o}(\omega) = q^u - 1$ and $\langle\omega\rangle = \mathrm{GF}(q^u)^*$.

Diagonal automorphisms of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ *and* $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$
(cf. [KL, p. 20-21, 23] and [BHR, p. 34])

Since $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$ are normal subgroups of $\mathrm{GL}(V)$ and $\mathrm{GU}(V)$, there are canonical automorphisms of $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$ induced via conjugation by the elements of $\mathrm{GL}(V)$ and $\mathrm{GU}(V)$, respectively. Next, we define diagonal matrices $W_{\mathrm{SL}} \in \mathrm{GL}_n(q)$ in the case **L** and $W_{\mathrm{SU}} \in \mathrm{GU}_n(q^2)$ in the case **U** with respect to (in the case **U** certain) fixed ordered bases of V. About the following defined matrices W_{SL} (or W_{SU}) we note that always $\mathrm{GL}_n(q)$ (or $\mathrm{GU}_n(q^2)$) is generated by $\mathrm{SL}_n(q)$ (or $\mathrm{SU}_n(q^2)$) and W_{SL} (or W_{SU}). For the following we recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1,\ldots,A_k)$ and the anti-diagonal matrix $\mathrm{antidiag}(A_1,\ldots,A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$ introduced on page 8.
In the case **L** we fix an arbitrary basis of V and define $W_{\mathrm{SL}} = \mathrm{diag}(\omega, \mathbb{1}_{n-1})$.

Obviously, we have $\det(W_{\mathrm{SL}}) = \omega$, and so we obtain $\mathrm{GL}_n(q) = \mathrm{SL}_n(q) : \langle W_{\mathrm{SL}} \rangle$. In the case **U** the choice of W_{SU} relies on the fixed ordered basis. In this thesis we will use several bases, and so we list the different chosen elements W_{SU} for them. For this, we recall Proposition 1.2.3 (i) for the existence of the following ordered bases, and we also recall that here we have $\mathrm{o}(\omega) = q^2 - 1$.
For an orthonormal basis of V we define $W_{\mathrm{SU}} = \mathrm{diag}(\omega^{q-1}, \mathbb{1}_{n-1})$. Note, that W_{SU} is a member of $\mathrm{GU}_n(q^2)$, by Lemma 1.2.8 (a). Since we have that $\det(W_{\mathrm{SU}}) = \omega^{q-1}$ is a primitive $(q+1)$-th root of unity, we obtain $\mathrm{GU}_n(q^2) = \mathrm{SU}_n(q^2) : \langle W_{\mathrm{SU}} \rangle$.
For a fixed basis, such that the matrix of the non-degenerate unitary form on V is $\mathrm{antidiag}(\mathbb{1}_{\frac{n}{2}}, \mathbb{1}_{\frac{n}{2}})$ (only if n is even), we define $W_{\mathrm{SU}} = \mathrm{diag}(\omega, \mathbb{1}_{\frac{n}{2}-1}, \omega^{-q}, \mathbb{1}_{\frac{n}{2}-1})$. Again, via Lemma 1.2.8 (a) we see that W_{SU} is a member of $\mathrm{GU}_n(q^2)$. Since we have $|\det(\mathrm{GU}_n(q^2))| = q + 1$ and $\det(W_{\mathrm{SU}}) = \omega^{1-q}$ is a primitive $(q+1)$-th root of unity, we obtain that $\langle \mathrm{SU}_n(q^2), W_{\mathrm{SU}} \rangle = \mathrm{GU}_n(q^2)$. Furthermore, we note that $\mathrm{GU}_n(q^2) = \mathrm{SU}_n(q^2) : \langle W_{\mathrm{SU}}^{q-1} \rangle$ if q is even.
For a positive integer k where $n > 2k$ and a fixed ordered basis, such that the matrix of the non-degenerate unitary form on V is $\mathrm{antidiag}(\mathbb{1}_k, \mathbb{1}_{n-2k}, \mathbb{1}_k)$, we define $W_{\mathrm{SU}} = \mathrm{diag}(\omega, \mathbb{1}_{k-1}, \mathbb{1}_{n-2k}, \omega^{-q}, \mathbb{1}_{k-1})$. Analogously to above, we obtain that $W_{\mathrm{SU}} \in \mathrm{GU}_n(q^2)$ and $\langle \mathrm{SU}_n(q^2), W_{\mathrm{SU}} \rangle = \mathrm{GU}_n(q^2)$.
The automorphism of $\mathrm{SL}_n(q)$ (or $\mathrm{SU}_n(q^2)$) obtained via conjugation by the element W_{SL} (or W_{SU}) is also denoted by W_{SL} (or W_{SU}), and we call it the diagonal automorphism W_{SL} (or W_{SU}) of $\mathrm{SL}_n(q)$ (or $\mathrm{SU}_n(q^2)$). The automorphisms of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$ induced by W_{SL} and W_{SU} are denoted by W_{PSL} and W_{PSU} and we call them the diagonal automorphisms W_{PSL} and W_{PSU} of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$, respectively. So, we clearly have (recall also Proposition 1.2.15)

$$\mathrm{PGL}_n(q) = \langle \mathrm{PSL}_n(q), W_{\mathrm{PSL}} \rangle \leq \mathrm{Aut}(\mathrm{PSL}_n(q)) \text{ and}$$
$$\mathrm{PGU}_n(q^2) = \langle \mathrm{PSU}_n(q^2), W_{\mathrm{PSU}} \rangle \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2)).$$

In the case **L** we also denote by W_{SL} an automorphism of $\mathrm{SL}(V)$ which corresponds to the automorphism W_{SL} of $\mathrm{SL}_n(q)$ in a representation with respect to an arbitrary ordered basis. Also, we call W_{SL} a diagonal automorphism of $\mathrm{SL}(V)$. In the case **U** for W_{SU} we do so as well, but for the sake of well-definedness we restrict to orthonormal bases of V. As above, the automorphisms of $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$ induced by W_{SL} and W_{SU} we denote by W_{PSL} and W_{PSU}, respectively; and we call them also diagonal automorphisms. Abusing slightly notation, we will also drop the subscript and only write W for the diagonal automorphism W_{SL}, W_{SU}, W_{PSL} or W_{PSU} of the respective group.

Remark 1.2.17. (a) In the books [KL, see p. 21, 23] and [BHR, see Subsection 1.7.1] diagonal automorphisms δ are introduced (although in [KL] they are not termed diagonal automorphisms, cf. also Remark 1.2.18 (below) for the approach in this book). We note that in the case **L** our introduced diagonal automorphism W coincides with δ. In [KL] and [BHR], in the case **U** an orthonormal basis is chosen as the standard basis in the definition of the generating automorphisms of the automorphism group. We note that our choice of the diagonal automorphism W (in the case **U** with respect to an orthonormal basis) coincides with δ. For the importance of the declaration of the chosen basis of V, see Remark 1.2.18, below.

(b) We note that in the literature often the automorphisms of $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q^2)$, $\mathrm{PSL}_n(q)$, or $\mathrm{PSU}_n(q^2)$ belonging to $\langle W_{\mathrm{SL}}\rangle$, $\langle W_{\mathrm{SU}}\rangle$, $\langle W_{\mathrm{PSL}}\rangle$, or $\langle W_{\mathrm{PSU}}\rangle$ or which are induced via conjugation by other diagonal matrices in $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q^2)$, $\mathrm{PGL}_n(q)$, or $\mathrm{PGU}_n(q^2)$ are called diagonal automorphisms of $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q^2)$, $\mathrm{PSL}_n(q)$, or $\mathrm{PSU}_n(q^2)$, respectively.

Field automorphisms of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ *and* $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$
(cf. [KL, p. 10, 20-21, 23] and [BHR, p. 26, 34])

From (1.2.1), we recall the canonical surjective homomorphism σ from $\Gamma\mathrm{L}(V)$ to $\mathrm{Aut}(\mathrm{GF}(q^u))$ with kernel $\mathrm{GL}(V)$. Let $B = (b_1, \ldots, b_n)$ be an ordered basis of V and in the case **U** let the matrix J_B of the non-degenerate unitary form on V have all entries in $\mathrm{GF}(p)$. Clearly, each element of $\mathrm{GL}(V)$ is determined by its action on B. Analogously, each element g of $\Gamma\mathrm{L}(V)$ is determined by its action on B together with $\sigma(g)$, via

$$\Big(\sum_{i=1}^{n} \lambda_i b_i\Big) g = \sum_{i=1}^{n} \lambda_i^{\sigma(g)} (b_i g) \text{ for } \lambda_i \in \mathrm{GF}(q^u). \tag{1.2.10}$$

For $\varphi \in \mathrm{Aut}(\mathrm{GF}(q^u))$ we define $\overline{\varphi}_B$ to be the unique element of $\Gamma\mathrm{L}(V)$ which lies in $\sigma^{-1}(\varphi)$ and which stabilizes each element of B. Hence, we have that $(\sum_{i=1}^{n} \lambda_i b_i)\overline{\varphi}_B = \sum_{i=1}^{n} \lambda_i^{\varphi} b_i$ for $\lambda_i \in \mathrm{GF}(q^u)$; and we set that $\overline{\varphi}_B$ also denotes the respective element in $\Gamma\mathrm{L}_n(q^u)$ ($\Gamma\mathrm{L}_n(q^u)$ considered with respect to the ordered basis B). We note that in the case **U** the map $\overline{\varphi}_B$ is a member of $\Gamma\mathrm{U}(V)$ and $\Gamma\mathrm{U}_n(q^2)$ (the latter case with respect to the fixed basis B), since we have that J_B has only entries in $\mathrm{GF}(p)$. Recall the notation of the Frobenius automorphism of $\mathrm{GF}(q^u)$ by φ_p. By the definition of $\overline{\varphi}_B$, we have that $\mathrm{GL}_n(q) : \langle \overline{\varphi_p}_B \rangle = \Gamma\mathrm{L}_n(q)$; and since $\Delta\mathrm{U}_n(q^2) = \mathrm{GU}_n(q^2) \cdot \mathrm{Z}(\mathrm{GL}_n(q^2))$ (see the tables from Proposition 1.2.14, or [KL, p. 23]), we also have that $\Delta\mathrm{U}_n(q^2) : \langle \overline{\varphi_p}_B \rangle = \Gamma\mathrm{U}_n(q^2)$.
We obtain automorphisms of $\mathrm{GL}(V) \cong \mathrm{GL}_n(q)$ and $\mathrm{GU}(V) \cong \mathrm{GU}_n(q^2)$ via the conjugation action by $\overline{\varphi}_B$ (cf. the $\mathrm{A}(V,\kappa)$-invariance in sequence (1.2.7)). Now, we consider how these automorphisms act on the respective group. For this, let $A = (a_{ij})_{n\times n}$ be an element of $\mathrm{GL}_n(q)$ in case **L** and $\mathrm{GU}_n(q^2)$ in case **U** (both, $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q^2)$ considered with respect to B). By A^{φ}, we denote the matrix where the entries of A are replaced by their image under φ. For the conjugation action by $\overline{\varphi}_B$ on A we obtain for all $v = \sum_{i=1}^{n} \lambda_i b_i \in V$

$$vA^{\overline{\varphi}_B} = \Big(\sum_{i=1}^{n} \lambda_i^{\varphi^{-1}} b_i\Big) A \overline{\varphi}_B = \sum_{i=1}^{n} \lambda_i \Big(\sum_{j=1}^{n} a_{ij}^{\varphi} b_j\Big) = vA^{\varphi}.$$

Hence, abusing notation, we denote by φ the automorphisms on $\mathrm{GL}_n(q)$, $\mathrm{GL}(V)$, $\mathrm{GU}_n(q^2)$ and $\mathrm{GU}(V)$ induced via the conjugation action by $\overline{\varphi}_B$. We note that without using the $\mathrm{A}(V,\kappa)$-invariance in sequence (1.2.7), we also see that A^φ is a member of $\mathrm{GU}_n(q^2)$ for $A \in \mathrm{GU}_n(q^2)$, by Lemma 1.2.8 (a). Obviously, we also get automorphisms on $\mathrm{SL}_n(q)$, $\mathrm{SL}(V)$, $\mathrm{SU}_n(q^2)$ and $\mathrm{SU}(V)$ by φ. The automorphisms on $\mathrm{PSL}_n(q)$, $\mathrm{PSL}(V)$, $\mathrm{PSU}_n(q^2)$ and $\mathrm{PSU}(V)$ induced by φ we will also denote by φ, and we will call φ a field automorphism of the respective group.
We note that $\mathrm{P\Delta U}_n(q^2) = \mathrm{PGU}_n(q^2)$. Hence, from our previous considerations we easily obtain

$$\mathrm{P\Gamma L}_n(q) = \mathrm{PGL}_n(q) : \langle \varphi_p \rangle \leq \mathrm{Aut}(\mathrm{PSL}_n(q)) \text{ and}$$

$$\mathrm{P\Gamma U}_n(q^2) = \mathrm{PGU}_n(q^2) : \langle \varphi_p \rangle \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))^3.$$

We note that the definition of φ relies heavily on the fixed ordered basis B. As a result, some classical unitary groups (which are described by field automorphisms) are not well-defined, see [BHR2] and the following Remark.

Remark 1.2.18. The paper of Bray, Holt and Roney-Dougal [BHR2] analyzes the mentioned problem about well-definedness of certain classical groups which comprise field automorphisms. More precisely, the extensions of $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q^2)$, $\mathrm{Sp}_n(q)$, $\Omega_n^\epsilon(q)$ and $\mathrm{SO}_n^\epsilon(q)$ (and of their projective versions) by φ_p are examined. In the case **L** there is no problem concerning well-definedness. In the case **U** they obtain that these groups are well-defined (up to isomorphism), except if q is odd and n is even where there are two isomorphism classes.
We will tackle this problem of well-definedness of the classical groups by using our notation introduced in Table 1.2.1. The notation from column three is used if the considered classical group is well-defined, i.e. unique up to isomorphism (so, not dependent on the choice of a certain basis), such as in $\langle \mathrm{PGU}(V), \varphi_p \rangle = \mathrm{P\Gamma U}(V)$. This notation is not exclusive, as we will also use the notation from column four without specifying the basis if the classical group is well-defined (cf. Remark 1.2.5 (a)). Clearly, we will use the notation from column four and indicate the specific basis if the considered classical group is not well-defined.
The book [BHR] handles this problem of well-definedness by choosing a certain fixed ordered basis for each considered case **U**, **Sp** and **O** (with additional conditions and cases in the case **O**), see [BHR, p. 18 and Table 1.1]. They define the classical groups with respect to the chosen bases (see [BHR, Subsection 1.6.2]), and do so as well in the introduction of the automorphisms (see [BHR, Subsection 1.7.1]). We note that the results of the book [BHR], presented in the tables of Chapter 8, are to be understood as related to the chosen bases for the different cases. Furthermore, we note that an orthonormal basis is chosen in the case **U**.

[3]Note, that this expression only holds for our dimension restriction $n \geq 3$ in case **U**, cf. Remark 1.2.19 (c), below.

In [KL], Kleidman and Liebeck introduce the classical groups not related to a specific basis. Then they define the generators for a group from sequence (1.2.7) modulo the preceding group. This corresponds to our approach of defining specific automorphisms (as diagonal, field and graph automorphisms) of $\Omega(V,\kappa)$ (respecting the dimension restrictions from [KL, Theorem 2.1.3.]), see [KL, p. 14, Theorem 2.1.4. and p. 20]. Defining these generators, Kleidman and Liebeck do choose a fixed basis, see [KL, p. 20-39]. In the case **U** they choose an orthonormal basis as the fixed basis (note that this is not the basis called standard, or unitary basis in the book, see [KL, p. 22]). Hence, in the case **U** they define the field automorphisms with respect to an orthonormal basis. The author could not find a statement which relates to the problem of well-definedness in that case. Moreover, in the book [KL] in the case **U**, the author could also not find mistakes in proofs or results which are presented ambiguously (cf. [BHR2, p. 172 (i)]) when another basis is chosen, such as in [KL, Lemma 4.1.9].
In this thesis it is not advantageous to fix a specific basis in the definition of the classical groups (such as an orthonormal basis), as well as choosing a standard basis in the definition of the automorphisms of the group $\mathrm{PSU}(V) \cong \mathrm{PSU}_n(q^2)$. Since we will calculate the centralizers in G of certain subgroups of G for almost simple groups G with socle isomorphic to $\mathrm{PSL}(V)$ or $\mathrm{PSU}(V)$, it is necessary to choose the basis for the individual situation suitable to obtain a beneficial representation for calculating.
Finally, we note that the field automorphisms in [KL] and [BHR] are denoted by ϕ, whereas we use the notation φ.

The graph automorphism of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ *for* $n \geq 3$

In the case **L** for $n \geq 3$ there is another type of automorphism of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ not belonging to the group generated by the automorphisms listed above. To describe it, let B be an ordered basis of V. For the representation of $\mathrm{GL}(V)$ with respect to B we obtain a non-trivial involutory automorphism τ via

$$\tau : \mathrm{GL}_n(q) \to \mathrm{GL}_n(q), A \mapsto A^{-1t}.$$

It is not hard to see that τ is also an automorphism of $\mathrm{SL}_n(q)$, and we call τ the graph or inverse transpose automorphism of $\mathrm{SL}(V) \cong \mathrm{SL}_n(q)$ (with respect to B). We may extend τ to a non-trivial involutory automorphism of $\Gamma\mathrm{L}_n(q)$ by defining that τ commutes with the field automorphisms, i.e. $(g\varphi_p^j)^\tau = g^\tau \varphi_p^j$ for $g \in \mathrm{GL}_n(q)$ and an integer j. Since τ is a non-trivial involutory automorphism of $\Gamma\mathrm{L}_n(q)$, we may form the split extension $\Gamma\mathrm{L}_n(q) : \langle\tau\rangle = \mathrm{A}\Gamma\mathrm{L}_n(q)$, cf. [KL, p. 21]. (For a more formally definition of τ using the dual space V^* of V, see [As, p. 507]).
We now exhibit an important property of the automorphism τ. For this purpose, let U be a non-trivial subspace of V of dimension k, i.e. $0 < k < n$. Choosing a suitable ordered basis B of V (where the first k elements form a basis of U),

we obtain that a subgroup H of $\mathrm{GL}(V)$ which stabilizes U can be represented (with respect to B) as

$$H \leq \left\{ \begin{pmatrix} A & 0_{k,n-k} \\ C & D \end{pmatrix} \middle| \begin{array}{l} A \in \mathrm{GL}_k(q), D \in \mathrm{GL}_{n-k}(q) \\ \text{and } C \in \mathrm{Mat}_{n-k,k}(q) \end{array} \right\} = K_k \leq \mathrm{GL}_n(q). \tag{1.2.11}$$

It is not hard to see that $H^\tau \leq K_k^\tau$ is a stabilizer of a subspace of V of dimension $n-k$. Hence, we obtain that the automorphism τ interchanges stabilizers of k-subspaces with stabilizers of $(n-k)$-subspaces. We consider the case $H = K_1 \cap \mathrm{SL}_n(q)$, i.e. H is the stabilizer of U in $\mathrm{SL}_n(q)$ where $\dim(U) = 1$. We easily see that the diagonal automorphism W_{SL} and the field automorphisms of $\mathrm{SL}_n(q)$ stabilize H. Now, elementary considerations show that no automorphism $g = g_{\mathrm{SL}} g_W g_\varphi$ of $\mathrm{SL}_n(q)$ where $g_{\mathrm{SL}} \in \mathrm{Inn}(\mathrm{SL}_n(q))$, $g_W \in \langle W_{\mathrm{SL}} \rangle$ and $g_\varphi \in \langle \varphi_p \rangle$ induces the graph automorphism of $\mathrm{SL}_n(q)$ (recall, that $n \geq 3$). (Here, we note that it is also possible to argue as in Remark 1.2.19 (d), below).
The automorphism of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ induced by τ we also denote by τ and call it the graph or inverse transpose automorphism of $\mathrm{PSL}(V) \cong \mathrm{PSL}_n(q)$ (with respect to B). By above considerations, we easily deduce that

$$\mathrm{PA\Gamma L}_n(q) = \mathrm{P\Gamma L}_n(q) : \langle \tau \rangle \leq \mathrm{Aut}(\mathrm{PSL}_n(q)).$$

Remark 1.2.19. (a) We note that we have introduced the graph automorphism analogously to [KL, see p. 21] and [BHR, see Subsection 1.7.1] (although we have introduced it more detailed). In [KL] it is denoted by ι whereas [BHR] denoted it by γ. Furthermore, we note that in [BHR] in case **U** if $n \geq 3$ a graph automorphism is also introduced, cf. also part (c) of this remark.

(b) The denotation of τ as a graph automorphism has its origin in Lie theory. The groups $\mathrm{PSL}_n(q)$ for $n \geq 2$ are isomorphic to a specific class of groups (denoted by $\mathrm{A}_{n-1}(q)$) of the finite Chevalley groups, see [Car, p. 184-185]. For each member of the finite Chevalley groups there is a specific diagram, called the Dynkin diagram. From symmetries of the Dynkin diagram, there arise automorphisms of the related Chevalley group, called graph automorphisms, see [Car, p. 200]. In case $\mathrm{PSL}_n(q)$ the (non-trivial) symmetry of the Dynkin diagram reflects the property that the associated graph automorphism maps stabilizers of the k-subspaces of the underlying vector space to stabilizers of the $(n-k)$-subspaces and vice versa, as observed above.
We note that the excluded cases $\mathrm{Sp}_4(q)$ and $\Omega_8^+(q)$ in Proposition 1.2.15 arise, because of the occurrence of such graph automorphisms (c.f. the remark following Proposition 1.2.15). For more information, see [Car, Chapter 12].

(c) In case $n = 2$ we have also an inverse transpose automorphism acting on $\mathrm{SL}_n(q)$. Observe that this automorphism coincides with the inner automorphism of $\mathrm{SL}_2(q)$ induced via conjugation by $x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider case **U** for $n \geq 2$ (here we relax our restriction on the dimension and consider also dimension two). If the matrix of the non-degenerate unitary form is $\mathbb{1}_n$, we have that $A^{\varphi_p^a} = A^{-1t}$ for $A \in \mathrm{SU}_n(q^2)$, see Lemma 1.2.8 (a) and recall that $q = p^a$. Hence, an inverse transpose automorphism of $\mathrm{SU}_n(q^2)$ also exists, which coincides with the field automorphism φ_p^a. For $n = 2$ it also coincides with the inner automorphism induced via conjugation by $x \in \mathrm{SU}_2(q^2)$ where x is defined as above.

(d) We note some observations arising from the property that τ maps the stabilizer of a k-subspace to a stabilizer of an $(n-k)$-subspace. These observations will be useful later. For this, we also provide the terminology of a maximal flag. A *maximal flag* $F_{(V_i)_{0\leq i\leq n}} = F$ in the vector space V is a chain of i-dimensional subspaces V_i of V where $\{0\} = V_0 < V_1 < \ldots < V_n = V$. We say that an element $g \in \mathrm{GL}(V)$ *stabilizes* $F_{(V_i)_{0\leq i\leq n}}$ if it stabilizes each subspace V_i of V. For $S \leq \mathrm{GL}(V)$ and a maximal flag F in V we call the subgroup of S consisting of all elements in S which stabilize F the *stabilizer* in S of F.

Now, consider in (1.2.11) that $H = K_k \cap \mathrm{SL}_n(q)$ is the stabilizer of U in $\mathrm{SL}_n(q)$. Recall, that $\mathrm{char}(\mathrm{GF}(q)) = p$. It is not hard to see that H has a subgroup N which is the normalizer of the Sylow p-subgroup S of $\mathrm{SL}_n(q)$ in $\mathrm{SL}_n(q)$ where S consists of the lower triangular matrices with 1 at the diagonal entries. Clearly, N is the stabilizer in $\mathrm{SL}_n(q)$ of a maximal flag in V. We define the element $A = \mathrm{antidiag}(a_1, \ldots, a_n)$ for $a_i \in \{\pm 1\}$ such that $A \in \mathrm{SL}_n(q)$. By elementary calculations, we see that $H^{\tau A}$ also possesses the subgroup N. If $n = 2k$ then clearly $H^{\tau A} = H$. But if $n \neq 2k$ we have that $H^{\tau A} \neq H$, and so $H^{\tau A}$ is not conjugate to H in $\mathrm{SL}_n(q)$, by Lemma 1.4.5, below.

From our above investigations, we note two corollaries. The first one concerns the automorphism groups of $\mathrm{PSL}(V)$ and $\mathrm{PSU}(V)$.

Corollary 1.2.20. *Let V be an n-dimensional vector space over a finite field of characteristic p. Let $n \geq 2$ in case* **L** *and $n \geq 3$ in case* **U**. *Then the following hold.*

$$\mathrm{Aut}(\mathrm{PSL}(V)) = \begin{cases} \langle \mathrm{PGL}(V), \varphi_p, \tau\rangle = (\mathrm{PGL}(V) : \langle \varphi_p\rangle) : \langle \tau \rangle = \\ \ \mathrm{PGL}(V) : (\langle \varphi_p\rangle \times \langle \tau\rangle) = \mathrm{PA\Gamma L}(V) & \text{if } n \geq 3, \\ \langle \mathrm{PGL}(V), \varphi_p\rangle = \mathrm{PGL}(V) : \langle \varphi_p \rangle = \mathrm{P\Gamma L}(V) & \text{if } n = 2. \end{cases}$$

$$\mathrm{Aut}(\mathrm{PSU}(V)) = \langle \mathrm{PGU}(V), \varphi_p\rangle = \mathrm{PGU}(V) : \langle \varphi_p\rangle = \mathrm{P\Gamma U}(V).$$

Proof. The assertion follows from above examinations, together with Proposition 1.2.15. Or, using Lie theory, see [Car, Chapter 12, esp. Theorem 12.5.1.]. □

Remark 1.2.21. We note some facts concerning the automorphism groups of $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$. In Corollary 1.2.20 we have only considered the automorphism groups of the projective versions of $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$. Recall, that we have also introduced the diagonal, field and graph automorphisms of $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$. We note that the automorphism groups for the quasisimple groups $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$ can be considered as subgroups of the respective automorphism group of their projective version (see [BHR, Lemma 1.3.4] for a reference). So, there are actually no "further" automorphisms on $\mathrm{SL}(V)$ and $\mathrm{SU}(V)$, as on their projective versions. Hence, we obtain that $\mathrm{Aut}(\mathrm{SL}(V))$ is generated by $\mathrm{Inn}(\mathrm{SL}(V)) \cong \mathrm{PSL}(V)$, W_{SL}, φ_p and τ (τ only if $n \geq 3$); and $\mathrm{Aut}(\mathrm{SU}(V))$ is generated by $\mathrm{Inn}(\mathrm{SU}(V)) \cong \mathrm{PSU}(V)$, W_{SU} and φ_p.

Next, we collect the information about the structure of the outer automorphism group of the simple groups $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$. For this purpose, let the notation $\ddot{\ }$ denote the reduction modulo $\mathrm{PSL}_n(q)$ in case **L** and $\mathrm{PSU}_n(q^2)$ in case **U**. We note that in case **U** we will use the diagonal automorphism W_{PSU} of $\mathrm{PSU}_n(q^2)$ for the description of the structure. So, let the representation of $\mathrm{SU}(V)$ be with respect to a basis which is provided in the definition of the diagonal automorphisms, above.

Corollary 1.2.22. *Let $q = p^a$ where p is a prime and a is a positive integer. Let $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q^2)$ be simple groups and in case **U** let $n \geq 3$. Then the following hold.*

$$\mathrm{Out}(\mathrm{PSL}_n(q)) = \begin{cases} \langle \ddot{W}_{\mathrm{PSL}}, \ddot{\varphi}_p, \ddot{\tau} \rangle = (\langle \ddot{W}_{\mathrm{PSL}} \rangle : \langle \ddot{\varphi}_p \rangle) : \langle \ddot{\tau} \rangle = \\ \langle \ddot{W}_{\mathrm{PSL}} \rangle : (\langle \ddot{\varphi}_p \rangle \times \langle \ddot{\tau} \rangle) \cong \boldsymbol{Z}_{(n,q-1)} : (\boldsymbol{Z}_a \times \boldsymbol{Z}_2) & \textit{if } n \geq 3, \\ \langle \ddot{W}_{\mathrm{PSL}}, \ddot{\varphi}_p \rangle = \langle \ddot{W}_{\mathrm{PSL}} \rangle \times \langle \ddot{\varphi}_p \rangle \cong \boldsymbol{Z}_{(2,q-1)} \times \boldsymbol{Z}_a & \textit{if } n = 2, \end{cases}$$

where $\mathrm{o}(\ddot{W}_{\mathrm{PSL}}) = (n, q-1)$, $\mathrm{o}(\ddot{\varphi}_p) = a$, $\mathrm{o}(\ddot{\tau}) = 2$, $\ddot{W}_{\mathrm{PSL}}^{\ddot{\varphi}_p} = \ddot{W}_{\mathrm{PSL}}^{p}$ *and* $\ddot{W}_{\mathrm{PSL}}^{\ddot{\tau}} = \ddot{W}_{\mathrm{PSL}}^{-1}$.

$$\mathrm{Out}(\mathrm{PSU}_n(q^2)) = \langle \ddot{W}_{\mathrm{PSU}}, \ddot{\varphi}_p \rangle = \langle \ddot{W}_{\mathrm{PSU}} \rangle : \langle \ddot{\varphi}_p \rangle \cong \boldsymbol{Z}_{(n,q+1)} : \boldsymbol{Z}_{2a},$$

where $\mathrm{o}(\ddot{W}_{\mathrm{PSU}}) = (n, q+1)$, $\mathrm{o}(\ddot{\varphi}_p) = 2a$ *and* $\ddot{W}_{\mathrm{PSU}}^{\ddot{\varphi}_p} = \ddot{W}_{\mathrm{PSU}}^{p}$.

Proof. Without using the tables from Proposition 1.2.14, the only mentionable is the proof of $|\langle \ddot{W}_{\mathrm{PSU}} \rangle| = (n, q+1)$ which is not executed in [KL]; the rest follows from above considerations and by elementary observations, cf. [KL, Propositions 2.2.3. and 2.3.5.]. To prove $|\langle \ddot{W}_{\mathrm{PSU}} \rangle| = (n, q+1)$, let Z denote $\mathrm{Z}(\mathrm{GL}_n(q^2))$. We note that $\mathrm{GU}_n(q^2) \cap Z = \langle \mathrm{diag}(\lambda, \dots, \lambda) \rangle$ where $\lambda \in \mathrm{GF}(q^2)$ with $\mathrm{o}(\lambda) = q+1$. Hence, $|\det(Z \cap \mathrm{GU}_n(q^2))| = \frac{q+1}{(n,q+1)}$. Furthermore, we have $(\mathrm{SU}_n(q^2) \cdot Z) \cap \mathrm{GU}_n(q^2) = \mathrm{SU}_n(q^2)(Z \cap \mathrm{GU}_n(q^2))$, by the Dedekind modular law. So, we obtain

$$\begin{aligned} |\langle \ddot{W}_{\mathrm{PSU}} \rangle| &= |\mathrm{P\ddot{G}U}_n(q^2)| = |\mathrm{GU}_n(q^2) \cdot Z / (\mathrm{SU}_n(q^2) \cdot Z)| \\ &= |\mathrm{GU}_n(q^2) / ((\mathrm{SU}_n(q^2) \cdot Z) \cap \mathrm{GU}_n(q^2))| \\ &= (q+1) / |\det(\mathrm{SU}_n(q^2)(Z \cap \mathrm{GU}_n(q^2)))| = (n, q+1). \end{aligned}$$

□

Next, we note an explicit example of an automorphism group which finds repeated application later, such as in Propositions 2.1.26 or 2.8.7.

Example 1.2.23. Let $X = \mathrm{PSL}_3(4)$. Then we have

$$\mathrm{Aut}(X) = ((X : \langle W_{\mathrm{PSL}} \rangle) : \langle \varphi_2 \rangle) : \langle \tau \rangle = \mathrm{PGL}_3(4) : (\langle \varphi_2 \rangle \times \langle \tau \rangle),$$

by Corollary 1.2.20. Using Corollary 1.2.22, we see that

$$\mathrm{Out}(X) = (\langle \ddot{W}_{\mathrm{PSL}} \rangle : \langle \ddot{\varphi}_2 \rangle) : \langle \ddot{\tau} \rangle = \langle \ddot{W}_{\mathrm{PSL}} \ddot{\varphi}_2 \ddot{\tau} \rangle : \langle \ddot{\tau} \rangle \cong D_{12}$$

where D_{12} denotes the dihedral group of order 12. Hence, by elementary considerations, we have that up to conjugacy in $\mathrm{Aut}(X)$ there are only the following subgroups G of $\mathrm{Aut}(X)$ which contain X.

$$(1.)\ G = X, \quad (2.)\ G = X : \langle \varphi_2 \rangle, \quad (3.)\ G = X : \langle \tau \rangle, \quad (4.)\ G = X : \langle \varphi_2 \tau \rangle,$$

$$(5.)\ G = \mathrm{PGL}_3(4), \quad (6.)\ G = X : \langle W_{\mathrm{PSL}} \varphi_2 \tau \rangle, \quad (7.)\ G = X : (\langle \varphi_2 \rangle \times \langle \tau \rangle),$$

$$(8.)\ G = \mathrm{P\Gamma L}_3(4), \quad (9.)\ G = \mathrm{PGL}_3(4) : \langle \tau \rangle, \quad (10.)\ G = \mathrm{Aut}(X).$$

Furthermore, among these G is not a normal subgroup of $\mathrm{Aut}(X)$ if and only if case (2.), (3.), or (7.) holds.

Finally, we provide the following definition for later use.

Definition 1.2.24. Let the conditions from Corollary 1.2.22 hold, and let X be either $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$. Then we define the following subgroup of $\mathrm{Aut}(X)$

$$K_{\mathrm{Aut}} = \begin{cases} \langle W_{\mathrm{PSL}}, \varphi_p \rangle & \text{if } X = \mathrm{PSL}_2(q), \\ \langle W_{\mathrm{PSL}}, \varphi_p, \tau \rangle & \text{if } X = \mathrm{PSL}_n(q) \text{ and } n \geq 3, \\ \langle W_{\mathrm{PSU}}, \varphi_p \rangle & \text{if } X = \mathrm{PSU}_n(q^2). \end{cases}$$

Remark 1.2.25. Using the notation from the last definition, we note the following. Considering Corollary 1.2.20, we see that $\mathrm{Aut}(X) = \langle X, K_{\mathrm{Aut}} \rangle = X \cdot K_{\mathrm{Aut}}$. So, we obtain

$$\ddot{K}_{\mathrm{Aut}} = (K_{\mathrm{Aut}} \cdot X)/X = \mathrm{Out}(X) \cong K_{\mathrm{Aut}}/(X \cap K_{\mathrm{Aut}}).$$

We note that in general $K_{\mathrm{Aut}} \cap X > 1$.

1.2.3 Standard notation

In this thesis we consider almost simple groups with socle isomorphic to $\mathrm{PSL}(V)$ or $\mathrm{PSU}(V)$. So, here we introduce an appropriate notation to provide facts as well as state and prove assertions for the two cases **L** and **U** (recall p. 15-16) simultaneously. We also recall the previously introduced notation which will be used frequently in this thesis.

Let $q > 1$ be a prime power and let $u \in \{1, 2\}$ where $u = 1$ in case **L** and $u = 2$ in case **U**. Let V be an n-dimensional vector space over a finite field of order q^u. We introduce a notation used also in the books [KL] and [BHR]. The symbols $+$ and $-$ are used to distinguish the two cases **L** and **U**. We shall write

$$\mathrm{GL}^{\pm}(V) = \begin{cases} \mathrm{GL}^{+}(V) = \mathrm{GL}(V) & \text{in case } \mathbf{L}, \\ \mathrm{GL}^{-}(V) = \mathrm{GU}(V) & \text{in case } \mathbf{U}. \end{cases}$$

$$\mathrm{GL}_n^{\pm}(q^u) = \begin{cases} \mathrm{GL}_n^{+}(q^u) = \mathrm{GL}_n(q) & \text{in case } \mathbf{L}, \\ \mathrm{GL}_n^{-}(q^u) = \mathrm{GU}_n(q^2) & \text{in case } \mathbf{U}. \end{cases}$$

For the groups $\mathrm{SL}(V)$, $\mathrm{SU}(V)$, $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q^2)$ we will introduce this notation analogously. We also define $\epsilon \in \{\pm\}$ where $\epsilon = +$ in case **L** and $\epsilon = -$ in case **U**, as it is useful for describing the structure of a group, cf. [KL, p. 86]. Note, that the previously introduced notation is available for all dimensions n and all prime powers $q > 1$.
As we will consider almost simple groups with socle isomorphic to $\mathrm{PSL}(V)$ or $\mathrm{PSU}(V)$, we shall introduce for this situation a specific notation for our work (esp. for the groups occurring in the sequences (1.2.7) and (1.2.9)). For the following let f be a non-degenerate unitary, or trivial form on V. Furthermore, let the following restrictions hold. We require for the dimension n of V that $n \geq 2$ in case **L** and $n \geq 3$ in case **U**, in view of Proposition 1.2.11. (We note that these dimension restrictions are also common in [KL, see p. 80] and [BHR, see p. 59]). Furthermore, let $\mathrm{P}\Omega(V, f)$ be a (non-abelian) simple group. So, $(n, q) \notin \{(2, 2), (2, 3)\}$ in case **L**, $(n, q^2) \neq (3, 2^2)$ in case **U** and $\Omega(V, f)$ is quasisimple, see Proposition 1.2.12. In view of Table 1.2.1, we now introduce the following notation which we call the *standard notation*.
By omitting to write the symbol f for the form on V, we denote the groups from column three of Table 1.2.1 by

$$\Omega(V) = \begin{cases} \mathrm{SL}(V) & \text{in case } \mathbf{L}, \\ \mathrm{SU}(V) & \text{in case } \mathbf{U}, \end{cases} \quad \mathrm{I}(V) = \begin{cases} \mathrm{GL}(V) & \text{in case } \mathbf{L}, \\ \mathrm{GU}(V) & \text{in case } \mathbf{U}, \end{cases}$$

$$\Delta(V) = \begin{cases} \mathrm{GL}(V) & \text{in case } \mathbf{L}, \\ \Delta\mathrm{U}(V) & \text{in case } \mathbf{U}, \end{cases} \quad \Gamma(V) = \begin{cases} \Gamma\mathrm{L}(V) & \text{in case } \mathbf{L}, \\ \Gamma\mathrm{U}(V) & \text{in case } \mathbf{U}, \end{cases}$$

$$\mathrm{A}(V) = \begin{cases} \mathrm{A}\Gamma\mathrm{L}(V) & \text{in case } \mathbf{L}, \\ \Gamma\mathrm{U}(V) & \text{in case } \mathbf{U}. \end{cases}$$

The notation from column four of Table 1.2.1 we will introduce analogously, by omitting to write the symbol V for the vector space. So, e.g. we will write $\Omega = \mathrm{SL}_n(q)$ in case **L** and $\Omega = \mathrm{SU}_n(q^2)$ in case **U**. The notation for the projective versions of the groups from Table 1.2.1, we will introduce analogously by preceding each notation from above with the symbol P. Here, we recall that the projection map is denoted by P, see (1.2.8). Hence, we may write $\mathrm{P}\Omega = \mathrm{PSL}_n(q)$ in case **L** and $\mathrm{P}\Omega = \mathrm{PSU}_n(q^2)$ in case **U**. Furthermore, recall

Convention 1.2.2 for the notation of the full preimage under P and the notation for the image of a matrix under P.
If we want to use the above notation (together with its restrictions) for a specific situation, say for $\Omega(V)$ in case **U**, we simply write $\Omega(V) = \mathrm{SU}(V)$. Sometimes, we will relax our restriction for the groups $\mathrm{P}\Omega(V)$ to be (non-abelian) simple, but still keep the dimension restrictions. For this situation we also introduce the above notation and call it *generalized standard notation*. When we use the generalized standard notation, we will note it explicitly; otherwise, using the above notation it is assumed that it represents the standard notation.
We note that the above introduced notation is based on the notation from [KL, see p. 14-15]. In contrast to [KL], we recall that we have introduced the two notations from column three and four of Table 1.2.1 to indicate if a group is considered with respect to an ordered basis, or not (see Remarks 1.2.5 and 1.2.18). Also recall Remarks 1.2.1 and 1.2.5 for the differences between our notation and the notation from the books [KL] and [BHR].
Recall from Proposition 1.2.15 that we can identify the group $\mathrm{PA}(V)$ with the group $\mathrm{Aut}(\mathrm{P}\Omega(V))$. So, our notation will also find application in the situation of describing subgroups of an automorphism group. Furthermore, we recall the notations for the diagonal, field and graph automorphisms of $\mathrm{P}\Omega(V) \cong \mathrm{P}\Omega$ and $\Omega(V) \cong \Omega$ by W, φ and τ (the graph automorphism only in case **L** of dimension $n \geq 3$), as introduced in the previous subsection. We also recall that the field automorphism of the respective group, induced by the Frobenius automorphism of $\mathrm{GF}(q^u)$, we denote by φ_p if the characteristic of $\mathrm{GF}(q^u)$ is p. Finally, we introduce that in case **U** we will also write φ_q for the non-trivial involutory field automorphism φ_p^a if $q^2 = p^{2a}$.

1.3 Linear algebra and lemmas about finite fields

In the first two subsections of this section, we provide and recall some terminology and results from linear algebra which are important in this thesis. Furthermore, we recall that linear maps act on the right in this thesis if nothing else is assumed. Finally, in the third subsection, we state some lemmas about finite fields.

1.3.1 Some basic results

In this subsection, we provide some basic results from linear algebra. Without citation, by the first two lemmas we recall two well-known facts. Here, we note that in the first lemma we assume that linear maps act on the left, since it is advantageous in a later consideration.

Lemma 1.3.1. *Let V be an n-dimensional vector space over the finite field $\mathrm{GF}(q)$, and let $B = (b_1, \ldots, b_n)$ be an ordered $\mathrm{GF}(q)$-basis of V. Let $\varphi : V \to V$ be a $\mathrm{GF}(q)$-endomorphism of V, and $g \in \mathrm{GL}(V)$. For the following let linear maps act on the left. We recall that $M_\varphi^{B,l} := (m_{ij})_{1 \leq i,j \leq n}$ where $\varphi(b_j) =$*

$\sum_{i=1}^{n} m_{ij} b_i$ *for* $j \in \{1, \ldots, n\}$ *is the matrix which represents* φ *with respect to* B *(or in short, the matrix of* φ *with respect to* B*). Define the ordered* GF(q)*-basis* $B' = (b'_1, \ldots, b'_n)$ *of* V *where* $b'_j = g b_j$ *for* $j \in \{1, \ldots, n\}$*. Then, if* $M_\varphi^{B',l}$ *denotes the matrix of* φ *with respect to* B'*, we have*

$$M_\varphi^{B',l} = (M_g^{B,l})^{-1} \cdot M_\varphi^{B,l} \cdot M_g^{B,l} = (M_\varphi^{B,l})^{M_g^{B,l}}$$

where $M_g^{B,l}$ *denotes the matrix of* g *with respect to* B*.*

Remark 1.3.2. It is not hard to deduce a version of the last lemma for the case that linear maps act on the right. For this, if $M_\varphi^{B,r}$ denotes the matrix of the endomorphism $\varphi : V \to V$ (acting on the right) with respect to B, we recall that $M_\varphi^{B,r} = (M_\varphi^{B,l})^t$.

Lemma 1.3.3. *Let* $M \in \mathrm{Mat}_{m,n}(q)$ *for integers* $m, n \geq 1$ *and a prime power* q*. Let the rank of* M *be* r*. Then there exist elements* $A \in \mathrm{GL}_n(q)$ *and* $B \in \mathrm{GL}_m(q)$ *such that*

$$BMA = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}.^{4}$$

For the following we recall the notation φ_q introduced in Remark 1.2.7 (a) (or, see the end of Subsection 1.2.3).

Lemma 1.3.4. *Let* $M \in \mathrm{Mat}_n(q^2)$ *for an integer* $n \geq 1$ *and a prime power* q*. Let the rank of* M *be* r*. If* $M = M^{t\varphi_q}$ *then there exists an element* $A \in \mathrm{GL}_n(q^2)$ *such that*

$$A^{t\varphi_q} M A = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. The assertion follows from [SS, p. 364 and 374-376, esp. 71.11 Satz] (use also [Hu, II. 8.7 Hilfssatz]). ☐

Corollary 1.3.5. *Let* GF(q^2) *be a finite field of odd order* q^2*. Let* λ *denote a primitive element of* GF(q^2)**, and set* $\omega = \lambda^{\frac{q+1}{2}}$ *(esp.* $\omega^q = -\omega$*). Define the subgroups* $T_1 = \{N \mid N \in \mathrm{Mat}_n(q^2), N^{t\varphi_q} = N\}$ *and* $T_2 = \{N \mid N \in \mathrm{Mat}_n(q^2), N^{t\varphi_q} = -N\}$ *of* $(\mathrm{Mat}_n(q^2), +)$ *where* n *is a positive integer. Then we have that*

$$\eta : T_1 \to T_2, \ N \mapsto (\omega \mathbb{1}_n) N$$

is a group isomorphism. Furthermore, if $M \in T_2$ *is of rank* r *then there exists an element* $A \in \mathrm{GL}_n(q^2)$ *such that*

$$A^{t\varphi_q} M A = \begin{pmatrix} \omega\mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

[4] From the beginning of this chapter we recall that $\begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}$ denotes the matrix $0_{m,n}$ if $r = 0$, $\mathbb{1}_r$ if $r = n = m$, $\begin{pmatrix} \mathbb{1}_r \\ 0_{m-r,r} \end{pmatrix}$ if $r = n < m$ and $\begin{pmatrix} \mathbb{1}_r & 0_{r,n-r} \end{pmatrix}$ if $r = m < n$.

Proof. By easy observations, we see that η is a well-defined group homomorphism, and obviously η is bijective. So, the assertion follows by Lemma 1.3.4, since there is an element $A \in \mathrm{GL}_n(q^2)$ such that $\begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} = A^{t\varphi_q}(M^{\eta^{-1}})A = A^{t\varphi_q}(\omega^{-1}\mathbb{1}_n)MA$. □

1.3.2 Tensor products

(Cf. [BHR, Section 1.9], [Wil2, p. 82-83], [KL, p. 47 and § 4.4] and [Hu, V. 9.11 Satz]). Next, we provide some facts about tensor products. In this subsection, let K be a finite field of prime power order q. Let W_1 and W_2 be K-vector spaces of dimensions d_1 and d_2 where $d_1, d_2 \geq 1$ and let $B_{W_1} = (b_{1,1}, \dots, b_{1,d_1})$ and $B_{W_2} = (b_{2,1}, \dots, b_{2,d_2})$ be ordered K-bases of W_1 and W_2, respectively. For the *tensor product space* $V = W_1 \otimes W_2$ (which is a K-vector space of dimension $d_1 \cdot d_2$) we have a canonical basis $B_V = \{b_{1,i} \otimes b_{2,j} \mid 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$. We introduce on B_V the lexicographical order, such as for $d_1, d_2 \geq 2$ we have $B_V = (b_{1,1} \otimes b_{2,1}, b_{1,1} \otimes b_{2,2}, \dots, b_{1,1} \otimes b_{2,d_2}, b_{1,2} \otimes b_{2,1}, \dots, b_{1,d_1} \otimes b_{2,d_2})$. In this thesis we will always use the lexicographical order for a basis of a tensor product space of two vector spaces, as described before. We note that for $w_1, w_1' \in W_1$, $w_2, w_2' \in W_2$ and $\lambda \in K$ we have

$$\begin{aligned} (w_1 + w_1') \otimes w_2 &= w_1 \otimes w_2 + w_1' \otimes w_2, \\ w_1 \otimes (w_2 + w_2') &= w_1 \otimes w_2 + w_1 \otimes w_2', \\ (\lambda w_1) \otimes w_2 &= \lambda(w_1 \otimes w_2) = w_1 \otimes (\lambda w_2). \end{aligned} \tag{1.3.1}$$

Let $\phi_i : W_i \to W_i$ for $i \in \{1, 2\}$ be linear maps. Then we obtain a linear map $\phi_1 \otimes \phi_2 : V \to V$ via $(w_1 \otimes w_2)(\phi_1 \otimes \phi_2) = w_1\phi_1 \otimes w_2\phi_2$ for $w_i \in W_i$ and then extending linearly. Let $A = (a_{ij})_{1 \leq i,j \leq d_1}$ and $C = (c_{ij})_{1 \leq i,j \leq d_2}$ be the corresponding $d_1 \times d_1$-matrix and $d_2 \times d_2$-matrix of ϕ_1 and ϕ_2 with respect to B_{W_1} and B_{W_2}, respectively. We obtain by elementary considerations (using (1.3.1)) that the matrix of the linear map $\phi_1 \otimes \phi_2$ on V with respect to the lexicographical ordered basis B_V is the *Kronecker product* $A \otimes C := (M_{ij})_{1 \leq i,j \leq d_1}$ where M_{ij} is the $d_2 \times d_2$-matrix $a_{ij}C$. (We note that the Kronecker product can be introduced more generally, see [BHR, Definition 1.9.1]). For $M_i \subseteq \mathrm{Mat}_{d_i}(q)$ $(i \in \{1, 2\})$ we introduce the notation $M_1 \otimes M_2 = \{m_1 \otimes m_2 \mid m_i \in M_i\}$.
We note the following lemma concerning the Kronecker product. For this, we recall that t denotes the transpose map on a matrix, and for $\varphi \in \mathrm{Aut}(K)$, by abusing notation, φ denotes the map on a matrix which replaces all matrix entries by their image under φ.

Lemma 1.3.6. *Let n and m be positive integers. For $A, B \in \mathrm{Mat}_n(q)$, $C, D \in \mathrm{Mat}_m(q)$ and $\varphi \in \mathrm{Aut}(K)$ the following hold.*

(i) $(A \otimes C)^t = A^t \otimes C^t$.

(ii) $(A \otimes C)^\varphi = A^\varphi \otimes C^\varphi$.

(iii) $(A \otimes C)(B \otimes D) = AB \otimes CD$.

(iv) *For invertible* A, C *we have that* $(A \otimes C)^{-1} = A^{-1} \otimes C^{-1}$, *and hence* $(B \otimes D)^{(A \otimes C)} = B^A \otimes D^C$.

(v) $\det(A \otimes C) = (\det A)^m (\det C)^n$.

For the following we need the terminology of a central product.

Definition 1.3.7. (cf. [Go, p. 29 and Chap. 3 Theorem 7.2], [Hu, I. 9.10 Satz], or [Su, Chapter 2 (4.17)])
Let G, H be finite groups. Let $Z_G \leq \mathrm{Z}(G)$, $Z_H \leq \mathrm{Z}(H)$ and η be an isomorphism from Z_G to Z_H. The *central product* of G and H (with respect to Z_G, Z_H and η) is $G \circ H = (G \times H)/Z$ where $Z = \{(z, (z^{-1})^\eta) \mid z \in Z_G\}$. If the choice of Z_G, Z_H, or η is clear we omit indicating it.

By our previous considerations, we have an action ρ of $\mathrm{GL}_{d_1}(q) \times \mathrm{GL}_{d_2}(q)$ on V, via $(v, (A, B)) \mapsto v(A \otimes B) = \sum k_{ij}((b_{1,i}A) \otimes (b_{2,j}B))$ for $A \in \mathrm{GL}_{d_1}(q)$, $B \in \mathrm{GL}_{d_2}(q)$ and $v = \sum k_{ij}(b_{1,i} \otimes b_{2,j}) \in V$ where $k_{ij} \in K$. Since for $\lambda \in K$ we have $(\lambda \mathbb{1}_{d_1}) \otimes \mathbb{1}_{d_2} = \lambda(\mathbb{1}_{d_1} \otimes \mathbb{1}_{d_2}) = \mathbb{1}_{d_1} \otimes (\lambda \mathbb{1}_{d_2})$, it is not hard to deduce that $M = \{(\lambda \mathbb{1}_{d_1}, \lambda^{-1}\mathbb{1}_{d_2}) \mid \lambda \in K^*\} \leq \mathrm{Z}(\mathrm{GL}_{d_1}(q)) \times \mathrm{Z}(\mathrm{GL}_{d_2}(q))$ coincides with the kernel of ρ. The quotient of $\mathrm{GL}_{d_1}(q) \times \mathrm{GL}_{d_2}(q)$ by M is a central product of $\mathrm{GL}_{d_1}(q)$ and $\mathrm{GL}_{d_2}(q)$. So, we obtain an embedding of $\mathrm{GL}_{d_1}(q) \circ \mathrm{GL}_{d_2}(q)$ in $\mathrm{GL}_{d_1 d_2}(q)$ and by identification we can write $\mathrm{GL}_{d_1}(q) \circ \mathrm{GL}_{d_2}(q) \leq \mathrm{GL}_{d_1 d_2}(q)$. By our previous observations, it is not hard to see that $\mathrm{Z}(\mathrm{GL}_{d_1}(q)) \circ \mathrm{Z}(\mathrm{GL}_{d_2}(q)) = \mathrm{Z}(\mathrm{GL}_{d_1 d_2}(q))$. So, we can consider the projective image under P and obtain that

$$\mathrm{PGL}_{d_1}(q) \times \mathrm{PGL}_{d_2}(q) \leq \mathrm{PGL}_{d_1 d_2}(q). \tag{1.3.2}$$

Naturally, we can extend the construction of the tensor product space of two vector spaces to several vector spaces. For this, let $t \geq 2$ be an integer and for $j \in \{1, \dots, t\}$ let W_j be a d_j-dimensional vector space over K where $d_j \geq 1$ with ordered K-bases $B_{W_j} = (b_{j,1}, \dots, b_{j,d_j})$. For the *tensor product space* $V = W_1 \otimes \ldots \otimes W_t$, which is a K-vector space of dimension $n = \prod_{j=1}^t d_j$, we have a canonical basis $B_V = \{b_{1,j_1} \otimes \ldots \otimes b_{t,j_t} \mid 1 \leq j_i \leq d_i\}$. Introducing on B_V the lexicographical order, we obtain analogous results as above. Particularly, we make explicit that we have canonical inclusions (by identification)

$$\mathrm{GL}_{d_1}(q) \circ \ldots \circ \mathrm{GL}_{d_t}(q) \leq \mathrm{GL}_n(q) \tag{1.3.3}$$

and

$$\mathrm{PGL}_{d_1}(q) \times \ldots \times \mathrm{PGL}_{d_t}(q) \leq \mathrm{PGL}_n(q). \tag{1.3.4}$$

For the following let V be an n-dimensional vector space over K. Let L be an extension field of K. Regarding L as a K-vector space, we can form the tensor product space $V_L = V \otimes L$. Naturally, V_L can also be considered as an n-dimensional vector space over L, via $\lambda(v \otimes \mu) = v \otimes \lambda\mu$ for $v \in V$ and $\lambda, \mu \in L$. For an ordered K-basis $B_V = (b_1, \dots, b_n)$ of V we obtain an ordered L-basis of

V_L by $B_{V_L} = (b_1 \otimes 1, \ldots, b_n \otimes 1)$. For $G \leq \mathrm{GL}(V, K)$ we have that G acts on V_L, via $(v \otimes \mu)g = vg \otimes \mu$ for $v \in V$, $g \in G$ and $\mu \in L$. Hence, we can write $G \leq \mathrm{GL}(V_L, L)$ (by identification).
Let G be irreducible on the K-vector space V. Then we call G *absolutely irreducible* in $\mathrm{GL}(V, K)$ if G remains irreducible in $\mathrm{GL}(V_L, L)$ for all extension fields L of K.

1.3.3 Some lemmas about finite fields

Next, we provide some useful facts about finite fields for later use.

Lemma 1.3.8. *Let r be a prime and $a > 0$ be an integer. Let $\mathrm{GF}(q)$ be a finite field of order $q = r^a$ and $\varphi \in \mathrm{Aut}(\mathrm{GF}(q))$. Then the following hold.*

(i) The equation

$$\lambda^2 = (\lambda^2)^\varphi \ (*)$$

holds for all $\lambda \in \mathrm{GF}(q)$ if and only if $\varphi = 1$.

(ii) The equation

$$\lambda^\varphi = \lambda^{-1} \ (**)$$

holds for all $\lambda \in \mathrm{GF}(q)^*$ if and only if $q \in \{2, 3\}$ and so $\varphi = 1$, or $q = 4$ and φ is the Frobenius automorphism of $\mathrm{GF}(4)$.

(iii) The equation

$$(\lambda^2)^\varphi = \lambda^{-2}$$

holds for all $\lambda \in \mathrm{GF}(q)^*$ if and only if $q \in \{2, 3, 5\}$ and so $\varphi = 1$, or $q \in \{4, 9\}$ and φ is the Frobenius automorphism of $\mathrm{GF}(q)$.

Proof. The if-parts of the three assertions are clear. Conversely, let $\varphi = \varphi_r^i$ for an $i \in \{0, \ldots, a-1\}$ where φ_r denotes the Frobenius automorphism of $\mathrm{GF}(q)$.
To prove assertion (i), let λ be a primitive element of $\mathrm{GF}(q)^*$ (so, $\mathrm{o}(\lambda) = q-1$). Considering $(*)$, we see that $\lambda^{2(r^i-1)} = 1$, hence $r^a - 1$ divides $2(r^i - 1)$. By elementary calculations (note, that $i < a$), we now can deduce that $i = 0$, so assertion (i) is established.
Next, we prove assertion (ii). From $(**)$ we directly obtain that $r \in \{2, 3\}$, since φ centralizes the subfield of $\mathrm{GF}(q)$ isomorphic to $\mathrm{GF}(r)$. Let $r = 3$ and λ_3 be a primitive element of $\mathrm{GF}(q)^*$. By $(**)$, we have that $\lambda_3^{3^i} = \lambda_3^{-1}$. Hence, $3^a - 1$ divides $3^i + 1$. For $i = 0$ we obviously obtain that $a = 1$ and for $i > 1$ we have a contradiction. Now, let $r = 2$ and λ_2 be a primitive element of $\mathrm{GF}(q)^*$. Analogously to above we obtain that $2^a - 1$ divides $2^i + 1$. For $i = 0$ we see that $a = 1$ and for $i = 1$ we obtain that $a = 2$. Since for $i > 1$ there occurs a contradiction (for $a \geq 3$ we have $2^a - 1 > 2^{a-1} + 1$), assertion (ii) easily follows.
Finally, we consider assertion (iii). By an analogous argument as above, we see that $r \in \{2, 3, 5\}$. Considering each characteristic r separately, we obtain assertion (iii) analogously to above, by elementary calculations. □

Lemma 1.3.9. *Let r be a prime and $a > 0$ an integer. Let $\mathrm{GF}(q^2)$ be a finite field of order $q^2 = r^{2a}$ and $\varphi \in \mathrm{Aut}(\mathrm{GF}(q^2))$. Define*

$$M = \{\lambda \mid \lambda \in \mathrm{GF}(q^2)^*, \mathrm{o}(\lambda) \text{ divides } q+1\} \subseteq \mathrm{GF}(q^2)^*.$$

Then the following hold.

(i) The equation

$$\lambda^{\varphi} = \lambda \ (*)$$

holds for all $\lambda \in M$ if and only if $\varphi = 1$.

(ii) The equation

$$\lambda^{\varphi} = \lambda^{-1} \ (**)$$

holds for all $\lambda \in M$ if and only if φ is the non-trivial involutory automorphism of $\mathrm{GF}(q^2)$, i.e. the (unique) automorphism of $\mathrm{GF}(q^2)$ of order 2.

Proof. Since the if-parts are clear, we only have to prove the only-if-parts of our assertions. For a subset A of $\mathrm{Aut}(\mathrm{GF}(q^2))$ we define $\mathrm{GF}(q^2)_A = \{x \mid x \in \mathrm{GF}(q^2), x^{\eta} = x \text{ for all } \eta \in A\}$. By $(*)$, we have that $M \subseteq \mathrm{GF}(q^2)_{\varphi}$, and clearly $\mathrm{GF}(q^2)_{\varphi} = \mathrm{GF}(q^2)_{\langle\varphi\rangle}$ is a subfield of $\mathrm{GF}(q^2)$. Since the only subfield of $\mathrm{GF}(q^2)$ which includes the set M is $\mathrm{GF}(q^2)$, we obtain assertion (i).
To prove assertion (ii), let $\varphi = \varphi_r^i$ for an i where $0 \leq i < 2a$. Let $\lambda_{q+1} \in M$ be an element of order $q+1$. Since $(**)$ holds for λ_{q+1}, we obtain that $r^a + 1$ divides $r^i + 1$. For $0 \leq i \leq a$ we easily obtain $i = a$. So, suppose that $a < i < 2a$. Because $-1 \equiv r^i \equiv (-1)r^{i-a} \pmod{r^a+1}$, we see that $r^{i-a} - 1 \equiv 0 \pmod{r^a+1}$ where $1 \leq i-a < a$. Hence, a contradiction occurs and our assertion follows. □

Lemma 1.3.10. *Let p be a prime and r, a be positive integers where r is odd and $r \mid a$. Let $\mathrm{GF}(p^{2a})$ be a finite field and let $\mathrm{GF}(p^{\frac{2a}{r}})$ be a subfield of index r. Then we have that the non-trivial involutory automorphism of $\mathrm{GF}(p^{2a})$ restricted to $\mathrm{GF}(p^{\frac{2a}{r}})$ induces the non-trivial involutory automorphism of $\mathrm{GF}(p^{\frac{2a}{r}})$.*

Proof. By φ_p, we denote the Frobenius automorphism of $\mathrm{GF}(p^{2a})$. Let $a = k \cdot \frac{2a}{r} + l$ where l, k are integers and $0 \leq l < \frac{2a}{r}$ (note, that $0 < l$, since r is odd). For $x \in \mathrm{GF}(p^{\frac{2a}{r}})$ we obtain

$$x^{\varphi_p^a} = x^{p^a} = (x^{p^{k\frac{2a}{r}}})^{p^l} = x^{p^l} = x^{\varphi_p^l}.$$

Since φ_p^l acts non-trivially on $\mathrm{GF}(p^{\frac{2a}{r}})$, we easily obtain our assertion by

$$(x^{\varphi_p^l})^{\varphi_p^l} = x^{p^{2l}} = x^{p^{2k\frac{2a}{r}+2l}} = x.$$

□

1.4 Finite group theory

Next, we collect further group theoretic terminology, notation and results used in this thesis. We start by recalling the basic facts of finite group theory, such as the (three) Sylow theorems, the (three) isomorphism theorems for groups or the Dedekind modular law.

Now, we provide important well-known facts.

Lemma 1.4.1. ***(Schur's Lemma)*** *Let V be a $\mathrm{GF}(q)$-vector space for a prime power q, and H be an irreducible subgroup of $\mathrm{GL}(V)$. Then $\mathrm{C}_{\mathrm{End}_{\mathrm{GF}(q)}(V)}(H)$ is a field. (As usual, $\mathrm{End}_{\mathrm{GF}(q)}(V)$ denotes the endomorphism ring of V consisting of all $\mathrm{GF}(q)$-linear maps from V to V).*

Proof. The assertion follows by [As2, (12.4) (4)] (recall the well-known result of Wedderburn that every finite division ring is a field). □

Theorem 1.4.2. ***(Frobenius)*** *Let G be a finite group and H be a non-trivial subgroup of G. Let $H \cap H^g = 1$ for all $g \in G \setminus H$. Then*

$$K = G \setminus \bigcup_{g \in G} (H \setminus 1)^g$$

is a normal subgroup of G where $G = HK$ and $K \cap H = 1$, so $G = K \rtimes H$.

Proof. See [Hu, V. 7.6 Hauptsatz]. □

Definition 1.4.3. Let the conditions from the last theorem hold. Then we call G a *Frobenius* group (to H). Moreover, we call K the *Frobenius kernel* of G and H a *Frobenius complement* of G.

Next, we note two elementary observations.

Lemma 1.4.4. *Let n be a positive integer, p be a prime and set $q = p^a$ for a positive integer a. Then there is no non-trivial normal p-subgroup of $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q^2)$.*

Proof. $\mathrm{GL}_n(q)$ acts faithfully and irreducibly on $\mathrm{GF}(q)^n$ and $\mathrm{GU}_n(q^2)$ acts faithfully and irreducibly on $\mathrm{GF}(q^2)^n$, so the assertion follows. □

Lemma 1.4.5. *Let G be a finite group, and let S be a Sylow p-subgroup of G for a prime p. Then the following hold.*

(i) For a subgroup H of G with $\mathrm{N}_G(S) \le H \le G$ we have $\mathrm{N}_G(H) = H$.

(ii) For subgroups H_1 and H_2 of G such that $\mathrm{N}_G(S) \le H_1, H_2 \le G$ with $H_1 \ne H_2$ we have that H_1 is not conjugate to H_2 in G.

Proof. The assertions follow by elementary considerations, using the Sylow theorems (see e.g. [Uf, Lemma 1.1.8.]). □

For the following we recall the standard notation, introduced in Subsection 1.2.3. Also, we recall the projection map P from (1.2.8) and the notation ˆ for the full preimage under P, introduced in Convention 1.2.2.
Next, we note an elementary but important fact concerning the projection map P, which we will also sometimes use without reference to it.

Lemma 1.4.6. *Let* $H, K \leq \mathrm{A}(V)$ *and* $\mathrm{Z}(\mathrm{GL}(V)) \leq H$. *Then we have*

$$\mathrm{P}(H \cap K) = \mathrm{P}H \cap \mathrm{P}K.$$

Proof. One inclusion is obvious, so let $y\mathrm{Z}(\mathrm{GL}(V)) \in \mathrm{P}H \cap \mathrm{P}K$. Here, we have that there are elements $h \in H, k \in K$ and $z_1, z_2 \in \mathrm{Z}(\mathrm{GL}(V))$ with $y = hz_1 = kz_2$. So, we obtain our assertion by $k = hz_1z_2^{-1} \in H$. □

Remark 1.4.7. (a) We note that the assertion of the last lemma does not hold in general if $\mathrm{Z}(\mathrm{GL}(V)) \not\leq H$. To see this, let e.g. ω be a primitive element of $\mathrm{GF}(4)^*$ and consider $H = \langle h \rangle$ and $K = \langle k \rangle$ where $h = \mathrm{diag}(1, \omega, \omega^2), k = \mathrm{diag}(\omega^2, 1, \omega) \in \mathrm{SL}_3(4)$.

(b) Recalling Remark 1.2.1 (d), we see that the following assertion also holds. If $H, K \leq \Omega(V)$ such that $\mathrm{Z}(\Omega(V)) \leq H$ then $\mathrm{P}(H \cap K) = \mathrm{P}H \cap \mathrm{P}K$.

By the following lemma, we provide an assertion concerning the centralizer of certain p-subgroups of $\mathrm{PA}(V)$ in $\mathrm{PA}(V)$ for a prime p.

Lemma 1.4.8. *Let* H *be a* p*-subgroup of* $\mathrm{PA}(V)$ *for a prime* p *where* $p \nmid q^u - 1 = |\mathrm{Z}(\mathrm{GL}(V))|$. *Let* $S \in \mathrm{Syl}_p(\hat{H})$. *Then we have* $\hat{H} = S \times \mathrm{Z}(\mathrm{GL}(V))$ *(esp.* $\mathrm{P}S = H$ *and* $|S| = |H|$*) and*

$$\mathrm{C}_{\mathrm{PA}(V)}(H) = \mathrm{P}\mathrm{C}_{\mathrm{A}(V)}(S).$$

Proof. Let $|H| = p^a$ for a non-negative integer a. For $a = 0$ the assertion is trivial, hence let $a \geq 1$. Since $p \nmid |\mathrm{Z}(\mathrm{GL}(V))|$, we obtain by elementary considerations that $|\hat{H}| = p^a \cdot |\mathrm{Z}(\mathrm{GL}(V))|$. By the Sylow theorems, we easily obtain that S is the only Sylow p-subgroup of $\hat{H}$, so $\hat{H} = S \times \mathrm{Z}(\mathrm{GL}(V))$. It is obvious that $\mathrm{P}\mathrm{C}_{\mathrm{A}(V)}(S) \leq \mathrm{C}_{\mathrm{PA}(V)}(\mathrm{P}S)$. To prove that equality holds, let $\mathrm{P}s \in \mathrm{P}S$ and $\mathrm{P}g \in \mathrm{C}_{\mathrm{PA}(V)}(\mathrm{P}S)$ where $s \in S$ and $g \in \mathrm{A}(V)$. Clearly, we have $\mathrm{P}s = \mathrm{P}s^{\mathrm{P}g} = \mathrm{P}s^g$ if and only if $z = s^{-1}s^g$ for an appropriate $z \in \mathrm{Z}(\mathrm{GL}(V))$. Because g normalizes $\hat{H}$, we have that $s^g \in S$, by our previous considerations. Hence, $z = 1$ and our assertion follows. □

Next, we note an elementary but important fact, which we will also use without reference to it.

Lemma 1.4.9. *Let* G *be a finite group and* K, N *be subgroups of* G *where* N *is a normal subgroup of* G. *Let* p *be a prime. Then we have* $\mathrm{O}_p(K) \cap N = \mathrm{O}_p(K \cap N)$.

Proof. We obtain our assertion by elementary considerations, cf. [Uf, Lemma 1.1.23.]. □

1.4.1 Strong (p-)constraint

In this subsection, we introduce an important terminology for this thesis and list some facts. For this purpose, we provide the following definition.

Definition 1.4.10. {layer, generalized Fitting subgroup} For a finite group G we define the following characteristic subgroups.

(a) The *layer* $\mathrm{E}(G)$ of G is the subgroup of G generated by its quasisimple subnormal subgroups.

(b) We denote by $\mathrm{F}(G)$ the *Fitting subgroup* of G, i.e. the unique largest normal nilpotent subgroup of G.

(c) By $\mathrm{F}^*(G)$, we denote the *generalized Fitting subgroup* of G, which is the subgroup generated by the Fitting subgroup of G and the layer of G. So, we have $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$.

The following terminology will be of significant importance for our research.

Definition 1.4.11. {strongly (p-)constrained} Let G be a finite group. We call G *strongly p-constrained* if and only if $\mathrm{F}^*(G) = \mathrm{O}_p(G)$ for a prime p. G is called *strongly constrained* if and only if there is a prime p such that G is strongly p-constrained.

Remark. We note that if G is strongly p-constrained then G is p-constrained in the sense of [Go, see p. 268] (cf. also Proposition 1.4.18, below).

We note three easy consequences of the definition, and list some easy examples of strongly constrained groups.

Lemma 1.4.12. *Let G be a finite almost simple group with socle S. If M is a strongly constrained subgroup of G then $S \nleq M$.*

Lemma 1.4.13. *Let G be a strongly constrained group and H be a normal abelian subgroup of G. Then $H \leq \mathrm{O}_p(G)$ for a prime p.*

Lemma 1.4.14. *Let G be a finite group and H, K be subgroups of G such that $G = H \times K$. For a prime p we have that G is strongly p-constrained if and only if H and K are strongly p-constrained.*

Proof. Our assertion follows by elementary considerations. (Note, that $\mathrm{O}_r(G) = \mathrm{O}_r(H) \times \mathrm{O}_r(K)$ for a prime r). □

Example 1.4.15. We have that

(a) $\mathrm{GL}_2(2)$, $\mathrm{GU}_2(2^2)$ and $\mathrm{GU}_3(2^2)$ are strongly 3-constrained groups, and

(b) $\mathrm{GL}_2(3)$ and $\mathrm{GU}_2(3^2)$ are strongly 2-constrained groups.

Next, we provide an equivalent definition for a strongly p-constrained group, which we will use in this thesis also without reference to it. This definition will be (amongst other things) advantageous, if we prove by calculation that a group is strongly p-constrained. We provide the following theorem about the centralizer of the generalized Fitting subgroup of a finite group.

Theorem 1.4.16. ***(Bender)*** *Let G be a finite group. Then we have*

$$\mathrm{C}_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G).$$

Proof. See [As2, (31.13)]. □

Lemma 1.4.17. *Let G be a finite group and Q be a quasisimple subnormal subgroup of G. Let A be a subnormal subgroup of G where $Q \nleq A$. Then we have $[Q, A] = 1$.*

Proof. See [As2, (31.4)]. □

Proposition 1.4.18. **{*strongly p-constrained*}** *Let G be a finite group and p be a prime. Then the following conditions are equivalent*

(i) *G is strongly p-constrained,*

(ii) $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$.

Proof. (The following proof is adopted from a lecture by W. Knapp). By Theorem 1.4.16, we directly obtain the implication (i) to (ii). Now, let condition (ii) hold. Regarding Lemma 1.4.17, we easily deduce that $\mathrm{E}(G) \leq \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$. Hence, we have $\mathrm{E}(G) = 1$. Let r be a prime different from p. Since $[\mathrm{O}_r(G), \mathrm{O}_p(G)] = 1$, we obtain that $\mathrm{O}_r(G) \leq \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$, and hence $\mathrm{O}_r(G) = 1$. So, we have that G is strongly p-constrained. □

As a direct consequence from Theorem 1.4.16, we note the following corollary concerning strongly p-constrained groups.

Corollary 1.4.19. *Let G be a finite strongly p-constrained group for a prime p. If G is non-trivial then $\mathrm{O}_p(G)$ is non-trivial, i.e. if $|G| > 1$ then $|\mathrm{O}_p(G)| > 1$.*

Remark. From Corollary 1.4.19 we obtain that a finite non-trivial strongly constrained group is strongly p-constrained for only one prime p.

Next, we make explicit an elementary, but important lemma.

Lemma 1.4.20. *Let G be a finite almost simple group with socle S, so $S \leq G \leq \mathrm{Aut}(S)$. Let M be a subgroup of G, $s \in \mathrm{Aut}(S)$ and p be a prime. Then we have that M is a strongly p-constrained subgroup of G if and only if M^s is a strongly p-constrained subgroup of G^s.*

Proof. The assertion follows by elementary considerations from Definition 1.4.11, or Proposition 1.4.18. □

More general than Lemma 1.4.14, we provide the following important fact about strongly p-constrained groups.

Lemma 1.4.21. *Let G be a finite group and p be a prime. Then G is strongly p-constrained if and only if all subnormal subgroups of G are strongly p-constrained.*

Proof. See [Kn, Lemma 1.10]. □

At the end of this subsection, we provide a lemma concerning the layer, which is useful for later examinations.

Lemma 1.4.22. *Let H be a subgroup of* $\mathrm{GL}(V)$ *and* $\mathrm{E}(H) \neq 1$*. Then we have that* $\mathrm{E}(\mathrm{P}H) \neq 1$*.*

Proof. Because $\mathrm{E}(H) \neq 1$, there is a subnormal quasisimple subgroup N of H. Clearly, $\mathrm{P}N$ is a subnormal subgroup of $\mathrm{P}H$. So, we obtain our assertion by showing that $\mathrm{P}N$ is quasisimple. We have $\mathrm{P}N \cong N/N_0$ where $N_0 = \mathrm{Z}(\mathrm{GL}(V)) \cap N \leq \mathrm{Z}(N)$, and we define the canonical epimorphism $\phi : N \to N/N_0, A \mapsto A \cdot N_0/N_0$. Obviously, $\mathrm{Z}(\phi(N)) \geq \phi(\mathrm{Z}(N))$. Suppose that $\mathrm{Z}(\phi(N)) > \phi(\mathrm{Z}(N))$, then $M = \phi^{-1}(\mathrm{Z}(\phi(N)))$ is a normal subgroup of N where $\mathrm{Z}(N) < M < N$. Hence, we easily obtain a contradiction. So, we have $\mathrm{Z}(\phi(N)) = \phi(\mathrm{Z}(N))$. Now, using the third isomorphism theorem, we obtain our assertion by $\phi(N)/\mathrm{Z}(\phi(N)) \cong N/\mathrm{Z}(N)$. □

In the following subsection, we provide the basic terminology and notation as well as some important facts from finite permutation group theory for our later investigations.

1.4.2 Finite permutation groups

The terminology and notation about finite permutation groups in this thesis will be standard and (so) we keep with the book [Wie]. We also introduce further terminology and notation from finite permutation group theory which is necessary for our work. We recall the basic facts about permutation group theory, such as the observations in [Wie, Propositions 3.1-3.3].

For the following introduction of further terminology and notation we orientate on [Kn4, Abschnitt 1 and 2] and [Wie4] and we refer to these references for further information. Let G be a finite group and X be a finite set for the following.
If $G \leq \mathrm{Sym}(\mathrm{X})$ then G is called a *permutation group* on X. An *action* w of G on X is a map $w : \mathrm{X} \times G \to \mathrm{X}, (\chi, g) \mapsto w(\chi, g) = \chi^g$, such that $\chi^1 = \chi$ and $(\chi^g)^h = \chi^{gh}$ for all $\chi \in \mathrm{X}$ and $g, h \in G$. If there is an action w of G on X we say that G *acts* on X with respect to w and we write $(G, \mathrm{X})_w$ to denote this structure. If the role of w is clear or we do not want to specify it we also simply say G *acts* on X and drop the subscript w of $(G, \mathrm{X})_w$. If G acts on X (with respect to w) we also say that X is a *G-set* (with respect to w). A homomorphism $\rho : G \to \mathrm{Sym}(\mathrm{X})$ we call a *permutation representation* of G on X. A permutation representation ρ is called *faithful* if the kernel of ρ is 1. There is a canonical unique correspondence between the actions of G on X and the permutation representations of G on X, see [Wie4, Theorem 2.3]. The relation between permutation representations and permutation groups is also clear. For a permutation representation ρ of G on X we can consider G modulo the kernel of ρ as a permutation group on X (via identification); and a permutation group $G \leq \mathrm{Sym}(\mathrm{X})$ induces naturally a faithful permutation representation of G on X.

So, terminologies about permutation groups (such as transitivity, constituents or orbits (see [Wie, p. 4])) can canonical be transferred to the concept of actions of G on X and permutation representations of G on X. For precise definitions we refer to [Kn4, Abschnitt 1].
Let G act on X and let $\Pi \subseteq \mathrm{X}$. Then G_Π denotes the pointwise stabilizer of Π in G. For $\Pi = \{\pi_1, \dots, \pi_n\}$ we also write $G_\Pi = G_{\pi_1 \dots \pi_n}$. If Π is fixed by G (i.e. $\Pi^G = \{\pi^g \mid g \in G, \pi \in \Pi\} = \Pi$) we denote by G^Π the constituent of G on Π (note, that $G/G_\Pi \cong G^\Pi \leq \mathrm{Sym}(\Pi)$).
For finite groups G_1, G_2 and finite sets X_1, X_2 such that G_i acts on X_i for $i = 1, 2$ we define the following (cf. [Kn4, 1.9 Definition]). A pair (φ_1, φ_2) consisting of a group homomorphism $\varphi_1 : G_1 \to G_2$ and a map $\varphi_2 : \mathrm{X}_1 \to \mathrm{X}_2$ such that $\chi^g \varphi_2 = (\chi\varphi_2)^{g\varphi_1}$ for all $\chi \in \mathrm{X}_1$ and $g \in G_1$ we call a *homomorphism* from (G_1, X_1) in (G_2, X_2). A homomorphism (φ_1, φ_2) from (G_1, X_1) in (G_2, X_2) is called an *isomorphism* from (G_1, X_1) on (G_2, X_2) if and only if φ_1 is a group isomorphism and φ_2 is a bijective map. We write $(G_1, \mathrm{X}_1) \cong (G_2, \mathrm{X}_2)$ if there is an isomorphism from (G_1, X_1) on (G_2, X_2). As a concrete and important example we provide the following (cf. [Kn4, 1.11 Lemma (4)]). Let $K \leq G$ and let $G : K = \{Kg \mid g \in G\}$ denote the set of right cosets of K in G. Then G acts naturally and transitively on $G : K$ via

$$(Kg, h) \mapsto (Kg)^h = Kgh \tag{1.4.1}$$

for $Kg \in G : K$ and $h \in G$. If G acts transitively on X then we see that

$$(G, G : G_\chi) \cong (G, \mathrm{X}) \tag{1.4.2}$$

for every $\chi \in \mathrm{X}$ (for the group isomorphism choose the identity homomorphism and consider the well-defined bijective map $G : G_\chi \to \chi^G = \mathrm{X}, G_\chi g \mapsto \chi^g$).
For the following let G act transitively on X. Naturally, we can consider the action of G on $\mathrm{X}^2 = \mathrm{X} \times \mathrm{X}$ via $((\alpha, \beta), g) \mapsto (\alpha, \beta)^g = (\alpha^g, \beta^g)$ for $\alpha, \beta \in \mathrm{X}$ and $g \in G$. An orbit $O = (\alpha, \beta)^G \subseteq \mathrm{X}^2$ of G is called an *orbital* of (G, X). We call an orbital $(\alpha, \beta)^G$ of (G, X) *non-trivial* if $\alpha \neq \beta$, whereas we call $(\alpha, \alpha)^G$ the *diagonal* of X. If O is an orbital of (G, X) then $O' = \{(\alpha, \beta) \mid (\beta, \alpha) \in O\}$ is also an orbital of (G, X) and we call it the orbital *paired* with O. We note that $|O| = |O'|$. O is called *self-paired* if $O = O'$. For every $\alpha \in \mathrm{X}$ the map

$$O \mapsto O(\alpha) = \{\beta \in \mathrm{X} \mid (\alpha, \beta) \in O\}$$

is a bijective correspondence between the orbitals of (G, X) and the G_α-orbits on X, see [Kn4, Abschnitt 2, esp. 2.4 Proposition]. For all $g \in G$ and $\alpha \in \mathrm{X}$ we have $O(\alpha^g) = O(\alpha)^g$, so $|O(\alpha)| = d$ is not dependent on the choice of α and we call d the *length* of O (recall that G acts transitively on X). Hence, we can deduce that $|O| = |\mathrm{X}||O(\alpha)|$ for every $\alpha \in \mathrm{X}$. By definition, $O(\alpha)$ coincides with β^{G_α} if $O = (\alpha, \beta)^G$. So, recalling $G_{\alpha\beta} = G_{\beta\alpha}$ or $|O| = |O'|$, for $O = (\alpha, \beta)^G$ we obtain that

$$|O(\alpha)| = |G_\alpha : G_{\alpha\beta}| = |G_\beta : G_{\alpha\beta}| = |O'(\beta)|. \tag{1.4.3}$$

Let O be an orbital of (G, X). For an $\alpha \in \mathrm{X}$ we call a G_α-orbit $O(\alpha)$ on X a *suborbit* of (G, X), and so we also call the length $|O(\alpha)|$ of O a *subdegree* of

(G, X). The (transitive) constituent $G_\alpha^{O(\alpha)}$ is called a *subconstituent* of (G, X). A suborbit $O(\alpha)$, subdegree $|O(\alpha)|$ or subconstituent $G_\alpha^{O(\alpha)}$ of (G, X) is called *non-trivial* if O is non-trivial. Note, that a non-trivial subdegree $|O(\alpha)|$ of (G, X) can also have the value 1.
We say that G acts *primitively* on X if and only if G acts transitively on X and $G_\chi \lessdot G$ for a $\chi \in \mathrm{X}$. We note that the corresponding terminology of a primitive permutation group is equivalent to the terminology introduced by Wielandt in [Wie, see Theorems 7.4 and 8.2].

Next, we provide some important observations about primitive permutation groups.

Lemma 1.4.23. ***(Rudio, Wielandt)*** *Let G be a primitive permutation group on a finite set* X *and let $\emptyset \subset \Delta \subset \mathrm{X}$. Then for any $\alpha, \beta \in \mathrm{X}$ where $\alpha \neq \beta$ there exists a $g \in G$ with $\alpha \in \Delta^g$ and $\beta \notin \Delta^g$.*

Proof. See [Wie, Theorem 8.1] and cf. [Ru]. □

Using the last lemma, the following assertion can be proved.

Lemma 1.4.24. *Let G be a primitive permutation group on a finite set* X*, $\alpha \in \mathrm{X}$ and let $O(\alpha) \neq \{\alpha\}$ be a G_α-orbit on* X*. If the subgroup $H \neq 1$ of G leaves a point of* X *fixed, then there exists a $g \in G$ with $H^g \leq G_\alpha$ and $(H^g)^{O(\alpha)} \neq 1$.*

Proof. See [Wie, Proposition 18.1]. □

By Lemma 1.4.24, one can prove the following important theorem.

Theorem 1.4.25. ***(Jordan, Rudio, Wielandt)*** *Let the assumptions in the last lemma hold. Then any composition factor group of G_α (in the sense of the theorem of Jordan and Hölder) is isomorphic to a composition factor group of a subgroup of $G_\alpha^{O(\alpha)}$.*

Proof. See [Wie, Theorem 18.2], [Jor] and cf. [Kn4, p. 5]. □

As a direct consequence of the last theorem, we note the following weaker assertion.

Corollary 1.4.26. *Let G be a primitive permutation group on a finite set* X *and $\alpha \in \mathrm{X}$. Let d_{min} denote the minimal non-trivial subdegree of G. Then for every prime factor p of $|G_\alpha|$ we have $p \mid (d_{min})!$, especially $p \leq d_{min}$.*

Remark 1.4.27. (a) By the last corollary we can deduce that $|G_\alpha| = 1$ for the case $d_{min} = 1$, and $|G_\alpha| = 2^a$ for an $a > 0$ for the case $d_{min} = 2$. (For more precise assertions concerning these two cases see Theorem 3.1.3, below).

(b) We note that Theorem 1.4.25 has many other consequences and useful applications. Another corollary is that if $G_\alpha^{O(\alpha)}$ is soluble then G_α is also soluble, see [Wie, Theorem 18.3].

Next, we provide another useful fact.

Lemma 1.4.28. *Let G be a primitive permutation group on a finite set* X. *If G has a subdegree d which is a prime then d^2 does not divide the order of the stabilizer of a point.*

Proof. See [Si, Corollary 2.5]. □

Finally, we provide further useful facts for our investigations. For a finite group G we define $\mathrm{d}(G)$ to denote the minimal degree of all non-trivial permutation representations of G where non-trivial means that the kernel of the permutation representation is a proper subgroup of G. By $\mathrm{d}_f(G)$, we denote the minimal degree of all faithful permutation representations of G. If G is simple then we obviously have that $\mathrm{d}_f(G) = \mathrm{d}(G)$ and $\mathrm{d}_f(G)$ is the index of a largest proper subgroup of G. For classical simple groups G the values $\mathrm{d}_f(G)$ have been determined completely by Cooperstein, see [Co]; for some historical notes we refer to [KL, p. 174-175].
In the following proposition, we provide the values $\mathrm{d}_f(G)$ for the simple linear, unitary and symplectic groups G.

Proposition 1.4.29. *(see [Co, Table 1][5], [KL, p. 175] and [Ma])*
Let q be a prime power and n, m be positive integers. If G is a simple linear, unitary or symplectic group then the value $\mathrm{d}_f(G)$ is provided in the following table (also recall Proposition 1.2.11 and see Remark 2.2.10 (d) and Lemma 3.1.8, below).

G	$\mathrm{d}_f(G)$	*Conditions*
$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	$(n,q) \neq (2,5), (2,7), (2,9),$ $(2,11), (4,2)$
$\mathrm{PSL}_2(q)$	q	$q = 5, 7$ *or* 11
$\mathrm{PSL}_2(9) (\cong \mathrm{A}_6)$	6	
$\mathrm{PSL}_4(2) (\cong \mathrm{A}_8)$	8	
$\mathrm{PSU}_3(q^2)$	q^3+1	$q \neq 5$
$\mathrm{PSU}_3(5^2)$	50	
$\mathrm{PSU}_4(q^2)$	$(q^3+1)(q+1)$	
$\mathrm{PSU}_n(q^2)$	$\frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1}$	$n \geq 5,\ (n,q) \neq (2m, 2)$
$\mathrm{PSU}_n(2^2)$	$2^{n-1}(2^n-1)/3$	$n \geq 6$, n *even*
$\mathrm{PSp}_{2m}(q)$	$\frac{q^{2m}-1}{q-1}$	$m \geq 2,\ q > 2,\ (m,q) \neq (2,3)$
$\mathrm{PSp}_4(3)$	27	
$\mathrm{Sp}_{2m}(2)$	$2^{m-1}(2^m-1)$	$m \geq 3$
$\mathrm{Sp}_4(2)' (\cong \mathrm{A}_6)$	6	

[5] We note that some of the values provided in [Co] are false, and were corrected in [KL, p. 175] and further in [Ma].

For any finite group G and a subgroup N of G it is clear that $\mathrm{d}_f(N) \leq \mathrm{d}_f(G)$ holds. If N is a normal subgroup of G then $\mathrm{d}_f(G/N)$ is not always lower or equal to $\mathrm{d}_f(G)$, see e.g. [Ne]. But, the following lemma states that this is true if $N = \mathrm{O}_\pi(G)$ where π is any set of primes and $\mathrm{O}_\pi(G)$ denotes the largest normal π-subgroup of G.

Lemma 1.4.30. *Let G be a finite group and π be an arbitrary set of primes. Then we have*

$$\mathrm{d}_f(G/\mathrm{O}_\pi(G)) \leq \mathrm{d}_f(G).$$

Proof. The assertion follows by [Ho, Theorem 2]. □

Remark. We note that in [Ho] more general results are obtained as provided in the last lemma. Other mentionable works concerning this issue are the papers [EP] and [KP2].

1.4.3 Two order bounding propositions

In our further work, we encounter the following problem. We have to estimate the order of an abelian p-subgroup A (for a prime p) of a symmetric group of degree n, in terms of n and the rank k of A. Here, the *rank* of an abelian p-group A is the rank of an elementary abelian subgroup of maximum order, and we denote it by $\mathrm{rank}(A)$.
There are several results which describe the structure and order of maximal order abelian subgroups of S_n, see e.g. the paper of Bercov and Moser [BM], Dixon [Dix, Lemma 6], or the more general results obtained by Kovács and Praeger [KP]. But there has been no result of such an upper bound needed for our purposes. In [KU][6], W. Knapp and the author have considered this problem and the following considerations and results are extracted from this paper.

First, we mention two lemmas. The following lemma is a well-known fact, established by elementary calculus arguments.

Lemma 1.4.31. ***(Arithmetic-Geometric-Mean inequality)*** *Let $(a_i)_{1\leq i\leq m}$ be a sequence of positive real numbers of finite length m. Then $\prod_{i=1}^m a_i \leq (\frac{1}{m}\sum_{i=1}^m a_i)^m$ holds, with equality if and only if all a_i are equal.*

Proof. See [Cau, p. 457-459]. □

The next lemma states a result about the minimal faithful permutation degree $\mathrm{d}_f(A)$ of a finite abelian group A.

[6] Unfortunately, we have to note that the paper which was published differs from that which was handed in by the authors. In the published version, some elements of the layout and the notation were dropped (such as the description of a theorem, or a bracket), or changed inadequately (such as a cite on the first page). Furthermore, in the proof of [KU, Proposition 4] an expression "$t = 1$" was changed to "$t = 10$". But, the presented results in [KU] are taken over correctly from the version which was handed in.

Lemma 1.4.32. *Let A be a finite abelian group. Express $A \cong \mathbf{Z}_{p_1^{a_1}} \times \ldots \times \mathbf{Z}_{p_t^{a_t}}$ as its direct product decomposition into cyclic non-trivial groups of prime power order. Then $\mathrm{d}_f(A) = \sum_{j=1}^t p_j^{a_j}$.*

Proof. See [Joh] (or, the older papers [Po] or [Ore, Theorem 4]). □

Now, we easily obtain an upper bound for the order of an abelian p-subgroup A of a symmetric group of degree n, in terms of n and the rank k of A.

Proposition 1.4.33. *(see [KU, Theorem 1])*
Let p be a prime and let Ω be a set of finite cardinality $n \geq p$. Let A be an abelian p-subgroup of $\mathrm{Sym}(\Omega)$ and $k = \mathrm{rank}(A)$. Then $|A| \leq (\frac{n}{k})^k$.

Proof. Let $A \cong \mathbf{Z}_{p^{a_1}} \times \ldots \times \mathbf{Z}_{p^{a_k}}$ be its direct product decomposition into cyclic non-trivial groups. Using Lemmas 1.4.31 and 1.4.32, we easily obtain the assertion by

$$|A| = \prod_{j=1}^{k} p^{a_j} \leq \left(\frac{\mathrm{d}_f(A)}{k}\right)^k \leq \left(\frac{n}{k}\right)^k.$$

□

By further considerations and elementary calculations, a more precise upper bound can be obtained. Here, the results depend in addition on the prime p and in case $p = 3$, or 2 on the congruence of n modulo 3, or 4, respectively.

Proposition 1.4.34. *(see [KU, Theorem 2])*
Let p be a prime and let Ω be a set of finite cardinality $n \geq p$. Let A be an abelian p-subgroup of $\mathrm{Sym}(\Omega)$, $k = \mathrm{rank}(A)$ and $k_0 = \lfloor \frac{n}{p} \rfloor$. Then the following hold.

(i) – *For $p \geq 5$, or $p = 3$ and $3 \mid n$ define $t_n = 2$.*

– *For $p = 3$ and $n \equiv 1 \pmod 3$ define* $t_n = \begin{cases} 2 & \text{if } 4 \leq n \leq 61, \\ 3 & \text{if } 64 \leq n \leq 445, \\ 4 & \text{if } n \geq 448. \end{cases}$

– *For $p = 3$ and $n \equiv 2 \pmod 3$ define* $t_n = \begin{cases} 2 & \text{if } 5 \leq n \leq 23, \\ 3 & \text{if } 26 \leq n \leq 56, \\ 4 & \text{if } 59 \leq n \leq 122, \\ 5 & \text{if } 125 \leq n \leq 281, \\ 6 & \text{if } 284 \leq n \leq 893, \\ 7 & \text{if } n \geq 896. \end{cases}$

Then we have $|A| \begin{cases} = p^k & \text{if } k_0 - 1 \leq k \leq k_0, \\ \leq p^{k_0 - 1} & \text{if } k_0 - t_n + 1 \leq k \leq k_0 - 2 \\ & \text{(only for } p = 3\text{)}, \\ \leq (\frac{n}{k})^k & \text{if } 1 \leq k \leq k_0 - t_n. \end{cases}$

(ii) For $p = 2$ *define* $t_n = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 1 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$

$$\textit{Then we have } |A| \begin{cases} = 2^{k_0} & \text{if } k = k_0, \\ \leq 2^{k_0} & \text{if } \lfloor \frac{n}{4} \rfloor + t_n \leq k \leq k_0 - 1, \\ \leq 2^{k_0 - 1} & \text{if } k = \lfloor \frac{n}{4} \rfloor + t_n - 1, \\ \leq (\frac{n}{k})^k & \text{if } 1 \leq k \leq \lfloor \frac{n}{4} \rfloor + t_n - 2. \end{cases}$$

Remark. The remark following [KU, Theorem 2] deals with sharpness of the upper bound given in Proposition 1.4.34. For $p = 3$ part (a) of this remark provides a more precise upper bound for certain $k = \text{rank}(A)$. Parts (b) and (c) consider the case for $p = 2$ and $\lfloor \frac{n}{4} \rfloor + t_n - 1 \leq k \leq k_0 - 1$, and show that here the upper bound is sharp, except in two low degree cases. In part (d) of this remark, the problem is considered to estimate the order of an abelian p-subgroup A of S_n of rank $k \geq 1$ by $(\frac{n}{k})^k$ (this coincides also with the upper bound in Proposition 1.4.33). Here, we have that this upper bound is sharp if $n = k \cdot p^a$ for a positive integer a; and for $n \neq k \cdot p^a$ it is noted that further considerations may lead to more precise upper bounds in Propositions 1.4.34 and (obviously) 1.4.33.

1.5 Number theory

In this section, we provide some number theoretic terminology and results which will be useful for our purposes. First, we provide an important theorem due to Zsigmondy. For this, we define the following.

Definition 1.5.1. {Zsigmondy primes} Let $m, a > 1$ be integers. Prime numbers which divide $m^a - 1$ but not $m^i - 1$ for all $1 \leq i \leq a - 1$ are called *Zsigmondy primes* for $m^a - 1$. We denote a Zsigmondy prime for $m^a - 1$ by $z_{m,a}$.

Theorem 1.5.2. ***(Zsigmondy)*** *Let* $m > 1$ *and* $a > 2$ *be integers where* $(m, a) \neq (2, 6)$. *Then there exists a Zsigmondy prime* $z_{m,a}$.

Proof. See [Zs] or [Ar, Corollary 2]. ☐

We note a direct consequence of Definition 1.5.1.

Lemma 1.5.3. *Let* $m, a > 1$ *be integers and* $p = z_{m,a}$ *a Zsigmondy prime. Then we have* $a \mid p - 1$. *Hence,* $a \leq p - 1$.

Proof. Since a is the smallest positive integer with $m^a \equiv 1 \pmod p$, a is the order of m in the multiplicative group of the field $\mathbf{Z}_p$. So, our assertion follows easily, by Lagrange's theorem. ☐

Remark 1.5.4. (a) For m being a primitive element of $\mathbf{Z}_p^*$ in the last lemma we obviously have $a = p-1$. So, the upper bound for a by $p-1$ is sharp and cannot be improved in general.

(b) Another elementary argumentation for the estimate $a < p$ in the previous lemma is provided in [BHR, Lemma 1.13.3 (iii)].

Lemma 1.5.5. *Let $q > 1$ be a prime power and $n \geq 3$ be an odd integer. If $z = z_{q,n}$ is a Zsigmondy prime then z does not divide $|\mathrm{GU}_n(q^2)|$.*

Proof. We recall $|\mathrm{GU}_n(q^2)| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - (-1)^i)$ from Proposition 1.2.13 (b). Suppose that $z \mid |\mathrm{GU}_n(q^2)|$. Clearly, $z \nmid q^{\frac{n(n-1)}{2}} \prod_{i=1,\ i \text{ even}}^{n} (q^i - (-1)^i)$. Hence, we obtain that $z \mid q^j + 1$ for an odd j with $1 \leq j \leq n$. It is not hard to see that $j > \frac{n}{2}$. So, we have $n - j < \frac{n}{2}$. Considering $1 \equiv q^n \equiv -q^{n-j} \pmod{z}$, we now easily obtain a contradiction. □

For our purposes the following well-known fact is useful.

Lemma 1.5.6. *Let q, p be primes and $a, b \geq 1$ be integers satisfying*

$$p^a = q^b - 1,$$

then exactly one of the following holds.

(1) We have $p = 2$ and

(a) $q = 3, b = 2$ and $a = 3$, or

(b) $b = 1, a = 2^n$ for an integer $n \geq 0$ and q is a Fermat prime.

(2) We have $q = 2, a = 1, b$ is a prime and p is a Mersenne prime.

Proof. The assertion follows from [Su3, Part III, Theorem 2] together with the well-known facts that a is a power of 2 for q a Fermat prime in case (1)(b) and b is a prime for p a Mersenne prime in case (2). □

Next, we establish four number theoretic propositions which are important for later use. We note that nearly all of the results listed in the first and the third proposition have already been obtained by the author in [Uf].

Proposition 1.5.7. *Let m, p be primes and $a \geq 1$, $b \geq 0$ be integers. Set $q = m^a$ and let $n \geq 2$ be an integer. If the equation*

$$\tfrac{q^n - 1}{q - 1} = p^b \cdot \gcd(q-1, n) \quad (\star)$$

holds then $b > 0$ and we have the following.

(i) (a) If $a > 1$ or $n > 2$ then $p = z_{q,n}$ is a Zsigmondy prime.

(b) If $a > 1$ or $n > 2$ and $(m, a, n) \neq (2, 3, 2), (2, 2, 3)$ then $p = z_{m,an}$ is a Zsigmondy prime.

(ii) *n is a prime.*

(iii) *For $n = 2$ one of the following holds.*

(a) *$(m, a, p, b) = (2, 3, 3, 2)$, or $m = 2$, $a = 2^k$ where $k \geq 0$ is an integer, p is a Fermat prime and $b = 1$.*

(b) *$m \neq 2$, $p \neq 2 = \gcd(q-1, 2)$ and $p > 2a$.*

(c) *m is a Mersenne prime, $a = 1$, $p = 2$ and $b + 1$ is a prime.*

(iv) *For $n \geq 3$ we have $p > na$. Hence, p does not divide $\gcd(q-1, n)$, n and a.*

(v) *a is odd if $n \geq 3$ and $(q, n) \neq (4, 3)$.*

Proof. Since $n \geq 2$, we easily see $b > 0$. Assertion (i) follows immediately from Theorem 1.5.2 (note, that ($\star$) does not hold for $(m, a, n) = (2, 1, 6)$).
To prove (ii), suppose that $n = n_1 n_2$ where n_1,$n_2 > 1$ are integers. From (i)(a) we obtain that $p = z_{q,n}$ is a Zsigmondy prime. We examine the equation $\frac{q^n-1}{q-1} = (\frac{q^{n_1}-1}{q-1}) \cdot \sum_{j=0}^{n_2-1} q^{n_1 j} = \gcd(q-1, n) \cdot p^b$. Since $p = z_{q,n}$ is a Zsigmondy prime, p does not divide $\frac{q^{n_1}-1}{q-1}$. So, we have $\frac{q^{n_1}-1}{q-1} \mid \gcd(q-1, n)$ and a contradiction occurs.
Next, let $n = 2$. For $m = 2$ we obtain assertion (iii)(a) by Lemma 1.5.6 (1). For $m \neq 2$ we have to consider the two cases $4 \nmid q+1$ and $4 \mid q+1$. In the first case we have $p \neq 2 = \gcd(q-1, 2)$. For $a = 1$ our assertion (iii)(b) easily follows. Therefore, let $a > 1$. From (i)(b) we obtain that $p = z_{m,2a}$ is a Zsigmondy prime. So, the rest of assertion (iii)(b) follows by Lemma 1.5.3. In the latter case we have that $p = 2$ and we obtain assertion (iii)(c) by Lemma 1.5.6 (2).
It is easy to see that assertion (iv) holds for $(m, a, n) = (2, 2, 3)$. For $(m, a, n) \neq (2, 2, 3)$ we obtain from (i)(b) that $p = z_{m,an}$ is a Zsigmondy prime. Hence, our assertion follows by Lemma 1.5.3.
To see that assertion (v) is valid, suppose that a is even. By (i)(b), we have that $p = z_{m,an}$ is a Zsigmondy prime. Our assertion now follows analogously to (ii), considering $\frac{m^{na}-1}{m^a-1} = \frac{m^{\frac{na}{2}}+1}{m^{\frac{a}{2}}+1} \cdot \frac{m^{\frac{na}{2}}-1}{m^{\frac{a}{2}}-1}$ (note, that n is odd by (ii)). □

Remark. For our purposes the information given in Proposition 1.5.7 for $n \geq 3$ is sufficient. Further considerations lead to more precise results of this number theoretic problem, such as the following observation obtained in [Uf, Satz 2.2.5. (v)]. In the case that the equation $q^2 + q + 1 = p^b$ holds we have $3 \nmid m-1$, $3 \mid p-1$ (hence $p \geq 7$), a is a power of 3, b is odd and $\gcd(3, q-1) = 1$.

Proposition 1.5.8. *Let m, p be primes and $a \geq 1$, $b \geq 0$ be integers. Set $q = m^a$ and let $n \geq 3$ be an odd integer. If the equation*

$$\tfrac{q^n+1}{q+1} = p^b \cdot \gcd(q+1, n) \quad (\star)$$

holds then $b = 0$ if and only if $(q, n) = (2, 3)$; for $(q, n) \neq (2, 3)$ the following hold.

(i) $p = z_{m,2an}$ *is a Zsigmondy prime (hence, also* $p = z_{q,2n}$*).*

(ii) n *is a prime.*

(iii) We have $p > 2an$*. Hence,* p *does not divide* $\gcd(q+1, n)$*,* n *and* a*.*

Proof. It is not hard to see that $q^2 - q + 1 > q + 1$ if and only if $q \neq 2$. Hence, $\frac{q^n+1}{q+1} \geq q^2 - q + 1 > \gcd(q+1, n)$ for $q \neq 2$. So, we obtain $b = 0$ if and only if $(q, n) = (2, 3)$.
Let $(q, n) \neq (2, 3)$. By Theorem 1.5.2, we obtain assertion (i). To prove (ii), suppose that $n = n_1 n_2$ where $n_1, n_2 > 1$ are odd integers with $n_1 \leq n_2$. By (i), $p = z_{q,2n}$ is a Zsigmondy prime. For $(n_2, q) \neq (3, 2)$ we have that $\gcd(q+1, n) < \frac{q^{n_2}+1}{q+1}$. Here, we obtain the contradiction $p \mid q^{n_2} + 1$, considering the equation $\gcd(q+1, n) \cdot p^b = \frac{q^n+1}{q+1} = (\frac{q^{n_1 n_2}+1}{q^{n_2}+1}) \cdot (\frac{q^{n_2}+1}{q+1})$. Since $(\star)$ does not hold for $(n_1, n_2, q) = (3, 3, 2)$, we now get assertion (ii). Assertion (iii) follows immediately from (i) together with Lemma 1.5.3. □

Proposition 1.5.9. *Let* m, p *be primes and* $a \geq 1$*,* $b \geq 0$ *be integers. Set* $q = m^a$ *and let* $n \geq 2$ *be an integer. If the equation*

$$(q-1)^{n-1} = p^b \cdot \gcd(q-1, n)$$

holds then we have $b = 0$ *if and only if* $(m, a, n) = (3, 1, 2)$ *or* $(m, a) = (2, 1)$*; for* $b > 0$ *we have the following.*

(i) If $n = 2$ *then one of the following holds.*

- *(a)* $m = 2$*,* a *is a prime,* p *is a Mersenne prime and* $b = 1$*.*
- *(b)* $m \neq 2$*,* $p \neq 2 = \gcd(q-1, 2)$*,* a *is odd and* $a \leq \frac{p-1}{2}$*.*
- *(c)* $(m, a, p, b) = (3, 2, 2, 2)$*, or* m *is a Fermat prime,* $a = 1$*,* $p = 2$ *and* $b + 1 = 2^k$ *for an integer* $k \geq 1$*.*

(ii) If $n \geq 3$ *we have* $\gcd(q-1, n) = p^{b_0}$ *where* $b_0 \geq 0$ *is an integer and one of the following holds.*

- *(a)* $m = 2$*,* a *is a prime,* p *is a Mersenne prime and* $b + b_0 = n - 1$*.*
- *(b)* $(m, a, p, \frac{b+b_0}{n-1}) = (3, 2, 2, 3)$*, or* m *is a Fermat prime,* $a = 1$*,* $p = 2$ *and* $\frac{b+b_0}{n-1} = 2^k$ *where* $k \geq 0$ *is an integer.*

Proof. Easily, we see that $b = 0$ if and only if $(m, a, n) = (3, 1, 2)$ or $(m, a) = (2, 1)$.
Let $b > 0$ and $n = 2$. For $m = 2$ we obtain from Lemma 1.5.6 (2) all assertions of (i)(a). If m is odd we have to consider the two cases $4 \nmid q - 1$ and $4 \mid q - 1$. In the first case we have $p \neq 2 = \gcd(q-1, 2)$ and a is odd. For $a > 2$ we obtain by Theorem 1.5.2 that $p = z_{m,a}$ is a Zsigmondy prime, and so assertion (i)(b) follows from Lemma 1.5.3. In the latter case we have $p = 2$, and by Lemma 1.5.6 (1) we easily obtain assertion (i)(c).

Next, consider that $b > 0$ and $n \geq 3$. Since we have $(q-1)^{n-2} \mid p^b$, we obtain $\gcd(q-1, n) = p^{b_0}$ for a non-negative integer b_0. Now, by Lemma 1.5.6 we can deduce assertion (ii). □

Remark 1.5.10. (a) In Proposition 1.5.9 (i)(a) and (ii)(a), we have that $p = z_{2,a}$ is a Zsigmondy prime, see Theorem 1.5.2. So, we obtain an upper bound for a by $a < p$ (for $a \neq 2$ clearly $a \leq \frac{p-1}{2}$), considering Lemma 1.5.3. Here, obviously the exact value of a is given by $a = \frac{\ln(p+1)}{\ln(2)}$.

(b) We note that there are examples in Proposition 1.5.9 (i)(b) with $a = \frac{p-1}{2}$ and $b > 1$, such as $m = 3, a = 5, p = 11$ and $b = 2$.

Proposition 1.5.11. *Let m, p be primes and $a \geq 1$, $b \geq 0$ be integers. Set $q = m^a$ and let $n \geq 2$ be an integer. If the equation*

$$(q+1)^{n-1} = p^b \cdot \gcd(q+1, n)$$

holds then $b > 0$ and we have the following.

(i) For $n = 2$ one of the following holds.

(a) $(m, a, p, b) = (2, 3, 3, 2)$, or $m = 2$, $a = 2^k$ where $k \geq 0$ is an integer, p is a Fermat prime and $b = 1$.

(b) $m \neq 2$, $p \neq 2 = \gcd(q+1, 2)$ and $a \leq \frac{p-1}{2}$.

(c) m is a Mersenne prime, $a = 1$, $p = 2$ and $b + 1$ is a prime.

(ii) For $n \geq 3$ we have $\gcd(q+1, n) = p^{b_0}$ where $b_0 \geq 0$ is an integer and one of the following holds.

(a) $(m, a, p, b + b_0) = (2, 3, 3, 2(n-1))$, or $m = 2$, $a = 2^k$ where $k \geq 0$ is an integer, p is a Fermat prime and $b + b_0 = n - 1$.

(b) m is a Mersenne prime, $a = 1$, $p = 2$ and $\frac{b+b_0}{n-1}$ is a prime.

Proof. Obviously, we have $b > 0$. We obtain assertions (i) and (ii) analogously to Proposition 1.5.9, using Theorem 1.5.2 and Lemmas 1.5.3 and 1.5.6. □

Remark 1.5.12. (a) In analogy to Remark 1.5.10 (a), we note that in Proposition 1.5.11 (i)(a) and (ii)(a) we have that $p = z_{2,2a}$ is a Zsigmondy prime if $(m, a) \neq (2, 3)$. So, for these cases we have $2a < p$, and the exact value of a is given by $a = \frac{\ln(p-1)}{\ln(2)}$.

(b) In Proposition 1.5.11 (i)(b), we see that there are examples with $a = \frac{p-1}{2}$, such as $m = 3, a = 2, p = 5$ and $b = 1$.

Next, we note some elementary considerations.

Lemma 1.5.13. *Let n, m and a be integers where $n \geq 3$ is odd, m is a prime and $a \geq 1$. If $m^a \geq 4$ and $(n, a, m) \notin \{(3,2,2),(3,2,3),(3,3,2)\}$ then*

$$\tfrac{m^{an}-1}{m^a-1} > 2n \cdot m^{\frac{na-1}{2}}. \quad (\star)$$

Proof. First, we consider the case $n = 3$. Here, clearly $(\star)$ holds if and only if $m^{\frac{a+1}{2}} + m^{\frac{-a+1}{2}} + m^{\frac{-3a+1}{2}} > 6$. So, our assertion follows by elementary calculations (we note that it is advisable to consider a case-by-case analysis with respect to a).
Now, let $n \geq 5$. By easy calculations, we see that $(\star)$ holds for $(a, m) = (2, 2)$. So, assume $(a, m) \neq (2, 2)$. It is not hard to see that $m^{a(n-1)} > 6 \cdot m^{\frac{na-1}{2}}$. Furthermore, by elementary calculations, we see that $m^{a(n-1-j)} \geq 4 \cdot m^{\frac{na-1}{2}}$ holds for $j \in \{1, \ldots, \frac{n-3}{2}\}$, since $(a, m) \neq (2, 2)$. Hence, we easily obtain our assertion in the actual case. □

Remark 1.5.14. (a) We note that the inequality $(\star)$ from the previous lemma does not hold in general if we increase the factor 2 (e.g. to 2.5) or the exponent $\frac{na-1}{2}$ (e.g. to $\frac{na}{2}$) on the right-hand side, for having (further) exceptions for small n, m and a.

(b) Using the Arithmetic-Geometric-Mean inequality 1.4.31, we may deduce the less precise estimate $\frac{m^{an}-1}{m^a-1} > n \cdot m^{\frac{a(n-1)}{2}}$.

Lemma 1.5.15. *Let n, m and a be integers where $n \geq 3$ is odd, m is a prime and $a \geq 1$. If $m^a \geq 4$ and $(n, a, m) \notin \{(3,2,2),(3,3,2)\}$ then*

$$\tfrac{m^{an}+1}{m^a+1} > \tfrac{6}{5} n \cdot m^{\frac{na-1}{2}}.$$

Proof. Considering

$$((m^{an}-1)/(m^a-1))/((m^{an}+1)/(m^a+1)) < (m^a+1)/(m^a-1) \leq 5/3,$$

we easily obtain our assertion from Lemma 1.5.13. □

Remark 1.5.16. In analogy to Remark 1.5.14 (a), we note that increasing the factor $\frac{6}{5}$ (e.g. to 1.5) or the exponent $\frac{na-1}{2}$ (e.g. to $\frac{na}{2}$) on the right-hand side of the inequality from the previous lemma may lead to (further) exceptions in its validity for small n, m and a.

Next, we recall the terminology of the *Legendre symbol* and provide a lemma. For a reference see e.g. [La, p. 76-79].

Definition 1.5.17. Let n be an integer and p be a prime. The *Legendre symbol* $\left(\frac{n}{p}\right)$ is defined for having value 0 if p divides n, value 1 if n modulo p is a square in $\mathrm{GF}(p)^*$ and value -1 if n modulo p is a non-square in $\mathrm{GF}(p)^*$.

Lemma 1.5.18. *For an odd prime p we have that $\left(\frac{2}{p}\right) = 1$ if $p \equiv \pm 1 \pmod 8$ and $\left(\frac{2}{p}\right) = -1$ if $p \equiv \pm 3 \pmod 8$.*

In the following two lemmas, we provide some elementary facts.

Lemma 1.5.19. *Let q and n be positive integers where n is odd. Then we have*

$$\gcd\Big(q^n+1, \frac{q^{2n}-1}{q^2-1}\Big) = \frac{q^n+1}{q+1}.$$

Proof. Let $g = \gcd\Big(q^n+1, \frac{q^{2n}-1}{q^2-1}\Big)$. Obviously, we have that $\frac{q^n+1}{q+1}$ divides g. Suppose that $\frac{q^n+1}{q+1} < g$, so there is a prime r such that $\frac{q^n+1}{q+1}r$ divides g. Since $\frac{q^n+1}{q+1}r$ divides q^n+1, we can deduce that $q \equiv -1 \pmod r$ $(\star)$. Because $\frac{q^n+1}{q+1}r$ divides $\frac{q^{2n}-1}{q^2-1}$, we now obtain $0 \equiv \frac{q^n-1}{q-1} = \sum_{j=0}^{n-1} q^j \overset{(\star)}{\equiv} 1 \pmod r$. □

Lemma 1.5.20. *Let p be a prime and a be a non-negative integer. For a positive integer d the following hold:*

(a) if $p^a \mid d!$ but $p^{a+1} \nmid d!$ then $a = \sum_{i=1}^{\infty} \lfloor d/p^i \rfloor$,

(b) if $p^a \mid d!$ then $(p-1)a < d$.

Proof. (cf. [Ar, p. 364]) Assertion (a) holds by elementary considerations. Since $\sum_{i=1}^{\infty} \lfloor d/p^i \rfloor < \sum_{i=1}^{\infty} d/p^i = \frac{d}{p-1}$, assertion (b) follows from (a). □

Finally, we provide a theorem about the occurrence of a prime number in a certain interval.

Theorem 1.5.21. ***(Bertrand's postulate)*** *For any integer $n \geq 4$ there is always a prime p such that $n < p < 2n-2$.*

Proof. See [Be, p. 129], [Če, p. 371-382], [Er] and [Er2]. □

Remark 1.5.22. (a) A weaker but more elegant formulation for Bertrand's postulate is: For any integer $n \geq 1$ there is always a prime p such that $n < p \leq 2n$.

(b) We note that the assertion of Bertrand's postulate has been sharpened by several authors. Such as by Felgner in [Fe, Theorem 4.1] where it was shown that for all integers $n \geq 8$ there is a prime p such that $n < p < \frac{3}{2}n$; or, by Nagura in [Na, Theorem] where it was shown that for all integers $n \geq 25$ there is a prime p such that $n < p < \frac{6}{5}n$.

(c) The history of Bertrand's postulate (compare the quotes in the latter proof) and more general the history of theorems and conjectures on the number of primes in certain intervals is interesting. Many theorems have been established and are improved up to today (for Bertrand's postulate recall e.g. part (b)). Although there has been much effort, some conjectures still remain undecided. An important conjecture which has neither been proved nor disproved till today is e.g. Legendre's conjecture which states that for every positive integer n there exists a prime number p such that $n^2 < p < (n+1)^2$.

Chapter 2

The strongly constrained maximal subgroups of the finite almost simple linear and unitary groups

In this chapter, we determine the pairs (G, M) where G is a finite almost simple linear or unitary group and M a strongly constrained maximal subgroup of G. Particularly, we determine all strongly constrained maximal subgroups of the almost simple linear and unitary groups. We provide a detailed description of these strongly constrained maximal subgroups and investigate also some issues concerning them. We note that the facts we will provide and the results we will obtain in this chapter will be important for Chapter 3.
For this chapter we recall Section 1.2 about the classical groups, and in particular we recall our standard notation, introduced in Subsection 1.2.3. Regarding our intended goal, it is sufficient to consider and investigate only the almost simple groups G with socle isomorphic to $\mathrm{P}\Omega(V)$, in particular $\dim(V) = n \neq 2$ in the case **U**. (Note, that in view of the isomorphism $\mathrm{PSL}_2(q) \cong \mathrm{PSU}_2(q^2)$, no finite almost simple unitary group is excluded from our investigations, and with little effort the results can be transferred to the omitted case).

2.0 Approach and introductory notes

Now, we will explain how to achieve our intended goal of this chapter and provide important facts for our work. In the fundamental paper of Aschbacher [As], the following theorem is stated.

Theorem 2.0.1. ***(Aschbacher's theorem)***[1] *A proper subgroup H of a finite almost simple classical group G (with an exception for certain extensions of $\mathrm{P}\Omega_8^+(q)$), where $G = \mathrm{soc}(G)H$, is contained in a member of a collection $\mathscr{C}(G)$ of (geometrically defined) subgroups of G, or fulfills some specific conditions.*

The collection $\mathscr{C}(G)$ of subgroups of G is partitioned into eight classes of subgroups which we denote by *Aschbacher classes* (in short: *A-classes*) $\mathscr{C}_1, \dots, \mathscr{C}_8$ of G, or simply $\mathscr{C}_1, \dots, \mathscr{C}_8$ of G. These eight classes consist (roughly described) of groups that preserve some kind of geometric structure. In Section $2.j$, we will provide an exact definition of A-class $\mathscr{C}_j$ of G for our intended situation (i.e. for G having a socle isomorphic to $\mathrm{P}\Omega(V)$), on the base of [As], [KL] and [BHR]. Subgroups of G which fulfill the mentioned specific conditions in the theorem of Aschbacher are collected in a ninth class of subgroups of G (as done in the books [KL] and [BHR]) which we denote by $\mathscr{S}(G)$. For a definition of the class $\mathscr{S}(G)$ of subgroups of G we refer to [KL, p. 3-4], or to the original paper of Aschbacher [As, Theorem]. Important for our investigations is the fact that all members of class $\mathscr{S}(G)$ are almost simple groups.
We provide the theorem of Aschbacher restricted to our intended situation, i.e. for almost simple groups with socle isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$, and add some remarks concerning the general theorem.

Theorem 2.0.2. ***(Aschbacher's theorem for finite almost simple linear and unitary groups)*** *(see [As, Theorem], or [KL, Theorem 1.2.1])*
Let G be a finite almost simple group with socle G_0 isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$. Let H be a proper subgroup of G where $G = G_0H$. Then either H is contained in some member of $\mathscr{C}(G)$, or $H \in \mathscr{S}(G)$.

Remark 2.0.3. (a) Concerning the proof of Aschbacher's theorem, we note that it is divided into several cases. In [As, Section 11], the cases are considered where $\mathrm{P}\Omega(V,\kappa) \leq G \leq \mathrm{P}\Gamma(V,\kappa)$ (using the notation from Table 1.2.1) where κ is a classical form on V. In view of Table 1.2.1 and Proposition 1.2.15, there are three cases left to consider to achieve an assertion for all finite almost simple classical groups G. For $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ and $G \not\leq \mathrm{P\Gamma L}_n(q)$ (for our situation the only relevant of the left three cases), in [As, Section 13] an extra family $\mathscr{C}_1'$ of subgroups of G is introduced, and Aschbacher's theorem is established with respect to $\mathscr{C}_1', \mathscr{C}_2, \dots, \mathscr{C}_8$ of G.
The case $\mathrm{soc}(G) \cong \mathrm{PSp}_4(2^a)$ and $G \not\leq \mathrm{P\Gamma Sp}_4(2^a)$ is considered in [As, Section 14]. There, $\mathscr{C}(G)$ is modified and Aschbacher's theorem is also proved for this case.
For the remaining case of almost simple groups which are extensions of $\mathrm{P}\Omega_8^+(q)$ including a graph automorphism of order three, Aschbacher's theorem [As, Theorem] does not include an assertion. In [Kle], this gap was closed by Kleidman. We note that in the paper of Aschbacher [As, Section 15] there are remarks provided which are relevant for this case.

[1] For a more precise formulation of the theorem for the finite almost simple linear and unitary groups, see Theorem 2.0.2 together with Definitions 2.1.2, 2.2.2, 2.3.1, 2.4.1, 2.5.2, 2.6.15, 2.7.1 and 2.8.1.

(b) We note that in the book [BHR] a non-projective version of Aschbacher's theorem is used, see [BHR, Theorems 2.1.5 and 2.2.19].

Clearly, by Aschbacher's theorem 2.0.1, we obtain that every maximal subgroup of a finite almost simple classical group G (with an exception for certain extensions of $\mathrm{P}\Omega_8^+(q)$), which does not contain the socle of G, is a member of $\mathscr{C}(G)$ or $\mathscr{S}(G)$. In the book of Kleidman and Liebeck [KL], the collection $\mathscr{C}(G)$ of subgroups of G is examined for all finite almost simple classical groups G with the exceptions for extensions of $\mathrm{P}\Omega_8^+(q)$ including a graph automorphism of order three and extensions of $\mathrm{PSp}_4(2^a)$ including a graph automorphism of order two (cf. Remark 1.2.16 (c)). Tables are provided, in [KL, Chapter 3], from which one can read off the group theoretic structure of the members of $\mathscr{C}(G)$ and the conjugacy amongst the members of $\mathscr{C}(G)$. (We note that these facts are provided for all cases which satisfy the dimension conditions from [KL, Theorem 2.1.3]). Furthermore, for a dimension of the associated vector space which is at least 13 the conditions are determined in which cases a member $H \in \mathscr{C}(G)$ is a maximal subgroup of G. These facts can be read off from the tables in [KL, Chapter 3]. Also, if H is not maximal in G one can read off from the tables the overgroups of H lying in $\mathscr{C}(G) \cup \mathscr{S}(G)$.

There is a long history about the study and classification of the maximal subgroups of the finite (almost) simple classical groups associated to vector spaces of low dimension. (E.g. for the linear and unitary cases of dimensions up to 3 see the papers [Dic], [Mi], [Ha] and [Blo], and for a short historical summary we refer to [BHR, p. xi-xii]). However, the question about the maximality of the members of $\mathscr{C}(G)$ in G for lower dimensions has remained long time answered incompletely. Written by the authors Bray, Holt and Roney-Dougal, the book [BHR] was published in the year 2013, which considered this question (amongst other things) and finally settled it.[2] In [BHR, Chapter 8], tables are provided from which one can read off the maximal subgroups of the finite almost simple classical groups in dimension up to 12 (which do not contain the socle). We note that also the maximal subgroups of these groups belonging to the ninth class $\mathscr{S}(G)$ are determined in this book. Also, there are tables displaying the maximal subgroups of all almost simple exceptional groups which occur as subgroups of these classical groups. We note that Bray, Holt and Roney-Dougal use the results about the conjugacy amongst the members of $\mathscr{C}(G)$, obtained in [KL] (cf. [BHR, p. 57]). Furthermore, we note that the authors transfer the information about the group theoretic structure of the members of $\mathscr{C}(\mathrm{P}\Omega(V))$ from [KL] to the non-projective case, see [BHR, Section 2.2].

[2] There is an interesting history concerning this gap in literature. In [KL, Theorem 1.2.2] it is claimed that this gap was closed, by the paper [Kle] and a book written by Kleidman which was said to appear in Longman Research Notes. Before [BHR] was published, the author has searched intensively for the mentioned book without success. Since other authors have referred to this book, it seemed to be mysterious not to obtain a copy. The appearance of the book [BHR] finally settled this question. In the foreword of this book, written by Liebeck, there is a summary about the historical background concerning this topic. It is said that Kleidman did not write the mentioned book, and so it was never published.

Our main goal of this chapter is to determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega(V)$ and M a strongly constrained maximal subgroup of G. (We provide also some additional facts about these M). In view of the considerations above, it seems reasonable to use Aschbacher's theorem 2.0.2 and the information provided in the books [KL] and [BHR]. As a first step, we easily deduce from Aschbacher's theorem the following corollary.

Corollary 2.0.4. *Let G be a finite almost simple group with socle isomorphic to $\mathrm{P}\Omega(V)$. Let M be a strongly constrained maximal subgroup of G. Then M is a member of $\mathscr{C}(G)$, so a member of an A-class $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of G.*

Remark. Concerning the last corollary, we note the obvious observation that a strongly constrained subgroup of a finite almost simple classical group cannot contain the socle of the group.

By the last corollary, we have to limit our attention to those maximal subgroups of an almost simple groups G with socle isomorphic to $\mathrm{P}\Omega(V)$ which are members of $\mathscr{C}(G)$. As noted above, one can read off the tables from the books [KL, in Chapter 3] and [BHR, in Chapter 8] the maximal subgroups of G belonging to $\mathscr{C}(G)$, and the information about their group theoretic structure and conjugacy. We will read off from these tables the relevant facts for our work and provide them properly. For this, we note that the facts in the tables from [BHR, Chapter 8] are provided with respect to a standard basis of the underlying vector space, see [BHR, Table 1.1 and p. 373]. So, for our purposes, we will transfer these facts to a version not depending on the choice of a specific basis if it is possible. (Here, we recall the problem of well-definedness, noted in Remark 1.2.18). For an introduction to read off the facts from the mentioned tables, we refer to [KL, Chapter 3] and [BHR, Chapter 8].
For the examination of $\mathscr{C}(G)$ it is advantageous to consider each A-class $\mathscr{C}_j$ of G separately. We will concern this in Section $2.j$, and provide in each section main theorems determining the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega(V)$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_j$ of G.
We note that we will use the standard notation (based on the book [KL] and introduced in Subsection 1.2.3) for the investigation of each A-class $\mathscr{C}_j$ of G, to work simultaneously with the two cases **L** and **U**. However, for the definition of the members of A-class $\mathscr{C}_j$ of G it is advantageous to choose the generalized standard notation (for its introduction see also Subsection 1.2.3 and cf. [KL, Chapter 4, esp. p. 80]). So, we will use the generalized standard notation for the introduction of the members of A-class $\mathscr{C}_j$ of G, and note it explicitly when we will use it.
As noted before, we will use the standard notation for the investigation of the A-classes $\mathscr{C}_j$ of G. But, we will present the results of our main theorems of each section using the more common notation concerning almost simple classical groups from Table 1.2.1 and usually consider the linear and unitary case separately. So, readers who are not familiar with the chosen standard notation

will be able to obtain these facts more easily. (Here, we recall also the notations for the diagonal, field and graph automorphisms of $\mathrm{P}\Omega(V) \cong \mathrm{P}\Omega$ by W, φ and τ, as introduced in Subsection 1.2.2).

To obtain our main theorems of each section, we will determine exact conditions for a maximal subgroup of G belonging to a certain A-class $\mathscr{C}_j$ of G to be strongly constrained. Sometimes, we will more generally investigate which members of a certain A-class $\mathscr{C}_j$ of G are strongly constrained without restricting to those which are maximal subgroups of G. Often, we will also provide additional information, such as the determination of the largest normal r-subgroup (for a prime r) of a strongly r-constrained maximal subgroup of G.

In the majority of cases, we do not start our investigations in each section by using the information about the maximal subgroups of G at once. Often, it is more advantageous first to reduce the cases to examine by the condition that the layer of a member $K \in \mathscr{C}_j$ of G is trivial if K is strongly constrained. (For this, the provided structure information of the members of $\mathscr{C}(G)$ from [KL] and [BHR] is essential). Then, for the remaining cases we can use the information about the maximality of K in G.

Remark 2.0.5. Let the terminology of a *local* (or *non-local*) maximal subgroup of a finite almost simple classical group be introduced as in [KL, p. 5]. (Note, that this is slightly different from the common use of the terminology *local*). In the book of Kleidman and Liebeck, the information is provided to read off the local maximal subgroups of the finite almost simple classical groups, see [KL, Corollary 1.2.4. and the related propositions in Chapter 4]. In view of Proposition 1.1.3 and Theorem 1.4.16, we easily see that every strongly constrained maximal subgroup of a finite almost simple classical group is also a local maximal subgroup. We note that it is not advantageous to create a list of the local maximal subgroups of the almost simple groups with a socle isomorphic to $\mathrm{P}\Omega(V)$, and then to examine this list with respect to strong constraint. By the previously described approach, we can easily obtain a more reduced list to use. E.g. compare the cases described in [KL, Proposition 4.1.4. (III)] with the cases in Corollary 2.1.6.

Concerning the A-classes $\mathscr{C}_1, \ldots, \mathscr{C}_8$

For the following we use the generalized standard notation (recall Subsection 1.2.3). Let G be a group where $\Omega(V) \leq G \leq \mathrm{A}(V)$. As mentioned above, the members of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of $\mathrm{P}G$ are defined by geometrical aspects. More precisely (but even rough), we note the following. For the non-projective case eight collections of subgroups of G are defined, called (*A-classes*) $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of G, whose members preserve some kind of geometric structure related to the vector space V. The collections of subgroups $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of $\mathrm{P}G$ are then defined related to the non-projective versions. We note that this approach is natural, regarding the proof of Aschbacher's theorem.

We introduce further terminology and notation. As above, let $\mathscr{C}(G)$ (or $\mathscr{C}(\mathrm{P}G)$) be the union of $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of G (or $\mathrm{P}G$). Furthermore, let $\mathscr{C}$ (or $\mathrm{P}\mathscr{C}$) denote the union of all $\mathscr{C}(G)$ (or $\mathscr{C}(\mathrm{P}G)$), and for a fixed $j \in \{1, \dots, 8\}$ let $\mathscr{C}_j$ (or $\mathrm{P}\mathscr{C}_j$) denote the union of all $\mathscr{C}_j$ of G (or $\mathrm{P}G$) for all G with $\Omega(V) \leq G \leq \mathrm{A}(V)$. The following division is introduced in [KL, see p. 58-59 and the related definitions in Chapter 4] and based on the action of $\Gamma(V)$ on $\mathscr{C}$. (For further information we refer to that reference). Each collection $\mathscr{C}_j$ (or $\mathrm{P}\mathscr{C}_j$) splits into several subcollections. These subcollections we call *types*. If T is a type in $\mathscr{C}_j$ (or $\mathrm{P}\mathscr{C}_j$) and $H \in T$ we call H *of type* T. We note that (in most cases) the notation of a certain type is chosen to describe the approximate group theoretic structure of a group H of that type and sometimes also to indicate which kind of geometric structure H stabilizes. By regarding the definitions of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of G and $\mathrm{P}G$ (below), we see that if $H \in \mathscr{C}$ (or $\mathrm{P}\mathscr{C}$) is of type T then every $\Gamma(V)$- (or $\mathrm{P}\Gamma(V)$-)conjugate of H is of type T. The authors of the book [BHR] adopt this division into types with a slight modification for $\mathscr{C}_1$, see Remark 2.1.3 (c), below.

As we wish to take advantage of all three sources [As], [KL] and [BHR], we have to point out the differences between them. A first step was done in Section 1.2 where the terminology and notation about the classical groups in these works were compared. Also, the definitions of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of G differ in these works. Kleidman and Liebeck pointed out and justified the differences of their definitions to the definitions in the paper of Aschbacher. See [KL, p. 58 and Table 3.5.J] for a collection of the differences. In the book [BHR], the differences of their definitions to the definitions in [KL] are also pointed out and justified (see [BHR, Section 2.2]). We note that some of the explanations for the different definitions in [KL] and [BHR] are very brief. Furthermore, some differences have not been pointed out (such as a modification which leads to a wrong definition of A-class $\mathscr{C}_5$ of G in [BHR]). We will base our definitions of the A-classes $\mathscr{C}_1, \dots, \mathscr{C}_8$ of G on the definitions in the mentioned three works (mainly to [KL] and [BHR]). But, we will not adopt the definitions solely from one of these works. So, also in view of the mentioned brief or missing explanations, it seems adequate to point out the differences between the definitions provided in the three works, cite or provide explanations for these differences, and set the definitions in this thesis in relation to the definitions in the three works. We explicit note that we will only consider and compare the definitions of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of G where $\Omega(V) \leq G \leq \mathrm{A}(V)$ (so, not for cases where G is another finite classical group also considered in the three works). The following remark lists three basic differences of the definitions in the three works which we will not further consider in our comparison.

Remark 2.0.6. (a) In this thesis, we introduce the members of $\mathscr{C}(G)$ using the generalized standard notation. There, we demand that $n \geq 3$ in case **U** (recall $\mathrm{PSL}_2(q) \cong \mathrm{PSU}_2(q^2)$). This dimension restriction is also demanded in the books [KL, see p. 57, 60 and Chapter 4] and [BHR, see Definition 1.6.20 and Section 2.2]. In the paper of Aschbacher, there is no such restriction on the dimension.

(b) For the following recall the notation ˆ for the (full) preimage under the projection map P. In the paper of Aschbacher, the members of $\mathscr{C}_1, \dots, \mathscr{C}_8$

of PG are introduced as the images of the members of $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of $\widehat{\mathrm{P}G}$ under P, see [As, p. 473]. The introduction of these members is done more generally in [KL]. There, the members of $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of PG are introduced as the images of the members of $\mathscr{C}_1, \ldots, \mathscr{C}_8$ of G under P, see [KL, p. 60]. Regarding the definitions of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of G in [KL], it is not hard to see that the definitions of the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$ of PG in this book are well-defined, use Lemma 1.4.6 and again see [KL, p. 60]. So, there is no difference in the two works concerning the members of $\mathscr{C}(\mathrm{P}G)$ arising from the before mentioned generalization in [KL]. We note that we have decided to follow the more general approach in [KL].
Concerning the book [BHR], we note that the authors (mainly) consider the non-projective case when working with the collections $\mathscr{C}_1$ to $\mathscr{C}_8$, cf. also Remark 2.0.3 (b).

(c) In the book [BHR], the introduction of the collections $\mathscr{C}_1$ to $\mathscr{C}_8$ of G is done with respect to a fixed standard basis of V (recall also Remark 1.2.1 (b)). The definitions in [As] and [KL] do not depend on the choice of a specific fixed basis of V. In this thesis, it is advantageous to follow the approach of [As] and [KL] (for reasons see Remark 1.2.1 (b)).

A necessary condition for maximality

Now, we use again the standard notation. The following is extracted from [KL, p. 59-65], and we refer to that reference for further details. Let H be a member of A-class $\mathscr{C}_j$ of $\mathrm{P}\Omega(V)$. By $X = \{H^g \mid g \in \mathrm{PA}(V)\}$, we denote the $\mathrm{PA}(V)$-conjugacy class of H. Under the action of the normal subgroup $\mathrm{P}\Omega(V)$ of $\mathrm{PA}(V)$, X splits into c $\mathrm{P}\Omega(V)$-conjugacy classes $X_1, \ldots, X_c$ of equal length.

Remark. Since $\mathrm{PA}(V)$ acts naturally on the set $X : \mathrm{P}\Omega(V) = \{X_1, \ldots, X_c\}$, we can deduce a homomorphism from $\mathrm{PA}(V)$ to $\mathrm{Sym}(X : \mathrm{P}\Omega(V)) \cong \mathrm{S}_c$. This homomorphism induces a homomorphism π from $\ddot{\mathrm{PA}}(V)$ to $\mathrm{Sym}(X : \mathrm{P}\Omega(V))$ where $\ddot{}$ denotes the reduction modulo $\mathrm{P}\Omega(V)$. We note that the value c, the homomorphism π and the stabilizer $\ddot{\mathrm{PA}}(V)_{X_i}$ of a point X_i in $\ddot{\mathrm{PA}}(V)$ (for an appropriate $i \in \{1, \ldots, c\}$) are determined in the book [KL, see Tables 3.5.A, 3.5.B and 3.5.G].

W.l.o.g. let X_1 be the $\mathrm{P}\Omega(V)$-conjugacy class of H. By elementary considerations from permutation group theory, we see that the stabilizer of the point X_1 in $\mathrm{PA}(V)$ is $S = \mathrm{N}_{\mathrm{PA}(V)}(H)\mathrm{P}\Omega(V)$. Now, let M be a member of A-class $\mathscr{C}_j$ of G where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$. In almost all cases, we have that $M \cap \mathrm{P}\Omega(V)$ is a member of A-class $\mathscr{C}_j$ of $\mathrm{P}\Omega(V)$, see [KL, Proposition 3.1.3]. Assume that $H = M \cap \mathrm{P}\Omega(V)$. Then G has to be a subgroup of S (hence, every element in G has to stabilize X_1) if M is a maximal subgroup of G. To see that the last assertion holds, we first note that $M = \mathrm{N}_G(H)$ if M is maximal in G. Suppose that $G \nleq S$. Then, using the Dedekind modular law, we easily obtain a contradiction by

$$M < \mathrm{N}_G(H)\mathrm{P}\Omega(V) = S \cap G < G.$$

Remark. We note that in the majority of cases we consider in this chapter we have $c = 1$. Here, the upper necessary condition trivially holds, since $S = \mathrm{PA}(V)$. For detailed examples when $c > 1$, see e.g. Section 2.5.

Finally, before we begin to investigate each A-class $\mathscr{C}_j$ separately, we recall the terminology of a novelty, see Definition 1.1.4 and also Remark 1.1.5.

2.1 A-class $\mathscr{C}_1$

In this section, we consider the members of A-class $\mathscr{C}_1$. Roughly described, the members of A-class $\mathscr{C}_1$ are the stabilizers of totally singular or non-degenerate subspaces of the underlying vector space V. (Recall from p. 21, that there are only totally singular subspaces of V in the case **L**).
We start by providing further appropriate notation for the description of the members of A-class $\mathscr{C}_1$.

Definition 2.1.1. Let V be a vector space, and let U and W be non-zero subspaces of V. For $G \leq \Gamma\mathrm{L}(V)$ we define $\mathrm{N}_G(U)$ (or $\mathrm{N}_G(U, W)$) to be the subgroup of G consisting of the elements in G stabilizing the subspace U (or the subspaces U and W).

Now, we define the members of A-class $\mathscr{C}_1$. For this, we recall the terminology and notation introduced in Subsections 1.2.1 and 1.2.2, esp. the terminology introduced after Lemma 1.2.8. Furthermore, we use the generalized standard notation, introduced in Subsection 1.2.3, see also the comments on p. 62.

Definition 2.1.2. $\{$***A-class*** $\mathscr{C}_1\}$ (cf. [BHR, p. 59-60] and [As, p. 472-473])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and let K be a subgroup of G. Let U and W be non-trivial subspaces of V (i.e. $\{0\} < U, W < V$) where $\dim(U) = k$ and $\dim(V) = n = k + \dim(W)$. For $G \leq \Gamma(V)$ the subgroup K belongs to *(A-class)* $\mathscr{C}_1$ of G if it appears in the following table. If $G \not\leq \Gamma(V)$ then K belongs to *(A-class)* $\mathscr{C}_1$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_1$ of $\Gamma(V)$.

Case	Type of K	Description of K	Conditions
L	P_k	$K = \mathrm{N}_G(U)$	
L	$P_{k,n-k}$	$K = \mathrm{N}_G(U, W)$, $U \leq W$	$k < n/2$
L	$\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$	$K = \mathrm{N}_G(U, W)$, $U \cap W = \{0\}$	$k < n/2$
U	P_k	$K = \mathrm{N}_G(U)$, U totally singular	$k \leq n/2$
U	$\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$	$K = \mathrm{N}_G(U)$, U non-degenerate	$k < n/2$

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_1$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_1$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$, P_k, $P_{k,n-k}$, or $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$ we call K of type $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$, P_k, $P_{k,n-k}$, or $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$, respectively.

Concerning the last definition and the members of A-class $\mathscr{C}_1$, we note the following remark.

Remark 2.1.3. (a) Here, we provide explanations for the conditions which are demanded in the table in the last definition for the different types. The condition in the second row of the table is clear. Corollary 1.2.10 yield the condition in case **U** of type P_k. Concerning the condition that $k = n/2$ is excluded in case **L** of type $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ and case **U** of type $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$, we note that this case is considered in A-class $\mathscr{C}_2$, see Definition 2.2.2, below (cf. also [KL, Remark on p. 83-84]).

(b) In this part of the remark, we note some considerations in case **L** concerning maximality of the members of A-class $\mathscr{C}_1$ of G of type P_k in G. For this, we consider a fixed ordered basis $B = (b_1, \ldots, b_n)$ of V where $\langle b_1, \ldots, b_k \rangle = U$ and the representation concerning B. We recall the introduction of the diagonal matrix W_{SL} and the semilinear transformation $\overline{\varphi_p}_B$ (here, also denoted as φ_p) in Subsection 1.2.2. It is not hard to see that W_{SL} and φ_p stabilize U. For the graph automorphism τ of $\mathrm{SL}_n(q)$ (only occurring for $n \geq 3$) the situation is different. As exposed in the definition of the graph automorphism of $\mathrm{SL}_n(q)$ (also in Subsection 1.2.2), τ maps stabilizers of k-subspaces of V to stabilizers of $(n-k)$-subspaces. Consider the case $n \neq 2k$ and $G \not\leq \Gamma$, and recall that for $G \not\leq \Gamma$ a member K of A-class $\mathscr{C}_1$ of G of type P_k is of shape $K = \mathrm{N}_{\mathrm{A}}(\mathrm{N}_{\Gamma}(U)) \cap G$. We note that $\mathrm{N}_{\Gamma}(U) = (\mathrm{N}_{\Omega}(U) : \langle W_{\mathrm{SL}} \rangle) : \langle \varphi_p \rangle$. By analogous considerations as in Remark 1.2.19 (d), we see that $K \leq \Gamma$. So, K cannot be a maximal subgroup of G, since $K < G \cap \Gamma < G$ (cf. also the considerations before this section and [KL, Table 3.5.G]). However, in the case $n = 2k > 2$ we see that there is an element $A \in \mathrm{SL}_n(q)$ such that τA normalizes the stabilizer of a k-subspace of V (again, see Remark 1.2.19 (d)). Hence, if $n = 2k$ and $G \not\leq \Gamma$ then $K \in \mathscr{C}_1$ of G may be a maximal subgroup of G.

(c) The above definition of the members of A-class $\mathscr{C}_1$ coincides with the definition in [BHR]. The definition of the members of A-class $\mathscr{C}_1$ in [BHR] in case **L** differs slightly from that in [KL]. As we have seen in part (b) (cf. also Remark 1.2.19 (d)), in case **L** if $n \geq 3$ the elements in $\mathrm{A}(V) \setminus \Gamma(V)$ interchange stabilizers of k-subspaces of V with stabilizers of $(n-k)$-subspaces. So, the groups in type P_k are interchanged with groups in type P_{n-k} via conjugation by these elements. Because of this fact, Kleidman and Liebeck identify the types P_k and P_{n-k}, and hence have the extra condition $k \leq n/2$ for this case (cf. [KL, p. 59 and 83] and [BHR, p. 60]). Apart from this extra condition, the definitions of the members of A-class

$\mathscr{C}_1$ in the cases **L** and **U** in [BHR] and [KL] coincide.
Recall from Remark 2.0.3 that Aschbacher has introduced in case **L** and if $G \not\leq \mathrm{P\Gamma L}_n(q)$ (respective $\hat{G} \not\leq \Gamma\mathrm{L}_n(q)$) an extra family $\mathscr{C}_1'$ of G. Kleidman and Liebeck have included this extra family in their definition of A-class $\mathscr{C}_1$, to obtain a uniform definition, cf. [KL, p. 4 and 58]. Except for this difference, the definition of the members of A-class $\mathscr{C}_1$ in the cases **L** and **U** in [KL] coincides with that in [As].

From now on we use the standard notation. By providing an important fact, we start the investigation for our intended goal.

Proposition 2.1.4. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* K *be a member of A-class* $\mathscr{C}_1$ *of* G. *Then* $K \cap \mathrm{P}\Omega(V)$ *is a member of A-class* $\mathscr{C}_1$ *of* $\mathrm{P}\Omega(V)$ *of the same type as* K.

Proof. Our assertion follows by [KL, Proposition 3.1.3]. □

2.1.1 $\mathscr{C}_1$ of types $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ and $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$

It is advantageous to investigate the members of A-class $\mathscr{C}_1$ of types $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ and $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$ separately. These types will be considered in this subsection. First, we provide important facts about the structure and conjugacy of the members of A-class $\mathscr{C}_1$ of $\mathrm{P}\Omega(V)$ of types $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ and $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$ in $\mathrm{P}\Omega(V)$.

Proposition 2.1.5. *(i)* $\mathrm{P}\Omega(V) = \mathrm{PSL}(V)$ *acts transitively (by conjugation) on the members of A-class* $\mathscr{C}_1$ *of* $\mathrm{P}\Omega(V)$ *of type* $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ *for each* k.

(ii) $\mathrm{P}\Omega(V) = \mathrm{PSU}(V)$ *acts transitively (by conjugation) on the members of A-class* $\mathscr{C}_1$ *of* $\mathrm{P}\Omega(V)$ *of type* $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$ *for each* k.

(iii) Let K *be a member of A-class* $\mathscr{C}_1$ *of* $\mathrm{P}\Omega(V)$ *of type* $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ *or* $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$. *Then* $K = \mathrm{P}H$ *where* H *is a member of A-class* $\mathscr{C}_1$ *of* $\Omega(V)$ *of the same type as* K, *and the structure of* H *is as provided in the following table.*

Case	*Type of* H, K	*Structure of* H
L	$\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$	$(\mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)) : (q-1)$
U	$\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$	$(\mathrm{SU}_k(q^2) \times \mathrm{SU}_{n-k}(q^2)).(q+1)$

Proof. See [KL, Proposition 4.1.4 (I) and (II)] and [BHR, Table 2.3]. □

Using the condition that the layer of a strongly constrained group is trivial, we see by the following corollary that there are only a small number of cases to examine.

Corollary 2.1.6. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and let* K *be a member of* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ *or* $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$*. If* $\mathrm{E}(K) = 1$ *then one of the following cases holds.*

(a) $\mathrm{P\Omega}(V) \cong \mathrm{PSL}_3(2)$ *and* K *is of type* $\mathrm{GL}_1(2) \oplus \mathrm{GL}_2(2)$*,*

(b) $\mathrm{P\Omega}(V) \cong \mathrm{PSL}_3(3)$ *and* K *is of type* $\mathrm{GL}_1(3) \oplus \mathrm{GL}_2(3)$*,*

(c) $\mathrm{P\Omega}(V) \cong \mathrm{PSU}_3(3^2)$ *and* K *is of type* $\mathrm{GU}_1(3^2) \perp \mathrm{GU}_2(3^2)$*,*

(d) $\mathrm{P\Omega}(V) \cong \mathrm{PSU}_4(2^2)$ *and* K *is of type* $\mathrm{GU}_1(2^2) \perp \mathrm{GU}_3(2^2)$*, or*

(e) $\mathrm{P\Omega}(V) \cong \mathrm{PSU}_5(2^2)$ *and* K *is of type* $\mathrm{GU}_2(2^2) \perp \mathrm{GU}_3(2^2)$*.*

Furthermore, one of these cases holds if K *is strongly constrained.*

Proof. Since $K \cap \mathrm{P\Omega}(V)$ is a normal (non-trivial) subgroup of K, the assertion follows in view of Propositions 1.2.11, 1.2.12, 2.1.4 and 2.1.5 and Lemmas 1.4.22 and 1.4.21. (We also recall that $\mathrm{PSU}_3(2^2)$ is not a simple group). □

Next, we will examine for the cases (a) to (e) from the last corollary in which cases K is strongly constrained. For this, we recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1, \ldots, A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$ introduced on page 8, and also we recall the notation $[a_{ij}]_{n\times n}$ for the projective version of the matrix $(a_{ij})_{n\times n} \in \mathrm{GL}_n(q)$ introduced in Convention 1.2.2.

Proposition 2.1.7. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and let* K *be a member of* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ *or* $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$*. Then* K *is strongly constrained if and only if one of the cases (a) - (e) from Corollary 2.1.6 holds and in case (a)* $G = \mathrm{P\Omega}(V)$*.*

Proof. First, we consider the if-part of the assertion in case **L**. If $G = \mathrm{P\Omega}(V) \cong \mathrm{PSL}_3(2)$ then $K \cong \mathrm{SL}_2(3)$, by Proposition 2.1.5. So, K clearly is strongly 3-constrained. Now, let $\mathrm{P\Omega}(V) \cong \mathrm{PSL}_3(3)$. Considering Propositions 2.1.4 and 2.1.5 (cf. also [KL, Theorem 3.1.2. (ii)]), w.l.o.g. we can assume that $K \cong H$ if $G = \mathrm{P\Omega}(V)$ and $K \cong H : Z$ if $G = \mathrm{PA}(V)$ where $H = \{\mathrm{diag}[\det(A)^{-1}, A] \mid A \in \mathrm{GL}_2(3)\} \cong \mathrm{GL}_2(3)$ and $Z = \langle \tau \rangle \cong \mathbf{Z}_2$. Now, it is not hard to see that $\mathrm{O}_2(K) > 1$ and $\mathrm{O}_3(K) = 1$, so K is strongly 2-constrained. Next, we consider the only-if-part of the assertion in case **L**. In view of Corollary 2.1.6, we have only to examine the case $G = \mathrm{PA}(V) \cong \mathrm{PSL}_3(2) : \langle \tau \rangle$. Here, w.l.o.g. we can assume that $K \cong \tilde{K} = \{\mathrm{diag}[1, A] \mid A \in \mathrm{GL}_2(2)\} : \langle \tau \rangle$, by an analogous argumentation as above. Clearly, $\mathrm{O}_3(\tilde{K}) > 1$. Since $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tau \in \mathrm{Z}(\tilde{K})$ and $\mathrm{o}(T) = 2$, we obtain that $\mathrm{O}_2(\tilde{K}) > 1$ (cf. also Remark 1.2.19 (c)). So, K is not strongly constrained.
In case **U** the assertion follows by analogous arguments as in case **L** together with elementary calculations. (We may use Lemma 1.4.14 and Example 1.4.15). □

In the following proposition, we will provide the information about the maximality of K in G for the cases (a) to (e) from Corollary 2.1.6.

Proposition 2.1.8. *Let one of the cases (a)-(e) from Corollary 2.1.6 hold. In case* $\mathbf{L}$ *we have that* K *is a maximal subgroup of* G *if and only if* $G = \mathrm{PA}(V) = \mathrm{PSL}(V) : \langle\tau\rangle$. *If case* $\mathbf{U}$ *holds then* K *is a maximal subgroup of* G.

Proof. The assertion follows by [BHR, Tables 8.3, 8.5, 8.10 and 8.20] (or, more precisely see [BHR, Definition 2.3.5, Propositions 2.3.2 and 3.2.1 and Theorem 6.3.10]). □

As a direct consequence of Propositions 2.1.7 and 2.1.8, we obtain the first two main theorems of this section. We determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_1$ of G of type $\mathrm{GL}_k(q)\oplus\mathrm{GL}_{n-k}(q)$ or $\mathrm{GU}_k(q^2)\perp\mathrm{GU}_{n-k}(q^2)$. As described at the beginning of the chapter, we will present the following results not using our standard notation.

Main Theorem 2.1.9. *Let* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ *where* $\mathrm{PSL}_n(q)$ *is simple. Let* M *belong to A-class* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GL}_k(q)\oplus\mathrm{GL}_{n-k}(q)$. *Then* M *is a strongly constrained maximal subgroup of* G *if and only if* $G = \mathrm{Aut}(\mathrm{PSL}_3(3))$ *(here,* M *is of type* $\mathrm{GL}_1(3) \oplus \mathrm{GL}_2(3)$*). Furthermore, a strongly constrained maximal subgroup* M *of* G *has order* $2^5 \cdot 3$.

Main Theorem 2.1.10. *Let* $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ *where* $\mathrm{PSU}_n(q^2)$ *is simple and* $n \geq 3$. *Let* M *belong to A-class* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GU}_k(q^2)\perp\mathrm{GU}_{n-k}(q^2)$. *Then* M *is a strongly constrained maximal subgroup of* G *if and only if one of the following holds.*

- *(a)* $\mathrm{PSU}_n(q^2) = \mathrm{PSU}_3(3^2)$ *(here,* M *is of type* $\mathrm{GU}_1(3^2)\perp\mathrm{GU}_2(3^2)$*),*
- *(b)* $\mathrm{PSU}_n(q^2) = \mathrm{PSU}_4(2^2)$ *(here,* M *is of type* $\mathrm{GU}_1(2^2)\perp\mathrm{GU}_3(2^2)$*), or*
- *(c)* $\mathrm{PSU}_n(q^2) = \mathrm{PSU}_5(2^2)$ *and* M *is of type* $\mathrm{GU}_2(2^2)\perp\mathrm{GU}_3(2^2)$.

Furthermore, for such a strongly constrained maximal subgroup M *of* G *we have* $|M| \leq 2^6 \cdot 3$ *in case (a),* $|M| \leq 2^4 \cdot 3^4$ *in case (b) and* $|M| \leq 2^5 \cdot 3^5$ *in case (c) and these upper bounds are sharp.*

2.1.2 $\mathscr{C}_1$ of types P_k and $P_{k,n-k}$

In this subsection, we investigate the members of A-class $\mathscr{C}_1$ of types P_k and $P_{k,n-k}$. We start by providing the facts about structure and conjugacy of the members of A-class $\mathscr{C}_1$ of $\mathrm{P}\Omega(V)$ of these types in $\mathrm{P}\Omega(V)$.

Proposition 2.1.11. *(i)* $\mathrm{P}\Omega(V)$ *acts transitively (by conjugation) on the members of A-class* $\mathscr{C}_1$ *of* $\mathrm{P}\Omega(V)$ *of type* P_k *or* $P_{k,n-k}$ *for each* k.

(ii) *Let K be a member of A-class $\mathscr{C}_1$ of $\mathrm{P}\Omega(V)$ of type P_k or $P_{k,n-k}$. Then $K = \mathrm{P}H$ where H is a member of A-class $\mathscr{C}_1$ of $\Omega(V)$ of the same type as K, and the structure of H is as described in the following table.*

Case	*Type of H*	*Structure of H*	*Cond.*
L	P_k	$[q^{k(n-k)}] : ((\mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)) : (q-1))$	
L	$P_{k,n-k}$	$[q^{k(2n-3k)}] : ((\mathrm{SL}_k(q)^2 \times \mathrm{SL}_{n-2k}(q)) : (q-1)^2)$	
U	P_k	$[q^{k(2n-3k)}] : ((\mathrm{SL}_k(q^2) \times \mathrm{SU}_{n-2k}(q^2)).(q^2-1))$	$k < \frac{n}{2}$
		$[q^{k(2n-3k)}] : (\mathrm{SL}_k(q^2).(q-1))$	$k = \frac{n}{2}$

Proof. See [KL, Propositions 4.1.17., 4.1.18. and 4.1.22.][3] together with [BHR, Table 2.3] (cf. also Remark 2.1.3 (c) and the following remark). □

Remark 2.1.12. (a) The set consisting of the members of A-class $\mathscr{C}_1$ of $\mathrm{PSL}(V)$ of the types P_k and P_{n-k} forms an $\mathrm{Aut}(\mathrm{PSL}(V))$-conjugacy class for each k. This $\mathrm{Aut}(\mathrm{PSL}(V))$-conjugacy class splits under the action of $\mathrm{PSL}(V)$ into two conjugacy classes (namely P_k and P_{n-k}) if and only if $n \neq 2k$, see Remarks 1.2.19 (d) and 2.1.3 (b). We note that this observation is described differently in [KL, Theorem 3.1.1], because of the different definition of A-class $\mathscr{C}_1$ of type P_k in the case **L** in this book, see Remark 2.1.3 (c).

(b) We note that the assertions of the last proposition still hold using the generalized standard notation.

In the following proposition, we provide a concrete representation (with respect to a fixed ordered basis of V) of a member of A-class $\mathscr{C}_1$ of $\Omega(V)$ and of $\mathrm{I}(V)$ of type P_k or $P_{k,n-k}$ for our further work. For this, we recall Lemma 1.2.8 (a), as well as the notation φ_q, introduced in Subsection 1.2.3. Also we recall that $u = 1$ in case **L** and $u = 2$ in case **U**. Furthermore, we recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1, \ldots, A_k)$ and the anti-diagonal matrix $\mathrm{antidiag}(A_1, \ldots, A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$ introduced on page 8.

Proposition 2.1.13. *Let V be an n-dimensional $\mathrm{GF}(q^u)$-vector space for a prime power q, and let U be a non-zero subspace of V of dimension k. Let $n \geq 2$ in case **L** and $n \geq 3$ in case **U**, and let f denote the non-degenerate unitary form on V in case **U**. In case **U** let U be totally singular (esp. $k \leq \frac{n}{2}$). With respect to a suitable ordered basis for V (depending on U), we have the following.*

(i) $\mathrm{N}_{\mathrm{GL}_n(q)}(U) =: H_k^{\mathrm{GL}} = U_k^{\mathrm{GL}} \rtimes L_k^{\mathrm{GL}} \in \mathscr{C}_1$ *of* $\mathrm{GL}_n(q)$ *of type* P_k *and* $\mathrm{N}_{\mathrm{SL}_n(q)}(U) =: H_k^{\mathrm{SL}} = U_k^{\mathrm{SL}} \rtimes L_k^{\mathrm{SL}} \in \mathscr{C}_1$ *of* $\mathrm{SL}_n(q)$ *of type* P_k *where*
$$U_k^{\mathrm{GL}} = U_k^{\mathrm{SL}} := \left\{ \begin{pmatrix} \mathbb{1}_k & 0_{k,n-k} \\ C & \mathbb{1}_{n-k} \end{pmatrix} \;\middle|\; C \in \mathrm{Mat}_{n-k,k}(q) \right\} \cong \mathrm{Mat}_{n-k,k}(q),$$

[3]We note that there is a mistake in [KL, Proposition 4.1.18.], mentioned and corrected in [BHR, p. 60 and Table 2.3].

$L_k^{\mathrm{GL}} := \left\{ \mathrm{diag}(A_1, A_2) \middle| \begin{array}{l} A_1 \in \mathrm{GL}_k(q), \\ A_2 \in \mathrm{GL}_{n-k}(q) \end{array} \right\} \cong \mathrm{GL}_k(q) \times \mathrm{GL}_{n-k}(q)$ *and*
$L_k^{\mathrm{SL}} := L_k^{\mathrm{GL}} \cap \mathrm{SL}_n(q) = \left\{ \mathrm{diag}(A_1, A_2) \middle| \begin{array}{l} A_1 \in \mathrm{GL}_k(q), A_2 \in \mathrm{GL}_{n-k}(q), \\ \det(A_1) \cdot \det(A_2) = 1 \end{array} \right\}.$

(ii) If $k < \frac{n}{2}$ *and* $U < U' < V$ *with* $\dim(U') = n - k$ *then we have that* $\mathrm{N}_{\mathrm{GL}_n(q)}(U, U') =: H_{k,n-k}^{\mathrm{GL}} = U_{k,n-k}^{\mathrm{GL}} \rtimes L_{k,n-k}^{\mathrm{GL}} \in \mathscr{C}_1$ *of* $\mathrm{GL}_n(q)$ *of type* $P_{k,n-k}$ *and* $\mathrm{N}_{\mathrm{SL}_n(q)}(U, U') =: H_{k,n-k}^{\mathrm{SL}} = U_{k,n-k}^{\mathrm{SL}} \rtimes L_{k,n-k}^{\mathrm{SL}} \in \mathscr{C}_1$ *of* $\mathrm{SL}_n(q)$ *of type* $P_{k,n-k}$ *where*

$$U_{k,n-k}^{\mathrm{GL}} = U_{k,n-k}^{\mathrm{SL}} := \left\{ \begin{pmatrix} \mathbb{1}_k & 0_{k,n-2k} & 0_{k,k} \\ B & \mathbb{1}_{n-2k} & 0_{n-2k,k} \\ C & D & \mathbb{1}_k \end{pmatrix} \middle| \begin{array}{l} B \in \mathrm{Mat}_{n-2k,k}(q), \\ C \in \mathrm{Mat}_k(q), \\ D \in \mathrm{Mat}_{k,n-2k}(q) \end{array} \right\},$$

$$L_{k,n-k}^{\mathrm{GL}} := \left\{ \mathrm{diag}(A_1, A_2, A_3) \middle| \begin{array}{l} A_1, A_3 \in \mathrm{GL}_k(q), \\ A_2 \in \mathrm{GL}_{n-2k}(q) \end{array} \right\} \cong \mathrm{GL}_k(q)^2 \times \mathrm{GL}_{n-2k}(q)$$

and $L_{k,n-k}^{\mathrm{SL}} := L_{k,n-k}^{\mathrm{GL}} \cap \mathrm{SL}_n(q)$

$$= \left\{ \mathrm{diag}(A_1, A_2, A_3) \middle| \begin{array}{l} A_1, A_3 \in \mathrm{GL}_k(q), A_2 \in \mathrm{GL}_{n-2k}(q), \\ \det(A_1 A_3) \cdot \det(A_2) = 1 \end{array} \right\}.$$

(iii) In case **U** *if* $n = 2k$ *and the matrix of* f *is* $\mathrm{antidiag}(\mathbb{1}_{\frac{n}{2}}, \mathbb{1}_{\frac{n}{2}})$, *we have* $\mathrm{N}_{\mathrm{GU}_n(q^2)}(U) =: H_k^{\mathrm{GU}} = U_k^{\mathrm{GU}} \rtimes L_k^{\mathrm{GU}} \in \mathscr{C}_1$ *of* $\mathrm{GU}_n(q^2)$ *of type* P_k *and* $\mathrm{N}_{\mathrm{SU}_n(q^2)}(U) =: H_k^{\mathrm{SU}} = U_k^{\mathrm{SU}} \rtimes L_k^{\mathrm{SU}} \in \mathscr{C}_1$ *of* $\mathrm{SU}_n(q^2)$ *of type* P_k *where*

$$U_k^{\mathrm{GU}} = U_k^{\mathrm{SU}} := \left\{ \begin{pmatrix} \mathbb{1}_k & 0_{k,k} \\ C & \mathbb{1}_k \end{pmatrix} \middle| \begin{array}{l} C \in \mathrm{Mat}_k(q^2), \\ C + C^{t\varphi_q} = 0 \end{array} \right\},$$

$L_k^{\mathrm{GU}} := \left\{ \mathrm{diag}(A, A^{-1t\varphi_q}) \mid A \in \mathrm{GL}_k(q^2) \right\} \cong \mathrm{GL}_k(q^2)$ *and*

$$L_k^{\mathrm{SU}} := L_k^{\mathrm{GU}} \cap \mathrm{SU}_n(q^2) = \left\{ \mathrm{diag}(A, A^{-1t\varphi_q}) \middle| \begin{array}{l} A \in \mathrm{GL}_k(q^2), \\ \mathrm{o}(\det(A)) \mid q - 1 \end{array} \right\}.$$

(iv) In case **U** *if* $n > 2k$ *and the matrix of* f *is* $\mathrm{antidiag}(\mathbb{1}_k, \mathbb{1}_{n-2k}, \mathbb{1}_k)$, *we have* $\mathrm{N}_{\mathrm{GU}_n(q^2)}(U) =: H_k^{\mathrm{GU}} = U_k^{\mathrm{GU}} \rtimes L_k^{\mathrm{GU}} \in \mathscr{C}_1$ *of* $\mathrm{GU}_n(q^2)$ *of type* P_k *and* $\mathrm{N}_{\mathrm{SU}_n(q^2)}(U) =: H_k^{\mathrm{SU}} = U_k^{\mathrm{SU}} \rtimes L_k^{\mathrm{SU}} \in \mathscr{C}_1$ *of* $\mathrm{SU}_n(q^2)$ *of type* P_k *where*

$$U_k^{\mathrm{GU}} = U_k^{\mathrm{SU}} := \left\{ \begin{pmatrix} \mathbb{1}_k & 0 & 0 \\ -D^{t\varphi_q} & \mathbb{1}_{n-2k} & 0 \\ C & D & \mathbb{1}_k \end{pmatrix} \middle| \begin{array}{l} D \in \mathrm{Mat}_{k,n-2k}(q^2) \\ C \in \mathrm{Mat}_k(q^2), \\ C + C^{t\varphi_q} + DD^{t\varphi_q} = 0 \end{array} \right\},$$

$L_k^{\mathrm{GU}} := \left\{ \mathrm{diag}(A_1, A_2, A_1^{-1t\varphi_q}) \middle| \begin{array}{l} A_1 \in \mathrm{GL}_k(q^2), A_2 \in \mathrm{GL}_{n-2k}(q^2) \\ \textit{with } A_2 A_2^{t\varphi_q} = \mathbb{1}_{n-2k} \end{array} \right\}$ *with*
$L_k^{\mathrm{GU}} \cong \mathrm{GL}_k(q^2) \times \mathrm{GU}_{n-2k}(q^2)$ *and*
$L_k^{\mathrm{SU}} := L_k^{\mathrm{GU}} \cap \mathrm{SU}_n(q^2)$

$$= \left\{ \mathrm{diag}(A_1, A_2, A_1^{-1t\varphi_q}) \middle| \begin{array}{l} A_1 \in \mathrm{GL}_k(q^2), A_2 \in \mathrm{GL}_{n-2k}(q^2) \\ \textit{with } A_2 A_2^{t\varphi_q} = \mathbb{1}_{n-2k} \textit{ and} \\ \det(A_2) = \det(A_1)^{q-1} \end{array} \right\}.$$

Proof. The assertion follows by elementary observations and calculations (cf. also Proposition 2.1.11 and Remark 2.1.12 (b)). □

Remark 2.1.14. (a) Concerning the assertions in part (iii) and (iv) of the last proposition, we note that if $G \leq \mathrm{GU}(V)$ stabilizes the totally singular subspace $U \leq V$ then G also stabilizes $U^\perp \geq U$ where $\dim(U^\perp) = n - k$.

(b) We note that the result in Proposition 2.1.11 (i) can also be easily deduced without using a reference, by the following elementary observations. By Witt's Lemma, see Lemma 1.2.9, we have that I acts transitively (by conjugation) on the members of A-class $\mathscr{C}_1$ of Ω of type P_k or $P_{k,n-k}$ for each k. In view of the last proposition, we see that the diagonal matrix W_{SL^ϵ} (recall Subsection 1.2.2) normalizes H_k^{SL} and $H_{k,n-k}^{\mathrm{SL}}$ in case **L** and H_k^{SU} in case **U**. Hence, also Ω acts transitively on the considered sets. Here, in case **U** we recall that the definition of the diagonal matrix W_{SU} depends on the matrix of the non-degenerate unitary form f on V.

(c) Let $\mathrm{char}(\mathrm{GF}(q)) = p$ and recall the semilinear transformation $\overline{\varphi_p}_B$ introduced in Subsection 1.2.2. By the last proposition, we see that the diagonal matrix W_{SL} and the semilinear transformation $\overline{\varphi_p}_B$ (where B denotes the suitable ordered basis of V in that proposition) normalize H_k^{SL} and $H_{k,n-k}^{\mathrm{SL}}$. So, since $H_{k,n-k}^{\mathrm{SL}} < H_k^{\mathrm{SL}}$ for $2k < n$, it can be deduced that for $G \leq \mathrm{P\Gamma L}_n(q)$ a member of A-class $\mathscr{C}_1$ of G of type $P_{k,n-k}$ cannot be a maximal subgroup of G (use Propositions 2.1.4, 2.1.11 (i) and 2.1.13).

Next, we provide the information from [BHR] and [KL] in which cases a member of A-class $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$ is a maximal subgroup of G. As we will see, the observations in Remarks 2.1.3 (b) and 2.1.14 (c) describe the only exceptions.

Proposition 2.1.15. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and let M be a member of $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Then M is a maximal subgroup of G except if one of the following holds.*

(i) Case **L** *holds, $n \neq 2k$, $G \not\leq \mathrm{P\Gamma L}(V)$ and M is of type P_k.*

(ii) Case **L** *holds, $G \leq \mathrm{P\Gamma L}(V)$ and M is of type $P_{k,n-k}$.*

In the excluded cases M is not a maximal subgroup of G.

Proof. The assertion follows by [KL, Tables 3.5.A, 3.5.B, 3.5.H and 3.5.G] together with [BHR, Propositions 2.3.1 and 2.3.4 and Theorem 6.3.10] (cf. also [BHR, Definition 2.3.5 or the related tables in Chapter 8][4]). □

Remark 2.1.16. In view of Proposition 2.1.4, we obtain by the last proposition typical examples for novelties (recall Definition 1.1.4).

Now, we provide further facts about the groups $H_k^{\mathrm{GL}^\epsilon}$, $H_k^{\mathrm{SL}^\epsilon}$, $H_{k,n-k}^{\mathrm{GL}}$ and $H_{k,n-k}^{\mathrm{SL}}$ from Proposition 2.1.13.

Remark. We note that some of the facts which we will encounter in the following investigations can also be obtained with a more general point of view, using Lie theory. In Lie theoretic terminology, $U_k^{\mathrm{GL}^\epsilon}$ is called the unipotent radical and $L_k^{\mathrm{GL}^\epsilon}$ a Levi factor of the maximal parabolic subgroup $H_k^{\mathrm{GL}^\epsilon}$ of $\mathrm{GL}^\epsilon(V)$. Analogous Lie theoretic descriptions can be formulated for $U_k^{\mathrm{SL}^\epsilon}$, $L_k^{\mathrm{SL}^\epsilon}$

[4] We have to mention a mistake in [BHR, first row of Table 8.3]. There, E_q^3 should be replaced by E_q^2.

and $H_k^{\mathrm{SL}^\epsilon} \leq \mathrm{SL}^\epsilon(V)$ and also for the subgroups $U_{k,n-k}^{\mathrm{SL}}$ ($U_{k,n-k}^{\mathrm{GL}}$) and $L_{k,n-k}^{\mathrm{SL}}$ ($L_{k,n-k}^{\mathrm{GL}}$) of the parabolic subgroup $H_{k,n-k}^{\mathrm{SL}}$ ($H_{k,n-k}^{\mathrm{GL}}$) of $\mathrm{SL}(V)$ ($\mathrm{GL}(V)$). We refer to [Car, Chapter 12] and [As2, Sections 43 and 47] for more information concerning this topic.
Since the facts for the here considered groups can be shown by elementary aspects and considerations, the author has decided that it is more adequate to follow this elementary path.

Lemma 2.1.17. *Let* $\mathrm{char}(\mathrm{GF}(q^u)) = p$. *Then, using the notation of Proposition 2.1.13, we have* $\mathrm{O}_p(H_k^{\mathrm{GL}^\epsilon}) = \mathrm{O}_p(H_k^{\mathrm{SL}^\epsilon}) = U_k^{\mathrm{SL}^\epsilon}$ *and* $\mathrm{O}_p(H_{k,n-k}^{\mathrm{GL}}) = \mathrm{O}_p(H_{k,n-k}^{\mathrm{SL}}) = U_{k,n-k}^{\mathrm{SL}}$.

Proof. Since $H_k^{\mathrm{SL}^\epsilon}$ and $H_{k,n-k}^{\mathrm{SL}}$ are normal subgroups of $H_k^{\mathrm{GL}^\epsilon}$ and $H_{k,n-k}^{\mathrm{GL}}$ of p'-index, respectively, it is sufficient to show that $\mathrm{O}_p(H_k^{\mathrm{GL}^\epsilon}) = U_k^{\mathrm{SL}^\epsilon}$ and $\mathrm{O}_p(H_{k,n-k}^{\mathrm{GL}}) = U_{k,n-k}^{\mathrm{SL}}$ hold. By Proposition 2.1.13, we see that $U_k^{\mathrm{SL}^\epsilon} \leq \mathrm{O}_p(H_k^{\mathrm{GL}^\epsilon})$ and $U_{k,n-k}^{\mathrm{SL}} \leq \mathrm{O}_p(H_{k,n-k}^{\mathrm{GL}})$. Suppose that inequality holds. Then, in view of Proposition 2.1.13 and Lemma 1.4.4, we obtain a contradiction by elementary observations. □

For the proof of the next lemma, we recall the notation $\mathrm{E}_{i,j}^{m,n}$ from page 8.

Lemma 2.1.18. *Using the notation of Proposition 2.1.13, we have that*

(i) U_k^{SL} *is a minimal normal subgroup of* H_k^{GL} *and of* H_k^{SL}, *and*

(ii) *if* $n = 2k$ *then* U_k^{SU} *is a minimal normal subgroup of* H_k^{GU} *and of* H_k^{SU}.

Proof. Since $H_k^{\mathrm{SL}^\epsilon}$ is a normal subgroup of $H_k^{\mathrm{GL}^\epsilon}$, it is sufficient to prove the assertion for the case $H_k^{\mathrm{SL}^\epsilon}$. First, let case **L** hold. Since H_{n-k}^{SL} is conjugate to H_k^{SL} in $\mathrm{Aut}(\mathrm{SL}_n(q))$ (cf. Remark 1.2.19 (d)), w.l.o.g. we can assume that $k \leq n-k$. Let $1 < N \leq U_k^{\mathrm{SL}}$ where $N \trianglelefteq H_k^{\mathrm{SL}}$. Regarding Proposition 2.1.13, we write for the elements $M \in U_k^{\mathrm{SL}}$ and $S \in L_k^{\mathrm{SL}}$

$$M = M(C) = \begin{pmatrix} \mathbb{1}_k & 0 \\ C & \mathbb{1}_{n-k} \end{pmatrix} \text{ and } S = S(A,B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $C \in \mathrm{Mat}_{n-k,k}(q)$, $A \in \mathrm{GL}_k(q)$ and $B \in \mathrm{GL}_{n-k}(q)$ with $\det(A)\det(B) = 1$. We note the obvious fact that $M(C_1)M(C_2) = M(C_1 + C_2)$ for $C_1, C_2 \in \mathrm{Mat}_{n-k,k}(q)$ (more general, $U_k^{\mathrm{SL}} \cong \mathrm{Mat}_{n-k,k}(q)$ can be considered as a $\mathrm{GF}(p)$-vector space where $p = \mathrm{char}(\mathrm{GF}(q))$), and we also note that

$$M(C)^{S(A,B)} = M(B^{-1}CA) \tag{2.1.1}$$

holds. Let $r = \min\{\mathrm{rk}(C') \mid M(C') \in N \setminus \{\mathbb{1}_n\}\} \geq 1$, and $M(C)$ be an element in $N \setminus \{\mathbb{1}_n\}$ where $\mathrm{rk}(C) = r$. First, we consider the case $n = 2$. Here, $C \in \mathrm{GF}(q)^*$ and if $S(A,B) \in L_k^{\mathrm{SL}}$ then $B = A^{-1}$. So, $M(C)^{S(A,A^{-1})} = M(A^2C)$. Define the homomorphism $\eta : \mathrm{GF}(q)^* \to \mathrm{GF}(q)^*, x \mapsto x^2$. For $\mathrm{char}(\mathrm{GF}(q)) = 2$ clearly η is an isomorphism, and so we easily deduce that $N = U_k^{\mathrm{SL}}$ and our assertion holds. If $\mathrm{char}(\mathrm{GF}(q)) \neq 2$ then the kernel of η is of order two. So, by easy observations, our assertion also follows for this case (cf. [Hu, II. 10.6 Hilfssatz]). Now, let $n > 2$. Suppose for the case $k = n - k$ that $r = k$. By Lemma 1.3.3, we see that there are elements $A', B' \in \mathrm{GL}_k(q)$ such that $B'^{-1}CA' = \mathbb{1}_r$. We set $A_1 = A' \cdot \mathrm{diag}(\mathbb{1}_{k-1}, \det(A')^{-1})$, $B_1 = B' \cdot \mathrm{diag}(\mathbb{1}_{k-1}, \det(B')^{-1})$, $A_2 = A_1 + A'\mathrm{E}^{k,k}_{1,k}$ and $B_2 = B_1$. Then we obtain a contradiction, since $M(\mathrm{E}^{k,k}_{1,k}) = M(C)^{S(A_2,B_2)}(M(C)^{S(A_1,B_1)})^{-1} \in N$. So, we have to consider the remaining case that $r < k$ or $k < n - k$. Again, by Lemma 1.3.3, we see that there are elements $A' \in \mathrm{GL}_k(q)$ and $B' \in \mathrm{GL}_{n-k}(q)$ such that $B'^{-1}CA' = \begin{pmatrix} \mathbb{1}_r & 0_{r,k-r} \\ 0_{n-k-r,r} & 0_{n-k-r,k-r} \end{pmatrix} = C_1$, and we note that $n - k - r \geq 1$. We set $A = A'$ and $B = B' \cdot \mathrm{diag}(\mathbb{1}_{n-k-1}, (\det(A')\det(B'))^{-1})$, and obtain that $M(C)^{S(A,B)} = M(C_1) \in N$. Hence, we can deduce that $T_r = \{M(C) \mid C \in \mathrm{Mat}_{n-k,k}(q) \text{ with } \mathrm{rk}(C) = r\} \subseteq N$ (note, that the elements in T_r form a single conjugacy class in H^{SL}_k). By easy observations, we now see that $r = 1$. So, we obtain our assertion, since obviously $\langle T_1 \rangle = U^{\mathrm{SL}}_k$.

Next, we prove assertion (ii); so let case **U** hold with $n = 2k \geq 4$. Let $1 < N \leq U^{\mathrm{SU}}_k$ where $N \trianglelefteq H^{\mathrm{SU}}_k$. In view of Proposition 2.1.13 (iii), we write for the elements $M \in U^{\mathrm{SU}}_k$ and $S \in L^{\mathrm{SU}}_k$

$$M = M(C) = \begin{pmatrix} \mathbb{1}_k & 0 \\ C & \mathbb{1}_k \end{pmatrix} \text{ and } S = S(A) = \begin{pmatrix} A & 0 \\ 0 & A^{-1t\varphi_q} \end{pmatrix}$$

where $C \in \mathrm{Mat}_k(q^2)$ with $-C = C^{t\varphi_q}$ and $A \in \mathrm{GL}_k(q^2)$ with $\mathrm{o}(\det(A)) \mid q - 1$. Clearly, we have

$$M(C)^{S(A)} = M(A^{t\varphi_q}CA). \tag{2.1.2}$$

As above, we define $r = \min\{\mathrm{rk}(C') \mid M(C') \in N \setminus \{\mathbb{1}_n\}\} \geq 1$, and let $M(C) \in N$ where $\mathrm{rk}(C) = r$. First, we consider the case $\mathrm{char}(\mathrm{GF}(q^2)) = 2$. Here, we have that $C \in J = \{C' \mid C' \in \mathrm{Mat}_k(q^2), C' = C'^{t\varphi_q}\}$, and we note that

$$J = \left\{ (c'_{ij})_{1 \leq i,j \leq k} \,\middle|\, \begin{array}{l} c'_{ll} \in \mathrm{GF}(q), c'_{ij} \in \mathrm{GF}(q^2) \text{ and } c'_{ji} = c'^{q}_{ij} \\ \text{for } 1 \leq l \leq k, 1 \leq i < j \leq k \end{array} \right\}. \tag{2.1.3}$$

For $0 \leq i \leq k$ we define the subsets $J_i = \{C' \mid C' \in J, \mathrm{rk}(C') = i\}$ of J. Furthermore, we set $M(J_i) = \{M(C') \mid C' \in J_i\} \subset U^{\mathrm{SU}}_k$. First, let $k \geq 3$. Suppose that $r = k$. By Lemma 1.3.4, we see that there is an element $A' \in \mathrm{GL}_k(q^2)$ where $A'^{t\varphi_q}CA' = \mathbb{1}_k$. We set $A_1 = A' \cdot \mathrm{diag}(\mathbb{1}_{k-1}, \det(A)^{-1})$ and $A_2 = A_1 + A'\mathrm{E}^{k,k}_{1,k}$. Then we easily obtain a contradiction, since $M(\mathrm{E}^{k,k}_{1,k} + \mathrm{E}^{k,k}_{k,1} + \mathrm{E}^{k,k}_{k,k}) = M(C)^{S(A_2)}(M(C)^{S(A_1)})^{-1} \in N$. Hence, we have $r < k$. Again, using Lemma 1.3.4, we have an element $A' \in \mathrm{GL}_k(q^2)$ where $A'^{t\varphi_q}CA' = \mathrm{diag}(\mathbb{1}_r, 0, \ldots, 0)$. We set $A = A' \cdot \mathrm{diag}(\mathbb{1}_{k-1}, \det(A)^{-1})$, and easily obtain that $M(\mathrm{diag}(\mathbb{1}_r, 0, \ldots, 0)) = M(C)^{S(A)} \in N$. By analogous arguments as done before, we now can deduce that the elements in $M(J_r)$ form a single conjugacy class in H^{SU}_k, so $M(J_r) \subseteq N$. Suppose that $r > 1$. Then we easily obtain a contradiction, since $M(\mathrm{E}_{1,1}^{k,k}) = M(C_1)(M(C_2))^{-1} \in N$ where $C_1, C_2 \in J_r$ with

$$C_1 = \operatorname{diag}\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \mathbb{1}_{r-2}, 0, \ldots, 0\right), C_2 = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbb{1}_{r-2}, 0, \ldots, 0\right).$$

Because $M(J_1) \subseteq N$, we obtain by analogous arguments as above $M(J_2) \subseteq N$ (recall that $k \geq 3$). Regarding (2.1.3), we see that every member of J can be expressed as a sum of matrices from J_1 and J_2. So, we easily obtain our assertion $N = U_k^{\mathrm{SU}}$.
Next, we consider the case $k = 2$. Here, suppose that $r = 2$. Again, by Lemma 1.3.4, we see that there is an $A' \in \mathrm{GL}_2(q^2)$ where $A'^{t\varphi_q} C A' = \mathbb{1}_2$. For $\mu = \det(A')$ we set

$$A_1 = A' \cdot \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix}, A_2 = A' \cdot \begin{pmatrix} 1 & 1 \\ 0 & \mu^{-1} \end{pmatrix} \text{ and } A_3 = A' \cdot \begin{pmatrix} 1 & 1 \\ \mu^{-1} & 0 \end{pmatrix}.$$

So, we obtain that $M(\operatorname{diag}(0, \mu^{-q-1}+1)) = M(C)^{S(A_1)} M(C)^{S(A_2)} M(C)^{S(A_3)} \in N$. If $\mu^{-q-1} \neq 1$ we obtain a contradiction. For $\mu^{-q-1} = 1$ we have that $M(C)^{S(A_1)} = M(\mathbb{1}_2) \in N$. (The following is adopted from the proof of [Hu, II. 10.4 Satz b][5]). Let $a \in \mathrm{GF}(q^2)$ be a solution of the equation $x^q + x = 1$ (such an element a exists, since the trace map $\mathrm{GF}(q^2) \to \mathrm{GF}(q), x \mapsto x^q + x$ is surjective). Set $A_4 = \begin{pmatrix} 1 & a \\ 1 & a^q \end{pmatrix} \in \mathrm{SL}_2(q^2)$. Then we have that $A_4^{t\varphi_q} \mathbb{1}_2 A_4 = \operatorname{antidiag}(1,1)$. So, we also obtain a contradiction for this case, since $M(\mathrm{E}_{2,2}^{2,2}) = M(C)^{S(A_3)} M(\mathbb{1}_2)^{S(A_4)} \in N$. Hence, we have $r = 1$. Now, by analogous arguments as above, we see that $M(J_1) \subseteq N$. So, we also have $M(\operatorname{antidiag}(1,1)) \in N$ (clearly $M(\mathbb{1}_2) \in N$, hence argue as above). Regarding (2.1.3), obviously every member in J can be expressed as a sum of diagonal matrices from J_1 and a matrix $C_0 = \begin{pmatrix} 0 & x \\ x^q & 0 \end{pmatrix}$ for some $x \in \mathrm{GF}(q^2)$. If $x \neq 0$ then there is an element $0 \neq b \in \mathrm{GF}(q^2)$ where $b^2 = x$ (recall that $\mathrm{char}(\mathrm{GF}(q^2)) = 2$). So, we set $A = \operatorname{antidiag}(b^q, b)$, and note that $\mathrm{o}(\det(A)) \mid q - 1$. Now, we obtain our assertion, since $M(\operatorname{antidiag}(1,1))^{S(A)} = M(C_0) \in N$.
Finally, we consider the case $\mathrm{char}(\mathrm{GF}(q^2)) \neq 2$. For this, let $\omega \in \mathrm{GF}(q^2)$ where $\omega^q = -\omega$. In view of Corollary 1.3.5, we see that

$$U_k^{\mathrm{SU}} = \left\{ \begin{pmatrix} \mathbb{1}_k & 0 \\ C & \mathbb{1}_k \end{pmatrix} \middle| \begin{array}{l} C \in \mathrm{Mat}_k(q^2), \\ C^{t\varphi_q} = -C \end{array} \right\} = \left\{ \begin{pmatrix} \mathbb{1}_k & 0 \\ \omega C & \mathbb{1}_k \end{pmatrix} \middle| \begin{array}{l} C \in \mathrm{Mat}_k(q^2), \\ C = C^{t\varphi_q} \end{array} \right\}.$$

So, for $k \geq 3$ we obtain the assertion analogous to the previous case (note, that for $k \geq 3$ no information was used about the characteristic of $\mathrm{GF}(q^2)$). For case $k = 2$ we obtain the assertion by similar arguments as above, using the information from the proof of [Hu, II. 10.4 Satz b] for odd characteristic. (We note to use [Hu, II. 10.6 Hilfssatz] for the final step). □

[5] We note a mistake in the proof of [Hu, II. 10.4 Satz b] for the case $\mathrm{char}(K) = 2$. There, the assertion is only proven if $a^q \neq a$; note, that $a^q \neq a \Leftrightarrow q = 2^e$ for odd e. So, for even e we have $a^q = a$ and the chosen elements w_1 and w_2 are linearly dependent. Furthermore, we note that this kind of mistake can also be found in several other books.

Remark. Using the notation from Proposition 2.1.13, it is not hard to see that

(a) $U^{\mathrm{SL}}_{k,n-k}$ is not a minimal normal subgroup of $H^{\mathrm{SL}}_{k,n-k}$ and of $H^{\mathrm{GL}}_{k,n-k}$, and

(b) U^{SU}_{k} is not a minimal normal subgroup of H^{SU}_{k} and of H^{GU}_{k} for $n > 2k$

(cf. also Proposition 2.1.21 (ii) and (iv) and Remark 2.1.22).

By the following lemma, we can transfer the facts provided above from the considered groups to their projective image under P. For this, we recall the notation $\hat{\ }$ from Convention 1.2.2.

Lemma 2.1.19. *Let $H \leq \mathrm{A}(V)$, and let p be a prime where $p \nmid |\mathrm{Z}(\mathrm{GL}(V))| = q^u - 1$. Then the following hold.*

(i) $\mathrm{P}(\mathrm{O}_p(H)) = \mathrm{O}_p(\mathrm{P}H)$.

(ii) If M is a minimal normal p-subgroup of H then $\mathrm{P}M$ is a minimal normal p-subgroup of $\mathrm{P}H$.

Proof. Set $Z = \mathrm{Z}(\mathrm{GL}(V))$. First, we prove assertion (i). It is not hard to see that $\mathrm{P}(\mathrm{O}_p(H)) \leq \mathrm{O}_p(\mathrm{P}H)$. So, we have only to prove the other inclusion. Because $|\widehat{\mathrm{O}_p(\mathrm{P}H)}| = |Z|\cdot|\mathrm{O}_p(\mathrm{P}H)|$, we obtain by the Sylow theorems that there is exactly one subgroup K of $\widehat{\mathrm{O}_p(\mathrm{P}H)}$ with $|K| = |\mathrm{O}_p(\mathrm{P}H)|$ (esp. $\widehat{\mathrm{O}_p(\mathrm{P}H)} = Z \times K$). Obviously, we have $\mathrm{P}K = \mathrm{O}_p(\mathrm{P}H)$. Since $K \operatorname{char} \widehat{\mathrm{O}_p(\mathrm{P}H)} \trianglelefteq \widehat{\mathrm{P}H} = HZ$, we obtain that K is a normal subgroup of H (note, that H is a normal subgroup of $\widehat{\mathrm{P}H}$ of p'-index). So, we deduce that $K \leq \mathrm{O}_p(H)$ and assertion (i) is established. To prove assertion (ii), let M be as assumed. Clearly, $\mathrm{P}M$ is a normal p-subgroup of $\mathrm{P}H$ of order $|M|$. Suppose that there is a normal p-subgroup N of $\mathrm{P}H$ with $1 < N < \mathrm{P}M$. Because $|\widehat{N}| = |Z| \cdot |N|$, we obtain by analogous arguments as above that there is a normal p-subgroup K of H of order $|N|$ where $\widehat{N} = K \times Z$. Since $K \leq \widehat{\mathrm{P}M} = M \times Z$, we obtain a contradiction by $K < M$. ☐

Remark 2.1.20. (a) Clearly, the assertion of the last lemma also holds using the generalized standard notation.

(b) If in addition $H \leq \mathrm{I}(V)$ (or $\Omega(V)$) in the last lemma, it is sufficient to demand that $p \nmid |\mathrm{Z}(\mathrm{I}(V))|$ (or $|\mathrm{Z}(\Omega(V))|$) to obtain the assertion (cf. Remark 1.2.1 (d)).

In the following proposition, we provide the information about the centralizer of $\mathrm{P}U^{\mathrm{SL}^\epsilon}_k$ in $\mathrm{Aut}(\mathrm{PSL}^\epsilon_n(q^u))$ and $\mathrm{P}U^{\mathrm{SL}}_{k,n-k}$ in $\mathrm{Aut}(\mathrm{PSL}_n(q))$. For this, recall the terminology and notation from Convention 1.2.2 and Subsection 1.2.2 (esp. the notation $[a_{ij}]_{n\times n}$ for the projective version of the matrix $(a_{ij})_{n\times n} \in \mathrm{GL}_n(q)$), as well as the notation $\mathrm{E}^{m,n}_{i,j}$ from page 8.

Proposition 2.1.21. *We use the notation introduced in Proposition 2.1.13. Then we have that*

(i) $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}U_k^{\mathrm{SL}}) = \mathrm{P}U_k^{\mathrm{SL}}$,

(ii) $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}U_{k,n-k}^{\mathrm{SL}}) = \mathrm{Z}(\mathrm{P}U_{k,n-k}^{\mathrm{SL}}) < \mathrm{P}U_{k,n-k}^{\mathrm{SL}}$,

(iii) if $n = 2k$ *then* $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_n(q^2))}(\mathrm{P}U_k^{\mathrm{SU}}) = \mathrm{P}U_k^{\mathrm{SU}}$ *and*

(iv) if $n > 2k$ *then* $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_n(q^2))}(\mathrm{P}U_k^{\mathrm{SU}}) = \mathrm{Z}(\mathrm{P}U_k^{\mathrm{SU}}) < \mathrm{P}U_k^{\mathrm{SU}}$.

Proof. We prove the assertion by elementary calculations.[6] Let $q = p^a$ for a prime p and a positive integer a, and let λ be a primitive element of $\mathrm{GF}(q^u)^*$. First, we prove assertion (i). Let $C \in \mathrm{C}_{\mathrm{P\Gamma L}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}})$, so $C = \varphi B$ where $\varphi \in \langle \varphi_p \rangle$ and $B \in \mathrm{PGL}_n(q)$. Since φ normalizes $\mathrm{P}U_k^{\mathrm{SL}}$, we have $B \in \mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}})$. By Lemmas 2.1.17 and 2.1.19 (i), we see $\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}H_k^{\mathrm{GL}}) \leq \mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}})$. So, we deduce $\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}}) = \mathrm{P}H_k^{\mathrm{GL}}$, by Propositions 2.1.13 (i) and 2.1.15. Hence, let $B = \begin{bmatrix} B_1 & 0 \\ D & B_2 \end{bmatrix}$ where $B_1 \in \mathrm{GL}_k(q), B_2 \in \mathrm{GL}_{n-k}(q)$ and $D \in \mathrm{Mat}_{n-k,k}(q)$. We define the subset $S = \{S_1', \dots, S_{n-1}'\}$ of $\mathrm{P}U_k^{\mathrm{SL}}$ consisting of the elements $S_l' = \begin{bmatrix} \mathbb{1}_k & 0 \\ S_l & \mathbb{1}_{n-k} \end{bmatrix}$ where $S_l = \mathrm{E}_{1,l}^{n-k,k}$ for $1 \leq l \leq k$ and $S_l = \mathrm{E}_{l-k+1,1}^{n-k,k}$ for $k+1 \leq l \leq n-1$. Since C centralizes S, we have that

$$S_l' = S_l'^C = S_l'^B \Leftrightarrow S_l B_1 = B_2 S_l. \qquad (2.1.4)$$

By elementary calculations, we now obtain $B_1 = \mu \mathbb{1}_k$ and $B_2 = \mu \mathbb{1}_{n-k}$ for a $\mu \in \mathrm{GF}(q)^*$. Hence, $B \in \mathrm{P}U_k^{\mathrm{SL}}$. Now, suppose that $\varphi \neq 1$ (esp. $a > 1$ and $\lambda^\varphi \neq \lambda$). Since C centralizes $\begin{bmatrix} \mathbb{1}_k & 0 \\ \lambda S_1 & \mathbb{1}_{n-k} \end{bmatrix} \in \mathrm{P}U_k^{\mathrm{SL}}$, we easily obtain a contradiction. So, we can deduce $\mathrm{C}_{\mathrm{P\Gamma L}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}}) = \mathrm{P}U_k^{\mathrm{SL}}$.
Now, let $n \neq 2k$. Suppose that there is an element in $\mathrm{N}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}U_k^{\mathrm{SL}}) \setminus \mathrm{P\Gamma L}_n(q)$. Then there is a $B \in \mathrm{PGL}_n(q)$ with $(\mathrm{P}U_k^{\mathrm{SL}})^\tau = (\mathrm{P}U_k^{\mathrm{SL}})^B$. Note, that $\mathrm{N}_{\mathrm{PGL}_n(q)}((\mathrm{P}U_k^{\mathrm{SL}})^\tau)$ is conjugate to $\mathrm{P}H_{n-k}^{\mathrm{GL}}$ by $\mathrm{antidiag}[1, \dots, 1] \in \mathrm{PGL}_n(q)$. We now obtain a contradiction by Lemma 1.4.5, since $\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U_k^{\mathrm{SL}}) = \mathrm{P}H_k^{\mathrm{GL}}$ is conjugate to $\mathrm{P}H_{n-k}^{\mathrm{GL}}$ in $\mathrm{PGL}_n(q)$ (cf. Remark 1.2.19 (d)). So, assertion (i) is established for this case. Let $n = 2k$, and suppose that there is a $C \in \mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}U_k^{\mathrm{SL}}) \setminus \mathrm{P\Gamma L}_n(q)$. Define $A_n = \mathrm{antidiag}[1, \dots, 1] \in \mathrm{PGL}_n(q)$, and write $C = \tau\varphi B_0$ where $\varphi \in \langle \varphi_p \rangle$ and $B_0 = A_n B$ with $B \in \mathrm{PGL}_n(q)$. Since $\tau\varphi A_n$ normalizes $\mathrm{P}U_k^{\mathrm{SL}}$, we have that $B = \begin{bmatrix} B_1 & 0 \\ D & B_2 \end{bmatrix}$ where $B_1, B_2 \in \mathrm{GL}_k(q)$ and $D \in \mathrm{Mat}_k(q)$, by analogous considerations as above. Since C centralizes $\begin{bmatrix} \mathbb{1}_k & 0 \\ \mathbb{1}_k & \mathbb{1}_k \end{bmatrix} \in \mathrm{P}U_k^{\mathrm{SL}}$, we have that $B_2 = -B_1$. Now, we easily obtain a contradiction, since C centralizes S_1' and S_2' where S_1' and S_2' are defined as above

[6] We note that we could use Lemma 1.4.8 and transfer our considerations to the non-projective case. But because of the structure of $\mathrm{P}U_k^{\mathrm{SL}^\epsilon}$ and $\mathrm{P}U_{k,n-k}^{\mathrm{SL}}$ there is no advantage for the calculations, as we will see.

(note, that $k \geq 2$). Hence, assertion (i) holds.
Next, we prove assertion (ii). First, let $C \in \mathrm{C}_{\mathrm{PGL}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k})$, and let $Q \in \mathrm{P}U^{\mathrm{SL}}_{k,n-k}$. By elementary considerations, similar to above, we see that $\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) = \mathrm{P}H^{\mathrm{GL}}_{k,n-k}$. So, we can write

$$C = \begin{bmatrix} B_1 & 0_{k,n-2k} & 0_{k,k} \\ M & B_2 & 0_{n-2k,k} \\ N & L & B_3 \end{bmatrix} \text{ and } Q = \begin{bmatrix} \mathbb{1}_k & 0_{k,n-2k} & 0_{k,k} \\ X & \mathbb{1}_{n-2k} & 0_{n-2k,k} \\ Z & Y & \mathbb{1}_k \end{bmatrix} \quad (2.1.5)$$

where $B_1, B_3 \in \mathrm{GL}_k(q), B_2 \in \mathrm{GL}_{n-2k}(q), X, M \in \mathrm{Mat}_{n-2k,k}(q), Z, N \in \mathrm{Mat}_k(q)$ and $Y, L \in \mathrm{Mat}_{k,n-2k}(q)$. By elementary calculations, we see that

$$C^{Q^{-1}} = C \Leftrightarrow \begin{cases} \text{(I)} & XB_1 = B_2X, \\ \text{(II)} & YB_2 = B_3Y, \\ \text{(III)} & ZB_1 + YM + B_3YX = B_3Z + LX + YB_2X. \end{cases} \quad (2.1.6)$$

The conditions (I) and (II) of (2.1.6) coincide with the condition in (2.1.4). Hence, by analogous considerations as above (i.e. choosing the appropriate matrices $\mathrm{E}^{n-2k,k}_{i,j}$ for X and set $Y = 0$, $Z = 0$, or $\mathrm{E}^{k,n-2k}_{i,j}$ for Y and set $X = 0$, $Z = 0$), we obtain that $B_1 = B_3 = \mu\mathbb{1}_k$ and $B_2 = \mu\mathbb{1}_{n-2k}$ for a $\mu \in \mathrm{GF}(q)^*$. So, condition (III) becomes $YM = LX$. Considering the last condition for $X \in \{\mathrm{E}^{n-2k,k}_{i,1} \mid 1 \leq i \leq n-2k\}$ and $Y = 0$, we easily see that $L = 0$. Analogously, we obtain that $M = 0$, by considering $Y \in \{\mathrm{E}^{k,n-2k}_{1,j} \mid 1 \leq j \leq n-2k\}$ and $X = 0$. Hence, we obtain

$$\mathrm{C}_{\mathrm{PGL}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) = \mathrm{Z}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) = \left\{ \begin{bmatrix} \mathbb{1}_k & 0 & 0 \\ 0 & \mathbb{1}_{n-2k} & 0 \\ Z & 0 & \mathbb{1}_k \end{bmatrix} \middle| Z \in \mathrm{Mat}_k(q) \right\}.$$

Now, let $C \in \mathrm{C}_{\mathrm{P\Gamma L}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k})$, so $C = \varphi B$ for a $\varphi \in \langle \varphi_p \rangle$ and $B \in \mathrm{PGL}_n(q)$. Since φ normalizes $\mathrm{P}U^{\mathrm{SL}}_{k,n-k}$, $B \in \mathrm{P}H^{\mathrm{GL}}_{k,n-k}$ (argue as above). Now, we obtain by analogous argumentations as done before that $B \in \mathrm{Z}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k})$ (note, that each matrix Q used above has only entries 1 and 0, so φ centralizes Q). Suppose that $\varphi \neq 1$, so $\lambda^\varphi \neq \lambda$. Then we easily obtain a contradiction, by considering that C centralizes Q with $Y = 0$, $Z = 0$ and $X = \lambda\mathrm{E}^{n-2k,k}_{1,1}$ (Q as in (2.1.5)).
Finally, suppose that $C \in \mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) \setminus \mathrm{P\Gamma L}_n(q)$. So, $C = \tau\varphi A_n B$ where $\varphi \in \langle \varphi_p \rangle$, $B \in \mathrm{PGL}_n(q)$ and A_n is defined as in the proof of part (i). Since $\tau\varphi A_n$ normalizes $\mathrm{P}U^{\mathrm{SL}}_{k,n-k}$, we have that $B \in \mathrm{P}H^{\mathrm{GL}}_{k,n-k}$. Now, considering that C centralizes the element Q, defined as in (2.1.5), for $Y = 0$, $Z = 0$ and a $X \neq 0$, we obtain by elementary calculations a contradiction. (Note, that $B \in \mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) = \mathrm{P}H^{\mathrm{GL}}_{k,n-k} < \mathrm{P}H^{\mathrm{GL}}_{n-k}$ and $\mathrm{O}_p(\mathrm{P}H^{\mathrm{GL}}_{n-k}) = \mathrm{P}U^{\mathrm{SL}}_{n-k}$). Hence, assertion (ii) is established.
Now, we prove part (iii). For this, let $C \in \mathrm{C}_{\mathrm{P\Gamma U}_n(q^2)}(\mathrm{P}U^{\mathrm{SU}}_k)$, so $C = \varphi B$ where $\varphi \in \langle \varphi_p \rangle$ and $B \in \mathrm{PGU}_n(q^2)$. Because φ normalizes $\mathrm{P}U^{\mathrm{SU}}_k$, we have that B normalizes $\mathrm{P}U^{\mathrm{SU}}_k$. By analogous arguments as in the proof of part (i), using

Lemmas 2.1.17 and 2.1.19 (i) and Propositions 2.1.13 (iii) and 2.1.15, we obtain that $B \in \mathrm{N}_{\mathrm{PGU}_n(q^2)}(\mathrm{P}U_k^{\mathrm{SU}}) = \mathrm{P}H_k^{\mathrm{GU}}$. Hence, we can write $B = \begin{bmatrix} B_1 & 0 \\ D & B_2 \end{bmatrix}$ where $B_1, B_2 \in \mathrm{GL}_k(q^2)$ and $D \in \mathrm{Mat}_k(q^2)$ (actually, we have $B_2 = B_1^{-1t\varphi_q}$, but there is not much advantage for our further calculations).
First, let $k \geq 3$. Define the subset $S = \{S'_1, \ldots, S'_{k+1}\} \subset \mathrm{P}U_k^{\mathrm{SU}}$ consisting of the elements $S'_l = \begin{bmatrix} \mathbb{1}_k & 0 \\ S_l & \mathbb{1}_k \end{bmatrix}$ where $S_l = \mathrm{E}^{k,k}_{l,l+1} - \mathrm{E}^{k,k}_{l+1,l}$ for $1 \leq l \leq k-1$, $S_k = \mathrm{E}^{k,k}_{1,3} - \mathrm{E}^{k,k}_{3,1}$ and $S_{k+1} = \mathrm{E}^{k,k}_{1,k} - \mathrm{E}^{k,k}_{k,1}$. Because C centralizes S, we have that

$$S'_l = S'^C_l = S'^B_l \Leftrightarrow S_l B_1 = B_2 S_l.$$

By elementary calculations, we obtain that $B_1 = \mu \mathbb{1}_k = B_2$ for a $\mu \in \mathrm{GF}(q^2)^*$. Hence, $B \in \mathrm{P}U_k^{\mathrm{SU}}$. Suppose that $\varphi \neq 1$ (so, $\lambda^\varphi \neq \lambda$). Considering that C centralizes the element $\begin{bmatrix} \mathbb{1}_k & 0 \\ \lambda \mathrm{E}^{k,k}_{1,k} - \lambda^q \mathrm{E}^{k,k}_{k,1} & \mathbb{1}_k \end{bmatrix} \in \mathrm{P}U_k^{\mathrm{SU}}$, we now obtain a contradiction. So, assertion (iii) follows for this case. Next, consider the remaining case $k = 2$ (note, that $n > 2$). If $p = 2$ the assertion follows by analogous arguments as before, considering the subset $T = \{T_1, T_2, T_3\}$ of $\mathrm{P}U_2^{\mathrm{SU}}$ where

$$T_1 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ \mathrm{E}^{2,2}_{1,1} & \mathbb{1}_2 \end{bmatrix}, T_2 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ \mathrm{E}^{2,2}_{2,2} & \mathbb{1}_2 \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ \mathrm{E}^{2,2}_{1,2} - \mathrm{E}^{2,2}_{2,1} & \mathbb{1}_2 \end{bmatrix}.$$

For $p \neq 2$ let $\zeta = \lambda^{\frac{q+1}{2}}$, so $\mathrm{o}(\zeta) = 2(q-1)$ and $\zeta + \zeta^q = 0$. Consider that C centralizes the subset $R = \{R_1, R_2, R_3\}$ of $\mathrm{P}U_2^{\mathrm{SU}}$ where

$$R_1 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ \zeta \mathrm{E}^{2,2}_{1,1} & \mathbb{1}_2 \end{bmatrix}, R_2 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ \zeta \mathrm{E}^{2,2}_{2,2} & \mathbb{1}_2 \end{bmatrix} \text{ and } R_3 = T_3.$$

Then, by elementary calculations, we obtain that $B_1 = \mathrm{diag}(b, \zeta^\varphi \zeta^{-1} b), B_2 = \mathrm{diag}(\zeta^\varphi \zeta^{-1} b, b)$ for a $b \in \mathrm{GF}(q^2)^*$, and $1 = (\zeta^\varphi \zeta^{-1})^2 = \zeta^{2(p^b - 1)}$ where $\varphi = \varphi_p^b$ for an integer $0 \leq b < a$. By the last equation, we easily deduce that $b = 0$. Hence, assertion (iii) follows.
Finally, we consider assertion (iv). Let $C \in \mathrm{C}_{\mathrm{P\Gamma U}_n(q^2)}(\mathrm{P}U_k^{\mathrm{SU}})$, so $C = \varphi B$ where $\varphi \in \langle \varphi_p \rangle$ and $B \in \mathrm{N}_{\mathrm{PGU}_n(q^2)}(\mathrm{P}U_k^{\mathrm{SU}}) = \mathrm{P}H_k^{\mathrm{GU}}$, by analogous arguments as in part (iii). Let $Q \in \mathrm{P}U_k^{\mathrm{SU}}$. Considering Proposition 2.1.13 (iv), we can write

$$B = \begin{bmatrix} B_1 & 0_{k,n-2k} & 0_{k,k} \\ M & B_2 & 0_{n-2k,k} \\ N & L & B_3 \end{bmatrix} \text{ and } Q = \begin{bmatrix} \mathbb{1}_k & 0_{k,n-2k} & 0_{k,k} \\ -D^{t\varphi_q} & \mathbb{1}_{n-2k} & 0_{n-2k,k} \\ C & D & \mathbb{1}_k \end{bmatrix}$$

with $C + C^{t\varphi_q} + DD^{t\varphi_q} = 0$ where $B_1, B_3 \in \mathrm{GL}_k(q^2)$, $B_2 \in \mathrm{GL}_{n-2k}(q^2)$, $D, L \in \mathrm{Mat}_{k,n-2k}(q^2)$, $M \in \mathrm{Mat}_{n-2k,k}(q^2)$ and $C, N \in \mathrm{Mat}_k(q^2)$. By elementary calculations, we see that

$$Q^C = Q \Leftrightarrow Q^\varphi B = BQ \Leftrightarrow \begin{cases} \text{(I)} & D^{t\varphi_q \varphi} B_1 = B_2 D^{t\varphi_q}, \\ \text{(II)} & D^\varphi B_2 = B_3 D, \\ \text{(III)} & C^\varphi B_1 + D^\varphi M + L D^{t\varphi_q} = B_3 C. \end{cases} \tag{2.1.7}$$

First, we consider the conditions (I) and (II). We choose $\eta \in \mathrm{GF}(q^2)^*$ with $\eta + \eta^q = -1$ (the existence of η is clear, see e.g. [Hu, p. 243]). Consider that conditions (I) and (II) hold for the elements Q where $(D, C) \in \{(\mathrm{E}_{1,j}^{k,n-2k}, \eta \mathrm{E}_{1,1}^{k,k}) \mid 1 \leq j \leq n-2k\} \cup \{(\mathrm{E}_{j,1}^{k,n-2k}, \eta \mathrm{E}_{j,j}^{k,k}) \mid 2 \leq j \leq k\}$. Then, by analogous calculations as for (2.1.4), we obtain that $B_1 = B_3 = \mu \mathbb{1}_k$ and $B_2 = \mu \mathbb{1}_{n-2k}$ for a $\mu \in \mathrm{GF}(q^2)^*$. Hence, $B \in \mathrm{P}U_k^{\mathrm{SU}}$ and $M = -L^{t\varphi_q}$. Now, suppose that $\varphi \neq 1$ (so, $\lambda^\varphi \neq \lambda$). Then we easily obtain a contradiction by condition (II) (or (I)), since C centralizes Q where $(D, C) = (\lambda \mathrm{E}_{1,1}^{k,n-2k}, \eta_1 \mathrm{E}_{1,1}^{k,k})$ for a $\eta_1 \in \mathrm{GF}(q^2)^*$ with $\eta_1 + \eta_1^q = -\lambda\lambda^q \in \mathrm{GF}(q)^*$ (again, see [Hu, p. 243] for the existence of η_1). Hence, $C = B \in \mathrm{P}U_k^{\mathrm{SU}}$. Now, condition (III) becomes (III') $LD^{t\varphi_q} = DL^{t\varphi_q}$. Let $L = (l_{ij})_{1 \leq i \leq k, 1 \leq j \leq n-2k}$. Considering that (III') holds for $D \in \{\mathrm{E}_{1,j}^{k,n-2k} \mid 1 \leq j \leq n-2k\} = T_1$, we obtain that $l_{1j} \in \mathrm{GF}(q)$ for $1 \leq j \leq n-2k$ and $l_{ij} = 0$ for $2 \leq i \leq k$, $1 \leq j \leq n-2k$. Suppose that $l_{1j} \neq 0$. Since (III') holds for $D \in \{\lambda \mathrm{E}_{1,j}^{k,n-2k} \mid 1 \leq j \leq n-2k\} = T_2$, we obtain that $l_{1j}\lambda^q = \lambda l_{1j}^q$. So, we easily obtain a contradiction, because $1 \neq \lambda^{q-1} = l_{1j}^{q-1} = 1$. (Here, we note that by analogous considerations as above we see that there are appropriate $C \in \mathrm{Mat}_k(q^2)$ (and hence $Q \in \mathrm{P}U_k^{\mathrm{SU}}$) such that $C + C^{t\varphi_q} + DD^{t\varphi_q} = 0$ holds for each matrix $D \in T_1 \cup T_2$). So, our proposition is established. □

Remark 2.1.22. Adopt the notation from Proposition 2.1.13 and let $p = \mathrm{char}(\mathrm{GF}(q^u))$. From the proof of the last proposition, we can also deduce that $\mathrm{Z}(\mathrm{P}U_{k,n-k}^{\mathrm{SL}}) = (\mathrm{P}U_{k,n-k}^{\mathrm{SL}})' = \Phi(\mathrm{P}U_{k,n-k}^{\mathrm{SL}})$ and, for $n > 2k$, $\mathrm{Z}(\mathrm{P}U_k^{\mathrm{SU}}) = (\mathrm{P}U_k^{\mathrm{SU}})' = \Phi(\mathrm{P}U_k^{\mathrm{SU}})$ are elementary abelian p-groups. Obviously, $\mathrm{P}U_k^{\mathrm{SL}}$ and, for $n = 2k$, $\mathrm{P}U_k^{\mathrm{SU}}$ are elementary abelian p-groups, cf. also Lemmas 2.1.18 and 2.1.19 (ii). Hence, we see that $\mathrm{P}U_k^{\mathrm{SL}^\epsilon}$ and $\mathrm{P}U_{k,n-k}^{\mathrm{SL}}$ are special groups, in the sense of [Hu, III. 13.1 Definition].

By the last proposition, we obtain the following corollary. Here, we use the standard notation and recall that in this notation V is a $\mathrm{GF}(q^u)$-vector space for a prime power q; also recall that $u = 1$ in case **L** and $u = 2$ in case **U**.

Corollary 2.1.23. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and let K be a member of $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Let $\mathrm{char}(\mathrm{GF}(q^u)) = p$. Then K is strongly p-constrained.*

Proof. In view of Propositions 2.1.4, 2.1.11 and 2.1.13, w.l.o.g. we can assume that $H = K \cap \mathrm{P}\Omega = \mathrm{P}H_k^{\mathrm{SL}^\epsilon}$ or $\mathrm{P}H_{k,n-k}^{\mathrm{SL}}$, by choosing an appropriate basis of V. Considering Lemmas 2.1.17 and 2.1.19 (i), we see that $\mathrm{O}_p(\mathrm{P}H_k^{\mathrm{SL}^\epsilon}) = \mathrm{P}U_k^{\mathrm{SL}^\epsilon}$ and $\mathrm{O}_p(\mathrm{P}H_{k,n-k}^{\mathrm{SL}}) = \mathrm{P}U_{k,n-k}^{\mathrm{SL}}$. Obviously, $\mathrm{P}U_k^{\mathrm{SL}^\epsilon} \leq \mathrm{O}_p(K)$ if $H = \mathrm{P}H_k^{\mathrm{SL}^\epsilon}$, and $\mathrm{P}U_{k,n-k}^{\mathrm{SL}} \leq \mathrm{O}_p(K)$ if $H = \mathrm{P}H_{k,n-k}^{\mathrm{SL}}$. So, our assertion follows easily from the last proposition. □

Using the information from Proposition 2.1.15 and Corollary 2.1.23, we now obtain the next two main theorems of this section. We determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Recalling Main Theorems 2.1.9 and 2.1.10, we note that finally we have determined all pairs (G, M) where G is as above and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_1$ of G. As usual, we present the following results not using the standard notation.

Main Theorem 2.1.24. *Let* $\mathrm{PSL}_n(p^a) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(p^a))$ *where* $\mathrm{PSL}_n(p^a)$ *is simple, p is a prime and a a positive integer. Let M belong to A-class $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Then the following hold.*

- *(i) If M is of type P_k then M is a strongly constrained maximal subgroup of G if and only if $G \leq \mathrm{P\Gamma L}_n(p^a)$ for $n \neq 2k$, or $G \leq \mathrm{Aut}(\mathrm{PSL}_n(p^a))$ for $n = 2k$.*
- *(ii) If M is of type $P_{k,n-k}$ then M is a strongly constrained maximal subgroup of G if and only if $G \not\leq \mathrm{P\Gamma L}_n(p^a)$.*

Furthermore, the subgroup M of G is strongly p-constrained.

Main Theorem 2.1.25. *Let* $\mathrm{PSU}_n(p^{2a}) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(p^{2a}))$ *where* $\mathrm{PSU}_n(p^{2a})$ *is simple, $n \geq 3$, p is a prime and a a positive integer. Then every member M of A-class $\mathscr{C}_1$ of G of type P_k is a strongly p-constrained maximal subgroup of G.*

As we have seen in Corollary 2.1.23, for $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ every member K of A-class $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$ is strongly p-constrained where $p = \mathrm{char}(\mathrm{GF}(q^u))$. So we close this section, by providing further information about $\mathrm{O}_p(K)$.

Proposition 2.1.26. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and K be a member of $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Let* $\mathrm{char}(\mathrm{GF}(q^u)) = p$. *Then we have that* $\mathrm{O}_p(K) = \mathrm{O}_p(K \cap \mathrm{P\Omega}(V))$ *except in case* **L** *if K is of type $P_{k,n-k}$ and with respect to an ordered* $\mathrm{GF}(q)$*-basis of V one of the following holds.*

- *(i)* $G = \mathrm{Aut}(\mathrm{PSL}_3(2))$ *(so, $k = 1$ and $n - k = 2$).*
- *(ii)* $G \leq \mathrm{Aut}(\mathrm{PSL}_3(4))$ *(so, $k = 1$ and $n - k = 2$), and G corresponds to a conjugate of* $\mathrm{PSL}_3(4) \rtimes \langle \tau \rangle$ *or* $\mathrm{PSL}_3(4) \rtimes (\langle \varphi_2 \rangle \times \langle \tau \rangle)$ *in* $\mathrm{Aut}(\mathrm{PSL}_3(4))$.
- *(iii)* $G = \mathrm{Aut}(\mathrm{PSL}_4(2))$ *(so, $k = 1$ and $n - k = 3$).*

In the excluded cases (i) to (iii) we have $\mathrm{O}_2(K) \not\leq \mathrm{P\Gamma}$ *and* $|\mathrm{O}_2(K)/\mathrm{O}_2(K \cap \mathrm{P\Omega})| = 2$.

Proof. Regarding Propositions 2.1.4, 2.1.11 and 2.1.13, w.l.o.g. we can choose an appropriate ordered $\mathrm{GF}(q^u)$-basis of V and assume that $H = K \cap \mathrm{P\Omega} = \mathrm{P}H_k^{\mathrm{SL}^\epsilon}$ if K is of type P_k and $H = K \cap \mathrm{P\Omega} = \mathrm{P}H_{k,n-k}^{\mathrm{SL}}$ if K is of type $P_{k,n-k}$. First, we consider the case **L** where K is of type P_k and the case **U** where K is of type $P_{\frac{n}{2}}$. Here, we have that $\mathrm{O}_p(H)$ is a minimal normal p-subgroup of H, by Lemmas 2.1.17, 2.1.18 and 2.1.19. So, $\mathrm{O}_p(H)$ is also a minimal normal p-subgroup of K. Since non-trivial normal subgroups of a finite p-group intersect the centre of the p-group non-trivially, we can deduce that $1 < \mathrm{O}_p(H) \cap \mathrm{Z}(\mathrm{O}_p(K)) \trianglelefteq K$, so $\mathrm{O}_p(H) \leq \mathrm{Z}(\mathrm{O}_p(K))$. Hence, we obtain

$$\mathrm{O}_p(H) \leq \mathrm{O}_p(K) \leq \mathrm{C}_K(\mathrm{Z}(\mathrm{O}_p(K))) \leq \mathrm{C}_K(\mathrm{O}_p(H)) = \mathrm{O}_p(H),$$

by Proposition 2.1.21 (i) and (iii). So, the assertion holds for these cases.
For the remaining cases we will obtain our assertion by elementary calculations. We will use the fact that the normal subgroups $\mathrm{O}_p(K)$ and H of K centralize each other modulo their intersection $\mathrm{O}_p(K) \cap H = \mathrm{O}_p(H)$ (use Lemma 1.4.9). We denote the reduction modulo $\mathrm{O}_p(H)$ by $^-$. Considering Proposition 2.1.13 and Lemmas 2.1.17 and 2.1.19, we note that $\mathrm{O}_p(H) = \mathrm{P}U^{\mathrm{SL}}_{k,n-k}$ in case **L** and $\mathrm{O}_p(H) = \mathrm{P}U^{\mathrm{SU}}_k$ where $n > 2k$ in case **U**. Furthermore, let λ denote a primitive element of $\mathrm{GF}(q^u)^*$.
First, we consider the case **L**. If $\mathrm{O}_p(K) \leq \mathrm{PGL}_n(q)$ then $\mathrm{O}_p(K) \leq \mathrm{PSL}_n(q)$, since $p \nmid |\mathrm{PGL}_n(q)/\mathrm{PSL}_n(q)|$. So, for $\mathrm{O}_p(K) \leq \mathrm{PGL}_n(q)$ our assertion obviously holds. Hence, it remains to consider that $\mathrm{O}_p(K) \not\leq \mathrm{PGL}_n(q)$. For this, let $h \in H$ and $t \in \mathrm{O}_p(K)$. In view of Proposition 2.1.13, we write

$$\overline{h} = \overline{\mathrm{diag}[A,B,C]} \in \overline{H} \cong \mathrm{P}L^{\mathrm{SL}}_{k,n-k}$$

where $A, C \in \mathrm{GL}_k(q)$ and $B \in \mathrm{GL}_{n-2k}(q)$ with $\det(AC) \cdot \det(B) = 1$. (Here, we note that $H = \mathrm{P}U^{\mathrm{SL}}_{k,n-k} \rtimes \mathrm{P}L^{\mathrm{SL}}_{k,n-k}$, because $\mathrm{P}U^{\mathrm{SL}}_{k,n-k} \cap \mathrm{P}L^{\mathrm{SL}}_{k,n-k} = \mathrm{P}(U^{\mathrm{SL}}_{k,n-k} \cap L^{\mathrm{SL}}_{k,n-k})$, see Lemma 1.4.6 and Remark 1.4.7 (b)). Now, suppose that $t \in (\mathrm{O}_p(K) \cap \mathrm{P\Gamma L}_n(q)) \setminus \mathrm{PGL}_n(q)$. Because t normalizes $\mathrm{O}_p(H)$, we have that $t = \varphi D$ with $1 \neq \varphi \in \langle \varphi_p \rangle$ and $D \in \mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}U^{\mathrm{SL}}_{k,n-k}) = \mathrm{P}H^{\mathrm{GL}}_{k,n-k}$, by analogous considerations as in Proposition 2.1.21. So, we can write

$$\overline{D} = \overline{\mathrm{diag}[X,Y,Z]} \in \overline{\mathrm{P}H^{\mathrm{GL}}_{k,n-k}} \cong \mathrm{P}L^{\mathrm{GL}}_{k,n-k} \tag{2.1.8}$$

where $X, Z \in \mathrm{GL}_k(q)$ and $Y \in \mathrm{GL}_{n-2k}(q)$. (As above, we see that $\mathrm{P}H^{\mathrm{GL}}_{k,n-k} = \mathrm{P}U^{\mathrm{SL}}_{k,n-k} \rtimes \mathrm{P}L^{\mathrm{GL}}_{k,n-k}$). We choose the element $\overline{h} \in \overline{H}$ where $A = \lambda \mathbb{1}_k$, $C = \lambda^{-1}\mathbb{1}_k$ and $B = \mathbb{1}_{n-2k}$. Since $\overline{h}^{\overline{t}} = \overline{h}^{\overline{\varphi D}} = \overline{h}$, we obtain a contradiction by easy calculations.
Next, assume that $t \in \mathrm{O}_p(K) \setminus \mathrm{P\Gamma L}_n(q)$. Since $|\mathrm{Aut}(\mathrm{PSL}_n(q))/\mathrm{P\Gamma L}_n(q)| = 2$, we obviously obtain a contradiction if $p \neq 2$. So, let $p = 2$. By analogous considerations as in Proposition 2.1.21, we see that we can write $t = \tau\varphi J_n D$ where $\varphi \in \langle \varphi_2 \rangle$, $J_n = \mathrm{antidiag}[1, \ldots, 1] \in \mathrm{PGL}_n(q)$ and $D \in \mathrm{P}H^{\mathrm{GL}}_{k,n-k}$. Hence, let $\overline{D}$ be as in (2.1.8). So, we have that

$$\overline{h}^{\overline{t}} = \overline{h}^{\overline{\tau\varphi J_n D}} = \overline{h} \quad (*)$$

holds. Suppose that $k > 1$. Then we choose $\overline{h} \in \overline{H}$ where $A = \mathbb{1}_k$, $B = \mathbb{1}_{n-2k}$ and $C = \mathbb{1}_k + \mathrm{E}^{k,k}_{k,k-1}$. Considering $(*)$, we now obtain by easy calculations a contradiction. Hence, let $k = 1$. Suppose that $n - 2 > 2$. We choose the elements $\overline{h} \in \overline{H}$ where $(A, B, C) \in \{(1, \mathbb{1}_{n-2} + \mathrm{E}^{n-2,n-2}_{2,1}, 1), (1, \mathbb{1}_{n-2} + \mathrm{E}^{n-2,n-2}_{n-2,1}, 1)\}$. Then, considering $(*)$, we obtain a contradiction, by elementary calculations

and observations. Hence, let $n \in \{3,4\}$. For $n=4$ we choose the element $\overline{h} \in \overline{H}$ where $A=1$, $B=\operatorname{diag}(1,\lambda)$ and $C=\lambda^{-1}$. By elementary calculations and considerations (recall that $p=2$), we see that only $q=2$ is possible. Hence, we have to examine $\mathrm{P}\Omega = \mathrm{PSL}_4(2)$ more precisely. Since $t \in \mathrm{O}_p(K) \setminus \mathrm{PSL}_4(2)$, G has to be $\mathrm{Aut}(\mathrm{PSL}_4(2))$. So, we have $K = H \rtimes \langle \tau J_4\rangle$. By elementary considerations and calculations (see also Remark 1.2.19 (c)), we now obtain that $\mathrm{O}_2(K) = \mathrm{O}_2(H) \rtimes \langle \tau J_4\rangle$. Hence, our assertion holds in case **L** if $n=4$.
So, we investigate the remaining case $n=3$.[7] Here, we choose $\overline{h} \in \overline{H}$ where $A=\lambda$, $B=\lambda^{-1}$ and $C=1$. Considering $(*)$, we easily obtain that $q=2$ or $q=4$ and $\varphi=1$. First, we examine the case $\mathrm{P}\Omega = \mathrm{PSL}_3(2)$, so $G=\mathrm{Aut}(\mathrm{PSL}_3(2))$. Here, our assertion obviously holds, since $K = H \rtimes \langle \tau J_3\rangle = \mathrm{O}_2(K)$. Now, let $\mathrm{P}\Omega = \mathrm{PSL}_3(4)$. Here, $\mathrm{Out}(\mathrm{PSL}_3(4))$ is isomorphic to the dihedral group of order 12. Example 1.2.23 list the possibilities for G by cases (1.) to (10.). Because $G \nleq \mathrm{P\Gamma L}_3(4)$, only the cases (3.), (4.), (6.), (7.), (9.) or (10.) are possible. So, let G be one of those (note, that we still can assume $H = \mathrm{P}H^{\mathrm{SL}}_{1,2}$). We have that $K = \mathrm{N}_G(H) = \mathrm{N}_G(\mathrm{O}_2(H))$ (use Proposition 2.1.15). Since t has to be an element in G, the cases (4.) and (6.) cannot occur. Because $D \in \mathrm{O}_2(H) \rtimes \mathrm{P}L^{\mathrm{GL}}_{1,2}$, we can assume that $t = \tau J_3 D$ where $D \in \mathrm{P}L^{\mathrm{GL}}_{1,2}$. We have $K = H \rtimes \langle W, \tau J_3\rangle$ in case (9.) and $K = H \rtimes \langle W, \varphi_2, \tau J_3\rangle$ in case (10.). So, we see that $t^{-1}t^{W} = W^{J_3}W \in \mathrm{O}_2(K)$. But $\mathrm{o}(W^{J_3}W) = 3$, hence we obtain a contradiction for the two cases. Finally, we consider the cases (3.) and (7.). Here, we have that $K = H \rtimes \langle \tau J_3\rangle$ in case (3.) and $K = H \rtimes (\langle \varphi_2\rangle \times \langle \tau J_3\rangle)$ in case (7.). By elementary observations, we now see that $\mathrm{O}_2(K) = \mathrm{O}_2(H) \rtimes \langle \tau J_3\rangle$. So, the assertion is established for case **L**.
Finally, we consider the remaining case for the case **U**. As above, we see that the assertion holds if $\mathrm{O}_p(K) \leq \mathrm{PGU}_n(q^2)$. So, suppose that there is an element $t \in \mathrm{O}_p(K) \setminus \mathrm{PGU}_n(q^2)$. By analogous considerations as in the proof of Proposition 2.1.21, we see that we can write $t = \varphi D$ where $1 \neq \varphi \in \langle \varphi_p\rangle$ and $D \in \mathrm{N}_{\mathrm{PGU}_n(q^2)}(\mathrm{P}U^{\mathrm{SU}}_k) = \mathrm{P}H^{\mathrm{GU}}_k$. Let $h \in H$. Regarding Proposition 2.1.13 (iv), we can write

$$\overline{h} = \overline{\operatorname{diag}[A, B, A^{-1t\varphi_q}]} \in \overline{H} \cong \mathrm{P}L^{\mathrm{SU}}_k \text{ and}$$

$$\overline{D} = \overline{\operatorname{diag}[X, Y, X^{-1t\varphi_q}]} \in \overline{\mathrm{P}H^{\mathrm{GU}}_k} \cong \mathrm{P}L^{\mathrm{GU}}_k$$

where $A, X \in \mathrm{GL}_k(q^2)$, $B, Y \in \mathrm{GL}_{n-2k}(q^2)$ with $BB^{t\varphi_q} = \mathbb{1}_{n-2k} = YY^{t\varphi_q}$ and $\det(B) = \det(A)^{q-1}$. (As above, we note that $H = \mathrm{P}U^{\mathrm{SU}}_k \rtimes \mathrm{P}L^{\mathrm{SU}}_k$ and $\mathrm{P}H^{\mathrm{GU}}_k = \mathrm{P}U^{\mathrm{SU}}_k \rtimes \mathrm{P}L^{\mathrm{GU}}_k$). So, we have

$$\overline{h}^{\overline{t}} = \overline{h}^{\overline{\varphi D}} = \overline{h} \quad (**).$$

We choose the element $\overline{h} \in \overline{H}$ where $B = \operatorname{diag}(\lambda^{q-1}, \mathbb{1}_{n-2k-1})$ and $A = \operatorname{diag}(\lambda, \mathbb{1}_{k-1})$. Considering $(**)$, we obtain a contradiction if $k>2$ or $n-2k>2$, by easy observations and calculations. Hence, let $k \leq 2$ and $n-2k \leq 2$. Suppose

[7] We note that this case was already considered in [Uf, Satz 2.4.7 and Beispiele 2.4.8], but here we determine precisely those G with $K \in \mathscr{C}_1$ of G of type $P_{1,2}$ and $\mathrm{O}_2(K) > \mathrm{O}_2(K \cap \operatorname{soc}(G))$.

that $k = 2$. Then we obtain that $(**)$ only holds for $\overline{h}$ and $\overline{t}$ if $n = 6$, $q = 2$ and $X = \text{antidiag}(x_{21}, x_{12})$ for some $x_{21}, x_{12} \in \text{GF}(q^2)^*$. But since $\overline{t}$ also centralizes $\overline{\text{diag}[\mathbb{1}_2 + \text{E}^{2,2}_{2,1}, \mathbb{1}_2, \mathbb{1}_2 + \text{E}^{2,2}_{1,2}]} \in \overline{H}$, this case can also be ruled out. Hence, let $k = 1$ and $n \leq 4$. Observing $(**)$ for $\overline{h}$ and $\overline{t}$, we obtain by elementary calculations that this condition only holds if $n = 3$ and $q = 2$. But, $\text{P}\Omega \neq \text{PSU}_3(2^2)$, because $\text{PSU}_3(2^2)$ is not simple. So, our assertion is established. □

Remark. Concerning the case which arose at the end of the last proof, we note the following. Let $\text{PSU}_3(2^2) \leq G \leq \text{Aut}(\text{PSU}_3(2^2))$ and $K \in \mathscr{C}_1$ of G of type P_1. Then, by elementary considerations, we see that $\text{O}_2(K \cap \text{PSU}_3(2^2)) < \text{O}_2(K)$ if and only if G is conjugate to $\text{PSU}_3(2^2) \rtimes \langle \varphi_2 \rangle$ in $\text{Aut}(\text{PSU}_3(2^2))$. If $\text{O}_2(K) > \text{O}_2(K \cap \text{PSU}_3(2^2))$ we have that $|K| = |\text{O}_2(K)| = 2^4 > 2^3 = |\text{O}_2(K \cap \text{PSU}_3(2^2))| = |K \cap \text{PSU}_3(2^2)|$. Note, that in view of this result (or, see [BHR2]) we do not have to specify the matrix of the non-degenerate unitary form in this remark for a unique description of the result.

2.2 A-class $\mathscr{C}_2$

In this section, we analyze the members of A-class $\mathscr{C}_2$. The members of A-class $\mathscr{C}_2$ are roughly described the stabilizers of certain subspace decompositions of a vector space V. First, we provide the necessary terminology for the introduction of this A-class. For this, we recall the terminology and notation introduced in Subsection 1.2.1, esp. the terminology of a totally singular or non-degenerate subspace (introduced after Lemma 1.2.8).

Definition 2.2.1. (cf. [KL, p. 99-100])

(i) Let V be a vector space of dimension n and $\mathcal{D} = \{V_1, \ldots, V_t\}$ be a set of non-trivial subspaces of V where $t \geq 2$. We call $\mathcal{D}$ a *subspace decomposition* of V if $V = V_1 \oplus \ldots \oplus V_t$.

(ii) If all subspaces in a subspace decomposition $\mathcal{D}$ of V have the same dimension m, we call $\mathcal{D}$ a *m-decomposition* of V. (Note, that $m = \frac{n}{t}$).

(iii) Let (V, f) be a vector space equipped with a non-degenerate unitary form f, and let $\mathcal{D} = \{V_1, \ldots, V_t\}$ be a subspace decomposition of V. We call $\mathcal{D}$ *totally singular* if the subspaces V_i are totally singular, and we call $\mathcal{D}$ *non-degenerate* if the subspaces V_i are non-degenerate and V_i is orthogonal to V_j for $i \neq j$.

(iv) For $G \leq \Gamma\text{L}(V)$ and a subspace decomposition $\mathcal{D}$ of V we call $\text{C}_G(\mathcal{D})$ the *centralizer* of $\mathcal{D}$ in G, which is the subgroup of G consisting of all elements in G fixing each V_i.
By $\text{N}_G(\mathcal{D})$, we denote the *stabilizer* of $\mathcal{D}$ in G, which is the subgroup of G consisting of all elements in G permuting the subspaces V_i among themselves.

Remark. We warn not to confuse the notation introduced in Definition 2.2.1 (iv) with the notation introduced in Definition 2.1.1.

Now, we are able to introduce the members of A-class $\mathscr{C}_2$. As described at the beginning of this chapter, for the following definition of A-class $\mathscr{C}_2$ we use the generalized standard notation where we recall Subsection 1.2.3 for its introduction.

Definition 2.2.2. {***A-class*** $\mathscr{C}_2$} (cf. [KL, p. 60, 100], [BHR, p. 62] and [As, p. 472-473, 507])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and let $\mathcal{D} = \{V_1, \ldots, V_t\}$ be a m-decomposition of V, so $m = \frac{n}{t}$. The subgroup $K \leq G$ belongs to *(A-class)* $\mathscr{C}_2$ of G if it appears in the following table.

Case	Type of K	Description of K	Conditions
L	$\mathrm{GL}_m(q) \wr \mathrm{S}_t$	$K = \mathrm{N}_G(\mathcal{D})$	$G \leq \Gamma(V)$
		$K = \mathrm{N}_{\mathrm{A}(V)}(\mathrm{N}_{\Gamma(V)}(\mathcal{D})) \cap G$	$G \not\leq \Gamma(V)$
U	$\mathrm{GU}_m(q^2) \wr \mathrm{S}_t$	$K = \mathrm{N}_G(\mathcal{D})$, $\mathcal{D}$ non-degenerate	
U	$\mathrm{GL}_{\frac{n}{2}}(q^2).2$	$K = \mathrm{N}_G(\mathcal{D})$, $m = \frac{n}{2}$, $\mathcal{D}$ totally singular	

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_2$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_2$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}^{\epsilon}_m(q^u) \wr \mathrm{S}_t$, or $\mathrm{GL}_{\frac{n}{2}}(q^2).2$ we call K of type $\mathrm{GL}^{\epsilon}_m(q^u) \wr \mathrm{S}_t$, or $\mathrm{GL}_{\frac{n}{2}}(q^2).2$, respectively.

Remark 2.2.3. (a) Our definition of A-class $\mathscr{C}_2$ coincides with the definitions in [As], [KL] and [BHR].

(b) We note that all members of A-class $\mathscr{C}_2$ of G for $G \leq \Gamma(V)$ are irreducible, except in the case **L** for type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$ for $q = 2$ (see [BHR, p. 62 and the proof of Proposition 2.3.6 (i)][8]).

By Kleidman and Liebeck [KL, Theorem 3.1.2. (ii)][9], we obtain the following information about the normalizer of a member of A-class $\mathscr{C}_2$ of $\mathrm{P}\Omega(V)$ in $\mathrm{P}\Omega(V)$ in case **L**.

Lemma 2.2.4. *Let K be a member of A-class $\mathscr{C}_2$ of $\mathrm{P}\Omega(V) = \mathrm{PSL}(V)$. Then $\mathrm{N}_{\mathrm{PSL}(V)}(K) = K$, except if K is of type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$ and one of the following holds*

(a) $q = 2$ and n is even (note, that $(q, n) \neq (2, 2)$), or

(b) $(q, n) \in \{(5, 2), (3, 4), (4, 3)\}$.

[8] We have to mention a mistake in [BHR, p. 62]. There, the exception $q = 2$ in case **L** for type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$ has not been mentioned.

[9] We note that [KL, Theorem 3.1.2. (ii)] only holds for simple $\overline{\Omega}$, which was not demanded (explicitly) when stating the theorem (cf. also the related propositions from [KL, Chapter 4] which are used to proof this theorem and which do have this restriction by [KL, p. 80]).

Remark 2.2.5. (cf. [Blo, p. 176] and [AD, p. 9-11])
We mention some facts about the exceptional case $(q, n) = (4, 3)$ in the previous lemma. Here, we have that

$$K = \left(\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \right\rangle \times \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \right) \rtimes \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

is a member of $\mathscr{C}_2$ of $\mathrm{PSL}_3(4)$ where ω denotes a primitive element of $\mathrm{GF}(4)^*$ (cf. Proposition 2.2.8, below). Easy calculations show that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, B = \begin{bmatrix} 1 & \omega & \omega \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{bmatrix} \in \mathrm{PSL}_3(4)$$

normalize K. So, we have that $\mathrm{N}_{\mathrm{PSL}_3(4)}(K) > K$ and K is not a maximal subgroup of $\mathrm{PSL}_3(4)$. Actually, we have

$$\mathrm{N}_{\mathrm{PSL}_3(4)}(K) = \left(\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \right\rangle \times \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \right) \rtimes \langle A, B \rangle \cong 3^2 \rtimes Q_8$$

where Q_8 denotes the quaternion group, defined in Lemma 2.6.2 (a), below (see also [Blo, p. 176] or cf. [KL, p. 61]).
In Example 2.8.6, we will see that $\mathrm{N}_{\mathrm{PSL}_3(4)}(K)$ is a member of $\mathscr{C}_8$ of $\mathrm{PSL}_3(4)$. Furthermore, we note that this exceptional case is in strong relation to the so called *Hessian group* where we refer to Example 2.8.6 and Remark 2.5.17 for more information.

Next, we provide an important fact about the intersection of a group $K \in \mathscr{C}_2$ of G (where $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$) with the socle of G, observed by Kleidman and Liebeck.

Lemma 2.2.6. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and* $K \in \mathscr{C}_2$ *of* G*. Then we have*

$$K \cap \mathrm{P\Omega}(V) \in \mathscr{C}_2 \textit{ of } \mathrm{P\Omega}(V),$$

except in case **L** *if* $G \not\leq \mathrm{P\Gamma L}(V)$ *and* K *is of type* $\mathrm{GL}_1(q) \wr \mathrm{S}_n$ *where we have* $q = 2$ *and* $n > 2$ *is even, or* $(q, n) \in \{(3, 4), (4, 3)\}$.

Proof. See [KL, Theorem 3.1.2. (ii) and Proposition 3.1.3.]. (Here, we recall that always $G \leq \mathrm{P\Gamma L}(V)$ in case **L** if $n = 2$). □

Remark. We note that the assertion of the previous lemma for $G \leq \mathrm{P\Gamma}(V)$ follows immediately from Definition 2.2.2, whereas Lemma 2.2.4 is needed to prove it for the case $G \not\leq \mathrm{P\Gamma}(V)$ (see [KL, proof of Proposition 3.1.3.]). We also note that the assertion still holds using the generalized standard notation.

The following table describes the structure of the members of A-class $\mathscr{C}_2$ of $\Omega(V)$. It is taken from [BHR, Table 2.5] which is deduced from the observations of Kleidman and Liebeck [KL, Propositions 4.2.4. and 4.2.9.]. We note that the results provided in the following table and Lemma 2.2.7 (below) also hold using the generalized standard notation.

Table 2.2.1 Structure of a member K of A-class $\mathscr{C}_2$ of $\Omega(V)$

Case	Type of K	Structure of K
L	$\mathrm{GL}_m(q) \wr \mathrm{S}_t$	$\mathrm{SL}_m(q)^t.(q-1)^{t-1}.\mathrm{S}_t$
U	$\mathrm{GU}_m(q^2) \wr \mathrm{S}_t$	$\mathrm{SU}_m(q^2)^t.(q+1)^{t-1}.\mathrm{S}_t$
U	$\mathrm{GL}_{\frac{n}{2}}(q^2).2$	$\mathrm{SL}_{\frac{n}{2}}(q^2).(q-1).2$

Next, we provide a concrete representation with respect to an ordered basis B of V for representatives of the $\mathrm{I}(V)$-conjugacy class of the members of A-class $\mathscr{C}_2$ of $\Omega(V)$ (and of $\mathrm{I}(V)$). For this, we choose B suitable, depending on the subspace decomposition $\mathcal{D}$ of V. We provide the following lemma.

Lemma 2.2.7. $\mathrm{I}(V)$ *acts transitively (by conjugation) on the members of each type in A-class* $\mathscr{C}_2$ *of* $\Omega(V)$ *(and of* $\mathrm{I}(V)$*).*

Proof. See [As, Theorems BΔ and BO]. (Or, more precisely the assertion follows by Witt's Lemma (see Lemma 1.2.9) and [As, (5.4)], cf. [As, p. 505]). □

For the following we recall Lemma 1.2.8 (a), and the notations $u \in \{1,2\}$, $\epsilon \in \{+,-\}$ and φ_q, introduced in Subsection 1.2.3. Furthermore, we recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1,\dots,A_k)$ and the anti-diagonal matrix $\mathrm{antidiag}(A_1,\dots,A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$ introduced on page 8.

Proposition 2.2.8. *Let V be a vector space of dimension n over a finite field of order q^u, and let $\mathcal{D} = \{V_1,\dots,V_t\}$ be an m-decomposition of V (hence, $m = \frac{n}{t}$ where $t \geq 2$). Let $n \geq 2$ in case* **L** *and $n \geq 3$ in case* **U**, *and let f denote the non-degenerate unitary form on V in case* **U**. *With respect to a suitable ordered basis for V (depending on $\mathcal{D}$), we have the following.*

(i) In case **L** *and case* **U** *where the matrix of f is $\mathbb{1}_n$, we have that $\mathrm{N}_{\mathrm{GL}_n^\epsilon(q^u)}(\mathcal{D}) =: K_{\mathrm{GL}^\epsilon,m} = C_{\mathrm{GL}^\epsilon,m} \rtimes S_{\mathrm{GL}^\epsilon,m}$ is a member of $\mathscr{C}_2$ of $\mathrm{GL}_n^\epsilon(q^u)$ of type $\mathrm{GL}_m^\epsilon(q^u) \wr \mathrm{S}_t$ where*
$C_{\mathrm{GL}^\epsilon,m} := \mathrm{C}_{\mathrm{GL}_n^\epsilon(q^u)}(\mathcal{D}) = \{\mathrm{diag}(A_1,\dots,A_t) \mid A_i \in \mathrm{GL}_m^\epsilon(q^u)\} \cong \mathrm{GL}_m^\epsilon(q^u)^t$ and

$$S_{\mathrm{GL}^\epsilon,m} := \left\langle \begin{pmatrix} 0 & \mathbb{1}_m & 0 \\ \mathbb{1}_m & 0 & 0 \\ 0 & 0 & \mathbb{1}_{n-2m} \end{pmatrix}, \dots, \begin{pmatrix} \mathbb{1}_{n-2m} & 0 & 0 \\ 0 & 0 & \mathbb{1}_m \\ 0 & \mathbb{1}_m & 0 \end{pmatrix} \right\rangle \cong \mathrm{S}_t.$$

(ii) In case **L** *and case* **U** *where the matrix of f is $\mathbb{1}_n$, we have that $\mathrm{N}_{\mathrm{SL}_n^\epsilon(q^u)}(\mathcal{D}) =: K_{\mathrm{SL}^\epsilon,m} = \langle C_{\mathrm{SL}^\epsilon,m}, S_{\mathrm{SL}^\epsilon,m}\rangle$ is a member of $\mathscr{C}_2$ of $\mathrm{SL}_n^\epsilon(q^u)$ of type $\mathrm{GL}_m^\epsilon(q^u) \wr \mathrm{S}_t$ where*

$$C_{\mathrm{SL}^\epsilon,m} := \mathrm{C}_{\mathrm{SL}_n^\epsilon(q^u)}(\mathcal{D}) = \left\{ \mathrm{diag}(A_1,\dots,A_t) \middle| \begin{array}{l} A_i \in \mathrm{GL}_m^\epsilon(q^u), \\ \prod_{i=1}^t \det(A_i) = 1 \end{array} \right\} \textit{ and}$$

$$S_{\mathrm{SL}^\epsilon,m} := \left\langle \begin{pmatrix} 0 & \mathbb{1}_m & 0 \\ \mathbb{1}_m & 0 & 0 \\ 0 & 0 & \mathbb{1}_{n-2m} \end{pmatrix}, \dots, \begin{pmatrix} \mathbb{1}_{n-2m} & 0 & 0 \\ 0 & 0 & \mathbb{1}_m \\ 0 & \mathbb{1}_m & 0 \end{pmatrix} \right\rangle \cong \mathrm{S}_t$$

for q or m even (here, we have $K_{\mathrm{SL}^\epsilon,m} = C_{\mathrm{SL}^\epsilon,m} \rtimes S_{\mathrm{SL}^\epsilon,m}$), or

$$S_{\mathrm{SL}^\epsilon,m} := \left\langle \begin{pmatrix} 0 & -\mathbb{1}_m & 0 \\ \mathbb{1}_m & 0 & 0 \\ 0 & 0 & \mathbb{1}_{n-2m} \end{pmatrix}, \dots, \begin{pmatrix} \mathbb{1}_{n-2m} & 0 & 0 \\ 0 & 0 & -\mathbb{1}_m \\ 0 & \mathbb{1}_m & 0 \end{pmatrix} \right\rangle$$

for q, m odd (here, we have $K_{\mathrm{SL}^\epsilon,m}/C_{\mathrm{SL}^\epsilon,m} \cong \mathrm{S}_t$).

(iii) In case **U** *for $2 \mid n$ with the matrix* $\mathrm{antidiag}(\mathbb{1}_{\frac{n}{2}}, \mathbb{1}_{\frac{n}{2}})$ *of f we have*

(a) $\mathrm{N}_{\mathrm{GU}_n(q^2)}(\mathcal{D}) =: K_{\mathrm{GU}} = C_{\mathrm{GU}} \rtimes S_{\mathrm{GU}}$ *is a member of $\mathscr{C}_2$ of* $\mathrm{GU}_n(q^2)$ *of type* $\mathrm{GL}_{\frac{n}{2}}(q^2).2$ *where*
$C_{\mathrm{GU}} := \mathrm{C}_{\mathrm{GU}_n(q^2)}(\mathcal{D}) = \left\{\mathrm{diag}(A, A^{-1t\varphi_q}) \mid A \in \mathrm{GL}_{\frac{n}{2}}(q^2)\right\}$ *with*
$C_{\mathrm{GU}} \cong \mathrm{GL}_{\frac{n}{2}}(q^2)$ *and* $S_{\mathrm{GU}} := \left\langle \mathrm{antidiag}(\mathbb{1}_{\frac{n}{2}}, \mathbb{1}_{\frac{n}{2}})\right\rangle \cong \mathrm{S}_2$, *and*

(b) $\mathrm{N}_{\mathrm{SU}_n(q^2)}(\mathcal{D}) =: K_{\mathrm{SU}} = \langle C_{\mathrm{SU}}, S_{\mathrm{SU}}\rangle$ *is a member of $\mathscr{C}_2$ of* $\mathrm{SU}_n(q^2)$ *of type* $\mathrm{GL}_{\frac{n}{2}}(q^2).2$ *where*

$$C_{\mathrm{SU}} := \mathrm{C}_{\mathrm{SU}_n(q^2)}(\mathcal{D}) = \left\{ \mathrm{diag}(A, A^{-1t\varphi_q}) \middle| \begin{array}{l} A \in \mathrm{GL}_{\frac{n}{2}}(q^2), \\ \mathrm{o}(\det(A)) | q-1 \end{array} \right\} \textit{ and}$$

$S_{\mathrm{SU}} := \left\langle \mathrm{antidiag}(\mathbb{1}_{\frac{n}{2}}, \mathbb{1}_{\frac{n}{2}})\right\rangle \cong \mathrm{S}_2$ *for q or $\frac{n}{2}$ even (here, we have $K_{\mathrm{SU}} = C_{\mathrm{SU}} \rtimes S_{\mathrm{SU}}$), or*
$S_{\mathrm{SU}} := \left\langle \mathrm{antidiag}(b \cdot \mathbb{1}_{\frac{n}{2}}, -b^{-1} \cdot \mathbb{1}_{\frac{n}{2}})\right\rangle$ *for $q, \frac{n}{2}$ odd where $b \in \mathrm{GF}(q^2)^*$ with $\mathrm{o}(b) = 2(q-1)$ (here, we have $K_{\mathrm{SU}}/C_{\mathrm{SU}} \cong \mathrm{S}_2$).*

(iv) $\mathrm{SL}_n^\epsilon(q^u)$ *acts transitively (by conjugation) on the members of each type in A-class $\mathscr{C}_2$ of* $\mathrm{SL}_n^\epsilon(q^u)$ *(and of* $\mathrm{GL}_n^\epsilon(q^u)$*).*

Proof. We obtain assertions (i) - (iii) immediately by Table 2.2.1 together with some calculations (or, compare [KL, Proposition 4.2.1., Corollary 4.2.2.] for (i) and (ii), and [KL, Corollary 4.2.2., Lemma 4.2.3., Proposition 4.2.4.] or [As, (5.4)] for (iii)). To prove assertion (iv), we recall the definition of the diagonal matrix W_{SL^ϵ} (in Subsection 1.2.2) which generates $\mathrm{GL}_n^\epsilon(q^u)$ by $\mathrm{SL}_n^\epsilon(q^u)$. Since W_{SL^ϵ} normalizes the groups $K_{\mathrm{GL},m}$ and $K_{\mathrm{SL},m}$ in case **L** and $K_{\mathrm{GU},m}$, $K_{\mathrm{SU},m}$, K_{GU} and K_{SU} in case **U** (see also the following remark), we easily obtain assertion (iv), by Lemma 2.2.7. □

Remark. Concerning the proof of Proposition 2.2.8 (iv), we recall that the definition of the diagonal matrix W_{SU} depends on the matrix of the non-degenerate unitary form on V. We also refer to [KL, Propositions 4.2.4. (I) and 4.2.9. (I)], for a more technical proof of the statement in Proposition 2.2.8 (iv).

Next, we provide the information in which cases a member of A-class $\mathscr{C}_2$ of G (where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$) is a maximal subgroup of G, taken from [BHR] and [KL].

Proposition 2.2.9. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* M *be a member of* A*-class* $\mathscr{C}_2$ *of* G*. Then* M *is a maximal subgroup of* G *if and only if one of the following holds.*

(i) For $n = 2$ *we have case* **L**, M *is of type* $\mathrm{GL}_1(q) \wr \mathrm{S}_2$ *and*

  (a) q *is even (note, that* $q \neq 2$*), or*

  (b) q *is odd (note, that* $q \neq 3$*) and* $q \geq 13$*, or* $q = 7, 11$ *and* $G = \mathrm{PGL}(V)$*, or* $q = 9$ *and* $G = \mathrm{PGL}(V)$*,* $\mathrm{PSL}(V)\langle W\varphi_3\rangle$ *or* $\mathrm{Aut}(\mathrm{PSL}(V))$*.*

(ii) For $n = 3$ *we have*

  (a) case **L**, M *is of type* $\mathrm{GL}_1(q) \wr \mathrm{S}_3$ *and* $q \neq 4$*; for* $q = 2$ *we have* $G = \mathrm{Aut}(\mathrm{PSL}(V))$*, or*

  (b) case **U**, M *is of type* $\mathrm{GU}_1(q^2) \wr \mathrm{S}_3$ *and* $q \neq 5$ *(note, that* $q > 2$*), or* $q = 5$ *and* $G = \mathrm{PGU}(V)$ *or* $G = \mathrm{P\Gamma U}(V)$*.*

(iii) For $n = 4$ *we have*

  (a) case **L**, M *is of type* $\mathrm{GL}_1(q) \wr \mathrm{S}_4$ *and* $q \geq 7$*, or* $q = 5$ *and we have* $G \nleq \mathrm{PSL}(V)\langle W^2, \tau\rangle$*,*

  (b) case **L**, M *is of type* $\mathrm{GL}_2(q) \wr \mathrm{S}_2$ *and* $q \geq 3$*; for* $q = 3$ *we have* $G \nleq \mathrm{PSL}(V)\langle \tau\rangle$*,*

  (c) case **U**, M *is of type* $\mathrm{GU}_1(q^2) \wr \mathrm{S}_4$ *and* $q \neq 3$*, or* $q = 3$ *and we have* $G_B \nleq \mathrm{PSU}_4(3^2)\langle W^2, \varphi_3\rangle$ *where* G_B *denotes* G *considered with respect to an orthonormal basis* B *of* V*,*

  (d) case **U**, M *is of type* $\mathrm{GU}_2(q^2) \wr \mathrm{S}_2$ *and* $q \geq 3$*, or*

  (e) case **U**, M *is of type* $\mathrm{GL}_2(q^2).2$ *and* $q \geq 3$*; for* $q = 3$ *we have* $G_B \nleq \mathrm{PSU}_4(3^2)\langle W^2, \varphi_3\rangle$ *where* G_B *denotes* G *considered with respect to an orthonormal basis* B *of* V*.*

(iv) For $n = 6$ *we have*

  (a) case **L**, M *is of type* $\mathrm{GL}_m(q) \wr \mathrm{S}_{\frac{6}{m}}$ *and*

    (α) for $m = 3$ *we have* $q \geq 2$*,*
    (β) for $m = 2$ *we have* $q \geq 3$ *and*
    (γ) for $m = 1$ *we have* $q \geq 5$*,*

  (b) case **U**, M *is of type* $\mathrm{GU}_m(q^2) \wr \mathrm{S}_{\frac{6}{m}}$ *and*

    (α) for $m = 3$ *we have* $q \geq 2$*,*
    (β) for $m = 2$ *we have* $q \geq 3$ *and*
    (γ) for $m = 1$ *we have* $q \geq 3$*, or* $q = 2$ *and* $G = \mathrm{PGU}(V)$ *or* $G = \mathrm{P\Gamma U}(V)$*, or*

  (c) case **U**, M *is of type* $\mathrm{GL}_3(q^2).2$ *and* $q \geq 2$*.*

(v) *For $n = 5$ or $n \geq 7$ we have*

(a) *case* **L**, *M is of type $\mathrm{GL}_m(q) \wr \mathrm{S}_{\frac{n}{m}}$ (note, that $m|n$) and*

(α) *for $m \geq 3$ we have $q \geq 2$,*
(β) *for $m = 2$ we have $q \geq 3$ and*
(γ) *for $m = 1$ we have $q \geq 5$,*

(b) *case* **U**, *M is of type $\mathrm{GU}_m(q^2) \wr \mathrm{S}_{\frac{n}{m}}$ (note, that $m|n$) and*

(α) *for $m = 1$ or $m \geq 3$ we have $q \geq 2$ and*
(β) *for $m = 2$ we have $q \geq 3$, or*

(c) *case* **U**, *M is of type $\mathrm{GL}_{\frac{n}{2}}(q^2).2$ (note, that n is even) and $q \geq 2$.*

Proof. We obtain (i) from [BHR, Lemma 3.1.3 and Proposition 6.3.11] (or, use [BHR, Table 8.1]). For (ii) see [BHR, Propositions 2.3.6, 3.2.1, 3.2.2, 3.2.6 and 6.3.12] (or, use [BHR, Tables 8.3 and 8.5]). Assertion (iii) follows after [BHR, Propositions 3.3.2, 3.3.3 and 6.3.13] (or, use [BHR, Tables 8.8 and 8.10]). We obtain (iv) from [BHR, Propositions 3.5.2 - 3.5.4 and 6.3.16] (or, use [BHR, Tables 8.24 and 8.26]).
To prove (v), we first handle the only-if-part. This follows directly from [BHR, Proposition 2.3.6 (i)-(v)], since there are no arguments in the proof which require a restriction of the dimension by $n \leq 12$. The if-part follows by observation of the tables for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q^2)$ for dimensions $n = 5$ and $7 \leq n \leq 12$ in [BHR, Chapter 8] together with [KL, Tables 3.5.A, 3.5.B and 3.5.H]. □

Remark 2.2.10. (a) In Proposition 2.2.9 (ii)(a) we have also listed the cases for $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(3)$. Note, that in [BHR, Table 8.3] the subgroups of $\mathrm{SL}_3(q)$ for $q = 2$ and $q = 3$ which belong to A-class $\mathscr{C}_2$ lie and are listed in A-class $\mathscr{C}_1$ and A-class $\mathscr{C}_8$ of $\mathrm{SL}_3(q)$, respectively (see [BHR, Note on p. 378]).

(b) For the only-if-part of (v) in the last proof we could also cite [BHR, Proposition 2.3.6 (i)-(v)] to obtain a proof for dimensions $n \leq 12$ and use [KL, Tables 3.5.A, 3.5.B and 3.5.H] for dimensions $n \geq 13$. Note, that the occurrence of a novelty in [KL, first line of Table 3.5.H] does not lead to further maximal subgroups, cf. Remark 1.1.5.

(c) Recall Remark 1.2.18. In case **U**, we have that the results of the book [BHR] are obtained and presented with respect to an orthonormal basis of the underlying vector space. Concerning well-definedness, it is not hard to see that we can transfer these results to the notation used in Proposition 2.2.9 (without specifying the basis), except in cases (iii)(c) and (e). Here, we have to consider G with respect to a fixed ordered basis, to obtain a unique description of the results, cf. [BHR2, Lemma 5 and Theorem 6].

(d) The occurrence of exceptions in Proposition 2.2.9 (ii)(b) for case $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(5^2)$ underlies a special combinatorial fact. The Hoffman-Singleton graph is a graph of 50 vertices, each joined to 7 others. Its automorphism

group is isomorphic to $\mathrm{PSU}_3(5^2).\langle\varphi_5\rangle$ (= $\mathrm{P\Sigma U}_3(5^2)$), see [At, p. xvii]. (Note, that we do not have to specify a basis in the last description, see [BHR2, p. 171-172 and Proposition 7]). The stabilizer of a vertex of the graph leads to the appearance of maximal subgroups of $\mathrm{PSU}_3(5^2)$ isomorphic to A_7, see [At, p. 34]. By [BHR, Proposition 6.3.12], we see that in the cases $\mathrm{PSU}_3(5^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_3(5^2))$ not listed in Proposition 2.2.9 (ii)(b), there is an inclusion by the members of A-class $\mathscr{C}_2$ of G of type $\mathrm{GU}_1(5^2) \wr \mathrm{S}_3$ in these subgroups isomorphic to A_7, or in their stabilizers in G (cf. [At, p. 34]).
By [BHR, Proposition 6.3.12] and [At, p. 34], we see that such inclusions also occur for other A-classes of G for some $\mathrm{PSU}_3(5^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_3(5^2))$.

From Lemma 2.2.6 and Proposition 2.2.9 we obtain immediately the following corollary.

Corollary 2.2.11. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and $M \in \mathscr{C}_2$ of G be a maximal subgroup of G. Then $M \cap \mathrm{P}\Omega(V) \in \mathscr{C}_2$ of $\mathrm{P}\Omega(V)$.*

Now, we have provided all necessary information to begin the investigation for our intended goal. We start with a necessary condition for a member K of A-class $\mathscr{C}_2$ of G to be strongly constrained. Using the condition that the layer of K is trivial if K is strongly constrained, we eliminate many cases of types in A-class $\mathscr{C}_2$ of G.

Proposition 2.2.12. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and $K \in \mathscr{C}_2$ of G. If $\mathrm{E}(K) = 1$ then we have that K is*

*(i) in case **L** of type*

(a) $\mathrm{GL}_1(q) \wr \mathrm{S}_n$, or
(b) $\mathrm{GL}_2(q) \wr \mathrm{S}_{\frac{n}{2}}$ for $q \in \{2, 3\}$ (note, that $2 \mid n$),

*(ii) in case **U** of type*

(a) $\mathrm{GU}_1(q^2) \wr \mathrm{S}_n$,
(b) $\mathrm{GU}_2(q^2) \wr \mathrm{S}_{\frac{n}{2}}$ for $q \in \{2, 3\}$ (note, that $2 \mid n$), or
(c) $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$ (note, that $3 \mid n$).

Furthermore, one of these cases holds if K is strongly constrained.

Proof. Let $K \in \mathscr{C}_2$ of G be not of a type listed above. Since $\mathrm{P}\Omega(V) \cap K$ is a normal subgroup of K, it is sufficient to show that $\mathrm{E}(\mathrm{P}\Omega(V) \cap K) \neq 1$. Following Lemma 2.2.6, we have $\mathrm{P}\Omega(V) \cap K \in \mathscr{C}_2$ of $\mathrm{P}\Omega(V)$. Hence, we can write $\mathrm{P}\Omega(V) \cap K = \mathrm{P}\widetilde{K}$ where $\widetilde{K} \in \mathscr{C}_2$ of $\Omega(V)$. Now, our assertion easily follows using Lemma 1.4.22, Propositions 1.2.11 and 1.2.12 together with Table 2.2.1 (or, more precisely together with Proposition 2.2.8). □

For the following corollary, we recall that $u = 1$ in case **L** and $u = 2$ in case **U**.

Corollary 2.2.13. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* $M \in \mathscr{C}_2$ *of* G *be a strongly constrained maximal subgroup of* G*. Then* M *is of type* $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$*,* $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_{\frac{n}{2}}$*, or* $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$*.*

Proof. The assertion is a direct consequence from Propositions 2.2.9 and 2.2.12. □

We proceed by considering the different types occurring in Corollary 2.2.13 separately. For this, it is advantageous from now on to consider a representation of $\Omega(V)$ with respect to a fixed ordered basis B of V where in case **U** we choose B to be an orthonormal basis.

2.2.1 $\mathscr{C}_2$ of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$

In this subsection, we investigate in which cases an almost simple group G with socle isomorphic to $\mathrm{P}\Omega$ has a maximal subgroup M belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ which is strongly constrained. We begin by providing a necessary number theoretic condition for the maximal subgroup M to be strongly constrained.

Lemma 2.2.14. *Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and let* M *be a maximal subgroup of* G *which belongs to type* $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ *of A-class* $\mathscr{C}_2$ *of* G*. If* M *is strongly constrained then* $\frac{(q-\epsilon 1)^{n-1}}{(q-\epsilon 1, n)}$ *is a prime power.*

Proof. Let $K_{\mathrm{SL}^\epsilon,1}$ and $C_{\mathrm{SL}^\epsilon,1}$ denote the groups from Proposition 2.2.8 (ii). In view of Corollary 2.2.11 and Proposition 2.2.8, w.l.o.g. we can assume that $M \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^\epsilon,1} \in \mathscr{C}_2$ of $\mathrm{P}\Omega$ (see Lemma 1.4.20). Since $\mathrm{P}K_{\mathrm{SL}^\epsilon,1}$ is a normal subgroup of M, we obtain from Lemma 1.4.21 that it is strongly constrained. It is easy to see that $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$ is a normal abelian subgroup of $\mathrm{P}K_{\mathrm{SL}^\epsilon,1}$. Hence, we obtain our assertion by Lemma 1.4.13 and the fact that $|\mathrm{P}C_{\mathrm{SL}^\epsilon,1}| = \frac{(q-\epsilon 1)^{n-1}}{(q-\epsilon 1, n)}$. □

In Propositions 1.5.9 and 1.5.11, we have investigated the number theoretic problems arising from the last lemma, and the results are important for our further research (especially in Chapter 3).
Next, we note a corollary from our previous considerations and Propositions 1.5.9 and 1.5.11.

Corollary 2.2.15. *Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and let* M *be a maximal subgroup of* G *which is a member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$*. Let* p *be a prime. If* M *is strongly* p*-constrained then* $\frac{(q-\epsilon 1)^{n-1}}{(q-\epsilon 1, n)} = p^b$ *for a positive integer* b*.*

Proof. Analogously to the proof of Lemma 2.2.14, w.l.o.g. we can assume that $M \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^{\epsilon},1}$, and we obtain that $|\mathrm{P}C_{\mathrm{SL}^{\epsilon},1}| = r^b$ for a prime r and an integer $b \geq 0$. For $b > 0$ our assertion easily follows by Lemma 1.4.21. Therefore, suppose that $b = 0$. By Propositions 1.5.9 and 1.5.11, we obtain that case **L** holds where $q = 2$ (note, that $(q, n) \neq (3, 2)$). In view of Proposition 2.2.9, we see that $G = \mathrm{PSL}_3(2) \rtimes \langle \tau \rangle$. Hence, we get

$$M = \left(\left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle \right) \rtimes \langle \tau \rangle.$$

It is not hard to see that $|\mathrm{O}_3(M)| = 3$ and $|\mathrm{O}_2(M)| = 2$; a contradiction. □

Remark 2.2.16. (a) By the proof of Corollary 2.2.15, we get an example for an almost simple group G (where $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$) which has a maximal subgroup belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_1^{\epsilon}(q^u) \wr \mathrm{S}_n$ which is not strongly constrained.

(b) Considering $\mathrm{PSL}_2(5) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_2(5))$, we obtain examples for almost simple groups G which have a strongly constrained subgroup M belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$ which is not a maximal subgroup of G. This is easy to see by Proposition 2.2.9 and the fact that M is a 2-group, which follows from Proposition 2.2.8.

Next, we provide two lemmas about the centralizer of the homogeneous diagonal matrices of $\mathrm{PSL}_n^{\epsilon}(q^u)$ in the full automorphism group of $\mathrm{PSL}_n^{\epsilon}(q^u)$.

Lemma 2.2.17. *Let r be a prime, a be a positive integer and set $q = r^a$. By $T = C_{\mathrm{GL},1}$ and $T_0 = C_{\mathrm{SL},1}$ we denote the groups in Proposition 2.2.8 (i) and (ii) (i.e. the subgroups consisting of the diagonal matrices in $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$, respectively). Let $q > 2$ for $n \geq 3$ and $q > 3$ for $n = 2$ (hence, $\mathrm{P}T_0 > 1$). Then the following hold.*

(i) For $n = 2$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_2(q))}(\mathrm{P}T_0) = \begin{cases} \mathrm{P}T \rtimes \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle & \text{for } q = 5, \\ \mathrm{P}T \rtimes \left\langle \varphi_r \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle & \text{for } q \in \{4, 9\}, \\ \mathrm{P}T & \text{otherwise.} \end{cases}$$

(ii) For $n = 3$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_3(q))}(\mathrm{P}T_0) = \begin{cases} \mathrm{P}T_0 \times \langle \tau \rangle & \text{for } q = 3, \\ \mathrm{P}T \rtimes \left\langle \tau \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \varphi_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle & \text{for } q = 4, \\ \mathrm{P}T & \text{otherwise.} \end{cases}$$

(iii) For $n = 4$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_4(q))}(\mathrm{P}T_0) = \begin{cases} \mathrm{P}T \rtimes (V_4 \times \langle\tau\rangle) & \text{for } q = 3, \\ \mathrm{P}T_0 \times \langle\varphi_2\tau\rangle & \text{for } q = 4, \\ \mathrm{P}T & \text{otherwise} \end{cases}$$

$$\text{where } V_4 = \left\langle \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\rangle \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2.$$

(iv) For $n \geq 5$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}T_0) = \begin{cases} \mathrm{P}T \rtimes \langle\tau\rangle & \text{for } q = 3, \\ \mathrm{P}T \rtimes \langle\varphi_2\tau\rangle & \text{for } q = 4, \\ \mathrm{P}T & \text{otherwise.} \end{cases}$$

Proof. First, we note that $\mathrm{P}T_0 > 1$, because of Proposition 1.5.9. Let $g = g_\tau g_\varphi g_{\mathrm{PGL}} \in \mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}T_0)$ for $n \geq 3$, or $g = g_\varphi g_{\mathrm{PGL}} \in \mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_2(q))}(\mathrm{P}T_0)$ for $n = 2$ where $g_\tau \in \langle\tau\rangle$, $g_\varphi \in \langle\varphi_r\rangle$ and $g_{\mathrm{PGL}} \in \mathrm{PGL}_n(q)$ (cf. Corollary 1.2.20). Since g_τ and g_φ normalize $\mathrm{P}T_0$, we obtain that $g_{\mathrm{PGL}} \in \mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}T_0) = \mathrm{P}T \rtimes S$ where $S = \mathrm{P}S_{\mathrm{GL},1}$ with $S_{\mathrm{GL},1}$ from Proposition 2.2.8 (i), so S denotes the subgroup of $\mathrm{PGL}_n(q)$ consisting of the monomial matrices having all non-zero entries equal to one. (We note that the given structure of $\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}T_0)$ is correct, since we know $\mathrm{N}_{\mathrm{PSL}_n(q)}(\mathrm{P}T_0)$ from [Hu, II 7.2 Satz (c)][10] and the fact that $|\mathrm{N}_{\mathrm{PGL}_n(q)}(\mathrm{P}T_0)/\mathrm{N}_{\mathrm{PSL}_n(q)}(\mathrm{P}T_0)|$ divides $(n, q-1) = |\mathrm{P}T/\mathrm{P}T_0|$). Hence, let $g_{\mathrm{PGL}} = g_T g_S$ where $g_T \in \mathrm{P}T$ and $g_S \in S$. Now, we prove each of the assertions separately; for this we note that the condition

$$t^g = t \quad (*)$$

holds for all $t = \mathrm{P}\tilde{t} \in \mathrm{P}T_0$ where $\tilde{t} \in T_0$.
For $n = 2$ we obtain assertion (i), by examining $(*)$ with elementary calculations, using Lemma 1.3.8 (i) and (iii).
Let $n = 3$. We consider the different possibilities for g separately. First, let $g_\tau = 1 = g_\varphi$, hence $g = g_T g_S$. We define the subset

$$C_{3,1} = \{\mathrm{diag}(\lambda, 1, \lambda^{-1}), \mathrm{diag}(1, \lambda, \lambda^{-1}) \mid \text{ for all } \lambda \in \mathrm{GF}(q)^*\} \subseteq T_0.$$

Since $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{3,1}$, we obtain by elementary calculations that g centralizes $\mathrm{P}T_0$ if and only if $q \neq 4$ and $g_S = 1$, or $q = 4$ and

$$g_S \in S_{3,1} = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle.$$

[10] We have to mention a slight mistake in part (c) of [Hu, II 7.2 Satz] where the case $q = 2$ for all dimensions n is not excluded (merely in part (a)).

Now, we consider the case $g_\tau = \tau$ and $g_\varphi = 1$. Again, using condition $(*)$ for $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{3,1}$, it follows by elementary calculations that g centralizes $\mathrm{P}T_0$ if and only if $q = 3$ and $g_S = 1$, or $q = 4$ and

$$g_S \in S_{3,2} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

Let $g_\tau = 1$ and $g_\varphi \neq 1$. We define the subset

$$C_{3,2} = \{\mathrm{diag}(1, \lambda, \lambda^{-1}) \mid \text{ for all } \lambda \in \mathrm{GF}(q)^*\} \subseteq T_0.$$

Using condition $(*)$ for $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{3,2}$, we obtain by elementary calculations together with Lemma 1.3.8 (ii) that g centralizes $\mathrm{P}T_0$ if and only if $q = 4$ and $g_S \in S_{3,2}$.
Now, let $g_\tau = \tau$ and $g_\varphi \neq 1$. Because $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{3,2}$, we obtain by elementary calculations together with Lemma 1.3.8 (ii) that g centralizes $\mathrm{P}T_0$ if and only if $q = 4$ and $g_S \in S_{3,1}$. To obtain assertion (ii), we note that for $q = 4$ we have

$$\langle S_{3,1}, \varphi_2 S_{3,2}, \tau S_{3,2}, \tau\varphi_2 S_{3,1} \rangle = \left\langle \tau \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \varphi_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

For $n = 4$ we first consider the subset

$$C_{4,1} = \{\mathrm{diag}(\lambda, \lambda^{-1}, 1, 1), \mathrm{diag}(\lambda, 1, \lambda^{-1}, 1), \mathrm{diag}(\lambda, 1, 1, \lambda^{-1})\} \subseteq T_0$$

for an element $1 \neq \lambda \in \mathrm{GF}(q)^*$. The element g_S acts on the diagonal matrices $\mathrm{P}\tilde{t}$ for $\tilde{t} \in T_0$ by permuting the entries. Since $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{4,1}$, it follows by elementary combinatorial considerations and easy calculations that g_S stabilizes all entries of the diagonal matrices $\mathrm{P}\tilde{t}$, or none. Again, considering $(*)$ for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{4,1}$, we obtain by elementary calculations that $\mathrm{o}(g_S) \neq 4$ (e.g. for $g_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ consider that g centralizes $\mathrm{P}\tilde{t}$ where $\tilde{t} = \mathrm{diag}(\lambda, \lambda^{-1}, 1, 1)$). Hence, it follows that $g_S \in V_4 \trianglelefteq S \cong \mathrm{S}_4$ where V_4 denotes the normal subgroup of S isomorphic to the Klein four-group.
The rest of assertion (iii) follows analogously to (ii), considering the different possibilities for g separately. E.g. let $g_\tau = 1 = g_\varphi$. Because $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{4,2} = \{\mathrm{diag}(\lambda, \lambda^{-1}, 1, 1) | \text{ for all } \lambda \in \mathrm{GF}(q)^*\} \leq T_0$, we obtain by elementary calculations that g centralizes $\mathrm{P}T_0$ if and only if $q \neq 3$ and $g_S = 1$, or $q = 3$ and $g_S \in V_4$.
Let $n \geq 5$. First, we choose the subset

$$C_{n,1} = \{\tilde{t}_2, \ldots, \tilde{t}_{n-1}\} \subseteq T_0$$

where $\tilde{t}_i = \operatorname{diag}(\lambda_{i_1}, \dots, \lambda_{i_n})$ for $i \in \{2, \dots, n-1\}$ with $1 \neq \lambda_{i_1} \in \mathrm{GF}(q)^*$, $\lambda_{i_i} = \lambda_{i_1}^{-1}$ and $\lambda_{i_j} = 1$ else. Since $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{n,1}$, we obtain by elementary combinatorial considerations and easy calculations that $g_S = 1$. Hence, we have $g = g_\tau g_\varphi g_T$. Now, we consider the different possibilities for g separately.

Let $g = \tau g_\varphi g_T$ where $1 \neq g_\varphi$ is induced by $\varphi \in \mathrm{Aut}(\mathrm{GF}(q))$. Because $(*)$ holds for all $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{n,2} = \{\operatorname{diag}(\lambda, \lambda^{-1}, 1, \dots, 1) | \text{ for all } \lambda \in \mathrm{GF}(q)^*\} \leq T_0$, we obtain that $\lambda^\varphi = \lambda^{-1}$ holds for all $\lambda \in \mathrm{GF}(q)^*$. By Lemma 1.3.8 (ii) and elementary calculations, it follows that g centralizes $\mathrm{P}T_0$ if and only if $q = 4$ and $g_\varphi = \varphi_2$.

We obtain analogously to the arguments above, using condition $(*)$ for $\mathrm{P}\tilde{t}$ where $\tilde{t} \in C_{n,2}$, that there is no element g centralizing $\mathrm{P}T_0$ if $g = g_\varphi g_T$ where $g_\varphi \neq 1$. For $g = \tau g_T$ we easily obtain by the same arguments that g centralizes $\mathrm{P}T_0$ if and only if $q = 3$. □

Lemma 2.2.18. *Let r be a prime, a be a positive integer and set $q = r^a$. We denote by $T = C_{\mathrm{GU},1}$ and $T_0 = C_{\mathrm{SU},1}$ the groups from Proposition 2.2.8 (i) and (ii) (i.e. the respective subgroups of $\mathrm{GU}_n(q^2)$ and $\mathrm{SU}_n(q^2)$ consisting of the diagonal matrices where the matrix of the non-degenerate unitary form is $\mathbb{1}_n$). Then the following hold.*

(i) For $n = 3$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_3(q^2))}(\mathrm{P}T_0) = \begin{cases} \mathrm{P}T \rtimes \left\langle \varphi_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \varphi_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle & \text{for } q = 2, \\ \mathrm{P}T & \text{otherwise.} \end{cases}$$

(ii) For $n \geq 4$ we have

$$\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_n(q^2))}(\mathrm{P}T_0) = \mathrm{P}T.$$

Proof. Our assertion follows analogously to Lemma 2.2.17, by combinatorial arguments and elementary calculations. We note that for some calculations we have to use Lemma 1.3.9. □

Remark. We note that the results for $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_n(2^2))}(\mathrm{P}T_0)$ in the last proposition can also be derived directly from Lemma 2.2.17, by intersection. Here, we use that the group of diagonal matrices in $\mathrm{PSU}_n(2^2)$ coincides with the group of diagonal matrices in $\mathrm{PSL}_n(2^2)$ for $\mathrm{PSU}_n(2^2) \leq \mathrm{PSL}_n(2^2)$.

Next, we will establish a proposition providing a necessary and sufficient condition for a member $M \in \mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ which is a maximal subgroup of G to be strongly constrained. For the following lemma we recall the introduction of the subgroup K_{Aut} of $\mathrm{Aut}(\mathrm{P}\Omega)$ from Definition 1.2.24, as well as the groups $K_{\mathrm{SL}^\epsilon,1}$ and $C_{\mathrm{SL}^\epsilon,1}$ from Proposition 2.2.8 (ii).

Lemma 2.2.19. *Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and* M *be a member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$*. Let* M *be a maximal subgroup of* G *and* $M \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^\epsilon,1}$*. Then* M *normalizes* $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$*.*

Proof. From the maximality of M in G together with the fact that $\mathrm{P}K_{\mathrm{SL}^\epsilon,1} \neq 1$ (see Proposition 2.2.8 or 1.1.3), we obtain $\mathrm{N}_G(\mathrm{P}K_{\mathrm{SL}^\epsilon,1}) = M$. So, we have that $\mathrm{N}_{\mathrm{P}\Omega}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1}) = \mathrm{P}K_{\mathrm{SL}^\epsilon,1}$ (cf. also Lemma 2.2.4 for the case **L**). It is easy to see that the diagonal automorphism W and the field automorphisms of $\mathrm{P}\Omega$ together with the graph automorphism of $\mathrm{P}\Omega$ (occurring if $\mathrm{P}\Omega = \mathrm{PSL}_n(q)$ for $n \geq 3$) normalize $\mathrm{P}K_{\mathrm{SL}^\epsilon,1}$. Hence, $K_{\mathrm{Aut}} \leq \mathrm{N}_{\mathrm{Aut}(\mathrm{P}\Omega)}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1})$. Obviously, $|\mathrm{N}_{\mathrm{Aut}(\mathrm{P}\Omega)}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1})/\mathrm{N}_{\mathrm{P}\Omega}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1})|$ divides $|\mathrm{Out}(\mathrm{P}\Omega)|$. Considering Remark 1.2.25, we see that

$$|(K_{\mathrm{Aut}} \cdot \mathrm{P}K_{\mathrm{SL}^\epsilon,1})/\mathrm{P}K_{\mathrm{SL}^\epsilon,1}| \geq |K_{\mathrm{Aut}}/(\mathrm{P}\Omega \cap K_{\mathrm{Aut}})| = |\mathrm{Out}(\mathrm{P}\Omega)|,$$

and so $\mathrm{N}_{\mathrm{Aut}(\mathrm{P}\Omega)}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1}) = \mathrm{P}K_{\mathrm{SL}^\epsilon,1} \cdot K_{\mathrm{Aut}}$. Now, we easily obtain our assertion, since we can deduce that $\mathrm{N}_{\mathrm{Aut}(\mathrm{P}\Omega)}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1})$ normalizes $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$. ☐

Remark. Without using the maximality of M in G, we note that the assertion of Lemma 2.2.19 for $n \geq 5$ also easily follows by the following argumentation. Since $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$ is an abelian normal subgroup of $\mathrm{P}K_{\mathrm{SL}^\epsilon,1}$, we have $\mathrm{F}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1}) \geq \mathrm{P}C_{\mathrm{SL}^\epsilon,1}$. Suppose that this inclusion is proper. Then we have that

$$1 < \mathrm{F}(\mathrm{P}K_{\mathrm{SL}^\epsilon,1})/\mathrm{P}C_{\mathrm{SL}^\epsilon,1} \trianglelefteq \mathrm{P}K_{\mathrm{SL}^\epsilon,1}/\mathrm{P}C_{\mathrm{SL}^\epsilon,1} \cong \mathrm{S}_n.$$

Since $n \geq 5$, we easily obtain a contradiction, and hence we have that $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$ is a characteristic subgroup of $\mathrm{P}K_{\mathrm{SL}^\epsilon,1}$.

Proposition 2.2.20. *Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and* M *be a maximal subgroup of* G *which is a member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$*. Let* p *be a prime. Then* M *is strongly* p*-constrained if and only if one of the following holds.*

(i) We have $n = 2$ *(hence case* **L** *holds) and* $\frac{q-1}{(q-1,2)} = p^b$ *for a positive integer* b*. For* $q \equiv 3 \pmod 4$ *we have that* G *does not include* $\mathrm{PGL}_2(q)$ *and for* $q = 4$ *we have* $G = \mathrm{PSL}_2(4)$*.*

(ii) We have $n \geq 3$ *and* $\frac{(q-\epsilon 1)^{n-1}}{(q-\epsilon 1,n)} = p^b$ *for a positive integer* b*.*

Proof. Let $C_{\mathrm{GL}^\epsilon,1}$, $K_{\mathrm{SL}^\epsilon,1}$ and $C_{\mathrm{SL}^\epsilon,1}$ denote the groups in Proposition 2.2.8 (i) and (ii). W.l.o.g. we can assume that $M \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^\epsilon,1}$, by Corollary 2.2.11 and Proposition 2.2.8 (see also Lemma 1.4.20). First, we consider $n \geq 3$. The only-if-part follows directly from Corollary 2.2.15. To prove the if-part, we first note that the order of $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$ is a non-trivial power of p, cf. proof of Lemma 2.2.14. So, we obtain that $1 < \mathrm{P}C_{\mathrm{SL}^\epsilon,1} \leq \mathrm{O}_p(\mathrm{P}K_{\mathrm{SL}^\epsilon,1}) \leq \mathrm{O}_p(M)$. We consider the inclusion

$$\mathrm{C}_M(\mathrm{O}_p(M)) \leq \mathrm{C}_M(\mathrm{P}C_{\mathrm{SL}^\epsilon,1}) = \mathrm{C}_{\mathrm{Aut}(\mathrm{P}\Omega)}(\mathrm{P}C_{\mathrm{SL}^\epsilon,1}) \cap M \quad (*).$$

Because, by Lemma 2.2.19, $\mathrm{P}C_{\mathrm{SL}^\epsilon,1}$ is a normal subgroup of M, we obtain that $\mathrm{C}_M(\mathrm{P}C_{\mathrm{SL}^\epsilon,1})$ is also a normal subgroup of M. Now, we show that $\mathrm{C}_M(\mathrm{P}C_{\mathrm{SL}^\epsilon,1})$ is a p-subgroup of M. Then M is strongly p-constrained following $(*)$ (recall Proposition 1.4.18).
Since $|\mathrm{P}C_{\mathrm{SL}^\epsilon,1}|$ is a non-trivial power of p, we obtain that this also holds for $|\mathrm{P}C_{\mathrm{GL}^\epsilon,1}|$, by Propositions 1.5.9 and 1.5.11. First, we consider the case $\mathrm{P}\Omega = \mathrm{PSU}_n(q^2)$. Because $(n,q) \neq (3,2)$, we obtain from Lemma 2.2.18 that the order of $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSU}_n(q^2))}(\mathrm{P}C_{\mathrm{SU},1})$ is a power of p. Hence, we easily obtain our assertion for this case. Now, let $\mathrm{P}\Omega = \mathrm{PSL}_n(q)$. Since M is a maximal subgroup of G and the order of $\mathrm{P}C_{\mathrm{SL},1}$ is not trivial, we obtain from Proposition 2.2.9 (ii)-(v) that $q > 2$ and $q \neq 4$. In view of Lemma 2.2.17 (ii)-(iv), we now obtain that the order of $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_n(q))}(\mathrm{P}C_{\mathrm{SL},1})$ is a power of p. Hence, we obtain that M is strongly p-constrained.
Now, we consider the case $n = 2$, so we have $\mathrm{P}\Omega = \mathrm{PSL}_2(q)$. First, we prove the if-part. Since $\mathrm{P}C_{\mathrm{SL},1}$ is a non-trivial normal p-subgroup of $\mathrm{P}K_{\mathrm{SL},1}$, we obtain as above $1 < \mathrm{P}C_{\mathrm{SL},1} \leq \mathrm{O}_p(M)$ (or, use Lemma 2.2.19). Hence, we have

$$\mathrm{C}_M(\mathrm{O}_p(M)) \leq \mathrm{C}_M(\mathrm{P}C_{\mathrm{SL},1}) = \mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_2(q))}(\mathrm{P}C_{\mathrm{SL},1}) \cap M.$$

Analogously to the arguments above, we have that $\mathrm{C}_M(\mathrm{P}C_{\mathrm{SL},1})$ is a normal subgroup of M. By Lemma 2.2.17 (i), we obtain that $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_2(q))}(\mathrm{P}C_{\mathrm{SL},1}) = \mathrm{P}C_{\mathrm{GL},1}$ for $q \notin \{4,5,9\}$. Hence, if $q \neq 4,5,9$, we easily obtain that $\mathrm{C}_M(\mathrm{P}C_{\mathrm{SL},1})$ is a normal p-subgroup of M (note, that by our condition for $q \equiv 3 \pmod 4$ we have $\mathrm{P}C_{\mathrm{GL},1} \cap M = \mathrm{P}C_{\mathrm{SL},1}$). So, M is strongly p-constrained. From Proposition 2.2.9 we have that $q \neq 5$. For the remaining cases $q = 4,9$ (where $G=\mathrm{PSL}_2(4)$ if $q = 4$) we obtain by Lemma 2.2.17 (i) that $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_2(q))}(\mathrm{P}C_{\mathrm{SL},1})$ has a prime power order. So, our assertion also follows for this cases.
Finally, we consider the only-if-part. From Corollary 2.2.15 we obtain $\frac{q-1}{(q-1,2)}$ is a non-trivial power of p. First, suppose that $q = 4$ and $G = \mathrm{Aut}(\mathrm{PSL}_2(4))$. By Lemma 2.2.17 (i) and Proposition 1.4.18, we easily obtain that M is not strongly constrained. Now, suppose that $q \equiv 3 \pmod 4$ and $G \geq \mathrm{PGL}_2(q)$. Since M is a maximal subgroup of G and $\mathrm{P}C_{\mathrm{SL},1}$ is a non-trivial normal subgroup of M (see Lemma 2.2.19), we obtain that $M = \mathrm{N}_G(\mathrm{P}C_{\mathrm{SL},1})$. Hence, we have $\mathrm{P}C_{\mathrm{GL},1} \leq M$. Analogously to above, we now obtain by Lemma 2.2.17 (i) that $M \trianglerighteq \mathrm{C}_M(\mathrm{P}C_{\mathrm{SL},1}) = \mathrm{P}C_{\mathrm{GL},1}$. Hence, there is a normal abelian subgroup of M which is not a p-subgroup. So, M is not strongly p-constrained and our assertion is established. □

Now, we easily can deduce the first two main theorems of this section, by including the information from Proposition 2.2.9 to the assertion of the last proposition. We determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$. As usual, we present the following results not using the standard notation.

Main Theorem 2.2.21. *Let* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ *where* $\mathrm{PSL}_n(q)$ *is simple. Let* M *be a member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_1(q) \wr \mathrm{S}_n$*. Let* p *be a prime. Then* M *is a strongly* p*-constrained maximal subgroup of* G *if and only if one of the following holds.*

(i) For $n = 2$ *we have*

- *(a)* $q \geq 17$, $\frac{q-1}{(q-1,2)} = p^b$ *for a positive integer* b *and for* $q \equiv 3 \pmod 4$ G *does not include* $\mathrm{PGL}_2(q)$,
- *(b)* $q = 4$ *and* $G = \mathrm{PSL}_2(4)$ *(here,* $p = 3$*),*
- *(c)* $q = 8$ *(here,* $p = 7$*), or*
- *(d)* $q = 9$ *and* $G \in \{\mathrm{PGL}_2(9),\ \mathrm{PSL}_2(9)\langle W\varphi_3\rangle,\ \mathrm{Aut}(\mathrm{PSL}_2(9))\}$ *(here,* $p = 2$*).*

(ii) For $n \geq 3$ *we have that* $\frac{(q-1)^{n-1}}{(q-1,n)} = p^b$ *for a positive integer* b *and*

- *(a) for* $n = 3$ *we have* $q \neq 4$,
- *(b) for* $n = 4$ *we have* $q \geq 8$*, or* $q = 5$ *and* $G \not\leq \mathrm{PSL}_4(5)\langle W^2, \tau\rangle$ *and*
- *(c) for* $n \geq 5$ *we have* $q \geq 5$.

Main Theorem 2.2.22. *Let* $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ *where the matrix of the non-degenerate unitary form is* $\mathbb{1}_n$*. Let* $\mathrm{PSU}_n(q^2)$ *be simple and* $n \geq 3$*. Let* M *be a member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GU}_1(q^2) \wr \mathrm{S}_n$*. Let* p *be a prime. Then* M *is a strongly* p*-constrained maximal subgroup of* G *if and only if* $\frac{(q+1)^{n-1}}{(q+1,n)} = p^b$ *for a positive integer* b *(note, that* $q > 2$ *for case* $n = 3$*) and*

- *(a) for* $(n, q) = (4, 3)$ *we have* $G \not\leq \mathrm{PSU}_4(3^2)\langle W^2, \varphi_3\rangle$ *and*
- *(b) for* $(n, q) = (6, 2)$ *we have* $G \in \{\mathrm{PGU}_6(2^2), \mathrm{P\Gamma U}_6(2^2)\}$.

Remark 2.2.23. (a) We recall Propositions 1.5.9 and 1.5.11 for the number theoretical consequences in Main Theorems 2.2.21 and 2.2.22 arising from the conditions that respective $\frac{(q-1)^{n-1}}{(q-1,n)}$ and $\frac{(q+1)^{n-1}}{(q+1,n)}$ are non-trivial powers of p.

(b) The results from Main Theorem 2.2.22 can also be presented without specifying the basis, except for one case. For the excluded case in dimension four we have to consider G with respect to a fixed ordered basis, for a unique description of the results, cf. Remark 2.2.10 (c).

2.2.2 $\mathscr{C}_2$ of types $\mathrm{GL}_2^{\epsilon}(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ and $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$

Now, we investigate for an almost simple group G with socle isomorphic to $\mathrm{P}\Omega$, in which cases a maximal subgroup of G belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_2^{\epsilon}(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$ is strongly constrained. Recall, that we consider a representation of $\Omega(V)$ with respect to a fixed ordered basis B of V where, in case **U**, B is an orthonormal basis of V (see the description just before Subsection 2.2.1). We also recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1, \ldots, A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$ introduced on p. 8 and the notation introduced in Convention 1.2.2.

Proposition 2.2.24. *Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and* M *be a maximal subgroup of* G *which belongs to A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_2^{\epsilon}(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ *or* $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. *Then* M *is strongly constrained.*

Proof. Let G and M be as assumed. Considering Proposition 2.2.8 and Corollary 2.2.11, w.l.o.g. we can assume that $K = M \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^{\epsilon},m}$ where $K_{\mathrm{SL}^{\epsilon},m}$ is described in Proposition 2.2.8 (ii) (cf. also Lemma 1.4.20). Moreover, from Proposition 2.2.8 (ii) we recall that $K_{\mathrm{SL}^{\epsilon},m} = C_{\mathrm{SL}^{\epsilon},m} \rtimes S_{\mathrm{SL}^{\epsilon},m}$, and we define the normal subgroup

$$C = \{\mathrm{diag}(A_1, \ldots, A_{\frac{n}{m}}) \mid A_i \in \mathrm{SL}_m^{\epsilon}(q^u)\} \cong \mathrm{SL}_m^{\epsilon}(q^u)^{\frac{n}{m}}$$

of $C_{\mathrm{SL}^{\epsilon},m}$. First, we consider the assertion for case **L**. Hence, let M be a member of $\mathscr{C}_2$ of G of type $\mathrm{GL}_2(3) \wr \mathrm{S}_{\frac{n}{2}}$. Here, we note that $Z = \mathrm{Z}(\Omega) = \mathrm{Z}(\mathrm{I}) \leq C$ is of order two. We consider the subgroup $\mathrm{P}C \leq K$. Clearly, we have that $\mathrm{P}C \cong \mathrm{SL}_2(3) \circ \ldots \circ \mathrm{SL}_2(3)$. Hence, it is also clear that $\mathrm{O}_2(\mathrm{P}C) > 1$ and that the layer $\mathrm{E}(\mathrm{P}C)$ of $\mathrm{P}C$ is trivial. Suppose that $\mathrm{O}_3(\mathrm{P}C) > 1$. Then $|\widehat{\mathrm{O}_3(\mathrm{P}C)}| = 2|\mathrm{O}_3(\mathrm{P}C)|$, and $S \in \mathrm{Syl}_3(\widehat{\mathrm{O}_3(\mathrm{P}C)})$ is a non-trivial normal subgroup of C, by the Sylow theorems. Since $\mathrm{O}_3(C) \cong \mathrm{O}_3(\mathrm{SL}_2(3))^{\frac{n}{2}} = 1$, we obtain a contradiction. So, we have shown that $\mathrm{P}C$ is strongly 2-constrained. Now, we can deduce that $\mathrm{P}C_{\mathrm{SL},2}$ also is strongly 2-constrained (note, that $\mathrm{O}_3(\mathrm{P}C_{\mathrm{SL},2}) = \mathrm{O}_3(\mathrm{P}C)$, since $|\mathrm{P}C_{\mathrm{SL},2}/\mathrm{P}C| = 2^{\frac{n}{2}-1}$).
Next, we show that K is strongly 2-constrained. For this, we prove that

$$\mathrm{C}_K(\mathrm{O}_2(K)) \leq \mathrm{P}C_{\mathrm{SL},2} \quad (*)$$

holds. It is not hard to see that $\mathrm{PO}_2(C) \leq \mathrm{O}_2(K)$ (note, that $\mathrm{O}_2(C) \cong \mathrm{O}_2(\mathrm{SL}_2(3))^{\frac{n}{2}}$). So, we define the subgroups $\mathrm{P}H_1, \ldots, \mathrm{P}H_{\frac{n}{2}}$ of $\mathrm{O}_2(K)$ where

$$\begin{aligned} H_1 &= \{\mathrm{diag}(A, \mathbb{1}_{n-2}) \mid A \in \mathrm{O}_2(\mathrm{SL}_2(3))\}, \\ H_2 &= \{\mathrm{diag}(\mathbb{1}_2, A, \mathbb{1}_{n-4}) \mid A \in \mathrm{O}_2(\mathrm{SL}_2(3))\}, \ldots, \\ H_{\frac{n}{2}} &= \{\mathrm{diag}(\mathbb{1}_{n-2}, A) \mid A \in \mathrm{O}_2(\mathrm{SL}_2(3))\} \leq \mathrm{O}_2(C). \end{aligned}$$

Suppose that $(*)$ does not hold. Then there is an element $g \in \mathrm{C}_K(\mathrm{O}_2(K))$ where $g = \mathrm{P}(g_C g_S)$ with $g_C \in C_{\mathrm{SL},2}$ and $g_S \in S_{\mathrm{SL},2} \setminus \{1\}$. Since g centralizes each $\mathrm{P}H_i$, we easily obtain a contradiction by elementary combinatorial considerations. So, $(*)$ is established. Now, suppose that $\mathrm{E}(K) > 1$. Then there is a quasisimple subnormal subgroup Q of K. Since Q centralizes $\mathrm{O}_2(K)$ (see Lemma 1.4.17), we easily get a contradiction by $(*)$. Next, suppose that $\mathrm{O}_r(K) > 1$ for a prime $r \neq 2$. Since $\mathrm{O}_r(K)$ and $\mathrm{O}_2(K)$ centralize each other, we similarly obtain a contradiction by $(*)$. So, K is strongly 2-constrained. Since $|M/K| = |G/\mathrm{P}\Omega|$

divides 4, we now easily obtain our assertion for this case.
Next, we consider the assertion in the case **U**. For $\mathrm{P}\Omega = \mathrm{PSU}_n(3^2)$ and $M \in \mathscr{C}_2$ of G of type $\mathrm{GU}_2(3^2) \wr \mathrm{S}_{\frac{n}{2}}$, the assertion follows analogously to the arguments above. (We note that in this case M is also strongly 2-constrained). Finally, let $\mathrm{P}\Omega = \mathrm{PSU}_n(2^2)$ and M be a member of $\mathscr{C}_2$ of G of type $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. First, we note some facts concerning the group $\mathrm{SU}_3(2^2)$ (for an explanation see Example 2.5.16, Definition 2.6.1 and Lemma 2.6.2, below). We have $\mathrm{SU}_3(2^2) \cong N \rtimes Q_8$ where N is an extraspecial 3-group of order 3^3 and Q_8 denotes the quaternion group. Furthermore, $\mathrm{O}_2(\mathrm{SU}_3(2^2)) = 1$ and $|\mathrm{O}_3(\mathrm{SU}_3(2^2))| = 3^3$.
It is not hard to see that $Z = \mathrm{Z}(\Omega) = \mathrm{Z}(\mathrm{I})$ is a subgroup of C of order 3 (cf. again Example 2.5.16), and $\mathrm{O}_3(\mathrm{P}C) > 1$. Analogously to the above considerations concerning case **L**, we obtain that $\mathrm{O}_2(\mathrm{P}C) = 1$. Since $\mathrm{P}C$ is soluble, $\mathrm{E}(\mathrm{P}C)$ is trivial, and thus $\mathrm{P}C$ is strongly 3-constrained. Now, we can deduce that $\mathrm{P}C_{\mathrm{SU},3}$ is also strongly 3-constrained, since $|\mathrm{P}C_{\mathrm{SU},3}/\mathrm{P}C| = 3^{\frac{n}{3}-1}$. Next, we consider the normal subgroup K of M. Analogous to the considerations concerning $(*)$, we obtain that $\mathrm{C}_K(\mathrm{O}_3(K)) \leq \mathrm{P}C_{\mathrm{SU},3}$ holds (note, that $\mathrm{PO}_3(C) \leq \mathrm{O}_3(K)$). Since $\mathrm{P}C_{\mathrm{SU},3}$ is strongly 3-constrained, we now can deduce that K is also strongly 3-constrained, by analogous considerations as above.
Now, we consider the maximal subgroup M of G. By Lemma 1.4.20 and since $\mathrm{Out}(\mathrm{P}\Omega) \cong \mathrm{S}_3$, it is sufficient to examine the cases $G \in \{\mathrm{P}\Omega, \mathrm{PI}, \mathrm{P}\Omega \rtimes \langle\varphi_2\rangle, \mathrm{PA}\}$. Case $G = \mathrm{P}\Omega$ was considered before. If $G = \mathrm{PI}$ then $|M/K| = 3$. Hence, it is easy to see that M is strongly 3-constrained, since K is strongly 3-constrained. In the case $G = \mathrm{P}\Omega \rtimes \langle\varphi_2\rangle$ we have that $M = K \rtimes \langle\varphi_2\rangle$. Suppose that M is not strongly 3-constrained. Since K is a strongly 3-constrained normal subgroup of M of index 2, we have that $\mathrm{O}_2(M) > 1$ (note, that $\mathrm{O}_r(M) = \mathrm{O}_r(K)$ for all primes $r \neq 2$ and $\mathrm{E}(M) = \mathrm{E}(K) = 1$). So, we have that $|\mathrm{O}_2(M)| = |(\mathrm{O}_2(M) \cdot K)/K| = |M/K| = 2$, and hence there is a non-trivial involutory element in $\mathrm{Z}(M)$. By elementary calculations, we see that such an element does not exist. Hence, M is strongly 3-constrained. The final case $G = \mathrm{PA}$ follows by analogous arguments as above, and so we have established our assertion. □

Remark. Concerning the case $\mathrm{PSL}_n(2) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(2))$ and K is a member of A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_2(2) \wr \mathrm{S}_{\frac{n}{2}}$, we note the following observations. (Recall that by Proposition 2.2.9, K is not a maximal subgroup of G). By analogous arguments as in the previous proof, we obtain that K is strongly 3-constrained if $G = \mathrm{PSL}_n(2)$. But, if $G = \mathrm{Aut}(\mathrm{PSL}_n(2))$ then K is not strongly constrained. To see this, w.l.o.g. let $K \cap \mathrm{PSL}_n(2) = \mathrm{P}K_{\mathrm{SL},2}$ as in the previous proof. Then we have that $\mathrm{P}(\mathrm{diag}(A, \ldots, A)\tau) \in \mathrm{Z}(K)$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence, $\mathrm{O}_2(K)$ and $\mathrm{O}_3(K)$ are not trivial.

Now, using the information in Propositions 2.2.9 and 2.2.24, we obtain the next two main theorems of this section. We determine all pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. Recalling Corollary 2.2.13 and Main Theorems 2.2.21 and 2.2.22, we note that we have determined the pairs (G, M) where G is as above and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_2$ of G. As usual, we present the following results not using the standard notation.

Main Theorem 2.2.25. *Let* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ *where* $\mathrm{PSL}_n(q)$ *is simple. Let* M *belong to A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_2(3) \wr \mathrm{S}_{\frac{n}{2}}$ *(esp.* n *is even and* $q = 3$*). Then* M *is a strongly constrained maximal subgroup of* G *if and only if* $G \not\leq \mathrm{PSL}_4(3) : \langle\tau\rangle$. *Furthermore, a strongly constrained maximal subgroup* M *of* G *is strongly* 2*-constrained.*

Main Theorem 2.2.26. *Let* $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ *where* $\mathrm{PSU}_n(q^2)$ *is simple and* $n \geq 3$. *Then every subgroup* M *of* G *belonging to A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GU}_2(3^2) \wr \mathrm{S}_{\frac{n}{2}}$, *or* $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$ *(occurring if and only if* n *is even and* $q = 3$, *or* $3 \mid n$ *and* $q = 2$, *respectively) is a strongly constrained maximal subgroup of* G. *Furthermore, we have that such a subgroup* M *of* G *is strongly* 2*-constrained if* M *is of type* $\mathrm{GU}_2(3^2) \wr \mathrm{S}_{\frac{n}{2}}$, *and strongly* 3*-constrained if* M *is of type* $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$.

2.3 A-class $\mathscr{C}_3$

In this section, we investigate the members of A-class $\mathscr{C}_3$. Roughly described, the members of A-class $\mathscr{C}_3$ are the stabilizers of certain linear or unitary groups over extension fields of $\mathrm{GF}(q^u)$ of prime index r dividing $n = \dim_{\mathrm{GF}(q^u)}(V)$. We will introduce the members of A-class $\mathscr{C}_3$ staying closely to the book of Kleidman and Liebeck [KL, § 4.3].

For the following we recall the notation introduced in Subsection 1.2.1, esp. the notation from Table 1.2.1. We also recall the generalized standard notation introduced in Subsection 1.2.3, esp. that V is an n-dimensional $\mathrm{GF}(q^u)$-vector space for a prime power q and $u = 1$ in case **L** and $u = 2$ in case **U**. For the following we use the generalized standard notation.
By $\mathrm{GF}(q')$, we denote a field extension of $\mathrm{GF}(q^u)$ of degree $r > 1$ where $r \mid n$ and we set $t = \frac{n}{r}$. In case **U** let r be odd. Let $B = (b_1, \ldots, b_r)$ be an ordered $\mathrm{GF}(q^u)$-basis for $\mathrm{GF}(q')$ where $b_j \in \mathrm{GF}(q')$ appropriate. In view of the $\mathrm{GF}(q^u)$-vector space isomorphism

$$\mathrm{GF}(q')^t \to \mathrm{GF}(q^u)^n \cong V, \quad (a_1, \ldots, a_t) \mapsto (\mu_{11}, \ldots, \mu_{1r}, \ldots, \mu_{t1}, \ldots, \mu_{tr})$$

where $a_i = \sum_{j=1}^r \mu_{ij} b_j \in \mathrm{GF}(q')$ for $\mu_{ij} \in \mathrm{GF}(q^u)$, we see that V can also be considered naturally as a $\mathrm{GF}(q')$-vector space of dimension t. Hence, in case **L** we obtain an embedding of the non-singular $\mathrm{GF}(q')$-linear transformations of V to the non-singular $\mathrm{GF}(q)$-linear transformations

$$\mathrm{GL}_t(q') \cong \mathrm{GL}(V, \mathrm{GF}(q')) \leq \mathrm{GL}(V, \mathrm{GF}(q)) = \mathrm{I}(V) \cong \mathrm{GL}_n(q) \qquad (2.3.1)$$

(cf. [KL, (4.3.1)]), and also that $\mathrm{GF}(q')^*$ embeds naturally in $\mathrm{GL}(V, \mathrm{GF}(q))$. We arrange that we write f in case **L** for the trivial form on $(V, \mathrm{GF}(q))$ and f' for the trivial form on $(V, \mathrm{GF}(q'))$.
Regarding the case **U**, we see that there exists a non-degenerate unitary form f' on $(V, \mathrm{GF}(q'))$ such that $f = Tf'$ where f denotes the non-degenerate unitary form on $(V, \mathrm{GF}(q^2))$ in case **U** and $T = T^{\mathrm{GF}(q')}_{\mathrm{GF}(q^2)}$ is the trace map from $\mathrm{GF}(q')$ to $\mathrm{GF}(q^2)$: Considering V as a t-dimensional $\mathrm{GF}(q')$-vector space equipped with a non-degenerate unitary form f', we obtain that Tf' is a non-degenerate unitary form on $(V, \mathrm{GF}(q^2))$, see [KL, p. 111, 116] and more detailed [Hu2, proof of Hilfssatz 1 b)]. Since all unitary geometries on $(V, \mathrm{GF}(q^2))$ are isometric (recall Proposition 1.2.3 (i)), w.l.o.g. we can assume that the non-degenerate unitary form f on $(V, \mathrm{GF}(q^2))$ in case **U** is Tf'. Hence, we obtain an embedding

$$\mathrm{GU}_t(q') \cong \mathrm{I}(V, \mathrm{GF}(q'), f') \leq \mathrm{I}(V, \mathrm{GF}(q^2), Tf') = \mathrm{I}(V) \cong \mathrm{GU}_n(q^2) \qquad (2.3.2)$$

(cf. also [Hu2, Hilfssatz 1 b)], [KL, p. 111] and note that $q' = q^{2r} = (q^r)^2$ is a square).

For the definition of A-class $\mathscr{C}_3$ we also have to introduce the following notation. We recall the map λ from (1.2.4) and (1.2.5), and write λ' for the corresponding map for $\Gamma(V, \mathrm{GF}(q'), f')$. Analogously to [KL, (4.3.3)], we define the subgroup

$$\Gamma^{\mathrm{GF}(q'), f'}_{\mathrm{GF}(q^u)} = \{g \in \Gamma(V, \mathrm{GF}(q'), f') \mid \lambda'(g) \in \mathrm{GF}(q^u)\}$$

of $\Gamma(V, \mathrm{GF}(q'), f')$, and we note that $\Gamma^{\mathrm{GF}(q'), f'}_{\mathrm{GF}(q^u)}$ is also a subgroup of $\Gamma(V)$ (cf. [KL, top of p. 112]).

Now, we are able to introduce the members of A-class $\mathscr{C}_3$. For this, we use the notation introduced above (esp. f denotes the trivial form on $(V, \mathrm{GF}(q))$ in case **L** or a non-degenerate unitary form on $(V, \mathrm{GF}(q^2))$ in case **U**). Furthermore, we recall that we use the generalized standard notation in the following definition.

Definition 2.3.1. {***A-class*** $\mathscr{C}_3$} (cf. [KL, p. 60 and Definition p. 112] and [BHR, Definition 2.2.5 and Table 2.6])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and let K be a subgroup of G. Let the degree r of the field extension $\mathrm{GF}(q')$ of $\mathrm{GF}(q^u)$ be a prime, and set $t = \frac{n}{r}$ (recall that r divides n). For $G \leq \Gamma(V)$ the subgroup K belongs to *(A-class)* $\mathscr{C}_3$ of G if $K = \Gamma^{\mathrm{GF}(q'), f'}_{\mathrm{GF}(q^u)} \cap G$ for a case arising in the following table. If $G \not\leq \Gamma(V)$ we define K belonging to *(A-class)* $\mathscr{C}_3$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_3$ of $\Gamma(V)$.

Case	Type	Description of f, f'	Conditions
L	$\mathrm{GL}_t(q^r)$	f and f' are trivial	
U	$\mathrm{GU}_t(q^{2r})$	f' is non-degenerate unitary, $f = Tf'$	$r \neq 2$

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_3$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_3$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}_t^{\epsilon}(q^{ur})$ we call K of type $\mathrm{GL}_t^{\epsilon}(q^{ur})$.

Remark 2.3.2. (a) It is not hard to see that r has to be a prime in the last definition, since otherwise there would be inclusions into other members of A-class $\mathscr{C}_3$ of G. In case **U** r is assumed to be odd, and for an explanation we refer to [KL, bottom of p. 116] and cf. also [Hu2, Hilfssatz 1 c)].

(b) The definition of A-class $\mathscr{C}_3$ in the paper of Aschbacher [As] differs from the definition provided in [KL]. In [KL, Remark p. 112 and Proposition 4.3.3], there is a detailed investigation showing that the collection of subgroups defined in the book of Kleidman and Liebeck is a subset of Aschbacher's collection. We note that for the cases **L** and **U** we are investigating in this thesis the definitions of A-class $\mathscr{C}_3$ in [As] and [KL] coincide.
Furthermore, we note that our definition of A-class $\mathscr{C}_3$ and that provided in the book [BHR] coincide with the definition in [KL].

From now on we use the standard notation. Next, an important fact is provided concerning the intersection of a member of A-class $\mathscr{C}_3$ of G (where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$) with the socle of G.

Proposition 2.3.3. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and let K be a member of A-class $\mathscr{C}_3$ of G. Then $K \cap \mathrm{P}\Omega(V)$ is a member of A-class $\mathscr{C}_3$ of $\mathrm{P}\Omega(V)$ of the same type as K.*

Proof. Our assertion follows from [KL, Proposition 3.1.3]. □

In the following proposition, we provide the facts about the structure and conjugacy of the members of A-class $\mathscr{C}_3$ of $\mathrm{P}\Omega(V)$. For this, we recall Definition 1.3.7 for the central product.

Proposition 2.3.4. *(i) $\mathrm{P}\Omega(V)$ acts transitively (by conjugation) on the members of A-class $\mathscr{C}_3$ of $\mathrm{P}\Omega(V)$ of type $\mathrm{GL}_t^{\epsilon}(q^{ur})$.*

(ii) Let K be a member of A-class $\mathscr{C}_3$ of $\mathrm{P}\Omega(V)$. Then $K = \mathrm{P}H$ where H is a member of A-class $\mathscr{C}_3$ of $\Omega(V)$ and the structure of H is as shown in the following table.

Case	*Type of H, K*	*Structure of H*
L	$\mathrm{GL}_t(q^r)$	$\left(\frac{(q-1,t)(q^r-1)}{q-1} \circ \mathrm{SL}_t(q^r)\right).\frac{(q^r-1,t)}{(q-1,t)}.r$
U	$\mathrm{GU}_t(q^{2r})$	$\left(\frac{(q+1,t)(q^r+1)}{q+1} \circ \mathrm{SU}_t(q^{2r})\right).\frac{(q^r+1,t)}{(q+1,t)}.r$

Proof. See [KL, Proposition 4.3.6 (I) and (II)] and [BHR, Table 2.6]. □

Now, we have provided the necessary information to start our investigations. Using the condition that the layer of $K \in \mathscr{C}_3$ of G is trivial if K is strongly constrained, we will obtain that there is only one case to consider.

Corollary 2.3.5. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* K *be a member of* $\mathscr{C}_3$ *of* G *of type* $\mathrm{GL}^{\epsilon}_t(q^{ur})$. *If* $\mathrm{E}(K) = 1$ *then* $t = 1$ *(esp.* $n = r$*). Furthermore, we have* $t = 1$ *if* K *is strongly constrained.*

Proof. Since there is a normal subgroup in $\frac{(q-\epsilon 1,t)(q^r-\epsilon 1)}{q-\epsilon 1} \circ \mathrm{SL}^{\epsilon}_t(q^{ur})$ isomorphic to $\mathrm{SL}^{\epsilon}_t(q^{ur})$, we obtain our assertion in view of Lemma 1.4.22 and Propositions 1.2.11, 1.2.12, 2.3.3 and 2.3.4 (cf. also [BHR, Lemma 2.2.7]). □

In the following proposition, we provide the information about the maximality in G of the members of A-class $\mathscr{C}_3$ of G of type $\mathrm{GL}^{\epsilon}_1(q^{un})$ where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$.

Proposition 2.3.6. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* M *be a member of* $\mathscr{C}_3$ *of* G *of type* $\mathrm{GL}^{\epsilon}_1(q^{un})$. *Then* M *is a maximal subgroup of* G *unless one of the following holds.*

(i) *Case* **L** *is given,*

  (a) $(n, q) \in \{(2, 7), (2, 9)\}$ *and* $G = \mathrm{P}\Omega(V)$ *if* $q = 7$, *or* $G \in \{\mathrm{P}\Omega(V), \mathrm{P}\Omega(V) : \langle\varphi_3\rangle\}$ *if* $q = 9$,

  (b) $(n, q) = (3, 4)$ *and* $G^x \leq \mathrm{P}\Omega(V) : \langle\varphi_2, \tau\rangle$ *for an* $x \in \mathrm{PA}(V)$.

(ii) *Case* **U** *is given,*

  (a) $(n, q^2) \in \{(3, 3^2), (3, 5^2)\}$ *and* $G \notin \{\mathrm{PI}(V), \mathrm{P}\Gamma(V)\}$ *if* $(n, q^2) = (3, 5^2)$,

  (b) $(n, q^2) = (5, 2^2)$.

In the excluded cases M *is not a maximal subgroup of* G.

Proof. The assertion follows by [BHR, Tables 8.1, 8.3, 8.5, 8.18, 8.20, 8.35, 8.37, 8.70 and 8.72] and [KL, Tables 3.5.A and 3.5.B]. For the exceptions see also [BHR, Propositions 3.1.4, 3.2.3, 6.3.11, 6.3.12 and 6.3.14]. □

Remark 2.3.7. (a) Concerning the exceptions occurring in the last proposition in case **U** for $(n, q^2) = (3, 5^2)$, see also Remark 2.2.10 (d).

(b) We note that a specification of a basis of V in case **U** is not necessary for a well-defined description of the results in the last proposition. (Recall the considerations in Section 1.2, esp. Subsection 1.2.2).

By the previous corollary, we only have to examine the cases where K is a member of A-class $\mathscr{C}_3$ of type $\mathrm{GL}^{\epsilon}_1(q^{un})$ for a prime n. These cases are in strong relation to the so called Singer subgroups which are well-known and investigated in several works, such as [Hu2].

Definition 2.3.8. (cf. [Ka] and [By])
An element s of $\mathrm{GL}^{\epsilon}_n(q^u)$ of order $q^n - \epsilon 1$ is called a *Singer cycle* of $\mathrm{GL}^{\epsilon}_n(q^u)$, and the cyclic subgroup of $\mathrm{GL}^{\epsilon}_n(q^u)$ generated by s is called a *Singer subgroup* of $\mathrm{GL}^{\epsilon}_n(q^u)$. The intersection of a Singer subgroup of $\mathrm{GL}^{\epsilon}_n(q^u)$ with $\mathrm{SL}^{\epsilon}_n(q^u)$ is called a *Singer subgroup* of $\mathrm{SL}^{\epsilon}_n(q^u)$.

In the following lemma, we provide some facts important for our further investigations.

Lemma 2.3.9. *Denote by* $S_{\mathrm{GL}_n^\epsilon(q^u)}$ *the image of the embedding of* $\mathrm{GL}_1^\epsilon(q^{un})$ *in* $\mathrm{GL}^\epsilon(V, \mathrm{GF}(q^u))$ *from* (2.3.1) *for* $\epsilon = +$ *and* (2.3.2) *for* $\epsilon = -$. *We define* $S_{\mathrm{SL}_n^\epsilon(q^u)} = S_{\mathrm{GL}_n^\epsilon(q^u)} \cap \mathrm{SL}^\epsilon(V, \mathrm{GF}(q^u))$. *Let* $\mathrm{SL}^\epsilon(V, \mathrm{GF}(q^u))$ *be quasisimple,* n *be a prime and note that* n *is odd by* (2.3.2) *in case* $\epsilon = -$. *Then the following hold.*

(a) $S_{\mathrm{GL}_n^\epsilon(q^u)}$ *is a Singer subgroup of* $\mathrm{GL}^\epsilon(V, \mathrm{GF}(q^u))$ *and* $S_{\mathrm{SL}_n^\epsilon(q^u)}$ *is a Singer subgroup of* $\mathrm{SL}^\epsilon(V, \mathrm{GF}(q^u))$.

(b) $|S_{\mathrm{GL}_n^\epsilon(q^u)}| = q^n - \epsilon 1$ *and* $|S_{\mathrm{SL}_n^\epsilon(q^u)}| = \frac{q^n - \epsilon 1}{q - \epsilon 1}$.

(c) $S_{\mathrm{GL}_n^\epsilon(q^u)}$ *and* $S_{\mathrm{SL}_n^\epsilon(q^u)}$ *act irreducibly on* $(V, \mathrm{GF}(q^u))$.

(d) $\mathrm{C}_{\mathrm{GL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{GL}_n^\epsilon(q^u)}\big) = S_{\mathrm{GL}_n^\epsilon(q^u)}$ *and* $\mathrm{C}_{\mathrm{GL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{SL}_n^\epsilon(q^u)}\big) = S_{\mathrm{GL}_n^\epsilon(q^u)}$.

(e) $\mathrm{N}_{\mathrm{GL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{GL}_n^\epsilon(q^u)}\big) \in \mathscr{C}_3$ *of* $\mathrm{GL}^\epsilon(V, \mathrm{GF}(q^u))$ *of type* $\mathrm{GL}_1^\epsilon(q^{un})$ *and* $\mathrm{N}_{\mathrm{SL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{SL}_n^\epsilon(q^u)}\big) \in \mathscr{C}_3$ *of* $\mathrm{SL}^\epsilon(V, \mathrm{GF}(q^u))$ *of type* $\mathrm{GL}_1^\epsilon(q^{un})$ *where* $\mathrm{N}_{\mathrm{GL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{GL}_n^\epsilon(q^u)}\big)/S_{\mathrm{GL}_n^\epsilon(q^u)}$ *and* $\mathrm{N}_{\mathrm{SL}^\epsilon(V,\mathrm{GF}(q^u))}\big(S_{\mathrm{SL}_n^\epsilon(q^u)}\big)/S_{\mathrm{SL}_n^\epsilon(q^u)}$ *are cyclic groups of order* n.

Proof. Since $\mathrm{GL}_1^\epsilon(q^{un})$ is a cyclic group of order $q^n - \epsilon 1$, assertion (a) is clear. We obtain assertion (b) for the case $S_{\mathrm{SL}_n(q)}$ analogously to [Hu, II. 7.3 Satz b)]. To prove assertion (b) for case $S_{\mathrm{SU}_n(q^2)}$, w.l.o.g. we can consider that $S_{\mathrm{GU}_n(q^2)} \leq S_{\mathrm{GL}_n(q^2)} \leq \mathrm{GL}(V, \mathrm{GF}(q^2))$. Since $S_{\mathrm{GL}_n(q^2)}$ is a cyclic group, we obtain that $|S_{\mathrm{SU}_n(q^2)}| = |S_{\mathrm{GU}_n(q^2)} \cap S_{\mathrm{SL}_n(q^2)}| = (q^n + 1, \frac{q^{2n}-1}{q^2-1}) = \frac{q^n+1}{q+1}$, by Lemma 1.5.19.
To prove assertion (c), it is sufficient to show that $S_{\mathrm{SL}_n^\epsilon(q^u)}$ acts irreducibly on $(V, \mathrm{GF}(q^u))$.[11] First, let $n = 2$, so case $\epsilon = +$ holds. Assume that $S_{\mathrm{SL}_2(q)}$ acts reducibly on $(V, \mathrm{GF}(q))$. Then there is a $\mathrm{GF}(q)$-basis of $(V, \mathrm{GF}(q))$ such that $S_{\mathrm{SL}_2(q)}$ is a subgroup of a member P of A-class $\mathscr{C}_1$ of $\mathrm{SL}_2(q)$ of type P_1, recall Definition 2.1.2. Hence, we obtain a contradiction by $|S_{\mathrm{SL}_2(q)}| = q + 1$ dividing $|P| = q(q-1)$. Assertion (c) for $n \geq 3$ follows analogously to [BHR, Lemma 2.3.14][12,13].
Next, we prove assertion (d) in case $\epsilon = +$. As in [Hu, II. 7.3 Satz a)], we obtain that $\mathrm{C}_{\mathrm{GL}(V,\mathrm{GF}(q))}\big(S_{\mathrm{GL}_n(q)}\big) = S_{\mathrm{GL}_n(q)}$. By Schur's Lemma (see Lemma 1.4.1) and part (c), we deduce that $C = \mathrm{C}_{\mathrm{GL}(V,\mathrm{GF}(q))}\big(S_{\mathrm{SL}_n(q)}\big)$ is isomorphic

[11] A direct proof for case $S_{\mathrm{GL}_n^\epsilon(q^u)}$ is also provided in [KL, Proposition 4.3.3. (i)], cf. also [Hu, II. 7.3 Satz a)]).

[12] Note, that the proof of [BHR, Lemma 2.3.14] also holds for $n \geq 13$, cf. also [KL, Lemma 7.3.2. (vii)].

[13] We note that there is another (more elegant) proof of [BHR, Lemma 2.3.14], and hence also for our assertion. Since there is a Zsigmondy prime $z_{q,un}$ dividing $|S_{\mathrm{SL}_n^\epsilon(q^u)}|$ (see Theorem 1.5.2), we can consider the unique subgroup S_0 of $S_{\mathrm{SL}_n^\epsilon(q^u)}$ of order $z_{q,un}$ acting on $(V, \mathrm{GF}(q^u))$. Using Maschke's Theorem (see [CR, (3.14)]), we now obtain an easier argumentation.

to the multiplicative group of a finite field $\mathrm{GF}(q^b)$ where $n \mid b$ (note that $S_{\mathrm{GL}_n(q)} \leq C$ and see [Hu2, Hilfssatz 2 b)]). Suppose that $n < b$. For $(n, q, b) \notin \{(2, 2, 6), (3, 2, 6)\}$ there is a Zsigmondy prime $z_{q,b}$ dividing $|C|$, see Theorem 1.5.2, which contradicts Proposition 1.2.13. Since $\mathrm{SL}_2(2)$ is not quasisimple and for $(n, q, b) = (3, 2, 6)$ we obtain a contradiction by $63 = |C| \mid |\mathrm{GL}(V, \mathrm{GF}(2))| = 168$, the assertion in case $\epsilon = +$ follows. To prove (d) in case $\epsilon = -$, w.l.o.g. we can assume $S_{\mathrm{SU}_n(q^2)} \leq S_{\mathrm{GU}_n(q^2)} \leq S_{\mathrm{GL}_n(q^2)}$. By analogous arguments as above, we obtain that $\mathrm{C}_{\mathrm{GL}(V,\mathrm{GF}(q^2))}\big(S_{\mathrm{GU}_n(q^2)}\big) = \mathrm{C}_{\mathrm{GL}(V,\mathrm{GF}(q^2))}\big(S_{\mathrm{SU}_n(q^2)}\big) = S_{\mathrm{GL}_n(q^2)}$. Considering the intersection with $\mathrm{GU}(V, \mathrm{GF}(q^2))$ now yields the assertion, using part (c) and [Hu2, Satz 4 a)].
Assertion (e) follows by elementary considerations, using Theorem 1.5.2, [Hu, II. 7.3 Satz] and Propositions 2.3.3 and 2.3.4. (Use also the elementary observations in the following remark). □

We note two easy conclusions from the last proof.

Remark 2.3.10. (a) We consider the situation of the last lemma and w.l.o.g. we assume that $S_{\mathrm{SU}_n(q^2)} \leq S_{\mathrm{GU}_n(q^2)} \leq S_{\mathrm{GL}_n(q^2)} \leq \mathrm{GL}(V, \mathrm{GF}(q^2))$. Then, analogously to the last proof, we obtain that $S_{\mathrm{GL}_n(q^2)} \cap \mathrm{GU}(V, \mathrm{GF}(q^2)) = S_{\mathrm{GU}_n(q^2)}$, and so $S_{\mathrm{GL}_n(q^2)} \cap \mathrm{SU}(V, \mathrm{GF}(q^2)) = S_{\mathrm{SU}_n(q^2)}$.

(b) Since $\mathrm{Z}(\mathrm{GL}^\epsilon(V, \mathrm{GF}(q^u))) \leq S_{\mathrm{GL}^\epsilon_n(q^u)}$ (see (2.3.1) and (2.3.2), or use Lemma 2.3.9 (d)), we can deduce that $\mathrm{Z}(\mathrm{SL}^\epsilon(V, \mathrm{GF}(q^u))) \leq S_{\mathrm{SL}^\epsilon_n(q^u)}$. Hence, by Lemma 2.3.9 (b), we obtain that $|\mathrm{P}S_{\mathrm{GL}^\epsilon_n(q^u)}| = \frac{q^n-\epsilon 1}{q-\epsilon 1}$ and $|\mathrm{P}S_{\mathrm{SL}^\epsilon_n(q^u)}| = \frac{q^n-\epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)}$, cf. also [KL, Proposition 4.3.6. (II)].

By the following proposition, we shall see that there is a necessary number theoretic condition for a member of A-class $\mathscr{C}_3$ to be strongly constrained.

Proposition 2.3.11. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and $K \in \mathscr{C}_3$ of G be strongly constrained (so, K is of type $\mathrm{GL}^\epsilon_1(q^{un})$, by Corollary 2.3.5). Then*

$$\frac{q^n - \epsilon 1}{(q - \epsilon 1)(n, q - \epsilon 1)} = p^b \tag{2.3.3}$$

holds for a prime p and a positive integer b. Moreover, K is strongly p-constrained.

Proof. We have that $K \cap \mathrm{P}\Omega(V) \in \mathscr{C}_3$ of $\mathrm{P}\Omega(V)$ is a normal subgroup of K, by Proposition 2.3.3. In view of Proposition 2.3.4 (i), Lemma 2.3.9 (e) and Remark 2.3.10 (b), we see that there is a cyclic normal subgroup S of $K \cap \mathrm{P}\Omega(V)$ of order $\frac{q^n-\epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)}$. By elementary calculations, we see that $|S| > 1$ (note, that $\mathrm{PSU}_3(2^2)$ is not simple). Using Lemmas 1.4.21 and 1.4.13, we now obtain our assertion. □

The number theoretic consequences arising from (2.3.3) are examined in Propositions 1.5.7 and 1.5.8. These observations will be useful for our further considerations.

Next, we provide the information concerning the largest normal p-subgroup of a strongly p-constrained member of A-class $\mathscr{C}_3$ for a prime p.

Proposition 2.3.12. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and* $K \in \mathscr{C}_3$ *of* G *of type* $\mathrm{GL}_1^{\epsilon}(q^{un})$. *Let* p *be a prime and* b *be a positive integer such that (2.3.3) holds (esp. these conditions are satisfied if* K *is strongly* p*-constrained). Then the following hold.*

(i) For $n \geq 3$ *we have that* $\mathrm{O}_p(K) = \mathrm{O}_p(K \cap \mathrm{P\Omega}(V))$ *is cyclic of order* $\frac{q^n - \epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)}$. *More precisely,* $\mathrm{O}_p(K)$ *is the projective image of a Singer subgroup of* $\Omega(V)$.

(ii) For $n = 2$ *(hence, we have case* **L***) one of the cases from Proposition 1.5.7 (iii) holds, and we have the following.*

(a) For the cases from Proposition 1.5.7 (iii)(a) and (b) and if $q \neq 8$ *we have that* $\mathrm{O}_p(K) = \mathrm{O}_p(K \cap \mathrm{PSL}(V))$ *is cyclic of order* $\frac{q+1}{(2,q-1)}$. *More precisely,* $\mathrm{O}_p(K)$ *is the projective image of a Singer subgroup of* $\mathrm{SL}(V)$.

(b) For the case of Proposition 1.5.7 (iii)(c) we have that $p = 2$ *and* $\mathrm{O}_2(K) = K$.

(c) For $q = 8$ *we have that* $p = 3$ *and* $\mathrm{O}_3(K)$ *is cyclic of order* 9 *if* $G = \mathrm{PSL}(V)$ *or* $\mathrm{O}_3(K) \cong \boldsymbol{Z}_9 \rtimes \boldsymbol{Z}_3$ *and* $\mathrm{O}_3(K) > \mathrm{O}_3(K \cap \mathrm{PSL}(V))$ *if* $G = \mathrm{P\Gamma L}(V)$.

Proof. Analogously to the proof of Proposition 2.3.11, we obtain $K \cap \mathrm{P\Omega}(V) = S.N \in \mathscr{C}_3$ of $\mathrm{P\Omega}(V)$ for a cyclic p-group S of order $\frac{q^n-\epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)}$ and a cyclic group N of prime order n. (Clearly, S is the projective image of a Singer subgroup of $\Omega(V)$).
First, we prove assertion (i). Since $|K/(K \cap \mathrm{P\Omega}(V))| \mid |\mathrm{Out}(\mathrm{P\Omega}(V))|$, we easily obtain $\mathrm{O}_p(K) = \mathrm{O}_p(K \cap \mathrm{P\Omega}(V)) = S$, using Propositions 1.5.7 (iv) and 1.5.8 (iii).
Now, let $n = 2$. Assertion (ii)(a) follows analogously to above, by Proposition 1.5.7 (iii)(a) and (b). As a direct consequence of Proposition 1.5.7 (iii)(c), we obtain assertion (ii)(b). So, it remains to examine the case $q = 8$. The assertion for $G = \mathrm{PSL}(V)$ is clear. Hence, let $G = \mathrm{P\Gamma L}(V)$. By x, we denote a primitive element of $\mathrm{GF}(8)^*$. In view of Proposition 2.3.4 (i), w.l.o.g. we can choose an appropriate $\mathrm{GF}(8)$-basis of V and consider the situation $K \cap \mathrm{PSL}_2(8) = S \rtimes N$ where

$$S = \langle s \rangle \text{ for } s = \begin{bmatrix} x^2 & x \\ x & x^4 \end{bmatrix} \text{ and } N = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle.$$

(Note, that $\mathrm{o}(s) = 9$ and S is isomorphic to a Singer subgroup of $\mathrm{SL}_2(8)$). Since we see by elementary calculations that $s^{\varphi_2} = s^7$, we obtain $K = (S \rtimes N) \rtimes \langle \varphi_2 \rangle$. Now, assertion (ii)(c) follows by elementary considerations (or using [GAP]). □

We will continue our investigations by generalizing the considerations from Lemma 2.3.9 (d). For this, we note the following lemma.

Lemma 2.3.13. *Let m be a prime and a be a positive integer. Let V be an n-dimensional $\mathrm{GF}(m^a)$-vector space. Then we have an embedding (via identification)*

$$\Gamma\mathrm{L}_n(m^a) \cong \Gamma\mathrm{L}(V, \mathrm{GF}(m^a)) \leq \mathrm{GL}(V, \mathrm{GF}(m)) \cong \mathrm{GL}_{na}(m).$$

Proof. From the beginning of this section we recall that the n-dimensional $\mathrm{GF}(m^a)$-vector space V can also be considered as an (na)-dimensional $\mathrm{GF}(m)$-vector space, and so we have an embedding

$$\mathrm{GL}_n(m^a) \cong \mathrm{GL}(V, \mathrm{GF}(m^a)) \leq \mathrm{GL}(V, \mathrm{GF}(m)) \cong \mathrm{GL}_{na}(m).$$

Recalling the introduction of a semilinear transformation of V from page 12 and the considerations following after that (cf. also the introduction of the field automorphisms of $\mathrm{PSL}(V)$ in Subsection 1.2.2 together with Proposition 1.2.15), we obtain our assertion. (We note that for $\eta \in \Gamma\mathrm{L}(V, \mathrm{GF}(m^a))$, $\lambda \in \mathrm{GF}(m) \leq \mathrm{GF}(m^a)$ and $v \in V$ we have $(\lambda v)^\eta = \lambda v^\eta$). □

Proposition 2.3.14. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and $K \in \mathscr{C}_3$ of G of type $\mathrm{GL}_1^\epsilon(q^{un})$. Let p be a prime and b be a positive integer such that (2.3.3) holds. Then we have the following.*

(i) For $n \geq 3$ and $\mathrm{P}\Omega(V) \not\cong \mathrm{PSL}_3(4)$ we have that

$$C = \mathrm{C}_{\mathrm{P}\Gamma(V)}(\mathrm{O}_p(K)) = \mathrm{O}_p(K) \times S$$

where $S \cong \boldsymbol{Z}_{(n,q-\epsilon 1)}$ and C is the projective image of a Singer subgroup of $\mathrm{I}(V)$. If $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_3(4)$ then $p = 7$ and

$$C = \mathrm{C}_{\mathrm{PA}(V)}(\mathrm{O}_7(K)) = (\mathrm{O}_7(K) \times S_1) \rtimes S_2$$

where $S_1 \cong \boldsymbol{Z}_3$, $S_2 \cong \boldsymbol{Z}_2$, $C \leq \mathrm{P}\Gamma(V)$ and $C \cap \mathrm{PI}(V) = \mathrm{O}_7(K) \times S_1$ is the projective image of a Singer subgroup of $\mathrm{I}(V)$.

(ii) For $n = 2$ and if q is not a Mersenne prime we have that

(a) $C = \mathrm{C}_{\mathrm{PA}(V)}(\mathrm{O}_p(K)) = \mathrm{O}_p(K) \times S$ where $S \cong \boldsymbol{Z}_{(2,q-1)}$ and C is the projective image of a Singer subgroup of $\mathrm{I}(V)$ excluding the case $G = \mathrm{PA}(V)$ for $q = 8$, or

(b) $q = 8$, $G = \mathrm{PA}(V)$, $p = 3$ and $C = \mathrm{C}_{\mathrm{PA}(V)}(\mathrm{O}_3(K)) < \mathrm{O}_3(K) \cap \mathrm{P}\Omega(V)$ where $C \cong \boldsymbol{Z}_3$.

Proof. First, in case **L** let $(n, q) \notin \{(2, 8), (3, 4)\}$, and note that the case of Proposition 1.5.7 (iii)(c) does not hold. In view of Lemma 2.3.9 (e) and Propo-

sitions 2.3.3, 2.3.4 and 2.3.12, w.l.o.g. we can assume that $\mathrm{O}_p(K) = \mathrm{P}S_{\mathrm{SL}^\epsilon_n(q^u)} \leq \mathrm{P}S_{\mathrm{GL}_n(q^u)}$, using the notation from Lemma 2.3.9. We note that p does not divide $|\mathrm{Z}(\mathrm{GL}(V, \mathrm{GF}(q^u)))|$, by Propositions 1.5.7 and 1.5.8. So, regarding Lemmas 1.4.6 and 1.4.8, we will obtain our assertion by determining $\mathrm{C}_{\Gamma(V)}(H)$ where H is the Sylow p-subgroup of $\widehat{\mathrm{O}_p(K)}$. (Note, that $|H| = |\mathrm{O}_p(K)| = \frac{q^n - \epsilon 1}{(q-\epsilon 1)(n, q-\epsilon 1)}$ and $H \leq S_{\mathrm{GL}_n(q^u)}$). Let the characteristic of $\mathrm{GF}(q^u)$ be m and $q = m^a$ for a positive integer a. We consider the embedding $H \leq \Gamma(V) \leq \Gamma\mathrm{L}(V, \mathrm{GF}(q^u)) \leq \mathrm{GL}(V, \mathrm{GF}(m)) \cong \mathrm{GL}_{nau}(m)$ (via identification) from Lemma 2.3.13.
We show that H acts irreducibly on the (nau)-dimensional $\mathrm{GF}(m)$-vector space V. Therefore, suppose that H acts reducibly. Using Maschke's Theorem (see [CR, (3.14)]), we obtain that $|H|$ divides $\prod_{i=1}^{l} |\mathrm{GL}_{k_i}(m)|$ for integers k_i where $1 \leq k_i < nau$ and $\sum_{i=1}^{l} k_i = nau$. For $n \geq 3$ we have by Propositions 1.5.7 (i)(b) and 1.5.8 (i) that $p = z_{m,nau}$ is a Zsigmondy prime (recall, that $(n, q) \neq (3, 4)$ in case **L**). This is a contradiction, regarding Proposition 1.2.13. For $n = 2$ (so, we have case **L**) one of the cases from Proposition 1.5.7 (iii) (a) or (b) holds (recall, that $q \neq 8$). Analogously to the previous case, we obtain a contradiction (note, that $\mathrm{PSL}_2(2)$ is not simple and for $a = 1$ see Lemma 2.3.9 (c)). Hence, H acts irreducibly on the (nau)-dimensional $\mathrm{GF}(m)$-vector space V. Using Schur's Lemma, see Lemma 1.4.1, we now obtain that $C_1 = \mathrm{C}_{\mathrm{GL}(V, \mathrm{GF}(m))}(H)$ is isomorphic to the multiplicative group of a finite field. Since $S_{\mathrm{GL}_n(q^u)}$ centralizes H, we obtain that $m^{nau} - 1$ divides $|C_1| = m^d - 1$. Suppose that $d > nau$, and note that $nau \mid d$, by [Hu2, Hilfssatz 2 b)]. Then there is a Zsigmondy prime $z = z_{m,d}$ dividing $|C_1|$ if $(n, q, d) \neq (3, 2, 6)$ (in case **L**), by Theorem 1.5.2. So, for $(n, q, d) \neq (3, 2, 6)$ we obtain a contradiction to z dividing $|\mathrm{GL}(V, \mathrm{GF}(m))|$. We now can deduce that $|C_1| = |S_{\mathrm{GL}_n(q^u)}|$ (for case $(n, q, d) = (3, 2, 6)$ see Lemma 2.3.9 (d)), and our assertion for the considered cases follows easily, in view of Remark 2.3.10.
Next, let $(q, n) = (4, 3)$ in case **L**. Here, we have $p = 7$. By analogous arguments as above, w.l.o.g. we can set $\mathrm{O}_7(K) = \langle A \rangle$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $K \cap \mathrm{P}\Omega(V) = \mathrm{N}_{\mathrm{P}\Omega(V)}(\mathrm{O}_7(K)) = \mathrm{O}_7(K) \rtimes \langle N \rangle \in \mathscr{C}_3$ of $\mathrm{P}\Omega(V)$ where $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. It is not hard to see that φ_2 centralizes $\mathrm{O}_7(K)$ and $Q = \tau \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ normalizes $\mathrm{O}_7(K)$. By elementary calculations and observations where we note that $A^N = A^4$ and $A^Q = A^{-1}$, we now obtain our assertion (or alternatively use [GAP]).
Finally, let $(n, q) = (2, 8)$. Here, we have that $G \in \{\mathrm{P}\Omega(V), \mathrm{PA}(V)\}$. By analogous considerations as above, w.l.o.g. we may write $K \cap \mathrm{P}\Omega(V) = S \rtimes N$ where $S = \langle s \rangle$ and N are as in the proof of Proposition 2.3.12 (ii)(c). Our assertion

now follows by elementary observations where we use that $s^3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (or alternatively use [GAP]). ☐

Next, we determine exact conditions for a member $K \in \mathscr{C}_3$ to be strongly constrained. We note that here we do not require K to be a maximal subgroup of the respective group.

Proposition 2.3.15. *Let* $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ *and let* K *be a member of A-class* $\mathscr{C}_3$ *of* G. *For a prime* p, K *is strongly* p*-constrained if and only if* K *is of type* $\mathrm{GL}_1^{\epsilon}(q^{un})$ *and one of the following holds.*

(i) We have $n = 2$ *(hence, case* **L** *holds) and*

- *(a)* $\frac{q+1}{(2,q-1)} = p^b$ *for a positive integer* b *and*
- *(b)* $G \cap \mathrm{PI}(V) = \mathrm{P\Omega}(V)$ *except if* q *is a Mersenne prime.*

(ii) We have $n \geq 3$ *and*

- *(a)* $\frac{q^n - \epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)} = p^b$ *for a positive integer* b,
- *(b)* $G \cap \mathrm{PI}(V) = \mathrm{P\Omega}(V)$ *and*
- *(c)* $G \cap \mathrm{P\Gamma}(V) = \mathrm{P\Omega}(V)$ *if* $(n,q) = (3,4)$ *and case* **L** *holds.*

Proof. First, recall that K has to be of type $\mathrm{GL}_1^{\epsilon}(q^{un})$ if K is strongly constrained, see Corollary 2.3.5.
Let $n = 2$. We note that if $\frac{q+1}{(2,q-1)} = p^b$ holds then one of the cases (a)-(c) from Proposition 1.5.7 (iii) occurs; moreover in the first two cases (a) and (b) p is an odd prime, and in the latter case (c) K is a 2-group which is always strongly 2-constrained. So, our assertion follows by Propositions 2.3.3 and 2.3.11 - 2.3.14.
Next, we consider the case $n \geq 3$. If $(n,q) = (3,4)$ and case **L** holds we obtain our assertion by Example 1.2.23 and Propositions 2.3.3, 2.3.4 (i) and 2.3.11 - 2.3.14, cf. also [KL, proof of Proposition 4.3.8.]. (Here, we note that $G \cap \mathrm{P\Gamma}(V) = \mathrm{P\Omega}(V)$ if and only if G is $\mathrm{PA}(V)$-conjugate to one of the groups occurring in Example 1.2.23 (1.), (3.), or (4.)). So, let $(n,q) \neq (3,4)$ in case **L**. The only-if-part is a consequence of Propositions 1.5.7 (i)(a), 1.5.8 (i) and 2.3.11 - 2.3.14.
It remains to establish the if-part. Again, by Propositions 2.3.12 and 2.3.14, we obtain this assertion in case **U** and if $K \leq \mathrm{P\Gamma}(V)$ also in case **L**. Hence, let case **L** hold and $K \not\leq \mathrm{P\Gamma}(V)$ (esp. $|K/(K \cap \mathrm{P\Gamma}(V))| = 2$). Let $q = m^a$ for a prime m and a positive integer a. Since $|K/(K \cap \mathrm{P\Omega}(V))|$ divides $|G/\mathrm{P\Omega}(V)| = 2a$, we obtain by Propositions 2.3.3 and 2.3.4 that $|K|$ divides $2anp^b$ (cf. also Lemma 2.3.9 (e) and Remark 2.3.10). We will prove that K is strongly p-constrained by showing that $r = p$ if $\mathrm{O}_r(K) > 1$ for a prime r. Suppose that $\mathrm{O}_r(K) > 1$ for a prime $r \neq p$. If r is odd then r divides na. Since $K \cap \mathrm{P\Gamma}(V)$ is a normal subgroup of K of index 2, we have that $\mathrm{O}_r(K) \leq K \cap \mathrm{P\Gamma}(V)$. Because $\mathrm{O}_r(K)$ and $\mathrm{O}_p(K)$ centralize each other, we easily obtain a contradiction by Proposition 2.3.14 (i). Now, suppose that $\mathrm{O}_2(K) > 1$. Since all Sylow 2-subgroups

of K have order 2 (see Proposition 1.5.7), we have that there is exactly one (esp. we have a non-trivial involutory element in K centralizing K). Obviously, $K \leq \mathrm{N}_G(\mathrm{O}_p(K)) = N$, and by analogous arguments as above we have that $|N|$ divides $2anp^b$ where we note that $N \cap \mathrm{P}\Omega(V) = \mathrm{N}_{\mathrm{P}\Omega(V)}(\mathrm{O}_p(K)) \in \mathscr{C}_3$ of $\mathrm{P}\Omega(V)$, by Lemma 2.3.9 (e), Remark 2.3.10 (b) and Propositions 2.3.4 (i) and 2.3.12. Regarding again Propositions 2.3.4 (i) and 2.3.12, we see that there is an element $A_0\tau \in N$ for an appropriate $A_0 \in \mathrm{P}\Omega(V)$ (cf. also [KL, (3.1.1)]). Clearly $(A_0\tau)^2 \in \mathrm{N}_{\mathrm{P}\Omega(V)}(\mathrm{O}_p(K))$, and so we can deduce by the Sylow theorems that there is an element $A \in \mathrm{P}\Omega(V)$ where $A\tau \in \langle A_0\tau\rangle \leq N$ and $\mathrm{o}(A\tau) = 2$ (note, that $|\mathrm{N}_{\mathrm{P}\Omega(V)}(\mathrm{O}_p(K))|$ is odd). Because N has Sylow 2-subgroups of order 2, w.l.o.g. we can consider $\mathrm{O}_2(K) = \langle A\tau\rangle$ (again by the Sylow theorems). In view of Lemma 1.4.8, we now obtain that there is an element $\tilde{A}\tau \in \widehat{A\tau}$ centralizing H where $\tilde{A} \in \Omega(V)$ appropriate and H is the Sylow p-subgroup of $\widehat{\mathrm{O}_p(K)}$ (of order $|\mathrm{O}_p(K)|$). Let $H = \langle S\rangle$ for $S \in \Omega(V)$. From now on we choose a $\mathrm{GF}(q)$-basis B of V and consider the representation with respect to B. By $\tilde{A}, S \in \Omega = \mathrm{SL}_n(q)$, we also denote the representations of $\tilde{A}$ and S with respect to B. Since $(\tilde{A}\tau)^2 = \lambda \mathbb{1}_n$ for a $\lambda \in \mathrm{GF}(q)^*$, we obtain $\tilde{A} = \tilde{A}^t \cdot \lambda \mathbb{1}_n$. So, we can deduce $\lambda \in \{\pm 1\}$. We now distinguish the two possibilities for q being odd or even. First, let q be an odd prime power. For $\lambda = -1$ we obtain a contradiction by $\det(\tilde{A}) = 0$, since $\det(\tilde{A}) = (-1)^n \det(\tilde{A}^t) = -\det(\tilde{A})$.[14] Next, we consider $\lambda = 1$, so $\tilde{A} = \tilde{A}^t$. Here, $\tilde{A}$ determines a non-degenerate symmetric bilinear form f on V, and furthermore f determines a non-degenerate quadratic from Q on V such that $f = f_Q$, see Remark 1.2.7 (c). Since $S^{\tilde{A}\tau} = S$, we can easily deduce $S\tilde{A}S^t = \tilde{A}$. So, regarding [KL, Lemma 2.1.8. and (2.5.2)], we see that $S \in \mathrm{O}_n^\circ(q) \cong \mathrm{I}(V, Q)$ (recall the notation from Table 1.2.1). Because a Zsigmondy prime $z_{q,n}$ divides $|H|$, see Proposition 1.5.7 (i)(a), we now obtain a contradiction to Proposition 1.2.13 (d).[15] Finally, let q be an even prime power. For this case we have the condition $\tilde{A} = \tilde{A}^t$. We define V' to be the $\mathrm{GF}(q^2)$-span of B where $\mathrm{GF}(q^2)$ denotes a field extension of $\mathrm{GF}(q)$ of degree 2. Clearly, V is a subset of V' and V' is an n-dimensional $\mathrm{GF}(q^2)$-vector space. Regarding Remark 1.2.7 (a), we see that $\tilde{A}$ induces on V' a non-degenerate unitary form f'. Because we have $S^{\tilde{A}\tau} = S$, we easily obtain $S\tilde{A}S^t = \tilde{A}$. Hence, $S \in \Omega = \mathrm{SL}_n(q)$ can be considered (extended linearly) as an element in $\mathrm{SU}_n(q^2) \cong \Omega(V', f')$, see Lemma 1.2.8 (a). As above, we have that a Zsigmondy prime $z_{q,n}$ divides $|H|$. But this leads to a contradiction, considering Lemma 1.5.5. So, our assertion is established. □

Summarizing the results of the last proposition and the facts from Proposition 2.3.6, we obtain the main theorems of this section. We determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_3$ of G. As usual, we present the following results not using our standard notation.

[14] The fact that $\det(\tilde{A}) = 0$ is also clear by considering Remark 1.2.7 (b). Since $\tilde{A}$ determines a symplectic form f on V, we obtain by Proposition 1.2.3 (ii) that f is degenerate.

[15] Here, cf. also [Hu2, Satz 3 a)] together with Lemma 2.3.9 (c) and Proposition 2.3.12.

Main Theorem 2.3.16. *Let* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ *where* $\mathrm{PSL}_n(q)$ *is simple. Let* M *be a member of A-class* $\mathscr{C}_3$ *of* G. *Let* p *be a prime. Then* M *is a strongly* p*-constrained maximal subgroup of* G *if and only if* M *is of type* $\mathrm{GL}_1(q^n)$ *and one of the following holds.*

(i) We have $n = 2$ *and*

- *(a)* $\frac{q+1}{(2,q-1)} = p^b$ *for a positive integer* b,
- *(b)* $G \cap \mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$ *if* q *is not a Mersenne prime,*
- *(c) if* $(n, q) = (2, 7)$ *then* $G = \mathrm{PGL}_2(7)$ *and*
- *(d) if* $(n, q) = (2, 9)$ *then* $G = \mathrm{PSL}_2(9)\langle \varphi_3 W \rangle$.

(ii) We have $n \geq 3$ *and*

- *(a)* $\frac{q^n-1}{(q-1)(n,q-1)} = p^b$ *for a positive integer* b,
- *(b)* $G \cap \mathrm{PGL}_n(q) = \mathrm{PSL}_n(q)$ *and*
- *(c)* $(n, q) \neq (3, 4)$.

Proof. Our assertion follows by Propositions 2.3.6 and 2.3.15; for the exception $(n, q) = (3, 4)$ see also Example 1.2.23. □

Main Theorem 2.3.17. *Let* $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ *where* $\mathrm{PSU}_n(q^2)$ *is simple and* $n \geq 3$. *Let* M *be a member of A-class* $\mathscr{C}_3$ *of* G. *Let* p *be a prime. Then* M *is a strongly* p*-constrained maximal subgroup of* G *if and only if* M *is of type* $\mathrm{GU}_1(q^{2n})$ *and*

- *(a)* $\frac{q^n+1}{(q+1)(n,q+1)} = p^b$ *for a positive integer* b,
- *(b)* $G \cap \mathrm{PGU}_n(q^2) = \mathrm{PSU}_n(q^2)$ *and*
- *(c)* $(n, q^2) \notin \{(3, 3^2), (3, 5^2), (5, 2^2)\}$.

Proof. The assertion is a direct consequence of Propositions 2.3.6 and 2.3.15. □

Remark. Concerning well-definedness in the last main theorem, we note that we do not have to specify a matrix of the non-degenerate unitary form for the description of the results. (Recall the considerations in Section 1.2, esp. Subsection 1.2.2).

2.4 A-class $\mathscr{C}_4$

Here, we consider the members of A-class $\mathscr{C}_4$. Roughly described, the members of A-class $\mathscr{C}_4$ are the stabilizers of tensor product decompositions $V = V_1 \otimes V_2$ where $\dim(V_1) < \dim(V_2)$. As we will see below, there are no members of A-class $\mathscr{C}_4$ of G (for $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$) which are strongly constrained maximal subgroups of G. So, we will introduce the members of this A-class more briefly and follow the introduction provided in [KL, p. 126-128]. For the following introduction of the members of A-class $\mathscr{C}_4$ we use again the generalized standard notation.

First, we recall the terminology introduced in Subsection 1.3.2, esp. the terminology of tensor products of $\mathrm{GF}(q)$-vector spaces and of linear maps. Furthermore, we recall the notation introduced in Subsection 1.2.1, esp. from Table 1.2.1.
Since it is advantageous for our further examinations, we will introduce the following terminology more general then needed for the definition of the members of A-class $\mathscr{C}_4$. Let $t \geq 1$ be an integer and for $1 \leq i \leq t$ let V_i be a $\mathrm{GF}(q^u)$-vector space of dimension $d_i \geq 1$. In this section we identify the $\mathrm{GF}(q^u)$-vector space V of dimension n in the (generalized) standard notation with the tensor product space $V_1 \otimes \ldots \otimes V_t$. So, we have that $n = \prod_{i=1}^{t} d_i$, and w.l.o.g. we can assume $d_i > 1$.
Let case **L** be given. Recalling Subsection 1.3.2 (esp. (1.3.3)), we see that there is a natural embedding

$$\begin{aligned} \mathrm{GL}_{d_1}(q) \circ \ldots \circ \mathrm{GL}_{d_t}(q) &\cong \mathrm{GL}(V_1, \mathrm{GF}(q)) \otimes \ldots \otimes \mathrm{GL}(V_t, \mathrm{GF}(q)) \\ &\leq \mathrm{GL}(V, \mathrm{GF}(q)) \cong \mathrm{GL}_n(q) \end{aligned} \tag{2.4.1}$$

(cf. also [KL, (4.4.1)]).
In case **U** let f_i be a non-degenerate unitary form on $(V_i, \mathrm{GF}(q^2))$. It is possible to define a non-degenerate unitary form $f = f_1 \otimes \ldots \otimes f_t$ on $(V, \mathrm{GF}(q^2))$ via $f(v_1 \otimes \ldots \otimes v_t, w_1 \otimes \ldots \otimes w_t) = \prod_{i=1}^{t} f_i(v_i, w_i)$ where $v_i, w_i \in V_i$ and then extending sesquilinearly (cf. [As, (9.1)]). So, we obtain an embedding

$$\begin{aligned} \mathrm{GU}_{d_1}(q^2) \circ \ldots \circ \mathrm{GU}_{d_t}(q^2) &\cong \mathrm{I}(V_1, \mathrm{GF}(q^2), f_1) \otimes \ldots \otimes \mathrm{I}(V_t, \mathrm{GF}(q^2), f_t) \\ &\leq \mathrm{I}(V, \mathrm{GF}(q^2), f) \cong \mathrm{I}(V) \cong \mathrm{GU}_n(q^2) \end{aligned} \tag{2.4.2}$$

(cf. [KL, (4.4.4)]). (Here, also recall the notation $\mathrm{I}(V)$ in our generalized standard notation, see Subsection 1.2.3).
Recalling Proposition 1.2.3 (i), w.l.o.g. we can assume that the non-degenerate unitary form on $(V, \mathrm{GF}(q^2))$ in case **U** coincides with the previously described form f. To obtain consistency, let in case **L** f_i denote the trivial form on $(V_i, \mathrm{GF}(q))$ such that $f = f_1 \otimes \ldots \otimes f_t$ is the trivial form on $(V, \mathrm{GF}(q))$. Hence, we can write

$$\begin{aligned} (V, \mathrm{GF}(q^u), f) &= (V_1 \otimes \ldots \otimes V_t, \mathrm{GF}(q^u), f_1 \otimes \ldots \otimes f_t) \\ &= (V_1, \mathrm{GF}(q^u), f_1) \otimes \ldots \otimes (V_t, \mathrm{GF}(q^u), f_t), \end{aligned} \tag{2.4.3}$$

and following Kleidman and Liebeck we call this expression a *tensor (product) decomposition* of $(V, \mathrm{GF}(q^u), f)$. We will also denote such a decomposition by $\mathcal{D}_t$. For the following we recall the notation $\Delta(V)$ in our generalized standard notation (see Subsection 1.2.3), and also the notation $\Delta(V, \mathrm{GF}(q^u), f)$ (see Subsection 1.2.1, esp. Table 1.2.1 and the note following (1.2.3)). As described in [KL, p. 128], there is a natural embedding

$$\begin{aligned} \Delta_t(V) := \Delta(V_1, \mathrm{GF}(q^u), f_1) \otimes \ldots \otimes \Delta(V_t, \mathrm{GF}(q^u), f_t) \\ \leq \Delta(V, \mathrm{GF}(q^u), f) = \Delta(V). \end{aligned} \tag{2.4.4}$$

(Note, that the last embedding coincides with (2.4.1) in case **L**). For the following we recall the homomorphism σ from (1.2.1) and (1.2.6). Let σ_f denote the respective homomorphism from $\Gamma(V, \mathrm{GF}(q^u), f)$ to $\mathrm{Aut}(\mathrm{GF}(q^u))$, and σ_{f_i} the respective homomorphism from $\Gamma(V_i, \mathrm{GF}(q^u), f_i)$ to $\mathrm{Aut}(\mathrm{GF}(q^u))$. Furthermore, by σ we denote the homomorphism from $\Gamma\mathrm{L}(V, \mathrm{GF}(q^u))$ to $\mathrm{Aut}(\mathrm{GF}(q^u))$. So, $\sigma_f = \sigma$ in case **L**. Let $\phi_i \in \Gamma(V_i, \mathrm{GF}(q^u), f_i)$ with $\sigma_{f_i}(\phi_i) = \varphi$ where $\mathrm{Aut}(\mathrm{GF}(q^u)) = \langle\varphi\rangle$. We define $\phi_{\mathcal{D}_t} = \phi_1 \otimes \ldots \otimes \phi_t$ as the unique element of $\Gamma\mathrm{L}(V, \mathrm{GF}(q^u))$ which satisfies $\sigma(\phi_{\mathcal{D}_t}) = \varphi$ and $(v_1 \otimes \ldots \otimes v_t)\phi_{\mathcal{D}_t} = v_1\phi_1 \otimes \ldots \otimes v_t\phi_t$ for $v_i \in V_i$. Then we have that $\phi_{\mathcal{D}_t} \in \Gamma(V, \mathrm{GF}(q^u), f) = \Gamma(V)$ (and so clearly $\sigma(\phi_{\mathcal{D}_t}) = \sigma_f(\phi_{\mathcal{D}_t})$) and $\phi_{\mathcal{D}_t}$ normalizes $\Delta_t(V)$, see [KL, p. 128]. So, we define

$$\Gamma(V)_{(\mathcal{D}_t)} := \Delta_t(V)\langle\phi_{\mathcal{D}_t}\rangle \leq \Gamma(V, \mathrm{GF}(q^u), f) = \Gamma(V) \qquad (2.4.5)$$

where we note that $\Gamma(V)_{(\mathcal{D}_t)}$ is independent of the choice of the ϕ_i. Finally, we define for $G \leq \Gamma(V)$ that $G_{(\mathcal{D}_t)} = G \cap \Gamma(V)_{(\mathcal{D}_t)}$.

Now, we have provided the terminology and notation to introduce the members of A-class $\mathscr{C}_4$. For the following definition we use the notation previously introduced in this section. Furthermore, we recall that we use the generalized standard notation.

Definition 2.4.1. $\{$***A-class*** $\mathscr{C}_4\}$ (cf. [KL, p. 60 and Definition p. 128])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and K be a subgroup of G. Let f denote the non-degenerate unitary form on V in case **U** or the trivial form on V in case **L**. If $G \leq \Gamma(V)$ the subgroup K belongs to *(A-class)* $\mathscr{C}_4$ of G if $K = G_{(\mathcal{D}_2)}$ where $(V, \mathrm{GF}(q^u), f) \cong (V_1 \otimes V_2, \mathrm{GF}(q^u), f_1 \otimes f_2)$ and $\mathcal{D}_2$ is as described in the following table. For $G \not\leq \Gamma(V)$ the subgroup K belongs to *(A-class)* $\mathscr{C}_4$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_4$ of $\Gamma(V)$.

Case	Type	Description of $\mathcal{D}_2$
L	$\mathrm{GL}_{d_1}(q) \otimes \mathrm{GL}_{d_2}(q)$	$1 < d_1 < d_2$, f_i is trivial
U	$\mathrm{GU}_{d_1}(q^2) \otimes \mathrm{GU}_{d_2}(q^2)$	$1 < d_1 < d_2$, f_i is unitary

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_4$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_4$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}^{\epsilon}_{d_1}(q^u) \otimes \mathrm{GL}^{\epsilon}_{d_2}(q^u)$ we call K of type $\mathrm{GL}^{\epsilon}_{d_1}(q^u) \otimes \mathrm{GL}^{\epsilon}_{d_2}(q^u)$.

Remark. We note that $(V_1, \mathrm{GF}(q^u), f_1)$ is not similar to $(V_2, \mathrm{GF}(q^u), f_2)$ in the last definition. So, the previous definition of A-class $\mathscr{C}_4$ coincides with the definition in [KL]. Moreover, the definition of A-class $\mathscr{C}_4$ in [KL] coincides with the definitions in [As] and [BHR].

For the remaining part of this section we use the standard notation. Now, we examine whether there are members of A-class $\mathscr{C}_4$ of G which are strongly constrained maximal subgroups of G where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$. The following two propositions concern the structure of the members of A-class $\mathscr{C}_4$ of G.

Proposition 2.4.2. *Let $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ and let K be a member of A-class $\mathscr{C}_4$ of G. Then $K \cap \mathrm{P\Omega}(V)$ is a member of A-class $\mathscr{C}_4$ of $\mathrm{P\Omega}(V)$ of the same type as K.*

Proof. The assertion follows by [KL, Proposition 3.1.3]. □

For the following proposition, we recall from the beginning of Chapter 1 that we write $[m]$ for a group of order m of unspecified structure.

Proposition 2.4.3. *Let H be a member of A-class $\mathscr{C}_4$ of $\mathrm{P\Omega}(V)$. Then we have*

$$H \cong (\mathrm{PSL}^{\epsilon}_{d_1}(q^u) \times \mathrm{PSL}^{\epsilon}_{d_2}(q^u)).[(q-\epsilon 1, d_1)(q-\epsilon 1, d_2)c/(q-\epsilon 1, n)]$$

where $c = (q - \epsilon 1, d_1, d_2)$.

Proof. See [KL, Proposition 4.4.10 (II)]. □

Remark. Concerning the structure information provided in the last proposition, we also recall the observations following Definition 1.3.7, esp. (1.3.2).

Using the fact that the layer of a strongly constrained group has to be trivial, we easily obtain the following corollary which state that there is only one case to consider.

Corollary 2.4.4. *Let $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ and let K be a member of $\mathscr{C}_4$ of G. If $\mathrm{E}(K) = 1$ then $\mathrm{P\Omega}(V) \cong \mathrm{PSU}_6(2^2)$. Especially, we have $\mathrm{P\Omega}(V) \cong \mathrm{PSU}_6(2^2)$ if K is strongly constrained.*

Proof. Obviously, $K \cap \mathrm{P\Omega}(V)$ is a normal subgroup of K. So, our assertion easily follows by Propositions 1.2.11, 1.2.12, 2.4.2 and 2.4.3. □

By the information provided in [BHR, Table 8.26] (or, more precisely see [BHR, Proposition 2.3.22]), we now easily obtain the main theorem of this section from the last corollary. As usual, we do not use the standard notation in the following main theorem.

Main Theorem 2.4.5. *Let G be an almost simple group with socle isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$ where $n \geq 3$ if $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$. Then no member of A-class $\mathscr{C}_4$ of G is a strongly constrained maximal subgroup of G.*

2.5 A-class $\mathscr{C}_5$

Roughly described, the members of A-class $\mathscr{C}_5$ are the stabilizers of subfields of $\mathrm{GF}(q^u)$ of prime index. To introduce the members of A-class $\mathscr{C}_5$, we have to provide some terminology and some introducing considerations. For this, we recall the notation $\mathrm{GL}(V, \mathrm{GF}(q))$ for the general linear group of a $\mathrm{GF}(q)$-vector space V, from page 12.

The following considerations are based on [KL, p. 139-140] and [BHR, Subsection 2.2.5]. For the following we use the generalized standard notation, and we recall that V denotes an n-dimensional vector space over a finite field $\mathrm{GF}(q^u)$ where $u = 1$ in case **L** and $u = 2$ in case **U**. By B, we denote a fixed ordered $\mathrm{GF}(q^u)$-basis of V. Let $\mathrm{GF}(q_0)$ denote a subfield of index r in $\mathrm{GF}(q^u)$ (hence, $q_0 = q^{\frac{u}{r}}$), and let V_0 be the $\mathrm{GF}(q_0)$-span of B, denoted by $\langle B\rangle_{\mathrm{GF}(q_0)}$. Clearly, V_0 is a subset of V and can also be regarded as an n-dimensional vector space over $\mathrm{GF}(q_0)$. An element $g \in \mathrm{GL}(V_0, \mathrm{GF}(q_0))$ can be extended $\mathrm{GF}(q^u)$-linear to a (unique) element of $\mathrm{GL}(V, \mathrm{GF}(q^u))$, and so we obtain (by identification) an inclusion

$$\mathrm{GL}_n(q_0) \cong \mathrm{GL}(V_0, \mathrm{GF}(q_0)) \leq \mathrm{GL}(V, \mathrm{GF}(q^u)) \cong \mathrm{GL}_n(q^u). \tag{2.5.1}$$

Also without further explanation, we will use inclusions (by identification) as in (2.5.1). We note the following lemma, analyzing the inclusion (2.5.1). For this, recall the notation $\overline{\varphi}_B$ for the specific element of $\Gamma\mathrm{L}(V)$ induced by $\varphi \in \mathrm{Aut}(\mathrm{GF}(q^u))$ from page 28. For two fields L, K where K is a subfield of L we denote the Galois group of L over K by $\mathrm{Gal}(L : K)$.

Lemma 2.5.1. (*see [KL, p. 139]*)
Concerning the inclusion (2.5.1) (and the previously introduced notation), we have that for $g \in \mathrm{GL}(V, \mathrm{GF}(q^u))$ the following assertions are equivalent.

(a) $g \in \mathrm{GL}(V_0, \mathrm{GF}(q_0))$.

(b) In a representation of $\mathrm{GL}(V, \mathrm{GF}(q^u))$ *with respect to B we have that* $g = (g_{ij})_{n\times n}$ *where* $g_{ij} \in \mathrm{GF}(q_0)$.

(c) g preserves V_0.

(d) g and $\overline{\varphi}_B$ commute where φ is a generator for $\mathrm{Gal}(\mathrm{GF}(q^u) : \mathrm{GF}(q_0))$.

Hence, we have $\mathrm{N}_{\mathrm{GL}(V,\mathrm{GF}(q^u))}(V_0) = \mathrm{GL}(V_0, \mathrm{GF}(q_0))$.

Proof. Since V_0 is defined with respect to the fixed ordered $\mathrm{GF}(q^u)$-basis B of V, we obtain the equivalence of the assertions by elementary considerations. (Recall, that for a representation of $\mathrm{GL}(V, \mathrm{GF}(q^u))$ with respect to B where $g = (g_{ij})_{n\times n}$, we have that $(g_{ij})_{n\times n}^{\overline{\varphi}_B} = (g_{ij}^{\varphi})$, by the observations in Subsection 1.2.2). □

We can naturally extend the embedding from (2.5.1) to $\Gamma\mathrm{L}_n(q_0) \leq \Gamma\mathrm{L}_n(q^u)$. Now, suppose that the subset $V_0 \subseteq V$ (viewed as an n-dimensional $\mathrm{GF}(q_0)$-vector space) is equipped with a classical form κ_0. We also recall the terminology of a semisimilarity from page 14. For a subgroup G of $\Gamma\mathrm{L}(V, \mathrm{GF}(q^u))$ one may consider the *stabilizer* in G of (V_0, κ_0), that is the subgroup of G consisting of the elements of G that preserve V_0 and also induce a semisimilarity (with respect to κ_0) on V_0. The stabilizer in G of (V_0, κ_0) we denote by $\mathrm{N}_G(V_0, \kappa_0)$, or simply by $\mathrm{N}_G(V_0)$ if the role of κ_0 is clear.
For $\Omega(V) \leq G \leq \Gamma(V)$ in [KL, Table 4.5.A] the possible cases are listed

where the intersection of G with the stabilizer in $\Gamma(V)$ of (V_0, κ_0) multiplied by $\mathrm{Z}(\mathrm{GL}(V))$ may lead to a maximal subgroup of G. (We note that sometimes (but not in general) this subgroup of G coincides with the stabilizer in G of (V_0, κ_0), as we will see below). The table given in our definition of the members of A-class $\mathscr{C}_5$ (below) coincides with [KL, Table 4.5.A]. To obtain the mentioned table, cases are deleted which do not lead to maximal subgroups and conditions are provided such that the described situation occurs. In Remark 2.5.6, we will provide information concerning these restrictions and conditions.

To introduce A-class $\mathscr{C}_5$, we need the following notation concerning a form κ on V inducing a form κ_0 on V_0. Let case **U** be given. So, let the $\mathrm{GF}(q^2)$-vector space V be equipped with a non-degenerate unitary form f. As above, we fix a $\mathrm{GF}(q^2)$-basis B of V and define $V_0 = \langle B \rangle_{\mathrm{GF}(q_0)}$ for a subfield $\mathrm{GF}(q_0)$ of $\mathrm{GF}(q^2)$ of index r. Restricting f to the elements of V_0, it induces a form on V_0 which we denote by f_{V_0}. We note that the induced form on V_0 has not to be a unitary form, even if f_{V_0} is a classical form (as we will see below).

Now, we are able to introduce the members of A-class $\mathscr{C}_5$ for our intended groups $\mathrm{P}G$ where $\Omega(V) \leq G \leq \mathrm{A}(V)$. For this definition we consider the stabilizers in $\Gamma(V)$ of (V_0, κ_0) where the subfield $\mathrm{GF}(q_0)$ of $\mathrm{GF}(q^u)$ has an index $r > 1$. In the following definition we use the terminology and notation as introduced above, and (as usual) we use the generalized standard notation.

Definition 2.5.2. {***A-class*** $\boldsymbol{\mathscr{C}_5}$} (cf. [KL, Definition p. 140] and [BHR, Definition 2.2.11 and Table 2.8])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and let K be a subgroup of G. By f, we denote the non-degenerate unitary form on V in case **U**. For $G \leq \Gamma(V)$ the subgroup K is a member of *(A-class)* $\mathscr{C}_5$ of G if $K = (\mathrm{N}_{\Gamma(V)}(V_0, \kappa_0) \cdot \mathrm{Z}(\mathrm{GL}(V))) \cap G$ where $V_0 \subset V$ and (V_0, κ_0) is a classical geometry over $\mathrm{GF}(q^{\frac{u}{r}})$ occurring in the following table. If $G \nleq \Gamma(V)$ then K belongs to *(A-class)* $\mathscr{C}_5$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_5$ of $\Gamma(V)$.

Table 2.5 Classical geometries (V_0, κ_0) over $\mathrm{GF}(q^{\frac{u}{r}})$ occurring in $\mathscr{C}_5$

Case	Type of K	Description of κ_0	Conditions
L	$\mathrm{GL}_n(q^{\frac{1}{r}})$	κ_0 is trivial	r prime
U	$\mathrm{GU}_n(q^{\frac{2}{r}})$	$\kappa_0 = f_{V_0}$	r odd prime
U	$\mathrm{O}_n^{\epsilon}(q)$	$\kappa_0 = f_{V_0}$	$r = 2$, q odd
U	$\mathrm{Sp}_n(q)$	$\kappa_0 = \zeta \cdot f_{V_0}$ where $\zeta \in \mathrm{GF}(q^2)^*$, $\zeta + \zeta^q = 0$	$r = 2$, n even

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_5$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_5$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}_n^{\epsilon}(q^{\frac{u}{r}})$, $\mathrm{O}_n^{\epsilon}(q)$, or $\mathrm{Sp}_n(q)$ we call K of type $\mathrm{GL}_n^{\epsilon}(q^{\frac{u}{r}})$, $\mathrm{O}_n^{\epsilon}(q)$, or $\mathrm{Sp}_n(q)$, respectively.

Remark 2.5.3. (a) Our definition of the members of A-class $\mathscr{C}_5$ of G coincides with the definition of [KL, see p. 60 and 140]. In [BHR, Subsection 2.2.5], the introduction of the members of A-class $\mathscr{C}_5$ is also adopted from [KL], but at this occurs a mistake. There, for $G \leq \Gamma(V)$ the subgroup $K \leq G$ belongs to A-class $\mathscr{C}_5$ of G if $K = \mathrm{N}_G(V_0, \kappa_0) \cdot (\mathrm{Z}(\mathrm{GL}(V)) \cap G)$. This definition of the members of A-class $\mathscr{C}_5$ of G coincides with our definition if $\mathrm{Z}(\mathrm{GL}(V)) \leq G$ (use the Dedekind modular law). But in general this definition is different from ours and may lead to proper subgroups of our members, which so have not to be maximal in G, see e.g. Examples 2.5.13 and 2.5.16, below. Although there is a mistake in the definition of the members of A-class $\mathscr{C}_5$ in [BHR], we note that the author could not find resulting wrong conclusions in the proofs of the results we shall cite from that book. We note especially that the assertions concerning the structure of the members of A-class $\mathscr{C}_5$ of $\Omega(V)$ provided in [BHR, Table 2.8] are valid.

(b) In the paper of Aschbacher [As, see p. 472], the definition of the members of A-class $\mathscr{C}_5$ of $\Gamma(V)$ is formulated differently as in the book of Kleidman and Liebeck [KL, see p. 140], and so it differs from our definition. Kleidman and Liebeck have not discussed equality or relations between the two definitions (merely a note concerning an extra condition not required by Aschbacher). Here, we will consider this issue. By [As, (8.1) and (8.2)], we see the relation between the two definitions, and also the cases occurring in [KL, Table 4.5.A] can be deduced. So, to see whether the two definitions coincide we only have to consider the following two points. First, as mentioned before, Kleidman and Liebeck have required an additional condition. Here, we refer to Remark 2.5.6 (e), below. Second, by the definition of Aschbacher it is required that V_0 is an absolutely irreducible $\mathrm{GF}(q^{\frac{u}{r}})N_{\mathrm{I}(V)}(V_0, \kappa_0)$-module (keep [As, (8.1) and (8.2)] in mind). This condition has not been mentioned or required by Kleidman and Liebeck. In view of the following part (c), we have that $N_{\mathrm{I}(V)}(V_0, \kappa_0) = \mathrm{I}(V_0, \kappa_0)$. Since in the definition of A-class $\mathscr{C}_5$ taken from Kleidman and Liebeck it is required that (V_0, κ_0) has to be a classical geometry and $n \neq 2$ in case **U**, we obtain by [KL, Proposition 2.10.6.] that the extra condition of Aschbacher is fulfilled. So, we have that the two definitions coincide (except the additional condition in [KL] mentioned before). (We note that the considerations concerning the absolute irreducibility have also be done in [BHR, p. 69], without discussing the relation between the two definitions in [As] and [KL]).

(c) For the following see [KL, p. 140] and recall the notation in Table 1.2.1. Concerning the cases in Table 2.5 and the listed classical geometries (V_0, κ_0), we note the following. Since we have that κ_0 is trivial in case **L** and $\kappa_0 = \zeta f_{V_0}$ for an appropriate $\zeta \in \mathrm{GF}(q^2)^*$ in case **U**, we can deduce that $\mathrm{N}_{\mathrm{I}(V)}(V_0, \kappa_0) = \mathrm{I}(V_0, \kappa_0)$ and $\Omega(V_0, \kappa_0) \leq \Omega(V)$. Cf. also Example 2.5.4 and Remark 2.5.5.

(d) The book of Kleidman and Liebeck [KL] provides explicit constructions

concerning an embedding of the isometry group $\mathrm{I}(V_0, \kappa_0)$ into $\mathrm{I}(V)$ (recall part (c)) for each of the occurring cases from Table 2.5. We want to provide references and some facts for those embeddings. In case **L** an embedding of $\mathrm{GL}_n(q^{\frac{1}{r}})$ in $\mathrm{GL}_n(q)$ has been considered at the beginning of this section, see also [KL, p. 139].
In case **U** the intended embeddings can be constructed in most cases by considering the stabilizer in $\mathrm{GU}_n(q^2)$ of (V_0, f_{V_0}). The case **U** of type $\mathrm{GU}_n(q^{\frac{2}{r}})$ is important for our further investigations. So, an embedding of $\mathrm{GU}_n(q^{\frac{2}{r}})$ in $\mathrm{GU}_n(q^2)$ will be discussed in the example following this remark in detail. The embedding of a group isomorphic to $\mathrm{Sp}_n(q)$ in $\mathrm{GU}_n(q^2)$ is provided in [KL, p. 143], or see [CK]. We note that in this case an embedding cannot always be obtained by arguing as described above. However, if q is even an argumentation as above is possible, cf. [KL, p. 143], or Remark 2.5.5, below. Case **U** of type $\mathrm{O}_n^{\epsilon}(q)$ is discussed in [KL, p. 142], or see [CK2]. For this case the terminology of the discriminant of a quadratic form Q is used for ensuring that both cases $\epsilon \in \{+, -\}$ occur if the dimension of V is even (cf. also [KL, Proposition 2.5.12.]). For more information concerning the discriminant of Q, see [KL, p. 31-33].

Example 2.5.4. $\mathrm{GU}_n(q^{\frac{2}{r}}) \leq \mathrm{GU}_n(q^2)$ (cf. [KL, p. 141] or [CS, p. 287])
Let $\mathrm{GF}(q_0)$ be a subfield of $\mathrm{GF}(q^2)$ of odd index $r > 1$, hence $q_0 = q^{\frac{2}{r}}$. (Note, that we do not require r to be a prime). Let V be an n-dimensional $\mathrm{GF}(q^2)$-vector space equipped with a non-degenerate unitary form f. Let $\mathrm{GU}_n(q^2)$ be the representation of $\mathrm{I}(V, f) = \mathrm{GU}(V)$ with respect to an orthonormal $\mathrm{GF}(q^2)$-basis B of V. Define $V_0 = \langle B \rangle_{\mathrm{GF}(q_0)}$ and let $\kappa_0 = f_{V_0}$. Then, by Lemma 1.3.10, it is not hard to see that κ_0 is a non-degenerate unitary form on the $\mathrm{GF}(q_0)$-vector space V_0.
Using Lemma 1.2.8 (a), we have that $\mathrm{GU}_n(q^2) = \{g \in \mathrm{GL}_n(q^2) \mid g\mathbb{1}_n g^{t\varphi_q} = \mathbb{1}_n\}$. We consider the subgroup

$$H = \{A \mid A = (a_{ij})_{n\times n} \in \mathrm{GU}_n(q^2) \text{ where } a_{ij} \in \mathrm{GF}(q_0)\} \leq \mathrm{GU}_n(q^2).$$

In view of Lemmas 1.3.10 and 2.5.1, we now obtain that $H = \mathrm{N}_{\mathrm{GU}_n(q^2)}(V_0, \kappa_0)$. Let $\mathrm{GU}_n(q_0)$ be the representation of $\mathrm{GU}(V_0, \kappa_0)$ with respect to the basis B of V_0 (viewed as a $\mathrm{GF}(q_0)$-basis). For $g \in \mathrm{GU}_n(q_0)$ the unique $\mathrm{GF}(q^2)$-linear extension to an element in $\mathrm{GL}_n(q^2)$ lies in $\mathrm{GU}_n(q^2)$ (again use Lemma 1.3.10). So, we obtain that $\mathrm{GU}_n(q_0) = H \leq \mathrm{GU}_n(q^2)$, by identification. Furthermore, it is not hard to see that we have a natural inclusion of $\mathrm{SU}_n(q_0)$ in $\mathrm{SU}_n(q^2)$, via identification (cf. also Remark 2.5.3 (c)).

Remark 2.5.5. For $r = 2$ the argumentation in the last example is not valid anymore (cf. e.g. Lemma 1.3.10), and for this situation we obtain other results. Here, we have that $\kappa_0 = f_{V_0}$ is a non-degenerate symmetric bilinear form on the $\mathrm{GF}(q)$-vector space $V_0 = \langle B \rangle_{\mathrm{GF}(q)}$ if $J_{f,B} \in \mathrm{Mat}_n(q)$ where $J_{f,B}$ denotes the matrix of f with respect to a $\mathrm{GF}(q^2)$-basis B of V. We recall Proposition 1.2.3 which provides specific bases related to a classical form on a vector space.

In view of this proposition, Lemma 1.2.8 and [KL, Lemma 2.1.8. (iii)], we see that the stabilizers in $\mathrm{GU}_n(q^2)$ of $(V_0 = \langle B\rangle_{\mathrm{GF}(q)}, \kappa_0 = f_{V_0})$ are of type $\mathrm{O}_n^\epsilon(q)$, or $\mathrm{Sp}_n(q)$ if q is even (see also [KL, p. 142-143]).

Unfortunately, some explanations for the restrictions in [KL, Table 4.5.A] in cases **L** and **U** are not described (in detail) by Kleidman and Liebeck, and so we will handle this in the following remark. (We note that in [BHR] the information from [KL, Table 4.5.A] was adopted without further explanations).

Remark 2.5.6. Concerning the conditions and cases occurring in Table 2.5 and [KL, Table 4.5.A in cases **L** and **U**], we provide the following explanations. Recalling Remark 2.5.3 (b), we will also consider some cases with regard to the definition of Aschbacher [As, p. 472].

(a) The condition that the index r of the subfield $\mathrm{GF}(q^{\frac{u}{r}})$ in $\mathrm{GF}(q^u)$ has to be a prime is clear, since otherwise there would exist proper overgroups (cf. Example 2.5.4, Remark 2.5.5 and the explicit constructions provided in [KL, p. 141-143]).

(b) In case **L** there is only one type listed in Table 2.5. The exclusion of other types is not hard to see. The stabilizer in G (where $\mathrm{SL}(V) \leq G \leq \Gamma\mathrm{L}(V)$) of a classical geometry (V_0, κ_0) where κ_0 is not a trivial form on V_0, would be a subgroup of the stabilizer in G of (V_0, κ_0') where κ_0' is trivial.

(c) In case **U** the condition that r has to be odd for type $\mathrm{GU}_n(q^{\frac{2}{r}})$ and two for types $\mathrm{O}_n^\epsilon(q)$ and $\mathrm{Sp}_n(q)$ is clear by Example 2.5.4, Remark 2.5.5 and also [KL, p. 143] for type $\mathrm{Sp}_n(q)$ when q is odd. Compare also [As, (8.1)], concerning the version of Aschbacher.

(d) We consider the condition that n has to be even in case **U** of type $\mathrm{Sp}_n(q)$. Since (V_0, κ_0) is a classical geometry, κ_0 is a non-degenerate symplectic form on V_0. By Proposition 1.2.3 (ii) or [KL, Proposition 2.4.1.], we obtain that this condition is always fulfilled. By Aschbacher's definition of A-class $\mathscr{C}_5$, this condition can also be easily derived (cf. also [As, (8.1)]).

(e) Concerning the condition that q is odd in case **U** for type $\mathrm{O}_n^\epsilon(q)$, Kleidman and Liebeck have provided a remark following their definition of A-class $\mathscr{C}_5$, see [KL, p. 140]. There, the authors argue that a subgroup H of type $\mathrm{O}_n^\epsilon(q)$ in $\mathrm{GU}_n(q^2)$ would normalize a subgroup of $\mathrm{GU}_n(q^2)$ isomorphic to $\mathrm{Sp}_n(q)$ if q is even. Hence, a stabilizer in $\mathrm{GU}_n(q^2)$ of a symplectic geometry would be an overgroup of a stabilizer in $\mathrm{GU}_n(q^2)$ of an orthogonal geometry. This argumentation is correct (cf. Remark 2.5.5 and [KL, the embeddings on p. 142-143]), but only valid if the dimension n of V is even (recall from part (d) that type $\mathrm{Sp}_n(q)$ only occurs for even dimensions). So, the condition that q is odd can be required for even n; and we note that it was not required in the paper of Aschbacher, cf. also [KL, p. 58 and Table 3.5.J].
Concerning the case of odd dimensions, we argue similarly to part (d). We have that (V_0, κ_0) is a classical geometry. In view of [KL, Proposition

2.5.1.], or Remark 1.2.5 (e), we obtain that q has to be odd. To see that this condition can also be derived by Aschbacher's definition of A-class $\mathscr{C}_5$, we recall Remark 2.5.3 (c) and the required condition that V_0 has to be an absolutely irreducible $\mathrm{GF}(q)N_{\mathrm{GU}_n(q^2)}(V_0, \kappa_0)$-module when κ_0 is an orthogonal form on V_0. In view of [BHR, Theorem 1.5.41], we now see that q is odd in this case.

From now on we use the standard notation. Next, we provide the information about the structure of the intersection of $K \in \mathscr{C}_5$ of G with the socle $\mathrm{P}\Omega(V)$ of G where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$.

Proposition 2.5.7. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and let* $K \in \mathscr{C}_5$ *of* G. *Then we have that* $K \cap \mathrm{P}\Omega(V) \in \mathscr{C}_5$ *of* $\mathrm{P}\Omega(V)$ *of the same type as* K. *Furthermore, we have that* $K \cap \mathrm{P}\Omega(V) = \mathrm{P}H$ *where* H *is a member of A-class* $\mathscr{C}_5$ *of* $\Omega(V)$ *of the same type as* K, *and the structure of* H *is as provided in the following table.*

Case	*Type of K*	*Structure of H*
L	$\mathrm{GL}_n(q^{\frac{1}{r}})$	$\mathrm{SL}_n(q^{\frac{1}{r}}).\left[\left(\frac{q-1}{q^{1/r}-1}, n\right)\right]$
U	$\mathrm{GU}_n(q^{\frac{2}{r}})$	$\mathrm{SU}_n(q^{\frac{2}{r}}).\left[\left(\frac{q+1}{q^{1/r}+1}, n\right)\right]$
U	$\mathrm{O}_n^{\epsilon}(q)$	$\mathrm{SO}_n^{\epsilon}(q).[(q+1, n)]$
U	$\mathrm{Sp}_n(q)$	$\mathrm{Sp}_n(q).\left[\left(q+1, \frac{n}{2}\right)\right]$

Proof. Our assertion follows by [KL, Proposition 3.1.3.] and [BHR, Table 2.8]. □

Now, we have provided the necessary information to begin the investigation for our intended goal. We start, by using the condition that the layer of a member $K \in \mathscr{C}_5$ has to be trivial if K is strongly constrained. Thereby, we will intensely reduce the cases to examine.

Proposition 2.5.8. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and* $K \in \mathscr{C}_5$ *of* G. *Let* $\mathrm{E}(K) = 1$. *Then* K *is*

(i) in case **L** *of type* $\mathrm{GL}_2(p)$ *for* $p \in \{2, 3\}$, *hence* $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_2(p^r)$ *for a prime* r,

(ii) in case **U** *of type*

- *(a)* $\mathrm{GU}_3(2^2)$, *hence* $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(2^{2r})$ *for an odd prime* r,
- *(b)* $\mathrm{O}_3^{\circ}(3)$, *hence* $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(3^2)$, *or*
- *(c)* $\mathrm{O}_4^{+}(3)$, *hence* $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_4(3^2)$.

Furthermore, K *is of one of these types if* K *is strongly constrained.*

Proof. Since we have that $K \cap \mathrm{P}\Omega(V)$ is a normal subgroup of K, our assertion follows by contradiction, using Propositions 1.2.11, 1.2.12 and 2.5.7 and Lemma 1.4.22. □

Next, we provide the information of [BHR] in which cases a member $K \in \mathscr{C}_5$ of G is a maximal subgroup of G for the cases occurring in Proposition 2.5.8. As a consequence, we will further reduce the cases to examine.

Lemma 2.5.9. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and $K \in \mathscr{C}_5$ of G. Then the following hold.*

(i) If $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_2(2^r)$ for a prime r and K is of type $\mathrm{GL}_2(2)$ then K is a maximal subgroup of G if and only if $r = 2$.

(ii) If $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_2(3^r)$ for a prime r and K is of type $\mathrm{GL}_2(3)$ then K is a maximal subgroup of G if and only if $r \geq 3$, or $r = 2$ and G is a subgroup of $\mathrm{P}\Omega(V) : \langle \varphi_3 \rangle$.

(iii) If $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(2^{2r})$ for an odd prime r and K is of type $\mathrm{GU}_3(2^2)$ then K is a maximal subgroup of G if and only if $9 \nmid 2^r + 1$, or $9 \mid 2^r + 1$ and there is an $\alpha \in \mathrm{PA}(V)$ such that $K^\alpha \leq G^\alpha \leq \mathrm{P}\Omega(V) : \langle \varphi_2 \rangle$ where φ_2 stabilizes the $\mathrm{P}\Omega(V)$-conjugacy class of $K^\alpha \cap \mathrm{P}\Omega(V)$.

(iv) K is not a maximal subgroup of G if K is of type $\mathrm{O}_3^\circ(3)$, or $\mathrm{O}_4^+(3)$.

Proof. Assertions (i) and (ii) follow by [BHR, Table 8.1], or more precisely [BHR, Lemmas 3.1.3, 3.1.5 and Proposition 6.3.11]. We obtain assertion (iii) by [BHR, Table 8.5] (cf. also [BHR, Proposition 3.2.4]). By [BHR, Tables 8.5 and 8.10] or [BHR, Propositions 3.2.4 and 3.3.5], assertion (iv) follows. (For the case $K \leq \mathrm{P}\Omega(V)$ where K is of type $\mathrm{GU}_3(2^2)$, or $\mathrm{O}_3^\circ(3)$, cf. also [Mi] and [Ha]). □

Remark 2.5.10. (a) Concerning well-definedness, we note that a specification of a basis of V in Lemma 2.5.9 (iii) is not necessary for a unique description of the results, cf. [BHR2, Lemma 5 and Proposition 7].

(b) We want to mention some other works which also consider maximality using other approaches as [KL] and [BHR], in more geometrical nature. The members of A-class $\mathscr{C}_5$ of $\mathrm{SL}_n^\epsilon(q^u)$ of type $\mathrm{GL}_n^\epsilon(q^{\frac{u}{r}})$ have also been examined in [CS]. By considering transvections, it has been shown that for $n \geq 3$ and $q^{\frac{u}{r}} \geq 3^u$ these members of A-class $\mathscr{C}_5$ of $\mathrm{SL}_n^\epsilon(q^u)$ are maximal subgroups of $\mathrm{SL}_n^\epsilon(q^u)$. For the considerations concerning the maximality in $\mathrm{SU}_n(q^2)$ of the members of A-class $\mathscr{C}_5$ of $\mathrm{SU}_n(q^2)$ of type $\mathrm{Sp}_n(q)$, or $\mathrm{O}_n^\epsilon(q)$ (for type $\mathrm{Sp}_n(q)$ if $n \geq 4$ and type $\mathrm{O}_n^\epsilon(q)$ if $n \geq 3$), we mention the two papers [CK] and [CK2] which use also geometrical aspects.

(c) We recall the considerations before Section 2.1 (esp. the notation c). By [KL, Proposition 4.5.3. (I)], we see that $c = 1$ for $p = 2$, or $p = 3$ and $r \neq 2$ in the case of Proposition 2.5.8 (i); $c = 2$ for $p = 3$ and $r = 2$ in the case of Proposition 2.5.8 (i); $c = 1$ if $9 \nmid 2^r + 1$ and $c = 3$ if $9 \mid 2^r + 1$ in the case of Proposition 2.5.8 (ii)(a).
Furthermore, we note the following. For the case $r = 2$ in Lemma 2.5.9 (ii) we do not have to demand that φ_3 stabilizes the $\mathrm{P}\Omega(V)$-conjugacy class of $K \cap \mathrm{P}\Omega(V)$, in view of Example 2.5.13, below. The condition that

φ_2 has to stabilize the $\mathrm{P}\Omega(V)$-conjugacy class of $K^\alpha \cap \mathrm{P}\Omega(V)$ in Lemma 2.5.9 (iii) for $9 \mid 2^r + 1$ is not redundant, see Remark 2.5.21, below.

In view of the previous proposition and lemma, we have only to consider the cases (i) and (ii)(a) of Proposition 2.5.8. We start by considering the case of Proposition 2.5.8 (i) for $p = 2$. For this, we note the following lemma.

Lemma 2.5.11. *The A-classes $\mathscr{C}_2$ and $\mathscr{C}_5$ of $\mathrm{SL}_2(4)$ coincide.*

Proof. First, we note that in each of the A-classes $\mathscr{C}_2$ and $\mathscr{C}_5$ of $\mathrm{SL}_2(4)$ there is only one type of members. We consider the subgroup

$$\mathrm{SL}_2(2) \cong H = \langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \rangle \rtimes \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \leq \mathrm{SL}_2(4).$$

Obviously, H is a member of $\mathscr{C}_5$ of $\mathrm{SL}_2(4)$. Let ω be a primitive element of $\mathrm{GF}(4)^*$ and let $g = \begin{pmatrix} \omega & \omega^{-1} \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(4)$. By Proposition 2.2.8, we obtain that $H^g \in \mathscr{C}_2$ of $\mathrm{SL}_2(4)$. Now, using Proposition 2.2.8 (iv) and [KL, Proposition 4.5.3. (I)], our assertion follows. □

Corollary 2.5.12. *Let $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ where $\mathrm{PSL}_n(q)$ is simple and q is even. Then G has a strongly constrained maximal subgroup which belongs to A-class $\mathscr{C}_5$ of G if and only if $G = \mathrm{PSL}_2(4)$. For $G = \mathrm{PSL}_2(4)$ we have that the members of A-class $\mathscr{C}_5$ of G are strongly 3-constrained maximal subgroups of G of type $\mathrm{GL}_2(2)$ which have order 6.*

Proof. The assertion follows by Proposition 2.5.8, Lemmas 2.5.9 and 2.5.11 together with Proposition 2.2.20 (i) or Theorem 2.2.21 (i). □

Remark. Without using Proposition 2.2.20 (i) or Theorem 2.2.21 (i) in the last corollary, we note that the exclusion of the case $G = \mathrm{Aut}(\mathrm{PSL}_2(4))$ can also easily be verified, since for this case we have $\mathrm{O}_3(K), \mathrm{O}_2(K) > 1$ for any member $K \in \mathscr{C}_5$ of G.

Now, we consider the case of Proposition 2.5.8 (i) for $p = 3$. For this, we provide the following elementary example.

Example 2.5.13. Let r be a prime and V be a 2-dimensional vector space over $\mathrm{GF}(3^r)$. For the following we provide the tower of fields $\mathrm{GF}(3) \leq \mathrm{GF}(3^r) \leq \mathrm{GF}(3^{2r})$. Let $\Omega = \mathrm{SL}_2(3^r)$ be the representation of $\mathrm{SL}(V)$ with respect to an ordered $\mathrm{GF}(3^r)$-basis B of V. We define the subset $V_0 = \langle B \rangle_{\mathrm{GF}(3)} \subset V$. By Lemma 2.5.1, we have that $\mathrm{N}_{\mathrm{GL}_2(3^r)}(V_0) = \mathrm{GL}_2(3)$, and hence $\mathrm{N}_{\mathrm{SL}_2(3^r)}(V_0) = \mathrm{SL}_2(3) = H_0$. We also obtain that $\mathrm{N}_{\Gamma\mathrm{L}_2(3^r)}(V_0) = \mathrm{GL}_2(3) \rtimes \langle \varphi_3 \rangle$.
Let $i \in \mathrm{GF}(3^{2r})$ denote a primitive 4-th root of 1, and note that $i \in \mathrm{GF}(3^r)$ if and only if $r = 2$. We define the element $W_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{SL}_2(3^{2r})$. By our

previous considerations (and recalling Definition 2.5.2), we obtain that

$$H = (\mathrm{N}_{\Gamma\mathrm{L}_2(3^r)}(V_0) \cdot \mathrm{Z}(\mathrm{GL}_2(3^r))) \cap \mathrm{SL}_2(3^r) = \begin{cases} H_0 & \text{if } r \neq 2, \\ \langle H_0, W_0 \rangle & \text{if } r = 2, \end{cases}$$

is a member of A-class $\mathscr{C}_5$ of $\mathrm{SL}_2(3^r)$ of type $\mathrm{GL}_2(3)$. Recall Remark 2.5.3 (a), concerning the wrong definition of A-class $\mathscr{C}_5$ in [BHR]. Following this definition, a member of A-class $\mathscr{C}_5$ of $\mathrm{SL}_2(3^r)$ of type $\mathrm{GL}_2(3)$ would be of the form $\mathrm{N}_{\mathrm{SL}_2(3^r)}(V_0) \cdot (\mathrm{Z}(\mathrm{GL}_2(3^r)) \cap \mathrm{SL}_2(3^r)) = H_0 \cdot \mathrm{Z}(\mathrm{SL}_2(3^r)) = H_0$, which is a proper subgroup of H for $r = 2$. So, in view of the following conjugacy considerations and Lemma 2.5.9 (ii) (which is derived by the correct information in [BHR, Table 8.1]), we see that the definition in [BHR] is wrong.
Now, we consider the projective case. Since $\mathrm{Z}(\mathrm{SL}_2(3^r)) = \mathrm{Z}(\mathrm{SL}_2(3))$, we clearly obtain that

$$\mathrm{P}H = \begin{cases} \mathrm{P}H_0 \cong \mathrm{PSL}_2(3) & \text{if } r \neq 2, \\ \mathrm{P}H_0 \rtimes \langle \mathrm{P}W_0 \rangle \cong \mathrm{PGL}_2(3) & \text{if } r = 2, \end{cases}$$

is a member of A-class $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$. Obviously, we have $\mathrm{E}(\mathrm{P}H) = 1$, $\mathrm{O}_3(\mathrm{P}H) = 1$ and $\mathrm{O}_2(\mathrm{P}H) \cong V_4$ where V_4 denotes the Klein four-group. So, $\mathrm{P}H$ is a strongly 2-constrained group.
We define the following subgroup of $\mathrm{N}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H_0)$

$$K = \begin{cases} \mathrm{P}H \rtimes \langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \rangle & \text{if } r \neq 2, \\ \mathrm{P}H & \text{if } r = 2. \end{cases}$$

Note, that by our previous considerations we have K is strongly 2-constrained where $\mathrm{O}_2(K) = \mathrm{O}_2(\mathrm{P}H)$. Furthermore, we have that $\mathrm{C}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H_0) = 1$, and so

$$\mathrm{PGL}_2(3) \cong K \mathrel{\widetilde{\leq}} \mathrm{N}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H_0)/\mathrm{C}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H_0) \mathrel{\widetilde{\leq}} \mathrm{Aut}(\mathrm{P}H_0) \cong \mathrm{PGL}_2(3).$$

Hence, we have that $K = \mathrm{N}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H_0)$, and since $\mathrm{P}H_0$ is a characteristic subgroup of $\mathrm{P}H$, clearly $K = \mathrm{N}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H)$. Now, we can obviously derive that $\mathrm{P}H = \mathrm{N}_{\mathrm{PSL}_2(3^r)}(\mathrm{P}H)$.
Finally, we consider the conjugacy of the members of $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$ in $\mathrm{PSL}_2(3^r)$. Obviously, φ_3 centralizes $\mathrm{P}H$. So, we obtain that $\mathrm{PGL}_2(3^r)$ acts transitively (by conjugation) on the members of A-class $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$, by [KL, Theorem 3.1.1] (or, see [As, Theorems BΓ and BΔ]). Hence, $m = |\mathrm{PGL}_2(3^r) : K|$ is the number of the members of $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$. Now, we can easily deduce that

$$|\mathrm{PSL}_2(3^r) : \mathrm{P}H| = \begin{cases} m & \text{if } r \neq 2, \\ \frac{m}{2} & \text{if } r = 2. \end{cases}$$

As a consequence, $\mathrm{PSL}_2(3^r)$ acts transitively on the members of $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$ if and only if $r > 2$. For $r = 2$ we see that there are precisely two $\mathrm{PSL}_2(3^r)$-conjugacy classes, each stabilized by φ_3 (cf. also [BHR, Table 8.1] and [KL, Proposition 4.5.3. (I)]). Furthermore, if $r = 2$ we note that $\mathrm{P}H$ and $(\mathrm{P}H)^W$ are representatives of these two $\mathrm{PSL}_2(3^r)$-conjugacy classes (see Subsection 1.2.2 for the notation W).

Now, we have provided all necessary information for examining the intended case.

Proposition 2.5.14. *Let r be a prime and $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ where $\mathrm{P}\Omega(V) \cong \mathrm{PSL}_2(3^r)$. Let M be a member of A-class $\mathscr{C}_5$ of G of type $\mathrm{GL}_2(3)$ which is a maximal subgroup of G. Then M is strongly constrained if and only if one of the following holds.*

(i) $r = 2$ (note, that by Lemma 2.5.9 G is a subgroup of $\mathrm{PSL}(V) : \langle\varphi_3\rangle$).

(ii) $r \geq 3$ and G is a subgroup of $\mathrm{PGL}(V)$.

Furthermore, if M is strongly constrained then M is strongly 2-constrained and we have that $|M| \leq 2^4 3$ where this upper bound is sharp.

Proof. To prove our assertion, w.l.o.g. we can consider a concrete representation of $\Omega(V)$ with respect to an ordered basis B of V, hence $\mathrm{PSL}_2(3^r) \leq G \leq \mathrm{P\Gamma L}_2(3^r)$. We use the notation introduced in Example 2.5.13. By Proposition 2.5.7, we have that $M \cap \mathrm{PSL}_2(3^r)$ is a member of A-class $\mathscr{C}_5$ of $\mathrm{PSL}_2(3^r)$ of type $\mathrm{GL}_2(3)$. Note, that $\mathrm{Out}(\mathrm{PSL}_2(3^r))$ is abelian, and hence G is a normal subgroup of $\mathrm{P\Gamma L}_2(3^r)$. In view of Example 2.5.13 (esp. the conjugacy considerations), w.l.o.g. we can now assume that $M \cap \mathrm{PSL}_2(3^r) = \mathrm{P}H$, see also Lemma 1.4.20. Hence, we obtain that $M = \mathrm{N}_G(\mathrm{P}H) = K_1 \cap G$ where $K_1 = \mathrm{N}_{\mathrm{P\Gamma L}_2(3^r)}(\mathrm{P}H) = K : \langle\varphi_3\rangle$ (recall by Example 2.5.13 that $\mathrm{N}_{\mathrm{PGL}_2(3^r)}(\mathrm{P}H) = K$).
First, we consider the case $r = 2$. In view of Lemma 2.5.9 (ii), we have that $G = \mathrm{PSL}_2(3^r)$, or $\mathrm{PSL}_2(3^r) : \langle\varphi_3\rangle$, and so $M = \mathrm{P}H$, or $\mathrm{P}H \times \langle\varphi_3\rangle$, respectively. The case $G = \mathrm{PSL}_2(3^r)$ is clear by Example 2.5.13. So, let $G = \mathrm{PSL}_2(3^r) : \langle\varphi_3\rangle$. Considering Example 2.5.13, it is not hard to see that $\mathrm{O}_2(M) = \mathrm{O}_2(\mathrm{P}H) \times \langle\varphi_3\rangle$, $\mathrm{O}_3(M) = 1$ and $\mathrm{E}(M) = 1$. Hence, M is strongly 2-constrained. (Here, we note that we have an explicit example where M is strongly 2-constrained and $|M| = 2^4 3$).
Next, let $r \geq 3$. We note that there is no restriction on the choice of G, arising from the condition $M \lessdot G$, see Lemma 2.5.9 (ii). Let M be strongly constrained. Since $\mathrm{O}_2(\mathrm{P}H)$ is a non-trivial normal subgroup of M, we have that M is strongly 2-constrained. Suppose that $\langle\varphi_3\rangle \leq G$ (so, $G = \mathrm{P\Gamma L}_2(3^r)$ or $\mathrm{PSL}_2(3^r) : \langle\varphi_3\rangle$). In view of Example 2.5.13 and our previous considerations, it is easy to see that $\langle\varphi_3\rangle$ is a normal subgroup of M. Hence, $\mathrm{O}_r(M) > 1$ and M is not strongly constrained. Now, there are only two possibilities left to consider, $G = \mathrm{PSL}_2(3^r)$ or $\mathrm{PGL}_2(3^r)$. Here, we have that $M = \mathrm{P}H$ or K, and so our assertion follows by Example 2.5.13. □

Considering the proof of the last proposition, we can easily derive the following corollary.

Corollary 2.5.15. *Let the situation of Proposition 2.5.14 be given. Let M be strongly 2-constrained. Then we have $\mathrm{O}_2(M) > \mathrm{O}_2(M \cap \mathrm{P}\Omega(V))$ if and only if $r = 2$ and $G = \mathrm{P}\Omega(V) : \langle \varphi_3 \rangle$. Furthermore, if M is strongly 2-constrained and $\mathrm{O}_2(M) > \mathrm{O}_2(M \cap \mathrm{P}\Omega(V))$ we have that $|\mathrm{O}_2(M)/\mathrm{O}_2(M \cap \mathrm{P}\Omega(V))| = 2$.*

Next, we consider the case of Proposition 2.5.8 (ii) (a). First, we provide detailed information by the following example.

Example 2.5.16. (see [Blo, p. 176], [AD, p. 9-11], [Uf, p. 79-86]; cf. [KL, p. 151])
Let r be an odd prime and $\Omega = \mathrm{SU}_3(2^{2r})$ where the matrix of the non-degenerate unitary form is $\mathbb{1}_3$. By Example 2.5.4, we have canonical inclusions $\mathrm{SU}_3(2^2) \leq \mathrm{SU}_3(2^{2r}) \leq \mathrm{SU}_3(2^{6r})$ for the tower of fields $\mathrm{GF}(4) \leq \mathrm{GF}(2^{2r}) \leq \mathrm{GF}(2^{6r})$. Note, that $\mathrm{Z}(\mathrm{SU}_3(2^2)) = \mathrm{Z}(\mathrm{SU}_3(2^{2r})) = \mathrm{Z}(\mathrm{SU}_3(2^{6r}))$. Let $\omega \in \mathrm{GF}(4)^*$ be a primitive 3-rd root of 1 and $\rho \in \mathrm{GF}(2^{6r})$ be a primitive 9-th root of 1 where $\rho^3 = \omega$. Obviously, we have $\rho \in \mathrm{GF}(2^{2r})$ if and only if $9 \mid 2^r + 1$.
We define the following elements (cf. also Lemma 1.2.8 (a))

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \in \mathrm{SU}_3(2^2),$$

$$E = \begin{pmatrix} \omega^2\rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix} \in \mathrm{SU}_3(2^{6r}) \text{ and } A^E = \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix} \in \mathrm{SU}_3(2^2).$$

By elementary calculations, we get the following (cf. [Uf, Definition und Satz 2.6.3.])

- $N = \langle X, Y \rangle$ is an extraspecial 3-group of order 3^3 with $\mathrm{Z}(N) = N' = \Phi(N) = \mathrm{Z}(\mathrm{SU}_3(2^2))$ (for the definition of an extraspecial 3-group, see Definition 2.6.1, below),
- $Q = \langle A, A^E \rangle$ is isomorphic to the quaternion group Q_8 (for the definition of the quaternion group Q_8, see Lemma 2.6.2, below),
- E normalizes Q, and $Q \rtimes \langle E \rangle$ normalizes N,
- $\langle N, Q \rangle = N \rtimes Q = \mathrm{SU}_3(2^2)$.

We define the subgroup H of $\mathrm{SU}_3(2^{2r})$ by

$$H = \begin{cases} N \rtimes Q & \text{if } 9 \nmid 2^r + 1, \\ \langle N \rtimes Q, E \rangle & \text{if } 9 \mid 2^r + 1. \end{cases}$$

Following Definition 2.5.2, we have that H is a member of $\mathscr{C}_5$ of $\mathrm{SU}_3(2^{2r})$ of type $\mathrm{GU}_3(2^2)$, cf. Example 2.5.4 and see also Proposition 2.5.7. (Here, we recall Remark 2.5.3 (a) and note that by the wrong Definition in [BHR] the proper subgroup $N \rtimes Q$ of H would be a member of $\mathscr{C}_5$ of $\mathrm{SU}_3(2^{2r})$ of type $\mathrm{GU}_3(2^2)$ if

$9 \mid 2^r+1$). It is not hard to see that $O_3(H) = N$. By Lemma 2.5.9 (iii), we can deduce that $H = N_{SU_3(2^{2r})}(H) = N_{SU_3(2^{2r})}(N)$ (cf. also [KL, Proposition 4.5.1.] to obtain $H = N_{SU_3(2^{2r})}(H)$ without using the argumentation of maximality). For the projective case we have that

$$\mathrm{P}H = \begin{cases} \mathrm{P}N \rtimes \mathrm{P}Q \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes Q_8 & \text{if } 9 \nmid 2^r+1, \\ (\mathrm{P}N \rtimes \mathrm{P}Q) \rtimes \langle \mathrm{P}E \rangle \cong ((\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes Q_8) \rtimes \mathbf{Z}_3 & \text{if } 9 \mid 2^r+1, \end{cases}$$

is a member of A-class $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$. By the previous considerations, we see that $\mathrm{P}H = N_{PSU_3(2^{2r})}(\mathrm{P}H)$. Since $O_3(\mathrm{P}H) = \mathrm{P}N$, $O_2(\mathrm{P}H) = 1$ and $E(\mathrm{P}H) = 1$, $\mathrm{P}H$ is a strongly 3-constrained group.
We next examine the conjugacy of the members of $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$ in $PSU_3(2^{2r})$. For this, we consider the normalizer of $\mathrm{P}N$ in $PGU_3(2^{2r})$. We define the subgroup $K \cong PGU_3(2^2)$ of $N_{PGU_3(2^{2r})}(\mathrm{P}N)$ via

$$K = \begin{cases} \mathrm{P}H \rtimes \langle \mathrm{P}W_0 \rangle & \text{if } 9 \nmid 2^r+1, \\ \mathrm{P}H & \text{if } 9 \mid 2^r+1, \end{cases}$$

where $W_0 = \mathrm{diag}(\omega, 1, 1) \in GU_3(2^{2r})$. Elementary calculations show that $C_{PGU_3(2^{2r})}(\mathrm{P}N) = \mathrm{P}N$, and so we have that $K/\mathrm{P}N$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{P}N) \cong GL_2(3)$. Actually, we have $K/\mathrm{P}N \cong SL_2(3)$. Let φ_2 denote the Frobenius automorphism of $GF(2^{2r})$. Then we have that the automorphism φ_2 of $PGU_3(2^{2r})$ (induced by φ_2) normalizes $\mathrm{P}N$ where $X^{\varphi_2} = X^2$ and $Y^{\varphi_2} = Y$. Elementary considerations show that none automorphism of N induced by conjugacy of elements of $PGU_3(2^{2r})$ coincide with the automorphism of N induced by φ_2. Hence, we obtain that $K = N_{PGU_3(2^{2r})}(\mathrm{P}N)$, and so $K = N_{PGU_3(2^{2r})}(\mathrm{P}H)$, since $O_3(\mathrm{P}H) = \mathrm{P}N$. We note that from $N_{PGU_3(2^{2r})}(\mathrm{P}H) = K$ the result $N_{PSU_3(2^{2r})}(\mathrm{P}H) = \mathrm{P}H$ obtained above can also be derived, without having used Lemma 2.5.9 (iii) or [KL, Proposition 4.5.1.].
Since φ_2 normalizes $\mathrm{P}H$, we obtain by [KL, Theorem 3.1.1.] (or, see [As, Theorems BΓ and BΔ]) that $PGU_3(2^{2r})$ acts transitively (by conjugation) on the members of A-class $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$. So, we have that $m = |PGU_3(2^{2r}) : K|$ is the number of the members of $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$. We easily obtain that

$$|PSU_3(2^{2r}) : \mathrm{P}H| = \begin{cases} m & \text{if } 9 \nmid 2^r+1, \\ \frac{m}{3} & \text{if } 9 \mid 2^r+1. \end{cases}$$

Hence, $PSU_3(2^{2r})$ acts transitively on the members of $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$ if and only if $9 \nmid 2^r+1$. For $9 \mid 2^r+1$ the members of $\mathscr{C}_5$ of $PSU_3(2^{2r})$ of type $GU_3(2^2)$ form three $PSU_3(2^{2r})$-conjugacy classes (cf. also [BHR, Table 8.5] and [KL, Proposition 4.5.3. (I)]). Furthermore, we obtain that $\mathrm{P}H, (\mathrm{P}H)^W$ and $(\mathrm{P}H)^{W^2}$ are representatives of the three $PSU_3(2^{2r})$-conjugacy classes if $9 \mid 2^r+1$ (see Subsection 1.2.2 for the notation W).

Finally, we note some facts concerning the group K. By above considerations, we easily obtain that $\mathrm{E}(K) = 1$, $\mathrm{O}_2(K) = \mathrm{O}_2(\mathrm{P}H) = 1$ and $\mathrm{O}_3(K) = \mathrm{O}_3(\mathrm{P}H) = \mathrm{P}N$. So, K is a strongly 3-constrained group.

Remark 2.5.17. The group K in the last example is also commonly known as *Hessian group*. The Hessian group is well investigated and has interesting properties (also accenting geometrical nature). A recommendable reference concerning this issue is the paper of Artebani and Dolgachev [AD]. We note that sometimes specific subgroups of K are also denoted as Hessian groups, see e.g. [Mi, p. 240-241] or [Ha, p. 158].

We provide the following lemma for our further investigations.

Lemma 2.5.18. *We use the notation introduced in Example 2.5.16. Let* $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ *and recall that for an odd prime* r, $\Omega = \mathrm{SU}_3(2^{2r})$ *where the matrix of the non-degenerate unitary form is* $\mathbb{1}_3$. *Let* $M \in \mathscr{C}_5$ *of* G *of type* $\mathrm{GU}_3(2^2)$ *be a maximal subgroup of* G *where* $M \cap \mathrm{P}\Omega = \mathrm{P}H$. *Then the following hold.*

(i) For $r > 3$ *the following three assertions are equivalent*

(a) $r \mid |M|$,

(b) $\langle \varphi_2^2 \rangle \leq M$,

(c) $\mathrm{O}_r(M) > 1$.

(ii) $\mathrm{O}_2(M) = 1$.

Proof. Since M is a maximal subgroup of G we have that $M = \mathrm{N}_G(\mathrm{P}H) = \mathrm{N}_{\mathrm{P\Gamma U}_3(2^{2r})}(\mathrm{P}H) \cap G$. By Example 2.5.16, we see $\mathrm{N}_{\mathrm{PGU}_3(2^{2r})}(\mathrm{P}H) = K$, and so we easily obtain $\mathrm{N}_{\mathrm{P\Gamma U}_3(2^{2r})}(\mathrm{P}H) = K \rtimes \langle \varphi_2 \rangle = K_1$. Note, that the order of K_1 is $3^3 2^4 r$.
First, we prove assertion (i). For this, we consider the subgroup $\langle \varphi_2^2 \rangle$ of K_1. Obviously, we have that $\langle \varphi_2^2 \rangle$ is a Sylow r-subgroup of K_1. In view of Example 2.5.16, it is not hard to see that φ_2^2 centralizes K_1. Hence, we obtain that $\langle \varphi_2^2 \rangle$ is the unique Sylow r-subgroup of K_1, and so assertion (i) follows by easy considerations using the Sylow theorems.
To prove assertion (ii), suppose that $\mathrm{O}_2(M) > 1$. Recall from Example 2.5.16 that $\mathrm{O}_2(\mathrm{P}H) = 1$. So, we obtain $\mathrm{O}_2(M) \cap \mathrm{P}H = 1$. This leads to

$$1 < |\mathrm{O}_2(M)| = |\mathrm{O}_2(M)/(\mathrm{O}_2(M) \cap \mathrm{P}H)| = |(\mathrm{O}_2(M) \cdot \mathrm{P}H)/\mathrm{P}H| \;\Big|\; |K_1/\mathrm{P}H| \;\Big|\; 6r.$$

Hence, $|\mathrm{O}_2(M)| = 2$ and there is a non-trivial involutory element C in $M \leq K_1$ which centralizes M and in particular $\mathrm{P}H$. We write $C = \varphi k \in M$ where $\varphi \in \langle \varphi_2 \rangle$ and $k \in K$. For the automorphism $\varphi \in \mathrm{Aut}(\mathrm{GF}(2^{2r}))$ which induces the automorphism φ (of $\mathrm{PGU}_3(2^{2r})$) we have that φ centralizes ω, or $\omega^\varphi = \omega^2$. Suppose that $\omega^\varphi = \omega^2$. For the following recall the elements $\mathrm{P}X$ and $\mathrm{P}Y$ of $\mathrm{P}H$ in Example 2.5.16. Since C centralizes $\mathrm{P}Y$, we have that k centralizes $\mathrm{P}Y$. By elementary calculations, we obtain that k is a member of $S, \mathrm{P}X \cdot S$ or $(\mathrm{P}X^2) \cdot S$ where

$$S = \left\{ \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \middle| a, b, c \in \mathrm{GF}(2^{2r}) \right\}.$$

Since C centralizes $\mathrm{P}X$, we have $\mathrm{P}X = (\mathrm{P}X)^C = (\mathrm{P}X^2)^k$. By easy calculations, we see that this condition leads to a contradiction.
So, we have that φ centralizes ω. In view of Example 2.5.16, we now easily obtain that φ centralizes $\mathrm{P}H$ (and K_1). Hence, C centralizes $\mathrm{P}H$ if and only if k does. Since K is a strongly 3-constrained group and $\mathrm{O}_3(K) = \mathrm{P}N = \mathrm{O}_3(\mathrm{P}H)$, we obtain that

$$k \in \mathrm{C}_K(\mathrm{P}H) \leq \mathrm{C}_K(\mathrm{P}N) = \mathrm{P}N = \langle \mathrm{P}X \rangle \times \langle \mathrm{P}Y \rangle.$$

Now, we consider the condition that the element $\mathrm{P}A \in \mathrm{P}H$ has to be centralized by k. By easy calculations, we see $\mathrm{P}X^{\mathrm{P}A} = \mathrm{P}Y^2$ and $\mathrm{P}Y^{\mathrm{P}A} = \mathrm{P}X$. So, we can deduce that $k = 1$. Because the order of $C = \varphi$ is two, we have that $C = \varphi_2^r$. But this leads to a contradiction, since φ centralizes ω. □

In view of Lemma 2.5.9 (iii) and Example 2.5.16, it is advantageous to distinguish the two cases $9 \mid 2^r + 1$ and $9 \nmid 2^r + 1$ and examine them separately.

Proposition 2.5.19. *Let r be an odd prime. Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ where $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(2^{2r})$ and $9 \mid 2^r + 1$. Let M be a maximal subgroup of G which belongs to A-class $\mathscr{C}_5$ of G of type $\mathrm{GU}_3(2^2)$. Then M is strongly constrained if and only if one of the following holds.*

(i) $r = 3$ (note, that by Lemma 2.5.9 G is a subgroup of a member of the $\mathrm{PA}(V)$-conjugacy class of $\mathrm{P}\Omega(V) : \langle \varphi_2 \rangle$).

(ii) $r > 3$ and G is a subgroup of a member of the $\mathrm{PA}(V)$-conjugacy class of $\mathrm{P}\Omega(V) : \langle \varphi_2^r \rangle$.

Furthermore, if M is strongly constrained then M is strongly 3-constrained and we have $|M| \leq 2^4 3^4$ where this upper bound is sharp.

Proof. In view of Lemmas 1.4.20 and 2.5.9 (iii), w.l.o.g. we can assume that $G \leq \mathrm{PSU}(V) : \langle \varphi_2 \rangle$ and φ_2 stabilizes the $\mathrm{PSU}(V)$-conjugacy class of $M \cap \mathrm{PSU}(V)$. Regarding [BHR2, Lemma 5 and Proposition 7], also w.l.o.g. we can consider a concrete representation of $\mathrm{SU}(V)$ with respect to an orthonormal basis B of V, hence we may assume $G \leq \mathrm{PSU}_3(2^{2r}) : \langle \varphi_2 \rangle$. So, we use the notation introduced in Example 2.5.16. Obviously, we have to consider only the four cases $G = G_1 = \mathrm{PSU}_3(2^{2r})$ or $G_j = \mathrm{PSU}_3(2^{2r}) : B_j$ where $2 \leq j \leq 4$ and $B_2 = \langle \varphi_2^2 \rangle$, $B_3 = \langle \varphi_2^r \rangle$ and $B_4 = \langle \varphi_2 \rangle$.
By Proposition 2.5.7, we have that $M \cap \mathrm{PSU}_3(2^{2r})$ is a member of $\mathscr{C}_5$ of $\mathrm{PSU}_3(2^{2r})$ of type $\mathrm{GU}_3(2^2)$, and obviously $M = \mathrm{N}_G(M \cap \mathrm{PSU}_3(2^{2r})) \lessdot G$. In view of the conjugacy considerations in Example 2.5.16, we see that w.l.o.g. we

can assume $M \cap \mathrm{PSU}_3(2^{2r}) \in \{\mathrm{P}H, (\mathrm{P}H)^W, (\mathrm{P}H)^{W^2}\}$. Hence, $M \cap \mathrm{PSU}_3(2^{2r}) = \mathrm{P}H$, by the condition that φ_2 stabilizes the $\mathrm{PSU}_3(2^{2r})$-conjugacy class of $M \cap \mathrm{PSU}_3(2^{2r})$. Now, we will consider each of the cases for G separately. If $G = G_1$ we have that $M = \mathrm{P}H$ and our assertion follows immediately, by Example 2.5.16. Next, let $G = G_3$. Then we have that $M = \mathrm{P}H : B_3$, and so $|M| = 2^4 3^3$. Because $\mathrm{E}(M) = 1$, M is strongly 3-constrained, in view of Lemma 2.5.18 (ii). Considering Example 2.5.16, we note that $\mathrm{O}_3(M) = \mathrm{O}_3(\mathrm{P}H) = \mathrm{P}N$, since $\mathrm{P}H$ is a normal subgroup of M of index 2.
Finally, we consider the cases $G = G_2$ or G_4. Here, we have $M = \mathrm{P}H : B_2$ or $\mathrm{P}H : B_4$. Let M be strongly constrained. Since $1 < \mathrm{O}_3(\mathrm{P}H) \leq \mathrm{O}_3(M)$, we have that M is strongly 3-constrained. Suppose that $r > 3$. Then, by Lemma 2.5.18 (i), we obtain a contradiction. Hence, let $r = 3$. Obviously, we have $2^3 3^4 \mid |M| \mid 2^4 3^4$. Since $\mathrm{E}(M) = 1$, we obtain by Lemma 2.5.18 (ii) that M is strongly 3-constrained. Furthermore, we have that $\langle \varphi_2^2 \rangle \leq \mathrm{Z}(M)$, see Example 2.5.16. So, $\mathrm{O}_3(\mathrm{P}H) \times \langle \varphi_2^2 \rangle \leq \mathrm{O}_3(M)$. Suppose that $\mathrm{O}_3(\mathrm{P}H) \times \langle \varphi_2^2 \rangle < \mathrm{O}_3(M)$. By Lemma 1.4.9, we have $\mathrm{O}_3(M) \cap \mathrm{P}H = \mathrm{O}_3(\mathrm{P}H)$. Hence, we obtain a contradiction by $9 = |\mathrm{O}_3(M)/\mathrm{O}_3(\mathrm{P}H)| = |(\mathrm{O}_3(M) \cdot \mathrm{P}H)/\mathrm{P}H| \bigm| |M/\mathrm{P}H| \bigm| 6$. □

We note the following corollary, which follows immediately by the proof of the previous proposition.

Corollary 2.5.20. *Let the situation in Proposition 2.5.19 be given. If M is strongly 3-constrained then $\mathrm{O}_3(M) > \mathrm{O}_3(M \cap \mathrm{P}\Omega(V))$ if and only if $r = 3$ and $G = \mathrm{P}\Omega(V) : \langle \varphi_2^2 \rangle$ or G belongs to the $\mathrm{PA}(V)$-conjugacy class of $\mathrm{P}\Omega(V) : \langle \varphi_2 \rangle$. Furthermore, if M is strongly 3-constrained and $\mathrm{O}_3(M) > \mathrm{O}_3(M \cap \mathrm{P}\Omega(V))$ then $|\mathrm{O}_3(M)/\mathrm{O}_3(M \cap \mathrm{P}\Omega(V))| = 3$.*

Remark 2.5.21. The condition in Lemma 2.5.9 (iii) that φ_2 has to stabilize the $\mathrm{P}\Omega(V)$-conjugacy class of $K^\alpha \cap \mathrm{P}\Omega(V)$ is not redundant. To see this, we use the notation in the proof of the last proposition. Consider the case $G = G_4$, and suppose that $M \cap \mathrm{PSU}_3(2^{2r}) = (\mathrm{P}H)^W$ or $(\mathrm{P}H)^{W^2}$. It is not hard to see that φ_2 maps the members of the $\mathrm{PSU}_3(2^{2r})$-conjugacy class represented by $(\mathrm{P}H)^W$ to that represented by $(\mathrm{P}H)^{W^2}$, and vice versa. Hence, we obtain that $\mathrm{N}_{G_4}((\mathrm{P}H)^W), \mathrm{N}_{G_4}((\mathrm{P}H)^{W^2}) < G_2$. So, M is not a maximal subgroup of G_4.

Next, we consider the case $9 \nmid 2^r + 1$ of Proposition 2.5.8 (ii) (a).

Proposition 2.5.22. *Let r be an odd prime. Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ where $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(2^{2r})$ and $9 \nmid 2^r + 1$ (esp. $r > 3$). Let $M \in \mathscr{C}_5$ of G of type $\mathrm{GU}_3(2^2)$, so M is a maximal subgroup of G (see Lemma 2.5.9 (iii)). Then M is strongly constrained if and only if $r \nmid |G : \mathrm{P}\Omega(V)|$.*
Furthermore, if M is strongly constrained then M is strongly 3-constrained and we have that $|M| \leq 2^4 3^3$ where this upper bound is sharp.

Proof. Regarding the assertion, w.l.o.g. we can consider a concrete representation of $\Omega(V)$ with respect to an ordered basis of V. So, let $\mathrm{P}\Omega \leq G \leq \mathrm{PA}$ where $\Omega = \mathrm{SU}_3(2^{2r})$ and the matrix of the non-degenerate unitary form is

$\mathbb{1}_3$. We use the notation introduced in Example 2.5.16. By Proposition 2.5.7, we have that $M \cap \mathrm{PSU}_3(2^{2r}) \in \mathscr{C}_5$ of $\mathrm{PSU}_3(2^{2r})$ of type $\mathrm{GU}_3(2^2)$. In view of the conjugacy considerations from Example 2.5.16, w.l.o.g. we can assume that $M \cap \mathrm{PSU}_3(2^{2r}) = \mathrm{P}H$, see Lemma 1.4.20. Hence, we have that $M = \mathrm{N}_G(\mathrm{P}H) = K_1 \cap G$ where $K_1 = \mathrm{N}_{\mathrm{P\Gamma U}_3(2^{2r})}(\mathrm{P}H) = K : \langle \varphi_2 \rangle$ (recall that $\mathrm{N}_{\mathrm{PGU}_3(2^{2r})}(\mathrm{P}H) = K$, by Example 2.5.16). We will show that

$$M \text{ is strongly constrained if and only if } r \nmid |M| \quad (*).$$

So, our assertion easily follows, since we have $|G/\mathrm{PSU}_3(2^{2r})| = |M/\mathrm{P}H|$, and $r \nmid |M|$ if and only if $r \nmid |M/\mathrm{P}H|$.
To prove assertion $(*)$, we first note that $\mathrm{O}_3(M) \geq \mathrm{O}_3(\mathrm{P}H) = \mathrm{P}N > 1$, by Example 2.5.16. So, if M is strongly constrained then M is strongly 3-constrained. Hence, the only-if-part of assertion $(*)$ is a direct consequence from Lemma 2.5.18 (i). Now, let $r \nmid |M|$. Since $|M| \mid |K_1|$, we easily obtain $|M| \mid 2^4 3^3$. Clearly, we have that $\mathrm{E}(M) = 1$, and so we obtain our assertion by Lemma 2.5.18 (ii). Furthermore, as a concrete example where M is strongly 3-constrained and $|M| = 2^4 3^3$ we can now provide a maximal subgroup of $G = \mathrm{PGU}_3(2^{2r}) : \langle \varphi_2^r \rangle$. □

We note the following corollary.

Corollary 2.5.23. *In the situation of Proposition 2.5.22, we have* $\mathrm{O}_3(M) = \mathrm{O}_3(M \cap \mathrm{P}\Omega(V))$.

Proof. Considering the proof of the last proposition, the assertion follows by elementary observations. (Note, that $\mathrm{O}_3(K_1) = \mathrm{O}_3(K) = \mathrm{O}_3(\mathrm{P}H)$). □

Summarizing our previous results, we obtain our main theorems of this section. We determine all pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P}\Omega$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_5$ of G. As usual, we present the following results not using the standard notation.

Main Theorem 2.5.24. *Let* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ *where* $\mathrm{PSL}_n(q)$ *is simple. Let* M *be a member of A-class* $\mathscr{C}_5$ *of* G*. Then* M *is a strongly constrained maximal subgroup of* G *if and only if one of the following holds.*

(i) $G = \mathrm{PSL}_2(4)$*. Here,* M *is of type* $\mathrm{GL}_2(2)$ *and strongly* 3*-constrained.*

(ii) $n = 2$, $q = 3^r$ *for a prime* r*,* M *is of type* $\mathrm{GL}_2(3)$ *and we have*

  (a) $r = 2$ *and* G *is a subgroup of* $\mathrm{PSL}_2(9) : \langle \varphi_3 \rangle$*, or*

  (b) $r \geq 3$ *and* G *is a subgroup of* $\mathrm{PGL}_2(3^r)$*.*

  In cases (a) and (b) M *is strongly* 2*-constrained.*

Furthermore, if M *is a strongly constrained maximal subgroup of* G *we have* $|M| \leq 2^4 3$ *where this upper bound is sharp.*

Proof. The assertion is a direct consequence from Lemma 2.5.9, Propositions 2.5.8 and 2.5.14 and Corollary 2.5.12. □

Main Theorem 2.5.25. *Let* $\mathrm{PSU}_n(q^2) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q^2))$ *where* $n \geq 3$ *and* $\mathrm{PSU}_n(q^2)$ *is simple. Let* M *be a member to A-class* $\mathscr{C}_5$ *of* G*. Then* M *is a strongly constrained maximal subgroup of* G *if and only if* $n = 3$, $q = 2^r$ *for an odd prime* r, M *is of type* $\mathrm{GU}_3(2^2)$ *and we have that*

(i) $9 \nmid 2^r + 1$ *and* $r \nmid |G : \mathrm{PSU}_3(2^{2r})|$, *or*

(ii) $9 \mid 2^r + 1$ *and one of the following holds.*

(a) $r = 3$ *and there is an* $\alpha \in \mathrm{Aut}(\mathrm{PSU}_3(2^6))$ *such that*

$$M^\alpha \leq G^\alpha \leq \mathrm{PSU}_3(2^6) : \langle \varphi_2 \rangle$$

where φ_2 *stabilizes the* $\mathrm{PSU}_3(2^6)$*-conjugacy class of* $M^\alpha \cap \mathrm{PSU}_3(2^6)$.

(b) $r > 3$ *and there is an* $\alpha \in \mathrm{Aut}(\mathrm{PSU}_3(2^{2r}))$ *such that*

$$M^\alpha \leq G^\alpha \leq \mathrm{PSU}_3(2^{2r}) : \langle \varphi_2^r \rangle$$

where φ_2 *(or, equivalently* φ_2^r*) stabilizes the* $\mathrm{PSU}_3(2^{2r})$*-conjugacy class of* $M^\alpha \cap \mathrm{PSU}_3(2^{2r})$.

In cases (i) and (ii) M *is strongly* 3*-constrained and we have* $|M| \leq 2^4 3^4$ *where this upper bound is sharp.*

Proof. Our assertion follows by Lemma 2.5.9 and Propositions 2.5.8, 2.5.19 and 2.5.22. □

Remark. Concerning well-definedness, we recall that we do not have to specify the basis in the previous main theorem for a unique description of the results. (Cf. Remark 2.5.10 (a) or Propositions 2.5.19 and 2.5.22).

2.6 A-class $\mathscr{C}_6$

Roughly described, the members of A-class $\mathscr{C}_6$ are the normalizers of certain absolutely irreducible symplectic-type r-groups for a prime r. For the introduction of the members of A-class $\mathscr{C}_6$ we follow the paper of Aschbacher [As] and the book [KL], but we will treat issues in more detail and fill some gaps.
First, we provide elementary facts concerning extraspecial r-groups which are important for the description of the members of A-class $\mathscr{C}_6$.

Definition 2.6.1. Let r be a prime and R be an r-group. Then R is called *extraspecial* if $\mathrm{Z}(R) = \Phi(R) = R' \cong \mathbf{Z}_r$.

Since an extraspecial r-group R (for a prime r) is not abelian, we have $|R| > r^2$. Furthermore, every non-abelian r-group of order r^3 is extraspecial, see [Su2, p. 67]. Using [Su2, Chapter 4 (4.13)], we obtain the following lemma.

Lemma 2.6.2. *Let r be a prime and R be an extraspecial r-group of order r^3. Then one of the following holds.*

(a) For $r = 2$, R is isomorphic to the dihedral group of order eight $D_8 = \langle x, y \mid x^y = x^{-1}, x^4 = y^2 = 1\rangle$, or the quaternion group $Q_8 = \langle x, y \mid x^y = x^{-1}, x^4 = 1, y^2 = x^2\rangle$; each of these groups has exponent 4.

(b) For $r > 2$, R is isomorphic to the group $R_1 = \langle x, y \mid x^r = y^r = [x, y]^r = 1, [x, y] \in \mathrm{Z}(R_1)\rangle$ of exponent r, or $L_1 = \langle x, y \mid x^{r^2} = y^r = 1, [x, y] = x^r\rangle$ of exponent r^2.

Recall Definition 1.3.7 of a central product of two finite groups. Considering the central product of two copies of Q_8 (with respect to $\mathrm{Z}(Q_8)$), we obtain by elementary observations that the resulting group is isomorphic to the central product of two copies of D_8 (with respect to $\mathrm{Z}(D_8)$), see e.g. [Hu, III. proof of 13.8 Satz] or [Su, p. 139-140]. For the central product of two copies of L_1 (with respect to $\mathrm{Z}(L_1)$) we obtain that the resulting group is isomorphic to the central product of R_1 and L_1 (with respect to $\mathrm{Z}(R_1)$ and $\mathrm{Z}(L_1)$), see [DH, p.79]. So, using structure investigations about extraspecial r-groups, see [DH, A. (20.4) Lemma] or [Hu, III. 13.7 Satz d)], we obtain the following proposition determining all isomorphism types of extraspecial r-groups by central products of extraspecial r-groups of order r^3.

Proposition 2.6.3. *(see [DH, A. (20.5) Theorem], [Su2, Chapter 4 Theorem 4.18], or cf. [Hu, p. 353-356])*
Let r be a prime and R be an extraspecial r-group. Then R is a central product of m extraspecial r-groups of order r^3 and $|R| = r^{2m+1}$. Furthermore, exactly one of the following holds.

(i) $r = 2$ and R is a central product of m groups isomorphic to D_8, esp. R has exponent 4. We denote this type by 2_+^{1+2m}.

(ii) $r = 2$ and R is a central product of $m - 1$ groups isomorphic to D_8 and one isomorphic to Q_8, esp. R has exponent 4. We denote this type by 2_-^{1+2m}.

(iii) $r > 2$ and R is a central product of m groups isomorphic to R_1, esp. R has exponent r. We denote this type by r^{1+2m}.

(iv) $r > 2$ and R is a central product of $m-1$ groups isomorphic to R_1 and one isomorphic to L_1, esp. R has exponent r^2.

Remark 2.6.4. (a) In view of the results of the last proposition, we note that for an extraspecial r-group R we also directly see by its definition that $\exp(R) \leq r^2$.

(b) In context to the above mentioned quotes [DH, A. (20.4) Lemma] and [Hu, III. 13.7 Satz d)], we note the following observation concerning extraspecial r-groups which is useful to keep in mind. By the definition of an extraspecial r-group R we have that $R/\mathrm{Z}(R)$ is elementary abelian, so $R/\mathrm{Z}(R)$ can be viewed as a $\mathrm{GF}(r)$-vector space (of dimension $2m$ for

$|R| = r^{2m+1}$). The commutator map of R induces a well-defined bilinear form from $R/\mathrm{Z}(R) \times R/\mathrm{Z}(R)$ to $\mathrm{GF}(r)$ which can be considered as a non-degenerate symplectic form on the vector space $R/\mathrm{Z}(R)$, see [DH, A. (20.4) Lemma] and [Hu, III. 13.7 Satz b)].

Next, we provide information about the automorphism group of an extraspecial r-group. For the determination of the structure of the automorphism group of an extraspecial r-group we refer to the book of Doerk and Hawkes [DH, A. Section 20][16] where this is accomplished in detail.

Proposition 2.6.5. *(see [DH, A. (20.8) Theorem] or [Win, Theorem 1]) Let r be a prime and R be an extraspecial r-group of order r^{2m+1}, so $m \geq 1$. Let R be of type 2_+^{1+2m}, 2_-^{1+2m} or r^{1+2m}. Then the following hold.*

(a) $\mathrm{Aut}(R) = B \rtimes T$ *where* $B = \mathrm{C}_{\mathrm{Aut}(R)}(\mathrm{Z}(R))$ *and T is a cyclic group of order $r-1$.*

(b) $\mathrm{Inn}(R) \cong R/\mathrm{Z}(R)$ *is an elementary abelian group of order r^{2m}.*

(c) If $r = 2$ we have $B/\mathrm{Inn}(R) \cong \mathrm{O}_{2m}^{\epsilon}(2)$ *for 2_ϵ^{1+2m} where $\epsilon \in \{+, -\}$.*

(d) For $r > 2$ we have that $B/\mathrm{Inn}(R) \cong \mathrm{Sp}_{2m}(r)$.

Remark. For an assertion concerning the automorphism group of an extraspecial r-group of type occurring in Proposition 2.6.3 (iv), see [Gr, p. 404] or [Win, Theorem 1 (b)].

In view of [KL, Proposition 2.10.6.][17] and our previous considerations, we note the following corollary.

Corollary 2.6.6. *We use the notation of Proposition 2.6.5. Then we have that* $\mathrm{Inn}(R)$ *is a minimal normal subgroup of B if and only if $(r, m, \epsilon) \neq (2, 1, +)$.*

In the following proposition, we provide further information concerning the automorphism group $\mathrm{Aut}(R)$ of an extraspecial r-group R of type 2_+^{1+2m}, 2_-^{1+2m} or r^{1+2m} (not considered in [As] or [KL]). In [Gr], it was (amongst other things) investigated in which cases the automorphism group $\mathrm{Aut}(R)$ is a split extension of $\mathrm{Inn}(R)$ by $\mathrm{Out}(R)$.

Proposition 2.6.7. *Let r be a prime and R be an extraspecial r-group of type 2_+^{1+2m}, 2_-^{1+2m} or r^{1+2m} (so $m \geq 1$). Then the following hold.*

(a) If $r \neq 2$ we have that $\mathrm{Aut}(R)$ *is a split extension of* $\mathrm{Inn}(R)$ *by* $\mathrm{Out}(R)$.

[16] Here, we note that [DH, A. (20.7) Proposition] is not correct and refer to [Hu3, 7.6 Examples d) and Exercise E7.2] for a correct version. Furthermore, we note that the assertion in [DH, A. (20.8) Theorem] (where [DH, A. (20.7) Proposition] is used) stays valid.

[17] We note that there is a mistake in part (ii) of [KL, Proposition 2.10.6.] for the case $\mathrm{SO}_2^+(2)$, cf. the assertion of part (iii). This mistake was already mentioned in [BHR, Proposition 1.12.2].

(b) If $r = 2$ *we have that*

(i) $\mathrm{Aut}(R)$ *is a split extension of* $\mathrm{Inn}(R)$ *by* $\mathrm{Out}(R)$ *for* $m \leq 2$*,*

(ii) the extension $\mathrm{Aut}(R)$ *of* $\mathrm{Inn}(R)$ *by* $\mathrm{Out}(R)$ *is non-split for* $m \geq 3$*.*

Proof. All assertions follow by [Gr, p. 404 and Theorem 1]. □

For the following observations we provide a lemma concerning a well-known fact. Unfortunately the author could not find an appropriate reference, so we will state the elementary proof.

Lemma 2.6.8. *(cf. [Hu, p. 361 Aufgabe 34])*
Let $Z = \langle z \rangle$ *be a cyclic group of order* 4*. Then the central product of* Z *and* Q_8 *(with respect to* $\Omega_1(Z)$ *and* $\mathrm{Z}(Q_8)$*) is isomorphic to the central product of* Z *and* D_8 *(with respect to* $\Omega_1(Z)$ *and* $\mathrm{Z}(D_8)$*).*

Proof. Let $Q_8 = \langle q_1, q_2 \rangle$ as in Lemma 2.6.2, esp. $\mathrm{o}(q_1) = \mathrm{o}(q_2) = 4$. Consider $G = Z \circ Q_8 = (Z \times Q_8)/Z_0 = \overline{Z \times Q_8}$ where $Z_0 = \langle (z^2, q_1^2) \rangle$. Set $\overline{Z} = \langle \overline{(z, 1)} \rangle \cong Z$. Let $\overline{d_j} = \overline{(z, q_j)} \in G$ for $j \in \{1, 2\}$. We have obviously $\mathrm{o}(\overline{d_j}) = 2$ and $\overline{d_3} = \overline{d_1 d_2} \in G$ is an element of order 4. By [Hu, I. 19.5 Beispiel], we obtain that $\overline{D} = \langle \overline{d_1}, \overline{d_2} \rangle \cong D_8$. It is clear that $[\overline{Z}, \overline{D}] = 1$, and since we have $\mathrm{Z}(\overline{D}) = \langle \overline{d_3}^2 \rangle = \Omega_1(\overline{Z})$ (note, that $(q_1 q_2)^2 = q_1^2 = q_2^2$), we obtain $G \cong Z \circ D_8$. □

For the introduction of the members of A-class $\mathscr{C}_6$, we need the terminology of an r-group of symplectic-type.

Definition 2.6.9. Let r be a prime and G be an r-group. G is called of *symplectic-type* if every characteristic abelian subgroup of G is cyclic.

Remark. The structure of symplectic-type r-groups is in strong relation to that of extraspecial r-groups, see a theorem of P. Hall [Su2, Chapter 4 Theorem 4.22.] or [Hu, III. 13.10 Satz].

Example 2.6.10. Let Z be a cyclic group of order 4 and $R^{\pm}$ an extraspecial 2-group of type $2_{\pm}^{1+2m}$. Then we have that $G^{\pm} = Z \circ R^{\pm}$ (with respect to $\Omega_1(Z)$ and $\mathrm{Z}(R^{\pm})$) is a 2-group of symplectic-type, see the theorem of P. Hall. In view of Proposition 2.6.3 and Lemma 2.6.8 we can deduce that G^+ is isomorphic to G^- (cf. also [KL, p. 149]), and groups isomorphic to G^+ (or G^-) will be called of type $4 \circ 2^{1+2m}$. Furthermore, we note that $\mathrm{Z}(G^{\pm}) = Z$, via identification.

Since 2-groups of type $4 \circ 2^{1+2m}$ will be important for our further investigations, we provide the following proposition concerning their automorphism group.

Proposition 2.6.11. *Let* R *be a* 2*-group of type* $4 \circ 2^{1+2m}$ *(so* $m \geq 1$*). Then the following hold.*

(a) $\mathrm{Inn}(R)$ *is an elementary abelian* 2*-group of order* 2^{2m}*.*

(b) $\mathrm{Out}(R) \cong \boldsymbol{Z}_2 \times \mathrm{Sp}_{2m}(2)$.

(c) *For* $B = \mathrm{C}_{\mathrm{Aut}(R)}(\mathrm{Z}(R))$ *we have* $B/\mathrm{Inn}(R) \cong \mathrm{Sp}_{2m}(2)$.

(d) *For* $m \geq 2$ *we have that the extension* $\mathrm{Aut}(R)$ *of* $\mathrm{Inn}(R)$ *by* $\mathrm{Out}(R)$ *and the extension* B *of* $\mathrm{Inn}(R)$ *by* $B/\mathrm{Inn}(R) \cong \mathrm{Sp}_{2m}(2)$ *are non-split.*

Proof. All assertions follow by [Gr, p. 403-404, Corollaries 2 and 3]. □

Remark. Exercise 8.5. in [As2] also handles the case of the last proposition. Kleidman and Liebeck have cited this reference for providing facts in [KL, p. 149]. We note that the mentioned exercise is not correct (cf. the last proposition), but the facts Kleidman and Liebeck have cited are valid; so there arises no mistake by this quote in their following discussions.

We note the following corollary from our previous considerations together with [KL, Proposition 2.10.6.].

Corollary 2.6.12. *Using the notation of Proposition 2.6.11,* $\mathrm{Inn}(R)$ *is a minimal normal subgroup of* B*.*

Now, we have provided enough information to start the examinations which lead to the definition of the members of A-class $\mathscr{C}_6$. For this, we take a closer look at one of the main proofs in Aschbacher's paper, [As, Section 11 Theorem Γ], for the cases **L** and **U** relevant in this thesis.
At [As, p. 504], we see that a relevant subgroup of $\Gamma(V)$, which leads to a member of A-class $\mathscr{C}_6$ of $\Gamma(V)$, is the normalizer of an r-group $R \leq \mathrm{I}(V)$ for a prime r. (We note that in Aschbacher's paper this subgroup is denoted by L). This r-group R has a condition concerning minimality that leads to the fact that $\mathrm{Z}(R)$ is the unique maximal abelian characteristic subgroup of R. Since R acts also irreducibly on the underlying vector space V over a finite field of characteristic p, we have that $r \neq p$ (note, that all Sylow p-subgroups of $\mathrm{I}(V)$ act reducibly on V) and the action of $\mathrm{Z}(R)$ on V corresponds to scalar matrices on V, cf. [Go, Chapter 3 proof of Theorem 2.2]. So, $\mathrm{Z}(R)$ is cyclic and R is an r-group of symplectic-type. In view of the theorem of P. Hall (see [Su2, Chapter 4 Theorem 4.22.] or [Hu, III. 13.10 Satz]) and the minimal condition put on R, we can deduce that R has to be an extraspecial r-group of exponent r if $r > 2$, or $R = \mathrm{Z}(R) \circ R_0$ (with respect to $\Omega_1(\mathrm{Z}(R))$ and $\mathrm{Z}(R_0)$) where R_0 is an extraspecial 2-group and R has exponent 4 if $r = 2$ (cf. [As, p. 504]). Recalling that $\mathrm{Z}(R)$ is cyclic, clearly $1 < |\mathrm{Z}(R)| \mid 4$ for $r = 2$. So, by Proposition 2.6.3 and Example 2.6.10, we obtain that there is exactly one isomorphism type of R to consider if $r \neq 2$, two isomorphism types if $r = 2$ and $|\mathrm{Z}(R)| = 2$ and one isomorphism type if $r = 2$ and $|\mathrm{Z}(R)| = 4$.
Summarizing the information provided above, we obtain the following table about the relevant possibilities for R. This table coincides with [KL, Table 4.6.A], except for the last column which indicates whether $\mathrm{C}_{\mathrm{Aut}(R)}(\mathrm{Z}(R))$ splits over $\mathrm{Inn}(R)$ or not.

Table 2.6.1

Type of R	$\|R\|$	$\|\mathrm{Z}(R)\|$, $\exp(R)$	$\mathrm{C}_{\mathrm{Aut}(R)}(\mathrm{Z}(R))$	Split
r^{1+2m}, r odd prime	r^{1+2m}	r, r	$r^{2m} : \mathrm{Sp}_{2m}(r)$	always split
2_ϵ^{1+2m}, $\epsilon \in \{+,-\}$	2^{1+2m}	2, 4	$2^{2m}.\mathrm{O}^\epsilon_{2m}(2)$	non-split for $m \geq 3$ split for $m \leq 2$
$4 \circ 2^{1+2m}$	2^{2+2m}	4, 4	$2^{2m}.\mathrm{Sp}_{2m}(2)$	non-split for $m \geq 2$

As a convenience, we (sometimes) also denote an r-group of type r^{1+2m} (r odd), $2_\pm^{1+2m}$, or $4 \circ 2^{1+2m}$ by r^{1+2m}, $2_\pm^{1+2m}$, or $4 \circ 2^{1+2m}$, respectively.

To obtain more information about the particular situation, we provide some facts concerning the representation theory of the groups in the previous table. The following proposition treats the cases for those groups which are extraspecial, i.e. the groups of type r^{1+2m} or $2_\pm^{1+2m}$.

Proposition 2.6.13. *Let r be a prime and R be an extraspecial r-group of Table 2.6.1, so R is of type $2_\pm^{1+2m}$ or r^{1+2m} (for r odd). Let p be a prime different from r. Then R has precisely $k = |\mathrm{Z}(R)| - 1$ inequivalent faithful absolutely irreducible representations over an algebraically closed field of characteristic p, which we denote by $\rho_1, \dots, \rho_k$. The representations ρ_j are quasiequivalent (see [BHR, Definitions 1.8.1, 1.8.4] or [KL, p. 55]), have degree r^m and can be realized over any field of order p^a where a is a positive integer with $p^a \equiv 1 \pmod r$.*

Proof. See [DH, B. (9.16), (9.17) Theorems], or [Su2, p. 335] and [Is, (15.13) Theorem]. □

Remark. The last proposition coincides with [KL, Proposition 4.6.3.], but it gives another reference which is more adequate for the situation. Moreover, there is a gap in [KL] for providing the necessary information. It is said that [KL, Proposition 4.6.3.] handles the representation theory for all groups of [KL, Table 4.6.A]; but actually only the information for the extraspecial groups (so, the groups from the first three rows of the mentioned table) is provided, cf. the results in the following Proposition 2.6.14. We note that there are no mistakes (resulting from this gap) in the further investigations in [KL], since the results for the remaining group of type $4 \circ 2^{1+2m}$ are similar to those for the extraspecial groups.

In the following proposition, we consider the representation theory of the remaining case of Table 2.6.1. Here, we obtain results by putting it down to the case of the representation theory of the extraspecial groups.

Proposition 2.6.14. *Let R_0 be an extraspecial 2-groups of order 2^{1+2m} and Y be a cyclic group of order 4. Consider the central product R of Y and R_0 (with respect to $\Omega_1(Y)$ and $\mathrm{Z}(R_0)$). Let p be an odd prime. Then R has precisely 2 inequivalent faithful absolutely irreducible representations ρ_1, ρ_2 over an* *algebraically closed field of characteristic p.* ρ_1, ρ_2 *are quasiequivalent, have degree* 2^m *and they can be realized over any field of order* p^a *where a is a positive integer with* $p^a \equiv 1 \pmod 4$.

Proof. (see [Gr, p. 416]) The irreducible representations of $Y \circ R_0$ over an algebraically closed field can be expressed as product of those of Y and R_0, see [Go, Chapter 3 Section 7]. Obviously, Y has exactly two faithful irreducible representations, each of degree 1. Using the information from Proposition 2.6.13 and [Go, Chapter 3 proof of Theorem 2.2.], we obtain our assertion. □

As we see by the last two propositions, the faithful absolutely irreducible representations of an r-group R of type r^{1+2m} (r odd), $2_{\pm}^{1+2m}$ or $4 \circ 2^{1+2m}$ of degree r^m can be realized over any field of order p^a with $p \neq r$ a prime and

$$p^a \equiv 1 \pmod{|\mathrm{Z}(R)|}. \tag{2.6.1}$$

By the proof of Aschbacher [As, p. 504], we have to consider those representations which can not be realized over a proper subfield, see [As, (11.7)]. So, a has to be minimal subject to (2.6.1). We note that the last condition is imposed for avoiding that the members of A-class $\mathscr{C}_6$ are subgroups of the members of A-class $\mathscr{C}_5$, see Definition 2.5.2 and Propositions 2.6.13 and 2.6.14 (cf. also [KL, p. 150]).

Keeping in mind the information so far provided in this section, we are now able to introduce the members of A-class $\mathscr{C}_6$. As usual, we use the generalized standard notation in the following definition.

Definition 2.6.15. {***A-class*** $\mathscr{C}_6$} (cf. [As, p. 472], [KL, p. 60, 150] and [BHR, p. 71])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and let K be a subgroup of G. Let p be the characteristic of $\mathrm{GF}(q^u)$. For $G \leq \Gamma(V)$ the subgroup K belongs to *(A-class)* $\mathscr{C}_6$ of G if $K = \mathrm{N}_G(R)$ where the following conditions hold.

(a) $R \leq \Delta(V)$ and R is an r-group (for a prime r) of type appearing in Table 2.6.1. So, R is of type r^{1+2m} (r odd), $2_{\pm}^{1+2m}$ or $4 \circ 2^{1+2m}$ for $m \geq 1$.

(b) R acts absolutely irreducibly on V, so $r \neq p$ and $\dim(V) = n = r^m$.

(c) $q^u = p^a$ where a is the smallest integer with $p^a \equiv 1 \pmod{|\mathrm{Z}(R)|}$.

(d) R appears in the following table.

Table 2.6.2

Case	Type	Description of R	Conditions
L	$r^{1+2m} : \mathrm{Sp}_{2m}(r)$	$R \cong r^{1+2m}$	a, r odd
L	$(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$	$R \cong 4 \circ 2^{1+2m}$	$a = 1$, $n \geq 4$
L	$2_-^{1+2} : \mathrm{O}_2^-(2)$	$R \cong 2_-^{1+2}$	$a = 1$, $n = 2$
U	$r^{1+2m} : \mathrm{Sp}_{2m}(r)$	$R \cong r^{1+2m}$	r odd, a even
U	$(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$	$R \cong 4 \circ 2^{1+2m}$	$a = 2$

If $G \not\leq \Gamma(V)$ then K belongs to *(A-class)* $\mathscr{C}_6$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_6$ of $\Gamma(V)$.
The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_6$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_6$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of a type occurring in Table 2.6.2 we call K of the same type.

Remark 2.6.16. (a) Our definition of A-class $\mathscr{C}_6$ coincides with the definitions in [KL, see p. 60 and 150] and [As, p. 472]; concerning the definition in Aschbacher's paper, we note that by the condition in [As, p. 472 ($C_6$1)] the field of the underlying vector space is a splitting field for R, cf. [As, p. 504], or [DH, B. (9.16), (9.17) Theorems] and [Go, Chapter 3 Section 7].
In [BHR, Definition 2.2.13] the definition of the members of A-class $\mathscr{C}_6$ is slightly different, but equivalent. See Lemma 2.6.18 and the ensuing remark for more details.

(b) Concerning the occurring possibilities and conditions in Table 2.6.2, we note the following comments (since some of them are not explained (in detail) in [KL] and [BHR]).
Clearly, it is necessary that a is even in case **U**. Moreover, as we will see in Constructions 2.6.25 and 2.6.38 (below), the absolutely irreducible r-group R fixes a non-degenerate unitary form on V if a is even. In view of [BHR, Lemma 1.8.9], we see that a has to be odd in case **L**, since otherwise there is an inclusion of the members of A-class $\mathscr{C}_6$ in the members of A-class $\mathscr{C}_8$, see Definition 2.8.1 (below).
Consider the case that $n \geq 4$ and $|\mathrm{Z}(R)| = 2$ (not appearing in the previous table). Then we have that n is even, q is odd and by condition (c) in the last definition we obtain $a = 1$, so case **L** holds. Here, R stabilizes a symplectic or quadratic form (respectively its associated bilinear form) on V, see [As, p. 472 ($C_6$2), ($C_6$3) and p. 504]. Again, by [BHR, Lemma 1.8.9], we see that these possibilities are not to consider in case **L**, in view of the members of $\mathscr{C}_8$.
For the case $n = 2$ we have by our restriction of the dimension that case **L** is given. Here, we obtain that we have to consider the case $R \cong 2_-^{1+2}$, by the isomorphism $\mathrm{SL}_2(q) \cong \mathrm{Sp}_2(q)$ and the considerations in [As, p. 504].
For more detailed information concerning these restrictions, see [As, p. 504-505].

(c) We note obvious (but important) observations concerning condition (c) in the above definition. For r being even the situation is clear, see part (b) of this remark. If r is odd we have the condition that a is the smallest (positive) integer with $p^a \equiv 1 \pmod r$. So, r is a Zsigmondy prime $z_{p,a}$ if $a \geq 2$, see Definition 1.5.1. In view of Lemma 1.5.3, we have that $a \mid r-1$. Since a is odd in case **L**, we obtain for this case that $a \mid \frac{r-1}{2^b}$ where 2^b is the highest 2-power dividing $r-1$. Moreover, we note that in case **U** r divides $p^{\frac{a}{2}} + 1$.

The properties of the centralizer of $\mathrm{Z}(R)$ in $\mathrm{Aut}(R)$ (for an r-group R occurring in Table 2.6.1) are important for the analysis of the members of A-class $\mathscr{C}_6$. By an observation of Aschbacher, the members of A-class $\mathscr{C}_6$ of $\mathrm{P}\Delta(V)$ are isomorphic to these centralizers. We note this fact in the following lemma.

Lemma 2.6.17. *(see [As, Theorem A (4)] and [KL, (4.6.1)])*
Let H be a member of A-class $\mathscr{C}_6$ of $\Delta(V)$ with corresponding subgroup R given in Table 2.6.2. Then we have $\mathrm{P}H \cong \mathrm{C}_{\mathrm{Aut}(R)}(\mathrm{Z}(R))$.

Using the previous lemma, we are able to simplify the definition of the members of A-class $\mathscr{C}_6$ which we will use in our further investigations. We will use this simplification sometimes without reference to it.

Lemma 2.6.18. *Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$. Let K be a member of A-class $\mathscr{C}_6$ of G and R be the corresponding r-subgroup in Table 2.6.2. Then we have $K = \mathrm{N}_G(R)$.*

Proof. Our assertion is clear for $G \leq \Gamma(V)$. Hence, let case **L** be given and $G \nleq \Gamma(V)$ (so, $n = \dim(V) > 2$). Following the definition of A-class $\mathscr{C}_6$ of G we have that $K = \mathrm{N}_{\mathrm{A}(V)}(K_1) \cap G$ where $K_1 = \mathrm{N}_{\Gamma(V)}(R)$. First, assume that $n \geq 4$. Since $\Gamma(V)/\Omega(V)$ is soluble, we obtain by Lemma 2.6.17, Table 2.6.1 and Propositions 1.2.11 and 1.2.12 that $K_1^\infty = H^\infty > 1$ where H is a member of A-class $\mathscr{C}_6$ of $\Omega(V)$, cf. also [KL, proof of Proposition 4.6.4.]. So, H^∞ is a non-trivial characteristic subgroup of K_1. Also in [KL, proof of Proposition 4.6.4.] it is shown that R is a characteristic subgroup of H^∞, so R is characteristic in K_1. Now, we easily can deduce $\mathrm{N}_{\mathrm{A}(V)}(K_1) = \mathrm{N}_{\mathrm{A}(V)}(R)$, and so our assertion holds in this case.
Let $n = 3$. Here, we have that $(q-1, 3) = 3$ and q is a prime, by Remark 2.6.16 (c). So, in view of Lemma 2.6.17, Construction 2.6.25 (c) (below) and elementary considerations, we obtain that $\mathrm{O}_3(K_1) = R$. Hence, our assertion follows. □

Remark. We note that in the book [BHR, see p. 70-71] the introduction of the members of A-class $\mathscr{C}_6$ is done very briefly and by the version coinciding to the previous lemma. Unfortunately, in this book there are no explanations given for that simplification of the general definitions given in [As] and [KL].

Next, we provide an important fact concerning the members of A-class $\mathscr{C}_6$ following from the last lemma. For this, we recall the notation $\hat{}$ introduced in Convention 1.2.2.

Corollary 2.6.19. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and K be a member of A-class $\mathscr{C}_6$ of G. Then we have that $K \cap \mathrm{P}\Omega(V)$ is a member of A-class $\mathscr{C}_6$ of $\mathrm{P}\Omega(V)$ of the same type as K.*

Proof. Our assertion follows from Lemma 2.6.18, cf. Lemma 1.4.6 and consider $K = \mathrm{PN}_{\widehat{G}}(R)$ for a suitable group R in Table 2.6.2. Or, see [KL, Proposition 3.1.3.]. □

The intended goal of this section is to determine the pairs (G, M) where G is an almost simple group with socle isomorphic to $\mathrm{P\Omega}(V)$ and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_6$ of G. Actually, we will see that every member of A-class $\mathscr{C}_6$ of G is a strongly constrained subgroup of G. So, we have decided to declare this more general observation to be the intended goal of this section and drop the previous intended goal. Nevertheless, we do provide the information from [KL] and [BHR] concerning maximality of the members of A-class $\mathscr{C}_6$ of G in G, since this information is important for the following chapter, and also such that the reader could easily read off the information about the previous intended goal.
In view of the definition of the members of A-class $\mathscr{C}_6$, it is advantageous to investigate the two cases for r being odd or even separately.

2.6.1 $\mathscr{C}_6$ of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$

In this subsection, we examine the cases of Table 2.6.2 for odd primes r. First, we provide the facts concerning the structure and conjugacy of the members of A-class $\mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ from the book [KL]. For this, we recall from the beginning of Subsection 1.2.3 that $\epsilon \in \{+, -\}$ where $\epsilon = +$ in case **L** and $\epsilon = -$ in case **U**. We also recall that V is a $\mathrm{GF}(q^u)$-vector space.

Proposition 2.6.20. *Let H be a member of A-class $\mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for an odd prime r, so $n = r^m$. Then the following hold.*

(i) $\mathrm{PI}(V) = \mathrm{P\Delta}(V)$ *acts transitively (by conjugation) on the members of A-class $\mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$. This $\mathrm{PI}(V)$-conjugacy class splits under the action of $\mathrm{P\Omega}(V)$ into c classes where*

$$c = \begin{cases} 1 & \text{if } n = 3 \text{ and } q \equiv \epsilon 4 \text{ or } \epsilon 7 \pmod 9, \\ (q - (\epsilon 1), n) & \text{otherwise.} \end{cases}$$

(ii) We have

$$H \cong \begin{cases} 3^2 : Q_8 & \text{if } n = 3 \text{ and } c = 1, \\ r^{2m} : \mathrm{Sp}_{2m}(r) & \text{otherwise.} \end{cases}$$

Proof. The assertion follows by [KL, Proposition 4.6.5.] and [As, Theorems BΔ,BO] (or [KL, Proposition 4.0.2.]) together with Lemma 2.6.17 and Table 2.6.1. □

Remark 2.6.21. In view of Lemma 2.6.17, Table 2.6.1 and part (ii) of the previous proposition, we see that for almost all cases we have $H \in \mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ for $H \in \mathscr{C}_6$ of $\mathrm{P\Delta}(V)$. Especially, we have $R \leq \Omega(V)$ for $H = \mathrm{PN}_{\Delta(V)}(R)$ and R as described in Table 2.6.2, as we shall also see in Construction 2.6.25, below.

Next, we provide the information about the maximality of the members of A-class $\mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ in G. For this, we recall Lemma 2.6.18.

Proposition 2.6.22. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and M be a member of A-class $\mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for an odd prime r. Let $M = \mathrm{PN}_{\hat{G}}(R)$ for a suitable group R as described in Table 2.6.2, and let the characteristic of $\mathrm{GF}(q^u)$ be p. Then M is a maximal subgroup of G if and only if one of the following holds.*

(a) *Case* **L** *is given and one of the following holds.*

   (i) *$n = 3$ (so $q = p$) and either $q \equiv 4$ or $7 \pmod 9$, or $q \equiv 1 \pmod 9$ and there is an $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega(V) : \langle\tau\rangle$ where τ stabilizes the $\mathrm{P}\Omega(V)$-conjugacy class of $\mathrm{P}R^\alpha$.*

   (ii) *$n > 3$ and there is an element $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega(V) : (\langle\varphi_p\rangle \times \langle\tau\rangle)$ where φ_p and τ stabilize the $\mathrm{P}\Omega(V)$-conjugacy class of $\mathrm{P}R^\alpha$.*

(b) *Case* **U** *is given and one of the following holds.*

   (i) *$n = 3$ (so $q^u = p^2$), $q \geq 11$ and either we have $q \equiv 2$ or $5 \pmod 9$, or $q \equiv 8 \pmod 9$ and there is an $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega(V) : \langle\varphi_p\rangle$ where φ_p stabilizes the $\mathrm{P}\Omega(V)$-conjugacy class of $\mathrm{P}R^\alpha$. Or, $n = 3$, $q = 5$ and $G = \mathrm{PI}(V)$ or $\mathrm{PA}(V)$.*

   (ii) *$n > 3$ and there is an $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega(V) : \langle\varphi_p\rangle$ where φ_p stabilizes the $\mathrm{P}\Omega(V)$-conjugacy class of $\mathrm{P}R^\alpha$.*

Proof. Our assertion follows by [BHR, Tables 8.3, 8.5, 8.18, 8.20, 8.35, 8.37, 8.54, 8.56, 8.70 [18] and 8.72] and [KL, Tables 3.5.A, 3.5.B, 3.5.G and 3.5.H and Proposition 4.6.5.(I)]. □

Remark 2.6.23. (a) We recall Remark 2.6.16 (c) to see that there are no further conditions put on $\mathrm{GF}(q^u)$ in the last proposition (note, that $\mathrm{PSU}_3(2^2)$ is not simple).

(b) Concerning the exceptions occurring for $\mathrm{P}\Omega(V) \cong \mathrm{PSU}_3(5^2)$ in the last proposition, we refer to Remark 2.2.10 (d), see also [BHR, Proposition 6.3.12].

(c) We note that in the above proposition we do not have to specify a basis for a well-defined description of the results, by [BHR2, Lemma 5 and Proposition 7] (note, that n is always odd).

In the next proposition we will determine important facts for our further considerations. For this, we recall the notation $\hat{}$ for the full preimage under P as introduced in Convention 1.2.2.

[18] We note a mistake in [BHR, Table 8.70] concerning the column 'Stab' of the members of A-class $\mathscr{C}_6$. There, the entry $\langle\gamma\rangle$ should be replaced by $\langle\phi, \gamma\rangle$, as we see by [KL, Table 3.5.G and Proposition 4.6.5.(I)] together with [BHR, Propositions 2.3.31 and 6.3.22]. Cf. also Construction 2.6.25 (c), below.

Proposition 2.6.24. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and* K *be a member of A-class* $\mathscr{C}_6$ *of* G *of type* $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ *for an odd prime* r*. Let* $K = \mathrm{PN}_{\hat{G}}(R)$ *for a suitable group* R *as described in Table 2.6.2, and* $H = K \cap \mathrm{P}\Omega(V) = \mathrm{PN}_{\Omega(V)}(R) \in \mathscr{C}_6$ *of* $\mathrm{P}\Omega(V)$*. Then* $\mathrm{O}_r(K) = \mathrm{O}_r(H) = \mathrm{P}R$ *is an elementary abelian* r*-group of order* r^{2m}*.*

Proof. First, we note that we can write $K = \mathrm{PN}_{\hat{G}}(R)$ for a suitable group R as described in Table 2.6.2, by Lemma 2.6.18. Hence, in view of Lemma 1.4.6, we obtain that $H = K \cap \mathrm{P}\Omega(V) = \mathrm{PN}_{\Omega(V)}(R) \in \mathscr{C}_6$ of $\mathrm{P}\Omega(V)$. Since we have $\mathrm{P}R = R/(R \cap \mathrm{Z}(\mathrm{GL}(V))) = R/\mathrm{Z}(R)$, we obtain that $\mathrm{P}R$ is a normal elementary abelian r-subgroup of H, cf. Lemma 2.6.17, Table 2.6.1 and Proposition 2.6.5.
To prove our assertion, let $q^u = p^a$ for a prime p and a positive integer a sufficing condition (c) of Definition 2.6.15. Since r is an odd prime, we have that $(r, 2a) = 1$ in case **L** and $(r, a) = 1$ in case **U**, in view of Remark 2.6.16 (c). Let $K_1 = K \cap \mathrm{P}\Delta(V)$. Since $|K/K_1|$ divides $|(G \cdot \mathrm{P}\Delta(V))/\mathrm{P}\Delta(V)|$, K_1 is a normal subgroup of K of index r'. Hence, we obtain $\mathrm{O}_r(K) = \mathrm{O}_r(K_1)$.
By the definition of A-class $\mathscr{C}_6$ (recall also Lemma 1.4.6), we have that $K_1 = \mathrm{PN}_{\hat{G}\cap\Delta(V)}(R) \leq M = \mathrm{PN}_{\Delta(V)}(R) \in \mathscr{C}_6$ of $\mathrm{P}\Delta(V)$. It is not hard to see that $\hat{G} \cap \Delta(V)$ is a normal subgroup of $\Delta(V)$, and hence K_1 is a normal subgroup of M. We will show that $\mathrm{O}_r(M) = \mathrm{P}R$, then our assertion follows easily. Obviously, $\mathrm{P}R$ is a normal r-subgroup of M of order r^{2m}, so $\mathrm{P}R \leq \mathrm{O}_r(M)$. Assume that $\mathrm{P}R < \mathrm{O}_r(M)$. In view of Lemma 2.6.17, Table 2.6.1 and Propositions 1.2.11, 1.2.12, 1.2.14 and 2.6.5, we obtain a contradiction (note, for $\dim(V) = n = r^m > 3$ we have that $\mathrm{Sp}_{2m}(r)$ is quasisimple and for $n = 3$ we have that $\mathrm{PSp}_2(3) \cong \mathrm{A}_4$). □

Remark. We note that the case $n = 3$ in the last proof can also be proven by elementary observations (without using the information from Lemma 2.6.17), cf. [Uf, Abschnitt 2.6, esp. Definition und Satz 2.6.3., Proposition 2.6.5. and Satz 2.6.17]. For this, we note that if $\mathrm{O}_3(K_1) \leq \mathrm{C}_{\mathrm{P}\Delta(V)}(\mathrm{P}R)$ (using the notation in the last proof) we have that $\mathrm{O}_3(K) = \mathrm{O}_3(K_1) = \mathrm{P}R = \mathrm{O}_3(K \cap \mathrm{P}\Omega(V))$, by [Uf, Lemma 2.6.2(4)], or Lemma 2.6.26, below. We also note that this elementary proof can be generalized to dimensions $n > 3$, by some effort.

For our further investigations it is necessary to consider a concrete construction of the representations from Proposition 2.6.13 for odd primes r; so provide a concrete representation of the r-groups R from Table 2.6.2 for odd primes r. For this, we recall the terminology and notation introduced in Subsection 1.3.2.

Construction 2.6.25. (cf. [KL, p. 151])

(a) For an odd prime r let R_m be an extraspecial r-group of order r^{1+2m} and exponent r (for $m \geq 1$). According to Proposition 2.6.3 (iii), we can assume that R_m is the central product of m copies of the group $R_1 = \langle x, y \rangle$ of Lemma 2.6.2 (b). First, we construct a faithful absolutely irreducible representation ρ_1 for R_1. For this, let U be a $\mathrm{GF}(p^a)$-vector space of dimension r where p is a prime and a is a positive integer with

$$p^a \equiv 1 \pmod{r} \quad (*).$$

(We note that we do not require that a is the smallest positive integer which satisfies the condition $(*)$). By $\omega \in \mathrm{GF}(p^a)^*$ we denote a primitive r-th root of 1. Choosing an ordered basis $B_U = (b_1, \ldots, b_r)$ of U, we define the following elements of $\mathrm{GL}_r(p^a)$ with respect to B_U. Let $y\rho_1$ be the element which satisfies $b_r(y\rho_1) = b_1$ and $b_j(y\rho_1) = b_{j+1}$ for $j \in \{1, \ldots, r-1\}$, and let $x\rho_1$ be the element which satisfies $b_i(x\rho_1) = w^{i-1}b_i$ for $i \in \{1, \ldots, r\}$. It is not hard to see that $R_1\rho_1 = \langle x\rho_1, y\rho_1 \rangle \cong R_1$ and ρ_1 is a faithful absolutely irreducible representation of R_1. Furthermore, we see that $R_1\rho_1 \leq \mathrm{SL}_r(p^a)$ and $\mathrm{Z}(R_1\rho_1) \leq \mathrm{Z}(\mathrm{SL}_r(p^a))$.
Naturally, we can consider the action of $\bigotimes_{j=1}^m (R_1\rho_1)$ on $V = \bigotimes_{j=1}^m U$. We have that $R_m \cong \bigotimes_{j=1}^m (R_1\rho_1)$ and so we obtain by [KL, Lemma 4.4.3. (vi)] a faithful absolutely irreducible representation ρ_m of R_m of degree r^m on V, cf. also the considerations following Definition 1.3.7 and [Go, Chapter 3 Section 7]. In view of Lemma 1.3.6 (v), we also obtain that $R_m\rho_m \leq \mathrm{SL}_{r^m}(p^a)$.

(b) Now, assume the situation introduced in part (a) and require that a is the smallest positive integer which satisfies $(*)$. Assume that a is even and consider that the $\mathrm{GF}(p^a)$-vector space U is equipped with a non-degenerate unitary form f_1 such that the ordered basis B_U is an orthonormal basis of U. By Lemma 1.2.8 (a), we obtain that $R_1\rho_1 \leq \mathrm{SU}_r(p^a)$ (note, that $\omega^{p^{\frac{a}{2}}} = \omega^{-1}$, see Remark 2.6.16 (c)). In view of [KL, (4.4.4)] (and using the notation introduced by Kleidman and Liebeck above this equation), we obtain that $\bigotimes_{j=1}^m (R_1\rho_1) \leq \mathrm{SU}_{r^m}(p^a)$ where the non-degenerate unitary form f_m on the vector space V is $f_m = \otimes_{j=1}^m f_1$, so the matrix of the form f_m is $\mathbb{1}_{r^m}$. Here, we recall Remark 2.6.16 (b) for the distinction that a is odd in case **L** and even in case **U**.

(c) In this part, we note some elementary but important observations concerning the action of the automorphisms W, φ_p and τ of $\mathrm{SL}^\epsilon_{r^m}(p^a)$ (or $\mathrm{PSL}^\epsilon_{r^m}(p^a)$) on $R_m\rho_m$ (or $\mathrm{P}R_m\rho_m$). (Recall that the field automorphism φ_p is of order a, induced by the Frobenius automorphism of $\mathrm{GF}(p^a)$ and the graph automorphism τ occurs only for $\epsilon = +$). We first consider the case $m = 1$ for the automorphisms φ_p and τ of $\mathrm{SL}^\epsilon_{r^m}(p^a)$. It is not hard to see that φ_p and τ stabilize $R_1\rho_1$. More specific, we have that $(y\rho_1)^{\varphi_p} = y\rho_1 = (y\rho_1)^\tau$, $(x\rho_1)^\tau = (x\rho_1)^{-1}$ and $(x\rho_1)^{\varphi_p} = (x\rho_1)^p$. Hence, in view of Lemma 1.3.6, we obtain that the automorphisms φ_p and τ of $\mathrm{SL}^\epsilon_{r^m}(p^a)$ stabilize $R_m\rho_m$. Moreover, we can deduce that φ_p and τ stabilize $\mathrm{N}_{\mathrm{SL}^\epsilon_{r^m}(p^a)}(R_m\rho_m) \in \mathscr{C}_6$ of $\mathrm{SL}^\epsilon_{r^m}(p^a)$. It is clear that these results transfer to the projective case.
For the diagonal automorphism W of $\mathrm{SL}^\epsilon_{r^m}(p^a)$ the situation is different. Here, we only consider the projective case $\mathrm{P}R_m\rho_m \leq \mathrm{PSL}^\epsilon_{r^m}(q^u)$ where $q^u = p^a$. Note, that $\mathrm{PSL}^\epsilon_{r^m}(q^u) < \mathrm{PGL}^\epsilon_{r^m}(q^u)$, since $(q - \epsilon 1, r^m) > 1$. We recall from Subsection 1.2.2 the definition of the diagonal automorphism W of $\mathrm{PSL}^\epsilon_{r^m}(q^u)$ and recall also that in case **U** the matrix of the non-degenerate unitary form is $\mathbb{1}_{r^m}$. For $n = r^m > 3$ it is not hard to check that every element $A \in \langle W \rangle$ with $A \neq 1$ does not normalize

$\mathrm{P}R_m\rho_m$ (consider e.g. the element $\mathrm{P}(y\rho_1 \otimes \mathbb{1}_{r^{m-1}}) \in \mathrm{P}R_m\rho_m$). In view of Proposition 2.6.24, we also can deduce that A does not normalize a member of A-class $\mathscr{C}_6$ associated to $\mathrm{P}R_m\rho_m$. Next, we examine the case $n = r^m = 3$. If $9 \nmid q - (\epsilon 1)$ we have that $\mathrm{PSL}_3^\epsilon(q^u) : \langle W_0 \rangle = \mathrm{PGL}_3^\epsilon(q^u)$ where $W_0 = \mathrm{P}(\mathrm{diag}(\omega, 1, 1))$. Here, we easily obtain that W_0 normalizes $\mathrm{P}R_1\rho_1$. For $9 \mid q - (\epsilon 1)$ we obtain by elementary calculations that for $A \in \langle W \rangle$ with $\mathrm{o}(A) \nmid 3$, A does not normalize $\mathrm{P}R_1\rho_1$. For $A \in \langle W \rangle$ with $\mathrm{o}(A) \mid 3$ we have $A \in \langle W_0 \rangle \leq \mathrm{PSL}_3^\epsilon(q^u)$ and A normalizes $\mathrm{P}R_1\rho_1$.
The described behavior concerning the action of W, φ_p and τ on $\mathrm{P}R_m\rho_m$ also shows up in Propositions 2.6.20 and 2.6.22, cf. also Remark 2.6.21. For more information concerning the case $n = 3$ see the following remark.

Remark. We refer to Examples 2.5.16 and 2.8.6 (below) for concrete investigations which are similar to considerations concerning the current case $n = 3$ in the last construction, cf. also Proposition 2.6.20 (ii). (Notice, that in Example 2.5.16 the subgroup $\mathrm{P}H \leq \mathrm{PSU}_3(2^{2r})$ and later in Example 2.8.6 the subgroup $\mathrm{P}H \leq \mathrm{PSL}_3(4)$ does not belong to A-class $\mathscr{C}_6$ of the respective group, for not fulfilling conditions of Definition 2.6.15). Furthermore, we note that the normalizer of $\mathrm{P}R_1\rho_1$ in $\mathrm{PGL}_3^\epsilon(q^u)$ (hence, a member of A-class $\mathscr{C}_6$ of $\mathrm{PGL}_3^\epsilon(q^u)$) is also known as Hessian group, see Remark 2.5.17. We also refer to [Blo, p. 176] and [Uf, Abschnitt 2.6] for more information concerning the case $n = 3$.

In our further investigations we will consider the subgroup $R_m\rho_m \leq \mathrm{SL}_{r^m}^\epsilon(p^a)$ occurring in the last construction. For this, we will identify R_m with $R_m\rho_m$ to simplify the notation. Next, we consider the centralizer of $\mathrm{P}R_m$ in $\mathrm{PGL}_{r^m}(p^a)$.

Lemma 2.6.26. *Let $R_m \leq \mathrm{SL}_{r^m}(p^a)$ from Construction 2.6.25 (a) where we note that a has not to be minimal with respect to Definition 2.6.15 (c). Then we have that*

$$\mathrm{C}_{\mathrm{PGL}_{r^m}(p^a)}(\mathrm{P}R_m) = \mathrm{P}R_m.$$

Proof. By Construction 2.6.25 (a), we directly obtain that $\mathrm{P}R_m$ is an elementary abelian r-group of order r^{2m}, cf. also Proposition 2.6.24. Hence, we have that $\mathrm{P}R_m \leq \mathrm{C}_{\mathrm{PGL}_{r^m}(p^a)}(\mathrm{P}R_m)$. To show that equality holds, we define the subset

$$M_m = \{x_1, y_1, \ldots, x_m, y_m\} \subset R_m \leq \mathrm{SL}_{r^m}(p^a)$$

where $x_1 = x \otimes \mathbb{1}_{r^{m-1}}$, $y_1 = y \otimes \mathbb{1}_{r^{m-1}}$, $x_2 = \mathbb{1}_r \otimes x \otimes Id_{r^{m-2}}$, ..., $x_m = \mathbb{1}_{r^{m-1}} \otimes x$, $y_m = \mathbb{1}_{r^{m-1}} \otimes y$. Note, that $R_m = \langle M_m \rangle$, and hence $\mathrm{P}R_m = \langle \mathrm{P}M_m \rangle$. So, we obtain that

$$\mathrm{C}_{\mathrm{PGL}_{r^m}(p^a)}(\mathrm{P}R_m) = \mathrm{C}_{\mathrm{PGL}_{r^m}(p^a)}(\mathrm{P}M_m) = \mathrm{P}C_m$$

where $C_m = \{K \mid K \in \mathrm{GL}_{r^m}(p^a) \text{ and } K^A \in K \cdot \mathrm{Z}(\mathrm{GL}_{r^m}(p^a)) \text{ for all } A \in M_m\}$. By induction, we will show that $C_m \leq \widehat{\mathrm{P}R_m} = \langle \mathrm{Z}(\mathrm{GL}_{r^m}(p^a)), R_m \rangle$, so our assertion will follow. The base case for $m = 1$ is done by elementary calculations, cf. e.g. [Uf, Lemma 2.6.2.]. As induction hypothesis assume that $C_{k-1} \leq \langle \mathrm{Z}(\mathrm{GL}_{r^{k-1}}(p^a)), R_{k-1} \rangle$ holds for some unspecified integer $k \geq 2$.

For the inductive step let $K \in \mathrm{GL}_{r^k}(p^a)$ and write $K = (K_{ij})_{1\le i,j\le r}$ where $K_{ij} \in \mathrm{Mat}_{r^{k-1}}(p^a)$. Following the notation from Construction 2.6.25 (a), let $\omega \in \mathrm{GF}(p^a)^*$ denote a primitive r-th root of 1. For the following investigations let all subscripts be read modulo r. By elementary calculations, we obtain that $K^{y_1} = (K_{(i-1)(j-1)})_{1\le i,j\le r}$. Hence, from the equation $K^{y_1} = K\cdot\lambda\mathbb{1}_{r^k}$ ($\lambda \in \mathrm{GF}(p^a)^*$) we can deduce by easy arguments that λ has to be an r-th root of 1 and $K \in \langle x_1\rangle \cdot S$ where $S = \{(A_{ij})_{1\le i,j\le r} \mid A_{ij} \in \mathrm{Mat}_{r^{k-1}}(p^a), A_{11} = A_{jj}, A_{12} = A_{j(j+1)}, \dots, A_{1r} = A_{j(j+r-1)}$ for $j \in \{1,\dots,r\}\}$. Now, let $K_S \in S$. Considering the equation $K_S^{x_1} = K_S\cdot\lambda\mathbb{1}_{r^k}$ ($\lambda \in \mathrm{GF}(p^a)^*$), we can deduce by elementary calculations that λ is an r-th root of 1 and $K_S \in \langle \mathbb{1}_r \otimes \mathrm{GL}_{r^{k-1}}(p^a), y_1\rangle$. Hence, we have $K \in \langle \mathbb{1}_r \otimes \mathrm{GL}_{r^{k-1}}(p^a), R_1 \otimes \mathbb{1}_{r^{k-1}}\rangle$, and so we obtain our assertion by the induction hypothesis. □

We need a further elementary lemma.

Lemma 2.6.27. *Let r be an odd prime and $\mathrm{GF}(p^a)$ be a finite field for a prime $p \neq r$ and a positive integer a such that a is the smallest integer with $p^a \equiv 1 \pmod r$. Let $\omega \in \mathrm{GF}(p^a)^*$ be a primitive r-th root of 1. Then the following hold.*

(a) For all $\varphi \in \mathrm{Aut}(\mathrm{GF}(p^a)) \setminus \{1\}$ we have that $\omega^\varphi \neq \omega$.

(b) If a is odd we have $\omega^\varphi \neq \omega^{-1}$ for all $\varphi \in \mathrm{Aut}(\mathrm{GF}(p^a))$.

Proof. Assertion (a) is a direct consequence of the assumption. To prove (b), suppose there is a $\varphi \in \mathrm{Aut}(\mathrm{GF}(p^a)) = \{\varphi_p^i \mid 0 \le i < a\}$ satisfying $\omega^\varphi = \omega^{-1}$ and let φ be minimal with respect to i. Note, that $1 \le i$. From $\omega^{-1} = \omega^\varphi = \omega^{p^i}$ we obtain that $r \mid p^i + 1$, hence $r \mid p^{2i} - 1$. Because of the minimality of the odd a, we get that $i < a < 2i$. So, we get a contradiction by $\omega = \omega^{p^a} = (\omega^{-1})^{p^{a-i}}$, since $0 < a - i < i$. □

Now, we have provided all necessary information to state the first main theorem of this section. For this, we recall the description just before Subsection 2.6.1, and we refer to Proposition 2.6.22 where the information concerning maximality is provided. We will consider the two cases **L** and **U** (not as usual) together, and therefore we recall from the beginning of Subsection 1.2.3 that the notation $\mathrm{PSL}^\epsilon = \mathrm{PSL}^\pm$ is PSL if $\epsilon = +$ and PSU if $\epsilon = -$.

Main Theorem 2.6.28. *For a prime p and a positive integer a let G be a group such that $\mathrm{PSL}_n^\epsilon(p^a) \le G \le \mathrm{Aut}(\mathrm{PSL}_n^\epsilon(p^a))$ for $\mathrm{PSL}_n^\epsilon(p^a)$ simple and $n \ge 3$ in case* **U**. *Let K be a member of A-class $\mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for an odd prime $r \neq p$ (esp. recall the conditions from Definition 2.6.15 (a)-(d)). Then K is strongly r-constrained.*

Proof. In view of Lemmas 1.4.20 and 2.6.18 and Propositions 2.6.20 (i) and 2.6.24, w.l.o.g. we can assume that $K = \mathrm{PN}_{\hat{G}}(R_m)$ for R_m as defined in Construction 2.6.25 and $\mathrm{O}_r(K) = \mathrm{P}R_m$. We will show that $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}^\epsilon_{r^m}(p^a))}(\mathrm{P}R_m) = \mathrm{P}R_m$, so our assertion easily follows. Recalling Lemma 2.6.26, we have to show

that there are no elements in $L = \mathrm{Aut}(\mathrm{PSL}^{\epsilon}_{r^m}(p^a)) \setminus \mathrm{PGL}^{\epsilon}_{r^m}(p^a)$ which centralize $\mathrm{P}R_m$.
First, we consider the case **L**. Suppose that $g \in L$ centralizes $\mathrm{P}R_m$, so we can write $g = \tau^j \varphi_p^i B$ for $B \in \mathrm{PGL}_{r^m}(p^a)$, $j \in \{0,1\}$ and $i \in \{0,\dots,a-1\}$ where $j \neq 0$ or $i \neq 0$. Recall the subsets $M_m = \{x_1, y_1, \dots, x_m, y_m\} \subset R_m$ and $S \subset \mathrm{Mat}_{r^m}(p^a)$ from the proof of Lemma 2.6.26. Since we have that τ and φ_p centralize $\mathrm{P}y_1$, see Lemma 1.3.6 and Construction 2.6.25 (c), we obtain analogously to Lemma 2.6.26 that $B = \mathrm{P}\tilde{B}$ where $\tilde{B} \in \langle x_1 \rangle \cdot S$. Write $\tilde{B} = \tilde{B}_x \tilde{B}_S$ where $\tilde{B}_x \in \langle x_1 \rangle$ and $\tilde{B}_S \in S$. From Lemma 2.6.27 we obtain that $\mathrm{P}x_1 = \mathrm{P}x_1^g = (\mathrm{P}x_1)^{\tau^j \varphi_p^i \mathrm{P}\tilde{B}} = (\mathrm{P}x_1^l)^{\mathrm{P}\tilde{B}_S}$ for a suitable $l \in \{2,\dots,r-1\}$ (note, that $l \neq 1$). Considering the last equation, $\tilde{B}_S$ has to satisfy $x_1^l \tilde{B}_S x_1^{-1} = \tilde{B}_S \cdot \lambda \mathbb{1}_{r^m}$ for a $\lambda \in \mathrm{GF}(p^a)^*$. By elementary considerations, we obtain that $\tilde{B}_S = 0$, and hence we have a contradiction.
The proof of case **U** is carried out analogously to case **L**, so our assertion is established. □

In view of Propositions 2.6.20 and 2.6.24 and the proof of the last main theorem, we can deduce the following corollary.

Corollary 2.6.29. *Let $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ and K be a member of A-class $\mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for an odd prime r. Let $\mathrm{P}R = \mathrm{O}_r(K)$ with R as defined in Table 2.6.2. Then we have that $\mathrm{C}_{\mathrm{PA}(V)}(\mathrm{P}R) = \mathrm{P}R$.*

Remark 2.6.30. We note that the result of the last corollary is consistent with the observation in [DH, A. (20.8) Theorem (a)] (to see this recall Lemma 2.6.17). Using these references, it would also be possible (by other efforts) to obtain the result of the last corollary, and deduce from that result the assertion of Main Theorem 2.6.28. The author has decided to present the approach above for proving Main Theorem 2.6.28 to provide an argumentation by elementary calculations and combinatorial arguments.

2.6.2 $\mathscr{C}_6$ of types $2_-^{1+2} : \mathrm{O}_2^-(2)$ and $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$

Next, we consider the cases in Table 2.6.2 where $r = 2$. It is advantageous to distinguish the two cases $n = 2$ and $n \geq 4$. We start our investigation with the case $n = 2$, so let case **L** be given. Using the information provided so far in this section, we can state the next main theorem of this section.

Main Theorem 2.6.31. *For a prime power $q > 3$ let G be a group such that $\mathrm{PSL}_2(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_2(q))$ (in particular $\mathrm{PSL}_2(q)$ is simple). Let K be a member of A-class $\mathscr{C}_6$ of G, so K is of type $2_-^{1+2} : \mathrm{O}_2^-(2)$ and $q = p \geq 5$ is a prime (recall Definition 2.6.15). Then K is strongly 2-constrained, $\mathrm{O}_2(K) = \mathrm{O}_2(K \cap \mathrm{PSL}_2(q))$ is an elementary abelian 2-group of order 4 and $|K| \bigm| 2^3 3$ where this upper bound is sharp.*

Proof. First, we note $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PGL}_2(p)$, so $G \in \{\mathrm{PSL}_2(p), \mathrm{PGL}_2(p)\}$. Let $\tilde{K} = \mathrm{PN}_{\mathrm{GL}_2(p)}(R)$ be an arbitrary member of A-class $\mathscr{C}_6$ of $\mathrm{PGL}_2(p)$ for a group R as described in Table 2.6.2. In view of Lemma 2.6.17 and Propositions 1.2.12, 1.2.14, 2.6.5 and 2.6.7, we obtain that $\tilde{K} \cong \mathrm{S}_4$, cf. also [BHR, Table 2.9] and [KL, Proposition 4.6.7. (II)]. Hence, $\tilde{K}$ is strongly 2-constrained and $\mathrm{O}_2(\tilde{K}) = \mathrm{P}R$, since $\mathrm{P}R \cong R/\mathrm{Z}(R) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ is a normal 2-subgroup of $\tilde{K}$. Since $H = \tilde{K} \cap \mathrm{PSL}_2(p)$ is a normal subgroup of $\tilde{K}$ of index dividing 2 and $\tilde{K} \cong \mathrm{S}_4$, we can deduce by elementary arguments that $\mathrm{P}R \leq H \leq \mathrm{PSL}_2(p)$. In particular, $\mathrm{O}_2(\tilde{K}) = \mathrm{O}_2(H) = \mathrm{P}R$. So, the assertion holds for the case $G = \mathrm{PGL}_2(p)$.
In view of Lemmas 1.4.6 and 1.4.21, we see that H is a strongly 2-constrained member of A-class $\mathscr{C}_6$ of $\mathrm{PSL}_2(p)$. Since the members of A-class $\mathscr{C}_6$ of $\mathrm{PSL}_2(p)$ are conjugate under $\mathrm{Aut}(\mathrm{PSL}_2(p))$, see [As, Theorems B$\Delta$,BO] (or [KL, Proposition 4.0.2.]), we obtain that all members of A-class $\mathscr{C}_6$ of $\mathrm{PSL}_2(p)$ are strongly 2-constrained. So, the assertion is also established for the case $G = \mathrm{PSL}_2(p)$ (recall from above that $\mathrm{O}_2(H) = \mathrm{P}R$). ☐

Remark 2.6.32. (a) (See [KL, 153-155]). We use the notation in the above proof. As the image under a faithful absolutely irreducible representation of Q_8 (i.e. an extraspecial group of type 2_-^{1+2}) in $\mathrm{SL}_2(p)$, w.l.o.g. we can consider R to be generated by x and y where $x = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $y = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ for $a^2 + b^2 = -1$. (Note, that there are always elements $a, b \in \mathrm{GF}(p)$ with $a^2 + b^2 = -1$, see e.g. [Hu, II. 10.6 Hilfssatz]). Hence, for the structure of a member T of A-class $\mathscr{C}_6$ of $\mathrm{PSL}_2(p)$ we have that

$$T \cong \mathrm{A}_4 : c \quad \text{where} \quad c = \begin{cases} 1 & \text{if } p \equiv \pm 3 \pmod 8, \\ 2 & \text{if } p \equiv \pm 1 \pmod 8. \end{cases}$$

This is clear, by observing $\mathrm{P}R = \langle \mathrm{P}x, \mathrm{P}y \rangle$ together with $\mathrm{P}g$ where $g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Obviously $\mathrm{P}g$ normalizes $\mathrm{P}R$ (more specific $x^g = x$ and $y^g = yx$). Since $\det(g) = 2$, we have that $\mathrm{P}g \in \mathrm{PGL}_2(p) \setminus \mathrm{PSL}_2(p)$ if and only if 2 is a non-square in $\mathrm{GF}(p)$. So, in view of Lemma 1.5.18, we see that $\mathrm{P}g \notin \mathrm{PSL}_2(p)$ if and only if $p \equiv \pm 3 \pmod 8$.

(b) By the last main theorem, a member K of A-class $\mathscr{C}_6$ of G is strongly 2-constrained if G is almost simple with $\mathrm{soc}(G) \cong \mathrm{PSL}_2(q)$. Concerning the information in which cases K is a maximal subgroup of G, we provide the reference [BHR, Table 8.1], see also [BHR, Lemma 3.1.6 and Proposition 6.3.11].

Next, we consider the case $r = 2$ and $n \geq 4$ in Table 2.6.2. First, we provide the facts about conjugacy and structure of the members of A-class $\mathscr{C}_6$ of $\mathrm{P}\Omega(V)$ of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$.

Proposition 2.6.33. *Let H be a member of A-class $\mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ (so, V is a $\mathrm{GF}(p^u)$-vector space of dimension $n = 2^m$ for an odd prime p and $m \geq 2$). Then the following hold.*

(i) $\mathrm{PI}(V) = \mathrm{P\Delta}(V)$ *acts transitively (by conjugation) on the members of A-class $\mathscr{C}_6$ of $\mathrm{P\Omega}(V)$ of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$. This $\mathrm{PI}(V)$-conjugacy class splits under the action of $\mathrm{P\Omega}(V)$ into c classes where*

$$c = \begin{cases} 2 & \text{if } n = 4 \text{ and } p \equiv \epsilon 5 \pmod 8, \\ (p - (\epsilon 1), n) & \text{otherwise.} \end{cases}$$

(ii) We have that

$$H \cong \begin{cases} 2^{4\cdot}\mathrm{A}_6 & \text{if } n = 4 \text{ and } p \equiv \epsilon 5 \pmod 8, \\ 2^{2m\cdot}\mathrm{Sp}_{2m}(2) & \text{otherwise.} \end{cases}$$

Proof. We obtain the assertion by [KL, Proposition 4.6.6.] and [As, Theorem BΔ,BO] (or [KL, Proposition 4.0.2.]) together with Lemma 2.6.17 and Table 2.6.1. □

Remark 2.6.34. Analogous considerations to Remark 2.6.21 concerning Proposition 2.6.20 can be done for the current case of the last proposition (cf. also Construction 2.6.38, below).

For $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ we provide in the following proposition the information about the maximality in G of the members of A-class $\mathscr{C}_6$ of G of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$. For this, we recall Lemma 2.6.18.

Proposition 2.6.35. *Let $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ and M be a member of A-class $\mathscr{C}_6$ of G of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ (so, $m \geq 2$ and for an odd prime p, V is a $\mathrm{GF}(p^u)$-vector space of dimension $n = 2^m$). Let $M = \mathrm{PN}_{\hat{G}}(R)$ for a suitable group R as described in Table 2.6.2. Then M is a maximal subgroup of G if and only if one of the following holds.*

(a) In case **L** *one of the following holds.*

(i) $n = 4$, and there is an element $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P\Omega}(V)\langle W^2, \tau\rangle$ where W^2, τ stabilize the $\mathrm{P\Omega}(V)$-conjugacy class of $\mathrm{P}R^\alpha$ if $p \equiv 5 \pmod 8$, or there is an element $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P\Omega}(V) : \langle\tau\rangle$ where τ stabilizes the $\mathrm{P\Omega}(V)$-conjugacy class of $\mathrm{P}R^\alpha$ if $p \equiv 1 \pmod 8$.

(ii) $n \geq 8$, $p \equiv 1 \pmod 4$ and there is an element $\alpha \in \mathrm{PA}(V)$ such that $M^\alpha \leq G^\alpha \leq \mathrm{P\Omega}(V) : \langle\tau\rangle$ where τ stabilizes the $\mathrm{P\Omega}(V)$-conjugacy class of $\mathrm{P}R^\alpha$.

(b) In case **U** *consider a representation* Ω *of* $\Omega(V)$ *with respect to an orthonormal basis of* V *and one of the following holds.*

(i) $n = 4$, *and there is an element* $\alpha \in \mathrm{PA}$ *such that* $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega\langle W^2, \varphi_p\rangle$ *where* W^2, φ_p *stabilize the* $\mathrm{P}\Omega$*-conjugacy class of* $\mathrm{P}R^\alpha$ *if* $p \equiv 3 \pmod 8$, *or there is an element* $\alpha \in \mathrm{PA}$ *such that* $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega : \langle\varphi_p\rangle$ *where* φ_p *stabilizes the* $\mathrm{P}\Omega$*-conjugacy class of* $\mathrm{P}R^\alpha$ *if* $p \equiv 7 \pmod 8$.

(ii) $n \geq 8$, $p \equiv 3 \pmod 4$ *and there is an element* $\alpha \in \mathrm{PA}$ *such that* $M^\alpha \leq G^\alpha \leq \mathrm{P}\Omega : \langle\varphi_p\rangle$ *where* φ_p *stabilizes the* $\mathrm{P}\Omega$*-conjugacy class of* $\mathrm{P}R^\alpha$.

Proof. Our assertion follows by [BHR, Tables 8.8, 8.10, 8.44 and 8.46] and [KL, Tables 3.5.A, 3.5.B, 3.5.G and 3.5.H and Proposition 4.6.6.(I)] □

Remark 2.6.36. In view of [BHR2, p. 172 (i) and Theorem 6], we have to present the results of the last proposition in case **U** with respect to an ordered basis of V for the sake of well-definedness.

Next, we consider the largest normal 2-subgroup of the members of A-class $\mathscr{C}_6$ of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$.

Proposition 2.6.37. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and* K *be a member of A-class* $\mathscr{C}_6$ *of* G *of type* $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$, *so* V *is a* $\mathrm{GF}(p^u)$*-vector space of dimension* $n = 2^m$ *for an odd prime* p. *Let* $K = \mathrm{PN}_{\hat{G}}(R)$ *for a suitable group* R *as described in Table 2.6.2, and* $H = K \cap \mathrm{P}\Omega(V) = \mathrm{PN}_{\Omega(V)}(R) \in \mathscr{C}_6$ *of* $\mathrm{P}\Omega(V)$. *Then we have that* $\mathrm{O}_2(H) = \mathrm{P}R$ *is an elementary abelian* 2*-group of order* 2^{2m} *and* $|\mathrm{O}_2(K)/\mathrm{P}R|$ *divides* 2. *Furthermore, we have that* $\mathrm{O}_2(K) = \mathrm{P}R$ *if* $K \leq \mathrm{P}\Delta(V)$.

Proof. First, we recall Lemmas 1.4.6 and 2.6.18 to see that we can generally write $K = \mathrm{PN}_{\hat{G}}(R)$ for a suitable group R as described in Table 2.6.2 and $H = K \cap \mathrm{P}\Omega(V) = \mathrm{PN}_{\Omega(V)}(R) \in \mathscr{C}_6$ of $\mathrm{P}\Omega(V)$, as assumed. Since $\mathrm{P}R = R/\mathrm{Z}(R)$, we obtain that $\mathrm{P}R$ is a normal elementary abelian 2-subgroup of H, cf. Lemma 2.6.17, Proposition 2.6.11 and Table 2.6.1.
Let $K_1 = K \cap \mathrm{P}\Delta(V) = \mathrm{PN}_{\hat{G}\cap\Delta(V)}(R)$ and note that K_1 is a normal subgroup of K of index dividing 2. Analogously to Proposition 2.6.24, we obtain that $\mathrm{O}_2(K_1) = \mathrm{P}R$, using the information provided in Lemma 2.6.17, Table 2.6.1 and Propositions 1.2.11, 1.2.12 and 1.2.14. Now, our assertion easily follows. □

Remark. Concerning the last proposition, we note that it actually occurs that $\mathrm{P}R$ is a proper subgroup of $\mathrm{O}_2(K)$, see the proof of Main Theorem 2.6.40, below.

To continue our investigation, we next provide a concrete construction of a representation from Proposition 2.6.14, cf. also Table 2.6.1. We note that some considerations are similar to those in Construction 2.6.25.

Construction 2.6.38. (cf. [KL, p. 152])

(a) Let R_m denote a 2-group of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ for $m \geq 2$. In view of Proposition 2.6.3, Lemma 2.6.8 and Example 2.6.10, we see that R_m can be considered as the central product of a cyclic group $Z = \langle z \rangle$ of order 4 and an extraspecial 2-group $R_{m,0}$ of order 2^{1+2m}. Note, that we do not have to specify the type of $R_{m,0}$, and so w.l.o.g. we can consider $R_{m,0}$ to be the central product of m copies of $Q_8 = \langle x, y \rangle$, defined as in Lemma 2.6.2 (a).
We first construct a faithful absolutely irreducible representation ρ_1 for Q_8. Let U be a 2-dimensional vector space over a finite field $\mathrm{GF}(p^a)$ for an odd prime p and a positive integer a where

$$p^a \equiv 1 \pmod 4 \quad (*).$$

(Note, that we do not require a to be the smallest positive integer satisfying this condition). By B_U, we denote an ordered basis for U. Let $i \in \mathrm{GF}(p^a)$ be a primitive 4-th root of 1. With respect to B_U, we define the following elements in $\mathrm{SL}_2(p^a)$

$$x\rho_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } y\rho_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Clearly, $Q_8\rho_1 = \langle x\rho_1, y\rho_1 \rangle \cong Q_8$ and $\mathrm{Z}(Q_8\rho_1) = \mathrm{Z}(\mathrm{SL}_2(p^a))$.
Naturally, we can consider the action of $\bigotimes_{j=1}^{m}(Q_8\rho_1)$ on $V = \bigotimes_{j=1}^{m} U$, and obtain by [KL, Lemma 4.4.3. (vi)] a faithful absolutely irreducible representation ρ_m of $R_{m,0}$ of degree 2^m on V, cf. also Construction 2.6.25. We note that $R_{m,0}\rho_m \leq \mathrm{SL}_{2^m}(p^a)$, by Lemma 1.3.6 (v). We can extend ρ_m to a faithful absolutely irreducible representation ρ of R_m on V, by defining $z\rho = i \cdot \mathbb{1}_{2^m}$, so $R_m\rho = \langle i, R_{m,0}\rho_m \rangle = \langle i \rangle \circ R_{m,0}\rho_m \leq \mathrm{SL}_{2^m}(p^a)$.

(b) From now on we consider the situation of part (a) and require that a is the smallest positive integer which satisfies $(*)$. Hence, we have $a = 1$ if $p \equiv 1 \pmod 4$ and $a = 2$ if $p \equiv 3 \pmod 4$. For $a = 2$ we may assume that the $\mathrm{GF}(p^2)$-vector space U is equipped with a non-degenerate unitary form f_1 such that B_U is an orthonormal basis of U. Analogously to Construction 2.6.25, we see that $Q_8\rho_1 \leq \mathrm{SU}_2(p^2)$ and so $R_m\rho \leq \mathrm{SU}_{2^m}(p^2)$ (note, that $i^p = -i$ and that the matrix of the non-degenerate unitary form on V is $\mathbb{1}_{2^m}$). Here, we recall from Remark 2.6.16 (b) and Table 2.6.2 that case **L** holds if $a = 1$ and case **U** if $a = 2$.

(c) Next, we consider the action of the automorphisms W, φ_p and τ of $\mathrm{PSL}^{\epsilon}_{2^m}(p^a)$ on $\mathrm{P}R_m\rho$. First, note that $\mathrm{P}R_m\rho$ is an elementary abelian 2-group of order 2^{2m}. In case **L** it is not hard to check that τ centralizes $\mathrm{P}R_m\rho$, and in case **U** we have that φ_p centralizes $\mathrm{P}R_m\rho$, cf. Lemma 1.3.6. Clearly, we can now deduce that τ normalizes $\mathrm{PN}_{\mathrm{SL}_{2^m}(p)}(R_m\rho) \in \mathscr{C}_6$ of $\mathrm{PSL}_{2^m}(p)$ in case **L** and φ_p normalizes $\mathrm{PN}_{\mathrm{SU}_{2^m}(p^2)}(R_m\rho) \in \mathscr{C}_6$ of $\mathrm{PSU}_{2^m}(p^2)$ in case **U**.
It remains to examine the action of the diagonal automorphism W on

$\mathrm{P}R_m\rho$. For this, recall the introduction of W in Subsection 1.2.2 and that in case **U** the matrix of the non-degenerate unitary form is $\mathbb{1}_{2^m}$. For $2^m \geq 8$ we obtain by elementary considerations that every element $A \in \langle W \rangle \setminus \{1\}$ does not normalize $\mathrm{P}R_m\rho$ (consider e.g. $\mathrm{P}(y\rho_1 \otimes \mathbb{1}_{2^{m-1}}) \in \mathrm{P}R_m\rho$). If $2^m = 4$ we have to distinguish two cases. For $p \equiv \epsilon 5 \pmod 8$ we have that $\mathrm{PSL}_4^\epsilon(p^a) : \langle W_0 \rangle = \mathrm{PGL}_4^\epsilon(p^a)$ where $W_0 = \mathrm{P}(\mathrm{diag}(i,1,1,1))$. It is not hard to see that $A \in \langle W_0 \rangle$ normalizes $\mathrm{P}R_2\rho$ if and only if $\mathrm{o}(A) \mid 2$. For $p \equiv \epsilon 1 \pmod 8$ we obtain by elementary calculations that an element $A \in \langle W \rangle$ with $\mathrm{o}(A) \nmid 2$ does not normalize $\mathrm{P}R_2\rho$. If $A \in \langle W \rangle$ and $\mathrm{o}(A) \mid 2$ we have that $A \in \langle W_0^2 \rangle \leq \mathrm{PSL}_4^\epsilon(p^a)$ and A normalizes $\mathrm{P}R_2\rho$.
We note that the described behavior concerning the action of W, φ_p and τ on $\mathrm{P}R_m\rho$ also shows up in Propositions 2.6.33 and 2.6.35.

In our further considerations we will work with the image under the constructed representation ρ of a 2-group R_m of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ from Construction 2.6.38 and simply write R_m for $R_m\rho$, via identification.
Next, we will provide a lemma concerning the centralizer of $\mathrm{P}R_m$ in $\mathrm{PGL}_{2^m}(p^a)$.

Lemma 2.6.39. *Let $R_m \leq \mathrm{SL}_{2^m}(p^a)$ as given in Construction 2.6.38 (a) and note that a has not to minimal with respect to Definition 2.6.15 (c). Then we have*
$$\mathrm{C}_{\mathrm{PGL}_{2^m}(p^a)}(\mathrm{P}R_m) = \mathrm{P}R_m.$$

Proof. Our assertion follows by elementary observations and calculations, analogously to Lemma 2.6.26. □

We have now provided all necessary information to state the last main theorem of this section. For this, we recall the description just before Subsection 2.6.1, and we recall Proposition 2.6.35 where the information concerning maximality is provided. As in Main Theorem 2.6.28, we will formulate the assertion for the two cases **L** and **U** (as an exception) together, hence we again recall from the beginning of Subsection 1.2.3 that the notation $\mathrm{PSL}^\epsilon = \mathrm{PSL}^\pm$ is PSL if $\epsilon = +$ and PSU if $\epsilon = -$.

Main Theorem 2.6.40. *For a prime p and a positive integer a let G be a group such that $\mathrm{PSL}_n^\epsilon(p^a) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n^\epsilon(p^a))$ for $\mathrm{PSL}_n^\epsilon(p^a)$ simple. Let K be a member of A-class $\mathscr{C}_6$ of G of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ (esp. $a = 1$ in case* **L** *and $a = 2$ in case* **U**, *$n = 2^m \geq 4$ and p is an odd prime). Then K is strongly 2-constrained.*

Proof. In view of Proposition 2.6.33 (i) and Lemmas 1.4.20 and 2.6.18, w.l.o.g. we can assume that $K = \mathrm{PN}_{\hat{G}}(R_m)$ for R_m as given in Construction 2.6.38 (b). Note, that $\mathrm{P}R_m$ is a normal non-trivial 2-subgroup of K, so $\mathrm{O}_2(K) \geq \mathrm{P}R_m > 1$. First, we will determine $\mathrm{C}_{\mathrm{Aut}(\mathrm{PSL}_{2^m}^\epsilon(p^a))}(\mathrm{P}R_m) = C$. In view of Construction 2.6.38 (c), we see that $\mathrm{P}R_m \times \langle \tau \rangle \leq C$ in case **L** and $\mathrm{P}R_m \times \langle \varphi_p \rangle \leq C$ in case **U**. Hence, $C \not\leq \mathrm{PGL}_{2^m}^\epsilon(p^a)$. Since we have $|\mathrm{Aut}(\mathrm{PSL}_{2^m}^\epsilon(p^a))/\mathrm{PGL}_{2^m}^\epsilon(p^a)| = 2$ (recall from Proposition 1.2.14 that $\mathrm{P}\Delta\mathrm{U}_{2^m}(p^2) = \mathrm{PGU}_{2^m}(p^2)$), we obtain that
$$|C/\mathrm{P}R_m| = |C/(\mathrm{PGL}_{2^m}^\epsilon(p^a) \cap C)| = |(\mathrm{PGL}_{2^m}^\epsilon(p^a) \cdot C)/\mathrm{PGL}_{2^m}^\epsilon(p^a)| = 2,$$

by Lemma 2.6.39 and recalling that $\mathrm{P}R_m \leq \mathrm{PSL}^{\epsilon}_{2^m}(p^a)$ from Construction 2.6.38 (a) and (b). So, we have

$$C = \begin{cases} \mathrm{P}R_m \times \langle \tau \rangle & \text{in case } \mathbf{L}, \\ \mathrm{P}R_m \times \langle \varphi_p \rangle & \text{in case } \mathbf{U}. \end{cases}$$

Moreover, C is an elementary abelian 2-group of order 2^{2m+1}.
Now, to prove that K is strongly 2-constrained, we consider $\mathrm{C}_K(\mathrm{O}_2(K)) \leq \mathrm{C}_K(\mathrm{P}R_m) = C \cap K$. For $C \cap K = \mathrm{P}R_m$ our assertion easily follows. So, we have to examine the remaining case $C \leq K$. Since $\mathrm{P}R_m$ is a normal subgroup of K, C is a normal subgroup of K. Hence, we also obtain our assertion in this case. Furthermore, we note that $C = \mathrm{O}_2(K)$, in view of Proposition 2.6.37. □

In the following corollary, we note a result from the last proof, using Lemma 2.6.18 and Propositions 2.6.33 (i) and 2.6.37.

Corollary 2.6.41. *Let* $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ *and* $K = \mathrm{PN}_{\hat{G}}(R) \in \mathscr{C}_6$ *of* G *of type* $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ *for a suitable* 2*-group* R *as described in Table 2.6.2. Then we have that* $\mathrm{C}_{\mathrm{PA}(V)}(\mathrm{P}R) > \mathrm{C}_{\mathrm{P}\Omega(V)}(\mathrm{P}R) = \mathrm{P}R$ *is an elementary abelian* 2*-group of order* 2^{2m+1}*. More specific, for the group* $R_m \leq \Omega$ *according to Construction 2.6.38 (b) we have that*

$$\mathrm{C}_{\mathrm{PA}}(\mathrm{P}R_m) = \begin{cases} \mathrm{P}R_m \times \langle \tau \rangle & \textit{in case } \mathbf{L}, \\ \mathrm{P}R_m \times \langle \varphi_p \rangle & \textit{in case } \mathbf{U}. \end{cases}$$

Remark 2.6.42. We note that the result of the last corollary can also be deduced by the information provided in [Gr, p. 404] and Lemma 2.6.17 and by other efforts. Hence, it would also be possible to obtain the assertion of Main Theorem 2.6.40 by this alternative way. The author has decided to present the approach above for proving Main Theorem 2.6.40 for conformity with the case considered in the previous subsection and to provide a proof based on elementary calculations and combinatorial arguments.

2.7 A-class $\mathscr{C}_7$

In this section, we analyze the members of A-class $\mathscr{C}_7$. Roughly described, the members of A-class $\mathscr{C}_7$ are the stabilizers of tensor product decompositions $V = \bigotimes_{i=1}^{t} V_i$ where $t \geq 2$ and $\dim(V_1) = \dim(V_i)$ for $i \in \{1, \dots, t\}$. Using the provided information in [KL], we will easily see that there are no members of A-class $\mathscr{C}_7$ of G (where $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$) which are strongly constrained. So, the introduction of this A-class will be handled more briefly, and we follow the introduction provided in [KL, p. 155-157]. As usual, we use the generalized standard notation.

First, we recall the terminology and notation introduced in Subsection 1.2.1, esp. the terminology of a classical geometry and of a similarity of two vector spaces with forms. Moreover, we recall the terminology and notation from Subsection 1.3.2 and also from the beginning of Section 2.4.
Let V_1 be a $\mathrm{GF}(q^u)$-vector space of dimension $m \geq 1$, and let f_1 be the trivial form on V_1 if case **L** holds or a non-degenerate unitary form on V_1 if case **U** holds. For $i \in \{1, \ldots, t\}$ let $(V_i, \mathrm{GF}(q^u), f_i)$ be a classical geometry which is similar to $(V_1, \mathrm{GF}(q^u), f_1)$; so, there is a similarity $\eta_i : (V_1, \mathrm{GF}(q^u), f_1) \to (V_i, \mathrm{GF}(q^u), f_i)$. Clearly, we have $\dim_{\mathrm{GF}(q^u)}(V_1) = \dim_{\mathrm{GF}(q^u)}(V_i) = m$. We recall the terminology of a tensor product decomposition $\mathcal{D}_t$ from (2.4.3) and consider the case

$$\begin{aligned}(V, \mathrm{GF}(q^u), f) &= (V_1 \otimes \ldots \otimes V_t, \mathrm{GF}(q^u), f_1 \otimes \ldots \otimes f_t) \\ &= (V_1, \mathrm{GF}(q^u), f_1) \otimes \ldots \otimes (V_t, \mathrm{GF}(q^u), f_t) = \mathcal{D}_t\end{aligned}$$

where f denotes the non-degenerate unitary form on $(V, \mathrm{GF}(q^2))$ in case **U** or the trivial form on $(V, \mathrm{GF}(q))$ in case **L**. Hence, $n = \dim_{\mathrm{GF}(q^u)}(V) = m^t$ where we have obviously $m \geq 2$.
The following considerations are provided briefly, and we refer to [KL, p. 155-156] for more details. We define $\alpha_i : \Gamma(V_1, \mathrm{GF}(q^u), f_1) \to \Gamma(V_i, \mathrm{GF}(q^u), f_i)$, $g \mapsto g\alpha_i$ where $(v\eta_i)(g\alpha_i) = (vg)\eta_i$ for $v \in V_1$. Note, that α_i is an isomorphism and that we have $(\Delta(V_1, \mathrm{GF}(q^u), f_1))\alpha_i = \Delta(V_i, \mathrm{GF}(q^u), f_i)$. For the following we recall the homomorphism σ from (1.2.1) and (1.2.6). Let σ_{f_1} denote the respective homomorphism from $\Gamma(V_1, \mathrm{GF}(q^u), f_1)$ to $\mathrm{Aut}(\mathrm{GF}(q^u))$. Choose $\phi_1 \in \Gamma(V_1, \mathrm{GF}(q^u), f_1)$ such that $\sigma_{f_1}(\phi_1) = \varphi$ where $\mathrm{Aut}(\mathrm{GF}(q^u)) = \langle \varphi \rangle$. We set $\phi_{\mathcal{D}_t} = \phi_1\alpha_1 \otimes \ldots \otimes \phi_1\alpha_t$, and hence we obtain

$$\Gamma(V)_{(\mathcal{D}_t)} = \Delta_t(V)\langle \phi_{\mathcal{D}_t} \rangle \leq \Gamma(V, \mathrm{GF}(q^u), f) = \Gamma(V)$$

(here, recall (2.4.4) and (2.4.5)). For a permutation $\rho \in \mathrm{S}_t$ we define the element $g_\rho \in \mathrm{I}(V, \mathrm{GF}(q^u), f) = \mathrm{I}(V)$ via

$$(v_1\eta_1 \otimes \ldots \otimes v_t\eta_t)g_\rho = v_{1\rho^{-1}}\eta_1 \otimes \ldots \otimes v_{t\rho^{-1}}\eta_t$$

for $v_1, \ldots, v_t \in V_1$ and then extending linearly. We obtain that $\mathrm{S}_t \cong J = \{g_\rho \mid \rho \in \mathrm{S}_t\} \leq \mathrm{I}(V, \mathrm{GF}(q^u), f) = \mathrm{I}(V)$, and that J permutes the subgroups $1 \otimes \ldots \otimes 1 \otimes \Delta(V_i, \mathrm{GF}(q^u), f_i) \otimes 1 \otimes \ldots \otimes 1 \leq \Delta_t(V)$. It is not hard to see that $[\langle \phi_{\mathcal{D}_t} \rangle, J] = 1$, and so we can set

$$\Gamma(V)_{\mathcal{D}_t} = \Gamma(V)_{(\mathcal{D}_t)}J = \Delta_t(V)(\langle \phi_{\mathcal{D}_t} \rangle \times J).$$

Here, we note that $\Gamma(V)_{\mathcal{D}_t}$ is independent of the choice of ϕ_1 (cf. (2.4.5)). Finally, for $G \leq \Gamma(V)$ we set $G_{\mathcal{D}_t} = G \cap \Gamma(V)_{\mathcal{D}_t}$.

Now, we can define the members of A-class $\mathscr{C}_7$. For the following definition use the previously introduced notation and the generalized standard notation.

Definition 2.7.1. $\{$***A-class*** $\mathscr{C}_7\}$ (cf. [KL, p. 60 and Definition p. 156-157] and [As, p. 472])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ and K be a subgroup of G. Let f denote the non-degenerate unitary form on V in case **U** or the trivial form on V in case **L**. If $G \leq \Gamma(V)$ then K belongs to *(A-class)* $\mathscr{C}_7$ of G if $K = G_{\mathcal{D}_t}$ for $t \geq 2$ where $(V, \mathrm{GF}(q^u), f) \cong (V_1 \otimes \ldots \otimes V_t, \mathrm{GF}(q^u), f_1 \otimes \ldots \otimes f_t)$, $\Omega(V_1, \mathrm{GF}(q^u), f_1) \cong \mathrm{SL}^{\epsilon}_m(q^u)$ is quasisimple and $\mathcal{D}_t$ is described in the following table. For $G \nleq \Gamma(V)$ we define that K belongs to *(A-class)* $\mathscr{C}_7$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H is a member of A-class $\mathscr{C}_7$ of $\Gamma(V)$.

Case	Type	Description of $\mathcal{D}_t$	Conditions
L	$\mathrm{GL}_m(q) \wr \mathrm{S}_t$	f_i is trivial	$m \geq 3$
U	$\mathrm{GU}_m(q^2) \wr \mathrm{S}_t$	f_i is unitary	$m \geq 3$

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_7$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_7$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GL}^{\epsilon}_m(q^u) \wr \mathrm{S}_t$ we call K of type $\mathrm{GL}^{\epsilon}_m(q^u) \wr \mathrm{S}_t$.

Remark. We note that the previously given definition of A-class $\mathscr{C}_7$ coincides with the definition in [KL], and also with the definition in [BHR]. The definition of A-class $\mathscr{C}_7$ in [KL] coincides with the definition in [As], except for the condition $m \geq 3$. This extra condition is justified, since otherwise those members would be subgroups of members of other A-classes, see [KL, p. 158-159].

After the introduction of the members of A-class $\mathscr{C}_7$, we can already state the main theorem for this section, using the information provided in [KL]. As usual, we do not use the standard notation in the following main theorem.

Main Theorem 2.7.2. *Let G be an almost simple group with socle isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$ where $n \geq 3$ if $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$. Then there is no member of A-class $\mathscr{C}_7$ of G which is strongly constrained.*

Proof. Let K be a member of A-class $\mathscr{C}_7$ of G, and suppose that K is strongly constrained. Using [KL, Proposition 3.1.3], we see that $H = K \cap \mathrm{soc}(G)$ is a member of A-class $\mathscr{C}_7$ of $\mathrm{soc}(G)$. Since $H > 1$ is a normal subgroup of K, we easily obtain a contradiction, regarding Lemma 1.4.21, Corollary 1.4.19 and [KL, Lemma 4.7.1]. $\square$

2.8 A-class $\mathscr{C}_8$

Finally, we introduce and examine the remaining A-class $\mathscr{C}_8$. Here, we consider the intersection of G where $\Omega(V) \leq G \leq \mathrm{A}(V)$ with the semisimilarity group of a classical geometry (V, κ) where κ is non-trivial. The collection of $\mathscr{C}_8$-subgroups of G is empty in case **U** (see [As, p. 473], [KL, p. 165] or [BHR, Definition 2.2.17]), and so we have only to consider the case **L**. For the following definition of the members of A-class $\mathscr{C}_8$ recall Subsection 1.2.1 (especially Table 1.2.1), and (for conformity) we use as usual the generalized standard notation.

Definition 2.8.1. {***A-class*** $\mathscr{C}_8$} (cf. [KL, p. 60 and 165], [BHR, Definition 2.2.17] and [As, p. 473])
Let G be a group such that $\Omega(V) \leq G \leq \mathrm{A}(V)$ where $\Omega(V) = \mathrm{SL}(V)$. Consider that the $\mathrm{GF}(q)$-vector space V is also equipped with a non-degenerate form κ as described in the following Table 2.8. Let K be a subgroup of G. If $G \leq \Gamma(V)$ then K is a member of *(A-class)* $\mathscr{C}_8$ of G if $K = G \cap \Gamma(V, \kappa)$. For $G \nleq \Gamma(V)$ we define that K belongs to *(A-class)* $\mathscr{C}_8$ of G if $K = \mathrm{N}_{\mathrm{A}(V)}(H) \cap G$ where H belongs to $\mathscr{C}_8$ of $\Gamma(V)$.

Table 2.8

Type	Description of κ	Conditions
$\mathrm{GU}_n(q_0^2)$	$\kappa = f$ unitary	$q_0^2 = q$, $n \geq 3$
$\mathrm{Sp}_n(q)$	$\kappa = f$ symplectic	n even, $n \geq 4$
$\mathrm{O}_n^\epsilon(q)$	$\kappa = Q$ quadratic	q odd, $n \geq 3$, $\epsilon = \mathrm{sgn}(Q)$

The subgroup $K \leq \mathrm{P}G$ belongs to *(A-class)* $\mathscr{C}_8$ of $\mathrm{P}G$ if there is a member $\widetilde{K}$ of A-class $\mathscr{C}_8$ of G such that $K = \mathrm{P}\widetilde{K}$. If $\widetilde{K}$ is of type $\mathrm{GU}_n(q_0^2)$, $\mathrm{Sp}_n(q)$, or $\mathrm{O}_n^\epsilon(q)$ we call K of type $\mathrm{GU}_n(q_0^2)$, $\mathrm{Sp}_n(q)$, or $\mathrm{O}_n^\epsilon(q)$, respectively.

Remark 2.8.2. (a) Our definition of A-class $\mathscr{C}_8$ coincides with the definition in [KL]. It also coincides with the definition in [BHR], except that there the cases for dimension $n = 2$ are not excluded. In the remark following the definition of A-class $\mathscr{C}_8$ in [KL, see p. 165] these restrictions of the dimension are explained. There, it is argued that the members of A-class $\mathscr{C}_8$ of types $\mathrm{GU}_2(q_0^2)$ and $\mathrm{O}_2^\pm(q)$ coincide with members of other A-classes (namely $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_5$), which so were already considered before. We note that these observations are considered in much greater detail in [BHR, Lemma 3.1.1] (cf. also [BHR, p. 74]).

(b) Considering the conditions in Table 2.8 we note the following. That q has to be a square for type $\mathrm{GU}_n(q_0^2)$ is clear by the definition of a unitary form. For the condition that n is even for type $\mathrm{Sp}_n(q)$, see Proposition 1.2.3 (ii). Concerning the condition that q has to be odd for type $\mathrm{O}_n^\epsilon(q)$, cf. Remark 2.5.6 (e) (we note that already Aschbacher has required this condition in his paper [As, see p. 473]).

(c) Keeping in mind part (a) and (b), we see that the definition of A-class $\mathscr{C}_8$ in [As] is equivalent to the definitions in [BHR] and [KL] (except for the dimension restrictions).

In the following proposition, we provide structure information about the members of A-class $\mathscr{C}_8$. For that, we introduce the notation that for a prime r and integers a, b where b is positive we denote by $r^a \mid\mid b$ that $r^a \mid b$ but $r^{a+1} \nmid b$.

Proposition 2.8.3. *Let* $\mathrm{PSL}(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(V))$ *where* $\mathrm{PSL}(V)$ *is simple. Let* $K \in \mathscr{C}_8$ *of* G *(so,* $\dim(V) = n \geq 3$*). Let* a *be an integer. Then we have that* $H = K \cap \mathrm{PSL}(V) \in \mathscr{C}_8$ *of* $\mathrm{PSL}(V)$ *of the same type as* K*. Furthermore, the following hold.*

(i) If K *is of type* $\mathrm{GU}_n(q_0^2)$ *then* $H \cong \mathrm{PSU}_n(q_0^2).m$ *where*

$$m = \begin{cases} 2 & \text{if } n \text{ is even, } q_0 \text{ is odd and for} \\ & 2^a \mid\mid n \text{ we have } 2^{a+1} \mid q_0^2 - 1, \\ 1 & \text{otherwise.} \end{cases}$$

(ii) If K *is of type* $\mathrm{Sp}_n(q)$ *then* $H \cong \mathrm{PSp}_n(q).k$ *where*

$$k = \begin{cases} 2 & \text{if } q \text{ is odd and } (q-1, \frac{n}{2}) = (q-1, n), \\ 1 & \text{otherwise.} \end{cases}$$

(iii) If K *is of type* $\mathrm{O}_n^\epsilon(q)$ *then* $H \cong \mathrm{PSO}_n^\epsilon(q).(n, 2)$*.*

Proof. By [KL, Proposition 3.1.3.], we obtain that H is a member of A-class $\mathscr{C}_8$ of $\mathrm{PSL}(V)$. Assertions (ii) and (iii) now follow from [KL, Propositions 4.8.3. (II) and 4.8.4. (II)]. So, assertion (i) is left to consider. By [KL, Proposition 4.8.5. (II)], we have that $H \cong \mathrm{PSU}_n(q_0^2).[m_0]$ where $m_0 = \frac{(q_0+1,n)(q_0^2-1)}{(q_0^2-1,n)\left[q_0+1, \frac{q_0^2-1}{(q_0^2-1,n)}\right]}$. Hence, we have to show that $m_0 = m$. In view of [BHR, Lemma 1.13.5 (iii)][19], we see that $m_0 = \frac{(q_0+1,n)(q_0-1,n)}{(q_0^2-1,n)}$. For odd n or even q_0 we have obviously $m_0 = 1$. If n is even and q_0 is odd we obtain that $(q_0^2-1,n) < (q_0+1,n)(q_0-1,n)$ if and only if $2(q_0^2-1,n) = (q_0+1,n)(q_0-1,n)$ if and only if for $2^a \mid\mid n$ we have that $2^a \mid q_0+1$ or $2^a \mid q_0-1$. So, our assertion follows. (Here, cf. also [Ki2, Theorem VII (iii)]). ☐

Now, we start the investigation for our intended goal. First, we use the fact that the layer of a strongly constrained group is trivial.

Proposition 2.8.4. *Let* $\mathrm{PSL}(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(V))$ *where* $\mathrm{PSL}(V)$ *is simple. Let* $K \in \mathscr{C}_8$ *of* G *(hence,* $\dim(V) = n \geq 3$*). If* $\mathrm{E}(K) = 1$ *then exactly one of the following holds.*

(a) K *is of type* $\mathrm{GU}_3(2^2)$ *and hence* $\mathrm{PSL}(V) \cong \mathrm{PSL}_3(4)$*.*
(b) K *is of type* $\mathrm{O}_3^\circ(3)$ *and hence* $\mathrm{PSL}(V) \cong \mathrm{PSL}_3(3)$*.*
(c) K *is of type* $\mathrm{O}_4^+(3)$ *and hence* $\mathrm{PSL}(V) \cong \mathrm{PSL}_4(3)$*.*

Furthermore, K *is of one of the types listed above if* K *is strongly constrained.*

[19] We note that there is a mistake in [BHR, Lemma 1.13.5 (iii)]. There, two times the expression $q^{\frac{e}{2}}$ should be replaced by $p^{\frac{e}{2}}$. Furthermore, we note that the provided structure information in [BHR, Table 2.11] (where the mentioned lemma finds application) in case **L** of type $\mathrm{GU}_n(q^{1/2})$ is correct, cf. [Ki2, Theorems I and VII].

Proof. Because $K \cap \mathrm{PSL}(V)$ is a normal subgroup of K, our assertion follows by contradiction, using Propositions 1.2.11, 1.2.12 and 2.8.3. □

Remark. Recalling Theorem 1.4.16, we could also directly obtain the assertion of the last proposition by [KL, Propositions 3.1.3., 4.8.3. (III), 4.8.4. (III) and 4.8.5. (III)][20] (cf. also Remark 2.0.5).

Next, we provide the information about the maximality of the members of A-class $\mathscr{C}_8$ of G in G for the cases obtained in Proposition 2.8.4.

Lemma 2.8.5. *Adopt the notation of Proposition 2.8.4 and let one of the cases (a)-(c) hold. Then K is a maximal subgroup of G.*

Proof. Our assertion follows from [BHR, Tables 8.3 and 8.8]. For the cases in Proposition 2.8.4 (a) and (b) cf. also [Ha] and [Mi]. □

Remark. Here, we want to mention also some other works (in more geometrical nature than [KL] and [BHR]), which examine maximality of the members of $\mathscr{C}_8$ of $\mathrm{SL}_n(q)$ in $\mathrm{SL}_n(q)$ (and also for the projective version). Type $\mathrm{Sp}_n(q)$ is examined in [Dy], type $\mathrm{O}_n^{\epsilon}(q)$ in [Ki] and type $\mathrm{GU}_n(q_0^2)$ in [Ki2]. Furthermore, more results are obtained in these works, such as information concerning maximality of the members of $\mathscr{C}_8$ of $\mathrm{GL}_n(q)$ in $\mathrm{GL}_n(q)$ (and also for the projective version).

According to Proposition 2.8.4 and Lemma 2.8.5, we only have to consider three cases. First, we examine the case of Proposition 2.8.4 (a). For this, we provide the needed information by the following example which is in relation to Example 2.5.16.

Example 2.8.6. (see Example 2.5.16 and cf. [BHR, proof of Proposition 2.3.6. (iii)])
Let V be a 3-dimensional GF(4)-vector space equipped with a non-degenerate unitary form f, and denote $\mathrm{SU}(V) = \Omega(V, f)$ (see Table 1.2.1). Let B be an ordered basis of V such that the matrix $J_{f,B}$ of f is $\mathbb{1}_3$, and let $\mathrm{SU}_3(2^2)$ and $\mathrm{SL}_3(4)$ be the representations of $\mathrm{SU}(V)$ and $\mathrm{SL}(V)$ with respect to B. Let $\omega \in \mathrm{GF}(4)^*$ be a primitive 3-rd root of 1. Analogously to Example 2.5.16, we define the elements X, Y, A, $A^E \in \mathrm{SU}_3(2^2)$. As in Example 2.5.16, we have that $N = \langle X, Y\rangle$ is an extraspecial 3-group of order 3^3, $Q = \langle A, A^E\rangle$ is isomorphic to the quaternion group Q_8 and $H = \langle N, Q\rangle = N \rtimes Q = \mathrm{SU}_3(2^2)$.
Since $\mathrm{Z}(\mathrm{GL}_3(4)) = \mathrm{Z}(\mathrm{GU}_3(2^2))$, we have $\Delta\mathrm{U}_3(2^2) = \mathrm{GU}_3(2^2)$ (cf. [KL, (2.3.3)]). Hence, we easily obtain $\Gamma\mathrm{U}_3(2^2) \cap \mathrm{SL}_3(4) = \mathrm{SU}_3(2^2) = H$. So, H is a member of A-class $\mathscr{C}_8$ of $\mathrm{SL}_3(4)$ of type $\mathrm{GU}_3(2^2)$ (cf. also [BHR, Table 2.11]). Considering the projective case, we have that $\mathrm{P}H = \mathrm{PSU}_3(2^2) \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \rtimes Q_8$ belongs to A-class $\mathscr{C}_8$ of $\mathrm{PSL}_3(4)$ of type $\mathrm{GU}_3(2^2)$ (cf. also Proposition 2.8.3 (i)). We easily see that $\mathrm{E}(\mathrm{P}H) = 1$, $\mathrm{O}_2(\mathrm{P}H) = 1$ and $\mathrm{O}_3(\mathrm{P}H) = \mathrm{P}N$. Hence, $\mathrm{P}H$ is a strongly 3-constrained group.

[20] We note a mistake in [KL, Proposition 4.8.5. (III)]. There, the cases of types $\mathrm{GU}_2(2^2)$ and $\mathrm{GU}_2(3^2)$ are listed, which do not exist according to the definition of A-class $\mathscr{C}_8$.

For the following, recall from Subsection 1.2.2 the projective diagonal matrix $W = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We consider $K = \mathrm{P}H \rtimes \langle W \rangle \leq \mathrm{N}_{\mathrm{PGL}_3(4)}(\mathrm{P}N)$. By analogous arguments as in Example 2.5.16, we obtain that $K = \mathrm{N}_{\mathrm{PGL}_3(4)}(\mathrm{P}N)$; and since $\mathrm{P}N$ is a characteristic subgroup of $\mathrm{P}H$, $K = \mathrm{N}_{\mathrm{PGL}_3(4)}(\mathrm{P}H)$. So, we have $\mathrm{P}H = \mathrm{N}_{\mathrm{PSL}_3(4)}(\mathrm{P}H)$. (We note that the last results could also easily be deduced using Lemma 2.8.5). As in Example 2.5.16, we now obtain that $\mathrm{PSL}_3(4)$ acts transitively on the members of A-class $\mathscr{C}_8$ of $\mathrm{PSL}_3(4)$ of type $\mathrm{GU}_3(2^2)$ (cf. also [KL, Proposition 4.8.5. (I)]).
Finally, we note some facts about K. It is not hard to see that $\mathrm{E}(K) = 1$, $\mathrm{O}_2(K) = 1$ and $\mathrm{O}_3(K) = \mathrm{P}N = \mathrm{O}_3(\mathrm{P}H)$. So, K is a strongly 3-constrained group.

Remark. We note that the group K of the last example is also known as Hessian group and refer to Remark 2.5.17 for more information.

Proposition 2.8.7. *Let* $\mathrm{PSL}(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(V))$ *where* $\mathrm{PSL}(V) \cong \mathrm{PSL}_3(4)$. *Let M be a member of $\mathscr{C}_8$ of G of type* $\mathrm{GU}_3(2^2)$ *(so, according to Lemma 2.8.5 M is a maximal subgroup of G). Then M is strongly constrained if and only if G is a subgroup of* $\mathrm{P\Gamma L}(V)$ *or* $\mathrm{PGL}(V) : \langle \tau \rangle$.
Furthermore, if M is strongly constrained then M is strongly 3*-constrained,* $\mathrm{O}_3(M) = \mathrm{O}_3(M \cap \mathrm{PSL}(V))$ *and* $|M| \leq 2^4 3^3$ *where this upper bound is sharp.*

Proof. W.l.o.g. we can consider a concrete representation of $\mathrm{SL}(V)$ with respect to an ordered basis of V. So, we use the notation introduced in Example 2.8.6. In view of Lemma 1.4.20, there are only the cases (1.)-(10.) of Example 1.2.23 to consider. Hence, let G be one of those cases. By Proposition 2.8.3, we have that $M \cap \mathrm{PSL}_3(4) \in \mathscr{C}_8$ of $\mathrm{PSL}_3(4)$ of type $\mathrm{GU}_3(2^2)$. Following Example 2.8.6, $\mathrm{PSL}_3(4)$ acts transitively (via conjugation) on the members of A-class $\mathscr{C}_8$ of $\mathrm{PSL}_3(4)$ of type $\mathrm{GU}_3(2^2)$. So, w.l.o.g. we can assume that $M \cap \mathrm{PSL}_3(4) = \mathrm{P}H$. It is not hard to check that $W, \varphi_2, \tau \in \mathrm{Aut}(\mathrm{PSL}_3(4))$ normalize $\mathrm{P}H$. Hence, $M = \mathrm{N}_G(\mathrm{P}H) = K_1 \cap G$ where $K_1 = \mathrm{N}_{\mathrm{Aut}(\mathrm{PSL}_3(4))}(\mathrm{P}H) = K : (\langle \varphi_2 \rangle \times \langle \tau \rangle)$.
We consider the element $\varphi_2\tau$ in $\mathrm{Aut}(\mathrm{PSL}_3(4))$. Easy calculations show that $\varphi_2\tau$ centralizes $\mathrm{P}H$, and since $\varphi_2\tau$ also centralizes W, φ_2 and τ, we have that $\varphi_2\tau$ centralizes K_1. Let M be strongly constrained. Since by Example 2.8.6 $\mathrm{P}H$ is strongly 3-constrained, M has to be strongly 3-constrained. So, $\varphi_2\tau \notin G$ since otherwise $\mathrm{O}_2(M) > 1$. In view of Example 1.2.23, we now see that G has to be a group in one of the cases (1.)-(3.), (5.), (8.), or (9.). Cases (1.) and (5.) are clear by Example 2.8.6. So, let one of the remaining cases hold. Suppose that $\mathrm{O}_2(M) > 1$. Then, by $\mathrm{O}_2(M) \cap \mathrm{P}H = 1$, we have that

$$1 < |\mathrm{O}_2(M)| = |(\mathrm{O}_2(M) \cdot \mathrm{P}H)/\mathrm{P}H| \bigm| |M/\mathrm{P}H| \bigm| 6.$$

So, $|\mathrm{O}_2(M)| = 2$ and there exists a non-trivial involutory element in M which centralizes M. Now, analogous considerations as in the proof of Lemma 2.5.18 (ii) lead to a contradiction (note, that the action of φ_2 on $\mathrm{P}H$ coincides with the action of τ on $\mathrm{P}H$). Since obviously $\mathrm{E}(M) = 1$, $1 < \mathrm{O}_3(\mathrm{P}H) \leq \mathrm{O}_3(M)$ and $|M| \mid 2^4 3^3$, we now obtain that M is strongly 3-constrained. To see that $\mathrm{O}_3(M) = \mathrm{O}_3(\mathrm{P}H)$, use Lemma 1.4.9 and Example 2.8.6. □

Next, we consider the case of Proposition 2.8.4 (b). The following example provides all necessary information about this case.

Example 2.8.8. (cf. [BHR, proof of Proposition 2.3.6 (ii)])
Let V be a 3-dimensional GF(3)-vector space. Let Q be a non-degenerate quadratic form on V such that for an appropriate basis B of V the matrix of the associated bilinear form f_Q is $J_{f_Q,B} = \mathbb{1}_3$. (To see that w.l.o.g. we can consider this situation and choose the described matrix of f_Q for the consideration of a general non-degenerate quadratic form on V, see [KL, Proposition 2.6.1. and the discussion following Proposition 2.5.4.]). We consider the representation of $\mathrm{GL}(V)$ with respect to B. In view of [KL, (2.5.2), Lemma 2.1.8. (i) and (iii)] and Proposition 1.2.14 we have that

$$H = \mathrm{GO}_3^\circ(3) = \mathrm{O}_3^\circ(3) = \{A \in \mathrm{GL}_3(3) \mid AA^t = \mathbb{1}_3\} \leq \mathrm{GL}_3(3).$$

So, H is a member of A-class $\mathscr{C}_8$ of $\mathrm{GL}_3(3)$ of type $\mathrm{O}_3^\circ(3)$. Furthermore, we have that $H \cap \mathrm{SL}_3(3) = \mathrm{SO}_3^\circ(3) \in \mathscr{C}_8$ of $\mathrm{SL}_3(3)$ and $\mathrm{P}H = \mathrm{PSO}_3^\circ(3) \in \mathscr{C}_8$ of $\mathrm{PSL}_3(3)$. By our previous considerations, we obtain that $\mathrm{P}H \cong \mathrm{S}_4$ (cf. also Propositions 1.2.11 and 1.2.12), and so $\mathrm{P}H$ is strongly 2-constrained.
In view of Proposition 2.2.8 and [KL, Proposition 4.8.4. (I)], we see that the members of A-class $\mathscr{C}_2$ and $\mathscr{C}_8$ of $\mathrm{SL}_3(3)$ coincide (and also their projective versions).
Finally, we state explicitly that the inverse transpose map centralizes H, since $AA^t = \mathbb{1}_3$ for $A \in H$.

In view of the previous example, we easily obtain the following result.

Proposition 2.8.9. *Let* $\mathrm{PSL}(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(V))$ *where* $\mathrm{PSL}(V) \cong \mathrm{PSL}_3(3)$. *Let M be a member of $\mathscr{C}_8$ of G, so M is of type $\mathrm{O}_3^\circ(3)$ and a maximal subgroup of G (see Lemma 2.8.5). Then M is strongly 2-constrained and $|M| \leq 2^4 3$ where this upper bound is sharp. Furthermore, we have that $\mathrm{O}_2(M \cap \mathrm{PSL}(V)) < \mathrm{O}_2(M)$ if and only if $G = \mathrm{Aut}(\mathrm{PSL}(V))$ where $|\mathrm{O}_2(M)/\mathrm{O}_2(M \cap \mathrm{PSL}(V))| = 2$.*

Finally, consider the case of Proposition 2.8.4 (c).

Proposition 2.8.10. *Let* $\mathrm{PSL}(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(V))$ *where* $\mathrm{PSL}(V) \cong \mathrm{PSL}_4(3)$. *Let M be a member of $\mathscr{C}_8$ of G of type $\mathrm{O}_4^+(3)$, so M is a maximal subgroup of G following Lemma 2.8.5. Let $H = M \cap \mathrm{PSL}(V)$. Then the following hold.*

(a) H is strongly 2-constrained.

(b) M is strongly 2-constrained, $|M| \leq 2^8 3^2$ and this upper bound is sharp.

Proof. In view of Proposition 2.8.3, we have that H is a member of $\mathscr{C}_8$ of $\mathrm{PSL}(V)$ of type $\mathrm{O}_4^+(3)$ and $H \cong \mathrm{PSO}_4^+(3).2$. Let H_1 be a normal subgroup of H where $H_1 \cong \mathrm{PSO}_4^+(3)$. We know that there is a unique subgroup H_2 of H_1 of index 2 which is isomorphic to $\mathrm{P\Omega}_4^+(3)$, by [KL, Proposition 2.5.7.] and [Hu, I 3.10 Satz] (cf. also Proposition 1.2.14). So, H_2 is a characteristic subgroup of H_1, and hence H_2 is a normal subgroup of H. Furthermore, we have that $H_2 \cong \mathrm{P\Omega}_4^+(3) \cong \mathrm{PSL}_2(3) \times \mathrm{PSL}_2(3)$, using Proposition 1.2.11. Since $\mathrm{O}^2(\mathrm{PSL}_2(3)) = \mathrm{PSL}_2(3)$, we obtain that $H_2 = \mathrm{O}^2(H)$ (cf. also [KL, proof of Proposition 4.8.2.]). It is easy to see that $1 < \mathrm{O}_2(H_2) \leq \mathrm{O}_2(H)$ and $\mathrm{E}(H) = 1$. Because $|H/H_2| = 4$, we have $\mathrm{O}_3(H) \leq H_2$. So, we can deduce that $\mathrm{O}_3(H) = 1$. Obviously $|H| = 2^6 3^2$, and hence we obtain assertion (a).
To prove assertion (b), we recall that H_2 is a characteristic subgroup of H. So, we obtain that H_2 is a normal subgroup of M (and hence even characteristic, since $H_2 = \mathrm{O}^2(M)$). Because $|M/H_2| \mid 2^4$, we now obtain our assertion analogously to part (a). □

We summarize the results of this section in the following main theorem where we determine the pairs (G, M) where G is an almost simple linear group and M a strongly constrained maximal subgroup of G belonging to A-class $\mathscr{C}_8$ of G.

Main Theorem 2.8.11. *Let $\mathrm{PSL}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_n(q))$ where $\mathrm{PSL}_n(q)$ is simple. Let M belong to A-class $\mathscr{C}_8$ of G. Then M is a strongly constrained maximal subgroup of G if and only if one of the following holds.*

- *(a) $n = 3$, $q = 4$, M is of type $\mathrm{GU}_3(2^2)$ and G is a subgroup of $\mathrm{P\Gamma L}_3(4)$ or $\mathrm{PGL}_3(4) : \langle\tau\rangle$. Here, M is strongly 3-constrained.*
- *(b) $n = 3$, $q = 3$ and M is of type $\mathrm{O}_3^\circ(3)$. Here, M is strongly 2-constrained.*
- *(c) $n = 4$, $q = 3$ and M is of type $\mathrm{O}_4^+(3)$. Here, M is strongly 2-constrained.*

Furthermore, if M is a strongly constrained maximal subgroup of G we have $|M| \leq 2^8 3^2$ and this upper bound is sharp.

Chapter 3

Stabilizer order bounds in case of almost simple primitive permutation groups with a socle isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$

In this chapter, we achieve the following goal, stated in Main Theorem 3.1.19: For an almost simple primitive permutation group G on a finite set X where $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$, we determine an explicit upper bound $h(d)$ in terms of an arbitrary non-trivial subdegree d of G for the order of a stabilizer of a point G_α ($\alpha \in \mathrm{X}$) if G_α is strongly constrained. For the case that G_α is not strongly constrained such an upper bound is already explicitly known, see Corollary 3.1.18. So, in Main Theorem 3.1.20, we can deduce an upper bound for the order of (any) G_α for the given situation.
For this chapter we recall Subsection 1.4.2 where some basic terminology and notation about finite permutation group theory is introduced. Furthermore, we use the standard notation in this chapter, as introduced in Subsection 1.2.3.

3.1 Historical notes, preliminary considerations and the main theorems

We begin by providing a historical overview about the background of the Main Theorems 3.1.19 and 3.1.20 of this chapter.

In the middle of the 1960s, Sims conjectured the following (see [Th, p. 135] or [CPSS, Theorem 1]):

Theorem 3.1.1. ***(Sims conjecture)*** *For a primitive permutation group G on a finite set* X *the order of G_α, the stabilizer in G of a point $\alpha \in \mathrm{X}$, is bounded by a function f in terms of an arbitrary non-trivial subdegree d of G.*

In the following years, much effort was invested to prove the conjecture of Sims. A first step was taken by Thompson in [Th] where the existence of a function $g(d)$ was proved such that $|G_\alpha/\mathrm{O}_p(G_\alpha)| \leq g(d)$ for some prime p in the given situation. An explicit function $g(d)$, bounding $|G_\alpha/\mathrm{O}_p(G_\alpha)|$ for some prime p, was given first by Wielandt. In [Wie2, Theorem 6.7 (regard also the notes following the proof)][1], it was shown that there is a prime p such that for the given situation

$$|G_\alpha/\mathrm{O}_p(G_\alpha)| \bigm| d!((d-1)!)^d =: \mathrm{wdt}(d). \tag{3.1.1}$$

Because of its importance (see e.g. Corollary 3.1.18), we call the previously defined function $\mathrm{wdt}(d)$ the *Wielandt order bound.*
Several authors investigated the situation for the case that G has a small non-trivial subdegree d. Here, explicit upper bounds for $|G_\alpha|$ were obtained, and for certain small d also the possible group theoretic structures of G and G_α were determined.[2] Not restricting on the situation that G has a small non-trivial subdegree d, there were also published several works in which an upper bound for $|G_\alpha|$ in terms of d was determined, by demanding additional conditions. Such as additional conditions for the subconstituent $G_\alpha^{O(\alpha)}$ if $d = |O(\alpha)| \geq 2$ (O an orbital of G), see e.g. [Kn3] or [Kn2, Sätze 2 and 3] (cf. also for a particular example [Wie5, 18.6 Satz]), or the additional condition to choose a certain subdegree d, see [Kn6, Corollary 1] (or [Kn5, Korollar 6.4]).
In the year 1983, the paper [CPSS] of Cameron, Praeger, Saxl and Seitz was published where the authors proved the conjecture of Sims, using the classification of finite simple groups (which was recently announced at that time), as well as the O'Nan-Scott theorem on the structure of primitive permutation groups, observations of Thompson and Wielandt and Lie theory. In a first step, using the O'Nan-Scott theorem (see e.g. [LPS]), the four authors showed that the conjecture of Sims is established if it is shown to hold for all almost simple primitive permutation groups. Then, using information of the classification of finite simple groups, the authors proved the conjecture of Sims for this reduced case.
For the following and for our investigation in this chapter it is useful to introduce the following notation. A function f which satisfies the conditions of the

[1] We note about [Wie2, Theorem 6.7] that by its proof it is also possible to state "$|G_\alpha : N|$ divides $d!(d-1)!^d$" in its assertion.

[2] The determination of the possible group theoretic structures of G and G_α (where G is a finite primitive permutation group on a finite set X and G_α the stabilizer in G of a point $\alpha \in \mathrm{X}$), if it is known that G has a non-trivial subdegree $d = |O(\alpha)|$ (O an orbital of G) or the group theoretic structure of the subconstituent $G_\alpha^{O(\alpha)}$ is known, can be seen as a motivation for the conjecture of Sims, cf. [Kn4, p. 5], [Kn5, p. 1-2] and [LLM, p. 750].

conjecture of Sims 3.1.1 (for a collection $\mathscr{G}$ of primitive permutation groups) we call a *Sims order bound* (for $\mathscr{G}$). In the paper [CPSS], the authors focused on proving the existence of a Sims order bound rather than on providing an explicit function. In his Wissenschaftliche Arbeit [St], Stolz reviewed the proof of the conjecture of Sims provided in [CPSS]. Unfortunately, this work was not published in a paper. Using results obtained by Wielandt, the mentioned reduction in [CPSS] to the case of almost simple primitive permutation groups was shortened and improved in [St]. It was used that the Wielandt order bound $\mathrm{wdt}(d)$ (recall (3.1.1)) is a Sims order bound for the collection $\mathscr{G}^{nc}$ where $\mathscr{G}^{nc}$ consists of the primitive permutation groups for which the stabilizer of a point is *n*ot strongly *c*onstrained, see [Kn, Proposition 4.1 and Theorem 4.2] or Corollary 3.1.18 (below). So, to determine a Sims order bound it is sufficient to determine a Sims order bound $f_{sc}(d)$ for the collection $\mathscr{G}^{sc}$ where $\mathscr{G}^{sc}$ consists of all almost *s*imple primitive permutation groups G which have a strongly *c*onstrained point stabilizer G_α.

Remark 3.1.2. (see [St, Folgerung 2.2.7 and Sätze 2.2.10, 2.2.11 and 2.3.5, cf. p. 25])
The results in [St] can be described in the following way. If a Sims order bound $f_{sc}(d)$ for $\mathscr{G}^{sc}$ is determined then $f_s(d) = \max\{f_{sc}(d), \mathrm{wdt}(d)\}$ is a Sims order bound for $\mathscr{G}^s$ consisting of all almost *s*imple primitive permutation groups. The function $f_s(d)$ can be chosen to be increasing (if necessary after some obvious modifications). Then we obtain a Sims order bound f by $f(d) = f_s(d)^{\lceil d/2 \rceil} \cdot (\lceil d/2 \rceil)!$.

By considering the classification of finite simple groups, it is reasonable to split up the collection $\mathscr{G}^{sc}$ into finitely many subcollections $\mathscr{G}_1^{sc}, \dots, \mathscr{G}_m^{sc}$ (e.g. depending on the type of socle of an almost simple primitive permutation group). Then, if for each subcollection $\mathscr{G}_j^{sc}$ a Sims order bound $f_{sc_j}(d)$ for $\mathscr{G}_j^{sc}$ is determined, we obtain by $f_{sc}(d) = \max\{f_{sc_1}(d), \dots, f_{sc_m}(d)\}$ a Sims order bound for $\mathscr{G}^{sc}$. In several works, Sims order bounds for subcollections $\mathscr{H}$ of $\mathscr{G}^{sc}$ have been determined:

- In [St, see Satz 3.2.1], for $\mathscr{H}$ consisting of the groups $G \in \mathscr{G}^{sc}$ with $\mathrm{soc}(G) \cong \mathrm{A}_n$ a Sims order bound h for $\mathscr{H}$ was determined by $h(d) = \max\{2^{15} \cdot 3^5, 4^{d+1} \cdot \mathrm{wdt}(d)\}$.
- In [St, see Satz 3.3.1 and cf. p. 34], for $\mathscr{H}$ consisting of the groups $G \in \mathscr{G}^{sc}$ with $\mathrm{soc}(G) \cong \mathrm{PSL}_2(q)$ a Sims order bound h for $\mathscr{H}$ was determined by $h(d) = \mathrm{wdt}(d)$.
- In [Uf, see Satz 3.1.1, Hauptsatz 3.1.2 and p. 2], for $\mathscr{H}$ consisting of the groups $G \in \mathscr{G}^{sc}$ with $\mathrm{soc}(G) \cong \mathrm{PSL}_3(q)$ a Sims order bound h for $\mathscr{H}$ was determined by $h(d) = \mathrm{wdt}(d) \cdot \lfloor ((3/2 \cdot \mathrm{wdt}(d))^{\frac{1}{2}} + 1)^3 \rfloor$.[3]

[3]We note that it is possible to deduce a more precise Sims order bound for $\mathscr{H}$ from the results of this work, by determining a more precise upper bound for $|G_\alpha/\mathrm{O}_p(G_\alpha)|$ than $\mathrm{wdt}(d)$ (where $G \in \mathscr{H}$ and the stabilizer of a point G_α is strongly p-constrained for the prime p), cf. also Remark 3.1.17 (below).

- Furthermore, in [Gü, see Hauptsätze (7.1) and (7.2)], for the collections $\mathscr{K}_1$ respectively $\mathscr{K}_2$ consisting of the almost simple primitive permutation groups with socle isomorphic to a Suzuki group $\mathrm{Sz}(2^{2m+1})$ respectively to a Ree group $\mathrm{R}(3^{2m+1})$ (not restricted to those which have a strongly constrained stabilizer of a point), a Sims order bound for $\mathscr{K}_1$ respectively $\mathscr{K}_2$ was determined by $k_1(d) = d!$ respectively $k_2(d) = d!(d-1)!$.

In this chapter, we consider the subcollection $\mathscr{H}$ consisting of the groups $G \in \mathscr{G}^{sc}$ with $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$. We determine a Sims order bound $h(d)$ for $\mathscr{H}$, stated in Main Theorem 3.1.19, and we have $h(d) \leq \mathrm{wdt}(d)$. Especially, this order bound is more precise than the order bound determined by the author in [Uf] (see the third item of the last list). Using Corollary 3.1.18 (below), following from [Kn, Proposition 4.1 and Theorem 4.2], we can then deduce in Main Theorem 3.1.20 that the Wielandt order bound $\mathrm{wdt}(d)$ is a Sims order bound for the collection consisting of the almost simple primitive permutation groups with socle isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$.

Primitive permutation groups G having a small non-trivial subdegree

As mentioned before, for a primitive permutation group G which has a small non-trivial subdegree there are several publications providing a list of possible group theoretic structures of G and the stabilizer of a point G_α, or providing an upper bound for the order of G_α. Next, we collect results from these publications for our further investigations. For this, let D_n denote the dihedral group of order n.

Theorem 3.1.3. ***(Miller, Wielandt)*** *Let G be a primitive permutation group on a finite set X and let $\alpha \in \mathrm{X}$. Let d denote a non-trivial subdegree of G. Then the following hold.*

(a) If $d = 1$ then $G_\alpha = 1$ and $G \cong \boldsymbol{Z}_p$ for a prime p.

(b) If $d = 2$ then $G_\alpha \cong \boldsymbol{Z}_2$ and $G \cong D_{2p}$ for a prime p.

Proof. See [Wie, Proposition 8.6, Theorem 18.7 and Exercise 18.8] or [Wie5, 18.7 Folgerung and 18.8 Übung] or [St, Satz 2.1.2]. □

Theorem 3.1.4. ***(Sims, Wong)*** *Let G be a primitive permutation group on a finite set X such that the stabilizer G_α in G of a point $\alpha \in \mathrm{X}$ has an orbit of length 3. Then G_α is isomorphic to $\boldsymbol{Z}_3$, S_3, D_{12}, S_4 or $\mathrm{S}_4 \times \boldsymbol{Z}_2$; especially we have $|G_\alpha| \mid 2^4 \cdot 3$ and this upper bound is sharp. Moreover, if G is insoluble then G is isomorphic to A_5, S_5, $\mathrm{PGL}_2(7)$, $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(p)$ for a prime p with $p \equiv \pm 1 \pmod{16}$, $\mathrm{PSL}_3(3)$ or $\mathrm{Aut}(\mathrm{PSL}_3(3))$.*

Proof. See [Wo, Lemma 6 and Theorem] and [Si, Section 5]. □

Remark. The primitive permutation groups $G \leq \mathrm{Sym}(\mathrm{X})$ (X a finite set) which have a subdegree 3 have been determined completely by Wong, see [Wo, p. 236-237]. Especially, we note that for each listed insoluble group in the last theorem there is a faithful primitive permutation representation on a finite set X such that the group has a subdegree 3.

Theorem 3.1.5. ***(Knapp, Li, Lu, Marušič, Quirin, Sims, Wang, Wong)*** *Let G be a primitive permutation group on a finite set* X *such that the stabilizer G_α in G of a point $\alpha \in \mathrm{X}$ has an orbit of length* 4*. Then $|G_\alpha| \leq 2^4 \cdot 3^2$ and this upper bound is sharp. If G is insoluble then the pair (G, G_α) is one of the following listed cases up to isomorphy.*

G	G_α	*Conditions*
$\mathrm{PGL}_2(p)$	S_4	p *prime,* $p \equiv \pm 3 \pmod 8$
$\mathrm{PSL}_2(p)$	S_4	$p > 7$ *prime,* $p \equiv \pm 1 \pmod 8$
$\mathrm{PSL}_2(p)$	A_4	$p \geq 5$ *prime,* $p \equiv \pm 3 \pmod 8$, $p \not\equiv \pm 1 \pmod{10}$
$\mathrm{PSL}_2(3^s)$	A_4	s *odd prime*
$\mathrm{PGL}_2(7)$	D_{16}	
$\mathrm{PGL}_2(9)$	D_{16}	
$\mathrm{PSL}_2(9)\langle W\varphi_3\rangle$	$\boldsymbol{Z}_8 : \boldsymbol{Z}_2$	
$\mathrm{PSL}_2(17)$	D_{16}	
$\mathrm{Aut}(\mathrm{A}_6)$	$[2^5]$	
$\mathrm{PSL}_2(3^3) : \langle\varphi_3\rangle$	$\mathrm{A}_4 \times \boldsymbol{Z}_3$	
$\mathrm{PSL}_3(3)$	S_4	
$\mathrm{PSL}_3(7)$	$(\mathrm{A}_4 \times \boldsymbol{Z}_3) : \boldsymbol{Z}_2$	
$\mathrm{PSL}_3(7) : \langle\tau\rangle$	$\mathrm{S}_4 \times \mathrm{S}_3$	
A_7	$(\mathrm{A}_4 \times \boldsymbol{Z}_3) : \boldsymbol{Z}_2$	
S_7	$\mathrm{S}_4 \times \mathrm{S}_3$	

Proof. See [Wa, Theorems 1.3 and 1.4] and [LLM, Theorem 3.4], cf. also [Si2], [Qu], [Kn3, Theorem 5.1] and [Wo, Theorem 4]. (Concerning [LLM, Theorem 3.4 (iv)], we note that always $3^s \equiv 3 \pmod 8$ and $3^s \not\equiv \pm 1 \pmod{10}$ for an odd prime s). □

Remark 3.1.6. Concerning the last theorem we note the following. In the paper [Wa, see Theorem 1.4] Wang claimed that all primitive permutation groups $G \leq \mathrm{Sym}(X)$ (X a finite set) which have a subdegree 4 are classified. For this, Wang uses the previously obtained results of Sims and Quirin (see [Si2][4] and [Qu]) which have provided lists for such G demanding additional conditions. Sims and Quirin only note that the listed groups in their papers do have faithful permutation representations of the required type, without providing a proof.

[4] We note that [Si2, Theorem] was quoted false by Wang. In [Wa, Theorems 1.1 (i)(b) and 1.4 (2)], the group $\mathrm{P\Sigma L}_2(9) \cong \mathrm{S}_6$ should be replaced by the Mathieu group M_{10} (isomorphic to $\mathrm{PSL}_2(9)\langle W\varphi_3\rangle$), see also Remark 3.1.9 (b) (below) and cf. [LLM, Table 3] where this mistake was corrected without a note.

In Wang's paper (in which the remaining cases for such G are investigated), proofs are provided to show that the groups he lists do have faithful permutation representations of the required type, such as in [Wa, Proposition 2.8]. Unfortunately, in the last mentioned proposition of Wang occurs a mistake. In [Wa, Proposition 2.8, cf. also Theorem 1.3 (3)], Wang lists a case where $|G_\alpha| = 2^4 \cdot 3^6$ and states a wrong proof for its existence by referring a false note from Quirin [Qu, p. 273], cf. [LLM, historical note on p. 750 and Lemma 3.3]. In the paper [LLM], three cases of the list [Wa, Theorem 1.4] are ruled out, including the previously mentioned case with $|G_\alpha| = 2^4 \cdot 3^6$, see [LLM, Subsection 3.1]. But, by ruling out cases in [LLM] there also occurs a mistake. In [LLM, Lemma 3.1], there is examined and ruled out a case which was wrongly listed in [Wa, Theorem 1.4 (6)] by a copying error from [Wa, Theorem 1.3 (2)] (cf. also [Wa, Proposition 2.8]). So, it is not justified to rule out the case of [Wa, Theorem 1.3 (2)]. We also note that the authors Li, Lu and Marušič claim that [LLM, Theorem 3.4] provides a "precise" list of such groups G (together with the isomorphism type of G_α) if G is insoluble, only by ruling out the mentioned three cases and unfortunately without providing explicit proofs that a faithful permutation representation of the required type exists for each group occurring in that list.

Theorem 3.1.7. ***(Knapp, Quirin)*** *Let G be a primitive permutation group on a finite set X and let $\alpha \in \mathrm{X}$. Let O be an orbital of G of length $|O(\alpha)| = 5$. Then we have*

(a) $|G_\alpha|$ divides $2^4 \cdot 5$ if the subconstituent $G_\alpha^{O(\alpha)}$ is soluble, and

(b) $|G_\alpha|$ divides $2^{14} \cdot 3^2 \cdot 5$ if the subconstituent $G_\alpha^{O(\alpha)}$ is insoluble.

Proof. See [Kn3, Theorem 5.2] and [Qu, Theorem 2.2]. □

For working with the above Theorems 3.1.4 and 3.1.5, we provide the following lemma.

Lemma 3.1.8. *The only alternating groups that are isomorphic to simple classical groups are A_5, A_6 and A_8. If a simple linear or unitary group is isomorphic to one of these three alternating groups it is contained in the following list.*

(i) $A_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{PSU}_2(4^2) \cong \mathrm{PSU}_2(5^2)$.

(ii) $A_6 \cong \mathrm{PSL}_2(9) \cong \mathrm{PSU}_2(9^2)$.

(iii) $A_8 \cong \mathrm{PSL}_4(2)$.

Proof. See [KL, Proposition 2.9.1] and cf. [BHR, Proposition 1.10.2]. □

Remark 3.1.9. (a) We note that there are further isomorphisms between simple classical groups and the alternating groups A_5, A_6 and A_8 as listed in Lemma 3.1.8 (i)-(iii), see e.g. Proposition 1.2.11.

(b) By the last lemma we see that $A_6 \cong \mathrm{PSL}_2(9)$. Recalling Corollary 1.2.20, we deduce $\mathrm{Aut}(A_6) \cong \mathrm{P\Gamma L}_2(9)$, esp. $\mathrm{Out}(A_6) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. There are precisely three subgroups of $\mathrm{P\Gamma L}_2(9)$ of index 2, namely $G_1 = \mathrm{PGL}_2(9)$, $G_2 = \mathrm{PSL}_2(9) : \langle\varphi_3\rangle$ and $G_3 = \mathrm{PSL}_2(9)\langle W\varphi_3\rangle$. Clearly, $A_6 < S_6 < \mathrm{Aut}(A_6)$, so S_6 is isomorphic to G_1, G_2 or G_3. Because G_1 has an element of order 8 (consider $\mathrm{Pdiag}(\omega, 1) \in G_1$ where $\langle\omega\rangle = \mathrm{GF}(9)^*$), we see that $S_6 \ncong G_1$. By some effort, one can show that G_3 is a non-split extension of $\mathrm{PSL}_2(9)$. (Actually, G_3 is isomorphic to the Mathieu group M_{10}, the stabilizer of a point in the Mathieu group M_{11}). Hence, we can deduce $S_6 \cong G_2$ (cf. also [At, p. 4]). Furthermore, $G_3 \ncong G_1$ (use e.g. [GAP]). So, $\mathrm{Aut}(A_6)$ has three non-isomorphic normal subgroups of index 2.

(c) Concerning the symmetric groups S_5, S_6 and S_8, we note that $S_5 \cong \mathrm{P\Gamma L}_2(4) \cong \mathrm{PGL}_2(5)$, $\mathrm{PSL}_2(9) : \langle\varphi_3\rangle \cong \mathrm{S}_6 < \mathrm{Aut}(\mathrm{A}_6) \cong \mathrm{P\Gamma L}_2(9)$ (recall part (b)) and $S_8 \cong \mathrm{PSL}_4(2) : \langle\tau\rangle$.

(d) In view of part (c), we note that [BHR, Proposition 1.10.2] is not quite correct, since the groups S_5 and S_8 have not been listed.

We provide another well-known fact.

Lemma 3.1.10. *Let $n \geq 5$. If $n \neq 6$ we have $\mathrm{Aut}(A_n) = S_n$, so $|\mathrm{Out}(A_n)| = 2$. For $n = 6$ we have $\mathrm{Out}(A_6) \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$.*

Proof. See [Pa, Theorem 5.7]. □

Remark. Concerning the case $n = 6$ in the last lemma, recall also Remark 3.1.9 (b).

Also mentioned above, for a primitive permutation group G on a finite set X there are publications in which an upper bound for $|G_\alpha|$ ($\alpha \in \mathrm{X}$) in terms of a subdegree $d = |O(\alpha)| \geq 2$ is determined, by demanding additional conditions (e.g. for the subconstituent $G_\alpha^{O(\alpha)}$). The following theorem provides such upper bounds, obtained by Knapp, for later use.

Theorem 3.1.11. ***(Knapp)*** *Let G be a primitive permutation group on a finite set X and let $\alpha \in \mathrm{X}$. Let O be an orbital of G of length $|O(\alpha)| = d \geq 2$, and let $\beta \in O(\alpha)$. Then the following hold.*

(a) If $\mathrm{F}(G_{\alpha\beta}^{O(\alpha)}) = 1$ then $\mathrm{F}(G_{\alpha\beta}) = 1$ and $|G_\alpha|$ divides $d!((d-1)!)^d$.

(b) If the subconstituent $G_\alpha^{\mathrm{O}(\alpha)}$ is 2-primitive and $\mathrm{F}(G_{\alpha\beta}^{O(\alpha)}) \neq 1$ then $|G_\alpha|$ divides $d(d-1)\log_2(d)$ where $d \geq 8$ is a power of 2, or $|G_\alpha|$ divides $d(d-1)^6(d-2)^2(\log_p(d-1))^2$ where $d-1$ is a power of a prime p. Especially, we always have $|G_\alpha| \leq d(d-1)^6(d-2)^2(\log_2(d-1))^2$.

Proof. Assertion (a) follows by [Kn2, Satz 1] and (b) by [Kn3, Theorem 4.2]. □

The following theorem is also useful for later investigations, classifying for a primitive permutation group G the structure of the stabilizer of a point G_α if a certain subconstituent $G_\alpha^{\mathrm{O}(\alpha)}$ is given.

Theorem 3.1.12. ***(Knapp)*** *Let G be a primitive permutation group on a finite set X, $\alpha \in \mathrm{X}$ and O be an orbital of G of length $d = |O(\alpha)| \geq 6$ such that $\mathrm{Alt}(O(\alpha)) \leq G_\alpha^{\mathrm{O}(\alpha)}$. If G_α acts not faithfully on $\mathrm{O}(\alpha)$ the following hold.*

(i) If $G_\alpha^{\mathrm{O}(\alpha)} = \mathrm{Alt}(O(\alpha))$ then G_α is isomorphic to $\mathrm{A}_d \times \mathrm{A}_{d-1}$.

(ii) If $G_\alpha^{\mathrm{O}(\alpha)} = \mathrm{Sym}(O(\alpha))$ then G_α is isomorphic to $\mathrm{S}_d \times \mathrm{S}_{d-1}$, $\mathrm{S}_d \times \mathrm{A}_{d-1}$ or $(\mathrm{A}_d \times \mathrm{A}_{d-1}) : \boldsymbol{Z}_2$ where $\boldsymbol{Z}_2$ acts as a transposition on both direct factors.

Proof. See [Kn, Theorem 6.1]. □

The main theorems of this chapter

Next, we will state the main theorems of this chapter. For this, some preliminary considerations are necessary.
For analyzing the stabilizer of a point of a primitive permutation group it is useful to work with methods developed by Wielandt, see [Kn, Sections 2 and 4] and [Kn4, Abschnitt 6]. We will do so and note that we use applications of these methods which are of more general nature. We begin by providing some useful notation for our investigation, by defining appropriate subgroups of a primitive permutation group, following [Kn, Theorem 2.1 and Proposition 4.1]. For this, we define that for a group X, a subgroup $U \leq X$ and a subset $Y \subseteq X$ we denote by $U_Y = \bigcap_{y \in Y} U^y$ the largest subgroup of U normalized by Y, cf. [Kn, p. 138].

Convention 3.1.13. Let G be a primitive permutation group on a finite set X, O be an orbital of G of length $d \geq 2$ and $(\alpha, \beta) \in O$ (esp. $\alpha, \beta \in \mathrm{X}$ with $\alpha \neq \beta$ and $d = |O(\alpha)| = |O'(\beta)|$, recall (1.4.3)). Then define the following subgroups of G:

$$K(\alpha) = (G_\alpha)_{O(\alpha)} \trianglelefteq G_\alpha, \quad K'(\beta) = (G_\beta)_{O'(\beta)} \trianglelefteq G_\beta,$$
$$E(\alpha, \beta) = K(\alpha) \cap K'(\beta),$$
$$L(\alpha) = (G_{\alpha\beta})_{G_\beta G_\alpha} \trianglelefteq G_\alpha \text{ and } L'(\beta) = (G_{\alpha\beta})_{G_\alpha G_\beta} \trianglelefteq G_\beta.$$

By elementary observations, also using the isomorphism theorems, we obtain the following, see [Kn, Proposition 4.1] or [St, Beobachtung 2.2.2 and Hilfssatz 2.2.3].

Lemma 3.1.14. *Assume the conditions and notations of the last convention. Then we have*

(i) $G = \langle G_\alpha, G_\beta \rangle$ and $(G_\alpha)_G = (G_\beta)_G = (G_{\alpha\beta})_G = 1$,

(ii) $K(\alpha) = (G_{\alpha\beta})_{G_\alpha} = (G_\beta)_{G_\alpha}$ and $K'(\beta) = (G_{\alpha\beta})_{G_\beta} = (G_\alpha)_{G_\beta}$,

(iii) $L(\alpha) = K'(\beta)_{G_\alpha} = E(\alpha, \beta)_{G_\alpha}$ and $L'(\beta) = K(\alpha)_{G_\beta} = E(\alpha, \beta)_{G_\beta}$,

(iv) $|G_\alpha : K(\alpha)|$ and $|G_\beta : K'(\beta)|$ both divide $d!$,

(v) $|G_\alpha : E(\alpha, \beta)|$ and $|G_\beta : E(\alpha, \beta)|$ both divide $d!(d-1)!$ and

(vi) $|G_\alpha : L(\alpha)|$ and $|G_\beta : L'(\beta)|$ both divide $d!((d-1)!)^d$.

In Figure 3.1 we present the subgroups defined in Convention 3.1.13.

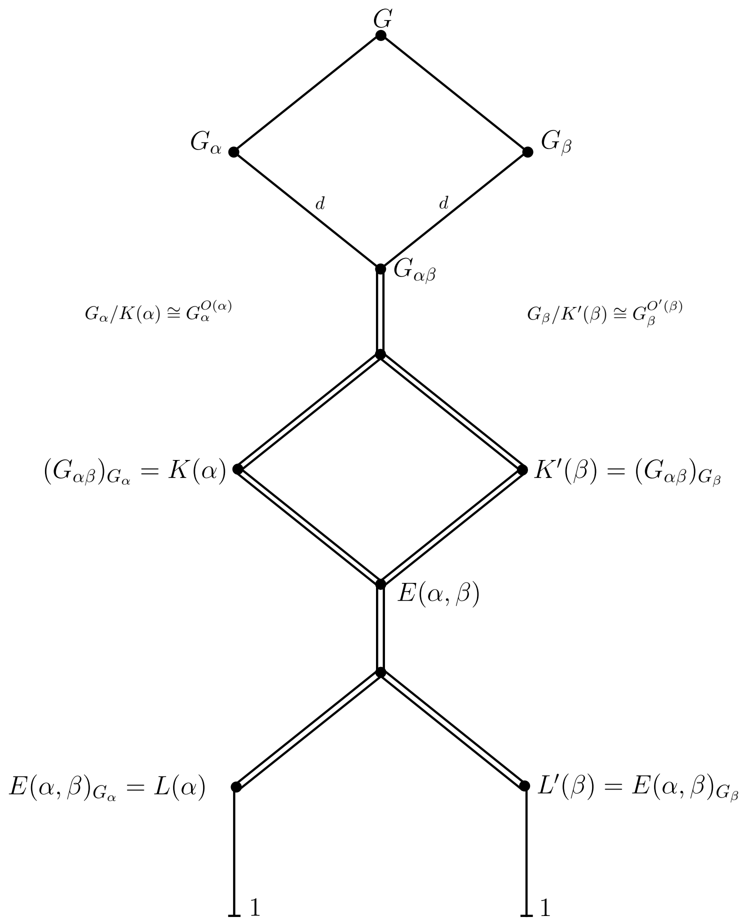

Figure 3.1: Subgroup diagram of $K(\alpha)$, $K'(\beta)$, $E(\alpha, \beta)$, $L(\alpha)$ and $L'(\beta)$.

By a theorem of Wielandt, see [Kn, Theorems 2.1 and 4.2] and [Wie2, Theorem 6.7], the following important facts are provided.

Theorem 3.1.15. ***(Wielandt)*** *Assume the conditions and notations of Convention 3.1.13. Then there exists a prime p such that the following hold.*

(i) The subgroup $L(\alpha)$ or $L'(\beta)$ of smaller or equal order is a p-group.

(ii) If the subgroup $L(\alpha)$ or $L'(\beta)$ of higher order is not a p-group then it contains the subgroup of smaller order.

(iii) *If $L(\alpha) \neq 1 \neq L'(\beta)$ then every subgroup Y of G with $K(\alpha) \leq Y \leq G_\alpha$ or $K'(\beta) \leq Y \leq G_\beta$ is strongly p-constrained (esp., G_α and $G_{\alpha\beta}$ are strongly p-constrained). (Note, that p is uniquely determined, see Corollary 1.4.19 and its following remark).*

(iv) *There exists a normal p-subgroup N of G_α with $N \leq G_{\alpha\beta}$ such that $|G_\alpha : N|$ divides $d!((d-1)!)^d$.*

Remark 3.1.16. (a) In [Bü, 2.16 Satz], a stronger version of the last theorem is proved. There it is shown that if $L(\alpha) \neq 1 \neq L'(\beta)$ then $L(\alpha)$ and $L'(\beta)$ are p-groups for the same prime p, cf. also [St, Satz 2.2.6].

(b) We note that the last theorem was crucial for the investigations in [St] which shortened and improved the reduction in [CPSS] to the case of almost simple primitive permutation groups, mentioned above in the historical overview.

(c) We also want to mention a further result obtained in [St]. Using the conditions and notations of Convention 3.1.13, in [St, Satz 2.2.9] it is shown that if $L(\alpha) \neq 1 \neq L'(\beta)$, so following Theorem 3.1.15 (iii) G_α is strongly p-constrained for some prime p, we have

(i) the Sylow p-subgroups of G_α are non-abelian, and

(ii) $O_p(G_\alpha)$ is not a minimal normal subgroup of G_α.

Remark 3.1.17. Consider the situation that G is a primitive permutation group, d a non-trivial subdegree of G and G_α the stabilizer of a point. Let G_α be strongly p-constrained for a prime p. If $d = 1$ then $|G_\alpha| = 1$ (recall Theorem 3.1.3 (a)). For $d \geq 2$ (esp. $|G_\alpha| > 1$ and for all primes $r \neq p$ we have $|O_r(G_\alpha)| = 1$) we obtain by Theorem 3.1.15 (iv) that $|G_\alpha/O_p(G_\alpha)|$ divides $\mathrm{wdt}(d) = d!((d-1)!)^d$. Hence, to obtain an upper bound for $|G_\alpha|$ in terms of d it is sufficient to estimate $|O_p(G_\alpha)|$ by an increasing function $h_0(y)$ in terms of $y = |G_\alpha/O_p(G_\alpha)|$. Then in any case $|G_\alpha| \leq \mathrm{wdt}(d) \cdot h_0(\mathrm{wdt}(d))$, and more precise upper bounds may be obtained by determining a more precise upper bound for $|G_\alpha/O_p(G_\alpha)|$ than $\mathrm{wdt}(d)$.
In [Uf], this approach was used for analyzing the case that the socle of G is isomorphic to $\mathrm{PSL}_3(q)$. We note that in this work arose cases where it was not possible to determine such an upper bound $h_0(y)$, because of the unanswered question whether there is only a finite number of Fermat primes (see [Uf, Bemerkung 2.5.4] and cf. Remark 3.4.3, below).
To determine more precise upper bounds for $|G_\alpha|$ in terms of d, we will not use the previously described (indirect) approach to achieve our intended goal in this chapter (see Main Theorem 3.1.19, below), and only provide further remarks concerning this method.

As a direct consequence of Lemma 3.1.14 (vi) and Theorem 3.1.15 (iii) we obtain the following corollary (cf. also [St, Folgerung 2.2.7]), which provides a Sims order bound for a large collection of primitive permutation groups.

Corollary 3.1.18. *Let G be a primitive permutation group on a finite set* X*, $\alpha \in X$ and d be a non-trivial subdegree of G. If G_α is not strongly constrained (esp. $d \geq 2$, recall Theorem 3.1.3 (a)) then*

$$|G_\alpha| \;\Big|\; d!((d-1)!)^d,$$

consequently the order of G_α is bounded by the Wielandt order bound $\mathrm{wdt}(d) = d!((d-1)!)^d$.

Remark. In view of the last corollary, to obtain an explicit Sims order bound it is sufficient to determine a Sims order bound for the collection $\mathscr{G}^c$ consisting of all primitive permutation groups G which have a strongly *c*onstrained stabilizer of a point. From the above historical overview, we also recall that by using the O'Nan-Scott theorem it is possible to obtain a further reduction to the cases of the collection $\mathscr{G}^{sc}$ (p. 167), see [St, Sätze 2.2.10, 2.2.11 and 2.3.5] or [CPSS, Section 1].

Now, we have provided the facts and the notation to state the intended goal of this chapter and to explain the approach how to reach it. We consider the collection $\mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$ consisting of all almost *s*imple primitive permutation groups G on a finite set X where $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$ and which have a strongly *c*onstrained stabilizer G_α of a point $\alpha \in$ X. The goal of this chapter is to determine a Sims order bound $h(d)$ for $\mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$. In view of Proposition 1.1.2, w.l.o.g. we can consider $\mathrm{PSL}^\epsilon(V) \leq G \leq \mathrm{Aut}(\mathrm{PSL}^\epsilon(V))$. We recall that all pairs (G, G_α) were determined in Chapter 2, and the facts collected there will be important for our further investigations. Define the subcollections $\mathscr{G}^{\mathbf{L}}$ and $\mathscr{G}^{\mathbf{U}}$ of $\mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$ where $\mathscr{G}^{\mathbf{L}}$ consists of all $G \in \mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$ with $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ and $\mathscr{G}^{\mathbf{U}}$ consists of all $G \in \mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$ with $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ and $n \geq 3$. In view of the isomorphism $\mathrm{PSL}_2(q) \cong \mathrm{PSU}_2(q^2)$, it is sufficient to determine Sims order bounds $h^{\mathbf{L}}(d)$ and $h^{\mathbf{U}}(d)$ for the collections $\mathscr{G}^{\mathbf{L}}$ and $\mathscr{G}^{\mathbf{U}}$, and then set $h(d) = \max\{h^{\mathbf{L}}(d), h^{\mathbf{U}}(d)\}$. Moreover, we divide the collections $\mathscr{G}^{\mathbf{L}}$ and $\mathscr{G}^{\mathbf{U}}$ further according to the A-classes $\mathscr{C}_1$ to $\mathscr{C}_8$. Let the collection $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ consist of all $G \in \mathscr{G}^{\mathbf{L}^\epsilon}$ where G_α is a member of A-class $\mathscr{C}_j$ of G. Clearly, if Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ for each $j \in \{1, \ldots, 8\}$ are determined, we obtain by $h^{\mathbf{L}^\epsilon}(d) = \max\{h^{\mathbf{L}^\epsilon}_{\mathscr{C}_1}(d), \ldots, h^{\mathbf{L}^\epsilon}_{\mathscr{C}_8}(d)\}$ a Sims order bound for $\mathscr{G}^{\mathbf{L}^\epsilon}$. Keeping in mind the previously introduced notation, we can state the first main theorem of this chapter.

Main Theorem 3.1.19. *Let G be an almost simple primitive permutation group on a finite set* X *where* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *or* $\mathrm{PSU}_n(q^2)$ *and let d be an arbitrary non-trivial subdegree of G. If the stabilizer G_α in G of a point $\alpha \in \mathrm{X}$ is strongly constrained then we have*

$$|G_\alpha| \leq h(d) = \max\{h^{\mathbf{L}}(d), h^{\mathbf{U}}(d)\}$$

where

$$h^{\mathbf{L}}(d) = \max\{h^{\mathbf{L}}_{\mathscr{C}_j}(d) \mid 1 \leq j \leq 8\}$$

and the functions $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$ are Sims order bounds for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_j}$ and are as defined in Theorems 3.2.1, 3.2.3, 3.2.6, 3.3.5, 3.4.6, 3.5.4 and 3.6.6, and

$$h^{\mathbf{U}}(d) = \max\{h^{\mathbf{U}}_{\mathscr{C}_j}(d) \mid 1 \leq j \leq 8\}$$

and the functions $h^{\mathbf{U}}_{\mathscr{C}_j}(d)$ are Sims order bounds for $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_j}$ and are as defined in Theorems 3.2.1, 3.2.5, 3.3.6, 3.4.7, 3.5.5 and 3.6.6. In particular, the function $h(d)$ is a Sims order bound for $\mathscr{G}^{sc}_{\mathbf{L},\mathbf{U}}$.
Furthermore, we have $h^{\mathbf{L}}_{\mathscr{C}_j}(d)$, $h^{\mathbf{U}}_{\mathscr{C}_j}(d) \leq d!((d-1)!)^d = \mathrm{wdt}(d)$ for each $j \in \{1, \ldots, 8\}$, so $h(d) \leq \mathrm{wdt}(d)$.

By Corollary 3.1.18 and Main Theorem 3.1.19, we now deduce the second main theorem of this chapter.

Main Theorem 3.1.20. *The Wielandt order bound $\mathrm{wdt}(d) = d!((d-1)!)^d$ is a Sims order bound for the collection consisting of all almost simple primitive permutation groups whose socle is isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q^2)$.*

Remark 3.1.21. Concerning Main Theorem 3.1.19, regarding Corollary 3.1.18 and the resulting Main Theorem 3.1.20, we note that it is a goal of this chapter to determine Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ which are smaller or equal than the Wielandt order bound $\mathrm{wdt}(d)$. But, we do not restrict to show that $\mathrm{wdt}(d)$ bounds the order of the stabilizer of a point. Our aim is to determine for each A-class $\mathscr{C}_j$ a more appropriate Sims order bound $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$. In view of the enormous amount of cases of pairs (G, G_α) we have to consider and since it is hard to determine the minimal non-trivial subdegree, within the scope of this thesis one should not expect to determine sharp Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$. So, we will investigate each occurring situation by an appropriate amount of work and provide remarks how to sharpen the obtained Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$. As we will see, the determined Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ are considerably smaller than $\mathrm{wdt}(d)$ for large d. But, for small d we have to consider the possible cases which may occur and use the facts provided above in this chapter to obtain Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ which are smaller than $\mathrm{wdt}(d)$.
Furthermore, we will often also mention further possibilities how to determine Sims order bounds for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ which are less precise than $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$.

In addition to the standard notation introduced in Subsection 1.2.3, we now introduce further notations for the following investigations in Sections 3.2 to 3.6. We use the notations introduced in the considerations before Main Theorem 3.1.19. Let $G \in \mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ where w.l.o.g. $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$, esp. the maximal strongly constrained subgroup G_α of G is a member of A-class $\mathscr{C}_j$ of G. In addition, adopt the notations from Convention 3.1.13, so let O be an orbital of G of length $d = |O(\alpha)| \geq 3$ (recall Theorem 3.1.3), $(\alpha, \beta) \in O$ and let $K(\alpha), K'(\beta), E(\alpha, \beta), L(\alpha)$ and $L'(\beta)$ denote the subgroups of G defined there. Since $K(\alpha) \trianglelefteq G_\alpha$, we define the reduction map $\rho_{K(\alpha)}$ as the reduction in G_α modulo $K(\alpha)$ by

$$\begin{aligned} \rho_{K(\alpha)} : G_\alpha \to \rho_{K(\alpha)}(G_\alpha) = G_\alpha^{\rho_{K(\alpha)}} = G_\alpha/K(\alpha) \cong G_\alpha^{O(\alpha)}, \\ g \mapsto \rho_{K(\alpha)}(g) = g^{\rho_{K(\alpha)}} = g \cdot K(\alpha)/K(\alpha). \end{aligned} \tag{3.1.2}$$

We will also often drop the subscript $K(\alpha)$ and only write ρ for $\rho_{K(\alpha)}$ if it is clear by the situation.
In the following sections, we determine Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ where $j \in \{1, \ldots, 8\}$. For this, we use the previously introduced notation and we set $G \in \mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ where w.l.o.g. $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$ as above (note, that G_α is a strongly constrained member of A-class $\mathscr{C}_j$ of G).

3.2 Sims order bounds for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_4}, \mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_5}, \mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_7}$ and $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_8}$

By our investigations in Chapter 2 and the facts provided in the previous section (recall also the notation introduced at the end of that section), we can already determine Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ for $j \in \{4, 5, 7, 8\}$.
We begin with the trivial cases. Considering Main Theorems 2.4.5 and 2.7.2 and Section 2.8, we see that the collections $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_4}$, $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_7}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_8}$ are empty. So, in these cases we may set $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_4}(d) = h^{\mathbf{L}^\epsilon}_{\mathscr{C}_7}(d) = h^{\mathbf{U}}_{\mathscr{C}_8}(d) = 0$. Hence, we obtain the following theorem.

Theorem 3.2.1. *The collections* $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_4}$, $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_7}$ *and* $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_8}$ *are empty, and we have*

$$h^{\mathbf{L}}_{\mathscr{C}_4}(d) = h^{\mathbf{L}}_{\mathscr{C}_7}(d) = h^{\mathbf{U}}_{\mathscr{C}_4}(d) = h^{\mathbf{U}}_{\mathscr{C}_7}(d) = h^{\mathbf{U}}_{\mathscr{C}_8}(d) = 0.$$

However, we obtain non-trivial Sims order bounds $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ in the following three theorems.

Convention 3.2.2. In the following investigations we set also $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}(d_0) = 0$ if $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$ is non-empty and we have proved that d_0 does not occur as a subdegree for any $G \in \mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_j}$.

Theorem 3.2.3. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_5$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_5}(d) = \begin{cases} 2 \cdot 3 & \text{for } d = 3, \\ 2^2 \cdot 3 & \text{for } d = 4, \\ 0 & \text{for } d = 5, \\ 2^4 \cdot 3 & \text{for } d \geq 6. \end{cases}$$

In particular, $h^{\mathbf{L}}_{\mathscr{C}_5}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_5}$ *and* $h^{\mathbf{L}}_{\mathscr{C}_5}(d) \leq \mathrm{wdt}(d)$.

Proof. Since all (almost simple) primitive permutation groups which have a subdegree 3 or 4 are known, we first consider the cases $d = 3, 4$ separately. Let $d = 3$. Comparing Main Theorem 2.5.24 with Theorem 3.1.4, also regarding Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (c), we see that only one case is possible, listed in Main Theorem 2.5.24 (i). This case coincides with the case listed in [Wo, (3) p. 236] (cf. also [Wo, Lemma 3]), and hence we have here $|G_\alpha| = 2 \cdot 3$. So, our assertion holds for $d = 3$.
Now, let $d = 4$. In view of Main Theorem 2.5.24 and Theorem 3.1.5 (also regard Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (b) and (c)), we see that only the cases with $G \cong \mathrm{PSL}_2(3^r)$ (and $|G_\alpha| = |\mathrm{A}_4|$) of Main Theorem 2.5.24 (ii) (b) are possible. Hence, the assertion also holds in this case.
Suppose that $d = 5$. Then $5 \mid |G_\alpha|$ which contradicts Main Theorem 2.5.24 (see also Proposition 2.5.7). Hence, we may assume that $d \geq 6$ and our assertion follows from Main Theorem 2.5.24. □

Remark 3.2.4. Recalling Remark 3.1.6, we will show that the possible cases for $d = 4$ in the last proof actually have a faithful permutation representation of the required type. In view of [KL, Proposition 4.5.3 (I)], w.l.o.g. we can set $G = \mathrm{PSL}_2(3^r)$ (for an odd prime r) and $G_\alpha = \mathrm{P}H$ according to Example 2.5.13. Consider the canonical faithful primitive permutation representation $(G : G_\alpha) \times G \to G : G_\alpha$, see (1.4.1). Let $S = \langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rangle \in \mathrm{Syl}_3(G_\alpha)$. It is not hard to see that there exists an element $g \in \mathrm{N}_G(S) \setminus G_\alpha$. Hence, we obtain $S \leq G_\alpha \cap G_\alpha^g < G_\alpha$. Since $G_\alpha \cong \mathrm{A}_4$ and $|S| = 3$, we obtain by Theorem 3.1.3 (b) that $S = G_\alpha \cap G_\alpha^g$. So, G_α has an orbit of length 4.
Considering the last theorem, we now see that $h^{\mathbf{L}}_{\mathscr{C}_5}(d)$ is a sharp upper bound for the order of G_α for $d \in \{3, 4\}$.

Theorem 3.2.5. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_5$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{U}}_{\mathscr{C}_5}(d) = \begin{cases} 0 & \text{for } 3 \leq d \leq 5, \\ 2^4 \cdot 3^4 & \text{for } d \geq 6. \end{cases}$$

In particular, $h^{\mathbf{U}}_{\mathscr{C}_5}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_5}$ *and* $h^{\mathbf{U}}_{\mathscr{C}_5}(d) \leq \mathrm{wdt}(d)$.

Proof. As in the proof of the previous theorem, we first consider the cases $d = 3, 4$. Comparing Main Theorem 2.5.25 with Theorems 3.1.4 and 3.1.5, we see that there is no case where $d = 3$ or 4 (recall also Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (b) and (c)). Because $5 \nmid |G_\alpha|$ (see Main Theorem 2.5.25 and also Proposition 2.5.7), we have $d \neq 5$. So, we can assume that $d \geq 6$ and we obtain our assertion by Main Theorem 2.5.25. □

Theorem 3.2.6. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_8$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_8}(d) = \begin{cases} 2^4 \cdot 3 & \text{for } d = 3, \\ 2^3 \cdot 3 & \text{for } d = 4, \\ 0 & \text{for } d = 5, \\ 2^8 \cdot 3^2 & \text{for } d \geq 6. \end{cases}$$

In particular, $h^{\mathbf{L}}_{\mathscr{C}_8}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_8}$ *and* $h^{\mathbf{L}}_{\mathscr{C}_8}(d) \leq \mathrm{wdt}(d)$*.*

Proof. As in the proofs of the previous two theorems, we begin by considering the cases $d = 3, 4$. First, let $d = 3$. Comparing Main Theorem 2.8.11 with Theorem 3.1.4 (regarding also Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (c)), we obtain that only the two cases listed in Main Theorem 2.8.11 (b) are possible. These cases coincide with the cases listed in [Wo, (8) and (9) p. 237], see also Example 2.8.8, Proposition 2.8.9 and [Wo, Lemma 3]. So, we have $|G_\alpha| \leq 2^4 \cdot 3$ if $d = 3$, and the assertion holds in this case.
Now, consider $d = 4$. Regarding Main Theorem 2.8.11 and Theorem 3.1.5 (also recall Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (b) and (c)), we can deduce that only the case with $G \cong \mathrm{PSL}_3(3)$ (and $|G_\alpha| = |\mathrm{S}_4|$) from Main Theorem 2.8.11 (b) is possible. So, our assertion also holds for $d = 4$.
We obviously have $d \neq 5$, since $5 \nmid |G_\alpha|$ by Main Theorem 2.8.11 (see also Proposition 2.8.3). Hence, we may assume that $d \geq 6$ and the assertion follows by Main Theorem 2.8.11. □

Remark 3.2.7. (a) As in Remark 3.2.4, we will show that the possible case for $d = 4$ in the last proof actually has a faithful permutation representation of the required type. Regarding [KL, Proposition 4.8.4 (I)], w.l.o.g. we can set $G = \mathrm{PSL}_3(3)$ and $G_\alpha = \mathrm{P}H$ according to Example 2.8.8. Consider the canonical faithful primitive permutation representation $(G : G_\alpha) \times G \to G : G_\alpha$, see (1.4.1). We set $\mathrm{S}_3 \cong S = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right\rangle \leq G_\alpha$ and $g = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in G$. By easy calculations, we see that $g \notin G_\alpha$ and g normalizes S, hence $S \leq G_\alpha \cap G_\alpha^g < G_\alpha$. So, since $G_\alpha \cong \mathrm{S}_4$, we can deduce that G_α has an orbit of length 4, by Theorem 3.1.3 (b).
Considering the above theorem, we now see that $h^{\mathbf{L}}_{\mathscr{C}_8}(d)$ is a sharp upper bound for $|G_\alpha|$ for $d \in \{3, 4\}$.

(b) Concerning the last three theorems, we note that investigations by further case-by-case analysis with respect to the values for d (for the cases listed in Main Theorems 2.5.24, 2.5.25 and 2.8.11) may lead to more precise upper bounds; such as the trivial observation that for primes $d \geq 7$ we can also set $h^{\mathbf{L}}_{\mathscr{C}_5}(d) = h^{\mathbf{U}}_{\mathscr{C}_5}(d) = h^{\mathbf{L}}_{\mathscr{C}_8}(d) = 0$.

3.3 Sims order bound for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_1}$

Next, we determine a Sims order bound $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_1}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_1}$. We recall that we use the notation introduced at the end of Section 3.1, and we also recall Convention 3.2.2. In the following proposition, we first consider the case that G_α is a member of A-class $\mathscr{C}_1$ of G of type $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ or $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$.

Proposition 3.3.1. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_1,\oplus}(d) = \begin{cases} 0 & \text{for } 3 \leq d \leq 5, \\ 2^5 \cdot 3 & \text{for } d \geq 6. \end{cases}$$

For the case that $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_1$ *of* G *of type* $\mathrm{GU}_k(q^2) \perp \mathrm{GU}_{n-k}(q^2)$*, the order of* G_α *is bounded by*

$$h^{\mathbf{U}}_{\mathscr{C}_1,\perp}(d) = \begin{cases} 0 & \text{for } 3 \leq d \leq 5, \\ 2^5 \cdot 3^5 & \text{for } d \geq 6. \end{cases}$$

Proof. Considering Proposition 2.1.5 and Main Theorems 2.1.9 and 2.1.10, the assertion follows analogously to the proof of Theorem 3.2.5. □

Remark. As in Remark 3.2.7 (b), we note concerning the upper bounds of the previous proposition that by further case-by-case analysis with respect to the values for d (observing the cases from Main Theorems 2.1.9 and 2.1.10) more precise upper bounds can be obtained.

Considering Definition 2.1.2, there are the cases left to examine where G_α is a member of A-class $\mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. By the following remark, we provide for these cases upper bounds for the order of G_α, based on the approach mentioned in Remark 3.1.17.

Remark 3.3.2. We recall the approach described in Remark 3.1.17 to estimate $|\mathrm{O}_p(G_\alpha)|$ by an increasing function $h_0(y)$ in terms of $y = |G_\alpha/\mathrm{O}_p(G_\alpha)|$ if G_α is strongly p-constrained. Here, we determine such upper bounds $h_0(y)$ for our intended case that $G_\alpha \in \mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$ and we recall that $\mathrm{P\Omega}(V) \leq G \leq \mathrm{PA}(V)$ where V is an n-dimensional $\mathrm{GF}(q^u)$-vector space. Let

$q = p^a$ for a prime p and a positive integer a (then G_α is strongly p-constrained, recall Main Theorems 2.1.24 and 2.1.25). First, consider the case **L**. Let $G_\alpha \in \mathscr{C}_1$ of G of type $P_{k,n-k}$ (esp. $G \nleq \mathrm{P\Gamma L}(V)$, again recall Main Theorem 2.1.24). Considering Propositions 2.1.13 and 2.1.26 together with Lemmas 2.1.17 and 2.1.19 (cf. also Proposition 2.1.11), we see that

$$|\mathrm{O}_p(G_\alpha)| = q^{k(2n-3k)} \quad \text{and} \quad \mu \;\big|\; |G_\alpha/\mathrm{O}_p(G_\alpha)| \;\big|\; \mu \cdot a \cdot (n, q-1)$$

for $\mu = 2 \cdot |\mathrm{SL}_k(q)|^2 \cdot |\mathrm{SL}_{n-2k}(q)| \cdot (q-1)^2/(n, q-1)$, excepting the three cases (i) to (iii) from Proposition 2.1.26. We consider the case $n = 3$. Here, we obtain that the "worst case" (for estimating $|\mathrm{O}_p(G_\alpha)|$ by a function in terms of $|G_\alpha/\mathrm{O}_p(G_\alpha)|$) occurs for $|\mathrm{O}_p(G_\alpha)| = q^3$ and $|G_\alpha/\mathrm{O}_p(G_\alpha)| = \frac{2}{3} \cdot (q-1)^2$. (Note, that this "worst case" occurs for infinitely many choices of q). So, here we obtain an upper bound for $|\mathrm{O}_p(G_\alpha)|$ in terms of $y = |G_\alpha/\mathrm{O}_p(G_\alpha)|$ by

$$h_0(y) = ((3/2 \cdot y)^{\frac{1}{2}} + 1)^3$$

which is sharp with respect to the mentioned "worst case" (cf. also [Uf, Satz 2.4.7 and Korollar 2.4.9]). By elementary calculations, using the facts provided in Propositions 2.1.13 and 2.1.26 together with Lemmas 2.1.17 and 2.1.19, it can be shown that the previously determined function $h_0(y)$ is also an upper bound for $|\mathrm{O}_p(G_\alpha)|$ in terms of $y = |G_\alpha/\mathrm{O}_p(G_\alpha)|$ for any other n, q and $G_\alpha \in \mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$ in the case **L**, except in the cases (i) and (ii) in Proposition 2.1.26 where we have $|G_\alpha| \leq 2^8 \cdot 3$. For the mentioned calculations we note that it is useful to use the estimate

$$|\mathrm{SL}_k(q)| = q^{\frac{k(k-1)}{2}} \prod_{i=2}^{k} (q^i - 1) \geq q^{k(k-1)} \cdot (q-1)^{k-1}. \tag{3.3.1}$$

In the case **U** (here, $G_\alpha \in \mathscr{C}_1$ of G of type P_k), we obtain by analogous considerations as in the case **L** the upper bound

$$h_0(y) = (3y + 1)^{\frac{3}{2}}$$

for $|\mathrm{O}_p(G_\alpha)|$ in terms of $y = |G_\alpha/\mathrm{O}_p(G_\alpha)|$. For the necessary calculations in this case we note that is useful to use (3.3.1) and the estimate

$$\begin{aligned} |\mathrm{SU}_k(q^2)| &= q^{\frac{k(k-1)}{2}} \prod_{i=2}^{k} (q^i - (-1)^i) \\ &\geq \begin{cases} q^{\frac{k(k-1)}{2}} \cdot (q-1)^{\frac{k}{2}} \cdot q^{\frac{k^2}{2}-1} = q^{k^2-1-\frac{k}{2}} \cdot (q-1)^{\frac{k}{2}} & \text{for } k \text{ even}, \\ q^{\frac{k(k-1)}{2}} \cdot (q-1)^{\frac{k-1}{2}} \cdot q^{\frac{k^2-1}{2}} = q^{k^2-\frac{k+1}{2}} \cdot (q-1)^{\frac{k-1}{2}} & \text{for } k \text{ odd}. \end{cases} \end{aligned}$$

Finally, we recall that $y = |G_\alpha/\mathrm{O}_p(G_\alpha)|$ is bounded by $\mathrm{wdt}(d)$, so $|G_\alpha| \leq \mathrm{wdt}(d) \cdot h_0(\mathrm{wdt}(d))$ in the above considered cases; and more precise upper bounds for y in terms of d than $\mathrm{wdt}(d)$ may lead to more precise upper bounds for $|G_\alpha|$ in terms of d.

Remark 3.3.3. Considering our intended situation for the cases $G_\alpha \in \mathscr{C}_1$ of G of type P_k in the case **L** or $G_\alpha \in \mathscr{C}_1$ of G of type $P_{\frac{n}{2}}$ in the case **U**, we obtain by Lemmas 2.1.17, 2.1.18 and 2.1.19 and Propositions 2.1.13 and 2.1.26 (cf. also its proof) that $\mathrm{O}_p(G_\alpha)$ is a minimal normal p-subgroup of G_α where $p = \mathrm{char}(\mathrm{GF}(q^u))$. Regarding Lemma 3.1.14 and Remark 3.1.16 (c)(ii), we now can deduce that for the described cases we have $|G_\alpha| \leq \mathrm{wdt}(d)$.
Furthermore, we note that for the cases $G_\alpha \in \mathscr{C}_1$ of G of type $P_{k,n-k}$ in the case **L** or $G_\alpha \in \mathscr{C}_1$ of G of type P_k with $2k < n$ in the case **U**, $\mathrm{O}_p(G_\alpha)$ is not a minimal normal subgroup of G_α, recall Propositions 2.1.13 and 2.1.26 and cf. Remark 2.1.22.

In the following theorems, we will determine more precise upper bounds as in the previous two remarks. For this, we recall our generalized notation of the diagonal matrix $\mathrm{diag}(A_1, \dots, A_k)$ where $A_i \in \mathrm{GL}_{n_i}(q)$, introduced on page 8, and the notation $\mathrm{d}_f(G)$ for the minimal degree of all faithful permutation representations of a finite group G, introduced in Subsection 1.4.2. Furthermore, we recall the following well-known facts.

Lemma 3.3.4. *(i) S_5 acts faithfully and sharply 3-transitively (by conjugation) on the set $\mathrm{Syl}_5(S_5)$. So, there is a sharply 3-transitive permutation group $H \cong S_5$ of degree 6.*

(ii) A_5 acts faithfully and 2-primitively (by conjugation) on the set $\mathrm{Syl}_5(A_5)$. So, there is a 2-primitive permutation group $H \cong A_5$ of degree 6.

Proof. The assertions follow by elementary considerations, using the Sylow theorems and the facts in [At, p. 2]. □

Theorem 3.3.5. *For the case that $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ and G_α is a strongly constrained member of A-class $\mathscr{C}_1$ of G, the order of G_α is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_1}(d) = \begin{cases} 0 & \text{for } d = 3, \\ 2^3 \cdot 3 & \text{for } d = 4, \\ 2^7 \cdot 3^2 \cdot 5 & \text{for } d = 5, \\ 2^{12} \cdot 3^{15} & \text{for } d = 6, \\ 2^{22} \cdot 3^3 \cdot 7 & \text{for } d = 7, \\ 2^{20} \cdot 3^{30} \cdot 5^3 & \text{for } d = 8, \\ 2^{67} \cdot 3^6 \cdot 5^3 \cdot 7^3 & \text{for } 9 \leq d \leq 14, \\ h_1(d) & \text{for } 15 \leq d \leq 30, \\ \max\{h_1(d), h_2(d)\} & \text{for } d \geq 31. \end{cases}$$

where $h_1(d) = (d-4)^{6\frac{\ln(d-4)}{\ln(3)}+14.5} \cdot \frac{\ln(d-4)}{\ln(2)}$ and $h_2(d) = 2^{6\left(\frac{\ln(d+1)}{\ln(2)}\right)^2-2}$. In particular, $h^{\mathbf{L}}_{\mathscr{C}_1}(d)$ is a Sims order bound for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_1}$ and $h^{\mathbf{L}}_{\mathscr{C}_1}(d) \leq \mathrm{wdt}(d)$.

Proof. Suppose that the assertion is false and let the pair (G, G_α) be a counterexample (also recall that $\mathrm{P}\Omega(V) \leq G \leq \mathrm{PA}(V)$). Regarding Proposition

3.3.1, we see that $h^{\mathbf{L}}_{\mathscr{C}_1,\oplus}(d) \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d)$, so we can assume that $G_\alpha \in \mathscr{C}_1$ of G of type P_k or $P_{k,n-k}$. Let $q = p^a$ for a prime p and a positive integer a, and note that G_α is strongly p-constrained, see Main Theorem 2.1.24. First, we consider the case where $G_\alpha \in \mathscr{C}_1$ of G of type $P_{k,n-k}$.
Considering Propositions 2.1.4, 2.1.11 and 2.1.13 (and using the notation from the last listed proposition), we choose an appropriate ordered GF(q)-basis of V and w.l.o.g. we may assume that

$$H_\alpha = G_\alpha \cap \mathrm{P}\Omega = \mathrm{P}H^{\mathrm{SL}}_{k,n-k}, \tag{3.3.2}$$

and note that H_α is a member of A-class $\mathscr{C}_1$ of $\mathrm{P}\Omega$ of type $P_{k,n-k}$. We define the following subgroups of H_α:

$$\begin{aligned} H_{\alpha,1} &= \mathrm{P}(\{\mathrm{diag}(A, \mathbb{1}_{n-k}) \mid A \in \mathrm{SL}_k(q)\}) \cong \mathrm{SL}_k(q), \\ H_{\alpha,2} &= \mathrm{P}(\{\mathrm{diag}(\mathbb{1}_k, A, \mathbb{1}_k) \mid A \in \mathrm{SL}_{n-2k}(q)\}) \cong \mathrm{SL}_{n-2k}(q). \end{aligned}$$

Let $l = \max\{k, n-2k\}$, and let H be one of the groups $H_{\alpha,1}$, or $H_{\alpha,2}$ which is isomorphic to $\mathrm{SL}_l(q)$. Now, we consider separately the two possibilities that G_α is soluble or insoluble.
First, suppose that G_α is insoluble. We note that $d \geq 5$ by Theorem 1.4.25 (also cf. Remark 1.4.27 (b)). Because $G_\alpha/H_\alpha \cong G/\mathrm{P}\Omega$ is soluble (recall Corollary 1.2.22), H_α is insoluble. Considering Propositions 1.2.12 and 2.1.11 (ii) (or, 2.1.13 (ii)) and (3.3.2), we can deduce that $\mathrm{PSL}_l(q)$ is simple. We note that the following considerations are based on [Wie, proof of Theorem 18.2], and we recall the reduction map $\rho_{K(\alpha)} = \rho$ from (3.1.2). By Lemma 1.4.24, there is an element $g \in G$ with $H^g \leq G_\alpha$ and $1 \neq (H^g)^\rho \cong (H^g)^{O(\alpha)} \leq \mathrm{Sym}(O(\alpha)) \cong \mathrm{S}_d$ (note, that $(H^g)^\rho = H^g \cdot K(\alpha)/K(\alpha) \cong H^g/(K(\alpha) \cap H^g) = H^g/H^g_{O(\alpha)}$). Since $H^g \cong \mathrm{SL}_l(q)$ is quasisimple and $K(\alpha) \cap H^g$ is a proper normal subgroup of H^g, we obtain by elementary considerations $K(\alpha) \cap H^g \leq \mathrm{Z}(H^g)$ (e.g. see [As2, (31.2)]). Using Lemma 1.4.30 (e.g. for $\pi = \{p\}'$), by our previous considerations it is not hard to deduce that

$$\mathrm{d}_f(\mathrm{PSL}_l(q)) \leq \mathrm{d}_f(H^g/(K(\alpha) \cap H^g)) \leq d \tag{3.3.3}$$

The values of $\mathrm{d}_f(\mathrm{PSL}_l(q))$ are known, and are provided in Proposition 1.4.29. Now, using this information together with (3.3.3) and the previously provided facts about primitive permutation groups related to a subdegree d, we will show that there is no counterexample (G, G_α) for our assertion in the actual case. We will do so by case-by-case analysis with respect to the possibilities for $H \cong \mathrm{SL}_l(q)$. For this, the estimate

$$|\mathrm{SL}_m(q)| = q^{\frac{m(m-1)}{2}} \prod_{i=2}^{m} (q^i - 1) \leq q^{\frac{m(m-1)}{2}} \cdot q^{\frac{m(m+1)}{2} - 1} = q^{m^2-1} \tag{3.3.4}$$

for a positive integer m is useful. We note that we will frequently use the information from Propositions 2.1.11 and 2.1.13 and Main Theorem 2.1.24, also without reference to them.

First, let $l = 2$ and $H \cong \mathrm{SL}_2(q)$ quasisimple where $q \notin \{4, 5, 7, 9, 11\}$. Here, by Proposition 1.4.29 and (3.3.3), we have

$$q + 1 \leq d, \tag{3.3.5}$$

especially $d \geq 9$. The "worst case" for the actual case (i.e. the case where G_α is of maximal order such that H is as assumed) occurs for $l = k = n - 2k$ (so, $n = 6$) and $|G_\alpha| = q^{12} \cdot |\mathrm{SL}_2(q)|^3 \cdot (q-1)^2/(6, q-1) \cdot (6, q-1) \cdot a \cdot 2$. So,

$$\begin{aligned} |G_\alpha| &\leq q^{15}(q+1)^3(q-1)^5 \cdot 2a \overset{(3.3.5)}{\leq} d^3(d-1)^{15}(d-2)^5 \cdot 2\frac{\ln(d-1)}{\ln(p)} \\ &\leq d^3(d-1)^{15}(d-2)^5 \cdot 2\frac{\ln(d-1)}{\ln(2)} \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d). \end{aligned}$$

Next, consider $H \cong \mathrm{SL}_2(4)$, and note that $d \geq 5$ by Proposition 1.4.29 and (3.3.3). Here, we have (recall Proposition 2.1.11 and Main Theorem 2.1.24)

$$4^5 \cdot |\mathrm{SL}_2(4)| \cdot 3^2 \cdot 2 = 2^{13} \cdot 3^3 \cdot 5 \;\Big|\; |G_\alpha| \text{ and} \tag{3.3.6}$$

$$|G_\alpha| \leq 4^{12} \cdot |\mathrm{SL}_2(4)|^3 \cdot 3^2 \cdot 2^2 = 2^{32} \cdot 3^5 \cdot 5^3. \tag{3.3.7}$$

Suppose that $d = 5$. Then, considering Theorem 1.4.25 and Lemma 3.1.8, we can deduce that $G_\alpha^{O(\alpha)}$ is isomorphic to A_5 or S_5. So, in any case we have $G_\alpha^{O(\alpha)}$ is 2-primitive and $\mathrm{F}(G_{\alpha\beta}^{O(\alpha)}) \neq 1$ (recall that we have assumed $(\alpha, \beta) \in O$). Hence, by Theorem 3.1.11 (b), we obtain $|G_\alpha| \;\big|\; 2^{14} \cdot 3^2 \cdot 5$ which contradicts (3.3.6). Next, suppose that $d = 6$. Since $\mathrm{PSL}_2(4) \cong \mathrm{A}_5$, by Theorem 1.4.25 and the facts provided in [At, p. 4], we can deduce that $G_\alpha^{O(\alpha)}$ is isomorphic to A_5, S_5, A_6 or S_6. The latter two cases cannot occur, since no composition factor group of G_α is isomorphic to A_6. Recalling that $G_\alpha^{O(\alpha)}$ acts transitively on $O(\alpha)$, we see that $G_\alpha^{O(\alpha)}$ is not a subgroup of the stabilizer of a point in $\mathrm{Sym}(O(\alpha))$. So, again considering [At, p. 4], we see that the permutation group $G_\alpha^{O(\alpha)} \leq \mathrm{Sym}(O(\alpha))$ coincides with one of the cases listed in Lemma 3.3.4. Hence, we can deduce that (in any case) $G_\alpha^{O(\alpha)}$ is 2-primitive and $\mathrm{F}(G_{\alpha\beta}^{O(\alpha)}) \neq 1$. Now, again using Theorem 3.1.11 (b), we obtain $|G_\alpha| \;\big|\; 2^5 \cdot 3 \cdot 5^6$ which contradicts (3.3.6). We now deduce $d \geq 8$ (note, that $7 \nmid |G_\alpha|$), and so we have $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d)$, by (3.3.7).

Now, we consider the case $H \cong \mathrm{SL}_2(5)$, and note that $d \geq 5$ by Proposition 1.4.29 and (3.3.3). Here, we have (recall Proposition 2.1.11 and Main Theorem 2.1.24)

$$5^5 \cdot |\mathrm{SL}_2(5)| \cdot 4^2/(4,4) \cdot 2 = 2^6 \cdot 3 \cdot 5^6 \;\Big|\; |G_\alpha| \text{ and} \tag{3.3.8}$$

$$|G_\alpha| \leq 5^{12} \cdot |\mathrm{SL}_2(5)|^3 \cdot 4^2/(6,4) \cdot (6,4) \cdot 2 = 2^{14} \cdot 3^3 \cdot 5^{15}. \tag{3.3.9}$$

By Lemma 1.4.28 and (3.3.8), we see $d > 5$. Suppose that $d = 6$. Since $\mathrm{PSL}_2(5) \cong \mathrm{A}_5$, by analogous arguments as above we obtain that $G_\alpha^{O(\alpha)}$ is 2-primitive and $\mathrm{F}(G_{\alpha\beta}^{O(\alpha)}) \neq 1$. So, by Theorem 3.1.11 (b), we obtain $|G_\alpha| \;\big|\; 2^5 \cdot 3 \cdot 5^6$ which contradicts (3.3.8). So, since $7 \nmid |G_\alpha|$, we have $d \geq 8$, and obtain $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d)$, by (3.3.9).

Now, we consider $H \cong \mathrm{SL}_2(q)$ where $q \in \{7, 11\}$. Here, using Proposition 1.4.29 and (3.3.3), we see $d \geq q$. Suppose that $q = d$. Then, regarding Lemma 1.4.28 and Proposition 2.1.11, we obtain a contradiction. So, $d \geq q+1$. In the considered case, we now see that

$$\begin{aligned} |G_\alpha| &\leq q^{12} \cdot |\mathrm{SL}_2(q)|^3 \cdot (q-1)^2/(6, q-1) \cdot (6, q-1) \cdot 2 \\ &= 2 \cdot q^{15}(q-1)^5(q+1)^3 \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d). \end{aligned}$$

Next, consider the case $H \cong \mathrm{SL}_2(9)$. Here, $d \geq 6$ by Proposition 1.4.29 and (3.3.3). Suppose that $d = 6$. Since $\mathrm{PSL}_2(9) \cong \mathrm{A}_6$, we obtain that $G_\alpha^{O(\alpha)}$ is isomorphic to A_6 or S_6, in view of Theorem 1.4.25. Clearly, G_α does not act faithfully on $O(\alpha)$. So, by Theorem 3.1.12, we obtain that G_α is isomorphic to $\mathrm{A}_6 \times \mathrm{A}_5$, $\mathrm{S}_6 \times \mathrm{S}_5$, $\mathrm{S}_6 \times \mathrm{A}_5$ or $(\mathrm{A}_6 \times \mathrm{A}_5) : \mathbf{Z}_2$. But this contradicts Proposition 2.1.11. Hence, $d \geq 8$ (note, that $7 \nmid |G_\alpha|$). Because we have that $|G_\alpha| \leq 9^{12} \cdot |\mathrm{SL}_2(9)|^3 \cdot 8^2/(6,8) \cdot (6,8) \cdot 2^2 = 2^{20} \cdot 3^{30} \cdot 5^3$, we obtain $|G_\alpha| \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d)$.
Now, we consider $H \cong \mathrm{SL}_l(q)$ where $l \geq 3$ and $q \neq 2$. Here, recalling Proposition 1.4.29 and (3.3.3), we have

$$d \geq \frac{q^l - 1}{q-1} \geq q^{l-1} + 4, \tag{3.3.10}$$

esp. $d \geq 13$. The "worst case" for our actual case (i.e. the case where $|G_\alpha|$ is maximal such that H is as assumed, also recalling (3.3.10) and that $l = \max\{k, n-2k\} \geq \frac{n}{3}$) occurs for $l = k = n - 2k$ (so, $n = 3k$, and note that $k \geq 3$) and $|G_\alpha| = q^{\frac{1}{3}n^2} \cdot |\mathrm{SL}_{\frac{n}{3}}(q)|^3 \cdot (q-1)^2/(n, q-1) \cdot (n, q-1) \cdot a \cdot 2$. So,

$$\begin{aligned} |G_\alpha| &\overset{(3.3.4)}{\leq} q^{\frac{1}{3}n^2+2} \cdot (q^{\frac{1}{9}n^2-1})^3 \cdot 2a = q^{\frac{2}{3}n^2-1} \cdot 2a \\ &\overset{(3.3.10)}{\leq} (d-4)^{(\frac{2}{3}n^2-1)/(\frac{n}{3}-1)} \cdot 2\frac{\ln(d-4)}{\ln(p)(l-1)} \\ &= (d-4)^{2n+6+5/(\frac{n}{3}-1)} \cdot 2\frac{\ln(d-4)}{\ln(p)(l-1)} \\ &\overset{(3.3.10)}{\leq} (d-4)^{2(3\frac{\ln(d-4)}{\ln(3)}+3)+8.5} \cdot 2\frac{\ln(d-4)}{2\ln(2)} \\ &= (d-4)^{6\frac{\ln(d-4)}{\ln(3)}+14.5} \cdot \frac{\ln(d-4)}{\ln(2)} \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d). \end{aligned} \tag{3.3.11}$$

Next, we consider $H \cong \mathrm{SL}_3(2)$ and note that $d \geq 7$, by Proposition 1.4.29 and (3.3.3). First, let $k < l = n - 2k = 3$ (so, $n \in \{5, 7\}$). Here, the "worst case" occurs for $n = 7$ and

$$|G_\alpha| = 2^{16} \cdot |\mathrm{SL}_2(2)|^2 \cdot |\mathrm{SL}_3(2)| \cdot 2 = 2^{22} \cdot 3^3 \cdot 7 \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d).$$

For $n - 2k \leq k = l$ (so, $n \in \{7, 8, 9\}$) we always have $7^2 \mid |G_\alpha|$. So, $d \geq 8$, by Lemma 1.4.28. For the actual case we now obtain that

$$|G_\alpha| \leq 2^{27} \cdot |\mathrm{SL}_3(2)|^3 \cdot 2 = 2^{37} \cdot 3^3 \cdot 7^3 \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d).$$

Now, consider $H \cong \mathrm{SL}_4(2)$ where $d \geq 8$, by Proposition 1.4.29 and (3.3.3). Suppose that $d = 8$. Since $\mathrm{PSL}_4(2) \cong \mathrm{A}_8$ (recall Lemma 3.1.8), we obtain that $G_\alpha^{O(\alpha)}$ is isomorphic to A_8 or S_8, considering Theorem 1.4.25. So, by Theorem 3.1.12, we obtain a contradiction to Proposition 2.1.11. Hence, $d \geq 9$. Recalling Proposition 2.1.11 and Main Theorem 2.1.24, for our actual case we have

$$|G_\alpha| \leq 2^{48} \cdot |\mathrm{SL}_4(2)|^3 \cdot 2 = 2^{67} \cdot 3^6 \cdot 5^3 \cdot 7^3 \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d).$$

Finally, for the considered case that G_α is insoluble, let $H \cong \mathrm{SL}_l(2)$ where $l \geq 5$. Here, by Proposition 1.4.29 and (3.3.3), we have

$$2^l - 1 \leq d, \tag{3.3.12}$$

especially $d \geq 31$. For the actual case the "worst case" occurs for $l = k = n-2k$ (so, $n = 3k$) and $|G_\alpha| = 2^{\frac{1}{3}n^2} \cdot |\mathrm{SL}_{\frac{n}{3}}(2)|^3 \cdot 2$. Hence, we can deduce

$$|G_\alpha| \overset{(3.3.4)}{\leq} 2^{\frac{2}{3}n^2-2} \overset{(3.3.12)}{\leq} 2^{\frac{2}{3}\left(3\frac{\ln(d+1)}{\ln(2)}\right)^2-2} = 2^{6\left(\frac{\ln(d+1)}{\ln(2)}\right)^2-2} \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d).$$

Now, we consider the case that G_α is soluble. As in the previous case, we will consider the possibilities for H separately. Recalling Proposition 1.2.12, $l \leq 2$. Let $l = 2$, then we have $H \cong \mathrm{SL}_2(q)$ where $q \in \{2,3\}$ and $n \in \{4,5,6\}$. Regarding Proposition 2.1.11 and Theorems 3.1.4 and 3.1.5, we see that in our considered case no G has a subdegree $d = 3$ or 4. So, $d \geq 6$, since $5 \nmid |G_\alpha|$. The "worst case" for our actual case occurs for $l = k = n - 2k$ (so, $n = 6$) and $|G_\alpha| = q^{12} \cdot |\mathrm{SL}_2(q)|^3 \cdot (q-1)^2/(6, q-1) \cdot (6, q-1) \cdot 2$. So, $|G_\alpha| \leq 2^{12} \cdot 3^{15} \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d)$. So, let $l = 1$ and H be trivial. Hence, $\mathrm{soc}(G) = \mathrm{PSL}_3(q)$ for an arbitrary prime power $q = p^a$. Here, recalling Proposition 2.1.11 and Main Theorem 2.1.24, we have

$$2 \cdot q^3 \cdot (q-1)^2/(3, q-1) \bigm| |G_\alpha| \bigm| 2a \cdot q^3 \cdot (q-1)^2. \tag{3.3.13}$$

Considering Propositions 1.2.11 and 2.1.11 and Theorems 3.1.4 and 3.1.5, we see that $d \neq 3$ and if $d = 4$ only $G = \mathrm{PSL}_3(2) : \langle \tau \rangle$ is possible. Here, G actually has a subdegree $d = 4$ (recall Remark 3.1.6), considering

$$G_\alpha = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \middle| x, y, z \in \mathrm{GF}(2) \right\} : \left\langle \tau \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle, g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $|G_\alpha : G_\alpha \cap G_\alpha^g| = 4$. Hence, for $d \leq 4$ we have $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_1}(d)$. By (3.3.13), we see that if $5 \mid |G_\alpha|$ then $5^2 \mid |G_\alpha|$. So, $d \neq 5$ according to Lemma 1.4.28.

Now, to obtain an adequate upper bound for $|G_\alpha|$ in terms of d in the actual case, we have to do some further considerations. We recall from (3.3.2) that $H_\alpha = G_\alpha \cap \mathrm{P}\Omega = \mathrm{P}(H^{\mathrm{SL}}_{1,2})$, and since there is no danger of ambiguity we simplify the notation by writing $H_{1,2}$ for $H^{\mathrm{SL}}_{1,2}$. Define the subgroups $H_\beta = G_\beta \cap \mathrm{P}\Omega$ and $H_{\alpha\beta} = H_\alpha \cap H_\beta$, and note that $G_\alpha = \mathrm{N}_G(H_\alpha), G_\beta = \mathrm{N}_G(H_\beta)$, since $G_\alpha, G_\beta \lessdot G$ and $1 < H_\alpha, H_\beta$. Clearly, $H_\beta = H_\alpha^g$ for all $g \in G$ with $\beta = \alpha^g$. Since $G_\beta \cap H_\alpha \leq \mathrm{P}\Omega$, we obtain $H_{\alpha\beta} \leq G_\beta \cap H_\alpha \leq (G_\beta \cap \mathrm{P}\Omega) \cap H_\alpha \leq H_{\alpha\beta}$, so

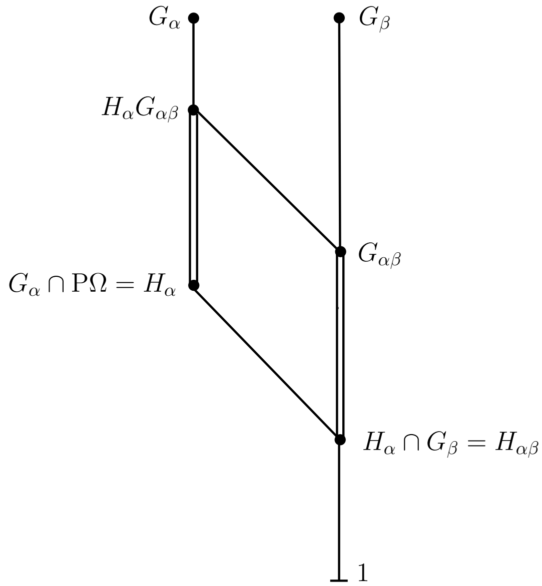

Figure 3.2: Subgroup diagram of G_α, $G_{\alpha\beta}$, H_α and $H_{\alpha\beta}$.

$G_\beta \cap H_\alpha = H_{\alpha\beta}$. Hence, it is not hard to deduce $|H_\alpha : H_{\alpha\beta}| \mid |G_\alpha : G_{\alpha\beta}| = d$, see Figure 3.2. So, we will consider $|H_\alpha : H_{\alpha\beta}|$, and determine the "worst case" for our intended goal, i.e. determining the highest possible order for $H_{\alpha\beta}$. For this, let $g \in G$ such that $G_\alpha^g = G_\beta$, and $g = \mathrm{P}\tilde{g}$ for an appropriate element $\tilde{g} \in \hat{G}$ (recall Convention 1.2.2). Recalling Remark 1.4.7 (b) and $\mathrm{Z}(\mathrm{SL}_3(q)) \leq H_{1,2}$, we see

$$H_{\alpha\beta} = H_\alpha \cap H_\alpha^g = \mathrm{P}(H_{1,2}) \cap \mathrm{P}(H_{1,2})^g = \mathrm{P}(H_{1,2}) \cap \mathrm{P}(H_{1,2}^{\tilde{g}}) = \mathrm{P}(H_{1,2} \cap H_{1,2}^{\tilde{g}}).$$

So, we have to determine the highest possible order of $H_{1,2} \cap H_{1,2}^{\tilde{g}}$. We recall the terminology of a maximal flag in V from Remark 1.2.19 (d). Considering Proposition 2.1.13 (ii), it is not hard to see that $H_{1,2}$ is the stabilizer in $\mathrm{SL}_3(q)$ of a certain maximal flag F_1 $\{0\} < \langle v_1 \rangle < \langle v_1, v_2 \rangle < V$ in V for some $v_1, v_2 \in V$. So, $H_{1,2}^{\tilde{g}}$ also is the stabilizer in $\mathrm{SL}_3(q)$ of a maximal flag F_2 $\{0\} < \langle w_1 \rangle < \langle w_1, w_2 \rangle < V$ in V for some $w_1, w_2 \in V$. Hence, the intersection $H_{1,2} \cap H_{1,2}^{\tilde{g}}$ consists of those elements in $\mathrm{SL}_3(q)$ which stabilize both maximal flags F_1 and F_2 (called the *stabilizer* in $\mathrm{SL}_3(q)$ of F_1 and F_2). Recalling that $G_\alpha = \mathrm{N}_G(H_\alpha)$ and $G_\alpha \neq G_\beta$, we see $H_\alpha \neq H_\alpha^g$, hence $H_{1,2} \neq H_{1,2}^{\tilde{g}}$ and the two maximal flags F_1 and F_2 cannot be identical. By elementary combinatorial considerations and some linear algebra, we see that the following two cases of maximal flags F_1 and F_2 leads to stabilizers in $\mathrm{SL}_3(q)$ of F_1 and F_2 of highest order.
In the first case, we have $\langle v_1 \rangle \neq \langle w_1 \rangle$ and $\langle v_1, v_2 \rangle = \langle w_1, w_2 \rangle$. Here, we choose $b_1 = v_1$, $b_2 = w_1$ and $b_3 \in V \setminus \langle v_1, w_1 \rangle$. With respect to the ordered basis

$B = (b_1, b_2, b_3)$ of V, we obtain that the stabilizer in $\mathrm{SL}_3(q)$ of F_1 and F_2 is the subgroup

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ x & y & (ab)^{-1} \end{pmatrix} \,\middle|\, \begin{array}{l} x, y \in \mathrm{GF}(q), \\ a, b \in \mathrm{GF}(q)^* \end{array} \right\} \leq \mathrm{SL}_3(q).$$

In the second case, we have $\langle v_1 \rangle = \langle w_1 \rangle$ and $\langle v_1, v_2 \rangle \neq \langle w_1, w_2 \rangle$. Here, we choose $b_1 = v_1$, $b_2 \in \langle v_1, v_2 \rangle \setminus \langle v_1 \rangle$ and $b_3 \in \langle w_1, w_2 \rangle \setminus \langle w_1 \rangle$. Then, with respect to the ordered basis $B = (b_1, b_2, b_3)$ of V, the stabilizer in $\mathrm{SL}_3(q)$ of F_1 and F_2 is the subgroup

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ x & b & 0 \\ y & 0 & (ab)^{-1} \end{pmatrix} \,\middle|\, \begin{array}{l} x, y \in \mathrm{GF}(q), \\ a, b \in \mathrm{GF}(q)^* \end{array} \right\} \leq \mathrm{SL}_3(q).$$

So, we obtain $|H_{1,2} \cap H_{1,2}^{\tilde{g}}| \leq q^2(q-1)^2$, hence $|\mathrm{P}(H_{1,2} \cap H_{1,2}^{\tilde{g}})| \leq q^2 \frac{(q-1)^2}{(3,q-1)}$ and this upper bound is sharp. Now, obviously we have $d \geq |H_\alpha : H_{\alpha\beta}| = |H_\alpha : \mathrm{P}(H_{1,2} \cap H_{1,2}^{\tilde{g}})| \geq q$. Hence, recalling (3.3.13), we obtain

$$|G_\alpha| \leq 2 \cdot d^3 \cdot (d-1)^2 \cdot \frac{\ln(d)}{\ln(2)} \leq h_{\mathscr{C}_1}^{\mathbf{L}}(d).$$

So, we have shown that there is no counterexample to the assertion in the case $G_\alpha \in \mathscr{C}_1$ of G of type $P_{k,n-k}$. For the remaining case $G_\alpha \in \mathscr{C}_1$ of G of type P_k the non-existence of a counterexample can be shown analogously as for the case of type $P_{k,n-k}$, and so our assertion is established. (For the case of type P_k we note that w.l.o.g. it is useful to assume $k \leq n-k$ (recall Remark 2.1.12 (a) and see Proposition 2.1.13), and we also recall the restriction $G \leq \mathrm{P\Gamma L}(V)$ if $n \neq 2k$ from Main Theorem 2.1.24). □

Theorem 3.3.6. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_1$ *of* G*, the order of* G_α *is bounded by*

$$h_{\mathscr{C}_1}^{\mathbf{U}}(d) = \begin{cases} 0 & \text{for } d \in \{3,4,7\}, \\ 2^{11} \cdot 3^2 \cdot 5 & \text{for } d = 5, \\ 2^{11} \cdot 3^4 & \text{for } d = 6, \\ 2^{11} \cdot 3^{15} \cdot 5 & \text{for } 8 \leq d \leq 10, \\ (d-1)^6 d^2 (d-2)^2 \cdot 2 \frac{\ln(d-1)}{\ln(2)} & \text{for } 11 \leq d \leq 16, \\ (d-1)^{7.5} d (d-2)^3 \cdot \frac{\ln(d-1)}{\ln(2)} & \text{for } 17 \leq d \leq 20, \\ (3d+1)^{\frac{3\ln(3d+1)}{4\ln(2)} + 3\frac{1}{3}} \cdot 3^4 & \text{for } 21 \leq d \leq 26, \\ (d-11)^{16.5} \cdot \frac{\ln(d-11)}{2\ln(2)} & \text{for } 27 \leq d \leq 238, \\ (d-37)^{\frac{3\ln(d-37)}{2\ln(2)} + 5.5} \cdot \frac{2\ln(d-37)}{7\ln(2)} & \text{for } 239 \leq d \leq 340, \\ (d-85)^{\frac{3\ln(d-85)}{2\ln(2)} + 6\frac{5}{8}} \cdot \frac{\ln(d-85)}{4\ln(2)} & \text{for } d \geq 341. \end{cases}$$

In particular, $h_{\mathscr{C}_1}^{\mathbf{U}}(d)$ *is a Sims order bound for* $\mathscr{G}_{\mathscr{C}_1}^{\mathbf{U}}$ *and* $h_{\mathscr{C}_1}^{\mathbf{U}}(d) \leq \mathrm{wdt}(d)$.

Proof. We obtain the assertion analogously to the proof of the last theorem, using the information provided in Propositions 1.4.29, 2.1.11, 2.1.13 and 3.3.1 and Main Theorem 2.1.25. We note about the case $G_\alpha \in \mathscr{C}_1$ of G of type P_k with $k < \frac{n}{2}$ that it is advisable to consider separately the case for G_α being soluble or insoluble. For the case that G_α is insoluble we also note that it is advisable to consider further divisions for cases of G_α: If G_α has only one non-abelian composition factor group then it is useful to distinguish if it is isomorphic to $\mathrm{PSL}_k(q^2)$ or $\mathrm{PSU}_{n-2k}(q^2)$ (recall Proposition 2.1.11); for the case that G_α has two composition factor groups isomorphic to $\mathrm{PSL}_k(q^2)$ and $\mathrm{PSU}_{n-2k}(q^2)$ it is useful to distinguish whether $\mathrm{d}_f(\mathrm{PSU}_{n-2k}(q^2)) \leq \mathrm{d}_f(\mathrm{PSL}_k(q^2))$ or $\mathrm{d}_f(\mathrm{PSL}_k(q^2)) \leq \mathrm{d}_f(\mathrm{PSU}_{n-2k}(q^2))$ (recall the different possibilities in Proposition 1.4.29).
Furthermore, since $(q^l - 1)(q^{l+1} + 1) \leq q^{l+(l+1)}$ for all non-negative integers l, we note that it is useful to work also with the estimate

$$|\mathrm{SU}_m(q^2)| = q^{\frac{m(m-1)}{2}} \prod_{i=2}^{m} (q^i - (-1)^i) \leq q^{m^2-1}$$

for all positive integers m, and we also recall (3.3.4). □

Remark. By using (more intensively) the methods in the proofs of the previous two theorems, we note that there are several possibilities to determine more precise upper bounds for $|G_\alpha|$. Using the notation from the proof of Theorem 3.3.5, we note the following examples.

- By further case-by-case analysis with respect to the possibilities for $H = \mathrm{SL}_l(q)$ depending on l or q, more precise upper bounds can be obtained. Such as to consider more cases separately for fixed small l or q, which leads to more precise estimates for d in (3.3.10), see also (3.3.11). Or, to consider separately the cases where q is a prime, cf. again (3.3.11) together with (3.3.10).
- By further analysis of the structures of G_α and $G_\alpha^{O(\alpha)} \leq \mathrm{Sym}(O(\alpha))$ for certain situations and for certain fixed subdegrees $d = |O(\alpha)|$ one can rule out further cases and obtain more precise upper bounds.
- By considering the situation for certain fixed subdegrees d separately, more precise upper bounds can be obtained; such as for small primes d to determine all groups G which have such a subdegree d (e.g. consider Propositions 2.1.11 and 2.1.13 and Main Theorems 2.1.24 and 2.1.25, and recall (3.3.3) as well as Lemma 1.4.28).

3.4 Sims order bound for $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{L}^\epsilon}$

In this section, we determine Sims order bounds $h_{\mathscr{C}_2}^{\mathbf{L}}(d)$ and $h_{\mathscr{C}_2}^{\mathbf{U}}(d)$ for $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{L}}$ and $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{U}}$. We recall that we use the notation introduced at the end of Section 3.1, and we also recall Convention 3.2.2. Regarding Corollary 2.2.13, we consider the cases that G_α is a member of A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$ separately. In the following remark, we present an approach how by elementary number theory and Corollary 1.4.26 one can easily deduce an upper bound for $|G_\alpha|$ in terms of a non-trivial subdegree d.

Remark 3.4.1. Consider the situation that G_α is a strongly constrained member of A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. Set $t = \frac{n}{2}$ in the case where G_α is of type $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ and $t = \frac{n}{3}$ in the case where G_α is of type $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. Regarding Corollaries 1.2.22 and 2.2.11, Table 2.2.1 (or, Proposition 2.2.8) and Main Theorems 2.2.25 and 2.2.26, we easily see that the order of G_α is bounded by an increasing function in terms of t. Especially, we have

$$|G_\alpha| \leq 2^{4t} \cdot 3^t \cdot t! \tag{3.4.1}$$

in case **L**, and

$$|G_\alpha| \leq 2^{3t+1} \cdot 3^{4t-1} \cdot t! \tag{3.4.2}$$

in case **U**. So, to obtain an upper bound for $|G_\alpha|$ in terms of d, we only have to estimate t by d. Regard the cases where $t \geq 5$ (otherwise, we have concrete fixed values as upper bounds for $|G_\alpha|$). Again, considering Corollary 2.2.11 and Proposition 2.2.8 (or, Table 2.2.1), we see that $t!$ divides $|G_\alpha|$ (recall, that G_α is of type $\mathrm{GL}_2^\epsilon(3^u) \wr \mathrm{S}_t$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_t$). Hence, in view of Corollary 1.4.26, we can deduce $d \geq r$ where r denotes the largest prime lower or equal to t. Now, by using Bertrand's postulate 1.5.21, we obtain a lower bound for r in terms of t by

$$r \geq \begin{cases} \frac{t+5}{2} & \text{if } t \text{ is odd,} \\ \frac{t}{2} + 2 & \text{if } t \text{ is even.} \end{cases} \tag{3.4.3}$$

Hence, in any case we have $t \leq 2d - 4$, and by (3.4.1) and (3.4.2) we can now deduce that

$$|G_\alpha| \leq 2^{8d-16} \cdot 3^{2d-4} \cdot (2d-4)! \tag{3.4.4}$$

in case **L**, and

$$|G_\alpha| \leq 2^{6d-11} \cdot 3^{8d-17} \cdot (2d-4)! \tag{3.4.5}$$

in case **U**.
By using more precise versions of Bertrand's postulate (recall Remark 1.5.22 (b)), it is possible to sharpen the estimate in (3.4.3), and hence to obtain more precise upper bounds for $|G_\alpha|$ in (3.4.4) and (3.4.5). But note that in the described approach the best possible estimate for t by d is $t \leq d$ which only can be assumed if t is a prime.

In the last remark, we have obtained an upper bound for $|G_\alpha|$ in terms of d by elementary number theory and Corollary 1.4.26. By using the stronger assertion of Theorem 1.4.25 and the facts provided in Section 3.1, we obtain a more precise upper bound in the following proposition.

Proposition 3.4.2. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GL}_2(3) \wr \mathrm{S}_{\frac{n}{2}}$*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_2,0}(d) = \begin{cases} 0 & \text{for } d \in \{3,4,5,7\}, \\ 2^{19} \cdot 3^5 & \text{for } d = 6, \\ 2^{4(d-1)} \cdot 3^{d-1} \cdot (d-1)! & \text{for } d \geq 8. \end{cases}$$

For the case that $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_2$ *of* G *of type* $\mathrm{GU}_2(3^2) \wr \mathrm{S}_{\frac{n}{2}}$ *or* $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$*, the order of* G_α *is bounded by*

$$h^{\mathbf{U}}_{\mathscr{C}_2,0}(d) = \begin{cases} 0 & \text{for } d \in \{3,4,5,7\}, \\ 2^{16} \cdot 3^{16} & \text{for } d = 6, \\ 2^{3d-2} \cdot 3^{4d-5} \cdot (d-1)! & \text{for } d \geq 8. \end{cases}$$

Proof. Suppose that there is a counterexample (G, G_α) to the assertion where $G_\alpha \in \mathscr{C}_2$ of G of type $\mathrm{GL}^\epsilon_2(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$. Set $t = \frac{n}{2}$ in the case that G_α is of type $\mathrm{GL}^\epsilon_2(3^u) \wr \mathrm{S}_{\frac{n}{2}}$ and $t = \frac{n}{3}$ in the case that G_α is of type $\mathrm{GU}_3(2^2) \wr \mathrm{S}_{\frac{n}{3}}$; note that $t \geq 2$.

First, suppose that $d \in \{3, 4\}$. Then, considering Proposition 2.2.8 (or Table 2.2.1) and Corollary 2.2.11 together with Theorems 3.1.4 and 3.1.5, we obtain a contradiction. Hence, we can assume $d \geq 5$. We will now separately consider the cases that G_α is soluble or insoluble. We note that in the following considerations we will often use Corollaries 1.2.22 and 2.2.11, Table 2.2.1 and Proposition 2.2.8 also without reference to it, and we also recall (3.4.1) and (3.4.2).

Let G_α be soluble which occurs if and only if $t \in \{2, 3, 4\}$. Since $5 \nmid |G_\alpha|$, we then can assume $d \geq 6$. The highest order for G_α in the actual case occurs for $t = 4$ and in the case **U** if G_α is of type $\mathrm{GU}_3(2^2) \wr \mathrm{S}_4$. So, considering Corollaries 1.2.22 and 2.2.11, Proposition 2.2.8 and Main Theorems 2.2.25 and 2.2.26, we obtain

$$|G_\alpha| \leq 2^{19} \cdot 3^5 \leq h^{\mathbf{L}}_{\mathscr{C}_2,0}(d)$$

in case **L**, and

$$|G_\alpha| \leq 2^{16} \cdot 3^{16} \leq h^{\mathbf{U}}_{\mathscr{C}_2,0}(d)$$

in case **U**. So, no counterexample exists for this case.

Next, consider the case that G_α is insoluble, so $t \geq 5$. In view of Theorem 1.4.25, we see that $\frac{t!}{2}$ divides $d!$, so $d \geq t$. First, let $t = 5$. Here, we have

$$2^{21} \cdot 3^6 \cdot 5 \mid |G_\alpha| \mid 2^{23} \cdot 3^6 \cdot 5 \tag{3.4.6}$$

in the case **L**,

$$2^{25} \cdot 3^6 \cdot 5 \mid |G_\alpha| \mid 2^{27} \cdot 3^6 \cdot 5 \tag{3.4.7}$$

in the case **U** and if G_α is of type $\mathrm{GU}_2(3^2) \wr \mathrm{S}_5$ and

$$2^{18} \cdot 3^{19} \cdot 5 \mid |G_\alpha| \mid 2^{19} \cdot 3^{20} \cdot 5 \tag{3.4.8}$$

in the case **U** and if G_α is of type $\mathrm{GU}_3(2^2) \wr \mathrm{S}_5$. Suppose that $d = 5$. Then, analogously to the proof of Theorem 3.3.5 (using Theorems 1.4.25 and 3.1.11 (b)), we obtain $|G_\alpha| \bigm| 2^{14} \cdot 3^2 \cdot 5$. But this contradicts (3.4.6) to (3.4.8). Next, suppose that $d = 6$. Again, analogously to the proof of Theorem 3.3.5 (using the facts from Theorems 1.4.25 and 3.1.11 (b), Lemma 3.3.4 and [At, p. 4]), we obtain $|G_\alpha| \bigm| 2^5 \cdot 3 \cdot 5^6$. Again, this contradicts (3.4.6) to (3.4.8). Hence, we can assume $d \geq 8$, since $7 \nmid |G_\alpha|$. Now, by (3.4.6) to (3.4.8), we see that there is no counterexample to the assertion in the actual case.
Finally, let $t \geq 6$. Suppose that $d = t$. Then, considering Proposition 2.2.8 (or, Table 2.2.1), Corollary 2.2.11 and Theorem 1.4.25, we can deduce that $G_\alpha^{O(\alpha)}$ is isomorphic to A_d or S_d. Obviously, G_α does not act faithfully on $O(\alpha)$. So, by Theorem 3.1.12, we obtain that G_α is isomorphic to $\mathrm{A}_d \times \mathrm{A}_{d-1}$, $\mathrm{S}_d \times \mathrm{S}_{d-1}$, $\mathrm{S}_d \times \mathrm{A}_{d-1}$ or $(\mathrm{A}_d \times \mathrm{A}_{d-1}) : \mathbf{Z}_2$. This clearly contradicts Proposition 2.2.8. Hence, $d \geq t+1$, and note that we can further assume $d \geq 8$, because $7 \nmid |G_\alpha|$ in the case $t = 6$. So, by Corollaries 1.2.22 and 2.2.11, Proposition 2.2.8 and Main Theorems 2.2.25 and 2.2.26, we obtain

$$|G_\alpha| \leq 2^{4t} \cdot 3^t \cdot t! \leq 2^{4(d-1)} \cdot 3^{d-1} \cdot (d-1)! \leq h^{\mathbf{L}}_{\mathscr{C}_2,0}(d)$$

in case **L**, and

$$|G_\alpha| \leq 2^{3t+1} \cdot 3^{4t-1} \cdot t! \leq 2^{3d-2} \cdot 3^{4d-5} \cdot (d-1)! \leq h^{\mathbf{U}}_{\mathscr{C}_2,0}(d)$$

in case **U**. Hence, there exists no counterexample to our assertion. □

Next, we consider the remaining case where G_α is a member of A-class $\mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ (recall Corollary 2.2.13). By the following remark, we first note an observation concerning the method described in Remark 3.1.17.

Remark 3.4.3. Recall from Remark 3.1.17 the approach to obtain an upper bound for $|G_\alpha|$ in terms of d by estimating $|\mathrm{O}_p(G_\alpha)|$ by an increasing function in terms of $|G_\alpha/\mathrm{O}_p(G_\alpha)|$ where G_α is strongly p-constrained for the prime p. Here, we consider the cases where $G_\alpha \in \mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ for a Fermat prime q in the case **L** and a Mersenne prime q in the case **U**. In view of Main Theorems 2.2.21 and 2.2.22 together with Propositions 1.5.9 (i)(c) and (ii)(b) and 1.5.11 (ii)(b), we note that these cases actually occur (except for a few exceptional cases) in the situation we are investigating (recall the conventions before Subsection 3.2). Furthermore, note that G_α is strongly 2-constrained. Regarding Corollary 2.2.11 and Proposition 2.2.8 (iv) and using the notation in that proposition, we choose an appropriate ordered $\mathrm{GF}(q^u)$-basis of V and with respect to that basis w.l.o.g. we may assume that $G_\alpha \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL}^\epsilon,1}$. By Lemma 2.2.19, we see $\mathrm{P}C_{\mathrm{SL}^\epsilon,1} \leq \mathrm{O}_2(G_\alpha)$. So, in view of Corollaries 1.2.22 and 2.2.11 and Proposition 2.2.8, we obtain

$$\frac{(q-\epsilon 1)^{n-1}}{(q-\epsilon 1, n)} \leq |\mathrm{O}_2(G_\alpha)| \tag{3.4.9}$$

and $|G_\alpha/\mathrm{O}_2(G_\alpha)| \bigm| 2\cdot(q-\epsilon 1,n)\cdot n!$. Now, let the dimension n of V be fixed, so $|G_\alpha/\mathrm{O}_2(G_\alpha)|$ is bounded by a constant integer. Since it is not known (till today) whether there is only a finite number of Fermat primes or Mersenne primes, it is not possible to decide whether $|\mathrm{O}_2(G_\alpha)|$ can be bounded by an increasing function in terms of $|G_\alpha/\mathrm{O}_2(G_\alpha)|$, see (3.4.9).[5] So, it is not possible to decide whether the approach described in Remark 3.1.17 for the cases considered above can be applied.

The mentioned cases in the last remark will also cause some difficulties in our further investigations. To tackle these, we provide the following lemma and remark.

Lemma 3.4.4. *We use the notation from Proposition 2.2.8.*

(a) If $T = C_{\mathrm{GL}^\epsilon,1} = \left\{ \mathrm{diag}(x_1,\ldots,x_n) \middle| \begin{array}{l} x_i \in \mathrm{GL}_1^\epsilon(q^u) \text{ for} \\ i \in \{1,\ldots,n\} \end{array} \right\} \cong (\boldsymbol{Z}_{q-\epsilon 1})^n$ *and* $g \in \mathrm{GL}_n^\epsilon(q^u)$ *then*

$$T \cap T^g \cong \left\{ \mathrm{diag}(\lambda_1 \mathbb{1}_{n_1},\ldots,\lambda_l \mathbb{1}_{n_l}) \middle| \begin{array}{l} \lambda_j \in \mathrm{GL}_1^\epsilon(q^u) \text{ for} \\ j \in \{1,\ldots,l\} \end{array} \right\} = T_0 \cong (\boldsymbol{Z}_{q-\epsilon 1})^l$$

for appropriate integers $l \geq 1$ *and* $n_1,\ldots,n_l \geq 1$ *where* $n = n_1 + \ldots + n_l$. *(The number* l *can be considered as the number of "blocks" in* T_0 *of the "lengths"* $n_1,\ldots,n_l$*).*

(b) If $T = C_{\mathrm{SL}^\epsilon,1} = \left\{ \mathrm{diag}(x_1,\ldots,x_n) \middle| \begin{array}{l} x_i \in \mathrm{GL}_1^\epsilon(q^u) \text{ for} \\ i \in \{1,\ldots,n\} \text{ and} \\ \prod_{i=1}^n x_i = 1 \end{array} \right\} \cong (\boldsymbol{Z}_{q-\epsilon 1})^{n-1}$ *and* $g \in \mathrm{GL}_n^\epsilon(q^u)$ *then*

$$T \cap T^g \cong \left\{ \mathrm{diag}(\lambda_1 \mathbb{1}_{n_1},\ldots,\lambda_l \mathbb{1}_{n_l}) \middle| \begin{array}{l} \lambda_j \in \mathrm{GL}_1^\epsilon(q^u) \text{ for} \\ j \in \{1,\ldots,l\} \text{ and} \\ \prod_{j=1}^l \lambda_j^{n_j} = 1 \end{array} \right\} = T_0$$

for appropriate integers $l \geq 1$ *and* $n_1,\ldots,n_l \geq 1$ *where* $n = n_1 + \ldots + n_l$.

Proof. To state the proof of the lemma, it is advantageous to agree that linear maps act on the left. We do so, and note that this assumption only holds for this proof.
First, we prove assertion (a) in the case **L**. Let $g = (g_{ij})_{1\leq i,j\leq n} \in \mathrm{GL}_n(q)$. Let $t \in T$ be an arbitrary element, so $t = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ for appropriate $\lambda_i \in \mathrm{GF}(q)^*$. Using analogous notation as in Lemma 1.3.1, we can consider $t = M_\varphi^{B,l}$, so t is the matrix of an endomorphism $\varphi : V \to V$ with respect to an ordered basis $B = (b_1,\ldots,b_n)$ of V. Clearly, $\varphi(b_i) = \lambda_i b_i$ for $i \in \{1,\ldots,n\}$.

[5] Note, that the decision of the existence of such a function in the case **L** (or **U**) is equivalent to the decision of the long time open question whether there is only a finite number of Fermat primes (or Mersenne primes). Furthermore, the author conjectures that no crucial advantage should arise from considering the problem of the number of Fermat primes or Mersenne primes by this group theoretic point of view.

Analogously, g can be considered as the matrix of an endomorphism of V, abusing slightly notation also called g, with respect to B. Now, we examine the element $t^g \in T^g$. Recalling Lemma 1.3.1, we can consider g as a change-of-basis matrix for V and $t^g = M_{\varphi}^{B',l}$ as the matrix of φ with respect to the ordered basis $B' = (b'_1, \dots, b'_n)$ of V where $b'_j = gb_j$ for $j \in \{1, \dots, n\}$. Since T consists of all diagonal matrices in $\mathrm{GL}_n(q)$ (and there are no further conditions put on T), we see that $t^g \in T^g \cap T$ if and only if t^g is a diagonal matrix. So, $t^g \in T^g \cap T$ if and only if there are elements $\lambda'_1, \dots, \lambda'_n \in \mathrm{GF}(q)^*$ with $\varphi(b'_j) = \lambda'_j b'_j$ for $j \in \{1, \dots, n\}$. (Clearly, the elements $\lambda_1, \dots, \lambda_n$ are the eigenvalues of φ. If $\varphi(b'_j) = \lambda'_j b'_j$ for $j \in \{1, \dots, n\}$, also the elements $\lambda'_1, \dots, \lambda'_n$ are the eigenvalues of φ, and so there is a permutation $\sigma \in \mathrm{S}_n$ such that $\lambda'_j = \lambda_{\sigma(j)}$ for $j \in \{1, \dots, n\}$). We define the set

$$\mathscr{S}_g = \{(r,s) \mid \text{ there is a } j \in \{1, \dots, n\} \text{ with } g_{rj} \neq 0 \neq g_{sj}\},$$

and note that $\{(s,s) \mid 1 \leq s \leq n\} \subseteq \mathscr{S}_g$. By easy considerations, we see

$$\varphi(b'_j) = \varphi(gb_j) = \varphi(\sum_{i=1}^n g_{ij} b_i) = \sum_{i=1}^n g_{ij} \lambda_i b_i \quad \text{and} \quad \lambda'_j b'_j = \sum_{i=1}^n g_{ij} \lambda'_j b_i.$$

Hence, $\varphi(b'_j) = \lambda'_j b'_j$ if and only if $g_{ij}\lambda_i = g_{ij}\lambda'_j$ for $i \in \{1, \dots, n\}$. So, we can deduce that there exist elements $\lambda'_1, \dots, \lambda'_n \in \mathrm{GF}(q)^*$ with $\varphi(b'_j) = \lambda'_j b'_j$ for $j \in \{1, \dots, n\}$ if and only if $\lambda_r = \lambda_s$ for all $(r,s) \in \mathscr{S}_g$. Now, by elementary considerations and the above investigations, we obtain our assertion for the actual case.
The assertion in the case **U** follows by analogous arguments as in the case **L**. Here, we note that $g \in \mathrm{GU}_n(q^2)$; so, for the matrix of the non-degenerate unitary form f on V with respect to B (respectively B') we have $J_{f,B} = J_{f,B'} = \mathbb{1}_n$ (recall Proposition 2.2.8). Hence, there arises no additional condition for $t^g \in T^g$ (than t^g has to be a diagonal matrix) if we consider t^g to be an element in $T^g \cap T$, cf. Lemma 1.2.8.
Assertion (b) follows analogously to assertion (a). Here, we note the obvious observation $\det(t^g) = 1$ if $\det(t) = 1$. So, also for this case no additional condition for $t^g \in T^g$ arises if we consider t^g to be an element in $T^g \cap T$. $\square$

Remark 3.4.5. (a) We recall Lemma 1.3.1 and Remark 1.3.2. Clearly, it is also possible to prove the assertion of the last lemma by considering that linear maps act on the right. For this, it is appropriate to consider $T \cap T^g = T \cap T^{(g^{-1})^{-1}}$, and to work with the inverse of g.

(b) We note two easy observations about the number l in Lemma 3.4.4. If $l = n$ (so, all $n_i = 1$) we have $T \cap T^g = T$. For $l = 1$ (so, $n = n_1$) we have $T \cap T^g = \begin{cases} \mathrm{Z}(\mathrm{GL}_n^{\epsilon}(q^u)) & \text{if } T = C_{\mathrm{GL}^{\epsilon},1}, \\ \mathrm{Z}(\mathrm{SL}_n^{\epsilon}(q^u)) & \text{if } T = C_{\mathrm{SL}^{\epsilon},1}. \end{cases}$ Note, that in general the converses of the above assertions do not hold. E.g. consider in case **L** the cases $q = 2$ in Lemma 3.4.4 (a) and (b) or $n = 2$ and $q = 3$ in Lemma 3.4.4 (b).

(c) The structure of T_0 in Lemma 3.4.4 (b) can be (very) different. Here, we provide three examples for structures of T_0 in the case $\mathbf{L}$.
First, let $n = 8$, $q = 5$ and $g = \mathrm{diag}(g_1, g_2)$ where $g_1 = g_2 \in \mathrm{GL}_4(5)$ with $g_1 = (g_{1_{ij}})_{1 \leq i,j \leq 4}$ and $g_{1_{ij}} = \begin{cases} 1 & \text{for } j = 1, \\ 1 & \text{for } i = j, \\ 0 & \text{else.} \end{cases}$ Here, $l = 2$, $n_1 = n_2 = 4$ and $T_0 \cong (\mathbf{Z}_4)^2$.
Now, let $n = 4$, $q = 5$ and $g = \mathrm{diag}(g_1, g_2)$ where $g_1 = g_2 \in \mathrm{GL}_2(5)$ with $g_1 = (g_{1_{ij}})_{1 \leq i,j \leq 2}$ and $g_{1_{ij}}$ defined as above. Here, we have $l = 2$, $n_1 = n_2 = 2$ and $T_0 \cong \mathbf{Z}_4 \times \mathbf{Z}_2$.
Next, let $n = 3$, q be arbitrary (e.g. $q = 5$ as above) and $g = \mathrm{diag}(g_1, g_2)$ where $g_2 = 1$ and $g_1 = (g_{1_{ij}})_{1 \leq i,j \leq 2}$ where $g_{1_{ij}}$ is defined as above. Here, we have $l = 2$, $n_1 = 2, n_2 = 1$ and $T_0 \cong \mathbf{Z}_{q-1}$.

(d) In this part, we note facts about the structure of T_0 in Lemma 3.4.4 (b) for certain cases we will encounter below. (We recommend to keep part (c) in mind). Consider that the case of Lemma 3.4.4 (b) holds where $T \cap T^g = T_0$ for appropriate integers $l, n_1, \ldots, n_l \geq 1$. Let $l > 1$ (otherwise, recall part (b)). Consider the case that $q - \epsilon 1$ is a non-trivial power of a prime p. W.l.o.g. let $(n_l, q - \epsilon 1) = \min\{(n_i, q - \epsilon 1) \mid 1 \leq i \leq l\}$, and note that $(n_l, q - \epsilon 1)$ divides $(n_i, q - \epsilon 1)$ for any $i \in \{1, \ldots, l\}$. Define the isomorphism $\rho : \mathrm{GL}_1^\epsilon(q^u) \to \mathrm{GL}_1^\epsilon(q^u), x \mapsto x^{\frac{n_l}{(n_l, q-\epsilon 1)}}$.
First, we consider the case $(n_l, q - \epsilon 1) = 1$ (esp. this case occurs if there is an $i \in \{1, \ldots, l\}$ with $n_i = 1$). In view of Lemma 3.4.4 (b) and recalling the isomorphism ρ, we see that $T_0 \cong (\mathbf{Z}_{q-\epsilon 1})^{l-1}$. (Note, that $\lambda_1, \ldots, \lambda_{l-1} \in \mathrm{GL}_1^\epsilon(q^u)$ in T_0 can be chosen arbitrary, and each choice determines λ_l uniquely).
Now, let $(n_l, q - \epsilon 1) > 1$. Recall the condition

$$\prod_{j=1}^{l} \lambda_j^{n_j} = 1 \quad (*)$$

in T_0. By considering

$$\prod_{j=1}^{l} \lambda_j^{n_j} = \left(\left(\prod_{j=1}^{l-1} \lambda_j^{\frac{n_j}{(n_l, q-\epsilon 1)}} \right) \cdot \lambda_l^{\frac{n_l}{(n_l, q-\epsilon 1)}} \right)^{(n_l, q-\epsilon 1)}$$

and recalling the isomorphism ρ, we see that for any choice of elements $\lambda_1, \ldots, \lambda_{l-1} \in \mathrm{GL}_1^\epsilon(q^u)$ there exists a $\lambda_l \in \mathrm{GL}_1^\epsilon(q^u)$ such that $(*)$ is satisfied. Clearly, λ_l is not uniquely determined by a choice of $\lambda_1, \ldots, \lambda_{l-1}$. More precisely, by further elementary considerations we see that $T_0 \cong (\mathbf{Z}_{q-\epsilon 1})^{l-1} \times \mathbf{Z}_{(n_l, q-\epsilon 1)}$.
Summarizing the previous examinations, for the considered case we have $T_0 \cong (\mathbf{Z}_{q-\epsilon 1})^{l-1} \times \mathbf{Z}_{p^b}$ where $p^b = \min\{(n_i, q - \epsilon 1) \mid 1 \leq i \leq l\}$. Especially, we note that $T_0 \ncong (\mathbf{Z}_{q-\epsilon 1})^l$ if $q - \epsilon 1 > p^b$.

Now, we have provided the facts to determine the intended Sims order bounds $h_{\mathscr{C}_2}^{\mathbf{L}}(d)$ and $h_{\mathscr{C}_2}^{\mathbf{U}}(d)$ for $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{L}}$ and $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{U}}$.

Theorem 3.4.6. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_2$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_2}(d) = \begin{cases} 2^4 \cdot 3 & \text{for } d = 3, \\ 2^5 & \text{for } d = 4, \\ 2^{12} \cdot 3 \cdot 5 & \text{for } d = 5, \\ 2^{19} \cdot 3^5 & \text{for } d = 6, \\ 2 \cdot 3 \cdot 7 & \text{for } d = 7, \\ 2^{4(d-1)} \cdot 3^{d-1} \cdot (d-1)! & \text{for } d \in \{8, 9\}, \\ 2^{40} \cdot 3^4 \cdot 5^9 \cdot 7 & \text{for } d = 10, \\ 2 \cdot (d(d-1))^{d-2} \cdot (d-1)! & \text{for } d \geq 11. \end{cases}$$

In particular, $h^{\mathbf{L}}_{\mathscr{C}_2}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_2}$ *and* $h^{\mathbf{L}}_{\mathscr{C}_2}(d) \leq \mathrm{wdt}(d)$.

Proof. Suppose that the pair (G, G_α) is a counterexample to our assertion. In view of Corollary 2.2.13, $G_\alpha \in \mathscr{C}_2$ of G is of type $\mathrm{GL}_2(3) \wr \mathrm{S}_{\frac{n}{2}}$ or $\mathrm{GL}_1(q) \wr \mathrm{S}_n$. By Proposition 3.4.2, we see $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$ if G_α is of type $\mathrm{GL}_2(3) \wr \mathrm{S}_{\frac{n}{2}}$. Hence, we may assume that G_α is of type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$. First, we consider the cases $d \in \{3, 4\}$ separately. Let $d = 3$. Comparing Theorem 3.1.4 with Main Theorem 2.2.21, also regarding Proposition 1.2.11, Lemma 3.1.8 and Remark 3.1.9 (c), we see that only the cases $G \cong \mathrm{PSL}_2(4)$, $\mathrm{PSL}_3(3)$ or $\mathrm{Aut}(\mathrm{PSL}_3(3))$ are possible. If $G \cong \mathrm{PSL}_2(4)$ then $G_\alpha \cong \mathrm{S}_3$ and this case coincides with the case listed in [Wo, (3) p. 236] (recall Proposition 2.2.8). For $G \cong \mathrm{PSL}_3(3)$ or $\mathrm{Aut}(\mathrm{PSL}_3(3))$ we have $G_\alpha \cong \mathrm{S}_4$ or $\mathrm{S}_4 \times \mathbf{Z}_2$ and these cases coincide with one of the cases listed in [Wo, (8) and (9) p. 237] (again recall Proposition 2.2.8 and Corollary 2.2.11). Hence, here we have $|G_\alpha| \leq 2^4 \cdot 3 \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$.
Now, let $d = 4$. In view of Theorem 3.1.5 and Main Theorem 2.2.21, also considering Propositions 1.2.11 and 2.2.8, Lemma 3.1.8 and Remark 3.1.9 (b) and (c), we see that only the cases $G \cong \mathrm{PGL}_2(9)$, $\mathrm{PSL}_2(9)\langle W\varphi_3\rangle$, $\mathrm{Aut}(\mathrm{PSL}_2(9))$, $\mathrm{PSL}_2(17)$ or $\mathrm{PSL}_3(3)$ are possible. For these cases the point stabilizer G_α of the highest order occurs in the case $G \cong \mathrm{Aut}(\mathrm{PSL}_2(9))$ where $|G_\alpha| = 2^5$. Hence, in any case we have $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$ if $d = 4$. So, for the rest of the proof we can assume $d \geq 5$.
Next, we provide further notation for our proof. Let $q = m^a$ for the appropriate prime m and positive integer a. Let p be the prime for which G_α is strongly constrained (recall Corollary 1.4.19 and that $|G_\alpha| > 1$). Regarding Corollary 2.2.11 and Proposition 2.2.8 and using the notation of that proposition, we choose an appropriate ordered $\mathrm{GF}(q)$-basis of V and with respect to that basis w.l.o.g. we may assume that

$$G_\alpha \cap \mathrm{P}\Omega = \mathrm{P}K_{\mathrm{SL},1} =: H_\alpha, \tag{3.4.10}$$

and note that $H_\alpha \in \mathscr{C}_2$ of $\mathrm{P}\Omega$ of type $\mathrm{GL}_1(q) \wr \mathrm{S}_n$. Set $T = C_{\mathrm{SL},1} \leq K_{\mathrm{SL},1}$, and recalling Proposition 2.2.8, Lemma 2.2.19 and Main Theorem 2.2.21, we note that

$$T = \left\{ \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \,\middle|\, \begin{matrix} \lambda_1, \ldots, \lambda_n \in \mathrm{GF}(q)^* \text{ with} \\ \lambda_n = (\lambda_1 \cdot \ldots \cdot \lambda_{n-1})^{-1} \end{matrix} \right\} \cong (\mathbf{Z}_{q-1})^{n-1} \tag{3.4.11}$$

and PT is an abelian normal p-subgroup of G_α of order $\frac{(q-1)^{n-1}}{(q-1,n)} > 1$. Let $g \in G$ where $\alpha^g = \beta$, and recall that $(\alpha, \beta) \in O$ according to the conventions before Section 3.2. So, we clearly have $(\mathrm{P}T)^g \trianglelefteq G_\beta = G_\alpha^g$, and we set $H_\beta = G_\beta \cap \mathrm{P}\Omega = (\mathrm{P}K_{\mathrm{SL},1})^g$. Note, that $H_\beta/(\mathrm{P}T)^g \cong H_\alpha/\mathrm{P}T \cong \mathrm{S}_n$, by Proposition 2.2.8. Since $1 < \mathrm{P}T \leq \mathrm{P}\Omega$, we see $G_\alpha = \mathrm{N}_G(\mathrm{P}T)$ and $G_\beta = \mathrm{N}_G((\mathrm{P}T)^g)$. So, $\mathrm{P}T \neq (\mathrm{P}T)^g$, because $G_\alpha \neq G_\beta$. Define the group $Y = (\mathrm{P}T \cap G_\beta)/(\mathrm{P}T \cap (\mathrm{P}T)^g)$ and set $y = |Y|$. By elementary considerations, we now obtain

$$\begin{aligned} Y &= (\mathrm{P}T \cap G_\beta)/(\mathrm{P}T \cap (\mathrm{P}T)^g) = (\mathrm{P}T \cap H_\beta)/((\mathrm{P}T \cap H_\beta) \cap (\mathrm{P}T)^g) \\ &\cong (\mathrm{P}T \cap H_\beta) \cdot (\mathrm{P}T)^g/(\mathrm{P}T)^g \leq H_\beta/(\mathrm{P}T)^g \cong \mathrm{S}_n. \end{aligned}$$

So, Y is isomorphic to an abelian p-subgroup of S_n. In Figure 3.3, we illustrate the position of certain previously defined groups for our further investigations.

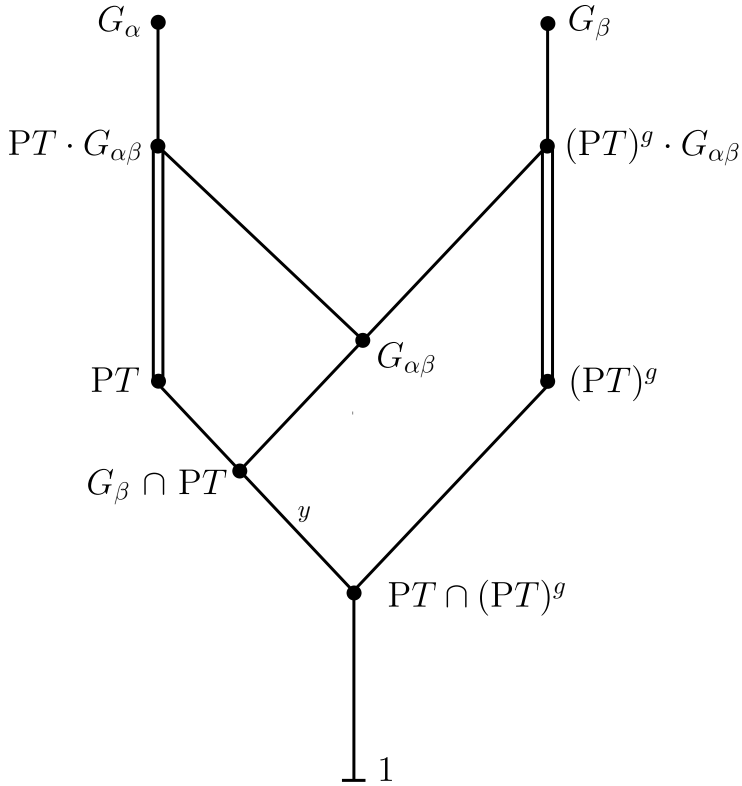

Figure 3.3: Subgroup diagram of G_α, G_β, PT, $(\mathrm{P}T)^g$, $G_\beta \cap \mathrm{P}T$ and $\mathrm{P}T \cap (\mathrm{P}T)^g$.

Now, it is not hard to deduce

$$\frac{|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)|}{y} = |\mathrm{P}T/(\mathrm{P}T \cap G_\beta)| \;\Big|\; |G_\alpha : G_{\alpha\beta}| = d. \qquad (3.4.12)$$

Finally, for providing the needed notation for our investigations, we note the following observation. Write $g = g_\tau g_{\varphi_m} g_{\mathrm{PGL}}$ for appropriate elements $g_\tau \in \langle\tau\rangle$, $g_{\varphi_m} \in \langle\varphi_m\rangle$ and $g_{\mathrm{PGL}} \in \mathrm{PGL}_n(q)$ (recall Corollary 1.2.20). Regarding Proposition 2.2.8, it is not hard to see that τ and φ_m normalize $\mathrm{P}T$. So, $\mathrm{P}T \cap (\mathrm{P}T)^g = \mathrm{P}T \cap (\mathrm{P}T)^{g_{\mathrm{PGL}}}$. Let $\tilde{g} \in \mathrm{GL}_n(q)$ such that $\mathrm{P}\tilde{g} = g_{\mathrm{PGL}}$. Since $\mathrm{Z}(\mathrm{SL}_n(q)) \leq T$, we have $\mathrm{P}T \cap (\mathrm{P}T)^g = \mathrm{P}(T \cap T^{\tilde{g}})$, by Remark 1.4.7 (b). So, we can deduce

$$\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g) = (T/\mathrm{Z}(\mathrm{SL}_n(q)))/((T \cap T^{\tilde{g}})/\mathrm{Z}(\mathrm{SL}_n(q))) \cong T/(T \cap T^{\tilde{g}}). \tag{3.4.13}$$

Now, using case-by-case analysis with respect to the dimension n of V, we will show that there is no counterexample to our assertion. First, let $n \geq 6$. Analogously to the considerations in the proof of Proposition 3.4.2 for $t \geq 6$, using Corollaries 1.2.22 and 2.2.11, Proposition 2.2.8 and Main Theorem 2.2.21 together with Theorems 1.4.25 and 3.1.12, we obtain

$$n + 1 \leq d. \tag{3.4.14}$$

Moreover, we have $d \geq 8$. To see this, consider the case $n = 6$ and suppose that $d = 7$. In view of Corollaries 1.2.22 and 2.2.11 and Proposition 2.2.8, we see $(q-1)^5/(6, q-1) \cdot 6! \mid |G_\alpha| \mid 2a \cdot (q-1)^5 \cdot 6!$. Since $7 \mid |G_\alpha|$, we obtain that 7 divides a or $q - 1$. In view of Lemma 1.4.28, we easily deduce $7 \mid a$. Now, by Theorem 1.5.2, we obtain that there is a Zsigmondy prime $z_{m,a}$ which divides $|G_\alpha|$. Using Lemma 1.5.3, we see $a < z_{m,a}$. But this is a contradiction to Corollary 1.4.26.

In view of Main Theorem 2.2.21 together with the facts provided in Proposition 1.5.9 (ii), we obtain that $q \geq 5$ and one of the following three cases holds:

$$\begin{aligned} &\text{(I) } m = 3, a = 2 \text{ and } p = 2,\\ &\text{(II) } m = 2, a \text{ is a prime}, p \text{ is a Mersenne prime and } q - 1 = p, \text{ or}\\ &\text{(III) } q = m \text{ is a Fermat prime (so, } a = 1) \text{ and } p = 2. \end{aligned} \tag{3.4.15}$$

We note that from now on we will often use the facts provided in Corollaries 1.2.22 and 2.2.11, Table 2.2.1, Proposition 2.2.8 and Main Theorem 2.2.21 also without always giving explicit reference to them. First, we consider the case (I). Here, using (3.4.14), we easily obtain

$$|G_\alpha| \leq 8^{n-1} \cdot n! \cdot 4 \leq 2^{3d-4} \cdot (d-1)! \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

Next, assume that case (II) holds. By Corollary 1.4.26, we see $q - 1 = p \leq d$. Hence, recalling (3.4.14) and also Remark 1.5.10 (a), for this case we can deduce

$$|G_\alpha| \leq (q-1)^{n-1} \cdot n! \cdot 2a \leq d^{d-2} \cdot (d-1)! \cdot 2\frac{\ln(d+1)}{\ln(2)} \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

As we have seen, by Theorems 1.4.25 and 3.1.12 and elementary number theory we have obtained an upper bound for $|G_\alpha|$ in terms of d in the previous two cases. For the remaining case (III) the situation is different. Here, we have

$$|H_\alpha| = (q-1)^{n-1}/(n, q-1) \cdot n! = 2^{m_1} \cdot n! \mid |G_\alpha| \mid 2^{m_2} \cdot n! = 2(q-1)^{n-1} \cdot n! \tag{3.4.16}$$

for appropriate positive integers m_1 and m_2. In this case, the only non-abelian composition factor group of G_α is isomorphic to A_n, and all other (abelian) composition factor groups of G_α are isomorphic to $\mathbf{Z}_2$. Since q can be chosen independently from n and it is not known whether there is only a finite number of Fermat primes q, only by using the above methods we cannot deduce an upper bound for $|G_\alpha|$ in terms of d in general.[6] Clearly, for the case that $q-1 \leq n$ we easily obtain by (3.4.14) $|G_\alpha| \leq 2(q-1)^{n-1} \cdot n! \leq 2(d-1)^{d-2} \cdot (d-1)! \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$. So, we may assume $q-1 > n$. (We note that this method can be extended to the cases $q-1 \leq h(n)$ where h is an increasing integer function, but does not apply for the general case). To obtain the intended upper bound for $|G_\alpha|$ for the cases $q-1 > n$, we have to estimate $q-1$ in terms of d. For this, we will use (3.4.12) and we recall that $|\mathrm{P}T| = \frac{(q-1)^{n-1}}{(q-1,n)} > 1$. So, we have to determine possible values for $|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)|$ and y. Regarding Figure 3.3, it is not hard to see that these two values are related to each other. We will now work out a relation of these two values and use it to obtain an estimate for $q-1$ in terms of d. We recall (3.4.13). Furthermore, for the following we recall the terminology of the rank of an abelian r-group (r a prime) from Subsection 1.4.3. Considering Main Theorem 2.2.21 together with Proposition 1.5.9 (ii), we obtain that $q-1$ is a non-trivial power of the prime 2. In view of Lemma 3.4.4 (b) (cf. also its proof) and Remark 3.4.5 (d) (and using the notation T_0 from Lemma 3.4.4 (b)), we see that there are positive integers $l, n_1, \ldots, n_l$ such that $T \cap T^{\tilde{g}} \cong T_0 \cong (\mathbf{Z}_{q-1})^{l-1} \times \mathbf{Z}_{2^b}$ where $2^b = \min\{(n_i, q-1) \mid 1 \leq i \leq l\}$. Also, it is not hard to see that l is uniquely determined by $\tilde{g}$ for the actual case (recall, that $q-1 > n$). (Note, by Remark 3.4.5 (b) and (c) we see that this does not hold in the general case). Again regarding Lemma 3.4.4 (b), Remark 3.4.5 (d) and (3.4.11), we can consider $T/(T \cap T^{\tilde{g}})$ as an abelian 2-group of rank $n-l =: k$. As noted above, we have $\mathrm{P}T \cap (\mathrm{P}T)^g \neq \mathrm{P}T$, so $T \cap T^{\tilde{g}} \neq T$. Hence, $l \neq n$ (recall Remark 3.4.5 (b)) and $k > 0$. Recalling (3.4.13), we have now seen that $\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)$ is an abelian 2-group of rank $k > 0$. Also clear by the above examinations is that the higher (lower) the value k is, the higher (lower) is the order of $\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)$. More precisely, the noted relation is proper, i.e. if $g_1, g_2 \in G$ with $\alpha^{g_1} = \alpha^{g_2} = \beta$ and associated positive integers l_1 and l_2 where $n - l_1 < n - l_2$, then

$$|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^{g_1})| < |\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^{g_2})| \quad (*).$$

We recall (3.4.12) and that Y is an abelian 2-subgroup of $\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)$ of order y which is isomorphic to a subgroup of S_n. Let the rank of Y be denoted by $\tilde{k}$, hence $\tilde{k} \leq k$. Now, it is clear that the higher (lower) the value k is, the higher (lower) is a possible value of y. So, we have a relation of the value k to both $|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)|$ and y. Now, to obtain a benefit of these relations, we

[6] We note that the analogous problem arises in the case $\mathbf{U}$ in Theorem 3.4.7, below. But there, q is a Mersenne prime. So, the fact that only five Fermat primes are known till today (maybe the only ones) has no influence on the importance of the method we will present in the following to solve this problem. (We note that this method can also be applied in the case $\mathbf{U}$).

need an upper bound for $|Y|$ in terms of $\tilde{k}$ and n. Such an upper bound was determined in [KU], and in Subsection 1.4.3 we have provided the main results of this paper. By Proposition 1.4.34 (ii), we obtain

$$|Y| \leq h(\tilde{k}) = \begin{cases} 2^{\lfloor \frac{n}{2} \rfloor} & \text{if } \lfloor \frac{n}{4} \rfloor + t_n \leq \tilde{k} \leq \lfloor \frac{n}{2} \rfloor, \\ 2^{\lfloor \frac{n}{2} \rfloor - 1} & \text{if } \tilde{k} = \lfloor \frac{n}{4} \rfloor + t_n - 1, \\ (\frac{n}{\tilde{k}})^{\tilde{k}} & \text{if } 1 \leq \tilde{k} \leq \lfloor \frac{n}{4} \rfloor + t_n - 2, \end{cases} \tag{3.4.17}$$

where $t_n = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 1 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$ Note, that we always have $\tilde{k} \leq \lfloor \frac{n}{2} \rfloor$, cf. [KU, Proposition 1]. Furthermore, we see that the highest possible value for $|Y|$ occurs if $\lfloor \frac{n}{4} \rfloor + t_n \leq \tilde{k} \leq \lfloor \frac{n}{2} \rfloor$ by $2^{\lfloor \frac{n}{2} \rfloor}$ (cf. [KU, Lemma 1 and Propositions 2 and 4]). Moreover, we note that the upper bound in Proposition 1.4.34 (ii) for $\lfloor \frac{n}{4} \rfloor + t_n \leq k \leq \lfloor \frac{n}{2} \rfloor$ is sharp, see [KU, Remark (b) following Theorem 2]. So, recalling $(*)$, we see that the "worst case" for our actually intended goal (that is to use (3.4.12) together with (3.4.17) to obtain an estimate for $q-1$ in terms of d) occurs if $k \leq \lfloor \frac{n}{4} \rfloor + t_n$. Hence, we may assume $k \leq \lfloor \frac{n}{4} \rfloor + t_n$. Suppose now that $n_i > 1$ for all $i \in \{1, \ldots, l\}$. Then $n = \sum_{i=1}^{l} n_i \geq 2l$, hence $k = n - l \geq \frac{n}{2}$. But, $k \leq \lfloor \frac{n}{4} \rfloor + t_n < \frac{n}{2}$ (recall, that $n \geq 6$). So, there is an $i \in \{1, \ldots, l\}$ with $n_i = 1$. Regarding Lemma 3.4.4 (b) and Remark 3.4.5 (d), we now can deduce $T \cap T^{\tilde{g}} \cong (\mathbf{Z}_{q-1})^{l-1}$ (note, that $n > l > 1$), and with (3.4.11) and (3.4.13) we see $\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g) \cong (\mathbf{Z}_{q-1})^k$ where $k > 0$. Next, by considering (3.4.12) and (3.4.17), we will determine the "worst case" for our actually intended goal. First, let $n \geq 10$, hence $1 \leq \lfloor \frac{n}{4} \rfloor + t_n - 2$. Consider the case $1 \leq k \leq \lfloor \frac{n}{4} \rfloor + t_n - 2$. By elementary calculations, we see that $h(\tilde{k})$ is increasing for $\tilde{k} \in \{1, \ldots, k\}$ (cf. [KU, Lemma 1]), so here we have

$$\frac{|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)|}{y} \geq \frac{(q-1)^k}{(\frac{n}{\tilde{k}})^{\tilde{k}}} \geq \left(\frac{(q-1)k}{n} \right)^k.$$

Define the real function $\rho :]0, \infty[\to \mathbb{R}, r \mapsto ((q-1)r/n)^r$. By easy calculations, we see that $\rho(r)$ has exactly one minimum, at $\frac{n}{(q-1)e}$, and it is strictly increasing in the interval $[\frac{n}{(q-1)e}, \infty[$. Since $q - 1 > n$, we now can deduce that here the "worst case" for our actually intended goal occurs for $k = 1$, so $(q-1)/n \leq d$, by (3.4.12). By further elementary calculations, we see that the case $k = 1$ stays the "worst case" if we compare it to the estimates which arise for the remaining cases $k = \lfloor \frac{n}{4} \rfloor + t_n$ or $\lfloor \frac{n}{4} \rfloor + t_n - 1$ (for the calculations it is useful to recall from Main Theorem 2.2.21 (ii)(c) that $q \geq 5$). So, using (3.4.14), we now can deduce that for the actual case we have $q - 1 \leq dn \leq d(d-1)$, so

$$|G_\alpha| \leq 2 \cdot (q-1)^{n-1} \cdot n! \leq 2 \cdot (d(d-1))^{d-2} \cdot (d-1)! \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

Now, let $n \in \{6, 7, 8, 9\}$. We recall $\tilde{k} \leq k \leq \lfloor \frac{n}{4} \rfloor + t_n = 2$, and by (3.4.17) we see

$$|Y| \leq h(\tilde{k}) = \begin{cases} 2^{\lfloor \frac{n}{2} \rfloor} & \text{if } \tilde{k} = 2, \\ 2^{\lfloor \frac{n}{2} \rfloor - 1} & \text{if } \tilde{k} = 1. \end{cases}$$

Since $(q-1)^2/2^{\lfloor \frac{n}{2} \rfloor} > (q-1)/2^{\lfloor \frac{n}{2} \rfloor -1}$, we obtain that the "worst case" for our actually intended goal occurs if $k = 1$, so $(q-1)/2^{\lfloor \frac{n}{2} \rfloor -1} \leq d$. Hence, by (3.4.14) and recalling from above that $d \geq 8$, we see

$$|G_\alpha| \leq 2 \cdot (q-1)^{n-1} \cdot n! \leq 2 \cdot (d \cdot 2^{\lfloor \frac{n}{2} \rfloor -1})^{n-1} \cdot n! \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d).$$

Now, we consider the remaining cases $n \in \{2,3,4,5\}$. For this, recall from the beginning of the proof that we can assume $d \geq 5$. We start with the case $n = 5$, and by Main Theorem 2.2.21 (ii)(c) we recall that $q \geq 5$. Here, we have

$$|H_\alpha| = (q-1)^4/(5, q-1) \cdot 5! \;\big|\; |G_\alpha| \;\big|\; 2a \cdot (q-1)^4 \cdot 5!, \tag{3.4.18}$$

G_α is insoluble and the only non-abelian composition factor group of G_α is isomorphic to A_5. Assume that $d = 5$. Then, by analogous arguments as in Theorem 3.3.5 for the case $H \cong SL_2(4)$, we obtain by Theorems 1.4.25 and 3.1.11 (b) that $|G_\alpha| \;\big|\; 2^{14} \cdot 3^2 \cdot 5$. Regarding (3.4.18), we see that only the case $q = 5$ is possible, and here we have

$$|G_\alpha| \leq 2^{12} \cdot 3 \cdot 5 \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d).$$

Now, suppose $d = 6$. Again, as in Theorem 3.3.5 for the case $H \cong SL_2(4)$, using Theorems 1.4.25 and 3.1.11 (b) together with the facts provided in Lemma 3.3.4 and [At, p. 4], we obtain $|G_\alpha| \;\big|\; 2^5 \cdot 3 \cdot 5^6$. But, this contradicts (3.4.18). Suppose that $d = 7$. Here, $7 \mid |G_\alpha|$, so 7 divides $q-1$ or a by (3.4.18). Analogously to the considerations above for the case $n = 6$ and $d = 7$, we now obtain a contradiction. Hence, we can assume $d \geq 8$. Considering Main Theorem 2.2.21 and Proposition 1.5.9 (ii), we obtain that one of the three cases (I) to (III) listed in (3.4.15) holds. For the case (I) we easily get

$$|G_\alpha| \leq 2^{17} \cdot 3 \cdot 5 \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d).$$

In the case (II), we have $q - 1 = p \leq d$, by Corollary 1.4.26. Hence, here we have

$$|G_\alpha| \leq (q-1)^4 \cdot 5! \cdot 2a \leq 2^4 \cdot 3 \cdot 5 \cdot \frac{\ln(d+1)}{\ln(2)} \cdot d^4 \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d).$$

Now, let the case (III) hold. Recall Figure 3.3 and that $y = |Y|$ where Y is isomorphic to an abelian 2-subgroup of S_5. Let $\tilde{k}$ denote the rank of Y, and note that $\tilde{k} \leq 2$, see e.g. [KU, Proposition 1]. In view of Proposition 1.4.34 (ii) (or, here by easy considerations), we obtain

$$|Y| \leq 4 \text{ for } \tilde{k} \in \{1,2\}, \tag{3.4.19}$$

and note that the upper bound in Proposition 1.4.34 (ii) for $\lfloor \frac{n}{4} \rfloor + t_n \leq k \leq \lfloor \frac{n}{2} \rfloor$ is sharp, see [KU, Remark (b) following Theorem 2]. As above, we will determine the "worst case" for estimating $q-1$ in terms of d with respect to (3.4.12) and (3.4.19) (recall also that $Y \leq PT/(PT \cap (PT)^g)$). Here, regarding (3.4.19), it is not hard to see that this "worst case" occurs if $|PT/(PT \cap (PT)^g)|$ has its smallest possible value. Considering Lemma 3.4.4 (b), Remark 3.4.5 (d), (3.4.11) and (3.4.13), we easily see that the smallest value for $|\mathrm{P}T/(\mathrm{P}T \cap (\mathrm{P}T)^g)|$ occurs by $q-1$, hence we can deduce $\frac{q-1}{4} \leq d$. So, for the actual case we have

$$|G_\alpha| \leq (q-1)^4 \cdot 5! \cdot 2 \leq 2^{12} \cdot 3 \cdot 5 \cdot d^4 \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

Next, assume that $n \in \{3,4\}$. Here, G_α is soluble and we have

$$|H_\alpha| = (q-1)^{n-1}/(n, q-1) \cdot n! \;\Big|\; |G_\alpha| \;\Big|\; 2a \cdot (q-1)^{n-1} \cdot n!. \tag{3.4.20}$$

Assume that $d = 5$ or 7. Then, by (3.4.20), we see that d divides $q-1$ or a. In analogy to above, we now obtain a contradiction by Lemmas 1.4.28 and 1.5.3, Corollary 1.4.26 and Theorem 1.5.2. So, we may assume $d = 6$ or $d \geq 8$. In view of Main Theorem 2.2.21 and Proposition 1.5.9 (ii), we see that then one of the cases (I) to (III) listed in (3.4.15) holds. In the case (I), we easily obtain

$$|G_\alpha| \leq 2^{14} \cdot 3 \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

Analogously as in the case $n = 5$, we obtain for the case (II)

$$|G_\alpha| \leq 2^4 \cdot 3 \cdot \frac{\ln(d+1)}{\ln(2)} \cdot d^3 \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

For the case (III) note that Y has rank $\tilde{k} \leq 2$ in the case $n = 4$ and rank $\tilde{k} \leq 1$ in the case $n = 3$. Clearly (see e.g. Proposition 1.4.34 (ii)), we have $|Y| \leq 4$ for $\tilde{k} \in \{1,2\}$ in the case $n = 4$ and $|Y| = 2$ for $\tilde{k} = 1$ in the case $n = 3$, and it is not hard to see that the upper bound in Proposition 1.4.34 (ii) for these $\tilde{k}$ are sharp. Hence, in analogy to the case $n = 5$, we can deduce $\frac{q-1}{4} \leq d$ in the case $n = 4$ and $\frac{q-1}{2} \leq d$ in the case $n = 3$. Hence, also for these cases we can deduce $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$.
Finally, let $n = 2$. Suppose that $d = 5$. Then, by

$$|H_\alpha| = (q-1)/(2, q-1) \cdot 2 \;\Big|\; |G_\alpha| \;\Big|\; 2a \cdot (q-1), \tag{3.4.21}$$

we see that 5 divides $q-1$ or a. As above, we can rule out the case $5 \mid a$. So, let $5 \mid q-1$. Now, by Main Theorem 2.2.21 (i) and Lemma 1.4.28, we obtain a contradiction. Next, assume that $d = 7$. Regarding (3.4.21), we can deduce $7 \mid q-1$ (the case $7 \mid a$ can be ruled out as before). Considering Main Theorem 2.2.21 (i) and Lemma 1.4.28, we see that only the case $q = 8$ is possible, and here we have

$$|G_\alpha| \leq 2 \cdot 3 \cdot 7 \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d).$$

So, we have $d = 6$ or $d \geq 8$. For the cases listed in Main Theorem 2.2.21 (i)(b) to (d) it is not hard to see that $|G_\alpha| \leq h^{\mathbf{L}}_{\mathscr{C}_2}(d)$. Hence, we can assume that the case of Main Theorem 2.2.21 (i)(a) is given, so $q \geq 17$ and $\frac{q-1}{(2,q-1)}$ is a power of the prime p (recall, that G_α is strongly p-constrained). (Note, that, opposite to the cases above, here $q-1$ has not to be a power of p in general). In view of Proposition 1.5.9 (i), we see that one of the following three cases holds:

(I) $m = 2, a$ is a prime and $p = q-1$ is a Mersenne prime,

(II) $m \neq 2$ and $p \neq 2$, or

(III) $q = m$ is a Fermat prime (so, $a = 1$) and $p = 2$.

In the case (I), we deduce from Corollary 1.4.26 and (3.4.21)

$$|G_\alpha| \leq 2\frac{\ln(d+1)}{\ln(2)} \cdot d \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d).$$

For the cases (II)[7] and (III) we first note that by Lemma 3.4.4 (b) and Remarks 1.4.7 (b) and 3.4.5 (b) we have $|\mathrm{P}T \cap (\mathrm{P}T)^g| = 1$ (recall, that $\mathrm{P}T \neq (\mathrm{P}T)^g$). So, in view of Figure 3.3 and (3.4.12), we see that $\frac{q-1}{2} \leq d$ in the case (II) (note, that here $y = 1$) and $\frac{q-1}{4} \leq d$ in the case (III). Hence, we can deduce

$$|G_\alpha| \leq 4d \cdot \frac{\ln(2d+1)}{\ln(3)} \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d)$$

in the case (II) and

$$|G_\alpha| \leq 2^3 d \leq h_{\mathscr{C}_2}^{\mathbf{L}}(d)$$

in the case (III). Since we have shown that no counterexample exists, we have established our assertion. □

Theorem 3.4.7. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_2$ *of* G*, the order of* G_α *is bounded by*

$$h_{\mathscr{C}_2}^{\mathbf{U}}(d) = \begin{cases} 0 & \text{for } d \in \{3,4,7\}, \\ 2^{12} \cdot 3 \cdot 5 & \text{for } d = 5, \\ 2^{16} \cdot 3^{16} & \text{for } d = 6, \\ 2^{3d-2} \cdot 3^{4d-5} \cdot (d-1)! & \text{for } 8 \leq d \leq 28, \\ 2 \cdot (d(d-1))^{d-2} \cdot (d-1)! & \text{for } d \geq 29. \end{cases}$$

In particular, $h_{\mathscr{C}_2}^{\mathbf{U}}(d)$ *is a Sims order bound for* $\mathscr{G}_{\mathscr{C}_2}^{\mathbf{U}}$ *and* $h_{\mathscr{C}_2}^{\mathbf{U}}(d) \leq \mathrm{wdt}(d)$.

Proof. The assertion follows analogously to the proof of Theorem 3.4.6, using the additional facts of Propositions 1.5.11 and 3.4.2 and Main Theorem 2.2.22. Furthermore, we recall our assumption $n \geq 3$. As mentioned in the proof of the last theorem, we note that for the cases arising in Proposition 1.5.11 (ii)(b) if $G_\alpha \in \mathscr{C}_2$ of G of type $\mathrm{GU}_1(q^2) \wr \mathrm{S}_n$ (here, q is a Mersenne prime), it is possible to argue analogously as in the proof of the last theorem, using Lemma 3.4.4 (b) and Remark 3.4.5 (d). □

Remark. By using (more intensively) the methods in the proofs of the previous two theorems and Proposition 3.4.2, it is possible to determine more precise upper bounds for $|G_\alpha|$; such as by further case-by-case analysis for small dimensions n of the vector space V, or by considering more situations for some fixed subdegrees d separately. About the latter point, we note that e.g. the case $d = 7$ has been investigated separately. More generally, for the case that $d \geq 7$ is a prime, we note the following.

[7]Recall by Remark 1.5.10 (b) that for the case listed in Proposition 1.5.9 (i)(b) it is possible that $b > 1$. So, in the case (II) we cannot argue as easily as in the previous case (I).

Let G_α be a strongly constrained member of $\mathscr{C}_2$ of G of type $\mathrm{GL}_1^\epsilon(q^u) \wr \mathrm{S}_n$ where $n \geq 3$. Let $q = m^a$ for the prime m and the positive integer a. Recalling Corollaries 1.2.22 and 2.2.11 and Proposition 2.2.8, we see that

$$(q-\epsilon 1)^{n-1}/(q-\epsilon 1, n) \cdot n! \;\Big|\; |G_\alpha| \;\Big|\; 2a \cdot (q-\epsilon 1)^{n-1} \cdot n!.$$

Consider the situation that $d \geq 7$ is a prime. Since $d \;\big|\; |G_\alpha|$, we see that d divides $n!$, $q - \epsilon 1$ or a. Obviously, $d \mid n!$ if and only if $d \leq n$. But, by (3.4.14), we have $n + 1 \leq d$ if $n \geq 6$ (note, that this assertion also holds in the case $\mathbf{U}$). So, we obtain $d > n$ and $d \nmid n!$. Next, suppose that $d \mid q - \epsilon 1$. Then $d^2 \;\big|\; |G_\alpha|$ which contradicts Lemma 1.4.28. Hence, d has to divide a. Considering Theorem 1.5.2, we see that there is a Zsigmondy prime $z_{m,ua}$ which divides $|G_\alpha|$. Regarding Lemma 1.5.3, we have $ua < z_{m,ua}$. But now, we obtain a contradiction by Corollary 1.4.26. So, for the actual case this situation cannot occur. By analogous considerations, we see that the above situation also does not occur if G_α is of a type treated in Proposition 3.4.2. So, if $G_\alpha \in \mathscr{C}_2$ of G, the only case left where the situation can occur is if G_α is of type $\mathrm{GL}_1(q) \wr \mathrm{S}_2$ (recall Corollary 2.2.13). For this case, now more precise upper bounds for $|G_\alpha|$ can be determined by analogous considerations as in Theorem 3.4.6.

3.5 Sims order bound for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_3}$

Next, we determine a Sims order bound $h^{\mathbf{L}^\epsilon}_{\mathscr{C}_3}(d)$ for $\mathscr{G}^{\mathbf{L}^\epsilon}_{\mathscr{C}_3}$. As we will see, by elementary group theoretic considerations and some elementary number theory it is possible to obtain an upper bound for $|G_\alpha|$ in terms of an arbitrary non-trivial subdegree d of G. We recall that we use the notation introduced at the end of Section 3.1. Especially, from Subsection 1.2.3 we recall that $\epsilon = +$ in case $\mathbf{L}$ and $\epsilon = -$ in case $\mathbf{U}$, and that V denotes an n-dimensional $\mathrm{GF}(q^u)$-vector space where q is a prime power and $u = 1$ in case $\mathbf{L}$ and $u = 2$ in case $\mathbf{U}$. Furthermore, we recall Convention 3.2.2. For the rest of this section we set that $q = m^a$ for a prime m and a positive integer a.

We begin by noting some basic facts about the actual case. Let G_α be a member of A-class $\mathscr{C}_3$ of G which is strongly p-constrained for the prime p. Then, in view of Main Theorems 2.3.16 and 2.3.17, we obtain that G_α is of type $\mathrm{GL}_1^\epsilon(q^{un})$ and

$$\frac{q^n - \epsilon 1}{(q-\epsilon 1)(n, q-\epsilon 1)} = p^b \tag{3.5.1}$$

holds for a positive integer b. Regarding Propositions 1.5.7 and 1.5.8, we see that $p = 2$ in (3.5.1) if and only if case $\mathbf{L}$ holds, $n = 2$ and $q = m$ is a Mersenne prime. For the case $p \neq 2$ we note the following observations. For this, we recall from our chosen notation (see the end of Section 3.1) that $d = |G_\alpha : G_{\alpha\beta}|$ $(\alpha, \beta \in \mathrm{X})$ denotes an arbitrary non-trivial subdegree of the primitive permutation group G.

Lemma 3.5.1. *Let G_α be a member of A-class $\mathscr{C}_3$ of G of type $\mathrm{GL}_1^\epsilon(q^{un})$. Let G_α be strongly p-constrained for the odd prime p, in particular $\frac{q^n-\epsilon 1}{(q-\epsilon 1)(n,q-\epsilon 1)} = p^b$ holds for an appropriate positive integer b (recall (3.5.1)). If $(n,q) = (2,8)$ let $G = \mathrm{P}\Omega(V)$. Then the following hold.*

(a) We have

$$|G_\alpha \cap \mathrm{P}\Omega(V)| = p^b n \mid |G_\alpha| \mid \begin{cases} p^b 2a & \text{for } n = 2, \\ p^b 2an & \text{for } n \geq 3. \end{cases}$$

(b) $\mathrm{O}_p(G_\alpha) = \mathrm{O}_p(G_\alpha \cap \mathrm{P}\Omega(V))$ is a cyclic normal Sylow p-subgroup of G_α of order p^b.

(c) We have

$$p^b \mid d = |G_\alpha : G_{\alpha\beta}|.$$

Proof. Assertion (a) follows from Corollary 1.2.22, Propositions 2.3.3 and 2.3.4, Lemma 2.3.9 (e), Remark 2.3.10 and Main Theorems 2.3.16 and 2.3.17. (Here, note that we have $|G_\alpha/(G_\alpha \cap \mathrm{P}\Omega(V))| = |G/\mathrm{P}\Omega(V)| = |G/(G \cap \mathrm{PI}(V))| = |G \cdot \mathrm{PI}(V)/\mathrm{PI}(V)| \mid |\mathrm{PA}(V)/\mathrm{PI}(V)|$, in view of Main Theorems 2.3.16 (i)(b) and (ii)(b) and 2.3.17 (b)). Considering assertion (a) and Propositions 1.5.7, 1.5.8 and 2.3.12, we obtain assertion (b). To prove assertion (c), suppose that p divides $|G_{\alpha\beta}| = |G_\alpha \cap G_\beta|$. Then there is a subgroup $P \in \mathrm{Syl}_p(G_{\alpha\beta})$ with $P > 1$. Since $\mathrm{O}_p(G_\alpha)$ is the only Sylow p-subgroup of G_α and $\mathrm{O}_p(G_\beta)$ the only of G_β (recall part (b)), we obtain that $P \leq \mathrm{O}_p(G_\alpha), \mathrm{O}_p(G_\beta)$. Hence, because $\mathrm{O}_p(G_\alpha)$ and $\mathrm{O}_p(G_\beta)$ are cyclic groups (again recall part (b)), P is a normal subgroup of G_α and G_β. Now, we easily obtain a contradiction, since $G = \langle G_\alpha, G_\beta \rangle$ is a primitive permutation group. So, $p \nmid |G_{\alpha\beta}|$ and assertion (c) follows from (a). □

Remark 3.5.2. Adopting the assumptions from Lemma 3.5.1, we note the following.

(a) The observation in the previous proof that p does not divide the order of $G_{\alpha\beta}$ follows from the fact that the cyclic groups $\mathrm{O}_p(G_\alpha)$ and $\mathrm{O}_p(G_\beta)$ intersect trivially. This fact can also be obtained without using the argument that G is a permutation group. To see this, recall the results of the previous lemma and note that $\mathrm{O}_p(G_\alpha) = \mathrm{O}_p(G_\alpha \cap \mathrm{P}\Omega(V))$ is also a cyclic Sylow p-subgroup of $\mathrm{P}\Omega(V)$, by Propositions 1.2.13, 1.2.14, 1.5.7 and 1.5.8 (for the case $n = 2$ recall that $p > 2$). In his mathematical diary, Wielandt has conjectured that two different cyclic Sylow subgroups of a finite simple group intersect trivially.[8] This conjecture was proven completely in [Bla]. (For a proof concerning the alternating groups, the finite classical simple groups and the 26 sporadic groups, using more elementary arguments, see [Oe]). Hence, we also see by this result that $\mathrm{O}_p(G_\alpha)$ and $\mathrm{O}_p(G_\beta)$ intersect trivially.

[8][Wie3, p. 6 ”Zu einer einfachen Gruppe sind je zwei verschiedene zyklische Sylowgruppen elementefremd [scheint für $\mathrm{PSL}(n,q)$ und A_n und Mathieu zu stimmen].”].

(b) For the following we recall the normal subgroup $K(\alpha) = (G_\alpha)_{O(\alpha)}$ of G_α (O an orbital of G of length $d = |O(\alpha)|$) from our chosen notation (see the end of Section 3.1). Suppose that $K(\alpha) > 1$. Then there is a minimal normal subgroup $N > 1$ of G_α in $K(\alpha)$. Since $\mathrm{F}^*(G_\alpha) = \mathrm{O}_p(G_\alpha)$ and in view of Lemma 3.5.1 (b), we see that $N \cong \mathbf{Z}_p$. Because $K(\alpha) \le G_{\alpha\beta}$, we now obtain a contradiction by analogous arguments as above. Hence, we may deduce that $K(\alpha) = 1$, so $G_\alpha \cong G_\alpha^{\mathrm{O}(\alpha)}$.
In particular, we obtain that there is a cyclic subgroup of $\mathrm{Sym}(O(\alpha)) \cong \mathrm{S}_d$ of order p^b. Hence, using Proposition 1.4.33, we also see by these considerations that p^b is a lower bound for d (recall Lemma 3.5.1 (c)).

For the case considered in Lemma 3.5.1 we can state the following upper bound for $|G_\alpha|$ in terms of d.

Corollary 3.5.3. *Adopt the assumptions from Lemma 3.5.1. Then we have*

$$|G_\alpha| \bigm| p^b(p-1) \le d(d-1).$$

Proof. By Main Theorem 2.3.16, we see that $(n, q) \neq (3, 4)$ in the case **L**. So, the assertion is a direct consequence of Propositions 1.5.7 and 1.5.8 and Lemmas 1.5.3 and 3.5.1. (Note, that for $n \ge 3$ in case **L**, a, n and p are odd). □

Remark. In view of Lemma 3.5.1 (c), we note the obvious observation that it is possible to state a more precise upper bound for $|G_\alpha|$ in terms of d in the previous corollary provided that $b > 1$.

In the following two theorems, by using further considerations from elementary number theory, we obtain more precise upper bounds for $|G_\alpha|$ in terms of d than in the previous corollary. We recall that we have $q = m^a$ for a prime m and a positive integer a.

Theorem 3.5.4. *For the case that* $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_3$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{L}}_{\mathscr{C}_3}(d) = \begin{cases} 0 & \text{for } d \in \{11, 12, 15\},\\ 2\cdot 3 & \text{for } d \in \{3, 6\},\\ 2^{l+2} & \text{for } d = 2^l \text{ where } l \in \{2,3,4\},\\ 2^2\cdot 5 & \text{for } d \in \{5, 10\},\\ 2\cdot 3\cdot 7 & \text{for } d \in \{7, 14\},\\ 2\cdot 3^3 & \text{for } d \in \{9, 18\},\\ 2\cdot 3\cdot 13 & \text{for } d = 13,\\ 2^3\cdot 17 & \text{for } d = 17,\\ 2d\left(2^{\frac{\ln(d)}{\ln(2)}} - 1\right) & \text{for } d \ge 19. \end{cases}$$

In particular, $h^{\mathbf{L}}_{\mathscr{C}_3}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_3}$ *and* $h^{\mathbf{L}}_{\mathscr{C}_3}(d) \le \mathrm{wdt}(d)$.

Proof. Suppose that the assertion is false and let the pair (G, G_α) be a counterexample. Let G_α be strongly p-constrained for the prime p. Regarding Main Theorem 2.3.16, we see that G_α is of type $\mathrm{GL}_1(q^n)$ and

$$\frac{q^n - 1}{(q-1)(n, q-1)} = p^b \tag{3.5.2}$$

for an appropriate positive integer b. Now, by considering the possible cases, we will show that there is no counterexample to our assertion.
First, we consider the case $n \geq 3$. Here, regarding Proposition 1.5.7, we note that p is odd. We begin our considerations in the actual case with the case $q = m^a \in \{2, 3\}$. By (3.5.2), here we easily obtain

$$n = \frac{\ln((q-1)p^b + 1)}{\ln(q)}. \tag{3.5.3}$$

Hence, using (3.5.3) and Lemma 3.5.1 (a) and (c), we may deduce

$$|G_\alpha| \leq 2np^b \leq 2d\frac{\ln((q-1)d + 1)}{\ln(q)} = h_1(d). \tag{3.5.4}$$

More precisely, considering the cases $n \in \{3, 5\}$ separately and using Lemma 3.5.1 (a) and (c), we have in the actual case

$$|G_\alpha| \leq h_2(d) = \begin{cases} 2 \cdot 3 \cdot 7 & \text{for } d \in \{7, 14, 21, 42\}, \\ 2 \cdot 3 \cdot 13 & \text{for } d \in \{13, 26, 39, 78\}, \\ 2 \cdot 5 \cdot 31 & \text{for } d \in \{31, 62\}, \\ 2 \cdot 5 \cdot 11^2 & \text{for } d = 121, \\ 2d\frac{\ln(d+1)}{\ln(2)} & \text{for } d \geq 127, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.5}$$

Hence, since $h_2(d) \leq h_{\mathscr{C}_3}^{\mathbf{L}}(d)$, there is no counterexample to the assertion in the actual case.
Next, we consider the case $q \geq 4$. In view of (3.5.2) and Lemma 3.5.1 (c), we note that here $d \geq 19$ (consider the case $(n, q) = (3, 7)$ and recall from Main Theorem 2.3.16 that $(n, q) \neq (3, 4)$). First, we consider the case $(n, q) = (3, 8)$ separately. Here, by Lemma 3.5.1 (c), we have $p^b = 73 \leq d$. Hence, using Lemma 3.5.1 (a), we easily see that $|G_\alpha| \leq 73 \cdot 3^2 \cdot 2 \leq h_{\mathscr{C}_3}^{\mathbf{L}}(d)$. So, let $(n, q) \neq (3, 8)$. Again recalling from Main Theorem 2.3.16 that $(n, q) \neq (3, 4)$ and because $(n, q) = (3, 9)$ does not satisfy (3.5.2), we obtain by Lemma 1.5.13 and (3.5.2) that

$$p^b = \frac{m^{an} - 1}{(m^a - 1)(n, m^a - 1)} \geq \frac{m^{an} - 1}{(m^a - 1)n} > 2 \cdot m^{\frac{na-1}{2}}. \tag{3.5.6}$$

So, by Lemma 3.5.1 (c), we may deduce

$$na < 2\frac{\ln(p^b/2)}{\ln(m)} + 1 \leq 2\frac{\ln(d/2)}{\ln(2)} + 1 = 2\frac{\ln(d)}{\ln(2)} - 1. \tag{3.5.7}$$

Regarding Lemma 3.5.1 (a) and (c), we now obtain

$$|G_\alpha| \;\big|\; 2p^b na < 2d\left(2\frac{\ln(d)}{\ln(2)} - 1\right) = h_3(d). \tag{3.5.8}$$

Recalling from above that $d \geq 19$, we see that in the actual case there exists no counterexample to the assertion, since $|G_\alpha| \leq h_3(d) \leq h^{\mathbf{L}}_{\mathscr{C}_3}(d)$.

Next, we investigate the case $n = 2$. In view of Propositions 2.3.12 and Main Theorem 2.3.16, it is advisable to consider the case $q \leq 9$ separately. So, we first assume that $q = m^a \geq 11$. By (3.5.2), we recall that

$$\frac{m^a + 1}{(2, m^a - 1)} = p^b. \tag{3.5.9}$$

Consider the case $p = 2$. Here, regarding Proposition 1.5.7 (iii), we have $a = 1$ and m is a Mersenne prime. So, $m+1 = 2^{b+1}$ and note that $b \geq 4$ since $m \geq 31$. Furthermore, we note that $G \in \{\mathrm{P}\Omega(V), \mathrm{PI}(V)\}$, cf. also Main Theorem 2.3.16. Considering Lemma 2.3.9 and Remark 2.3.10 (b) (recall also Propositions 2.3.3 and 2.3.4), we see that there is a Singer subgroup S of $\Omega(V)$ (respectively $\mathrm{I}(V)$) such that $\mathrm{P}S$ is a normal cyclic 2-subgroup of G_α of index 2 if $G = \mathrm{P}\Omega(V)$ (respectively $G = \mathrm{PI}(V)$). Furthermore, note that the order of G_α is 2^{b+1} (respectively 2^{b+2}) if $G = \mathrm{P}\Omega(V)$ (respectively $G = \mathrm{PI}(V)$). Let $g \in G$ such that $\alpha^g = \beta$ and recall from our chosen notation (at the end of Section 3.1) that $d = |G_\alpha : G_{\alpha\beta}|$. Next, we will show that the order of $G_{\alpha\beta}$ divides 4. If $G_{\alpha\beta} \leq \mathrm{P}S$ or $(\mathrm{P}S)^g$ then we obtain $G_{\alpha\beta} = 1$, by [Kn2, Satz 1 (a)]. So, assume $G_{\alpha\beta} \nleq \mathrm{P}S, (\mathrm{P}S)^g$. Then we easily deduce that $|G_{\alpha\beta}/(\mathrm{P}S \cap G_{\alpha\beta})| = |G_\alpha/\mathrm{P}S| = 2 = |G_\beta/(\mathrm{P}S)^g| = |G_{\alpha\beta}/((\mathrm{P}S)^g \cap G_{\alpha\beta})|$. It is not hard to see that $\mathrm{P}S \cap (\mathrm{P}S)^g = 1$; otherwise, $\mathrm{P}S \cap (\mathrm{P}S)^g$ would be a non-trivial normal subgroup of G_α and G_β which contradicts to the assumption that $G = \langle G_\alpha, G_\beta\rangle$ is a primitive permutation group. So, $(\mathrm{P}S \cap G_{\alpha\beta}) \cap ((\mathrm{P}S)^g \cap G_{\alpha\beta}) = 1$ and we easily may deduce that $|G_{\alpha\beta}|$ divides 4. Hence, we obtain that $|G_\alpha|/4$ divides d and

$$|G_\alpha| \leq 4d \leq h^{\mathbf{L}}_{\mathscr{C}_3}(d) \tag{3.5.10}$$

in the actual case. (Note, that $|G_\alpha|/4 \geq 2^{b-1} \geq 8$, since $b \geq 4$).

Next, consider the case $p \neq 2$ in (3.5.9) and recall that we assume $q = m^a \geq 11$. First, let $m = 2$. Here, in view of Proposition 1.5.7 (iii), we see that $b = 1$ and p is a Fermat prime. Especially, note that $d \geq 17$ and $d \neq 18$, regarding Lemma 3.5.1 (c). By (3.5.9), we easily deduce $a = \frac{\ln(p-1)}{\ln(2)}$. Hence, using Lemma 3.5.1, we now obtain that

$$|G_\alpha| \leq 2pa \leq 2d\frac{\ln(d-1)}{\ln(2)} \leq h^{\mathbf{L}}_{\mathscr{C}_3}(d). \tag{3.5.11}$$

So, in the actual case there exists no counterexample to the assertion. Now, assume $m \neq 2$. Here, regarding (3.5.9), we have $a = \frac{\ln(2p^b-1)}{\ln(m)}$. So, in view of Lemma 3.5.1, we have

$$|G_\alpha| \leq 2p^b a \leq 2d\frac{\ln(2d-1)}{\ln(3)}. \tag{3.5.12}$$

For $11 \leq q \leq 37$ we see that only $q \in \{13, 17, 25, 37\}$ satisfy (3.5.9) in the actual case. Hence, again using Lemma 3.5.1, we may deduce

$$|G_\alpha| \leq h_4(d) = \begin{cases} 2 \cdot 7 & \text{for } d \in \{7, 14\}, \\ 2 \cdot 3^2 & \text{for } d \in \{9, 18\}, \\ 2^2 \cdot 13 & \text{for } d = 13, \\ 2d^{\frac{\ln(2d-1)}{\ln(3)}} & \text{for } d \geq 19, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.13}$$

So, since $h_4(d) \leq h_{\mathscr{C}_3}^{\mathbf{L}}(d)$, there exists no counterexample to the assertion in the actual case.
Finally, we consider the case $n = 2$ and $q \leq 9$. In this case we have

$$|G_\alpha| \leq h_5(d) = \begin{cases} 2 \cdot 3 & \text{for } d \in \{3, 6\}, \\ 2^2 \cdot 5 & \text{for } d \in \{5, 10, 20\}, \\ 2^4 & \text{for } d \in \{4, 8, 16\}, \\ 2 \cdot 3^3 & \text{for } d \in \{9, 18, 27, 54\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.14}$$

by using analogous arguments as before (also use Lemma 3.5.1 and recall the conditions in Main Theorem 2.3.16). Only for the case $q = 8$ and $G = \mathrm{PA}(V)$ where $\mathrm{O}_3(G_\alpha)$ is not cyclic (recall Proposition 2.3.12 (ii)(c)) the argumentation is slightly different and we note the following. In the described case $S = \mathrm{O}_3(G_\alpha \cap \mathrm{P}\Omega(V))$ is a cyclic normal subgroup of G_α of order 3^2 (recall Propositions 2.3.3 and 2.3.4, Lemma 2.3.9 (e) and Remark 2.3.10). By analogous arguments as before, we see that $S \cap S^g = 1$ where $g \in G$ with $\alpha^g = \beta$. Hence, we may deduce that

$$\begin{aligned} |S \cap G_\beta| &= |(S \cap G_\beta)/(S \cap S^g)| = |(S \cap G_\beta)/((S \cap G_\beta) \cap S^g)| \\ &= |(S \cap G_\beta) \cdot S^g/S^g| \bigm| |(G_\beta \cap \mathrm{P}\Omega(V))/S^g| = 2 \end{aligned}$$

(recall Lemma 3.5.1 (a)). So, $S \cap G_\beta = 1$ and we obtain

$$\begin{aligned} |G_{\alpha\beta}| &= |(G_\alpha \cap G_\beta)/(G_\beta \cap S)| = |(G_\alpha \cap G_\beta)/((G_\alpha \cap G_\beta) \cap S)| \\ &= |(G_\alpha \cap G_\beta) \cdot S/S| \bigm| |G_\alpha/S| = 6. \end{aligned}$$

Now, it is easy to see that $9 = \frac{|G_\alpha|}{6} \bigm| |G_\alpha : G_{\alpha\beta}| = d$ and $|G_\alpha| \leq h_5(d)$ in the actual case.
Since we have $|G_\alpha| \leq h_5(d) \leq h_{\mathscr{C}_3}^{\mathbf{L}}(d)$, no counterexample exists and our assertion is established. □

Theorem 3.5.5. *For the case that* $\operatorname{soc}(G) \cong \mathrm{PSU}_n(q^2)$ *and* G_α *is a strongly constrained member of A-class* $\mathscr{C}_3$ *of* G*, the order of* G_α *is bounded by*

$$h^{\mathbf{U}}_{\mathscr{C}_3}(d) = \begin{cases} 0 & \text{for } 3 \le d \le 36 \text{ and } d \notin \{13, 19, 26\}, \\ 2^2 \cdot 3 \cdot 13 & \text{for } d \in \{13, 26\}, \\ 2 \cdot 3^2 \cdot 19 & \text{for } d = 19, \\ 2d\left(2^{\frac{\ln(5d/6)}{\ln(2)}} + 1\right) & \text{for } d \ge 37. \end{cases}$$

In particular, $h^{\mathbf{U}}_{\mathscr{C}_3}(d)$ *is a Sims order bound for* $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_3}$ *and* $h^{\mathbf{U}}_{\mathscr{C}_3}(d) \le \operatorname{wdt}(d)$*.*

Proof. Suppose that there is a counterexample (G, G_α) to the assertion. Let G_α be strongly p-constrained for the prime p. In view of Main Theorem 2.3.17, we see that G_α is of type $\mathrm{GU}_1(q^{2n})$ and

$$\frac{q^n + 1}{(q+1)(n, q+1)} = p^b \tag{3.5.15}$$

holds for an appropriate positive integer b. Regarding Proposition 1.5.8, we note that p is odd. In the following, we will show that no counterexample to our assertion exists.
First, we consider the cases $q = m^a \in \{2, 3\}$. We note that $(n, q) \notin \{(3, 2), (3, 3), (5, 2)\}$, considering Main Theorem 2.3.17 and since $\mathrm{PSU}_3(2^2)$ is not simple. Easily, we obtain by (3.5.15) that

$$n = \frac{\ln((q+1)p^b - 1)}{\ln(q)} \tag{3.5.16}$$

(note, that $(n, q+1) = 1$ since n is an odd prime, recall Definition 2.3.1 or see Proposition 1.5.8). Furthermore, in view of Lemma 3.5.1 (c), we note that $d \ge 43$. Now, by (3.5.16) and Lemma 3.5.1 (a) and (c), we obtain

$$|G_\alpha| \le 2np^b \le 2d\frac{\ln((q+1)d - 1)}{\ln(q)} = h_1(d). \tag{3.5.17}$$

More precisely, considering the cases $n \in \{5, 7\}$ separately and using Lemma 3.5.1 (a) and (c), we may deduce in the actual case

$$|G_\alpha| \le h_2(d) = \begin{cases} 2 \cdot 7 \cdot 43 & \text{for } d \in \{43, 86, 301, 602\}, \\ 2 \cdot 5 \cdot 61 & \text{for } d \in \{61, 122, 305, 610\}, \\ 2 \cdot 7 \cdot 547 & \text{for } d = 547, \\ 2d^{\frac{\ln(3d-1)}{\ln(2)}} & \text{for } d \ge 683, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.18}$$

So, because $h_2(d) \le h^{\mathbf{U}}_{\mathscr{C}_3}(d)$, there exists no counterexample to the assertion in the actual case.
Next, we consider the case $q \ge 4$. In this case, we note that the smallest

values p^b in (3.5.15) occur for $(n, q) = (3, 4), (3, 8)$ and $(3, 11)$ by $13, 19$ and 37, respectively (note, that $(n, q) \neq (3, 5)$ by Main Theorem 2.3.17). First, we consider the cases $(n, q) \in \{(3, 4), (3, 8)\}$ separately. By Lemma 3.5.1 (c), we see that $13 \mid d$ in the case $(n, q) = (3, 4)$ and $19 \mid d$ in the case $(n, q) = (3, 8)$. Since d divides $|G_\alpha|$, by Lemma 3.5.1 (a) it is now not hard to see that $|G_\alpha| \leq h^{\mathbf{U}}_{\mathscr{C}_3}(d)$ in the actual cases. So, let $(n, q) \notin \{(3, 4), (3, 8)\}$, in particular note that now $d \geq 37$, by Lemma 3.5.1 (c). In view of Lemma 1.5.15 and (3.5.15), we may deduce

$$p^b = \frac{m^{an} + 1}{(m^a + 1)(n, m^a + 1)} \geq \frac{m^{an} + 1}{(m^a + 1)n} > \frac{6}{5} \cdot m^{\frac{na-1}{2}}. \tag{3.5.19}$$

Hence, regarding Lemma 3.5.1 (c), we obtain

$$na < 2\frac{\ln(5p^b/6)}{\ln(m)} + 1 \leq 2\frac{\ln(5d/6)}{\ln(2)} + 1. \tag{3.5.20}$$

Again considering Lemma 3.5.1, we now may deduce

$$|G_\alpha| \mid 2p^b na < 2d \left(2\frac{\ln(5d/6)}{\ln(2)} + 1 \right) = h_3(d). \tag{3.5.21}$$

Recalling that $d \geq 37$ in the actual case, we see $|G_\alpha| \leq h_3(d) \leq h^{\mathbf{U}}_{\mathscr{C}_3}(d)$; so there is no counterexample to the assertion. □

Remark 3.5.6. By an extended case-by-case analysis with respect to the values n or q, it is possible to determine more precise upper bounds for $|G_\alpha|$ in terms of d in the previous two theorems (see e.g. (3.5.5) or (3.5.18)). Also, more precise upper bounds may be obtained by considering separately the cases of small fixed values m or a. Furthermore, by an extended case-by-case analysis it would also be possible to obtain a more precise estimate in Lemma 1.5.13 or 1.5.15 for a certain case (cf. also Remarks 1.5.14 and 1.5.16), which also may lead to more precise upper bounds (recall (3.5.6) and (3.5.19)).

In most cases it is hard to analyze a determined upper bound for $|G_\alpha|$ in terms of d for quality, because the group theoretic structure of G_α is complex and it is hard to state assertions about the occurring non-trivial subdegrees d of G, such as to determine the minimal possible non-trivial subdegree of G. In the actual case where G_α is a member of A-class $\mathscr{C}_3$ of G, the group theoretic structure of G_α is (pretty) easy, and so here it is possible to analyze the upper bounds determined in the proofs of the previous two theorems for quality. For this, we provide the following lemma and remark, and we recall our chosen notation introduced at the end of Section 3.1.

Lemma 3.5.7. *Let G_α be a strongly constrained member of A-class $\mathscr{C}_3$ of G. Then there exists an element $g \in G \setminus G_\alpha$ such that $\gamma = \alpha^g \in \mathrm{X}$ and $n = \dim(V) \leq |G_{\alpha\gamma}|$. In particular, there exists a non-trivial subdegree d_0 of G where*

$$d_0 = |G_\alpha : G_{\alpha\gamma}| \leq |G_\alpha|/n.$$

Proof. G_α is of type $\mathrm{GL}_1^\epsilon(q^{un})$, following Main Theorems 2.3.16 and 2.3.17. Let G_α be strongly p-constrained for the prime p, and recall that (3.5.1) holds for an appropriate positive integer b. First, let $n \geq 3$. Here, in view of Propositions 1.5.7 and 1.5.8, p is odd. By Proposition 2.3.3 and Lemma 1.4.21, we obtain that $G_\alpha \cap \mathrm{P}\Omega(V) = \mathrm{P}\Omega(V)_\alpha \in \mathscr{C}_3$ of $\mathrm{P}\Omega(V)$ of type $\mathrm{GL}_1^\epsilon(q^{un})$ and $\mathrm{P}\Omega(V)_\alpha$ is strongly p-constrained. Regarding Proposition 2.3.6 and Main Theorems 2.3.16 and 2.3.17, we see that $\mathrm{P}\Omega(V)_\alpha$ is a maximal subgroup of $\mathrm{P}\Omega(V)$ (recall that we actually assume $n \neq 2$). Since $\mathrm{P}\Omega(V) > 1$ is a normal subgroup of the primitive permutation group G, we obtain by elementary considerations from permutation group theory that $\mathrm{P}\Omega(V)$ acts transitively on X. Hence, we can deduce that $\mathrm{P}\Omega(V) \leq \mathrm{Sym(X)}$ is primitive. (Note, that we have shown that we also may apply Lemma 3.5.1 on the case $(\mathrm{P}\Omega(V), \mathrm{P}\Omega(V)_\alpha)$). Now, suppose that for every $g \in \mathrm{P}\Omega(V) \setminus \mathrm{P}\Omega(V)_\alpha$ we have $\mathrm{P}\Omega(V)_\alpha \cap (\mathrm{P}\Omega(V)_\alpha)^g = \mathrm{P}\Omega(V)_{\alpha\alpha^g} = 1$. (Note, that $\alpha^g \neq \alpha$). Then $\mathrm{P}\Omega(V)$ is a Frobenius group to $\mathrm{P}\Omega(V)_\alpha$, and $\mathrm{P}\Omega(V) = K \rtimes \mathrm{P}\Omega(V)_\alpha$ where K denotes the Frobenius kernel of $\mathrm{P}\Omega(V)$, see Theorem 1.4.2. Since $\mathrm{P}\Omega(V)$ is simple, we obtain a contradiction, and so there is an element $g \in \mathrm{P}\Omega(V) \setminus \mathrm{P}\Omega(V)_\alpha$ with $\mathrm{P}\Omega(V)_{\alpha\alpha^g} > 1$. Set $\gamma = \alpha^g$ and recall that $\gamma \neq \alpha$. In view of Lemma 3.5.1 (c), we see that p^b divides $|\mathrm{P}\Omega(V)_\alpha : \mathrm{P}\Omega(V)_{\alpha\gamma}|$. By Lemma 3.5.1 (a), we obtain $|\mathrm{P}\Omega(V)_\alpha| = p^b n$, and we note that n is an odd prime (recall Definition 2.3.1 or see Propositions 1.5.7 and 1.5.8). So, by the previous considerations, we may deduce $|\mathrm{P}\Omega(V)_{\alpha\gamma}| = n$. Hence, our assertion is established for the actual case, because $|G_{\alpha\gamma}| \geq |\mathrm{P}\Omega(V)_{\alpha\gamma}| = n$.
Now, consider the other case $n = 2$. Suppose that our assertion is false, hence $G_\alpha \cap (G_\alpha)^g = 1$ for all $g \in G \setminus G_\alpha$. Then G is a Frobenius group to G_α, and $G = K \rtimes G_\alpha$ where K is the Frobenius kernel of G, see Theorem 1.4.2. Clearly, $K \cap \mathrm{P}\Omega(V)$ is a normal subgroup of $\mathrm{P}\Omega(V)$ where $K \cap \mathrm{P}\Omega(V) \neq \mathrm{P}\Omega(V)$. Because $\mathrm{C}_G(\mathrm{P}\Omega(V)) = 1$, we see that $K \cap \mathrm{P}\Omega(V) > 1$. Since $\mathrm{P}\Omega(V)$ is simple, we now obtain a contradiction; so our assertion is established. □

Remark 3.5.8. Adopt the notation from the previous lemma and its proof. Consider the case $G = \mathrm{P}\Omega(V)$ and let p be odd. Note, that here we have $|G_\alpha| = p^b n$ where n is a prime, regarding Definition 2.3.1, Proposition 2.3.4, Lemma 2.3.9 (e) and Remark 2.3.10. Then, in view of Lemmas 3.5.1 (c) and 3.5.7, we see that the minimal non-trivial subdegree d_0 of G is $d_0 = p^b$.

Remark 3.5.9. Let us exemplarily investigate some of the upper bounds $h_j(d)$ for $|G_\alpha|$ in terms of d for quality, which were determined in the proofs of Theorems 3.5.4 and 3.5.5. For this, we use the notation in these proofs, especially recall that then (3.5.2) and (3.5.15) hold. The investigation for quality we will do for two cases. In the first case (I), we consider the "worst case" for $h_j(d)$ overestimating the actual order of G_α: For $d_1 = |G_\alpha|$, by $h_j(d_1)$, we obtain the highest possible value of $h_j(d)$ for a given case. So, in the case (I), by the factor $\frac{h_j(d_1)}{|G_\alpha|} = \frac{h_j(|G_\alpha|)}{|G_\alpha|}$ we will determine the highest possible factor of $h_j(d)$ overestimating $|G_\alpha|$. In the second case (II), we consider the concrete case that $G = \mathrm{P}\Omega(V)$ and p is odd. Here, we recall from Remark 3.5.8 that the minimal non-trivial subdegree d_0 of G is $d_0 = p^b$. Clearly, by $h_j(d_0)$, we obtain the lowest value of $h_j(d)$ in the considered case. So, recalling from Remark 3.5.8 that $|G_\alpha| = p^b n$, in the case (II) we will determine the factor $\frac{h_j(d_0)}{|G_\alpha|} = \frac{h_j(p^b)}{p^b n}$ which indicates how $h_j(d)$ overestimates $|G_\alpha|$ at least in the considered concrete case. (Note, that by the considerations in the proofs of Theorems 3.5.4 and 3.5.5 it is easy also to determine a more precise upper bound for $|G_\alpha|$ in the case $G = \mathrm{P}\Omega(V)$, by considering this concrete case separately).

(a) Consider the case $n \geq 3$ and $q \in \{2,3\}$, investigated in the proofs of Theorems 3.5.4 and 3.5.5. Here, recalling (3.5.4) and (3.5.17), we see that $|G_\alpha| \leq h_1(d) = 2d\frac{\ln((q-\epsilon 1)d+\epsilon 1)}{\ln(q)}$. Hence, in the case (I) we have

$$\frac{h_1(d_1)}{|G_\alpha|} = \frac{h_1(|G_\alpha|)}{|G_\alpha|} = 2\frac{\ln((q-\epsilon 1)|G_\alpha| + \epsilon 1)}{\ln(q)}.$$

Clearly, the higher the order of G_α the smaller is the previous factor compared to the order of G_α.
In the case (II), recalling (3.5.3) and (3.5.16), we have

$$\frac{h_1(d_0)}{|G_\alpha|} = \frac{h_1(p^b)}{p^b n} = \frac{2}{n} \cdot \frac{\ln((q-\epsilon 1)p^b + \epsilon 1)}{\ln(q)} = 2.$$

(b) In this part, we consider the case in the proof of Theorem 3.5.4 for $n \geq 3$ and $q \geq 4$. Here, recalling (3.5.8), we have $|G_\alpha| < h_3(d)$ and note that this estimate also holds in the separately considered case $(n,q) = (3,8)$. So, in case (I) we have

$$\frac{h_3(d_1)}{|G_\alpha|} = \frac{h_3(|G_\alpha|)}{|G_\alpha|} = 4\frac{\ln(|G_\alpha|)}{\ln(2)} - 2. \tag{3.5.22}$$

It is easy to see that in general the higher the values n and q the higher is the order of G_α, and the higher the order of G_α the smaller is the factor (3.5.22) compared to the order of G_α (cf. also the values provided in Table 3.5.1, below).
In case (II), we obtain that

$$\frac{h_3(d_0)}{|G_\alpha|} = \frac{h_3(p^b)}{p^b n} = \frac{2}{n}\left(2\frac{\ln(p^b)}{\ln(2)} - 1\right). \tag{3.5.23}$$

Regarding (3.5.2), we see that for a fixed value n in general the smaller the value q the smaller is the factor (3.5.23) the smaller is the order of G_α (cf. also the values provided in Table 3.5.1, below). Furthermore, using (3.5.7), we note the observation $\frac{h_3(d_0)}{|G_\alpha|} > 2a$ (note, that this is also clear by recalling (3.5.8) and that $|G_\alpha| = p^b n$ in the actual case).
By way of illustration, in the following table we list the factors from (3.5.22) and (3.5.23) for certain values of pairs (n,q) (note, that all these pairs (n,q) satisfy the condition (3.5.2)). In the case of (3.5.22) we consider the case where the order of G_α is maximal for a given situation by (n,q), i.e. we consider the case $|G_\alpha| = p^b 2an$ (recall Lemma 3.5.1 (a)).

Table 3.5.1

(n,q)	(3.5.22)	(3.5.23)
$(3,7)$	25.33	5.00
$(3,13)$	32.06	7.24
$(3,31)$	41.82	10.49
$(3,2^9)$	93.03	23.34
$(3,11^3)$	97.71	27.01
$(3,13^3)$	97.15	26.82
$(5,11)$	57.90	8.92
$(7,29)$	118.79	14.79
$(11,23)$	183.20	15.03

Recalling the notes at the beginning of the remark, we see by our previous considerations that the order bound $h_3(d)$ estimates the order of G_α tight in the actually considered cases.

(c) Next, consider the case in the proof of Theorem 3.5.5 for $q \geq 4$ where $(n,q) \neq (3,4),(3,8)$. This case is in analogy to the case in part (b) and the notes below (3.5.22) and (3.5.23) provided there are also valid in the actual corresponding cases.
Regarding (3.5.21), we have $|G_\alpha| < h_3(d)$. So, we obtain in case (I)

$$\frac{h_3(d_1)}{|G_\alpha|} = \frac{h_3(|G_\alpha|)}{|G_\alpha|} = 4\frac{\ln(5|G_\alpha|/6)}{\ln(2)} + 2. \tag{3.5.24}$$

In case (II), we see that

$$\frac{h_3(d_0)}{|G_\alpha|} = \frac{h_3(p^b)}{p^b n} = \frac{2}{n}\left(2\frac{\ln(5p^b/6)}{\ln(2)} + 1\right). \tag{3.5.25}$$

By way of illustration, we give in the following table the factors from (3.5.24) and (3.5.25) for certain values of pairs (n,q) (note, that each pair (n,q) satisfies the condition (3.5.15)). In analogy to part (b), in the case of (3.5.24) we consider the case $|G_\alpha| = p^b 2an$ (recall Lemma 3.5.1 (a)).

Table 3.5.2

(n,q)	(3.5.24)	(3.5.25)
$(3,11)$	32.13	7.26
$(3,41)$	47.67	12.44
$(3,5^3)$	66.97	16.76
$(3,2^7)$	72.13	16.85
$(3,11^3)$	94.31	25.88
$(5,4)$	39.67	4.48
$(5,59)$	98.97	17.14
$(7,17)$	113.95	14.10
$(11,13)$	166.38	13.50

In view of our previous considerations, we see that the order bound $h_3(d)$ estimates the order of G_α tight in the actually considered cases.

3.6 Sims order bound for $\mathscr{G}_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}$

Finally, in this section we determine the remaining intended Sims order bound $h_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}(d)$ for $\mathscr{G}_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}$. Again, we recall that we use the notation introduced at the end of Section 3.1, and we also recall Convention 3.2.2. We begin by the following remark where we present an approach how by Theorem 1.4.25 (respectively its Corollary 1.4.26) and elementary number theory an upper bound for $|G_\alpha|$ in terms of an arbitrary non-trivial subdegree d of G can be deduced.

Remark 3.6.1. In this remark, let the dimension n of the vector space V be at least 5. Let G_α be a strongly constrained member of A-class $\mathscr{C}_6$ of G of type $r^{1+2m}:\mathrm{Sp}_{2m}(r)$ or $(4\circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ where r is an odd prime and m a positive integer (note, that $m \geq 3$ if G_α is of type $(4\circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$, because $n \geq 5$). In view of Definition 2.6.15, Corollaries 1.2.22 and 2.6.19, Remark 2.6.16 (c), Propositions 2.6.20, 2.6.22, 2.6.33 and 2.6.35 and Main Theorems 2.6.28 and 2.6.40, we see that

$$|G_\alpha| \leq \begin{cases} r^{2m}\cdot|\mathrm{Sp}_{2m}(r)|\cdot(r-1) & \text{for } G_\alpha \text{ of type } r^{1+2m}:\mathrm{Sp}_{2m}(r),\\ 2^{2m+1}\cdot|\mathrm{Sp}_{2m}(2)| & \text{for } G_\alpha \text{ of type } (4\circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2).\end{cases}$$

By easy calculations and Proposition 1.2.13, we obtain that for a prime power $\tilde{q}$ and a positive integer $\tilde{m}$

$$|\mathrm{Sp}_{2\tilde{m}}(\tilde{q})| = \tilde{q}^{\tilde{m}^2}\prod_{j=1}^{\tilde{m}}(\tilde{q}^{2j}-1) < \tilde{q}^{\tilde{m}^2+\sum_{j=1}^{\tilde{m}}2j} = \tilde{q}^{2\tilde{m}^2+\tilde{m}}. \tag{3.6.1}$$

Hence, we can deduce

$$|G_\alpha| \leq \begin{cases} r^{2m^2+3m}\cdot(r-1) & \text{for } G_\alpha \text{ of type } r^{1+2m}:\mathrm{Sp}_{2m}(r),\\ 2^{2m^2+3m+1} & \text{for } G_\alpha \text{ of type } (4\circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2).\end{cases} \tag{3.6.2}$$

Now, to determine an upper bound for $|G_\alpha|$ in terms of an arbitrary non-trivial subdegree d, it will be sufficient to estimate r and m in terms of d. Next, we present two possibilities to obtain such estimates.

(a) If G_α is of type $r^{1+2m}:\mathrm{Sp}_{2m}(r)$ we see by Proposition 2.6.20 and Corollary 2.6.19 that r^2 divides $|G_\alpha|$. So, in this case we obtain

$$r+1 \leq d, \tag{3.6.3}$$

by Corollary 1.4.26 and Lemma 1.4.28. Again considering Propositions 1.2.13, 2.6.20 and 2.6.33 and Corollary 2.6.19, we see that $r^{2m}-1$ (respectively $2^{2m}-1$) divides $|G_\alpha|$ if G_α is of type $r^{1+2m}:\mathrm{Sp}_{2m}(r)$ (respectively $(4\circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$). Using Theorem 1.5.2 and Lemma 1.5.3 together with Corollary 1.4.26, we obtain $2m \leq d-1$ (here, we note that $7 \mid |G_\alpha|$ if G_α is of type $(4\circ 2^{1+6})^{\cdot}\mathrm{Sp}_6(2)$ and we recall in the case of type $r^{1+2}:\mathrm{Sp}_2(r)$ that always $d \geq 3$). Now, in view of (3.6.2), we easily deduce

$$|G_\alpha| \leq \begin{cases} (d-1)^{\frac{d^2+d-2}{2}} \cdot (d-2) & \text{for } G_\alpha \text{ of type } r^{1+2m} : \mathrm{Sp}_{2m}(r), \\ 2^{\frac{(d-1)^2+3(d-1)}{2}+1} = 2^{\frac{d^2+d}{2}} & \text{for } G_\alpha \text{ of type } (4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2). \end{cases} \tag{3.6.4}$$

(b) Using the stronger assertion of Theorem 1.4.25, the upper bound (3.6.4) for $|G_\alpha|$ can be sharpened. By Propositions 2.6.20 and 2.6.33 and Corollary 2.6.19, we see that G_α has a composition factor group isomorphic to $\mathrm{PSp}_{2m}(r)$ (respectively $\mathrm{PSp}_{2m}(2)$) if G_α is of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ (respectively $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$), recall from Proposition 1.2.12 that $\mathrm{PSp}_{2m}(2)$ is always simple since $m \geq 3$. So, by Theorem 1.4.25 and Propositions 1.2.13 and 1.2.14, we see that r^{m^2} (respectively 2^{m^2}) divides $d!$ if G_α is of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ (respectively $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$). In view of Lemma 1.5.20, we obtain $(r-1)m^2 < d$ (respectively $m^2 < d$) if G_α is of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ (respectively $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$). Hence, by (3.6.2) and (3.6.3), we now can deduce

$$|G_\alpha| \leq \begin{cases} (d-1)^{d+3\sqrt{\frac{d-1}{2}}-1} \cdot (d-2) & \text{for } G_\alpha \text{ of type } r^{1+2m} : \mathrm{Sp}_{2m}(r), \\ 2^{2d+3\sqrt{d-1}-1} & \text{for } G_\alpha \text{ of type } (4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2). \end{cases} \tag{3.6.5}$$

By using the assertion of Theorem 1.4.25 more precisely, further results of permutation group theory and observations of Section 2.6, it is possible to determine a more precise upper bound for $|G_\alpha|$ in terms of an arbitrary non-trivial subdegree d of G than in (3.6.4) and (3.6.5). The following two propositions will lead to the intended goal of this section.

Proposition 3.6.2. *Let G_α be a strongly constrained member of A-class $\mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for an odd prime r and a positive integer m. Then the order of G_α is bounded by*

$$h^{\mathbf{L}^\epsilon}_{\mathscr{C}_6,1}(d) = \begin{cases} 0 & \text{for } d \in \{3,4,5\}, \\ d(d-1)^3(d-2)^2 & \text{for } 6 \leq d \leq 333, \\ (d-31)^{\frac{\ln(d-31)}{2\ln(3)}+3\frac{1}{2}} & \text{for } d \geq 334. \end{cases}$$

Proof. Suppose that the assertion is false and that the pair (G, G_α) is a counterexample (so, $G_\alpha \in \mathscr{C}_6$ of G of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$ for some odd prime r and some positive integer m). We recall that we use the notation introduced at the end of Section 3.1. Furthermore, in view of Definition 2.6.15, we note that V is an n-dimensional $\mathrm{GF}(q^u)$-vector space (q an appropriate prime power) where $n = r^m$.

By Proposition 2.6.20 and Corollary 2.6.19, we see that r^2 divides $|G_\alpha|$. Hence, in view of Corollary 1.4.26 and Lemma 1.4.28, we have

$$r + 1 \leq d. \tag{3.6.6}$$

First, we show that $d \notin \{3,4,5\}$. Since r is an odd prime, by (3.6.6) we can deduce that $d \neq 3$, and $r = 3$ if $d = 4$ or 5. If $m > 1$ and $r = 3$ (esp.

$n = 3^m \geq 9$) we obtain by the considerations in Remark 3.6.1 (b) that $d > 8$. So, for $d \in \{4, 5\}$ we have $n = 3$. In the case $n = 3$, considering Corollaries 1.2.22 and 2.6.19, Propositions 2.6.20 and 2.6.22 and Main Theorem 2.6.28 (cf. also Construction 2.6.25 (c)), we see that

$$2^3 \cdot 3^2 \mid |G_\alpha| \mid 2^4 \cdot 3^3 \tag{3.6.7}$$

and G_α is strongly 3-constrained. (Note, that in case $n = 3$ the prime power q is a prime by Definition 2.6.15). Since for $n = 3$ we have $5 \nmid |G_\alpha|$, we now see that $d \neq 5$. So, assume that $d = 4$. By the above considerations and regarding the table in Theorem 3.1.5, this case can also be ruled out easily (note, that $\mathrm{O}_2(G_\alpha)$ has to be trivial). So, for the rest of the proof we can assume $d \geq 6$.
Next, we consider the above mentioned case $n = r^m = 3$. Here, recalling (3.6.7), we see that $|G_\alpha| \leq 2^4 \cdot 3^3 \leq h^{\mathbf{L}^{\epsilon}}_{\mathscr{C}_6,1}(d)$, since $d \geq 6$. So, we may assume $n = r^m \geq 5$. Considering Corollary 2.6.19 and Proposition 2.6.20, we see that

$$H := G_\alpha \cap \mathrm{P}\Omega(V) \cong r^{2m} : \mathrm{Sp}_{2m}(r) \tag{3.6.8}$$

is a member of A-class $\mathscr{C}_6$ of $\mathrm{P}\Omega(V)$ of type $r^{1+2m} : \mathrm{Sp}_{2m}(r)$. Let H_0 be a subgroup of H isomorphic to $\mathrm{Sp}_{2m}(r)$. Note, that H_0 is quasisimple, since $n = r^m \geq 5$ (recall Propositions 1.2.11 and 1.2.12). The following considerations are similar to the proof of Theorem 3.3.5. From our chosen notation (at the end of Section 3.1) we recall that $d = |O(\alpha)|$ for an orbital O of G ($\alpha \in \mathrm{X}$) and we recall $K(\alpha) = (G_\alpha)_{O(\alpha)}$. Furthermore, we recall the reduction map $\rho_{K(\alpha)} = \rho$ from (3.1.2). We note that the following is based on [Wie, proof of Theorem 18.2]. By Lemma 1.4.24, there exists an element $g \in G$ such that $H_0^g \leq G_\alpha$ and $1 \neq (H_0^g)^\rho \cong (H_0^g)^{O(\alpha)} \leq \mathrm{Sym}(O(\alpha)) \cong \mathrm{S}_d$ (note, that $(H_0^g)^\rho = H_0^g \cdot K(\alpha)/K(\alpha) \cong H_0^g/(K(\alpha) \cap H_0^g) = H_0^g/(H_0^g)_{O(\alpha)}$). Since $(H_0^g)_{O(\alpha)}$ is a proper normal subgroup of the quasisimple group H_0^g, we obtain by elementary considerations (e.g. see [As2, (31.2)]) that $(H_0^g)_{O(\alpha)} \leq \mathrm{Z}(H_0^g)$. It is well-known that $\mathbf{Z}_2 \cong \mathrm{Z}(\mathrm{Sp}_{2m}(r)) \cong \mathrm{Z}(H_0^g)$ (e.g. see Proposition 1.2.14). So, we obtain that $\mathrm{Sp}_{2m}(r)$ or $\mathrm{PSp}_{2m}(r)$ is isomorphic to a subgroup of S_d. In view of Lemma 1.4.30 (e.g. for $\pi = \{2\}$), we now can easily deduce

$$\mathrm{d}_f(\mathrm{PSp}_{2m}(r)) \leq d. \tag{3.6.9}$$

Now, using (3.6.9), we will show that there is no counterexample (G, G_α) to our assertion. Based on the values of $\mathrm{d}_f(\mathrm{PSp}_{2m}(r))$ provided in Proposition 1.4.29, we will do a case-by-case analysis with respect to the possibilities for r and m. First, let $m \geq 2$ and $(m, r) \neq (2, 3)$. By Proposition 1.4.29 and (3.6.9), we obtain

$$d \geq \frac{r^{2m} - 1}{r - 1} = r^{2m-1} + \frac{r^{2m-1} - 1}{r - 1} \geq r^{2m-1} + 31. \tag{3.6.10}$$

Hence, we can deduce

$$r \leq (d - 31)^{\frac{1}{2m-1}} \tag{3.6.11}$$

and

$$m \leq \frac{\ln(d-31)}{2\ln(3)} + \frac{1}{2}. \tag{3.6.12}$$

So, using (3.6.2), we obtain that

$$|G_\alpha| < r^{2m^2+3m+1} \overset{(3.6.11)}{\leq} (d-31)^{m+2+\frac{3}{2m-1}} \leq (d-31)^{m+3} \overset{(3.6.12)}{\leq} (d-31)^{\frac{\ln(d-31)}{2\ln(3)}+3\frac{1}{2}}.$$

Hence, $|G_\alpha| \leq h^{\mathbf{L}^\epsilon}_{\mathscr{C}_6,1}(d)$; note, that $d \geq 156$ in the actual case (recall (3.6.10)).
Next, consider the case $(m, r) = (2, 3)$. (Here, note that the prime power q is a prime by Definition 2.6.15). By Proposition 1.4.29 and (3.6.9), we have $d \geq 27$. Hence, in view of Corollaries 1.2.22 and 2.6.19 and Propositions 1.2.13, 2.6.20 and 2.6.22 (recall also Main Theorem 2.6.28), we see that $|G_\alpha| \leq 2^8 \cdot 3^8 \cdot 5 \leq h^{\mathbf{L}^\epsilon}_{\mathscr{C}_6,1}(d)$.
So, it remains to consider the case $m = 1$ (here, recall that $n = r > 3$). For this case it is advantageous to use the estimate in (3.6.6) (note, that this estimate coincides with the estimate which follows by Proposition 1.4.29 and (3.6.9) if $r \geq 13$). In the actual case, regarding again Corollaries 1.2.22 and 2.6.19, Remark 2.6.16 (c) and Propositions 2.6.20 and 2.6.22 (recall also Main Theorem 2.6.28), we have

$$|G_\alpha| \leq r^3(r+1)(r-1)^2 \overset{(3.6.6)}{\leq} d(d-1)^3(d-2)^2 \leq h^{\mathbf{L}^\epsilon}_{\mathscr{C}_6,1}(d).$$

So, because no counterexample exists, we have established our assertion. □

Remark 3.6.3. We note that more precise upper bounds for $|G_\alpha|$ can be determined in the previous proposition, by using (more intensively) the methods used in its proof; such as by further case-by-case analysis with respect to the possibilities for r and m, or by considering the situation for further fixed subdegrees d separately.

Proposition 3.6.4. *Let G_α be a strongly constrained member of A-class $\mathscr{C}_6$ of G of type $2^{1+2}_- : \mathrm{O}_2^-(2)$ or $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ for an integer $m \geq 2$. Then the following hold.*

(i) If $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ the order of G_α is bounded by

$$h^{\mathbf{L}}_{\mathscr{C}_6,2}(d) = \begin{cases} 2^3 \cdot 3 & \text{for } d \in \{3,4,6\}, \\ 0 & \text{for } d \in \{5,7\}, \\ 2^9 \cdot 3^2 \cdot 5 & \text{for } 8 \leq d \leq 27, \\ 2^{\frac{1}{2}\cdot\left(\frac{\ln(d-12)}{\ln(2)}\right)^2 + \frac{7}{2}\cdot\frac{\ln(d-12)}{\ln(2)}+6} & \text{for } d \geq 28. \end{cases}$$

(ii) If $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ the order of G_α is bounded by

$$h^{\mathbf{U}}_{\mathscr{C}_6,2}(d) = \begin{cases} 0 & \text{for } 3 \leq d \leq 7, \\ 2^9 \cdot 3^2 \cdot 5 & \text{for } 8 \leq d \leq 27, \\ 2^{\frac{1}{2}\cdot\left(\frac{\ln(d-12)}{\ln(2)}\right)^2 + \frac{7}{2}\cdot\frac{\ln(d-12)}{\ln(2)}+6} & \text{for } d \geq 28. \end{cases}$$

Proof. Suppose that the pair (G, G_α) is a counterexample to our assertion. One after another, we will now consider the different possibilities for G_α. First, let $G_\alpha \in \mathscr{C}_6$ of G of type $2_-^{1+2} : \mathrm{O}_2^-(2)$. Recall from Definition 2.6.15 that this case only occurs in case $\mathbf{L}$ for $n = \dim(V) = 2$. In view of Main Theorem 2.6.31, we see that $|G_\alpha|$ divides $2^3 \cdot 3$. Since 5 and 7 do not divide $|G_\alpha|$, we have $d \neq 5, 7$. So, we see that no counterexample to our assertion exists in the actual case, since $|G_\alpha| \leq h_{\mathscr{C}_6,2}^{\mathbf{L}}(d)$.[9]
Now, we consider the case that $G_\alpha \in \mathscr{C}_6$ of G of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$ for an integer $m \geq 2$. We recall that we use the notation introduced at the end of Section 3.1. Furthermore, in view of Definition 2.6.15, we note that V is an n-dimensional $\mathrm{GF}(q^u)$-vector space where q is an odd prime and $n = 2^m \geq 4$. It is advantageous to consider the case $m = 2$ separately, so let $m = 2$. Here, in view of Corollaries 1.2.22 and 2.6.19 and Propositions 1.2.12 and 2.6.33 we see that G_α is insoluble, and there is precisely one non-abelian composition factor group of G_α, which is isomorphic to A_6. From Theorem 1.4.25 we now can deduce that $|A_6| \bigm| d!$, hence $d \geq 6$. Suppose that $d = 6$. Again regarding Theorem 1.4.25, we see that $G_\alpha^{O(\alpha)}$ is isomorphic to A_6 or S_6. It is easy to see that G_α does not act faithfully on $O(\alpha)$ (recall Corollary 2.6.19 and Proposition 2.6.33). So, by Theorem 3.1.12, we obtain that G_α is isomorphic to $\mathrm{A}_6 \times \mathrm{A}_5$, $\mathrm{S}_6 \times \mathrm{S}_5$, $\mathrm{S}_6 \times \mathrm{A}_5$ or $(\mathrm{A}_6 \times \mathrm{A}_5) : \mathbf{Z}_2$. But this contradicts the fact that G_α has precisely one non-abelian composition factor group. So, we may assume $d \geq 7$. In view of Corollaries 1.2.22 and 2.6.19 and Propositions 2.6.33 and 2.6.35 (recall also Main Theorem 2.6.40), we obtain that

$$|G_\alpha| \bigm| 2^9 \cdot 3^2 \cdot 5.$$

Since $7 \nmid |G_\alpha|$, we can deduce that $d \geq 8$. So, no counterexample to our assertion exists in the actual case, because $|G_\alpha| \leq 2^9 \cdot 3^2 \cdot 5 \leq h_{\mathscr{C}_6,2}^{\mathbf{L}^\epsilon}(d)$.
Next, consider the case $m \geq 3$. Here, in view of Corollary 2.6.19 and Proposition 2.6.33, we see that

$$H := G_\alpha \cap \mathrm{P}\Omega(V) \cong 2^{2m}{}^{\cdot}\mathrm{Sp}_{2m}(2) \tag{3.6.13}$$

is a member of A-class $\mathscr{C}_6$ of $\mathrm{P}\Omega(V)$ of type $(4 \circ 2^{1+2m})^{\cdot}\mathrm{Sp}_{2m}(2)$. For our further considerations, we note the following. Since $m \geq 3$, $\mathrm{Sp}_{2m}(2) = \mathrm{PSp}_{2m}(2)$ is simple (recall Proposition 1.2.12). In view of Corollary 2.6.12, Table 2.6.1, Lemma 2.6.17 and Propositions 2.6.33 and 2.6.37 (recall also Remark 2.6.34), we see that $\mathrm{O}_2(H) = \mathrm{P}R \cong 2^{2m}$ is a minimal normal subgroup of H where R is a suitable group as described in Table 2.6.2. Next, we will argue similarly as in the proof of Proposition 3.6.2 (but recall from (3.6.13) that here H is a non-split extension of $\mathrm{O}_2(H) = \mathrm{P}R$). Again, we recall the notations $d = |O(\alpha)|$ (O an orbital of G), $K(\alpha) = (G_\alpha)_{O(\alpha)}$ and the reduction map $\rho_{K(\alpha)} = \rho$ from our chosen notation (see the end of Section 3.1). We note that the following is based

[9] Concerning the cases $d \in \{3, 4\}$ in the actually considered case, see also [Wo, p. 237 (7) and Lemma 3] and [LLM, Theorem 3.4] together with the proof of Main Theorem 2.6.31 and Remark 2.6.32.

on [Wie, proof of Theorem 18.2]. In view of Lemma 1.4.24, there is an element $g \in G$ with $H^g \leq G_\alpha$ and $1 \neq (H^g)^\rho \cong (H^g)^{O(\alpha)} \leq \mathrm{Sym}(O(\alpha)) \cong \mathrm{S}_d$. Clearly, $(H^g)^\rho \cong H^g/(K(\alpha) \cap H^g)$ and $K(\alpha) \cap H^g = H^g_{O(\alpha)}$ is a proper normal subgroup of H^g. Since $(\mathrm{P}R)^g$ is a minimal normal subgroup of H^g, we can deduce that $H^g_{O(\alpha)} \cap (\mathrm{P}R)^g = (\mathrm{P}R)^g$ or 1. In the case $H^g_{O(\alpha)} \cap (\mathrm{P}R)^g = (\mathrm{P}R)^g$, by the above considerations, we easily see that $H^g_{O(\alpha)} = (\mathrm{P}R)^g$. Hence, here $\mathrm{Sp}_{2m}(2)$ is isomorphic to a subgroup of S_d. In the other case $H^g_{O(\alpha)} \cap (\mathrm{P}R)^g = 1$, $H^g_{O(\alpha)}$ centralizes $(\mathrm{P}R)^g$. We have $\mathrm{C}_{H^g}((\mathrm{P}R)^g) = \mathrm{C}_{\mathrm{P}\Omega(V)}((\mathrm{P}R)^g) \cap H^g = (\mathrm{P}R)^g$, in view of Corollary 2.6.41. Hence, in this case we easily can deduce that $H^g_{O(\alpha)} = 1$, so H^g is isomorphic to a subgroup of S_d. Summarizing, we have $\mathrm{Sp}_{2m}(2)$ or $2^{2m} {}^{\cdot}\mathrm{Sp}_{2m}(2)$ is isomorphic to a subgroup of S_d. In view of Lemma 1.4.30 (e.g. for $\pi = \{2\}$) and Proposition 1.4.29, we now can easily deduce

$$2^{m-1}(2^m - 1) = \mathrm{d}_f(\mathrm{Sp}_{2m}(2)) \leq d. \tag{3.6.14}$$

Since $2^m - 1 \geq 2^{m-1} + 3$, by (3.6.14) we obtain $d \geq 2^{m-1}(2^m - 1) \geq 2^{2m-2} + 12$, so $m \leq \frac{\ln(d-12)}{2\ln(2)} + 1$. Hence, in view of (3.6.2), we see that also no counterexample to our assertion exists in the actual case, since

$$|G_\alpha| \leq 2^{2m^2+3m+1} \leq 2^{\frac{1}{2}\cdot\left(\frac{\ln(d-12)}{\ln(2)}\right)^2 + \frac{7}{2}\cdot\frac{\ln(d-12)}{\ln(2)} + 6} \leq h^{\mathbf{L}^\epsilon}_{\mathscr{C}_6,2}(d).$$

(Here, note that by (3.6.14) we have $d \geq 28$). So, we have established our assertion. □

Remark 3.6.5. (a) By using (more intensively) the methods used in the proof of the previous proposition, it is possible to determine more precise upper bounds for $|G_\alpha|$; such as by further case-by-case analysis with respect to m (respectively $n = 2^m$), or by considering the situation for further fixed subdegrees d separately.

(b) By the previous proof, it is not hard to see that in the case $G_\alpha \in \mathscr{C}_6$ of G of type $(4 \circ 2^{1+2m}){}^{\cdot}\mathrm{Sp}_{2m}(2)$ for $m \geq 3$ it is also possible to state an upper bound $h(d)$ for $|G_\alpha|$ in terms of d by claiming $h(d) = 2^{2t+1} \cdot |\mathrm{Sp}_{2t}(2)|$ if $2^{t-1}(2^t - 1) \leq d < 2^t(2^{t+1} - 1)$ for integers $t \geq 3$. The author has decided to state an upper bound for $|G_\alpha|$ as done in Proposition 3.6.4 for conformity with the result in Proposition 3.6.2 and to formulate an upper bound which is easy to handle (cf. also Theorem 3.6.6, below).

Summarizing the results of the previous two propositions, we obtain the following theorem in which we determine the last two intended Sims order bounds $h^{\mathbf{L}}_{\mathscr{C}_6}(d)$ and $h^{\mathbf{U}}_{\mathscr{C}_6}(d)$ for $\mathscr{G}^{\mathbf{L}}_{\mathscr{C}_6}$ and $\mathscr{G}^{\mathbf{U}}_{\mathscr{C}_6}$ in this chapter.

Theorem 3.6.6. *Let G_α be a strongly constrained member of A-class $\mathscr{C}_6$ of G. Then the following hold.*

(i) If $\mathrm{soc}(G) \cong \mathrm{PSL}_n(q)$ the order of G_α is bounded by

$$h_{\mathscr{C}_6}^{\mathbf{L}}(d) = \begin{cases} 2^3 \cdot 3 & \text{for } 3 \leq d \leq 4, \\ 0 & \text{for } d = 5, \\ d(d-1)^3(d-2)^2 & \text{for } 6 \leq d \leq 29, \\ 2^{\frac{1}{2}\cdot\left(\frac{\ln(d-12)}{\ln(2)}\right)^2 + \frac{7}{2}\cdot\frac{\ln(d-12)}{\ln(2)} + 6} & \text{for } d \geq 30. \end{cases}$$

(ii) If $\mathrm{soc}(G) \cong \mathrm{PSU}_n(q^2)$ the order of G_α is bounded by

$$h_{\mathscr{C}_6}^{\mathbf{U}}(d) = \begin{cases} 0 & \text{for } 3 \leq d \leq 5, \\ d(d-1)^3(d-2)^2 & \text{for } 6 \leq d \leq 29, \\ 2^{\frac{1}{2}\cdot\left(\frac{\ln(d-12)}{\ln(2)}\right)^2 + \frac{7}{2}\cdot\frac{\ln(d-12)}{\ln(2)} + 6} & \text{for } d \geq 30. \end{cases}$$

In particular, $h_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}(d)$ is a Sims order bound for $\mathscr{G}_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}$ and $h_{\mathscr{C}_6}^{\mathbf{L}^\epsilon}(d) \leq \mathrm{wdt}(d)$.

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